Topic 1 part 1

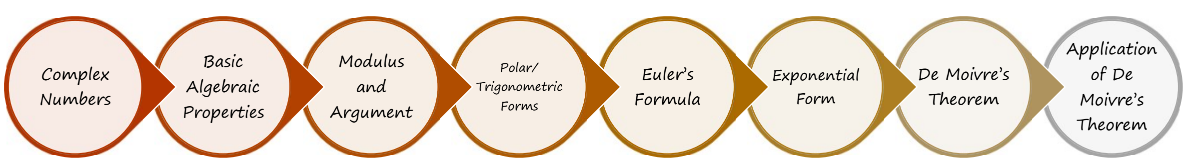
MAT 215: Complex Variables and Laplace Transformations

Inspiring Excellence

MAT 215: Complex Variables and Laplace Transformations

Key Points

Introduction to Complex Numbers



Complex Numbers

- *Solving the equation:* $x^2 + 1 = 0$

  $\Rightarrow x = \pm\sqrt{-1} = \pm i$

Note: The imaginary unit (i) *is defined such that* $i^2 = -1$.

Form of complex numbers: Let $z = x + iy$ *be any complex number where* $x, y \in \mathbb{R}$.

Real part=$Re(z) = x$ *Imaginary part*=$Im(z) = y$

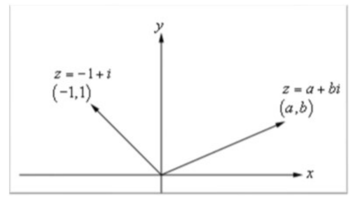


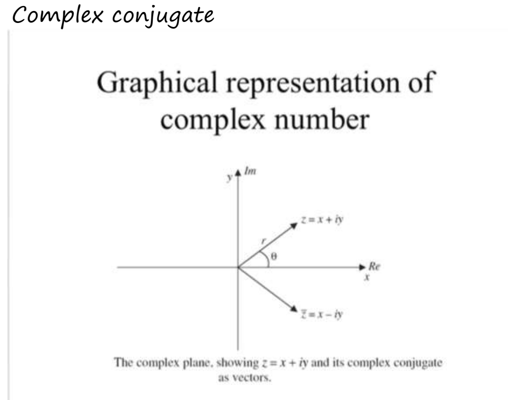
Fig: Graphical representation of complex numbers

Basic Algebraic Properties

Let z_1, z_2 *and* z_3 *be any three complex numbers.*

- ***Commutative Law:***
  - ✓ $z_1 + z_2 = z_2 + z_1$ *[for addition]*
  - ✓ $z_1.z_2 = z_2.z_1$ *[for multiplication]*
- ***Associative Law:***
  - ✓ $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ *[for addition]*
  - ✓ $z_1.(z_2.z_3) = (z_1.z_2).z_3$ *[for multiplication]*
- ***Distributive Law:***
  - ✓ $z_1.(z_2 + z_3) = z_1.z_2 + z_1.z_3$
- ***Identity Law:***
  - ✓ $z_1 + 0 = z_1 = 0 + z_1$ *[Additive identity:* 0*]*
  - ✓ $z_1.1 = z_1 = 1.z_1$ *[Multiplicative identity:* 1*]*

Complex conjugate



Modulus and Argument

- *Modulus of* $z = |z| = r$:

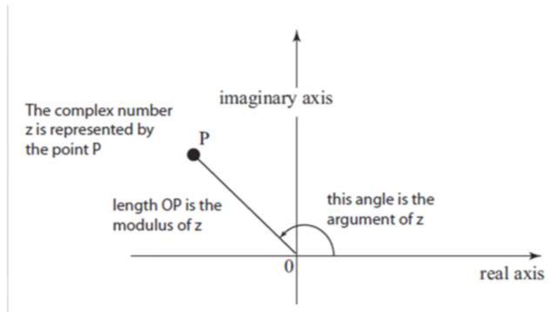
  Let $z = x + iy, x, y \in \mathbb{R}$

  $|z| = \sqrt{(Re(z))^2 + (Im(z))^2} = \sqrt{x^2 + y^2}$

  Complex Conjugate of $z = \bar{z} = x - iy, x, y \in \mathbb{R}$

  $|\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2}$

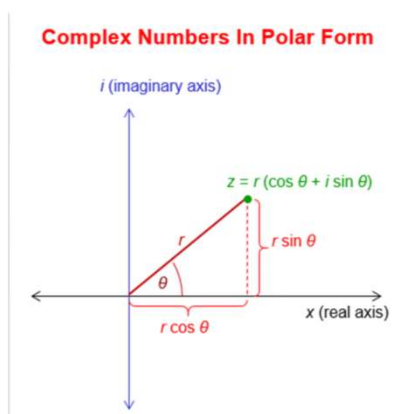
- *Properties*:
  - ✓ $|z| = |\bar{z}|$
  - ✓ $\bar{\bar{z}} = z$
  - ✓ $z.\bar{z} = |z|^2 = |\bar{z}|^2$
- *Argument of* $z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$



Polar/ Trigonometric Forms

- $z = x + iy$ $[x = r\cos\theta$ *and* $y = r\sin\theta]$

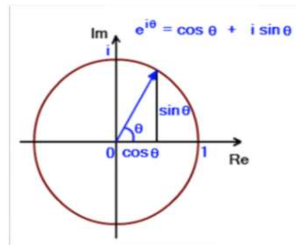
=> $z = r(\cos\theta \ + i\sin\theta)$



Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

It is a unit circle
r=1



Exponential Form

- *Applying Euler's Identity in Polar form of complex numbers:*

  $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

- *Note:*
  - ✓ *Let* $z_1 = r_1e^{i\theta_1}$ *and* $z_2 = r_2e^{i\theta_2}$ *be two complex numbers.*

    $z_1 . z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$
  - ✓ $\frac{1}{z} = \frac{1}{re^{i\theta}}$

    => $z^{-1} = \frac{1}{r}e^{-i\theta}$

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ *then for all values of* n,

$z^n = r^n(\cos n\theta + i\sin n\theta) = r^ne^{in\theta}$*[Using Euler's formula].*

De Moivre's Theorem : (Proof)

Statement: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ is known as De Moivre's Theorem/Formula

$n = 1, 2, 3, \ldots$ any +ve integer

Proof: We use the principle of Mathematical induction.

Step 1: Show that the statement is true for $n = 1$

If $n = 1$, then $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$ and it is true

Step 2:

Assume if the result is true for $n = k$, $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

then it is also true for $n = k+1$

$$\therefore (\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k (\cos\theta + i\sin\theta)$$

$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

Trig id:

→ $\cos A\cos B - \sin A\sin B = \cos(A+B)$

→ $\sin A\cos B + \cos A\sin B = \sin(A+B)$

$$= \cos k\theta\cos\theta + i\sin k\theta\cos\theta + i\cos k\theta\sin\theta + i^2\sin k\theta\sin\theta$$

$$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$

$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

$\therefore$ the statement is true for any +ve integer 'n'.

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Application of De Moivre's Theorem

$z = r(\cos\theta + i\sin\theta)$

$z^n = r^n(\cos\theta + i\sin\theta)^n$

$= r^n(\cos n\theta + i\sin n\theta)$

$= r^n e^{in\theta}$

↳ Euler's formula

$\text{cis}\,\theta = \cos\theta + i\sin\theta$

$\boxed{1}$

$$\frac{(8\,\text{cis}\,40^\circ)^3}{(2\,\text{cis}\,60^\circ)^4} = \frac{8^3[\cos 3(40^\circ) + i\sin 3(40^\circ)]}{2^4[\cos 4(60^\circ) + i\sin 4(60^\circ)]}$$

$$= \frac{2^9(\cos 120^\circ + i\sin 120^\circ)}{2^4(\cos 240^\circ + i\sin 240^\circ)}$$

$$= \frac{2^5\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}$$

$$= \frac{2^5\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{2^5\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2}{\left(-\frac{1}{2}\right)^2 - \left(i\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2^5\left[\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right)\left(i\frac{\sqrt{3}}{2}\right) + \left(i\frac{\sqrt{3}}{2}\right)^2\right]}{\frac{1}{4} - i^2\frac{3}{4}}$$

$$= \frac{2^5\left(\frac{1}{4} - i\frac{\sqrt{3}}{2} - \frac{3}{4}\right)}{\frac{1}{4} + \frac{3}{4}}$$

