

Iterated Dominance Solvability

- ▶ A game can be *solved via iterated strict dominance (ISD)* if only a single action profile survives the iterated elimination of dominated actions.
 - ▶ A two-player game can be solved via ISD if both players have only one rationalizable action.
- ▶ Can this game be solved via ISD?

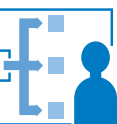
	<i>x</i>	<i>y</i>	<i>z</i>
<i>a</i>	3, 1	0, 0	0, 0
<i>b</i>	1, 1	2, 1	0, 5
<i>c</i>	0, 1	4, 0	0, 0



Iterated Dominance (ctd.)

- ▶ The set of actions that survive iterated dominance can be computed in **polynomial time** by solving a polynomial number of LPs.
 - ▶ The set of rationalizable actions of a two-player game can be found efficiently.
- ▶ How about weak dominance?
 - ▶ Action a_i is weakly dominated iff there is a strategy s_i such that the following LP has a solution with **positive value**.

$$\begin{array}{ll}
 \max & \sum_{a_{-i} \in A_{-i}} \left(\left(\sum_{b_i \in A_i} s_i(b_i) u_i(b_i, a_{-i}) \right) - u_i(a_i, a_{-i}) \right) \\
 \text{s.t.} & \sum_{b_i \in A_i} s_i(b_i) u_i(b_i, a_{-i}) \geq u_i(a_i, a_{-i}) & \forall a_{-i} \in A_{-i} \\
 & s_i(b_i) \geq 0 & \forall b_i \in A_i \\
 & \sum_{b_i \in A_i} s_i(b_i) = 1
 \end{array}$$



Example

- ▶ Can this game be solved via iterated weak dominance (IWD)?

	<i>x</i>	<i>y</i>	<i>z</i>
<i>a</i>	2, 1	1, 1	0, 0
<i>b</i>	1, 1	1, 2	0, 1
<i>c</i>	0, 0	1, 0	1, 1

- ▶ IWD is order-dependent!



Iterated Weak Dominance

- ▶ Theorem: Deciding whether a game can be solved via IWD is **NP-complete** (Conitzer & Sandholm, 2005).
 - ▶ even when only considering dominance by pure strategies
 - ▶ even when there are only two different utility values (say, 0 and 1)
 - even with only three outcomes ((0,0), (1,0), and (0,1)) (B. et al., 2009)
 - ▶ also deciding whether a given action can be eliminated via IWD is NP-complete

	p	$\neg p$	q	$\neg q$	r	$\neg r$	a	b	c	d^*
p	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 0)
$\neg p$	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 0)
$p \downarrow$	(1, 0)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 1)
q	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 0)
$\neg q$	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 0)
$q \downarrow$	(0, 1)	(0, 1)	(1, 0)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 1)
r	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 0)
$\neg r$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(1, 0)	(0, 1)	(0, 0)	(0, 0)
$r \downarrow$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(1, 0)	(0, 1)	(0, 1)	(0, 0)	(0, 1)
p	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 0)
q	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 0)
$\neg r$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 0)	(0, 0)
$p \vee q \vee \neg r$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 1)
$\neg p$	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 0)
q	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 0)
r	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 0)
$\neg p \vee q \vee r$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 1)
$\neg p$	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 0)
$\neg q$	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 0)
$\neg r$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 0)	(0, 0)
$\neg p \vee \neg q \vee \neg r$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 1)
e	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(0, 1)	(0, 1)	(0, 0)

	a_2^1	\dots	a_2^m	c	d^*	f	g^*	x_2^1	\dots	y_2^4	z_2^1	z_2^2	z_2^3
a_1^1	.	\dots	\dots	.	(0, 1)	(0, 1)	(0, 1)
\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots
a_1^n	.	\dots	\dots	.	(0, 1)	(0, 1)	(0, 1)
e	.	\dots	.	(0, 1)	(0, 0)	(0, 1)	(0, 0)	.	\dots	.	(0, 1)	(0, 1)	(0, 1)
x_1^1	.	\dots	\dots	.	(0, 1)	(0, 1)	(0, 1)
\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots
y_1^4	.	\dots	\dots	.	(0, 1)	(0, 1)	(0, 1)
z_1^1	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 1)	(1, 0)	(1, 0)
z_1^2	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)
z_1^3	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(1, 0)	(0, 1)
z_1^4	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 0)	(0, 1)	(1, 0)



Solution Concepts

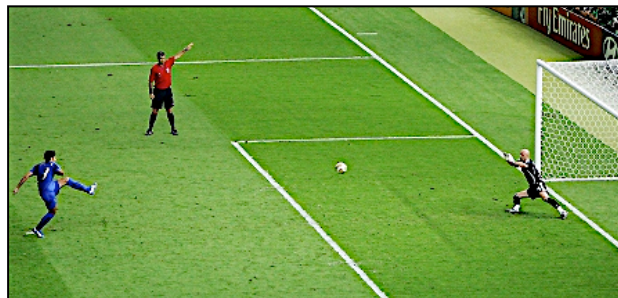
- ▶ A solution concept identifies reasonable, desirable, or otherwise significant strategy profiles of a game.
 - ▶ In order to be meaningful, solution concepts should be **invariant under positive affine transformations**.
- ▶ Important properties of solution concepts:
 - ▶ **Existence**
 - Games may not be solvable via ISD or IWD.
 - ▶ **Uniqueness**
 - If a game is solvable via ISD, the resulting action profile is unique.
 - If a game is solvable via IWD, there may be different resulting action profiles.
 - ▶ **Efficient computability**
 - It can be decided in polynomial time whether a game can be solved via ISD.
 - Whether a game can be solved via IWD cannot be decided in polynomial time unless $P=NP$.



