

Laplace Transformation

Part A

Laplace Transformation is an integral transformation which is useful in solving linear ODE such as those arising in the analysis of electronic circuits.

Definition

Let  $F(t)$  be a function of  $t$ , for  $t > 0$ . Then the Laplace transformation of  $F(t)$  denoted by  $\mathcal{L}\{F(t)\}$ , where 
$$\mathcal{L}\{F(t)\} = f(s) = \int_0^{\infty} e^{-st} F(t) dt \quad (i)$$

" $s$ " is a complex argument.

→ parameter;

→ initially assume ' $s$ ' is real

→ later, will consider ' $s$ ' as complex, which is useful.

{ At present we will assume the parameter " $s$ " is real.  
Later it will be found useful to consider " $s$ " complex.

$F(t)$  exists if integral (i) converges for some real " $s$ ".

Otherwise it does not exist

It is a useful technique for dealing with linear systems described by ODE.

The table of Laplace transformation of some elementary functions can be found in the corresponding folder of the Google drive.

Find the Laplace Transformation of the following functions:

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t) dt$$

**1**  $\mathcal{L}\{1\} = \frac{1}{s}, s > 0$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} (1) dt \quad (\text{improper integral})$$

$$= \lim_{l \rightarrow \infty} \int_0^l e^{-st} dt$$

$$= \lim_{l \rightarrow \infty} \left[ \frac{e^{-st}}{-s} \right]_0^l = \lim_{l \rightarrow \infty} \left[ \frac{e^{-sl}}{-s} - \frac{e^0}{(-s)} \right]$$

$$= \lim_{l \rightarrow \infty} \frac{1 - e^{-sl}}{s}$$

$$= \frac{1}{s} - \frac{e^{-\infty}}{s}$$

$$= \frac{1}{s} - \frac{1}{s e^\infty} \quad \because e^\infty = \infty ; \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

$$= \frac{1}{s} - 0$$

$$= \frac{1}{s}$$

$$\therefore \mathcal{L}\{F(t)\} = f(s)$$

$$\Rightarrow \mathcal{L}\{1\} = \frac{1}{s}$$

[2]

$$[2] \quad \mathcal{L}\{t\} = \lim_{l \rightarrow \infty} \int_0^l e^{-st} t dt, \quad s > 0$$

$$F(t) \xrightarrow{t \rightarrow \infty} f(s)$$

$$\left( \int_0^\infty e^{-st} F(t) dt \right) \mathcal{L}\{F(t)\}$$

$$\begin{aligned} & \int e^{-st} t dt \\ &= t \int e^{-st} dt - \int \left\{ \frac{d}{dt}(t) \int e^{-st} dt \right\} dt \\ &= t \left( \frac{e^{-st}}{-s} \right) - \int \left( 1 \right) \left( \frac{e^{-st}}{-s} \right) dt \\ &= -\frac{t}{s} e^{-st} + \frac{1}{s} \int e^{-st} dt \\ &= -\frac{t}{s} e^{-st} + \frac{1}{s} \left( \frac{e^{-st}}{-s} \right) \\ &= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \end{aligned}$$

$$\therefore \mathcal{L}\{t\} = \lim_{l \rightarrow \infty} \left[ -\frac{t}{s} e^{-st} - \frac{e^{-st}}{s^2} \right]_0^l$$

$$= \lim_{l \rightarrow \infty} \left[ -\frac{l}{s} e^{-sl} - \frac{e^{-sl}}{s^2} + \frac{0}{s} e^{-0} + \frac{1}{s^2} e^{-0} \right]$$

"0" by  
L'Hospital's  
rule

$$= \lim_{l \rightarrow \infty} \left[ -\frac{l}{s} e^{-sl} - \frac{e^{-sl}}{s^2} + 0 + \frac{1}{s^2} \right]$$

$$\lim_{l \rightarrow \infty} \frac{-l}{s e^{sl}} = \lim_{l \rightarrow \infty} \left[ \frac{-l}{s e^{sl}} - \frac{1}{s^2 e^{sl}} \right] + \frac{1}{s^2}$$

$$\begin{aligned} &= \lim_{l \rightarrow \infty} \frac{-1}{s e^{sl} \cdot s} \\ &= \frac{-1}{s^2 e^{\infty}} = \frac{1}{\infty} \\ &= 0 \end{aligned}$$

$$\therefore \mathcal{L}\{F(t)\} = f(s)$$

$$\Rightarrow \mathcal{L}\{t\} = \frac{1}{s^2}$$

as "l"  
approaches  
to "∞"  
 $\frac{1}{e^{sl}} \approx 0$

Consider

④  $F(t) = e^{at}$   
show  $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ ,  $s > a \because (s-a) > 0$

by definition,  $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$

$$\mathcal{L}\{e^{at}\} = \lim_{l \rightarrow \infty} \int_0^l e^{-st} e^{at} dt$$

$$= \lim_{l \rightarrow \infty} \int_0^l e^{(a-s)t} dt$$

$$= \lim_{l \rightarrow \infty} \left[ \frac{e^{(a-s)t}}{a-s} \right]_0^l$$

$$= \lim_{l \rightarrow \infty} \left[ \frac{e^{(a-s)l}}{(a-s)} - \frac{e^{(a-s)0}}{(a-s)} \right] \quad \begin{cases} \because s > a \\ \therefore a-s < 0 \end{cases}$$

$$= \frac{e^{-\infty}}{a-s} - \frac{1}{a-s}$$

$$= \frac{1}{e^\infty(a-s)} - \frac{1}{a-s}$$

$$= 0 - \frac{1}{a-s}$$

$$= -\frac{1}{a-s}$$

$$= -\frac{1}{s-a}, \quad s > a$$

$$\therefore \mathcal{L}\{F(t)\} = f(s)$$

$$\Rightarrow \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

④

[5] Consider:  
 $F(t) = \sin at$

Show:  $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t) dt = f(s)$

$$\Rightarrow \mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at dt = \frac{a}{s^2 + a^2}, s > 0$$

$$\begin{aligned} \int_0^\infty e^{-st} \sin at dt &= \sin at \int e^{-st} dt - \int \{ \cancel{\cos at} \} \{ e^{-st} dt \} dt \\ &= \sin at \left( \frac{e^{-st}}{-s} \right) - a \int \{ \cos at \left( \frac{e^{-st}}{-s} \right) \} dt \\ &\quad \text{Integrate by parts} \\ &= u \int v dx - \left[ \int u' \{ v dx \} dx \right] \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{s} \sin at e^{-st} + \frac{a}{s} \int \cancel{\cos at} \frac{e^{-st}}{-s} dt \\ &= -\frac{1}{s} \sin at e^{-st} + \frac{a}{s} \left[ \cos at \left( \frac{e^{-st}}{-s} \right) - \int (\sin at)(a) \left( \frac{e^{-st}}{-s} \right) dt \right] \end{aligned}$$

$$= -\frac{1}{s} \sin at e^{-st} - \frac{a}{s^2} \cos at e^{-st} - \frac{a^2}{s^2} \int \sin at e^{-st} dt$$

$$\Rightarrow \int_0^\infty e^{-st} \sin at dt = -\frac{1}{s} \sin at e^{-st} - \frac{a}{s^2} \cos at e^{-st} - \frac{a^2}{s^2} \int \sin at e^{-st} dt$$

$$\Rightarrow \int_0^\infty e^{-st} \sin at dt + \frac{a^2}{s^2} \int \sin at e^{-st} dt = -\frac{1}{s} \sin at e^{-st} - \frac{a}{s^2} \cos at e^{-st}$$

$$\Rightarrow \left(1 + \frac{a^2}{s^2}\right) \int_0^\infty e^{-st} \sin at dt = -\frac{1}{s} \sin at e^{-st} - \frac{a}{s^2} \cos at e^{-st}$$

$$\Rightarrow \int_0^\infty e^{-st} \sin at dt = \frac{1}{1 + \frac{a^2}{s^2}} \left( -\frac{1}{s} \sin at e^{-st} - \frac{a}{s^2} \cos at e^{-st} \right)$$

[5]

$$= \frac{1}{s^2+a^2} \left( -\frac{1}{s} \sin at - \frac{a}{s^2} \cos at e^{-st} \right)$$

$$= \frac{s^2}{s^2+a^2} e^{-st} \left( -\frac{1}{s} \sin at - \frac{a}{s^2} \cos at \right)$$

$$= \frac{e^{-st}}{s^2+a^2} (-s \sin at - a \cos at)$$

$$\mathcal{L}\{\sin at\} = \lim_{l \rightarrow \infty} \left[ \frac{e^{-st}}{s^2+a^2} (-s \sin at - a \cos at) \right]_0^l$$

$e^{-st} (\sin at + a \cos at)$

$$= -\frac{1}{s^2+a^2} \lim_{l \rightarrow \infty} \left[ e^{-sl} (\sin at + a \cos at) - e^{-0} (\sin 0 + a \cos 0) \right]$$

$$= -\frac{1}{s^2+a^2} \lim_{l \rightarrow \infty} \left[ \frac{\sin at}{e^{st}} + \frac{a \cos at}{e^{st}} - \frac{1}{e^0} (0+a) \right]$$

$$= -\frac{1}{s^2+a^2} \left[ s \frac{\sin at}{\infty} + a \frac{\cos at}{\infty} - a \right]$$

$\text{any no.}$   
 $b/w -1 & +1$

$$= -\frac{1}{s^2+a^2} [0+0-a] = \frac{a}{s^2+a^2}$$

Similarly by squeeze Thm

$\lim_{l \rightarrow \infty} \cos at = \text{any number}$   
between -1 & 1

$$\therefore -1 \leq \cos at \leq 1.$$

Consider squeeze Thm:  
 $g(x) \leq f(x) \leq h(x)$

$$-1 \leq \sin x \leq 1$$

$$\therefore -1 \leq \sin at \leq 1$$

$$\lim_{l \rightarrow \infty} (-1) = -1$$

∴ by Squeeze Thm  $\lim_{l \rightarrow \infty} \sin at =$   
any no. between -1 & 1

$$\int_a^b f(x) dx$$

$= F(b) - F(a)$

3, 6, 7, 8

The proof of rest of the functions are available  
on the Reference Book "Schaum's outline Laplace  
Transformation" page 10, 11, 12.

