The Indifference Principle

- The following characterization of (mixed) Nash equilibria will turn out to be very useful.
- Lemma: A strategy profile *s* is a Nash equilibrium iff for every player *i*, assuming that the other players play s_{-i},
  - all actions in the support of s_i yield the same expected payoff, and
  - no action outside the support of s_i yields more expected payoff.
- Any randomization of player *i* among actions in the support of s_i yields the same expected payoff.
  - "You randomize for the other players."
- Among other things, the indifference principle permits the efficient verification of potential Nash equilibria.

Equilibria of Standard Examples

	shoot left	*shoot right*
jump left	1, 0	0, 1
jump right	0, 1	1, 0

"Penalty shootout"

([*1/2*: left, *1/2*: right], [*1/2*: left, *1/2*: right])

	yield	*straight*
yield	2, 2	1, 3
straight	3, 1	0, 0

"Chicken"

([*1*: yield], [*1*: straight])

([*1*: straight], [*1*: yield])

([*1/2*: yield, *1/2*: straight], [*1/2*: yield, *1/2*: straight])

	boxing	*ballet*
boxing	2, 1	0, 0
ballet	0, 0	1, 2

"Battle of the Sexes"

([*1*: boxing], [*1*: boxing])

([*1*: ballet], [*1*: ballet])

([*2/3*: boxing, *1/3*: ballet], [*1/3*: boxing, *2/3*: ballet])

More on Nash Equilibria

- Some (not all) Nash equilibria of the introductory example:
  - e_1: ([$1{:}a$], [$1{:}x$]), utility (0, 0)
  - e_2: ([$1/2{:}b$, $1/2{:}c$], [$1{:}x$]), utility (0, $3/2$)
  - e_3: ([$1{:}a$], [$1/2{:}y$, $1/2{:}z$]), utility ($3/2$, 0)
  - e_4: ([$1/2{:}b$, $1/2{:}c$], [$1/2{:}y$, $1/2{:}z$]), utility ($3/2$, $3/2$)

	x	*y*	*z*
a	0, 0	1, 0	2, 0
b	0, 1	2, 0	1, 2
c	0, 2	0, 3	3, 1

- Simple facts about Nash equilibria:
  - Only rationalizable actions can be in the support of an equilibrium.
  - The payoff in any equilibrium is always at least as large as the player's security level.
  - In general, the set of equilibria is *not convex* (e.g., BoS, Chicken).

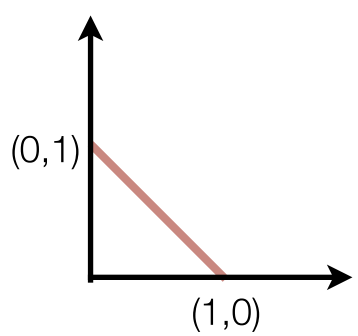
Existence of Nash Equilibria


L. E. J. Brouwer

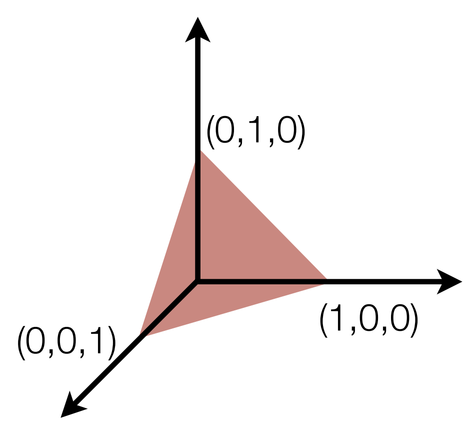
- Does every game contain a Nash equilibrium?
- Theorem (Nash, 1950): Every normal-form game contains a Nash equilibrium.
  - non-constructive proof using Brouwer's fixed point theorem
- Theorem (Brouwer, 1909/1912): If $S \subset \mathbb{R}^n$ is compact and convex and $f : S \rightarrow S$ is continuous, then there exists $x \in S$ such that $f(x) = x$.
  - A set $S \subset \mathbb{R}^n$ is compact if every sequence in S has a convergent subsequence, whose limit lies in S.
  - A set $S \subset \mathbb{R}^n$ is convex if for all $x,y \in S$ and every $\alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in S$.

Nash's Proof Sketch

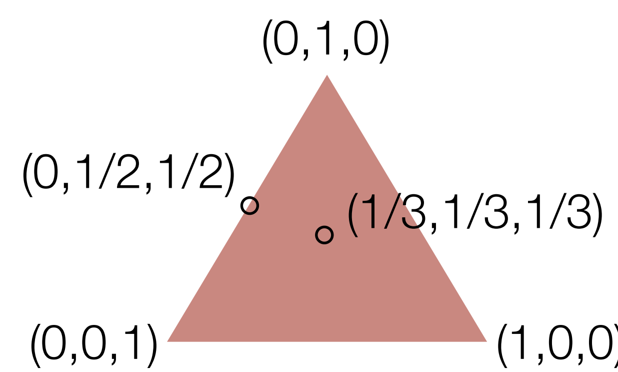
- A *k-simplex* is the set $\left\{ x \in \mathbb{R}^k \mid \sum_{i=1}^{k} x_i = 1 \;\wedge\; \forall i : x_i \geq 0 \right\}$



2-Simplex



3-Simplex



3-Simplex

- Observations
  - A *k*-Simplex is compact and convex.
  - Every strategy set S_i for k actions is a k-simplex.
  - The set of strategy profiles S is compact and convex.

