

MAT 215

TOPIC 4

ANTIDERIVATIVE

PART A

If $F'(z) = f(z)$ then $\int_{z_1}^{z_2} f(z) dz = F(z_2) - F(z_1)$

$$\text{Example: } \int_0^{2+i} z^3 dz = \left[\frac{z^4}{4} \right]_0^{2+i}$$

$$= \frac{1}{4} \left[(2+i)^4 - 0^4 \right]$$

$$= \frac{1}{4} (16 + 32i - 24 - 8i + 1)$$

Considered
 $i^2 = -1$

$$(x+y)^4 \\ = x^4 + 6x^3y + 10x^2y^2 \\ + 6xy^3 + y^4$$

$$= \frac{1}{4} (24i - 7)$$

Let a function f of a complex variable is continuous in a domain D . Then there exists a function F such that

$F'(z) = f(z)$, $\forall z \in D$ is called an antiderivative

of f . i.e. $\int f(z) dz = F(z)$.

Complex Valued function on Real Domain:

$\{ f: D \rightarrow \mathbb{C} \text{ while } D \subset \mathbb{R} \}$

↳ domain ↳ complex variable

$$f(t) = u(t) + iv(t), t \in [a, b]$$

Definite Integral

$$\int_a^b f(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt$$

Example 1. $f(t) = t^2 + 1 + it^3$; $0 \leq t \leq 1$

$$\begin{aligned} & \int_0^1 (t^2 + 1) dt + i \int_0^1 t^3 dt \\ &= \left[\frac{t^3}{3} + t \right]_0^1 + i \left[\frac{t^4}{4} \right]_0^1 \\ &= \frac{1}{3} + 1 + i \cdot \frac{1}{4} = \frac{4}{3} + \frac{1}{4}i \end{aligned}$$

2. $f(t) = e^{i2t}$; $0 \leq t \leq \pi/6$

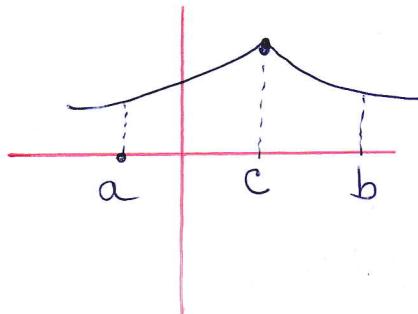
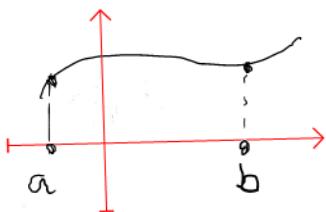
$$\begin{aligned} \int_0^{\pi/6} e^{i2t} dt &= \int_0^{\pi/6} \cos 2t dt + i \int_0^{\pi/6} \sin 2t dt \\ &= \left[\frac{\sin 2t}{2} \right]_0^{\pi/6} + i \left[-\frac{\cos 2t}{2} \right]_0^{\pi/6} \\ &= \frac{1}{2} \left[\sin 2\left(\frac{\pi}{6}\right) - \sin 2(0) \right] + i \left(-\frac{1}{2} \right) \left[\cos 2\left(\frac{\pi}{6}\right) - \cos 2(0) \right] \\ &= \frac{1}{2} \sin \frac{\pi}{3} - \frac{i}{2} \cos \frac{\pi}{3} + \frac{i}{2} \cos(0) \\ &= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{i}{2} \cdot \frac{1}{2} + \frac{i}{2} \cdot 1 \\ &= \frac{\sqrt{3}}{4} - i \frac{1}{4} + i \frac{1}{2} = \frac{\sqrt{3}}{4} + i \frac{1}{4}. \end{aligned}$$

Properties:

Let $f(t) = u(t) + iv(t)$, $g(t) = p(t) + iq(t)$
 $t \in [a, b]$;

$$\textcircled{i} \quad \int_a^b [f(t) \pm g(t)] dt = \int_a^b f(t) dt \pm \int_a^b g(t) dt$$

$$\textcircled{ii} \quad \int_a^b f(t) dt = \int_a^c f(t) dt + \int_c^b f(t) dt, \quad c \in [a, b]$$



{ when there is
no direct
path between
a and b.

$$\textcircled{iii} \quad \int_a^b f(t) dt = - \int_b^a f(t) dt$$

$$\begin{aligned} \textcircled{iv} \quad \int_a^b f(t) g(t) dt &= \int_a^b (u+iv)(p+iq) dt \\ &= \int_a^b (up + ivp + iq + i^2 vq) dt \\ &= \int_a^b (up - vq) dt + i \int_a^b (vp + uq) dt \end{aligned}$$

$$\textcircled{v} \quad \operatorname{Re} \int_a^b f(t) dt = \int_a^b \operatorname{Re}(f(t)) dt$$

$$\begin{aligned} \textcircled{vi} \quad \int_a^b z_0 f(t) dt &= \int_a^b (x_0 + iy_0)(u + iv) dt \\ &= \int_a^b (x_0 u + iy_0 u + ix_0 v + i^2 y_0 v) dt \\ &= \int_a^b [u(x_0 + iy_0) + iv(x_0 + iy_0)] dt \\ &= (x_0 + iy_0) \left[\int_a^b u dt + i \int_a^b v dt \right] \end{aligned}$$

complex
constant
 $z_0 = x_0 + iy_0$

(3)

vii Absolute Value of integral:

$$f(t) = u + iv$$

$$|f(t)| = |u + iv| = \sqrt{u^2 + v^2}$$

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt$$

Improper Integrals

while there is no direct path or there exist asymptote

$$f: \stackrel{\text{domain}}{D} \longrightarrow \stackrel{\text{complex Variable}}{C}, D \subset \mathbb{R}$$

$$f(t) = u(t) + iv(t), t \in \mathbb{R}$$

$$\int f(t) dt = \int u(t) dt + i \int v(t) dt$$

provided that the right side integral exist

All properties of real integrals hold true

Improper to Definite integral

$$\text{Let } U'(t) = u(t), V'(t) = v(t)$$

$$f(t) = u(t) + iv(t)$$

$$\text{Then } \int u(t) dt = U(t), \int v(t) dt = V(t)$$

$$\therefore \int f(t) dt = U(t) + iV(t)$$

$$\text{Then } \int_a^b f(t) dt = \left[U(t) + iV(t) \right]_a^b$$

$$= U(b) - U(a) + i(V(b) - V(a))$$

$$= U(b) + iV(b) - (U(a) + iV(a))$$

$$= F(b) - F(a)$$

$$\int f(t) dt = F(t)$$

Paths: arcs & curves

An arc C in complex plane

$z(x, y)$, $x = x(t)$, $y = y(t)$, $a \leq t \leq b$

- t is a free parameter
- t represents characteristic of an equation.

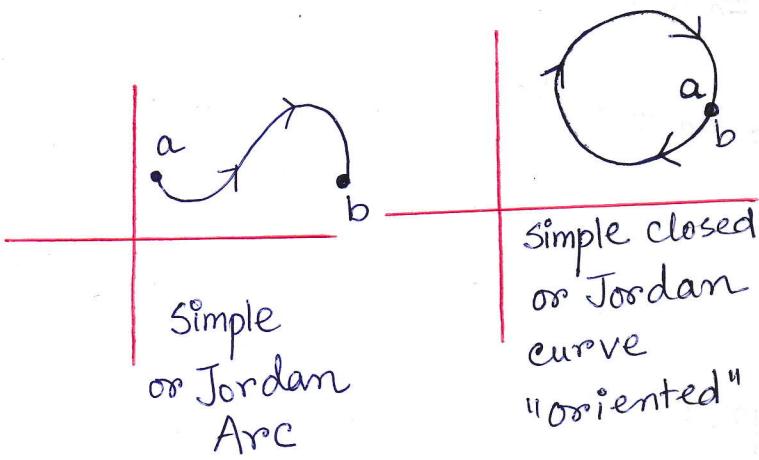
Eqn of arc : $z = z(t) = x(t) + iy(t)$, $a \leq t \leq b$ ex

$$x^2 + y^2 = r^2$$

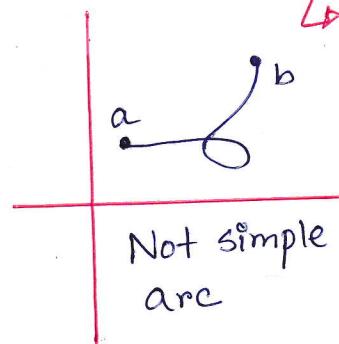
$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

↳ eqn of circle.

