Example (Exercise Sheet 1)

Using properties of conjugate & modulus show that:

$|2z+3\bar{z}| \leq 4|Re(z)| + |z|$

$$|2z+3\bar{z}| = |2x+2iy+3x-3iy|$$

$$= |5x-iy|$$

$$= |4x+x-iy|$$

$$= |4x+(x-iy)|$$

$$\leq |4x| + |x-iy|$$

$\left\{ \begin{array}{l} \text{Triangular inequality} \\ |z_1+z_2| \leq |z_1|+|z_2| \end{array} \right.$

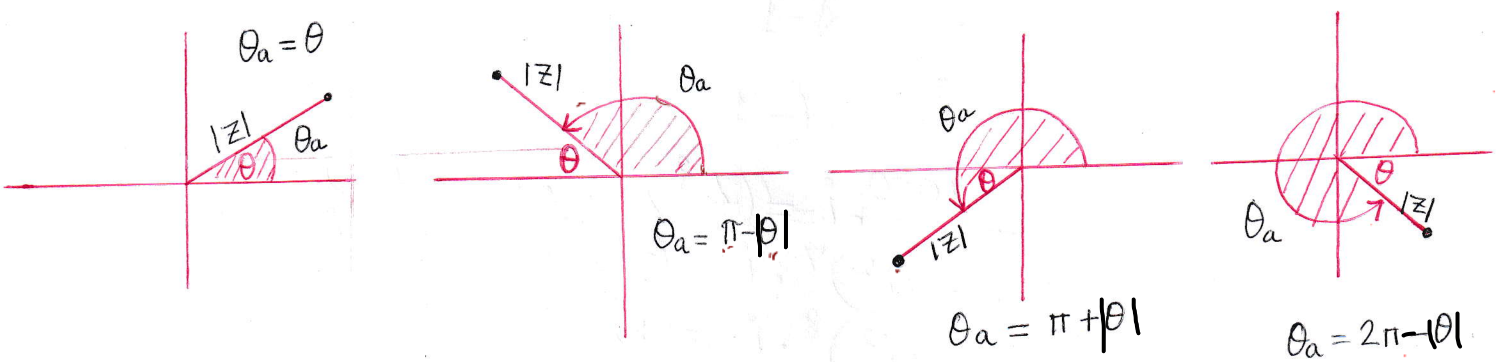
$$= 4|x| + \sqrt{x^2+(-y)^2}$$

$$= 4|Re(z)| + \sqrt{x^2+y^2}$$

$$= 4|Re(z)| + |z|$$

$$\therefore |2z+3\bar{z}| \leq 4|Re(z)| + |z|$$

Position of 'θ'



Find the modulus & argument of the following complex number. (Exercise sheet # 1)

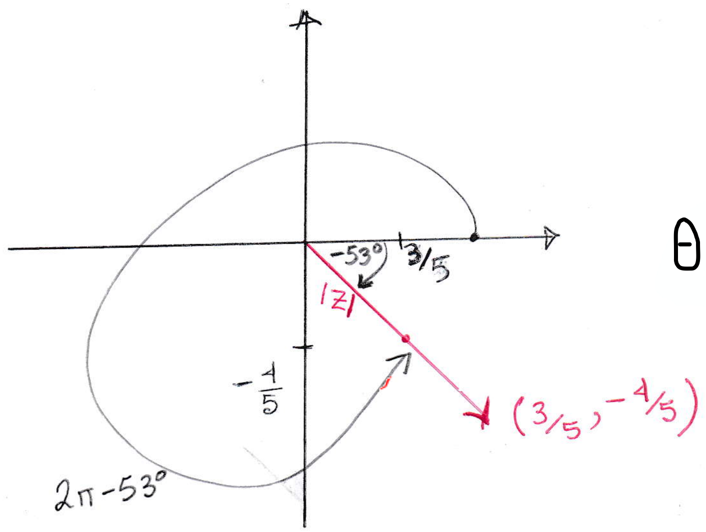
(i) $\dfrac{2-i}{2+i}$

$$= \frac{(2-i)(2-i)}{(2+i)(2-i)}$$

$$= \frac{2^2 - 2.2.i + i^2}{2^2 - i^2}$$

$$= \frac{3-4i}{5}$$

$$z = \frac{3}{5} - \frac{4}{5}i$$



$$|z| = \left|\frac{3}{5} - \frac{4}{5}i\right|$$

$$= \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{-4}{5}\right)^2}$$

$$= \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= 1$$

$$\arg z = \theta = \tan^{-1}(y/x)$$ $\therefore \theta_a = 360^\circ - |-53^\circ|$ $= 307^\circ$