$$= \frac{2^5\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)}{1} = 2^5\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$\boxed{2}$ $$\frac{3e^{\frac{\pi i}{6}}\ 2e^{-\frac{5\pi i}{4}}\ 6e^{\frac{5\pi i}{3}}}{\left(4e^{\frac{2\pi i}{3}}\right)^2}$$

$$= \frac{36\, e^{\frac{\pi i}{6} - \frac{5\pi i}{4} + \frac{5\pi i}{3}}}{16\, e^{\frac{4\pi i}{3}}}$$

$$= \frac{9}{4}\, e^{\frac{\pi i}{6} - \frac{5\pi i}{4} + \frac{5\pi i}{3} - \frac{4\pi i}{3}}$$

$$= \frac{9}{4}\, e^{\frac{2\pi i - 15\pi i + 20\pi i - 16\pi i}{12}}$$

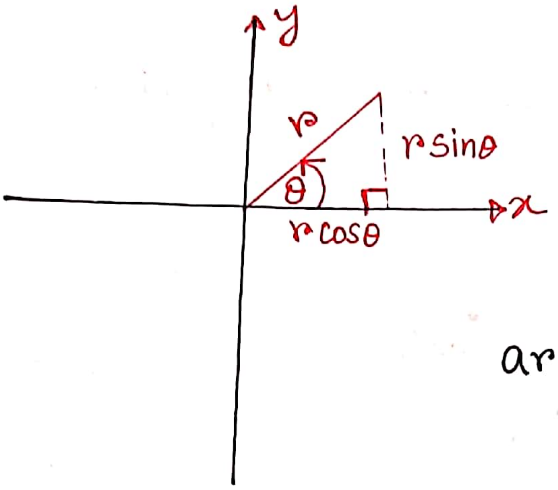
$$= \frac{9}{4}\, e^{-\frac{9}{12}\pi i}$$

$$= \frac{9}{4}\, e^{\left(-\frac{3\pi}{4}\right)i}$$

$$= \frac{9}{4}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$$

$$= \frac{9}{4}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$= \frac{9}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$



$r = |z| =$ modulus of z

$\theta =$ argument of z

$\arg z = \mathrm{Arg}(\theta) \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$z = 2+3i \longrightarrow$ single solution

$\left\{ \begin{array}{l} z^5 = 2+3i \\ z = (2+3i)^{1/5} \end{array} \right.$ 5 solutions

** For 1st degree eqution $\arg z = \mathrm{Arg}(\theta)$

$\boxed{\text{For higher degree equation } \arg z = \mathrm{Arg}(\theta + 2k\pi)}$

if $z^n = r(\cos(\theta+2k\pi) + i\sin(\theta+2k\pi))$

then $z = r^{1/n}\left(\cos(\theta+2k\pi) + i\sin(\theta+2k\pi)\right)^{1/n}$

$= r^{1/n}\left(\cos\frac{1}{n}(\theta+2k\pi) + i\sin\frac{1}{n}(\theta+2k\pi)\right)$

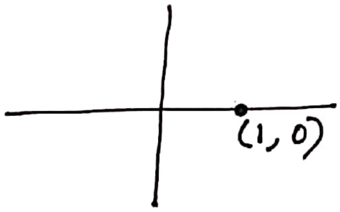
$k = 0, 1, 2, 3, \ldots, (n-1)$

Ex: $z = (2+3i)^{1/5}$, 5 Solutions, $k = 0, 1, 2, 3, 4$

** $z = 2+3i$, 1 solution, $k = 0$; $\arg z = \mathrm{Arg}(\theta)$

$\boxed{4}$ Solve: $z^3-1=0$, show the roots graphically

$$z^3-1=0$$
$$z^3=1$$
$$z=(1)^{1/3}$$



$\longrightarrow x+iy = 1+0i$

$$r=\sqrt{x^2+y^2}=\sqrt{1^2+0^2}=1$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)$$
$$=\tan^{-1}\left(\frac{0}{1}\right)$$
$$=\tan^{-1}0$$
$$=0$$

