Week 1 **Topic 1 part 2**

Properties of complex conjugate & Absolute Value

① $z\bar{z} = (x+iy)(x-iy)$

$= x^2 - i^2 y^2$

$= x^2 + y^2$

$= |z|^2 \quad \because |z| = \sqrt{x^2+y^2}$

② $|z| = |\bar{z}| = \sqrt{x^2+y^2}$ $\quad$ * $|z| = \sqrt{x^2+y^2}$; $\bar{z} = x - iy$

$\because |z| = |-z| = |\bar{z}|$ = Real numbers $\quad$ * $|\bar{z}| = \sqrt{x^2+(-y)^2}$

$= \sqrt{x^2+y^2}$

③ $\bar{\bar{z}} = z$ $\quad$ $-z = -x - iy$

* $|-z| = \sqrt{(-x)^2+(-y)^2}$

$= \sqrt{x^2+y^2}$

④ $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

⑤ $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

⑥ $z + \bar{z} = 2\,Re(z)$ $\quad$ Similarly, $z - \bar{z} = 2iy$

$= 2i\,Im(z)$

L.S. $= z + \bar{z}$

$= x + iy + x - iy$

$= 2x$

$= 2\,Re(z)$

⑦ if $z = x + iy$ $\quad \because |x+iy| = \sqrt{x^2+y^2}$ as $|z| = |x+iy|$

$|z|^2 = x^2 + y^2$

$= \{Re(z)\}^2 + \{Im(z)\}^2$

$|z|^2 \geqslant \{Re(z)\}^2$, $\quad |z|^2 \geqslant \{Im(z)\}^2$

$\Rightarrow |z| \geqslant Re(z)$, $\quad |z| \geqslant Im(z)$

Triangular Inequality (Exercise Sheet #1)

Q5. (iii) $|z_1 \pm z_2| \leqslant |z_1| + |z_2|$ Similarly (iv) $|z_1 \pm z_2| \geqslant |z_1| - |z_2|$

(iii) Proof $|z_1 \pm z_2| \leqslant |z_1| + |z_2|$

$$|z_1 + z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) \qquad \because z\bar{z} = |z|^2$$

$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$

$$= z_1\bar{z}_1 + \bar{z}_1 z_2 + z_1\bar{z}_2 + z_2\bar{z}_2$$

$$= |z_1|^2 + \underbrace{\overline{z_1\bar{z}_2}} + \underbrace{z_1\bar{z}_2} + |z_2|^2 \qquad \because \overline{z_1\bar{z}_2} = \bar{z}_1 z_2$$

$z = z_1\bar{z}_2$

$\bar{z} = \overline{z_1\bar{z}_2}$

$= \bar{z}_1 z_2$

We conveyed option blue in the calculation. You may alternatively consider option green

$$= |z_1|^2 + 2\,\mathrm{Re}\, z_1\bar{z}_2 + |z_2|^2 \qquad \because z + \bar{z} = 2\mathrm{Re}(z)$$

$$\leqslant |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 \qquad \because \mathrm{Re}(z) \leqslant |z|$$

OR

$z = \bar{z}_1 z_2$

$\bar{z} = \overline{\bar{z}_1 z_2}$

$= z_1\bar{z}_2$

$$= |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2 \qquad \because |z_1 z_2| = |z_1||z_2|$$

$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \qquad \because |z| = |\bar{z}|$$

$$\therefore |z_1+z_2|^2 \leqslant (|z_1| + |z_2|)^2$$

$$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\Rightarrow |z_1 - z_2| \leq |z_1| + |-z_2| \qquad \text{Replacing } z_2 \text{ by } -z_2$$

$$\because |z| = |-z|$$

$$\Rightarrow |z_1 - z_2| \leq |z_1| + |z_2|$$

(vi) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

$$|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2}) \qquad \because |z|^2 = z\bar{z}$$

$$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \qquad \because \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

$$= z_1\bar{z}_1 - \bar{z}_1 z_2 - z_1\bar{z}_2 + z_2\bar{z}_2$$

$$= |z_1|^2 - (\overline{z_1\bar{z}_2} + z_1\bar{z}_2) + |z_2|^2 \qquad \because \overline{z_1\bar{z}_2} = \bar{z}_1 z_2$$

