Central: $\dfrac{f(x_2)-f(x_0)}{2h} - \dfrac{f'''(\xi)}{6}h^2$

$x_0 \to x_0 - h$

$x_1 \to x_0$

$x_2 \to x_0 + h$

$$\frac{f(x_0-h)-f(x_0+h)}{2h} - \frac{f'''(\xi)}{6}h^2$$

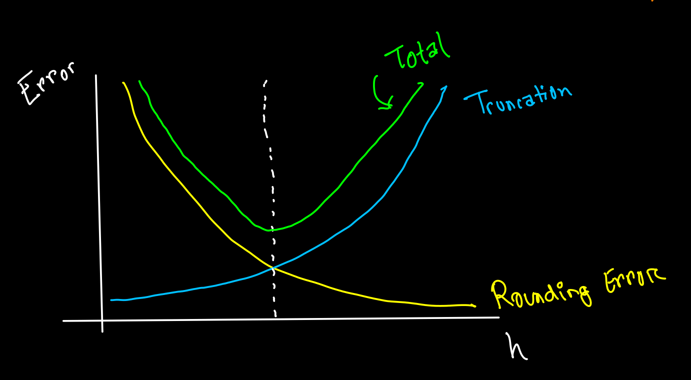
Upper Bound of Total Error = Truncation Error + Rounding Error

$$= \frac{f'''(x)}{6}h^2 + \epsilon_m \cdot \frac{f(x+h)-f(x-h)}{2h}$$

$$\text{Error} \leq \left|\frac{f'''(x)}{6}h^2\right| + \epsilon_m \cdot \left|\frac{f(x+h)-f(x-h)}{2h}\right|$$

h ↑ error ↑

h ↑ error ↓



As h increases, truncation error increases and rounding error decreases.
But at a certain value of h,the overall error is the lowest.
So we will have to change the value of h to reach that point.

Central : $\dfrac{f'''(\xi)}{6}h^2 \Rightarrow O(h^2)$

Forward: $\dfrac{f''(\xi)}{2}h \Rightarrow O(h)$

$$D_h = \frac{f(x+h) - f(x-h)}{2h}$$

$$f(x+h) = \cancel{f(x)} + f'(x).h + \cancel{\frac{f^2(x).h^2}{2!}} + \frac{f^3(x).h^3}{3!} + \cancel{\frac{f^4(x)h^4}{4!}} + \frac{f^5(x)h^5}{5!} + \cancel{O(h^6)}$$

$$f(x-h) = \cancel{f(x)} - f'(x).h + \cancel{\frac{f^2(x).h^2}{2!}} - \frac{f^3(x).h^3}{3!} + \cancel{\frac{f^4(x)h^4}{4!}} - \frac{f^5(x)h^5}{5!} + \cancel{O(h^6)}$$

$$\boxed{D_h = \frac{f(x+h)-f(x-h)}{2h}}$$

$$f(x+h) - f(x-h) = 2f'(x).h + 2\frac{f^3(x).h^3}{3!} + 2\frac{f^5(x)h^5}{5!} + O(h^7)$$

$$\frac{f(x+h)-f(x-h)}{2h} = \frac{1}{\cancel{2}h}\left(\cancel{2}f'(x).\cancel{h} + \cancel{2}\frac{f^3(x).h^{\cancel{3}\,2}}{3!} + \cancel{2}\frac{f^5(x)h^{\cancel{5}\,4}}{5!} + O(h^{\cancel{7}\,6})\right)$$

$$D_h = f'(x) + \underbrace{\frac{f^3(x).h^2}{3!} + \frac{f^5(x)h^4}{5!} + O(h^6)}_{\text{Truncation Error}}$$

$O(h^2)$ $O(h^4)$

First we have to find $D_h^{(1)}$

$$D_{\frac{h}{2}} = f'(x) + \underbrace{\frac{f^3(x)}{3!}.\frac{h^2}{4}}_{\text{We need to remove this part}} + \frac{f^5(x)}{5!}\frac{h^4}{16} + O(h^6)$$

$$4D_{\frac{h}{2}} - D_h = \underbrace{3f'(x)}_{\text{We have to remove the 3 on front. And get 1 as a co-effecient}} + 0 + \left(\frac{1}{4}-1\right)\frac{f^5(x)}{5!}h^4 + O(h^6)$$

Here the denominator under h^2 is being multiplied

$$D_h^{(1)} = \frac{4D_{\frac{h}{2}} - D_h}{3} = f'(x) - \frac{1}{4}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

Now, we have to find $D_h^{(2)}$

$$D_{\frac{h}{2}}^{(1)} = f'(x) \underbrace{- \frac{1}{4} \times \frac{f^5(x)}{5!}\frac{h^4}{16}}_{\text{We need to remove this part}} + O(h^6)$$

$$16D_{\frac{h}{2}}^{(1)} - D_h^{(1)} = \underbrace{15f'(x)}_{\text{We have to remove the 15 on front.}} + O(h^6)$$

