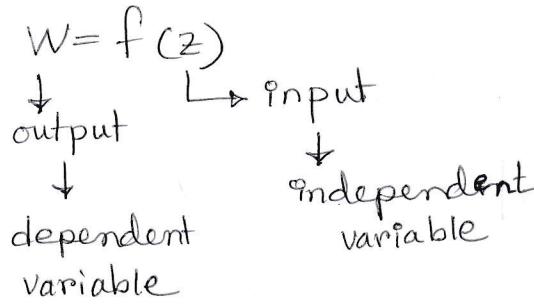


Functions of complex Variables:Example

$$\textcircled{1} \quad f(z) = z^2, \quad z = x + iy$$

$$\begin{aligned} f(x+iy) &= (x+iy)^2 = x^2 + 2xiy + i^2 y^2 \\ &= (x^2 - y^2) + i(2xy) \quad \because i^2 = -1 \\ &= u(x, y) + i v(x, y) \end{aligned}$$

$$u = \underbrace{x^2 - y^2}_{\text{Real part of } f}, \quad v(x, y) = \underbrace{2xy}_{\text{Imaginary part of } f}$$

$$\textcircled{2} \quad f(z) = |z|^2 = x^2 + y^2 = u(x, y) + iv(x, y)$$

$$\Rightarrow u(x, y) = x^2 + y^2 \quad \& \quad v(x, y) = 0$$

Domain:

Example: Find the domain for each of the functions:

$$\text{a) } f(z) = \frac{1}{z^2 + 1}$$

$$z^2 + 1 \neq 0$$

$$z^2 \neq -1$$

$$z^2 \neq i^2$$

$z \neq \pm i \rightarrow \text{domain}$

$$\text{b) } f(z) = \frac{z}{z + \bar{z}}$$

$$= \frac{x+iy}{x+iy+\bar{x}-iy}$$

$$= \frac{x+iy}{2x} = \frac{x+iy}{2\operatorname{Re}(z)}$$

domain: $\operatorname{Re}(z) \neq 0$

③ Prove that $\lim_{z \rightarrow 0} \frac{\bar{z}}{z}$ does not exist.

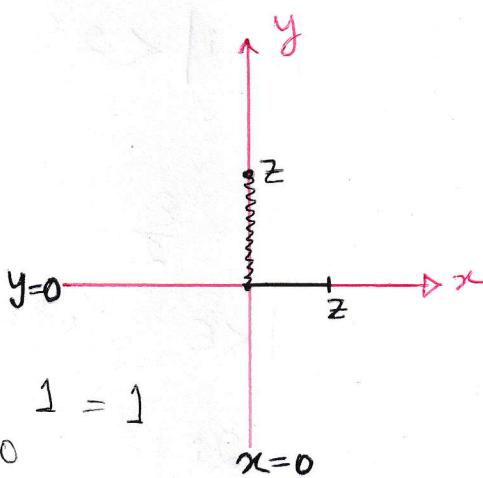
Let $z \rightarrow 0$, along x -axis, then $y=0$

$$z = x+iy = x$$

$$\bar{z} = x-iy = x$$

$$\frac{\bar{z}}{z} = \frac{x}{x} = 1$$

$$\therefore \lim_{z \rightarrow 0} \frac{\bar{z}}{z} = \lim_{z \rightarrow 0} 1 = 1$$



Let $z \rightarrow 0$ along y -axis, then $x=0$

$$z = x+iy = iy$$

$$\frac{\bar{z}}{z} = \frac{iy}{-iy} = -1$$

$$\bar{z} = x-iy = -iy$$

$$\therefore \lim_{z \rightarrow 0} \frac{\bar{z}}{z} = -1$$

\therefore the two sided approaches do not provide same result,
the limit does not exist.

Note

If the limit exists it must be independent of the manner in which z approaches 0.

In this case

z approaches '0' at x -axis $\Rightarrow y=0$

z approaches '0' at y -axis $\Rightarrow x=0$

Example Show that $f(z) = \frac{xy}{x^2+y^2}$ is not continuous

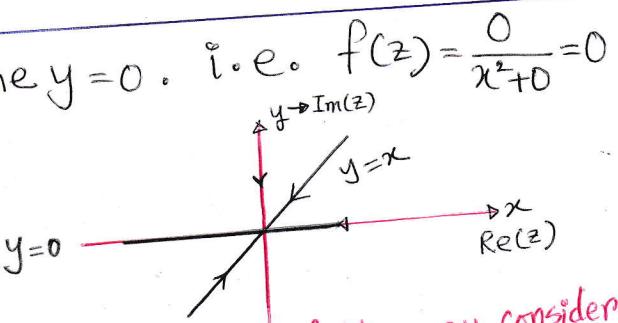
at $z=0$.

$\xrightarrow{z_0}$

NOTE: In order to prove something is not continuous, simply find two lines going through the desired limit point & show that along these two lines, the limits you get are different. On the other hand it is much harder to prove that something is continuous.

First, take the limit along the line $y=0$. i.e. $f(z) = \frac{0}{x^2+0} = 0$

$$\lim_{z \rightarrow 0} \frac{xy}{x^2+y^2} = \lim_{z \rightarrow 0} 0 = 0$$



But now take the limit along the line $y=x$
i.e. $f(z) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$

$$\lim_{z \rightarrow 0} \frac{xy}{x^2+y^2} = \lim_{z \rightarrow 0} \frac{1}{2} = \frac{1}{2}$$

$$0 \neq \frac{1}{2}$$

∴ $f(z)$ is not continuous at $z=0$, since the results are different

You may consider any one of the following graphs as they are going through $(0,0)$

$y=0$
 $y=x$

L'Hôpital's Rule: (Indeterminant form $\frac{0}{0}, \frac{\infty}{\infty}$)

The L'Hôpital's rule states that

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \quad \text{with Indeterminant form such as } \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

Note: $0 \cdot \infty, \infty^0, 0^0, 1^\infty, \infty \pm \infty \Rightarrow$ All imply ∞ .

