# Theory of Machines and Languages

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# Closure Properties of Regular Languages

- **Example**
  - Show that the family of regular languages is closed under difference

$$L_1 - L_2 = L_1 \cap \overline{L_2}$$

$L_1$ and $L_2$ are regular $\Rightarrow$ $\overline{L_2}$ and $L_1 \cap \overline{L_2}$ are regular $\Rightarrow$ $L_1 - L_2$ *is regular*

- **Theorem**
  - The family of regular languages is closed under reversal

# Closure Properties of Regular Languages

- Suppose $\Sigma$ and $\Gamma$ are alphabets. Then a function

$$h : \Sigma \to \Gamma^*$$

is called a **homomorphism**. In words, a homomorphism is a substitution in which a single letter is replaced with a string. The domain of the function $h$ is extended to strings in an obvious fashion; if

$$w = a_1 a_2 \cdots a_n,$$

then

$$h(w) = h(a_1) h(a_2) \cdots h(a_n).$$

If $L$ is a language on $\Sigma$, then its **homomorphic image** is defined as

$$h(L) = \{h(w) : w \in L\}.$$

# Closure Properties of Regular Languages

- **Example**

Let $\Sigma = \{a, b\}$ and $\Gamma = \{a, b, c\}$ and define $h$ by

$$\begin{aligned} h(a) &= ab, \\ h(b) &= bbc. \end{aligned}$$

Then $h(aba) = abbbcab$. The homomorphic image of $L = \{aa, aba\}$ is the language $h(L) = \{abab, abbbcab\}$.

- **Example**

Take $\Sigma = \{a, b\}$ and $\Gamma = \{b, c, d\}$. Define $h$ by

$$\begin{aligned} h(a) &= dbcc, \\ h(b) &= bdc. \end{aligned}$$

$\Rightarrow$

If $L$ is the regular language denoted by

$$r = (a + b^*)(aa)^*,$$

then

$$r_1 = (dbcc + (bdc)^*)(dbccdbcc)^*$$

denotes the regular language $h(L)$.

# Closure Properties of Regular Languages

- **Theorem**
  - **The family of regular languages is closed under arbitrary homomorphisms**

# Closure Properties of Regular Languages

- Let $L_1$ and $L_2$ be languages on the same alphabet. Then the **right quotient** of $L_1$ with $L_2$ is defined as

$$L_1/L_2 = \{x : xy \in L_1 \text{ for some } y \in L_2\}.$$

- We take all the strings in $L_1$ that have a suffix belonging to $L_2$
- Every such string, after removal of this suffix, belongs to $L_1/L_2$

