# Theory of Machines and Languages

Fatemeh Deldar

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# Decidability

- **Theorem**
  - $A_{DFA}$ **is a decidable language.**

$$A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

  - **Proof**
    1. **Simulate $B$ on input $w$**
    2. **If the simulation ends in an accept state, *accept*. If it ends in a nonaccepting state, *reject*.**

- **Theorem**
  - $A_{NFA}$ **is a decidable language.**

$$A_{\mathsf{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}$$

  - **Proof**
    1. **Convert NFA $B$ to an equivalent DFA $C$**
    2. **Run TM $M$ from previous theorem on input $\langle C, w\rangle$**
    3. **If M accepts, *accept*; otherwise, *reject***

# Decidability

- **Theorem**
  - $E_{DFA}$ **is a decidable language.**

$$E_{\mathsf{DFA}} = \{\langle A\rangle |\ A \text{ is a } \mathsf{DFA} \text{ and } L(A) = \emptyset\}$$

  - **Proof**
    1. **Mark the start state of A**
    2. **Repeat until no new states get marked:**
       - **Mark any state that has a transition coming into it from any state that is already marked**
    3. **If no accept state is marked, *accept*; otherwise, *reject***

- **Theorem**
  - $EQ_{DFA}$ **is a decidable language.**

$$EQ_{\mathsf{DFA}} = \{\langle A, B\rangle |\ A \text{ and } B \text{ are } \mathsf{DFAs} \text{ and } L(A) = L(B)\}$$

  - **Proof**
    1. **Construct DFA $C$ as:** $L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right)$
    2. **Run TM $T$ from previous theorem on input** $\langle C\rangle$
    3. **If $T$ accepts, *accept*. If $T$ rejects, *reject*.**

# Decidability

- **Theorem**
  - $A_{CFG}$ **is a decidable language.**

$$A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$$

  - **Proof**
    1. **Convert $G$ to an equivalent grammar in Chomsky normal form**
    2. **List all derivations with $2n-1$ steps, where $n$ is the length of $w$**
    3. **If any of these derivations generate $w$, *accept*; if not, *reject***

- **Theorem**
  - $E_{CFG}$ **is a decidable language.**

$$E_{\mathsf{CFG}} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$$

  - **Proof**
    1. **Mark all terminal symbols in $G$**
    2. **Repeat until no new variables get marked:**
       - **Mark any variable $A$ where $G$ has a rule $A \rightarrow U_1 U_2 \ldots U_k$ and each symbol $U_1, \ldots, U_k$ has already been marked**
    3. **If the start variable is not marked, accept; otherwise, reject**

# Undecidability

- **Theorem**
  - $A_{TM}$ **is undecidable.**

$$A_{\mathsf{TM}} = \{\langle M, w\rangle |\ M \text{ is a TM and } M \text{ accepts } w\}$$

  - **Proof**
    - **We assume that $A_{TM}$ is decidable and obtain a contradiction**
    - **Suppose that $H$ is a decider for $A_{TM}$**

$$H\big(\langle M, w\rangle\big) = \begin{cases} \mathit{accept} & \text{if } M \text{ accepts } w \\ \mathit{reject} & \text{if } M \text{ does not accept } w. \end{cases}$$

    - **Now we construct a new Turing machine $D$ on input $\langle M\rangle$, where $M$ is a TM**
      - **Run $H$ on input $\langle M, \langle M\rangle\rangle$**
      - **Output the opposite of what $H$ outputs**

$$D\big(\langle M\rangle\big) = \begin{cases} \mathit{accept} & \text{if } M \text{ does not accept } \langle M\rangle \\ \mathit{reject} & \text{if } M \text{ accepts } \langle M\rangle. \end{cases}$$

# Undecidability

- **Theorem**
  - $A_{TM}$ **is undecidable.**

$$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$

  - **Proof (Cont.)**
    - **What happens when we run $D$ with its own description $\langle D\rangle$ as input?**

$$D(\langle D\rangle) = \begin{cases} \mathit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \mathit{reject} & \text{if } D \text{ accepts } \langle D\rangle. \end{cases}$$

- **$H$ accepts $\langle M, w\rangle$ exactly when $M$ accepts $w$.**
- **$D$ rejects $\langle M\rangle$ exactly when M accepts $\langle M\rangle$ .**
- **$D$ rejects $\langle D\rangle$ exactly when D accepts $\langle D\rangle$.**

# Undecidability

- **Theorem**
  - **$A_{TM}$ is undecidable.**

$$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$

| | $\langle M_1\rangle$ | $\langle M_2\rangle$ | $\langle M_3\rangle$ | $\langle M_4\rangle$ | $\cdots$ | $\langle D\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $M_1$ | *accept* | *reject* | *accept* | *reject* | | *accept* | |
| $M_2$ | *accept* | *accept* | *accept* | *accept* | $\cdots$ | *accept* | $\cdots$ |
| $M_3$ | *reject* | *reject* | *reject* | *reject* | | *reject* | |
| $M_4$ | *accept* | *accept* | *reject* | *reject* | | *accept* | |
| $\vdots$ | | $\vdots$ | | | $\ddots$ | | |
| $D$ | *reject* | *reject* | *accept* | *accept* | | ? | |
| $\vdots$ | | $\vdots$ | | | | | $\ddots$ |