CSE330

Spring 2024

Assignment 03

01. (a) Forward Difference :

$$f'(x) = \frac{f(x+h) - f(x)}{h} = \frac{f(1+0.1) - f(1)}{0.1}$$

$$= \frac{f(1.1) - f(1)}{0.1}$$

$$= \frac{1.1 \ln(1.1) - 1 \ln(1)}{0.1}$$

$$= 1.0484 \quad (5 \text{ s.f.})$$

Central Difference :

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(1+0.1) - f(1-0.1)}{2 \times 0.1}$$

$$= \frac{f(1.1) - f(0.9)}{0.2}$$

$$= \frac{1.1 \ln(1.1) - 0.9 \ln(0.9)}{0.2}$$

$$= 0.99833 \quad (5 \text{ s.f.})$$

(b) Backward Difference :

$$f(x) = x \ln x$$

$$f'(x) = \ln x + 1$$

$$f''(x) = 1/x$$

Interval : $[x-h, x] = [1-0.1, 1] = [0.9, 1]$

$f''(x=0.9) = 1.1111$ (5 s.f.) ✓

$f''(x=1) = 1.0000$ (5 s.f.)

Upper bound of Truncation Error $= \frac{f''(\xi)}{2} h$

$$= \frac{1.1111}{2} \times 0.1$$

$$= 0.055555 \text{ (5 s.f.)}$$

Central Difference ;

$$f'''(x) = -1/x^2$$

Interval ; $[x-h, x+h] = [1-0.1, 1+0.1] = [0.9, 1.1]$

$f'''(x=0.9) = -1.2346$ (5 s.f.)

$f'''(x=1.1) = -0.82645$ (5 s.f.) ✓

Upper bound of Truncation Error $= \frac{f'''(\xi)}{3!} h^2$

$$= \frac{-0.82645}{6} \times (0.1)^2$$

$$= -1.3774 \times 10^{-3}$$

(c)

$$D_h = \frac{f(x+h) - f(x-h)}{2h}$$

$$f(x+h) = f(x) + f'(x)h + \frac{f^2(x)}{2!} h^2 + \frac{f^3(x)}{3!} h^3 + \frac{f^4(x)}{4!} h^4 + \frac{f^5(x)}{5!} h^5 + \dots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f^2(x)}{2!} h^2 - \frac{f^3(x)}{3!} h^3 + \frac{f^4(x)}{4!} h^4 - \frac{f^5(x)}{5!} h^5 + \dots$$

$$\therefore D_h = \frac{1}{2h} \left[2f'(x)h + 2\frac{f^3(x)}{3!} h^3 + 2\frac{f^5(x)}{5!} h^5 + O(h^7) \right]$$

$$= f'(x) + \frac{f^3(x)}{3!} h^2 + \frac{f^5(x)}{5!} h^4 + O(h^6)$$

Replacing h by $4h/3$,

$$D_{4h/3} = f'(x) + \frac{f^3(x)}{3!}\left(\frac{4h}{3}\right)^2 + \frac{f^5(x)}{5!}\left(\frac{4h}{3}\right)^4 + O(h^6)$$

$$\Rightarrow \left(\frac{3}{4}\right)^2 D_{4h/3} = \left(\frac{3}{4}\right)^2 f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{f^5(x)}{5!}\left(\frac{4h}{3}\right)^4 \times \left(\frac{3}{4}\right)^2$$
$$+ O(h^6)$$

$$= \left(\frac{3}{4}\right)^2 f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{f^5(x)}{5!}\frac{4^2h^4}{3^2}$$
$$+ O(h^6)$$

$$\Rightarrow \left(\frac{3}{4}\right)^2 D_{4h/3} - D_h = \left(\frac{3}{4}\right)^2 f'(x) - f'(x) + \frac{f^5(x)}{5!}\frac{4^2h^4}{3^2}$$
$$- \frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$= f'(x)\left(\frac{3^2}{4^2} - 1\right) + \frac{f^5(x)h^4}{5!}\left(\frac{4^2}{3^2} - 1\right)$$
$$+ O(h^6)$$

$$\Rightarrow \frac{\left(\frac{3}{4}\right)^2 D_{4h/3} - D_h}{\left(\frac{3}{4}\right)^2 - 1} = f'(x) + \frac{f^5(x)h^4}{5!}\frac{\left(\frac{4^2}{3^2} - 1\right)}{\left(\frac{3}{4}\right)^2 - 1} + O(h^6)$$

$$\Rightarrow D_h^{(1)} = f'(x) - \frac{16}{9}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

02. (a) Central Difference :

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(1 \cdot 2 + 0 \cdot 1) - f(1 \cdot 2 - 0 \cdot 1)}{2 \times 0 \cdot 1}$$

$$= \frac{f(1 \cdot 3) - f(1 \cdot 1)}{0 \cdot 2}$$

$$= \frac{0 \cdot 01131 - 0 \cdot 29027}{0 \cdot 2}$$

$$= -1 \cdot 394 \quad (4 \text{ s.f.})$$

$$f(x) = x\cos(x) - x + \sin(x)$$

$$\Rightarrow f'(x) = \cos(x) - x\sin(x) - 1 + \cos(x)$$

$$= -x\sin(x) + 2\cos(x) - 1$$

$$\Rightarrow f'(x = 1.2) = -1.2\sin(1.2) + 2\cos(1.2) - 1$$

$$= -1.394 \quad (4 \text{ s.f.})$$

$\therefore$ Relative error $= \left| \dfrac{f'(x) - \text{central difference}}{f'(x)} \right|$

$$= \left| \frac{-1.394 - (-1.394)}{-1.394} \right|$$

$$= 0$$

03. (a)

$$D_h = D_{0.2} = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(2.7+0.2) - f(2.7-0.2)}{2\times 0.2}$$

$$= \frac{f(2.9) - f(2.5)}{0.4}$$

$$= -1.384 \times 10^{10} \quad (4 \text{ s.f})$$

$$D_{h/2} = D_{0.1} = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(2.7+0.1) - f(2.7-0.1)}{2\times 0.1}$$

$$= \frac{f(2.8) - f(2.6)}{0.2}$$

$$= -1.103 \times 10^{10} \quad (4 \text{ s.f.})$$

$$\therefore D_{0.2}^{(1)} = \frac{2^r D_{0.1} - D_{0.2}}{2^r - 1}$$

$$= \frac{2^r(-1.103\times10^{10}) - (-1.384\times10^{10})}{2^2 - 1}$$

$$= -1.009\times10^{10} \quad (4 \text{ s.f.})$$

(b)

$$D_{h/4} = D_{0.05} = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(2.7+0.05) - f(2.7-0.05)}{2\times0.05}$$

$$= \frac{f(2.75) - f(2.65)}{0.1}$$

$$= -1.038\times10^{10} \quad (4 \text{ s.f.})$$

$$\therefore D_{0.1}^{(1)} = \frac{2^r D_{0.05} - D_{0.1}}{2^r - 1}$$

$$= \frac{2^r\times(-1.038\times10^{10}) - (-1.103\times10^{10})}{2^r - 1}$$

$$= -1.016\times10^{10} \quad (4 \text{ s.f.})$$

$$\therefore \quad D_{0.2}^{(2)} = \frac{2^4 D_{0.1}^{(1)} - D_{0.2}^{(1)}}{2^4 - 1}$$

$$= \frac{2^4 \times (-1.016 \times 10^{10}) - (-1.009 \times 10^{10})}{2^4 - 1}$$

$$= -1.016 \times 10^{10} \quad (4 \text{ s.f.})$$