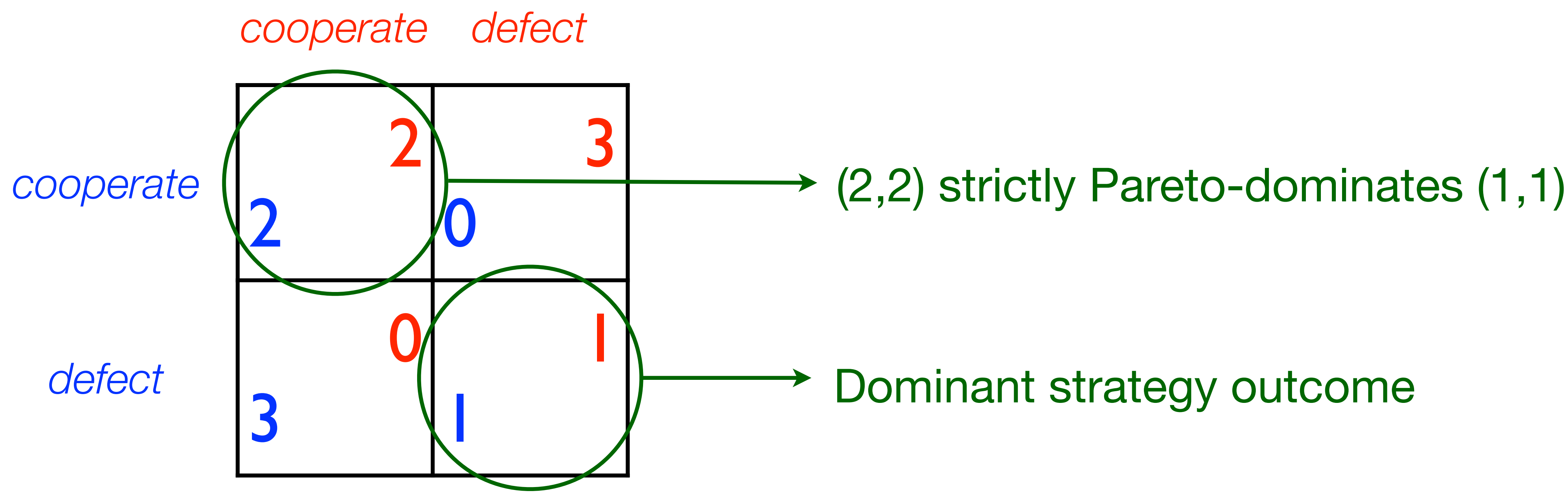
Prisoner's Dilemma (ctd.)



- An action is *dominant* if it always yields strictly more utility than every other action of the same player.
- An outcome is strictly *Pareto-dominated* if there exists another outcome in which every player is better off.

Pareto-Optimality


Vilfredo Pareto

- An outcome is (weakly) *Pareto-dominated* if there exists another outcome in which all players obtain at least as much utility and one player is strictly better off.
  - Pareto dominance usually refers to *weak* Pareto dominance.
- An outcome is *Pareto-optimal* if it is not Pareto-dominated.
  - It is impossible to increase the utility of one player without reducing the utility of another.
- All outcomes in the prisoner's dilemma except (defect,defect) are Pareto-optimal.

2 2	3 0
0 3	1 1

Dominance

- Strict dominance
  - a_i *(strictly) dominates* b_i if for all $a_{-i} \in A_{-i}$, $u_i(a_i,a_{-i}) > u_i(b_i,a_{-i})$.
- Weak and very weak dominance
  - a_i *weakly dominates* b_i if for all $a_{-i} \in A_{-i}$, $u_i(a_i,a_{-i}) \geq u_i(b_i,a_{-i})$ and for at least one $a_{-i} \in A_{-i}$, $u_i(a_i,a_{-i}) > u_i(b_i,a_{-i})$.
  - a_i *very weakly dominates* b_i if for all $a_{-i} \in A_{-i}$, $u_i(a_i,a_{-i}) \geq u_i(b_i,a_{-i})$.
  - Dominance usually refers to *strict* dominance.
- An action is *dominated* if there exists another action that dominates it.
- An action is *dominant* if it dominates all other actions (of the same player).
  - To defect is a dominant action in the prisoner's dilemma.

2 2	3 0
0 3	1 1

Split or Steal

(UK Game Show, 2008)

Example

- Which outcomes are *Pareto-optimal*?
- Are there *dominated* actions?
- What if players may *randomize* and aim at maximizing their *expected* payoff?
- What if players may assume that their *opponents are rational*?
- What if players may assume that their *opponents know that they are rational*?
- What if players have doubts about the rationality of their opponents?