$$= \tan^{-1}\left(\frac{-4/5}{3/5}\right)$$

$$= \tan^{-1}(-4/3)$$

$$= -53^\circ$$

$$\theta_a \; 360^\circ - |-53^\circ|$$

$$= 307^\circ$$

Continued with Example

(iii) $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^2$

$$= \frac{1+2\sqrt{3}i+3i^2}{1-2\sqrt{3}i+3i^2}$$

$$= \frac{-2+2\sqrt{3}i}{-2-2\sqrt{3}i}$$

$$= \frac{-(1-\sqrt{3}i)}{-(1+\sqrt{3}i)}$$

$$= \frac{(1-\sqrt{3}i)(1-\sqrt{3}i)}{(1+\sqrt{3}i)(1-\sqrt{3}i)}$$

$$= \frac{1+2\sqrt{3}i+3i^2}{1-3i^2}$$

$$= \frac{-2-2\sqrt{3}i}{4}$$

$$= -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$Z = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$|Z| = \sqrt{\left(-\frac{1}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{3}{4}}$$

$$= 1$$

Argument:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

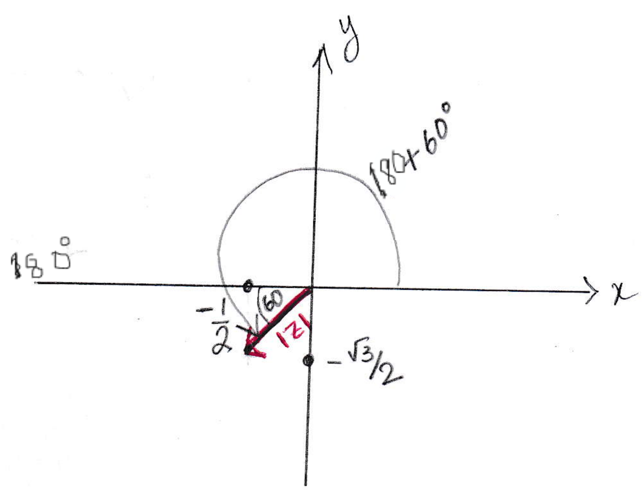
$$= \tan^{-1}\left(\frac{-\sqrt{3}/2}{-1/2}\right)$$

$$= \tan^{-1}(\sqrt{3})$$

$$= 60^\circ$$

$$= 180^\circ + 60^\circ$$

$$= 240^\circ$$



Example (Practice Sheet #1)

Prove that $|z+2i|+|z-2i|=6$ represents an ellipse

$$|x+iy+2i|+|x+iy-2i|=6$$

$$|x+i(y+2)|+|x+i(y-2)|=6$$

$$\sqrt{x^2+(y+2)^2}+\sqrt{x^2+(y-2)^2}=6$$

$$\sqrt{x^2+(y+2)^2}=6-\sqrt{x^2+(y-2)^2}$$

Square both sides:

$$x^2+(y+2)^2=36-2\cdot6\cdot\sqrt{x^2+(y-2)^2}+x^2+(y-2)^2$$

$$(y+2)^2=36-12\sqrt{x^2+(y-2)^2}+(y-2)^2$$

$$y^2+4y+4=36-12\sqrt{x^2+(y-2)^2}+y^2-4y+4$$

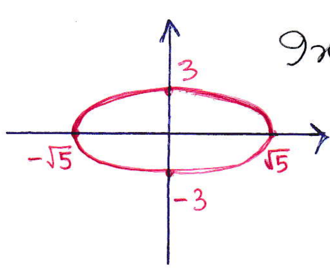
$$12\sqrt{x^2+(y-2)^2}=36-8y$$

$$3\sqrt{x^2+(y-2)^2}=9-2y \quad (\div \text{ by } 4)$$

$$9\left(x^2+(y-2)^2\right)=81-36y+4y^2 \quad (\text{Square both sides})$$

$$9(x^2+y^2-4y+4)=81-36y+4y^2$$

$$9x^2+9y^2-\cancel{36y}+36=81-\cancel{36y}+4y^2$$



$$9x^2+5y^2=45 \Rightarrow \frac{x^2}{5}+\frac{y^2}{9}=1 \quad (\div \text{ by } 45$$

