第三章多元函数微分学部分习题解答或答案

习题3.1

(A)

3.1.1. 设E为$\mathbb{R}^n$的点集, 证明: E为闭集的充要条件是$E=\overline{E}$.

证 若$E'\subset E$, 则$\bar{E}=E'\cup E=E$; 反之, 若$\bar{E}=E$, 则由$\bar{E}=E'\cup E$, 得到$E'\subset E$.

3.1.2. 设$E,F\subset\mathbb{R}^n$为有界闭集, 证明$E\cap F$和$E\cup F$都为有界闭集.

3.1.3. 设$E,F\subset\mathbb{R}^n$. 若E为开集, F为闭集, 证明: $E\setminus F$为开集, $F\setminus E$为闭集. (提示: $E\setminus F=E\cap F^c$)

证 (1) F闭$\Rightarrow$ F^c开, 从而$E\setminus F=E\cap F^c$开.

(2) E开$\Rightarrow$ E^c闭, 从而$F\setminus E=F\cap E^c$闭.

3.1.4. 求下列平面点集的导集, 闭包, 并说明是否为闭集:

(1) $E=\{(x,y)|\ x^2+y^2>3\}$;

(2) $E=\{(x,y)|\ x,\ y$ 为有理数 $\}$;

(3) $E=\left\{\left(\cos\dfrac{2k\pi}{5},\sin\dfrac{2k\pi}{5}\right)\Big|\ k=1,2,\cdots\right\}$;

(4) $E=\{(x,y)|\ (x^2+y^2)(y^2-x^2+1)\le 0\}$;

(5) $E=\{(x,y)|\ y=\sin(1/x),\ x\in(0,1]\}$.

解 (1) $E'=\{(x,y):\ x^2+y^2\ge 3\}=\bar{E}$, E非闭.

(2) $E'=\mathbf{R}^2=\bar{E}$, E非闭.

(3) $E'=\emptyset$, $\bar{E}=E$, E闭.

(4) $E'=\{(x,y):\ y^2-x^2+1\le 0\}$, $\bar{E}=E$, E闭.

(5) $E'=\{(x,y):\ y=\sin(1/x),x\in(0,1]\ \ or\ \ x=0,y\in[-1,1]\}$. $\bar{E}=E'$, E非闭.

3.1.5. 设$\vec{x}=(x_1,\cdots,x_n)\in\mathbb{R}^n$, $\vec{y}=(y_1,\cdots,y_n)\in\mathbb{R}^n$, $1\le p<\infty$. 定义两

距离:

$$\rho_1(\vec{x},\vec{y}) = \max\{|x_j - y_j|:\ 1 \le j \le n\}, \qquad \rho_2(\vec{x},\vec{y}) = \Big(\sum_{j=1}^{n} |x_j - y_j|^p\Big)^{1/p}.$$

证明: $\rho_1(\vec{x},\vec{y}) \le \rho_2(\vec{x},\vec{y}),\quad \rho_2(\vec{x},\vec{y}) \le \sqrt[p]{n}\rho_1(\vec{x},\vec{y})$.

证 (1) 由$\forall k \in \{1,2,\cdots,n\}$, 有

$$|x_k - y_k| \le \Big(\sum_{j=1}^{n} |x_j - y_j|^p\Big)^{1/p}$$

得证$\rho_1(\vec{x},\vec{y}) \le \rho_2(\vec{x},\vec{y})$.

(2) 因为

$$|x_1 - y_1|^p + \cdots + |x_n - y_n|^p \le n \cdot \max_{1\le j\le n} |x_j - y_j|^p = n \cdot \Big(\max_{1\le j\le n} |x_j - y_j|\Big)^p,$$

所以$\rho_2(\vec{x},\vec{y}) \le n^{1/p}\rho_1(\vec{x},\vec{y})$.

3.1.6. 设A是n维欧氏空间的有界闭集, 映射$F: A \to A$满足如下条件: $\forall \vec{x},\vec{y} \in A\ (\vec{x} \ne \vec{y})$, 有$|F\vec{x} - F\vec{y}| < |\vec{x} - \vec{y}|$. 证明: 存在常数$a$使得$\forall \vec{x},\vec{y} \in A\ (\vec{x} \ne \vec{y})$, 有$|F\vec{x} - F\vec{y}| < a|\vec{x} - \vec{y}|$.

a\in (0,1)

证 用反证法. 假设$\forall a \in (0,1)$, 存在$\vec{x}_a$, $\vec{y}_a \in A$, $\vec{x}_a \ne \vec{y}_a$, 使得

$$|F\vec{x}_a - F\vec{y}_a| \ge a|\vec{x}_a - \vec{y}_a|.$$

取$a = 1 - \dfrac{1}{n}$时, 相应有$\vec{x}_n$, $\vec{y}_n \in A$, 使得

$$|F\vec{x}_n - F\vec{y}_n| \ge (1 - \frac{1}{n})|\vec{x}_n - \vec{y}_n|.$$

由A为n维欧氏空间的有界闭集可知, $\{\vec{x}_n\}$和$\{\vec{y}_n\}$相应有收敛子列$\{\vec{x}_{n_k}\}$和$\{\vec{y}_{n_k}\}$, 记它们的极限依次为$\vec{x}_0$, $\vec{y}_0$, 则$\vec{x}_0 \in A$, $\vec{y}_0 \in A$. 最后由

$$|F(\vec{x}_{n_k}) - F(\vec{y}_{n_k})| \ge (1 - \frac{1}{n_k})|\vec{x}_{n_k} - \vec{y}_{n_k}|$$

和映射F的条件(推出F连续), 让$k \to \infty$, 得到$|F\vec{x}_0 - F\vec{y}_0| \ge |\vec{x}_0 - \vec{y}_0|$. 这与已知矛盾.

习题3.2

(A)

3.2.1. 确定下列函数的定义域:

(1) $z = \arccos \dfrac{y}{x}$; (3) $z = \sqrt{\dfrac{2x - x^2 - y^2}{x^2 + y^2 - x}}$.

解 (1) 依题意知: 所求的定义域为$\{(x, y) \in \mathbb{R}^2 : |y| \le |x|,\ x \ne 0\}$.

(3) 依题意知: $\dfrac{2x - x^2 - y^2}{x^2 + y^2 - x} \ge 0$且$x^2 + y^2 - x \ne 0$. 于是, $2x - x^2 - y^2 \ge 0,\ x^2 + y^2 - x > 0$或$2x - x^2 - y^2 \le 0,\ x^2 + y^2 - x < 0$. 设

$$D_1 = \{(x, y)|(x-1)^2 + y^2 \le 1\} \cap \{(x, y)|(x - \frac{1}{2})^2 + y^2 > \frac{1}{4}\},$$

$$D_2 = \{(x, y)|(x-1)^2 + y^2 \ge 1\} \cap \{(x, y)|(x - \frac{1}{2})^2 + y^2 < \frac{1}{4}\} = \emptyset,$$

即得所求的定义域$D = D_1 = \{(x, y)|\ x < x^2 + y^2 \le 2x\}$.

3.2.2. 求下列函数极限：

(1) $\lim\limits_{(x,y)\to(0,1)} \dfrac{x + e^y}{x^2 + y^2}$; (2) $\lim\limits_{(x,y)\to(2,0)} \dfrac{1}{x^2 + y^2}$;

(3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x}$; (4) $\lim\limits_{(x,y)\to(+\infty,+\infty)} (x^2 + y^2)e^{-(x+y)}$.

解 (1) $\lim\limits_{(x,y)\to(0,1)} \dfrac{x + e^y}{x^2 + y^2} = \dfrac{0 + e^1}{0^2 + 1^2} = e$;

(2) $\lim\limits_{(x,y)\to(2,0)} \dfrac{1}{x^2 + y^2} = \dfrac{1}{2^2 + 0^2} = \dfrac{1}{4}$;

(3) 因为对$x \ne 0$有$\left|\dfrac{\sin(xy)}{x}\right| \le \dfrac{|xy|}{|x|} = |y|$, 所以$\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x} = 0$;

(4) 令$x = r\cos\theta,\ y = r\sin\theta\ (0 \le \theta \le \pi/2)$, 则

$$\text{原式} = \lim_{r\to+\infty} \frac{r^2}{\exp\left(r \cdot \sqrt{2}\sin(\theta + (\pi/4))\right)} = 0,$$

这是因为$0 \le \theta \le \pi/2$, 从而有$\sqrt{2}\sin(\theta + (\pi/4)) \ge 1$. 由$\lim\limits_{r\to+\infty} r^2 e^{-r} = 0$即得.