### First Translation or Shifting Property

If  $\mathcal{L}\{F(t)\} = f(s)$  by definition of equation(i) in  
page ①, then  $\mathcal{L}\{e^{at} F(t)\} = f(s-a)$  → Refer to  
exercise no. 2 to  
Problem Sheet #7

#### (Exercise Sheet #7)

1. Find the Laplace transformation of each of the  
following function.

iii)  $7 \sin 2t - 3 \cos 2t$

$$= 7 \mathcal{L}\{\sin 2t\} - 3 \mathcal{L}\{\cos 2t\}$$

$$= 7 \frac{2}{s^2 + 2^2} - 3 \frac{s}{s^2 + 2^2}, s > 0$$

$$= \frac{14}{s^2 + 4} - \frac{3s}{s^2 + 4}$$

$$= \frac{14 - 3s}{s^2 + 4}$$

$$\sin at = \frac{a}{s^2 + a^2}, \cos at = \frac{s}{s^2 + a^2}$$

{using the properties  
of L.T. of  
some elementary  
function}

$$1. \text{iv} \quad (t^2+1)^2$$

$$t^n = \frac{n!}{s^{n+1}}, n=0,1,2,\dots \\ s>0$$

$$= t^4 + 2t^2 + 1$$

$$= L\{t^4\} + 2L\{t^2\} + L\{1\}$$

$$= \frac{4!}{s^{4+1}} + 2 \cdot \frac{2!}{s^{2+1}} + \frac{1}{s}, s>0$$

$$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$$

{ by the properties  
of L.T. of  
some elementary  
functions }

### First Translation or Shifting Property

If  $L\{F(t)\} = f(s)$  by definition of equation(i) in page 4, then  $L\{e^{at} F(t)\} = f(s-a)$  → Refer to exercise no. 2 to Problem sheet #7

### (Exercise Sheet # 7)

2. Evaluate each of the following:

$$\textcircled{i} \quad L\{5e^{3t} \sin 4t\}$$

$$= 5 L\{e^{3t} \underbrace{\sin 4t}_{F(t)}\}$$

$$= 5 \left[ \frac{4}{s^2 - 6s + 25} \right]$$

$$= \frac{20}{s^2 - 6s + 25}$$

shifting property:

$$\begin{aligned} & \text{sin} at \\ &= \frac{a}{s^2 + a^2} \quad L\{F(t)\} = f(s) \\ & \Rightarrow L\{e^{at} F(t)\} = f(s-a) \end{aligned}$$

$$f(s) = L\{F(t)\}$$

$$= L\{\sin 4t\} = \frac{4}{s^2 + 4^2} = \frac{4}{s^2 + 16}, s>0$$

$$\therefore f(s-a) = f(s-3) = \frac{4}{(s-3)^2 + 16}, (s-3)>0 \text{ or } s>3$$

∴  $e^{3t}$  is given

$$= \frac{4}{s^2 - 6s + 9 + 16}$$

$$= \frac{4}{s^2 - 6s + 25}$$

$$2.(iii) \mathcal{L} \{(t+2)^2 e^t\}$$

$$= \mathcal{L} \{(t^2 + 4t + 4)e^t\}$$

$$= \mathcal{L} \{t^2 e^t + 4te^t + 4e^t\}$$

$$= \mathcal{L} \{e^t t^2\} + 4 \mathcal{L} \{e^t t\} + 4 \mathcal{L} \{e^t\}$$

(I)  $f(s) = \mathcal{L} \{t^n\} = \frac{n!}{s^{n+1}}$

$\overset{a=1}{\nearrow} \quad \overset{F(t)}{\nearrow} \quad \overset{a=1}{\nearrow}$

$= \frac{2!}{s^3}, s > 0$

(II)

$e^{at} = \frac{1}{s-a}$

$\therefore e^{st} = \frac{1}{s-1} = \frac{2}{s^3} = f(s)$

$f(s-a) = f(s-1) = \frac{2}{(s-1)^3}, s > 1$

$$= \frac{2}{(s-1)^3} + \frac{4}{(s-1)^2} + \frac{4}{s-1}, s > 1$$

(III)  $f(s) = \mathcal{L} \{t\}$

$$f(s) = \frac{1}{s^2}, s > 0$$

$$f(s-a) = f(s-1) = \frac{1}{(s-1)^2}, s > 1$$

$$2.9iv) L\{e^{-t}(3\sinh 2t - 5\cosh 2t)\}$$

$$= L\{3e^{-t}\sinh 2t - 5e^{-t}\cosh 2t\}$$

$$= 3L\{e^{-t} \underbrace{\sinh 2t}_{F(t)}\} - 5L\{e^{-t} \underbrace{\cosh 2t}_{F(t)}\}$$

①  $a = -1$   
for e at

$$= 3 \cdot \frac{2}{(s+1)^2 - 4} - 5 \frac{s+1}{(s+1)^2 - 4}$$

$$= \frac{6 - 5s - 5}{(s+1)^2 - 4}$$

$$= \frac{1 - 5s}{(s+1)^2 - 4}$$

①  $f(s) = L\{F(t)\}$   
 $= L\{\sinh 2t\}$   
 $= \frac{2}{s^2 - 2^2}, s > |2|$

$$f(s) = \frac{2}{s^2 - 4}$$

$$f(s-a) = f(s-(-1)) = f(s+1)$$

$$= \frac{2}{(s+1)^2 - 4}, s > -1$$

⑪  $f(s) = L\{\cosh 2t\}$

$$= \frac{s}{s^2 - 2^2}, s > |2|$$

$$f(s) = \frac{s}{s^2 - 4}$$

$$f(s-a) = f(s-(-1)) = f(s+1)$$

$$= \frac{s+1}{(s+1)^2 - 4}, s > -1$$

## Inverse Laplace Transformation

$F(t) = \mathcal{L}^{-1}\{f(s)\}$  is the inverse Laplace Transformation  
of  $\mathcal{L}\{F(t)\} = f(s)$ .

$\mathcal{L}^{-1}$  → inverse Laplace Transformation operator.

The inverse Laplace transformation of a function  
 $f(s)$  is the piecewise continuous real function  $F(t)$ .

### (Exercise Sheet #7)

3. Determine each of the following:

$$\begin{aligned} \textcircled{1} \quad & \mathcal{L}^{-1}\left\{\frac{12}{4-3s}\right\} \\ &= -4 \mathcal{L}^{-1}\left\{\frac{1}{s-\frac{4}{3}}\right\} \\ &= -4 e^{4/3 t} \\ &\quad \begin{aligned} & \frac{12}{4-3s} \\ &= \frac{12}{-3s+4} \\ &= \frac{12}{-3(s-\frac{4}{3})} \\ &= \frac{-4}{s-\frac{4}{3}} \\ &= -4\left(\frac{1}{s-\frac{4}{3}}\right) \end{aligned} \end{aligned}$$

$$3. \text{iii} \quad \mathcal{L}^{-1} \left\{ \frac{23s - 15}{s^2 + 8} \right\}$$

$$= 23 \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + (2\sqrt{2})^2} \right\}$$

$$- \frac{15}{2\sqrt{2}} \mathcal{L}^{-1} \left\{ \frac{2\sqrt{2}}{s^2 + (2\sqrt{2})^2} \right\}$$

$$\frac{23s - 15}{s^2 + 8}$$

$$= \frac{23s - 15}{s^2 + (2\sqrt{2})^2} = \frac{23s}{s^2 + (2\sqrt{2})^2} - \frac{15}{s^2 + (2\sqrt{2})^2}$$

$$= 23 \left( \frac{s}{s^2 + (2\sqrt{2})^2} \right) - \frac{15}{2\sqrt{2}} \left( \frac{2\sqrt{2}}{s^2 + (2\sqrt{2})^2} \right)$$

$$= 23 \cos 2\sqrt{2} t - \frac{15}{2\sqrt{2}} \sin 2\sqrt{2} t$$

$$\begin{cases} \frac{a}{s^2 + a^2} = \sin at \\ \frac{s}{s^2 + a^2} = \cos at \end{cases}$$

$$3.\text{iv} \quad \mathcal{L}^{-1} \left\{ \frac{1}{s^{3/2}} \right\}$$

$$= \frac{1}{(\frac{1}{2})!} \mathcal{L}^{-1} \left\{ \frac{(\frac{1}{2})!}{s^{\frac{1}{2}+1}} \right\}$$

$$= \frac{1}{(\frac{1}{2})!} t^{\frac{1}{2}}$$

$$= \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}$$

$$t^n = \frac{n!}{s^{n+1}}$$

$$\begin{aligned} & \frac{1}{s^{3/2}} \\ &= \frac{1}{s^{\frac{1}{2}+1}} \\ &= \frac{1}{(\frac{1}{2})!} \left( \frac{(\frac{1}{2})!}{s^{\frac{1}{2}+1}} \right) \end{aligned}$$

$$1! = 1$$

$$0! = 1$$

$$(-\frac{1}{2})! = \sqrt{\pi}$$

$$(\frac{1}{2})! = \frac{1}{2} \sqrt{\pi}$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$3. \text{ V} \text{ } \mathcal{L}^{-1} \left\{ \frac{1}{\sqrt{2s+3}} \right\}$$

$$= \frac{1}{\sqrt{2}} \mathcal{L}^{-1} \left\{ \frac{1}{(s + \frac{3}{2})^{\frac{1}{2}}} \right\}$$

$$= \frac{1}{\sqrt{2}} \mathcal{L}^{-1} \left\{ \frac{1}{(s - (-\frac{3}{2}))^{\frac{1}{2}}} \right\}$$

