Practice Questions for Mid

Chapter - 03

① $f(x) = e^x + x\ln(x)$

ⓐ Using Forward Difference, [Given $h = 0.1$]

$$f'(3) = \frac{f(x+h) - f(x)}{h}$$

$$f'(3) = \frac{f(3+0.1) - f(3)}{0.1}$$

$$f'(3) = \frac{f(3.1) - f(3)}{0.1}$$

$f(3.1) = e^{3.1} + (3.1)\ln(3.1)$ [Calculation in radian mode]

$= 25.705$

$f(3) = e^3 + 3\ln(3)$

$= 23.381$

$$f'(3) = \frac{25.705 - 23.381}{0.1} = 23.240 \text{ [5sf]}$$

Using Backward Difference,

$$f'(3) = \frac{f(x) - f(x-h)}{h}$$

$$f'(3) = \frac{f(3) - f(3-0.1)}{0.1}$$

$$= \frac{f(3) - f(2.9)}{0.1}$$

$$= \frac{23.381 - [e^{2.9} + 2.9\ln(2.9)]}{0.1}$$

$= 21.190$ [5sf]

Using Central Difference,

$$f'(3) = \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(3) = \frac{f(3+0.1) - f(3-0.1)}{(2 \times 0.1)}$$

$$f'(3) = \frac{f(3.1) - f(2.9)}{0.2}$$

$$f'(3) = \frac{25.705 - 21.262}{0.2}$$

$$f'(3) = 22.215 \quad [5\text{ sf}]$$

(b) Truncation Error for forward Difference, Backward Difference & Central Difference

Actual value at $x = 3$,

$$f(x) = e^x + x\ln(x)$$

$$f'(x) = e^x + x \cdot \frac{1}{x} + \ln(x) \cdot 1$$

$$f'(x) = e^x + \ln(x) + 1$$

$$f'(3) = e^3 + \ln(3) + 1$$

Forward Difference $= |(e^3 + \ln(3) + 1) - 23.240|$

$= 1.056$

Backward Difference $= |(e^3 + \ln(3) + 1 - 21.190|$

$= 0.9941$

Central Difference $= |e^3 + \ln(3) + 1 - 22.215|$

$= 0.03085$

ⓒ Upper bound of truncation error for Forward and Backward Difference,

$$\frac{|f''(\xi)|}{2}(-h)$$

The range for ξ for forward difference is

$[x, x+h] = [2, 2.2]$

The range for ξ for backward difference is,

$[x-h, x] = [1.8, 2]$

$f(x) = e^x + x\ln(x)$
$f'(x) = e^x + \ln(x) + 1$
$f''(x) = e^x + \frac{1}{x}$

For forward differerce,
$f''(2) = e^2 + \frac{1}{2} = 7.889$

$f''(2.2) = e^{(2.2)} + \frac{1}{2.2} = 9.480$ ✓

$$\left|\frac{9.480}{2} + (-0.2)\right| = 0.948$$

For backward differerce, the upper bound of truncation error is same as forward difference.

Using Central Difference

$$\frac{|f'''(\xi)| h^2}{3!}$$

The range for ξ will be $[x-h, x+h]$
$[1.8, 2.2]$

$$f'''(x) = e^x - \frac{1}{x^2}$$

$$f'''(1.8) = e^{1.8} - \frac{1}{(1.8)^2} = 5.741$$

$$f'''(2.2) = e^{2.2} - \frac{1}{(2.2)^2} = 8.818$$

$$\frac{8.818}{3!} * (0.2)^2 = 0.059.$$

(d) upper bound of truncation error when $x = 2.3$ and $h = 0.1$ within $[1.3, 1.7]$

$$\frac{|f''(\xi)| (-h)}{2}$$

$$f''(x) = e^x + \frac{1}{x}$$

$$f''(1.3) = e^{1.3} + \frac{1}{1.3} = 4.4385$$

$$f''(1.7) = e^{1.7} + \frac{1}{1.7} = 6.0622$$

$$\left|\frac{6.0622}{2} * (-0.2)\right| = 0.6062.$$

② Using Richardson Extrapolation Method

$$D_h = f^{(1)}(x_1) + \frac{f^{(3)}(x_1)}{3!} h^2 + \frac{f^{(5)}(x_1)}{5!} h^4 + O(h^6)$$

Replacing h by 2h/3,

$$D_{2h/3} = f^{(1)}(x_1) + \frac{f^{(3)}(x_1)}{3!}\left(\frac{2h}{3}\right)^2 + \frac{f^{(5)}(x_1)}{5!}\left(\frac{2h}{3}\right)^4 + O(h^6)$$