3.2.3. 讨论函数$f(x, y) = \begin{cases} \dfrac{\sin(xy)}{\sqrt{x^2 + y^2}}, & x^2 + y^2 \ne 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$ 的连续性：

解 $|\sin(xy)| \le |xy|$. 由

$$\Big|\frac{xy}{\sqrt{x^2+y^2}}\Big| \le \frac{x^2+y^2}{2\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2}}{2} \to 0\ ((x,y)\to(0,0)),$$

得到$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$. 因此,

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{\sqrt{x^2+y^2}} = 0 = f(0,0).$$

故, $f(x,y)$在$(0,0)$点连续. 又, 当$x_0^2+y_0^2 \ne 0$时, $f(x,y)$在点(x_0,y_0)连续. 因此, $f(x,y)$在$\mathbb{R}^2$上连续.

3.2.5. 设$f: D\subseteq \mathbb{R}^2 \to \mathbb{R}$, 若$f(x,y)$在区域$D$内对变量$x$连续，变量$y$满足Lipschitz条件, 即对$D$内任意两点$(x,y_1),(x,y_2)$, 有

$$|f(x,y_1)-f(x,y_2)| \le L|y_1-y_2|,$$

其中L为常数, 证明：$f(x,y)$在区域D内连续.

证 任取$(x_0,y_0)\in D$, 考虑

$$|f(x,y)-f(x_0,y_0)| \le |f(x,y)-f(x_0,y)| + |f(x_0,y)-f(x_0,y_0)|.$$

依题意知, 对每个y, $\forall \varepsilon>0$, $\exists \delta_1>0$, 使当$|x-x_0|<\delta_1$时有

$$|f(x,y)-f(x_0,y)| < \frac{\varepsilon}{2};$$

另一方面, 由于$|f(x_0,y)-f(x_0,y_0)| \le L|y-y_0|$, 所以当$|y-y_0| < \dfrac{\varepsilon}{2L}$时有

$$|f(x_0,y)-f(x_0,y_0)| \le \frac{\varepsilon}{2}.$$

综上知: 若取$\delta = \min\{\delta_1, \dfrac{\varepsilon}{2L}\}$, 则$\delta>0$, 且当$|x-x_0|<\delta$, $|y-y_0|<\delta$时有

$$|f(x,y)-f(x_0,y_0)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

由此得证$f(x,y)$在区域D内连续.

3.2.6. 证明下列极限不存在：

(2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$;　　　　(4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^3+y^3}$.

证　(2)依题意知:令$y=kx$得

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-(kx)^2}{x^2+(kx)^2} = \frac{1-k^2}{1+k^2},$$

与k值有关, 故极限不存在.

(4)依题意知: 令$y=kx$得

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{x^2y^2}{x^3+y^3} = \lim_{x\to 0} \frac{x^2\cdot k^2x^2}{x^3+k^3x^3} = 0;$$

而当$y=x\sqrt[3]{x-1}$时有

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x\sqrt[3]{x-1}}} \frac{x^2y^2}{x^3+y^3} = \lim_{x\to 0} \frac{x^2\cdot x^2(x-1)^{\frac{2}{3}}}{x^3+x^4-x^3} = 1,$$

故极限不存在. (也可直接取曲线$x^3+y^3=kx^4$而得函数沿该曲线的极限为$1/k$)

(B)

3.2.2. 用定义证明(2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{xy+1}-1}{xy} = \dfrac{1}{2}$.

证　因为

$$\Big|\frac{\sqrt{xy+1}-1}{xy}-\frac{1}{2}\Big| = \Big|\frac{1}{\sqrt{xy+1}+1}-\frac{1}{2}\Big| = \frac{|xy|}{2(1+\sqrt{xy+1})^2} \le \frac{x^2+y^2}{4},$$

所以$\forall\varepsilon>0$, 取$\delta=2\sqrt{\varepsilon}$, 则当$0<\sqrt{x^2+y^2}<\delta$时, 有

$$\Big|\frac{\sqrt{xy+1}-1}{xy}-\frac{1}{2}\Big| \le \frac{x^2+y^2}{4} < \frac{\delta^2}{4} = \frac{4\varepsilon}{4} = \varepsilon.$$

故$\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{xy+1}-1}{xy} = \dfrac{1}{2}$.

3.2.3. 讨论极限$\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{\sqrt{|x|}}{3x+2y}$的存在性. 若存在求此极限, 若不存在则说明理由.

解 极限不存在. 理由如下: 取$y=\dfrac{1}{2}(\sqrt[3]{x}-3x)$, 则当$x\to+\infty$时有$y\to-\infty$, 且$\lim\limits_{\substack{x\to+\infty\\ y=\frac{1}{2}(\sqrt[3]{x}-3x)}}\dfrac{\sqrt{|x|}}{3x+2y}=\lim\limits_{x\to+\infty}\dfrac{\sqrt{x}}{\sqrt[3]{x}}=+\infty$, 因此原极限不存在.

习题3.3

(A)

3.3.1. 求下列函数的偏导数:

(2) $z=x^yy^x$; (5) $z=\dfrac{\cos(x^2)}{y}$; (8) $z=\ln\sqrt{x^2+y^2}$;

(11) $u=(xy)^z$; (14) $u=xye^{\sin(yz)}$.

解 (2) $z_x=x^{y-1}y^{x+1}+x^yy^x\ln y,\quad z_y=x^yy^x\ln x+x^{y+1}y^{x-1}$.

(5) $z_x=-\dfrac{2x\sin(x^2)}{y},\quad z_y=-\dfrac{\cos(x^2)}{y^2}$.

(8) $z_x=\dfrac{x}{x^2+y^2},\quad z_y=\dfrac{y}{x^2+y^2}$.

(11) $u_x=zx^{z-1}y^z,\quad u_y=zx^zy^{z-1},\quad u_z=(xy)^z\ln(xy)$.

(14) $u_x=ye^{\sin(yz)},\quad u_y=xe^{\sin(yz)}+xyze^{\sin(yz)}\cos(yz)$,

$u_z=xy^2e^{\sin(yz)}\cos(yz)$.

3.3.3. 求下列函数的全微分:

(4) $u=x^{yz}$; (5) $z=\dfrac{y}{\sqrt{x^2+y^2}}$; (6) $z=x^2y+\dfrac{x}{y}$.

解 (4) $\mathrm{d}u=yzx^{yz-1}\mathrm{d}x+zx^{yz}\ln x\mathrm{d}y+yx^{yz}\ln x\mathrm{d}z$.

(5) $\mathrm{d}z=\dfrac{-xy}{(x^2+y^2)\sqrt{x^2+y^2}}\,\mathrm{d}x+\dfrac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}\,\mathrm{d}y$.

(6) $\mathrm{d}z=(2xy+\dfrac{1}{y})\mathrm{d}x+(x^2-\dfrac{x}{y^2})\mathrm{d}y$.

3.2.5. (1) 研究$f(x,y)=\begin{cases}x\sin\frac{1}{x^2+y^2}, & x^2+y^2\neq0\\ 0, & x^2+y^2=0\end{cases}$在点$(0,0)$是否存在偏导数$f_x(0,0)$及$f_y(0,0)$; (2) 设函数$f(x,y)=|x-y|g(x,y)$, 其中函数$g(x,y)$在点$(0,0)$的某邻域内连续，试问$g(0,0)$为何值时, f在点$(0,0)$的两个偏导数均存在? $g(0,0)$为何值时, f在点$(0,0)$处可微?

解 (1) 因为极限

$$\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x\sin\frac{1}{(\Delta x)^2}}{\Delta x}=\lim_{\Delta x\to 0}\sin\frac{1}{(\Delta x)^2}$$

不存在, 所以$f_x(0,0)$不存在; 因为

$$\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0\cdot\sin\frac{1}{(\Delta y)^2}}{\Delta y}=0,$$

所以$f_y(0,0)=0$.

