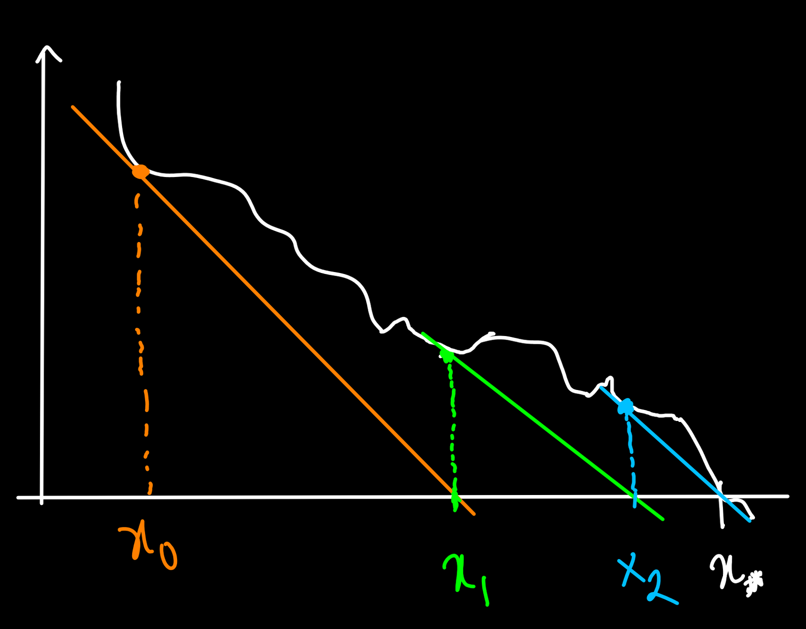
Newton Raphson Method

This method always results in SUPER LINEAR CONVERGENCE



$$\left[x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \right]$$

Using x_k, find $f(x_k)$

if $|f(x_k)| <$ Error Bound [STOP, ans reached]

else find $f'(x_k)$. Then find x_{k+1}

$$\left[x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \right]$$

x_{k+1} for this interation is x_k for next iteration

Q1) Using Newton Raphson method and a given Error Bound find the root

$$f(x) = x^2 - 2xe^{-x} + e^{-2x}$$

given, $x_0 = 1$

$$f'(x) = 2x - \left[2e^{-x} + 2xe^{-x} \cdot (-1)\right] + e^{-2x} \cdot (-2)$$

$$= 2x - 2e^{-x} + 2xe^{-x} - 2e^{-2x}$$

Iteration 1

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1 - \frac{(1)^2 - 2(1)e^{-1} + e^{-2(1)}}{2(1) - 2e^{-(1)} + 2(1)e^{-(1)} - 2e^{-2(1)}}$$

$$= 1 - \frac{0.3995}{1.7293} = 0.7689$$

Iteration 2

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad x_1 = 0.7689$$

$$x_2 = 0.7689 - \frac{(x_1)^2 - 2(x_1)e^{-x_1} + e^{-2x_1}}{2(x_1) - 2e^{-(x_1)} + 2(x_1)e^{-(x_1)} - 2e^{-2(x_1)}}$$

$$= 0.7689 - \frac{0.033}{} = 0.6648$$

Error bound $= 1 \times 10^{-5}$

k	x_k	$f(x_k)$	$\lvert f(x_k) \rvert <$ Error Bound
0	1	0.3995	No
1	0.7689	0.0933	No
2	0.6646	0.02253	No
3	0.6150	0.0055	No
4	0.5989	0.0014	No
5	0.5790	0.0003	No
6	0.5730	0.00085	No
7	0.5701	2×10^{-5}	No
8	0.5686	0.5×10^{-5}	Yes

Our Root

0.5×10^{-5} is less than 1×10^{-5}, so we STOP

With every iteration, out f(x) becomes closer to 0, since we are moving closer to the root.
If we decrease the error bound, then we are increasing the precision. So we will need more iteration until we stop. This will result in a value of root which is more accurate (closer to actual root.)

Q2) Show the first 3 iteration and relative error for the following f(x)

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

$$f'(x) = 3x^2 - 0.33x$$

Iteration 1

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.05 - \frac{f(0.05)}{f'(0.05)}$$

