Technische Universität München **SS 2015**
Institut für Informatik (I18) **Solution hints for exercise sheet 1**
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Algorithmic Game Theory (IN2239)

G-exercise due: April 18, 2015, 23:59 (Moodle, https://www.moodle.tum.de)

Exercise 2 *Transitivity* (**H**)

Let $\succsim$ be a complete preference relation.
Show that $\succsim$ is transitive if and only if $\succ$ and $\sim$ are both transitive.

Solution hints:

$\Rightarrow$: Assume $\succsim$ is transitive. Consider arbitrary x, y, z such that $x \sim y \sim z$. Then, $x \succsim y$ as well as $y \succsim z$. Hence by transitivity of $\succsim$, $x \succsim z$. Moreover, also $z \succsim y$ and $y \succsim x$. Again by transitivity of $\succsim$, $z \succsim x$. By definition of $\sim$, it now follows that $x \sim z$, as desired.
Now, consider arbitrary x, y, z such that $x \succ y \succ z$. Then again, $x \succsim y \succsim z$ and by transitivity of $\succsim$, $x \succsim z$. Assume for contradiction that $z \succsim x$. By transitivity of $\succsim$, we get $z \succsim y$ which contradicts $y \succ z$. Therefore $x \succ z$.

$\Leftarrow$: Assume that both $\succ$ and $\sim$ are transitive. Consider arbitrary x, y, z and assume that $x \succsim y \succsim z$. Assume for a contradiction that it is not the case that $x \succsim z$. By completeness, then, $z \succsim x$. Hence, $z \succ x$. We distinguish four cases:

i) $x \succ y$ and $y \succ z$,
ii) $x \succ y$ and $y \sim z$,
iii) $x \sim y$ and $y \succ z$,
iv) $x \sim y$ and $y \sim z$.

If i), by transitivity of $\succ$, $x \succ z$ and hence $x \succsim z$, a contradiction. If iv), by transitivity of $\sim$, $x \sim z$ and hence $x \succsim z$, a contradiction. If ii), both $z \succ x$ and $x \succ y$. Hence, by transitivity of $\succ$, $z \succ y$, which contradicts $y \sim z$. Finally, if (iii), $y \succ z$ and $z \succ x$, by transitivity of $\succ$, $y \succ x$, contradicting $x \sim y$.

Exercise 3 *Extending rational preferences* (**H**)

Let X, Y be disjoint sets and $R \subseteq X \times X, R' \subseteq Y \times Y$ complete and transitive relations on X and Y, respectively. Show that there is a complete and transitive extension of $R \cup R'$ on $X \cup Y$.
Solution hints: Consider $R'' = R \cup R' \cup (X \times Y)$. R'' is obviously complete.

For transitivity, consider arbitrary $x, y, z \in X \cup Y$ and, for contradiction, assume that $x\ R''\ y\ R''\ z$ but not $x\ R''\ z$. Either $z \in X$ or $z \in Y$. If the former, also $x \in X$ and $y \in X$. But then $x\ R\ y\ R\ z$ and by transitivity of R on X, also $x\ R\ z$. It follows that $x\ R''\ z$ as well, a contradiction.
In the latter case, also $x \in Y$ (because of $zR''x$). Hence, $y \in Y$ as well. Then, $x\ R'\ y\ R'\ z$ and by transitivity of R' on Y, moreover, $x\ R'\ z$. By definition of R'', also $x\ R''\ z$, a contradiction.

Exercise 4 *Representing preferences by utility functions* **(T)**

(a) Show that rational preferences over a countable set A of alternatives can be represented by a utility function $u\colon A \to [-1, 1]$.

(b) Let $A = [0, 1] \times [0, 1]$. Define lexicographic preferences $\succsim$ over A such that for all $x, y \in A$,

$$x \succsim y \text{ iff } x_1 > y_1 \text{ or both } x_1 = y_1 \text{ and } x_2 \geqslant y_2.$$

Show that this preference relation cannot be represented by a utility function.

Solution hints: For the solution of this exercise, we will use that

- $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e., for any two reals, there is a rational number between them, and
- the cardinality of $\mathbb{R}$ is strictly greater than the cardinality of $\mathbb{Q}$.