(2)依题意知:

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{|\Delta x|g(\Delta x,0)}{\Delta x}=\lim_{\Delta x\to 0}\pm g(\Delta x,0)=\pm g(0,0),$$

其中$\Delta x\to 0^+$时取正号, $\Delta x\to 0^-$时取负号; 下式类似.

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{|\Delta y|g(0,\Delta y)}{\Delta y}=\pm g(0,0).$$

所以当$g(0,0)=0$时, f在点$(0,0)$的两个偏导数均存在, 均为0.

当$g(0,0)=0$时, 函数f在点$(0,0)$处也可微. 事实上,

$$\lim_{(x,y)\to(0,0)}g(x,y)=g(0,0)=0,$$

而在极坐标系下

$$\frac{|x-y|}{\sqrt{x^2+y^2}}=|\cos\theta-\sin\theta|\le 2,$$

所以

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-xf_x(0,0)-yf_y(0,0)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{|x-y|g(x,y)}{\sqrt{x^2+y^2}}=0.$$

3.3.6. 设x,y的绝对值都很小, 利用全微分概念推出下列各式的近似计算公式:

(1) $(1+x)^m(1+y)^n$;　　　　(2) $\arctan\dfrac{x+y}{1+xy}$.

解 (1) 设$f(s,t)=s^mt^n$, 则$f(1,1)=1$, $f_s(1,1)=m$, $f_t(1,1)=n$. 所以由$f(1+x,1+y)\approx f(1,1)+f_s(1,1)x+f_t(1,1)y$, 得到

$$(1+x)^m(1+y)^n\approx 1+mx+ny.$$

(2) 设$f(x,y)=\arctan\dfrac{x+y}{1+xy}$, 则$f(0,0)=0$, $f_x(0,0)=1=f_y(0,0)$. 所以由$f(x,y)\approx f(0,0)+xf_x(0,0)+yf_y(0,0)$, 得到

$$\arctan\frac{x+y}{1+xy}\approx x+y.$$

3.3.7 近似计算下列数值：(1) $\sin 29^\circ\tan 46^\circ$.

解 (1) 设$f(x,y)=\sin x\tan y$, 令 $x_0=30^\circ$, $y_0=45^\circ$, $\Delta x=-1^\circ$, $\Delta y=1^\circ$, 由全微分的近似公式知:

$$\begin{aligned}\sin 29^\circ\tan 46^\circ &= f(x_0+\Delta x, y_0+\Delta y)\\ &\approx f(30^\circ,45^\circ)+f_x(30^\circ,45^\circ)\Delta x+f_y(30^\circ,45^\circ)\Delta y\\ &= \frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{-\pi}{180}+\frac{1}{2}\cdot 2\cdot\frac{\pi}{180}\\ &\approx 0.5023.\end{aligned}$$

3.3.8. 设函数$f(t)$有二阶连续导数, $r=\sqrt{x^2+y^2}$, $g(x,y)=f(\dfrac{1}{r})$, 求$\dfrac{\partial^2 g}{\partial x^2}+\dfrac{\partial^2 g}{\partial y^2}$.

解 $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$, $\dfrac{\partial r}{\partial y}=\dfrac{y}{r}$, 于是

$$\frac{\partial g}{\partial x}=-\frac{x}{r^3}f'(\frac{1}{r}),\quad \frac{\partial^2 g}{\partial x^2}=\frac{x^2}{r^6}f''(\frac{1}{r})+\frac{2x^2-y^2}{r^5}f'(\frac{1}{r}).$$

由对称性, 有

$$\frac{\partial^2 g}{\partial y^2}=\frac{y^2}{r^6}f''(\frac{1}{r})+\frac{2y^2-x^2}{r^5}f'(\frac{1}{r}).$$

故

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}=\frac{1}{r^4}f''(\frac{1}{r})+\frac{1}{r^3}f'(\frac{1}{r}).$$

3.3.9 求下列函数的高阶偏导数(假设函数f具有二阶连续偏导数或二阶连续导数, 函数g具有二阶连续导数)：

(1) $z=e^x(\cos y+x\sin y)$　　所有二阶偏导数;

(3) $z=f(xy^2,x^2y)$　　所有二阶偏导数;

(5) $z = f\big(xy, \dfrac{x}{y}\big) + g\big(\dfrac{y}{x}\big)$, $\qquad \dfrac{\partial^2 z}{\partial x \partial y}$;

(7) $z = f(x^2 - y^2, xy)$, $\qquad \dfrac{\partial^2 z}{\partial x \partial y}$.

解 (1) 由$z_x = e^x(\cos y + x\sin y) + e^x \sin y$, $z_y = e^x(-\sin y + x\cos y)$知:

$$\begin{aligned} z_{xx} &= e^x(\cos y + x\sin y + 2\sin y), \\ z_{xy} &= z_{yx} = e^x(-\sin y + x\cos y + \cos y), \\ z_{yy} &= -e^x(\cos y + x\sin y). \end{aligned}$$

(3) 由$z_x = y^2 f_1 + 2xyf_2$, $z_y = 2xyf_1 + x^2 f_2$知:

$$\begin{aligned} z_{xx} &= y^2(y^2 f_{11} + 2xyf_{12}) + 2yf_2 + 2xy(y^2 f_{21} + 2xyf_{22}) \\ &= y^4 f_{11} + 4xy^3 f_{12} + 4x^2y^2 f_{22} + 2yf_2, \\ z_{xy} &= z_{yx} = 2yf_1 + y^2(2xyf_{11} + x^2 f_{12}) + 2xf_2 + 2xy(2xyf_{12} + x^2 f_{22}) \\ &= 2yf_1 + 2xf_2 + 2xy^3 f_{11} + 5x^2y^2 f_{12} + 2x^3 y f_{22} \\ z_{yy} &= 2xf_1 + 2xy(2xyf_{11} + x^2 f_{12}) + x^2(2xyf_{21} + x^2 f_{22}) \\ &= 2xf_1 + 4x^2y^2 f_{11} + 4x^3 y f_{12} + x^4 f_{22}. \end{aligned}$$

(5) 由$z_x = yf_1 + \dfrac{1}{y} f_2 - \dfrac{y}{x^2} g'$知:,

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= f_1 + y(xf_{11} - \frac{x}{y^2} f_{12}) + (xf_{21} - \frac{x}{y^2} f_{22})\frac{1}{y} - \frac{1}{y^2} f_2 - \frac{y}{x^3} g'' - \frac{1}{x^2} g' \\ &= f_1 + xyf_{11} - \frac{x}{y^3} f_{22} - \frac{1}{y^2} f_2 - \frac{y}{x^3} g'' - \frac{1}{x^2} g'. \end{aligned}$$

(7) 由$z_x = 2xf_1 + yf_2$知:

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= 2x(-2yf_{11} + xf_{12}) + (-2yf_{21} + xf_{22})y + f_2 \\ &= -4xyf_{11} + 2(x^2 - y^2) f_{12} + xyf_{22} + f_2. \end{aligned}$$

3.3.10. 利用一阶全微分形式不变性和微分运算法则, 求下列函数的全微分和偏导数(设φ与f均可微):

(2) $z = e^{xy}\sin(x+y)$; $\qquad$ (4) $u = f(x^2 - y^2, e^{xy}, z)$.

解 (2) 函数的全微分

$$\begin{aligned}
\mathrm{d}z &= e^{xy}\mathrm{d}(\sin(x+y)) + \sin(x+y)\mathrm{d}(e^{xy}) \\
&= e^{xy}\cos(x+y)\mathrm{d}(x+y) + e^{xy}\sin(x+y)\mathrm{d}(xy) \\
&= e^{xy}\cos(x+y)(\mathrm{d}x+\mathrm{d}y) + e^{xy}\sin(x+y)(y\mathrm{d}x + x\mathrm{d}y) \\
&= e^{xy}(\cos(x+y) + y\sin(x+y))\mathrm{d}x + e^{xy}(\cos(x+y) + x\sin(x+y))\mathrm{d}y.
\end{aligned}$$

所以, $z_x = e^{xy}(\cos(x+y) + y\sin(x+y))$, $z_y = e^{xy}(\cos(x+y) + x\sin(x+y))$.

