بسم اللّه الرّحمن الرّحیم

دانشگاه صنعتی اصفهان ــ دانشکدهٔ مهندسی برق و کامپیوتر
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نظریهٔ زبان‌ها و ماشین‌ها

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Part 1: Removing Useless Productions

One invariably wants to remove productions from a grammar that can never take part in any derivation.

For example, in the grammar whose entire production set is

$$S \to aSb|\varepsilon|A, \quad A \to aA,$$

the production $S \to A$ clearly plays no role, as A cannot be transformed into a terminal string. While A can occur in a string derived from S, this can never lead to a sentence. Removing this production leaves the language unaffected and is a simplification by any definition.

Definition: ***Let $G = (V, \Sigma, R, S)$ be a context-free grammar. A variable $A \in V$ is said to be useful if and only if there is at least one $w \in L(G)$ such that***

$$S \Rightarrow^* xAy \Rightarrow^* w,$$

with x, y in $(V \cup \Sigma)^*$. In words, a variable is useful if and only if it occurs in at least one derivation. A variable that is not useful is called useless. A production is useless if it involves any useless variable.

Evidently, omitting useless productions from a grammar will not change the language generated, so we may as well detect and eliminate all useless productions.

Two things a variable has to be able to do to be useful

1. *We say a variable A is generating if $A \Rightarrow^* w$ for some terminal string w.*
2. *We say a variable A is reachable if there is a derivation $S \Rightarrow^* \alpha X \beta$ for some $\alpha \in (V \cup \Sigma)^*$ and $\beta \in (V \cup \Sigma)^*$.*

Two reasons why a variable is useless: either because it cannot be reached from the start symbol or because it cannot derive a terminal string. A procedure for removing useless variables and productions is based on recognizing these two situations.

Surely a variable that is useful will be both generating and reachable. If we eliminate the variables that are not generating first, and then eliminate from the remaining grammar those variables that are not reachable, we shall, as will be proved, have only the useful symbols left.

Example 1: ***In a grammar with start symbol S and productions***

$$S \to A, \quad A \to aA|\varepsilon, \quad B \to bA,$$

the variable B is useless and so is the production $B \to bA$. Although B can derive a terminal string, there is no way we can achieve $S \Rightarrow^* xBy$.

Example 2: ***Eliminate useless symbols and productions from $G = (V, \Sigma, R, S)$, where $V = \{S, A, B, C\}$ and $\Sigma = \{a, b\}$, with R consisting of***

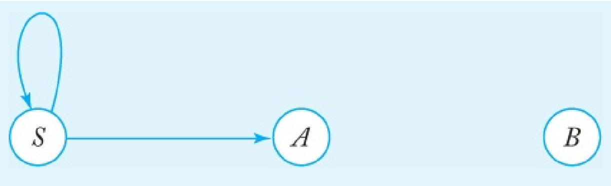
$$S \to aS|A|C,\ A \to a,\ B \to aa,\ C \to aCb.$$

First, ***we identify the set of variables that can lead to a terminal string. Because $A \to a$ and $B \to aa$, the variables A and B belong to this set. So does S, because $S \Rightarrow A \Rightarrow a$. However, this argument cannot be made for C, thus identifying it as useless. Removing C and its corresponding productions, we are led to the grammar G_1 with variables $V_1 = \{S, A, B\}$, terminals $\Sigma_1 = \{a\}$, and productions***

$$S \to aS|A,\ A \to a, B \to aa.$$

ادامهٔ مثال ۲

Next, we want to eliminate the variables that cannot be reached from the start variable. For this, we can draw a dependency graph for the variables. Dependency graphs are a way of visualizing complex relationships and are found in many applications. For CFGs, a dependency graph has its vertices labeled with variables, with an edge between vertices C and D if and only if there is a production of the form $C \to xDy$. A dependency graph for V_1 is shown in the following figure.



ادامهٔ مثال ۲

A variable is useful only if there is a path from the vertex labeled S to the vertex labeled with that variable. In our case, the figure shows that B is useless. Removing it and the affected productions and terminals, we are led to the final answer with productions

$$S \to aS|A,\ A \to a.$$

بیان روند حذف متغیرها و رول‌های بی‌فایده

Let $G = (V, \Sigma, R, S)$ be a context-free grammar. Then there exists an equivalent grammar $\widehat{G} = (\widehat{V}, \widehat{\Sigma}, \widehat{R}, S)$ that does not contain any useless variables or productions.

