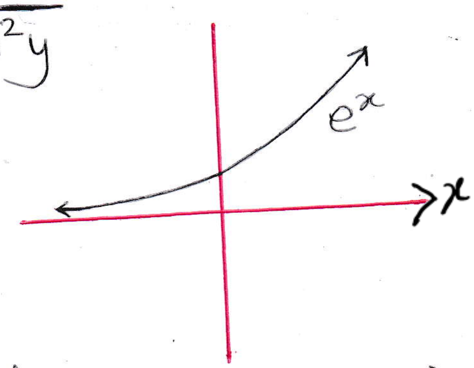
MAT 215 TOPIC 3

EXPONENTIAL FUNCTION

$$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$$
$$= e^x(\cos y + i\sin y)$$

$$|f(z)| = |e^z| = |e^x||\cos y + i\sin y|$$
$$= |e^x|\sqrt{\cos^2 y + \sin^2 y}$$
$$= |e^x|\sqrt{1}$$
$$= e^x$$



$x \in (-\infty, +\infty)$

$|e^z| \neq 0$ for any complex number

$\because |e^z| = e^x > 0 \quad \forall x \in \mathbb{R}$

Properties of exponential function

1. $e^{z_1} e^{z_2} = e^{z_1+z_2}$

<u>Proof</u> let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$$e^{z_1} e^{z_2} = e^{x_1+iy_1} e^{x_2+iy_2}$$
$$= e^{x_1} e^{iy_1} e^{x_2} e^{iy_2}$$
$$= e^{x_1+iy_1+x_2+iy_2}$$
$$= e^{(x_1+iy_1)+(x_2+iy_2)}$$
$$= e^{z_1+z_2}$$

2. $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

3. $\frac{1}{e^z} = e^{-z}$

4. $(e^z)^n = e^{nz}$, (n is an integer
i.e. $n = 0, \pm 1, \pm 2, \pm 3, \cdots$)

5. $e^{z+2\pi i} = e^z e^{2\pi i}$

$= e^z (\cos 2\pi + i \sin 2\pi)$

$= e^z (1 + i \cdot 0)$

$= e^z$

$\therefore e^{z+2\pi i} = e^z$

Example

Show that $e^{(2 \pm 3\pi i)} = -e^2$

Proof $e^{2+3\pi i} = e^2 e^{3\pi i}$

$= e^2 (\cos 3\pi + i \sin 3\pi)$

$= e^2(-1+0) = -e^2$

$e^{2-3\pi i} = e^2 e^{-3\pi i}$

$= e^2 [\cos(-3\pi) + i \sin(-3\pi)]$

$= e^2 (\cos 3\pi - i \sin 3\pi)$

$= e^2 (-1 - i \cdot 0)$

$= -e^2$

$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$

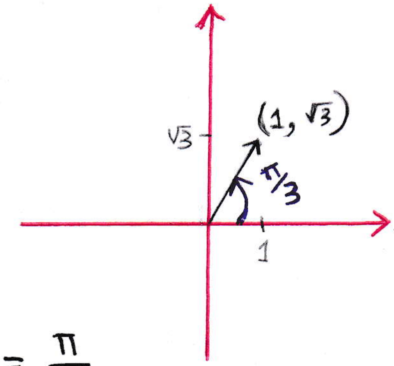
$\therefore e^{2 \pm 3\pi i} = -e^2$ (proved)

(Exercise Sheet #3)

2. Find all values of z such that

(ii) $e^z = 1+\sqrt{3}\, i$

Consider $e^z = 1+\sqrt{3}\, i$ (Given)