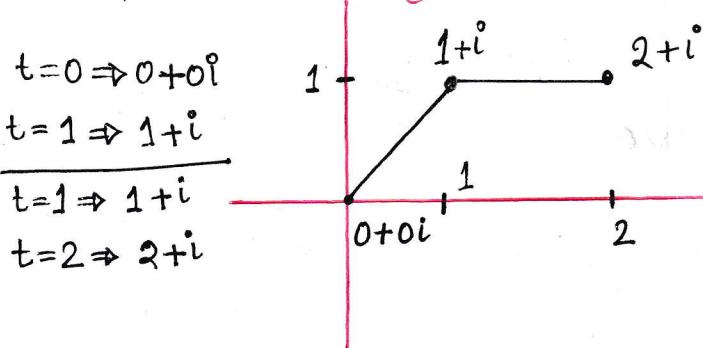


Simple closed or Jordan curve "oriented"

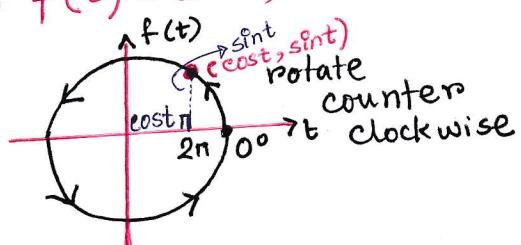


Examples

$$1. f(t) = \begin{cases} t + it, & 0 \leq t \leq 1 \\ t + i^t, & 1 \leq t \leq 2 \end{cases}$$

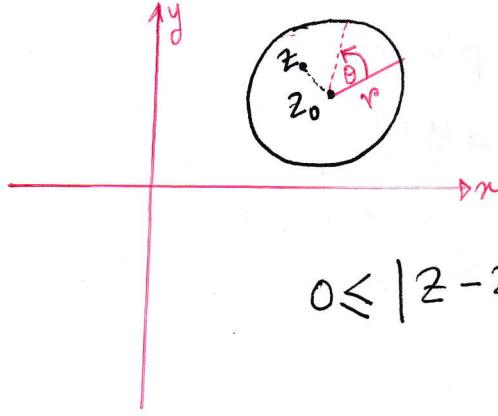


$$2. f(t) = e^{it}, 0 \leq t \leq 2\pi$$



$$e^{it} = \cos t + i \sin t$$

$$3. f(t) = z_0 + re^{it}, \quad 0 \leq t \leq 2\pi$$



$$0 \leq |z - z_0| \leq R \quad \text{by def}^n \text{ of neighbourhood}$$

$$\Rightarrow z - z_0 \leq re^{i\theta}$$

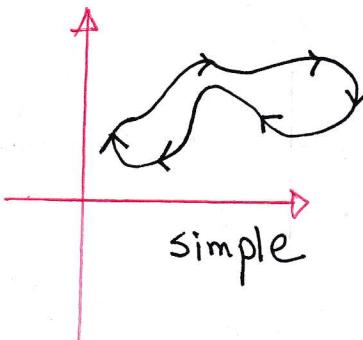
$$\Rightarrow z \leq z_0 + re^{i\theta}$$

$z_0 \rightarrow$ center
of circle

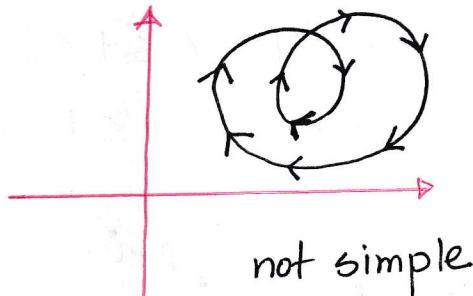
Curves

Consider parametric eqn:

$$z = z(t) = x(t) + iy(t); \quad t_1 \leq t \leq t_2$$



simple



not simple

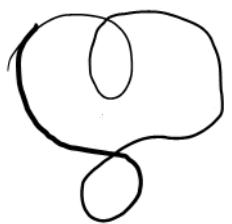
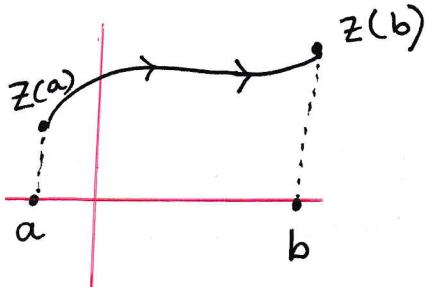
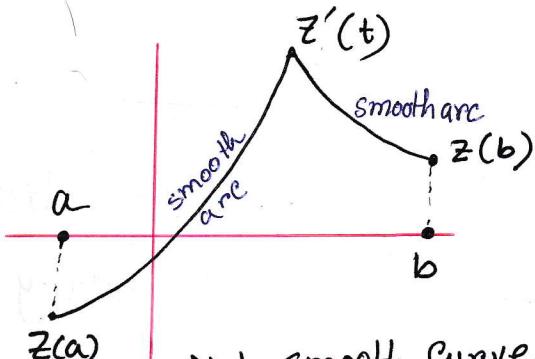


Figure: Closed Curve

A closed curve which does not intersect itself anywhere is called a simple closed curve.



smooth curve



Not smooth curve

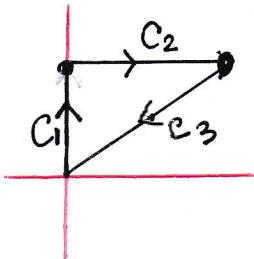
$z(t)$ or $(x(t), y(t))$
has continuous derivatives
in $a \leq t \leq b$.

It is a piecewise
or sectionally smooth
curve.

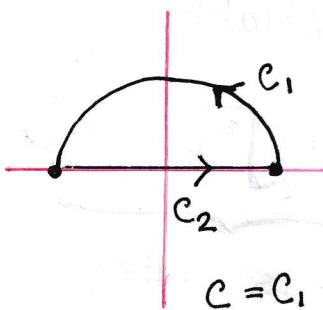
Smooth arc:
 $\rightarrow z(t)$ is smooth if $z'(t)$ is continuous and $z'(t) \neq 0$,
 $\forall t \in [a, b]$

Contour or Path

When C is constructed by joining finitely many smooth curves end to end is called a contour.



$$C = C_1 + C_2 + C_3$$



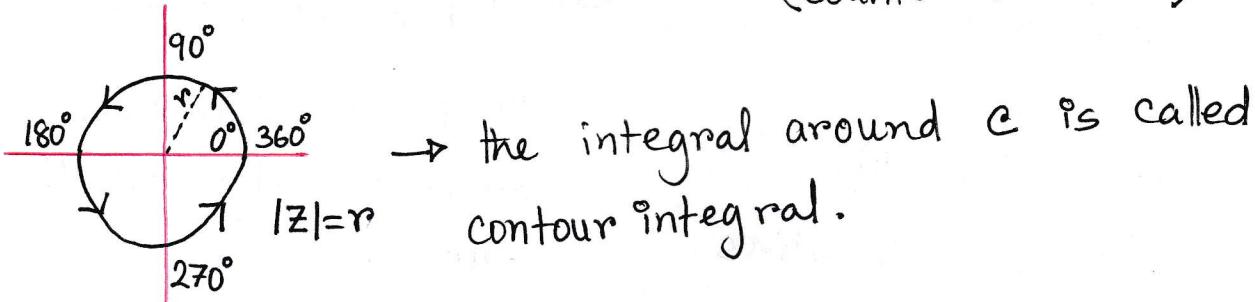
$$C = C_1 + C_2$$

$$C = C_1 + C_2 + C_3 + \dots + C_n$$

C could be a combination of n many smooth curves

Contour Integral

$\oint_C f(z) dz \rightarrow$ integration of $f(z)$ around the boundary C in a 've' sence.
(counter clockwise)



Length of Arc

Let $z = z(t) = x(t) + iy(t); a \leq t \leq b$

arc on curve

Suppose $x'(t)$ and $y'(t)$ exist and are continuous throughout the interval $a \leq t \leq b$

$\therefore z'(t) = x'(t) + iy'(t); a \leq t \leq b$

$\therefore z'(t)$ is differentiable over $[a, b]$

$$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

It is integrable on $[a, b]$

$$\begin{aligned} &|x'(t) + iy'(t)| \\ &\therefore z = x + iy \\ &\Rightarrow |z| = \sqrt{x^2 + y^2} \end{aligned}$$

$$\therefore \text{Length of arc } L = \int_a^b |z'(t)| dt$$

Recall MAT 120
Length of parametric curve

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

LINE Integral

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$

$$\int_{C_1 + C_2} f(z) dz = \int_{C_3} f(z) dz$$

$$= \int_{C_4} f(z) dz$$

$C_1 \leq C_2$

C_1, C_2 are independent path

Independent PATH

$C_1 + C_2 \geq C_3$

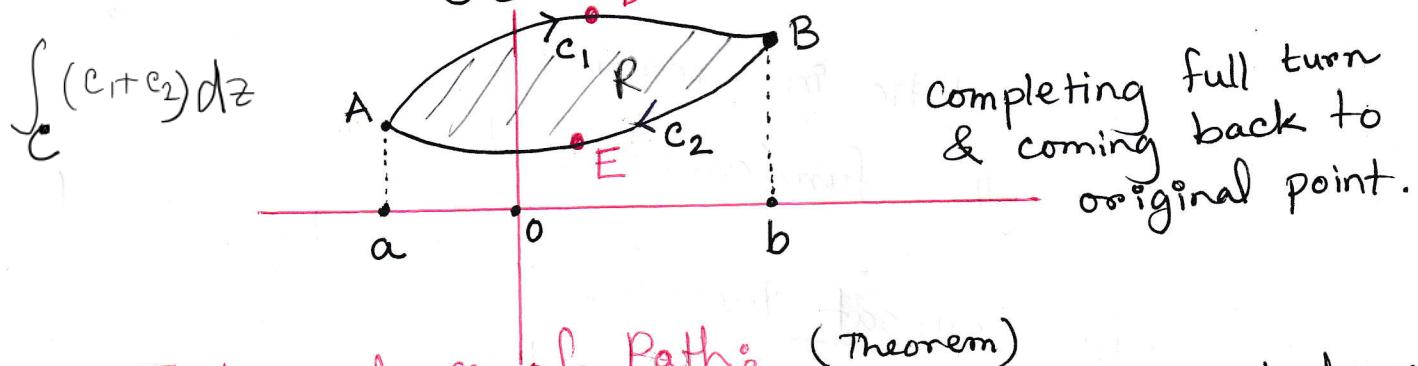
$C_1 + C_2, C_3, C_4$ are all independent of path

Cauchy's Theorem or Cauchy Goursat Theorem

If $f(z)$ is analytic everywhere within a simply connected region then $\int_C f(z) dz = 0$ for every

simple curve 'C' lying in the region R,

since $\oint_C f(z) dz = \int_{C_1 + C_2 + C_3 + \dots + C_n}$



Independence of Path: (Theorem)

If $f(z)$ is analytic in a simply connected region R, then the line integral $\int_a^b f(z) dz$ is independent of the path in R joining any two points "a" and "b" in R.