(4) 函数的全微分

$$\begin{aligned}
\mathrm{d}u &= f_1 \cdot \mathrm{d}(x^2 - y^2) + f_2 \cdot \mathrm{d}(e^{xy}) + f_3\mathrm{d}z \\
&= f_1 \cdot (2x\mathrm{d}x - 2y\mathrm{d}y) + e^{xy}f_2 \cdot (y\mathrm{d}x + x\mathrm{d}y) + f_3\mathrm{d}z \\
&= (2xf_1 + ye^{xy}f_2)\mathrm{d}x + (-2yf_1 + xe^{xy}f_2)\mathrm{d}y + f_3\mathrm{d}z.
\end{aligned}$$

所以, $u_x = 2xf_1 + ye^{xy}f_2$, $u_y = -2yf_1 + xe^{xy}f_2$, $u_z = f_3$.

3.3.11. 设函数$u = u(x, y)$具有二阶连续偏导数. 试求常数a和b, 使得在变换

$$\xi = x + ay, \qquad \eta = x + by$$

之下, 可将方程$\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + 3\dfrac{\partial^2 u}{\partial y^2} = 0$ 化为$\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$.

解 $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial x} = \dfrac{\partial u}{\partial \xi} + \dfrac{\partial u}{\partial \eta}$, $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial y} = a\dfrac{\partial u}{\partial \xi} + b\dfrac{\partial u}{\partial \eta}$; 所以

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}, \\
\frac{\partial^2 u}{\partial x \partial y} &= a\frac{\partial^2 u}{\partial \xi^2} + (a+b)\frac{\partial^2 u}{\partial \xi \partial \eta} + b\frac{\partial^2 u}{\partial \eta^2}, \\
\frac{\partial^2 u}{\partial y^2} &= a^2\frac{\partial^2 u}{\partial \xi^2} + 2ab\frac{\partial^2 u}{\partial \xi \partial \eta} + b^2\frac{\partial^2 u}{\partial \eta^2}.
\end{aligned}$$

由方程$\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + 3\dfrac{\partial^2 u}{\partial y^2} = 0$在变换下可化为$\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$得

$$1 + 4a + 3a^2 = 0, \qquad 1 + 4b + 3b^2 = 0,$$

所以$a = -1$或 $-\dfrac{1}{3}$ $b = -1$或 $-\dfrac{1}{3}$; 经检验知:$a = -1$, $b = -\dfrac{1}{3}$; 或 $a = -\dfrac{1}{3}$, $b = -1$. (最后一步的说明: 由二阶混合偏导数$u_{\xi\eta}$的系数之和$\alpha := 2 + 4(a+b) + 6ab$知$a \neq b$, 否则系数$\alpha = 0$, 不合要求)

3.3.14. 设$u=u(\sqrt{x^2+y^2})$具有二阶连续偏导数, 且满足$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}-\dfrac{1}{x}\dfrac{\partial u}{\partial x}+u=x^2+y^2$, 试求函数$u$的表达式.

解 设$x=r\cos\theta,\ y=r\sin\theta$, 则

$$u=u(r),\qquad \frac{\partial u}{\partial x}=\frac{x}{r}u'(r),\qquad \frac{\partial u}{\partial y}=\frac{y}{r}u'(r).$$

于是

$$\frac{\partial^2 u}{\partial x^2}=\frac{x^2}{r^2}u''(r)+\frac{y^2}{r^3}u'(r),\qquad \frac{\partial^2 u}{\partial y^2}=\frac{y^2}{r^2}u''(r)+\frac{x^2}{r^3}u'(r).$$

方程$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}-\dfrac{1}{x}\dfrac{\partial u}{\partial x}+u=x^2+y^2$化为方程

$$u''(r)+u(r)=r^2.$$

齐次方程的通解为$u=C_1\cos r+C_2\sin r$. 为求一特解$\tilde{u}$, 设$\tilde{u}(r)=ar^2+br+c$. 代入上式得: $2a+ar^2+br+c=r^2$, 所以$a=1,\ b=0,\ c=-2$, $\tilde{u}(r)=r^2-2$, 即$\tilde{u}(\sqrt{x^2+y^2})=x^2+y^2-2$. 故$u=C_1\cos r+C_2\sin r+r^2-2$, 即

$$u=C_1\cos\sqrt{x^2+y^2}+C_2\sin\sqrt{x^2+y^2}+x^2+y^2-2,$$

其中$C_1,\ C_2$为任意常数.

3.3.17. 若$u=f(xyz),\ f(0)=0,\ f'(1)=1$且$\dfrac{\partial^3 u}{\partial x\partial y\partial z}=x^2y^2z^2f'''(xyz)$, 求$u$.

解 依题意知$\dfrac{\partial u}{\partial x}=yzf'$, 且

$$\frac{\partial^2 u}{\partial x\partial y}=zf'+xyz^2f'',\quad \frac{\partial^3 u}{\partial x\partial y\partial z}=f'+3xyzf''+x^2y^2z^2f'''=x^2y^2z^2f''',$$

所以$f'+3xyzf''=f'+3tf''=0$ (令 $t=xyz$); 解得

$$f'(t)=c_1t^{-\frac{1}{3}},\ f(t)=\frac{3}{2}c_1t^{\frac{2}{3}}+c_2;$$

又$f(0)=0,\ f'(1)=1$, 所以$c_1=1,\ c_2=0$, 即$u(xyz)=\dfrac{3}{2}(xyz)^{\frac{2}{3}}$.

3.3.21. 设函数$f(x,y)$有二阶连续偏导数，满足$f_y\neq 0$, 且

$$f_x^2f_{yy}-f_xf_yf_{xy}+f_y^2f_{xx}=0,$$

$y=y(x,z)$是由方程$z=f(x,y)$所确定的函数，求$\dfrac{\partial^2 y}{\partial x^2}$.

解 方程$z=f(x,y)$两边对x求两次偏导，依次得到

$$0=f_x+f_y\frac{\partial y}{\partial x},\quad 0=f_{xx}+2f_{xy}\frac{\partial y}{\partial x}+f_{yy}\Big(\frac{\partial y}{\partial x}\Big)^2+f_y\frac{\partial^2 y}{\partial x^2}.$$

由第一个式子得到$\dfrac{\partial y}{\partial x}=-\dfrac{f_x}{f_y}$, 代入第二个式子并用已知条件, 得

$$f_y\frac{\partial^2 y}{\partial x^2}=0,\qquad \therefore\quad \frac{\partial^2 y}{\partial x^2}=0.$$

(B)

3.3.3. 若函数$f(x,y)$对任意正实数t满足关系$f(tx,ty)=t^nf(x,y)$, 则称$f(x,y)$为n次齐次函数. 设$f(x,y)$可微, 试证明$f(x,y)$为n次齐次函数的充要条件是

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y).$$

该题称为Euler定理.

证 必要性. 已知$f(tx,ty)=t^nf(x,y)$, 则对式子两边关于t求导, 得到

$$xf_{(tx)}(tx,ty)+yf_{(ty)}(tx,ty)=nt^{n-1}(x,y).$$

令$t=1$, 得

$$xf_x+yf_y=nf.$$

充分性. 令$x=u,\ y=uv$, 并记$f(x,y)=f(u,uv):=g(u,v)$, 则

$$g_u(u,v)=f_x+vf_y.$$

在其两端同乘以x, 并注意到$x=u,\ y=uv$, 得到$ug_u=xf_x+yf_y=nf$, 即$ug_u=ng$. 由此解得$g=\pm u^ne^{\alpha(v)}$, 其中$\alpha(v)$为仅与v有关的式子. 因此, $f(x,y)=\pm x^ne^{\alpha(y/x)}$, 从而有$f(tx,ty)=t^nf(x,y)$. 故$f(x,y)$为$n$次齐次函数.