Nash's Proof Sketch (ctd.)

- Consider the following function $f : S \to S$, which maps the probability $s_i(a_i^h)$ of each action a_i^h to the probability $f_i^h(s)$.

$$f_i^h(s) = \frac{s_i(a_i^h) + \max(u_i(a_i^h, s_{-i}) - u_i(s), 0)}{\sum_{a_i^\ell \in A_i} \left(s_i(a_i^\ell) + \max(u_i(a_i^\ell, s_{-i}) - u_i(s), 0) \right)}$$

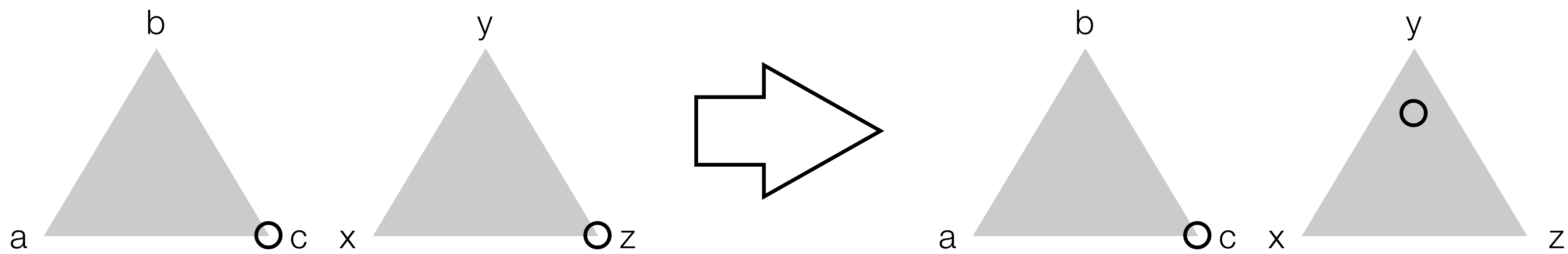
- $\max(u_i(a_i^h, s_{-i}) - u_i(s), 0) > 0$ iff a_i^h is a better response than s_i.
- The denominator normalizes output such that $f(s) \in S$.
- f is continuous because it is composed of continuous functions.

Nash's Proof Sketch (ctd.)

$$f_i^h(s) = \frac{s_i(a_i^h) + \max(u_i(a_i^h, s_{-i}) - u_i(s), 0)}{\sum_{a_i^\ell \in A_i} \left(s_i(a_i^\ell) + \max(u_i(a_i^\ell, s_{-i}) - u_i(s), 0)\right)}$$

- Claim: $f(s)=s$ iff s is a Nash equilibrium.
  - Suppose s is an equilibrium, then no pure strategy can be a better response than s_i. Hence, $f(s)=s$.
  - Suppose s is *not* an equilibrium, then at least one pure strategy yields strictly more payoff (denominator > 1). At least one pure strategy (in $supp(s_i)$) yields less or equal payoff. The probability of the corresponding action decreases. Hence, $f(s) \neq s$.
- Since S is compact and convex and f is continuous, Brouwer's fixed point theorem implies that there exists an $s \in S$ such that $f(s) = s$.

Nash's Proof Sketch Illustration



$$f_i^h(s) = \frac{s_i(a_i^h) + \max(u_i(a_i^h, s_{-i}) - u_i(s), 0)}{\sum_{a_i^\ell \in A_i} \left(s_i(a_i^\ell) + \max(u_i(a_i^\ell, s_{-i}) - u_i(s), 0)\right)}$$

	x	y	z
a	0, 0	1, 0	2, 0
b	0, 1	2, 0	1, 2
c	0, 2	0, 3	3, 1

$$s = ([1 : c], [1 : z])$$

$$f_2^x(s) = \frac{1}{4} \qquad f_2^y(s) = \frac{2}{4} \qquad f_2^z(s) = \frac{1}{4}$$

$$f(c, z) = \left([1 : c], \left[\frac{1}{4} : x, \frac{1}{2} : y, \frac{1}{4} : z\right]\right)$$

Nash's Proof Sketch (ctd.)

- Alternatively, one may use Kakutani's fixed point theorem (as also observed by Nash).
- Theorem (Kakutani, 1941): If $S \subset \mathbb{R}^n$ is compact and convex, $f : S \rightarrow 2^S$ is upper semi-continuous, and $f(x)$ is convex for all $x \in S$, then there exists $x \in S$ such that $x \in f(x)$.
  - letting $f(s)=(B_1(s_{-1}), \ldots, B_n(s_{-n}))$ yields Nash's theorem
  - numerous other applications in economics
- The availability of mixed strategies is essential for the existence of Nash equilibria.
  - Randomization may also be interpreted as beliefs on the actions to be played by the other players.