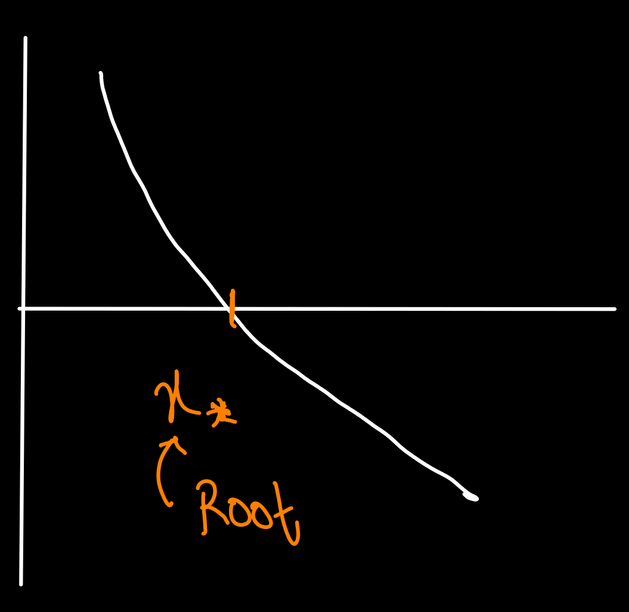
Fixed Point Iteration



$f(x_*) = 0$

$\downarrow$

$\boxed{g(x) = x} \rightarrow$ fixed point

$\underset{\text{input}}{g(x)} = \underset{\text{ouput}}{x}$

When input = output.
We stop iteration as we got our root.

Root = input/output

Step 1: In fixed point iteration, our target is to form different g(x) where x is the subject. We have to create multiple different g(x)

$$f(x) = x^2 - 2x - 3$$

$x^2 - 2x - 3 = 0$

$\Rightarrow x^2 = 2x+3$

$\Rightarrow x = \sqrt{2x+3}$

$g_1(x) = \sqrt{2x+3}$

$x^2 - 2x - 3 = 0$

$\Rightarrow x^2 - x - x - 3 = 0$

$\Rightarrow -x = -x^2 + x - 3$

$\Rightarrow x = x^2 - x + 3$

$g_2(x) = x^2 - x + 3$

$x^2 - 2x - 3 = 0$

$\Rightarrow 2x^2 - x^2 - 2x - 3 = 0$

$\Rightarrow 2x^2 - 2x = x^2 + 3$

$\Rightarrow x(2x-2) = x^2 + 3$

$\Rightarrow x = \frac{(x^2+3)}{(2x-2)}$

$g_3(x) = \frac{(x^2+3)}{(2x-2)}$

Step 2: Use the given starting point of x, x_0 to start the iteration

$x_0 = 0 \leftarrow$ given in question.

$g_1(x) = \sqrt{2x+3}$

$g_1(0) = \sqrt{2(0)+3} = 1.73$

$g_1(1.73) = \sqrt{2(1.73)+3} = 2.54$

$g_1(2.54) = \sqrt{2(2.54)+3} = 2.84$

$g_1(2.84) = \sqrt{2(2.84)+3} = 2.95$

$g_1(2.95) = \sqrt{2(2.95)+3} = 2.98$

$g_1(2.98) = \sqrt{2(2.98)+3} = 3.00$

$g_1(\underbrace{3.00}) = \sqrt{2(3.00)+3} = \underbrace{3.00}$

Since our input and output matches, we got our root.
Root = 3.00.

$g_2(x) = x^2 - x - 3$

$g_2(0) = (0)^2 - (0) - 3 = 9.00$

$g_2(9.00) = (9)^2 - (9) - 3 = 69.0$

$g_3(69.0) = (69)^2 - (69) - 3 = 4.69 \times 10^3$

We stop iteration as this g(x) is divergent.
As with each iteration , difference between input and output INCREASES. Hence it is divergent

$g_3(x) = \frac{x^2+3}{2x-2}$

$g_3(0) = \frac{0^2 + 3}{2(0) - 2} = -1.50$

$g_3(-1.50) = -1$

$g_3(\underbrace{-1}) = \underbrace{-1}$

Root = -1.

So, finally ROOT = -1, 3

Showing g(x) is Convergent or Divergent

$$\lambda = |g'(x)|$$

$\lambda = 0$ [super linear Convergence]

$0 < \lambda < 1$ [Linear Convergence]

$\lambda \geq 1$ [Divergence]

Q2.a) Find the exact roots of f(x) = $x^3 - 2x^2 - x + 2$

When we get roots, $f(x) = 0$

$f(x) = 0$

$x^3 - 2x^2 - x + 2 = 0$

$x^2(x-2) - 1(x-2) = 0$

$(x^2-1)(x-2) = 0$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm\sqrt{1}$

$x = +1, -1$

$x - 2 = 0$

$x = 2$

Roots:- $-1, 1, 2$

b) Construct three g(x) from f(x):

$g_1(x) = \sqrt{\frac{1}{2}(x^3 - x + 2)}$

$g_2(x) = -\frac{2}{x^2 - 2x - 1}$

$g_3(x) = x^3 - 2x + 2 = 0$

$g_1(x) = \frac{1}{\sqrt{2}}(x^3 - x + 2)^{\frac{1}{2}}$

$\lambda_1 = |g_1'(x)| = \frac{1}{\sqrt{2}} \times \frac{1}{2}(x^3 - x + 2)^{-\frac{1}{2}} \times (3x^2 - 1)$

$= \frac{(3x^2-1)}{2\sqrt{2}(x^3 - x + 2)^{\frac{1}{2}}}$

$\lambda_1 = g_1(x) = \left| \frac{(3x^2-1)}{2\sqrt{2}(x^3-x+2)^{\frac{1}{2}}} \right|$

Root	λ_1	
-1	0.6	$\rightarrow 0 < \lambda_1 < 1 \rightarrow$ Linear Convergence
1	0.5	$\rightarrow 0 < \lambda_1 < 1 \rightarrow$ linea Conv.
2	3.75	$\rightarrow \lambda \geq 1 \rightarrow$ Divergence

$g_2(x) = -2(x^2 - 2x - 1)^{-1}$

$\lambda_2 = |g_2'(x)| = -2 \times -1 (x^2 - 2x - 1)^{-2} \times (2x-2)$

$= 2(x^2 - 2x - 1)^{-2} \times (2x - 2)$

$= \frac{2(2x-2)}{(x^2 - 2x - 2)^2}$

$\lambda_2 = |g_2'(x)| = \left| \frac{2(2x-2)}{(x^2-2x-2)^2} \right|$

Root	λ_1	
-1	2	$\rightarrow \lambda \geq 1 \rightarrow$ Divergence
1	0	$\rightarrow \lambda = 0 \rightarrow$ Super Linear Convergence
2	4	$\rightarrow \lambda \geq 1 \rightarrow$ Divergence

$\lambda_3 = \left| g_3'(x) \right| = \left| 3x^2 - 4x \right|$

Root	λ_1		
-1	7	$\rightarrow \lambda \geq 1$	$\rightarrow$ Divergence
1	1	$\rightarrow \lambda \geq 0$	$\rightarrow$ Divergence
2	4	$\rightarrow \lambda \geq 1$	$\rightarrow$ Divergence