Chapter 6
Non-sinusoidal periodic waves

Day 39

Harmonics

ILOs – Day 39

- Define harmonics
- Identify the sources of harmonics
- Recognize how different harmonics can cause distorted complex signal
- Express complex signals mathematically
- Derive RMS value of a complex signal
- Derive average value of a complex signal

Harmonics

- AC signals analyzed so far are assumed to be of purely sinusoidal in nature
- Analysis of such signals and circuits using such signals is straight forward
- However, in real life, attaining such an ideal signal is rarely possible
- All real life AC signals deviate from this ideal sinusoidal shape to some extent
- Such signals are referred to as **distorted**, **noisy**, **non-sinusoidal signals** or **complex waves**.

Harmonics

- Such signals, when analyzed carefully, are found to have several other spurious signal components in addition to the main sinusoidal signal

- Such waves occur in speech, music, TV, rectifier outputs and many other applications of electronics that have *non-linear* circuit elements

- In opposition to linear-loads, a non-linear load changes its impedance with instantaneous applied voltage that will lead to a non-sinusoidal current being drawn

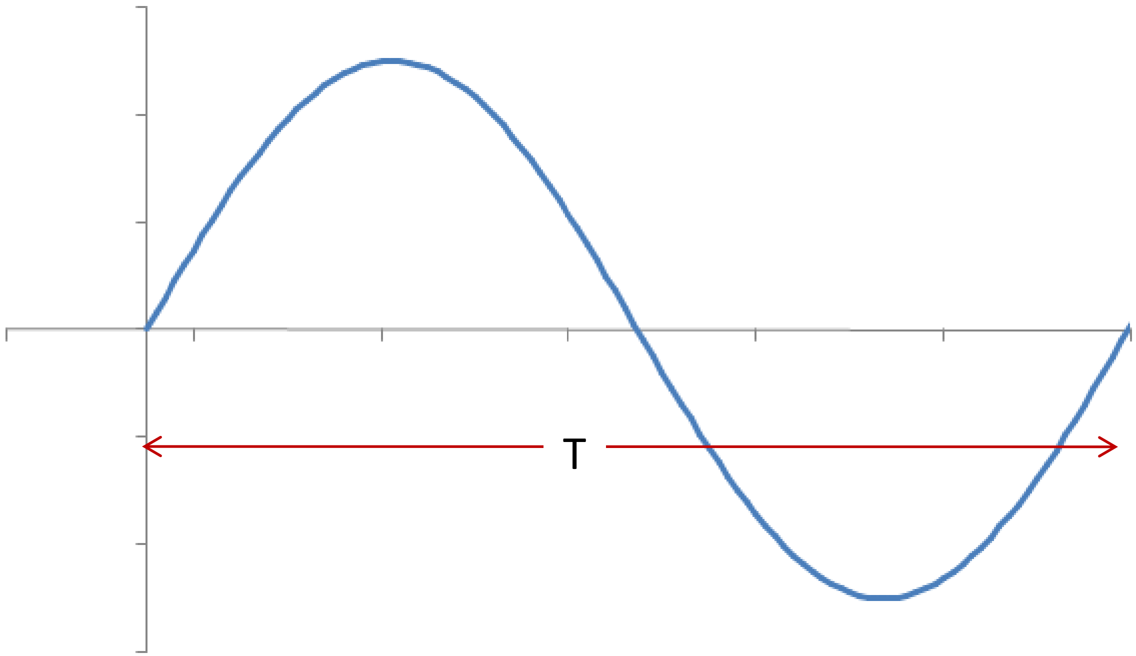
Harmonics - sources

- The simplest circuit to represent a non-linear load is a diode-rectifier
- Some other examples of non-linear loads, capable of injecting harmonics into an electrical distribution, are:
  - industrial equipments (welding, arc furnace)
  - variable frequency drives (VFD)
  - line-switched rectifiers
  - switch-mode power supplies (SMPS)
  - lighting ballasts (tube light choke)
- All these circuits can contain semiconductor power devices such as:
  - diodes
  - thyristors (SCR's)
  - transistors

Harmonics

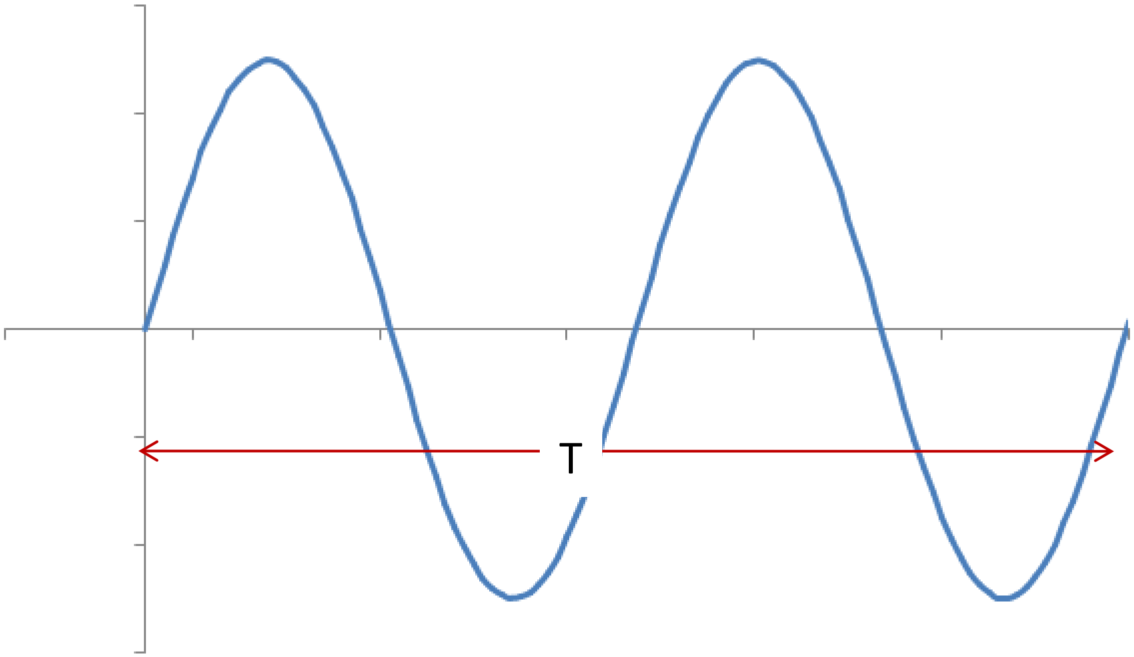
- On analysis, it is found that a complex wave essentially consists of:
  - a fundamental wave – it has the lowest frequency, say 'f'
  - a number of other sinusoidal waves whose frequencies are an integral multiple of the fundamental or basic frequency like 2f, 3f and 4f etc.
  - Such higher frequency components are called **harmonics.**
- The fundamental wave itself is called the first harmonic
- The second harmonic has frequency twice that of the fundamental, the third harmonic has frequency thrice that of the fundamental and so on
- Waves having frequencies of 2f, 4f and 6f etc. are called even harmonics
- and those having frequencies of 3f, 5f and 7f etc. are called odd harmonics.