Take combination in such a way that the h^2 term goes away,

$$\left(\frac{3}{2}\right)^2 D_{2h/3} - D_h = \left(\frac{3}{2}\right)^2 f^{(1)}(x_1) + \left(\frac{3}{2}\right)^2 \frac{f^{(3)}(x_1)}{3!}\left(\frac{2}{3}\right)^2 h^2 + \left(\frac{3}{2}\right)^2 \frac{f^{(5)}(x_1)}{5!}\left(\frac{2}{3}\right)^4 (h)^4 - f^{(1)}(x) - \frac{f^{(3)}(x_1)}{3!} h^2 - \frac{f^{(5)}(x_1)}{5!} h^4 + O(h^6) - O(h^6)$$

$$\left(\frac{3}{2}\right)^2 D_{2h/3} - D_h = \left[\left(\frac{3}{2}\right)^2 - 1\right] f^{(1)}(x) + \left[\frac{2^2}{3^2} - 1\right] \frac{f^{(5)}(x_1)}{5!} h^4 + O(h^7)$$

$$\frac{\frac{9}{4} D_{2h/3} - D_h}{\left(\frac{9}{4} - 1\right)} = f^{(1)}(x) + \frac{\left(\frac{4}{9} - 1\right)}{\left(\frac{9}{4} - 1\right)} * \frac{f^{(5)}(x_1)}{5!} h^4 + O(h^7)$$

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$$D^{(1)}h = \frac{\frac{9}{4} D_{h/2} - D_h}{\frac{5}{4}}.$$

(3) $f(x) = x^3 \cos(x) - e^x + \sin(x) - \ln(x)$

(a) $D^{(1)}h = \frac{4 D_{h/2} - D_h}{3}$

$h = 0.2$, $h/2 = 0.1$

At $h = 0.2$,

$$f'(3.4) = \frac{f(3.4+0.2) - f(3.4-0.2)}{(2\times 0.2)}$$

$$= \frac{f(3.6) - f(3.2)}{0.4}.$$

$f(3.6) = (3.6)^3 \cos(3.6) - e^{(3.6)} + \sin(3.6) - \ln(3.6)$

$= -80.1608$

$f(3.2) = (3.2)^3 \cos(3.2) - e^{(3.2)} + \sin(3.2) - \ln(3.2)$

$= -58.4662.$

$$f'(3.4) = \frac{-80.1608 + 58.4662}{0.4}$$

$$= -54.2365.$$

At $h = 0.1$

$$f'(3.4) = \frac{f(3.4+0.1) - f(3.4-0.1)}{(2\times 0.1)}$$

$$f'(3.4) = \frac{f(3.5) - f(3.3)}{0.2}$$

$$= \frac{-74.8696 + 63.9514}{0.2} = -54.59$$

$$f(3.5) = (3.5)^3 \cos(3.5) - e^{3.5} + \sin(3.5) - \ln(3.5)$$
$$= -74.8696$$

$$F(3.3) = (3.3)^3 \cos(3.3) - e^{3.3} + \sin(3.3) - \ln(3.3)$$
$$= -63.9514$$

$$D^{(1)}\ 0.2 = \frac{[4 * (-54.591)] - [-54.2365]}{3}$$
$$= -54.7092$$
$$= -54.71\ [4sf]$$

(b) $$D^{2}_{0.2} = \frac{2^4 D'(h/2) - D'(h)}{2^4 - 1}$$

At $x = 3.1$ and $h = 0.2$

$$\cancel{D'h} = f'(3.1) = \frac{f(3.1 + 0.2) - f(3.1 - 0.2)}{(2 * 0.2)}$$

$$f'(3.1) = \frac{f(3.3) - f(2.9)}{0.4}$$

$$F(2.9) = (2.9)^3 \cos(2.9) - e^{2.9} + \sin(2.9) - \ln(2.9)$$
$$f(2.9) = -42.6803$$

$$f'(3.1) = \frac{-63.9514 - (-42.6803)}{0.4} = -53.17775$$

At $x = 3.1$ and $h/2 = 0.1$

$$f'(3.1) = \frac{f(3.1 + 0.1) - f(3.1 - 0.1)}{(2 * 0.2)}$$

$$f'(3.1) = \frac{f(3.2) - f(3.0)}{0.4}$$

$$= \frac{-58.4622 - (-47.7728)}{0.4} = -26.7285$$

$$D^{(1)}h = \frac{4 * D(h/2) - D_h}{3}$$

$$= \frac{4 * (-26.7285) - (-53.17775)}{3}$$

$$= -17.91208.$$

At $x = 3.1$ and $h/4 = 0.05$,

$$f'(3.1) = \frac{f(3.1+0.05) - f(3.1-0.05)}{(2 * 0.05)}$$

$$= \frac{f(3.15) - f(3.05)}{0.1}$$

$$= \frac{-55.7466 - (-50.3927)}{0.1}$$

$$= -53.539.$$

h	D_h		
h = 0.2	-53.1775		
h = 0.1	-26.7285	$D'h = -17.91208$	$D^2(h)$
h = 0.05	-53.539	$D'h = -62.47583$	= -65.45

At h = 0.1 and h = 0.05

$$D^{(1)}h = \frac{4 * (-53.539) - (-26.7285)}{3}$$

$$= -62.47583$$

$$D^{(2)}(h) = \frac{2^4 * (-62.47583) - (-17.91208)}{2^4 - 1}$$

$$= -65.45 \; [4sf]$$