ELECTROSTATICS

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Contents

Electrostatics

Electrostatics is that branch of science which deals with the phenomena associated with electricity at rest.

Absolute and Relative Permittivity of a Medium

Permittivity is the property of a medium that determines its response in an electric field. Every medium is supposed to possess two permittivities: ***(i)***absolute permittivity (ε) and ***(ii)***relative permittivity (ε_r).

For measuring relative permittivity, vacuum or free space is chosen as the reference medium. It has an absolute permittivity of 8.854×10^{-12} F/m

Absolute permittivity $\varepsilon_0 = 8.854 \times10^{-12}$F/m

Relative permittivity, $\varepsilon_r= 1$

Being a ratio of two similar quantities, ε_r has no units.

Now, take any other medium. If its relative permittivity, as compared to vacuum is ε_r, then its absolute permittivity is $\varepsilon= \varepsilon_0\varepsilon_r$ F/m

If, for example, relative permittivity of mica is 5, then, its absolute permittivity is:

$\varepsilon= \varepsilon_0\varepsilon_r= 8.854\times10^{-12}\times5 = 44.27 \times10^{-12}$F/m

Laws of Electrostatics

First Law: Like charges of electricity repel each other, whereas unlike charges attract each other.

Second Law: According to this law, the force exerted between two *point* charges ***(i)***is directly proportional to the product of their strengths ***(ii)***is inversely proportional to the square of the distance between them.

This law is known as **Coulomb's Law** and can be expressed mathematically as:

$$F \propto \frac{Q_1Q_2}{d^2} \text{ or, } F = k\frac{Q_1Q_2}{d^2} \qquad (1)$$

Coulomb's Law thus states that the force between two point charges is proportional to each charge and inversely proportional to the square of the distance between them.It is the law that describes the electrostatic interaction between electrically charged particles.

In vector form, Coulomb's law is expressed as:

$$\vec{F}_{21} = k\frac{Q_1Q_2}{{d_{12}}^2}\hat{u} \qquad (2)$$

Where, $\vec{F}_{21}$ is the force on charge Q_2 by the charge Q_1 and $\hat{u}$ is the unit vector in direction from Q_1 to Q_2. The electrostatic force of interaction acts along the straight line joining the two point charges

The value of the constant of proportionality k (Coulomb's constant) in SI system is:

$$k = \frac{1}{4\pi\varepsilon} \text{Nm}^2/\text{C}^2 \qquad (3)$$

Hence mathematical expression for the Coulomb's law becomes:

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon d^2} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 \varepsilon_r d^2} \qquad (4)$$

Hence, in a medium with relative permittivity ε_r, Coulbo's law can be written as:

$$F = \frac{Q_1 Q_2}{4\pi 8.854 \times 10^{-12} \varepsilon_r d^2} = 9 \times 10^9 \frac{Q_1 Q_2}{\varepsilon_r d^2} \qquad (5)$$

In vacuum (or air): $F = 9 \times 10^9 \dfrac{Q_1 Q_2}{d^2}$ (6)

If $Q_1 = Q_2 = 1$ C (say) and d = 1 m (say), then $F = 9 \times 10^9$ N

Hence, one coulomb of charge may be defined as ***that charge (or quantity of electricity) which when placed at 1 m distance in air (strictly vacuum) from an equal and similar charges repels it with a force of 9*** $\times 10^9$***N.***

Electric Field

It is found that in the medium around a charge a force acts on a positive or negative charge when placed in that medium. If the charge is sufficiently large, then it may create such a huge stress as to cause the electrical breakdown of the medium, followed by the passage of an arc discharge. The region around a charge where such electric forces are present is called electric field or electrostatic field. The stress is represented by imaginary lines of forces. The direction of the lines of force at any point is the direction along which a unit positive charge placed at that point would move if free to do so. Like charges repel each other and unlike charges attract each other along these lines of forces as shown in Fig 1.

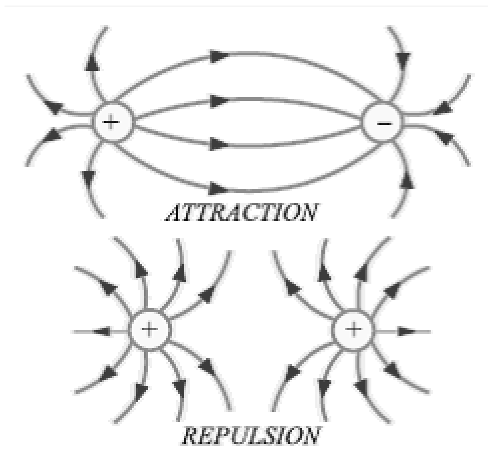


Fig. 1 Force of attraction and repulsion between charges along the lines of force

Properties of lines of forces:

- These lines are supposed to start from a positive charge and end on a negative charge as shown by Fig 2
- These lines always leave or enter a conducting surface perpendicularly

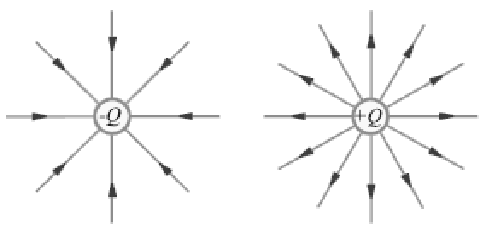


Fig 2. Line of forces for +ve and –ve charges

Electric Flux and Electric Flux Density

Consider the case of air coming in through a window. The amount of air that comes through the window depends on the speed of the air, the direction of the air and the area of the window. The air that comes through the window may be called the *air flux*. Similarly, the amount of electric field lines that pass through an area is the electric flux through that area. Consider the case of a source point charge of positive polarity. If the source charge magnitude is Q C, then the total amount of electric field lines coming out of the source charge will be also Q C. Now, if a fictitious sphere of radius d is considered such that the source charge is located at the centre of the sphere, then the electric flux through the surface of the sphere will be Q C, as the surface of the sphere completely encloses the source charge, and all the electric field lines coming out radially from the source point charge passes through the spherical surface. Electric flux is typically denoted by ψ. Since electric flux is numerically equal to the charge, it is measured in coulombs.

