An Introduction to Cooperative Game Theory

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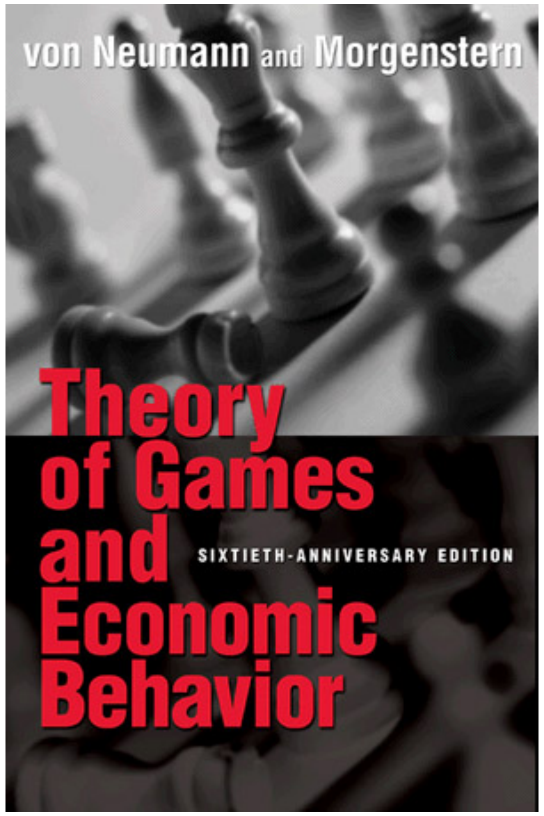
Cooperative Games

- *'Our subsequent discussion of "games of strategy" will show that the role and size of "coalitions" is decisive throughout the entire subject. […] Any satisfactory theory […] will have to explain under what circumstances such big coalitions will or will not be formed'* — von Neumann and Morgenstern



- Noncooperative games:
  - No binding agreements
  - Modeling unit is the individual player
- Cooperative games:
  - Binding agreements possible
  - Modeling unit is the coalition

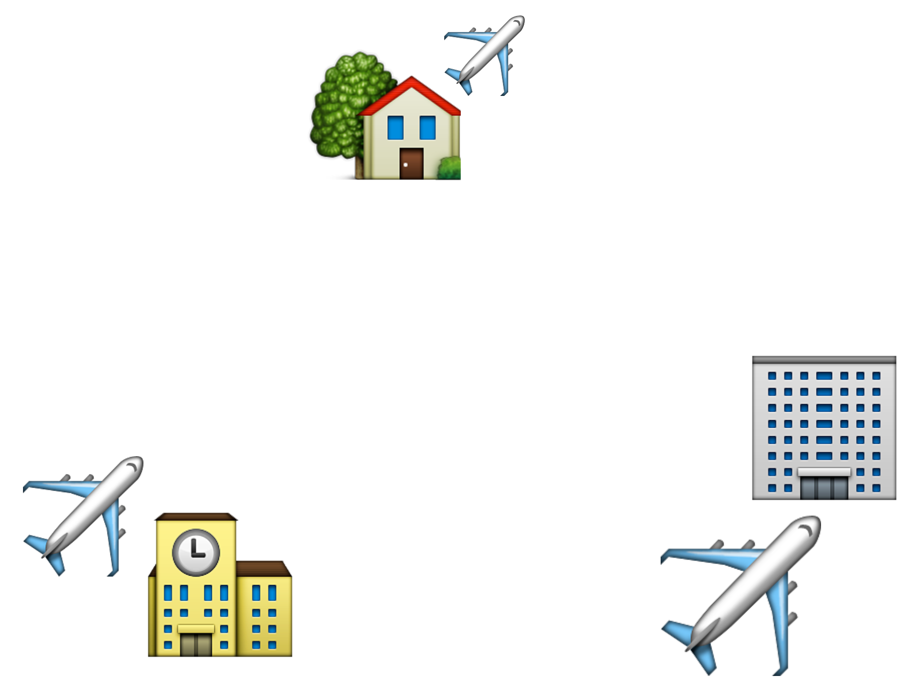
Outline

- Games in characteristic function form
  - Different classes of games
- Solution concepts
  - Stability: the core
  - Fairness: the Shapley Value
- Outlook

