MAT 110 & 120 REVIEW

Parametric Equation

Examples $x(t) = x, \quad y(t) = y$

① $x = 2t, \quad y = t^2 \rightarrow$ } • pair of these together called parametric eqns
• t = parameter

$x = 2t$ — (i)

$y = t^2$ — (ii)

from (i) $t = \frac{x}{2}$ — (iii)

Substitute (iii) into (ii) $\Rightarrow y = t^2$

$\Rightarrow y = \left(\frac{x}{2}\right)^2$

$\Rightarrow y = \frac{x^2}{4}$

↳ This is a Cartesian eqn

② $x = \sin\theta + 2, \quad y = \cos\theta - 3$

Say, if $x = 2$

then $x = \sin\theta + 2$

$\Rightarrow 2 = \sin\theta + 2$

$\Rightarrow \sin\theta = 0$

$\Rightarrow \theta = \sin^{-1} 0 = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

if $\theta = 0$ then $y = \cos(0) - 3 = -2$

if $\theta = \pi$ then $y = \cos\pi - 3 = -4$

etc.

$x = \sin\theta + 2$ — (i)

$\sin\theta = x - 2$

$\sin^2\theta = (x-2)^2$ — (i)

$y = \cos\theta - 3$ — (ii)

$\cos\theta = y + 3$

$\cos^2\theta = (y+3)^2$ — (ii)

(i) + (ii) $\Rightarrow \sin^2\theta + \cos^2\theta = (x-2)^2 + (y+3)^2$

$1 = (x-2)^2 + (y+3)^2 \longrightarrow \theta$ is eliminated

Cartesian eqn (eqn of circle)

$(h, k) = (2, -3), \quad r = 1$

Differentiation

$x = 2t, \quad y = t^2 \qquad \frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ (Chain Rule)

$= 2t \cdot \frac{1}{2} = t$

Arc Length of Parametric Curve

Over the interval $t \in [a, b]$ the arc length of parametric curve $x = x(t)$, $y = y(t)$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

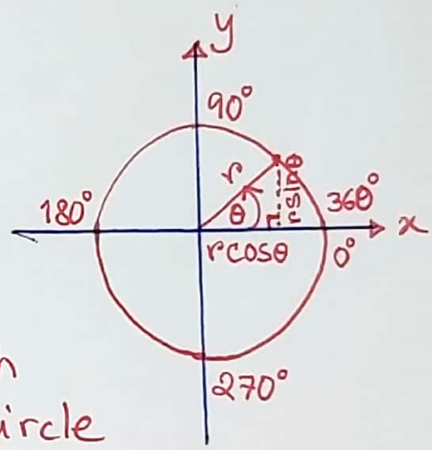
Ex: $x(\theta) = \cos\theta$, $y(\theta) = \sin\theta$, $0 \le \theta \le 2\pi$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{\sin^2\theta + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1}\, d\theta$$

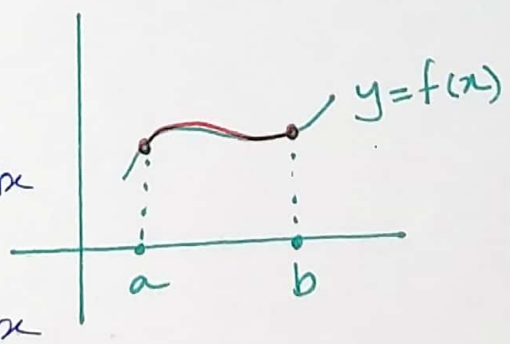
$$= [\theta]_0^{2\pi} = 2\pi$$ = arc length of full circle



On x-axis:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx$$

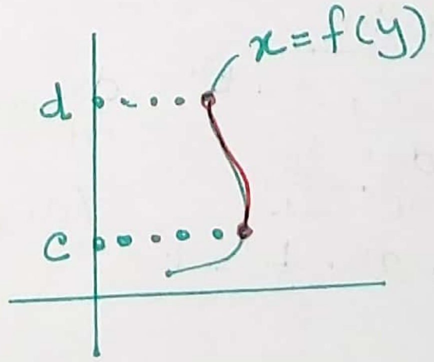
$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



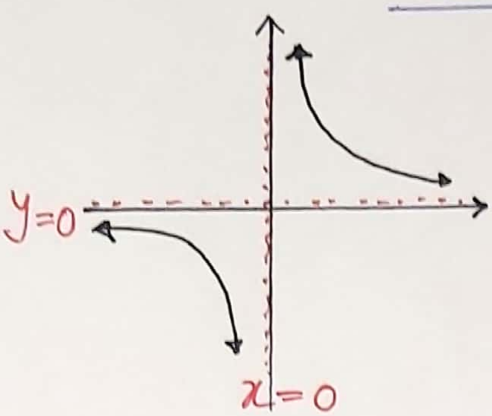
On y-axis:

$$L = \int_c^d \sqrt{\left(\frac{dy}{dy}\right)^2 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$



Improper Integral



$y = \frac{1}{x}$ continuous if $x \in [1, +\infty)$
or $x \in (-\infty, -1]$
discontinuous at $x = 0$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{l \to \infty} \int_1^l \frac{1}{x}\,dx$$

⋮

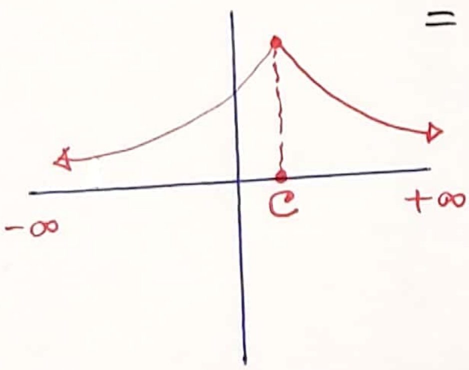
4 possible cases of improper integral:

- $\int_a^\infty f(z)\,dz = \lim_{l \to \infty} \int_a^l f(z)\,dz$

- $\int_{-\infty}^b f(z)\,dz = \lim_{K \to -\infty} \int_K^b f(z)\,dz$

- $\int_{-\infty}^\infty f(z)\,dz = \int_{-\infty}^c f(z)\,dz + \int_c^{+\infty} f(z)\,dz$

$$= \lim_{K \to -\infty} \int_K^c f(z)\,dz + \lim_{l \to \infty} \int_c^l f(z)\,dz$$



- $\int_a^3 \frac{1}{z-3}\,dz, \quad z \neq 3$

$$= \lim_{K \to 3} \int_a^K \frac{1}{z-3}\,dz$$