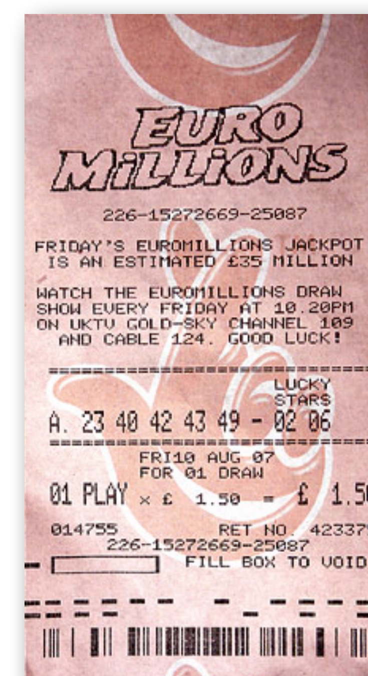
Preferences under Uncertainty

- Frequently, the consequences of a decision are not deterministic but stochastic.
  - The agent chooses a “lottery ticket” rather than an alternative.
- Defining preferences over lotteries is not straightforward.
- Which lottery would you prefer?
  - L_1: €1million guaranteed
  - L_2: €3million with 98% probability
- Which lottery would you prefer?
  - L_3: €1million with 5% probability
  - L_4: €3million with 4.9% probability



Lotteries

- We denote the set of all lotteries over a finite set of alternatives $x_1, \ldots, x_k$ by
$$\mathcal{L}(\{x_1, \ldots, x_k\}) = \{p \in [0,1]^k \mid \textstyle\sum_{i=1}^k p_i = 1\}.$$
- Lotteries
  - Simple lotteries $L = [p_1 : x_1, \ldots, p_k : x_k]$
    - A simple lottery is *degenerate* if it puts probability 1 on one alternative.
- Lotteries over lotteries
  - Compound lotteries $L = [p_1 : L_1, \ldots, p_k : L_k]$
  - Compound lotteries can be simplified to simple lotteries by multiplying probabilities ("Consequentialist premise")
  - Example: $L=[0.5 : L_1, 0.5 : L_2]$, $L_1=[1 : a]$, $L_2=[0.5 : b, 0.5 : c]$
    $L=[0.5 : a, 0.5 : [0.5{:}b, 0.5{:}c]] = [0.5 : a, 0.25 : b, 0.25 : c]$

Preferences over Lotteries

- Obviously the set of lotteries is infinite (whenever $|A|>1$).
- Preferences over lotteries may involve many different factors such as
  - most likely outcomes
    (e.g., the most likely outcome of which lottery is preferred?),
  - most desirable and/or least desirable outcomes
    (e.g., the probability of the least desirable outcome of which lottery is lower?),
  - uniformity of probabilities,
  - size of support (i.e., the set of alternatives with positive probability),
  - expected utility (requires the existence of a utility function),
  - etc.
- Let us employ the axiomatic method to get a clearer picture of preferences over lotteries.

Continuity and Independence

- Two important axioms are *continuity* and *independence*.
- Continuity:
  For all $L_1 > L_2 > L_3$, there is some $p \in (0,1)$ such that
  $$L_2 \sim [p : L_1, (1\text{-}p) : L_3].$$
  - Alternatively, one can demand that for all $L_1 > L_2 > L_3$, there is some $\varepsilon \in (0,1)$ such that $[(1\text{-}\varepsilon) : L_1, \varepsilon : L_3] > L_2 > [(1\text{-}\varepsilon) : L_3, \varepsilon : L_1]$.
- Independence:
  For all L_1, L_2, L_3 and all $p \in (0,1)$,
  $$L_1 \succsim L_2 \Leftrightarrow [p : L_1, (1\text{-}p) : L_3] \succsim [p : L_2, (1\text{-}p) : L_3]$$
  - also known as Savage's *sure thing principle*

vNM Utility Functions

- Theorem (von Neumann & Morgenstern, 1947):
  - A preference relation $\succsim$ on $\mathscr{L}(A)$ is rational, continuous, and independent iff there exists a utility function u on A such that for two lotteries $L_1 = [p_1 : x_1, \ldots, p_k : x_k]$ and $L_2 = [q_1 : x_1, \ldots, q_k : x_k]$:

$$L_1 \succsim L_2 \Longleftrightarrow \sum_{i=1}^{k} p_i u(x_i) \geq \sum_{i=1}^{k} q_i u(x_i).$$

- This theorem provides an axiomatic foundation of expected utility theory.
- For every positive affine transformation $f(x)=\alpha x+\beta$ with $\alpha>0$, $f(u(\cdot))$ is a new vNM utility function which represents the same preference relation.
  - comparable to temperature, which can, for example, be measured in Celsius, Fahrenheit, or Kelvin.