$1+\sqrt{3}\, i$ is a complex number

$$r = \sqrt{1^2+(\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$1+\sqrt{3}\, i = r e^{i(\theta+2n\pi)} \qquad n \geq 0$$

$$e^z = 2e^{\left(\frac{\pi}{3}+2n\pi\right)i}$$

$$= 2e^{\left(2n+\frac{1}{3}\right)\pi i}$$

$$\ln e^z = \ln\left[2e^{\left(2n+\frac{1}{3}\right)\pi i}\right] \quad \text{take } \ln \text{ on both sides}$$

$$z = \ln 2 + \ln e^{\left(2n+\frac{1}{3}\right)\pi i}$$

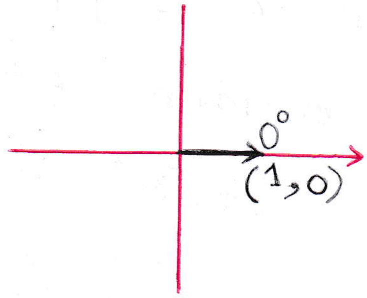
$$\therefore\ z = \ln 2 + \left(2n+\frac{1}{3}\right)\pi i$$

(iii) $e^{2z-1} = 1$ (Given)

Consider $1 = 1 + 0i \longrightarrow$ Complex number

$r = \sqrt{1^2 + 0^2} = 1$

$\theta = \tan^{-1}\left(\frac{0}{1}\right) = \tan^{-1} 0 = 0$



$e^{2z-1} = 1 = r e^{i(\theta + 2n\pi)} \quad n \geq 0$

$= 1 e^{i(0 + 2n\pi)}$

$= e^{2n\pi i}$

$\ln e^{2z-1} = \ln e^{2n\pi i}$ (take ln on both sides)

$2z - 1 = i\, 2n\pi$

$2z = 1 + i\, 2n\pi$

$\therefore z = \frac{1}{2} + i n\pi$

TRIGONOMETRIC FUNCTION

From Euler's Formula:

$$e^{iz} = \cos z + i\sin z \quad , \quad e^{-iz} = \cos z - i\sin z \qquad \forall x \in \mathbb{R}$$

$$e^{iz} + e^{-iz} = 2\cos z \qquad \text{and} \qquad e^{iz} - e^{-iz} = 2i\sin z$$

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

e^{iz} and e^{-iz} are entire functions (always differentiable)

$\therefore$ $\cos z$ and $\sin z$ are entire functions

$$\frac{d}{dz}(\sin z) = \cos z \quad ; \quad \frac{d}{dz}(\cos z) = -\sin z$$

$$\sin(-z) = -\sin z \qquad \cos(-z) = \cos z$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z \qquad \cos\left(z + \frac{\pi}{2}\right) = -\sin z$$

$$\sin(z - \pi) = -\sin z \qquad \cos(z - \pi) = -\cos z$$

Example

① Prove that $\sin^2 z + \cos^2 z = 1$ (by Substituting Trigonometric complex function)

$$\text{L.S.} = \sin^2 z + \cos^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2$$

$$= \frac{e^{i2z} - 2e^{iz}e^{-iz} + e^{-2iz}}{-4} + \frac{e^{iz} + 2e^{iz}e^{-i2z} + e^{-2iz}}{4}$$

$$= \frac{e^{i2z} + e^{-2iz} - 2e^{0}}{-4} + \frac{e^{i2z} + e^{-2iz} + 2e^{0}}{4}$$

$$= \frac{1}{4}\left[-e^{2iz} - e^{-2iz} + 2 + e^{2iz} + e^{-2iz} + 2\right]$$

$$= \frac{1}{4}[4]$$

$$= 1$$

$$= \text{R.S.}$$

(2) Show that $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

by substituting trigonometric complex function

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \therefore (\cos z)^4 = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^4$$

$$L.S. = \cos^4\theta = (\cos\theta)^4 = \frac{1}{16}\Big[e^{4iz} + 4e^{3iz}e^{-iz} + 6e^{2iz}e^{-2iz} + 4e^{iz}e^{-3iz} + e^{-4iz}\Big]$$

Replacing 'θ' with 'z'

$$= \frac{1}{16}\Big[e^{4iz} + e^{-4iz} + 4e^{2iz} + 4e^{-2iz} + 6\Big]$$

$$= \frac{1}{16}\left[2 \cdot \frac{e^{(4)iz} + e^{-(4)iz}}{2} + 2\times 4\, \frac{e^{(2)iz} + e^{-(2)iz}}{2} + 6\right]$$

$$= \frac{1}{16}\Big[2(\cos 4z) + 8(\cos 2z) + 6\Big]$$

$$= \frac{1}{8}\cos 4z + \frac{1}{2}\cos 2z + \frac{3}{8}$$

$$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

Replacing 'z' with 'θ'.