$\mathcal{L}^{-1} \left\{ \frac{1}{s^{\frac{1}{2}}} \right\}$   
while  $s = s + \frac{3}{2}$

inner part  
 $\frac{1}{s-a} = e^{at}$   
 $\therefore \frac{1}{s - (-\frac{3}{2})} = e^{-\frac{3}{2}t}$

$$\frac{1}{\sqrt{2s+3}}$$

$$= \frac{1}{(2s+3)^{\frac{1}{2}}}$$

$$= \frac{1}{\{(2s+3)^{\frac{1}{2}}\}^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2}(s + \frac{3}{2})^{\frac{1}{2}}}$$

(chain Rule)

inner part  
 outer part

outer part  
 $\frac{1}{(s)^{\frac{1}{2}}}$

$$= \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} \mathcal{L}^{-1} \left\{ \frac{1}{s^{\frac{1}{2}}} \right\}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} \cdot \frac{2t^{-\frac{1}{2}}}{\sqrt{\pi}}$$

$$= \frac{2}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-\frac{3}{2}t}$$

$$= \sqrt{\frac{2}{\pi}} t^{-\frac{1}{2}} e^{-\frac{3}{2}t}$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^{\frac{1}{2}}} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{1}{s^{-\frac{1}{2}+1}} \right\}$$

$$= \frac{t^{-\frac{1}{2}}}{(-\frac{1}{2}+1)!}$$

$$= \frac{t^{-\frac{1}{2}}}{(\frac{1}{2})!}$$

$$(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$$

$$= \frac{t^{-\frac{1}{2}}}{\frac{\sqrt{\pi}}{2}} = \frac{2t^{-\frac{1}{2}}}{\sqrt{\pi}}$$

Cannot take formula ③

Consider:

$$\frac{n!}{s^{n+1}} = t^n, n > 0$$

$$\Rightarrow \frac{1}{s^{n+1}} = \frac{t^n}{n!}$$

$$\therefore n = -\frac{1}{2}$$

$$\therefore \frac{1}{s^{n+1}} = \frac{t^n}{(n+1)!}$$

$$\therefore n > -1$$

&  $n < 0$

Using Formula ⑨

## (Exercise Sheet #7)

## Reading

4. Evaluate each of the following using partial fraction

$$\textcircled{1} \quad L^{-1} \left\{ \frac{6s-4}{s^2-4s+20} \right\}$$

$$\begin{aligned} & s^2 - 4s + 20 \\ & = s^2 - 2 \cdot s \cdot 2 + 2^2 + 16 \\ & = (s-2)^2 + 16 \end{aligned}$$

$$= L^{-1} \left\{ \frac{6s-4}{(s-2)^2 + 16} \right\}$$

$$\frac{a}{s^2+a^2} = \sin at$$

$$= L^{-1} \left\{ \frac{6s-4}{(s-2)^2 + 4^2} \right\}$$

$$\frac{s}{s^2+a^2} = \cos at$$

$$= L^{-1} \left\{ \frac{6(s-2)+8}{(s-2)^2 + 4^2} \right\}$$

$$L^{-1} \left\{ \frac{4}{(s-2)^2 + 4^2} \right\}$$

$$\left\{ \begin{array}{l} s=s-a \\ \therefore e^{at} \\ = e^{2t} \end{array} \right. = 6 L^{-1} \left\{ \frac{s-2}{(s-2)^2 + 4^2} \right\} + 2 L^{-1} \left\{ \frac{4}{(s-2)^2 + 4^2} \right\}$$

$$= 6 e^{2t} \cos 4t + 2 e^{2t} \sin 4t$$

$$= 2 e^{2t} (3 \cos 4t + \sin 4t)$$

$$4(iv) L^{-1} \left\{ \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} \right\}$$

$$\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{s-2}$$

$$5s^2 - 15s - 11 = A(s-2)^3 + B(s+1) + C(s+1)(s-2) + D(s+1)(s-2)^2 \quad \text{①}$$

In eqn ①:

$$\text{if } s=2 : -21 = 3B \Rightarrow B = -7$$

$$\text{if } s=-1 : 9 = -27A \Rightarrow A = -1/3$$

We <sup>also</sup> need to find C & D, hence expand eqn ①

$$5s^2 - 15s - 11 = A(s^3 - 6s^2 + 12s - 8) + Bs + B + C(s^2 - s - 2) + (Ds + D)(s^2 - 4s + 4)$$

$$= As^3 - 6As^2 + 12As - 8A + Bs + B + Cs^2 - Cs - 2C + Ds^3 \\ + Ds^2 - 4Ds^2 - 4Ds + 4Ds + 4D$$

$$\Rightarrow 5s^2 - 15s - 11 = s^3(A+D) + s^2(-6A+C+D-1D) \\ + s(12A+B-C-4D+4D) + (-8A+B-2C+4D)$$

Equating factors of like terms:	"S <sup>3</sup> "	"S <sup>2</sup> "	"S"	"Constant"
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$$A+D=0$$

$$-6A+C+D-4D=5$$

$$12A+B-C=15$$

$$-8A+B-2C+4D \\ = -11$$

$$A=-D$$

$$-6A+C-3D=5$$

$$-\frac{1}{3} = -D$$

$$-6\left(-\frac{1}{3}\right) + C - 3\left(\frac{1}{3}\right) = 5$$

$$D = \frac{1}{3}$$

$$C = 4$$

$$\mathcal{L}^{-1} \left\{ \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{-1/3}{s+1} + \frac{-7}{(s-2)^3} + \frac{4}{(s-2)^2} + \frac{1/3}{s-2} \right\}$$

$$= -\frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} - 7 \mathcal{L}^{-1} \left\{ \underbrace{\frac{1}{(s-2)^3}}_{\text{chain rule}} \right\} + 4 \mathcal{L}^{-1} \left\{ \underbrace{\frac{1}{(s-2)^2}}_{\text{inner, outer part}} \right\} + \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\}$$

$$= -\frac{1}{3} e^{-t} - 7 \cdot \frac{t^2}{2!} e^{2t} + 4 t e^{2t} + \frac{1}{3} e^{2t}$$

$\left\{ \begin{array}{l} f(s) \\ = \mathcal{L}\{F(t)\} \end{array} \right.$	$F(t)$
$\frac{1}{s-a}$	$e^{at}$
$\frac{n!}{s^{n+1}}$	$t^n$
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
$\frac{1}{s^2}$	$t$

Using shifting law:

$$\mathcal{L}\{e^{at} F(t)\} = f(s-a)$$

$$\text{while } \mathcal{L}\{F(t)\} = f(s).$$

$$4. \text{(VII)} \quad \mathcal{L}^{-1} \left\{ \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} \right\}$$

$$\begin{aligned} & s^2 + 2s + 2 \\ & = s^2 + 2 \cdot s \cdot 1 + 1^2 + 1 \\ & = (s+1)^2 + 1^2 \end{aligned}$$

$$\begin{aligned} & s^2 + 2s + 5 \\ & = s^2 + 2 \cdot s \cdot 1 + 1^2 + 4 \\ & = (s+1)^2 + 2^2 \end{aligned}$$

Now

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

:

$$\Rightarrow s^2+2s+3 = (A+C)s^3 + (2A+B+2C+D)s^2 + (5A+2B+2C+2D)s + (5B+2D)$$

Equating Factors of like terms:

" $s^3$ "	" $s^2$ "	" $s$ "
$A+C=0$ -①	$2A+B+2C+D=1$	$5A+2B+2C+2D=2$ -③
	$2(A+C)+B+D=1$	"constant"
	$2(0)+B+D=1$	$5B+2D=3$ -④
	$B+D=1$ -②	

Solve ② & ④ we have  $B = \frac{1}{3}$ ,  $D = \frac{2}{3}$

Substitute  $B$  &  $D$  into ③

$$5A + 2C + 2\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) = 2$$

$$5A + 2C = 2 - \frac{2}{3} - \frac{4}{3}$$

$$5A + 2C = 0 \quad \text{--- (5)}$$

Solve ① & ⑤  $\Rightarrow A=0$ ,  $C=0$

$$\begin{aligned}
 & \frac{s^2+2s+2}{(s^2+2s+2)(s^2+2s+5)} \\
 &= \frac{s^2+2 \cdot s \cdot 1 + 1^2 + 1}{(s+1)^2 + 1^2} \\
 &= \frac{s^2+2s+5}{(s+1)^2 + 2^2}
 \end{aligned}$$

$$\begin{aligned}
 & L^{-1} \left\{ \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} \right\} \\
 &= L^{-1} \left\{ \frac{\frac{1}{3}}{s^2+2s+2} + \frac{\frac{2}{3}}{s^2+2s+5} \right\}
 \end{aligned}$$

$$= \frac{1}{3} L^{-1} \left\{ \frac{1}{s^2+2s+2} \right\} + \frac{2}{3} L^{-1} \left\{ \frac{1}{s^2+2s+5} \right\}$$

$$= \frac{1}{3} L^{-1} \left\{ \frac{1}{(s+1)^2 + 1^2} \right\} + \frac{2}{3} L^{-1} \left\{ \frac{1}{(s+1)^2 + 2^2} \right\}$$

$$= \frac{1}{3} e^{-t} \frac{\sin t}{1} + \frac{2}{3} e^{-t} \frac{\sin 2t}{2} \quad e^{at} = \frac{1}{s-a}$$

$$= \frac{1}{3} e^{-t} (\sin t + \sin 2t)$$

$$\begin{aligned}
 \sin at &= \frac{a}{s^2+a^2} \\
 \Rightarrow \frac{\sin at}{a} &= \frac{1}{s^2+a^2}
 \end{aligned}$$

# Applications to Ordinary Differential Equations (ODE)

## with constant coefficients:

In mathematics, an ODE is a differential equation containing one or more functions of one independent variable and its derivatives.