Harmonics



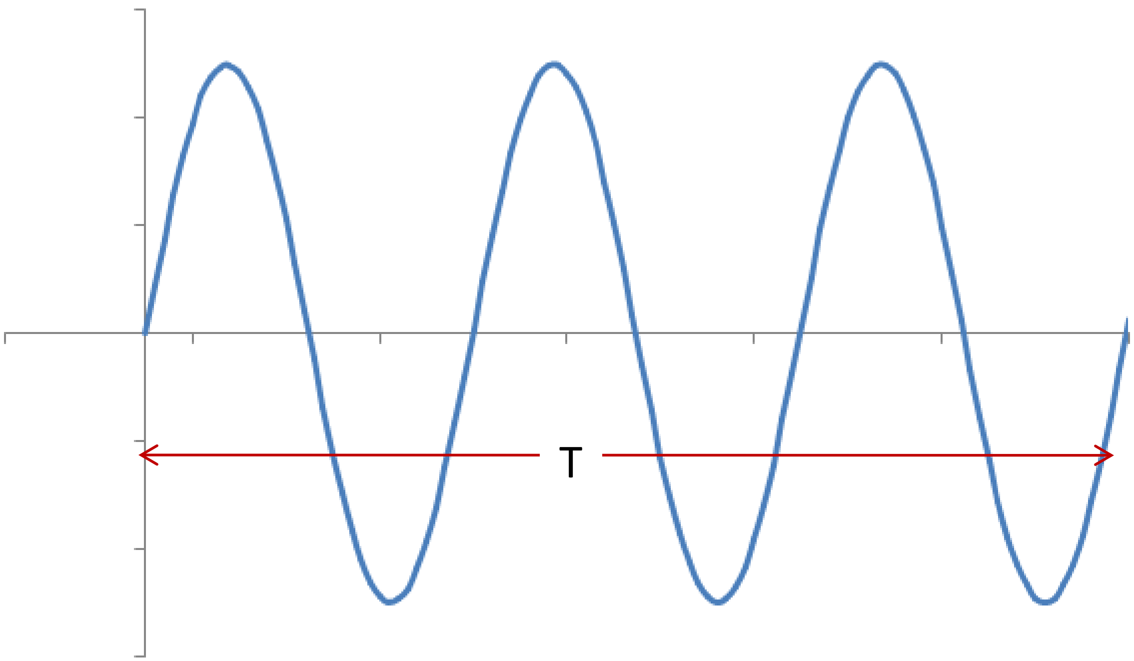
Fundamental signal having frequency *f* has one complete cycle within a time *T*

Harmonics



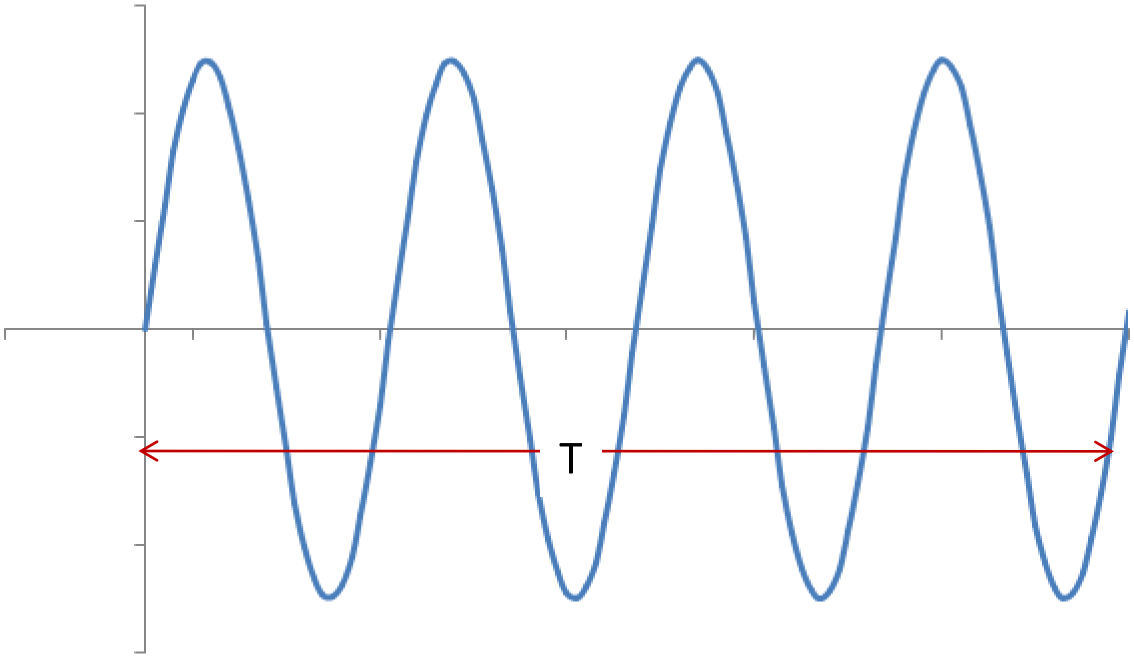
2^{nd} harmonic signal having frequency $2f$ has two complete cycles within the same time T

