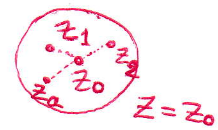
MAT 215 Topic 5

SINGULARITIES

A singularity of an analytic function $f(z)$ is the point at which the function ceases to be analytic

ex the function $f(z) = \frac{1}{z-a}$ is analytic except at the point $z = a$. Thus the point $z = a$ is singularity of $f(z)$.

Types of Singularities:



1) Isolated singularities:

Let $z = z_0$ be a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = z_0$, then this point is said to be an isolated singularity and otherwise it is termed as non-isolated singularity.

example: (i) $f(z) = \frac{z+1}{(z-1)(z-2)}$ is analytic every where except at $z = 1, 2$ which are its singularities.

Also there are no other singularities of $f(z)$ in the neighbour of these points and as such these are isolated singularities.

__example__ (ii) $f(z) = \dfrac{z+3}{z^4(z^2+2)}$ possesses three isolated (belongs to)

singularities at $z=0$, $z=\sqrt{2}\,i$ and $z=-\sqrt{2}\,i$

2] Poles:

If we can find a positive integer n such that $\lim\limits_{z \to z_0} (z-z_0)^n f(z) = A \neq 0$, then $z=z_0$ is called a pole of order 'n'.

If $n=1$, z_0 is called a simple pole.

__example__ $f(z) = \dfrac{3z-2}{(z-1)^2(z+1)(z-4)}$ has a pole of order 2 at $z=1$ & simple poles at $z=-1, 4$.

3] Branch Points:

Branch points of multiple valued functions are singular points.

__example:__ $f(z) = \ln(z^2+z-2)$

$$f'(z) = \frac{2z+1}{z^2+z-2} \; ; \quad \begin{aligned} &z^2+z-2 \neq 0 \\ &z^2+2z-z-2 \neq 0 \\ &(z+2)(z-1) \neq 0 \\ &z \neq 1, -2 \end{aligned}$$

$f'(z)$ is not analytic at $z=1,-2$

Hence $f(z)$ has branch points at $z=1,-2$

4| Removable Singularities:

The singular points z_0 is called a Removable singularity of $f(z)$ if $\lim_{z \to z_0} f(z)$ exists.

<u>example</u> The singular point $z=0$ is a removable singularity of $f(z) = \frac{\sin z}{z}$ since $\lim_{z \to 0} \frac{\sin z}{z} = 1$

<u>i.e.</u> $\lim_{z \to 0} \frac{\sin z}{z} = \frac{0}{0}$

$$\therefore \lim_{z \to 0} \frac{\frac{d}{dz} \sin z}{\frac{d}{dz} z} = \lim_{z \to 0} \frac{\cos z}{1} = \cos 0^\circ = 1$$

(L, Hôspital's rule)

5| Essential Singularities:

A singularity which is not a pole, or removable singularity is called an essential singularity.

<u>example</u>: (i) $z=0$ for $\log(z) = f(z)$

(ii) $z=0$ for $f(z) = e^{1/z}$

The easiest way to define essential singularity of a function involves a "Laurent Series" which we will discuss at the end of Topic 5.

Examples

Locate and name the singularity:

1. $f(z) = \dfrac{z}{(z+1)(z+2)}$

$$\lim_{z \to -1} \cancel{(z+1)} \frac{z}{\cancel{(z+1)}(z+2)} = \frac{-1}{-1+2} = -1$$

$$\lim_{z \to -2} \cancel{(z+2)} \frac{z}{(z+1)\cancel{(z+2)}} = \frac{-2}{-2+1} = 1$$

$\therefore$ $z = -1$ and -2 are poles of order 1.

2. $f(z) = \dfrac{z}{(z^2+4)^2}$

$$\frac{z}{(z^2+4)^2} = \frac{z}{(z^2 - 2^2 i^2)^2} = \frac{z}{(z+2i)^2 (z-2i)^2}$$

$$\therefore \lim_{z \to 2i} (z-2i)^2 \cdot \frac{z}{(z+2i)^2(z-2i)^2} = \lim_{z \to 2i} \frac{z}{(z+2i)^2}$$

$$= \frac{2i}{16 i^2} = -\frac{i}{8} \neq 0$$

$\therefore$ $z = 2i$ is a pole of order 2.

Similarly $z = -2i$ is a pole of order 2.

3. $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$

$\therefore \lim_{z \to 0} \dfrac{\sin\sqrt{z}}{\sqrt{z}} = 1$, $z = 0$ is a removable singularity

$$\lim_{z \to 0} \frac{\sin\sqrt{z}}{\sqrt{z}} = \frac{0}{0} \Rightarrow \lim_{z \to 0} \frac{(\sin\sqrt{z})'}{(\sqrt{z})'} = \lim_{z \to 0} \frac{\cos\sqrt{z} \cdot \frac{1}{2\sqrt{z}}}{\frac{1}{2\sqrt{z}}}$$

$$= \lim_{z \to 0} \cos\sqrt{z} = \cos\sqrt{0} = 1$$

The Residue Theorem

Residues:

Let $f(z)$ be single-valued and analytic inside and on a circle 'C' except at the point $z = z_0$ chosen as the centre of 'C'. Then, $f(z)$ has a Laurent series about $z = z_0$ given by

Res or R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{....} \; (i)$$

$$\text{where} \quad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

see Cauchy's Integral Formula "Topic 4" Page 31

when $n = 1$, this expression for b_n can be written:

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz \Rightarrow R = \frac{1}{2\pi i} \int_C f(z) dz$$

$$\Rightarrow 2\pi i R = \int_C f(z) dz$$

The complex number b_1, which is the coefficient of $\frac{1}{z-z_0}$ in expansion (i), is called the residue of f at the isolated singular point $z = z_0$.

