Newton's Polynomial:

$n_0(x) = 1$

$n_1(x) = (x-x_0)$

$n_2(x) = (x-x_0)(x-x_1)$

$n_3(x) = (x-x_0)(x-x_1)(x-x_2)$

$\vdots$

→ x is variable

→ $x_0, x_1, x_2 \ldots$ are values/constants which can be found from the given nodes

Creating $P_n(x)$ using Newton's polynomial as basis:

$$P_n(x) = \sum_{k=0}^{n} a_k \, n_k(x)$$

$$= a_0 \underbrace{n_0(x)}_{1} + a_1 \underbrace{n_1(x)}_{(x-x_0)} + a_2 \underbrace{n_2(x)}_{(x-x_0)(x-x_1)} + \cdots + a_n n_n(x)$$

$$= a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

How to find a?

Lets introduce a new notation

$a_0 = f[x_0]$

$a_1 = f[x_0, x_1]$

$a_2 = f[x_0, x_1, x_2]$

$\vdots$

$a_n = f[x_0, x_1 \ldots x_n]$

$$P_n(x) = \overset{f[x_0]}{\cancel{a_0}} + \overset{f[x_0,x_1]}{\cancel{a_1}}(x-x_0) + \overset{f[x_0,x_1,x_2]}{\cancel{a_2}}(x-x_0)(x-x_1) + \cdots$$

$$= f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots$$

<u>Example:</u>

$x_0 = -1$ $\quad x_1 = 0$ $\quad x_2 = 1$ $\quad x_3 = 2$

$f(x_0) = 5$ $\quad f(x_1) = 1$ $\quad f(x_2) = 1$ $\quad f(x_3) = 11$

4 nodes $\longrightarrow P_3(x)$

$\downarrow$

find till a_3

$$P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= f[x_0] + f[x_0,x_1](x-x_0)$$
$$+ f[x_0,x_1,x_2](x-x_0)(x-x_1)$$
$$+ f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

Need to find $f[x_0]$

$f[x_0,x_1]$

$f[x_0,x_1,x_2]$

$f[x_0,x_1,x_2,x_3]$

$x_0 = -1 \quad f[x_0] = 5$

$f[x_0, x_1] = \frac{1-5}{0-(-1)} = -4$

$x_1 = 0 \quad f[x_1] = 1$

$f[x_0, x_1, x_2] = \frac{0-(-4)}{1-(-1)} = 2$

$f[x_1, x_2] = \frac{1-1}{1-0} = 0$

$f[x_0, x_1, x_2, x_3] = \frac{5-2}{2-(-1)} = 1$

$x_2 = 1 \quad f[x_2] = 1$

$f[x_1, x_2, x_3] = \frac{10-0}{2-0} = 5$

$f[x_2, x_3] = \frac{11-1}{2-1} = 10$

$x_3 = 2 \quad f[x_3] = 11$

$$
\begin{aligned}
P_3(x) = {} & f[x_0] + f[x_0, x_1](x-x_0) \\
& + f[x_0, x_1, x_2](x-x_0)(x-x_1) \\
& + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)
\end{aligned}
$$

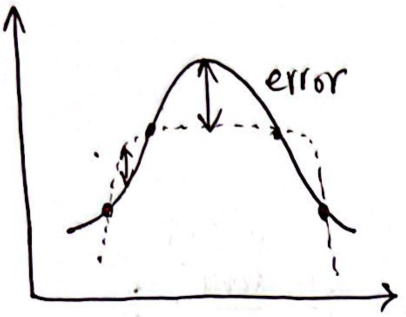
$$= 5 + (-4)(x+1) + 2(x+1)(x-0) + 1(x+1)(x-0)(x-1)$$

degree 3 polynomial.

Advantage of Newton's divided difference

→ can keep on adding nodes, but no need to do calculations from very beginning.