$$= 0.0624$$

$$\text{Relative error} = |X_{new} - X_{old}| = |0.0624 - 0.05| = 0.0124$$

Iteration 2

$$X_{1+1} = X_1 - \frac{f(x)}{f'(x)}$$

$$X_2 = 0.0624 - \frac{f(0.0624)}{f'(0.0624)}$$

$$= 0.0623$$

$$R.E = |0.0623 - 0.0624| = 0.0001$$

Iteration 3

$$X_{2+1} = X_2 - \frac{f(x)}{f'(x)}$$

$$X_3 = 0.0623 - \frac{f(0.0623)}{f'(0.0623)}$$

$$= 0.0623$$

$$R.E = |0.0623 - 0.0623| = 0.0000$$

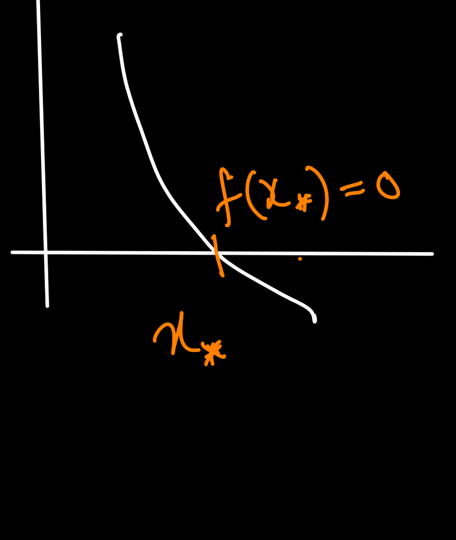
Showing Newton Raphson will always have SUPER LINEAR CONVERGENCE

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$\lambda = |g'(x)| = \left[1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}\right]$$

$$\lambda = |g'(x)| = \frac{f(x)f''(x)}{[f'(x)]^2}$$

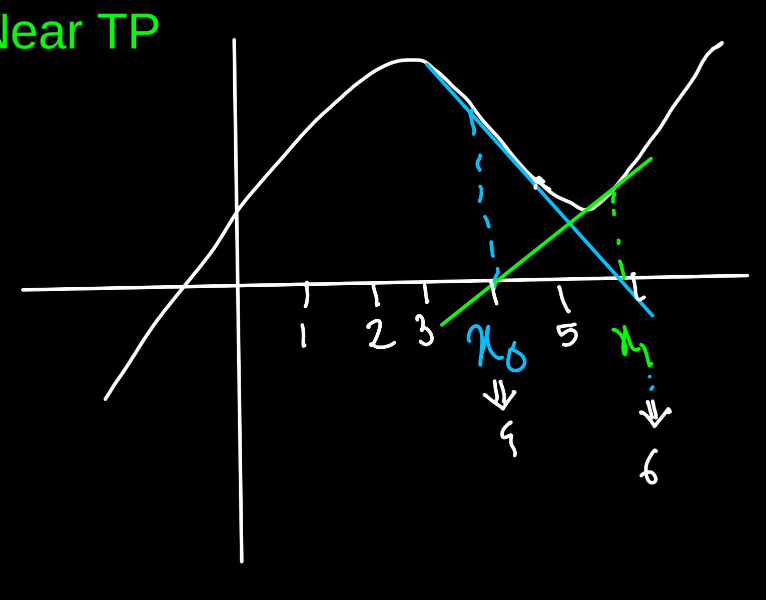


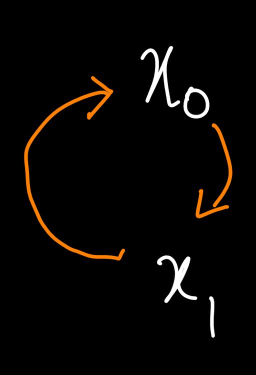
Since at root, f(x) = 0, so our above formulae becomes 0. Hence the entire formulae becomes 0. So Super Linear Convergence

$\lambda = 0$. (shown)

Issues with Newton Raphson Method near or at turning Point

1. Near TP





Turning point = 5.5

$x_0 = 4$

$$x_1 = x_0 + \frac{f(x_0)}{f'(x_0)} = 4 + \frac{f(4)}{f'(4)}$$

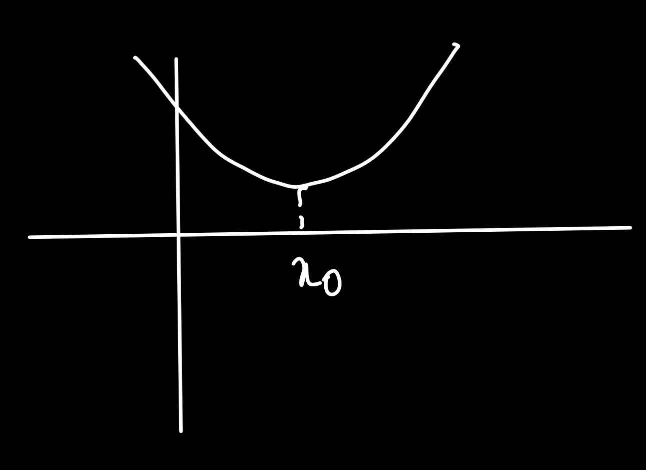
$$= 6$$

$$x_2 = x_1 + \frac{f(x_1)}{f'(x_1)} = 6 + \frac{f(6)}{f'(6)}$$

$$= 4$$

Since turning point (5.5) exists within 4 and 6 so our x will oscillate within 4 and 6 indefini

2. At TP



$$x_1 = x_0 + \frac{f(x_0)}{\underbrace{f'(x_0)}} =$$

The derivative result in 0 as we are AT the turning point. So this will result in infinity for x_1