CH 7.5 Integrating Rational Function by Partial Fraction

Anton's Calculus 10th Ed

Decompositions:

(i) $\frac{3x-1}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4} \rightarrow \frac{Ax^0}{x-3} + \frac{Bx^0}{x+4}$

(ii) $\frac{5}{x(x^2-4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$

(iii) $\frac{2x-3}{x^3-x^2} = \frac{2x-3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

(iv) $\frac{x^2}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$

$x^2 = x, x^2, 1$

$x^5 = x, x^2, x^3, x^4, x^5, 1$

(v) $\frac{1-x^2}{x^3(x^2+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+2}$

(vi) $\frac{3x}{(x-1)(x^2+6)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+6}$

$\frac{Ax^2+Bx+C}{x^3+2}$

(vii) $\frac{4x^3-x}{(x^2+5)^2} = \frac{Ax+B}{x^2+5} + \frac{Dx+E}{(x^2+5)^2}$

$(x^2+5)^2 = (x^2+5), (x^2+5)^2$

(viii) $\frac{1-3x^4}{(x-2)(x^2+1)^2} = \frac{A}{x-2} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$

a) $\frac{\cdots}{x^3+3} = \frac{Ax^2+Bx+C}{x^3+3}$

b) $\frac{\cdots}{x^2+3} = \frac{Ax+B}{x^2+3}$

c) $\frac{\cdots}{x+3} = \frac{A}{x+3}$

Improper ~~Integrate~~ fraction

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$$

$\int \frac{x^3}{x^2+2}\,dx$, $\int \frac{x^2}{x^2+2}\,dx$ → improper

$\int \frac{x}{x^2+2}\,dx$ → proper

$$\begin{array}{r} 3x^2 \quad +1 \\ x^2+x-2 \overline{)\,3x^4+3x^3-5x^2+x-1} \\ 3x^4+3x^3-6x^2 \\ \hline x^2+x-1 \\ x^2+x-2 \\ \hline 1 \end{array}$$

$3x^2 \times x^2 = 3x^4$

$1 \times x^2 = x^2$

$$\int \left[3x^2+1+\frac{1}{x^2+x-2}\right]dx$$

$$= \frac{3x^3}{3} + x + \int \frac{1}{x^2+x-2}\,dx$$

$$= x^3+x+\int \left(\frac{A}{x+1}+\frac{B}{x-2}\right)dx$$

$$= x^3+x+\frac{1}{3}\int\left(\frac{1}{x+1}-\frac{1}{x-2}\right)dx$$

$$= x^3+x+\frac{1}{3}\left[\ln|x+1|-\ln|x-2|\right]+C$$

$$= x^3+x+\frac{1}{3}\ln\left|\frac{x+1}{x-2}\right|+C$$

$\frac{3}{2} = 1\frac{1}{2}$,

$\frac{2}{2} = 1$

$\frac{1}{2}$ →

$x^2+x-2 = x^2-2x+x-2$

$= (x-2)(x+1)$

$$\frac{1}{x^2+x-2} = \frac{1}{(x+1)(x-2)}$$

$$\frac{1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{x-2}$$

$$\frac{1}{(x+1)(x-2)} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

$$1 = A(x-2)+B(x+1)$$

$0 \cdot x + 1$

Equating factor of x: $0 = A+B$

Equating like terms (constant): $1 = -2A+B$

$= Ax-2A+Bx+B$

$A = \frac{1}{3}$, $B = -\frac{1}{3}$

$\frac{13}{6} = 2\frac{1}{6}$

$6\overline{)13}$, 12, 1

$2 \times 6 + 1 = 13$

$2 + \frac{1}{6} = \frac{13}{6}$

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(17) $\dfrac{3x^2-10}{x^2-4x+4}$

$$\begin{array}{r} 3 \\ x^2-4x+4 \overline{)\,3x^2+0x-10} \\ 3x^2-12x+12 \\ \hline 12x-22 \Rightarrow R \end{array}$$

$$\frac{12x-22}{x^2-4x+4} = \frac{12x-22}{(x-2)^2}$$

$$3+\frac{12x-22}{x^2-4x+4}$$

Now $\int \left(3+\frac{12x-22}{x^2-4x+4}\right)dx$

$= 3x+\int\left(\frac{A}{(x-2)}+\frac{B}{(x-2)^2}\right)dx$

$= 3x+\int\left\{\frac{12}{x-2}+\frac{2}{(x-2)^2}\right\}dx$

$\frac{12x-22}{(x-2)^2} = \frac{A}{(x-2)}+\frac{B}{(x-2)^2}$

$\frac{12x-22}{(x-2)^2} = \frac{A(x-2)+B}{(x-2)^2}$

$12x-22 = A(x-2)+B = Ax-2A+B$

Equating factors of like terms:

$12x = Ax$; $12 = A$

$-22 = -2A+B$

$-22 = -2(12)+B$

$B = -22+24 = 2$

$B = 2$

$= 3x+12\ln|x-2|+\frac{2(x-2)^{-2+1}}{-2+1}+C$

$= 3x+12\ln|x-2|-\frac{2}{x-2}+C$

(29) $\int \frac{2x^2-1}{(4x-1)(x^2+1)}\,dx$

$= \int \frac{A}{4x-1}+\frac{Bx+C}{x^2+1}\,dx$

$= \int\left(\frac{A}{4x-1}+\frac{Bx}{x^2+1}+\frac{C}{x^2+1}\right)dx$

$= \int\left(\frac{-14/17}{4x-1}+\frac{\frac{12}{17}x}{x^2+1}+\frac{\frac{3}{17}}{x^2+1}\right)dx$

$= -\frac{14}{17}\cdot\frac{\ln|4x-1|}{4}+\frac{12}{17}\int\frac{x}{x^2+1}\,dx+\frac{3}{17}\tan^{-1}x+C$

$= \frac{-7}{34}\ln|4x-1|+\frac{12}{17}\cdot\frac{1}{2}\ln|x^2+1|+\frac{3}{17}\tan^{-1}x+C$

$= \frac{-7}{34}\ln|4x-1|+\frac{6}{17}\ln|x^2+1|+\frac{3}{17}\tan^{-1}x+C$

$\frac{2x^2-1}{(4x-1)(x^2+1)} = \frac{A}{4x-1}+\frac{Bx+C}{x^2+1}$

$\frac{2x^2-1}{(4x-1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(4x-1)}{(4x-1)(x^2+1)}$

$2x^2-1 = A(x^2+1)+(Bx+C)(4x-1)$

$2x^2-1 = Ax^2+A+4Bx^2+4Cx-Bx-C$

Equating factor of like terms:

x^2: $2 = A+4B$ (I)

x: $0 = 4C-B$ (II)

constant: $-1 = A-C$ (III)

(I) − (III) $\Rightarrow 3 = 4B+C$ (IV)

(II) × 4 $\Rightarrow -4B+16C = 0$

(IV) × 1 $\Rightarrow 4B+C = 3$

Add $\Rightarrow 17C = 3$

$C = \frac{3}{17}$

(III) $A-C = -1$

$A-\frac{3}{17} = -1$

$A = -1+\frac{3}{17}$

$= \frac{-14}{17}$

(II) $4C-B = 0$

$4C = B$

$B = 4\left(\frac{3}{17}\right) = \frac{12}{17}$

$\int\frac{x}{x^2+1}\,dx$ let $x^2+1 = z$

$= \int\frac{\frac{1}{2}dz}{z}$ $2x\,dx = dz$

$= \frac{1}{2}\ln z = \frac{1}{2}\ln(x^2+1)$ $x\,dx = \frac{1}{2}dz$