We often use the notation,

$$\text{residue at } z_0 = \operatorname*{Res}_{z=z_0} f(z)$$

Calculation of Residues

To obtain the residue of a function $f(z)$ in the case of $z = z_0$ is a pole of order k, there is a simple formula for b_K given by

$$b_K = \lim_{z \to z_0} \frac{1}{(K-1)!} \cdot \frac{d^{K-1}}{dz^{K-1}} \left\{ (z-z_0)^K f(z) \right\}$$

Note $0! = 1$

$f(z) = \frac{g(z)}{(z-z_0)^K}$

if $K = 1$

then $b_1 = \lim_{z \to z_0} (z-z_0) f(z)$ $\longrightarrow$ $f(z) = \frac{g(z)}{(z-z_0)}$

Examples

For the following functions, determine the poles and the residues at the poles:

(i) $f(z) = \frac{2z+1}{z^2-z-2} = \frac{2z+1}{(z-2)(z+1)}$

$f(z)$ has simple poles at $z = 2$ and $z = -1$

Residue at $z = 2$ is

$$\lim_{z \to 2} \cancel{(z-2)} \cdot \frac{2z+1}{\cancel{(z-2)}(z+1)} = \frac{5}{3}$$

Residue at $z = -1$ is

$$\lim_{z \to -1} (z+1) \frac{2z+1}{(z-2)(z+1)} = \frac{-2+1}{-1-2} = \frac{-1}{-3} = \frac{1}{3}$$

(ii) $f(z) = \left(\frac{z+1}{z-1}\right)^2$

$$f(z) = \frac{(z+1)^2}{(z-1)^2}$$

$f(z)$ has a double pole at $z = 1$

Residue at $z = 1$ is:

$$\lim_{z \to 1} \frac{1}{1!} \frac{d}{dz} \left\{ (z-1)^2 \frac{(z+1)^2}{(z-1)^2} \right\}$$

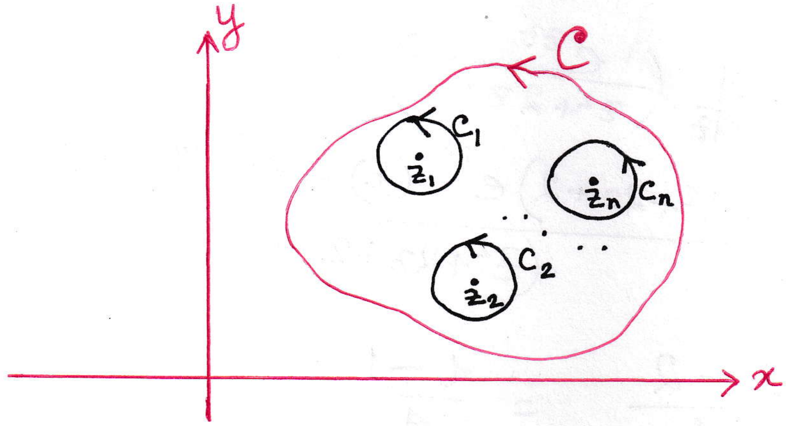
$$= \lim_{z \to 1} 2(z+1) = 4$$

Residue Theorem

If $f(z)$ is analytic inside and on a simple closed curve 'C' except at a finite number of singular points z_K $(K = 1, 2, \dots, n)$ inside 'C', then

$$\oint_C f(z)\,dz = 2\pi i \sum_{K=1}^{n} R_K$$

$$= 2\pi i \,(R_1 + R_2 + \dots + R_n)$$



Poles

If $\lim_{z \to z_0} (z-z_0)^n F(z) = A \neq 0$ while $F(z) = \frac{f(z)}{(z-z_0)^n}$ — Analytic

then $F(z)$ has a pole of order n at $z = z_0$ or $z = z_0$ is called a pole of order 'n'.

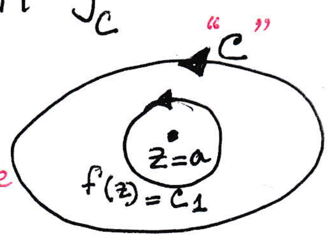
Theorem of Residue

(remains / rest / remainder)

If $f(z)$ is analytic on and within C (closed curve) except for a pole of order m at $z = a$ inside C
where $f(z)$ is undefined

then $\frac{1}{2\pi i} \oint_C F(z)\, dz = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m F(z) \right\}$

"C" is a closed curve



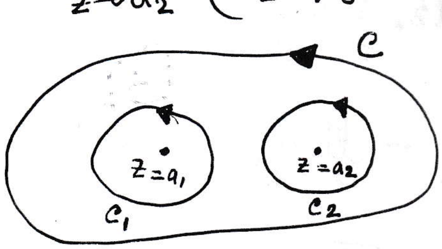
$F(z) = \frac{f(z)}{(z-a)^m}$

$f(z) = C_1$ represents a circle inside "C".

If more than one pole inside $C \Rightarrow z = a_1, a_2$
i.e. pole of order m_1, m_2

$$\frac{1}{2\pi i} \oint_C F(z)\, dz = \lim_{z \to a_1} \frac{1}{(m_1-1)!} \cdot \frac{d^{m_1-1}}{dz^{m_1-1}} \left\{ (z-a_1)^{m_1} F(z) \right\}$$
$$+ \lim_{z \to a_2} \frac{1}{(m_2-1)!} \frac{d^{m_2-1}}{dz^{m_2-1}} \left\{ (z-a_2)^{m_2} F(z) \right\}$$

$\Rightarrow \oint_C F(z)\, dz = 2\pi i [\,"\,]$
$= 2\pi i (R_1 + R_2)$



C_1, C_2 are non overlaping simple closed curve inside C

limits are denoted by R_1 & R_2 which represents residues of $F(z)$.

