Basics of Quantum Error Correction

Part 1

Correcting errors with the Shor code

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The need for error correction

Quantum computers are highly susceptible to errors:

- Unwanted interactions with the environment cause disturbances, including *decoherence.*
- Quantum operations can only be implemented with *limited accuracy.*

Classical error correction has many uses and applications — but isn't really essential for classical computation today.

In contrast, it is widely believed that error correction will be essential for large-scale quantum computing.

Classical repetition codes

Repetition codes are very basic examples of error correcting codes. The idea is simply to repeat each bit multiple times.

3-bit repetition code

Encoding	Decoding
$0 \mapsto 000$ $1 \mapsto 111$	$abc \mapsto \text{majority}(a, b, c)$

This code corrects up to one *bit flip* on any of the three bits used for encoding.

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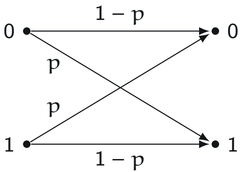
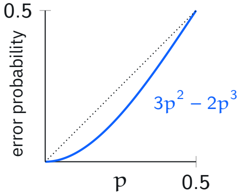
This code corrects up to one *bit flip* on any of the three bits used for encoding.

Suppose each bit is sent through a *binary symmetric channel* that flips a bit with probability p.



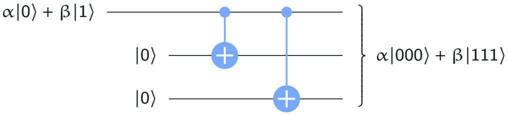


Repetition code for qubits

The 3-bit repetition code can be used to encode a qubit:

$$\alpha|0\rangle + \beta|1\rangle \mapsto \alpha|000\rangle + \beta|111\rangle$$

This circuit performs the encoding:



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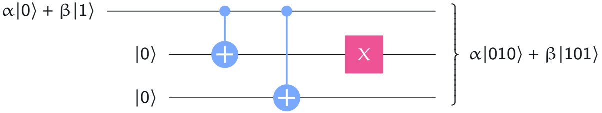
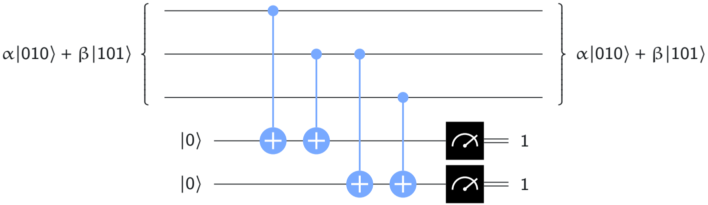


We can identify the location of a single bit-flip with this circuit:



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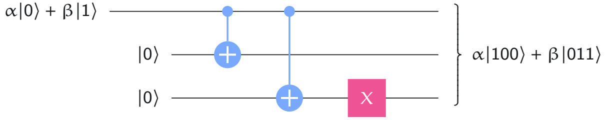

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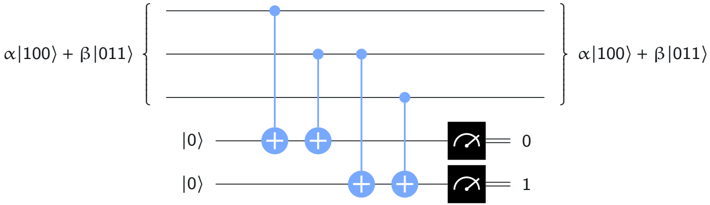

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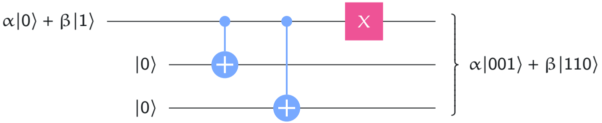

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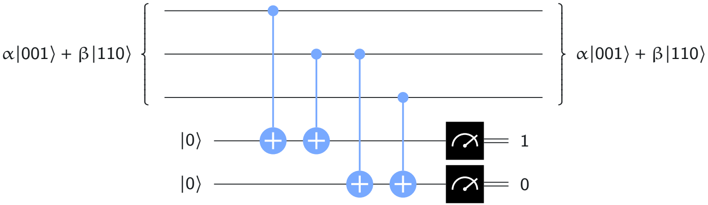

