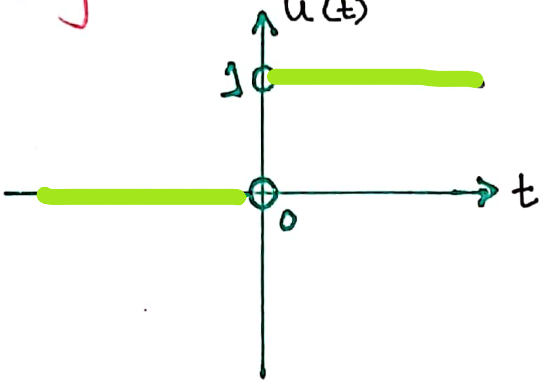
Laplace Transformation (Part B)

→ UNIT STEP FUNCTION or HEAVISIDE FUNCTION

Say we have the following signal:

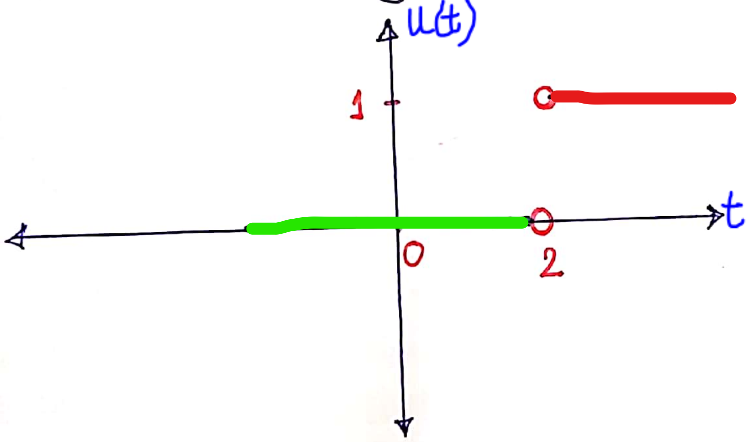


$$u(t) = \begin{cases} 0 & , t < 0 \\ 1 & , t > 0 \end{cases}$$

This is unit step function

$$u(t) \Rightarrow u(t-0)$$

Write the following graph in terms of function:

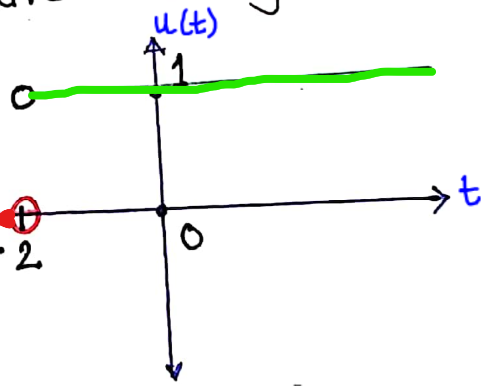


We are shifting towards right side

$$u(t-2) = \begin{cases} 0 & , \boxed{t<2} \\ 1 & , \boxed{t>2} \end{cases}$$

$$\begin{cases} t<2 \Rightarrow t-2<0 \\ t>2 \Rightarrow t-2>0 \end{cases}$$

Graph the following function while the signal is shifting two units towards left:



$$u(t+2) = \begin{cases} 0 & , \boxed{t<-2} \\ 1 & , \boxed{t>-2} \end{cases}$$

$$\begin{cases} t<-2 \Rightarrow t+2<0 \\ t>-2 \Rightarrow t+2>0 \end{cases}$$

Consider the following step function:

"a" is an arbitrary +ve number

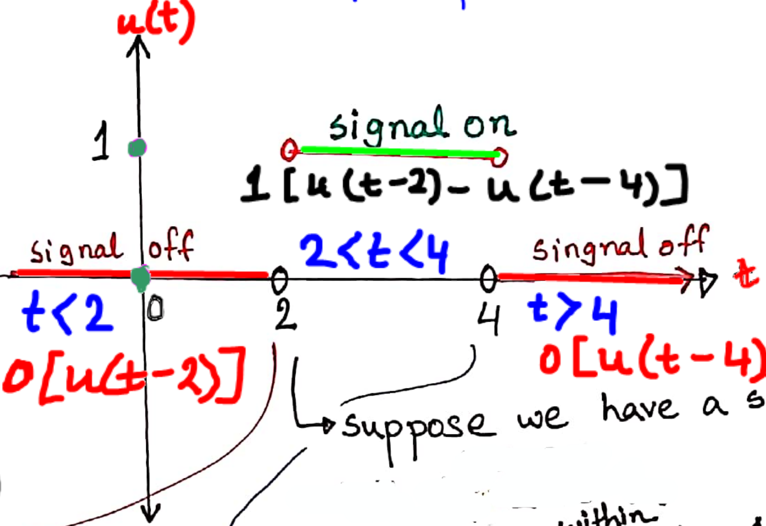
General structure of Unit Step Function:

$$u(t-a) = \begin{cases} 0, & \underbrace{t<a}_{} \text{ -ve input} \\ 1 & \underbrace{t>a}_{} \text{ +ve input} \end{cases} \qquad a \geq 0$$

$t-a<0 \Rightarrow t<a$

$t-a>0 \Rightarrow t>a$

$f(t)$
$= 0[u(t-2)]$
$+ 1[u(t-2) - u(t-4)]$
$+ 0[u(t-4)]$
$\therefore f(t) = u(t-2) - u(t-4)$



Suppose we have a signal while t within 2 to 4 signal is on & when t > 4, the signal is off.

Lets express this in terms of step function.

$$\boxed{f(t) = u(t-2) - u(t-4)}$$

The signal of the above graph can be presented mathematically in this way.

let t = 1

$u(t-2)$ -ve input, $u(t-4)$ -ve input

$\underbrace{t<2}_{} \Rightarrow f(t) = 0 - 0 = 0$

graphically output should be "0"

let t = 3

$\underbrace{2<t<4}_{} \Rightarrow f(t) = u(t-2) - u(t-4)$ (+ve input, -ve input)

graphically output should be "1" $= 1 - 0 = 1$

$t>2$ & $t<4 \Rightarrow u(t-2)$ is +ve

$t>2$ & $t<4 \Rightarrow u(t-4)$ is -ve

$\begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$

let t = 5

$\underbrace{t>4}_{} \Rightarrow f(t) = u(t-2) - u(t-4)$ (+ve input, +ve input)

$= 1 - 1 = 0$

graphically output should be "0"

it has to be "−" so the final answer is "0"

$t>4 \Rightarrow u(t-2)$ is +ve output = 1

$t>4 \Rightarrow u(t-4)$ is +ve $\therefore$ output = 1

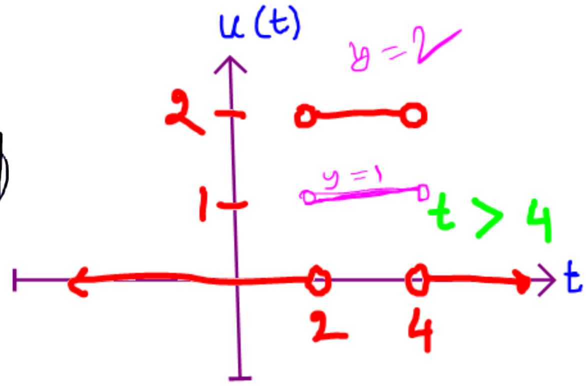
If we change the function

$$f(t) = 2\,[\underbrace{u(t-2)}_{\text{+ve input}} - \underbrace{u(t-4)}_{\text{+ve input}}]$$

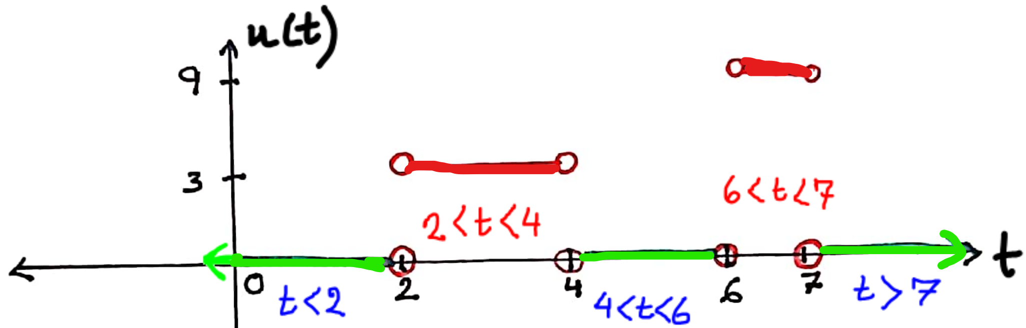
$t>4$

let $t=5$

$$= 2(1-1) = 0$$



Consider another signal:



$$f(t) = 0[u(t-2)] + 3[u(t-2) - u(t-4)] + 0[u(t-4) - u(t-6)] + 9[u(t-6) - u(t-7)] + 0[u(t-7)]$$

$$\therefore f(t) = 3[u(t-2) - u(t-4)] + 9[u(t-6) - u(t-7)]$$

Represent the above function:

