MLE of GARCH (p, q)

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(using modified version of Bollerslev et all)

Conditional Mean

$r_t = \gamma + \varepsilon_t$ where r_t = returns & ε_t = conditional variance

for simplification, we assume $\gamma = 0$ as all GARCH models are variance centr

$$\Rightarrow \boxed{r_t = \varepsilon_t} \quad -(1)$$

also, in order to handle heteroskedasticity, we assume that -
time varying variation comes from another exogenous variable z_t i.e.

$$\boxed{\varepsilon_t = \eta_t z_t} \quad -(2)$$; where η_t is white noise

for GARCH (p, q)

Conditional Variance →

$$\boxed{\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2} \quad -(3)$$

where $\forall \beta_j \geq 0$

Quasi MLE

Joint density ⇒ $$\boxed{p(x_1, \dots, x_n) = p(x_n \mid x_{n-1}, \dots x_1) \cdot p(x_{n-1} \mid x_{n-2} \dots x_1) \dots p(x_1)} \quad -(4)$$

Now,
- $\eta_t \sim N(0,1)$ [property of white noise
- $\varepsilon_t \mid \varepsilon_{t-1}, \dots \varepsilon_{t-} \sim N(0, \sigma_t^2)$ using eq^n (4)

Suppose we are given obs $\{r_0, r_1, r_2 \dots r_n\}$

for some densities like $p(\varepsilon_1)$

initial obs i.e. $y_0, \dots y_{1-q}$ are unavailable

$\sigma_0, \dots, \sigma_{1-p}$ are unavailable

So, likelihood condition of these variables is set to

$\star\ y_0, \ldots, y_{1-q} = 0$

$\star\ \sigma_0 = \dfrac{\alpha}{1-(\alpha_1+\beta_1)}$

$\sigma_1 = \cancel{alpha}\ \alpha_0 + \alpha_1 \varepsilon_0^2 + \beta_1 \sigma_0^2$

$\sigma_2 = \alpha_0 + \alpha_1 \varepsilon_0^2 + \alpha_2 \varepsilon_1^2 + \beta_1 \sigma_0^2 + \beta_2 \sigma_1^2$

so on till σ_p

following all the above points & eq (1), (2), (3) & (4)

log likelihood: $\boxed{\mathcal{L}_n(\theta) = \mathcal{L}(r_1, \ldots r_n; \theta) = -\frac{1}{2}\left[\sum_{t=1}^{n}\left(\frac{r_t^2}{\sigma_t^2} + \log \sigma_t^2\right)\right]}$

where $\theta = \{\alpha_0, \alpha_1, \alpha_2 \ldots \alpha_q, \beta_1, \beta_2 \ldots \beta_p\}$

$\therefore\ \boxed{\hat{\theta}_{MLE} = \underset{\theta \in \Theta}{\operatorname{argmax}}\ \mathcal{L}_n(\theta)}$

Note: use any differential based optimization methods