Electric flux density is then defined as the electric flux per unit area normal to the direction of electric flux. In the case of a point charge the electric field lines are directed radially from the source charge and hence the electric field lines are always normal to the surface of the sphere having the point charge at its center. Hence, for a point source charge of magnitude Q C, the electric flux that passes through the spherical surface area of magnitude $4\pi d^2$ is Q. Then the electric flux density ($\vec{D}$) at a radial distance d from the point charge is given by:

$$\vec{D} = \frac{Q}{4\pi d^2}\hat{u} \tag{7}$$

However, it is not necessary that electric flux will always be normal to the area A under consideration. In such cases, the component of the area that is normal to electric flux has to be taken for computing electric flux density. The following Fig 3 shows such a case, where an electric flux of magnitude ψ passes through an area of magnitude A, which is not normal to the direction of electric flux. In this case, electric flux and electric flux density are related as follows:

$$\psi = \left|\vec{D}\right| A\cos\theta$$

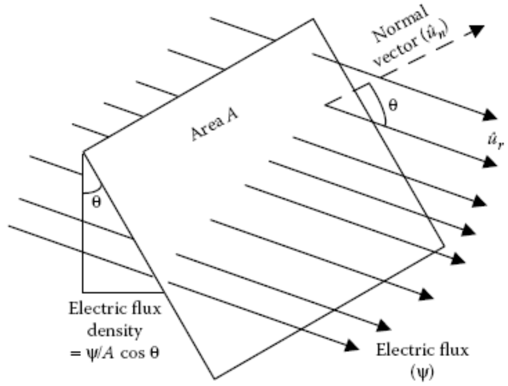


Fig. 3 Area not perpendicular to electric flux lines

Electrostatic Induction

It is found that when an uncharged body is brought near a charged body, it acquires some charge. This phenomenon of an uncharged body getting charged merely by the nearness of a charged body is known as ***induction***.It is found that: **(*i*)**a positive charge induces a negative charge and ***vice-versa*** **(*ii*)**each of the induced charges is equal to the inducing charge.

Field Strength or Field Intensity or Electric Intensity (E)

Electric intensity at any point within an electric field may be defined as the force experienced by a unit positive charge placed at that point. Its direction is the direction along which the force acts. The unit of E is Newton/coulomb (N/C).
For example, if a charge of Q coulombs placed at a particular point P within an electric field experiences a force of F, then electric field at that point is given by:

$$E = \frac{F}{Q} \tag{8}$$

The value of E within the field due to a point charge can be found with help of Coulomb's laws. Suppose it is required to find the electric field at any point "A" situated at a distance of d metres from a charge of Q coulombs. The force experienced by an unit positive charge placed at that point "A" according to Coulomb's law is:

$$F = \frac{Q \times 1}{4\pi\varepsilon_0\varepsilon_r d^2}\,\text{N} \tag{9}$$

Thus, electric field at the point A is:

$$E = \frac{Q}{4\pi\varepsilon_0\varepsilon_r d^2}\,\text{N/C} \tag{10}$$

Then, the force on a test charge of magnitude q could be written as follows:

$$F = qE = \frac{qQ}{4\pi\varepsilon_0\varepsilon_r d^2} \tag{11}$$

A relationship between the electric field intensity E and the electric flux density D can now be derived from equations (7) and (10) as:

$$\vec{D} = \varepsilon_0\varepsilon_r\vec{E} \tag{12}$$

Electric flux density is a vector quantity because it has a direction along the electric field lines at the position where electric flux density is being computed.

Electric Potential

Consider that a test charge of magnitude q is located at a given position within an electric field produced by a system of charges. The test charge will experience a force due to the source charges. If the test charge moves in the direction of the field forces, then the work is done by the field forces in moving the test charge from position 1 to position 2. In other words, energy is spent by the electric field. Hence, the potential energy of test charge at position 2 will be lower than that at position 1. On the other hand, if the charge is moved against the field forces by an external agent, then the work done by the external agent will be stored as potential energy of the test charge. Hence, the potential energy of the test charge at position 2 will be higher than that at position 1. Here, it is to be noted that the force experienced by the test charge within an electric field is dependent on the magnitude of the test charge. Hence, the potential energy of the charge at any position is dependent on its magnitude and the distance by which it moves within the electric field.

In the gravitational field, usually 'sea level' is chosen as the place of 'zero' potential. The force acting on a charge at infinity is zero, hence 'infinity' is chosen as the theoretical place of zero electric potential. In electric field, though infinity is chosen as the theoretical place of 'zero' potential, in practice, earth is chosen as 'zero' potential, because earth is such a large conductor that its potential remains practically constant although it keeps on losing and gaining electric charge every day. Therefore, potential at any point in an electric field may be defined as ***numerically equal to the work done in bringing a positive charge of one coulomb from infinity to that point against the electric field.***

The electric potentialat any point within an electric field is mathematically defined as potential energy perunit charge at that point and hence it is a scalar quantity. If, in shifting one coulomb from infinity to a certain point in the electric field, the work done is one joule, then potential of that ponit is one volt. Thus, the unit of electricpotential is volt (V), which is equivalent to joules per coulomb (J/C).

It is interesting to note that electric field intensity is defined as force perunit charge and electric potential is defined as potential energy per unitcharge. The work done in moving a unit positive charge from one point to the other within an electric field is equal to the difference in potential energies and hence difference in electric potentials at the two points. Hence, potential difference (p.d.) of one volt exists between two points if one joule of work is done in shifting a charge of one coulomb from one point to the other.

As shown in Fig 4, consider that a unit positive charge is moved from point 1 to point 2 by a small distance *dl*. The force experienced by the unit positive charge at point 1 is the electric field intensity *E*. Then the potential difference between the ending point 2 and starting point 1 is given by:

$$V_2 - V_1 = -\vec{E}.d\vec{l} \qquad (13)$$

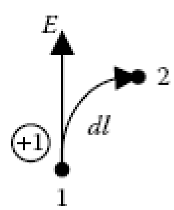


Fig. 4 Definition of electric potential

Equation (13) gives the idea of potential gradient which is defined as the rate of change of potential with distance in the direction of electric force, i.e. dV/dx

From (13), we obtain an alternate definition for electric field intensity *E* as ***the potential gradient at that point.***

$$\vec{E} = -\frac{dV}{d\vec{l}} \qquad (14)$$

The unit of electric field intensity is thus popularly expressed in terms of [volts/meter], V/malthough volt/cm is generally used in practice.