$\hookrightarrow$ Eqn of Ellipse

Reading

Examples (Practice Sheet #1)

Perform each of the indicated operations:

(v) $$\frac{3i^{10} - i^{19}}{2i-1}$$

$$= \frac{3(i^2)^5 - i^{18}\cdot i}{2i-1}$$

$$= \frac{3(-1)^5 - (i^2)^9\,(i)}{2i-1}$$

$$= \frac{-3 - (-1)^9 i}{2i-1}$$

$$= \frac{-3+i}{2i-1}$$

$$= \frac{i-3}{2i-1}$$

$$= \frac{i-3}{2i-1}\cdot\frac{-2i-1}{-2i-1}$$

$$= \frac{-2i^2+6i-i+3}{(-1)^2-(2i)^2}$$

$$= \frac{+5+5i}{+5}$$

$$= \frac{1+i}{1}$$

$$= 1+i$$

(iv) $$3\left(\frac{1+i}{1-i}\right)^2 - 2\left(\frac{1-i}{1+i}\right)^3$$

$$= \frac{3(1+i)^2}{(1-i)^2} - \frac{2(1-i)^3}{(1+i)^3}$$

$$= \frac{3(1+2i-1)}{1-2i-1} - \frac{2(1-3i+3i^2-i^3)}{1+3i+3i^2+i^3}$$

$$= \frac{3(2i)}{-2i} - \frac{2(-2-2i)}{(-2+2i)}$$

$$= -3 - \frac{(-4)(1+i)}{(-2)(1-i)}$$

$$= -3 - \frac{2(1+i)}{1-i}$$

$$= -3 - \frac{2(1+i)(1+i)}{(1-i)(1+i)}$$

$$= -3 - \frac{2(1+2i+i^2)}{1^2-i^2}$$

$$= -3 - \frac{2(1+2i-1)}{2}$$

$$= -3-2i$$

If $z_1 = 1-i$, $z_2 = -2+4i$, $z_3 = \sqrt{3} - 2i$, evaluate each of the following:

(iii) $\overline{(z_2+z_3)(z_1-z_3)} = \overline{(-2+4i+\sqrt{3}-2i)(1-i-\sqrt{3}+2i)}$

Reading

$$= \overline{(-2+\sqrt{3}+2i)(1-\sqrt{3}+i)}$$

$$= \overline{-\cancel{2}+\sqrt{3}+2i+2\sqrt{3}-3-2\sqrt{3}-2i-\sqrt{3}i-\cancel{2}i^2}$$

$$= \overline{-7+3\sqrt{3}-\sqrt{3}i}$$

$$= (3\sqrt{3}-7)+\sqrt{3}i$$

(v) $\operatorname{Im}\left\{\dfrac{z_1 z_2}{z_3}\right\} = \operatorname{Im}\left\{\dfrac{(1-i)(-2+4i)}{\sqrt{3}-2i}\right\}$

$$= \operatorname{Im}\left\{\frac{-2+2i+4i-4i^2}{\sqrt{3}-2i}\right\}$$

$$= \operatorname{Im}\left\{\frac{2+6i}{\sqrt{3}-2i}\right\}$$

$$= \operatorname{Im}\left\{\frac{(2+6i)(\sqrt{3}+2i)}{(\sqrt{3}-2i)(\sqrt{3}+2i)}\right\}$$