Proof

By Cauchy's Theorem:

$$\int_{ADB\rightarrow EA} f(z) dz = 0$$

$$\Rightarrow \int_{ADB} f(z) dz + \int_{BEA} f(z) dz = 0$$

$$\Rightarrow \int_{ADB} f(z) dz = - \int_{BEA} f(z) dz = \int_{AEB} f(z) dz$$

$$\Rightarrow \int_{C_1} f(z) dz = \int_{C_2} f(z) dz = \int_a^b f(z) dz$$

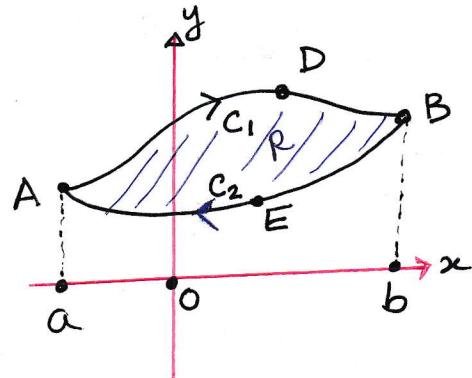
$C_1 \rightarrow$ heading to +ve direction

$C_2 \rightarrow$ heading to -ve direction

Hence the integral is independent of path.

Let $f(z)$ be analytic in a region 'R' and on its boundary 'C'. Then $\oint_C f(z) dz = 0$. This fundamental theorem often

called Cauchy-Goursat Theorem.



Integrals (Recall)

$$\omega(t) = u(t) + i v(t)$$

t is a real variable
 $\omega(t)$ is a complex valued function

$$\Rightarrow \omega'(t) = u'(t) + i v'(t)$$

$$\Rightarrow \frac{d}{dt} [z_0 \omega(t)] = z_0 \omega'(t) \quad z_0 = x_0 + i y_0$$

$$\Rightarrow \frac{d}{dt} [e^{z_0 t}] = z_0 e^{z_0 t}$$

$$\Rightarrow \int_a^b \omega(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt$$

$$\Rightarrow \int_a^b \omega(t) dt = \int_a^c \omega(t) dt + \int_c^b \omega(t) dt \quad \text{for } c \in [a, b]$$

Example 8
 1. Evaluate $\int_1^2 \left(\frac{1}{t} - i \right)^2 dt$

$$\int_1^2 \left(\frac{1}{t} - i \right)^2 dt$$

$$= \int_1^2 \left[\frac{1}{t^2} - 2 \cdot i \cdot \frac{1}{t} + i^2 \right] dt$$

$$= \int_1^2 \left(\frac{1}{t^2} - i \right) dt - 2i \int_1^2 \frac{1}{t} dt$$

$$= \left[-\frac{1}{t} - t \right]_1^2 - i \left[2 \ln t \right]_1^2$$

$$= \left[\left(-\frac{1}{2} - 2 \right) - \left(-1 - 1 \right) \right] - i \left[2 \ln 2 - 2 \ln 1 \right]$$

$$= -\frac{1}{2} - 2 + 2 - i \left(2 \ln \left(\frac{2}{1} \right) \right) = -\frac{1}{2} - i 2 \ln 2$$

$$2. \int_0^\infty e^{-zt} dt \quad (\operatorname{Re} z > 0)$$

$$= \lim_{l \rightarrow \infty} \int_0^l e^{-zt} dt \quad l \in \mathbb{C}$$

$$= \lim_{l \rightarrow \infty} \left[\frac{e^{-zt}}{-z} \right]_0^l$$

$$= \lim_{l \rightarrow \infty} \left[-\frac{e^{-zl}}{z} + \frac{e^0}{z} \right]$$

$$= -\frac{e^{-\infty}}{z} + \frac{1}{z}$$

$$= -\frac{1}{ze^\infty} + \frac{1}{z}$$

$$= -\frac{1}{\infty} + \frac{1}{z}$$

$$= 0 + \frac{1}{z}$$

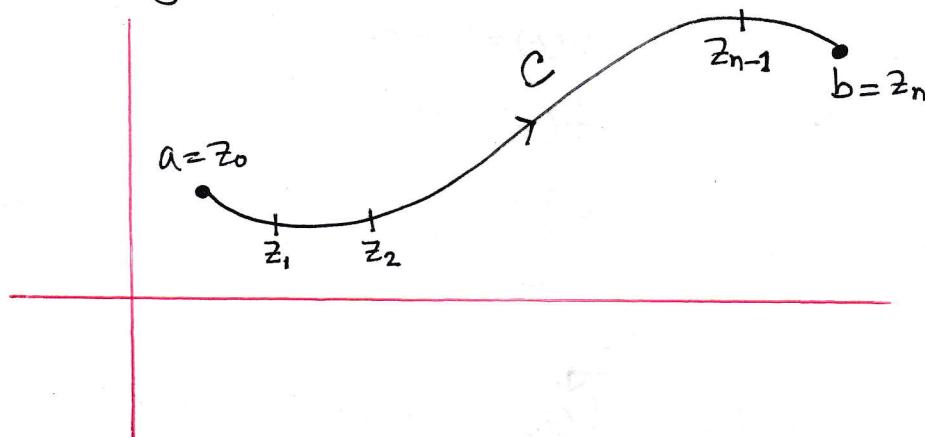
$$= \frac{1}{z}$$

Complex Line Integral

The definite integral of $f(z)$ from a to b along curve C is:

$$\int_a^b f(z) dz \quad \text{or} \quad \oint_C f(z) dz$$

It is also known as path integral, contour integral, curve integral.



$$\int_a^b f(z) dz = \oint_C f(z) dz = \int_{t_1}^{t_2} f(z(t)) z'(t) dt$$

$$f(z) = f(z(t)) \quad \frac{z = z(t)}{dz = z'(t) dt} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

(considering parametric eqn
 $z = z(t) = x(t) + iy(t)$)

Theorem \rightarrow PAGE 8

$$t_1 \leq t \leq t_2$$

$f(z)$ is continuous on C . Let $z(t)$ be any parametric function of C for $a \leq t \leq b$, then

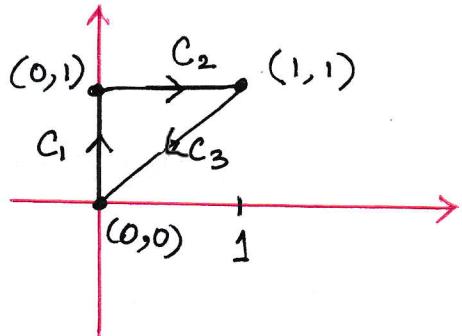
$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$ \rightarrow (It is known as chain rule)

$$= \int_a^b |z'(t)| dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \text{arc length}$$

Examples

① Evaluate $\int_C f(z) dz$ over the close triangular contour with vertices $(0,0)$, $(0,1)$, $(1,1)$ where

$$f(z) = y - x - 3ix^2$$



$$C_1 + C_2 \geq C_3 \quad (\text{triangular inequality})$$

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz + \left(-\int_{C_3} f(z) dz \right)$$

$$C_1: x=0 \Rightarrow z = iy \quad f(z) = y - x - 3ix^2 = y - 0 - 3i(0)^2 = y$$

$$dz = idy \quad f(z) = y$$

$$\int_{C_1} f(z) dz = \int_0^1 y idy = i \left[\frac{y^2}{2} \right]_0^1 = \frac{i}{2}$$

$$C_2: y=1 \Rightarrow z = x + iy \quad f(z) = y - x - 3ix^2 = 1 - x - 3ix^2$$

$$= x + i \quad dz = dx$$

$$\int_{C_2} f(z) dz = \int_0^1 (1 - x - 3ix^2) dx$$

$$= \left[x - \frac{x^2}{2} - 3i \frac{x^3}{3} \right]_0^1 = 1 - \frac{1}{2} - i = \frac{1}{2} - i$$

$$C_3: y=x \Rightarrow z = x + iy \quad f(z) = y - x - 3ix^2 = x - x - 3ix^2 = -3ix^2$$