$$= |z_1|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2 \qquad \because z + \bar{z} = 2\,\mathrm{Re}(z)$$

$$\geq |z_1|^2 - 2|z_1\bar{z}_2| + |z_2|^2 \qquad \because \mathrm{Re}(z) \leq |z| \Rightarrow -\mathrm{Re}(z) \geq -|z|$$

$$= |z_1|^2 - 2|z_1||\bar{z}_2| + |z_2|^2$$

$$= |z_1|^2 - 2|z_1||z_2| + |z_2|^2$$

$$= (|z_1| - |z_2|)^2$$

$$\therefore |z_1 - z_2|^2 \geq (|z_1| - |z_2|)^2$$

$$\Rightarrow |z_1 - z_2| \geq |z_1| - |z_2|$$

$$\Rightarrow |z_1 + z_2| \geq |z_1| - |-z_2| \qquad \text{Replacing } z_2 \text{ with } -z_2$$

$$\Rightarrow |z_1 + z_2| \geq |z_1| - |z_2| \qquad \because |z| = |-z|$$

Eqn of Circle

Let $z = x+iy$ & $z_0 = x_0+iy_0$

then $z - z_0 = (x-x_0) + i(y-y_0)$

$|z-z_0| = \sqrt{(x-x_0)^2+(y-y_0)^2}$

Let $|z-z_0| = R$

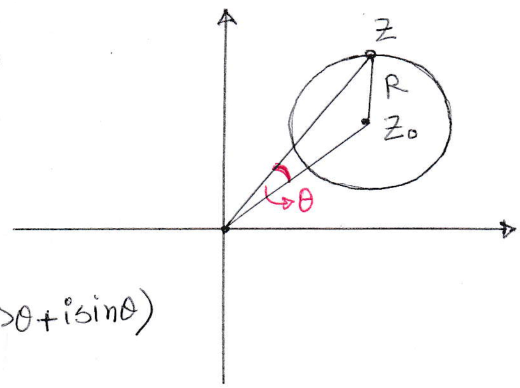
$\Rightarrow \sqrt{(x-x_0)^2+(y-y_0)^2} = R$

$(x-x_0)^2 + (y-y_0)^2 = R^2$ → eqn of circle

center at (x_0, y_0) or it implies z_0

$\because z_0 = x_0 + iy_0$

radius $= R$



$|z| = \sqrt{x^2+y^2} = r$

$z = x+iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$\therefore$ $|z-z_0| = R$

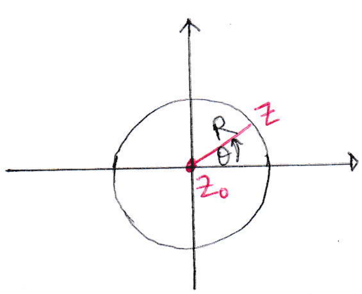
$z - z_0 = Re^{i\theta}$ — (a) $\quad \because z = re^{i\theta} = r(\cos\theta+i\sin\theta)$

$= R(\cos\theta + i\sin\theta)$

similarly $|z| = \sqrt{x^2+y^2} = r = R$

$z = Re^{i\theta}$ — (b) center at the origin

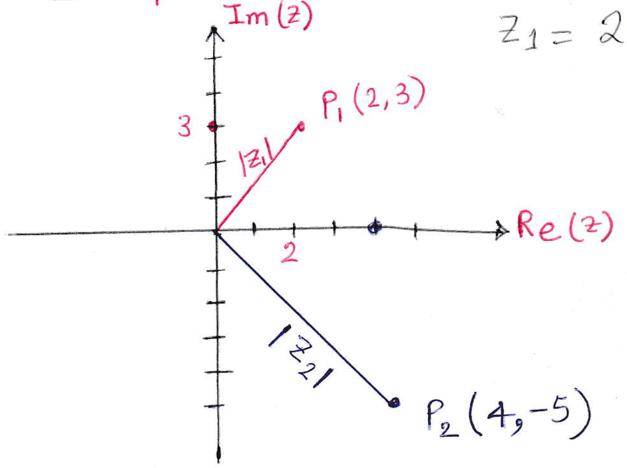
$0 \leq \theta \leq 2\pi$ $\qquad z_0 = 0 = (x_0, y_0)$



