Problems with Nash Equilibrium (and Potential Fixes)

	x	y
a	0, 0	1, 0
b	0, 1	2, 2

- Multiplicity of equilibria
  - require robustness with respect to small strategy deviations
- Coalitions of players may deviate simultaneously
  - consider deviations by coalitions of players
- Players may be indifferent between actions inside and outside the support of a Nash equilibrium
  - require that equilibrium actions yield strictly more payoff
- Computational hardness
  - heuristics, approximation, restricted classes of games

0, 0	1, 0
0, 1	2, 2

Equilibrium Refinements (1)


Reinhard Selten

- Robustness with respect to small strategy deviations
- Trembling-hand perfect equilibrium (Selten, 1975)
  - Strategy profile s such that there is a sequence $(s^h)_{h=1}^{\infty}$ of full-support strategy profiles that converges to s and for all i and $h \geq 1$, s_i is a best response to s_{-i}^h.
  - Example: $s^h=([1/(h+1){:}a,\ 1-1/(h+1){:}b], [1/(h+1){:}x,\ 1-1/(h+1){:}y])$
  - Existence can be shown using Brouwer's theorem.
  - Finding a trembling-hand perfect equilibrium of a two-player game is PPAD-complete.
    - Deciding whether a pure Nash equilibrium of a three-player game is trembling-hand perfect is NP-hard (Hansen et al., 2010).
  - can be further refined to proper equilibrium which considers payoff-dependent trembles (Myerson, 1978)

0 0	0 1
1 0	2 2

Equilibrium Refinements (2)


Robert Aumann

- Deviations by coalitions of players
- Strong equilibrium (Aumann, 1959)
  - Strategy profile s such that for all $C \subseteq N$ there is no $t_C \neq s_C$ with $u_i(t_C, s_{-C}) > u_i(s)$ for all $i \in C$.
  - always yields weakly Pareto-optimal outcomes
  - very rarely exists, e.g., there is no strong equilibrium in the prisoner's dilemma

2 2	3 0
0 3	1 1

  - Deciding whether a game contains a strong Nash equilibrium is NP-complete (Conitzer & Sandholm, 2008).
    - Finding a strong Nash equilibrium is NP-hard.

Equilibrium Refinements (3)

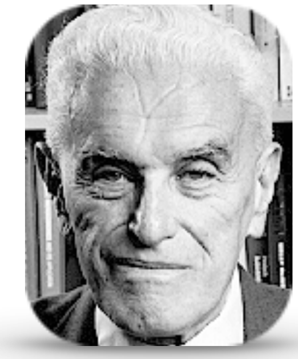
- Coalition-proof equilibrium (Bernheim et al., 1987)
    - generalizes strong equilibrium by only considering coalitional deviations that lead to an outcome in which no *subcoalition* has an incentive to deviate
    - Every strong equilibrium is coalition-proof.
    - (defect,defect) is a coalition-proof equilibrium in the prisoner's dilemma.

2, 2	0, 3
3, 0	1, 1

- In general, coalition-proof equilibria rarely exist.
- Finding a coalition-proof equilibrium is NP-hard.

0 0	1 0
0 1	2 2

Equilibrium Refinements (4)


John Harsanyi

- Strictly more payoff via actions in equilibrium support
- Quasi-strict equilibrium (Harsanyi, 1973)
  - A Nash equilibrium s such that for all i, $a_i \in supp(s_i)$, and $b_i \notin supp(s_i)$, $u_i(a_i, s_{-i}) > u_i(b_i, s_{-i})$.
  - Existence in two-player games can be shown using Brouwer's theorem (Norde, 1999).
    - very complicated proof
    - first shown for zero-sum games, $2{\times}k$ games, non-degenerate games, etc.
  - "Isolated" quasi-strict equilibria are proper and hence trembling-hand perfect.
  - Deciding whether a three-player game contains a quasi-strict equilibrium is NP-complete (B. & Fischer, 2008)
    - Finding a quasi-strict equilibrium is NP-hard.

Equilibrium Refinements (5)

- For two players, a support profile $B=(B_1,B_2)$ can be checked for quasi-strict equilibria by verifying whether the following LP has a solution with positive value ($\varepsilon>0$).

$$
\begin{array}{llrcll}
\text{max} & & & \varepsilon & & \\
\text{s.t.} & & \sum_{a_{-i}\in A_{-i}} s_{-i}(a_{-i})u_i(a_i,a_{-i}) & = & U_i^* & \forall i\in N, a_i\in B_i \\
& & \sum_{a_{-i}\in A_{-i}} s_{-i}(a_{-i})u_i(a_i,a_{-i}) + \varepsilon & \leq & U_i^* & \forall i\in N, a_i\notin B_i \\
& & s_i(a_i) & \geq & 0 & \forall i\in N, a_i\in B_i \\
& & s_i(a_i) & = & 0 & \forall i\in N, a_i\notin B_i \\
& & \sum_{a_i\in A_i} s_i(a_i) & = & 1 & \forall i\in N
\end{array}
$$

Example

- Alice, Bob, and Charlie decide who has to take out the garbage by independently and simultaneously raising a hand or not.
  - Alice loses if exactly one player raises his hand.
  - Bob loses if exactly two players raise their hands.
  - Charlie loses if either all or no players raise their hand.
- Which strategy would you recommend to the players?
  - There is a unique Nash equilibrium, which also happens to be trembling-hand perfect and strong.
  - The equilibrium fails to be quasi-strict.
  - Alice and Bob can obtain the same expected payoff by playing their maximin strategies.
  - If everybody plays maximin, Alice and Bob may gain by deviating.
  - Even though there is a *unique* equilibrium (which is even strong and trembling-hand perfect), it's unclear what to play!

