Axiomatic Characterization

- The epistemic foundations for Nash equilibrium play are quite demanding.
  - Nevertheless, there is a nice axiomatic characterization of Nash equilibrium.
- Axioms
  - Utility maximization: In a one-player-game, only expected utility-maximizing strategies are returned.
  - Consistency: Let *s* be the solution of an *n*-player game *G* and define *G'* as the (*n*-*k*)-player game that results when *k* players invariably play their strategies from *s*. Then, the remaining players' strategies from *s* constitute a solution of *G'*.
  - Existence: Every game has at least one solution.

Axiomatic Characterization (ctd.)

- Theorem (Norde et al., 1996): If a solution concept satisfies utility maximization, consistency, and existence, then it is Nash equilibrium.
- Lemma (Peleg & Tijs, 1996): If a solution concept satisfies utility maximization and consistency, it maps to a subset of Nash equilibria.
  - Proof: *By contradiction*.
    Let s be a solution of some n-player game G that is *not* a Nash equilibrium. Then, there has to be some player i who obtains more payoff by deviating from s_i.

    Fix the strategies of all players except i to obtain the 1-player game G'. Consistency implies that s_i has to be a solution of G'. However, s_i is not utility-maximizing in G'.

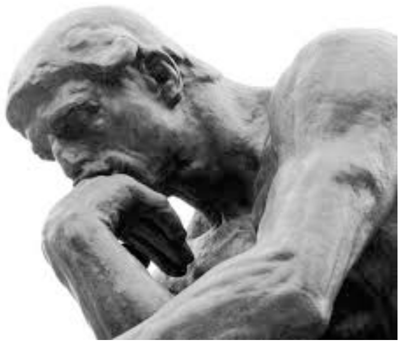
Computing Nash Equilibria

"The complexity of the mathematical work needed for a complete investigation increases rather rapidly, however, with increasing complexity of the game; so that analysis of a game much more complex than the example given here might only be feasible using approximate computational methods."

John F. Nash (1951)

- Deciding whether a game contains a Nash equilibrium is trivial.
- Pure Nash equilibria can be found efficiently.
  - Note, however, that the size of a normal-form game is exponential in the number of players.