The grammar $\widehat{G}$ can be generated from G by an algorithm consisting of two parts.

In the first part, we construct an intermediate grammar $G_1 = (V_1, \Sigma_1, R_1, S)$ such that V_1 contains only variables A for which $A \Rightarrow^* w \in \Sigma^*$ is possible. The steps in the algorithm are

Construction of the set of variables that derive terminal strings:

1. Set V_1 to $\varnothing$.

2. Repeat the following step until no more variables are added to V_1. For every $A \in V$ for which R has a production of the form $A \to \alpha$ and $\alpha \in (V_1 \cup \Sigma)^*$, add A to V_1. (Note that this rule includes the case where $\alpha = \varepsilon$; all variables that have ε as a production body are surely generating.)

3. Take R_1 as all the productions in R whose symbols are all in $(V_1 \cup \Sigma)$.

In the second part of the construction, we get the final answer $\widehat{G}$ from G_1. We draw the variable dependency graph for G_1 and from it find all variables that cannot be reached from S. These are removed from the variable set, as are the productions involving them. We can also eliminate any terminal that does not occur in some useful production.

برای گام دوم، راه دیگر، بهره‌گیری از روند استقرایی زیر است (مثل الگوریتم قبل):

An inductive algorithm by which we can find the set of reachable symbols for the grammar $G = (V_1, \Sigma_1, R_1, S)$:

BASIS: S is surely reachable.
INDUCTION: Suppose we have discovered that some variable $A \in V_1$ is reachable. Then for all productions with A in the head, all the symbols of the bodies of those productions are also reachable.

Example 3:

$$
\begin{array}{lcl}
S & \rightarrow & AC|BS|B \\
A & \rightarrow & aA|aF \\
B & \rightarrow & CF|b \\
C & \rightarrow & cC|D \\
D & \rightarrow & aD|BD|C \\
E & \rightarrow & aA|BSA \\
F & \rightarrow & bB|b
\end{array}
$$

در گام اول، همهٔ متغیرهای غیرسازنده باید حذف شوند:

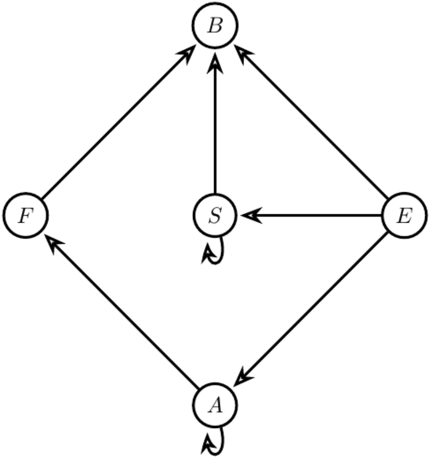
$\{B, F\}$ ☞ $\{B, F, A, S\}$ ☞ $\{B, F, A, S, E\}$ ☞ $\{B, F, A, S, E\}$

$$V_1 = \{S, A, B, E, F\}$$

$$\Sigma_1 = \{a, b\}$$

$$R_1 = \{S \rightarrow BS|B, A \rightarrow aA|aF, B \rightarrow b, E \rightarrow aA|BSA, F \rightarrow bB|b\}$$

در گام دوم، همهٔ متغیرهای غیرقابل‌دسترسی باید حذف شوند:



گرامر نهایی $\widehat{G}$، فقط دارای دو رول $S \to SB|B$ و $B \to b$ است. نیز داریم $L(\widehat{G}) = L(b^+)$.

نکتهٔ نهایی: حتماً ابتدا باید غیرسازنده‌ها را حذف کرد، سپس غیرقابل‌دسترس‌ها را.
اگر برعکس عمل شود، ممکن است اشکال پیش آید؛مانند مثال زیر.

Example 4: *Consider the grammar $S \to AB|a,\ A \to b$. All variables but B are generating; S generates a, and A generates b. If we eliminate B, we must eliminate the production $S \to AB$, leaving the grammar $S \to a,\ A \to b$. Now we find that only S is reachable from S. Eliminating A and b leaves only the production $S \to a$. That production by itself is a grammar whose language is $\{a\}$, just as is the language of the original grammar.*

Note that if we start by checking for reachability first, we find that all symbols of the grammar

$$S \to AB|a,\ A \to b$$

are reachable. If we then eliminate the symbol B because it is not generating, we are left with a grammar that still has useless symbols: $S \to a,\ A \to b$.