$$= x + ix \quad f(z) = -3ix^2$$

$$z = (1+i)x \quad \int_{C_3} f(z) dz = \int_0^1 (-3ix^2)(1+i) dx$$

$$dz = (1+i)dx \quad = \int_0^1 (-3ix^2 - 3i^2x^2) dx$$

$$= \int_0^1 (-3ix^2 + 3x^2) dx$$

$$= \int_0^1 3x^2(1-i) dx$$

$$= 3(1-i) \left[\frac{x^3}{3} \right]_0^1$$

$$= (1-i)(1-0)$$

$$= 1 - i$$

$$\therefore \int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz - \int_{C_3} f(z) dz$$

$$= \frac{i}{2} + \frac{1}{2} - i - (1-i)$$

$$= \frac{i}{2} - \frac{1}{2} = \frac{1}{2}(-1+i)$$

② Evaluate $\int_C \bar{z} dz$ given by

(a) from $z=0$ to $z=4+2i$ along the curve C

$z = t^2 + it$ from $z=0$ to $z=2i$ and then the

(b) the line from $z=0$ to $z=4+2i$
line from $z=2i$ to $z=4+2i$

a) when $z=0$; $\rightarrow 0 = 0+0i = t^2 + it$
by equating factors of like terms: $t=0$

when $z=4+2i$; $4+2i = t^2 + it$
by equating factors of like terms: $t=2$

$\therefore 0 \leq t \leq 2$ or $t \in [0, 2]$

$$2t dt + i dt = (2t+i) dt$$

$$z = t^2 + it \implies dz = (2t+i) dt \xrightarrow{z'(t)}$$

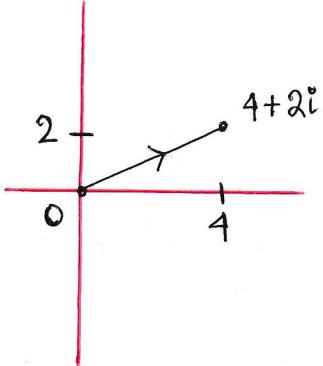
$$\bar{z} = t^2 - it = f(z(t)) \therefore f(z) = \bar{z} \text{ (given)}$$

$$\int_C f(z) dz = \int_C \bar{z} dz = \int_C f(z(t)) z'(t) dt$$

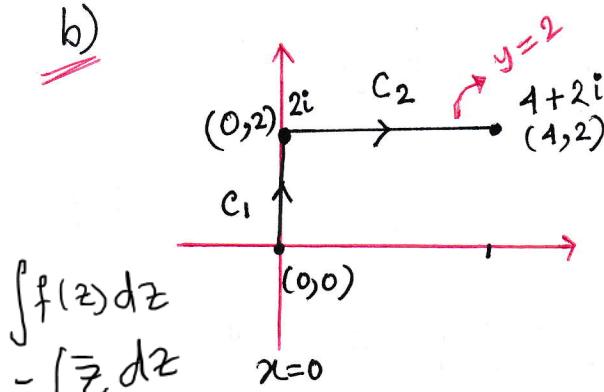
$$= \int_0^2 (t^2 - it)(2t+i) dt = \int_0^2 (2t^3 + it^2 - 2it^2 - i^2 t) dt$$

$$= \int_0^2 (2t^3 - it^2 + t) dt$$

$$= \left[\frac{2t^4}{4} - \frac{it^3}{3} + \frac{t^2}{2} \right]_0^2$$

$$= \frac{16}{2} - i \frac{8}{3} + \frac{4}{2} - 0 = 10 - \frac{8}{3}i$$


b)



$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$$

$$C_1: x=0, \Rightarrow z=x+iy \quad \bar{z} = -iy = f(z)$$

$$\Rightarrow \bar{z} = iy$$

$$dz = idy$$

$$0 \leq y \leq 2$$

$$\int_{C_1} \bar{z} dz = \int_{C_1} f(z) dz = \int_0^2 (-iy) idy = \int_0^2 -i^2 y dy = \left[\frac{y^2}{2} \right]_0^2 = 2$$

$$C_2: y=2, \quad z = x+iy \quad f(z) = \bar{z} = x-2i$$

$$z = x+2i \quad \int_{C_2} \bar{z} dz = \int_{C_2} f(z) dz = \int_0^4 (x-2i) dx$$

$$0 \leq x \leq 4 \quad dz = dx \quad = \left[\frac{x^2}{2} - 2ix \right]_0^4$$

$$\int_C f(z) dz = \int_C \bar{z} dz = \int_{C_1} dz + \int_{C_2} dz = 2 + 8 - 8i = 10 - 8i$$

③ Evaluate $\int \bar{z} dz$ where c is a right handed half of circle $|z|=2$ or $z=2e^{i\theta}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The contour is given by

$$|z|=2$$

$$\Rightarrow |z|=R$$

$$z = 2e^{i\theta}$$

$$\Rightarrow z = re^{i\theta}$$

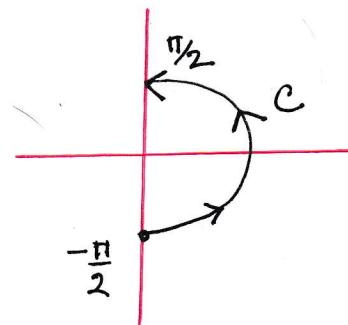
$$\bar{z} = 2e^{-i\theta}$$

$$\int_c f(z) dz$$

$$z = 2e^{i\theta}$$

$$= \int_c \bar{z} dz$$

$$\left\{ \begin{array}{l} \Rightarrow dz = 2i e^{i\theta} d\theta \\ \Rightarrow \int_c \bar{z} dz = \int_{-\pi/2}^{\pi/2} (2e^{-i\theta})(2i e^{i\theta} d\theta) \\ = 4i \left[\theta \right]_{-\pi/2}^{\pi/2} = 4i \left[\frac{\pi}{2} - (-\frac{\pi}{2}) \right] = 4\pi i \end{array} \right.$$



(Exercise Sheet #4)

① A counter clockwise oriented contour is called

a positively oriented contour.

Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)dx + (2y-x)dy$ along

- (a) the curve $y=x^2+1$
- (b) the straight line joining $(0,1)$ & $(2,5)$
- (c) the straight lines from $(0,1)$ to $(0,5)$ and then from $(0,5)$ to $(2,5)$.
- (d) the straight lines from $(0,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,5)$.

a)

$$y = x^2 + 1$$

$$dy = 2x \, dx$$

$$\int_{(0,1)}^{(2,5)} ((3x+y)dx + (2y-x)dy)$$

$$= \int_0^2 ((3x+x^2+1)dx + (2(x^2+1)-x)2x \, dx)$$

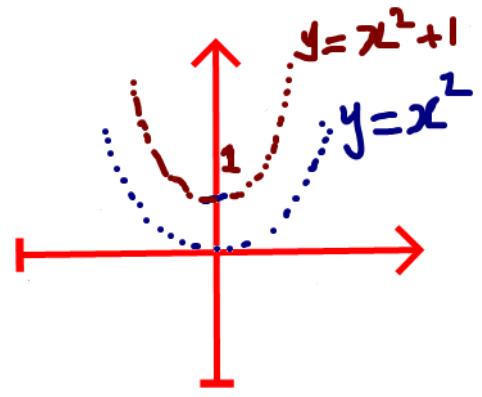
$$= \int_0^2 ((3x+x^2+1)dx + (2x^2+2-x)2x \, dx)$$

$$= \int_0^2 (3x+x^2+1 + 4x^3+4x-2x^2) \, dx$$

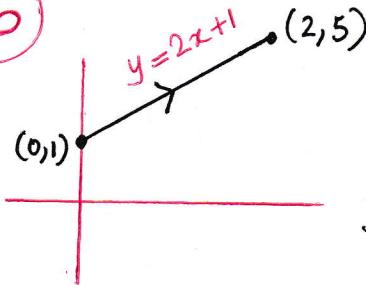
$$= \int_0^2 (4x^3-x^2+7x+1) \, dx$$

$$= \left[\frac{4x^4}{4} - \frac{x^3}{3} + \frac{7x^2}{2} + x \right]_0^2$$

$$= 16 - \frac{8}{3} + 7 \cdot \frac{4}{2} + 2 = 16 - \frac{8}{3} + 14 + 2 = \frac{88}{3}$$



b)



$$m = \frac{dy}{dx} = \frac{4}{2} = 2 \quad \int_{(0,1)}^{(2,5)} (3x+y)dx + (2y-x)dy$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - 0)$$

