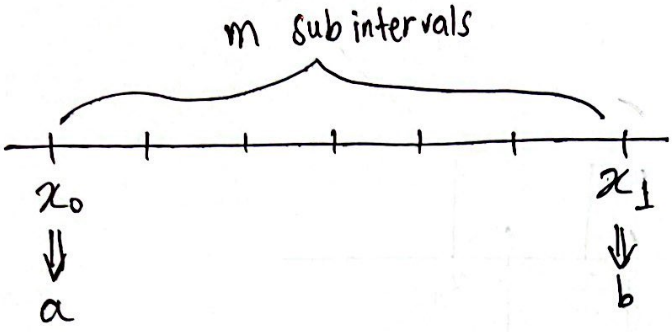
Composite Newton - Cotes Formula:

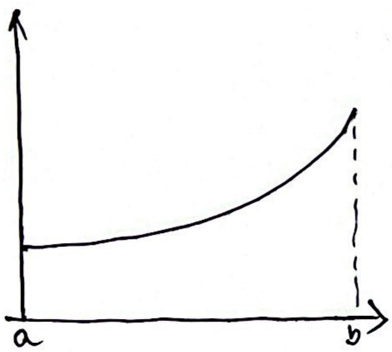
→ This method improves result without increasing num. of nodes.

→ Basic idea is to divide the interval $[a, b]$ into m sub intervals.

<u>when $n = 1$:</u>

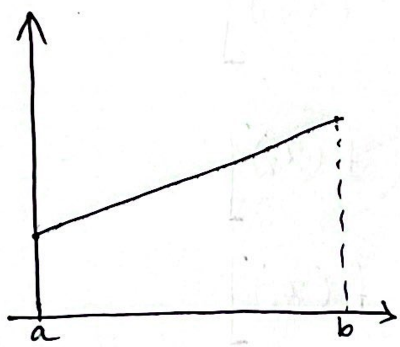


→ For each sub interval, we apply trapezium rule, then add them up.



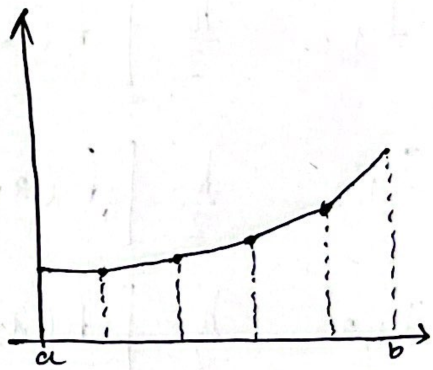
Actual integration

$I(f)$



Newton-cotes with $n=1$

$I_1(f)$



Composite Newton Cotes with $n = 1$

$C_{1,m}(f)$

→ Total sum is denoted by $C_{1,m}(f)$ and called composite Newton Cote's.

for degree 1   for m sub interval

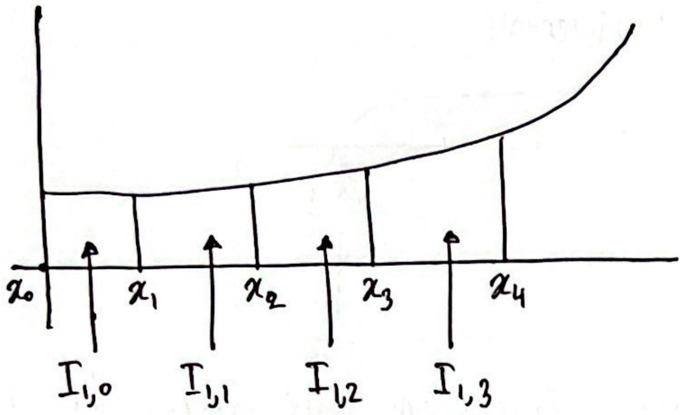
→ For m sub intervals, we define

$$\boxed{h = \frac{b-a}{m}}$$

Apply Trapezium Rule for each sub interval

$$I_1(f) = \text{Trapezium Rule} = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$= \frac{h}{2}\left[f(a) + f(b)\right]$$



$$I_{1,0} = \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$

$$I_{1,1} = \frac{h}{2}\left[f(x_1) + f(x_2)\right]$$

$$I_{1,2} = \frac{h}{2}\left[f(x_2) + f(x_3)\right]$$

$$\vdots$$

$$I_{1,m_1} = \frac{h}{2}\left[f(x_{m-2}) + f(x_{m-1})\right]$$

$$I_{1,m} = \frac{h}{2}\left[f(x_{m-1}) + f(x_m)\right]$$

$$C_{1,m}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \dots + 2f(x_{m-1}) + f(x_m)\right]$$

