Gaussian Elimination

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}}_{b}$$

Augmented Matrix(A),

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right]$$

$m_{21} = \frac{a_{21}}{a_{11}} = 1$ $\qquad R_2 = R_2 - \left(\frac{1}{1}\right)R_1$

$m_{31} = \frac{a_{31}}{a_{11}} = 2$ $\qquad R_3 = R_3 - \left(\frac{2}{1}\right)R_1$

Round 1

Making everything under row 1, col 1 --> 0

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 8 & -4 & 4 \end{array}\right]$$

$R_3 = R_3 - \left(\frac{8}{-4}\right)R_2$

$m_{32} = \frac{a_{32}}{a_{22}} = -2$

Round 2

Making everything under row 2, col 2 --> 0

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 0 & -2 & 12 \end{array}\right]$$

Backward Substitution

① $-2x_3 = 12$

$x_3 = \frac{12}{-2}$

$x_3 = -6$

② $-4x_2 + 1x_3 = 4$

$-4x_2 + 1(-6) = 4$

$-4x_2 = 4 + 6$

$x_2 = \frac{10}{-4}$

$x_2 = -2.5$

③ $x_1 + 2x_2 + x_3 = 0$

$x_1 = 11$

$$x = \begin{pmatrix} 11 \\ -2.5 \\ -6 \end{pmatrix}$$

Pivoting

$$2x_2 + 3x_3 = 1$$
$$1x_1 + 2x_3 = 2$$
$$5x_2 + 1x_3 = 3$$

$$\underbrace{\begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 5 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{b}$$

Here a_{11} and a_{22} are 0. Hence while finding the multiplier we will get infinity.

$m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{0} = \propto$ $\qquad m_{32} = \frac{a_{32}}{a_{22}} = \frac{1}{0} = \propto$

$m_{31} = \frac{a_{31}}{a_{11}} = \frac{1}{0} = \propto$

This is known pivoting problem.

Solution 1: Row Swap

$$2x_2 + 3x_3 = 1$$
$$1x_1 + 2x_3 = 2$$
$$5x_2 + 1x_3 = 3$$

$$\underbrace{\begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 5 & 1 \end{pmatrix}}_{A} \quad \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{b}$$

Row 1 and 2 Swapped

A & b swapped

Final Result

$$1x_1 + 2x_3 = 2$$
$$2x_2 + 3x_3 = 1$$
$$5x_2 + 1x_3 = 3$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 5 & 1 \end{pmatrix}}_{A} \quad \underbrace{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}}_{b}$$

Solution 2: Column Swap

$$0x_1 + 2x_2 + 3x_3 = 1$$
$$1x_1 + 0x_2 + 2x_3 = 2$$
$$0x_1 + 5x_2 + 1x_3 = 3$$

$$\underbrace{\begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 5 & 1 \end{pmatrix}}_{A} \quad \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x}$$

Column 1 and 2 Swapped

A & x swapped

Final Result

$$2x_2 + 0x_1 + 3x_3 = 1$$
$$0x_2 + 1x_1 + 2x_3 = 2$$
$$5x_2 + 0x_1 + 1x_3 = 3$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}}_{A} \quad \underbrace{\begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix}}_{x}$$

Can we find $x = A^{-1}b$ using this A

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}_{b}$$

$$x = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1/7 & 2/7 \\ 4/7 & -1/7 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9/7 \\ 6/7 \end{bmatrix}$$

So Solvable