Fictitious Play

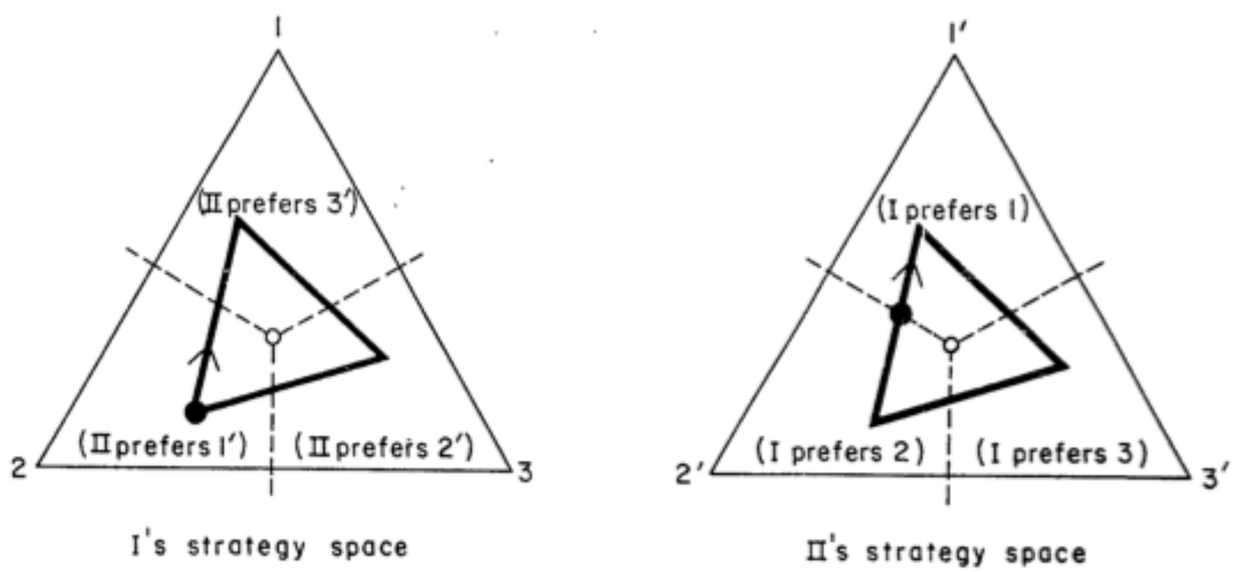


- Fictitious Play (FP) belongs to the class of so-called best response dynamics.
  - FP was proposed by Brown in 1951.
  - FP proceeds in rounds.
    - In the first round, each player arbitrarily chooses one of his actions.
    - In subsequent rounds, each player simultaneously looks at the empirical frequency of actions played by his opponents in previous rounds, interprets it as a probability distribution, and plays a pure best response against this distribution.
- If FP converges, then it converges to a Nash equilibrium.
  - FP has been proven to converge in 2×2 games by Miyasawa (1961), in games solvable via ISD by Nachbar (1990), and in 2×*k* games by Berger (2005).

	x	*y*
a	1, 0	0, 1
b	0, 1	1, 0

Fictitious Play (ctd.)

- Shapley (1964) showed that FP does not converge in general using a *3×3* game.



	x	*y*	*z*
a	0, 0	1, 0	0, 1
b	0, 1	0, 0	1, 0
c	1, 0	0, 1	0, 0

- Even for games in which FP is guaranteed to converge, it may take an exponential number of rounds before *some* equilibrium action is eventually played (B. et al., 2013; Exercise).

Decision Problems

- Each of the following problems is NP-complete (Gilboa & Zemel, 1989):
  - Is there a Nash equilibrium
    - that yields total utility of at least x?
    - that yields at least utility x for player i?
    - whose expected outcome is Pareto-optimal?
    - in which player i plays action a_i with positive probability?
    - in which player i never plays action a_i?
  - Is there more than one Nash equilibrium?
- All these results just require two players and also hold in binary games (Abbott et al., 2005; Biro et al., 2012).
  - None of these results implies the computational hardness of finding an equilibrium!

Finding all Equilibria

Robert Wilson

- A two-player game is degenerate if there is a strategy s_{-i} and $s_i \in B(s_{-i})$ such that $|supp(s_i)| > |supp(s_{-i})|$.
  - All Nash equilibria in non-degenerate games have same-size supports for both players.
- Theorem (Wilson, 1971): The number of Nash equilibria in every non-degenerate two-player game is finite and odd.
- Finding all Nash equilibria of a (non-degenerate) normal-form game requires exponential time in the worst case.
  - Proof: Common-payoff game defined by identity matrix of size k contains 2^k-1 Nash equilibria.
- Theorem (Du, 2013): Checking whether a two-player game is degenerate is NP-complete.

Nash Equilibrium Algorithms

- Lemke-Howson algorithm (1964)
  - only for 2-player games
  - search in strategy sets (simplices)
  - exponential worst-case running time (Savani & v. Stengel, 2004)
  - finding a “Lemke-Howson equilibrium” is PSPACE-complete (Goldberg et al., 2011)
- Simplicial subdivision algorithms
  - fixed point approximation
  - exponential worst-case running time
- Support enumeration algorithms
  - check all support profiles for equilibria using indifference principle
  - support profiles may be sorted by balancedness and size
  - exponential worst-case running time

Support Enumeration Algorithms (1)

- In two-player games, a *support profile B* ($\forall i \in N$: $B_i \subseteq A_i$) can be checked for Nash equilibria by solving a *linear feasibility program* (an LP without a maximization objective).