Part 2: Preview of Undecidable CFL Problems

In the next sessions we shall develop a remarkable theory that lets us prove formally that there are problems we cannot solve by any algorithm that can run on a computer. We shall use it to show that a number of simple-to-state questions about grammars and CFL's have no algorithm; they are called "undecidable problems." For example, the following are undecidable:

1. *Is a given CFG G ambiguous? (AMBIG$_{CFG}$)*
2. *Is a given CFL inherently ambiguous?*
3. *Is the intersection of two CFL's empty?*
4. *Are two CFL's the same? (EQ$_{CFG}$)*
5. *Is a given CFL equal to Σ^*, where Σ is the alphabet of this language? (ALL$_{CFG}$)*

Part 3: Some Decidable Properties of Context-Free Languages

☞ ***A_{CFG}: Given a string w of terminals, we want to know whether or not w is in $L(G)$. An algorithm that can tell us whether w is in $L(G)$ is a membership algorithm. We will establish the existence of a membership algorithm for context-free languages.***

☞ ***E_{CFG}: Given a context-free grammar G, there exists an algorithm for deciding whether or not $L(G)$ is empty.***

☞ ***$INFINITE_{CFG}$: Given a context-free grammar G, there exists an algorithm for determining whether or not $L(G)$ is infinite.***

Part 4: The CYK (Cocke-Younger-Kasami) Algorithm

In general, for a given language L, the problem of deciding whether a string w belongs to the language or not is called the membership problem.

Throughout this part, a context-free language is assumed to be given in terms of a context-free grammar $G = (V, \Sigma, R, S)$ in Chomsky normal form.

چرا رویکرد brute-force برای حل این مسئله بسیار ناکارآمد است؟

In order to generate a string of length n, we apply rules of the form $A \to BC$, $n-1$ times and rules of the form $A \to a$, n times. So we apply rules $2n-1$ times in total to generate a string of length n. Hence, given a string w of length n, we enumerate all the sequences of $2n-1$ rules $r_1 r_2 \cdots r_{2n-1}$ and check to see if $S \Rightarrow^* w$ for each of the sequences. If $S \Rightarrow^* w$ for at least one of these sequences, the grammar generates w, and does not generate w otherwise. This is a typical example of a brute-force search. When the number of rules in the grammar is denoted by m, then the total number of sequences of rules of length $2n-1$ is given by m^{2n-1}, which is an exponential function in the length n of an input string and becomes enormously large as n becomes large. Hence this way of solving the membership problem cannot be practical.

We shall describe the CYK (Cocke-Younger-Kasami) algorithm which efficiently solves the membership problem, based on dynamic programming. In $O(n^3)$ time, the algorithm constructs a triangular table that tells whether w is in L.

In the CYK algorithm we construct a triangular table. The horizontal axis corresponds to the positions of the string $w = a_1a_2 \cdots a_n$, which we have supposed has length 5.

The table entry X_{ij} is the set of variables A such that

$$A \Rightarrow^* a_i a_{i+1} \cdots a_j.$$

Note in particular that we are interested in whether S is in the set X_{1n} because that is the same as saying $S \Rightarrow^* w$, i.e., w is in L.

X_{15}				
X_{14}	X_{25}			
X_{13}	X_{24}	X_{35}		
X_{12}	X_{23}	X_{34}	X_{45}	
X_{11}	X_{22}	X_{33}	X_{44}	X_{55}
a_1	a_2	a_3	a_4	a_5

To fill the table we work row by row upwards. Notice that each row corresponds to one length of substrings; the bottom row is for strings of length 1, the second-from-bottom row for strings of length 2, and so on, until the top row corresponds to the one substring of length n, which is w itself.

It takes $O(n)$ time to compute any one entry of the table, by a method we shall discuss next. Since there are $n(n+1)/2$ table entries, the whole table construction process takes $O(n^3)$ time.