(Exercise Sheet #3)

3. Prove that: (iii) $\sin(z+2\pi) = \sin z$ (substitute trigonometric complex function)

$$L.S. = \sin(\underbrace{z+2\pi}_{z}) = \frac{1}{2i}\left(e^{i(z+2\pi)} - e^{-i(z+2\pi)}\right)$$

$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$$= \frac{1}{2i}\left(e^{iz}e^{2\pi i} - e^{-iz}e^{-i2\pi}\right)$$

$$= \frac{1}{2i}\left(e^{iz}\left(\cos 2\pi + i\sin 2\pi\right) - e^{-iz}\left(\cos(-2\pi) + i\sin(-2\pi)\right)\right)$$

$$= \frac{1}{2i}\left(e^{iz}(1+i\cdot 0) - e^{-iz}(1+i\cdot 0)\right)$$

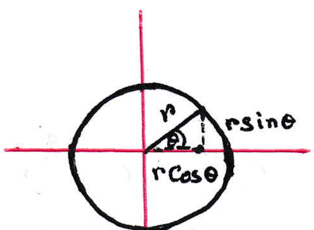
$$= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$= \sin z = R.S. \quad \text{(Proved)}$$

HYPERBOLIC FUNCTION

Hyperbolic functions are analogs (similar algebric structure) of ordinary trigonometric functions but different slightly in composition

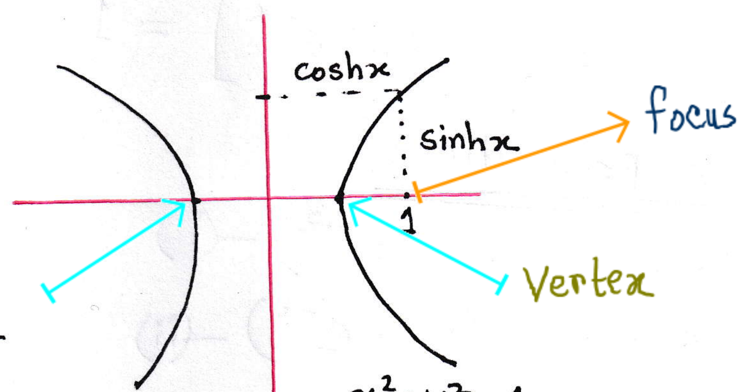
Also known as circular function



$x^2+y^2=r^2$

$r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$\cos^2\theta + \sin^2\theta = 1$



$x^2 - y^2 = 1$

$\cosh^2 x - \sinh^2 x = 1$

The basic hyperbolic functions are:

Hyperbolic sine "sinh"

Hyperbolic cosine "cosh"

Hyperbolic tangent "tanh"

The hyperbolic functions are similar to the trigonometric functions or circular functions. Generally, the hyperbolic functions are defined through the algebric expressions that include e^x and e^{-x}. "e" is the Euler's constant.

Hyperbolic Functions occur in:

→ linear differential equation

→ calculation of distance & angles in the hyperbolic geometry

→ laplace's eqn in the cartesian coordinate

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

From these basic functions, the other functions such as

Hyperbolic cosecant "cosech"
Hyperbolic secant "sech"
Hyperbolic cotangent "coth"

} functions are defined.

Properties:

$\sinh(-x) = -\sinh x$ $\qquad \sinh 2x = 2\sinh x \cosh x$

$\cosh(-x) = \cosh x$ $\qquad \cosh 2x = \cosh^2 x + \sinh^2 x$

$\frac{d}{dx}\cosh = \sinh x$

$\frac{d}{dx}\sinh x = \cosh$

Relation between hyperbolic & trigonometrics

$\cosh x = \cos(ix)$

$\sinh x = -i\sin(ix)$

Identities: $\boxed{\cosh^2 x - \sinh^2 x = 1}$

$\tanh^2 x + \operatorname{sech}^2 x = 1$

$\coth^2 x - \operatorname{cosech}^2 x = 1$

Complex Hyperbolic Functions are similar to standard Hyperbolic Functions but are defined over the complex number field.