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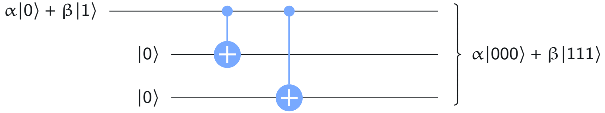

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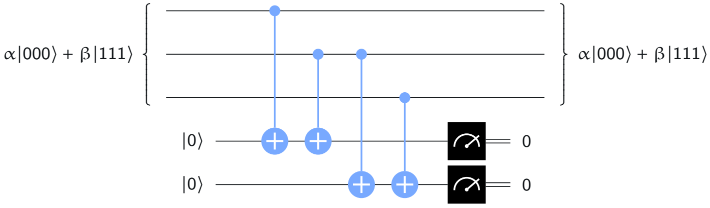

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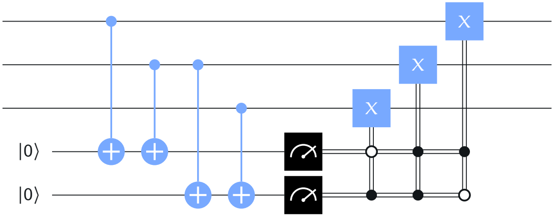
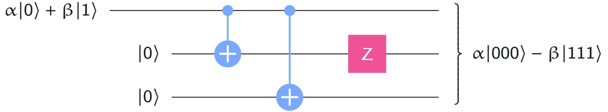


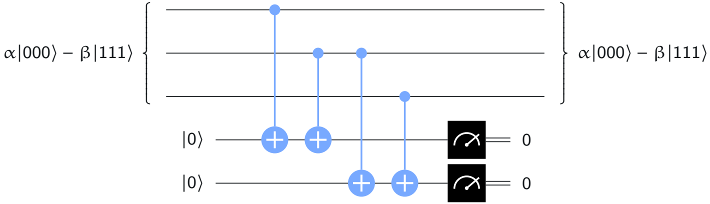
State	Syndrome	Correction
$\alpha\|000\rangle + \beta\|111\rangle$	00	$\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$
$\alpha\|100\rangle + \beta\|011\rangle$	10	$X \otimes \mathbb{1} \otimes \mathbb{1}$
$\alpha\|010\rangle + \beta\|101\rangle$	11	$\mathbb{1} \otimes X \otimes \mathbb{1}$
$\alpha\|001\rangle + \beta\|110\rangle$	01	$\mathbb{1} \otimes \mathbb{1} \otimes X$

Phase-flip errors

Bit-flip errors aren't the only quantum errors we need to worry about. For instance, we also have *phase-flip errors,* which are described by Z gates.

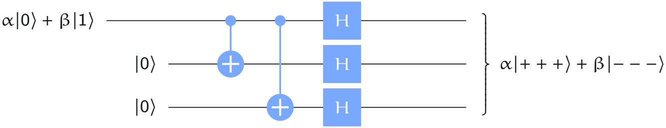
Unfortunately, the 3-bit repetition code fails to detect phase-flip errors.



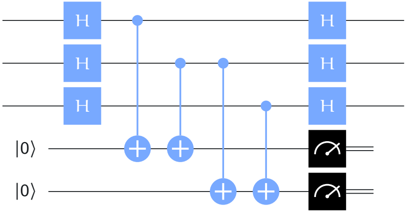


Correcting phase-flip errors

A modified version of the 3-bit repetition code allows for a correction of phase-flip errors.



Modifying the error detection circuit allows for the location of a phase-flip error.