Since Chomsky normal form is assumed, $A \Rightarrow^* a_i a_{i+1} \cdots a_j$ holds if $a_i a_{i+1} \cdots a_j$ is divided into two substrings $a_i a_{i+1} \cdots a_k$ and $a_{k+1} a_{k+2} \cdots a_j$ in such a way that $B \Rightarrow^* a_i a_{i+1} \cdots a_k$ and $C \Rightarrow^* a_{k+1} a_{k+2} \cdots a_j$ holds for some nonterminals B, C and for some rule $A \to BC$.

In other words, to decide whether or not $A \in X_{ij}$, we only have to check whether or not there exists an integer k such that $i \leq k \leq j-1$ and nonterminals B and C such that $B \in X_{ik}$, $C \in X_{k+1,j}$, and $A \to BC \in R$, where R denotes the set of rules.

6	X_{16}					
5	X_{15}	X_{26}				
4	X_{14}	X_{25}	X_{36}			
3	X_{13}	X_{24}	X_{35}	X_{46}		
2	X_{12}	X_{23}	X_{34}	X_{45}	X_{56}	
1	X_{11}	X_{22}	X_{33}	X_{44}	X_{55}	X_{66}
Row No./String	a_1	a_2	a_3	a_4	a_5	a_6

☞ ***The variables $X_{11}, \ldots, X_{nn}$ in row 1 of the table contain nonterminals that generate $a_1, \ldots, a_n$, respectively.***
☞ ***The variables in $X_{12}, \ldots, X_{n-1,n}$ in row 2 contain nonterminals that generate $a_1a_2, a_2a_3, \ldots, a_{n-1}a_n$, respectively.***
☞ ***Similarly, nonterminals in rows numbered $3, \ldots, n$ are specified.***
☞ ***The algorithm to solve the membership problem somehow computes the nonterminals of the variables from the bottom row toward the top row and decides that $S \Rightarrow^* a_1a_2\cdots a_n$ holds if the variable X_{1n} contains the start symbol S.***

The problem left is how to compute the nonterminals of each variable X_{ij}, which will now be explained. First, let element X_{ii} be a set of all A such that $A \to a_i \in R$. Suppose that nonterminals in the variables that are placed in row 1 up to row $l-1$ have been computed so far. We shall describe how to compute nonterminals in X_{ij}'s that are placed on row l, where $n \geq l = j-i+1 \geq 2$.

☞ *Clearly we have $A \Rightarrow^* a_i a_{i+1} \cdots a_j$ if there exists an integer k such that $i \leq k \leq j-1$ and nonterminals B and C such that*

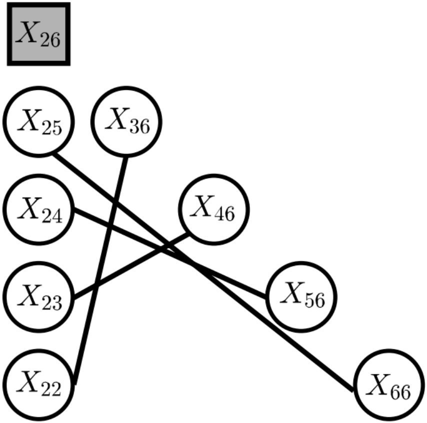
(1) $B \in X_{ik}$

(2) $C \in X_{k+1,j}$

(3) $A \to BC \in R$

☞ *Note that, since $k-i+1 \leq j-1-i+1 = j-i < l$ and $j-(k+1)+1 \leq j-(i+1)+1 = j-i < l$, both X_{ik} and $X_{k+1,j}$ are placed in row $1, \ldots,$ and row $l-1$. Since nonterminals in X_{ik} and those in $X_{k+1,j}$ are already computed, we can check the conditions (1), (2), and (3).*

How to compute X_{26} from the four pairs (X_{22}, X_{36}), (X_{23}, X_{46}), (X_{24}, X_{56}), and (X_{25}, X_{66})?



روند الگوریتم CYK

1. *Accept* if $a_1a_2\cdots a_n = \varepsilon$ and $S \to \varepsilon$ is a rule.
2. (computing the variables in the first row)
For each rule of type $A \to a$.
For each i such that $a = a_i$.
Add A to X_{ii}.
3. (computing the variables in the lth row where $l \geq 2$)
Repeat the following for $l = 2, \ldots, n$.
Repeat the following for each X_{ij} in the lth row such that $j - i + 1 = l$.
Repeat the following for each rule $A \to BC$ and each $i \leq k \leq j - 1$.
If $B \in X_{ik}$ and $C \in X_{k+1,j}$, then add A to X_{ij}.
4. *Accept* if $S \in X_{1n}$.

