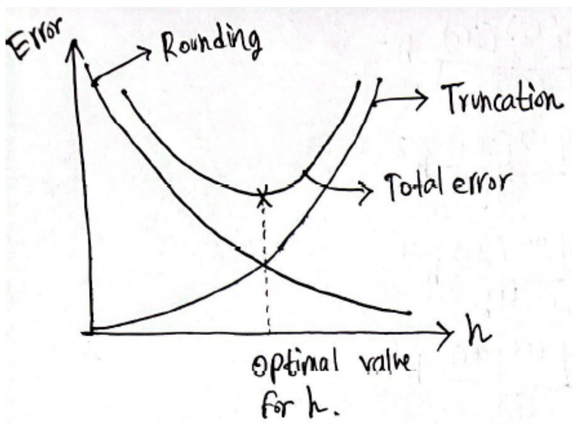
1. **CO-1:** Answer any one from Questions-(1a-1b).

(a) i. (4 marks) Given two nodes, **explain** why the degree of the Hermite interpolation polynomial will be three.

Solution: In hermite interpolation, we look for a polynomial that matches both $f(x_i)$ and $f^{/}(x_i)$ at the nodes $x_i = x_0$, ..., x_n. So, there are $(n + 1) + (n + 1) = 2n + 2$ conditions. This suggests that we will need a polynomial of degree $2n + 1$. Given two nodes, $n = 1$. So the degree will be $2 * 1 + 1 = 3$.

ii. (2+2 marks) In numerical differentiation, the total error involves both the rounding error and truncation error. **State** how the rounding error and truncation error terms for the central difference method changes with respect to the step size h. Finally **explain** how to optimize the total error.

Solution: The first term is the truncation error, which tends to zero as h -> 0. But the second term is the rounding error, which tends to infinity as h -> 0.



(b) i. (4 marks) **Explain** what would be the degree of the Hermite interpolation polynomial if we add the second derivative conditions in addition to first derivative conditions.

Solution: In hermite interpolation, we look for a polynomial that matches both $f(x_i)$ and $f^{/}(x_i)$ at the nodes $x_i = x_0$, ..., x_n. So, there are $(n + 1) + (n + 1) = 2n + 2$ conditions. If we add the second derivative conditions, total conditions will be $(2n + 2) + (n + 1) = 3n + 3$. This suggests that we will need a polynomial of degree $3n + 2$.

ii. (2+2 marks) **Explain** what is loss of significance? **State** how the loss of significance is resolved for a quadratic equation that we used during the lecture.

Solution: Loss of significance is a major cause of errors in floating point calculations. When we subtract two numbers that are very close to each other, there can be an arbitrarily large relative error in the result. This relative error can be much larger than machine epsilon. This is called loss of significance.
If the roots of a quadratic equation($ax^2 + bx + c$) are x_1 and x_2, we computed x_2 from $x_2 = (c/a)/x_1$, which has given us the correct answer.

c.i) $f(x) = \sin x - \cos x$ $\left\{ -\frac{\pi}{4}, 0, \frac{\pi}{4} \right\}$

$$|f(x) - P_2(x)| = \frac{f^{n+1}(\xi)}{(n+1)!} (x-x_0)(x-x_1)(x-x_2)$$

$$= \underbrace{\frac{f^3(\xi)}{3!}}_{=\text{ Part}A} \underbrace{\left(x + \frac{\pi}{4}\right)(x-0)\left(x - \frac{\pi}{4}\right)}_{w(x)}$$

$f'(x) = \cos x + \sin x$

$f''(x) = -\sin x + \cos x$

$f'''(x) = -\cos x - \sin x$

$$|f''' - \cos x - \sin x| = |-\cos x| + |-\sin x| = \cos x + \sin x$$
$$= 1 + 1$$
$$= 2$$

$$w(x) = \left(x + \frac{\pi}{4}\right)(x)\left(x - \frac{\pi}{4}\right)$$
$$= x^3 - \frac{\pi^2}{16} x$$

$$w'(x) = 3x^2 - \frac{\pi^2}{16}$$

~~Part B~~

$\omega'(x) = 0$

$3x^2 - \frac{\pi^2}{16} = 0$

$x = \pm \frac{\pi}{4\sqrt{3}}$

x	$\lvert w(x)\rvert$
$-\frac{\pi}{4\sqrt{3}}$	0.1864738
$\frac{\pi}{4\sqrt{3}}$	0.1864738

