AC Fundamentals

Day 13

Complex notation applied to AC circuits

ILOs – Day 13

- Introduce the use of complex notations for AC circuits
- Perform addition and subtraction of complex quantities
- Perform multiplication and division of complex quantities
- Represent RL circuit by complex quantities
- Represent RC circuit by complex quantities
- Define and explain Admittance, conductance, susceptance

Complex notation applied to AC circuits

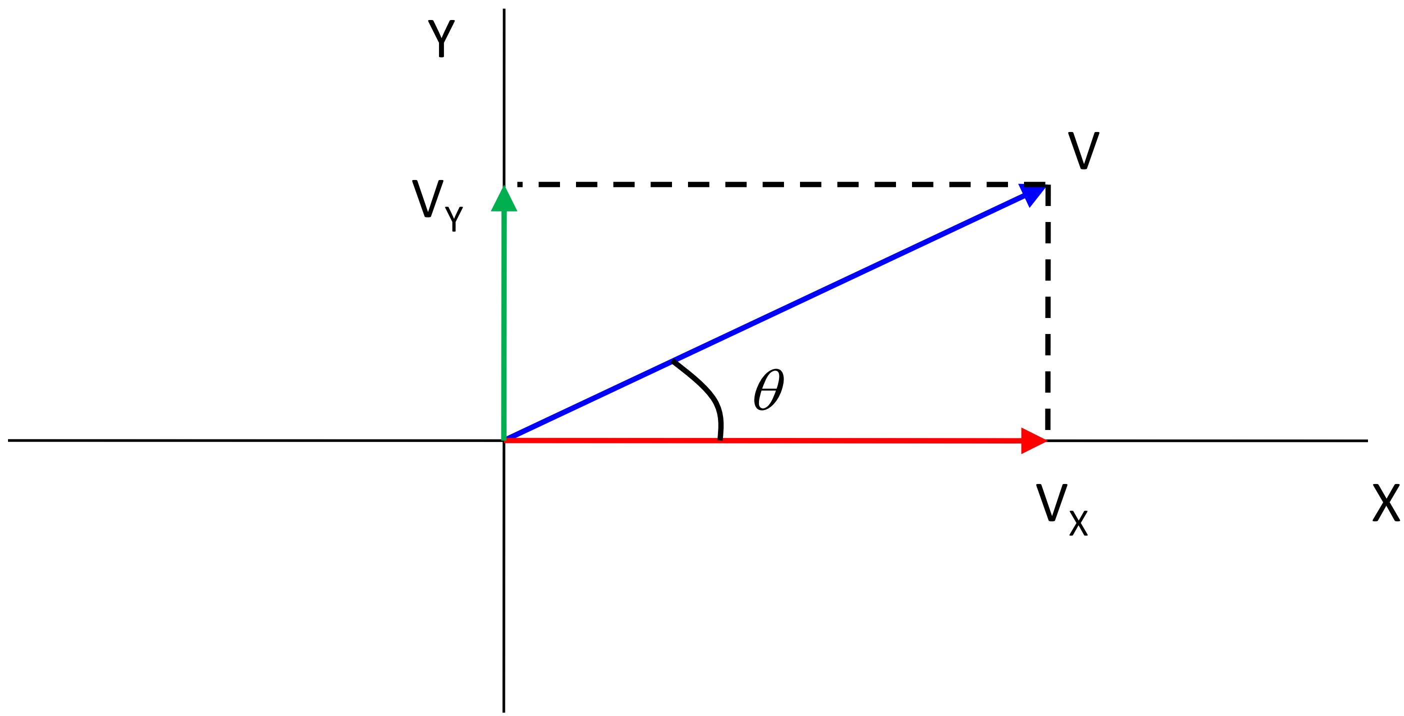
- Recapitulate basic concepts of complex numbers, their representations, and mathematical operations

- Use complex numbers to denote RL and RC circuits

- Define impedance, conductance, admittance, susceptance of circuit elements and obtain expressions for these in terms of complex numbers

Complex notation applied to AC circuits

- For example, the voltage phasor V in the following diagram is resolved into two components
- V_X along the positive X-axis (horizontal component)
- V_Y along the positive Y axis (vertical component)

