derivace v bodě

derivace fce v bodě

př: $f: y = x$; určit $f(x_0)$; $x_0 \in \mathbb{R}$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$$

př: $f: y = 3x - 1$; určit $f(x_0)$

$$f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{(3x-1) - (3x_0 - 1)}{x - x_0} = \frac{3x - 1 - 3x_0 + 1}{x - x_0} = \frac{3(x - x_0)}{x - x_0} = 3$$

př: $f: y = x^2$, určit $f(x_0)$

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x + x_0)(x - x_0)}{x - x_0} =$$

$$= \lim_{x \to x_0} (x + x_0) = 2x_0$$

př: $f: y = x^2$; $x_0 = 5$, $f'(x_0) = ?$

$$\lim_{x \to 5} \frac{x^2 - 5^2}{x - 5} = \lim_{x \to 5} (x + 5) = 10$$

př: $f: y = \sqrt[3]{x}$; $f'(0) = ?$

$$f'(0) = \lim_{x \to 0} \frac{\sqrt[3]{x} - 0}{x - 0} = \frac{\sqrt[3]{x}}{x} = \frac{x^{\frac{1}{3}}}{x^1} = x^{\frac{1}{3} - 1} = x^{-\frac{2}{3}} = \frac{1}{x^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{x^2}} = +\infty$$

nevlastní limita → derivace neexistuje

0^+ 0^+, ale hodně větší

př: Kde protne osa x tečnu s derivací v bodě -1?

$f: y = x^2$ $\quad f'(x_0) = -1$

$$\lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \frac{(x + x_0)(x - x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0$$

$2x_0 = -1$

$x_0 = \underline{-0{,}5}$

basic derivace

př: $f: y = \frac{1}{2}x^3 - 5x^2 + 7x - 3$

$f'(x) = \frac{1}{2} \cdot 3x^2 - 5 \cdot 2x + 7 = \frac{3}{2}x^2 - 10x + 7$

př: $f: y = 7x^5 - 2x^3 + 27$

$f'(x) = 35x^4 - 6x^2$

př: $f: y = 2 - \frac{4}{3}x^3 - \frac{3}{4}x^8$

$f'(x) = -\frac{12}{3}x^2 - \frac{24}{4}x^7 = -4x^2 - 6x^7$

př: $f: y = -x^4 + 7\sin x - 2\cos x + 3e^x$ $f'(x) = -4x^3 + 7\cos x + 2\sin x + 3e^x$

př: $f: y = x(x^2 - 1)(x + 2) = x(x^4 + 2x^3 - x - 2) = x^5 + 2x^4 - x^2 - 2x$

$f'(x) = 5x^4 + 8x^3 - 2x - 2$

př: $f: y = 3x - 2\ln(x)$

$f'(x) = 3 - 2 \cdot \frac{1}{x}$

derivace součinu

př: $f: y = x^3 \cdot \ln(x)$

$f'(x) = (x^3)' \cdot \ln x + x^3 \cdot (\ln x)' = 3x^2 \cdot \ln x + x^3 \cdot \frac{1}{x} = 3x^2 \ln x + x^2$

př: $f: y = \ln(x) \cdot 2^x$

$f'(x) = \frac{1}{x} \cdot 2^x + \ln x \cdot 2^x \cdot \ln 2 = 2^x \left(\frac{1}{x} + \ln x \cdot \ln 2\right)$

př: $f: y = x \cdot \sin x + \cos x$

$f'(x) = (x)' \cdot \sin x + x \cdot (\sin x)' + (\cos x)' = \sin x + \cos x \cdot x - \sin x = x \cos x$

př: $f: y = 3x - 2\ln(x)$

$f'(x) = 3 - \frac{2}{x}$

př: $f: y = \operatorname{tg} x - x = \frac{1}{\cos^2 x} - 1$

pr: $f: y = \mathrm{tg}\, x - \mathrm{cotg}\, x$ $\quad D_f = D_{f'} = \mathbb{R} - \{k\cdot\frac{\pi}{2}\}$

$f'(x) = \frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}$

pr: $f: y = 2^x - \log x$

$f'(x) = 2^x \cdot \ln 2 - \frac{1}{x\cdot \ln 10}$ $\quad D_f = D_{f'} = \mathbb{R}^+$

pr: $f: y = \sqrt{x}\cdot \ln x$

$f'(x) = (x^{\frac{1}{2}})' \cdot \ln x + x^{\frac{1}{2}} \cdot (\ln x)' = \frac{1}{2} x^{-\frac{1}{2}} \cdot \ln x + \sqrt{x} \cdot \frac{1}{x} = \frac{1}{2\sqrt{x}} \cdot \ln x + \frac{\sqrt{x}}{x}$