$$y = 2x + 1$$

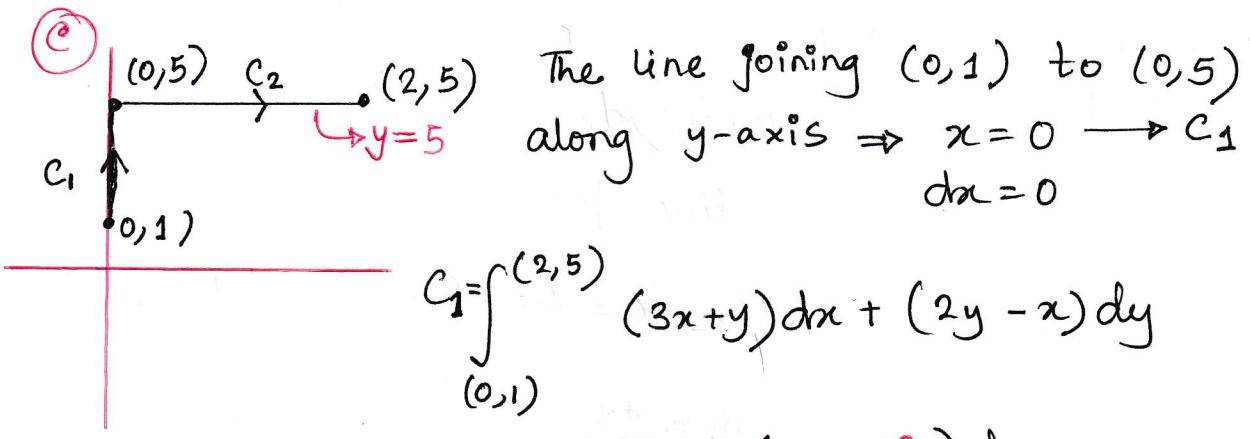
$$dy = 2 \, dx$$

$$= \int_0^2 ((3x+2x+1)dx + (2(2x+1)-x)2 \, dx)$$

$$= \int_0^2 ((5x+1)+(6x+4)) \, dx$$

$$= \int_0^2 (11x+5) \, dx = \left[11 \cdot \frac{x^2}{2} + 5x \right]_0^2$$

$$= 11 \cdot \frac{4}{2} + 5(2) - 0 = 32$$



$$\begin{aligned}
 C_1 &= \int_{(0,1)}^{(2,5)} (3x+y) dx + (2y-x) dy \\
 &= \int_1^5 (0+y) \cdot 0 + (2y-0) dy \\
 &= \int_1^5 2y dy = 2 \left[\frac{y^2}{2} \right]_1^5 = 25 - 1 = 24
 \end{aligned}$$

The line joining $(0,5)$ to $(2,5)$ is :

x -axis : $y = 5 \rightarrow C_2$

$$dy = 0$$

$$\begin{aligned}
 C_2 : & \int_{(0,1)}^{(2,5)} (3x+y) dx + (2y-x) dy \\
 &= \int_0^2 (3x+5) dx + (10-x)(0) \\
 &= \int_0^2 (3x+5) dx \\
 &= \left[\frac{3x^2}{2} + 5x \right]_0^2 = \frac{3}{2} \cdot 4 + 5(2) - 0 = 16
 \end{aligned}$$

$$\begin{aligned}
 \text{∴ The required value} &= \oint_C f(z) dz = \int_{C_1} \dots dz + \int_{C_2} \dots dz \\
 &= 24 + 16 = 40
 \end{aligned}$$

(Exercise Sheet #4)

6. Evaluate $\int_{i}^{2-i} (3xy + iy^2) dz$

a) along the straight line joining $z = i$ and $z = 2 - i$

b) along the parabola $x = 2t - 2$, $y = 1 + t - t^2$

a) $z = i \Rightarrow (0, 1)$ $m = \frac{dy}{dx}$

$$z = 2 - i \Rightarrow (2, -1) \quad = -\frac{2}{2} = -1$$

$$z = x + iy$$

$$y - y_1 = m(x - x_1)$$

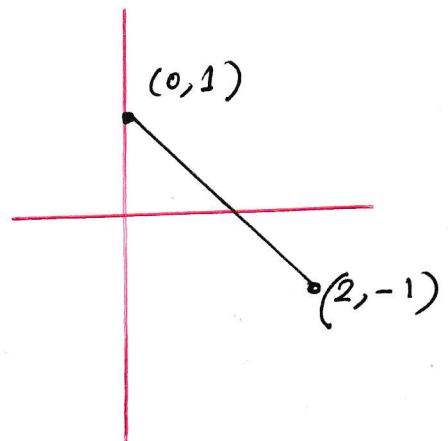
$$dz = dx + idy$$

$$y - 1 = -1(x - 0)$$

$$y - 1 = -x$$

$$y = -x + 1$$

$$dy = -dx$$



$$\int_i^{2-i} (3xy + iy^2) dz$$

$$\int_{(0,1)}^{(2,-1)} (3xy + iy^2) (dx + idy)$$

$$= \int_0^2 \left\{ 3x(-x+1) + i(-x+1)^2 \right\} \left\{ dx + i(-dx) \right\}$$

$$= \int_0^2 \left\{ -3x^2 + 3x + i(x^2 - 2x + 1) \right\} (1-i) dx$$

$$= \int_0^2 (-3x^2 + 3x + ix^2 - 2ix + i) (1-i) dx$$

$$= \left[-\frac{3x^3}{3} + \frac{3x^2}{2} + i\frac{x^3}{3} - 2i\frac{x^2}{2} + ix \right]_0^2$$

$$= (1-i) \left[-\frac{3x^3}{3} + \frac{3x^2}{2} + i\frac{x^3}{3} - 2i\frac{x^2}{2} + ix \right]_0^2$$

$$= (1-i) \left[-8 + 6 + i \frac{8}{3} - 4i + 2i - 0 \right]$$

$$= (1-i) \left(-8 + 6 + i \left(\frac{8-12+6}{3} \right) \right)$$

$$= (1-i) \left(-2 + \frac{2i}{3} \right) = -2 + 2i + \frac{2}{3}i - \frac{2}{3}i^2 = \frac{8i}{3} - \frac{4}{3}$$

$$= \frac{4}{3} (-1 + 2i)$$

(b) $x = 2t - 2$ where $0 \leq x \leq 2$

$$\therefore 0 = 2t - 2 \quad \begin{cases} 2 = 2t - 2 \\ t = 2 \end{cases}$$

$(2, -1)$

$$\int_{i}^{2-i}$$

$$(3xy + iy^2) dz$$

Given:

$$\begin{aligned} x &= 2t - 1 \\ y &= 1 + t - t^2 \end{aligned}$$

$$\therefore x \in [0, 2]$$

$$z = x + iy$$

$$z(t) = (2t-1) + i(-t^2+t+1)$$

$$z'(t) = 2 + i(1-2t) \quad \leftarrow$$

$$f(z) = 3xy + iy^2$$

$$f(z(t)) = 3(2t-1)(1+t-t^2) + i(1+t-t^2)$$

$$= (6t-3)(1+t-t^2) + i(1+t-t^2)^2$$

$$= (6t^3 + 9t^2 + 3t - 3) + i(1+t-t^2)^2$$

$$\oint_C f(z) dz$$

$$= \int_1^2 \underbrace{f(z(t))}_{\text{blue}} \underbrace{z'(t) dt}_{\text{pink}}$$

$$= \int_1^2 \left\{ (-6t^3 - 9t^2 + 3t - 3) + i(1+t-t^2)^2 \right\} (2+i(1-2t)) dt$$

$$= \int_1^2 \left\{ (-6t^3 - 9t^2 + 3t - 3) + i[(1+t)^2 - 2(1+t)t^2 + (t^2)^2] \right\} (2+i-2ti) dt$$

$$= \int_1^2 \left\{ (-6t^3 - 9t^2 + 3t - 3) + i[1 + 2t + t^2 - 2t^2 - 2t^3 + t^4] \right\} (2+i-2ti) dt$$

$$= \int_1^2 (-6t^3 - 9t^2 + 3t - 3 + i + 2ti + \underbrace{it^2 - i2t^2 - i2t^3 + it^4}_{-it^2}) (2+i-2ti) dt$$

$$= \int_1^2 \left[\begin{aligned} & -12t^3 + 24t^2 - 12 + 2i + 4ti - 2it^2 - i4t^3 + 2it^4 \\ & - 6it^3 + 12i^2t^2 - 6i + i^2 + 2t^2 - i^2t^2 - i^22t^3 + i^2t^4 \\ & - 12t^4i - 24t^3i + 12ti - 2ti^2 - 4t^2i^2 + 2t^3i^2 + 4t^4i^2 \end{aligned} \right] dt$$

$$= \int_1^2 \left[\begin{aligned} & -12t^3 + 29t^2 - 13 - 4i + 16ti + 10t^2i - 34t^3i - 10t^4i \\ & - 5t^4 + 2t^5 \end{aligned} \right] dt$$

$$= \left[-\frac{12}{4} t^4 + 29 \frac{t^3}{3} - 13t - 4it + \frac{16i}{2} \frac{t^2}{2} + 10i \frac{t^3}{3} \right. \\ \left. - \frac{17}{2} i \frac{t^4}{4} - 10i \frac{t^5}{5} - \frac{5t^5}{5} + 2 \frac{t^6}{6} \right]_1^2$$

$$= \left[-3t^4 + \frac{29}{3}t^3 - 13t - 4it - 8it^2 + \frac{10}{3}it^3 - \frac{17}{2}it^4 - 2it^5 \right. \\ \left. - t^5 + \frac{1}{3}t^6 \right]_1^2$$

$$= (-48 + 77.33 - 26 - 8i - 32i + 26.67i - 136i - 64i - 32) \\ + 21.33) - \left(-3 + \frac{29}{3} - 13 - 4i - 8i + \frac{10}{3}i - \frac{17}{2}i - 2i - 1 - \frac{1}{3} \right)$$

$$= -7.34 - 213.33i - (-7.67 - 19.17i)$$

$$= -7.34 - 213.33i + 7.67 + 19.17i$$

$$= 0.33 - 194.16i$$

There might be some minor calculation error in the process of basic integration but the concept of the application is correct in this sum.