Example:

$f(x) = e^x$

$a = 0$
$b = 2$

→ Exact result $= I(f) = \int_0^2 e^x \, dx = 6.389056$

→ Composite Newton Cotes with num of subintervals $= 2$ $(m=2)$:

Step 1: Find h

$$h = \frac{b-a}{m} = \frac{2-0}{2} = 1$$

Step 2: Find $x_0, x_1, x_2 \cdots x_m$

→ Remember: If $m=2$, find x_0 to x_2
If $m=3$, find x_0 to x_3
If $m=4$, find x_0 to x_4.

$X_0 = a = 0$ [since trapezium rule follows closed newton cotes]

$X_1 = X_0 + h = 0 + 1 = 1$

$X_2 = X_1 + h = 1 + 1 = 2$

Step 3: Find $C_{1,m}(f)$

$$C_{1,2}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$
$$= \frac{1}{2}\left[e^0 + 2e^1 + e^2\right]$$
$$= 6.91281$$

→ Composite Newton Cotes with num of sub intervals = 3 ($m=3$):

$$h = \frac{b-a}{m} = \frac{2-0}{3} = \frac{2}{3}$$

Find x_0 to x_3:

$x_0 = a = 0$

$x_1 = x_0 + h = 0 + \frac{2}{3} = \frac{2}{3}$

$x_2 = x_1 + h = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

$x_3 = x_2 + h = \frac{4}{3} + \frac{2}{3} = 2$

$$C_{1,3}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{2/3}{2}\left[e^{0} + 2e^{2/3} + 2e^{4/3} + e^{2}\right]$$

$$= 6.62395$$

→ Composite Newton Cotes with $m=4$:

$$C_{1,4} = \frac{0.5}{2}\left[e^{0} + 2e^{0.5} + 2e^{1} + 2e^{1.5} + e^{2}\right] = 6.52161$$

$\boxed{\text{Error decreases as } m \text{ increases}}$

Simpson's Rule:

Trapezium Rule $= \int_a^b P_1(x)\,dx$

Simpson's Rule $= \int_a^b P_2(x)\,dx$

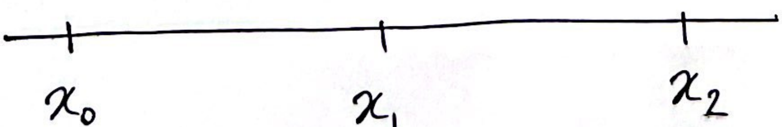
$$I_2(f) = \int_a^b \underbrace{P_2(x)}_{P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)}\,dx$$

$$I_2(f) = \int_a^b \left[l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2) \right] dx$$

$$= \underbrace{\int_a^b l_0(x)\,dx}_{\sigma_0} \cdot f(x_0) + \underbrace{\int_a^b l_1(x)\,dx}_{\sigma_1} \cdot f(x_1) + \underbrace{\int_a^b l_2(x)\,dx}_{\sigma_2} \cdot f(x_2)$$

$$\therefore\ I_2(f) = \sigma_0 f(x_0) + \sigma_1 f(x_1) + \sigma_2 f(x_2)$$

Here, since $n = 2$, number of nodes $= n+1 = 3 \rightarrow \{x_0, x_1, x_2\}$



$x_0 \Downarrow a$

$x_1 \Downarrow x_1 = x_0 + h$

$m = x_1 = \dfrac{a+b}{2}$

$x_2 \Downarrow b$

$$\sigma_0 = \int_a^b l_0(x)\,dx$$

$$= \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx$$

$$= \int_a^b \frac{(x-m)(x-b)}{(a-m)(a-b)}\,dx$$

$$= \frac{1}{(a-m)(a-b)} \int_a^b (x-m)(x-b)\,dx$$

$$= \frac{1}{6}(b-a)$$

$$\sigma_1 = \int_a^b l_1(x)\,dx$$

$$= \vdots$$

$$= \frac{2}{3}(b-a)$$

$$\sigma_2 = \int_a^b l_2(x)\,dx$$

$$= \vdots$$

$$= \frac{1}{6}(b-a)$$

$$
\begin{aligned}
I_2(f) &= \sigma_0 f(x_0) + \sigma_1 f(x_1) + \sigma_2 f(x_2) \\
&= \sigma_0 f(a) + \sigma_1 f(m) + \sigma_2 f(b) \\
&= \frac{1}{6}(b-a) f(a) + \frac{2}{3}(b-a) f(m) + \frac{1}{6}(b-a) f(b) \\
&= \frac{b-a}{6}\left[f(a) + 4f(m) + f(b) \right] \\
&= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\end{aligned}
$$