Example Evaluate $\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz$ around the circle 'C' with equation $|z|=3$.

$$f(z)=\frac{e^{zt}}{z^2(z^2+2z+2)}$$

$$=\frac{e^{zt}}{z^2(z+1-i)(z+1+i)}$$

$$z^2+2z+2=0$$

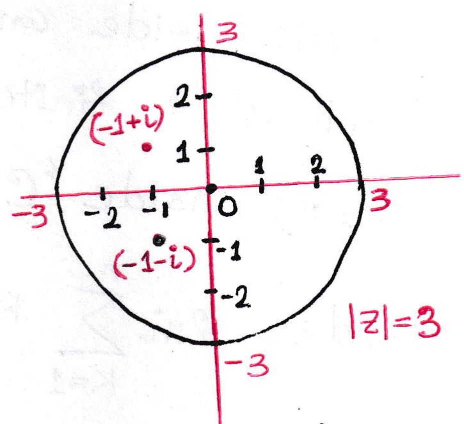
$$z^2+2\cdot z\cdot 1+1^2+1=0$$

$$(z+1)^2-i^2=0 \quad \begin{cases} i^2=-1 \\ -i^2=+1 \end{cases}$$

$$(z+1-i)(z+1+i)=0$$

$f(z)$ has a double pole at $z=0$ and two simple poles at $z=-1-i$, $z=-1+i$.

all these poles are inside 'C'.



$$\underset{z=0}{R_1}\, f(z)=\lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left\{z^2\cdot\frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$$

$$=\lim_{z\to 0}\frac{d}{dz}\left(\frac{e^{zt}}{z^2+2z+2}\right)$$

$$=\lim_{z\to 0}\frac{(z^2+2z+2)e^{zt}(t)-e^{zt}(2z+2)}{(z^2+2z+2)^2}$$

$$=\frac{2t-2}{2^2}=\frac{t-1}{2}$$

$$\underset{z=-1+i}{R_2} f(z) = \lim_{z \to -1+i} \left\{ (z+1-i) \frac{e^{zt}}{z^2(z^2+2z+2)} \right\}$$

$$= \lim_{z \to -1+i} \left\{ (z+1-i) \frac{e^{zt}}{z^2(z+1-i)(z+1+i)} \right\}$$

$$= \lim_{z \to -1+i} \frac{e^{zt}}{z^2(z+1+i)}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2(-1+i+1+i)}$$

$$= \frac{e^{(-1+i)t}}{(1-2i+i^2)(2i)} = \frac{e^{(-1+i)t}}{(-2i)(2i)} = \frac{e^{(-1+i)t}}{4}$$

$$\underset{z=-1-i}{R_3} f(z) = \lim_{z \to -1-i} \left\{ (z+1+i) \frac{e^{zt}}{z^2(z^2+2z+2)} \right\}$$

$$= \lim_{z \to -1-i} \left\{ (z+1+i) \frac{e^{zt}}{z^2(z+1-i)(z+1+i)} \right\}$$

$$= \lim_{z \to -1-i} \frac{e^{zt}}{z^2(z+1-i)}$$

$$= \frac{e^{(-1-i)t}}{(-1-i)^2(-1-i+1-i)}$$

$$= \frac{e^{(-1-i)t}}{(1+2i+i^2)(-2i)}$$

$$= \frac{e^{(-1-i)t}}{(2i)(-2i)} = \frac{e^{(-1-i)t}}{4}$$

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = 2\pi i\left[R_1+R_2+R_3\right]$$

$$= 2\pi i\left[\frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}\right]$$

$$\therefore \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{t-1}{2} + \frac{1}{4}\left[e^{-t}e^{it} + e^{-t}e^{-it}\right]$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}\left[\cos t + i\sin t + \cos(-t) + i\sin(-t)\right]$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}\left[\cos t + i\sin t + \cos t - i\sin t\right]$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}\left[2\cos t\right]$$

$$= \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t.$$

Overview

Cauchy Riemann Equations

Refer to Topic 2

If $f(z) = u(x,y) + iv(x,y)$ then $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$ known as necessary condition

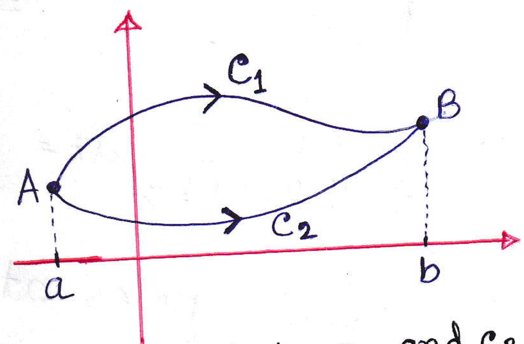
$\left.\begin{array}{l} f'(z_0) = u_x + i v_x \\ \text{or } f'(z_0) = v_y - i u_y \end{array}\right\}$ known as Sufficient Condition

Cauchy's Integral / Cauchy-Goursat Theorem

Refer to Topic 4 Page 9 and 10

$$\int_C f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \left[-\int_{C_2} f(z)\,dz\right] = 0$$