The Final Word on Equilibrium Refinements

- Recall Norde et al.'s characterization theorem:
  If a solution concept satisfies utility maximization, consistency, and existence, then it is Nash equilibrium.
- Clearly, all refinements satisfy utility maximization.
- Consequences
  - Every refinement that always exists fails to satisfy consistency.
    - E.g., trembling-hand perfect and proper equilibrium
  - Every refinement that satisfies consistency cannot always exist.
    - E.g., strong, coalition-proof, and quasi-strict equilibrium

Beyond the Normal Form

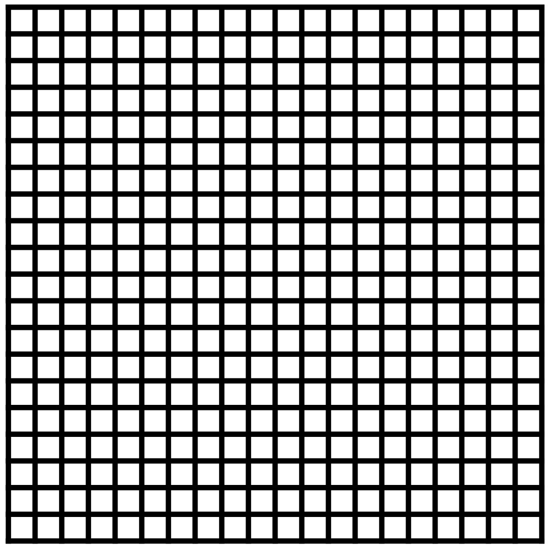
- Reasons for studying other representations
  - Restriction
    - Useful results (e.g., existence, uniqueness, efficient computability of solution concepts) might only hold in restricted classes.
    - E.g., symmetric games, zero-sum games, confrontation games
  - Compactness
    - Normal-form game representation of an n-player game with k actions per player requires the specification of k^n outcomes.
    - E.g., circuit games, anonymous games, graphical games
  - Representation
    - Other representation might be more natural and/or capture additional aspects of a strategic situations
    - E.g., extensive-form games, stochastic games, Bayesian games

Zero-Sum Games

- A two-player game is a *zero-sum game* if for every action profile $a \in A$, $u_1(a)+u_2(a) = 0$.
  - Since $u_2=-u_1$, zero-sum games can be represented using a single matrix (typically u_1).
  - Every “constant-sum” game is strategically equivalent to a zero-sum game.
- The interests of both players are diametrically opposed.
  - Examples: penalty shootout, rock-paper-scissors, chess, etc.

1	-1
-1	1

0	1	-1
-1	0	1
1	-1	0



Example

- Alice and Bob, once again, play a game by raising (one of) their hands. Alice wins if only one hand is raised, Bob if both raise their hands. Otherwise there is a tie.
  - Pure strategy security level
    - *0* for Alice (via *a*), *-1* for Bob (via any strategy)
    - Alice can enforce that her payoff is at least *0*. Bob can enforce that *her* payoff is at most *1*.
    - Alice's payoff will be between *0* and *1*.
  - Mixed strategy security level
    - *1/3* for Alice (via [*2/3:a, 1/3:b*]), *-1/3* for Bob (via [*2/3:x, 1/3:y*])
    - **The best randomized outcome that Alice can guarantee (1/3, -1/3) coincides with the best one that Bob can guarantee!**
  - Unique Nash equilibrium
    - [*2/3:a, 1/3:b*], [*2/3:x, 1/3:y*], expected outcome (*1/3, -1/3*)

	x	*y*
a	0	1
b	1	-1

Minimax Theorem

John v. Neumann

- Let's generalize the previous observation.
  - Let $v_1 = \max_{s_1} \min_{s_2} u_1(s_1,s_2)$ be player 1's security level and $v_2 = \min_{s_2} \max_{s_1} u_1(s_1,s_2)$ [$=-\max_{s_2} \min_{s_1} u_2(s_1,s_2)$] be the negative of player 2's security level.
  - Clearly, $v_1 \leq v_2$.
  - Both players can enforce that u_1 lies in [v_1,v_2] (and u_2 in [$-v_2$,$-v_1$]).
- Minimax Theorem (von Neumann, 1928): In every zero-sum game, $v_1 = v_2$.
- Von Neumann's original proof made involved use of topology, functional calculus, and Brouwer's theorem.
  - A much simpler proof can be given by relying on Nash's existence theorem from 1951.