Consider

$$y'' + \alpha y' + \beta y = Y(t) \quad \text{--- (1)}$$

initial values:

$$Y(0) = A, \quad Y'(0) = B \quad \text{--- (2)}$$

A, B are given constants (arbitrary values)

Taking the Laplace transformation of both sides of (1) and using (2), we obtain an algebraic equation for

determination of  $\mathcal{L}\{Y(t)\} = y(s)$ .

$$\text{Hence } \mathcal{L}\{y'' + \alpha y' + \beta y\} = \mathcal{L}\{Y(t)\} = y(s) = y$$

$\mathcal{L}\{y''\} + \alpha \mathcal{L}\{y'\} + \beta \mathcal{L}\{y\} = \boxed{\text{ }} \rightarrow \text{taken L.T. in eqn (1)}$

$$\text{Formula: } \mathcal{L}\{Y^{(n)}\} = s^n \mathcal{L}\{Y\} - s^{n-1} Y(0) - s^{n-2} Y'(0) \\ - s^{n-3} Y''(0) - \dots - s^1 Y^{(n-2)}(0) \\ - s^0 Y^{(n-1)}(0).$$

$$\left[ \text{while } Y^{(n)} = \frac{d^n Y}{dx^n} \quad \& \quad \mathcal{L}\{Y\} = y \right]$$

(Exercise Sheet #7)

5. Solve the given differential equation:

①  $y'' - 3y' + 2y = 4e^{2t}$ ,  $y(0) = -3$ ,  $y'(0) = 5$ ;  $y = ?$

$$\Rightarrow \mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{4e^{2t}\} \quad (\text{taking L.T. on both sides})$$

$$\Rightarrow \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 4\mathcal{L}\{e^{2t}\}$$

$$\Rightarrow [s^2 \mathcal{L}\{y\} - s^1 y(0) - s^0 y'(0)] - 3[s^1 \mathcal{L}\{y\}] - 3[s^0 y(0)]$$

$$+ 2[\mathcal{L}\{y\}] = 4\left(\frac{1}{s-2}\right)$$

$$\Rightarrow [s^2 y - s(-3) - 5] - 3[sy - (-3)] + 2y = \frac{4}{s-2} \quad \{\mathcal{L}\{y\} = y\}$$

$$\Rightarrow s^2 y + 3s - 5 - 3sy + 9 + 2y = \frac{4}{s-2}$$

$$\Rightarrow s^2 y - 3sy + 3s + 2y - 14 = \frac{4}{s-2}$$

$$\Rightarrow s^2 y - 3sy + 2y = \frac{4}{s-2} - 3s + 14$$

$$\Rightarrow y(s^2 - 3s + 2) = \frac{4}{s-2} - 3s + 14$$

$$\Rightarrow y = \left(\frac{1}{s^2 - 3s + 2}\right) \left(\frac{4}{s-2} - 3s + 14\right)$$

$$\Rightarrow \mathcal{L}\{y\} = \left(\frac{1}{s^2 - 3s + 2}\right) \left(\frac{4}{s-2} - 3s + 14\right) \therefore \mathcal{L}\{y\} = y$$

$$Y = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 - 3s + 2} \left( \frac{4}{s-2} - 3s + 14 \right) \right\}$$

$\begin{aligned} &= s^2 - 3s + 2 \\ &= s^2 - 2s - s + 2 \\ &= (s-2)(s-1) \end{aligned}$

$$= \mathcal{L}^{-1} \left\{ \frac{4}{(s-2)(s^2 - 3s + 2)} - \frac{3s}{s^2 - 3s + 2} + \frac{14}{s^2 - 3s + 2} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{4}{(s-2)(s-2)(s-1)} - \frac{3s}{(s-2)(s-1)} + \frac{14}{(s-2)(s-1)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{4}{(s-2)^2(s-1)} - \frac{3s}{(s-2)(s-1)} + \frac{14}{(s-2)(s-1)} \right\}$$

$$= 4 \mathcal{L}^{-1} \left\{ \frac{1}{(s-2)^2(s-1)} \right\} - 3 \mathcal{L}^{-1} \left\{ \frac{s}{(s-2)(s-1)} \right\} + 14 \mathcal{L}^{-1} \left\{ \frac{1}{(s-2)(s-1)} \right\}$$

①      ②      ③

$$\textcircled{1} \quad \frac{1}{(s-2)^2(s-1)} = \frac{A}{(s-2)^2} + \frac{B}{s-2} + \frac{C}{s-1}$$

$$1 = A(s-1) + B(s-2)(s-1) + C(s-2)^2 \quad \text{--- } \textcircled{i}$$

$$1 = As - A + Bs^2 - 2Bs - Bs + 2B + Cs^2 - 4Cs + 4C$$

$$1 = As - A + Bs^2 - 2Bs - Bs + 2B + Cs^2 - 4Cs + 4C \quad \text{--- } \textcircled{ii}$$

$$1 = s^2(B+C) + s(A-2B-B-4C) + (-A+2B+4C)$$

Equating factors of like terms

" $s^2$ "

" $s$ "

$$B+C=0 \quad \text{--- } \textcircled{0}$$

$$A-3B-4C=0 \quad \text{--- } \textcircled{**}$$

"constant"

$$-A+2B+4C=1$$

$\hookrightarrow \text{--- } \textcircled{***}$

in eqn  $\textcircled{1}$

$$\text{if } s=1 \Rightarrow 1 = C(-1)^2 \Rightarrow C=1$$

$$\text{if } s=2 \Rightarrow 1 = A(1) \Rightarrow A=1$$

$$\therefore \frac{1}{(s-2)^2(s-1)} = \frac{1}{(s-2)^2} + \frac{(-1)}{s-2} + \frac{1}{s-1}$$

Substitute A & C  
into  $\textcircled{***}$

$$-(1) + 2B + 4(1) = 1$$

$$2B = 1 - 4 + 1$$

$$2B = -2$$

$$B = -1$$

$$\textcircled{2} \quad \frac{s}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1}$$

$$\Rightarrow s = A(s-1) + B(s-2) \quad \text{--- } \textcircled{1}$$

in eqn \textcircled{1}

$$\text{if } s=1 \Rightarrow 1 = B(-1) \Rightarrow B = -1$$

$$\text{if } s=2 \Rightarrow 2 = A(1) \Rightarrow A = 2$$

$$\therefore \frac{s}{(s-2)(s-1)} = \frac{2}{s-2} + \frac{(-1)}{s-1}$$

$$\textcircled{3} \quad \frac{1}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1}$$

$$1 = A(s-1) + B(s-2) \quad \text{--- } \textcircled{1}$$

in eqn \textcircled{1}

$$\text{if } s=1 \Rightarrow 1 = B(-1) \Rightarrow B = -1$$

$$\text{if } s=2 \Rightarrow 1 = A(1) \Rightarrow A = 1$$

$$\therefore \frac{1}{(s-2)(s-1)} = \frac{1}{s-2} + \frac{-1}{s-1}$$

$$Y = 4L^{-1}\left\{\frac{1}{(s-2)^2(s-1)}\right\} - 3L^{-1}\left\{\frac{s}{(s-2)(s-1)}\right\} + 14L^{-1}\left\{\frac{1}{(s-2)(s-1)}\right\}$$

$$= 4L^{-1}\left\{\frac{1}{(s-2)^2} - \frac{1}{s-2} + \frac{1}{s-1}\right\} - 3L^{-1}\left\{\frac{2}{s-2} - \frac{1}{s-1}\right\} + 14L^{-1}\left\{\frac{1}{s-2} - \frac{1}{s-1}\right\}$$

$$= 4\left[L^{-1}\left\{\frac{1}{(s-2)^2}\right\} - L^{-1}\left\{\frac{1}{s-2}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\}\right] - 3\left[2L^{-1}\left\{\frac{1}{s-2}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}\right]$$

chain rule

$$+ 14\left[L^{-1}\left\{\frac{1}{s-2}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}\right]$$

$$= 4\left[e^{2t}L^{-1}\left\{\frac{1}{s^2}\right\} - e^{2t} + e^t\right] - 3\left[2e^{2t} - e^t\right] + 14\left[e^{2t} - e^t\right]$$

$$\begin{aligned}
 Y &= 4[e^{2t} \cdot t - e^{2t} + e^t] - 6e^{2t} + 3e^t + 14e^{2t} - 14e^t \\
 &= 4te^{2t} - 4e^{2t} + 4e^t + 8e^{2t} - 11e^t \\
 &= 4te^{2t} - 7e^t + 4e^{2t}
 \end{aligned}$$