Risk Aversion

- Utility can be very different from monetary value.
  - Which lottery would you prefer?
    - L_1 = [*1: €1million*]
    - L_2 = [*0.5: €2million, 0.5: nothing*]
- People buy insurances because their utility is concave in value. They are risk-averse.
  - Insurance premium is higher than expected loss.
- People buy lottery tickets because their utility is convex in value. They are risk-seeking.
  - Lottery ticket price is higher than expected gain.
- Expected utility theory is compatible with both notions by appropriately defining utility functions.

Allais Paradox (1979)


Maurice Allais

- Recall the introductory example from today's lecture.
  - Which lottery would you prefer?
    - $L_1 = [1\text{: €1}million]$
    - $L_2 = [0.98\text{: €3}million, 0.02\text{: }nothing]$
  - Which lottery would you prefer?
    - $L_3 = [0.050\text{: €1}million, 0.950\text{: }nothing]$
    - $L_4 = [0.049\text{: €3}million, 0.951\text{: }nothing]$
- Numerous experiments have shown that most people prefer L_1 to L_2 and L_4 to L_3.
  - These preferences cannot be explained by expected utility!
  - Let $L_0 = [1\text{: €0}]$. Then, $L_3 = [0.05\text{: }L_1, 0.95\text{: }L_0]$ and
    $L_4 = [0.05\text{: }L_2, 0.95\text{: }L_0]$.
    The independence axiom implies that $L_1 > L_2 \Leftrightarrow L_3 > L_4$.

Conclusion on Utility Representations

- Expected utility maximization is the predominant model in economic theory for decision-making under uncertainty.
- We will assume throughout this course that preference relations can be represented by utility functions and that agents are expected utility maximizers.
- In other words, we assume that the preferences of agents satisfy
  - completeness and transitivity, and
  - continuity and independence (in stochastic settings).

Dice Game

- The are four dice *A, B, C, D* with six faces each:

  A 4,4,4,4,0,0

  B 3,3,3,3,3,3

  C 6,6,2,2,2,2

  D 5,5,5,1,1,1

  - You get to pick a dice, then I have to pick one.
  - Whoever rolls a higher number with his dice wins.
- I can guarantee that I will win twice as often as you!
  - $p(A > B) = p(B > C) = p(C > D) = p(D > A) = 2/3$
  - These dice are known as Efron's dice or non-transitive dice.
  - Max. winning probability with n dice approaches 3/4.
  - Max. winning probability with 3 non-transitive lotteries: $1/\phi \sim 0.62$

Normal-form Games

- A natural and very general way to represent strategic interaction are games in normal form.
  - players choose actions simultaneously and independently
    - Interactions with sequential moves can be brought into the normal form by defining actions as *plans* that take every possible contingency into account.
  - utility defined for every combination of actions
- A *normal-form game* is a tuple $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$ where $N=\{1,\dots,n\}$ is a set of players, $A_i=\{a_{i1},\dots,a_{ik}\}$ are sets of actions, and $u_i : A \to \mathbb{R}$ are utility (or payoff) functions.
  - *Set of action profiles:* $A = A_1 \times \dots \times A_n$
  - $A_{-i} = A_1 \times \dots \times A_{i-1} \times A_{i+1} \times \dots \times A_n$
  - Utility vectors $(u_1(a), \dots, u_n(a))$ are sometimes called outcomes.
  - We focus on games with a finite number of actions.

Bimatrix Representation

- Normal-from games are typically represented using payoff matrices.
  - Two-player games are also called bimatrix games.

	a_2^1	a_2^2
a_1^1	$u_2(a_1^1,a_2^1)$ $u_1(a_1^1,a_2^1)$	$u_2(a_1^1,a_2^2)$ $u_1(a_1^1,a_2^2)$
a_1^2	$u_2(a_1^2,a_2^1)$ $u_1(a_1^2,a_2^1)$	$u_2(a_1^2,a_2^2)$ $u_1(a_1^2,a_2^2)$

Prisoner's Dilemma

- perhaps the most famous normal-form game
  - formulated by members of the RAND corporation in 1950
- Two guilty suspects are interrogated separately.
- There is insufficient evidence.
  - If both remain silent, they are put in short-term pre-trial custody.
  - If one testifies against the other, he will be released immediately and the silent accomplice receives a 10-year sentence.
  - If both betray each other, each suspect receives a 5-year sentence.

Prisoner's Dilemma (ctd.)

	cooperate	defect
cooperate	2 2	3 0
defect	0 3	1 1

Verdict	Utility
10 years	0
5 years	1
1 week	2
Freedom	3