$D_f = D_{f'} = \mathbb{R}^+$

pr: $f: y = 2 \sin x \cos x$

$f'(x) =$ $\quad D_f = D_{f'} = \mathbb{R}$

$= 2 \cdot (\sin x)' \cdot \cos x + \sin x \cdot (\cos x)' = 2 \cdot (\cos^2 x - \sin^2 x) = 2 \cdot \cos 2x$

pr: $f: y = (x^2+1)\cdot e^x \cdot \sin x$

$f'(x) = (2x \cdot e^x + (x^2+1)\cdot e^x) \cdot \sin x = [e^x(x^2+2x+1)]' \sin x + [e^x(x^2+2x+1)] \cdot (\sin x)'$

$= e^x(2x+2) \cdot \sin x + e^x(x^2+2x+1) \cdot \cos x$ $\quad D_f = D_f = \mathbb{R}$

pr: $f: y = x^3 \cdot \sqrt[7]{x} = x^3 \cdot x^{\frac{1}{7}}$

$f'(x) = 3x^2 \cdot x^{\frac{1}{7}} + x^3 \cdot \frac{1}{7} x^{-\frac{6}{7}} = 3x^{\frac{15}{7}} + \frac{1}{7} x^{\frac{15}{7}} = 3\frac{1}{7} x^{\frac{15}{7}}$ $\quad D_f = D_{f'} = \mathbb{R}_0^+$

pr: $f: y = \frac{x^4+1}{x^2} = (x^4+1)\cdot x^{-2}$

$f'(x) = (x^4+1)' \cdot x^{-2} + (x^4+1)(x^{-2})' =$

$=$ error 404 result not found

$= 4x^3 \cdot x^{-2} + (x^4+1)\cdot(-2)\cdot x^{-3} =$

$= 4x^{-1} - 2(x^4+1)\cdot x^{-3} = \frac{4}{x} - 2x^{-3} + x^{4-3} = \frac{4}{x} - \frac{2}{x^3} + x$

aii idk

derivace podílu

pi: $f: y = \frac{x^4+1}{x^2}$

$$f'(x) = \frac{(x^4+1)' \cdot x^2 - (x^4+1) \cdot (x^2)'}{(x^2)^2} = \frac{4x^3 \cdot x^2 - (x^4+1) \cdot 2x}{x^4} =$$

$$= \frac{4x^5 - 2x^5 \cancel{+} - 2x}{x^4} = \frac{2x^5 - 2x}{x^4} = \frac{2x^5}{x^4} - \frac{2x}{x^4} = 2x - \frac{2}{x^3}$$

$D_f = D_{f'} = \mathbb{R} - \{0\}$

pi: $f: y = \frac{x}{x^3+2}$

$$f'(x) = \frac{x' \cdot (x^3+2) - x \cdot (x^3+2)'}{(x^3+2)^2} = \frac{x^3+2 - x \cdot (3x^2)}{x^6 + 2 \cdot 2 \cdot x^3 + 4} = \frac{x^3+2-3x^3}{x^6+4x^3+4} = \frac{-2x^3+2}{x^6+4x^3+4} =$$

$\cancel{=}$ ende.

pi: $f: y = \frac{x^3}{x^3-1}$

$$f'(x) = \frac{(x^3)' \cdot (x^3-1) - x^3 \cdot (x^3-1)'}{(x^3-1)^2} = \frac{3x^2(x^3-1) - x^3(3x^2)}{(x^3-1)^2} =$$

$$= \frac{3x^5 - 3x^2 - 3x^5}{(x^3-1)^2} = \frac{-3x^2}{(x^3-1)^2}$$

$D_f = D_{f'} = \mathbb{R} - \{1\}$

pi: $f: y = \frac{\sin x}{\cos x}$

$$f'(x) = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$D_f = D_{f'} = \mathbb{R} - \{\frac{\pi}{2} \cdot k\}$

pi: $f: y = \frac{x}{\ln x}$

$$f'(x) = \frac{(x)' \cdot \ln x - x \cdot (\ln x)}{\ln^2 x} = \frac{\ln x - \frac{x}{x}}{\ln^2 x} = \frac{\ln x - 1}{\ln^2 x}$$