5. ii)  $y'' + 9y = \cos 2t$ ,  $y(0) = 1$ ,  $y(\pi/2) = -1$   
 $\therefore y'(0)$  is unknown, assume  $y'(0) = c$

$$\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{\cos 2t\}$$

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \frac{s}{s^2+2^2}$$

$$[s^2 \mathcal{L}\{y\} - s^1 y(0) - s^0 y'(0)] + 9y = \frac{s}{s^2+4}$$

$$s^2 y - s - c + 9y = \frac{s}{s^2+4}$$

$$y(s^2+9) = \frac{s}{s^2+4} + s + c$$

$$y = \frac{1}{s^2+9} \left( \frac{s}{s^2+4} + s + c \right)$$

$$\mathcal{L}\{y\} = \frac{s}{(s^2+9)(s^2+4)} + \frac{s}{s^2+9} + \frac{c}{s^2+9}$$

$$\begin{aligned}
 y &= \mathcal{L}^{-1} \left\{ \frac{s}{(s^2+9)(s^2+4)} + \frac{s}{s^2+9} + \frac{c}{s^2+9} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{s}{(s^2+9)(s^2+4)} \right\} + \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} + c \mathcal{L}^{-1} \left\{ \frac{1}{s^2+9} \right\}
 \end{aligned}$$

$$\frac{s}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4} = \frac{(As+B)(s^2+4) + (Cs+D)(s^2+9)}{(s^2+9)(s^2+4)}$$

$$s = As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + 9Cs + 9D$$

$$s = s^3(A+C) + s^2(B+D) + s(4A+9C) + (4B+9D)$$

Equating factor of like terms:

$"s^3"$	$"s^2"$	$"s"$	"constant"
$A+C=0$	$B+D=0$	$4A+9C=1$	$4B+9D=0$
①	②	③	④

Solve ① & ③

$$\begin{array}{rcl} \textcircled{1} \times 4 & \rightarrow & 4A + 4C = 0 \\ \textcircled{3} & \rightarrow & (-) 4A + 9C = 1 \\ \hline & & -5C = -1 \end{array}$$

$$\boxed{C = \frac{1}{5}}$$

$$\textcircled{1} \Rightarrow A + C = 0$$

$$A + \frac{1}{5} = 0$$

$$\boxed{A = -\frac{1}{5}}$$

Solve ② & ④

$$B = D = 0$$

$$\begin{aligned} \therefore \frac{s}{(s^2+9)(s^2+4)} &= \frac{-\frac{1}{5}s + 0}{s^2+9} + \frac{\frac{1}{5}s + 0}{s^2+4} \\ &= \frac{-\frac{1}{5}s}{s^2+9} + \frac{\frac{1}{5}s}{s^2+4} \end{aligned}$$

from page 23

$$Y = \mathcal{L}^{-1} \left\{ \frac{s}{(s^2+9)(s^2+4)} \right\} + \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} + C \mathcal{L}^{-1} \left\{ \frac{1}{s^2+9} \right\}$$

$$= \mathcal{L}^{-1} \left\{ -\frac{1/5}{s^2+9} + \frac{1/5}{s^2+4} \right\} + \cos 3t + C \cdot \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{3}{s^2+3^2} \right\}$$

$$= -\frac{1}{5} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} + \frac{1}{5} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+4} \right\} + \cos 3t + \frac{C}{3} \sin 3t$$

$$y = -\frac{1}{5} \cos 3t + \frac{1}{5} \cos 2t + \cos 3t + \frac{C}{3} \sin 3t$$

$$y = \cos 3t \left( 1 - \frac{1}{5} \right) + \frac{1}{5} \cos 2t + \frac{C}{3} \sin 3t$$

$$y = \frac{4}{5} \cos 3t + \frac{C}{3} \sin 3t + \frac{1}{5} \cos 2t \quad \rightarrow "t" \text{ is the variable}$$

$$\therefore y(t) = \frac{4}{5} \cos 3t + \frac{C}{3} \sin 3t + \frac{1}{5} \cos 2t \quad \text{--- (*)}$$

$$y\left(\frac{\pi}{2}\right) = \frac{4}{5} \cos 3\left(\frac{\pi}{2}\right) + \frac{C}{3} \sin 3\left(\frac{\pi}{2}\right) + \frac{1}{5} \cos 2\left(\frac{\pi}{2}\right) = -1$$

( Given  $y\left(\frac{\pi}{2}\right) = -1$ )

$$\Rightarrow \frac{4}{5}(0) + \frac{C}{3}(-1) + \frac{1}{5}(-1) = -1$$

$$\Rightarrow -\frac{C}{3} - \frac{1}{5} = -1$$

$$\Rightarrow \frac{C}{3} + \frac{1}{5} = 1 \Rightarrow \frac{C}{3} = 1 - \frac{1}{5}$$

$$\frac{C}{3} = \frac{4}{5}$$

$$\Rightarrow C = \frac{3 \times 4}{5}$$

$$\Rightarrow C = \frac{12}{5}$$

Substitute "C" into eqn (\*)

$$y = \frac{4}{5} \cos 3t + \frac{12}{5} \sin 3t + \frac{1}{5} \cos 2t \Rightarrow y = \frac{4}{5} \cos 3t + \frac{4}{5} \sin 3t + \frac{1}{5} \cos 2t$$

# Ordinary Differential Equations with Variable Coefficients

The Laplace transform can be used in solving ODE in which the coefficients are variable.  
A particular differential equation where the method proves useful is one in which the terms have the form

$$t^m y^{(n)}(t) = \mathcal{L} \{ t^m y^{(n)}(t) \}$$

the Laplace transform of which is:

$$(-1)^m \frac{d^m}{ds^m} \mathcal{L} \{ y^{(n)}(t) \}$$

Examples on ODE with variable coefficients:

[1]  $ty'' + (1-2t)y' - 2y = 0 ; Y(0) = 1, Y'(0) = 2$

$$\Rightarrow ty'' + y' - 2t y' - 2y = 0$$

$$\Rightarrow \mathcal{L} \{ ty'' + y' - 2t y' - 2y \} = \mathcal{L} \{ 0 \}$$

$$\Rightarrow \mathcal{L} \{ ty'' \} + \mathcal{L} \{ y' \} - 2 \mathcal{L} \{ t y' \} - 2 \mathcal{L} \{ y \} = 0$$

$$\Rightarrow (-1)^1 \frac{d}{ds} \mathcal{L} \{ y'' \} + \mathcal{L} \{ y' \} - 2(-1)^1 \frac{d}{ds} \mathcal{L} \{ y' \} - 2y = 0$$

$$\Rightarrow -\frac{d}{ds} [s^2 \mathcal{L} \{ y \} - s^1 Y(0) - s^0 Y'(0)] + [s^1 \mathcal{L} \{ y \} - s^0 Y(0)]$$

$$+ 2 \frac{d}{ds} [s^1 \mathcal{L} \{ y \} - s^0 Y(0)] - 2y = 0$$

$$\Rightarrow -\frac{d}{ds} [s^2 y - s - 2] + [sy - 1] + 2 \frac{d}{ds} [sy - 1] - 2y = 0$$

$$\Rightarrow \frac{d}{ds}(-s^2y + s + 2) + sy - 1 + 2 \frac{d}{ds}(sy - 1) - 2y = 0$$

$$\Rightarrow \frac{d}{ds}(-s^2y) + \frac{d}{ds}(s) + \frac{d}{ds}(2) + sy - 1 + 2 \left[ \frac{d}{ds}(sy) - \frac{d}{ds}(1) \right] - 2y = 0$$

$$\frac{d}{ds}(uv) = uv' + vu'$$

$$\Rightarrow -s^2 \frac{dy}{ds} + (-2s)y + 1 + 0 + sy - 1 + 2 \left[ y + s \frac{dy}{ds} - 0 \right] - 2y = 0$$

$$\Rightarrow -s^2 \frac{dy}{ds} - 2sy + sy + 2y + 2s \frac{dy}{ds} - 2y = 0$$

$$\Rightarrow s^2 \frac{dy}{ds} + 2sy - sy - 2s \frac{dy}{ds} = 0 \quad [ \div by -1 ]$$

$$\Rightarrow s^2 \frac{dy}{ds} - 2s \frac{dy}{ds} + sy = 0$$

$$\Rightarrow s \frac{dy}{ds} - 2 \frac{dy}{ds} + y = 0 \quad (\div by s)$$

$$\Rightarrow (s-2) \frac{dy}{ds} + y = 0$$

$$\Rightarrow (s-2) \frac{dy}{ds} = -y \Rightarrow (s-2)dy = -yds \quad [\text{separate variable}]$$

$$\Rightarrow \frac{dy}{y} = \frac{ds}{s-2}$$

$$\rightarrow \frac{dy}{y} + \frac{ds}{s-2} = 0$$

$$\rightarrow \int \frac{1}{y} dy + \int \frac{1}{s-2} ds = 0$$

$$\log_e \underline{x = y} \\ \underline{x = e^y}$$

$$\Rightarrow \ln y + \ln(s-2) = C \rightarrow (\text{integrate both sides})$$

$$\Rightarrow \ln(y(s-2)) = C$$

$$\Rightarrow \log_e [y(s-2)] = C$$

Re-label the constant  
 $e^C = C$

$$Y = \frac{C}{s-2}$$

$$\mathcal{L}\{Y\} = \frac{C}{s-2}$$

$$Y = C \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$Y = C e^{2t}$$

$$\Rightarrow Y(t) = C e^{2t} \quad \text{--- (i)}$$

Given  $Y(0) = 1$   
Substitute  $t=0, Y=1$  into (i)