Potential at a Point

Consider a positive point charge of Q coulombs is placed at the origin of Fig 5. The electrostatic force on one coulomb unit positive charge placed at the point A at a distance x metres from the charge Q is:

$$F = \frac{Q \times 1}{4\pi\varepsilon_0\varepsilon_r x^2}$$

Suppose, this one coulomb charge is moved towards Q through a small distance dx from A to B. Then, work done is:

$$dW = F.(-dx) = \frac{Q}{4\pi\varepsilon_0\varepsilon_r x^2}.(-dx) \qquad (15)$$

The negative sign is taken because dx is considered along the negative direction of x. The total work done in bringing this coulomb of positive charge from infinity to any point D which is d metres from Q is given by:

$$W = \int dW = -\int_{\infty}^{d} \frac{Qdx}{4\pi\varepsilon_0\varepsilon_r x^2} = -\frac{Q}{4\pi\varepsilon_0\varepsilon_r}\int_{\infty}^{d} \frac{dx}{x^2}$$

$$= -\frac{Q}{4\pi\varepsilon_0\varepsilon_r}\left| -\frac{1}{x} \right|_{\infty}^{d} = -\frac{Q}{4\pi\varepsilon_0\varepsilon_r}\left| -\frac{1}{d} + \frac{1}{\infty} \right| = \frac{Q}{4\pi\varepsilon_0\varepsilon_r d} \text{ Joules} \qquad (16)$$

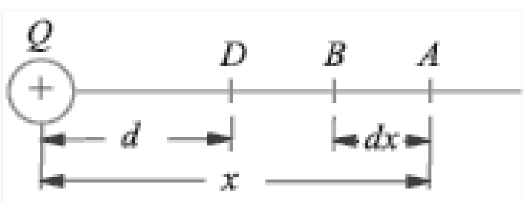


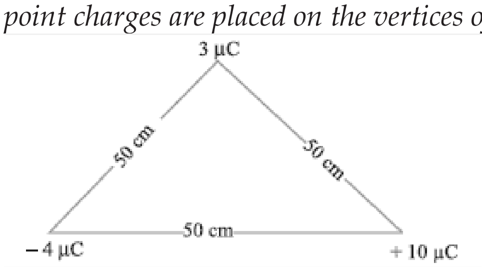
Fig. 5. Potential at a point

By definition, this work in joules in numerically equal to the potential of that point in volts.

$$\therefore V = \frac{Q}{4\pi\varepsilon_0\varepsilon_r d} = 9\times10^9 \frac{Q}{\varepsilon_r d} \text{ Volts}$$

It can be seen that as d increases, V decreases till it becomes zero at infinity.

Example 1: Determine resultant force on a 3 μC charge due to - 4μC and 10 μC charges. All these three point charges are placed on the vertices of equilateral triangle ABC of side 50 cm.



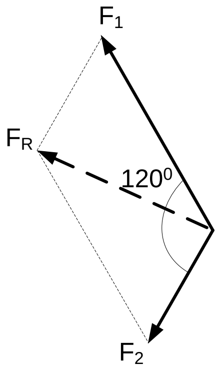
Solution.

$F = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 \varepsilon_r d^2}$ assuming the charges are placed in air, $F = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 d^2}$

$$\therefore F_1 = \frac{3\times10^{-6}\times10\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(0.5)^2} = 1.08\,\mathrm{N}$$

Similarly, $F_2 = \dfrac{3\times10^{-6}\times4\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(0.5)^2} = 0.432\,\mathrm{N}$

Directions of these two forces are shown below:



The resultant force on 3 μC charge is thus:

$$F_R = \sqrt{F_1^{\,2} + F_2^{\,2} + 2F_1F_2\cos120^0} = 0.942\,\text{N-m}$$

Example 2: Three charges of magnitude +2.0 X 10^{-10} C, -8.0 X 10^{-10} C, +4.0 X 10^{-10} C are placed at three corners B, C, D respectively of a rectangle ABCD whose side AB = 5 cm and BC = 7 cm. Calculate the field intensity and electric potential at corner A of the rectangle.

Solution.

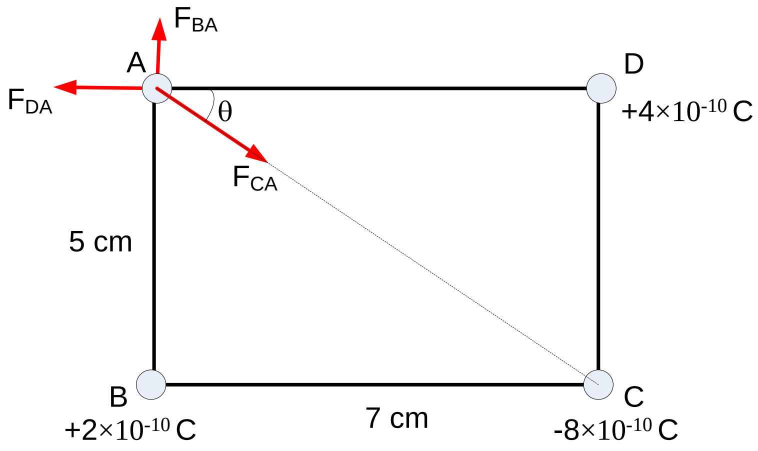
Assuming the medium to be air, the field intensities at A due to the other three charges are calculated as follows:

$E_{BA} = 9\times10^9\times\dfrac{Q}{d^2} = 9\times10^9\times\dfrac{2\times10^{-10}}{0.05^2} = 720\,V/m$, directed along B to A

$E_{DA} = 9\times10^9\times\dfrac{Q}{d^2} = 9\times10^9\times\dfrac{4\times10^{-10}}{0.07^2} = 734.7\,V/m$, directed along D to A

$E_{CA} = 9\times10^9\times\dfrac{Q}{d^2} = 9\times10^9\times\dfrac{8\times10^{-10}}{0.05^2+0.07^2} = 973\,V/m$, directed along A to C

$$\theta = \tan^{-1}\frac{5}{7} = 35.54^0$$



Resolving the field intensity vectors in real and imaginary components (rectangular form):

$$\vec{E}_{BA} = 0 + j720$$

$$\vec{E}_{DA} = -734.7 + j0$$

$$\vec{E}_{CA} = 973\cos 35.54^0 - j973\sin 35.54^0 = 791.7 - j565.6$$

$\therefore$ Resultant filed intensity at A:

$$\vec{E}_A = \vec{E}_{BA} + \vec{E}_{DA} + \vec{E}_{CA} = 0 + j720 - 734.7 + j0 + 791.7 - j565.6 = 57 + j154.4 = 164.6\angle 69.7^0 \; V/m$$

To find electric potential, we make note of the fact that potential is a scalar quantity and thus only algebraic summation will be applicable.

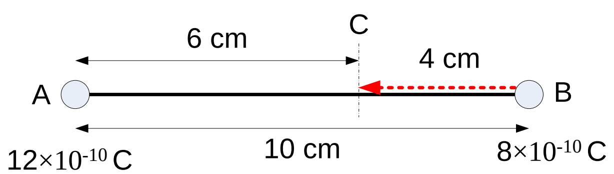
Thus,

$$V_A = V_{BA} + V_{DA} + V_{CA} = 720\times .05 + 734.7\times 0.07 - 973\times\sqrt{0.05^2 + 0.07^2} = 36 + 51.4 - 83.7 = 3.73\,V$$

Example 3: Two positive point charges of 12 ×10-10 C and 8 ×10-10 C are placed 10 cmapart in air. Find the work done in bringing the two charges 4 cm closer.