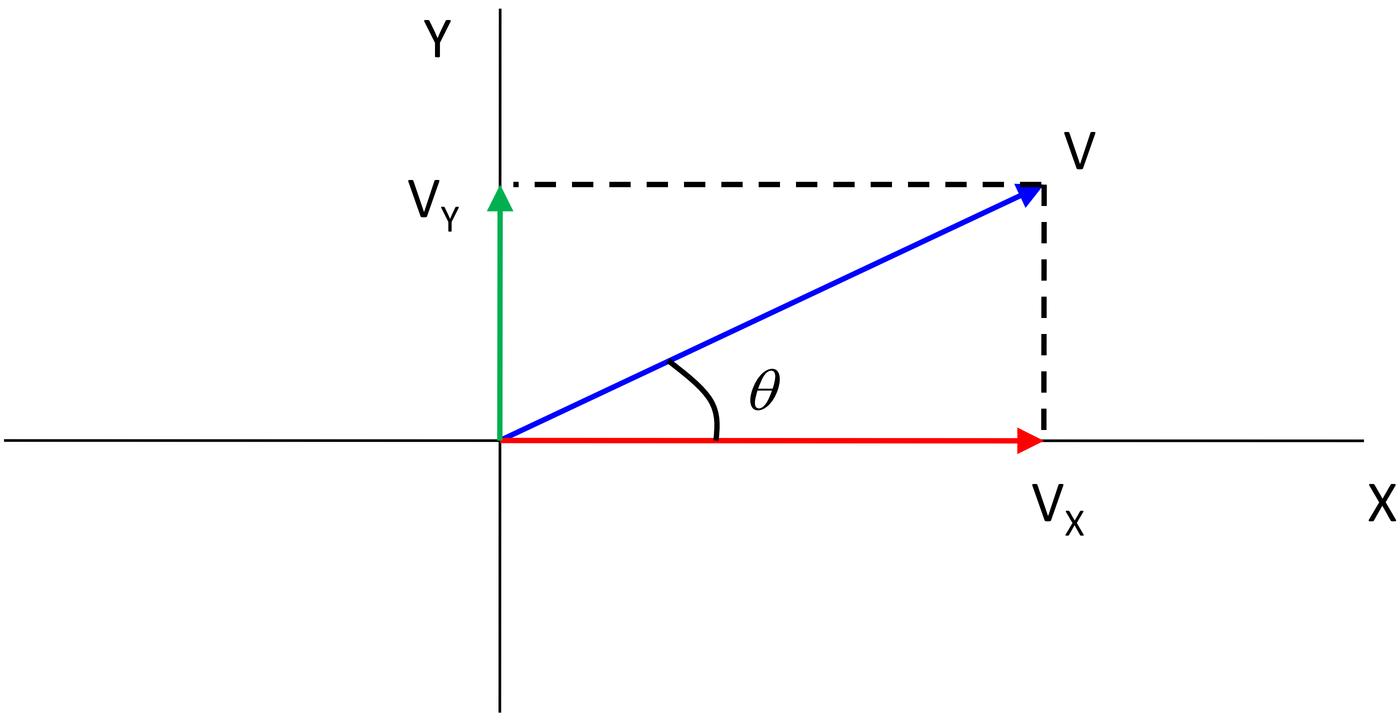


Complex notation applied to AC circuits

- We have the relation: $V^2 = V_X{}^2 + V_Y{}^2$

$$V_X = V\cos\theta$$

$$V_Y = V\sin\theta$$

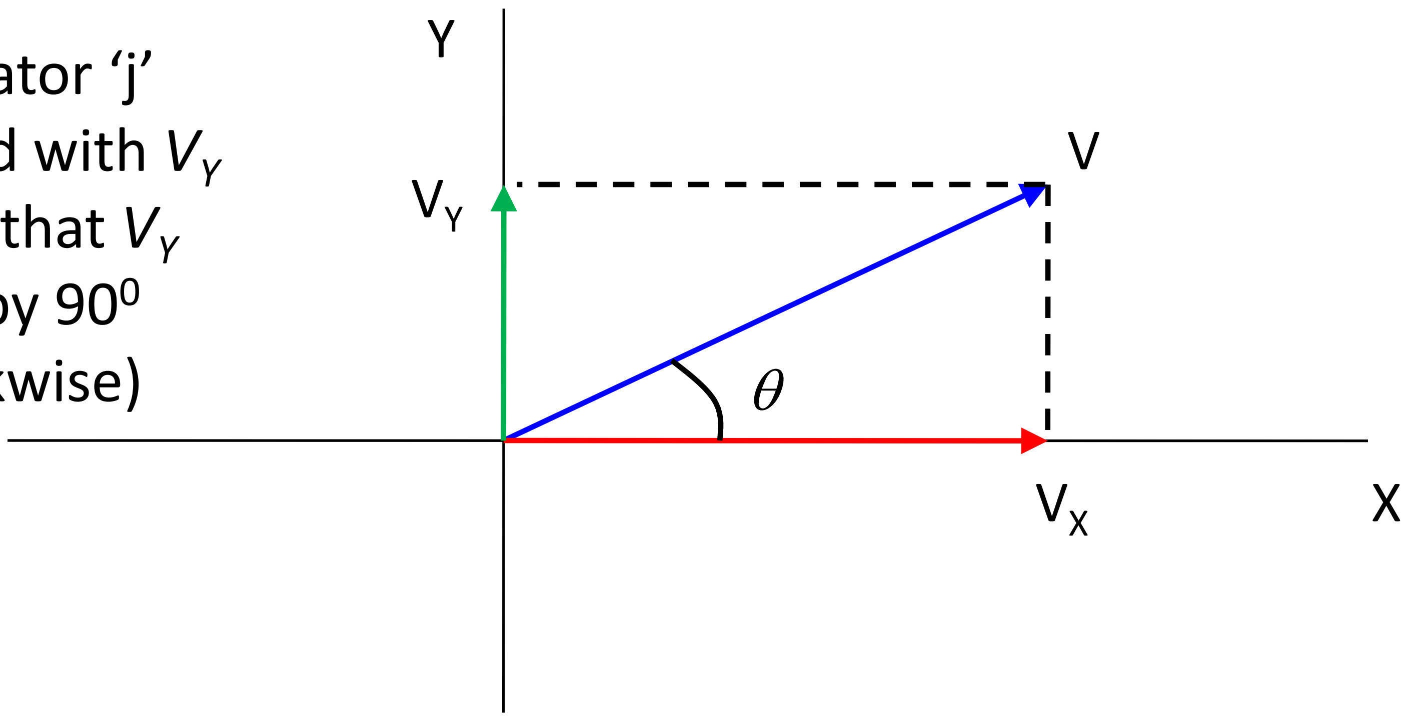


Complex notation applied to AC circuits

$V_X = V\cos\theta \qquad V_Y = V\sin\theta$