注: 充分性的另一证法. 令$F(t)=\dfrac{f(tx,ty)}{t^n}$, 则

$$F'(t)=\frac{t^n(xf_{(tx)}(tx,ty)+yf_{(ty)}(tx,ty))-nt^{n-1}f(tx,ty)}{t^{2n}}$$

$$= \frac{(tx)f_{(tx)}(tx,ty)+(ty)f_{(ty)}(tx,ty))-nf(tx,ty)}{t^{n+1}}$$
$$= 0.$$

于是F与t无关, 可设$F(t)=\varphi(x,y)$, 从而得到$\dfrac{f(tx,ty)}{t^n}=\varphi(x,y)$. 令$t=1$, 得到$f(x,y)=\varphi(x,y)$. 故$f(tx,ty)=t^n\varphi(x,y)=t^nf(x,y)$, 即$f(x,y)$为$n$次齐次函数.

习题3.4

(A)

3.4.2. (1)设$f_x(a,b,c)=2, f_y(a,b,c)=3, f_z(a,b,c)=1$, 求三个不同的单位向量$\vec{l}$, 使得$\frac{\partial f(a,b,c)}{\partial l}$为零. (2) 有多少个单位向量$\vec{l}$使$\frac{\partial f}{\partial l}$在点$(a,b,c)$的值为零?

解 设所求单位向量为(x,y,z), 则x,y,z满足$\begin{cases}2x+3y+z=0\\x^2+y^2+z^2=1\end{cases}$. 所以这样的单位向量有无穷多个, 比如可取$(1,0,-2)/\sqrt{5}$, $(0,1,-3)/\sqrt{10}$, $(1,1,-5)/\sqrt{27}$.

3.4.4. 求下列函数在指定点处函数值增加最快的方向:

(1) $f(x,y)=e^x(\cos y+\sin y),(0,0)$; (3) $f(x,y,z)=\cos(xyz),(\frac{1}{3},\frac{1}{2},\pi)$.

解 依题意, 所求为梯度.

(1) $\nabla f(0,0)=(e^x(\cos y+\sin y),e^x(-\sin y+\cos y))\Big|_{(0,0)}=(1,1)$.

(2) $$\nabla f(\frac{1}{3},\frac{1}{2},\pi)=-(yz,xz,xy)\sin(xyz)\Big|_{(\frac{1}{3},\frac{1}{2},\pi)}$$
$$=-\frac{1}{2}(\frac{\pi}{2},\frac{\pi}{3},\frac{1}{6})=(-\frac{\pi}{4},-\frac{\pi}{6},-\frac{1}{12}).$$

3.4.5. 求下列函数在指定点处函数值减小最快的方向: (1) $f(x,y)=\sin(\pi xy),(\frac{1}{2},\frac{2}{3})$.

解 所求为梯度的反方向: $-\nabla f(\dfrac{1}{2},\dfrac{2}{3})=-\pi(y,x)\cos(\pi xy)\Big|_{\frac{1}{2},\frac{2}{3}}=-\dfrac{\pi}{2}(\dfrac{2}{3},\dfrac{1}{2})=(-\dfrac{\pi}{3},-\dfrac{\pi}{4})$.

3.4.7. 设向量$\vec{u}=3\vec{i}-4\vec{j},\vec{v}=4\vec{i}+3\vec{j}$, 且二元可微函数在点$P$处有$\frac{\partial f}{\partial\vec{u}}|_P=-6,\frac{\partial f}{\partial\vec{v}}|_P=17$, 求$\mathrm{d}f|_P$.

解 由题意知: $\begin{cases} 3f_x(P)-4f_y(P)=-30 \\ 4f_x(P)+3f_y(P)=85 \end{cases}$. 解得$f_x(P)=10, f_y(P)=15$. 故$\mathrm{d}f|_P=10\mathrm{d}x+15\mathrm{d}y$.

(B)

3.4.1 设$\vec{l}_j$, $j=1,2,\cdots,n$是平面上点P_0处的n个单位向量, $n\geq 2$, 相邻两个向量之间的夹角为$\dfrac{2\pi}{n}$. 证明：若函数$f(x,y)$在点P_0有连续偏导数，则$\displaystyle\sum_{j=0}^{n}\frac{\partial f(P_0)}{\partial l_j}=0$.

证 由已知得$\sum\limits_{j=1}^{n}\vec{l}_j=\vec{0}$. 因为函数$f(x,y)$在点$P_0$有连续偏导数，所以$f$在$P_0$可微，从而有

$$\frac{\partial f(P_0)}{\partial l_j}=\nabla f(P_0)\cdot\vec{l}_j,\quad j=1,2,\cdots,n.$$

故
$$\sum_{j=0}^{n}\frac{\partial f(P_0)}{\partial l_j}=\sum_{j=0}^{n}\nabla f(P_0)\cdot\vec{l}_j=\nabla f(P_0)\cdot\sum_{j=1}^{n}\vec{l}_j=0.$$

习题3.5

(A)

3.5.2. 求$f(x,y)=\sin x\sin y$在点$(\frac{\pi}{4},\frac{\pi}{4})$的二阶Taylor公式.

解 因为$f_x(x,y)=\cos x\sin y$, $f_y(x,y)=\sin x\cos y$,

$f_{xx}(x,y)=-\sin x\sin y$, $\quad f_{xy}=\cos x\cos y$, $\quad f_{yy}(x,y)=-\sin x\sin y$,

所以

$$f(\frac{\pi}{4},\frac{\pi}{4})=f_x(\frac{\pi}{4},\frac{\pi}{4})=f_y(\frac{\pi}{4},\frac{\pi}{4})=-f_{xx}(\frac{\pi}{4},\frac{\pi}{4})=f_{xy}(\frac{\pi}{4},\frac{\pi}{4})=-f_{yy}(\frac{\pi}{4},\frac{\pi}{4})=\frac{1}{2}.$$

故

$$f(x,y)=\frac{1}{2}+\frac{1}{2}(x-\frac{\pi}{4})+\frac{1}{2}(y-\frac{\pi}{4})-\frac{1}{4}\Big[(x-\frac{\pi}{4})^2-2(x-\frac{\pi}{4})(y-\frac{\pi}{4})+(y-\frac{\pi}{4})^2\Big]+o(\rho^2),$$

其中$\rho=\sqrt{(x-\frac{\pi}{4})^2+(y-\frac{\pi}{4})^2}$.

3.5.4. 求下列函数的极值: (2) $f(x,y)=e^x(x+y^2+2y)$; (4) $f(x,y)=\sin x+\sin y+\sin(x+y), 0<x<\pi, 0<y<\pi$.

解 这两个函数的可能极值点只有驻点.

(2) 由$\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases}$知: $\begin{cases} x+y^2+2y+1=0 \\ x+y^2+4y+2=0 \end{cases}$, 从而得驻点$P(-\dfrac{1}{4},-\dfrac{1}{2})$. 经计算得到$A=f_{xx}(P)=\exp(-\frac{1}{4})>0, B=f_{xy}(P)=\exp(-\frac{1}{4}), C=f_{yy}(P)=3\exp(-\frac{1}{4})$, $\Delta=AC-B^2=2\exp(-\frac{1}{2})>0$. 因此点$P$为函数的极小值点, 极小值为$f(P)=-2\exp(-\frac{1}{4})$.

(4) 由

$$\begin{cases} f_x(x,y)=\cos x+\cos(x+y)=0 \\ f_y(x,y)=\cos y+\cos(x+y)=0 \end{cases}$$

解得$x=y$, 进而得到唯一驻点$P(\dfrac{\pi}{3},\dfrac{\pi}{3})$. 经计算得到$A=C=-\sqrt{3}<0, B=-\dfrac{\sqrt{3}}{2}$, 从而有$\Delta=AC-B^2=3-\dfrac{3}{4}>0$. 因此点$P$为函数的极大值点, 极大值为$f(P)=\dfrac{3\sqrt{3}}{2}$.

3.5.5. 求下列函数在指定区域D上的最大值与最小值:

(1) $z=x^2y(4-x-y), \quad D=\{(x,y)|x\geq 0, y\geq 0, x+y\leq 4\}$;

(3) $z=x^2+y^2-12x+16y, \quad D=\{(x,y)|x^2+y^2\leq 25\}$.

解 (1) 函数在区域D内仅一个驻点$P(1,2)$, 得函数值$f(P)=4$.

