Rounding Error:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

→ smaller step size, h, better result.

→ However, if h is very small, $f(x+h)$ and $f(x-h)$ will have similar values. Example $f(2+0.001) \approx f(2-0.001)$

→ In floating point chapter, we learned about ~~too~~ loss of significance

→ When we subtract 2 numbers which are close to each other, there are large errors.

→ When h tends to 0 (very small), $f(x+h)$ & $f(x-h)$ are numbers which are closer to each other.

→ Therefore $\frac{f(x+h) - f(x-h)}{2h}$ will give large error (large Rounding error)

$$\boxed{fl(x) = (1+\delta_1)\,x} \leftarrow \text{From chapter 1.}$$

$$fl[f(x_1+h)] = (1+\delta_1)\, f(x_1+h)$$

$$fl[f(x_1-h)] = (1+\delta_2)\, f(x_1-h)$$

$$\boxed{|\delta_1|, |\delta_2| \leq \epsilon_M} \leftarrow \text{from chapter 1}$$

Error:

$$\left| \text{Actual value of differentiation} - \begin{array}{c}\text{Value of differentiation}\\ \text{by numerical Approach}\end{array} \right|$$

$$= \left| \frac{f(x_1+h) - f(x_1-h)}{2h} - \frac{f'''(\xi)}{3!}h^2 - \frac{fl[f(x_1+h)] - fl[f(x_1-h)]}{2h} \right|$$

cancel

$$= \left| \frac{f(x_1+h) - f(x_1-h)}{2h} - \frac{f'''(\xi)}{3!}h^2 - \frac{(1+\delta_1)f(x_1+h) - (1+\delta_2)f(x_1-h)}{2h} \right|$$

$$= \left| -\frac{f'''(\xi)}{6}h^2 - \frac{\delta_1 f(x_1+h) - \delta_2 f(x_1-h)}{2h} \right|$$

$$|a+b| \leq |a| + |b| \longrightarrow \begin{array}{l} |5+(-1)| \leq |5| + |-1| \\ |4| \leq 5+1 \\ |4| \leq 6 \end{array}$$

$$\leq \left|\frac{f'''(\xi)}{6}\right| h^2 + \frac{|\delta_1 \cdot f(x_1+h) - \delta_2 \cdot f(x_1-h)|}{2h}$$

$$|\delta_1|, |\delta_2| \leq \epsilon_M$$

$$\leq \underbrace{\frac{|f'''(\xi)|}{6} h^2}_{} + \underbrace{\epsilon_M \frac{|f(x_1+h)| + |f(x_1-h)|}{2h}}_{}$$

→ This term comes from the truncation of the series.
→ smaller h, lesser error

→ smaller h, larger error (Rounding error).
→ since h is in the denominator.

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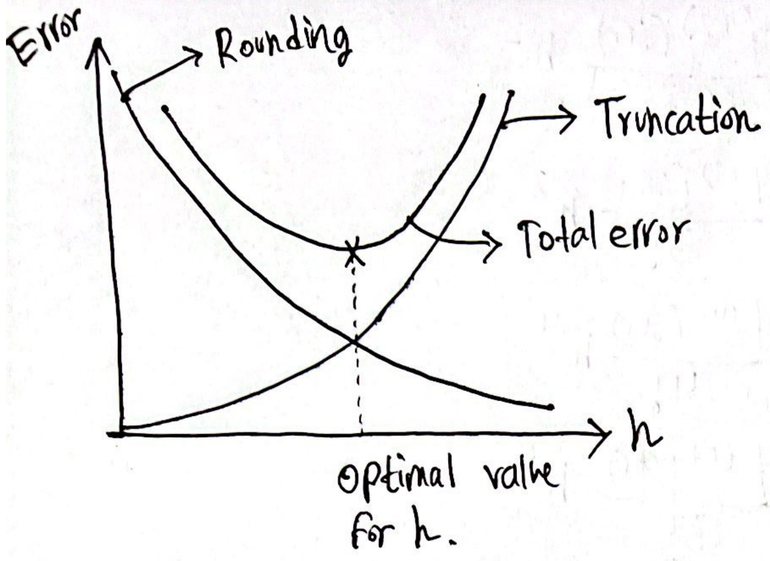

Richardson Extrapolation:

$$D_h := \frac{f(x_1+h) - f(x_1-h)}{2h}$$ ← central differentiation (D_h)

$$\boxed{D_{\frac{h}{2}} = \frac{f(x_1+\frac{h}{2}) - f(x_1-\frac{h}{2})}{2(\frac{h}{2})}}$$

Taylor Series:

$$f(x) = f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}}{2!}(x_0)(x-x_0)^2 + \cdots$$