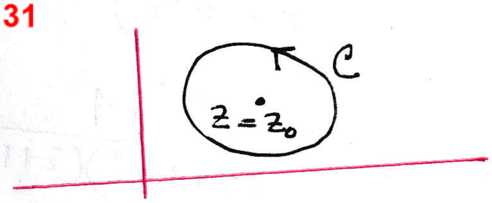


From a to b, C_1 and C_2 are independent of path

Cauchy's Integral Formula

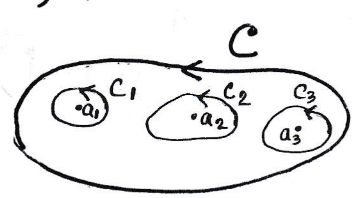
Topic 4 Page 31

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz$$



n^{th} derivative:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad n = 1, 2, 3, \ldots$$



Theorem of Residue

$$\int_C F(z)\,dz = 2\pi i\,[R_1 + R_2 + \cdots + R_n]$$

$$R = \lim_{z \to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m F(z)\right\}$$

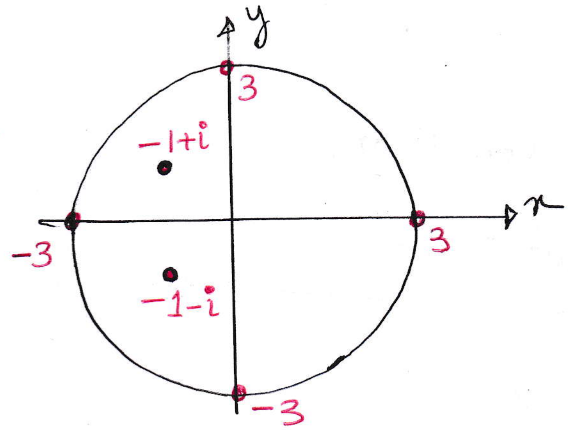
$$F(z) = \frac{f(z)}{(z-a)^m}$$

$f(z)$ has pole of order m at $z = a$.

$f(z)$ is analytic everywhere except $z = a$.

(Exercise Sheet # 5)

5) Evaluate $\oint_C \frac{z^2+4}{z^3+2z^2+2z}\,dz$ using the residue at the poles, around the circle $|z|=3$.



$$z^3+2z^2+2z$$
$$= z(z^2+2z+2)$$
$$= z(z^2+2z+1+1)$$
$$= z((z+1)^2+1)$$
$$= z[(z+1)^2-i^2]$$
$$= z[(z+1+i)(z+1-i)]$$

$$\oint_C \frac{z^2+4}{z^3+2z^2+2z}\,dz = \int_C \frac{z^2+4}{(z-0)\{z-(-1-i)\}\{z-(-1+i)\}}\,dz$$

$f(z)$ has simple poles at $\underbrace{z=0,\ -1-i,\ -1+i}_{\text{all inside the region}}$

$$R_1 = \lim_{z\to 0} z\cdot f(z)$$
$$= \lim_{z\to 0} \frac{z^2+4}{(z+1+i)(z+1-i)}$$
$$= \frac{4}{(1+i)(1-i)} = \frac{4}{1-i^2} = \frac{4}{2} = 2$$

$$R_2 = \lim_{z \to -1-i} (z+1+i)\frac{z^2+4}{z(z+1+i)(z+1-i)}$$

$$= \lim_{z \to -1-i} \frac{z^2+4}{z(z+1-i)}$$

$$= \frac{(-1-i)^2+4}{(-1-i)(-1-i+1-i)} = \frac{1+2(-1)(-i)+(-i)^2+4}{(-1-i)(-2i)}$$

$$= \frac{1+2i-1+4}{2i+2i^2}$$

$$= \frac{2i+4}{2i-2} = \frac{i+2}{i-1} \cdot \frac{-i-1}{-i-1}$$

$$= \frac{-i^2-2i-i-2}{(-1)^2-i^2}$$

$$= \frac{-1-3i}{2}$$

$$= -\frac{1}{2} - \frac{3}{2}i$$

$$R_3 = \lim_{z \to -1+i} (z+1-i)f(z)$$

$$= \lim_{z \to -1+i} \frac{z^2+4}{z(z+1+i)}$$

$$= \frac{(-1+i)^2+4}{(-1+i)(-1+i+1+i)}$$

$$= \frac{1-2i-1+4}{(-1+i)(2i)}$$

$$= \frac{-2i+4}{-2i+2i^2}$$

$$= \frac{-2i+4}{-2i-2}$$

$$= \frac{i-2}{i+1} \cdot \frac{-i+1}{-i+1}$$

$$= \frac{-i^2+2i+i-2}{1^2-i^2}$$

$$= \frac{-1+3i}{2} = -\frac{1}{2} + \frac{3i}{2}$$

$$\therefore \oint_C \frac{z^2+4}{z^3+2z^2+2z}\,dz = 2\pi i(R_1+R_2+R_3)$$

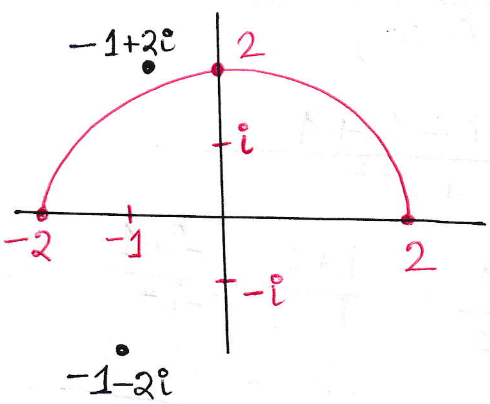
$$= 2\pi i\left(2 - \frac{1}{2} - \frac{3i}{2} - \frac{1}{2} + \frac{3i}{2}\right)$$

$$= 2\pi i(2-1)$$

$$= 2\pi i$$

(Exercise Sheet # 5)