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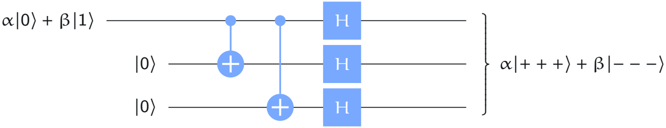

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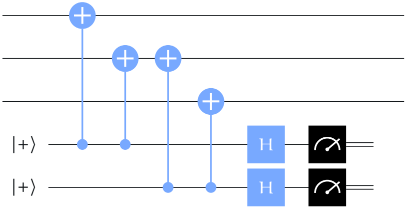

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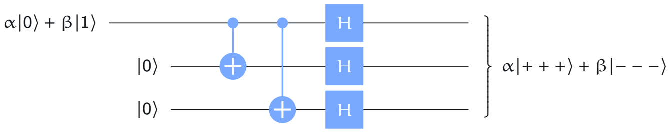

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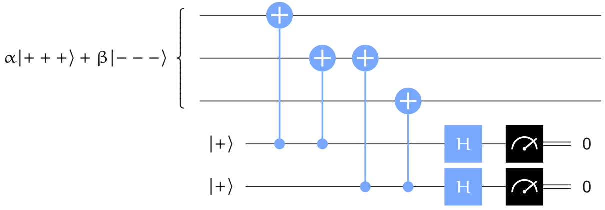

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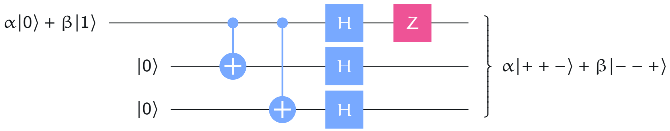

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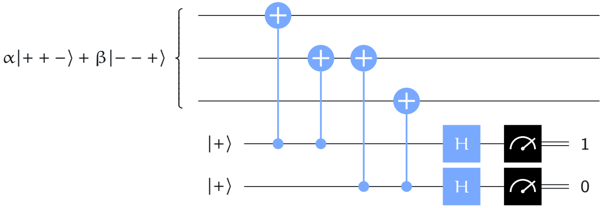
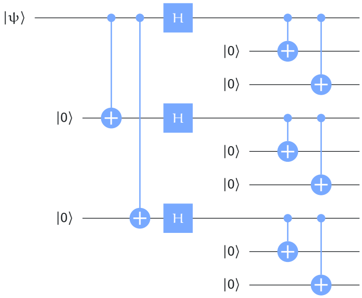


Unfortunately this code fails to detect bit-flip errors.

9-qubit Shor code

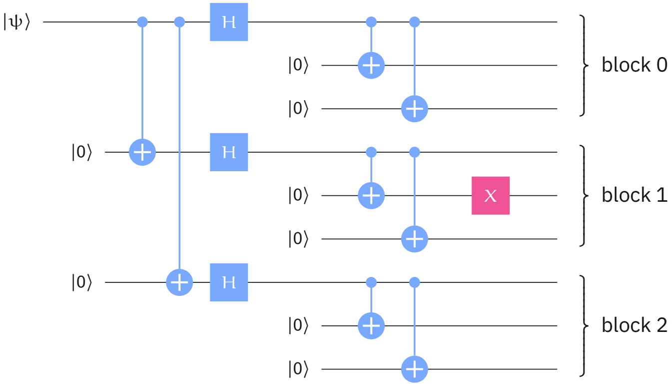
This is the *concatenation* of the 3 qubit phase-flip and bit-flip repetition codes.



$$|0\rangle \;\mapsto\; \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$$

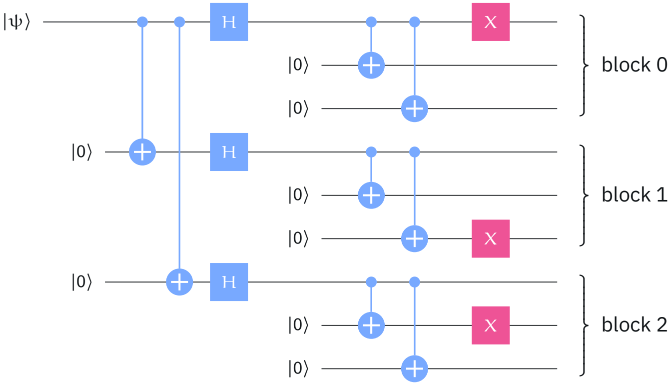
$$|1\rangle \;\mapsto\; \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)$$

Correcting bit-flip errors



Bit-flip errors can be detected/corrected independently on each block by means of the *inner code* (the ordinary 3-bit repetition code).