- $\cosh z = \frac{1}{2}(e^{z}+e^{-z})$ — (i)
- $\sinh z = \frac{1}{2}(e^{z}-e^{-z})$ — (ii)

They are entire functions $\because e^{z}$ and e^{-z} are entire (always differentiable)

In complex analysis these functions arise as the imaginary parts of sine and cosine.

$$\cosh^2 z - \sinh^2 z = 1$$

$$\frac{d}{dz}(\sinh z) = \cosh z \;;\; \frac{d}{dz}(\cosh z) = \sinh z$$

Example

Prove: $\cosh^2 z - \sinh^2 z = 1$ (Substitute complex hyperbolic function)

$$L.S. = \cosh^2 z - \sinh^2 z$$

$$= \left(\frac{e^{z}+e^{-z}}{2}\right)^2 - \left(\frac{e^{z}-e^{-z}}{2}\right)^2$$

$$= \frac{1}{4}\left[(e^{2z}+2e^{z}e^{-z}+e^{-2z}) - (e^{2z}-2e^{z}e^{-z}+e^{-2z})\right]$$

$$= \frac{1}{4}\left[4e^{z}e^{-z}\right] = 1 = R.S. \quad \text{(proved)}$$

Few Properties:

$\cosh z = \frac{1}{2}(e^{z}+e^{-z})$ — (i)

$\sinh z = \frac{1}{2}(e^{z}-e^{-z})$ — (ii)

$\cos z = \frac{1}{2}(e^{iz}+e^{-iz})$ — (iii)

$\sin z = \frac{1}{2i}(e^{iz}-e^{-iz})$ — (iv)

Example: Evaluate a) $\cos iz$ and b) $\sin iz$

a) $\cos \underbrace{iz}_{z} = \frac{1}{2}\left(e^{i(iz)} + e^{-i\cdot(iz)}\right)$ $\quad \because \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$

$= \frac{1}{2}\left(e^{-z} + e^{z}\right)$

$\therefore \boxed{\cos iz = \cosh z}$ — (a)

b) $\sin \underbrace{iz}_{z} = \frac{1}{2i}\left(e^{i(iz)} - e^{-i(iz)}\right)$ $\quad \because \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$= \frac{1}{2i}\left(e^{-z} - e^{z}\right)$

$= \frac{1}{2i}\cdot\frac{-i}{-i}\left(e^{-z} - e^{z}\right)$

$= \frac{-i}{-2i^2}\left\{-\left(e^{z} - e^{-z}\right)\right\}$

$= -\frac{i}{(-2)}\left(e^{z} - e^{-z}\right)$

$= \frac{i}{2}\left(e^{z} - e^{-z}\right)$

$= i\cdot\frac{1}{2}\left(e^{z} - e^{-z}\right)$ $\quad \sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)$

$\therefore \boxed{\sin iz = i \sinh z}$ — (b)

Now, $\cos z = \cos(x+iy)$

$= \cos x \cos iy - \sin x \sin iy$

$= \cos x \cosh y - \sin x \, i \sinh y$

$\left\{ \begin{array}{l} \cos iz = \cosh z \\ \sin iz = i \sinh z \end{array} \right.$

$\therefore \boxed{\cos z = \cos x \cosh y - \sin x \, i \sinh y}$

Also, $\sin z = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + \cos x \, i \sinh y$

$\therefore \boxed{\sin z = \sin x \cosh y + \cos x \, i \sinh y}$

(Exercise Sheet #3)