$D_f = D_{f'} = \mathbb{R}^+$

derivace složené fce

př: $f: y=\sin(3x^2+2)$

$f'(x)=(\sin(3x^2+2))'\cdot(3x^2+2)'=\cos(3x^2+2)\cdot 6x$

$D_f=D_{f'}=\mathbb{R}$

př: $f: y=\ln(3x+1)$

$f'(x)=\frac{1}{3x+1}\cdot 3=\frac{3}{3x+1}$

$D_f > -\frac{1}{3}$

$D_{f'}=\mathbb{R}-\{-\frac{1}{3}\}$

př: $f: y=e^{tg\,x}$

$f'(x)=(e^{tg\,x})'\cdot(tg\,x)'=e^{tg\,x}\cdot\frac{1}{\cos^2 x}$ $D_f \in \mathbb{R}$

$D_{f'}=\mathbb{R}-\{k\frac{\pi}{2}\}$

př: $f: y=(x^3+1)^8$

$f'(x)=8(x^3+1)^7\cdot(3x^2)=24x^2(x^3+1)^7$

př: $f: y=e^{x^2+3x+1}$

$f'(x)=e^{x^2+3x+1}\cdot(2x+3)$ $D_f=D_{f'}=\mathbb{R}$

př: $f: y=\sin 2x$

$f'(x)=\cos 2x\cdot 2$ $D_f=D_{f'}=\mathbb{R}$

př: $f: y=e^{\cos(\ln(x))}$

$(\cos(\ln(x)))'\cdot(\ln x)'=-\sin(\ln x)\cdot\frac{1}{x}$ $D_f=D_{f'}=\mathbb{R}^+$

$f'(x)=-\sin(\ln x)\cdot\frac{1}{x}\cdot e^{\cos(\ln x)}$

př: $f: y=\sin^2(x^2-3x)$ nejdříve $\sin^2$, pak $\sin x$, pak x

$f'(x)=2\sin(x^2-3x)\cdot\cos(x^2-3x)\cdot(2x-3)$ $D_f=D_{f'}=\mathbb{R}$

př: $f: y=\sqrt{x^2+4x}=(x^2+4x)^{\frac{1}{2}}$

$f'(x)=\frac{1}{2}(x^2+4x)^{-\frac{1}{2}}\cdot(2x+4)$

$x^2+4x\geq 0$

$x(x+4)\geq 0$

L'Hospitalovo pravidlo

př: $\lim\limits_{x\to 0}\frac{\sin x}{x} = \lim\limits_{x\to 0}\frac{(\sin x)'}{(x)'} = \lim\limits_{x\to 0}\frac{\cos x}{1} = 1$

$\sin x \to 0$, $x \to 0$, $\cos x \to 1$

př: $\lim\limits_{x\to 1}\frac{x^2-1}{x^2-3x+2} = \frac{2x}{2x-3} = \frac{2}{-1} = -2$

$1-3+2=0$

př: $\lim\limits_{x\to 7}\frac{x^2-8x+7}{x^2-49} = \frac{2x-8}{2x} = \frac{6}{14} = \frac{3}{7}$

př: $\lim\limits_{x\to 0}\frac{x-\sin x}{x^3} = \lim\limits_{x\to 0}\frac{1-\cos x}{3x^2} = \frac{1}{3}\cdot\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$ ... atd.

$1-\cos x \to 0$, $3x^2 \to 0$

užití v praxi

př: $f: y = x + \frac{1}{x} = x + x^{-1}$ $\qquad D_f = \mathbb{R} - \{0\}$

$f'(x) = 1 + (-1x^{-2}) = 1 - x^{-2}$

$f'(x) = 0$

$1 - x^{-2} = 0$

$x^{-2} = 1$

$x = \pm 1$

		-1		0		1	
$f'(x)$	$+$		$-$		$-$		$+$
$f(x)$	↗		↘		↘		↗

$y_1 = -1 + \frac{1}{-1} = -1-1 = -2$ $\qquad [-1; -2]$

$y_2 = 1 + \frac{1}{1} = 2$ $\qquad [1; 2]$

$\lim\limits_{x\to 0+}\left(x + \frac{1}{x}\right) = +\infty$

$\frac{1}{x} \to \infty$

př: $f: y = e^{-x^2}$

$f'(x) = e^{-x^2}\cdot(-2)x$

$f'(x) = 0 \iff x = 0$

		0	
f'	$+$		$-$
f	↗		↘

pr: $f: y = x + \sin x$

$f'(x) = 1 + \cos x$

	$\mathbb{R} - \{\pi + 2k\pi\}$	$\pi + 2k\pi$
~~$f'(x)$~~	~~+~~	~~0~~
	~~↗~~	~~-~~

$f'(x) = 0 \Leftrightarrow 1 + \cos x = 0$

$\cos x = -1$

$x = \pi + 2k\pi$

$\pi, 3\pi, -\pi, 7\pi \ldots$

	$-\pi$	π	3π	7π
$f'(x)$	+	+	+	+
$f(x)$	↗	↗	↗	↗

... je to furt kladný, ale je to ∿∿∿ ...