Step Function

$$f(t) = \begin{cases} 3, & 2<t<4 \\ 9, & 6<t<7 \\ 0, & o/w \end{cases}$$

$t<2$ *, $t>7$
$4<t<6$ **
Not given
$\therefore$ assuming over those intervals the signal is off.

Express this function in one line: (Unit step function)

$$f(t) = 3[u(t-2) - u(t-4)] + 9[u(t-6) - u(t-7)]$$

* $t<2 \Rightarrow f(t) = 3[\underbrace{u(t-2)}_{\substack{-ve \\ \text{"0"}}} - \underbrace{u(t-4)}_{\substack{-ve \\ \text{"0"}}}] + 9[\underbrace{u(t-6)}_{\substack{-ve \\ \text{"0"}}} - \underbrace{u(t-7)}_{\substack{-ve \\ \text{"0"}}}] = 0$

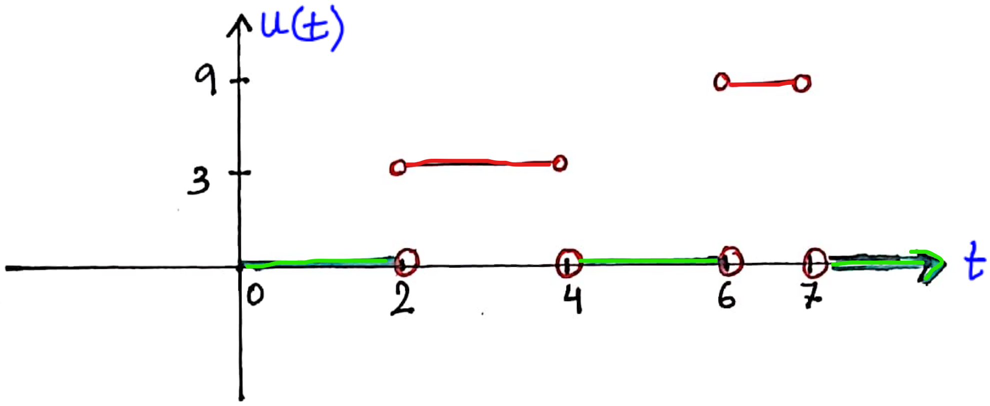
* let "t = 1" $\therefore$

$2<t<4 \Rightarrow f(t) = 3[(\underbrace{u(t-2)}_{\substack{+v \\ \text{"1"}}} - \underbrace{u(t-4)}_{\substack{-ve \\ \text{"0"}}}] + 9[\underbrace{u(t-6)}_{\substack{-ve \\ \text{"0"}}} - \underbrace{u(t-7)}_{\substack{-ve \\ \text{"0"}}}] = 3$

let "t = 3"

** $4<t<6 \Rightarrow f(t) = 3[\underbrace{u(t-2)}_{\substack{+ve \\ \text{"1"}}} - \underbrace{u(t-4)}_{\substack{+ve \\ \text{"1"}}}] + 9[\underbrace{u(t-6)}_{\substack{-ve \\ \text{"0"}}} - \underbrace{u(t-7)}_{\substack{-ve \\ \text{"0"}}}] = 0$

let "t = 5"



$$6<t<7 \Rightarrow f(t) = 3[u(\underbrace{t-2}_{\substack{+ve \\ \text{"}1\text{"}}}) - u(\underbrace{t-4}_{\substack{+ve \\ \text{"}1\text{"}}})] + 9[u(\underbrace{t-6}_{\substack{+ve \\ \text{"}1\text{"}}}) - u(\underbrace{t-7}_{\substack{-ve \\ \text{"}0\text{"}}})] = 9$$

let "$t = 6.5$"

Unit step function helps us to write the **step** function in one line.