Solution.

Suppose the *12 ×10-10* C charge is fixed and the other charge is being moved.



Due to the charge at A, there will be some potential at the points B and C

$$V_B = 9\times10^9 \times \frac{Q}{d_1} = 9\times10^9 \times \frac{12\times10^{-10}}{0.1} = 108\,V$$

$$V_C = 9\times10^9 \times \frac{Q}{d_2} = 9\times10^9 \times \frac{12\times10^{-10}}{0.06} = 180\,V$$

$\therefore$ Potential difference $V_{CB} = 180 - 108 = 72\,V$

$\therefore$ Work done = Charge ×p.d. = $8\times10^{-10} \times 72 = 5.76\times10^{-8}\;J$

Potential of a Charged Conducting Sphere

The formula $V = \dfrac{Q}{4\pi\varepsilon_0\varepsilon_r d}$ applies only to a charge concentrated at a point. The problem of finding potential at a point outside a charged sphere sounds difficult, because the charge on the sphere is distributed over its entire surface and so, is not concentrated at a point. But the problem is easily solved by imagining that the charge on the sphere is concentrated at its centre *O*. If *r* is the radius of sphere in metres and *Q* its charge in coulomb then, potential of its surface is: $V = \dfrac{Q}{4\pi\varepsilon_0\varepsilon_r r}$

At any other point '*d*' ($d > r$) metres from the centre of the sphere, the potential is:

$$V = \frac{Q}{4\pi\varepsilon_0\varepsilon_r d}$$

The variations of the potential with distance from the centre for a charged sphere are shown in Fig 6.

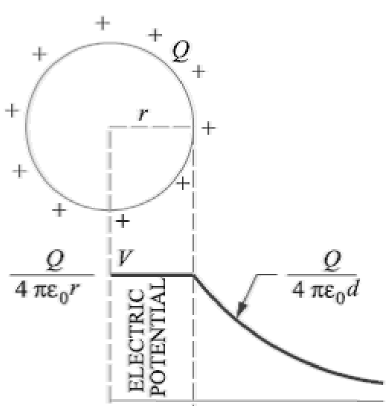


Fig. 6. Potential variation due to a charged sphere

It has been experimentally found that when charge is given to a conducting body say, a sphere then it resides entirely on its outer surface *i.e.*, within a conducting body whether hollow or solid, the charge is zero. Hence, **(*i*)**flux is zero **(*ii*)**field intensity is zero **(*iii*)**all points within the conductor are at the same potential as at its surface.

Equipotential Surfaces

It is common that in an electric field, there could be several points which are at the same potential. If all these points are joined together then one may get a line or a surface on which every point has the same electric potential. Such a line or surface is called an equipotential. An equipotential surface is thus a surface in an electric field such that all points on it are at the same potential. For example, different spherical surfaces around a charged sphere are equipotential surfaces. Fig 7 shows typical examples of equipotentials in two-dimensional systems, where these will be lines. In the case of three-dimensional systems, such equipotentials will be surfaces.

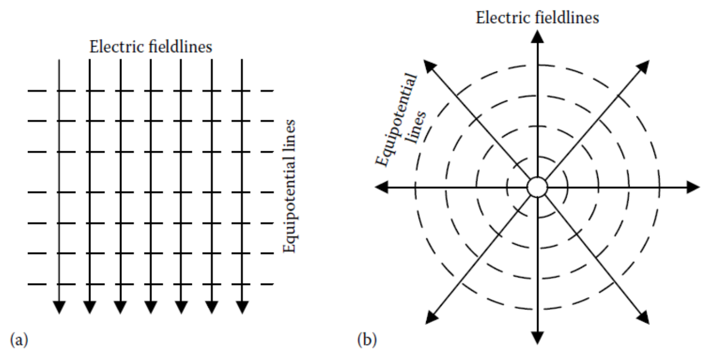


Fig 7 (a) uniform longitudinal field due to a line charge (b) radial field due to appoint charge

One important property of an equipotential surface is that the direction of the electric field strength and flux density is always at right angles to the surface. Also, electric flux emerges out normal to such a surface. If, it is not so, then there would be some component of E along the surface resulting in potential difference between various points lying on it which is contrary to the definition of an equipotential surface.

To explain further, let us consider an equipotential of any electric field, as shown in Fig. 7. At any point P on this equipotential, consider that the electric field line makes an angle θ with the tangent to the equipotential at that point. If an elementary length dl is considered along the equipotential at P, then the potential difference between the two extremities of dl will be given by $\vec{E}\cos\theta.\vec{dl}$. But ifdl lies on the equipotential, then there should not be any potential differenceacross dl. Again, the magnitude of electric field intensity is not zero at P anddl is also a non-zero quantity. Hence, the potential difference across dl couldonly be zero if cos θ is zero (i.e. if θ is 90^0).

Thus, a basic constraint of electric field distribution is that the electric fieldlinesare always normal to the equipotential surface. A practical example ofthis constraint is that the electric fieldlines will always leave or enter conductorsurfaces at 90^0.

Breakdown Voltage and Dielectric Strength

An insulator or dielectric is a substance within which there are no mobile electrons necessary for electric conduction. However, when the voltage applied to such an insulator exceeds a certain value, then it breaks down and allows a heavy electric current (much larger than the usual leakage current) to flow through it. If the insulator is a solid medium, it gets punctured or cracked. The disruptive or breakdown voltage of an insulator is the minimum voltage required to break it down.

Dielectric strength of an insulator or dielectric medium is given by the ***maximum potential difference which a unit thickness of the medium can withstand without breaking down***.

In other words, the dielectric strength is given by the potential gradient necessary to cause breakdown of an insulator. Its unit is volt/metre (V/m) although it is usually expressed in kV/mm. For example, when we say that the dielectric strength of air is 3 kV/mm, then it means that the maximum p.d. which one mm thickness of air can withstand across it without breaking down is 3 kV or 3000 volts. If the p.d. exceeds this value, then air insulation breaks down allowing large electric current to pass through.

Dielectric strength of various insulating materials is very important factor in the design of high voltage generators, motors and transformers. Its value depends on the thickness of the insulator, temperature, moisture, content, shape and several other factors.

Gauss Law

Gauss's law states that the net electric flux through any closed surface enclosinga homogeneous volume of material is equal to the net electric chargeenclosed by that closed surface. In other words, the surface integral of electricflux density over a closed surface is equal to the total charge enclosed (volume integralof charge densities within the volume) by that surface.