WTF

~~A~~ extrémy

pr: $f: y = 3x^4 - 4x^3$ $\qquad D_f = D_{f'} = \mathbb{R}$

$f'(x) = 12x^3 - 12x^2$

$12x^3 - 12x^2 = 0$

$x^3 - x^2 = 0$

$x^3 = x^2$

$x_1 = 0;\ x_2 = 1$

		0	1
$f'(x)$	−	−	+
f	↘	↘	↗

$\Rightarrow$ 1 - lok. min.

$12 - 12 = 0 \Rightarrow [1; 0]$

pr: $f: y = x^2 - 2x + 2$ - extrémy, vrchol

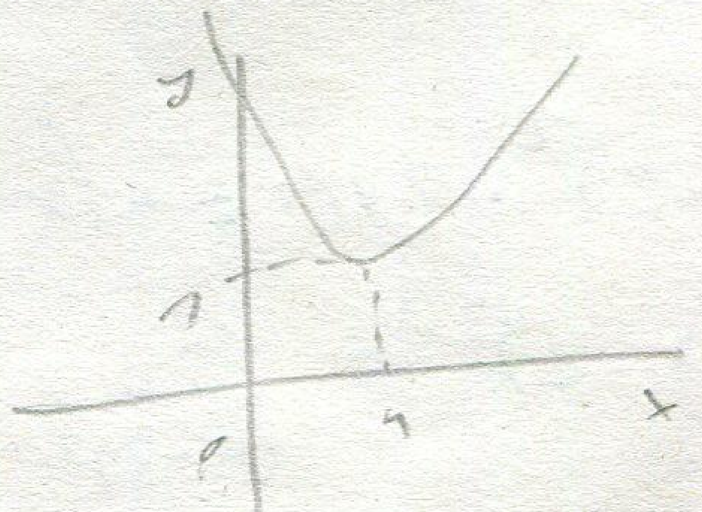
$f'(x) = 2x - 2$

$2x - \cancel{20}2 = 0$

$2x = 2$

$x = 1$

		1
$f'(x)$	−	+
$f(x)$	↘	↗



$1 - 2 + 2 = 1 \Rightarrow [1; 1]$ vrchol

pr: $f: y = x^3 - 12x + 20$

$f'(x) = 3x^2 - 12$

$3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

		−2	2
$f'(x)$	+	−	+
f	↗	↘	↗

lok. max. $\qquad$ lok. min.

$\lim_{x \to +\infty} (x^3 - 12x + 20) = +\infty$

$\lim_{x \to -\infty} (x^3 - 12x + 20) = -\infty$

⇓

nemá glob. ex.

př: $f: y = x^5 - 10x^3 + 40x$, lok. a gl. extrémy = ?

a) $D_f = \mathbb{R}$

b) $D_f = \langle -3; 3 \rangle$

$f'(x) = 5x^4 - 30x^2 + 40$

$x^2 = t$

$5t^2 - 30t + 40 = 0$

~~[illegible]~~

$t_1 = 4;\ t_2 = 2$

a)

$x_1^2 = 4 \Rightarrow x_1 = \pm 2$

$x_2^2 = 2 \Rightarrow x_2 = \pm\sqrt{2}$

	-2	$-\sqrt{2}$	$\sqrt{2}$	2	
$f'(x)$	+	−	+	−	+
f	↗	↘	↗	↘	↗

$\Rightarrow$ lok. max. $-2;\ \sqrt{2}$

lok. min $-\sqrt{2};\ 2$

$\lim_{x \to \infty} (f(x)) = +\infty$

$\lim_{x \to -\infty} (f(x)) = -\infty$ $\Rightarrow$ glob. extr. nejsou

b) $D_f = \langle -3; 3 \rangle$

$f(3) = 93$ ... 3 ... lok. i glob. max.