$$= \operatorname{Im}\left\{\frac{2\sqrt{3}+6\sqrt{3}i+4i-12}{3-4i^2}\right\}$$

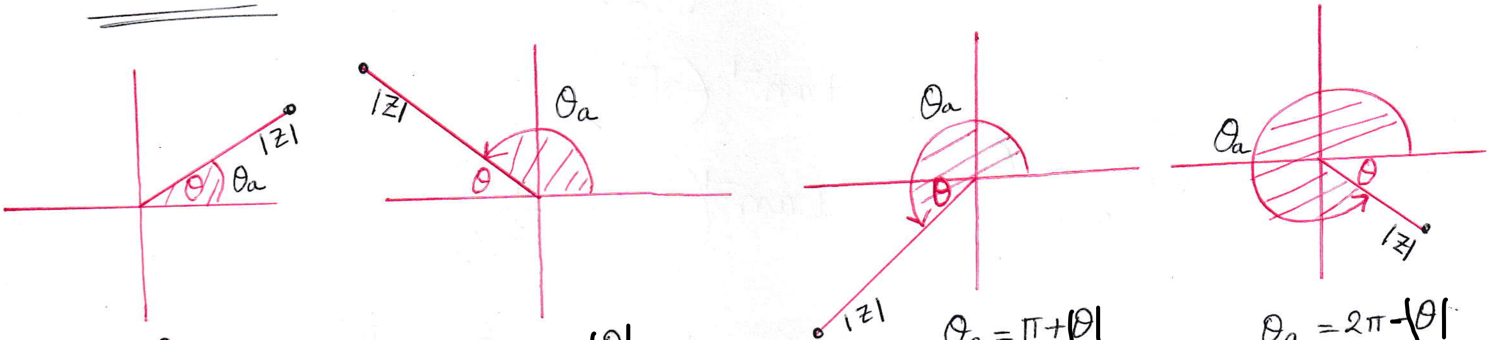
$$= \operatorname{Im}\left\{\frac{2(\sqrt{3}-6)+i2(3\sqrt{3}+2)}{7}\right\} = \frac{6\sqrt{3}+4}{7}$$

(Practice Exercise # 1)

Express each of the following complex numbers in Polar form and show them graphically

[Note that 'z' does not have root in the following problems]

Recall



$\theta_a = \theta$ $\qquad$ $\theta_a = \pi - |\theta|$ $\qquad$ $\theta_a = \pi + |\theta|$ $\qquad$ $\theta_a = 2\pi - |\theta|$

(i) $2+2\sqrt{3}\,i$ $\qquad x = 2,\ y = 2\sqrt{3}$ $\qquad$ **Reading**

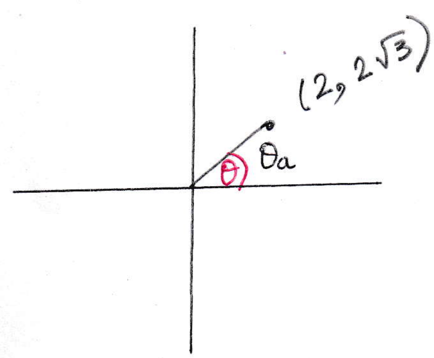
$$r = \sqrt{2^2+(2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$

$$= \tan^{-1}\left(\sqrt{3}\right)$$

$$= \frac{\pi}{3}$$



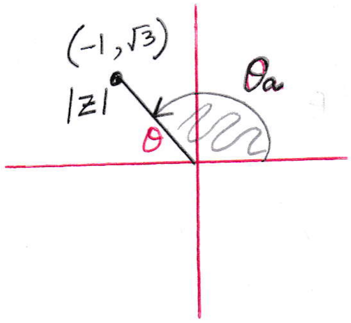
$$z = r(\cos\theta + i\sin\theta) = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

(iv) $-1+\sqrt{3}i$, $x=-1$, $y=\sqrt{3}$

$$r = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$

$$= \tan^{-1}(-\sqrt{3})$$



$$= -\frac{\pi}{3}$$

$$\theta_a = \pi - \left|-\frac{\pi}{3}\right|$$

$$= \frac{2\pi}{3}$$

$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

(Practice Sheet #1)

Find each of the indicated roots:

(iv) $z^6 = 64$

$$z = (64)^{1/6} = (64 + 0i)^{1/6}$$

$x = 64$, $y = 0$ $\quad r = \sqrt{64^2} = 64$

$$\theta = \tan^{-1}\left(\frac{0}{64}\right) = 0^\circ$$

$$z = r^{1/6}\left(\cos\frac{1}{6}(\theta + 2k\pi) + i\sin\frac{1}{6}(\theta + 2k\pi)\right)$$

$$= (64)^{1/6}\left(\cos\frac{1}{6}(0 + 2k\pi) + i\sin\frac{1}{6}(0 + 2k\pi)\right)$$

$$= 2\left(\cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}\right)$$

$$= 2\left(\cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3}\right), \quad k = 0, 1, 2, 3, 4, 5$$