Interpolation Error:



error $\rightarrow$ $|f(x) - P_n(x)|$

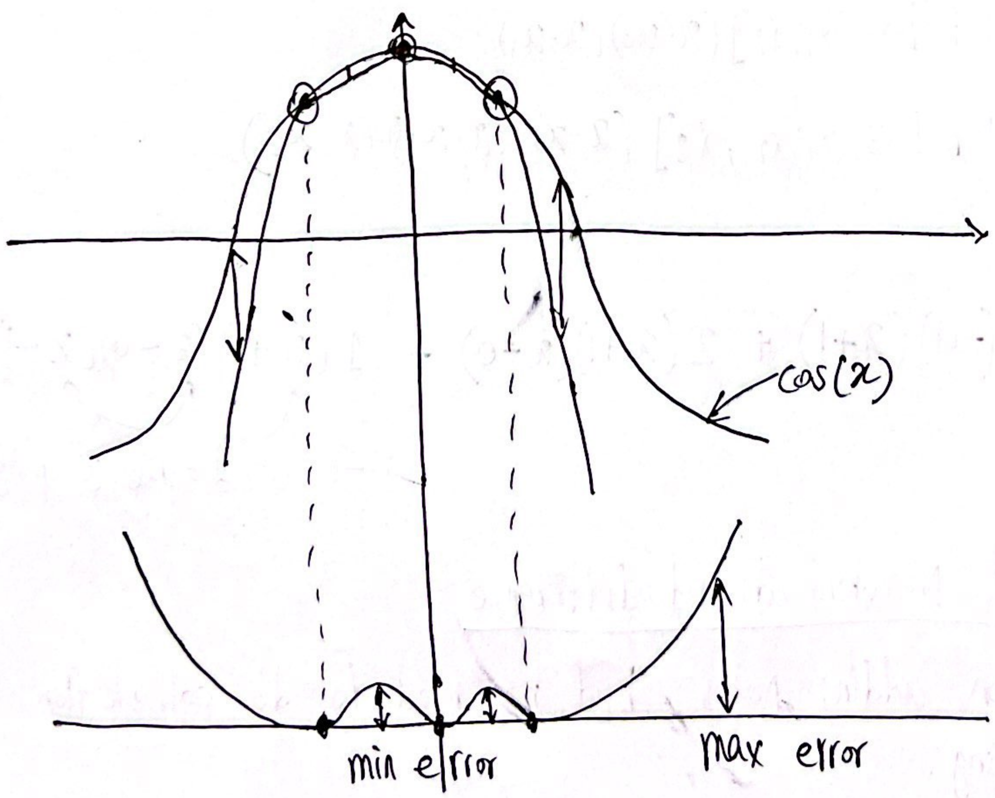
error at nodes $= 0$

Example

$f(x) = \cos x$

$P_2(x) = \frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}} - 1\right)x^2 + 1$

$$|f(x) - P_2(x)| = \left|\cos(x) - \frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}} - 1\right)x^2 - 1\right|$$



error goes to zero at the nodes

Cauchy's Theorem

Let $P_n \in \mathcal{P}_n$ be the unique polynomial interpolating $f(x)$ at the $(n+1)$ distinct nodes $x_0, x_1 \cdots x_n \in [a,b]$ and let f be continuous on $[a,b]$ with $(n+1)$ continuous derivatives on (a,b). Then for each $x \in [a,b]$, there exists a $\xi \in (a,b)$, such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{(x-x_0)(x-x_1) \cdots (x-x_n)}$$

→ error goes to zero at the nodes $(x_0, x_1 \cdots x_n)$

→ $(x-x_0)(x-x_1) \cdots (x-x_n)$
so when $x = x_0$ or x_1 or x_2, error goes to zero

Similar to taylors theorem, we could've found the exact error at any point if we knew the value of ξ. But all we know is that ξ is a value between a & b. So we ~~have~~ can find the maximum value of the function ~~at~~ betwen a & b instead.

$$\left| f(x) - P_n(x) \right| = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)(x-x_1) \cdots (x-x_n)$$

Example

$f(x) = \cos(x)$

$x_0 = -\frac{\pi}{4}$ $\quad x_1 = 0$ $\quad x_1 = \frac{\pi}{4}$

$f(x_0) = \frac{1}{\sqrt{2}}$ $\quad f(x_1) = 1$ $\quad f(x_2) = \frac{1}{\sqrt{2}}$

3 nodes $\rightarrow P_2(x)$ [find $P_2(x)$ using either vandermonde matrix, lagrange, or newtons divided difference form.

$$\left| f(x) - P_2(x) \right| = \frac{f^{(3)}(\xi)}{3!} \left(x + \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)(x)$$

$$= \boxed{\frac{\sin(\xi)}{6} (x)\left(x + \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)}$$

We need to find the maximum of this whole thing inside of an interval.