4. Prove that (i) $\sinh z = \sinh x \cos y + i \cosh x \sin y$

$i \sinh z = \sin iz$ $\longrightarrow$ Provided from eqn (b) / we know

$= \sin i(x+iy)$

$= \sin(ix - y)$

$= \sin ix \cos y - \sin y \cos ix$

$= i \sinh x \cos y - \sin y \cosh x$ $\quad \because \left\{ \begin{array}{l} \sin ix = i \sinh x \\ \cos ix = \cosh x \end{array} \right.$

$\therefore i \sinh z = i \sinh x \cos y + i^2 \sin y \cosh x$ $\quad \because i^2 = -1$

$\Rightarrow \sinh z = \sinh x \cos y + i \sin y \cosh x$ ($\div$ by i)

TRY 4 (ii)

Examples

1. Prove that $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

$i\sinh z = \sin iz \longrightarrow$ we know

$|i\sinh z|^2 = |\sin iz|^2$

$|i|^2 |\sinh z|^2 = |\sin(ix - y)|^2 \quad \{ iz = i(x+iy) = ix - y \}$

$|i^2| \; |\sinh z|^2 = |\sin ix \cos y - \cos ix \sin y|^2$

$|-1| \, |\sinh z|^2 = |\underbrace{i \sinh x}_{+(iy)} \cos y \; \boxed{-} \underbrace{\cosh x \sin y}_{(-x)}|^2 \quad \begin{cases} \sin ix = i\sinh x \\ \cos ix = \cosh x \end{cases}$

$|\sinh z|^2 = \left(\sqrt{(\sinh x \cos y)^2 + (-\cosh x \sin y)^2}\right)^2$

$|z| = |\bar{z}|$
$= |-z|$

$= \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y$

$= \sinh^2 x \cos^2 y + (1 + \sinh^2 x)\sin^2 y \quad (\cosh^2 x - \sinh^2 x = 1)$

$= \sinh^2 x \cos^2 y + \sin^2 y + \sin^2 y \sinh^2 x$

$= \sinh^2 x (\cos^2 y + \sin^2 y) + \sin^2 y$

$\therefore |\sinh z|^2 = \sinh^2 x + \sin^2 y$

2. Prove that $|\cosh z|^2 = \sinh^2 x + \cos^2 y$

↳ TRY THIS

INVERSE TRIGONOMETRIC FUNCTION

(Exercise Sheet #3)

5(ii) Prove that $\cos^{-1} z = -i \ln\left[z \pm i(1-z^2)^{1/2}\right]$

Let $w = \cos^{-1} z$

$$z = \cos w$$
$$= \frac{1}{2}\left(e^{iw} + e^{-iw}\right)$$

$$2z = e^{iw} + e^{-iw}$$
$$2ze^{iw} = \left(e^{iw}\right)^2 + 1$$ [multiply by e^{iw} to avoid the $-ve$ power]

$$\left(e^{iw}\right)^2 - 2ze^{iw} + 1 = 0$$

$$e^{iw} = \frac{2z \pm \sqrt{4z^2 - 4}}{2}$$
$$= \frac{1}{2}\left(2z \pm 2\sqrt{z^2-1}\right)$$
$$= z \pm \sqrt{z^2-1}$$
$$= z \pm \sqrt{-(1-z^2)}$$
$$= z \pm i\sqrt{1-z^2}$$

$$\ln e^{iw} = \ln\left(z \pm i\sqrt{1-z^2}\right)$$ (take ln on both sides)
$$iw = \ln\left(z \pm i\sqrt{1-z^2}\right)$$
$$w = \frac{\ln\left(z \pm i\sqrt{1-z^2}\right)}{i} = \frac{\ln\left(z \pm i\sqrt{1-z^2}\right)}{i} \cdot \frac{(-i)}{(-i)}$$

$$= \frac{-i\ln\left(z \pm i\sqrt{1-z^2}\right)}{-i^2}$$

$$= \frac{-i\ln\left(z \pm i\sqrt{1-z^2}\right)}{-(-1)}$$

$$w = -i\ln\left(z \pm i\sqrt{1-z^2}\right)$$

$$\cos^{-1} z = -i\ln\left(z \pm i\sqrt{1-z^2}\right) \quad \text{(proved)}$$

6(i) Solve the following equation:

$\cosh z = \frac{1}{2}$

$$\cosh z = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2}\left(e^z + e^{-z}\right) = \frac{1}{2}$$

$$\Rightarrow e^z + e^{-z} = 1$$

$$\Rightarrow (e^z)^2 + 1 = e^z \quad \text{(multiply by } e^z \text{ to avoid the -ve power)}$$

$$(e^z)^2 - e^z + 1 = 0$$

$$e^z = \frac{1 \pm \sqrt{1-4}}{2}$$

$$e^z = \frac{1 \pm i\sqrt{3}}{2}$$

$$e^z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Consider the complex number $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

for $e^z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$$

$$= \tan^{-1}\sqrt{3}$$

$$= \frac{\pi}{3}$$

for $e^z = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2} = 1$$

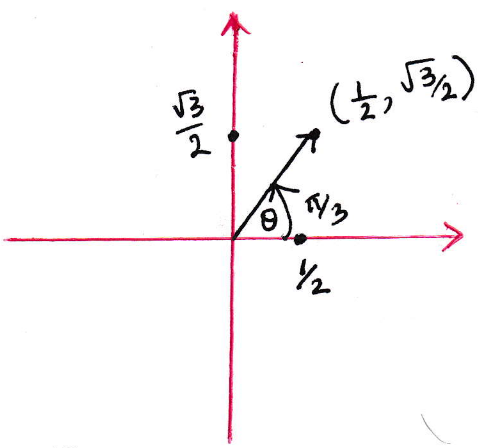
$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}/2}{1/2}\right)$$

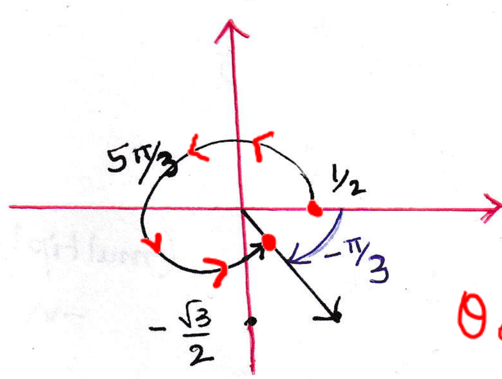
$$= \tan^{-1}(-\sqrt{3})$$

$$= -\pi/3$$

$$\theta_a = 2\pi - \left|-\frac{\pi}{3}\right|$$

$$= \frac{5\pi}{3}$$





$\theta = \frac{\pi}{3}$

$\theta_a = 2\pi - |\theta|$

$= 2\pi - \theta$

For:

$$e^z = \frac{1}{2} + i\frac{\sqrt{3}}{2} = re^{i(\theta + 2n\pi)}, \quad n \geq 0$$

$$= 1e^{i(\pi/3 + 2n\pi)}$$

$$e^z = e^{i(\frac{\pi}{3} + 2n\pi)}$$

$$e^{z} = e^{i\left(\frac{\pi}{3}+2n\pi\right)}$$

$$z = i\left(\frac{\pi}{3}+2n\pi\right) = \pi i\left(\frac{1}{3}+2n\right); \quad n \geqslant 0$$

For:

$$e^{z} = \frac{1}{2} - i\frac{\sqrt{3}}{2} = re^{i(\theta+2n\pi)}, \quad n \geqslant 0$$

$$= 1e^{i\left(\frac{5\pi}{3}+2n\pi\right)}$$

$$e^{z} = e^{i\left(\frac{5\pi}{3}+2n\pi\right)}$$

$$z = i\left(\frac{5\pi}{3}+2n\pi\right)$$

$$= \pi i\left(\frac{5}{3}+2n\right); \quad n \geqslant 0$$

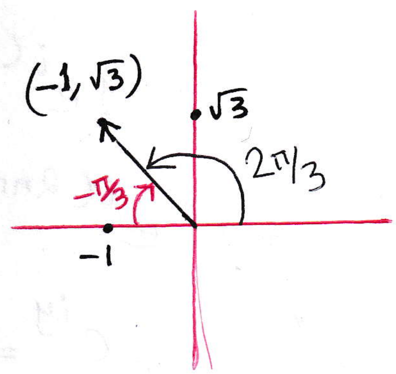
LOGARITHMIC FUNCTION

(Exercise Sheet #3)