In general: $z=r(\cos\theta+i\sin\theta)$

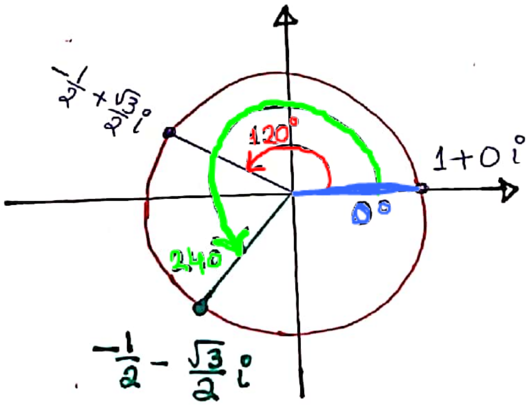
In this case: $z=\left[r(\cos\underbrace{\theta}_{(\theta+2k\pi)}+i\sin\theta)\right]^{1/3}$ $\quad k=0,1,2$

$$=1^{1/3}\left(\cos(0+2k\pi)+i\sin(0+2k\pi)\right)^{1/3}$$
$$=1\left[\cos\tfrac{1}{3}(2k\pi)+i\sin\tfrac{1}{3}(2k\pi)\right]$$

for $k=0$: $z_0=\cos\frac{1}{3}(0)+i\sin\frac{1}{3}(0)=1+i\cdot 0=1$

$k=1$: $z_1=\cos\frac{1}{3}(2(1)\pi)+i\sin\frac{1}{3}(2(1)\pi)=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$

$k=2$: $z_2=\cos\frac{1}{3}(2(2)\pi)+i\sin\frac{1}{3}(2(2)\pi)=-\frac{1}{2}+i\left(-\frac{\sqrt{3}}{2}\right)$



$\boxed{3}$ $z^5 = -4+4i$

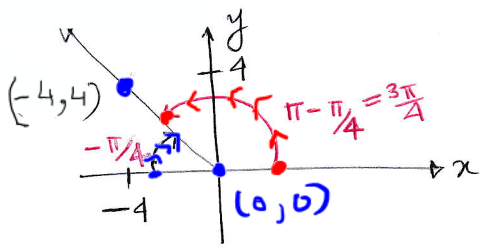
$\Rightarrow z = (-4+4i)^{1/5}$

consider $-4+4i$: $\overbrace{x=-4,\ y=4}^{\to\ 2^{nd}\ \text{Quad}}$ $\to (-4,4)$

$r = \sqrt{(-4)^2+4^2} = \sqrt{32} = \sqrt{2^5} = \sqrt{2^4\,2} = 2^2\sqrt{2} = 4\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{-4}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$\therefore \theta = \pi - \left|-\frac{\pi}{4}\right| = \frac{3\pi}{4}$ $\because \theta$ is producing -ve orientation



no. of roots = 5

K = 0, 1, 2, 3, 4

$\theta = \theta + 2\pi k$

$(-4+4i)^{1/5} = (4\sqrt{2})^{1/5}\left(\cos\frac{1}{5}\left(\frac{3\pi}{4}+2k\pi\right) + i\sin\frac{1}{5}\left(\frac{3\pi}{4}+2k\pi\right)\right)$

For $K=0$, $z_0 = (4\sqrt{2})^{1/5}\left(\cos\frac{3\pi}{20} + i\sin\frac{3\pi}{20}\right) = 1.26 + i0.64$ $\to 27^\circ$

$K=1$, $z_1 = (4\sqrt{2})^{1/5}\left(\cos\frac{11\pi}{20} + i\sin\frac{11\pi}{20}\right) = -0.22 + 1.397i$ $\to 99^\circ$

$K=2$, $z_2 = (4\sqrt{2})^{1/5}\left(\cos\frac{19\pi}{20} + i\sin\frac{19\pi}{20}\right) = -1.4 + 0.22i$ $\to 171^\circ$

$K=3$, $z_3 = (4\sqrt{2})^{1/5}\left(\cos\frac{27\pi}{20} + i\sin\frac{27\pi}{20}\right) = -0.64 - 1.26i$ $\to 243$

$K=4$, $z_4 = (4\sqrt{2})^{1/5}\left(\cos\frac{35\pi}{20} + i\sin\frac{35\pi}{20}\right) = 1 - i$ $\to 315$