Theory of Machines and Languages

Fatemeh Deldar

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Properties of Context-Free Languages

A Pumping Lemma for Context-Free Languages

- Let L be an infinite context-free language. Then there exists some positive integer m such that any $w \in L$ with $|w| \geq m$ can be decomposed as

$$w = uvxyz,$$

with

$$|vxy| \leq m,$$

and

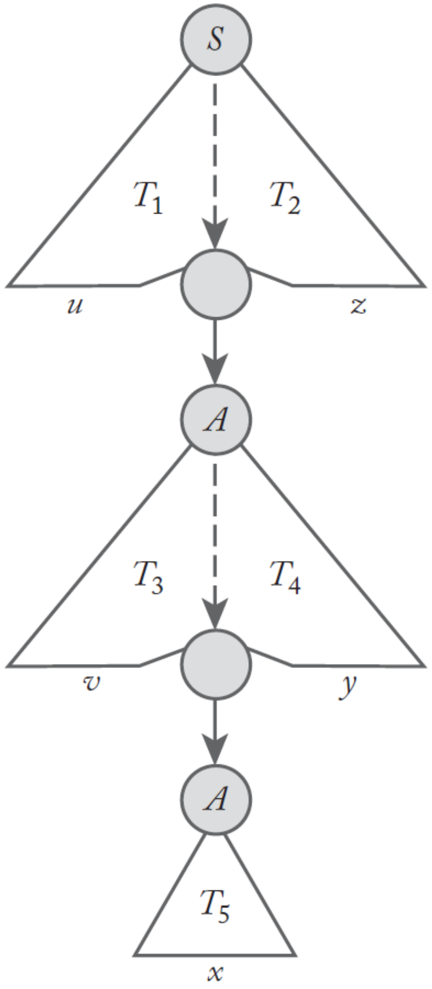
$$|vy| \geq 1,$$

such that

$$uv^ixy^iz \in L,$$

for all $i = 0, 1, 2, ...$ This is known as the pumping lemma for context-free languages.

A Pumping Lemma for Context-Free Languages



- Consider a derivation tree and some sufficiently long path from the root to a leaf
  - Since
    - The length of the string on the right side of any production is bounded
    - The number of variables in G is finite
  - There must be some variable that repeats on this path

$$S \stackrel{*}{\Rightarrow} uAz \stackrel{*}{\Rightarrow} uvAyz \stackrel{*}{\Rightarrow} uvxyz,$$

- So all the strings $uv^ixy^iz, i = 0, 1, 2, \ldots$, can be generated by the grammar

A Pumping Lemma for Context-Free Languages

- **Example**

  Show that the language

  $$L = \{a^n b^n c^n : n \geq 0\}$$

  is not context-free.

- **We pick the string $a^m b^m c^m$, which is in L**
  - **If vxy contains only a's, then the pumped string will obviously not be in L**
  - **If v and y are composed of an equal number of a's and b's, then the pumped string $a^k b^k c^m$ with $k \neq m$ can be generated, and the generated string not in L**

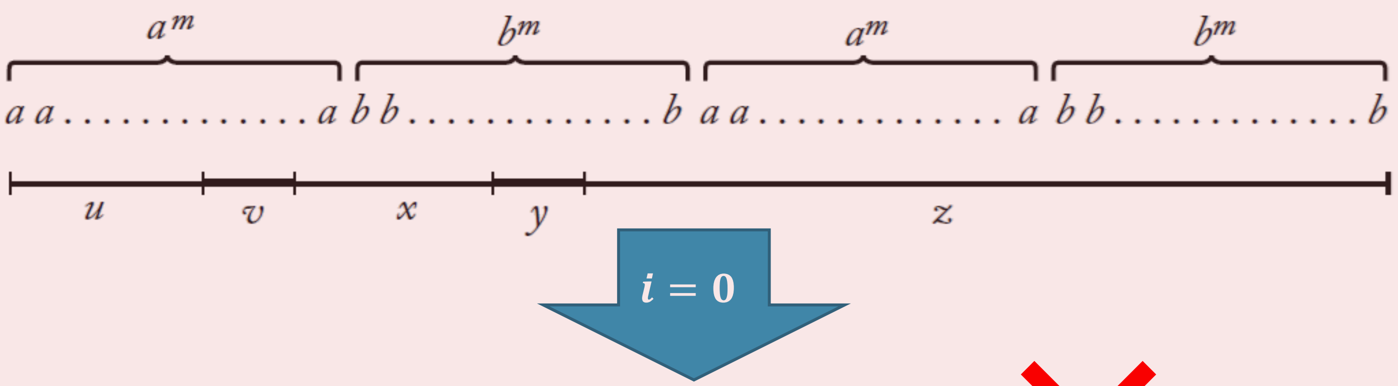
- **In fact, the only way is to pick vxy so that vy has the same number of a's, b's, and c's. But this is not possible because of the restriction $|vxy| \leq m$**
  - **Therefore, L is not context-free**

A Pumping Lemma for Context-Free Languages

- **Example**
  - **Show that the following language is not context-free.**

$$L = \{ww : w \in \{a,b\}^*\}$$

- We pick the string $a^m b^m a^m b^m$, which is in L
- There are many ways in which we can pick vxy, but for all of them we can obtain a pumped string not in L



$$a^k b^j a^m b^m, k < m \text{ or } j < m$$

A Pumping Lemma for Context-Free Languages

- **Example**
  - **Show that the following language is not context-free.**

$$L = \{a^{n!} : n \geq 0\}$$

- We pick the string $a^{m!}$, which is in L
- $v = a^k, \quad y = a^l$

$$\Big\downarrow \; i = 0$$

- This string is in L only if $m! - (k + l) = j!$ for some j
- Since $k + l \leq m \implies m! - (k + l) > m! - m > (m - 1)!$