在两坐标轴上函数恒为0. 在直线段$y=4-x$ $(0\leq x\leq 4)$上, $f(x,4-x)=0$ $(0\leq x\leq 4)$. 故, 函数在D上的最大值为4, 最小值为0.

(3) 由$\begin{cases} f_x(x,y)=2x-12=0 \\ f_y(x,y)=2y+16=0 \end{cases}$解得$x=6$, $y=-8$, 点$(6,-8)\notin D$. 下面考虑函数在边界∂D的取值情况. 令

$$L=x^2+y^2-12x+16y+\lambda(x^2+y^2-25) \quad 或 \quad L=25-12x+16y+\mu(x^2+y^2-25).$$

则由$\nabla L=\vec{0}$, 得到$\lambda+1=\pm 2$(或$\mu=\pm 2$), $x=\dfrac{6}{\lambda+1}=\pm 3$(或$x=\dfrac{6}{\mu}=\pm 3$), 同时$y=-\dfrac{8}{\lambda+1}=\mp 4$(或$y=-\dfrac{8}{\mu}=\mp 4$). 因为$f(3,-4)=-75$, $f(-3,4)=125$, 所以函数在D上的最大值为125, 最小值为-75.

3.5.6 求原点到曲线$\begin{cases} x^2+y^2=z, \\ x+y+z=1 \end{cases}$的最长和最短距离.

解 设目标函数$d^2=f(x,y,z)=x^2+y^2+z^2$, 令

$$L=x^2+y^2+z^2+\lambda(x^2+y^2-z)+\mu(x+y+z-1).$$

由$\begin{cases} L_x=2x+2\lambda x+\mu=0, \\ L_y=2y+2\lambda y+\mu=0, \\ L_z=2z-\lambda+\mu=0 \end{cases}$的前两式相减, 得到$(\lambda+1)(x-y)=0$. 由于$\lambda\neq -1$(否则有$\mu=0$, $z=-1/2<0$, 不合约束条件), 所以$x=y$. 再联立约束条件$z=x^2+y^2$与$x+y+z=1$, 可解出$x=y=\dfrac{1}{2}(-1\pm\sqrt{3})$, $z=2x^2=2\mp\sqrt{3}$. 于是得到$d^2=9\mp 5\sqrt{3}$. 故, 最长距离$=\sqrt{9+5\sqrt{3}}$, 最短距离$=\sqrt{9-5\sqrt{3}}$.

注: 可由$\nabla f,\nabla g,\nabla h$的混合积$2(2z+1)(x-y)=0$得到$x=y$, 其中$g(x,y,x)=x^2+y^2-z$, $h(x,y,z)=x+y+z-1$.

3.5.9. 求函数$f(x,y,z)=x+2y+3z$在圆柱$x^2+y^2=2$与平面$y+z=1$的交线椭圆上的最大值与最小值.

解 目标函数为$f(x,y,z)=x+2y+3z$, 此时有两个约束条件$g_1=x^2+y^2-2=0$与$g_2=y+z-1=0$. 作Lagrange函数

$$L(x,y,z,\lambda,\mu)=x+2y+3z+\lambda(x^2+y^2-2)+\mu(y+z-1).$$

由方程组

$$\begin{cases} L_x=1+2\lambda x=0, \\ L_y=2+2\lambda y+\mu=0, \\ L_z=3+\mu=0, \\ L_\lambda=x^2+y^2-2=0, \\ L_\mu=y+z-1=0 \end{cases}$$

解得Lagrange函数L有两个驻点$(1,-1,2)$和$(-1,1,0)$. 由于函数的最大值和最小值存在, 故所求最大值为$f(1,-1,2)=5$, 最小值为$f(-1,1,0)=1$.

(B)

3.5.1. 求曲线$\begin{cases} z=\sqrt{x}, \\ y=0 \end{cases}$与曲线$\begin{cases} x+2y-3=0, \\ z=0 \end{cases}$之间的距离.

解 在第一条曲线上任取一点$(x,0,\sqrt{x})$, 在第二条曲线上任取一点$(3-2v,v,0)$, 设它们距离的平方为目标函数$f(x,v)=(x+2v-3)^2+v^2+x\ (x\geq 0)$. 由$\begin{cases} f_x(x,v)=2(x+2v-3)+1=0 \\ f_v(x,v)=4(x+2v-3)+2v=0 \end{cases}$解得唯一驻点$x=1/2,\ v=1$. 由几何意义知$f$的最小值存在, 故在两曲线上对应点$(1/2,0,1/\sqrt{2})$与$(1,1,0)$的距离最小, 其值为$\sqrt{7}/2$.

3.5.2. 设椭球面$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$被通过原点的平面$lx+my+nz=0$截成一个椭圆, 求这个椭圆的面积.

解 设目标函数$f(x,y,z)=x^2+y^2+z^2$, 则考虑函数$f(x,y,z)$在约束条件

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1, \\ lx+my+nz=0 \end{cases}$$

下的极值. 令

$$L=x^2+y^2+z^2-\lambda(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1)+\mu(lx+my+nz).$$

由$(L_x,L_y,L_Z)=(0,0,0)$得到

$$\begin{cases} L_x=2x-\frac{2\lambda x}{a^2}+l\mu=0, & (1) \\ L_y=2y-\frac{2\lambda y}{b^2}+m\mu=0, & (2) \\ L_z=2z-\frac{2\lambda z}{c^2}+n\mu=0. & (3) \end{cases}$$

$(1)\times x+(2)\times y+(3)\times z$, 并利用约束条件, 得到$x^2+y^2+z^2=\lambda$. 联立(1), (2), (3)与$lx+my+nz=0$, 由于$(x,y,z,\mu)$为方程组的非零解, 所以系数行列式为0, 即

$$\begin{aligned} 0 &= \begin{vmatrix} 2-\frac{2\lambda}{a^2} & 0 & 0 & l \\ 0 & 2-\frac{2\lambda}{b^2} & 0 & m \\ 0 & 0 & 2-\frac{2\lambda}{c^2} & n \\ l & m & n & 0 \end{vmatrix} \\ &= -4m^2(1-\frac{\lambda}{a^2})(1-\frac{\lambda}{c^2})-4n^2(1-\frac{\lambda}{a^2})(1-\frac{\lambda}{b^2})-4l^2(1-\frac{\lambda}{b^2})(1-\frac{\lambda}{c^2}). \end{aligned}$$

因此, 有

$$(\frac{m^2}{a^2c^2}+\frac{n^2}{a^2b^2}+\frac{l^2}{b^2c^2})\lambda^2-(\frac{a^2+c^2}{a^2c^2}m^2+\frac{a^2+b^2}{a^2b^2}n^2+\frac{b^2+c^2}{b^2c^2}l^2)\lambda+m^2+n^2+l^2=0.$$

得到: $\lambda_1\lambda_2 = \dfrac{m^2+n^2+l^2}{\frac{m^2}{a^2c^2}+\frac{n^2}{a^2b^2}+\frac{l^2}{b^2c^2}}$, 故所求面积

$$S = \pi\sqrt{\lambda_1\lambda_2} = \pi\sqrt{\frac{m^2+n^2+l^2}{\frac{m^2}{a^2c^2}+\frac{n^2}{a^2b^2}+\frac{l^2}{b^2c^2}}} = \pi abc\sqrt{\frac{m^2+n^2+l^2}{b^2m^2+c^2n^2+a^2l^2}}.$$

3.5.5. 设函数$f(x)$在$[1+\infty)$内有二阶连续导数, $f(1)=0, f'(1)=1$且$z=(x^2+y^2)f(x^2+y^2)$满足$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$, 求$f(x)$在$[1+\infty)$上的最大值.

