Gram - Schmidt Process:

Lets have a basis $(u_1, u_2, \dots u_n)$ from a vector space. Gram - Schmidt process takes the basis $(u_1, u_2 \dots u_n)$ and forms a new orthogonal basis $(P_1, P_2 \dots P_n)$. We can later transform these orthogonal basis into orthonormal basis $(q_1, q_2 \dots q_n)$.

Original basis : u_1, u_2, u_3

$$\Big\downarrow \text{Gram Schmidt Process}$$

Orthogonal basis : P_1, P_2, P_3

$$\Big\downarrow \text{normalization}$$

Orthonormal basis : q_1, q_2, q_3

1) $P_1 = u_1$

2) $$P_2 = u_2 - \frac{u_2 \cdot P_1}{P_1 \cdot P_1} P_1$$

3) $$P_3 = u_3 - \frac{u_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{u_3 \cdot P_2}{P_2 \cdot P_2} P_2$$

Example:

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

1) $P_1 = u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2) $$P_2 = u_2 - \frac{u_2 \cdot P_1}{P_1 \cdot P_1} P_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{(1 \times 1) + (0 \times -1) + (1 \times 1)}{(1 \times 1) + (-1 \times -1) + (1 \times 1)} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

③ $P_3 = U_3 - \frac{U_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{U_3 \cdot P_2}{P_2 \cdot P_2} P_2$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{(1\times1)+(1\times-1)+(2\times1)}{(1\times1)+(-1\times-1)+(1\times1)} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{(1\times\frac{1}{3})+(1\times\frac{2}{3})+(2\times\frac{1}{3})}{(\frac{1}{3}\times\frac{1}{3})+(\frac{2}{3}\times\frac{2}{3})+(\frac{1}{3}\times\frac{1}{3})} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} - \begin{bmatrix} 5/6 \\ 5/3 \\ 5/6 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

P_1, P_2, P_3 form an orthogonal basis

If we want orthonormal basis, we can devide each vectors by its length. (normalization).

① $q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

② $q_2 = \frac{P_2}{|P_2|} = \frac{1}{\sqrt{6}/3} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix}$

③ $q_3 = \frac{P_3}{|P_3|} = \frac{1}{\sqrt{2}/2} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$

QR Decomposition:

Any real $(m \times n)$ matrix "A" with $m > n$ can be written in the form:

$$A = QR$$

Where Q is a $(m \times n)$ matrix with orthonormal columns

R is an upper triangular matrix of shape $(n \times n)$

orthonormal matrix. Therefore has the properties:

$$QQ^T = I_{nn}$$
$$Q^TQ = I_{nn}$$

$$\underset{m \times n}{A} = \underset{m \times n}{Q}\ \underset{n \times n}{R}$$

→ Multiplying Q^T on both sides.

$$Q^T A = \boxed{Q^T Q} R$$

$$Q^T A = \cancelto{1}{I} R$$

$$Q^T A = R$$

$$\boxed{R = Q^T A}$$

Example of QR Decomposition:

$$A = QR$$

Starting with A

$$A = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & & & \\ u_{m1} & u_{m2} & & u_{mn} \end{bmatrix}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$u_1 \quad u_2 \quad u_n$$

convert u into orthogonal vectors p (Gram-Smidt process)

convert p into orthonormal vectors q (normalization)

$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\downarrow \quad \downarrow$$

$$u_1 \quad u_2$$

$$P_1 = u_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$P_2 = U_2 - \frac{U_2 \cdot P_1}{P_1 \cdot P_1} P_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{(1\times3)+(2\times6)+(2\times0)}{(3\times3)+(6\times6)+(0\times0)} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\therefore P_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

P_1 and P_2 are orthogonal.

→ Now convert P_1 and P_2 into unit vectors. We will call the unit vectors q_1 and q_2.

$$q_1 = \frac{P_1}{|P_1|} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \\ 0 \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore q_1 = \begin{bmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \\ 0 \end{bmatrix} \qquad q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

q_1 and q_2 are orthonormal

$$\therefore Q = \begin{bmatrix} \sqrt{5}/5 & 0 \\ 2\sqrt{5}/5 & 0 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix} \sqrt{5}/5 & 2\sqrt{5}/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\sqrt{5} & \sqrt{5} \\ 0 & 2 \end{bmatrix}$$

$$A = Q \quad R$$

$$\begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5}/5 & 0 \\ 2\sqrt{5}/5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3\sqrt{5} & \sqrt{5} \\ 0 & 2 \end{bmatrix}$$

Check if $Q \cdot R = A$ in your calculator.

Example 2

$x_0 = -3$ $\quad x_1 = 0$ $\quad x_2 = 6$

$f(x_0) = 0$ $\quad f(x_1) = 0$ $\quad f(x_2) = 2$

$$\underset{A}{\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}} \cdot \underset{x}{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

$u_1 \quad u_2$

$$P_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_2 = u_2 - \frac{u_2 \cdot P_1}{P_1 \cdot P_1} P_1$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - \frac{(-3\times1)+(0\times1)+(6\times1)}{(1\times1)+(1\times1)+(1\times1)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

P_1 and P_2 are orthogonal vectors.

$$q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{1}{\sqrt{42}} \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{42} \\ -1/\sqrt{42} \\ 5/\sqrt{42} \end{bmatrix}$$

$$\therefore Q = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{bmatrix}$$ $\longrightarrow$ matrix Q has orthonormal columns.

$$R = Q^T A$$

$$= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix}$$

$$Ax = b$$

Applying Least square approximation method by applying A^T on both sides.

$$A^T A x = A^T b$$

$$(QR)^T (QR)\, x = (QR)^T b$$

$$R^T \boxed{Q^T Q} R\, x = R^T Q^T b$$

I_{nn} [because Q = orthonormal matrix]

$$\cancel{R^T} R x = \cancel{R^T} Q^T b$$

$$\boxed{R x = Q^T b}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ 5\sqrt{2}/\sqrt{21} \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 5/21 \end{bmatrix}$$