⑥ Evaluate $\oint_C \frac{z e^{i\pi z}}{(z^2+2z+5)(z^2+1)^2}\,dz$ using the residue at the poles, where C is the upper half circle of the equation $|z| = 2$.



$$(z^2+2z+5)(z^2+1)^2$$
$$= (z^2+2z+1+4)(z^2-i^2)^2$$
$$= [(z+1)^2-(2i)^2]\,(z+i)^2(z-i)^2$$
$$= \underbrace{(z+1+2i)}_{z-(-1-2i)}\underbrace{(z+1-2i)}_{z-(-1+2i)}(z-i)^2(z+i)^2$$

$f(z)$ has simple poles at $z = \underbrace{-1-2i,\ -1+2i}_{\text{outside region}}$

$f(z)$ has poles of order 2 at $z = i, \underbrace{-i}_{\text{outside region}}$

$$R_1 = \lim_{z\to i} \frac{1}{(2-1)!} \cdot \frac{d}{dz}\left\{(z-i)^2 f(z)\right\}$$
$$= \lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \cdot \frac{z e^{i\pi z}}{(z^2+2z+5)(z-i)^2(z+i)^2}\right\}$$
$$= \lim_{z\to i} \frac{d}{dz}\left\{\frac{z e^{i\pi z}}{(z^2+2z+5)(z+i)^2}\right\}$$
$$= \lim_{z\to i} \frac{d}{dz}\left\{\frac{z e^{i\pi z}}{(z^2+2z+5)(z^2+2zi-1)}\right\}$$

$$= \lim_{z \to i} \frac{d}{dz} \left\{ \frac{z e^{i\pi z}}{z^4 + 2z^3 + 5z^2 + 2z^3 i + 4z^2 i + 10zi - z^2 - 2z - 5} \right\}$$

$$= \lim_{z \to i} \frac{d}{dz} \left\{ \frac{z e^{i\pi z}}{z^4 + 2z^3 + 4z^2 - 2z - 5 + 2z^3 i + 4z^2 i + 10zi} \right\}$$

$$= \lim_{z \to i} \frac{(z^4 + 2z^3 + 4z^2 - 2z - 5 + 2z^3 i + 4z^2 i + 10zi)\left[e^{i\pi z} + zi\pi e^{i\pi z}\right] + z e^{i\pi z}(4z^3 + 6z^2 + 8z - 2 + 6iz^2 + 8iz + 10i)}{(z^4 + 2z^3 + 4z^2 - 2z - 5 + 2z^3 i + 4z^2 i + 10zi)^2}$$

$$= \frac{(1 - 2i - 4 - 2i - 5 + 2 - 4i - 10)(e^{-\pi} - \pi e^{-\pi}) + ie^{-\pi}(-4i - 6 + 8i - 2 - 6i - 8 + 10i)}{(1 - 2i - 4 - 2i - 5 + 2 - 4i - 10)^2}$$

$$= \frac{(-20 - 8i)(e^{-\pi} - \pi e^{-\pi}) - ie^{-\pi}(-16 + 8i)}{(-20 - 8i)^2}$$

$$= \frac{20e^{-\pi} - 8ie^{-\pi} - 11\pi e^{-\pi} - 8\pi i e^{-\pi} + 16ie^{-\pi} - \underbrace{8i^2 e^{-\pi}}_{+8e^{-\pi}}}{(-20 - 8i)^2}$$

$$= \frac{-12e^{-\pi} + 8ie^{-\pi} - 11\pi e^{-\pi} - 8\pi i e^{-\pi}}{(-20 - 8i)^2}$$

$$= \frac{-12e^{-\pi} - 11\pi e^{-\pi} + 8ie^{-\pi} - 8\pi i e^{-\pi}}{400 + 2(-20)(-8i) + (-8i)^2} = \frac{e^{-\pi}(12 - 11\pi) + i8e^{-\pi}(1 - \pi)}{400 + 320i + 64}$$

$$= \frac{-2.01 - i0.74}{464 + 320i}$$

$$= \frac{(-2.01 - 0.74i)}{(464 + 320i)} \cdot \frac{(464 - 320i)}{(464 - 320i)}$$

$$= \frac{-933 - 343i + 322i + 237i^2}{215296 + 102400}$$

$$= \frac{1170 - 21i}{317696}$$

$$\oint_c F(z)\,dz = 2\pi i \; R_1$$

$$= 2\pi i \left(\frac{1170 - 21i}{317696} \right)$$

There might be some numerical error but the process is correct

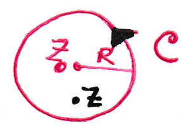
Taylor Series

If a function $f(z)$ is analytic at all the points inside a circle 'C', with its center z_0 and radius R

then

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!}f^{(n)}(z_0)$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!}$$



$n = 0, 1, 2, 3, \cdots$

(Exercise Sheet # 5)

1. Expand each of the following functions in a Taylor series about the indicated points:

(ii) $f(z) = \cos z$ at $z = \frac{\pi}{2}$

$f(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$f'(z) = -\sin z$ $\qquad$ $f'(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

$f''(z) = -\cos z$ $\qquad$ $f''(\pi/2) = -\cos\frac{\pi}{2} = 0$

$f'''(z) = \sin z$ $\qquad$ $f'''(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