Example: $f(t) = \begin{cases} t & ; \ 1<t<3 \\ \sin t & ; \ 6<t<7 \\ e^{2t} & ; \ t>7 \end{cases}$

Reading

$\rightarrow \because t<1 \rightarrow$ output $= 0$
$3<t<6 \rightarrow$ " $= 0$

$\because$ this intervals are missing in the information

Unit step function will be as follows:

$$f(t) = t[u(t-1) - u(t-3)] + \sin t[u(t-6) - u(t-7)] + e^{2t}[u(t-7)]$$

$$t<1 \Rightarrow f(t) = t[u(\underbrace{t-1}_{\substack{-ve \\ \text{"}0\text{"}}}) - u(\underbrace{t-3}_{\substack{-ve \\ \text{"}0\text{"}}})] + \sin t[u(\underbrace{t-6}_{\substack{-ve \\ \text{"}0\text{"}}}) - u(\underbrace{t-7}_{\substack{-ve \\ \text{"}0\text{"}}})] + e^{2t}(u(\underbrace{t-7}_{\substack{-ve \\ \text{"}0\text{"}}}))$$

$$= 0$$

$$1<t<3 \Rightarrow f(t) = t[u(\underbrace{t-1}_{\substack{+ve \\ \text{"}1\text{"}}}) - u(\underbrace{t-3}_{\substack{-ve \\ \text{"}0\text{"}}})] + \sin t[u(\underbrace{t-6}_{\substack{-ve \\ \text{"}0\text{"}}}) - u(\underbrace{t-7}_{\substack{-ve \\ \text{"}0\text{"}}})] + e^{2t}[u(\underbrace{t-7}_{\substack{-ve \\ \text{"}0\text{"}}})]$$

$$= t$$

$$6<t<7 \Rightarrow f(t) = t\,[u(\underbrace{t-1}_{\substack{+ve \\ "1"}}) - u(\underbrace{t-3}_{\substack{+ve \\ "1"}})] + \sin t\,[u(\underbrace{t-6}_{\substack{+ve \\ "1"}}) - u(\underbrace{t-7}_{\substack{-ve \\ "0"}})] + e^{2t}\,[u(\underbrace{t-7}_{\substack{-ve \\ "0"}})]$$

$$= t\,[1-1] + \sin t\,[1-0] + e^{2t}\,[0]$$

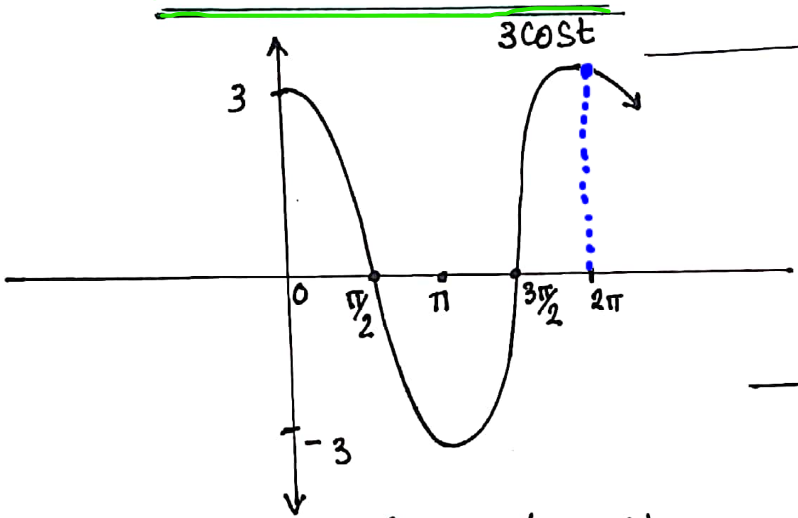
$$= \sin t$$

$$t>7 \Rightarrow f(t) = t\,[u(\underbrace{t-1}_{+ve}) - u(\underbrace{t-3}_{+ve})] + \sin t\,[u(\underbrace{t-6}_{+ve}) - u(\underbrace{t-7}_{+ve})] + e^{2t}[u(\underbrace{t-7}_{+ve})]$$