In simple form: $\int_A \vec{D}.d\vec{A} = \sum q$ (17)

In integral form, $\int_A \vec{D}.d\vec{A} = \int_V \rho_V \, dV$ (18)

Fig. 8 shows a certain volume V of homogeneous material enclosed by a surface A. This volume is charged by N number of discrete point charges spread randomly within the volume.

Take a point charge q_1 located within V. Then consider an elementary area dA on A, as shown in Fig 8. The distance of dA from q_1 is, say, r.

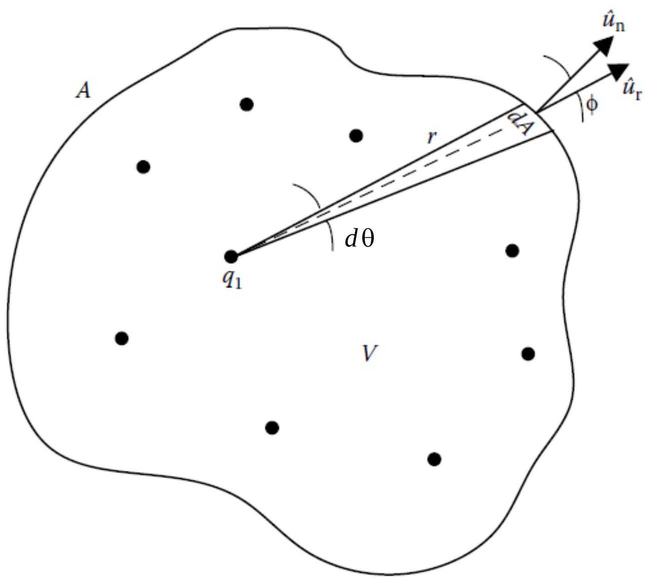


Fig. 8. Proof of Gauss law

Then at a distance r from q_1, the electric field intensity and electric flux density vectors are given according to Coulomb's law as:

$$\vec{E}_1 = \frac{q_1}{4\pi\varepsilon_0 r^2}\hat{u}_r \text{ and } \vec{D}_1 = \frac{q_1}{4\pi r^2}\hat{u}_r \qquad (19)$$

Then, the total flux through A due to $q1$ could be obtained as follows:

$$\int_A \vec{D}_1.d\vec{A} = \int_A \frac{q_1}{4\pi}\frac{\hat{u}_r.d\vec{A}}{r^2} = \frac{q_1}{4\pi}\int_A \frac{\hat{u}_r.d\vec{A}}{r^2} \qquad (20)$$

From the properties of *solid angle*, the quantity $\frac{\hat{u}_r.d\vec{A}}{r^2}$ is equal to the incremental solid angle $d\theta$ as shown in Fig 7.

Because the surface area A completely encloses the charge q_1, the total solid angle subtended by A at thelocation of q_1 is 4 π, i.e. $\int_A \frac{\hat{u}_r.d\vec{A}}{r^2} = \int_A d\theta = 4\pi$.

Thus,

$$\int_A \vec{D}_1.d\vec{A} = \frac{q_1}{4\pi}\times 4\pi = q_1 \qquad (21)$$

Considering, N number of point charges within V, total electric flux densityat the location of dA, as shown in Fig 7, will be as follows:

$$\vec{D} = \vec{D}_1 + \vec{D}_2 + \vec{D}_3 + \dots\dots\dots + \vec{D}_N$$

Therefore, the total flux through A will be as follows:

$$\int_A \vec{D}.d\vec{A} = \int_A \left(\vec{D}_1 + \vec{D}_2 + \vec{D}_3 + \ldots.. + \vec{D}_N\right).d\vec{A}$$

$$= \int_A \vec{D}_1.d\vec{A} + \int_a \vec{D}_2.d\vec{A} + \int_a \vec{D}_3.d\vec{A} + \ldots\ldots\ldots + \int_a \vec{D}_N.d\vec{A}$$

$$= q_1 + q_2 + q_3 + \ldots\ldots\ldots. + q_N \qquad (22)$$

The right-hand side (RHS) of Equation (22) is equal to the total electric chargeenclosed within V. Considering continuously distributed charge within V,the total charge within is:

$$Q = \int_V \rho_V \, dV$$

where: ρ_V is the volume charge density within V

Hence, Equation (22) can be rewritten as:

$$\int_A \vec{D}.d\vec{A} = \sum q = \int_V \rho_V \, dV$$

Hence proved

Capacitor

A capacitor essentially consists of two conducting surfaces separated by a layer of an insulating medium called ***dielectric***. The conducting surfaces may be in the form of either circular (or rectangular) plates or be of spherical or cylindrical shape. The purpose of a capacitor is to store electrical energy by electrostatic stress in the dielectric.

A parallel-plate capacitor is shown in Fig 9. One plate is joined to the positive end of the supply and the other to the negative end or is earthed. When such a capacitor is put across a battery, there is a momentary flow of electrons from A to B. As negatively-charged electrons are withdrawn from A, it becomes positive and as these electrons collect on B, it becomesnegative. Hence, a p.d. is established between plates A and B. The transientflow of electrons gives rise to charging current. The strength of the chargingcurrent is maximum when the two plates are uncharged but it then decreases and finally ceases whenp.d. across the plates becomes slowly and slowly equal and opposite to the battery e.m.f.

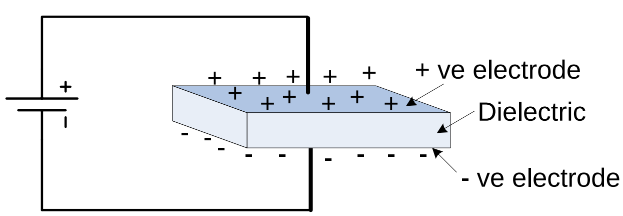


Fig 9. Parallel plate capacitor

Capacitance

The property of a capacitor to 'store electricity' may be called its capacitance. The capacitance of a capacitor is defined as ***"the amount of charge required to create a unit p.d. between its plates."***

Suppose we give Q coulomb of charge to one of the two plates of capacitor and if a p.d. of V volts is established between the two, then its capacitance is:

$$C = \frac{Q}{V} = \frac{\text{charge}}{\text{potential difference}} \tag{23}$$

Hence, capacitance is the ***charge required per unit potential difference.***

By definition, the unit of capacitance is coulomb/volt which is also called ***farad***(in the honour of Michael Faraday)

$\therefore$1 farad = 1 coulomb/volt

One farad is defined as ***the capacitance of a capacitor which requires a charge of one coulomb to establish a p.d. of one volt between its plates***.

Capacitance of an Isolated Sphere

Consider anisolated charged sphere of radius r metres having a charge of Q coulomb placed in a medium of relative permittivity ε_r as shown in Fig 10.