$\lim_{x \to -3} (f(x)) = -93$ ... −3 ... lok. i glob. min.

druhá derivace

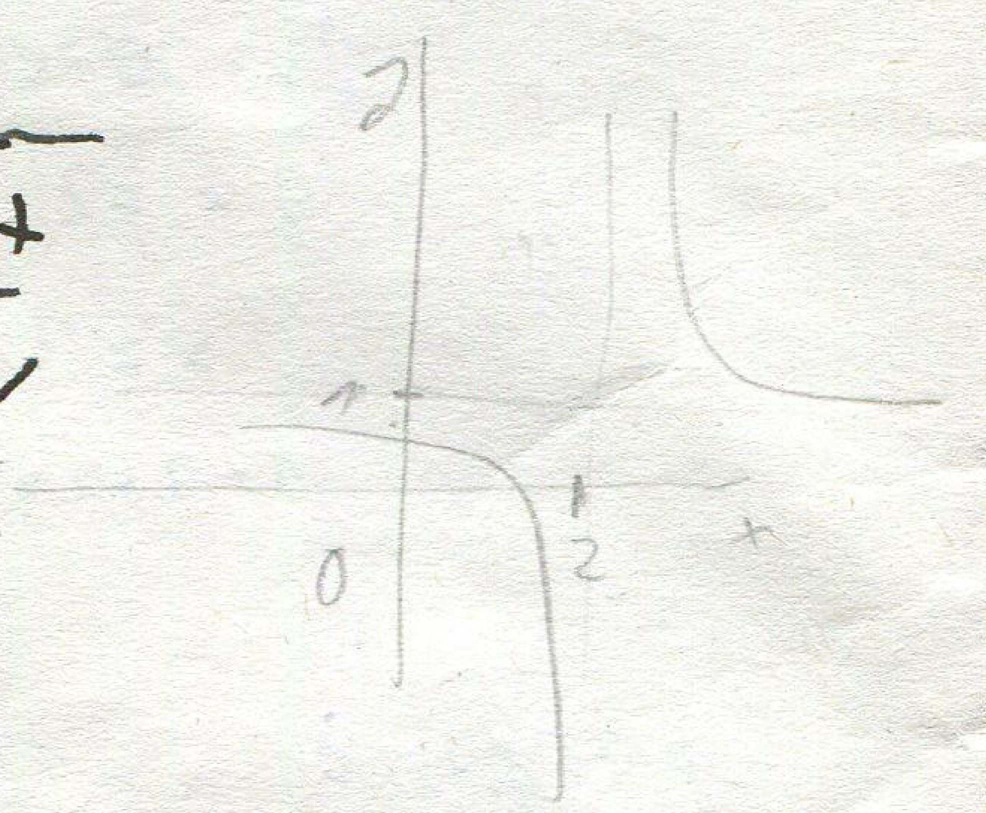
př: $f: y = \frac{1}{x-2} + 1$

$f'(x) = 1 \cdot (x-2)^{-2} \cdot 1 + 0 = -(x-2)^{-2}$

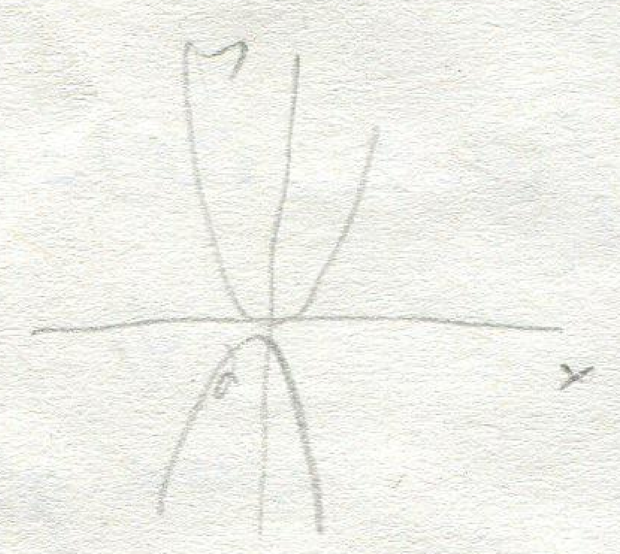
$f''(x) = 2(x-2)^{-3} = \frac{2}{(x-2)^3}$

		2	
$f''(x)$	−		+
f	⌒		⌣

$D_{f''} = \mathbb{R} - \{2\} = D_f = D_{f'}$



př: $f: y = a \cdot x^2$; $a \in \mathbb{R}$; $D_f = \mathbb{R}$



$a > 0$ ... konv.

$a < 0$ ... konk.