Here the denominator under h^4 is being multiplied

$$D_h^{(2)} = \frac{16D_{\frac{h}{2}}^{(1)} - D_h^{(1)}}{15} = f'(x) + O(h^6)$$

Q1) By replacing h with $\frac{h}{4}$. From D_h derive $D_h^{(1)}$

$$D_h = f'(x) + \frac{f^3(x).h^2}{3!} + \frac{f^5(x)h^4}{5!} + O(h^6)$$

$$D_{\frac{h}{4}} = f'(x) + \frac{f^3(x)}{3!}\frac{h^2}{16} + \frac{f^5(x)}{5!}\frac{h^4}{256} + O(h^6)$$

$$16D_{\frac{h}{4}} - D_h = 15f'(x) + 0 + \underbrace{\left(\frac{1}{16}-1\right)}_{=-\frac{15}{16}}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\frac{16D_{\frac{h}{4}} - D_h}{15} = f'(x) + 0 - \frac{1}{16}\frac{f^5(x)h^4}{5!} + O(h^6)$$

$$D_h^{(1)} = f'(x) - \frac{1\,f^5(x)h^4}{16 \times 5!} + O(h^6)$$

Q2) Find $D_{0.2}^{(1)}$ at $x=1$ using Richardson Extrapolation

Given,

$h=0.2$ $f'(x) = 0.5$

$\frac{h}{2} = 0.1$ $f'(x) = 0.7$

$$D_{h=1}^{(1)} = \frac{4D_{\frac{h}{2}} - D_h}{3} = \frac{4(0.7)-0.5}{3}$$

$$= 0.77$$

Q3) Find $D_{1.2}^{(1)}$ & $D_{1.2}^{(2)}$ at $x=2$ using Richardson Extrapolation

$$f(x) = e^{2x+3x}$$

Given,

$h=1.2$, $f'(2) = ?$

$h=0.6$, $f'(2) = ?$

For $h=1.2$,

$$D_{1.2} = f'(2) = \frac{f(x+h) - f(x-h)}{2h}$$

$$= \frac{e^{2(x+h)} + 3(x+h) - [e^{2(x-h)} + 3(x-h)]}{2(h)}$$

$$= \frac{e^{2(2+1.2)} + 3(2+1.2) - [e^{2(2-1.2)} + 3(2-1.2)]}{2(1.2)}$$

$$= 251.705$$

For $h=0.6$,

$$D_{0.6}=f'(2) = \frac{f(x+h)-f(x-h)}{2h}$$

$$= \frac{e^{2(x+h)}+3(x+h)-\left[e^{2(x-h)}+3(x-h)\right]}{2(h)}$$

$$= \frac{e^{2(2+0.6)}+3(2+0.6)-\left[e^{2(2-0.6)}+3(2-0.6)\right]}{2(0.6)}$$

$$= 140.356$$

$$D^{(1)}_{h=1.2} = \frac{4D_{h/2}-D_h}{3}$$

$$= \frac{4(140.356)-251.705}{3}$$

$$= 103.230.$$

- - - - - - -

$$D_h^{(2)} = \frac{16D^{(1)}_{0.6}-D^{(1)}_{1.2}}{15}$$

We have to find $D^{(1)}_{0.6}$

$$D^{(1)}_{0.6} = \frac{4D_{0.3}-D_{0.6}}{3}$$

We have to find $D_{0.3}$

For $h=0.3$,

$$D_{0.3}=f'(2) = \frac{f(x+h)-f(x-h)}{2h}$$

$$= \frac{e^{2(x+h)}+3(x+h)-\left[e^{2(x-h)}+3(x-h)\right]}{2(h)}$$

$$= \frac{e^{2(2+0.3)}+3(2+0.3)-\left[e^{2(2-0.3)}+3(2-0.3)\right]}{2(0.3)}$$

$$= 118.867$$

$$D^{(1)}_{0.6} = \frac{4(118.867)-140.356}{3}$$

$$= 111.696.$$

$$D^{(2)}_{h=1.2} = \frac{16D^{(1)}_{0.6}-D^{(1)}_{1.2}}{15} = \frac{16(111.696)-103.230}{15}$$

$$= 112.2604.$$

Q4) Consider the following table and find:

x	1.1	1.2	1.3
$f(x)$	0.52	0.92	1.29

a) Using the above data, compute $f'(1.2)$ using the central difference method

Here $x_0 = 1.2$, so $h = 0.1$

$$f'(1.2) = \frac{f(x_0+h)-f(x_0-h)}{2h}$$

$$= \frac{f(1.2+0.1)-f(1.2-0.1)}{2(0.1)}$$

$$= \frac{1.29-0.52}{2(0.1)}$$

$$= 5.45.$$

b) Find the absolute error upto 4 significant figures if ,

$f(x) = \cos x + 2\sin x + x^4$

$$f'(x) = -\sin x + 2\cos x + 4x^3$$

Calculations in rad mode

$$f'(1.2) = -\sin(1.2) + 2\cos(1.2) + 4(1.2)^3$$

$$= 6.704678$$

Absolute Error = |Actual value - Aproximate value|

$$= |6.704678 - 5.45|$$

$$= 1.254678$$

$$\simeq 1.255 \text{ (upto 4 s.f.)}$$

c) Find the Relative Error upto 4 significant figures.

Relative Error = $\dfrac{|\text{Actual value - Aproximate value}|}{|\text{Actual Value}|}$

$$= \frac{|6.704678 - 5.45|}{|6.704678|}$$

$$= 0.187135$$

$$\simeq 0.1871 \text{ (upto 4 s.f.)}$$

d) Compute the upper bound of the truncation error if the above data is generated from the function to 4 s.f.

$f(x)=\cos x - 2\sin x + x^4$

The interval is [1.1, 1.3]

$$\left|\frac{h^2}{6}f'''(\xi)\right|$$

So first we need to find $f'''(x)$

$$f(x) = \cos x + 2\sin x + x^4$$

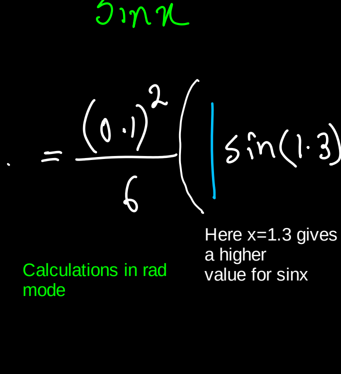
$$f'(x) = -\sin x + 2\cos x + 4x^3$$

$$f''(x) = -\cos x + 2(-\sin x) + 12x^2 = -\cos x - 2\sin x + 12x^2$$

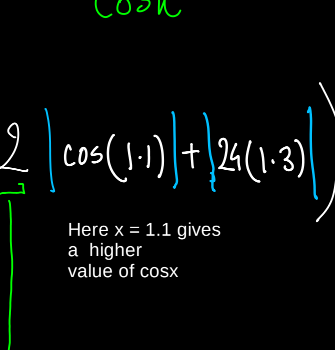
$$f'''(x) = -(-\sin x) - 2\cos x + 24x' = \sin x - 2\cos x + 24x'$$

$$\left|\frac{h^2}{6}f'''(\xi)\right| = \frac{h^2}{6}\left|\sin x - 2\cos x + 24x\right|$$

$$= \frac{(0.1)^2}{6}\ |\sin(1.3)| + |-2\cos(1.1)| + |24(1.3)|$$



sinx



cosx

$$= \frac{(0.1)^2}{6}\left(|\sin(1.3)| + 2\,|\cos(1.1)| + |24(1.3)|\right)$$

Here x=1.3 gives a higher value for sinx

Here x = 1.1 gives a higher value of cosx

Calculations in rad mode

Modulus of -2 gives 2.

$$= 0.0551179$$

$$\simeq 0.05512 \text{ (4 s.f.)}$$

Q5) Compute the upper bound of the truncation error for forward and backward difference for h=0.1 if the above data is generated from the function to 4 s.f.

$f(x)=\cos x-2\sin x+x^4$

The interval is [1.1, 1.3]

Formulae Upper bound of the truncation error for forward / backward difference:

$$\left|\frac{h}{2}f''(\xi)\right|$$

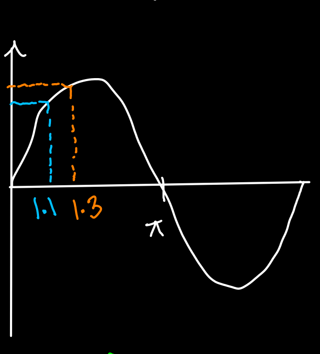
So first we need to find $f''(x)$

$$f(x)=\cos x+2\sin x+x^4$$
$$f'(x)=-\sin x+2\cos x+4x^3$$
$$f''(x)=-\cos x+2(-\sin x)+12x^2=\underline{-\cos x-2\sin x+12x^2}$$