(Exercise Sheet #4)

8. Evaluate $\int_C \frac{dz}{z-2}$ around (a) the circle $|z-2|=4$

(b) the circle $|z-1|=9$

$$(a) |z-2|=4$$

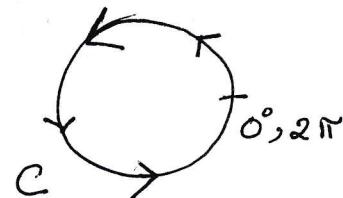
$$z-2 = 4e^{i\theta}$$

$$z = 2 + 4e^{i\theta}$$

$$dz = 4ie^{i\theta} d\theta$$

$$|z|=R$$

$$z=Re^{i\theta}$$



$$\int \frac{dz}{z-2} = \int_0^{2\pi} \frac{4ie^{i\theta}}{4e^{i\theta}} d\theta = i \int_0^{2\pi} e^{\circ} d\theta = i \int_0^{2\pi} 1 d\theta = i [0]^{2\pi} = 2\pi i$$

$$(b) |z-1|=9$$

$$z-1 = 9e^{i\theta} \quad \left\{ \begin{array}{l} |z|=R \\ z=re^{i\theta} \end{array} \right.$$

$$z = 1 + 9e^{i\theta}$$

$$dz = 9ie^{i\theta} d\theta$$

$$u = 9e^{i2\pi} - 1$$

$$= 9(\cos 2\pi + i \sin 2\pi) - 1$$

$$= 9(1 + i \cdot 0) - 1 = 8$$

$$\int_C \frac{dz}{z-2} = \int_C \frac{dz}{z-1-1}$$

$$= \int_0^{2\pi} \frac{9ie^{i\theta}}{9e^{i\theta}-1} d\theta$$

Let $u = 9e^{i\theta} - 1$
 $du = 9ie^{i\theta} d\theta$
 limits:
 $\theta=0 \Rightarrow u=8$
 $\theta=2\pi \Rightarrow u=8$

$$= \int_8^8 \frac{du}{u} = 0$$

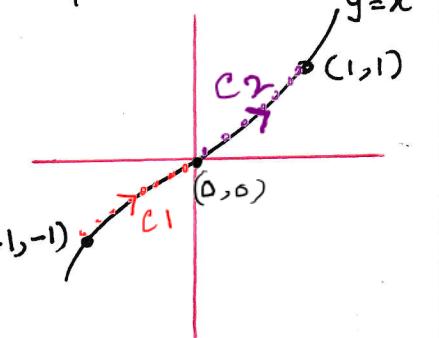
Example $f(z)$ is defined by equations

$$f(z) = \begin{cases} 1, & y < 0 \\ 4y, & y > 0 \end{cases} \quad c \text{ is the arc from } z = -1 - i \text{ to } z = 1 + i \text{ along the curve } y = x^3$$

$f(z)$ is changing the ^{concavity} at $y=0$ {^{so} the conditions are provided as $y > 0$ or $y < 0$ } which implies $y \neq 0$

$z = -1 - i \Rightarrow (-1, -1); z = 1 + i \Rightarrow (1, 1)$

$$\therefore \int_C f(z) dz = \int_{C_1}^{(0,0)} f(z) dz + \int_{C_2}^{(1,1)} f(z) dz$$



$$C_1: y = x^3, z = x + iy = x + ix^3 \quad \because y = x^3 \text{ over } -1 \leq x \leq 0.$$

\downarrow

$$(-1, -1) \text{ to } (0, 0) \quad dz = (1 + 3x^2i) dx \quad \text{when } y < 0$$

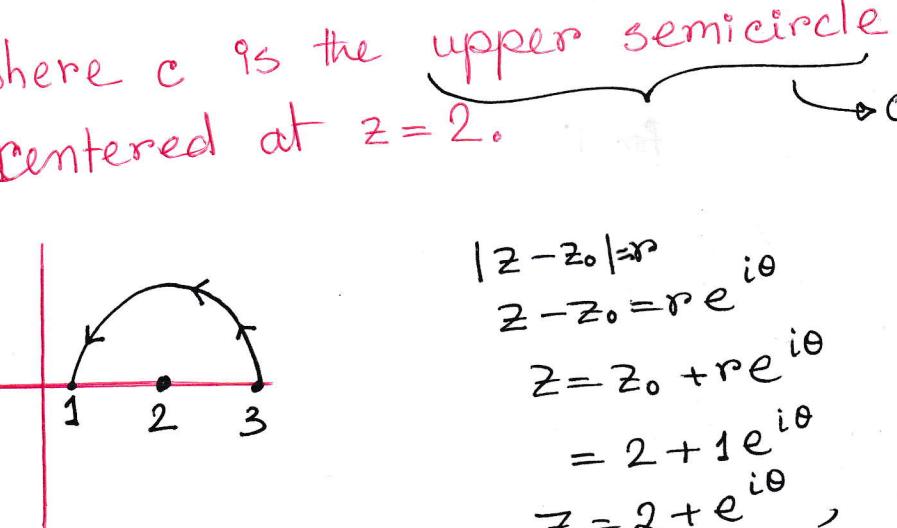
$$\begin{aligned} \int_{C_1} f(z) dz &= \int_{C_1} 1 dz = \int_{-1}^0 (1 + 3x^2i) dx \\ &= \left[x + 3 \frac{x^3}{3} i \right]_0^{-1} \\ &= -(-1 - i) = 1 + i \end{aligned}$$

$C_2 : (0,0) \text{ to } (1,1)$

$$\begin{aligned}\int_{C_2} f(z) dz &= \int_{C_2} 4y \, dz \\&= \int_0^1 4x^3 (1+3x^2 i) dx \\&\quad \begin{array}{l} \text{y = } x^3 \\ z = x + iy \\ = x + ix^3 \\ dz = (1+3x^2 i) dx \end{array} \\&= \int_0^1 (4x^3 + 12x^5 i) dx \\&= \left[\frac{4x^4}{4} + i \frac{12x^6}{6} \right]_0^1 \\&= \left[x^4 + 2i x^6 \right]_0^1 \\&= 1 + 2i\end{aligned}$$

$$\begin{aligned}\therefore \int_C f(z) dz &= \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \\&= 1 + i + 1 + 2i \\&= 2 + 3i\end{aligned}$$

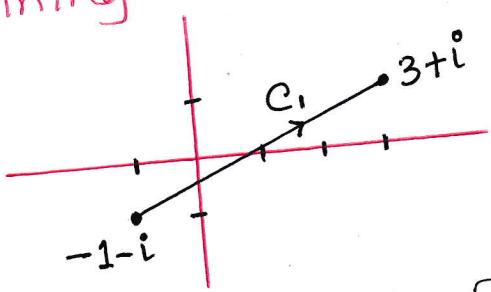
Example Evaluate the contour integral $\int_C \frac{dz}{z-2}$

where C is the upper semicircle with radius 1
centered at $z=2$. 

$$\begin{aligned} |z - z_0| &= r \\ z - z_0 &= r e^{i\theta} \\ z &= z_0 + r e^{i\theta} \\ &= 2 + 1 e^{i\theta} \\ z &= 2 + e^{i\theta}, \quad 0 \leq \theta \leq \pi \\ dz &= i e^{i\theta} d\theta \end{aligned}$$

$$\begin{aligned} \int_C \frac{1}{z-2} dz &= \int_0^\pi \frac{i e^{i\theta}}{2 + e^{i\theta} - 2} d\theta \\ &= \int_0^\pi i d\theta \\ &= i [\theta]_0^\pi = \pi i \end{aligned}$$

Example Show that $\int_{C_1} z dz = \int_{C_2} z dz = 4+2i$,
where C_1 is the line segment from $-1-i$ to $3+i$,
and C_2 is the portion of the parabola $x=y^2+2y$
going from $-1-i$ to $3+i$.



a) The line segment that contains the points

$$m = \frac{4y}{4x} = \frac{2}{4} = \frac{1}{2}$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{1}{2}(x - 3)$$

$$y = 1 + \frac{x}{2} - \frac{3}{2}$$

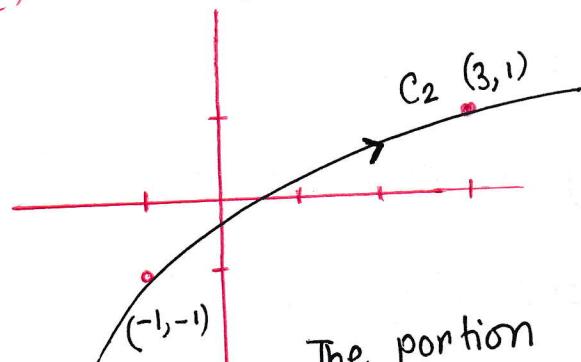
$$y = \frac{1}{2}x - \frac{1}{2}$$

OR $x = 2y + 1$

$$\text{along } C_1 : z = x + iy = 2y + 1 + iy$$
$$dz = (2 + i)dy$$

$$\begin{aligned}\int_{C_1} zdz &= \int_{-1}^1 (2y + 1 + iy)(2 + i)dy \\ &= \int_{-1}^1 (4y + 2 + 2iy + 2iy + i - y)dy \\ &= \int_{-1}^1 (3y + 4iy + i + 2)dy \\ &= \left[\frac{3y^2}{2} + 4iy^2 + iy + 2y \right]_{-1}^1 \\ &= \frac{3}{2} + 2i + i + 2 - \left(\frac{3}{2} + 2i - i - 2 \right) \\ &= 4 + 2i\end{aligned}$$