$$= t\,[1-1] + \sin t\,[1-1] + e^{2t}\,(1)$$

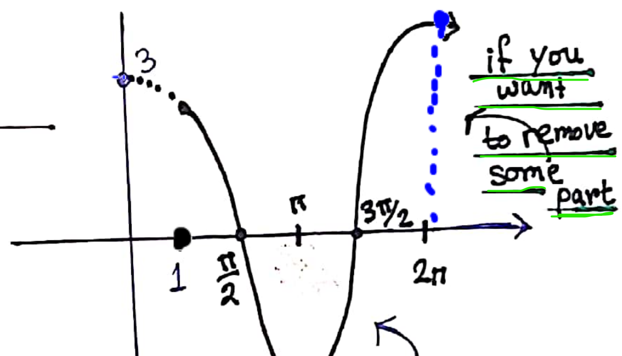
$$= e^{2t}$$

Omit theses Graphs → TRANSLATIONS

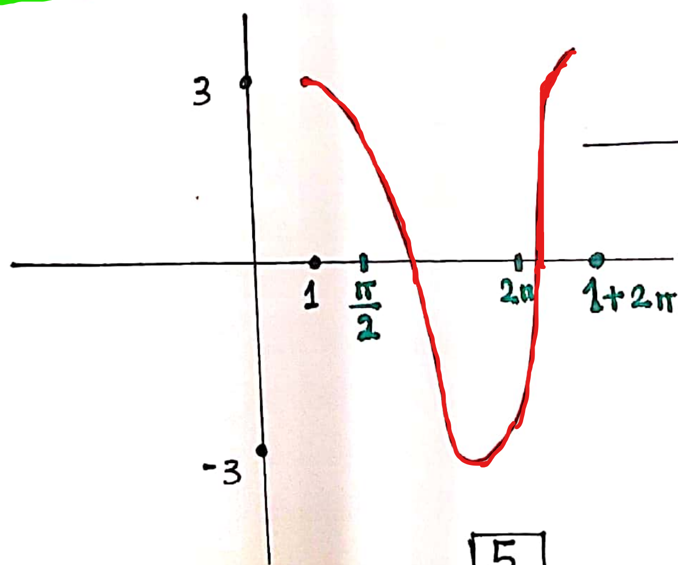


$\rightarrow f(t) = 3\cos t$

shift it forward by 1 unit



$f(t) = 3\cos t\,[u(t-1)]$
$= f(t)\,u(t-1)$



$\rightarrow f(t) = 3\cos(t-1)[u(t-1)]$
$= f(t-1)\,u(t-1)$
Translation

Example ①

Piecewise function

$$f(t) = \begin{cases} 0 \; ; & 0<t<1 \\ t-1 \; ; & 1<t<2 \\ t+1 \; ; & t>2 \end{cases}$$

Find the laplace transformation of the above function

Unit step function

$$f(t) = 0\left[u(t-0) - u(t-1)\right] + (t-1)\left[u(t-1) - u(t-2)\right] + (t+1)\left[u(t-2)\right]$$

$$= t\,u(t-1) - \cancel{t\,u(t-2)} - u(t-1) + u(t-2) + \cancel{t\,u(t-2)} + u(t-2)$$

$$= \underbrace{t\,u(t-1)}_{\text{(i) } f(t)=t} - \underbrace{u(t-1)}_{\text{(ii) } f(t)=1} + \underbrace{2u(t-2)}_{\text{(iii) } f(t)=1}$$

Formula

$$\mathcal{L}(u(t-a)f(t)) = e^{-sa}\mathcal{L}\{f(t+a)\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = e^{-s(1)}\mathcal{L}\{t+1\} - e^{-s(1)}\mathcal{L}\{1\} + 2e^{-s(2)}\mathcal{L}\{1\}$$

(i) $f(t) = t$

$f(t+a) = f(t+1) = t+1$

$\because a = 1$

(ii) $f(t) = 1$

$f(t+a) = f(t+1) = 1$

$\because a = 1$

(iii) $f(t) = 1$

$f(t+a) = f(t+2) = 1$

$\because a = 2$

$$\mathcal{L}\{f(t)\} = e^{-s}\mathcal{L}\{t+1\} - e^{-s}\mathcal{L}\{1\} + 2e^{-2s}\mathcal{L}\{1\}$$

$$= e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-s}\cdot\frac{1}{s} + 2e^{-2s}\cdot\frac{1}{s}$$

$$= e^{-s}\frac{1}{s^2} + e^{-s}\cdot\frac{1}{s} - e^{-s}\cdot\frac{1}{s} + 2e^{-2s}\cdot\frac{1}{s}$$

$$= \frac{1}{s^2}e^{-s} + \frac{2}{s}e^{-2s}$$

Example ② Find the Laplace Transformation of

$$f(t) = \begin{cases} 2 \;; & 0<t<3 \\ t^2 \;; & 3<t<5 \\ t+1 \;; & t>5 \end{cases}$$