0^0 is a mathematical expression that is defined as 1 or left undefined, depending on context.

In algebra and combinatorics one typically defines $0^0 = 1$.

In mathematical analysis $0^0 = 0^{1-1} = 0^1 \cdot 0^{-1} = \frac{0^1}{0^1} = \frac{0}{0} = \text{undefined}$.

In the case of limit it is considered as an indeterminate form.

Example:

$$\begin{aligned} \lim_{z \rightarrow i} \frac{z^{10} + 1}{z^6 + 1} &= \lim_{z \rightarrow i} \frac{10z^9}{6z^5} \\ &= \lim_{z \rightarrow i} \frac{5}{3} z^4 = \frac{5}{3} i^4 = \frac{5}{3} (i^2)^2 \\ &= \frac{5}{3} (-1)^2 = \frac{5}{3} \end{aligned}$$

Theorems on Limits

$$w = f(z) \\ (u, v) \rightarrow (x, y)$$

Theorem 1, suppose $f(z) = u(x, y) + i v(x, y)$

$$\begin{cases} z_0 = x_0 + iy_0 \\ w_0 = u_0 + iv_0 \end{cases}$$

$$u_0(x, y) + iv_0(x, y)$$

$$\text{then } \lim_{z \rightarrow z_0} f(z) = w_0 \quad \text{--- (i)}$$

$$\text{iff } \lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0 \quad \& \quad \lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = v_0. \quad \text{--- (ii)}$$

Theorem 2

Suppose $\lim_{z \rightarrow z_0} f(z) = w_0$ & $\lim_{z \rightarrow z_0} F(z) = W_0$

Then $\lim_{z \rightarrow z_0} [f(z) + F(z)] = w_0 + W_0$

$$\lim_{z \rightarrow z_0} [f(z) \cdot F(z)] = w_0 \cdot W_0$$

$$\lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}, \text{ if } W_0 \neq 0$$

Examples:

① $\lim_{z \rightarrow 1+i} (z^2 - 5z + 10)$

$$= \lim_{z \rightarrow 1+i} z^2 - 5 \lim_{z \rightarrow 1+i} z + \lim_{z \rightarrow 1+i} 10$$

$$= (1+i)^2 - 5(1+i) + 10$$

$$= 1+2i+i^2 - 5 - 5i + 10$$

$$= -3i + 5$$

$$\textcircled{2} \quad \lim_{z \rightarrow -2i} \frac{(2z+3)(z-1)}{z^2 - 2z + 4}$$

$$= \frac{\{2(-2i)+3\}\{-2i-1\}}{(-2i)^2 - 2(-2i) + 4}$$

$$= \frac{(-4i+3)(-2i-1)}{-4+4i+4}$$

$$= \frac{8i^2 - 6i + 4i - 3}{4i} = \frac{-8 - 2i - 3}{4i} \quad \because i^2 = -1$$

$$= \frac{-8 - 2i - 3}{(0+4i)} \cdot \frac{(0-4i)}{(0-4i)}$$

$$= \frac{-(2i+11)(-4i)}{(4i)(-4i)}$$

$$= \frac{(2i+11)(4i)}{-16i^2}$$

$$= \frac{8i^2 + 44i}{16}$$

$$= \frac{-8 + 44i}{16}$$

$$= -\frac{1}{2} + \frac{11}{4}i$$

(Exercise Sheet #2)

1. Evaluate the following limits:

① $\lim_{z \rightarrow 1+i}$

$$\frac{z^2 - z + 1 - i}{z^2 - 2z + 2}$$

$$\frac{(1+i)^2 - 1 - i + 1 - i}{(1+i)^2 - 2 - 2i + 2} = \frac{1+2i-1-2i}{1+2i-1-2i}$$

$$= \lim_{z \rightarrow 1+i} \frac{2z-1}{2z-2} \quad \text{Apply L'Hospital's rule} \quad = \frac{0}{0}$$

$$= \frac{2(1+i) - 1}{2(1+i) - 2}$$

$$= \frac{2i + 1}{2i}$$

$$= \frac{(2i+1)}{2i} \cdot \frac{(-2i)}{-2i}$$

$$= \frac{-4i^2 - 2i}{-4i^2}$$

$$= \frac{4 - 2i}{4}$$

$$= 1 - \frac{1}{2}i$$

Continuity

$f(z)$ is continuous at a point z_0 if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0) \neq \infty$$

① $\lim_{z \rightarrow z_0} f(z)$ exist

② $f(z_0)$ exist

③ $\lim_{z \rightarrow z_0} f(z) = f(z_0)$; $f(z)$ continuous iff $u(x,y)$ & $v(x,y)$ are continuous.