$|f(x) - P_2(x)| = \frac{2}{3!} \times 0.1864738 = 0.062158.$ (Ans)

c(ii)

		$f(x)$	$f'(x)$
x_0	$-\frac{\pi}{4}$	$-\sqrt{2}$	0
x_1	$\frac{\pi}{4}$	0	$\sqrt{2}$

$$P_{2n+1} = P_3 = h_0(x) f(x_0) + h_1(x) \cancel{f(x_1)} + \cancel{\hat{h}_0(x) f'(x_0)} + \hat{h}_1(x) f'(x_1)$$

$\therefore f(x_1) = 0$

$f'(x_0) = 0$

$$= h_0(x) f(x_0) + \hat{h}_1(x) f'(x_1)$$

$\underline{h_0(x)}$

$$l_0(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) = \left(\frac{x - \frac{\pi}{4}}{-\frac{\pi}{4} - \frac{\pi}{4}}\right) = -\frac{2}{\pi}\left(x - \frac{\pi}{4}\right)$$

$$= \frac{-2x}{\pi} + \frac{1}{2}$$

$$l_0'(x) = -\frac{2}{\pi}$$

$$h_0(x) = \left[1 - 2(x - x_0)(l_0'(x))\right](l_0(x))^2$$

$$= \cancel{\left[1 - 2\left(x + \frac{\pi}{4}\right)\left(-\frac{2}{\pi}\right)\right]\left(\frac{-2x}{\pi} + \frac{1}{2}\right)^2}$$

$$= \left[1 - 2\left(x + \frac{\pi}{4}\right)\left(-\frac{2}{\pi}\right)\right]\left(-\frac{2x}{\pi} + \frac{1}{2}\right)^2$$

$\hat{h}_1(x) -$

$$l_0(x) = \left(\frac{x-x_0}{x_1-x_0}\right) = \left(\frac{x+\frac{\pi}{4}}{\frac{\pi}{4}+\frac{\pi}{4}}\right)$$

$$= \left(\frac{2}{\pi}x + \frac{1}{2}\right)$$

$$l_1'(x) = \cancel{\frac{2}{\pi}}$$

$$\hat{h}_1(x) = (x-x_1)\left(l_1(x)\right)^2$$

$$= \cancel{(x+\frac{\pi}{4}}$$

$$= \left(x-\frac{\pi}{4}\right)\cancel{\left(\frac{2}{\pi}\right)^2}\left(\frac{2}{\pi}x+\frac{1}{2}\right)^2$$

$$P_3(x) = -\sqrt{2}\left[\left\{1-2\left(x+\frac{\pi}{4}\right)\cancel{\left(-\frac{2}{\pi}\right)}\right\}\left(-\frac{2x}{\pi}+\frac{1}{2}\right)^2\right]$$

$$+\sqrt{2}\left[\left(x-\frac{\pi}{4}\right)\left(\frac{2}{\pi}x+\frac{1}{2}\right)^2\right]$$

Ans to the question No. 3

a

i) $x = (6.25)_{10} = 110.01$

Expressing in ~~de~~ normalized form in the given system,

$$110.01 \rightarrow (1.1001)_2 \times 2^2$$

$$\therefore x = (1.1001)_2 \times 2^2$$

$$y = (7.75)_{10} = 111.11$$

Expressing in normalized form in the given system

$$111.11 \rightarrow (1.1111)_2 \times 2^2$$

$$\therefore y = (1.1111)_2 \times 2^2 .$$

ii) $x \times y = (48.4375)_{10} = (110000.11)_2$

$\rightarrow 1.10000011 \times 2^5$

$\therefore$ in normalized form, $\cancel{(1.10000011)_2 \times 2^5}$ $(1.1000)_2 \times 2^5$

xy

1.10001

$1.1000 \qquad\qquad 1.1001$

$\rightarrow \frac{3}{2} \qquad\qquad \rightarrow \frac{25}{16}$

$$\text{mid} = \frac{\frac{3}{2} + \frac{25}{16}}{2} = \frac{49}{16} \times \frac{1}{2} = \frac{49}{32}$$