Cooperative Games with Transferable Utility

- A cooperative game with transferable utility (TU game) is a pair (N,v) where
  - N is a finite set of players, $|N| = n$, and
  - $v : 2^N \to \mathbb{R}$ is the characteristic function assigning a value to every subset $S \subseteq N$ with $v(\varnothing) = 0$.
- Example: Construction of an airport
  - $N = \{1,2,3\}$,

S	$v(S)$
$\{1\}$	-5
$\{2\}$	-7
$\{3\}$	-10
$\{1,2\}$	-9
$\{1,3\}$	-12
$\{2,3\}$	-12
N	-14



Classes of TU Games

- A game (N,v) is additive if for all $S,T \subseteq N$ where $S \cap T = \emptyset$ we have that $v(S \cup T) = v(S) + v(T)$.
- Example: Due to large distances, sharing an airport is not possible.
  - $N = \{1,2,3\}$,

S	$v(S)$
$\{1\}$	-5
$\{2\}$	-7
$\{3\}$	-10
$\{1,2\}$	-12
$\{1,3\}$	-15
$\{2,3\}$	-17
N	-22

  - There are no synergy effects.

Classes of TU Games

- A game (N,v) is convex if for all $S,T \subseteq N$ we have that $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$.
- Example: The introductory airport example.
  - $N = \{1,2,3\}$,

S	$v(S)$
$\{1\}$	-5
$\{2\}$	-7
$\{3\}$	-10
$\{1,2\}$	-9
$\{1,3\}$	-12
$\{2,3\}$	-12
N	-14

  - If there are synergy effects, they have to be nonnegative and fulfill some minimal requirements.
  - Every additive game is convex (proof: exercises).

Classes of TU Games

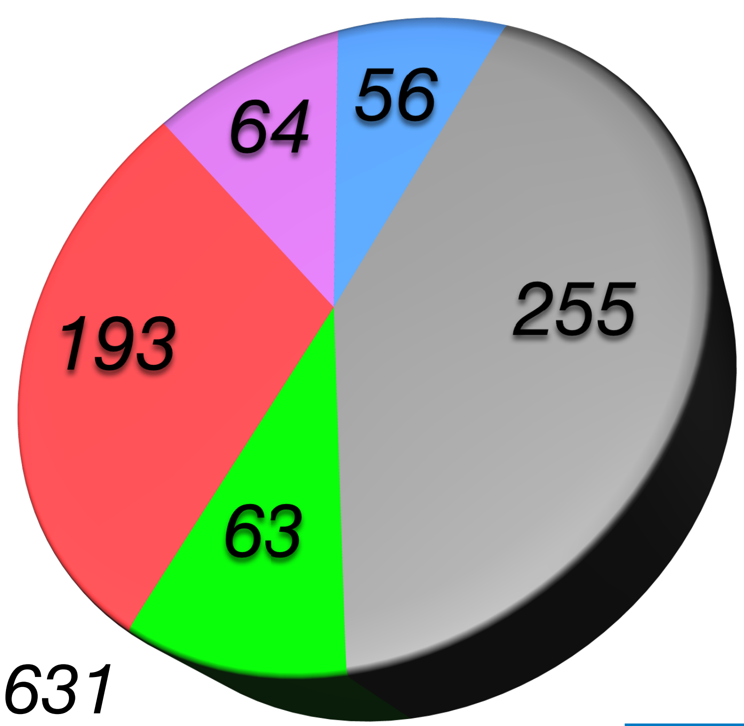
- A game (N,v) is superadditive if for all $S,T \subseteq N$ where $S \cap T = \emptyset$ we have that $v(S \cup T) \geq v(S) + v(T)$.
- Example: The building ground at the center is more expensive.
    - $N = \{1,2,3\}$,

S	$v(S)$
$\{1\}$	-5
$\{2\}$	-7
$\{3\}$	-10
$\{1,2\}$	-9
$\{1,3\}$	-12
$\{2,3\}$	-12
N	-15

    - Synergy effects have to be nonnegative.
    - Every convex game is superadditive (proof: exercises).
    - Most 'realistic' examples of TU games are superadditive.