(b) similarly along C_2



$$z = x + iy$$

$$= (y^2 + 2y) + iy \quad \text{given} \\ x = y^2 + 2y$$

$$dz = (2y + 2 + i)dy$$

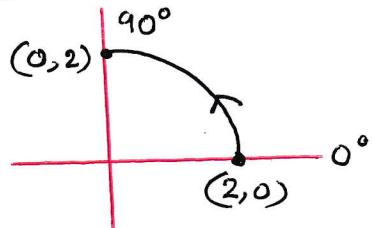
$$\begin{aligned} \int_{C_2} z dz &= \int_{-1}^1 [(y^2 + 2y) + iy](2y + 2 + i) dy \\ &= \int_{-1}^1 (2y^3 + 4y^2 + 2iy^2 + 2y^2 + 4y + 2iy + iy^2 + 2iy - y) dy \\ &= \int_{-1}^1 (2y^3 + 6y^2 + 3iy^2 + 3y + 4iy) dy \\ &= \left[\frac{2y^4}{4} + \frac{6y^3}{3} + 3i \cdot \frac{y^3}{3} + \frac{3y^2}{2} + 4i \cdot \frac{y^2}{2} \right]_{-1}^1 \\ &= \frac{1}{2} + 2 + i + \frac{3}{2} + 2i - \left(\frac{1}{2} - 2 - i + \frac{3}{2} + 2i \right) \\ &= 4 + 2i \end{aligned}$$

This outcome does not hold in general for arc length while C_1, C_2 are independent of paths.

(Exercise Sheet #4)

5. Evaluate $\int_C (z^2 + 3z) dz$

a) along the circle $|z|=2$ from $(2,0)$ to $(0,2)$ in a counter clockwise direction.



$$|z|=2 \Rightarrow z=2e^{i\theta}$$

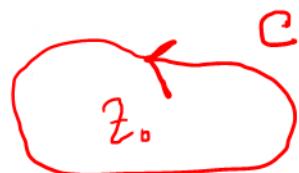
$$\therefore dz = 2ie^{i\theta} d\theta$$

$$\begin{aligned}
 \int_C (z^2 + 3z) dz &= \int_0^{\pi/2} (4e^{2i\theta} + 6e^{i\theta}) 2ie^{i\theta} d\theta \\
 &= 2i \int_0^{\pi/2} (4e^{3i\theta} + 6e^{2i\theta}) d\theta \\
 &= 2i \left[\frac{4e^{3i\theta}}{3i} + \frac{6e^{2i\theta}}{2i} \right]_0^{\pi/2} \\
 &= 2 \left[\frac{4}{3} e^{3i\theta} + 3e^{2i\theta} \right]_0^{\pi/2} \\
 &= 2 \left[\left(\frac{4}{3} e^{i\frac{3\pi}{2}} + 3e^{i\pi} \right) - \left(\frac{4}{3} e^0 + 3e^0 \right) \right] \\
 &= 2 \left[\left\{ \frac{4}{3} \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) + 3(\cos \pi + i \sin \pi) \right\} \right. \\
 &\quad \left. - \left\{ \frac{4}{3} + 3 \right\} \right] \\
 &= 2 \left[\frac{4}{3} (-i) + 3(-1 - 0) - \frac{4}{3} - 3 \right] \\
 &= 2 \left[-\frac{4}{3}i - 3 - \frac{13}{3} \right] \\
 &= 2 \left[-\frac{4}{3}i - \frac{22}{3} \right] = -\frac{44}{3} - \frac{8}{3}i
 \end{aligned}$$

Cauchy Integral Formula

Suppose $f(z)$ is analytic everywhere on and within a closed (+vely oriented) contour C , (counter clockwise).

If z_0 is any arbitrary but fixed point inside the contour C , then



$$(i) f(z_0) = \frac{1}{2\pi i} \oint_C \left(\frac{f(z)}{z-z_0} \right) dz$$

The n th derivative of $f(z)$ at $z=z_0$ is:

$$(ii) a) f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz, n=1, 2, 3, \dots$$

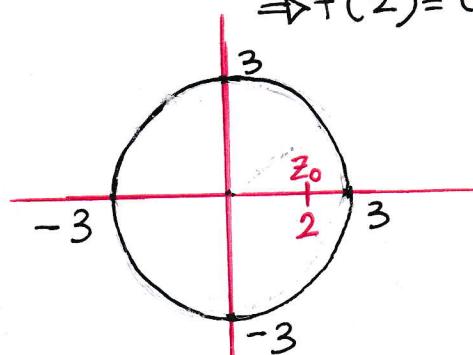
$$b) \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Examples ① Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz$ if C is

(a) the circle $|z|=3$, (b) the circle $|z|=1$

(a) $|z|=3$, $z_0=2$ is inside the circle and $f(z)=e^z$

① formula
 $\therefore \frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz = f(2) = e^2$

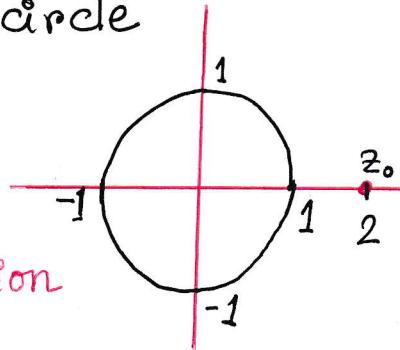


(b) $|z|=1$, $z_0 = 2$ is outside the circle

formula ①

$$\therefore \frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz = 0$$

$\because z_0$ is outside region



(Exercise Sheet # 4) $f(z)$

3. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i} dz$ where C is the circle $|z-1|=4$

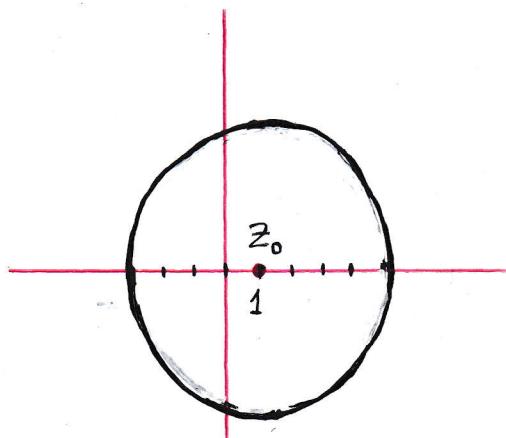
$$f(z_0) = \frac{1}{2\pi i} \oint_C \left(\frac{f(z)}{z-z_0} \right) dz$$

$$\begin{aligned} \oint_C \frac{e^{3z}}{z-\pi i} dz &= 2\pi i f(\pi i) \\ &= 2\pi i (-1) \\ &= -2\pi i \end{aligned}$$

— formula(i)

$$\begin{aligned} f(z) &= e^{3z} \\ f(\pi i) &= e^{3\pi i} \end{aligned}$$

$$\begin{aligned} &= \cos 3\pi + i \sin 3\pi \\ &= -1 \end{aligned}$$

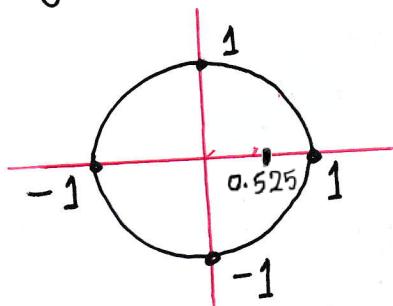


Example Find the values of

(a) $\int_C \frac{\sin^6 z}{z - \frac{\pi}{6}} dz$ (b) $\int \frac{\sin^6 z}{(z - \frac{\pi}{6})^3} dz$

if C is the circle $|z|=1$

(a) $z_0 = \frac{\pi}{6}$ which is inside the circle as $\frac{\pi}{6} = \frac{3.14}{6} = 0.525$



$$0.525 \in |z|=1$$

$$\int_C \frac{\sin^6 z}{z - \frac{\pi}{6}} dz = 2\pi i f(\pi/6)$$

formula (i)

$$\therefore f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

$$\Rightarrow \int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

$$f(z) = \sin^6 z = (\sin z)^6$$

$$\therefore f(\pi/6) = \left(\sin \frac{\pi}{6}\right)^6 = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$\text{from (i)} \quad \int_C \frac{\sin^6 z}{z - \frac{\pi}{6}} dz = 2\pi i \frac{1}{64} = \frac{\pi i}{32}$$

$$(b) \int \frac{\sin^6 z}{(z - \frac{\pi}{6})^{2+1}} dz = \frac{2\pi i}{2!} f''(\pi/6) \quad \text{formula (ii)}$$

$$= \pi i f''(\pi/6)$$

$$\begin{aligned} \therefore \int \frac{f(z)}{(z - z_0)^{n+1}} dz \\ &= \frac{2\pi i}{n!} f^{(n)}(z_0) \end{aligned}$$

$$f(z) = \sin^6 z$$

$$f'(z) = 6\sin^5 z \cos z$$

$$\begin{aligned} f''(z) &= 30 \sin^4 z \cos z (\cos z) + 6 \sin^5 z (-\sin z) \\ &= 30 \sin^4 z \cos^2 z - 6 \sin^6 z \end{aligned}$$

$$\begin{aligned} f''\left(\frac{\pi}{6}\right) &= 30 \left(\sin \frac{\pi}{6}\right)^4 \left(\cos \frac{\pi}{6}\right)^2 - 6 \left(\sin \frac{\pi}{6}\right)^6 \\ &= \frac{21}{16} \end{aligned}$$