The Taylor Series:

$$f(z) = f(\tfrac{\pi}{2}) + (z-\tfrac{\pi}{2})f'(\tfrac{\pi}{2}) + \frac{(z-\pi/2)^2}{2!}f''(\tfrac{\pi}{2}) + \frac{(z-\frac{\pi}{2})^3}{3!}f'''(\tfrac{\pi}{2}) + \cdots$$

$$= 0 + (z-\tfrac{\pi}{2})(-1) + \frac{(z-\pi/2)^2}{2!}(0) + \frac{(z-\pi/2)^3}{3!}(1) + \cdots$$

$$= -\underline{\frac{(z-\frac{\pi}{2})}{1!}} + \frac{(z-\pi/2)^3}{3!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(z-\pi/2)^{2n+1}}{(2n+1)!}(-1)^{n+1}$$

$2n+1 = 1, 3, 5, 7, \cdots$ $\boxed{n = 0, 1, 2, 3, \cdots}$

$(-1)^{n+1} = -1, 1, -1, 1, -1, \cdots$

$(-1)^{n-1} = -1, 1, -1, 1, \cdots$

(iii) $f(z) = z^3 - z^2 + 4z - 2$ at $z = 2$

$f(z) = z^3 - z^2 + 4z - 2$ $\quad f(2) = 10$

$f'(z) = 3z^2 - 2z + 4$ $\quad f'(2) = 12$

$f''(z) = 6z - 2$ $\quad f''(2) = 10$

$f'''(z) = 6$ $\quad f'''(2) = 6$

$f^{(4)}(z) = 0$ $\quad f^{(4)}(2) = 0$

$$f(z) = f(2) + (z-2)f'(2) + \frac{(z-2)^2}{2!}f''(2) + \frac{(z-2)^3}{3!}f'''(2) + \frac{(z-2)^4}{4!}f^{(4)}(2)$$

$$= 10 + 12.(z-2) + 10.\frac{(z-2)^2}{2} + 6.\frac{(z-2)^3}{6} + 0.\frac{(z-2)^4}{24}$$

$$= 10 + 12(z-2) + 5(z-2)^2 + (z-2)^3$$

It is a finite series

(iv) ze^{2z} at $z = -1$

$f(z) = ze^{2z} \longleftrightarrow f(-1) = -e^{-2}$

$f'(z) = 2ze^{2z} + e^{2z} \longleftrightarrow f'(-1) = -2e^{-2} + e^{-2} = -e^{-2}$

$f''(z) = 2e^{2z} + 4ze^{2z} + 2e^{2z} \longleftrightarrow f''(-1) = 4e^{-2} - 4e^{-2} = 0$

$= 4e^{2z} + 4ze^{2z} = 4e^{2z}(1+z)$

$f'''(z) = 8e^{2z} + 4e^{2z} + 8ze^{2z} \longleftrightarrow f'''(-1) = 12e^{-2} - 8e^{-2} = 4e^{-2}$

$= 12e^{2z} + 8ze^{2z}$

$f^{(4)}(z) = 24e^{2z} + 8e^{2z} + 16ze^{2z} \longleftrightarrow f^{(4)}(-1) = 32e^{-2} - 16e^{-2}$

$= 32e^{2z} + 16ze^{2z}$ $\quad = 16e^{-2}$

$$f(z) = f(-1) + (z+1)f'(-1) + \frac{(z+1)^2}{2!}f''(-1) + \frac{(z+1)^3}{3!}f'''(-1)$$

$$+ \frac{(z+1)^4}{4!}f^{(4)}(-1) + \cdots$$

$$= -e^{-2} + (z+1)(-e^{-2}) + \frac{(z+1)^2}{2!}(0) + \frac{(z+1)^3}{3!}4e^{-2} + \frac{(z+1)^4}{4!}16e^{-2}$$

$$+ \cdots$$

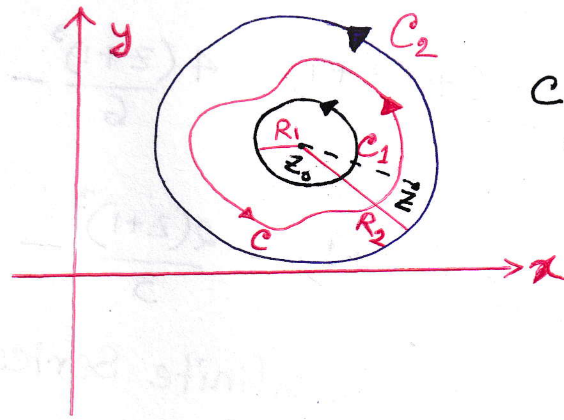
$$= -e^{-2}\left[1 + (z+1) - \frac{4(z+1)^3}{6} - \frac{16(z+1)^4}{24} + \cdots\right]$$

$$= -\frac{1}{e^2}\left[1 + (z+1) - \frac{2(z+1)^3}{3} - \frac{2(z+1)^4}{3} + \cdots\right]$$

It is an infinite Series

Laurent Expansion:

Suppose a function 'f' is analytic throughout the region $R_1 < |z-z_0| < R_2$, and let c denote any positively oriented simple closed contour around z_0 between c_1 and c_2, then $f(z)$ has the series representation:



C is between C_1 & C_2

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\substack{\text{analytic part} \\ \text{(Taylor series)}}} + \underbrace{\sum_{n=0}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{principal part}}, \quad R_1 < |z-z_0| < R_2$$

over the domain
(neighbourhood law)

where $$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}, \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}}, \quad n = 0, 1, 2, \dots$$

This series is called Laurent series.

If the principal part is zero, the Laurent series reduces to a Taylor series.