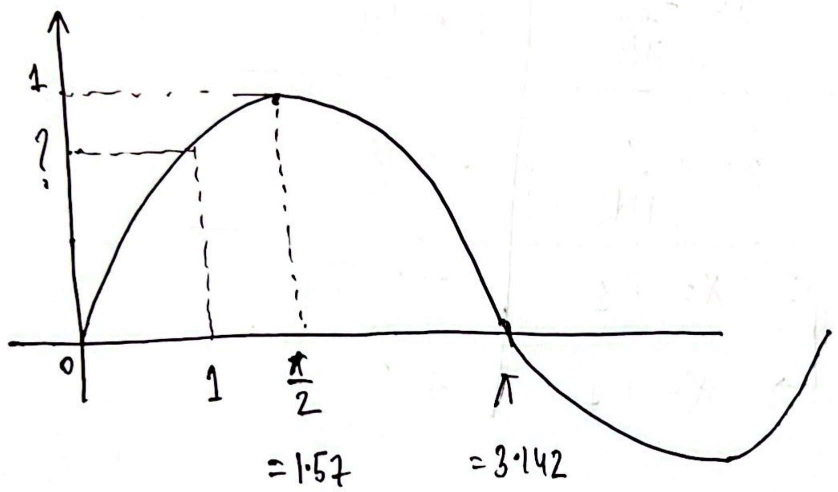
example $\xi \in [-1, 1] \leftarrow$ interval will be given in the question.

$\rightarrow$ if interval is not given, use the lowest value of the nodes as 'a', highest value of the nodes as 'b'

Important: nodes x_0, x_1, x_2 must lie within interval, example $[-1, 1]$

We will use the interval $\xi \in [-1, 1]$ for the ~~problem~~ rest of the problem.

$$\underbrace{\frac{\sin(\xi)}{6}}_{\text{Need to find max}^m \text{ of this in } [-1,1] \text{ interval}} \qquad \underbrace{(x)\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)}_{\text{Need to find max}^m \text{ of this in } [-1,1] \text{ interval}}$$



Max^m value of $\sin(\xi)$ in interval $[-1, 1]$ is $\sin(1)$

$$\frac{\sin(1)}{6} =$$

$$W(x) = x\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)$$

$$= x\left(x^2 - \frac{\pi^2}{16}\right)$$

$$= x^3 - \frac{\pi^2}{16}x$$

$$W'(x) = 3x^2 - \frac{\pi^2}{16} = 0$$

$$x = \pm \frac{\pi}{4\sqrt{3}}$$

General rule.

Find maximum of $W(x)$ in interval.

1. set $W'(x) = 0$
2. solve for x
3. if $W''(x) < 0$, maxima
   $W''(x) > 0$, minima

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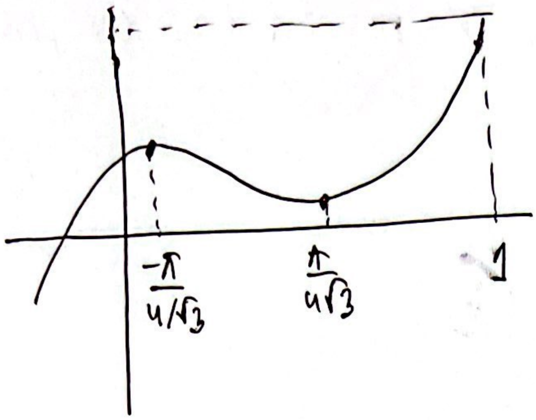

$\therefore$ find $W(x)$ at

X	W(x)
$x = -\frac{\pi}{4\sqrt{3}}$	$+0.186$
$x = +\frac{\pi}{4\sqrt{3}}$	-0.186
$x = -1$	-0.383
$x = +1$	$+0.383$

include the intervals too

$$\therefore \left| f(x) - P_2(x) \right| = \frac{\sin(\xi)}{6} (x) \left(x + \frac{\pi}{4}\right) \left(x - \frac{\pi}{4}\right)$$

$$\text{Max error} = \frac{\sin(1)}{6} * 0.383$$

If we did not have a sin(x) function in the first part, but we got another polynomial function instead, we would have to do the same calculation of W(x) twice for both part.