(Exercise Sheet #2) $\frac{1}{z} \underset{z=0}{\cancel{\neq 0}}$

3. Let $f(z) = \frac{z^2 + 4}{z - 2i}$ if $z \neq 2i$, while $\underbrace{f(2i)}_{f(z_0)} = 3 + 4i$. Is

$f(z)$ continuous at $z = 2i$?

$$\begin{aligned} & z < 2i \\ & \lim_{z \rightarrow 2i} f(z) \quad \left(\begin{array}{l} z = 2i \\ f(z) = 3 + 4i \end{array} \right) \\ & z > 2i \\ & \lim_{z \rightarrow 2i} z + 2i + f(z) = ? = 4i \\ & = ? = 4i \end{aligned}$$

\downarrow

$$f(z) = \begin{cases} \frac{z^2 + 4}{z - 2i}, & z \neq 2i \\ 3 + 4i, & z = 2i \end{cases}$$

Given in the question

Algebraic manipulation

$$\begin{aligned} \textcircled{1} \quad \lim_{z \rightarrow 2i} f(z) &= \lim_{z \rightarrow 2i} \frac{z^2 + 4}{z - 2i} = \lim_{z \rightarrow 2i} \frac{z^2 - 2^2 i^2}{z - 2i} \\ &= \lim_{z \rightarrow 2i} \frac{(z + 2i)(z - 2i)}{(z - 2i)} \\ &= \lim_{z \rightarrow 2i} z + 2i \\ &= 2i + 2i = 4i \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad f(2i) &\stackrel{?}{=} 3 + 4i \quad (\text{Given}) \\ \textcircled{3} \quad \lim_{z \rightarrow 2i} f(z) &\stackrel{?}{=} f(2i) \quad (\Rightarrow 4i \neq 3 + 4i) \end{aligned}$$

$f(z)$ is not continuous at $z = 2i$

Theorem: If z_0 and w_0 are points in the z and w planes, respectively, then

- $\lim_{z \rightarrow z_0} f(z) = \infty$ if and only if $\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$
- $\lim_{z \rightarrow \infty} f(z) = w_0$ if and only if $\lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0$
- $\lim_{z \rightarrow \infty} f(z) = \infty$ if and only if $\lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = \infty$.

Limits:

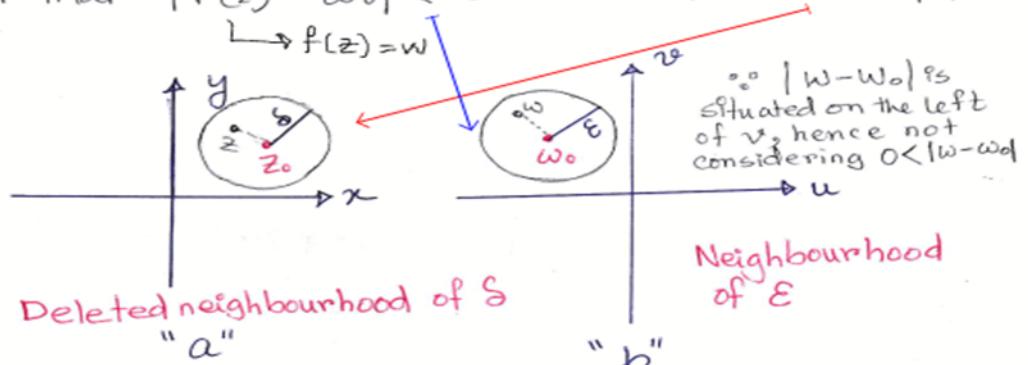
$$\lim_{z \rightarrow z_0} f(z) = w_0 \quad \text{eqn(1)}$$

$w = f(z)$
 (u, v) $\xrightarrow{\text{Image}}$
 (x, y) $\xrightarrow{\text{original}}$

eqn(1) means for each +ve no. ϵ , there is a +ve no. δ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$
or $|w - w_0| < \epsilon$

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

$\therefore |w - w_0| < \epsilon$
 $0 < |z - z_0| < \delta$



"b" is the image of "a"

For each ϵ -neighbourhood $|w - w_0| < \epsilon$ of w_0 there is a deleted δ -neighbourhood of $0 < |z - z_0| < \delta$ of z_0 such that

\downarrow
limit point

every point "z" in it has an image "w" lying in the ϵ -neighbourhood.

② Show that $\lim_{z \rightarrow i} 2z = 2i$ whenever $0 < |z - i| < s$ for given any $\epsilon > 0$, $s > 0$

$$|2z - 2i| < \epsilon \quad \text{when } z_0 \text{ whenever } 0 < |z - i| < s \quad \text{for given any } \epsilon > 0, s > 0$$

Now $|2z - 2i| = 2|z - i| < 2s$

$$\begin{cases} f(z) = w \\ |f(z) - w_0| < \epsilon \\ 0 < |z - z_0| < s \end{cases}$$

Let $2s = \epsilon \Rightarrow s = \frac{\epsilon}{2}$

we have $|2z - 2i| < \epsilon$ whenever $0 < |z - i| < \frac{\epsilon}{2}$

Hence $\lim_{z \rightarrow i} 2z = 2i$

Show that if $f(z) = \frac{i\bar{z}}{2}$ then $\lim_{z \rightarrow 1} f(z) = \frac{i}{2}$

Show that $|w - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$
while $\varepsilon > 0, \delta > 0$

$$\lim_{\substack{z \rightarrow 1 \\ \downarrow z_0}} f(z) = \frac{i}{2} \xrightarrow[w \rightarrow w_0]{} w_0$$

$\Rightarrow |f(z) - \frac{i}{2}| < \varepsilon$ whenever $0 < |z - 1| < \delta$

$\Rightarrow \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| < \varepsilon$ whenever $0 < |z - 1| < \delta$ $\xrightarrow{\textcircled{i}}$

$$\begin{aligned} \text{Now } \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| &= \left| \frac{i\bar{z} - i}{2} \right| \\ &= \frac{|i\bar{z} - i|}{|2|} \\ &= \frac{1}{2} |z - 1| \end{aligned}$$

$$\therefore \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| < \frac{1}{2} \delta \xrightarrow{\textcircled{ii}}$$