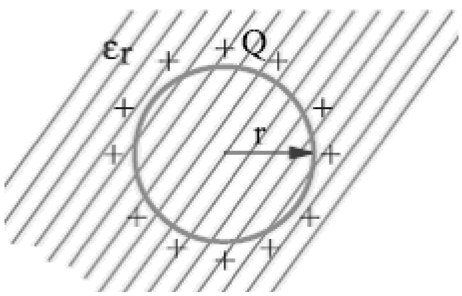


Fig 10. Spherical capacitor

Surface potential $V$of such a sphere with respect to infinity (in practice, earth) is given by:

$$V = \frac{Q}{4\pi\varepsilon_0\varepsilon_r r} \tag{24}$$

$$\therefore \frac{Q}{V} = 4\pi\varepsilon_0\varepsilon_r r$$

Thus, from (23), we have capacitance of the sphere as:

$$C = \frac{Q}{V} = 4\pi\varepsilon_0\varepsilon_r r \text{ F} \tag{25}$$

If the medium is air, $C = 4\pi\varepsilon_0 r$ F

Spherical Capacitor

(Outer sphere is earthed)

Consider a spherical capacitor consisting of two concentric spheres of radii '*a*' and '*b*' metres as shown in Fig 11.

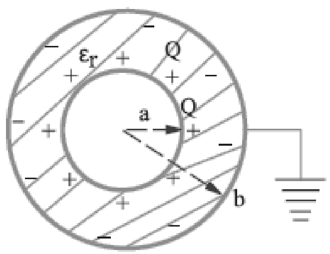


Fig. 11. Capacitance of concentric spheres

Suppose, the inner sphere is given a charge of + Q coulombs. It will induce a charge of -Q coulombs on the inner surface of the outer sphere which will go to earth. If the dielectric medium between the two spheres has a relative permittivity of ε_r, then the free surface potential of the inner sphere due to its own charge is:

$$V_a = \frac{Q}{4\pi\varepsilon_0\varepsilon_r a}$$

The potential of the inner sphere due to -Q charge on the outer sphere is:

$$V_b = \frac{Q}{4\pi\varepsilon_0\varepsilon_r b}$$

(remembering that potential anywhere inside a sphere is the same as thatits surface).

$\therefore$ Total potential difference between two surfaces is:

$$V = V_a - V_b = \frac{Q}{4\pi\varepsilon_0\varepsilon_r a} - \frac{Q}{4\pi\varepsilon_0\varepsilon_r b} = \frac{Q}{4\pi\varepsilon_0\varepsilon_r}\left[\frac{1}{a} - \frac{1}{b}\right] = \frac{Q}{4\pi\varepsilon_0\varepsilon_r}\left(\frac{b-a}{ab}\right)$$

$$\therefore \text{Capacitance, } C = \frac{Q}{V} = 4\pi\varepsilon_0\varepsilon_r \frac{ab}{b-a} \qquad (26)$$

Parallel-plate Capacitor

(*i*) Uniform Dielectric-Medium

A parallel-plate capacitor consisting of two plates M and N each of area A m^2 separated by a thickness d metres of a
medium of relative permittivity ε_r is shown in Fig 12.

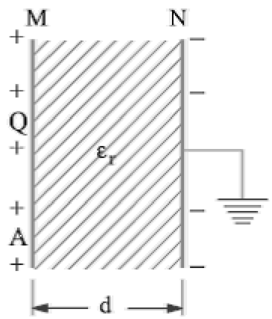


Fig 12. Parallel plate capacitor with single dielectric

If a charge of + Q coulomb is given to plate M, then flux passing through the medium is $\psi = Q$ coulomb. Flux density in the medium is:

$$D = \frac{\psi}{A} = \frac{Q}{A}$$

Electric field intensity: $E = \dfrac{V}{d}$ and $D = \varepsilon E$

Thus, $\dfrac{Q}{A} = \varepsilon E = \varepsilon \dfrac{V}{d} \therefore \dfrac{Q}{V} = \varepsilon \dfrac{A}{d}$

Thus, capacitance, $C = \dfrac{Q}{V} = \varepsilon \dfrac{A}{d} = \varepsilon_0 \varepsilon_r \dfrac{A}{d}$

For air as dielectric medium, $C = \varepsilon_0 \dfrac{A}{d}$ (27)

(*ii*) Medium Partly Air

As shown in Fig. 13, the medium consists partly of air and partlyof parallel-sided dielectric slab of thickness t and relative permittivityε_r .

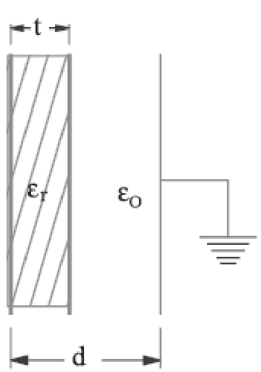


Fig. 13. Parallel plate capacitor with two dielectrics

Since the same flux passes through the material and also the air, the electric flux density $D = Q/A$ is the same in both media. Butelectric field intensities are different.

$E_1 = \dfrac{D}{\varepsilon_0 \varepsilon_r}$ in the medium

$E_2 = \dfrac{D}{\varepsilon_0}$ in air

Potential difference between the two electrodes:

$$V = E_1 \times t + E_2 \times (d - t)$$

$$= \frac{D}{\varepsilon_0 \varepsilon_r} \times t + \frac{D}{\varepsilon_0} \times (d - t)$$

$$= \frac{D}{\varepsilon_0}\left(\frac{t}{\varepsilon_r} + d - t\right)$$

$$= \frac{Q}{A\varepsilon_0}\left[d - \left(t - \frac{t}{\varepsilon_r}\right)\right]$$

$$\therefore \text{Capacitance, } C = \frac{Q}{V} = \frac{\varepsilon_0 A}{\left[d - \left(t - \frac{t}{\varepsilon_r}\right)\right]} \quad (28)$$

If the medium were totally air, then capacitance would have been:

$$C = \frac{\varepsilon_0 A}{d} \quad (29)$$

From **(28)** and **(29)**, it is obvious that when a dielectric slab of thickness t and relative permittivity$\varepsilon_r$is introduced between the plates of an air capacitor, then its capacitance increases because as seenfrom **(28)**, the denominator decreases.

iii) Composite Medium

Extending the above discussion to a more generalized case when the space between the two electrodes is filled up with more than one dielectric mediums with different values of relative permittivities as shown in Fig 14.

