Theory of Machines and Languages

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Closure Properties and Decision Algorithms for Context-Free Languages

- The family of context-free languages is closed under union, concatenation, and star-closure.

$$G_1 = (V_1, T_1, S_1, P_1)$$

$$G_2 = (V_2, T_2, S_2, P_2)$$

$$G_3 = (V_1 \cup V_2 \cup \{S_3\}, T_1 \cup T_2, S_3, P_3),$$
$$P_3 = P_1 \cup P_2 \cup \{S_3 \rightarrow S_1 | S_2\}.$$
$$L(G_3) = L_1 \cup L_2.$$

$$G_4 = (V_1 \cup V_2 \cup \{S_4\}, T_1 \cup T_2, S_4, P_4).$$
$$P_4 = P_1 \cup P_2 \cup \{S_4 \rightarrow S_1 S_2\}.$$
$$L(G_4) = L(G_1) L(G_2)$$

$$G_5 = (V_1 \cup \{S_5\}, T_1, S_5, P_5),$$
$$P_5 = P_1 \cup \{S_5 \rightarrow S_1 S_5 | \lambda\}$$
$$L(G_5) = L(G_1)^*.$$

Closure Properties and Decision Algorithms for Context-Free Languages

- The family of context-free languages is not closed under intersection and complementation.

$L_1 = \{a^n b^n c^m : n \geq 0, m \geq 0\}$ is context-free

$L_2 = \{a^n b^m c^m : n \geq 0, m \geq 0\}$ is context-free

$L_1 \cap L_2 = \{a^n b^n c^n : n \geq 0\}$ is not context-free

Thus, the family of context-free languages is not closed under intersection

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

If the family of context-free languages were closed under complementation, then the right side of the above expression would be a context-free language for any context-free L_1 and L_2

Closure Properties and Decision Algorithms for Context-Free Languages

Closure under regular intersection

- Let L_1 be a context-free language and L_2 be a regular language. Then $L_1 \cap L_2$ is context-free.

Example

Show that the language

$$L = \{a^n b^n : n \geq 0, n \neq 100\}$$

is context-free.

$L_1 = \{a^{100} b^{100}\}$ **is regular**

$L = \{a^n b^n : n \geq 0\} \cap \overline{L_1}$

Because of:

- **The closure of regular languages under complementation**
- **The closure of context-free languages under regular intersection**

L is context-free

Closure Properties and Decision Algorithms for Context-Free Languages

- **Example**

Show that the language

$$L = \{w \in \{a, b, c\}^* : n_a(w) = n_b(w) = n_c(w)\}$$

is not context-free.

- **Suppose that L were context-free, then** $L \cap L(a^*b^*c^*) = \{a^n b^n c^n : n \geq 0\}$ **would also be context-free**
- **We conclude that L is not context-free**

Closure Properties and Decision Algorithms for Context-Free Languages

- Given a context-free grammar $G = (V, T, S, P)$, there exists an algorithm for deciding whether or not $L(G)$ is empty.

- **Proof**
  - **We first remove useless symbols and productions**
  - **If S is found to be useless, then $L(G)$ is empty**
  - **If not, then $L(G)$ contains at least one element**

Closure Properties and Decision Algorithms for Context-Free Languages

- Given a context-free grammar $G = (V, T, S, P)$, there exists an algorithm for determining whether or not $L(G)$ is infinite.

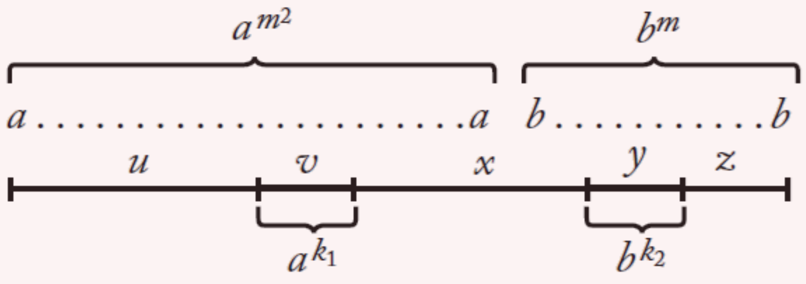
- **Proof**
  - **We need only to determine whether the grammar has some repeating variables**
    - **This can be done simply by drawing a dependency graph for the variables**
  - **If no variable can ever repeat, then the length of any derivation is bounded by $|V|$**
    - **In that case, $L(G)$ is finite**

A Pumping Lemma for Context-Free Languages

- **Example**
  - **Show that the following language is not context-free.**

$$L = \{a^n b^j : n = j^2\}$$

- We pick the string $a^{m^2} b^m$, which is in L
- The most challenging way in picking vxy is:



- Pumping i times will yield a string with $m^2 + (i-1)k_1$, a's and $m + (i-1)k_2$, b's.

$$\begin{aligned}(m-k_2)^2 &\le (m-1)^2 \\ &= m^2 - 2m + 1 \\ &< m^2 - k_1,\end{aligned}$$

$i = 0$