Correcting bit-flip errors



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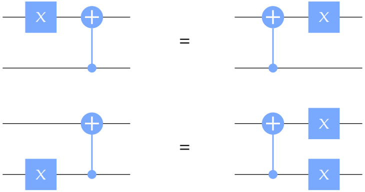
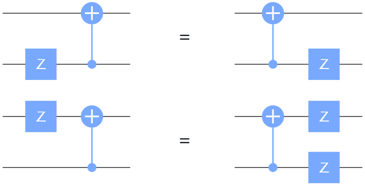
Errors and CNOTs

X and CNOT relations



Z and CNOT relations



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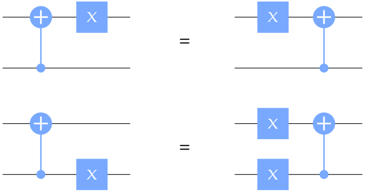

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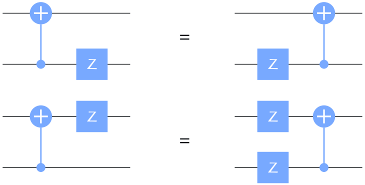

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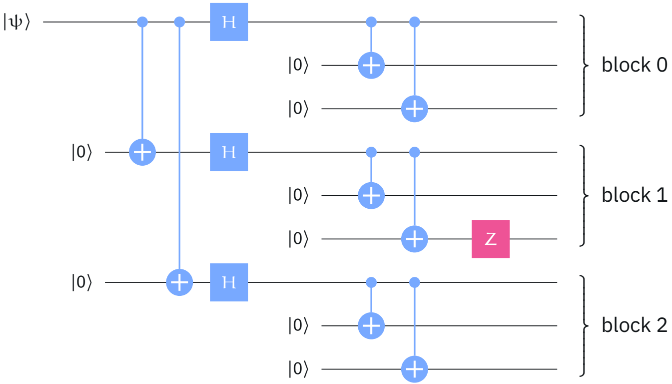

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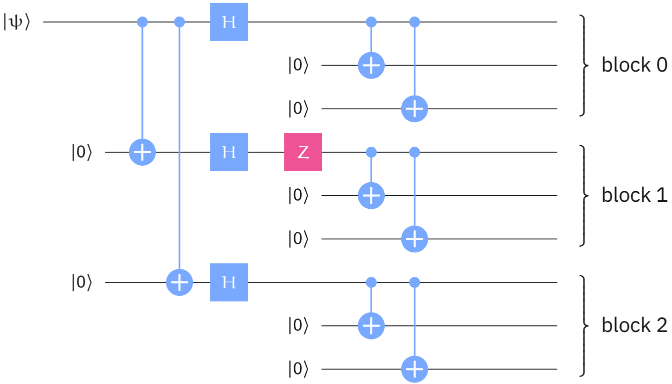
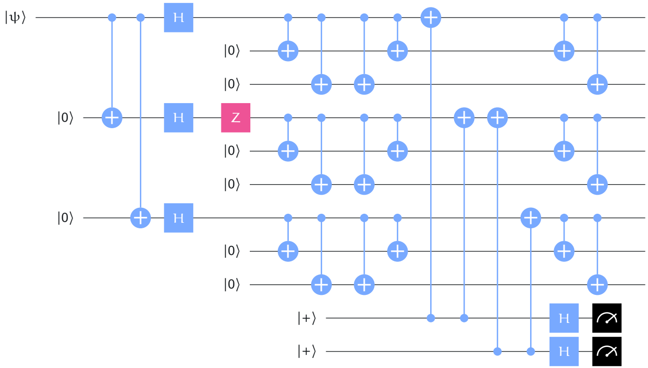


Phase-flip errors within each block have the same effect as phase-flip errors prior to the inner encoding.

