Chapter 6
Non-sinusoidal periodic waves

Day 40

Power in Harmonics
+ Tutorial 1

ILOs – Day 40

- Derive expressions for
  - Active power supplied by a complex wave
  - Power factor of a circuit with complex wave
- Solve numerical problems related to complex waves and harmonics

Power supplied by a complex wave

- For single frequency sinusoidal AC signals, the active power supplied is: $P = VI\cos\phi$
- Where, *V* and *I* are RMS values of voltage and current with ϕ being the phase angle between them
- But since complex waves contain signals of different frequencies, calculation of active power needs special attention
- Let the complex voltage wave applied to a circuit is given by:

$$e = E_{m1}\sin(\omega t + \phi_1) + E_{m2}\sin(2\omega t + \phi_2) + \dots\dots\dots\dots + E_{mn}\sin(n\omega t + \phi_n)\dots\dots$$

- The resulting current, which is also a complex waveform, has an equation such as:

$$i = I_{m1}\sin(\omega t + \psi_1) + I_{m2}\sin(2\omega t + \psi_2) + \dots\dots\dots + I_{mn}\sin(n\omega t + \psi_n)\dots\dots$$

Power supplied by a complex wave

$$e = E_{m1}\sin(\omega t+\phi_1)+E_{m2}\sin(2\omega t+\phi_2)+\dots\dots\dots\dots+E_{mn}\sin(n\omega t+\phi_n)\dots\dots$$

$$i = I_{m1}\sin(\omega t+\psi_1)+I_{m2}\sin(2\omega t+\psi_2)+\dots\dots\dots\dots+I_{mn}\sin(n\omega t+\psi_n)\dots\dots$$

- The instantaneous value of the power in the circuit is $p = ei$ watt
- For obtaining the value of this product, we will have to multiply every term of the voltage wave, in turn, by every term in the current wave
- The average power supplied during a cycle:

$$P = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T ei\,dt$$

$$= \frac{1}{T}\int_0^T \left[\sum_x E_{mx}\sin(x\omega t+\phi_x)\right]\times\left[\sum_y I_{my}\sin(y\omega t+\psi_y)\right]dt$$

Power supplied by a complex wave

$$P = \frac{1}{T}\int_0^T \left[\sum_x E_{mx}\sin\left(x\omega t + \phi_x\right)\right] \times \left[\sum_x I_{mx}\sin\left(x\omega t + \psi_x\right)\right] dt$$

- When expanded, we get the following terms:
  - Product of two sine functions of different frequencies, for example the term:

$$E_{m1}\sin\left(\omega t + \phi_1\right) \times I_{m2}\sin\left(2\omega t + \psi_1\right)$$

  - Product of two sine functions of same frequency, for example the term:

$$\boxed{E_{m3}\sin\left(3\omega t + \phi_1\right) \times I_{m3}\sin\left(3\omega t + \psi_1\right)}$$

- We already have seen earlier that the average value of all product terms involving harmonics of different frequencies will be zero over one cycle
- So that we need consider only the products of current and voltage harmonics of the same frequency

Power supplied by a complex wave

$$E_{m3}\sin(3\omega t+\phi_1)\times I_{m3}\sin(3\omega t+\psi_1)$$

- Let us consider a general term of this nature i.e.

$$E_{mn}\sin(n\omega t+\phi_n)\times I_{mn}\sin(n\omega t+\psi_n)$$

- And find its average value over one cycle

$$P_n=\frac{1}{T}\int_0^T E_{mn}\sin(n\omega t+\phi_n)\times I_{mn}\sin(n\omega t+\psi_n)dt$$

$$=\frac{1}{T}\int_0^T E_{mn}\sin\left(n\frac{2\pi}{T}t+\phi_n\right)\times I_{mn}\sin\left(n\frac{2\pi}{T}t+\psi_n\right)dt$$

$$=\frac{E_{mn}I_{mn}}{T}\int_0^T \frac{1}{2}\left[\cos(\phi_n-\psi_n)-\cos\left(\frac{4n\pi}{T}t+\phi_n+\psi_n\right)\right]dt$$