1 3	0 0
3 0	2 3
1 1	2 1

Mixed Strategies

- A (mixed) strategy $s_i \in S_i = \mathscr{L}(A_i)$ is a lottery over actions.
  - Action a_i is played with probability $s_i(a_i)$.
  - The *support* of a strategy s_i is $supp(s_i)=\{a_i \in A_i \mid s_i(a_i)>0\}$.
- The *expected utility* of a player in a given strategy profile $s \in S = S_1 \times \ldots \times S_n$ is

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j=1}^{n} s_j(a_j).$$

- The utility function is assumed to be a vNM utility function.
- Degenerate strategy lotteries (i.e., actions) are sometimes also called pure strategies.

Mixed Dominance

- All dominance definitions carry over from actions to strategies.
  - s_i *dominates* t_i if for all $s_{-i} \in S_{-i}$, $u_i(s_i,s_{-i}) > u_i(t_i,s_{-i})$.
- Since $u_i(s_i,s_{-i})$ as a (multilinear) function of s_{-i} takes its extreme values at the vertices of S_{-i}, it suffices to consider pure strategies of the opponents.
- Hence, s_i *dominates* t_i if for all $a_{-i} \in A_{-i}$, $u_i(s_i,a_{-i}) > u_i(t_i,a_{-i})$.
  - We will usually consider the domination of actions by strategies, i.e., $s_i \in S_i$ and $t_i \in A_i$.
- Weak and very weak dominance can be extended analogously.

David Pearce

Rationalizability

Douglas Bernheim

- An action is *rationalizable* if a rational player could justifiably play it against rational opponents when the rationality of all players is common knowledge.
  - The formal definition using epistemic belief structures is involved.
- Strategy s_i is a *best response* to the strategy profile s_{-i}, denoted by $s_i \in B_i(s_{-i})$, if $u_i(s_i,s_{-i}) \geq u(t_i,s_{-i})$ for all $t_i \in S_i$.
  - s_{-i} admits either one or infinitely many best responses.
  - Theorem: In two-player games, s_i is never a best response iff it is dominated.
- Theorem (Pearce, Bernheim; 1984): In two-player games, the set of rationalizable actions consists of all actions that survive the iterated elimination of dominated actions.

Iterated Dominance

- When a dominated action is eliminated (from consideration), other actions might become dominated.
- This yields a polynomial-time algorithm because at least one action is eliminated at each step and the total number of actions $\sum_{i \in N} |A_i|$ is polynomial.
  - `REPEAT remove dominated actions`
  - `UNTIL no more actions are dominated`
- The order in which actions are eliminated is irrelevant because a dominated action will always remain dominated.
- How can we efficiently check whether an action is dominated by some *mixed* strategy?

Linear Programming (LP)

- Standard form:

$$\begin{array}{llcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s.t.} & \mathbf{A}\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

- Almost all linear optimization problems can be brought into this form.
  - minimization objective: max $-c^Tx$ (instead of min c^Tx)
  - equalities: $ax \leq b$ and $-ax \leq -b$ (instead of $ax = b$)
- Example:

$$\begin{array}{llcl} \max & x_1 + x_2 & & \\ \text{s.t.} & x_1 + 2x_2 & \leq & 8 \\ & x_1 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$

  - Solution: $x_1=4$, $x_2=2$

Linear Programming (ctd.)

- Linear programming algorithms
  - Simplex (Dantzig, 1947): exponential worst-case running time
  - Ellipsoid (Khachiyan, 1979): polynomial worst-case running time
  - Interior-point (Karmarkar, 1984): polynomial worst-case running time
- Linear programming is P-complete.
  - Every problem in P can be solved by an LP.
- LP solvers
  - glpk (GNU Linear Programming Kit)
  - CPLEX (IBM)
  - Gurobi

Linear Program for Dominance

- a_i is dominated iff there is a strategy s_i such that

$$
\begin{array}{rcll}
\sum_{b_i \in A_i} s_i(b_i) u_i(b_i, a_{-i}) & > & u_i(a_i, a_{-i}) & \forall a_{-i} \in A_{-i} \\
s_i(b_i) & \geq & 0 & \forall b_i \in A_i \\
\sum_{b_i \in A_i} s_i(b_i) & = & 1 &
\end{array}
$$

  - This is not an LP (due to the strict inequality)!

- a_i is dominated iff there is a strategy s_i such that the following LP has a solution with positive value.

$$
\begin{array}{lrcll}
\text{max} & & \varepsilon & & \\
\text{s.t.} & \sum_{b_i \in A_i} s_i(b_i) u_i(b_i, a_{-i}) & \geq & u_i(a_i, a_{-i}) + \varepsilon & \forall a_{-i} \in A_{-i} \\
& s_i(b_i) & \geq & 0 & \forall b_i \in A_i \\
& \sum_{b_i \in A_i} s_i(b_i) & = & 1 &
\end{array}
$$