Harmonics



3^{rd} harmonic signal having frequency $3f$ has three complete cycles within the same time T

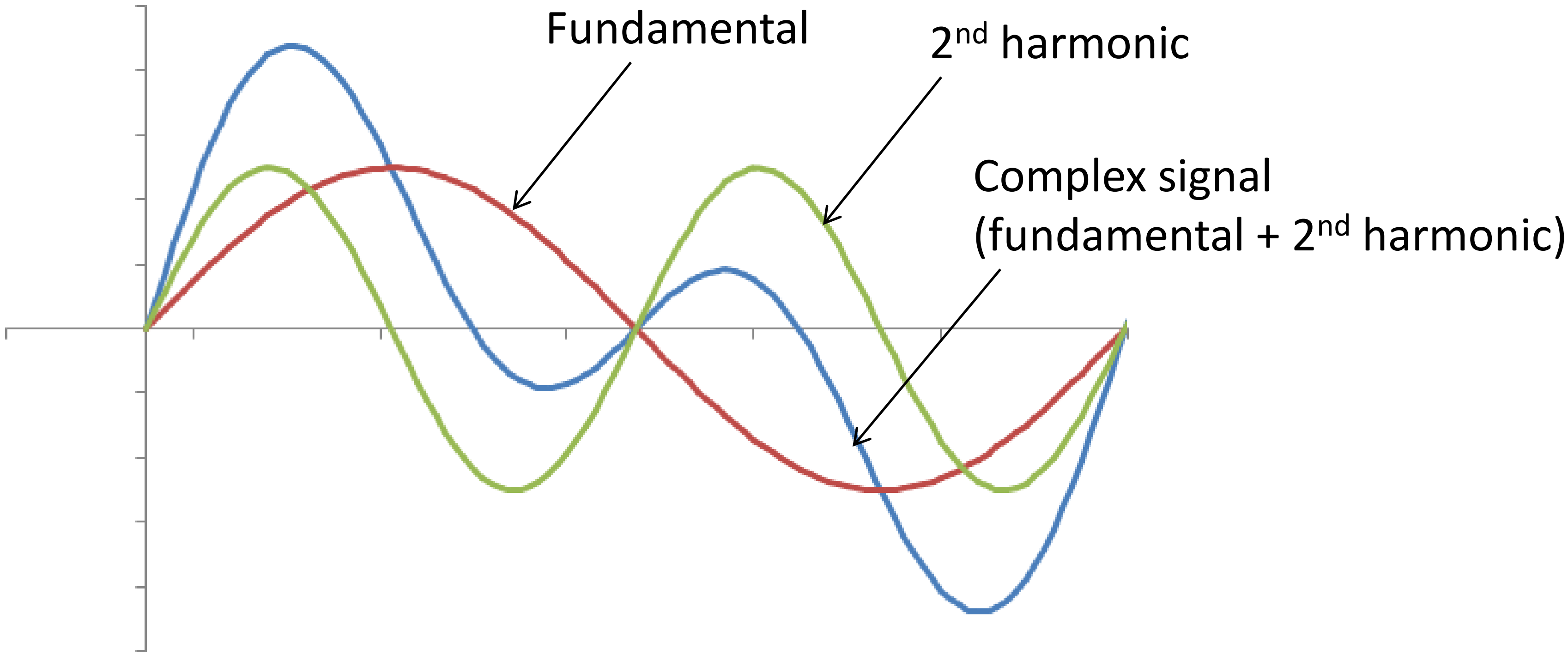
Harmonics



4^{th} harmonic signal having frequency *4f* has four complete cycles within the same time *T*

