Planetary Motion around a Star

Abstract

This document explains the equations of motion for a planet moving around a star, and provides a numerical solution to the equations using Python code. The code is included and the results are plotted.

1 Introduction

The Hamiltonian for a planet moving around a heavy star is given by:

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) \tag{1}$$

where $V(r) = -\frac{GMm}{r}$, G is the gravitational constant, M is the mass of the star, m is the mass of the planet, and r is the distance between the planet and the star.

Using the Hamiltonian equations of motion:

$$\frac{\partial H}{\partial p} = \dot{q} \tag{2}$$

$$\frac{\partial H}{\partial q} = -\dot{p} \tag{3}$$

we have:

$$\dot{r} = \frac{p_r}{m} \tag{4}$$

$$\dot{\theta} = \frac{p_\theta}{mr^2} \tag{5}$$

$$\dot{p_r} = -\frac{GMm}{r^2} + \frac{p_\theta}{mr^3} \tag{6}$$

$$\dot{p_\theta} = 0 \tag{7}$$

Using these equations, we have:

$$mr^2\dot{\theta} = L_0 \tag{8}$$

$$\ddot{r} = -\frac{GM}{r^2} + \frac{L_0^2}{2m^2r^3} \tag{9}$$

where $L_0 = mv_0r_0$.

Now, we can solve the last equation to get r, $\dot{r}$ as a function of time and hence, θ as a function of time and also energy as a function of time. Our initial conditions are, $\dot{r} = 0$, $r = 1.47\text{e}11$ m

We can make the substitution $t' = \frac{t}{z_0}$ and $y = \frac{r}{r_0}$ where $z_0 = 31536000$ sec (1 year converted to seconds) to make the equation dimensionless. The final equation is:

$$\ddot{y} = -\frac{GMz_0^2}{r_0^3 y^2} + \frac{v_0^2 z_0^2}{r_0^2 y^3} \tag{10}$$

with initial conditions $y(0) = 1$ and $y'(0) = 0$