- The voltage V and its two components can be represented in **Cartesian** or **complex** or **rectangular** form as: $V = V_X + jV_Y = V\cos\theta + jV\sin\theta$

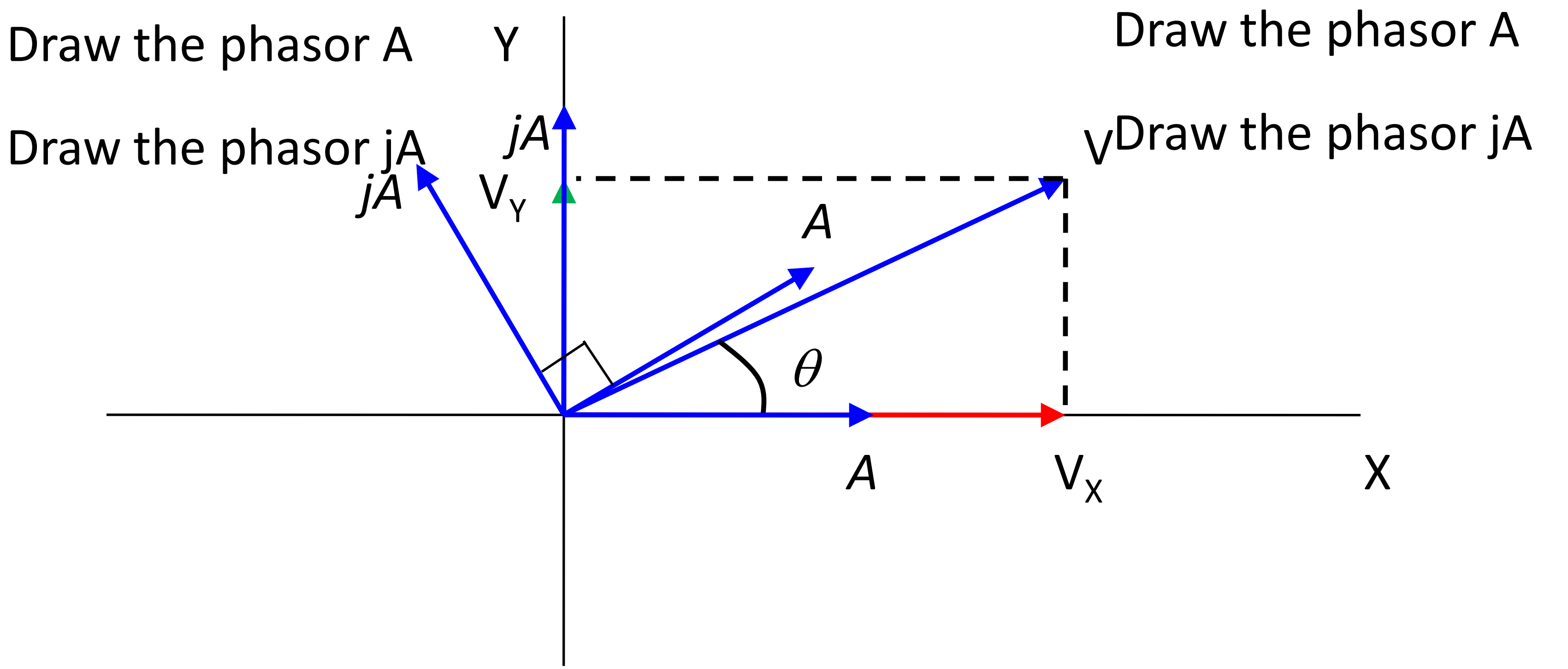
The operator 'j' multiplied with V_Y indicates that V_Y leads V_X by 90^0 (anticlockwise)