(a) In the lecture, the statement was made for a *finite* number of alternatives, we proof the stronger statement for a *countable* number of alternatives. First, enumerate the alternatives $a_1, \ldots, a_n$. This can be done as A is countable. We will subsequently assign to each a_i a utility $u(a_i)$ in $\mathbb{R}$, starting with a_1.

Construction hypothesis At step n, we have that $u(a_1), \ldots, u(a_n)$ are already fixed in a way s.t. for all $i, j \in \{1, \ldots, n\}$

$$u(a_i) \notin \{-1, 1\} \text{ and}$$
$$u(a_i) \geqslant u(a_j) \text{ iff } a_i \succsim a_j.$$

Construction Basis:

$$u(a_1) = 0$$

Construction Step: Assume construction hypothesis holds for n. Need to fix a value for $u(a_{n+1})$.

- If $a_{n+1} \sim a_k$ for some $k \in \{1, \ldots, n\}$, set $u(a_{n+1} = u(a_k))$.
- If not, consider

$$A_n^+ = \{u(a_i) \mid i \in \{1, \ldots, n\} \text{ and } a_{n+1} \succsim a_i\} \cup \{-1\} \text{ and}$$
$$A_n^- = \{u(a_i) \mid i \in \{1, \ldots, n\} \text{ and } a_i \succsim a_{n+1}\} \cup \{1\}$$

We claim that A_n^+ and A_n^- form discrete intervals of $[-1, 1]$ that do not overlap, i.e., for all $u(a^+) \in A_n^+, u(a^-) \in A_n^-$ we have $u(a^+) < u(a^-)$. If this was not the case, there would be $u(a^+) \in A_n^+$ and $u(a^-) \in A_n^-$ with $u(a^-) < u(a^+)$. But this implies (by construction hypothesis) that $a^+ \succ a^-$. Also, by definition of A_n^-, $a^- \succ a_{n+1}$. By transitivity, we get $a^+ \succ a_{n+1}$, violating $a^+ \in A_n^+$, a contradiction. Given that either $|A_n^+| = 1$ or $|A_n^-| = 1$, the discrete intervals do not overlap by the hypothesis. Thus, the claim holds. Therefore, there is "space" between A^+ and A^- from where we can choose the utility for a_{n+1}, e.g.,

$$u(a_{n+1}) = \frac{\max A_n^+ + \min A_n^-}{2}.$$

Now, we need to check whether this utility function does actually represent the preferences. To this end, let $b, c \in A$. By the enumeration, there are k and ℓ such that $b = a_k$ and $c = a_\ell$. W.l.o.g $k > l$. It is straightforward to check that the construction determined $u(a_\ell)$ in a way that $u(a_\ell) \geqslant u(a_k)$ if and only if $a_\ell \succsim a_k$. This concludes the proof.

(b) The high-level idea is as follows: For each value in the first dimension, we need to reserve some space in the reals for a utility representation. Reserving space uncountably many times is problematic because in each such space lies a rational number, entailing a contradiction.

Assume that there is a utility representation of $\succsim$. Observe that for all $a \in [0, 1]$, we have $(a, 1) \succ (a, 0)$, implying $u(a, 1) > u(a, 0)$.

As $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a $q(a) \in \mathbb{Q} \cap [u(a, 0), u(a, 1)]$ for every $a \in [0, 1]$. This defines a function $q : [0, 1] \to \mathbb{Q}$. If we can show that q is injective, we have a contradiction because the cardinality of $[0, 1]$ is strictly larger than the cardinality of $\mathbb{Q}$. (This function q has no additional meaning, we just use it for the contradiction.)

Assume that q was not injective, i.e., there exist distinct $b, c \in [0, 1]$ such that $q(b) = q(c)$. W.l.o.g., $c > b$. But

$$\Bigg(u(b, 0) \leqslant \Bigg) q(b) \leqslant u(b, 1) < u(c, 0 \leqslant q(c) \Bigg(\leqslant u(c, 1) \Bigg).$$

A contradiction and q is indeed injective. As said before, such a function cannot exist and therefore the initial assumption on the existence of such a utility function was wrong and we proved that no such utility function exists.