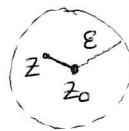
$$\begin{aligned} &\left| \frac{i\bar{z} - i}{2} \right| \quad \left| z - 1 \right| \\ &= \left| i(x - iy) - i \right| \quad \left| x + iy - 1 \right| \\ &= \left| ix + y - i \right| \quad \left| \sqrt{(x-1)^2 + y^2} \right| \\ &= \sqrt{(x-1)^2 + y^2} \end{aligned}$$

Compare \textcircled{i} & \textcircled{ii} we have $\varepsilon = \frac{1}{2} \delta$ or $\delta = 2\varepsilon$

Thus the condition of the limits is satisfied by points in the region $|z - 1| < \delta$ when $\delta = 2\varepsilon$

Derivatives

Let $f(z)$ be defined in a neighbourhood around z_0 .



$$0 < |z - z_0| < \epsilon$$

The derivative of $f(z)$ at z_0 is defined by

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Recall 1st Principle of derivative MAT 110

$$f'(x) = \lim_{x \rightarrow w} \frac{f(x) - f(w)}{x - w}$$

Alternatively

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The function $f(z)$ is said to be differentiable at z_0 if the derivative $f'(z) = \frac{dw}{dz}$ is defined as

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$\therefore f(z) = w$

Rules of differentiation in complex numbers are same as real numbers.

(Exercise Sheet #2)

5. Using the definitions, find the derivative of each function at the indicated points.

$$\textcircled{i} \quad f(z) = \frac{2z - i}{z + 2i} \quad \text{at } z = -i \quad \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

$\begin{matrix} z \rightarrow z \\ \Delta z = h \rightarrow \Delta z \end{matrix}$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\therefore f'(-i) = \lim_{\Delta z \rightarrow 0} \frac{f(-i + \Delta z) - f(-i)}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} [f(-i + \Delta z) - f(-i)]$$

$$= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left[\frac{2(-i + \Delta z) - i}{-i + \Delta z + 2i} - \frac{2(-i) - i}{-i + 2i} \right]$$

$$= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left[\frac{-3i + 2\Delta z}{i + \Delta z} + \frac{3i}{i} \right]$$

$$= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left[\frac{-3i + 2\Delta z + 3(i + \Delta z)}{(i + \Delta z)} \right]$$

$$= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left[\frac{-3i + 2\Delta z + 3i + 3\Delta z}{(i + \Delta z)} \right]$$

$$= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \cdot \frac{5\Delta z}{i + \Delta z}$$

$$= \frac{5}{i} = \frac{5}{i} \cdot \frac{(-i)}{(-i)} = \frac{-5i}{-i^2} = \frac{-5i}{1} = -5i$$

$$f(z) = \frac{2z - i}{z + 2i}$$

$$\therefore f(-i + \Delta z) = \frac{2(-i + \Delta z) - i}{-i + \Delta z + 2i}$$

$$f'(-i) = \frac{2(-i) - i}{-i + 2i}$$

Examples

① Using the definition of derivative, find the limit of $f(z) = z^2$

$$f(z) = z^2$$

$$f'(z) = \lim_{\Delta z \rightarrow 0}$$

$$\frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{z^2 + 2z\Delta z + \Delta z^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{(2z + \Delta z)\Delta z}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} 2z + \Delta z$$

$$= 2z$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$x \rightarrow z$$

$$\Delta z = h \rightarrow \Delta z$$

$$\checkmark f(z) = z^2$$

$$\checkmark f(z + \Delta z) = (z + \Delta z)^2$$

② Let $f(z) = |z|^2$. Show that the derivative of $f(z) = |z|^2$ exists only at $z = 0$.

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

by definition

$$\checkmark f(z) = |z|^2$$

$$\checkmark f(z + \Delta z) = |z + \Delta z|^2$$

$$\begin{cases} |z|^2 = x^2 + y^2 \\ \therefore |z|^2 = z\bar{z} \end{cases}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)(\bar{z} + \bar{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)(\bar{z} + \bar{\Delta z}) - z\bar{z}}{\Delta z}$$

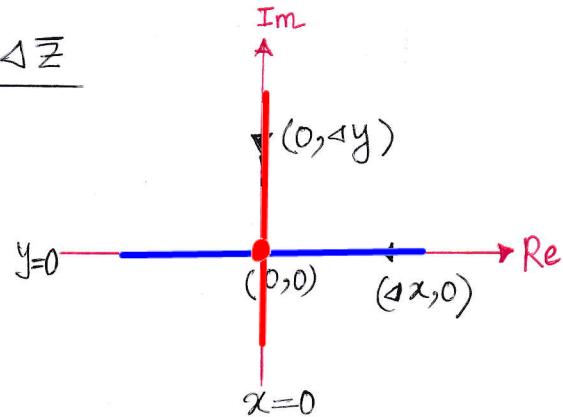
$$= \lim_{\Delta z \rightarrow 0} \frac{\bar{z}\bar{z} + \bar{z}\Delta z + z\bar{\Delta z} + \Delta z\bar{z} - z\bar{z}}{\Delta z}$$

$$\therefore \bar{z}_1 + \bar{z}_2 = \bar{z}_1 + \bar{z}_2$$

$$= \lim_{\Delta z \rightarrow 0} \frac{\bar{z}\Delta z + z\Delta \bar{z} + \Delta z\Delta \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \left(\bar{z} + z \frac{\Delta \bar{z}}{\Delta z} + \Delta \bar{z} \right)$$