Complex notation applied to AC circuits

$$V = V_X + jV_Y = V\cos\theta + jV\sin\theta$$

- Here the quantity ***'j'*** is an operator which when multiplied to any phasor, the phasor is rotated by 90^0 anticlockwise



Complex notation applied to AC circuits

- Similarly, when a phasor is to be rotated by 90^0 clockwise, it is to be multiplied by the quantity $-j$
- Mathematically, $j = \sqrt{-1}$
- In mathematics, $\sqrt{-1}$ is denoted by i but in electrical engineering j is adopted because letter i is reserved for representing current
- This helps to avoid confusion

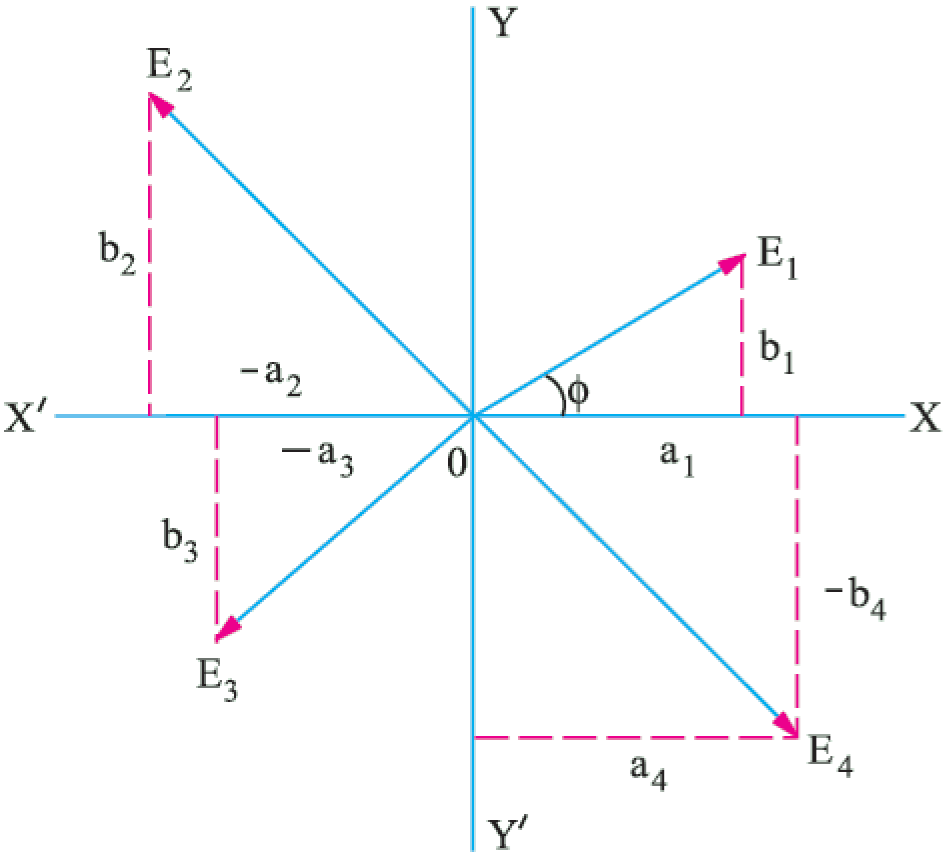
Complex notation applied to AC circuits

$$E_1 = a_1 + jb_1$$

$$E_2 = -a_2 + jb_2$$

$$E_3 = -a_3 - jb_3$$

$$E_4 = a_4 - jb_4$$



Complex notation applied to AC circuits

- When operator *j* is multiplied to a vector *E*, we get the new vector *jE* which is displaced by 90^0 in CCW direction from *E*
- If we apply the operator *j* once again, the vector will be rotated further by 90^0 CCW thus giving a total 180^0 rotation CCW from its original position, hence we have $j^2E = -E$
- If the operator *j* is again applied to the vector j^2E, the result is $j^3E = -jE$
- The vector j^3E is now 270^0 CCW away from the reference axis
- If the vector j^3E is operated on by *j* again , the result will be:

$$j^4E = \left(\sqrt{-1}\right)^4 E = E$$

- Hence, it is seen that successive applications of the operator *j* to the vector *E* produce successive 90^0 steps of rotation of the vector in the CCW direction without in anyway affecting the magnitude of the vector