Alternative (Expansion of Laurent Series)

$$\frac{1}{1-z} = (1-z)^{-1} = \sum_{n=0}^{\infty} z^n \; ; \quad |z|<1$$

Domain

$$R_1 < \underbrace{|z-z_0|}_{z} < R_2$$

$$\Rightarrow \frac{R_1}{R_2} < \frac{|z-z_0|}{R_2} < 1$$

$$\frac{1}{1+z} = (1+z)^{-1} = \sum_{n=0}^{\infty} (-1)^n z^n \; ; \quad |z|<1$$

The upper limit should be '1' $\therefore |z|<1$

(Exercise Sheet # 5)

3. Expand $f(z) = \dfrac{1}{z(z-2)}$ in a Laurent series valid for (i) $0<|z|<2$, (ii) $|z|>2$

Given $f(z) = \dfrac{1}{z(z-2)}$

$$\frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

(partial fraction of Rational function)

$$\frac{1}{z(z-2)} = \frac{A(z-2)+Bz}{z(z-2)}$$

$$1 = A(z-2) + Bz$$

$$1 = Az - 2A + Bz$$

Equating factor of like terms:

$$Az + Bz = 0 \; ; \quad -2A = 1$$

$$\boxed{A = -\tfrac{1}{2}}$$

$$A + B = 0$$

$$-\tfrac{1}{2} + B = 0$$

$$\boxed{B = \tfrac{1}{2}}$$

$$\therefore \frac{1}{z(z-2)} = \frac{-1/2}{z} + \frac{1/2}{z-2}$$

$$= -\frac{1}{2z} + \frac{1}{2(z-2)}$$

Alternatively:

$$\frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$f(z)$ is analytic everywhere except $z=0, 2$

$$\Rightarrow 1 = A(z-2)+B(z)$$

for $z=0 \Rightarrow 1 = A(0-2) \Rightarrow A = -\frac{1}{2}$

for $z=2 \Rightarrow 1 = B(2) \Rightarrow B = \frac{1}{2}$

$$\therefore \frac{1}{z(z-2)} = \frac{-1/2}{z} + \frac{1/2}{z-2}$$

$$= -\frac{1}{2z} + \frac{1}{2(z-2)}$$

ⓘ if $0<|z|<2 \Rightarrow \boxed{0 < \frac{|z|}{2} < 1} \rightarrow$ domain

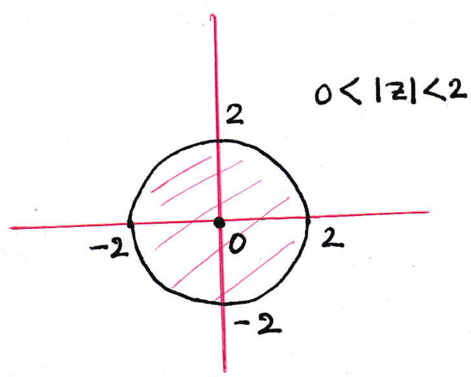
$$f(z) = -\frac{1}{2z} + \frac{1}{2(z-2)}$$

$$= -\frac{1}{2z} + \frac{1}{2\left[2\left(\frac{z}{2}-1\right)\right]}$$

$$= -\frac{1}{2z} + \frac{1}{2\left[-2\left(1-\frac{z}{2}\right)\right]}$$

$$= -\frac{1}{2z} - \frac{1}{4}\cdot\frac{1}{1-\frac{z}{2}}$$

$$= -\frac{1}{2z} - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$



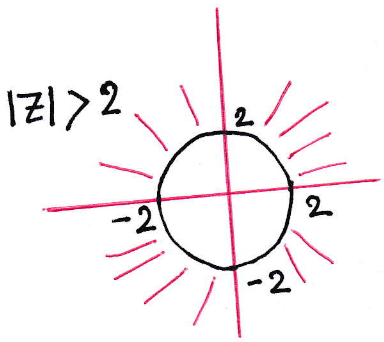
$\because (1-z)^{-1} = \sum_{n=0}^{\infty} z^n, \; |z|<1$

Domain

(ii) $|z| > 2$

$\Rightarrow 2 < |z|$

$\Rightarrow \dfrac{2}{|z|} < 1$



$$f(z) = -\frac{1}{2z} + \frac{1}{2(z-2)}$$

$$= -\frac{1}{2z} + \frac{1}{2\left\{z\left(1-\frac{2}{z}\right)\right\}}$$

$$= -\frac{1}{2z} + \frac{1}{2z} \cdot \frac{1}{(1-2/z)}$$

$$= -\frac{1}{2z} + \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \qquad \because (1-z)^{-1} = \sum_{n=0}^{\infty} z^n, \; |z|<1$$

$$= -\frac{1}{2z} + \sum_{n=0}^{\infty} \frac{2^{n-1}}{z^{n+1}}$$

(Exercise Sheet #5)

2. Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for (i) $|z|<1$, (ii) $1<|z|<2$, (iii) $|z|>2$, (iv) $|z-1|>1$, (v) $0<|z-2|<1$

$$f(z) = \frac{z}{(z-1)(2-z)} = \frac{A}{z-1} + \frac{B}{2-z}$$

$f(z)$ is analytic everywhere except $z = 1, 2$

$$z = A(2-z) + B(z-1)$$

if $z=1 \Rightarrow 1 = A(2-1) \Rightarrow A = 1$

if $z=2 \Rightarrow 2 = B(2-1) \Rightarrow B = 2$

$$\therefore f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

(i) $\boxed{\text{Domain}}$ $|z| < 1$

$$\Rightarrow \frac{|z|}{2} < \frac{1}{2} \quad \text{but } \frac{1}{2} < 1$$

$$\therefore \frac{|z|}{2} < 1$$

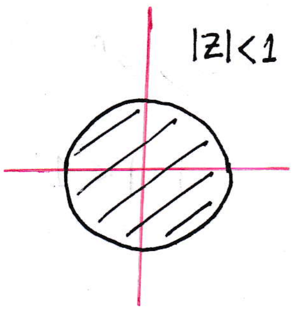
$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= -\frac{1}{(1-z)} + \frac{2}{2(1-\frac{z}{2})}$$

$$= -\frac{1}{(1-z)} + \frac{1}{1-\frac{z}{2}}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(-1 + \frac{1}{2^n}\right) z^n$$