$o+0i$



Case I: Approaching along real line, $\Delta z \rightarrow 0$ implies $\Delta y = 0, \Delta x \rightarrow 0$

Then $\lim_{\Delta z \rightarrow 0} \left(\bar{z} + z \frac{\Delta \bar{z}}{\Delta z} + \Delta \bar{z} \right) = \bar{z} + z \quad \text{--- (i)}$

$y=0$

1 $\because \Delta \bar{z} = \Delta z$
 $\& \Delta z = 0 \therefore \Delta \bar{z} = 0$

$\begin{aligned} z &= x+i y = x+i \cdot 0 = x \\ \bar{z} &= x-i y = x-i \cdot 0 = x \end{aligned}$
 when $y=0$
 then $\bar{z} = z$
 $\Rightarrow \Delta z = \Delta \bar{z}$

Case II: Approaching along imaginary line, $\Delta z \rightarrow 0$ implies $\Delta x = 0, \Delta y \rightarrow 0$

then $\lim_{\Delta z \rightarrow 0} \left(\bar{z} + z \frac{\Delta \bar{z}}{\Delta z} + \Delta \bar{z} \right) = \bar{z} - z \quad \text{--- (ii)}$

$x=0$

(-1) $\because \Delta \bar{z} = -\Delta z$
 $\& \Delta z = 0 \therefore \Delta \bar{z} = 0$

$\begin{aligned} z &= x+i y = 0+i y = iy \\ \bar{z} &= x-i y = 0-i y = -iy \end{aligned}$
 when $x=0$
 then $\bar{z} = -z$
 $\Rightarrow \Delta z = -\Delta \bar{z}$

\therefore the limits are unique, then compare (i) & (ii)

~~$$z + \bar{z} = \bar{z} - z$$~~

$$2z = 0$$

$$z = \bar{z} \quad \left\{ \begin{array}{l} z = 0 = 0+0i = 0 \\ \bar{z} = 0 = 0-0i = 0 \end{array} \right.$$

$\therefore z = 0 \Rightarrow \bar{z} = 0$ as well as $\Delta z \rightarrow 0$

$$\left. \begin{array}{l} \Rightarrow z = 0+0i \\ \bar{z} = 0-0i \end{array} \right\}$$

$$\begin{aligned} \therefore \lim_{\Delta z \rightarrow 0} & \left(\bar{z} + z \frac{\Delta \bar{z}}{\Delta z} + \Delta \bar{z} \right) \\ & = \lim_{\Delta z \rightarrow 0} (0+0+\Delta \bar{z}) = \lim_{\Delta z \rightarrow 0} \Delta \bar{z} \\ & = \lim_{\Delta z \rightarrow 0} (\Delta z) = \lim_{\Delta z \rightarrow 0} \Delta \bar{z} = 0 \end{aligned}$$

Hence $f'(z)$ exists only at $z=0$.

(Exercise sheet #2)

4. Find all points of discontinuity for the function

$$f(z) = \frac{2z-3}{z^2+2z+2}.$$

$$z^2+2z+2 \neq 0$$

$$f(z) = \frac{2z-3}{z^2+2z+2}$$

$$\text{consider } z^2+2z+2=0$$

$$z = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 2}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

$\therefore f(z)$ is discontinuous at $z = -1 \pm i$.

Cauchy Riemann Equations

Named after French Mathematician Augustin Cauchy (1789-1857)
and German Mathematician Bernhard Riemann (1826-66).

$$u_x = v_y, \quad u_y = -v_x \quad (\text{Partial derivative})$$

Every single function which is complex differentiable must satisfy these equations while $f(z) = u(x,y) + iv(x,y)$

and $z_0 = (x_0, y_0)$ be a fixed point

$$\begin{array}{ccc} u_x & \cancel{v_x} \\ \cancel{u_y} & v_y \end{array}$$

Analytic function:



If $f'(z)$ exists at all points 'z' of a region R , then $f(z)$ is said to be analytic in R .

Necessary & Sufficient Condition

Suppose $f(z) = u(x, y) + iv(x, y)$ and $f'(z)$ exists at $z_0 = x_0 + iy_0$. Then the 1st order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy-Riemann equations:

$$U_x = V_y \quad ; \quad U_y = -V_x$$

→ This is known as Necessary condition

$$\begin{array}{ccc} U_x & \xrightarrow{(-)} & V_x \\ \cancel{U_y} & \cancel{\longrightarrow} & V_y \end{array}$$

Also $f'(z_0)$ can be written as:

$$f'(z_0) = U_x + iV_x$$

Sufficient condition

Alternatively

$$f'(z_0) = V_y - iU_y$$

where these partial derivatives are to be evaluated at (x_0, y_0)

For 2nd Partial Derivative

$$\begin{array}{c} u_{xx} \leftarrow (-) \\ u_{xy} \leftarrow \\ v_{xy} \leftarrow \\ v_{xx} \end{array}$$

$$u_{xx} = v_{xy} \quad \& \quad u_{xy} = -v_{xx} \quad \rightarrow \text{Necessary condition}$$

$$f''(z) = u_{xx} + i v_{xx} \quad \rightarrow \text{sufficient condition}$$

$$\text{or } f''(z) = v_{yy} - i u_{yy}$$

Examples:

① Show that if $f(z) = z - \bar{z}$ then $f'(z)$ does not exist.

$$\text{Given } f(z) = z - \bar{z} = x + iy - (x - iy) = 2iy$$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \begin{cases} \text{Necessary} \\ \text{Condition} \end{cases}$$

$$\begin{cases} f'(z_0) = u_x + i v_x \\ f'(z_0) = v_y - i u_y \end{cases} \quad \begin{cases} \text{Sufficient} \\ \text{Condition} \end{cases}$$

$$\therefore u(x, y) + i v(x, y) = 0 + i 2y$$

$$u(x, y) = 0 \quad ; \quad v(x, y) = 2y$$

$$u_x = 0, v_x = 0$$

$$u_y = 0, v_y = 2$$

$$u_x \neq v_y, u_y \neq -v_x$$

$\therefore f'(z)$ does not exist

by Necessary condition / Cauchy-Riemann Eqn.