Now from (ii) we have:

$$\begin{aligned} \int \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^{2+1}} dz &= \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right) \\ &= \pi i \cdot \frac{21}{16} \\ &= \frac{21\pi i}{16} \end{aligned}$$

(Exercise Sheet #4) (Cauchy's Integral Formula)

4. Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z^2+1)^2} dz$ if $t > 0$ and C is

the circle $|z| = 3$.

$$\because z^2 + 1 = z^2 - i^2 = (z-i)(z+i)$$

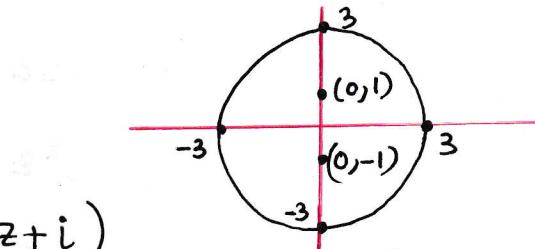
$\underbrace{z=i}, \underbrace{z=-i}$

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

$$= \frac{A}{z+i} + \frac{B}{z-i}$$

$$= \frac{-\frac{1}{2}i}{z+i} + \frac{\frac{1}{2}i}{z-i}$$

$$= \frac{1}{2i} \left(\frac{1}{z-i} - \frac{1}{z+i} \right)$$



Partial fraction of rational function:

$$\frac{1}{(z-i)(z+i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

$$1 = A(z-i) + B(z+i)$$

Equating Factors of like terms:

$$Az + Bi = 0$$

$$A + Bi = 0$$

$$-Ai + Bi = 1$$

$$-A + B = \frac{1}{i}$$

$$2A = -\frac{1}{i} \quad \rightarrow B = \frac{1}{2i}$$

$$A = -\frac{1}{2i}$$

$$\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2+1} dz = \frac{1}{2\pi i} \oint_C \frac{1}{2i} \left[\frac{-1}{z+i} + \frac{1}{z-i} \right] e^{zt} dz$$

$$= \frac{1}{4\pi i^2} \left(\oint_C -\frac{e^{zt}}{z+i} dz + \oint_C \frac{e^{zt}}{z-i} dz \right)$$

$\downarrow z_0 = -i$
 $\downarrow z_0 = i$
both z_0 's are inside the circle

$$f(z_0) = \frac{1}{2\pi i} \oint_C \left(\frac{f(z)}{z-z_0} \right) dz = \frac{1}{4\pi i^2} [-2\pi i f(-i) + 2\pi i f(i)] \quad \text{by Thm ①}$$

$$= \frac{2\pi i}{4\pi i^2} [-f(-i) + f(i)]$$

$$\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2+1} dz = \frac{1}{2i} (-f(-i) + f(i))$$

$$\therefore \frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z^2+1)^{1+1}} dz = \frac{1}{2i} (-f'(-i) + f'(i)) \quad \text{by Thm ②}$$

$$f(z) = e^{zt}$$

$$f'(z) = te^{zt}$$

$$f'(-i) = -te^{-it}$$

$$= -t(\cos t - i \sin t)$$

$$f'(i) = te^{it}$$

$$= t(\cos t + i \sin t)$$

$$= \frac{1}{2i} [t \cos t + it \sin t + t \cos t + it \sin t]$$

$$= \frac{1}{2i} [2it \sin t]$$

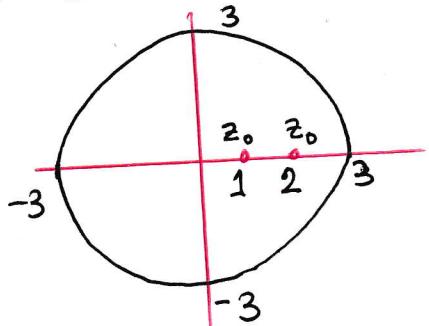
$$= t \sin t$$

(Exercise sheet #4)

1. Evaluate (a) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$

$$(b) \oint_C \frac{e^{2z}}{(z+1)^4} dz$$

where C is the circle $|z| = 3$



$$a) \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

Partial fraction of rational function.

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C \left(\frac{1}{z-2} - \frac{1}{z-1} \right) (\sin \pi z^2 + \cos \pi z^2) dz$$

$$f(z) = \sin \pi z^2 + \cos \pi z^2$$

$$f(2) = \sin \pi 2^2 + \cos \pi 2^2$$

$$= 1$$

$$f(1) = \sin \pi 1^2 + \cos \pi 1^2$$

$$= -1$$

$$= 2\pi i f(2) - 2\pi i f(1)$$

$$= 2\pi i [1 - (-1)]$$

$$= 4\pi i$$

$$(b) \int_C \frac{e^{2z}}{(z+1)^4} dz, \quad z_0 = -1$$

$$\int \frac{e^{2z}}{(z-(-1))^{3+1}} dz$$

$$\int \frac{e^{2z}}{(z-(-1))^{3+1}} dz = \frac{2\pi i}{3!} f'''(-1) \quad f(z) = e^{2z}$$

$$= \frac{2\pi i}{6} \cdot \frac{8}{e^2} \quad f'(z) = 2e^{2z}$$

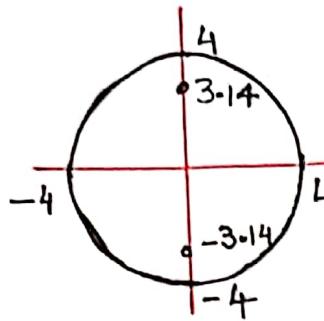
$$= \frac{8\pi i}{3e^2} \quad f''(z) = 4e^{2z}$$

$$f'''(z) = 8e^{2z}$$

$$f'''(-1) = 8e^{-2} = \frac{8}{e^2}$$

(Exercise Sheet # 4 Cauchy's Integral Formula)

[2] Evaluate $\int_C \frac{e^z}{(z^2 + \pi^2)^2} dz$ where C is the circle $|z|=1$.



$$\pi = 3.14$$

$$\left\{ \begin{array}{l} z = \pm \pi i \\ \text{inside } C \end{array} \right.$$

$$\begin{aligned} \frac{1}{(z^2 + \pi^2)^2} &= \frac{1}{(z^2 - i^2 \pi^2)^2} = \frac{1}{[(z + \pi i)(z - \pi i)]^2} \\ &= \frac{1}{(z + \pi i)^2(z - \pi i)^2} \end{aligned}$$

$$\begin{aligned} \frac{1}{z^2 + \pi^2} &= \frac{1}{(z + \pi i)(z - \pi i)} \quad \frac{1}{(z + \pi i)(z - \pi i)} = \frac{A}{z + \pi i} + \frac{B}{z - \pi i} \\ &= \frac{A}{z + \pi i} + \frac{B}{z - \pi i} \quad 1 = A(z - \pi i) + B(z + \pi i) \\ &= -\frac{1}{2\pi i} + \frac{1}{2\pi i} \quad 1 = Az - A\pi i + Bz + B\pi i \\ &= \frac{1}{2\pi i} \left(-\frac{1}{z + \pi i} + \frac{1}{z - \pi i} \right) \quad \text{Equating factors of like terms:} \\ &\quad Az + Bz = 0 \quad -A\pi i + B\pi i = 1 \\ &\quad A + B = 0 \quad -A + B = \frac{1}{\pi i} \\ &\quad 2B = \frac{1}{\pi i} \quad \therefore A + B = 0 \\ &\quad B = \frac{1}{2\pi i} \quad \Rightarrow A + \frac{1}{2\pi i} = 0 \\ &\quad \Rightarrow A = -\frac{1}{2\pi i} \end{aligned}$$

$$\begin{aligned} \oint \frac{e^z}{(z^2 + \pi^2)} dz &= \oint_C \frac{1}{2\pi i} \left(-\frac{1}{z + \pi i} + \frac{1}{z - \pi i} \right) e^z dz \\ &= \frac{1}{2\pi i} \left(\int_C -\frac{e^z}{z + \pi i} dz + \int_C \frac{e^z}{z - \pi i} dz \right) \quad z_0 = \pi i \\ &\quad \hookrightarrow z_0 = -\pi i \\ &= \frac{1}{2\pi i} [2\pi i f(-\pi i) + 2\pi i f(\pi i)] \end{aligned}$$

$$\oint \frac{e^z}{z^2 + \pi^2} dz = f(-\pi i) + f(\pi i)$$

$$\begin{aligned}\therefore \oint_C \frac{e^z}{(z^2 + \pi^2)^{1+1}} dz &= \frac{2\pi i}{1!} \left[-f'(-\pi i) + f'(\pi i) \right] \\ &= 2\pi i \left[-(-1) + (1) \right]\end{aligned}$$

$$f(z) = e^z = 0$$

$$f'(z) = e^z$$

$$\begin{aligned}f'(-\pi i) &= e^{-\pi i} \\ &= \cos(-\pi) + i \sin(-\pi)\end{aligned}$$

$$= \cos \pi - i \sin \pi$$

$$= -1 - i \cdot 0$$

$$= -1$$

$$f'(\pi i) = e^{\pi i}$$

$$= \cos \pi + i \sin \pi$$

$$= -1 + i \cdot 0$$

$$= -1$$