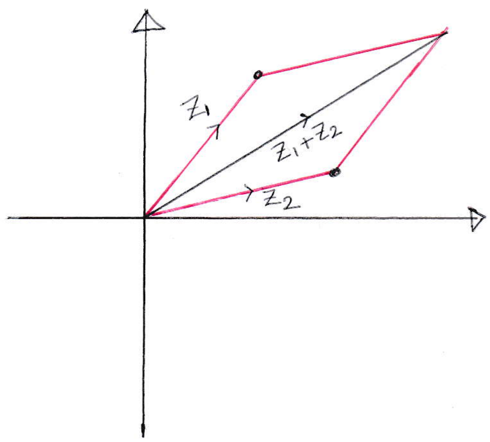
Compare eqn (a) & (b) states that $z_0 = 0$ in eqn (b)

Graphs

$z_1 = 2+3i$; $z_2 = 4-5i$



Vector presentation of complex Number:



z_1, z_2 are unit vectors

z_1+z_2 creates parallelogram

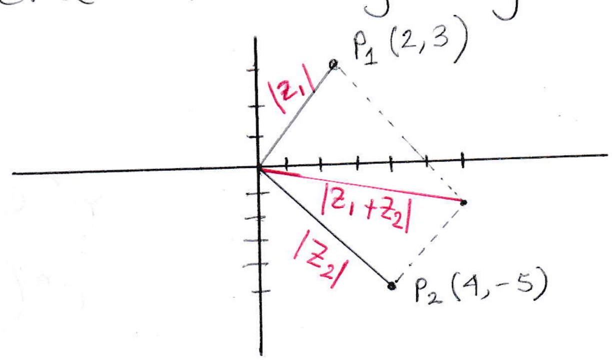
Example

Perform the indicated operations analytically & graphically.

a) $\overset{\to z_1}{(2+3i)} + \overset{\to z_2}{(4-5i)}$

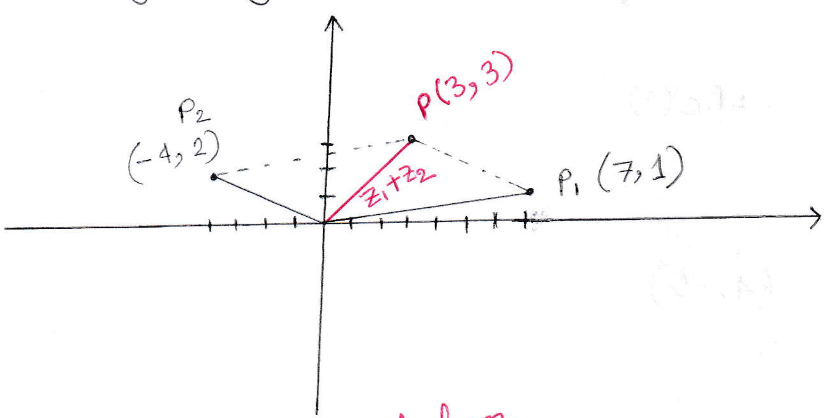
Analytically $= 2+3i+4-5i$

$= 6-2i = z_1+z_2$



b) $(7+i)-(4-2i) = \underbrace{(7+i)}_{z_1} + \underbrace{(-4+2i)}_{z_2}$

Anallytically $= 7+i-4+2i = 3+3i = z_1+z_2$



$$Z = \overbrace{re^{i\theta}}^{\text{exponential form}} = \overbrace{r(\cos\theta + i\sin\theta)}^{\text{polar form}}$$

$$Z^2 = r^2(\cos\theta + i\sin\theta)^2$$

$$= r^2(\cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta)$$

$$= r^2\left[(\cos^2\theta - \sin^2\theta) + i\,2\sin\theta\cos\theta\right]$$

$$= r^2(\cos 2\theta + i\sin 2\theta)$$

$$= r^2 e^{i2\theta}$$

$$= r^2(e^{i\theta})^2$$

$$Z^2 = (re^{i\theta})^2$$

$$\Rightarrow Z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$= r^n(\cos n\theta + i\sin n\theta)$$