Standard Examples

	shoot left	shoot right
jump left	1, 0	0, 1
jump right	0, 1	1, 0

“Penalty Shootout”
or “Matching Pennies”



	yield	straight
yield	2, 2	1, 3
straight	3, 1	0, 0

“Chicken”
or “Hawks and Doves”



	boxing	ballet
boxing	2, 1	0, 0
ballet	0, 0	1, 2

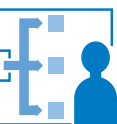
“Battle of the Sexes”
or “Bach or Stravinsky”



- ▶ All these games are neither solvable via ISD nor via IWD.
 - ▶ Every action is rationalizable.

Maximin and Security Level

- ▶ In face of uncertainty, a player may **maximize his worst-case utility**.
 - ▶ “Worst-case” refers to the choice of strategy of the opponents, not the outcome of the randomization.
- ▶ The set of **maximin strategies** of player i is given by $\arg \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.
- ▶ The **security level** of player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.
 - ▶ The security level is the minimal utility that the player can enforce.
 - Independent of the opponents’ rationality.
 - ▶ Again, it suffices to consider pure strategies of the opponents.

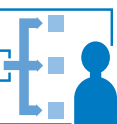


Maximin (Linear Program)

- ▶ Any solution s_i of the following LP is a maximin strategy.

$$\begin{array}{ll} \max & U_i^* \\ \text{s.t.} & \sum_{b_i \in A_i} s_i(b_i) u_i(b_i, a_{-i}) \geq U_i^* \quad \forall a_{-i} \in A_{-i} \\ & s_i(b_i) \geq 0 \quad \forall b_i \in A_i \\ & \sum_{b_i \in A_i} s_i(b_i) = 1 \end{array}$$

- ▶ The value of the LP (U_i^*) is the security level of player i .
- ▶ Maximin strategies (and security levels) can be computed in **polynomial time**.
- ▶ The set of maximin strategies is **convex**, i.e., a mixture of maximin strategies is again a maximin strategy.



Standard Examples Revisited

	<i>shoot left</i>	<i>shoot right</i>
<i>jump left</i>	1, 0	0, 1
<i>jump right</i>	0, 1	1, 0

“Penalty Shootout”
or “Matching Pennies”

	<i>yield</i>	<i>straight</i>
<i>yield</i>	2, 2	1, 3
<i>straight</i>	3, 1	0, 0

“Chicken”
or “Hawks and Doves”

	<i>boxing</i>	<i>ballet</i>
<i>boxing</i>	2, 1	0, 0
<i>ballet</i>	0, 0	1, 2

“Battle of the Sexes”
or “Bach or Stravinsky”

► Maximin strategies

- Penalty Shootout: $[1/2: \text{left}, 1/2: \text{right}]$, security level $1/2$
- Chicken: $[1: \text{yield}]$, security level 1
- Battle of the Sexes: $[1/3: \text{boxing}, 2/3: \text{ballet}]$ for player 1 and $[2/3: \text{boxing}, 1/3: \text{ballet}]$ for player 2, security level $2/3$



Preliminary Summary

	<i>existence</i>	<i>uniqueness</i>	<i>efficient computability</i>
<i>dominant strategy</i>	–	✓	✓ (has to be pure)
<i>weakly dominant strategy</i>	–	✓	✓ (has to be pure)
<i>ISD-solvability</i>	–	✓	✓
<i>IWD-solvability</i>	–	–	–
<i>maximin</i>	✓	✓ (security level)	✓



Example

- ▶ Maximin strategies always exist, but often they do not permit significant statements about games because a lot of the structure is ignored.

	<i>x</i>	<i>y</i>	<i>z</i>
<i>a</i>	0 0	1 0	2 0
<i>b</i>	0 1	2 0	1 2
<i>c</i>	0 2	0 3	3 1

- ▶ Which strategies should the players play?
 - *all* strategies are maximin strategies, security level 0 for both players
 - *all* actions are rationalizable, no action is dominated (not even weakly)





John Nash

Nash Equilibrium



- ▶ A strategy profile $s = (s_1, \dots, s_n)$ is a *Nash equilibrium* if for all players i and all strategies $t_i \neq s_i$, $u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i})$.
 - ▶ No player can increase his payoff by deviating unilaterally.
 - ▶ Nash equilibrium is perhaps the most widely known solution concept in game theory.
 - ▶ A *pure Nash equilibrium* is a Nash equilibrium in pure strategies.
- ▶ A Nash equilibrium is a steady state of mutual best responses.
 - ▶ Every strategy is a best response to the strategies of the others:
 $s_i \in B_i(s_{-i})$ for all $i \in N$
- ▶ (defect, defect) is the only Nash equilibrium in the prisoner's dilemma.

	2	3
2	0	
3	0	1