Complex notation applied to AC circuits

Also note:

$$\frac{1}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$$

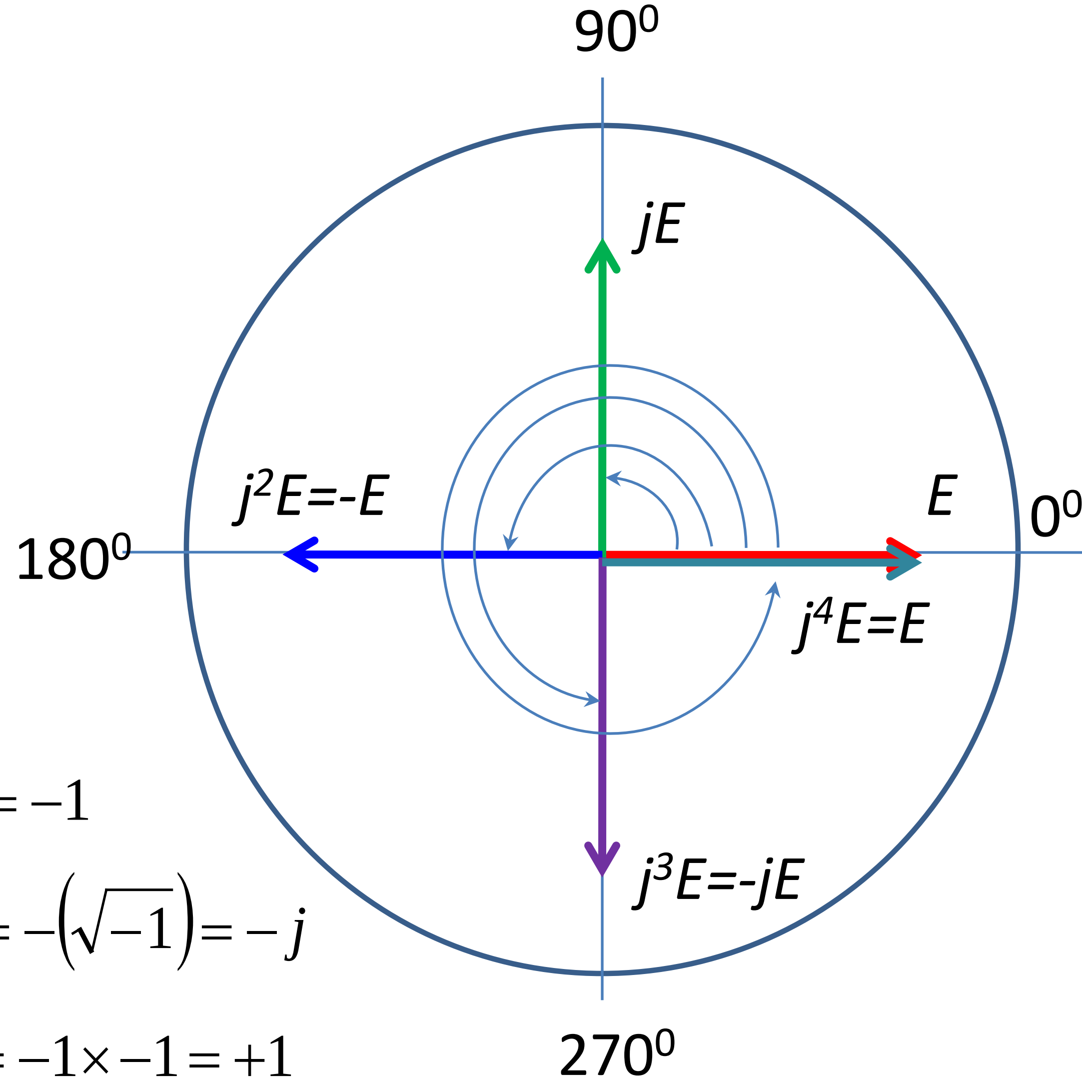
Summary of *j* operation

$$j = 90^0 \text{ CCW rotation} = \sqrt{-1}$$

$$j^2 = 180^0 \text{ CCW rotation} = \left(\sqrt{-1}\right)^2 = -1$$

$$j^3 = 270^0 \text{ CCW rotation} = \left(\sqrt{-1}\right)^3 = -\left(\sqrt{-1}\right) = -j$$

$$j^4 = 360^0 \text{ CCW rotation} = \left(\sqrt{-1}\right)^4 = -1 \times -1 = +1$$