$$Y(0) = C e^{2(0)} = 1 \quad \text{Given}$$

$$\Rightarrow C e^0 = 1$$

$$\Rightarrow C = 1$$

Substitute "C" into (i)

$$Y = e^{2t}$$

2]  $Y'' - tY' + Y = 1 ; \quad Y(0) = 1, \quad Y'(0) = 2$

Ans:  $Y = 1 + 2t$

TRY

**3** Solve  $tY'' + Y' + 4tY = 0$ ,  $Y(0) = 3$ ,  $Y'(0) = 0$ .

take L.T. on both sides

$$t^m = (-1)^m \frac{d^m}{ds^m}$$

$$\mathcal{L}\{tY''\} + \mathcal{L}\{Y'\} + \mathcal{L}\{4tY\} = \mathcal{L}\{0\}$$

$$\Rightarrow (-1)^1 \frac{d}{ds} \mathcal{L}\{Y''\} + \mathcal{L}\{Y'\} + 4(-1)^1 \frac{d}{ds} \mathcal{L}\{Y\} = 0$$

$$\Rightarrow -\frac{d}{ds} [S^2 \mathcal{L}\{Y\} - S^1 Y(0) - S^0 Y'(0)] + [S^1 \mathcal{L}\{Y\} - S^0 Y(0)]$$

$$-4 \frac{d}{ds} [y] = 0$$

$$\Rightarrow -\frac{d}{ds} [S^2 y - 3S - (1)(0)] + [sy - 1(3)] - 4 \frac{dy}{ds} = 0$$

$$\Rightarrow \frac{d}{ds} (S^2 y + 3S) + sy - 3 - 4 \frac{dy}{ds} = 0$$

$$\Rightarrow \frac{d}{ds} S^2 y - \frac{d}{ds} 3S - sy + 3 + 4 \frac{dy}{ds} = 0 \quad (\text{by } -1)$$

$$\Rightarrow S^2 \frac{dy}{ds} + 2sy - 3 - sy + 3 + 4 \frac{dy}{ds} = 0$$

$$\Rightarrow (S^2 + 4) \frac{dy}{ds} + sy = 0$$

$$\Rightarrow (S^2 + 4) \frac{dy}{ds} = -sy \Rightarrow (S^2 + 4) dy = -sy ds$$

$$\Rightarrow \frac{dy}{y} = -ds/(S^2 + 4) \quad (\text{separate variable})$$

$$\Rightarrow \frac{dy}{y} + \frac{s}{S^2 + 4} ds = 0$$

$$\Rightarrow \int \frac{1}{y} dy + \int \frac{s}{S^2 + 4} ds = \int 0 = C$$

$$\Rightarrow \ln y + \int \frac{s}{s^2+4} ds = C$$

$$\Rightarrow \ln y + \int \frac{\frac{1}{2} du}{u} = C$$

Let  $s^2+4=u$   
 $2sds=du$

$$sds = \frac{1}{2} du$$

$$\Rightarrow \ln y + \frac{1}{2} \ln u = C$$

$$\Rightarrow \ln y + \frac{1}{2} \ln(s^2+4) = C$$

$$\Rightarrow \ln y + \ln \sqrt{s^2+4} = C$$

$$\Rightarrow \ln(y\sqrt{s^2+4}) = C$$

$$\Rightarrow \log_e(y\sqrt{s^2+4}) = C$$

$$\Rightarrow y\sqrt{s^2+4} = e^C = C$$

Relabel the constant  
 $e^C = C$

$$\Rightarrow y = \frac{C}{\sqrt{s^2+4}}$$

$$Y(0) = C L^{-1} \left\{ \frac{1}{s^2+4} \right\} = 3 \cdot 1$$

Given  $Y(0)=3$

$$\Rightarrow L\{Y\} = \frac{C}{\sqrt{s^2+4}}$$

$$\Rightarrow Y = C L^{-1} \left\{ \frac{1}{(s^2+4)^{1/2}} \right\}$$

$$\frac{(n+i)!}{s^{n+i}} = t^n$$

by equating factors of like terms

$$\Rightarrow Y(0) = 3 \frac{\sin 2t}{2}$$

$$= \frac{3 \sin 2t}{2}$$

$$= \frac{3 \sin 2t}{2} \cdot \frac{t^{-1/2}}{(-\frac{1}{2}+1)!}$$

$$\frac{1}{s^{1/2}} = \frac{1}{s^{-\frac{1}{2}+1}} \quad n = -\frac{1}{2}$$

$$= \frac{1}{s^{n+1}}$$

use formula

$$\Rightarrow \frac{t^n}{(n+1)!}$$

$$= \frac{3 \sin 2t}{2} \cdot \frac{1}{(+\frac{1}{2})!} \cdot t^{-\frac{1}{2}}$$

$$Y = \frac{3 \sin 2t}{2} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{t^{\frac{1}{2}}} \rightarrow Y(t)$$

t is the variable

$$Y = \frac{3 \sin 2t}{\sqrt{\pi t}}$$

Examples on IVP (Initial value problem)

[1]  $y''' - y'' - y' + y = 8te^{-t}$

while  $y(0) = 0, y'(0) = 1, y''(0) = 0$

$$\mathcal{L}\{y'''\} - \mathcal{L}\{y''\} - \mathcal{L}\{y'\} + \mathcal{L}\{y\} = 8 \mathcal{L}\{te^{-t}\}$$

$$\Rightarrow [s^3 \mathcal{L}\{y\} - s^2 y(0) - \frac{s^1}{s} y'(0)] - [s^2 \mathcal{L}\{y\} - s^1 y(0) - s^0 y'(0)] - [s^1 \mathcal{L}\{y\} - s^0 y(0)] + y = 8 \cdot \frac{1}{(s+1)^2}$$

$f(s) = \mathcal{L}\{t\} = \frac{1}{s^2}$

$$\begin{aligned} f(s-a) &= f(s-(-1)) \\ &= f(s+1) = \frac{1}{(s+1)^2} \end{aligned}$$

$$\Rightarrow [s^3 y - s^2(0) - \frac{s^1(1)}{1}] - [s^2 y - s(0) - 1] - [s y - 0] + y = \frac{8}{(s+1)^2}$$

$$\Rightarrow s^3 y - s - s^2 y + 1 - s y + y = \frac{8}{(s+1)^2}$$

$$\Rightarrow y (s^3 - s^2 - s + 1) = \frac{8}{(s+1)^2} + s - 1$$

$$\Rightarrow y (s^2(s-1) - 1(s-1)) = \frac{8}{(s+1)^2} + s - 1$$

$$y(s^2-1)(s-1) = \frac{8}{(s+1)^2} + s - 1$$

$$\begin{aligned} y &= \frac{8}{(s+1)^2(s^2-1)(s-1)} + \frac{s}{(s^2-1)(s-1)} - \frac{1}{(s^2-1)(s-1)} \\ &= \frac{8}{(s+1)^3(s-1)^2} + \frac{s}{(s+1)(s-1)^2} - \frac{1}{(s+1)(s-1)^2} \end{aligned}$$

$\downarrow \textcircled{A}$        $\downarrow \textcircled{B}$        $\downarrow \textcircled{C}$

(a):

$$\frac{8}{(s+1)^3(s-1)^2} = \frac{A}{(s+1)^3} + \frac{B}{(s+1)^2} + \frac{C}{s+1} + \frac{D}{(s-1)^2} + \frac{E}{s-1}$$

$$8 = A(s-1)^2 + B(s+1)(s-1)^2 + C(s+1)^2(s-1)^2 + D(s+1)^3 + E(s+1)^3(s-1)$$

$\downarrow \textcircled{i}$

$$\text{if } s = 1 \text{ then } 8 = D(2)^3 \Rightarrow D = 1$$

$$\text{if } s = -1 \text{ then } 8 = A(-2)^2 \Rightarrow A = 2$$

Substitute  $A=2$ ,  $D=1$  into i

$$\therefore 8 = 2(s-1)^2 + B(s+1)(s-1)^2 + C(s+1)^2(s-1)^2 + 1(s+1)^3 + E(s+1)^3(s-1)$$

$$\begin{aligned} 8 &= 2s^2 - 4s + 1 + (Bs+B)(s^2-2s+1) + (s^2+2s+1)(s^2-2s+1)C \\ &\quad + s^3 + 3s^2 + 3s + 1 + (Es^3 + 3Es^2 + 3Es + E)(s-1) \end{aligned}$$

$$8 = 5s^2 - s^3 + Bs^3 + Bs - Bs^2 + B + s^3 + s^4E + 2s^3E + 2Es + B$$

$$+ s^4C - 2s^2C + C + 2$$

$"s^4"$        $\left\{ \begin{array}{l} "s^3" \\ B+1+2E=0 \\ B+2(-C)=0 \\ B=3 \end{array} \right.$        $\left\{ \begin{array}{l} "s^2" \\ 5-B-2C=0 \\ -1-B-2E=0 \\ -B-2(-C)=1 \\ -B+2C=1 \end{array} \right.$        $\left\{ \begin{array}{l} "s" \\ B-E+C+2=8 \\ B-E+C=6 \\ B-(-C)+C=6 \\ B+2C=6 \end{array} \right.$       "constant"