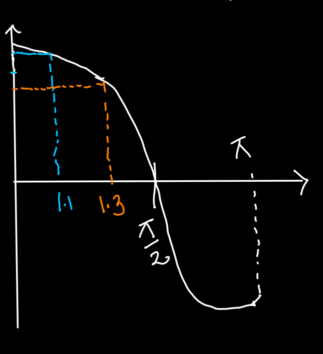
$$\left|\frac{h}{2}f''(\xi)\right|=\frac{0.1}{2}\left|-\cos x-2\sin x+12x^2\right|$$

$$\frac{0.1}{2}\left(\left|-\cos x\right|+\left|-2\sin x\right|+\left|12x^2\right|\right)$$

$$=\frac{0.1}{2}\left(\left|\cos x\right|+2\left|\sin x\right|+\left|12x^2\right|\right)$$



sinx



cosx

$$=\frac{(0.1)}{2}\left(\left|\cos(1.1)\right|+\underbrace{2}\left|\sin(1.3)\right|+\left|12(1.3)^2\right|\right)$$

$$=1.133036$$

Calculations in rad mode

$$\simeq 1.133 \quad (\text{to } 4 \text{ s.f.})$$

Since the interval is already given, hence the upper bound value of forward and backward is the same

Q6) Compute the upper bound of the truncation error for forward, backward and central difference for x =1.2 h=0.1 if the above data is generated from the function to 4 s.f.

$f(x)=\cos x-2\sin x+x^4$

Formulae Upper bound of the truncation error for forward / backward difference:

$$\left|\frac{h}{2}f''(\xi)\right|$$

Formulae Upper bound of the truncation error for central difference:

$$\left|\frac{h^2}{6}f'''(\xi)\right|$$

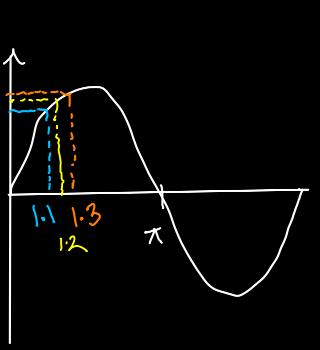
Finding Intervals:

Forward: $[x, x+h]=[1.2, 1.3]$

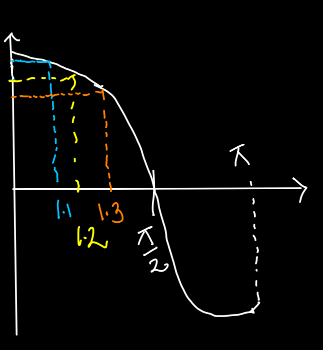
Backward: $[x-h, x]=[1.1, 1.2]$

Central: $[x-h, x+h]=[1.1, 1.3]$

$$f(x)=\cos x+2\sin x+x^4$$
$$f'(x)=-\sin x+2\cos x+4x^3$$
$$f''(x)=-\cos x+2(-\sin x)+12x^2=-\cos x-2\sin x+12x^2$$
$$f'''(x)=-(-\sin x)-2\cos x+24x=\sin x-2\cos x+24x^1$$



sinx



cosx

$$\left|\frac{h}{2}f''(\xi)\right|=\frac{0.1}{2}\left|-\cos x-2\sin x+12x^2\right|$$

$$\frac{0.1}{2}\left(\left|-\cos x\right|+\left|-2\sin x\right|+\left|12x^2\right|\right)$$

$$=\frac{0.1}{2}\left(\left|\cos x\right|+2\left|\sin x\right|+\left|12x^2\right|\right)$$

Upper bound for forward difference:

$$=\frac{(0.1)}{2}\left(\left|\cos(1.2)\right|+2\left|\sin(1.3)\right|+\left|12(1.3)^2\right|\right)$$

$$=1.12847$$

Calculations in rad mode

$$\simeq 1.1285 \quad (4 \text{ s.f})$$

Upper bound for backward difference:

$$=\frac{(0.1)}{2}\left(\left|\cos(1.1)\right|+2\left|\sin(1.2)\right|+\left|12(1.2)^2\right|\right)$$

Calculations in rad mode

$$=0.979884$$
$$\simeq 0.9799 \quad (4 \text{ s.f})$$

Upper bound for central difference:

$$\left|\frac{h^2}{6}f'''(\xi)\right|=\frac{h^2}{6}\left|\sin x-2\cos x+24x\right|$$

$$=\frac{(0.1)^2}{6}\left|\sin(1.3)\right|+\left|-2\cos(1.1)\right|+\left|24(1.3)\right|$$

$$=\frac{(0.1)^2}{6}\left(\left|\sin(1.3)\right|+\underbrace{2}_{\text{Modulus of -2 gives 2.}}\left|\cos(1.1)\right|+\left|24(1.3)\right|\right)$$

Calculations in rad mode

Here x=1.3 gives a higher value for sinx

Here x = 1.1 gives a higher value of cosx

Modulus of -2 gives 2.

$$=0.0551179$$
$$\simeq 0.05512 \quad (4 \text{ s.f}).$$