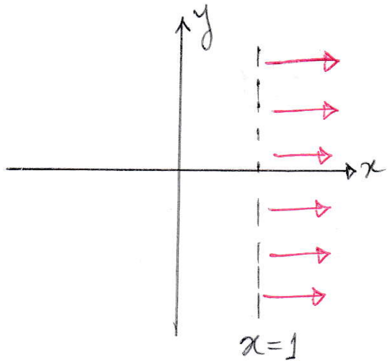
GRAPHS (Exercise Sheet #1)

Describe geometrically the set of pts z satisfying the following conditions.

a) $Re(z) > 1$

$R(x+iy) > 1$

$x > 1$



b) $|2z+3| > 4$

$\Rightarrow |2(x+iy)+3| > 4$

$\Rightarrow |\underbrace{2x+3}_{Re} + i\cdot\underbrace{2y}_{Im}| > 4$

$\Rightarrow \sqrt{(2x+3)^2+(2y)^2} > 4$

$\Rightarrow 4x^2+12x+9+4y^2 > 16$

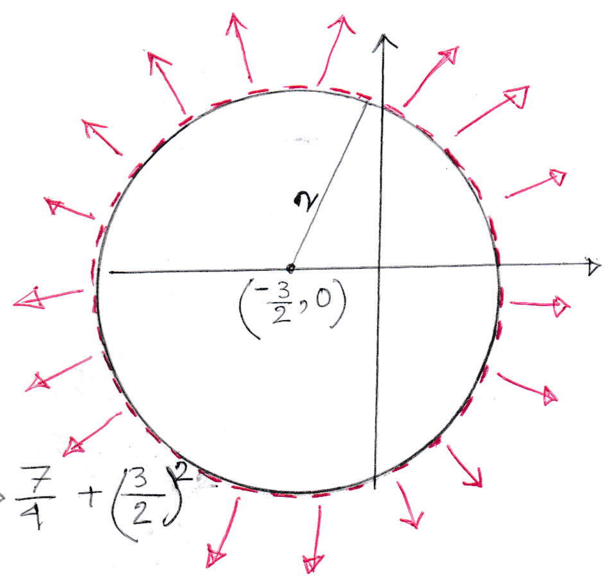
$\Rightarrow 4x^2+12x+4y^2 > 7$

$\Rightarrow x^2+3x+y^2 > 7/4$

$\Rightarrow x^2+2\cdot x\cdot\frac{3}{2}+\left(\frac{3}{2}\right)^2+y^2 > \frac{7}{4}+\left(\frac{3}{2}\right)^2$

$\Rightarrow (x+\frac{3}{2})^2+y^2 > \frac{16}{4} = 4$

$\Rightarrow (x+\frac{3}{2})^2+(y-0)^2 > 4$



c) $\mathrm{Re}\left(\frac{1}{z}\right) > 1$

$\Rightarrow \mathrm{Re}\left(\frac{1}{x+iy}\right) > 1$

$\Rightarrow \mathrm{Re}\left(\frac{1(x-iy)}{(x+iy)(x-iy)}\right) > 1$

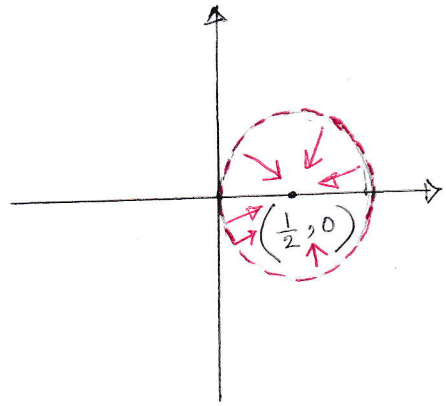
$\Rightarrow \mathrm{Re}\left(\frac{x-iy}{x^2+y^2}\right) > 1$