解 $z_x = 2xf + 2x(x^2+y^2)f'$, 于是

$$z_{xx} = 2f+4x^2f'+2(x^2+y^2)f'+4x^2f'+4x^2(x^2+y^2)f'' = 2f+2(5x^2+y^2)f'+4x^2(x^2+y^2)f''.$$

同理可得 $z_{yy} = 2f+2(x^2+5y^2)f'+4y^2(x^2+y^2)f''$. 因此

$$0 = \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = 4(x^2+y^2)^2f''(x^2+y^2)+12(x^2+y^2)f'(x^2+y^2)+4f(x^2+y^2).$$

若记$t=x^2+y^2$, 则得到Euler方程

$$t^2f''(t)+3tf'(t)+f(t)=0.$$

其对应的常系数方程为$z''+2z'+z=0$(令$s=\ln t$, 则$z=f(e^s)$), 通解为$z=(C_1+C_2s)e^{-s}$. 故, $f(t)=(C_1+C_2\ln t)t^{-1}$. 由初值条件解得$C_1=0$, $C_2=1$. 因此$f(t)=\dfrac{\ln t}{t}$, 有唯一驻点$t=e$, 且$1<t<e$时$f'(t)>0$; $t>e$时$f'(t)<0$, 从而知$f(e)=e^{-1}$为极大值. 又$\lim\limits_{t\to+\infty}f(t)=0$, $f(1)=0$, 故$f(e)=e^{-1}$是$f(t)$在$[1+\infty)$上的最大值.

习题3.6

(A)

3.6.1. 求下列曲线在给定点的切线和法平面方程:

(1) $\vec{r}=(t,2t^2,t^2)$, 在$t=1$处; (2) $\vec{r}=(3\cos\theta,3\sin\theta,4\theta)$,在点$(\frac{3}{\sqrt2},\frac{3}{\sqrt2},\pi)$处.

解 (1) 切线方程: $\dfrac{x-1}{1}=\dfrac{y-2}{4}=\dfrac{z-1}{2}$; 法平面方程: $x+4y+2z=11$.

(2) 切线方程: $\dfrac{x-\frac{3}{\sqrt2}}{-3}=\dfrac{y-\frac{3}{\sqrt2}}{3}=\dfrac{z-\pi}{4\sqrt2}$;

法平面方程: $3x-3y-4\sqrt{2}z=-4\pi\sqrt{2}$(或$\dfrac{3}{\sqrt{2}}x-\dfrac{3}{\sqrt{2}}y-4z=-4\pi$).

3.6.2. 求下列平面曲线的弧长:
(1) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},(a>0)$的全长; (2) $\rho=a(1+\cos\theta)$的全长.

解 (1) $6a$; (2) $8a$.

3.6.3. 求下列空间曲线的弧长:
(1) $\vec{r}=(e^t\cos t,e^t\sin t,e^t)$介于点(1,0,1)与点$(0,e^{\frac{\pi}{2}},e^{\frac{\pi}{2}})$之间的弧长;
(3) $\begin{cases} x^2=3y, \\ 2xy=9z \end{cases}$ 介于点$(0,0,0)$与点$(3,3,2)$之间的弧长.

解 (1) 弧长为: $\sqrt{3}\displaystyle\int_0^{\frac{\pi}{2}} e^t\mathrm{d}t=\sqrt{3}(e^{\frac{\pi}{2}}-1)$.
(3) 曲线以x为参数, 得到$y=\dfrac{x^2}{3}$, $z=\dfrac{2x^3}{27}$. 所以弧长为:
$$\int_0^3 \sqrt{1+(\frac{2}{3}x)^2+(\frac{2}{9}x^2)^2}\ \mathrm{d}x=\int_0^3(1+\frac{2}{9}x^2)\ \mathrm{d}x=5.$$

3.6.4. 求下列曲面在给定点的切平面与法线方程:
(2) $z^2=\frac{x^2}{4}+\frac{y^2}{9}$在$(6,12,5)$处; (3) $x^3+y^3+z^3+xyz-6=0$在$(1,2,-1)$处.

解 对曲面$F(x,y,z)=0$, 法向量$\vec{n}=(F_x,F_y,F_z)\big|_{p_0}$.
(2) 令$F(x,y,z)=\dfrac{x^2}{4}+\dfrac{y^2}{9}-z^2$, 得法向量$\vec{n}=(3,8/3,-10)$.
切平面方程: $9x+8y-30z=0$; 法线方程: $\dfrac{x-6}{9}=\dfrac{y-12}{8}=\dfrac{z-5}{-30}$.
(3) 令$F(x,y,z)=x^3+y^3+z^3+xyz-6$, 法向量$\vec{n}=(1,11,5)$.
切平面方程: $x+11y+5z=18$; 法线方程: $\dfrac{x-1}{1}=\dfrac{y-2}{11}=\dfrac{z+1}{5}$.

3.6.6. (1) 求曲面$x^2+y^2+z^2=x$的切平面, 使它垂直于平面$x-y-\frac{1}{2}z=2$和平面$x-y-z=2$; (2) 过直线$\begin{cases} 10x+2y-2z=27, \\ x+y-z=0 \end{cases}$,作曲面$3x^2+y^2-z^2=27$的切平面, 求此切平面的方程.

解 (1) 依题意, 所求切平面的法向量为$(1,-1,-1/2)\times(1,-1,-1)=(1/2,1/2,0)$, 取$\vec{n}=(1,1,0)$. 另一方面, 曲面$x^2+y^2+z^2=x$上点$(x_0,y_0,z_0)$处

的法向量为$(2x_0-1, 2y_0, 2z_0)$, 从而有$2x_0-1=2y_0$, $z_0=0$, 将其代入曲面方程$x^2+y^2+z^2=x$, 得到$x_0=\dfrac{2\pm\sqrt{2}}{4}$, $y_0=\dfrac{\pm\sqrt{2}}{4}$, $z_0=0$. 故所求切平面为

$$(x-\frac{2+\sqrt{2}}{4})+(y-\frac{\sqrt{2}}{4})=0 \qquad \text{和} \quad (x-\frac{2-\sqrt{2}}{4})+(y+\frac{\sqrt{2}}{4})=0,$$

即$x+y=\dfrac{1}{2}(1+\sqrt{2})$和$x+y=\dfrac{1}{2}(1-\sqrt{2})$.

(2) $9x+y-z=27$与$9x+17y-17z=-27$.

3.6.8. 求曲面$x^2+2y^2+z^2=22$的法线, 使它与直线$\begin{cases} x+3y+z=3, \\ x+y=0 \end{cases}$平行.

解 依题意, 所求法线方向向量为$(1,3,1)\times(1,1,0)=(-1,1,-2)$, 取$\vec{n}=(1,-1,2)$. 另一方面, 曲面$x^2+2y^2+z^2=22$上点$(x_0,y_0,z_0)$处的法向量为$(x_0,2y_0,z_0)$, 所以$x_0=-2y_0=z_0/2$. 将其代入曲面方程$x^2+2y^2+z^2=22$, 得到$x_0=\pm2$, $y_0=\mp1$, $z_0=\pm4$. 故所求法线为

$$\frac{x\pm2}{1}=\frac{y\mp1}{-1}=\frac{z\pm4}{2}.$$

3.6.9. 求旋转抛物面$S: z=x^2+y^2$和平面$\pi: x+y-2z=2$平行的切平面的方程.

解 设S上点$P_0(x_0,y_0,z_0)$处的切平面与平面π平行. 由于S上点P_0处法向量为

$$\vec{n}|_{P_0}=(2x_0,2y_0,-1),$$

平面π的法向量为$(1,1,-2)$, 按照平面平行的条件, 应该有

$$\frac{2x_0}{1}=\frac{2y_0}{1}=\frac{-1}{-2}=\frac{1}{2},$$

从而求得$P_0(1/4,1/4,1/8)$. 因此, 所求切平面方程是

$$(x-1/4)+(y-1/4)-2(z-1/8)=0, \text{或 } x+y-2z=1/4.$$

(B)

3.6.2. 若可微函数$f(x,y)$对任意x,y,t满足$f(tx,ty)=t^2f(x,y)$, $P_0(1,-2,2)$是曲面$z=f(x,y)$上的一点, 且$f'_x(1,-2)=4$, 求曲面在P_0处的切平面方程.

解　所求切平面方程为: $4x-z=2$.

3.6.3. 设函数$f(u,v)$在全平面上有连续的偏导数，取S由方程$f(\frac{x-a}{z-c},\frac{y-b}{z-c})=0$确定. 证明: 该曲面的所有切平面都过点$(a,b,c)$.