$\mathcal{L}\{u(t-a)f(t)\}$
$= e^{-sa}\mathcal{L}\{f(t+a)\}$

$$f(t) = 2[u(t-0) - u(t-3)] + t^2[u(t-3) - u(t-5)] + (t+1)[u(t-5)]$$

$$= 2u(t-0) - 2u(t-3) + t^2u(t-3) - t^2u(t-5) + t(u(t-5)) + 1u(t-5)$$

$$= \underbrace{2u(t-0)}_{\substack{a=0 \\ f(t)=1}}^{\text{i}} - \underbrace{2u(t-3)}_{f(t)=1,\ a=3}^{\text{ii}} + \underbrace{t^2u(t-3)}_{f(t)=t^2,\ a=3}^{\text{iii}} - \underbrace{t^2u(t-5)}_{\substack{a=5 \\ f(t)=t^2,}}^{\text{iv}}$$

$\mathcal{L}(u(t-a)f(t))$
$= e^{-sa}\mathcal{L}\{f(t+a)\}$

$$+ \underbrace{t\,u(t-5)}_{\substack{a=5 \\ f(t)=t}}^{\text{v}} + \underbrace{u(t-5)}_{\substack{a=5 \\ f(t)=1}}^{\text{vi}}$$

i $f(t) = 1$
$f(t+a) = f(t+0) = 1$

ii $f(t) = 1$
$f(t+a) = f(t+3) = 1$

iii $f(t) = t^2$
$f(t+a) = f(t+3) = (t+3)^2$

iv $f(t) = t^2$
$f(t+a) = f(t+5) = (t+5)^2$

v $f(t) = t$
$f(t+a) = f(t+5) = t+5$

vi $f(t) = 1$
$f(t+a) = f(t+5) = 1$

Take Laplace on both sides:

$$\therefore \mathcal{L}\{f(t)\} = 2e^{-s(0)}\mathcal{L}\{1\} - 2e^{-s(3)}\mathcal{L}\{1\} + e^{-s(3)}\mathcal{L}\{(t+3)^2\} + e^{-s(5)}\mathcal{L}\{1\}$$
$$+ e^{-s(5)}\mathcal{L}\{(t+5)^2\} + e^{-s(5)}\mathcal{L}\{t+5\}$$

$$= 2\cdot\frac{1}{s} - 2e^{-3s}\frac{1}{s} + e^{-3s}\mathcal{L}\{t^2+6t+9\} + e^{-5s}\left(\frac{1}{s^2}+\frac{5}{s}\right) + e^{-5s}\left(\frac{1}{s}\right)$$
$$+ e^{-5s}\mathcal{L}\{t^2+10t+25\}$$

$$= \frac{2}{s} - \frac{2}{s}e^{-3s} + e^{-3s}\left(\frac{2!}{s^{2+1}} + 6\cdot\frac{1}{s^2} + \frac{9}{s}\right)$$

$$+ e^{-5s}\left(\frac{2!}{s^{2+1}} + 10\cdot\frac{1}{s^2} + 25\frac{1}{s}\right) + e^{-5s}\left(\frac{1}{s^2} + \frac{5}{s}\right)$$

$$+ \frac{e^{-5s}}{s}$$

$$= \frac{2}{s} - \frac{2e^{-3s}}{s} + \frac{2e^{-3s}}{s^3} + \frac{6e^{-3s}}{s^2} + \frac{9e^{-3s}}{s}$$

$$+ \frac{2e^{-5s}}{s^3} + \frac{10}{s^2} + \frac{25}{s} + \frac{e^{-5s}}{s^2} + \frac{5e^{-5s}}{s} + \frac{e^{-5s}}{s}$$

$$= \frac{27}{s} + \frac{7e^{-3s}}{s} + \frac{6e^{-5s}}{s} + \frac{2e^{-3s}}{s^3} + \frac{2e^{-5s}}{s^3}$$

$$+ \frac{6e^{-3s}}{s^2} + \frac{e^{-5s}}{s^2} + \frac{10}{s^2}.$$