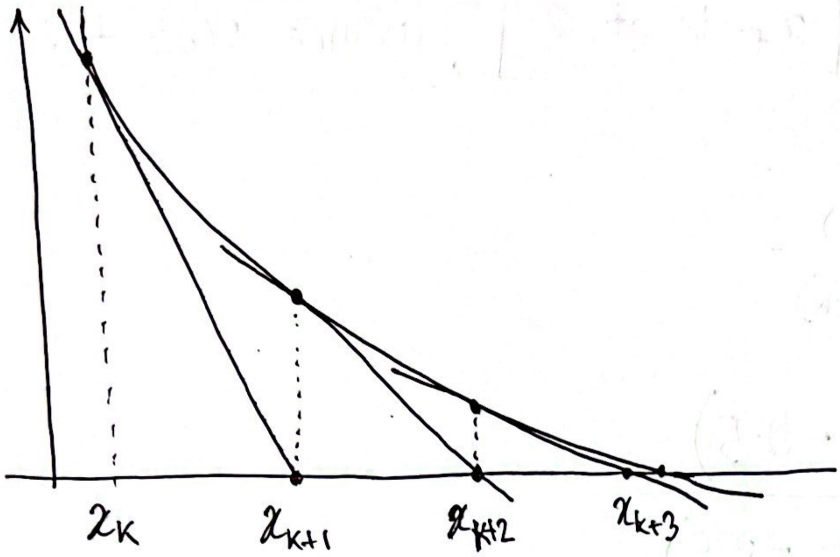
Newton's Method:



Slope of the tangent line:

$$\text{slope} = \frac{\cancelto{0}{f(x_{k+1})} - f(x_k)}{x_{k+1} - x_k} = \frac{-f(x_k)}{x_{k+1} - x_k} \quad \text{—— (I)}$$

$$\text{slope} = f'(x_k) \quad \text{—— (II)}$$

Equating the two equations

$$f'(x) = \frac{-f(x_k)}{x_{k+1} - x_k}$$

$$\boxed{x_{k+1} = \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{g(x)}}$$

Example:

$$f(x) = \frac{1}{x} - 0.5 \qquad [x_* \text{ is at } 2], \text{ assume } x_0 = 1.$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

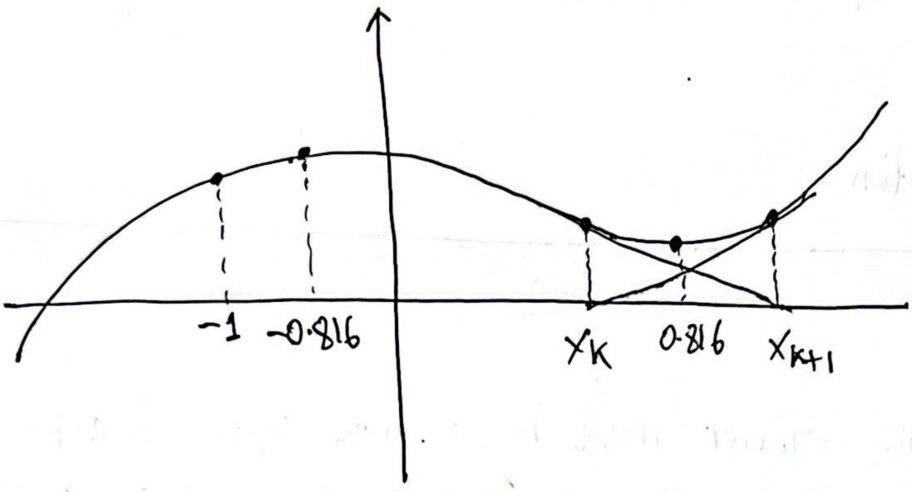
$$= x_k - \frac{\left(\frac{1}{x_k} - 0.5\right)}{\frac{d}{dx}\left(\frac{1}{x_k} - 0.5\right)}$$

$$x_{k+1} = 2x_k - 0.5x_k^2$$

k	x_k
0	1
1	1.5
2	1.875
3	1.9921875
4	1.999969482
5	2
6	2

Newton's Method will not work if there is a turning point between x_K and x_{K+1}.

Example:



Lets say the above function is $f(x) = x^3 - 2x + 2$

Finding the turning points:

$$f'(x) = 3x^2 - 2 = 0$$

$$\Rightarrow x = \pm\sqrt{\frac{2}{3}}$$

$$= \pm 0.816$$

Make sure $x_0 < -0.816$

Example, $x_0 = -1$.

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Example:

$$f(x) = x^2 - 2xe^{-x} + e^{-2x}, \quad x_0 = 1$$

Find the solution of this function within 10^{-5} using

a) Newton's Method

b) Aitken Acceleration.

Note:

0.00001

→ Within 10^{-5} means answer must be accurate upto 5 d.p.
[Calculate answers upto 6 dp, then round upto 5 dp at the last]

→ In the above question, what should we compare our answers to? Should we compare our answers to $f(x)=0$ or compare our answer to the actual root, x_*?

→ If we compare to $f(x)=0$, we will reach our answer when $f(x_k) < 0.00001$, where k is the iteration number.

→ If we compare to the actual root, x_*, we will reach our answer when $|x_* - x_k| < 0.00001$.

→ This means that we compare to $f(x)=0$ when x_* is not given, or x_* cannot be found numerically.

Solution:

$$f(x) = x^2 - 2xe^{-x} + e^{-2x}$$

$$f'(x) = 2x - 2e^{-x} - 2x(-e^{-x}) - 2e^{-2x}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{x_k^2 - 2x_k e^{-x_k} + e^{-2x_k}}{2x_k - 2e^{-x_k} + 2x_k e^{-x_k} - 2e^{-2x_k}}$$

K	x_k	$f(x_k)$	if $\lvert f(x_k)\rvert < 0.0001$?
0	1 $\xrightarrow{\text{put } f(1)}$ ↓ put (1) in iteration formula	0.399576	NO
1	0.768941 →	0.093292	NO
2	0.664590	0.022532	NO
3	0.615033	0.005537	NO
4	0.590884	0.00137	NO
5	0.578963	0.000342	NO
6	0.573041	0.000085	NO
7	0.570089	2×10^{-5}	NO
8	0.568615	0.5×10^{-5}	Yes

Answer: $x_* = 0.56862$ (upto 5 d.p)