Exercise Sheet 2

1. Evaluate the following limits:

(i) $\lim\limits_{z\to 1+i} \dfrac{z^2 - z + 1 - i}{z^2 - 2z + 2}$ (ii) $\lim\limits_{z\to 1+i} \left\{\dfrac{z-1-i}{z^2 - 2z + 2}\right\}^2$ (iii) $\lim\limits_{z\to i} \dfrac{z^2+1}{z^6+1}$.

2. If $f(z) = \dfrac{2z-1}{3z+2}$, prove that $f'(z_0) = \lim\limits_{h\to 0} \dfrac{f(z_0+h) - f(z_0)}{h} = \dfrac{7}{(3z_0+2)^2}$ provided $z_0 \neq -\dfrac{2}{3}$.

3. Let $f(z) = \dfrac{z^2+4}{z-2i}$ if $z \neq 2i$, while $f(2i) = 3+4i$. Is $f(z)$ continuous at $z = 2i$?

4. Find all points of discontinuity for the function $f(z) = \dfrac{2z-3}{z^2+2z+2}$.

5. Using the definitions, find the derivative of each function at the indicated points

(i) $f(z) = \dfrac{2z-i}{z+2i}$ at $z = -i$

(ii) $f(z) = 3z^{-2}$ at $z = 1+i$.

6. Evaluate the following limits using L' Hôpital's rule

(i) $\lim\limits_{z\to 2i} \dfrac{z^2+4}{2z^2+(3-4i)z-6i}$ (ii) $\lim\limits_{z\to 0} \dfrac{z-\sin z}{z^3}$.

7. Determine which of the following functions u are harmonic. For each harmonic function find the conjugate harmonic function v and express $u + iv$ as an analytic function of z

i) $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ ii) $u(x,y) = xe^x \cos y - ye^x \sin y$

iii) $u(x,y) = e^{-x}(x \sin y - y \cos y)$