Power supplied by a complex wave

$$= \frac{E_{mn} I_{mn}}{T} \int_0^T \frac{1}{2}\left[\cos(\phi_n - \psi_n) - \cos\left(\frac{4n\pi}{T}t + \phi_n + \psi_n\right)\right] dt$$

$$= \frac{E_{mn} I_{mn}}{T} \frac{1}{2}\left[\cos(\phi_n - \psi_n) t - \frac{T}{4n\pi}\sin\left(\frac{4n\pi}{T}t + \phi_n + \psi_n\right)\right]_0^T$$

$$= \frac{E_{mn} I_{mn}}{2T}\left[\cos(\phi_n - \psi_n) T - 0\right]$$

$$= \frac{E_{mn} I_{mn}}{2} \cos(\phi_n - \psi_n)$$

$$= \frac{E_{mn}}{\sqrt{2}} \frac{I_{mn}}{\sqrt{2}} \cos(\phi_n - \psi_n)$$

$$\boxed{= E_n I_n \cos\theta_n}$$

= Active power of the nth harmonic component

Power supplied by a complex wave

Active power of the n[th] harmonic component

$$\boxed{P_n = \frac{E_{mn}}{\sqrt{2}}\frac{I_{mn}}{\sqrt{2}}\cos(\phi_n - \psi_n) = E_n I_n \cos\theta_n}$$

Where, $E_n = \frac{E_{mn}}{\sqrt{2}}$ is RMS value of nth voltage harmonic

$I_n = \frac{I_{mn}}{\sqrt{2}}$ is RMS value of nth current harmonic

$\theta_n = (\phi_n - \psi_n)$ is the phase angle between nth voltage and current harmonic waves

Thus, total active power, i.e. average power over one complete cycle due to all the harmonics taken together is:

$$\boxed{P = \sum_i P_i = \sum_i E_i I_i \cos\theta_i = E_1 I_1 \cos\theta_1 + E_2 I_2 \cos\theta_2 + \ldots\ldots.. + E_n I_n \cos\theta_n}$$

Power supplied by a complex wave

$$\boxed{P=\sum_i P_i=\sum_i E_i I_i \cos\theta_i = E_1 I_1 \cos\theta_1 + E_2 I_2 \cos\theta_2 + \ldots\ldots + E_n I_n \cos\theta_n}$$

- Hence, total average power supplied by a complex wave is the sum of the average power supplied by each harmonic component acting independently

- *No power results from voltages and currents of different frequencies*

Power factor of a complex wave

The overall power factor of the system is given by:

$$pf = \frac{\text{Total Watt}}{\text{Total apparent power}} = \frac{E_1 I_1 \cos\theta_1 + E_2 I_2 \cos\theta_2 + \ldots\ldots.. + E_n I_n \cos\theta_n}{EI}$$

Where,
E = RMS value of the overall complex voltage wave
I = RMS value of the overall complex current wave

- *Note that when harmonics are present, the overall p.f. of the circuit cannot be stated lagging or leading*

- *It is simply the ratio of power in watts to volt-amperes*

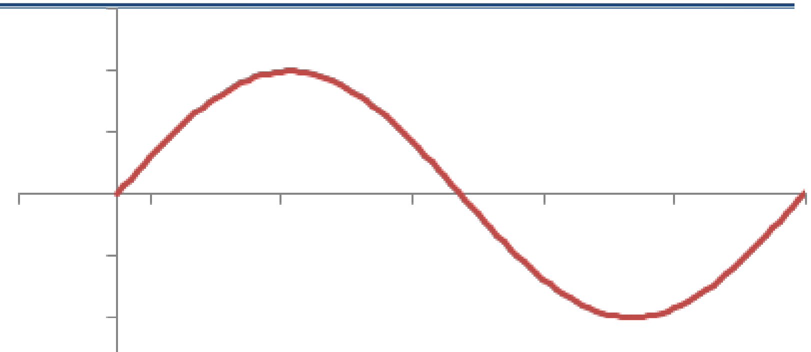
Tutorial 1

#1) Draw one complete cycle of the following wave.