$$= 1 + \frac{16}{32} + \frac{1}{32}$$

$$= 1.10001$$

So, $fl(xy) = (1.1000)_2 \times 2^5$

iii) For the above system,

without negative support,

highest number in de-normalized form is $(0.111111)_2 \times 2^5$

lowest number in de-normalized form is $(0.100000)_2 \times 2^{-}$

with negative support,

highest number in de-normalized form is $(0.11111)_2 \times 2^5$

lowest number in de-normalized form is $-(0.11111)_2 \times 2^5$

←

B

i) Vandermonde matrix,

$$V = \begin{bmatrix} 1 & x_0^1 & x_0^2 \\ 1 & x_1^1 & x_1^2 \\ 1 & x_2^1 & x_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 16 \\ 1 & 8 & 64 \\ 1 & 12 & 144 \end{bmatrix}$$

ii)

$x_0 = 4 \quad f[x_0] = 20$

$f[x_0,x_1] = \frac{30-20}{8-4} = 2.5$

$x_1 = 8 \quad f[x_1] = 30$

$f[x_0,x_1,x_2] = \frac{1.25-2.5}{12-4} = -5/32$

$f[x_1,x_2] = \frac{35-30}{12-8} = 1.25$

$x_2 = 12 \quad f[x_2] = 35$

$$P_2(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$

$$= 20 + 2.5(x-4) + \left(-5/32\right)(x-4)(x-8)$$

$$= 20 + 2.5x - 10 - 5/32\,(x^2-12x+32)$$

$$= 10 + 2.5x - 5/32\,x^2 + 15/8\,x - 5$$

$= 5 + 4.375x - \frac{5}{32}x^2$

$\therefore P_2'(x) = 4.375 - \frac{5}{16}x$

$\therefore P_2'(6) = 4.375 - 1.875$

$= 2.5$

iii) $\begin{bmatrix} 1 & 4 & 16 \\ 1 & 8 & 64 \\ 1 & 12 & 144 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ 35 \end{bmatrix}$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 16 \\ 1 & 8 & 64 \\ 1 & 12 & 144 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 30 \\ 35 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 1 \\ -0.625 & 1 & -0.375 \\ 0.0312 & -0.062 & 0.0312 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \\ 35 \end{bmatrix} = \begin{bmatrix} 5 \\ 4.375 \\ -0.144 \end{bmatrix}$$

$\therefore P_2(x) = 5 + 4.375x - 0.144x^2$

Set 02

02. (a)

i) The method of choosing the nodes to avoid Runge phenomena is called Chebyshev nodes.

In case of chebyshev nodes, we use equally angled nodes instead of equally spaced nodes. Moreover, we choose more nodes on the edges to mitigate the error caused by spikes on the edges for Runge functions. Mainly the equally angled node selection property of chebyshev node allows us to avoid the Runge phenomemon from occuring.

ii) $f(x) = \dfrac{1}{1+36x^2}$, $I = [-3, 3]$, $n = 3$

$$x_j = r \cos\left[\frac{(2j+1)\pi}{2(3+1)}\right]$$

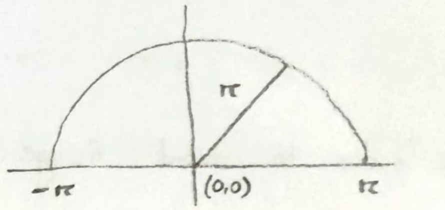
$$\therefore\ x_0 = 3 \cos\left[\frac{(2\times 0+1)\pi}{8}\right] = 3 \cos \frac{\pi}{8}$$

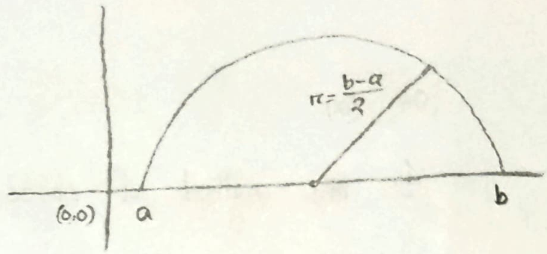
$$x_1 = 3 \cos\left[\frac{(2\times 1+1)\pi}{8}\right] = 3 \cos \frac{3\pi}{8}$$

$$x_2 = 3 \cos\left[\frac{(2\times 2+1)\pi}{8}\right] = 3 \cos \frac{5\pi}{8}$$

$$x_3 = 3 \cos\left[\frac{(2\times 3+1)\pi}{8}\right] = 3 \cos \frac{7\pi}{8}$$

(iii)





Within symmetric interval $[-\pi, \pi]$, the formula for Chebyshev node is, $x_j = \pi \cos\left[\frac{(2j+1)\pi}{2(n+1)}\right]$ where the radius is π. However, in case of asymmetric interval $[a, b]$ where $a \neq -b$, we have to shift the interval.