- **Example**

If

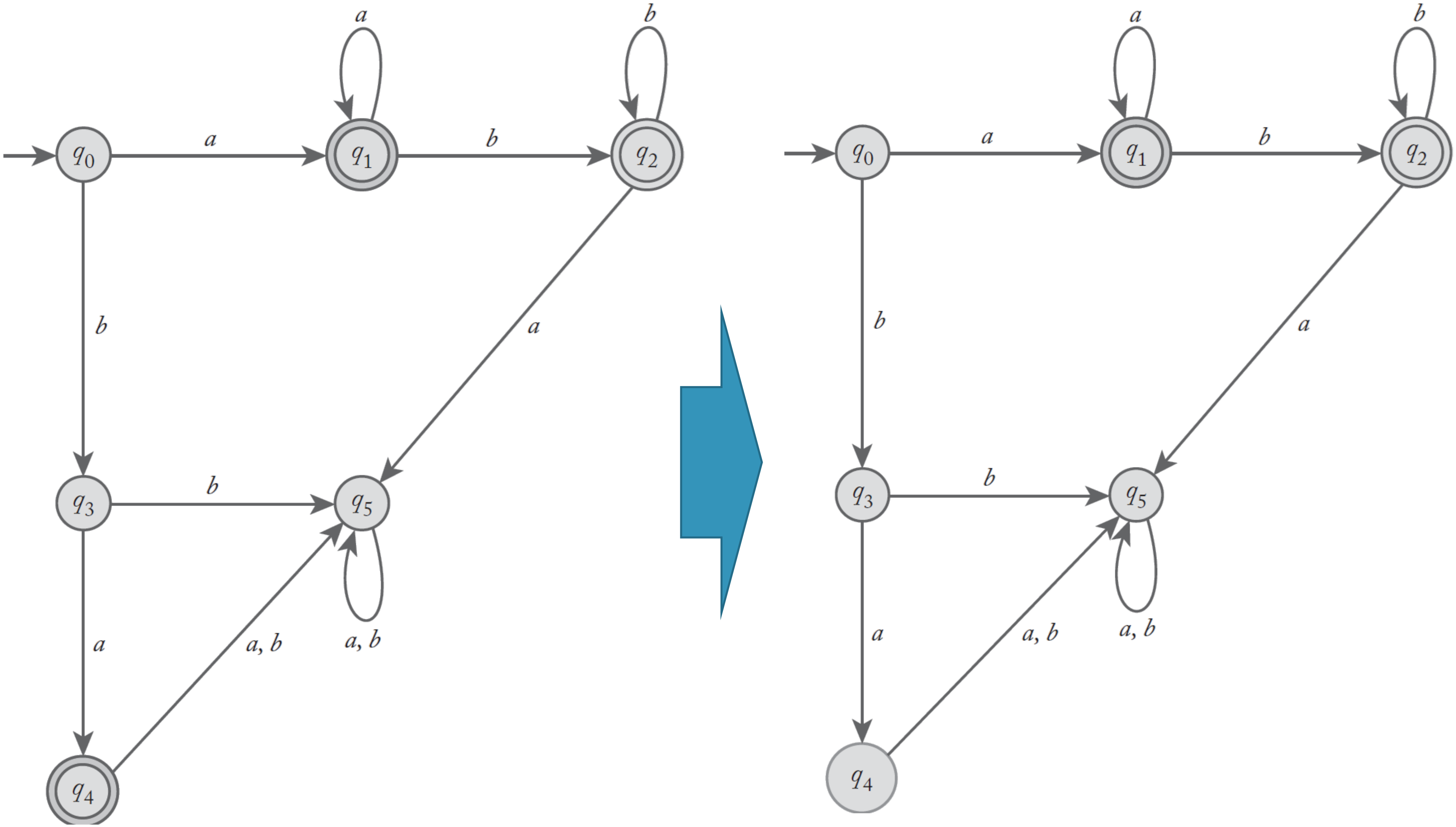
$$L_1 = \{a^n b^m : n \geq 1, m \geq 0\} \cup \{ba\}$$

and

$$L_2 = \{b^m : m \geq 1\},$$

then

$$L_1/L_2 = \{a^n b^m : n \geq 1, m \geq 0\}.$$


q0
q1
q2
q3
q4
q5
a
b
a, b

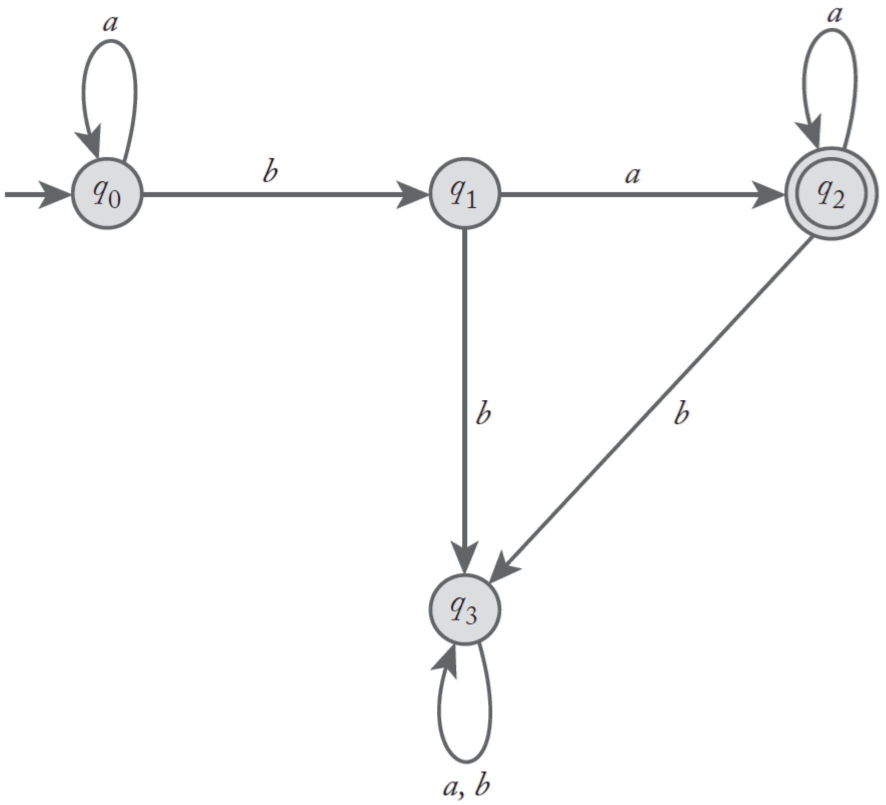
# Closure Properties of Regular Languages