② Show that $f'(z)$ and $f''(z)$ exist everywhere and find $f''(z)$ where $f(z) = z^3$.

$$f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2iy + 3x^2y^2 + i^3y^3 \\ = x^3 + i(3x^2y - 3xy^2 - iy^3) \\ = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$\begin{cases} i^2 = -1 \\ i^3 = -i \end{cases}$

$$u(x,y) = x^3 - 3xy^2 \quad \left\{ \begin{array}{l} v(x,y) = 3x^2y - y^3 \\ (-) \\ u_x = 3x^2 - 3y^2 \\ u_y = -6xy \end{array} \right. \quad \left\{ \begin{array}{l} v_x = 6xy \\ v_y = 3x^2 - 3y^2 \end{array} \right.$$

Here $u_x = v_y$; $u_y = -v_x$

Also, u_x, u_y, v_x, v_y are continuous everywhere
Hence $f'(z)$ exist and $f'(z) = u_x + iv_x \rightarrow$ sufficient condition

$$u_{yx} = u_{xy} \quad \left(\begin{array}{c} u_y \\ u_{yy} \end{array} \right) \quad \left(\begin{array}{c} u_x \\ u_{xx} \end{array} \right) \quad \begin{aligned} &= (3x^2 - 3y^2) + i6xy \\ &= 3(x^2 - y^2) + i6xy \end{aligned}$$

Similarly $u_{xx} = 6x \quad \left\{ \begin{array}{l} v_{xx} = 6y \\ v_{xy} = -6x \end{array} \right.$

Hence $u_{xx} = v_{xy}$ & $u_{xy} = -v_{xx}$
 \rightarrow Necessary Condition

Also $u_{xx}, u_{xy}, v_{xx}, v_{xy}$ are continuous everywhere,
hence $f''(z)$ exist and $f''(z) = u_{xx} + iv_{xx} = 6x + i6y = 6(x+iy)$

Harmonic Functions

A real valued function u of two real variables ' x ' and ' y ' is said to be harmonic in a given domain of the xy -plane if throughout that domain, it has continuous partial derivatives of the 1st and 2nd order and satisfies the partial differential eqn:

$$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

$$f(z) = u(x,y) + i v(x,y)$$

(Exercise Sheet #2)

7. Determine which of the following functions u are harmonic. For each harmonic function. Find the conjugate harmonic function v and express $u+iv$ as an analytic function of z :

$$\textcircled{1} \quad u = x e^x \cos y - y e^x \sin y$$

$$(uv)' = u'v + v'u$$

$$u = x e^x \cos y - y e^x \sin y$$

$$u_x = \cos y (x e^x + e^x) - y e^x \sin y$$

$$u_{xx} = \cos y (x e^x + e^x + e^x) - y e^x \sin y$$

$$= x e^x \cos y + \underline{2e^x \cos y} - \underline{y e^x \sin y}$$

$$u_y = -x e^x \sin y - e^x (y \cos y + \sin y)$$

$$= -e^x x \sin y - y e^x \cos y - e^x \sin y$$

$$u_{xx} + u_{yy} = 0$$

$$u_{yy} = -x e^x \cos y - e^x (-y \sin y + \cos y) - e^x \cos y$$

$$= -x e^x \cos y + e^x y \sin y - e^x \cos y - e^x \cos y$$

$$= \underline{-x e^x \cos y} - \underline{2e^x \cos y} + \underline{y e^x \sin y}$$

$$U_{xx} + U_{yy} = 0$$

∴ It is harmonic

$$\begin{matrix} U_x & -V_x \\ U & \times \\ & y \end{matrix}$$

Now we have to obtain $V(x, y)$

$$\text{by Necessary condition } \quad U_x(x, y) = x e^x \cos y + e^x \cos y - y e^x \sin y = V_y(x, y) \quad \textcircled{i}$$

$$\text{condition } \quad U_y(x, y) = -e^x x \sin y - y e^x \cos y - e^x \sin y = -V_x(x, y) \quad \textcircled{ii}$$

holding 'x' constant, integrate eqn \textcircled{i} with respect to 'y'.

$$\int V_y(x, y) dy = \int (x e^x \cos y + e^x \cos y - y e^x \sin y) dy$$

$$\int u dv = u v - \int v du$$

$$\begin{aligned} V(x, y) &= x e^x \sin y + e^x \sin y - e^x \int y \sin y dy \\ &= x e^x \sin y + e^x \sin y - e^x [-y \cos y - \int [1(-\cos y)] dy] \\ &= x e^x \sin y + e^x \sin y - e^x [-y \cos y + \sin y] + \phi(x) \end{aligned}$$

↑ Constant from
Integration ∵ x-fixed,
y-variable.