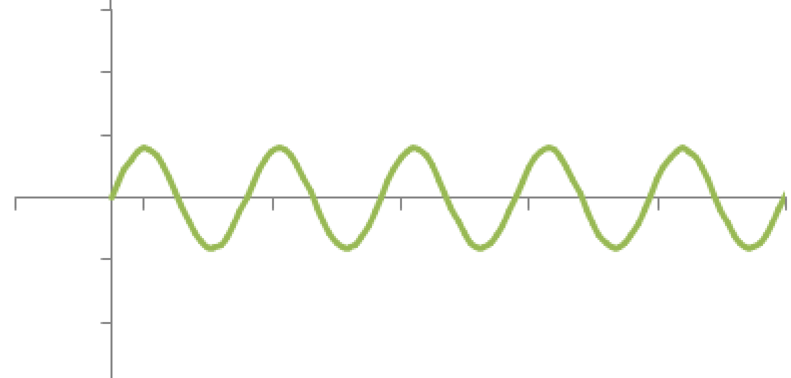
$$i = 100 \sin \omega t + 40 \sin 5\omega t$$

Determine the average value, the r.m.s. value and form factor of the wave.

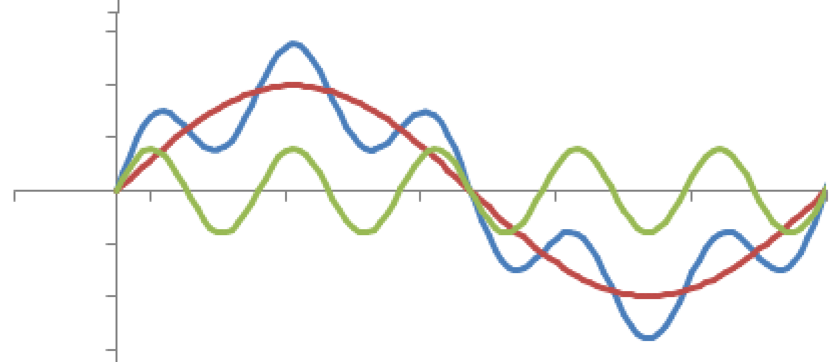
One complete cycle of the signal $i_1 = 100 \sin \omega t$, which we take as the fundamental, looks like:



One complete cycle of the signal $i_5 = 40 \sin 5\omega t$, which is the 5th harmonic signal, looks like:



Now the combined signal $i = 100 \sin \omega t + 40 \sin 5\omega t$ will look like:



#1) Draw one complete cycle of the following wave.

$$i = 100\sin\omega t + 40\sin 5\omega t$$

Determine the average value, the r.m.s. value and form factor of the wave.

Average value:

$$i_{av} = \frac{2}{\pi}\left(I_{m1} + \frac{I_{m5}}{5}\right) = \frac{2}{\pi}\left(100 + \frac{40}{5}\right) = 68.7\ A$$

RMS value:

$$I_{RMS} = \sqrt{\frac{{I_{m1}}^2}{2} + \frac{{I_{m5}}^2}{2}} = \sqrt{\frac{100^2}{2} + \frac{40^2}{2}} = 76.16\ A$$

$\therefore$ Form factor: $= \dfrac{I_{RMS}}{i_{av}} = \dfrac{76.16}{68.7} = 1.109$

#2) A single-phase voltage source 'e' is given by

$$e = 141\sin\omega t + +42.3\sin 3\omega t + 28.8\sin 5\omega t$$

The corresponding current in the load circuit is given by

$$i = 16.5\sin\left(\omega t + 54.5^0\right) + 8.43\sin\left(3\omega t - 38^0\right) + 4.65\sin\left(5\omega t - 34.3^0\right)$$

Find the power supplied by the source.

It is best to deal with each harmonic separately.

Power at fundamental,

$$P_1 = E_1 I_1 \cos\phi_1 = \frac{E_{m1}}{\sqrt{2}}\frac{I_{m1}}{\sqrt{2}}\cos\phi_1 = \frac{141}{\sqrt{2}} \times \frac{16.5}{\sqrt{2}}\cos\left(0^0 - 54.5^0\right) = 675.5\,W$$

Power at 3rd harmonic,

$$P_3 = E_3 I_3 \cos\phi_3 = \frac{E_{m3}}{\sqrt{2}}\frac{I_{m3}}{\sqrt{2}}\cos\phi_3 = \frac{42.3}{\sqrt{2}} \times \frac{8.43}{\sqrt{2}}\cos\left(0^0 + 38^0\right) = 140.5\,W$$

Power at 5th harmonic,

$$P_5 = E_5 I_5 \cos\phi_5 = \frac{E_{m5}}{\sqrt{2}}\frac{I_{m5}}{\sqrt{2}}\cos\phi_5 = \frac{28.8}{\sqrt{2}} \times \frac{4.65}{\sqrt{2}}\cos\left(0^0 - 34.3^0\right) = 55.5\,W$$

#2) A single-phase voltage source 'e' is given by

$$e = 141\sin\omega t + +42.3\sin 3\omega t + 28.8\sin 5\omega t$$

The corresponding current in the load circuit is given by

$$i = 16.5\sin\left(\omega t + 54.5^0\right) + 8.43\sin\left(3\omega t - 38^0\right) + 4.65\sin\left(5\omega t - 34.3^0\right)$$

Find the power supplied by the source.