Exactness:

For numerical integration, upper bound of error =

$$|I - I_n| = \left|\frac{1}{(n+1)!} f^{(n+1)}(\xi)\right| \int_a^b \left|(x-x_0)(x-x_1)\ldots(x-x_n)\right| dx$$

→ If $f^{(n+1)}(\xi) = 0$, error $= 0$

→ In that case, Newton Cotes will give exact answers

→ The above formula was derived using Cauchy's Theorem.

Cauchy's Theorem:

$$|f(x) - P_n(x)| = \underbrace{\left|\frac{1}{(n+1)!} f^{(n+1)}(\xi) \quad (x-x_0)(x-x_1)\ldots(x-x_n)\right|}_{\text{error}}$$

$$f(x) - P_n(x) = \text{error}$$

$$f(x) = P_n(x) + \text{error}$$

→ If error $= 0$,

$$\boxed{f(x) = P_n(x)}$$

- → $f(x)$ is a n-degree polynomial
- → $f(x)$ is a polynomial itself.

→ If $f(x)$ itself is a polynomial, $I_n(f)$ will give exact result since error $=0$.

→ ~~The~~ This implies that trapezium rule $I_1(f)$ is exact for all functions $f(x) = P_1(x)$

→ In other words, if we have a degree 1 polynomial, $P_1(x)$ and we apply both the actual integration, $I(f)$, and numerical integration, $I_1(f)$, we will get the exact result.

Definition:

The degree of exactness is the largest integer, n, for which the formula is exact for all polynomials, $P_n(x)$.

Example:

Find (a) Actual integration, $I(f)$

(b) Newton Cote's integral using $n=2$, $I_2(f)$

for the following functions:

① $f(x) = 1$

② $f(x) = x$

③ $f(x) = x^2$

④ $f(x) = x^3$

⑤ $f(x) = x^4$

① $f(x) = 1$

(a) Exact $= I(f) = \int_a^b 1\,dx = b-a$

match / zero error

(b) Newton cotes $= I_2(f) = \frac{b-a}{6}\left[1+4+1\right] = b-a$

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② $f(x) = x$

(a) Exact $= I(f) = \int_a^b x\,dx = \frac{1}{2}(b^2-a^2)$

match/ zero error

(b) Newton Cotes $= I_2(f) = \frac{b-a}{6}\left[a+4\left(\frac{a+b}{2}\right)+b\right] = \frac{1}{2}(b^2-a^2)$

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③ $f(x) = x^2$

(a) Exact $= I(f) = \int_a^b x^2\,dx = \frac{1}{3}(b^3-a^3)$

match

(b) Newton Cotes $= I_2(f) = \frac{b-a}{2}\left[a^2+4\left(\frac{a+b}{2}\right)^2+b^2\right] = \frac{1}{3}(b^3-a^3)$

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④ $f(x) = x^3$

(a) Exact $= I(f) = \int_a^b x^3\,dx = \frac{1}{4}(b^4-a^4)$

Match

(b) Newton cotes $= I_2(f) = \frac{b-a}{6}\left[a^3+4\left(\frac{a+b}{2}\right)^3+b^3\right] = \frac{1}{4}(b^4-a^4)$

⑤ $f(x) = x^4$

(a) Exact $= I(f) = \int_a^b x^4 \, dx = \frac{1}{5}(b^5 - a^5)$

(b) Newton Cotes $= I_2(f) = \frac{b-a}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right] \neq \frac{1}{5}(b^5 - a^5)$

$\therefore$ Above result shows that simpson's formula, $I_2(f)$, gives exact result upto degree 3 polynomial, and error becomes non-zero from degree 4 polynomial and higher.

$\rightarrow$ Degree of exactness is exactly 3 for Simpson's Rule, $I_2(f)$.