$$= x e^x \sin y + e^x \sin y + e^x y \cos y - e^x \sin y + \phi(x)$$

$$V(x, y) = x e^x \sin y + e^x y \cos y + \phi(x) \quad \textcircled{iii}$$

Now differentiate (iii) w.r.t. x (so that we can compare with eqn (ii) and evaluate $\phi(x)$)

$$V_x(x, y) = \sin y (xe^x + e^x) + e^x y \cos y + \phi'(x)$$

$$V_x(x, y) = xe^x \sin y + e^x \sin y + e^x y \cos y + \phi'(x) - \textcircled{iv}$$

Compare (ii) & (iv) { multiply (ii) with (-1)}

$$\begin{aligned} & e^x x \sin y + e^x y \cos y + e^x \sin y \\ & = \underline{\underline{e^x x \sin y}} + \underline{\underline{e^x y \cos y}} + \underline{\underline{e^x \sin y}} + \phi'(x) \end{aligned}$$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \int \phi'(x) dx = \int 0 dx$$

$$\Rightarrow \phi(x) = c$$

Substitute $\phi(x)$ into (iii)

$$\therefore V(x, y) = xe^x \sin y + e^x y \cos y + c$$

$$\therefore u + iv = xe^x \cos y - ye^x \sin y + i(xe^x \sin y + e^x y \cos y + c)$$

Example (Analytic function)

[i.e. $f(z) = \bar{z}$ is non-analytic everywhere]

$$f'(z) = \frac{d}{dz} f(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \text{by definition}$$

$$\checkmark f(z) = \bar{z}$$

$$\checkmark f(z + \Delta z) = \bar{z} + \bar{\Delta z}$$

$$\begin{aligned} \frac{d}{dz} \bar{z} &= \lim_{\Delta z \rightarrow 0} \frac{\bar{z} + \bar{\Delta z} - \bar{z}}{\Delta z} = \frac{\cancel{\bar{z}} + i\cancel{y} + \cancel{\Delta x} + i\cancel{\Delta y} - \cancel{\bar{x}} - i\cancel{\bar{y}}}{\Delta z} \\ &= \frac{\cancel{x} - i\cancel{y} + \cancel{\Delta x} - i\cancel{\Delta y} - (\cancel{x} - i\cancel{y})}{\cancel{\Delta x} + i\cancel{\Delta y}} \\ &= \lim_{\Delta z \rightarrow 0} \frac{\cancel{\Delta x} - i\cancel{\Delta y}}{\cancel{\Delta x} + i\cancel{\Delta y}} \end{aligned}$$



Δz approaches Re-axis $\Rightarrow \Delta y = 0$ & $\Delta x \rightarrow 0$

$$\lim_{\Delta x \rightarrow 0} \frac{\cancel{\Delta x} - i\cancel{\Delta y}}{\cancel{\Delta x} + i\cancel{\Delta y}} = \lim_{\Delta z \rightarrow 0} \frac{\cancel{\Delta x}}{\cancel{\Delta z}} = \lim_{\Delta z \rightarrow 0} \frac{1}{1} = 1$$

Δz approaches Im-axis $\Rightarrow \Delta x = 0$ & $\Delta y \rightarrow 0$

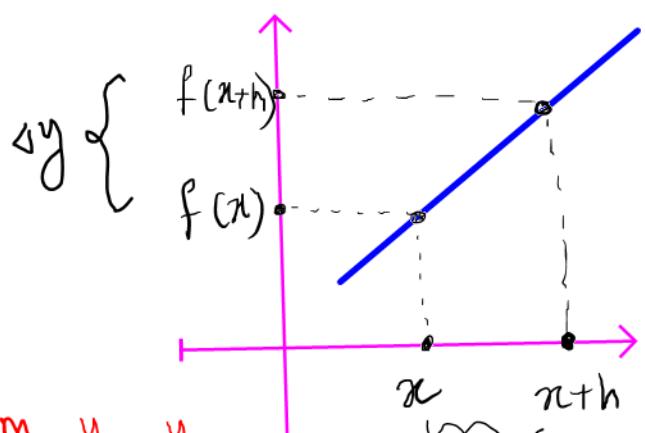
$$\lim_{\Delta y \rightarrow 0} \frac{\cancel{\Delta x} - i\cancel{\Delta y}}{\cancel{\Delta x} + i\cancel{\Delta y}} = \lim_{\Delta z \rightarrow 0} \frac{-i\cancel{\Delta y}}{i\cancel{\Delta y}} = \lim_{\Delta z \rightarrow 0} -1 = -1$$

Two sided limits are not equal, the derivative

does not exist.

$\Rightarrow f(z) = \bar{z}$ is non-analytic everywhere

Derivatives



$$M = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

$\therefore \Delta x = (x + h) - x$

$\therefore \Delta x = h$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{def of slope}$$

For Complex

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\boxed{\begin{array}{l} x \rightarrow z \\ \Delta x = h \rightarrow \Delta z \end{array}}$$