$E+C=0$   
 $E=-C$

$\downarrow \textcircled{IV}$        $\downarrow \textcircled{V}$

$$\textcircled{1} + \textcircled{2} \Rightarrow 4C = 7$$

$$C = \frac{7}{4}$$

$$\begin{aligned} \therefore E &= -C \\ \therefore E &= -\frac{7}{4} \end{aligned}$$

$$\begin{aligned} \therefore B + 2C &= 6 \quad \textcircled{3} \\ B &= 6 - 2C = 6 - 2\left(\frac{7}{4}\right) = 6 - \frac{7}{2} = \frac{5}{2} \end{aligned}$$

$$\therefore B = \frac{5}{2}$$

$$\frac{8}{(s+1)^3(s-1)^2} = \frac{2}{(s+1)^3} + \frac{\frac{5}{2}}{(s+1)^2} + \frac{\frac{7}{4}}{s+1} + \frac{1}{(s-1)^2} + \frac{-\frac{7}{4}}{s-1}$$

(b)

$$\frac{s}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)}$$

$$s = A(s-1)^2 + B(s+1) + C(s+1)(s-1)$$

$$s=1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$$

$$s=-1 \Rightarrow -1 = A(-2)^2 \Rightarrow 4A = -1 \Rightarrow A = -\frac{1}{4}$$

$$\therefore s = -\frac{1}{4}(s-1)^2 + \frac{1}{2}(s+1) + C(s^2-1) \rightarrow \text{Substitute } B = \frac{1}{2}, A = -\frac{1}{4}$$

$$s = -\frac{s^2}{4} - C + s + \frac{1}{4} + s^2 C$$

$$\begin{aligned} "s^2" \\ -\frac{1}{4} + C = 0 \end{aligned}$$

$$C = \frac{1}{4}$$

$$\begin{aligned} "s" \\ 1=1 \end{aligned} \quad \begin{aligned} \text{"Constant"} \\ 0 = -C + \frac{1}{4} \end{aligned} \quad \left. \begin{aligned} \therefore C &= \frac{1}{4} \\ \therefore \frac{s}{(s+1)(s-1)^2} &= \frac{-\frac{1}{4}}{s+1} + \frac{\frac{1}{2}}{(s-1)^2} + \frac{\frac{1}{4}}{s-1} \end{aligned} \right.$$

(c):

$$\frac{1}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{(s-1)^2} + \frac{C}{s-1}$$

$$1 = A(s-1)^2 + B(s+1) + C(s-1)(s+1)$$

$$\Rightarrow 1 = A(s-1)^2 + B(s+1) + C(s^2-1) = \frac{1}{4}(s-1)^2 + \frac{1}{2}(s+1) + Cs^2 - C$$

$$= \frac{1}{4}(s^2 - 2s + 1) + \frac{s}{2} + \frac{1}{2} + Cs^2 - C$$

$$= \frac{s^2}{4} - \frac{s}{2} + \frac{1}{4} + \frac{s}{2} + \frac{1}{2} + Cs^2 - C$$

$$= s^2(\frac{1}{4} + C) + \frac{3}{4} - C$$

$$\therefore 1 = s^2(\frac{1}{4} + C) + \frac{3}{4} - C$$

"s<sup>2</sup>"

"Constant"

$$C = \frac{3}{4} - 1 = -\frac{1}{4}$$

$$0 = +\frac{1}{4} + C$$

$$C = -\frac{1}{4}$$

$$C = 0 - \frac{1}{4} \Rightarrow$$

$$\therefore \frac{1}{(s+1)(s-1)^2} = \frac{1}{s+1} + \frac{1/2}{(s-1)^2} + \frac{-1/4}{s-1}$$

$$y = \frac{8}{(s+1)^3(s-1)^2} + \frac{s}{(s+1)(s-1)^2} - \frac{1}{(s+1)(s-1)^2}$$

$$= \frac{2}{(s+1)^3} + \frac{5/2}{(s+1)^2} + \frac{7/4}{s+1} + \frac{1}{(s-1)^2} + \frac{-7/4}{s-1} + \frac{-1/4}{s+1} + \frac{1/2}{(s-1)^2} + \frac{1/4}{s-1}$$

$$= \frac{2}{(s+1)^3} + \frac{5/2}{(s+1)^2} + \frac{7/4}{s+1} + \frac{2}{(s-1)^2} - \frac{5/4}{s-1}$$

$$\mathcal{L}\{Y\} = \frac{2}{(s+1)^3} + \frac{5/2}{(s+1)^2} + \frac{7/4}{s+1} + \frac{2}{(s-1)^2} - \frac{5/4}{s-1}$$

$$Y = \mathcal{L}^{-1} \left\{ \frac{2}{(s+1)^3} + \frac{5/2}{(s+1)^2} + \frac{7/4}{s+1} + \frac{2}{(s-1)^2} - \frac{5/4}{s-1} \right\}$$

$$Y = 2 \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^3} \right\} + \frac{5}{2} \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2} \right\} + \frac{7}{4} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\}$$

↓  
 $\frac{e^{-t}}{s^3}$   
 $\frac{5}{s^2+1}$

$$+ 2 \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2} \right\} - \frac{7}{4} \mathcal{L}^{-1} \left\{ \frac{5}{s-1} \right\}$$

↓  
 $\frac{e^t}{s^2}$

$$= 2 e^{-t} \frac{t^2}{2!} + \frac{5}{2} e^{-t} t + \frac{7}{4} e^{-t} + 2 e^t t - \frac{7}{4} e^t$$

$$= t^2 e^{-t} + \frac{5t}{2} t e^{-t} + \frac{7}{4} e^{-t} + 2 t e^t - \frac{7}{4} e^t$$

### Examples on IVP

**2**  $Y''' + 4Y'' + 5Y' + 2Y = 10 \cos t; Y(0) = Y'(0) = 0$  Reading

$$\mathcal{L}\{Y'''\} + 4\mathcal{L}\{Y''\} + 5\mathcal{L}\{Y'\} + 2\mathcal{L}\{Y\} = 10 \mathcal{L}\{\cos t\}$$

$$\Rightarrow [S^3 \mathcal{L}\{Y\} - S^2 Y(0) - S^1 Y'(0)] + 4[S^2 \mathcal{L}\{Y\} - S^1 Y(0) - S^0 Y'(0)] + 5[S^1 \mathcal{L}\{Y\} - S^0 Y(0)] + 2Y = 10 \cdot \frac{S}{S^2 + 1^2} = \frac{10S}{S^2 + 1}$$

$$\Rightarrow S^3 y + 4S^2 y + 5S y + 2y = \frac{10S}{S^2 + 1}$$

Rational Roots

$$S^3 + 4S^2 + 5S + 2 = 0$$

$$S = -1, S = -2$$

$$\Rightarrow y(S^3 + 4S^2 + 5S + 2) = \frac{10S}{S^2 + 1}$$

Polynomial eqns

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 x^0 = 0$$

if  $a_0$  &  $a_n$  are integers, then if there is a rational solution, it could be found by checking all the numbers produced for  $\frac{\pm \text{divider of } a_0}{\text{dividers of } a_n}$

$$a_0 = 2, \quad a_n = 1$$

The dividers of  $a_0: 1, 2$  The dividers of  $a_n: 1$

The following rational numbers are candidate roots:

$$\begin{array}{c} [1, 2] \\ \hline 1 \end{array}$$

$$\begin{array}{ll} s+1=0; & s+2=0 \\ s=-1 & s=-2 \end{array}$$

$$s^3 + 1s^2 + 5s + 2$$

$$= (s+1) \frac{s^3 + 1s^2 + 5s + 2}{s+1}$$

$$= (s+1)(s^2 + 3s + 2)$$

$$= (s+1)(s+1)(s+2)$$

$$= (s+1)^2(s+2)$$

$$\Rightarrow y(s^3 + 1s^2 + 5s + 2) = \frac{10s}{s^2 + 1}$$

$$\Rightarrow y(s+1)^2(s+2) = \frac{10s}{s^2 + 1}$$

$$\Rightarrow y = \frac{10s}{(s^2 + 1)(s+1)^2(s+2)}$$

$$\frac{10s}{(s^2+1)(s+1)^2(s+2)} = \frac{As+B}{s^2+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)} + \frac{E}{s+2}$$

$$10s = (As+B)(s+1)^2(s+2) + C(s^2+1)(s+2) + D(s^2+1)(s+1)(s+2) \\ + E(s^2+1)(s+1)$$

$$\text{if } s = -1 \Rightarrow s^3 + 10 = C(2) \Rightarrow C = -5$$

$$\text{if } s = -2 \Rightarrow -20 = E(5)(-1) \Rightarrow E = 4$$

$$\therefore 10s = (As+B)(s+1)^2(s+2) - 5(s^2+1)(s+2) \\ + D(s^2+1)(s+1)(s+2) + 4(s^2+1)(s+1)$$

$$= -s^3 + As^4 + s^3B - 6s^2 - s + s^4D + 5As^2 + 5Bs + 4As^3 \\ + 4s^2B + 2As + 2B + 3s^3D + 3s^2D + 3Ds + 2D - 6$$

$$\begin{array}{l} "S^4" \\ A + D = 0 \\ A = -D \end{array} \quad \left\{ \begin{array}{l} "S^3" \\ -1 + B + 4A + 3D = 0 \\ B - 4D + 3D = 1 \\ B - D = 1 \end{array} \right. \quad \left\{ \begin{array}{l} "S^2" \\ -6 + 5A + 4B + 3D = 0 \end{array} \right. \quad \left\{ \begin{array}{l} "S" \\ -1 + 5B + 2A + 3D = 10 \end{array} \right.$$

