Iterated Dominance Solvability

- A game can be *solved* via iterated strict dominance (ISD) if only a single action profile survives the iterated elimination of dominated actions.
  - A two-player game can be solved via ISD if both players have only one rationalizable action.
- Can this game be solved via ISD?

	x	y	z
a	3, 1	0, 0	0, 0
b	1, 1	1, 2	5, 0
c	0, 1	4, 0	0, 0

Iterated Dominance (ctd.)

- The set of actions that survive iterated dominance can be computed in polynomial time by solving a polynomial number of LPs.
  - The set of rationalizable actions of a two-player game can be found efficiently.
- How about weak dominance?
  - Action a_i is weakly dominated iff there is a strategy s_i such that the following LP has a solution with positive value.

$$
\begin{array}{llcll}
\text{max} & \multicolumn{4}{l}{\sum_{a_{-i}\in A_{-i}} \left(\left(\sum_{b_i\in A_i} s_i(b_i) u_i(b_i, a_{-i}) \right) - u_i(a_i, a_{-i}) \right)} \\
\text{s.t.} & \sum_{b_i\in A_i} s_i(b_i) u_i(b_i, a_{-i}) & \geq & u_i(a_i, a_{-i}) & \forall a_{-i} \in A_{-i} \\
 & s_i(b_i) & \geq & 0 & \forall b_i \in A_i \\
 & \sum_{b_i\in A_i} s_i(b_i) & = & 1 &
\end{array}
$$

Example

- Can this game be solved via iterated weak dominance (IWD)?

	x	y	z
a	2, 1	1, 1	0, 0
b	1, 1	1, 2	0, 1
c	0, 0	1, 0	1, 1

- IWD is order-dependent!

Iterated Weak Dominance

- Theorem: Deciding whether a game can be solved via IWD is NP-complete (Conitzer & Sandholm, 2005).
  - even when only considering dominance by pure strategies
  - even when there are only two different utility values (say, 0 and 1)
    - even with only three outcomes ((0,0), (1,0), and (0,1)) (B. et al., 2009)
  - also deciding whether a given action can be eliminated via IWD is NP-complete

	p	$\neg p$	q	$\neg q$	r	$\neg r$	a	b	c	d^*
p	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg p$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$p\downarrow$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$
q	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg q$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$q\downarrow$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$
r	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$r\downarrow$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$
p	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
q	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$p \vee q \vee \neg r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$
$\neg p$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
q	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
r	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg p \vee q \vee r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$
$\neg p$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg q$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg p \vee \neg q \vee \neg r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$
e	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,0)$

	a_2^1	$\cdots$	a_2^m	c	d^*	f	g^*	x_2^1	$\cdots$	y_2^4	z_2^1	z_2^2	z_2^3
a_1^1	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,1)$	$(0,1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_1^n	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,1)$	$(0,1)$
e	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,1)$	$(0,1)$
x_1^1	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,1)$	$(0,1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_1^4	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,1)$	$(0,1)$
z_1^1	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,0)$
z_1^2	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$
z_1^3	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(1,0)$	$(0,1)$
z_1^4	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$	$(1,0)$

Solution Concepts

- A solution concept identifies reasonable, desirable, or otherwise significant strategy profiles of a game.
  - In order to be meaningful, solution concepts should be invariant under positive affine transformations.
- Important properties of solution concepts:
  - Existence
    - Games may not be solvable via ISD or IWD.
  - Uniqueness
    - If a game is solvable via ISD, the resulting action profile is unique.
    - If a game is solvable via IWD, there may be different resulting action profiles.
  - Efficient computability
    - It can be decided in polynomial time whether a game can be solved via ISD.
    - Whether a game can be solved via IWD cannot be decided in polynomial time unless P=NP.