$P_1 = 675.5\,W$ $\quad\quad$ $P_3 = 140.5\,W$ $\quad\quad$ $P_5 = 55.5\,W$

Total power supplied = 675.5 + 140.5 + 55.5 = 871.5 W

#3) A single-phase voltage source 'e' is given by

$$e = 50 + 50\sin 5000t + 30\sin 1000t$$

The corresponding current in the load circuit is given by

$$i = 11.2\sin\left(5000t + 63.4^0\right) + 10.6\sin\left(1000t + 45^0\right)$$

Find the power supplied by the source.

Power at DC, $P_0 = E_0 I_0 = 50 \times 0 = 0\,W$

Power at 5000 rad/s harmonic:

$$P_5 = E_5 I_5 \cos\phi_5 = \frac{E_{m5}}{\sqrt{2}}\frac{I_{m5}}{\sqrt{2}}\cos\phi_5 = \frac{50}{\sqrt{2}} \times \frac{11.2}{\sqrt{2}}\cos\left(0^0 - 63.4^0\right) = 125.37\,W$$

Power at 1000 rad/s harmonic:

$$P_1 = E_1 I_1 \cos\phi_1 = \frac{E_{m1}}{\sqrt{2}}\frac{I_{m1}}{\sqrt{2}}\cos\phi_1 = \frac{30}{\sqrt{2}} \times \frac{10.6}{\sqrt{2}}\cos\left(0^0 - 45^0\right) = 112.43\,W$$

Total power supplied = 0 + 125.37 + 112.43 = 237.8 W

#4) A complex voltage is given by

$$e = 60\sin\omega t + 24\sin\left(3\omega t + \frac{\pi}{6}\right) + 12\sin\left(5\omega t + \frac{\pi}{3}\right)$$

The corresponding current in the load circuit is given by

$$i = 0.6\sin\left(\omega t - \frac{2\pi}{10}\right) + 0.12\sin\left(3\omega t - \frac{2\pi}{24}\right) + 0.1\sin\left(5\omega t + \frac{3\pi}{4}\right)$$

Find (i) r.m.s value of current and voltage (ii) total power supplied and (iii) the overall power factor.

RMS values of voltage components:

$$E_1 = \frac{E_{m1}}{\sqrt{2}} = \frac{60}{\sqrt{2}} = 42.43\,V \qquad E_3 = \frac{E_{m3}}{\sqrt{2}} = \frac{24}{\sqrt{2}} = 16.97\,V \qquad E_5 = \frac{E_{m5}}{\sqrt{2}} = \frac{12}{\sqrt{2}} = 8.49\,V$$

$\therefore$ RMS value of total voltage,

$$E = \sqrt{E_1^{\ 2} + E_3^{\ 2} + E_5^{\ 2}} = \sqrt{42.43^2 + 16.97^2 + 8.49^2} = 46.47\,V$$

#4) A complex voltage is given by

$$e = 60\sin\omega t + 24\sin\left(3\omega t + \frac{\pi}{6}\right) + 12\sin\left(5\omega t + \frac{\pi}{3}\right)$$

The corresponding current in the load circuit is given by

$$i = 0.6\sin\left(\omega t - \frac{2\pi}{10}\right) + 0.12\sin\left(3\omega t - \frac{2\pi}{24}\right) + 0.1\sin\left(5\omega t + \frac{3\pi}{4}\right)$$

Find (i) r.m.s value of current and voltage (ii) total power supplied and (iii) the overall power factor.