Correcting phase-flip errors



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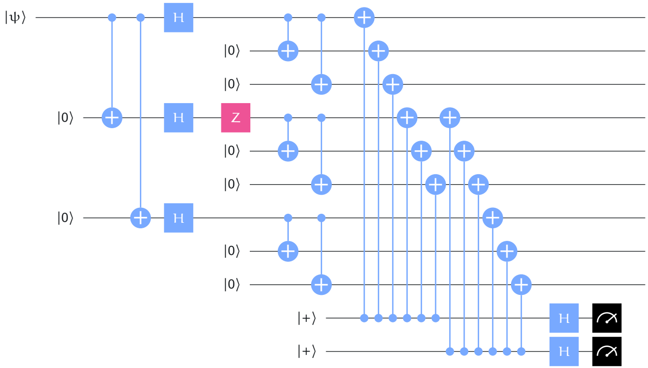

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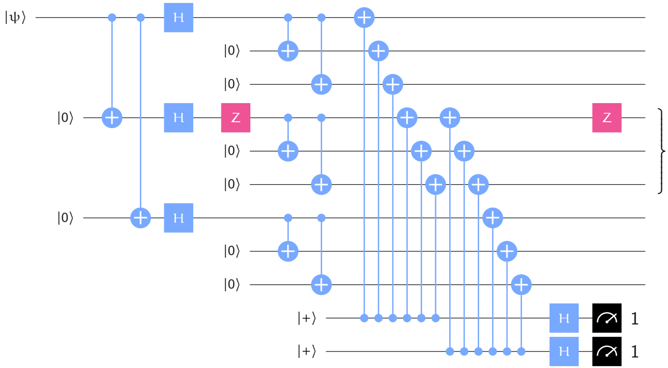
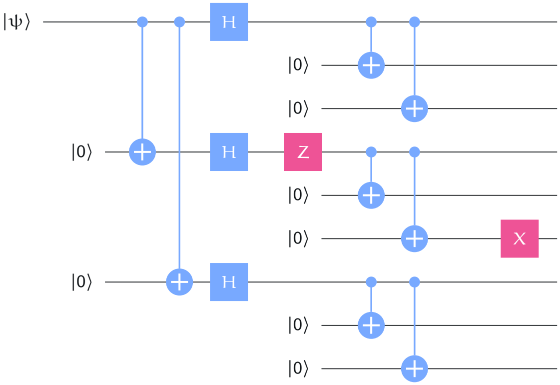


If a Z-error has occured, the syndrome indicates *which block* it occurred on. It can be corrected by applying a Z gate to *any qubit* within that block.

Correcting bit- and phase-flips



Bit-flip and phase-flip errors can be detected and corrected *completely independently.*

Random errors

A simple noise model

Errors occur *independently* on qubits. For each qubit, an error (X, Y, or Z) occurs with probability p, otherwise the qubit is unaffected.

Suppose Q is a qubit that we wish to protect against errors — and imagine we have the option to use the 9-qubit Shor code. Should we use it?

The Shor code corrects any Pauli error on a single qubit. The probability of successfully protecting Q against Pauli errors using the code is therefore as follows.

$$\mathrm{Pr}(\text{no errors}) + \mathrm{Pr}(\text{one error}) = (1-p)^9 + 9p(1-p)^8$$

Without the code, Q is unaffected with probability $1-p$. The code helps if

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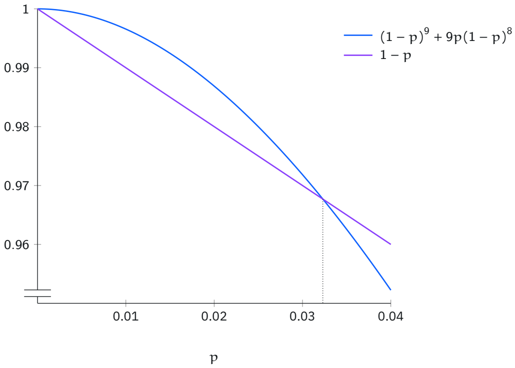

Unitary errors

Suppose that we encode one qubit into 9 using the 9-qubit Shor code, and a *unitary error* U occurs on one of the qubits.

We can express U as a linear combination of Pauli matrices (including the identity).

$$U = \alpha\mathbb{1} + \beta X + \gamma Y + \delta Z$$

Notation: write U_k to denote U applied to qubit k (and likewise for X, Y, and Z).