Standard Examples

	shoot left	*shoot right*
jump left	0 / 1	1 / 0
jump right	1 / 0	0 / 1

“Penalty Shootout”
or “Matching Pennies”

	yield	*straight*
yield	2 / 2	3 / 1
straight	1 / 3	0 / 0

“Chicken”
or “Hawks and Doves”

	boxing	*ballet*
boxing	1 / 2	0 / 0
ballet	0 / 0	2 / 1

“Battle of the Sexes”
or “Bach or Stravinsky”

- All these games are neither solvable via ISD nor via IWD.
  - Every action is rationalizable.

Maximin and Security Level

- In face of uncertainty, a player may maximize his worst-case utility.
  - "Worst-case" refers to the choice of strategy of the opponents, not the outcome of the randomization.
- The set of *maximin strategies* of player i is given by $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i,s_{-i})$.
- The *security level* of player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i,s_{-i})$.
  - The security level is the minimal utility that the player can enforce.
    - Independent of the opponents' rationality.
  - Again, it suffices to consider pure strategies of the opponents.

Maximin (Linear Program)

- Any solution s_i of the following LP is a maximin strategy.

$$
\begin{array}{llrcll}
\text{max} & & & & U_i^* & \\
\text{s.t.} & \sum_{b_i \in A_i} s_i(b_i) u_i(b_i, a_{-i}) & & \geq & U_i^* & \forall a_{-i} \in A_{-i} \\
& & s_i(b_i) & \geq & 0 & \forall b_i \in A_i \\
& & \sum_{b_i \in A_i} s_i(b_i) & = & 1 &
\end{array}
$$

- The value of the LP (U_i^*) is the security level of player i.
- Maximin strategies (and security levels) can be computed in polynomial time.
- The set of maximin strategies is convex, i.e., a mixture of maximin strategies is again a maximin strategy.

Standard Examples Revisited

	shoot left	*shoot right*
jump left	1, 0	0, 1
jump right	0, 1	1, 0

"Penalty Shootout"
or "Matching Pennies"

	yield	*straight*
yield	2, 2	1, 3
straight	3, 1	0, 0

"Chicken"
or "Hawks and Doves"

	boxing	*ballet*
boxing	2, 1	0, 0
ballet	0, 0	1, 2

"Battle of the Sexes"
or "Bach or Stravinsky"

- Maximin strategies
  - Penalty Shootout: [*1/2:* left, *1/2:* right], security level *1/2*
  - Chicken: [*1:* yield], security level *1*
  - Battle of the Sexes: [*1/3:* boxing, *2/3:* ballet] for player 1 and [*2/3:* boxing, *1/3:* ballet] for player 2, security level *2/3*

Preliminary Summary

	existence	*uniqueness*	*efficient computability*
dominant strategy	-	✓	✓ (has to be pure)
weakly dominant strategy	-	✓	✓ (has to be pure)
ISD-solvability	-	✓	✓
IWD-solvability	-	-	-
maximin	✓	✓ (security level)	✓

Example

- Maximin strategies always exist, but often they do not permit significant statements about games because a lot of the structure is ignored.

	x	*y*	*z*
a	0, 0	1, 0	2, 0
b	0, 1	2, 0	1, 2
c	0, 2	0, 3	3, 1

- Which strategies should the players play?
  - *all* strategies are maximin strategies, security level 0 for both players
  - *all* actions are rationalizable, no action is dominated (not even weakly)

John Nash

Nash Equilibrium

- A strategy profile $s = (s_1, \ldots, s_n)$ is a *Nash equilibrium* if for all players i and all strategies $t_i \neq s_i$, $u_i(s_i,s_{-i}) \geq u_i(t_i,s_{-i})$.
  - No player can increase his payoff by deviating unilaterally.
  - Nash equilibrium is perhaps the most widely known solution concept in game theory.
  - A pure Nash equilibrium is a Nash equilibrium in pure strategies.
- A Nash equilibrium is a steady state of mutual best responses.
  - Every strategy is a best response to the strategies of the others: $s_i \in B_i(s_{-i})$ for all $i \in N$
- (defect,defect) is the only Nash equilibrium in the prisoner's dilemma.

2, 2	0, 3
3, 0	$\boxed{1, 1}$