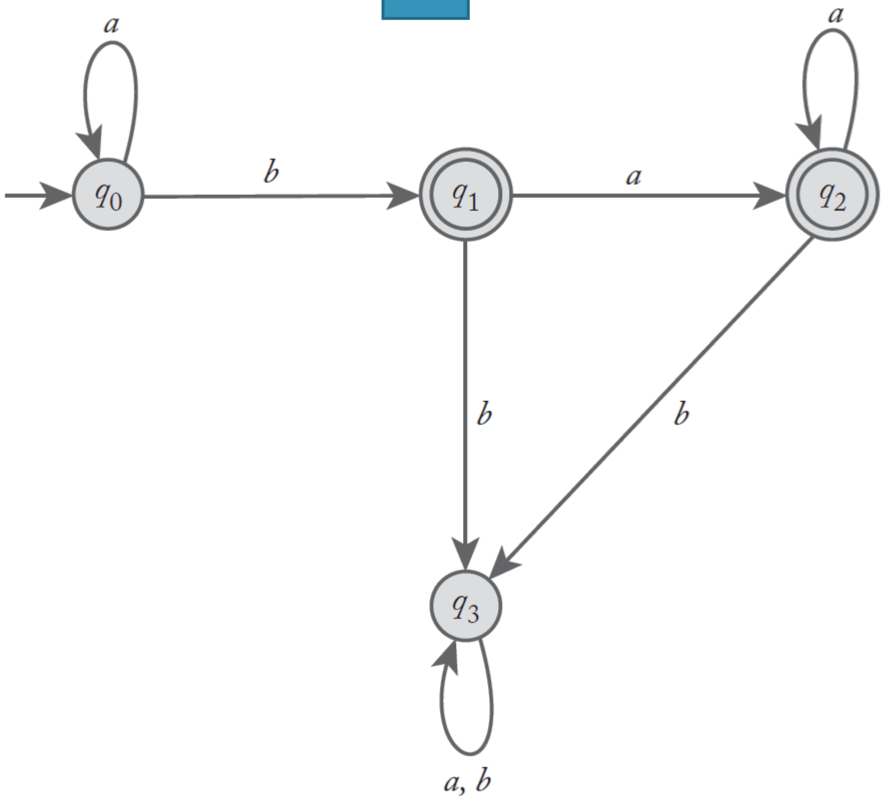
- **Example:** **Find $L_1/L_2$ for**

$$L_1 = L(a^*baa^*),$$

$$L_2 = L(ab^*).$$

$$\boldsymbol{L_1/L_2 = L(a^*ba^*)}$$



# Closure Properties of Regular Languages

- **Theorem**

  If $L_1$ and $L_2$ are regular languages, then $L_1/L_2$ is also regular. We say that the family of regular languages is closed under right quotient with a regular language.

# Closure Properties of Regular Languages

- **Exercise**

The *symmetric difference* of two sets $S_1$ and $S_2$ is defined as

$$S_1 \ominus S_2 = \{x : x \in S_1 \text{ or } x \in S_2, \text{ but } x \text{ is not in both } S_1 \text{ and } S_2\}.$$

Show that the family of regular languages is closed under symmetric difference.

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The *nor* of two languages is

$$nor\,(L_1, L_2) = \{w : w \notin L_1 \text{ and } w \notin L_2\}.$$

Show that the family of regular languages is closed under the *nor* operation.

# Identifying Nonregular Languages

- **Pigeonhole Principle**
  - If we put $n$ objects into $m$ boxes (pigeonholes), and if $n > m$, then at least one box must have more than one item in it

- **Example**
  - Is the language $L = \{a^n b^n \colon n \geq 0\}$ regular?
  - Suppose $L$ is regular, then some dfa $M = (Q, \{a, b\}, \delta, q_0, F)$ exists for it
  - The pigeonhole principle tells us that there must be some state $q$ such that

$$\delta^*(q_0, a^n) = q \quad \text{and} \quad \delta^*(q_0, a^m) = q \quad \text{with } n \neq m$$

  - Since $M$ accepts $a^n b^n$ we must have

$$\delta^*(q, b^n) = q_f \in F$$

  - We can conclude that

$$\delta^*(q_0, a^m b^n) = q_f$$

# Identifying Nonregular Languages

- **A Pumping Lemma**
  - The pumping lemma for regular languages, uses the pigeonhole principle in another form
  - The proof is based on the observation that in a transition graph with $n$ vertices, any walk of length $n$ or longer must repeat some vertex, that is, contain a cycle