RMS values of current components:

$$I_1 = \frac{I_{m1}}{\sqrt{2}} = \frac{0.6}{\sqrt{2}} = 0.424\ A \qquad I_3 = \frac{I_{m3}}{\sqrt{2}} = \frac{0.12}{\sqrt{2}} = 0.085\ A \qquad I_5 = \frac{I_{m5}}{\sqrt{2}} = \frac{0.1}{\sqrt{2}} = 0.071\ A$$

$\therefore$ RMS value of total current,

$$I = \sqrt{I_1{}^2 + I_3{}^2 + I_5{}^2} = \sqrt{0.424^2 + 0.085^2 + 0.071^2} = 0.438\ A$$

#4) A complex voltage is given by

$$e = 60\sin\omega t + 24\sin\left(3\omega t + \frac{\pi}{6}\right) + 12\sin\left(5\omega t + \frac{\pi}{3}\right)$$

The corresponding current in the load circuit is given by

$$i = 0.6\sin\left(\omega t - \frac{2\pi}{10}\right) + 0.12\sin\left(3\omega t - \frac{2\pi}{24}\right) + 0.1\sin\left(5\omega t + \frac{3\pi}{4}\right)$$

Find (i) r.m.s value of current and voltage (ii) total power supplied and (iii) the overall power factor.

Total power supplied:

$$P = E_1 I_1 \cos\phi_1 + E_3 I_3 \cos\phi_3 + E_5 I_5 \cos\phi_5$$

$$= 42.43 \times 0.424 \times \cos\left(0 + \frac{2\pi}{10}\right) + 16.97 \times 0.085 \times \cos\left(\frac{\pi}{6} + \frac{2\pi}{24}\right) + 8.49 \times 0.071 \times \cos\left(\frac{\pi}{3} - \frac{3\pi}{4}\right)$$

$$= 42.43 \times 0.424 \times \cos 36^0 + 16.97 \times 0.085 \times \cos 45^0 + 8.49 \times 0.071 \times \cos 75^0$$

$$= 14.55 + 1.02 + 0.16$$

$$= 15.73\, W$$

#4) A complex voltage is given by

$$e = 60\sin\omega t + 24\sin\left(3\omega t + \frac{\pi}{6}\right) + 12\sin\left(5\omega t + \frac{\pi}{3}\right)$$

The corresponding current in the load circuit is given by

$$i = 0.6\sin\left(\omega t - \frac{2\pi}{10}\right) + 0.12\sin\left(3\omega t - \frac{2\pi}{24}\right) + 0.1\sin\left(5\omega t + \frac{3\pi}{4}\right)$$

Find (i) r.m.s value of current and voltage (ii) total power supplied and (iii) the overall power factor.

Overall power factor:

$$pf = \frac{\text{Total Watt}}{\text{Total apparent power}} = \frac{15.73}{46.47 \times 0.438} = 0.773$$

#5) The following voltage is applied to a series circuit of resistance 20 Ω and inductance 0.05 H.

$$e = 250\sin \omega t + 50\sin\left(3\omega t + \frac{\pi}{3}\right) + 20\sin\left(5\omega t + \frac{5\pi}{6}\right)$$

Derive (a) an expression for the current (b) the RMS value of voltage (c) the RMS value of current (d) the total power supplied (e) the power factor. Take ω=314 rad/s.

For fundamental:

Inductive reactance: $X_1 = \omega L = 314 \times 0.05 = 15.7\ \Omega$

Impedance: $Z_1 = 20 + j15.7 = 25.43\angle 38.13^0\ \Omega$

For 3rd harmonic:

Inductive reactance: $X_3 = 3\omega L = 3 \times 314 \times 0.05 = 47.1\ \Omega$

Impedance: $Z_3 = 20 + j47.1 = 51.17\angle 67^0\ \Omega$

#5) The following voltage is applied to a series circuit of resistance 20 Ω and inductance 0.05 H.

$$e = 250\sin\omega t + 50\sin\left(3\omega t + \frac{\pi}{3}\right) + 20\sin\left(5\omega t + \frac{5\pi}{6}\right)$$

Derive (a) an expression for the current (b) the RMS value of voltage (c) the RMS value of current (d) the total power supplied (e) the power factor. Take ω=314 rad/s.

For 5th harmonic:

Inductive reactance: $X_5 = 5\omega L = 5 \times 314 \times 0.05 = 78.5\,\Omega$

Impedance: $Z_5 = 20 + j78.5 = 81\angle 75.7^0\ \Omega$

#5) The following voltage is applied to a series circuit of resistance 20 Ω and inductance 0.05 H.

$$e = 250\sin\omega t + 50\sin\left(3\omega t + \frac{\pi}{3}\right) + 20\sin\left(5\omega t + \frac{5\pi}{6}\right)$$

Derive (a) an expression for the current (b) the RMS value of voltage (c) the RMS value of current (d) the total power supplied (e) the power factor. Take ω=314 rad/s.