If $k = 0$, $z_0 = 2(\cos 0^\circ + i\sin 0^\circ) = 2(1 + 0i)$

If $k = 1$, $z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$

If $k = 2$, $z_2 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$

If $k = 3$, $z_3 = 2(\cos\pi + i\sin\pi) = 2(-1 + i0) = -2 + i0$

If $k = 4$, $z_4 = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1 - i\sqrt{3}$

If $k = 5$, $z_5 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - i\sqrt{3}$

(v) $z^4 + z^2 + 1 = 0$ (Polynomial Equation)

Let $x = z^2$

$$x^2 + x + 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$z^2 = \frac{-1 \pm \sqrt{-3}}{2}$$

$$z^2 = \frac{-1 \pm i\sqrt{3}}{2}$$

$$z = \left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)^{1/2}$$

$$z_1^* = \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{1/2} \quad ; \quad z_2^* = \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{1/2}$$

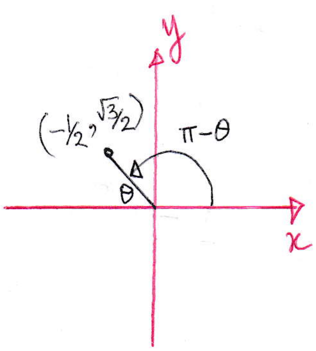
$x = -\frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$

$$r = \sqrt{\left(-1/2\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{1/4 + 3/4} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right)$$

$$= \tan^{-1}(-\sqrt{3})$$

$$= -\frac{\pi}{3}$$



$$\theta_a = \pi - \left|-\frac{\pi}{3}\right|$$

$$= \frac{2\pi}{3}$$

$x = -1/2$, $y = -\sqrt{3}/2$

$$r = \sqrt{(-1/2)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = 1$$

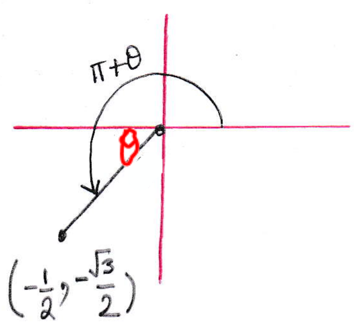
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sqrt{3}/2}{-1/2}\right)$$

$$= \tan^{-1}(\sqrt{3})$$

$$= \frac{\pi}{3}$$

$$\theta_a = \pi + \frac{\pi}{3}$$

$$= \frac{4\pi}{3}$$



$$Z_1^* = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{1/2}$$

$$Z = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{1/2} = r^{1/2}\left(\cos\frac{1}{2}(\theta + 2k\pi) + i\sin\frac{1}{2}(\theta + 2k\pi)\right) ; k = 0, 1$$

$$= 1^{1/2}\left(\cos\frac{1}{2}(\theta + 2k\pi) + i\sin\frac{1}{2}(\theta + 2k\pi)\right)$$

$$= \cos\frac{1}{2}\left(\frac{2\pi}{3} + 2k\pi\right) + i\sin\frac{1}{2}\left(\frac{2\pi}{3} + 2k\pi\right)$$

If $k = 0$, $z_0 = \cos\frac{1}{2}\left(\frac{2\pi}{3}\right) + i\sin\frac{1}{2}\left(\frac{2\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

If $k = 1$, $z_1 = \cos\frac{1}{2}\left(\frac{8\pi}{3}\right) + i\sin\frac{1}{2}\left(\frac{8\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$$Z_2^* = \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{1/2}$$

$$Z = \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{1/2} = r^{1/2}\left(\cos\frac{1}{2}(\theta + 2k\pi) + i\sin\frac{1}{2}(\theta + 2k\pi)\right) ; k = 0, 1$$

$$= 1^{1/2}\left(\cos\frac{1}{2}\left(\frac{4\pi}{3} + 2k\pi\right) + i\sin\frac{1}{2}\left(\frac{4\pi}{3} + 2k\pi\right)\right)$$

If $k = 0$, $z_0 = \cos\frac{1}{2}\left(\frac{4\pi}{3}\right) + i\sin\frac{1}{2}\left(\frac{4\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

If $k = 1$, $z_1 = \cos\frac{1}{2}\left(\frac{10\pi}{3}\right) + i\sin\frac{1}{2}\left(\frac{10\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$