$$
\begin{array}{rcll}
\sum_{a_{-i} \in A_{-i}} s_{-i}(a_{-i}) u_i(a_i, a_{-i}) & = & U_i^* & \forall i \in N, a_i \in B_i \\
\sum_{a_{-i} \in A_{-i}} s_{-i}(a_{-i}) u_i(a_i, a_{-i}) & \leq & U_i^* & \forall i \in N, a_i \notin B_i \\
s_i(a_i) & \geq & 0 & \forall i \in N, a_i \in B_i \\
s_i(a_i) & = & 0 & \forall i \in N, a_i \notin B_i \\
\sum_{a_i \in A_i} s_i(a_i) & = & 1 & \forall i \in N
\end{array}
$$

  - In non-degenerate games, solutions to the linear feasibility problems are always unique.
  - When there are more than two players, the constraints are not linear anymore.

Support Enumeration Algorithms (2)

- There are several tricks to improve the runtime.
  - If an action is strictly dominated, then it is not a best response to any strategy.
    - Use ISD as preprocessing technique to reduce the game size.
  - If an action is weakly dominated, then it is not a best response to any full-support strategy
    - Check weak dominance for fixed supports.
    - For small supports, there are many weak dominations.
- Support enumeration algorithms outperform most other algorithms on sample distributions of games.
- They can find all Nash equilibria in non-degenerate games.

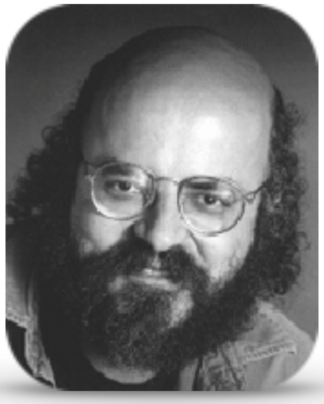
Support Enumeration Algorithms (3)

- Algorithm for *non-degenerate two-player* game *G*
  - REDUCE G via **ISD**
  - FOR EACH h∈{1,...,k}
  - FOR EACH $B_1 \subseteq A_1$ SUCH THAT $|B_1|=h$
  - $A_2' = \{a_2 \in A_2 \mid a_2$ **not weakly dominated** in $G|_{(B1\times A2)}\}$
  - IF $\forall a_1 \in B_1$: a_1 **not dominated** in $G|_{(A1\times A2')}$
  - FOR EACH $B_2 \subseteq A_2'$ SUCH THAT $|B_2|=h$
  - $A_1' = \{a_1 \in A_1 \mid a_1$ **not weakly dominated** in $G|_{(A1\times B2)}\}$
  - IF $B_1 \subseteq A_1'$
  - **check whether $G|_{(A1'\times A2')}$ contains an equilibrium with support B1,B2**
  - END IF
  - END FOR
  - END IF
  - END FOR
  - END FOR

ignore never-best-responses

ignore non-best-responses for given support

Equilibrium Complexity



Christos Papadimitriou

- Theorem (Daskalakis et al., Chen & Deng; 2005): The problem of finding a Nash equilibrium is PPAD-complete.
  - Finding a Nash equilibrium is as hard as finding a Brouwer fixed point in general.
  - It is believed that P≠PPAD and that PPAD-hardness is evidence that no efficient algorithm exists.
  - Theorem (Daskalakis, 2011): Computing a constant-factor *approximate Nash equilibrium* is PPAD-complete.
- History of the proof
  - Membership in PPAD: Papadimitriou (1991)
  - PPAD-hardness
    - 4 players: October 2005 (Daskalakis, Goldberg, & Papadimitriou)
    - 3 players: November 2005 (Daskalakis & Papadimitriou)
    - 2 players: December 2005 (Chen & Deng)

Preliminary Summary

	existence	*uniqueness*	*efficient computability*
dominant strategy	-	✓	✓
weakly dominant strategy	-	✓	✓
ISD-solvability	-	✓	✓
IWD-solvability	-	-	-
maximin	✓	✓ (security level)	✓
Nash equilibrium	✓	-	-