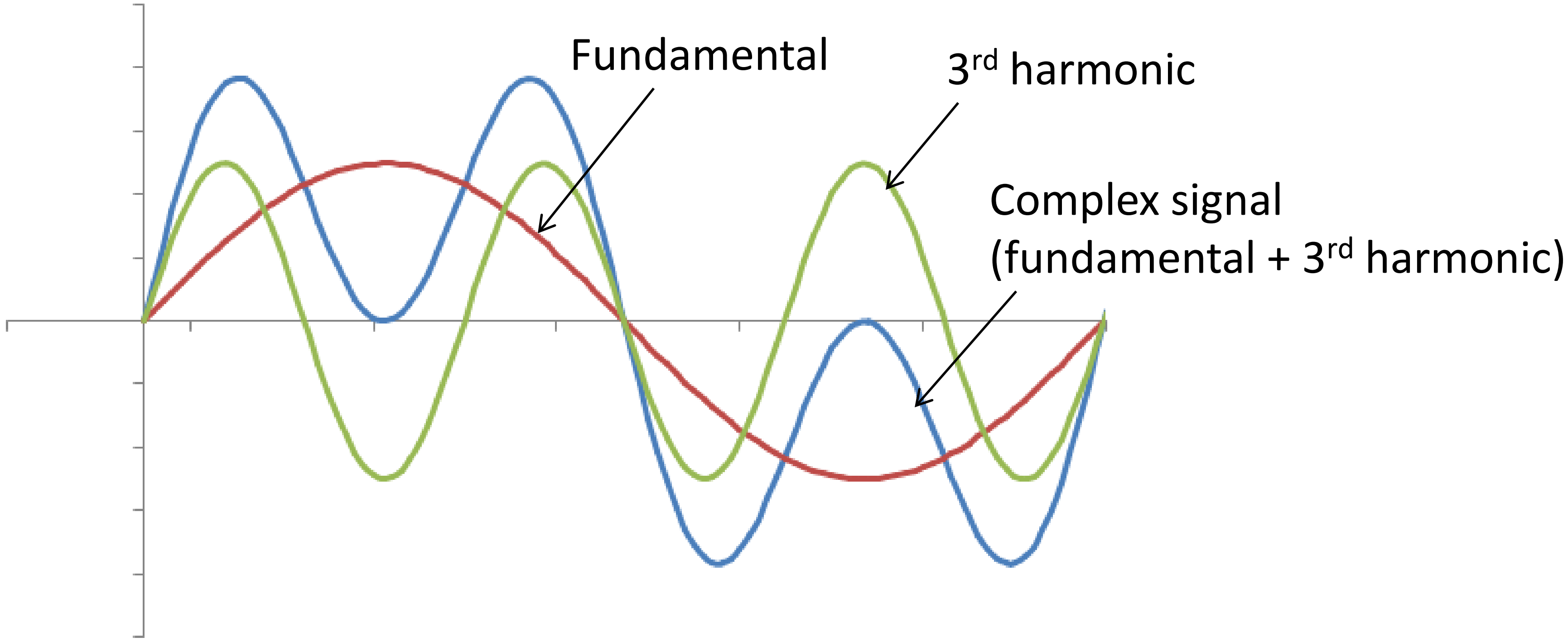
Complex signals

- A complex signal can have fundamental component superimposed with other harmonic(s)
- For example, when a complex signal has fundamental and the 2nd harmonic combined together, the resultant will be distorted as shown:



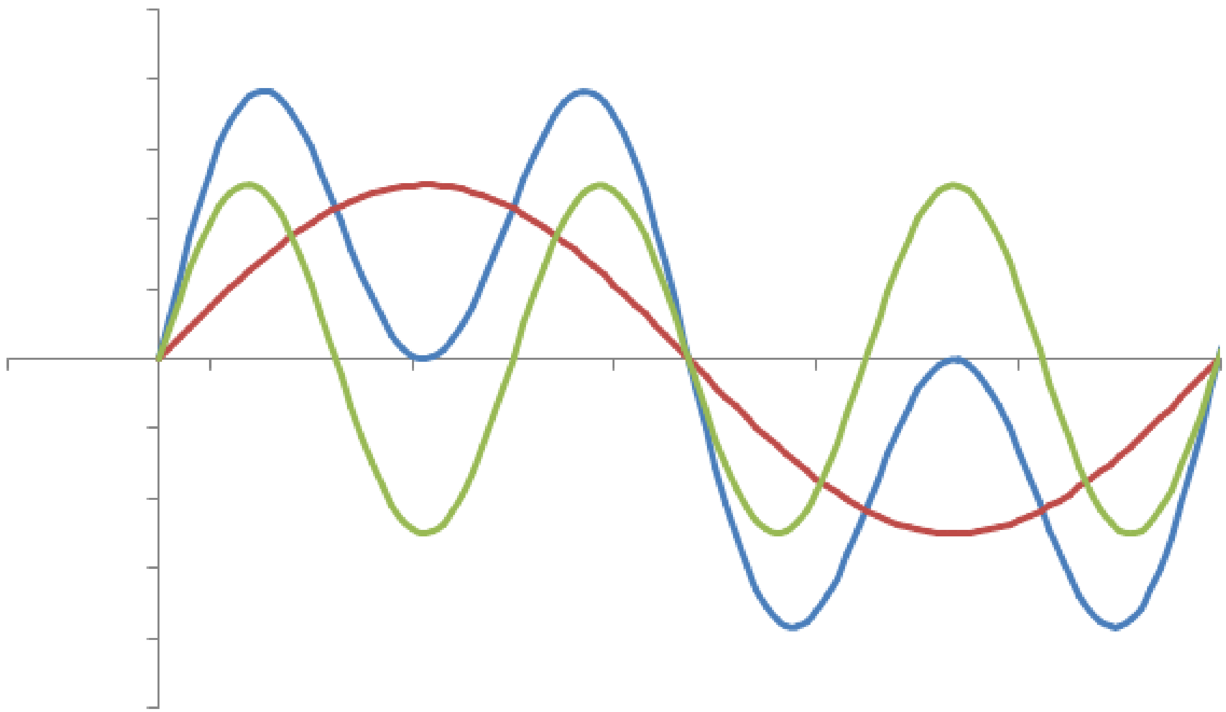
Complex signals

- A complex signal can have fundamental component superimposed with other harmonic(s)
- Similarly, when a complex signal has fundamental and the 3^{rd} harmonic combined together, the resultant will be distorted as shown:



Complex signals

- In all the above complex waveforms, the fundamental and its harmonics are assumed to have same peak amplitude and same phase angle