Example

Using Qiskit's numbering convention $(Q_8, Q_7, \ldots, Q_0)$, we have these expressions:

$$\begin{aligned}
X_0 &= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes X \\
Z_4 &= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes Z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \\
U_7 &= \mathbb{1} \otimes U \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}
\end{aligned}$$

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$$U_k = \alpha \mathbb{1} + \beta X_k + \gamma Y_k + \delta Z_k$$

Suppose $|\psi\rangle$ is the 9-qubit *encoding* of a qubit state. Applying the error U to qubit k has this action:

$$|\psi\rangle \xmapsto{\text{error}} U_k|\psi\rangle = \alpha|\psi\rangle + \beta X_k|\psi\rangle + \gamma Y_k|\psi\rangle + \delta Z_k|\psi\rangle$$

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$$|\psi\rangle \xmapsto{\text{error}} U_k|\psi\rangle = \alpha|\psi\rangle + \beta X_k|\psi\rangle + i\gamma X_k Z_k|\psi\rangle + \delta Z_k|\psi\rangle$$

Computing the syndrome yields this state:

$$\begin{gathered}\alpha|\mathbb{1}\text{ syndrome}\rangle \otimes |\psi\rangle \\ +\beta|X_k\text{ syndrome}\rangle \otimes X_k|\psi\rangle \\ +i\gamma|X_k Z_k\text{ syndrome}\rangle \otimes X_k Z_k|\psi\rangle \\ +\delta|Z_k\text{ syndrome}\rangle \otimes Z_k|\psi\rangle\end{gathered}$$

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Measuring the syndrome and correcting X and Z errors yields this state:

$$\xi\otimes|\psi\rangle\langle\psi|$$

$$\begin{aligned}\xi = {} & |\alpha|^2\,|\mathbb{1}\text{ syndrome}\rangle\langle\mathbb{1}\text{ syndrome}|\\ & +|\beta|^2\,|X_k\text{ syndrome}\rangle\langle X_k\text{ syndrome}|\\ & +|\gamma|^2\,|X_kZ_k\text{ syndrome}\rangle\langle X_kZ_k\text{ syndrome}|\\ & +|\delta|^2\,|Z_k\text{ syndrome}\rangle\langle Z_k\text{ syndrome}|\end{aligned}$$

Arbitrary errors

Suppose that we encode one qubit into 9 using the 9-qubit Shor code, and an *arbitrary error* — represented by a qubit channel Φ — occurs on one of the qubits.

Consider any *Kraus representation* of Φ.

$$\Phi(\sigma) = \sum_j A_j \sigma A_j^\dagger$$

Each Kraus matrix can be written as a linear combination of Pauli matrices.

$$A_j = \alpha_j \mathbb{1} + \beta_j X + \gamma_j Y + \delta_j Z$$

We can express the action of Φ on qubit k as follows.

$$\Phi_k(|\psi\rangle\langle\psi|) = \sum_j (\alpha_j \mathbb{1} + \beta_j X_k + \gamma_j Y_k + \delta_j Z_k)\,|\psi\rangle\langle\psi|\,(\alpha_j \mathbb{1} + \beta_j X_k + \gamma_j Y_k + \delta_j Z_k)^\dagger$$

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Computing and measuring the syndrome, followed by correcting Pauli errors, yields a state as follows.

$$\xi \otimes |\psi\rangle\langle\psi|$$

$$\begin{aligned}\xi = \sum_j \Big(& |\alpha_j|^2\, |\mathbb{1} \text{ syndrome}\rangle\langle \mathbb{1} \text{ syndrome}| \\ & + |\beta_j|^2\, |X_k \text{ syndrome}\rangle\langle X_k \text{ syndrome}| \\ & + |\gamma_j|^2\, |X_k Z_k \text{ syndrome}\rangle\langle X_k Z_k \text{ syndrome}| \\ & + |\delta_j|^2\, |Z_k \text{ syndrome}\rangle\langle Z_k \text{ syndrome}| \Big)\end{aligned}$$