The above-described algorithm is of order $O(n^3)$, which is a great improvement over the number of steps, given by $O(m^{2n-1})$, of the algorithm based on brute-force search described at the beginning of this part.

Example 5: ***We try to apply the algorithm described above to a context-free grammar given as follows:***

$$
\begin{aligned}
S &\rightarrow AB|AC|AA,\\
A &\rightarrow CB|a,\\
B &\rightarrow AC|b,\\
C &\rightarrow CC|b.
\end{aligned}
$$

5	$\{S, A, B\}$				
4	$\{S, A\}$	$\{S, A, B\}$			
3	$\{S\}$	$\{A\}$	$\{S, B\}$		
2	$\{A, C\}$	$\emptyset$	$\{S, B\}$	$\{A, C\}$	
1	$\{B, C\}$	$\{B, C\}$	$\{A\}$	$\{B, C\}$	$\{B, C\}$
Row No./Column No.	1	2	3	4	5
String	b	b	a	b	b

For example, let us see how X_{25} is computed. Variable X_{25} comes from three pairs (X_{22}, X_{35}), (X_{23}, X_{45}), and (X_{24}, X_{55}), which correspond to $(\{B, C\}, \{S, B\}), (\varnothing, \{A, C\})$, and $(\{A\}, \{B, C\})$, respectively. Since we have

$\{B, C\} \cdot \{S, B\} = \{BS, BB, CS, CB\}$,
$\varnothing \cdot \{A, C\} = \varnothing$,
$\{A\} \cdot \{B, C\} = \{AB, AC\}$,

and the rules such that these strings of length 2 appear on the right-hand side of the rules are

$$S \to AB,\ S \to AC,\ A \to CB \text{ and } B \to AC,$$

we have $X_{25} = \{S, A, B\}$. Other elements are similarly obtained. Since $X_{15} = \{S, A, B\}$, which includes the start symbol S, it is concluded that the string $bbabb$ is generated by the context-free grammar.

Example 6: ***The following are the productions of a CNF grammar G:***

$$
\begin{aligned}
S &\rightarrow AB|BC \\
A &\rightarrow BA|a \\
B &\rightarrow CC|b \\
C &\rightarrow AB|a
\end{aligned}
$$

We shall test for membership in $L(G)$ ***the string*** $baaba$

{S,A,C}				
–	{S,A,C}			
–	{B}	{B}		
{S,A}	{B}	{S,C}	{S,A}	
{B}	{A,C}	{A,C}	{B}	{A,C}
b	a	a	b	a

☞ *In order for a variable to generate ba, it must have a body whose first variable is in $X_{11} = \{B\}$ (i.e., it generates the b) and whose second variable is in $X_{22} = \{A, C\}$ (i.e., it generates the a). This body can only be BA or BC. If we inspect the grammar, we find that the productions $A \to BA$ and $S \to BC$ are the only ones with these bodies. Thus the two heads A and S constitute X_{12}.*

☞ *For a more complex example consider the computation of X_{24}. We can break the string aab that occupies positions 2 through 4 by ending the first string after position 2 or position 3. That is we may choose $k = 2$ or $k = 3$ in the definition of X_{24}. Thus we must consider all bodies in (X_{22}, X_{34}) and (X_{23}, X_{44}).*

$$\{A, C\} \cdot \{S, C\} = \{AS, AC, CS, CC\}$$

$$\{B\} \cdot \{B\} = \{BB\}$$

Of the five strings in this set only CC is a body, and its head is B. Thus $X_{24} = \{B\}$.

تمرین:

$$
\begin{aligned}
G:\ & S \rightarrow aA \mid BD \\
& A \rightarrow aA \mid aAB \mid aD \\
& B \rightarrow aB \mid aC \mid BF \\
& C \rightarrow Bb \mid aAC \mid E \\
& D \rightarrow bD \mid bC \mid b \\
& E \rightarrow aB \mid bC \\
& F \rightarrow aF \mid aG \mid a \\
& G \rightarrow a \mid b
\end{aligned}
$$