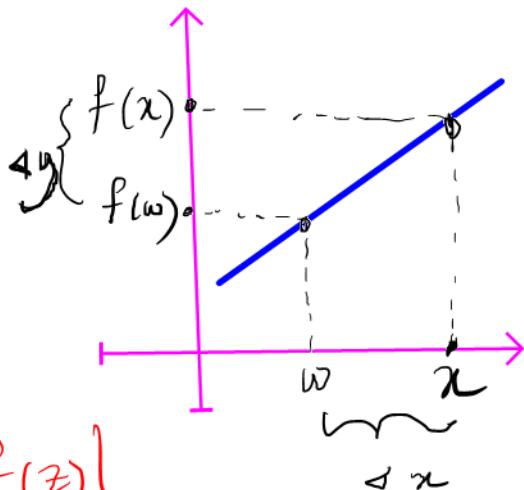
Alternatively,

$$\left\{ \begin{array}{l} y = f(x) \\ \frac{dy}{dx} = f'(x) \end{array} \right. \quad f'(x) = \lim_{x \rightarrow w} \frac{f(x) - f(w)}{x - w}$$

for Complex

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$\left. \begin{array}{l} w = f(z) \\ \frac{dw}{dz} = f'(z) \end{array} \right\}$$



ii) $\lim_{z \rightarrow \infty} f(z) = \infty$ iff $\lim_{z \rightarrow 0} \frac{1}{f(\frac{1}{z})} = 0$

$$\Rightarrow f(\infty) = \infty$$

$$= \frac{1}{f(\frac{1}{0})}$$

$$= \frac{1}{f(\infty)}$$

$$= \frac{1}{\infty} = 0$$

iii) $\lim_{z \rightarrow \infty} f(z) = \omega_0$ iff $\lim_{z \rightarrow 0} f(\frac{1}{z}) = \omega_0$

$$\Rightarrow f(\infty) = \omega_0$$

$$= f(\frac{1}{0})$$

$$= f(\infty)$$

$$= \omega_0$$

i) $\lim_{z \rightarrow z_0} f(z) = \infty$ iff $\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$

$$\Rightarrow f(z_0) = \infty$$

$$= \frac{1}{f(z_0)}$$

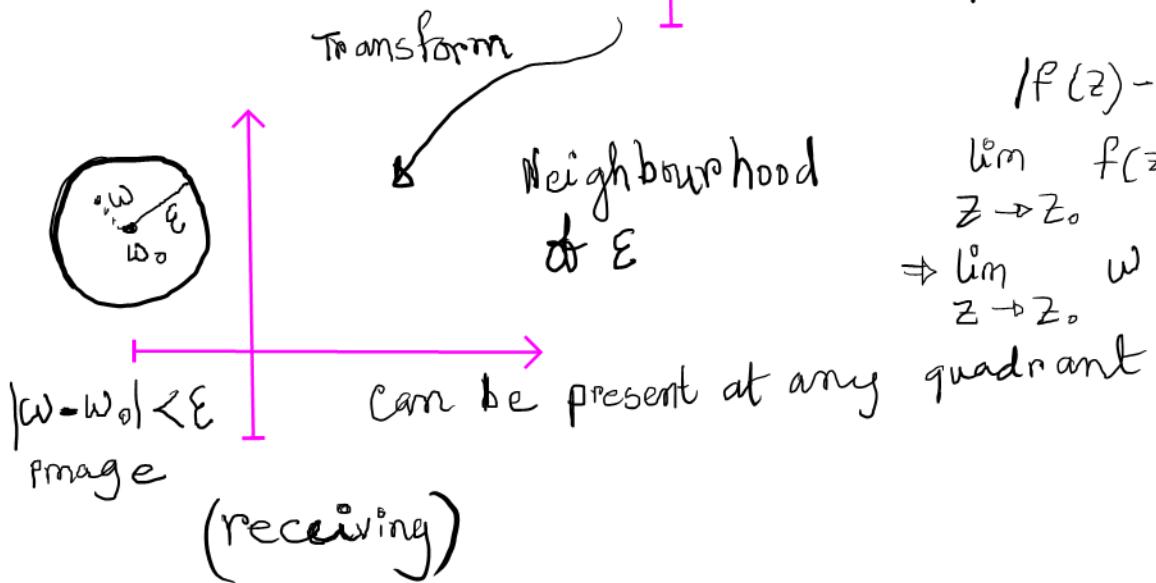
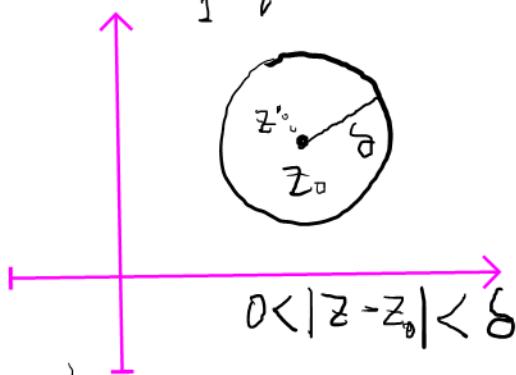
$$= \frac{1}{\infty} = 0$$

$$f(z) = w \xrightarrow{x+iy} u + iv$$

(x, y) (u, v)

1st quadrant

deleted
Neighbourhood
of δ



$$|f(z) - w_0| \Rightarrow |w - w_0|$$

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) &= w_0 \quad \because f(z) = w \\ \Rightarrow \lim_{z \rightarrow z_0} w &= w_0 \end{aligned}$$

$$\lim_{x \rightarrow \infty} 1^x = ?$$

$$\text{let } y = 1^x$$

$$\begin{aligned} \ln y &= \ln 1^x \\ &= x \ln 1 \end{aligned}$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} x \ln 1$$

$$\ln y = \infty \cdot 0$$

$$\log_e y = \infty$$

$$y = e^\infty = \infty$$

$$\lim_{x \rightarrow \infty} 1^x$$

$$= \lim_{x \rightarrow \infty} y$$

$$= \lim_{x \rightarrow \infty} \infty$$

$$= \infty$$

$$\therefore \lim_{x \rightarrow \infty} 1^x = \infty$$