To shift the interval $[a,b]$ from $[-\pi, \pi]$, we have to add $a-(-\pi) = a+\pi$ or $b-\pi$ amount of displacement with the existing formula. Because of the shift, the radius will also change to $\frac{b-a}{2}$. Here,

$$a+\pi = a + \frac{b-a}{2} = \frac{2a+b-a}{2} = \frac{a+b}{2}$$

$$b-\pi = b - \frac{b-a}{2} = \frac{2b-b+a}{2} = \frac{a+b}{2}$$

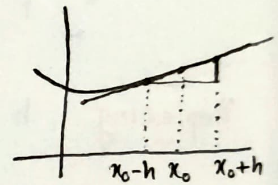
We can see both $a+\pi$ and $b-\pi$ gives the same value. So, after adding the displacement, ultimately, the formula would be,

$$x_j = \frac{b-a}{2} \cos\left[\frac{(2j+1)\pi}{2(n+1)}\right] + \frac{a+b}{2}$$

(b)

(i) Mahdi used Central Differentiation method.

Central differentiation method approximate the derivative by considering the slope between points on both sides of the ~~evalut~~ evaluation point. So it becomes almost parallel to the answer line making the error less. Also the ~~erro~~ truncation error term of central difference is proportional to h^2 instead of h, which is the case for forward and backward difference method's truncation error. Because of this, for $h < 1$, the truncation error of central difference becomes very ~~less~~ small swiftly.



ii) $f(x) = 3x + 6e^{-3x} \Rightarrow f'(x) = 3 - 18e^{-3x} \Rightarrow f'(1'2) = 2'5082$

$$f'(1'2) = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(1'2+0'5) - f(1'2-0'5)}{2\times 0'5}$$

$$= \frac{f(1'7) - f(0'7)}{1}$$

$$= 2'3018$$

Percentage Relative error $= \left| \frac{f'(x) - 2'3018}{f'(x)} \right| \times 100\,\%$

$$= \left| \frac{2'5082 - 2'3018}{2'5082} \right| \times 100\,\%$$

$$= 8'2290\,\%$$

iii) $D_h = \dfrac{f(x+h) - f(x-h)}{2h}$

$$f(x+h) = f(x) + f'(x)h + \frac{f^2(x)}{2!}h^2 + \frac{f^3(x)}{3!}h^3 + \frac{f^4(x)}{4!}h^4 + \frac{f^5(x)}{5!}h^5 + \dots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f^2(x)}{2!}h^2 - \frac{f^3(x)}{3!}h^3 + \frac{f^4(x)}{4!}h^4 - \frac{f^5(x)}{5!}h^5 + \dots$$

$$\therefore D_h = \frac{1}{2h}\left[2f'(x)h + \frac{2f^3(x)}{3!}h^3 + \frac{2f^5(x)}{5!}h^5 + O(h^7)\right]$$

$$= f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{f^5(x)}{5!}h^4 + O(h^6)$$

Replacing $h \to h/4$,

$$D_{h/4} = f'(x) + \frac{f^3(x)}{3!}\left(\frac{h}{4}\right)^2 + \frac{f^5(x)}{5!}\left(\frac{h}{4}\right)^4 + O(h^6)$$

$$\Rightarrow 4^2 D_{h/4} = 4^2 f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{1}{16}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\Rightarrow 4^2 D_{h/4} - D_h = f'(x)(4^2-1) + \frac{f^5(x)}{5!}h^4\left(\frac{1}{16} - 1\right) + O(h^6)$$

$$\Rightarrow \frac{4^2 D_{h/4} - D_h}{4^2-1} = f'(x) + \frac{f^5(x)}{5!}h^4\left(\frac{\frac{1}{16}-1}{4^2-1}\right) + O(h^6)$$

$$\Rightarrow D_h^{(1)} = f'(x) - \frac{1}{16}\frac{f^5(x)}{5!}h^4 + O(h^6)$$