$\Rightarrow \frac{x}{x^2+y^2} > 1$

$\Rightarrow x > x^2+y^2$

$\Rightarrow x^2 - x + y^2 < 0$

$\Rightarrow \left(x-\frac{1}{2}\right)^2 + y^2 < \left(\frac{1}{2}\right)^2$

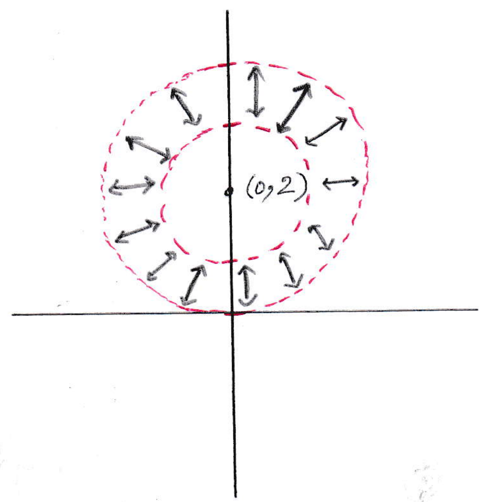


(d) $1 < |z-2i| < 2$

$\Rightarrow 1 < |x+iy-2i| < 2$

$\Rightarrow 1 < \sqrt{x^2+(y-2)^2} < 2$

$\Rightarrow 1^2 < x^2+(y-2)^2 < 2^2$



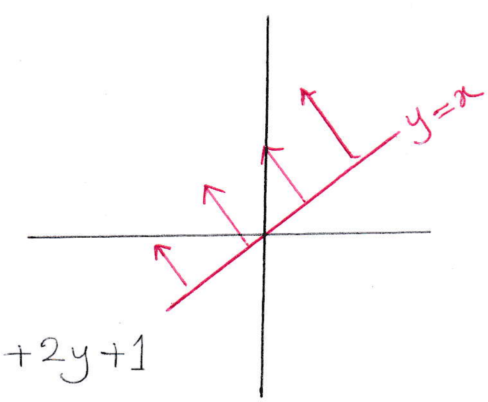
e) $|z+1-i| \leq |z-1+i|$

$\Rightarrow |x+iy+1-i| \leq |x+iy-1+i|$

$\Rightarrow \sqrt{(x+1)^2+(y-1)^2} \leq \sqrt{(x-1)^2+(y+1)^2}$

$\Rightarrow x^2+2x+1+y^2-2y+1 \leq x^2-2x+1+y^2+2y+1$

$\Rightarrow 4x \leq 4y \Rightarrow y \geqslant x$



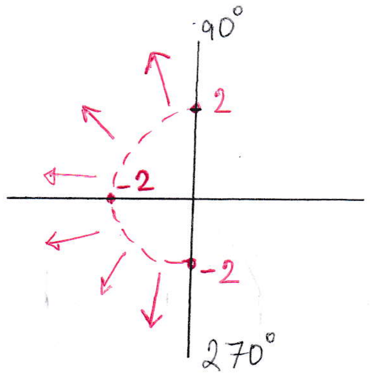
(e) $\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$, $|z| > 2$

$\Rightarrow 90° < \arg z < 270°$

Given $|z| > 2$

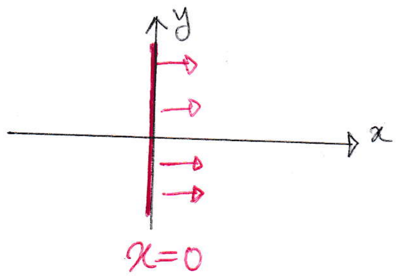
$\Rightarrow \sqrt{x^2+y^2} > 2$

$\Rightarrow x^2+y^2 > 2^2$



(f) $Re(z) \geq 0$

$x \geq 0$



(g) $|z-4| \geq |z|$

$|x+iy-4| \geq |z|$

$\sqrt{(x-4)^2+y^2} \geq \sqrt{x^2+y^2}$

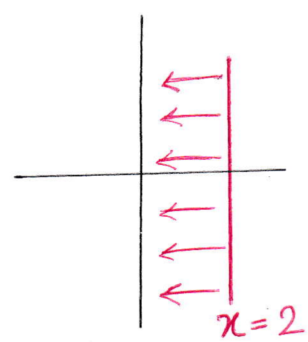
$x^2-8x+16+y^2 \geq x^2+y^2$

$-8x+16 \geq 0$

$-8x \geq -16$

$-x \geq -2$

$x \leq 2$



(h) $|z-2| \leq |z+2|$

$|x+iy-2| \leq |x+iy+2|$

$\sqrt{(x-2)^2+y^2} \leq \sqrt{(x+2)^2+y^2}$

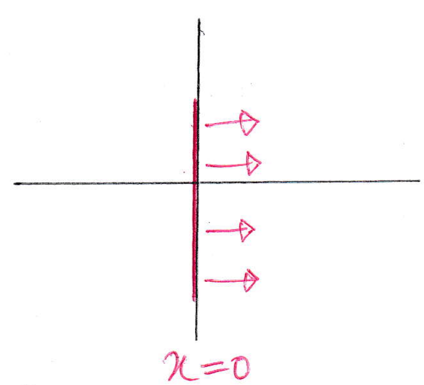
$\sqrt{x^2-4x+4+y^2} \leq \sqrt{x^2+4x+4+y^2}$