②

$$\begin{array}{l} \text{"constant"} \\ 2B + 2D - 6 = 0 \\ B + D = 3 \end{array} \quad \left. \begin{array}{l} \text{from ②} \\ B - D = 1 \\ B - 2 = 1 \\ B = 3 \end{array} \right. \quad \boxed{A = -2}$$

⑤

$$\textcircled{2} + \textcircled{5} \Rightarrow 2D = 4 \\ \boxed{D = 2}$$

$$y = \frac{10s}{(s^2+1)(s+1)^2(s+2)}$$

$$= \frac{As+B}{s^2+1} + \frac{C}{(s+1)^2} + \frac{D}{s+1} + \frac{E}{s+2}$$

$$\mathcal{L}\{Y\} = \frac{-2s+3}{s^2+1} + \frac{-5}{(s+1)^2} + \frac{2}{s+1} + \frac{4}{s+2}$$

$$\mathcal{L}\{Y\} = -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$+ 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= -2\cos t + 3\sin t - 5e^{-t} + 2e^{-t} + 4e^{-2t}$$

Example: solve the IVP

$$\boxed{3} \quad Y'' - 4Y' + 4Y = 4\cos 2t; \quad Y(0) = 2, \quad Y'(0) = 5$$

Reading

$$\mathcal{L}\{Y''\} - 4\mathcal{L}\{Y'\} + 4\mathcal{L}\{Y\} = 4\mathcal{L}\{\cos 2t\}$$

$$\Rightarrow [s^2\mathcal{L}\{Y\} - s^2Y(0) - s^0Y'(0)] - 4[s^1\mathcal{L}\{Y\} - s^0Y(0)]$$

$$+ 4Y = 4\left(\frac{s}{s^2+4}\right)$$

$$\Rightarrow s^2Y - 2s - 5 - 4sY + 8 + 4Y = \frac{4s}{s^2+4}$$

$$\Rightarrow Y(s^2 - 4s + 4) = \frac{4s}{s^2+4} + 2s - 3 = \frac{4s + 2s^3 + 8s - 3s^2 - 12}{s^2+4}$$

$$y(s-2)^2 = \frac{4s+2s^3+8s-3s^2-12}{s^2+4}$$

$$y = \frac{2s^3-3s^2+12s-12}{(s^2+4)(s-2)^2}$$

$$\therefore y = \frac{2s^3-3s^2+12s-12}{(s^2+4)(s-2)^2} = \frac{As+B}{s^2+4} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)}$$

$$2s^3-3s^2+12s-12 = (As+B)(s-2)^2 + C(s^2+4) + D(s^2+4)(s-2)$$

$$\text{if } s=2 \Rightarrow 16 = C(8) \Rightarrow C = 2$$

$$\therefore 2s^3-3s^2+12s-12 = (As+B)(s-2)^2 + 2(s^2+4) + D(s^2+4)(s-2)$$

$$\begin{aligned} 2s^3-3s^2+12s-12 &= As^3+Bs^2-4As^2-4Bs+4As+4B+2s^2+8 \\ &\quad +Ds^3-2Ds^2+4sD-8D \end{aligned}$$

$$\begin{array}{l} \text{"S"} \quad \left\{ \begin{array}{l} \text{"S"} \\ -3=B-1A+2-2D \end{array} \right. \quad \left\{ \begin{array}{l} \text{"S"} \\ 12=-4B+4A+4D \end{array} \right. \quad \left. \begin{array}{l} 12=4B+8-8D \\ 4B-8D=-20 \end{array} \right. \\ 2=A+D \quad \left. \begin{array}{l} B-4A-2D=-5 \\ B-\frac{1}{4}(2-D)-2D=-5 \\ B-\frac{8}{4}+\frac{4}{4}D-2D=-5 \\ B+2D=3-\text{II} \end{array} \right. \quad \left. \begin{array}{l} 4A-4B+4D=12 \\ A-B+D=3 \\ A-(-D)+2=3 \\ A=0 \end{array} \right. \quad \left. \begin{array}{l} B-2D=-5-\text{IV} \\ \text{Solve II \& IV} \\ B=-1 \quad D=2 \end{array} \right. \end{array}$$

$$\therefore y = \frac{0(s)-1}{s^2+4} + \frac{2}{(s-2)^2} + \frac{2}{(s-2)}$$

$$\Rightarrow \mathcal{L}\{Y\} = -\frac{1}{s^2+4} + \frac{2}{(s-2)^2} + \frac{2}{s-2}$$

$$\Rightarrow Y = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$Y = -\frac{1}{2}\sin 2t + 2e^{-2t}t + 2e^{-2t}$$

(Example)

Solve the following system:

$$\begin{cases} \frac{dx}{dt} + y = \sin t \\ \frac{dy}{dt} + x = \cos t \end{cases} \quad x(0)=2, \quad y(0)=0 \quad -\textcircled{1}$$

let  $\frac{dx}{dt} + y = \sin t \quad -\textcircled{1}$        $\frac{dy}{dt} + x = \cos t \quad -\textcircled{2}$

let  $\mathcal{L}\{x(t)\} = X(s)$  ;  $\mathcal{L}\{y(t)\} = Y(s) \Rightarrow \mathcal{L}\{Y\} = y$   
 $\Rightarrow \mathcal{L}\{X\} = x$

$$\textcircled{1} \quad x' + y = \sin t$$

$$\mathcal{L}\{x'\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin t\}$$

$$[s^2 \mathcal{L}\{x\} - s^0 x(0)] + y = \frac{1}{s^2 + 1}$$

$$sx - 2 + y = \frac{1}{s^2 + 1}$$

$$sx + y = \frac{1}{s^2 + 1} + 2$$

$$sx + y = \frac{1+2s^2+2}{s^2+1} = \frac{2s^2+3}{s^2+1} \quad -\textcircled{a}$$

$$\textcircled{a} \times 1 \Rightarrow sx + y = \frac{2s^2+3}{s^2+1}$$

$$\textcircled{b} \times s \Rightarrow sx + s^2 y = \frac{s^2}{s^2+1}$$

$$\underline{\underline{\text{Subtract}}} \quad \underline{\underline{y(1-s^2)}} = \frac{2s^2+3-s^2}{s^2+1} = \frac{s^2+3}{s^2+1}$$

$$y = \frac{-(s^2+3)}{(s^2+1)(s^2-1)} \rightarrow Y = ?$$

$$@ \times s \Rightarrow s^2 x + s y = \frac{2s^3 + 3s}{s^2 + 1}$$

$$b) \times 1 \Rightarrow x + s y = \frac{s}{s^2 + 1}$$

~~(-)~~      ~~(-)~~      ~~(-)~~

**Subtract**       $x(s^2 - 1) = \frac{2s^3 + 3s - s}{s^2 + 1}$

$$x = \frac{2s^3 + 2s}{(s^2 + 1)(s^2 - 1)}$$

$$= \frac{2s(s^2 + 1)}{(s^2 + 1)(s^2 - 1)}$$

$$x = \frac{2s}{s^2 - 1}, X = ?$$

| a : |

$$y = \frac{-s^2 - 3}{(s^2 + 1)(s^2 - 1)} = \frac{As + B}{s^2 + 1} + \frac{C}{(s+1)} + \frac{D}{(s-1)} \quad s^2 - 1 = (s+1)(s-1)$$

$$\Rightarrow -s^2 - 3 = (As + B)(s^2 - 1) + C(s^2 + 1)(s - 1) + D(s^2 + 1)(s + 1) \quad \text{L (iii)}$$

$$\text{if } s = 1 \Rightarrow -4 = 4D \Rightarrow D = -1$$

$$\text{if } s = -1 \Rightarrow -4 = -4C \Rightarrow C = 1$$

In eqn (iii)

$$-s^2 - 3 = (As + B)(s^2 - 1) + 1(s^2 + 1)(s - 1) + (-1)(s^2 + 1)(s + 1)$$

$$-s^2 - 3 = -2s^2 + As^3 - As + s^2B - B - 2$$

$$-1 = -2 + B$$

$$B = 1$$

$$\begin{matrix} "S" \\ 0 = -A \\ \boxed{A = 0} \end{matrix}$$

$$\therefore y = \frac{1}{s^2 + 1} + \frac{1}{s+1} - \frac{1}{s-1}$$

$$\therefore Y = ?$$

42

$$\therefore y = \frac{1}{s^2+1} + \frac{1}{s+1} - \frac{1}{s-1} \quad \text{from pg 42}$$

$$\therefore L\{Y\} = L\left\{\frac{1}{s^2+1} + \frac{1}{s+1} - \frac{1}{s-1}\right\} \quad \therefore L\{Y\} = y$$

$$Y = L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$Y = \sin t + e^{-t} - e^t$$

b

$$x = \frac{2s}{s^2-1} = \frac{A}{s+1} + \frac{B}{s-1} \Rightarrow 2s = A(s-1) + B(s+1)$$

$$\text{if } s=1 \Rightarrow 2 = B(2) \Rightarrow B = 1$$

$$\text{if } s=-1 \Rightarrow -2 = -2A \Rightarrow A = 1$$

$$\therefore x = \frac{1}{s+1} + \frac{1}{s-1}$$

$$L\{X\} = \frac{1}{s+1} + \frac{1}{s-1} \quad \therefore L\{X\} = x$$

$$X = L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= e^{-t} + e^t$$

$$\therefore X(t) = e^{-t} + e^t$$

$$Y(t) = \sin t + e^{-t} - e^{-t}$$