Centering at x_1:

$$f(x) = f(x_1) + f^{(1)}(x_1)(x-x_1) + \frac{f^{(2)}}{2!}(x_1)(x-x_1)^2 + \cdots$$

$$f(x_1+h) = f(x_1) + f^{(1)}(x_1)(x_1+h-x_1) + \frac{f^{(2)}}{2!}(x_1)(x_1+h-x_1)^2 + \cdots$$

$$= f(x_1) + f^{(1)}(x_1)(h) + \frac{f^{(2)}(x_1)}{2!}(h)^2 + \frac{f^{(3)}(x_1)}{3!}(h^3) + \frac{f^{(4)}(x_1)}{4!}(h^4) + \frac{f^{(5)}(x_1)}{5!}(h^5) + O(h^6) \quad \text{— (1)}$$

$$f(x_1-h) = f(x_1) - f^{(1)}(x_1)h + \frac{f^{(2)}(x_1)}{2!}h^2 - \frac{f^{(3)}(x_1)}{3!}h^3 + \frac{f^{(4)}(x_1)}{4!}h^4 - \frac{f^{(5)}(x_1)}{5!}h^5 + O(h^6) \quad \text{—— (II)}$$

$$D_h = \frac{\overset{\text{(I)}}{f(x_1+h)} - \overset{\text{(II)}}{f(x_1-h)}}{2h}$$

$$D_h = \frac{1}{2h}(\text{I} - \text{II})$$

$$= \frac{1}{\cancel{2}\cancel{h}}\left(\cancel{2}f^{(1)}(x_1)\cancel{h} + \frac{\cancel{2}f^{(3)}(x_1)h^{\cancel{3}}}{3!} + \frac{\cancel{2}f^{(5)}(x_1)h^{\cancel{5}}}{5!} + O(h^7)\right)$$

$$= \underbrace{\boxed{f^{(1)}(x_1)}}_{\text{exact value}} + \underbrace{\boxed{\frac{f^{(3)}(x_1)}{3!}h^2 + \frac{f^{(5)}(x_1)}{5!}h^4 + O(h^6)}}_{\text{error}}$$

→ Error is of order h^2, because $h^4, h^6 \ldots$ are less dominant than h^2.

→ Hence proving again that error $\sim h^2$ for central difference

→ Can we make it better?

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$$D_h = f^{(1)}(x_1) + \boxed{\frac{f^{(3)}(x_1)}{3!} h^2} + \frac{f^{(5)}(x_1)}{5!} h^4 + O(h^6)$$

$$D_{\frac{h}{2}} = f^{(1)}(x_1) + \boxed{\frac{f^{(3)}(x_1)}{3!} \left(\frac{h}{2}\right)^2} + \frac{f^{(5)}(x_1)}{5!} \left(\frac{h}{2}\right)^4 + O(h^6)$$

→ Take combination in such a way that h^2 term goes away.

→ So that we are left with h^4.

$$2^2 D_{\frac{h}{2}} - D_h = 2^2 f^{(1)}(x_1) - f^{(1)}(x_1) + 2^2 \frac{f^{(5)}(x_1)}{5!} \times \frac{1}{2^4} \times h^4 - \frac{f^{(5)}(x_1)}{5!} h^4 + O(h^6)$$

$$2^2 D_{\frac{h}{2}} - D_h = (2^2 - 1) f^{(1)}(x_1) + \left(\frac{1}{2^2} - 1\right) \frac{f^{(5)}(x_1)}{5!} h^4 + O(h^6)$$

$$\boxed{\frac{2^2 D_{\frac{h}{2}} - D_h}{2^2 - 1}} = f^{(1)}(x_1) + \frac{\left(\frac{1}{2^2} - 1\right)}{(2^2-1)\, 5!} f^{(5)}(x_1) h^4 + O(h^6)$$

↑

If we take this combination, error gets reduced to an order of 4.

→ Lets consider this as $D_h^{(1)}$.

→ calculate $D_{\frac{h}{2}}^{(1)}$

→ Then take combination in such a way that h^4 goes away. So now we can have an error of order h^6. It will be called $D_h^{(2)}$ then.

$$D_h = f'(x_1) + \boxed{c}h^n + O(h^{n+1})$$

could be anything like $\dfrac{f^{(n)}(x_1)}{n!}$

$$D_{\frac{h}{2}} = f'(x_1) + c\left(\frac{h}{2}\right)^n + O(h^{n+1})$$

multiply $D_{\frac{h}{2}}$ with 2^n

$$D_h^{(\ )} = \frac{2^n D_{\frac{h}{2}} - D_h}{2^n - 1}$$

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