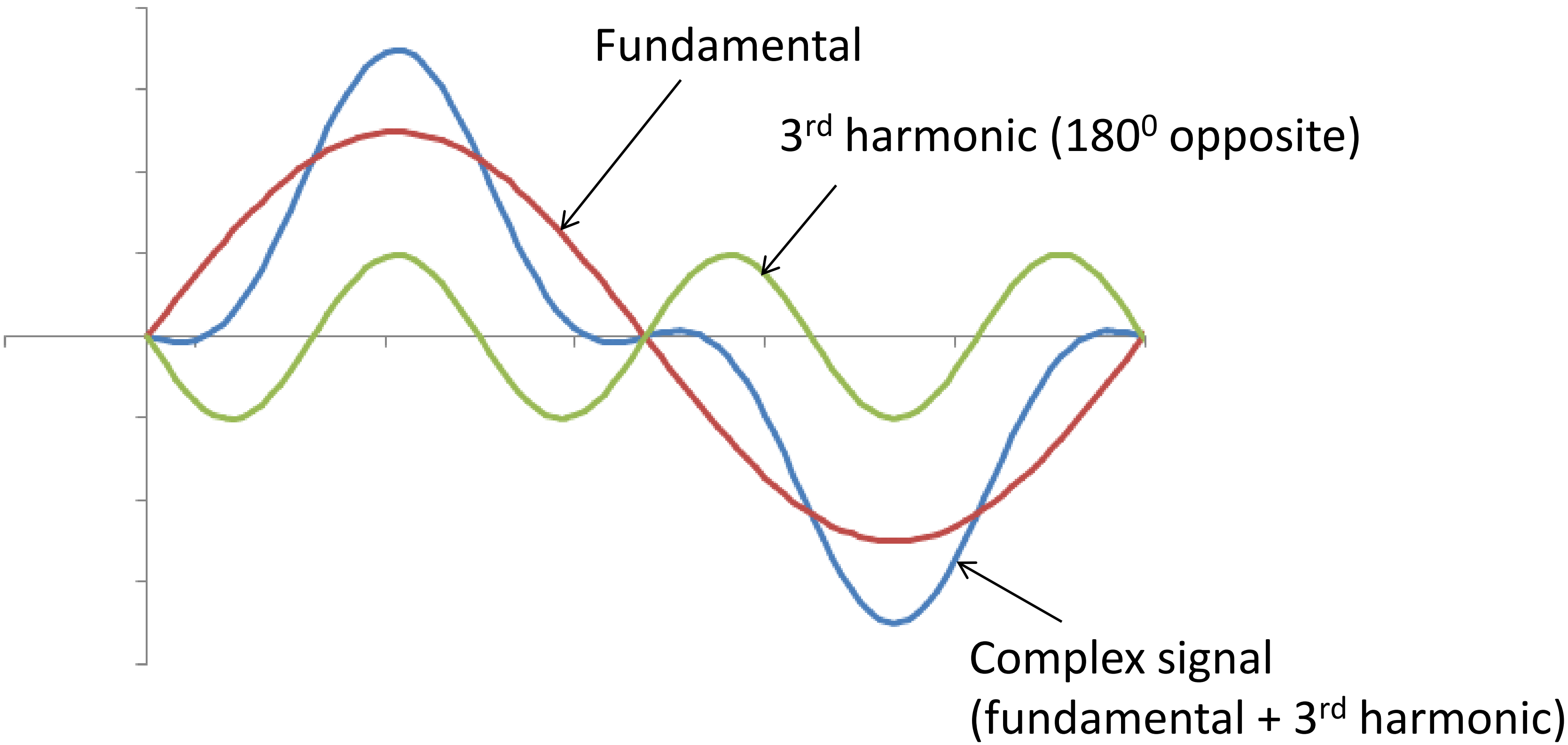


Complex signals

- In all the above complex waveforms, the fundamental and its harmonics are assumed to have same peak amplitude and same phase angle
- However, in reality it is common that the fundamental component has highest amplitude whereas the signal amplitude gradually reduces as the harmonic order goes up
- Also under certain circumstances there may be phase difference between the fundamental and its harmonics