Classes of TU Games

- A game (N,v) is simple if for all $S \subseteq N$ we have that $v(S) \in \{0,1\}$.
  - Simple games are often also called voting games.
  - A player i such that $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$ is called a vetoer.
- Example: Right now, five political parties are present in the Bundestag: CSU, CDU, B'90/Grüne, SPD, Linke
  - In order to pass a law changing the constitution, they need ⅔ of the votes.
  - Coalitions with ⅔ of the votes shall have value 1.
- Is there a vetoer?
  - Yes, CDU.
- What are the winning coalitions?
  - {CSU, CDU, B'90/Grüne, Linke},
    all $S \subseteq N$ such that {CDU, SPD} $\subseteq S$



Total: *631*

⅔: *421*

Solution Concepts

- A solution concept $\varphi : N \times \mathbb{R}^{2^N} \rightarrow \mathbb{R}^n$ maps every TU game to a payoff vector where $\varphi_i(N,v)$ is the payoff awarded to player i.
  - Whenever (N,v) is clear from the context, we write $x_i = \varphi_i(N,v)$ and $x(S) = \sum_{i \in S} x_i$.
- A payoff vector $x \in \mathbb{R}^n$ is called feasible if $\sum_{i \in N} x_i \leq v(N)$.
- $x \in \mathbb{R}^n$ is called individually rational if $x_i \geq v(\{i\})$ for all $i \in N$.
- $x \in \mathbb{R}^n$ is called efficient if $\sum_{i \in N} x_i = v(N)$.

The Core

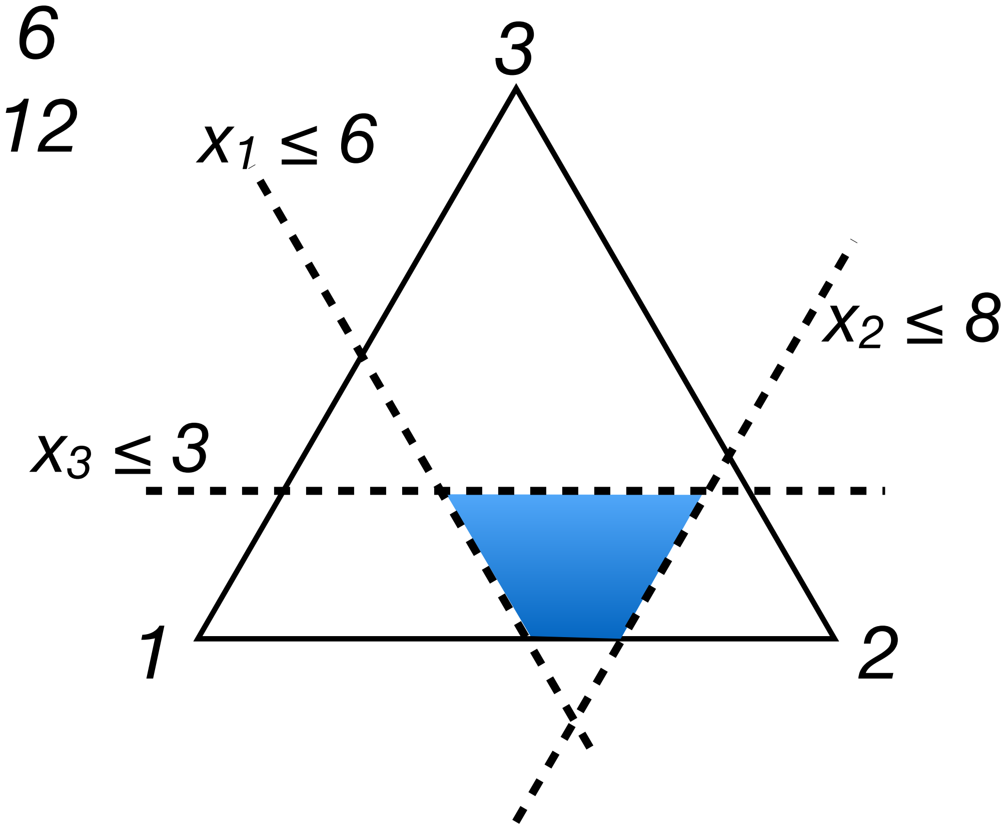
- The core is a measure of stability, i.e., how willing is the grand coalition to stay together.
- A payoff vector x is in the core iff $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$.
  - We will additionally assume efficiency for x.
- Being a notion of stability, the core is an analog to Nash equilibria in non-cooperative game theory (or, more precisely, to strong equilibria).
- Computation is straightforward via a linear feasibility problem.
- Two questions arise:
  - Is the core always nonempty?
  - Is the core always unique?

