# Cauchy's Theorem

$$\left|f(x) - P_n(x)\right| = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\ldots\ldots(x-x_n)$$

Ex 1. Using Cauchy's theorem find the upper bound of the error.

$f(x) = \cos x \qquad \left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$

$$\left|f(x) - P_2(x)\right| = \frac{f^{3}(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$= \underbrace{\left|\frac{\sin(\xi)}{3!}\right|}_{\text{Part A}}\underbrace{\left|\left(x - -\frac{\pi}{4}\right)(x-0)\left(x-\frac{\pi}{4}\right)\right|}_{\text{Part B }[w(x)]} \quad - (A)$$

Maximizing Part A,

The higest value of sin(x) is 1, so we have to replace sin(x) with 1.

$$\left|\frac{\sin(\xi)}{3!}\right| = \frac{1}{3!} = \frac{1}{6}$$

Maximizing Part B,

$$w(x) = \left(x+\frac{\pi}{4}\right)(x)\left(x-\frac{\pi}{4}\right)$$

$$= x^3 - \frac{\pi^2}{16}x$$

$$w'(x) = 3x^2 - \frac{\pi^2}{16}$$

$$w'(x) = 0$$

$$3x^2 - \frac{\pi^2}{16} = 0$$

$$x = \pm\frac{\pi}{4\sqrt{3}}$$

| $x$ | $\lvert w(x)\rvert$ |
|---|---|
| $\frac{-\pi}{4\sqrt{3}}$ | 0.186 |
| $\frac{\pi}{4\sqrt{3}}$ | 0.186 |

From eqn A, we get

$$= \frac{1}{6} \times 0.186 = \frac{0.186}{6} \; [\text{Ans}]$$

Ex 2. Using Cauchy's theorem find the upper bound of the error using limit

$x \,|\, \xi \in [-1, 1]$

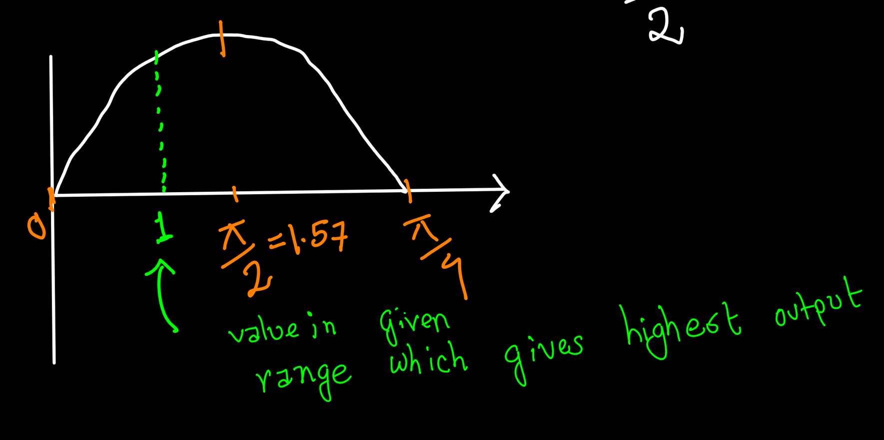
$f(x) = \cos x \qquad \left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$

$$\left|f(x) - P_2(x)\right| = \frac{f^{3}(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$= \underbrace{\left|\frac{\sin(\xi)}{3!}\right|}_{\text{Part A}}\underbrace{\left|\left(x - -\frac{\pi}{4}\right)(x-0)\left(x-\frac{\pi}{4}\right)\right|}_{\text{Part B }[w(x)]} \quad - (B)$$

Maximizing Part A,

The higest value of sin(x) is 1, but for that to happen $x = \frac{\pi}{2}$. We have to check that our x is within the range




$$\left|\frac{\sin(1)}{3!}\right| = \frac{\sin(1)}{3!} = \frac{\sin(1)}{6}$$

Maximizing Part B,

$$w(x) = \left(x+\frac{\pi}{4}\right)(x)\left(x-\frac{\pi}{4}\right)$$

$$= x^3 - \frac{\pi^2}{16}x$$

$$w'(x) = 3x^2 - \frac{\pi^2}{16}$$

$$w'(x) = 0$$

$$3x^2 - \frac{\pi^2}{16} = 0$$

$$x = \pm\frac{\pi}{4\sqrt{3}}$$

| $x$ | $\lvert w(x)\rvert$ |
|---|---|
| $\frac{-\pi}{4\sqrt{3}}$ | 0.186 |
| $\frac{\pi}{4\sqrt{3}}$ | 0.186 |
| −1 | 0.383 |
| 1 | 0.383 |

From eqn B, we get

$$= \frac{\sin(1)}{6} \times 0.383 \quad [\text{Ans}]$$

Ex 3. Using Cauchy's theorem find the upper bound of the error using limit

$f(x) = -\sin x - \cos x \qquad \left\{-\frac{\pi}{4}, 0\right\}$

node = 2

n = 1

$$\left|f(x) - P_1(x)\right| = \left|\frac{f^{1+1}(\xi)}{(1+1)!}(x-x_0)(x-x_1)\right|$$

$$= \underbrace{\left|\frac{f^{2}(\xi)}{2!}\right|}_{\text{Part A}}\underbrace{\left|\left(x - -\frac{\pi}{4}\right)(x-0)\right|}_{\text{Part B }(w(x))}$$

$$f'(x) = -\cos x + \sin x$$

$$f''(x) = \sin x + \cos x$$

Maximizing Part A,

The higest value of sin(x) is 1, so we have to replace sin(x) with 1 and cos(x) with 1.

$$\frac{\left|\sin x + \cos x\right|}{2!} = \frac{|\sin x| + |\cos x|}{2} = \frac{1+1}{2}$$

$$= \frac{2}{2} = 1$$

Maximizing Part B,

$$w(x) = \left(x+\frac{\pi}{4}\right)(x)$$

$$= x^2 + \frac{\pi}{4}x$$

$$w'(x) = 2x + \frac{\pi}{4}$$

$$w'(x) = 0$$

$$2x = -\frac{\pi}{4}$$

$$x = -\frac{\pi}{8}$$

| $x$ | $\lvert w(x)\rvert$ |
|---|---|
| $-\frac{\pi}{8}$ | 0.15421 |

$$\left|f(x) - P_1(x)\right| = \underbrace{1}_{\text{Part A}} \times \underbrace{0.15421}_{\text{Part B}}$$