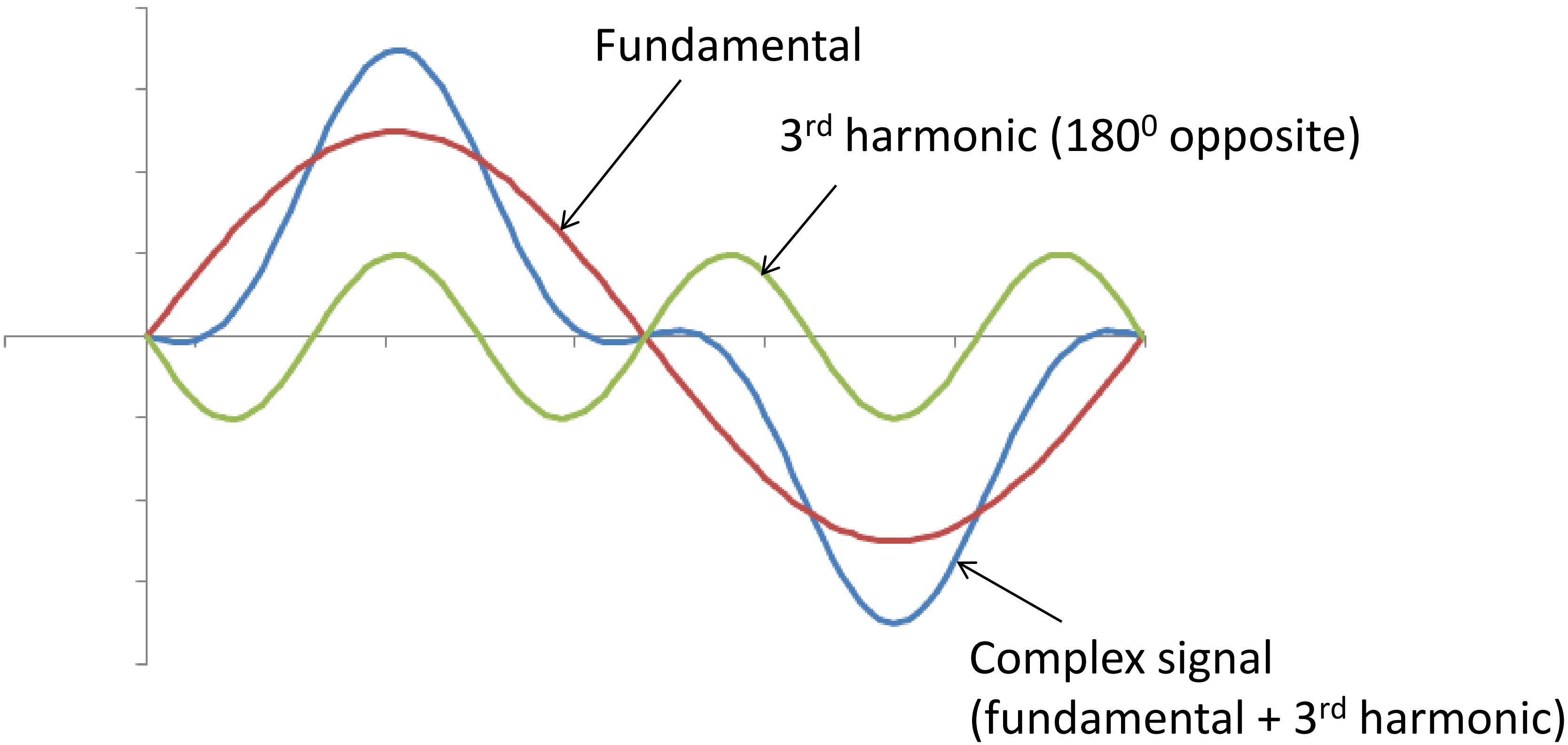
Complex signals

- One such example is shown below where the 3rd harmonic component is having lower peak amplitude than the fundamental and it is 180^0 out of phase with the fundamental

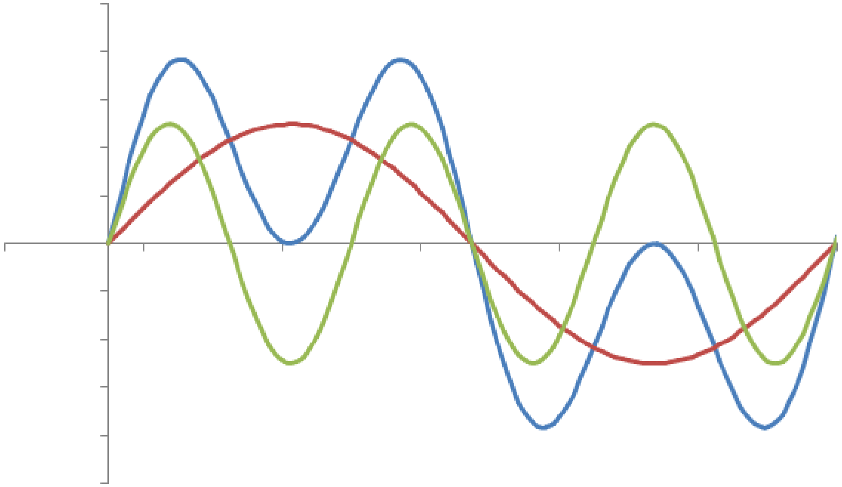


Complex signals

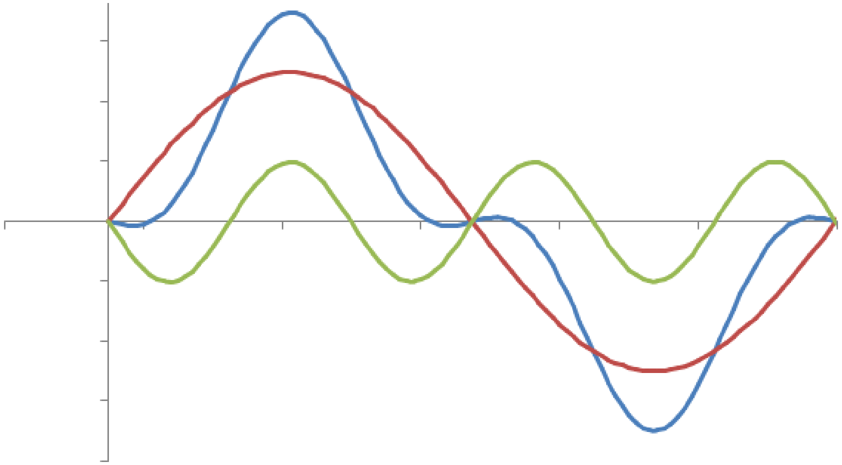
- Note that the combined complex wave-shape is markedly different from that when the fundamental and 3rd harmonic were in the same phase.