Example: Computing the Core

- Consider the following game:
  - $N = \{1,2,3\}$
S	$v(S)$
$\{i\}$	0
$\{1,2\}$	9
$\{1,3\}$	4
$\{2,3\}$	6
N	12

Recall: x is in the core iff $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$.

$$x_i \geq 0$$
$$\sum_{i \in N} x_i = 12$$
$$x_1 + x_2 \geq 9 \quad \Rightarrow x_3 \leq 3$$
$$x_1 + x_3 \geq 4 \quad \Rightarrow x_2 \leq 8$$
$$x_2 + x_3 \geq 6 \quad \Rightarrow x_1 \leq 6$$



What if we add 2 to every 2-coalition's payoff?

Example: Computing the Core

- Consider the following game:
  - $N = \{1,2,3\}$
S	$v(S)$
$\{i\}$	0
$\{1,2\}$	11
$\{1,3\}$	6
$\{2,3\}$	8
N	12

Recall: x is in the core iff $\sum_{i\in S} x_i \geq v(S)$ for all $S \subseteq N$.

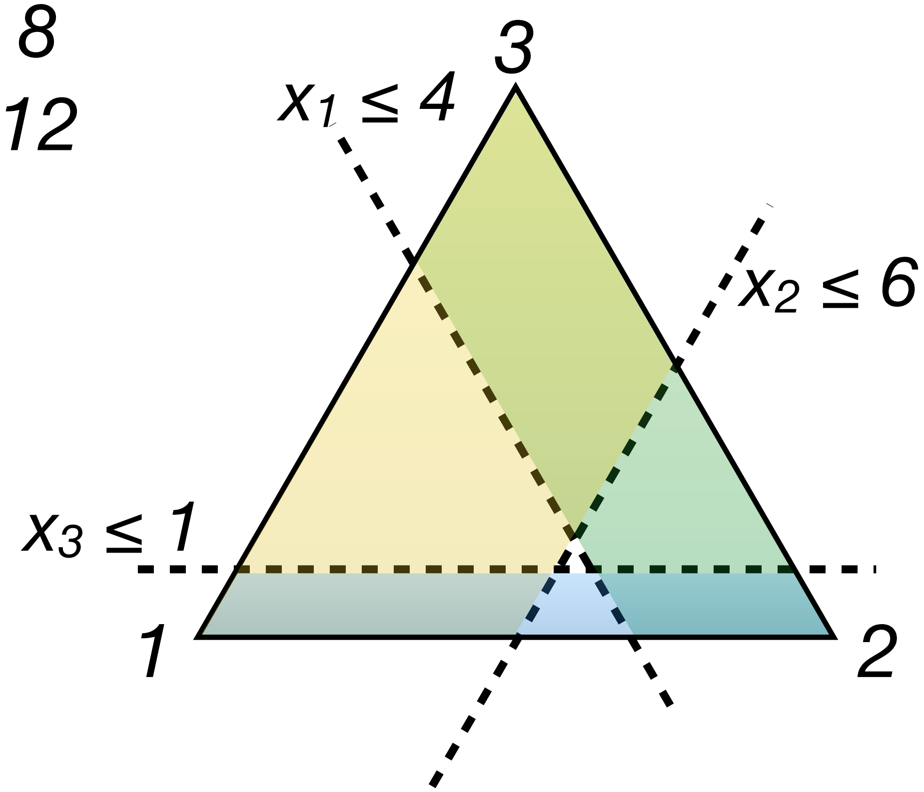
$$x_i \geq 0$$

$$\sum_{i\in N} x_i = 12$$

$$x_1 + x_2 \geq 11 \quad \Rightarrow x_3 \leq 1$$

$$x_1 + x_3 \geq 6 \quad \Rightarrow x_2 \leq 6$$

$$x_2 + x_3 \geq 8 \quad \Rightarrow x_1 \leq 4$$



$\Rightarrow$ The core is empty.

The Core

- However: we can say something about restricted classes of TU games:
- Theorem: A simple game has a nonempty core iff it has a vetoer.
  - ‘$\Leftarrow$’: Assume there exists at least one vetoer i and let $x_i = 1$. Now consider any $S \subseteq N$: if $v(S) = 0$, S has no incentive to deviate. If $v(S) = 1$, then $i \in S$ and therefore $x(S) = v(S)$.
  - ‘$\Rightarrow$’: Assume there is no vetoer. Now consider any x and note that $x_i > 0$ for some i. Since i is not a vetoer, $v(N \setminus \{i\}) = 1$. Consequently, $x(N \setminus \{i\}) < v(N \setminus \{i\})$.