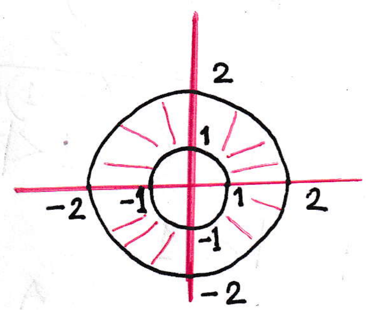


$$\left\{ \begin{aligned} \frac{1}{1-z} &= (1-z)^{-1} \\ &= \sum_{n=0}^{\infty} z^n \\ |z| &< 1 \end{aligned} \right.$$

(ii) $1 < |z| < 2 \leftarrow \boxed{\text{Domain}}$

$|z| > 1$ and $|z| < 2$

$\Rightarrow 1 < |z|$ $\qquad \frac{|z|}{2} < 1$



$$\Rightarrow \frac{1}{|z|} < 1$$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{z(1-\frac{1}{z})} + \frac{2}{2(1-\frac{z}{2})}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$\because \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$|z| < 1$

$$\frac{1}{z-1}$$

$$= \frac{1}{-(1-z)} \to |z| > 1$$

Given domain

$\to 1 < z$

$\to \frac{1}{z} < 1$

domain does not match. So this is incorrect iii $|z| > 2$

$\Rightarrow 2 < |z|$

$\Rightarrow \frac{2}{|z|} < 1$

$\therefore \frac{1}{|z|} < 1$

If the given domain was $|z|<1$, then we wouldn't need to change. $\frac{1}{z-1} \xrightarrow{\text{Convert}} \frac{1}{-(1-z)}$

only convertion could have been suffitient

Otherwise do change.

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{z(1-\frac{1}{z})} + \frac{2}{z(\frac{2}{z}-1)}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{2}{z(1-\frac{2}{z})}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} - \sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n+1}$$

iv $|z-1| > 1$

Let $z-1 = u$ $\quad \{ 2-z = 1-u$

$\therefore z = 1+u$

$\therefore |z-1|>1 \Rightarrow |u|>1$

or $1 < |u|$

$\Rightarrow \frac{1}{|u|} < 1$

$$f(z) = \frac{z}{(z-1)(2-z)}$$

$$= \frac{1+u}{u(1-u)}$$

$\frac{1+u}{u(1-u)} = \frac{A}{u} + \frac{B}{1-u}$ → This function is analytic everywhere except $u=0,1$

$\frac{1+u}{u(1-u)} = \frac{A(1-u)+Bu}{u(1-u)}$

$1+u = A(1-u) + Bu$

$u=0 \Rightarrow 1 = A$

$u=1 \Rightarrow 2 = B$

$$\therefore \frac{1+u}{u(1-u)} = \frac{A}{u} + \frac{B}{1-u} = \frac{1}{u} + \frac{2}{1-u}$$

$$f(z) = \frac{z}{(z-1)(2-z)}$$

$$= \frac{1+u}{u(1-u)}$$

domain:

$$\frac{1}{|u|} < 1$$

$$= \frac{A}{u} + \frac{B}{1-u}$$

$$= \frac{1}{u} + \frac{2}{1-u}$$

$$= \frac{1}{u} + \frac{2}{u(\frac{1}{u}-1)}$$

$$= \frac{1}{u} - \frac{2}{u(1-\frac{1}{u})}$$

$$= \frac{1}{u} - \frac{2}{u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^n$$

$$= \frac{1}{u} - 2\sum_{n=0}^{\infty}\frac{1}{u^{n+1}}$$

$$= \frac{1}{z-1} - 2\sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}$$

do not change
∵ we converted
our domain to u.
∴ after substitution
keep it as it is.

(v) $0<|z-2|<1$ **Reading**

Let $z-2=u$

$z = 2+u$

$\therefore z-1 = 1+u$

& $2-z = -u$

$\therefore 0<|u|<1$

$$\frac{-2-u}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u}$$

$f(u)$ is analytic everywhere except $u=0,-1$

$$\frac{-2-u}{u(1+u)} = \frac{A(1+u)+Bu}{u(1+u)}$$

$$-2-u = A(1+u)+Bu$$

if $u=0 \Rightarrow \boxed{-2=A}$

if $u=-1 \Rightarrow -1=-B$

$\Rightarrow \boxed{B=1}$

$$f(z) = \frac{z}{(z-1)(2-z)}$$

$$= \frac{2+u}{(1-u)(-u)}$$

$$= \frac{-2-u}{u(1+u)}$$

$$= \frac{A}{u} + \frac{B}{1+u}$$

$$= \frac{-2}{u} + \frac{1}{1+u}$$

$$= -\frac{2}{u} + \sum_{n=0}^{\infty} (-1)^n u^n$$

$$= \frac{-2}{(z-2)} + \sum_{n=0}^{\infty} (-1)^n (z-2)^n$$