Complex signals



Fundamental and 3rd harmonic have same magnitude and are in the same phase



3rd harmonic has lower amplitude than fundamental and are 180^0 opposite in phase

Resultant complex wave looks entirely different

General equation of a complex wave

- Let us consider a complex voltage wave that is built up due to combination of the fundamental signal and several harmonics, each of which has its own peak value and phase angle

- The fundamental may be represented by: $e_1 = E_{m1} \sin\left(\omega t + \phi_1\right)$

- The second harmonic by: $e_2 = E_{m2} \sin\left(2\omega t + \phi_2\right)$

- Note that the angular frequency of 2nd harmonic signal has been represented as 2ω, where ω is angular frequency of the fundamental

- The third harmonic by: $e_3 = E_{m3} \sin\left(3\omega t + \phi_3\right)$, and so on

General equation of a complex wave

- The equation for the instantaneous value of the combined complex voltage wave is given by:

$$
\begin{aligned}
e &= e_1 + e_2 + e_3 + \ldots\ldots \\
&= E_{m1}\sin(\omega t + \phi_1) + E_{m2}\sin(2\omega t + \phi_2) + E_{m3}\sin(3\omega t + \phi_3)\ldots\ldots
\end{aligned}
$$

- Here E_{m1}, E_{m2} and E_{mn} etc. denote the maximum values or the amplitudes of the fundamental, 2nd harmonic and nth harmonic etc.
- ϕ_1, ϕ_2 , and ϕ_n represent the phase differences of respective harmonic signals with respect to a reference wave

- The number of terms in the series depends on the shape of the complex wave
- In relatively simple waves, the number of terms in the series would be less, in others, more

General equation of a complex wave

- Complex voltage wave:

$$e = E_{m1}\sin(\omega t + \phi_1) + E_{m2}\sin(2\omega t + \phi_2) + E_{m3}\sin(3\omega t + \phi_3)\ldots\ldots$$

- Similarly, the instantaneous value of the complex current wave may be given by:

$$\begin{aligned} i &= i_1 + i_2 + i_3 + \ldots\ldots \\ &= I_{m1}\sin(\omega t + \psi_1) + I_{m2}\sin(2\omega t + \psi_2) + I_{m3}\sin(3\omega t + \psi_3)\ldots\ldots \end{aligned}$$

- Thus, $(\phi_1 - \psi_1)$ is the phase difference between the voltage and current for the fundamental, $(\phi_2 - \psi_2)$ for the 2nd harmonic and $(\phi_n - \psi_n)$ for the nth harmonic, and so on

RMS value of a complex wave

- Let the equation of a complex voltage wave is given by:

$$e = E_{m1}\sin(\omega t + \phi_1) + E_{m2}\sin(2\omega t + \phi_2) + \ldots\ldots\ldots\ldots + E_{mn}\sin(n\omega t + \phi_n)\ldots\ldots$$

- As per definition, its RMS value is given by:

$$E = \sqrt{\frac{1}{T}\int_0^T e^2 dt} = \sqrt{\text{Average value of } e^2 \text{ over the entire cycle}}$$

RMS value of a complex wave

$$E = \sqrt{\frac{1}{T}\int_0^T e^2 dt} = \sqrt{\text{Average value of } e^2 \text{ over the entire cycle}}$$

Now,

$$e^2 = \left[E_{m1}\sin(\omega t + \phi_1) + E_{m2}\sin(2\omega t + \phi_2) + \ldots\ldots\ldots\ldots E_{mn}\sin(n\omega t + \phi_n)\ldots\ldots\right]^2$$

$$= E_{m1}^{\;2}\sin^2(\omega t + \phi_1) + E_{m2}^{\;2}\sin^2(2\omega t + \phi_2) + \ldots\ldots\ldots\ldots E_{mn}^{\;2}\sin^2(n\omega t + \phi_n)\ldots\ldots$$

$$+ 2E_{m1}E_{m2}\sin(\omega t + \phi_1)\sin(2\omega t + \phi_2) + 2E_{m1}E_{m3}\sin(\omega t + \phi_1)\sin(3\omega t + \phi_3) + \ldots\ldots$$

- The right-hand side of the above equation consists of two types of terms
  - i. harmonic self-products (squares), the general expression for which for the p^th harmonic is:

    $$E_{mp}^{\;2}\sin^2(p\omega t + \phi_p)$$

  - ii. the products of different harmonics of the general form:

    $$2E_{mp}E_{mq}\sin(p\omega t + \phi_p)\sin(q\omega t + \phi_q)$$

RMS value of a complex wave

$$E = \sqrt{\frac{1}{T}\int_0^T e^2 dt} = \sqrt{\text{Average value of } e^2 \text{ over the entire cycle}}$$

i. harmonic self-products (squares), the general expression for which for the p^{th} harmonic is:

$$E_{mp}{}^2 \sin^2\left(p\omega t + \phi_p\right)$$

ii. the products of different harmonics of the general form:

$$2E_{mp}E_{mq}\sin\left(p\omega t + \phi_p\right)\sin\left(q\omega t + \phi_q\right)$$

- The average value of e^2 is the sum of the average values of these individual terms
- First we find average value of (i)
- Then we find average value of (ii)

RMS value of a complex wave

Average value of the general term ${E_{mp}}^2 \sin^2\left(p\omega t + \phi_p\right)$ over a whole cycle:

$$\text{Average value} = \frac{1}{2\pi}\int_0^{2\pi} {E_{mp}}^2 \sin^2\left(p\omega t + \phi_p\right) d\omega t$$

$$= \frac{{E_{mp}}^2}{2\pi}\int_0^{2\pi} \sin^2\left(p\omega t + \phi_p\right) d\omega t$$

$$= \frac{{E_{mp}}^2}{2\pi}\int_0^{2\pi} \frac{1-\cos 2\left(p\omega t + \phi_p\right)}{2} d\omega t$$

$$= \frac{{E_{mp}}^2}{4\pi}\left[\omega t - \frac{\sin 2\left(p\omega t + \phi_p\right)}{2p}\right]_0^{2\pi}$$