Complex notation applied to AC circuits

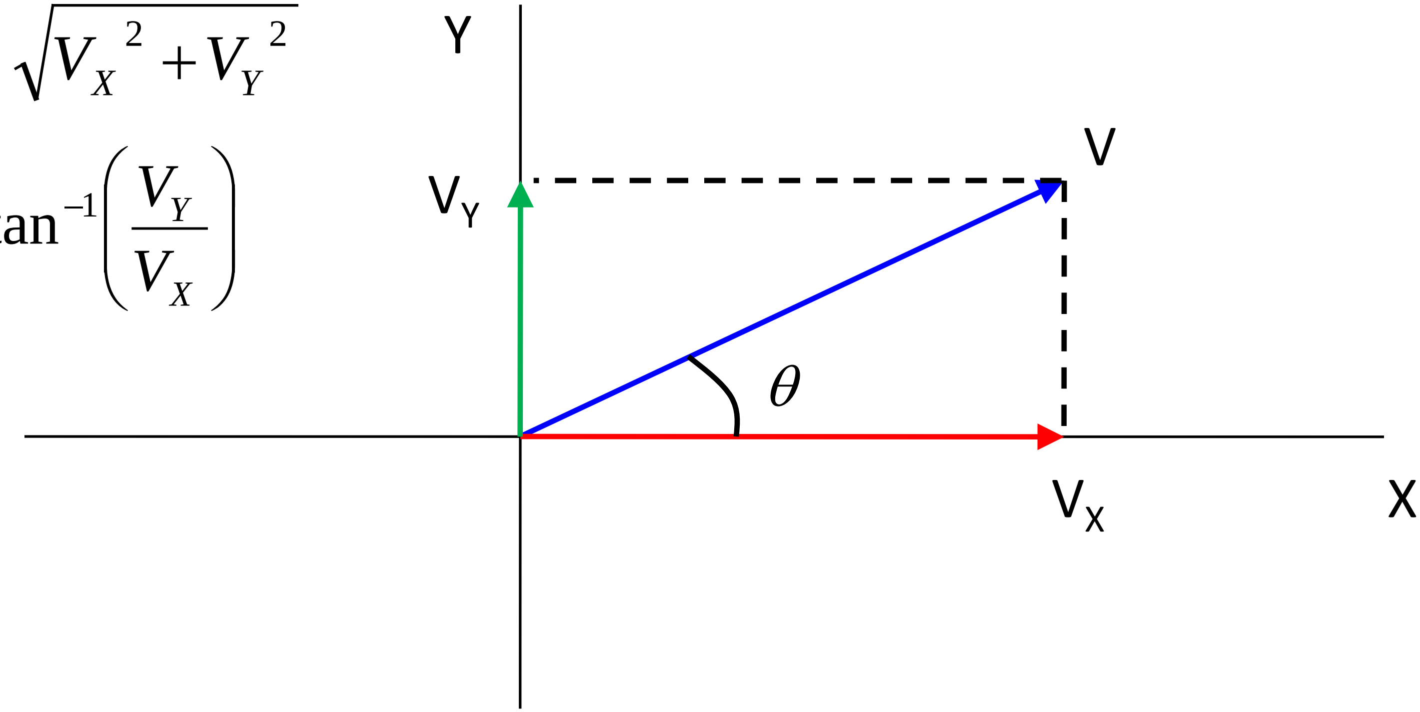
$$V = V_X + jV_Y = V\cos\theta + jV\sin\theta$$

- The same voltage phasor V in the diagram can be represented in **polar form** also as:

$$V = |V|\angle\theta$$

$$|V| = \sqrt{V_X^{\,2} + V_Y^{\,2}}$$

$$\theta = \tan^{-1}\left(\frac{V_Y}{V_X}\right)$$



Addition and subtraction of complex quantities

- ***Rectangular form is best suited for addition and subtraction of vector quantities***
- *Suppose we* are given two vector quantities $\boldsymbol{E_1 = a_1 + jb_1}$ *and* $\boldsymbol{E_2 = a_2 + jb_2}$ *and it is required to find their sum and* difference