$f'(x) = 0 \cdot x^2 + a \cdot 2x = 2x \cdot a$

$f''(x) = 2 \cdot a + 0 \cdot 2 = 2a$

Sečna ke grafu

př: $f: y = -x^2 - 6x - 4$ $\quad x_1 = -6 ; x_2 = -1$ — sečna

$[-6; -4] \quad [-1; 1]$

$-4 = -6a + b \quad b = -4 + 6a$

$1 = -1a + b \quad b = 1 + a$

$-4 + 6a = 1 + a$

$6a - a = 1 + 4 \quad b = 1 + 1$

$5a = 5 \quad b = 2 \qquad y = x + 2$

$a = 1$

- tečna v bodě $[-2; ?]$

$f'(x) = -2x - 6$

$f'(-2) = -2 \cdot (-2) - 6 = -2 \ldots k$

$-(-2)^2 - 6 \cdot (-2) - 4 = -4 + 12 - 4 = 12 \Rightarrow [-2; 12]$

$y = kx + q$

$12 = (-2)(-2) + q \qquad y = -2x + 8$

$q = 8$

př: $f: y = \dfrac{2x-1}{x+1}$; $T[-2; ?]$ + normála

$$f'(x) = \frac{2(x+1) - 1(2x-1)}{(x+1)^2} = \frac{\cancel{2x} + 2 - \cancel{2x} + 1}{(x+1)^2} = \frac{3}{(x+1)^2}$$

$f'(-2) = \dfrac{3}{(-2+1)^2} = \dfrac{3}{1} = 3$

$y = \dfrac{2 \cdot (-2) - 1}{-2+1} = \dfrac{-4-1}{-1} = \dfrac{-5}{-1} = 5 \Rightarrow [-2; 5]$

$5 = 3 \cdot (-2) + q$

$5 = -6 + q \qquad y = 3x + 11 \qquad \vec{n}_n(1; 3) \Rightarrow x + 3y + c = 0$

$q = 11 \qquad 3x - y + 11 = 0 \qquad -2 + 3 \cdot 5 + c = 0$

$\vec{n}_t(3; -1) \qquad 13 + c = 0 \Rightarrow x + 3y - 13 = 0$

$c = -13$

$y = -2x + 8$

$-2x - y + 8 = 0$

$2x + y - 8 = 0$

$\vec{n} = (2;1)$

$\vec{n}_n = (-1;2)$

$[-2;12]$

$-(-2) + 2 \cdot 12 + c = 0$

$2 + 24 = -c$

$c = -26$

$-x + 2y - 26 = 0$

sečny ke grafu

příř: $y = \log x$ $D_f = \mathbb{R}^+$, najít sečnu, která protíná graf v bodech $[1; f(1)]$ a $[10; f(10)]$

$y_1 = \log 1 = 0 \to [1; 0]$

$y_2 = \log 10 = 1 \to [10; 1]$

$y = ax + b$

$1 = a \cdot 10 + b \Rightarrow b = 1 - 10a$

$0 = a \cdot 1 + b \Rightarrow b = -a$

$1 - 10a = -a$

$1 = 9a$

$a = \frac{1}{9}$

$b = -\frac{1}{9}$

$\underline{y = \frac{1}{9}x - \frac{1}{9}}$

př: $f: y = \sin x$, sečna v $x_1 = \frac{\pi}{6}$; $x_2 = \frac{3}{2}\pi$

~~[illegible]~~

$[\frac{\pi}{6}; \frac{1}{2}]$

$[\frac{3\pi}{2}; -1]$

$\frac{1}{2} = a\frac{\pi}{6} + b$

$-1 = a \cdot \frac{3\pi}{2} + b$

$b = \frac{1}{2} - \frac{\pi a}{6}$

$b = -1 - \frac{3\pi a}{2}$

$\frac{9}{8} : \frac{6}{1} = \frac{9}{8} \cdot \frac{1}{6}$

$\frac{1}{2} - \frac{\pi a}{6} = -1 - \frac{3\pi a}{2} \quad / \cdot 6$

$3 - \pi a = -6 - 9\pi a$

$-\pi a + 9\pi a = -9$

$8\pi a = -9$

$a = -\frac{9}{8\pi}$

$b = \frac{1}{2} - \frac{\pi \cdot \frac{9}{8\pi}}{6} = \frac{1}{2} - \frac{9}{48} = \frac{15}{48} = \frac{5}{16}$

$\underline{y = -\frac{9}{8\pi}x + \frac{5}{16}}$