$Z_1 = 25.43\angle 38.13^0\ \Omega$ $\qquad$ $Z_3 = 51.17\angle 67^0\ \Omega$ $\qquad$ $Z_5 = 81\angle 75.7^0\ \Omega$

(a) Expression for current is:

$$i = \frac{250}{25.43}\sin\left(\omega t - 38.1^0\right) + \frac{50}{51.17}\sin\left(3\omega t + \frac{\pi}{3} - 67^0\right) + \frac{20}{81}\sin\left(5\omega t + \frac{5\pi}{6} - 75.7^0\right)$$

$$\therefore i = 9.84\sin\left(\omega t - 38.1^0\right) + 0.9\sin\left(3\omega t - 7^0\right) + 0.25\sin\left(5\omega t + 74.3^0\right)$$

#5) The following voltage is applied to a series circuit of resistance 20 Ω and inductance 0.05 H.

$$e = 250\sin\omega t + 50\sin\left(3\omega t + \frac{\pi}{3}\right) + 20\sin\left(5\omega t + \frac{5\pi}{6}\right)$$

Derive (a) an expression for the current (b) the RMS value of voltage (c) the RMS value of current (d) the total power supplied (e) the power factor. Take ω=314 rad/s.

(b) RMS value of voltage:

$$V = \sqrt{\frac{V_{m1}{}^2}{2} + \frac{V_{m3}{}^2}{2} + \frac{V_{m5}{}^2}{2}}$$

$$= \sqrt{\frac{250^2}{2} + \frac{50^2}{2} + \frac{20^2}{2}}$$

$$= 180.8\,V$$

#5) The following voltage is applied to a series circuit of resistance 20 Ω and inductance 0.05 H.

$$e = 250\sin\omega t + 50\sin\left(3\omega t + \frac{\pi}{3}\right) + 20\sin\left(5\omega t + \frac{5\pi}{6}\right)$$

Derive (a) an expression for the current (b) the RMS value of voltage (c) the RMS value of current (d) the total power supplied (e) the power factor. Take ω=314 rad/s.

(b) RMS value of current:

$$i = 9.84\sin\left(\omega t - 38.1^0\right) + 0.9\sin\left(3\omega t - 7^0\right) + 0.25\sin\left(5\omega t + 74.3^0\right)$$

$$I = \sqrt{\frac{I_{m1}^{\ 2}}{2} + \frac{I_{m3}^{\ 2}}{2} + \frac{I_{m5}^{\ 2}}{2}}$$

$$= \sqrt{\frac{9.84^2}{2} + \frac{0.9^2}{2} + \frac{0.25^2}{2}}$$

$$= 6.99\ A$$

#5) The following voltage is applied to a series circuit of resistance 20 Ω and inductance 0.05 H.

$$e = 250\sin\omega t + 50\sin\left(3\omega t + \frac{\pi}{3}\right) + 20\sin\left(5\omega t + \frac{5\pi}{6}\right)$$

Derive (a) an expression for the current (b) the RMS value of voltage (c) the RMS value of current (d) the total power supplied (e) the power factor. Take ω=314 rad/s.

$$V = 180.8\,V \qquad I = 6.99\ A$$

(d) Total power supplied:

$$P = I^2R = 6.99^2 \times 20 = 978\,W$$

You can check the result by alternate method of power calculation

#5) The following voltage is applied to a series circuit of resistance 20 Ω and inductance 0.05 H.

$$e = 250\sin\omega t + 50\sin\left(3\omega t + \frac{\pi}{3}\right) + 20\sin\left(5\omega t + \frac{5\pi}{6}\right)$$

Derive (a) an expression for the current (b) the RMS value of voltage (c) the RMS value of current (d) the total power supplied (e) the power factor. Take ω=314 rad/s.

$V = 180.8\,V \qquad I = 6.99\ A \qquad P = 978\,W$

(e) Power factor:

$$pf = \frac{\text{Active power}}{\text{Apparent power}} = \frac{Watt}{VI} = \frac{978}{180.8 \times 6.99} = 0.773$$