<u>Addition:</u>

$$E = E_1 + E_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

The magnitude of resultant vector *E* is

$$|E| = |(a_1 + a_2) + j(b_1 + b_2)| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$$

The angle of resultant vector *E* is

$$\angle E = \theta = \tan^{-1}\left(\frac{b_1 + b_2}{a_1 + a_2}\right)$$

Addition and subtraction of complex quantities

- ***Rectangular form is best suited for addition and subtraction of vector quantities***
- *Suppose we* are given two vector quantities $\boldsymbol{E_1 = a_1 + jb_1}$ *and* $\boldsymbol{E_2 = a_2 + jb_2}$ *and it is required to find their sum and* difference

<u>Subtraction:</u>

$$E = E_1 - E_2 = (a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2)$$

The magnitude of resultant vector *E* is

$$|E| = |(a_1 - a_2) + j(b_1 - b_2)| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

The angle of resultant vector *E* is

$$\angle E = \theta = \tan^{-1}\left(\frac{b_1 - b_2}{a_1 - a_2}\right)$$

Multiplication and division of complex quantities

- ***Polar form is best suited for multiplication and division of vector quantities***
- *Suppose we* are given two vector quantities $\boldsymbol{E_1 = a_1 + jb_1}$ *and* $\boldsymbol{E_2 = a_2 + jb_2}$

$$E_1 = |E_1|\angle\alpha = \sqrt{\left(a_1{}^2 + b_1{}^2\right)}\angle\tan^{-1}\frac{b_1}{a_1} \qquad E_2 = |E_2|\angle\beta = \sqrt{\left(a_2{}^2 + b_2{}^2\right)}\angle\tan^{-1}\frac{b_2}{a_2}$$

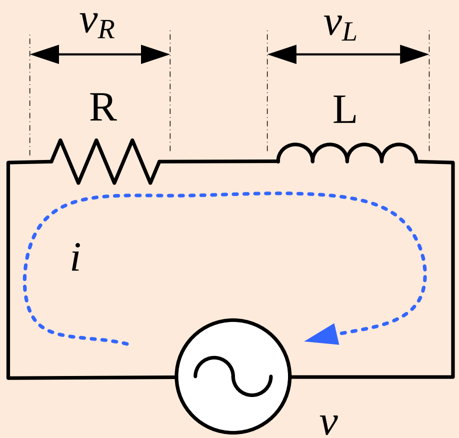
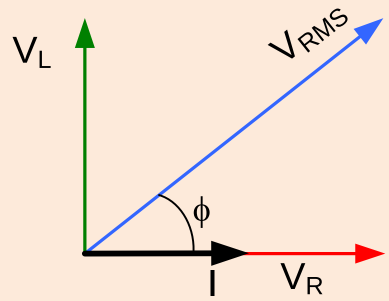
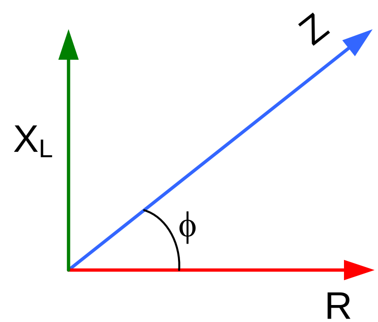
Multiplication:

$$E = |E_1|\angle\alpha \times |E_2|\angle\beta = |E_1E_2|\angle(\alpha+\beta)$$

Division:

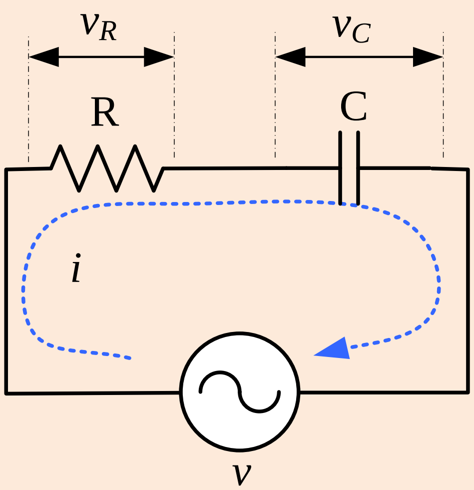
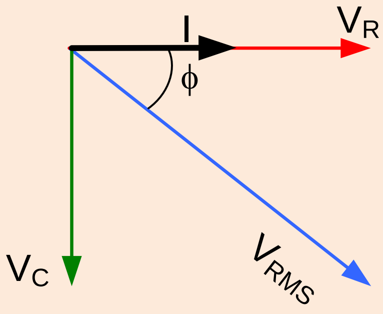
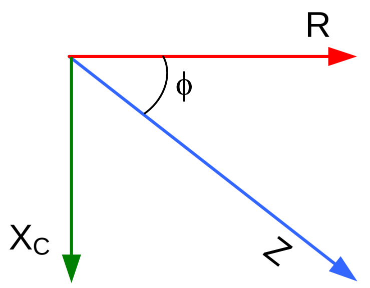
$$E = \frac{|E_1|\angle\alpha}{|E_2|\angle\beta} = \frac{|E_1|}{|E_2|}\angle(\alpha-\beta)$$