$x^2-4x+4+y^2 \leq x^2+4x+4+y^2$

$-4x \leq 4x$

$8x \geq 0$

$x \geq 0$



(i) $\operatorname{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}$ $\longmapsto$ **Reading**

$$\operatorname{Re}\left(\frac{1}{x+iy}\right) \leq \frac{1}{2}$$

$$\operatorname{Re}\left(\frac{1(x-iy)}{x^2+y^2}\right) \leq \frac{1}{2}$$

$$\frac{x}{x^2+y^2} \leq \frac{1}{2}$$

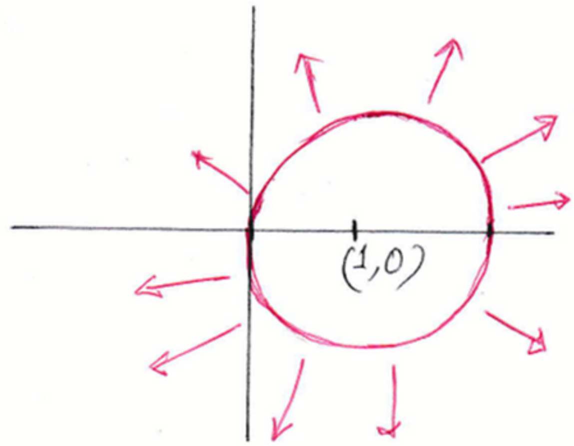
$$2x \leq x^2+y^2$$

$$x^2+y^2 \geq 2x$$

$$x^2-2x+y^2 \geq 0$$

$$x^2-2x+1+y^2 \geq 1$$

$$(x-1)^2+(y-0)^2 \geq 1^2$$



Polar & Exponential Form of Complex Number (Euler's Formula)

$$z = x + iy$$

$$= r(\cos\theta + i\sin\theta) \quad \text{(Polar form}$$

$$\cos\theta + i\sin\theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\cdots\right)$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} - \cdots\cdots\cdots$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots\cdots$$

$$= e^{i\theta}$$

(Euler's Formula)

$$\therefore \quad e^{i\theta} = \cos\theta + i\sin\theta$$

$$\left.\begin{array}{l} z = \underbrace{r(\cos\theta + i\sin\theta)}_{\to \text{Polar form}} \\ \Rightarrow z = \underbrace{re^{i\theta}}_{\to \text{Exponential Form}} \end{array}\right\} \quad 0 \leq \theta \leq 2\pi$$

Powers of 'i'

$$i^3 = i^2 i = -i$$

$$i^2 = -1 \quad \Rightarrow i = \sqrt{-1}$$

$$i^{1000} = (i^2)^{500} = (-1)^{500} = 1$$

$$i^{1001} = i^{1000} \cdot i = (i^2)^{500} \cdot i = (-1)^{500} \cdot i = i$$

$$i^{15} = i^{14} \cdot i = (i^2)^7 \cdot i = (-1)^7 \cdot i = (-1)i = -i$$

$$i^{17} = i^{16} \cdot i = (i^2)^8 \cdot i = (-1)^8 \cdot i = i$$

Apply DeMoivre's Thm to find $(1+i)^{20}$

$r = \sqrt{1^2+1^2} = \sqrt{2}$ $\quad \therefore z = 1+i$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ$

$z^{20} = (1+i)^{20} = r^{20}\left(\cos 20(45^\circ) + i \sin 20(45^\circ)\right)$

$= (\sqrt{2})^{20}\left(-1 + i \cdot 0\right)$

$= 1024(-1)$

$= -1024$

$\therefore z = x+iy = -1024 + i \cdot 0$