7.(i) Show that : $\ln(-1+\sqrt{3}i) = \ln 2 + 2(n+\frac{1}{3})\pi i$

$-1+\sqrt{3}i \Rightarrow x = -1, \; y = \sqrt{3}$

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$



$\theta = \tan^{-1}(-\sqrt{3})$

$= -\pi/3$

$\theta_a = \pi - \left|-\frac{\pi}{3}\right|$

$= \frac{2\pi}{3}$

Now,

$$-1+\sqrt{3}i = re^{i(\theta+2n\pi)}, \quad n \geq 0$$

$$= 2e^{i(\frac{2\pi}{3}+2n\pi)}$$

$$\ln(-1+\sqrt{3}i) = \ln\left(2e^{i(\frac{2\pi}{3}+2n\pi)}\right)$$

$$= \ln 2 + \ln e^{2(\frac{\pi}{3}+n\pi)i}$$

$$= \ln 2 + 2(\frac{\pi}{3}+n\pi)i$$

$\therefore \ln(-1+\sqrt{3}i) = \ln 2 + 2(n+\frac{1}{3})\pi i$ (proved)

7(iii) Show that $\ln(i^{1/2}) = \left(n+\frac{1}{4}\right)\pi i$

$$\ln(i^{1/2}) = \left(n+\tfrac{1}{4}\right)\pi i$$

$$z = 0 + i \rightarrow (0,1)$$

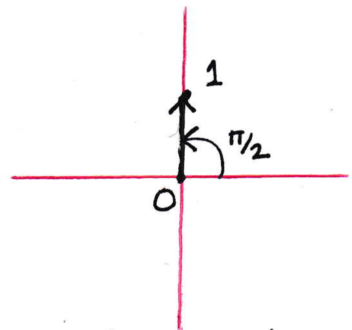
$$r = \sqrt{0^2+1^2} = 1$$

$$\Rightarrow \frac{1}{2}\ln(\underset{\downarrow\, z}{i}) = \left(n+\tfrac{1}{4}\right)\pi i$$

$$\theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2}$$

$$z = i = re^{i(\theta+2n\pi)} \; ; \; n \geq 0$$

$$= 1e^{i\left(\frac{\pi}{2}+2n\pi\right)}$$



$$i = e^{i\pi\left(\frac{1}{2}+2n\right)}$$

$$\ln(i) = \ln e^{i\pi\left(\frac{1}{2}+2n\right)} \quad \text{(take ln on both sides)}$$

$$\ln(i) = i\pi\left(\tfrac{1}{2}+2n\right)$$

$$\frac{1}{2}\ln(i) = i\pi\left(\tfrac{1}{4}+n\right) \quad (\div \text{ by } 2)$$

$$\ln(i^{1/2}) = \left(n+\tfrac{1}{4}\right)\pi i$$

Euler's formula : $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{iz} = \cos z + i\sin z$ — (i)

$e^{-iz} = \cos(-z) + i\sin(-z) = \cos z - i\sin z$ — (ii)

Hyperbolic function eliminate "i"

Add: $e^{iz} + e^{-iz} = 2\cos z \Rightarrow \cos z = \frac{1}{2}(e^{iz}+e^{-iz}) \rightarrow \cosh z = \frac{1}{2}(e^{z}+e^{-z})$

Subtract: $e^{iz} - e^{-iz} = 2i\sin z \Rightarrow \sin z = \frac{1}{2i}(e^{iz}-e^{-iz}) \rightarrow \sinh z = \frac{1}{2}(e^{z}-e^{-z})$

* $\cos iz = \cosh z$
* $\sin iz = i\sinh z$