Aitken's Acceleration

Aitken's acceleration works on top of other method (bisection, fixed point or newton raphson). Aitken's acceleration can not work on its own

Aitken's acceleration speeds up the process so the convergence happens even quicker ie we find the roots more quickly / early.

$$\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

Usually Aitken's Acceleration runs on Newton Raphson Method (NR) . The following is the format used by Aitken's Acceleration.

Every 3rd iteration is Aitken's Acceleration

$$x_0 \xrightarrow{NR} x_1 \xrightarrow{NR} x_2$$

$\Downarrow$ Aitken's Acceleration

$$\hat{x}_2 \xrightarrow{NR} x_3 \xrightarrow{NR} x_4$$

$\Downarrow$ Aitken's Acceleration

$$\hat{x}_4$$

Q) $f(x) = x^2 - 2xe^{-x} + e^{-2x}$

given, $x_0 = 1$

$$f'(x) = 2x - \left[2e^{-x} + 2xe^{-x} \cdot (-1)\right] + e^{-2x} \cdot (-2)$$
$$= 2x - 2e^{-x} + 2xe^{-x} - 2e^{-2x}$$

Iteration 1

$$x_{0+1} = x_0 + \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1 - \frac{(1)^2 - 2(1)e^{-1} + e^{-2(1)}}{2(1) - 2e^{-(1)} + 2(1)e^{-(1)} - 2e^{-2(1)}}$$

$$= 1 - \frac{0.3995}{1.7293} = 0.7689$$

Iteration 2

$$x_{1+1} = x_1 + \frac{f(x_1)}{f'(x_1)} \qquad x_1 = 0.7689$$

$$x_2 = 0.7689 - \frac{(x_1)^2 - 2(x_1)e^{-x_1} + e^{-2x_1}}{2(x_1) - 2e^{-(x_1)} + 2(x_1)e^{-(x_1)} - 2e^{-2(x_1)}}$$

$$= 0.7689 - \frac{0.033}{} = 0.6648$$

Error bound = 1×10^{-5}

k	x_k	Value x_k	$f(x_k)$	$\lvert f(x_k) \rvert <$ Error Bound
0	x_0	1	0.3995	No
1	x_1	0.7689	0.0933	No
2	x_2	0.6646	0.02253	No
2	$\hat{x}_2$	0.5788	0.000323	No
3	x_3	0.5730	0.000085	No
3	x_4	0.5701	0.000085	No
4	$\hat{x}_4$	0.5672	0.789×10^{-8}	Yes

Aitken's Acceleration (first time)

Here for the value of x we are using $\hat{x}_2$ not x_2

Aitken's Acceleration (second time)

Here also, we use $\hat{x}_2$ not x_2.

STOP

$$\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

Aitken's Acceleration (First Time) - Iteration 3

$$\hat{x}_{k+2} = \hat{x}_2$$

$\therefore k = 0$

Value of k is marked in blue

$$\hat{x}_{\underbrace{0}_{k}+2} = x_0 - \frac{(x_{0+1} - x_0)^2}{x_{0+2} - 2x_{0+1} + x_0}$$

$$= 1 - \frac{(0.7689 - 1)^2}{\left[0.6646 - 2(0.7689) + 1\right]}$$

$$= 0.5788$$

Iteration 4

We are using $\hat{x}_2$ instead of x_2 to accelerate the process.

$\hat{x}_2 = 0.5788$

$$x_{2+1} = x_1 + \frac{f(\hat{x}_2)}{f'(\hat{x}_2)}$$

$$x_3 = \hat{x}_2 - \frac{(\hat{x}_2)^2 - 2(\hat{x}_2)e^{-x_1} + e^{-2(\hat{x}_2)}}{2(\hat{x}_2) - 2e^{-(\hat{x}_2)} + 2(\hat{x}_2)e^{-(\hat{x}_2)} - 2e^{-2(\hat{x}_2)}}$$

$$= 0.5730$$

Iteration 5

Same as iteration 1 and 2.

Aitken's Acceleration (Second Time)

$$\hat{x}_{k+2} = \hat{x}_4$$

$\therefore k = 2$

Here we are using $\hat{x}_2$ not x_2.

Value of k is marked in blue

$$\hat{x}_{\underbrace{2}_{k}+2} = \hat{x}_2 - \frac{(x_{2+1} - \hat{x}_2)^2}{x_{2+2} - 2x_{2+1} + \hat{x}_2}$$

$$= 0.5788 - \frac{(0.5730 - 0.5788)^2}{0.5701 - 2(0.5730) + 0.5788}$$

$$= 0.5672$$

Secant

- Recall the Newton's method of finding the root. The iteration is:
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$
- The above requires that at each iteration point, we need to calculate the first derivative and check if it is zero. This is a drawback. Moreover if $f'(x_k)$ is very small, then loss of significance may occur.
- To avoid this drawback, we replace $f'(x_k)$ by an easily computable function g_k which is approximately equal to $f'(x_k)$. This technique is known as the Quasi-Newton Method.
- In particular if we choose g_k to be the backward difference between the nearest iteration points, then it is called the Secant Method.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$g_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{g_k}$$

$$\left[\, x_{k+1} = x_k - \frac{f(x_k)\,(x_k - x_{k-1})}{f(x_k)\, f(x_{k-1})} \,\right]$$

$$f(x) = \frac{1}{x} - 0.5 \qquad x_* = 2$$

$$x_0 = 0.25 \qquad \frac{|x_* - x_k|}{|x_* - x_{k-1}|}$$

$$x_1 = 0.5$$

$$x_{\underbrace{1+1}_{=2}} = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)} = 0.6875 \qquad 0.75$$

$$x_{\underbrace{2+1}_{=3}} = x_2 - \frac{f(x_2)\,(x_2 - x_1)}{f(x_2) - f(x_1)} = 1.01562 \qquad 0.65625$$

$\vdots \qquad \vdots$

$x_8 =$ 1.99916

$\vdots \qquad \vdots$

$x_{11} =$ 2 $\qquad$ 0