The Core

- Theorem (Shapley, 1971): Every convex game has a nonempty core.
  - The definition of a convex game is equivalent to $v(A \cup \{i\}) - v(A) \geq v(B \cup \{i\}) - v(B)$ for all $A, B \subseteq N \setminus \{i\}$ such that $B \subseteq A$ (proof: exercises).
  - Set $x_1 = v(\{1\})$, $x_2 = v(\{1,2\}) - (\{1\})$, … , $x_n = v(N) - v(N \setminus \{n\})$.
  - Consider any coalition $C = \{j_1, \ldots, j_k\} \subseteq N$ such that $j_1 < \ldots < j_k$:

$$
\begin{aligned}
\sum_{i \leq k} x_{j_i} &= \sum_{i \leq k} ([v(\{1, \ldots, j_i\}) - v(\{1, \ldots, j_i - 1\})] \\
&\geq \sum_{i \leq k} ([v(\{j_1, \ldots, j_i\}) - v(\{j_1, \ldots, j_{i-1}\})] \\
&= [v(j_1) - v(\varnothing)] + [v(\{j_1, j_2\}) - v(\{j_1\})] + \ldots + [v(\{j_1, \ldots, j_k\}) - v(\{j_1, \ldots, j_{k-1}\})] \\
&= v(\{j_1, \ldots, j_k\}) \\
&= v(C)
\end{aligned}
$$

  - x is in the core.

The Shapley Value

- Three axioms regarding fairness
  - φ is additive if for all v_1, v_2, N, $i \in N$, we have

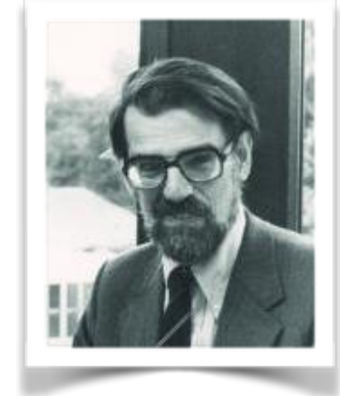
    $\varphi_i(N,v_1+v_2) = \varphi_i(N,v_1) + \varphi_i(N,v_2)$ where (N,v_1+v_2) is defined by $(v_1+v_2)(S) = v_1(S) + v_2(S)$ for all $S \subseteq N$.
  - φ is symmetric if it holds for all v, N:

    $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\} \Rightarrow \varphi_i(N,v) = \varphi_j(N,v)$.
  - φ satisfies nullity if it holds for all $i \in N$, v:

    $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\} \Rightarrow \varphi_i(N,v) = 0$

    In this case, i is called a null-player.
- Also recall:
  - efficiency: For all v, N, we have $\sum_{i \in N} \varphi_i(N,v) = v(N)$.

The Shapley Value



- Theorem (Shapley, 1953): There is a unique solution satisfying efficiency, additivity, symmetry and nullity: the Shapley Value.

- Given a TU game (N,v), the Shapley Value of player i is computed via

$$Sh_i(N,v) = \sum_{S \subseteq N,\, i \notin S} \frac{s!\,(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

where $s = |S|$.

The Shapley Value

- Call $v(S \cup \{i\})$ - $v(S)$ player i's marginal contribution (to S).
- The Shapley Value captures the average marginal contribution of player i with respect to all possible orderings of players joining the empty coalition in order to form the grand coalition.
  - This offers a different way of computing the Shapley Value via enumeration of all permutations (which can be easier for a small number of players).
- Does the Shapley Value always lie within the core?
  - In general: No, the core may be empty.
  - Theorem: In every convex game, the Shapley Value lies within the core.

Core & Shapley Value: Overview

	existence	uniqueness	computable in poly. time
Core	❌	❌	✅
Shapley Value	✅	✅	✅

- Regarding existence and uniqueness, there are various refinements of the core (ε-core, least core, nucleolus)
- Regarding computation, the real problem is the size of the input.
  - The representation by a characteristic function is exponential in the number of players.
- Possible way out: compact representations
  - Weighted graph games
  - Marginal contribution nets

What more is out there?

- Refinements of the core (e.g., ε-core, nucleolus)
- Variations of the Shapley Value (e.g., min/max payoff)
- Compact representations (e.g., games on graphs)
- Solutions focusing on voting power (e.g., Banzhaf Value)
- Games with coalition structures/in partition function form
- Fair division (e.g., cake cutting)
- Coalition formation (hedonic games)