证　记$F(x,y,z)=f(\frac{x-a}{z-c},\frac{y-b}{z-c})$, 则

$$(F_x,F_y,F_z)=\Big(\frac{f_1}{z-c},\frac{f_2}{z-c},-\frac{(x-a)f_1+(y-b)f_2}{(z-c)^2}\Big).$$

取曲面S的法向量

$$\vec{n}=\Big((z-c)f_1,(z-c)f_2,-(x-a)f_1-(y-b)f_2\Big).$$

记(x,y,z)为曲面S上的点, (X,Y,Z)为切平面上的点，则曲面S上过点x,y,z 的切平面为

$$(z-c)f_1(X-x)+(z-c)f_2(Y-y)-[(x-a)f_1+(y-b)f_2](Z-z)=0.$$

对应任意的$(x,y,z)(z\neq c)$, $(X,Y,Z)=(a,b,c)$都满足切平面方程. 证毕.

习题3.7

(A)

3.7.1. 求下列平面曲线在给定点的曲率: (2) $y=\sin x$,在点$(\frac{\pi}{2},1)$处.

解　$\kappa=\dfrac{|\sin x|}{(1+\cos^2 x)^{3/2}}\Big|_{\frac{\pi}{2}}=1.$

3.7.2. 求下列平面曲线的曲率:

(1) $y=ax^2$;　　　(3) $\vec{r}=(a\cosh t,a\sinh t)$.

解　(1) $\kappa=\dfrac{2|a|}{(1+4a^2x^2)^{3/2}}$.　(3) $\kappa=\dfrac{1}{a(\cosh(2t))^{3/2}}$.

3.7.3. 求下列曲线的曲率$(a>0)$:

(1) $\vec{r}=(a\cosh t,a\sinh t,bt)$;　(3) $\vec{r}=(a(1-\sin t),a(1-\cos t),bt)$.

解　(1) $\kappa=\dfrac{a\sqrt{b^2\cosh(2t)+a^2}}{(a^2\cosh(2t)+b^2)^{3/2}}$.　　　(2) $\kappa=\dfrac{1}{a^2(a^2+b^2)}$.

3.7.4. 曲线$y=\ln x$上哪一点处的曲率半径最小？求出该点处的曲率半径.

解 曲率$\kappa(x)=\dfrac{x}{(1+x^2)^{3/2}}$. 由$\kappa'(x)=0$得到驻点$x_0=1/\sqrt{2}$, 经检验它也是$\kappa(x)$的最大值点. 因此, 曲线$y=\ln x$上点$(1/\sqrt{2},\ln(1/\sqrt{2}))$处的曲率半径最小, 该点处的曲率半径为$1/\kappa(x_0)=3\sqrt{3}/2$.

(B)

3.7.1. 求曲率$\kappa(s)=\frac{a}{a^2+s^2}$的平面曲线.($s$是弧长参数)

解

$$\kappa(s)=\frac{d\theta}{\mathrm{d}s}\Rightarrow d\theta=\kappa(s)\mathrm{d}s=\frac{a}{a^2+s^2}\mathrm{d}s,$$

$$\theta(s)=\int\kappa(s)\mathrm{d}s=\int\frac{a}{a^2+s^2}\mathrm{d}s=\arctan\frac{s}{a}+C.$$

设曲线的参数方程为

$$\begin{cases}x=x(s),\\ y=y(s),\end{cases} s\in[0,l]$$

由弧微分公式

$$\mathrm{d}s=\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}$$

得

$$\mathrm{d}x=\cos\theta\mathrm{d}s,\ \mathrm{d}y=\sin\theta\mathrm{d}s.$$

不防设

$$x(0)=0,\ y(0)=0,\ \theta(0)=0,$$

则

$$\begin{aligned}x(s)&=x(0)+\int_0^s\cos(\theta(s))\mathrm{d}s=\int_0^s\cos(\arctan\frac{s}{a})\mathrm{d}s\\&=\int_0^s\frac{a}{\sqrt{a^2+s^2}}\mathrm{d}s=a[\ln(s+\sqrt{a^2+s^2})-\ln a],\end{aligned}$$

$$\begin{aligned}y(s)&=y(0)+\int_0^s\sin(\theta(s))\mathrm{d}s=\int_0^s\sin(\arctan\frac{s}{a})\mathrm{d}s\\&=\int_0^s\frac{s}{\sqrt{a^2+s^2}}\mathrm{d}s=\sqrt{a^2+s^2}-a,\end{aligned}$$

所以曲线的方程为

$$\mathbf{r}(s)=(a\ln(s+\sqrt{a^2+s^2})-a\ln a,\ \sqrt{a^2+s^2}-a).$$

消去s, 可得: $y=a(\cosh\dfrac{x}{a}-1)$. 事实上, 由$x=a\ln(s+\sqrt{a^2+s^2})-a\ln a$知

$$\frac{x}{a}=\ln\frac{s+\sqrt{a^2+s^2}}{a}=\ln\frac{a}{\sqrt{a^2+s^2}-s},$$

所以

$$e^{\frac{x}{a}}=\frac{s+\sqrt{a^2+s^2}}{a},\quad e^{-\frac{x}{a}}=\frac{\sqrt{a^2+s^2}-s}{a}.$$

于是, 注意到$y=\sqrt{a^2+s^2}-a$, 我们有

$$\frac{e^{\frac{x}{a}}+e^{-\frac{x}{a}}}{2}=\frac{\sqrt{a^2+s^2}}{a}\Rightarrow y=a\cosh\frac{x}{a}-a.$$

3.7.3. 设$\vec{r}(t)$是空间曲线, 曲率为$\kappa(t)$. 求曲线$\vec{\tilde{r}}=\vec{r}(-t)$的曲率.

解 因为$\vec{\tilde{r}}'(t)=-\vec{r}'(-t)$, $\vec{\tilde{r}}''(t)=\vec{r}''(-t)$, $\vec{\tilde{r}}'''(t)=-\vec{r}'''(-t)$, 所以曲线$\vec{\tilde{r}}=\vec{r}(-t)$的曲率为

$$\tilde{\kappa}(t)=\frac{\|-\vec{r}'(-t)\times\vec{r}''(-t)\|}{\|-\vec{r}'(-t)\|^3}=\frac{\|\vec{r}'(-t)\times\vec{r}''(-t)\|}{\|\vec{r}'(-t)\|^3}=\kappa(-t).$$

习题3.8

(A)

3.8.1. 求下列向量值函数的Jacobi矩阵：

(1) $\vec{f}(x,y)=(x^2+\sin y,2xy)^T$; (3) $\vec{f}(x,y,z)=(x\cos y,ye^x,\sin(xz))^T$.

解 (1) $\mathrm{D}\vec{f}=\begin{pmatrix}2x&\cos y\\2y&2x\end{pmatrix}$. (3) $\mathrm{D}\vec{f}=\begin{pmatrix}\cos y&-x\sin y&0\\ye^x&e^x&0\\z\cos(xz)&0&x\cos(xz)\end{pmatrix}$.

3.8.3. 求向量值函数$\vec{f}(x,y)=(\arctan x,e^{xy})^T$的导数$D\vec{f}(x,y)$.

解 $\mathrm{D}\vec{f}=\begin{pmatrix}\frac{1}{1+x^2}&0\\ye^{xy}&xe^{xy}\end{pmatrix}$.

3.8.6. 设向量值函数$\vec{f}: \mathbb{R}^3 \to \mathbb{R}^2$定义$\vec{f}(x,y,z) = (e^x \cos y + e^y z^2, 2x \sin y - 3yz^3)^T$, 求$\mathrm{D}\vec{f}(0, \dfrac{\pi}{2}, 1)$.

解　原式$= \begin{pmatrix} e^x \cos y & -e^x \sin y + e^y z^2 & 2ze^y \\ 2\sin y & 2x\cos y - 3z^3 & -9yz^2 \end{pmatrix}_{(0,\frac{\pi}{2},1)} = \begin{pmatrix} 0 & -1 + e^{\frac{\pi}{2}} & 2e^{\frac{\pi}{2}} \\ 2 & -3 & -\frac{9}{2}\pi \end{pmatrix}$.