Complex representation of RL circuit

Circuit	Phasor	Rectangular form	Polar form
		$V_{RMS} = V_R + jV_L$	$\lvert V_{RMS} \rvert = \sqrt{V_R{}^2 + V_L{}^2}$ $\phi = \angle V_{RMS} = \tan^{-1}\left(\frac{V_L}{V_R}\right)$
		$Z = R + jX_L$	$\lvert Z \rvert = \sqrt{R^2 + X_L{}^2}$ $\phi = \angle Z = \tan^{-1}\left(\frac{X_L}{R}\right)$

Power factor angle ϕ calculated from Voltage or Impedance, both gives the same value

Complex representation of RC circuit

<table>
<tr><th>Circuit</th><th>Phasor</th><th>Rectangular form</th><th>Polar form</th></tr>
<tr><td rowspan="2">
</td>
<td>
</td>
<td>$V_{RMS} = V_R - jV_C$</td>
<td>$|V_{RMS}| = \sqrt{V_R{}^2 + V_C{}^2}$
$\phi = \angle V_{RMS} = \tan^{-1}\left(\frac{V_C}{V_R}\right)$</td></tr>
<tr><td>
</td>
<td>$Z = R - jX_C$</td>
<td>$|Z| = \sqrt{R^2 + X_C{}^2}$
$\phi = \angle Z = \tan^{-1}\left(\frac{X_C}{R}\right)$</td></tr>
</table>

Power factor angle ϕ calculated from Voltage or Impedance, both gives the same value

Conductance, Susceptance, and Admittance

- The reciprocal of resistance is called **conductance**
- It is represented by the symbol G $$G = \frac{1}{R}$$
- It is expressed in the unit ‘Mho’ or ‘siemens’ (S)
- It is convenient to use conductance when a number of resistances are connected in parallel

Conductance, Susceptance, and Admittance

- Conductance is used when a number of resistances are connected in parallel
- The equivalent of a number of resistances connected in parallel is obtained using the relation:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots\ldots\ldots + \frac{1}{R_n}$$

- Instead, the equivalent conductance can be determined simply by:

$$G = G_1 + G_2 + G_3 + \ldots\ldots\ldots + G_n$$

$$G = \frac{1}{R}, G_1 = \frac{1}{R_1}, G_2 = \frac{1}{R_2} \ldots\ldots$$

Conductance, Susceptance, and Admittance

- Similarly, the reciprocal of reactance is termed as **susceptance**

- It is represented by the symbol B

$$B = \frac{1}{X}$$

- It is also expressed in the unit ‘Mho’ or ‘siemens’ (S)

Conductance, Susceptance, and Admittance

- The reciprocal of impedance is called **admittance**
- It is denoted by the symbol Y
- The unit of admittance is also 'Mho'

$$Y = \frac{1}{Z}$$

- Like impedance, the admittance is also a complex quantity $Y = a + jb$
- Real part of Y is the conductance (a)
- Imaginary part of Y is the susceptance (b)

Admittance of series RL circuit

- For a series R-L circuit, the impedance is: $Z = R + jX_L$
- Thus, admittance:

$$Y = \frac{1}{Z} = \frac{1}{R + jX_L}$$

$$Y = \frac{R - jX_L}{\left(R + jX_L\right) \times \left(R - jX_L\right)}$$

$$Y = \frac{R - jX_L}{R^2 - \left(jX_L\right)^2}$$

$$Y = \frac{R - jX_L}{R^2 + X_L{}^2}$$

$$Y = \frac{R - jX_L}{Z^2}$$

$$\boxed{Y = \frac{R}{Z^2} - j\frac{X_L}{Z^2}}$$

Admittance of series RL circuit

$$\boxed{Y = \frac{R}{Z^2} - j\frac{X_L}{Z^2}}$$

- The real part of Y is the conductance (G) and imaginary part is susceptance (B)
- Thus for a series R-L circuit we have: $Y = G - jB$
- Conductance $G = \frac{R}{Z^2}$
- Susceptance $B = \frac{X_L}{Z^2}$