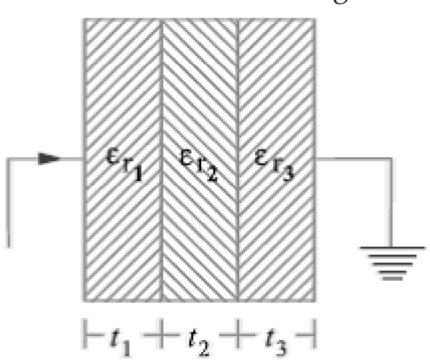
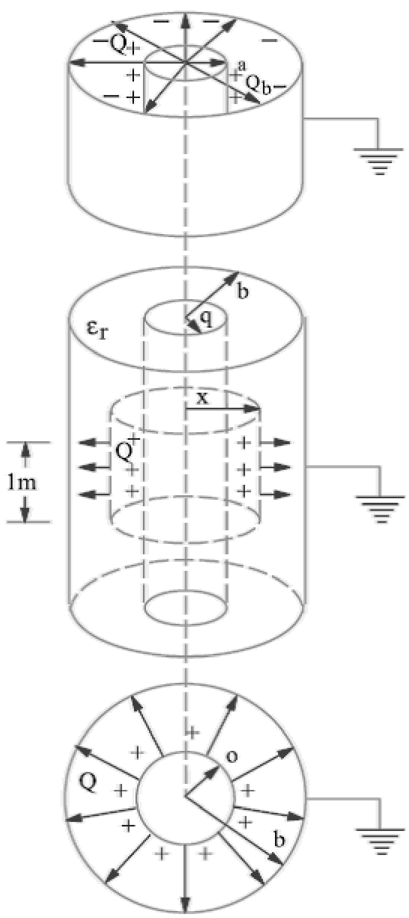


Fig. 14. Parallel plate capacitor with composite dielectrics

In such a case, though the flux through the composite medium remain same, the electric field intensities will be different in different medium.

If V is the total potential difference across the capacitor plates and $V1$, $V2$, $V3$, the potential differences across the three dielectric slabs, then:

$$V = V_1 + V_2 + V_3 = E_1 \times t_1 + E_2 \times t_2 + E_3 \times t_3$$
$$= \frac{D}{\varepsilon_0 \varepsilon_{r1}} \times t_1 + \frac{D}{\varepsilon_0 \varepsilon_{r2}} \times t_2 + \frac{D}{\varepsilon_0 \varepsilon_{r3}} \times t_3$$
$$= \frac{D}{\varepsilon_0}\left(\frac{t_1}{\varepsilon_{r1}} + \frac{t_2}{\varepsilon_{r2}} + \frac{t_3}{\varepsilon_{r3}}\right)$$
$$= \frac{Q}{\varepsilon_0 A}\left(\frac{t_1}{\varepsilon_{r1}} + \frac{t_2}{\varepsilon_{r2}} + \frac{t_3}{\varepsilon_{r3}}\right)$$
$$\therefore \text{Capacitance, } C = \frac{Q}{Q} = \frac{\varepsilon_0 A}{\left(\frac{t_1}{\varepsilon_{r1}} + \frac{t_2}{\varepsilon_{r2}} + \frac{t_3}{\varepsilon_{r3}}\right)} \qquad (30)$$



Cylindrical Capacitor

Cylindrical capacitor consisting two co-axial cylinders of radii a and b metres, is shown in Fig 15. Let the charge per metre length of the cable on the outer surfaceof the inner cylinder be + Q coulomb and that on the inner surface ofthe outer cylinder be -Q coulomb. For all practical purposes,the charge + Q coulomb/metre on the surface of the inner cylindercan be supposed to be located along its central vertical axis. Let $\varepsilon_r$be therelative permittivity of the medium between the two cylinders.The outer cylinder is earthed.

Now, let us find the value of electric intensity at any pointdistant x metres from the axis of the inner cylinder. As shown in Fig15, consider an imaginary co-axial cylinder of radius $x$metres and length one metre between the two given cylinders.

The electric field between the two cylinders is radial as shown.Total flux coming out radially from the curved surface of thisimaginary cylinder is Q coulomb.

Area of the curved surface $= 2\pi x \times 1 = 2\pi x \; m^2$

Hence, the value of electric flux density on the surface of theimaginary cylinder is:

$$D = \frac{\psi}{A} = \frac{Q}{A} = \frac{Q}{2\pi x} C/m^2 \quad (31)$$

Value of the electric field intensity is:

$$E = \frac{D}{\varepsilon_0 \varepsilon_r} = \frac{Q}{2\pi\varepsilon_0\varepsilon_r x} V/m \quad (32)$$

Now, $dV = -Edx$

$$\text{or, } V = \int_a^b -Edx = \int_a^b -\frac{Qdx}{2\pi\varepsilon_0\varepsilon_r x}$$

$$= -\frac{Q}{2\pi\varepsilon_0\varepsilon_r}\int_a^b \frac{dx}{x}$$

$$= -\frac{Q}{2\pi\varepsilon_0\varepsilon_r}\left|\log x\right|_a^b$$

$$= -\frac{Q}{2\pi\varepsilon_0\varepsilon_r}\left(\log a - \log b\right)$$

$$= \frac{Q}{2\pi\varepsilon_0\varepsilon_r}\log\left(\frac{b}{a}\right)$$

$$\therefore \text{Capacitance, } C = \frac{Q}{V} = \frac{2\pi\varepsilon_0\varepsilon_r}{\log\left(\frac{b}{a}\right)} \text{ F/m} \quad (33)$$

Thus, if the cylindrical conductor is of L meter length, then its total capacitance is:

$$C = \frac{2\pi\varepsilon_0\varepsilon_r L}{\log\left(\frac{b}{a}\right)} \text{ F} \quad (34)$$

Capacitors in Series

If a number of capacitors are connected in series as shown in Fig 16, then their overall equivalent capacitance is calculated as follows:

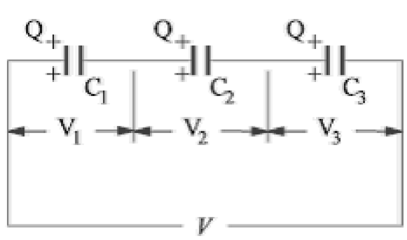


Fig. 16. Three capacitors in series

$C1$, $C2$, $C3$ = Capacitances of three capacitors
$V1$, $V2$, $V3$ = p.ds. across three capacitors.
V = applied voltage across combination
C = combined or equivalent or joining capacitance.

In series combination, charge on all capacitors is the same but p.d. across each is different.