$$= \frac{{E_{mp}}^2}{4\pi} \times 2\pi$$

$$\boxed{= \frac{{E_{mp}}^2}{2}}$$

RMS value of a complex wave

Average value of the general term $2E_{mp}E_{mq}\sin\left(p\omega t+\phi_p\right)\sin\left(q\omega t+\phi_q\right)$

$$\text{Average value} = \frac{1}{2\pi}\int_0^{2\pi} 2E_{mp}E_{mq}\sin\left(p\omega t+\phi_p\right)\sin\left(q\omega t+\phi_q\right)d\omega t$$

$$= \frac{E_{mp}E_{mq}}{2\pi}\int_0^{2\pi} 2\sin\left(p\omega t+\phi_p\right)\sin\left(q\omega t+\phi_q\right)d\omega t$$

$$= \frac{E_{mp}E_{mq}}{2\pi}\int_0^{2\pi}\left[\cos\left(p\omega t+\phi_p-q\omega t-\phi_q\right)-\cos\left(p\omega t+\phi_p+q\omega t+\phi_q\right)\right]d\omega t$$

$$= \frac{E_{mp}E_{mq}}{2\pi}\int_0^{2\pi}\left[\cos\left(\left(p-q\right)\omega t+\left(\phi_p-\phi_q\right)\right)-\cos\left(\left(p+q\right)\omega t+\left(\phi_p+\phi_q\right)\right)\right]d\omega t$$

$$= \frac{E_{mp}E_{mq}}{2\pi}\left[\frac{\sin\left(\left(p-q\right)\omega t+\left(\phi_p-\phi_q\right)\right)}{p-q}-\frac{\sin\left(\left(p+q\right)\omega t+\left(\phi_p+\phi_q\right)\right)}{p+q}\right]_0^{2\pi}$$

$$= 0$$

RMS value of a complex wave

Thus, total average value of e^2 is:

$$= \frac{E_{m1}{}^2}{2} + \frac{E_{m2}{}^2}{2} + \frac{E_{m3}{}^2}{2} + \ldots.. + \frac{E_{mn}{}^2}{2}$$

Hence, **RMS value:**

$$E = \sqrt{\frac{E_{m1}{}^2}{2} + \frac{E_{m2}{}^2}{2} + \frac{E_{m3}{}^2}{2} + \ldots.. + \frac{E_{mn}{}^2}{2}}$$

$$= 0.707\sqrt{E_{m1}{}^2 + E_{m2}{}^2 + E_{m3}{}^2 + \ldots.. + E_{mn}{}^2}$$

RMS value of a complex wave

The RMS value can also be expressed in the form:

$$E = \sqrt{\frac{E_{m1}{}^2}{2} + \frac{E_{m2}{}^2}{2} + \frac{E_{m3}{}^2}{2} + \ldots.. + \frac{E_{mn}{}^2}{2}}$$

$$= \sqrt{\left(\frac{E_{m1}}{\sqrt{2}}\right)^2 + \left(\frac{E_{m2}}{\sqrt{2}}\right)^2 + \left(\frac{E_{m3}}{\sqrt{2}}\right)^2 + \ldots.. + \left(\frac{E_{mn}}{\sqrt{2}}\right)^2}$$

$$= \sqrt{E_1{}^2 + E_2{}^2 + E_3{}^2 + \ldots.. + E_n{}^2}$$

Where, $E_1 = \dfrac{E_{m1}}{\sqrt{2}}$ = RMS value of fundamental

$E_2 = \dfrac{E_{m2}}{\sqrt{2}}$ = RMS value of 2^{nd} harmonic

$E_n = \dfrac{E_{mn}}{\sqrt{2}}$ = RMS value of n^{th} harmonic

RMS value of a complex wave

$$E = \sqrt{E_1{}^2 + E_2{}^2 + E_3{}^2 + \ldots\ldots + E_n{}^2}$$

Hence, the rule is that the **RMS value of a complex voltage (or current) wave is given by the square-root of the sum of the squares of the RMS values of its individual components**.

Now, if complex current wave contains a DC component of constant value E_D then its equation is given by:

$$e = E_D + E_{m1}\sin(\omega t + \phi_1) + E_{m2}\sin(2\omega t + \phi_2) + \ldots\ldots\ldots\ldots + E_{mn}\sin(n\omega t + \phi_n)\ldots\ldots$$

It's RMS value is then:

$$E = \sqrt{E_D{}^2 + \left(\frac{E_{m1}}{\sqrt{2}}\right)^2 + \left(\frac{E_{m2}}{\sqrt{2}}\right)^2 + \left(\frac{E_{m3}}{\sqrt{2}}\right)^2 + \ldots\ldots + \left(\frac{E_{mn}}{\sqrt{2}}\right)^2}$$

$$= \sqrt{E_D{}^2 + E_1{}^2 + E_2{}^2 + E_3{}^2 + \ldots\ldots + E_n{}^2}$$

Average value of a complex wave

As long as the signal is periodic, be it the fundamental or any other harmonic component, its average value is always **zero** when computed over one complete cycle

$$E_{av} = \text{Average value of } e \text{ over the entire cycle} = \frac{1}{T}\int_0^T e\,dt = 0$$

- Thus, as often is done for pure sinusoids, for complex signals also the average value could be defined for half the cycle
- In that case, the average value is calculated as the simple summation of average value of all the harmonic components:

$$E_{av} = \text{Average value of } e \text{ over half the cycle} = \sum \left(E_{av1} + E_{av2} + \ldots + E_{avn} + \ldots\ldots\right)$$

$$= \frac{2}{\pi}\left(E_{m1} + \frac{E_{m2}}{2} + \ldots\ldots\ldots\ldots \frac{E_{mn}}{n} \ldots\ldots\right)$$