$\therefore V = V_1 + V_2 + V_3$

or, $\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$

or, $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$ (35)

$\therefore C = \frac{C_1C_2C_3}{C_1C_2 + C_2C_3 + C_3C_1}$ (36)

We can also find the values of $V1$, $V2$, $V3$ in terms of V as:

$Q = C_1V_1 = C_2V_2 = C_3V_3 = CV$ (37)

Capacitors in Parallel

If a number of capacitors are connected in parallel as shown in Fig 17, then their overall equivalent capacitance is calculated as follows:

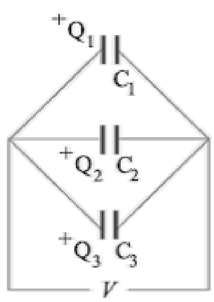


Fig. 17. Three capacitors in parallel

In this case, p.d. across each is the same but charge on each is different:

$\therefore Q = Q_1 + Q_2 + Q_3$

or, $CV = C_1V + C_2V + C_3V$

or, $C = C_1 + C_2 + C_3$ (38)

Energy Stored in a Capacitor (Energy stored in Electrostatic field)

Charging of a capacitor always involves some expenditure of energy by the charging agency. This energy is stored up in the electrostatic field set up in the dielectric medium. On discharging the capacitor, the field collapses and the stored energy is released.

To begin with, when the capacitor is uncharged, little work is done in transferring charge from one plate to another. But further instalments of charge have to be carried against the repulsive force due to the chargealready collected on the capacitor plates. Let us find theenergy spent in charging a capacitor of capacitance C toa voltage V.

Suppose at any stage of charging, the p.d. across theplates is *v*. By definition, it is equal to the work done inshifting one coulomb from one plate to another. If '*dq*'is charge next transferred, the work done is:

$dW = vdq$

Now, $q = Cv$

$\therefore dq = Cdv$

$\therefore dW = vdq = Cvdv$

Total work done in giving *V* units of potential is:

$$W = \int dW = \int_0^V Cvdv = C\left|\frac{v^2}{2}\right|_0^V$$

$$\therefore W = \frac{1}{2}CV^2$$

If *C* is in farads and *V* is in volts, then $W = \frac{1}{2}CV^2$ Joules = $\frac{1}{2}QV$ Joules = $\frac{Q^2}{2C}$ Joules

If *Q* is in coulombs and *C* is in farads, the energy stored is given in joules.

Now, for a capacitor of plate area *A* m^2 and dielectric of thickness *d* metre, energy per unit volume of dielectric medium:

$$W_V = \frac{1}{2}\frac{CV^2}{Ad} = \frac{1}{2}\varepsilon\frac{A}{d}\frac{V^2}{Ad} = \frac{1}{2}\varepsilon\frac{V^2}{d^2} = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}DE = \frac{1}{2\varepsilon}D^2$$

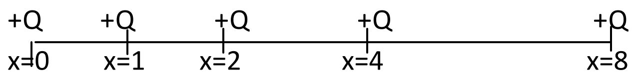
Question bank

Theory questions

1. State and explain Coulomb's law in Electrostatics and hence define "Coulomb", the unit for electric charge.
2. What is permittivity? What do you mean by relative permittivity of a medium? Why it does not have any unit?
3. What is meant by electric field intensity? Discuss the various factors on which it depends.
4. Find the expression of electric field intensity and electric potential of an isolated point charge in vector form.
5. Derive an expression of potential energy in an electric field.
6. Derive an expression for energy stored in an electric field.
7. Find a relationship between electric field strength and electric potential
8. State Gauss' Law and derive it from Coulomb's law.
9. Define electric capacitance and derive an expression for the capacitance of a parallel plate capacitor.
10. Find the capacitance of an isolated sphere.
11. Derive the expression to find capacitance of concentric spheres.
12. Deduce an equation capacitance of the parallel plate capacitor with (i) uniform dielectric medium, (ii) compound dielectric medium.
13. Derive an expression for capacitance of a cylindrical capacitor, assuming outer surface to be grounded
14. Derive an expression for the capacitance of a cylindrical capacitor consisting of the infinitely long axial cylinders of radii R_1 and R_2 ($R_2 > R_1$).
15. Explain the following terms: Electric field intensity, Potential difference.
16. Deduce an expression showing the relation between electric filed strength and potential.
17. What do you understand by the terms electric potential and electric potential difference?
18. Find an expression for potential at a point within an electric field. What is equipotential surface?
19. Discuss the various factors upon which the value of capacitance parallel plate capacitor depends.
20. Derive expression for the equivalent capacitance for a number of capacitors connected in (i) series, (ii) parallel.
21. Derive an expression for the energy stored in a charged capacitor.
22. State and prove Gauss law
23. Define the following terms:
    i) Electric flux
    ii) Electric flux density
    iii) Electric potential
    iv) Electric potential difference
    v) Dielectric strength

Numerical questions

1. Find the potential and field intensity at x=0 due to these set of charges shown in the figure below; x represents the distance from origin in x axis. Q is the magnitude of charge.



[Ans: $V = \dfrac{0.15Q^2}{\varepsilon_0}$ **,** $E = \dfrac{0.1Q}{\varepsilon_0}$ **]**

2. Three equal charges each of magnitude 3.0 X 10^{-6} C are placed at three corners of a right angled triangle of sides 3 cm and 4 cm. Find the force on the charge at the apex corner if 4 cm side is the base of the triangle.
[Ans: 112.3 N]

3. Three charges of magnitude +2.0 X 10^{-10} C, -8.0 X 10^{-10} C, +4.0 X 10^{-10} C are placed at three corners B, C, D respectively of a rectangle ABCD whose side AB = 5 cm and BC = 7 cm. Calculate the potential at corner A of the rectangle.
[Ans: V_A = 3.73 V]

4. A parallel plate capacitor has plate area of 0.1m^2 and plate separation 0.015cm. The dielectric medium between the plates has relative permittivity 3. The capacitor retains a charge of 0.1μC when placed across voltage source. Find the flux density, electric field strength and voltage across the plates. Assume the permittivity of space as 8.854 x 10^{-12} F/m.
[Ans: D = 0.98 C/m^2, E = 37.07 kV/m, V = 5.65 V]

5. Calculate the capacitance between two parallel plates of the area 0.4m^2 separated by a di-electric of 0.1 mm thick & of relative permittivity 5. If the voltage across the capacitor is 50 V, find the energy stored in the capacitor & the voltage gradient in the dielectric.
[Ans: C = 177.1 nF, Energy stored = 2.22×10^{-4} J, Voltage gradient = 500 kV/m]

6. A capacitor is made of two plates with an area of 10 cm^2 separated by a mica sheet of 1 mm thickness. Find the capacitance taking ε_r = 6 for mica. If one of the plates is moved to provide an air gap of 0.25 mm thickness between the upper plate and mica, calculate the change in the value of capacitance.
[Ans: C = 0.053 nF, New capacitance C' = 0.011 nF]