Nov 21, 2025
IIT Jodhpur

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End Semester Exam
CSL7130 - Mathematical Foundation for Computer Science

NOTE:

This is Question-cum-Answer sheet. Maximum Points: 100, Total Questions: 30, Total Pages: 8 (16 sides), Total Time: 3 Hours. **If there is anything not clear in the problems, go ahead with your own assumptions but state them clearly. No doubts will be entertained during the exam.** Be precise and write the answer in the box provided. Verbosity will be penalized. Use the other answer sheet for rough work and submit both.

Name: **Roll Number:** **Signature:**

	Question Number	**Topic**	**Total Marks**	**Marks Obtained**
Rishi	1-20	Short Answers	40	
AM	21-22	Linear Algebra	10	
	23-26	Probability and Random Process	25	
Avi	27-28	Optimization	15	
	29-30	Discrete Maths	10	
		Total	**100**	

Short Answers

1. The probability density function (PDF) of a one-dimensional Gaussian (normal) distribution is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

For which values of the standard deviation σ does the Gaussian PDF attain a value greater than 1? or it can never happen. Provide your reasoning.**(2 points)**

possible when $x = \mu$ and

$$\frac{1}{\sqrt{2\pi\sigma^2}} > 1$$

$$\sqrt{2\pi\sigma^2} < 1$$

$$\sigma < \frac{1}{\sqrt{2\pi}}$$

1/2

2. If we throw a fair six-sided die 100 times and each time note the number that appears on the die, let X be the sum of all the outcomes. Compute the expected value $\mathbb{E}[X]$.**(2 points)**

$$E[X_i] = \cancel{\frac{1}{6} + \frac{2}{6} + \frac{3}{6}} \sum_{i=1}^{6} i \frac{1}{6} = \frac{21}{6} = 3.5$$

$$E[X] = 100\, E[X_i] = 350$$

2

3. Compute sample covariance for the following two-dimensional points: (2,3), (3,2), (1,2), (2,1).**(2 points)**

$$\begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix}$$

Covariance computation is enough.

2/2

4. Which of the following has a greater chance of success? A. Six fair dice are tossed independently and at least one 6 appears. B. Twelve fair dice are tossed independently and at least two 6 s appear. Explain your answer.**(2 points)**

$P(A) = 1 - P(\text{no } 6) = 1 - \left(\frac{5}{6}\right)^6 = \underline{0.665}$

$P(B) = 1 - P(\text{no } 6) - P(\text{one } 6)$

$= 1 - \left(\frac{5}{12}\right)^{12} - {}^{12}C_1 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^{11}$

$= \underline{0.62}$

$\boxed{P(A) > P(B)}$

5. Suppose you have a biased coin ($P(H) \neq P(T)$). How will you use it to make unbiased decision. (hint: you can toss the coin multiple times)**(2 points)**

Toss two times HH – discard

HT → 0

TH → 1

TT → discard

6. Which of these is a convex function: $Sin(x)$ or $x^2 + 5$; $x \in \mathbb{R}$? Why? **(2 points)**

x^2+5 is convex
$\sin x$ is not convex

7. Compute the gradient of the following function:

$$f(x,y,z) = 3x^2y + 2yz^3 - 4xz + 5e^{xy}.$$

(2 points)

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 6xy - 4z + 5e^{xy}y \\ 3x^2 + 2z^3 + 5e^{xy}x \\ 6yz^2 - 4x \end{bmatrix}$$

8. Give an example English sentence that is not a proposition. **(2 points)**

Get out of this room.

9. Write truth table for $x \cap y$.**(2 points)**

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

10. Write negation for: All birds are cats. **(2 points)**

There exist a bird that is not cat.

11. In a room, there are five married couples. What is the minimum number of people you must select to guarantee that at least one husband–wife pair is included? **(2 points)**

6

12. Define an injective function (one-to-one) and give one example. **(2 points)**

$f(x) = x,\ x \in \mathbb{R}$ $\quad f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

(many other example)

13. Find the column space and row space of the following matrix: $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$. **(2 points)**

$\mathrm{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\mathrm{Row}(A) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

14. Let $X \in \mathbb{R}^{m \times n}$ be a data matrix where each of the m rows is an n-dimensional sample. Let $\mu \in \mathbb{R}^n$ be the mean vector computed across the rows of X.

Write a **one-line command** that subtracts the mean vector μ from **every row** of X to obtain the mean-centered matrix.

$m \times n$ **Hint:** You may use the all-ones vector $\mathbf{1}_m \in \mathbb{R}^{m \times 1}$. **(2 points)**

$X_c = \underset{m \times n}{X} - \underset{m \times n}{1_m \mu^T}$

15. Compute the product of the following two matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

Find AB. **(2 points)**

$\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

16. If v is and Eigenvector of A and B, then it is also an Eigenvector of $A + B$. (True/False, why) **(2 points)**

True $Av = \lambda_1 v$

$Bv = \lambda_2 v$

$(A+B)v = \lambda_1 v + \lambda_2 v$

$= (\lambda_1 + \lambda_2) v$

17. The eigenvalues of A^T are always the same as that of A. (True/False, why).**(2 points)**

True. $\det(A^T - \lambda I) = \det((A^T - \lambda I)^T) = \det(A - \lambda I)$

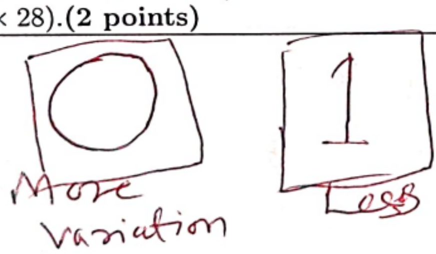
18. If v is a Eigen vector of AA^T, then what will be Eigen vector of $A^T A$? Show how? **(2 points)**

$AA^T v = \lambda v$

$\Rightarrow A^T A \underline{A^T v} = \lambda \underline{A^T v}$

$A^T v$ will be eigen vector of v

19. Today morning I wrote a code to check average Rank of MNIST handwritten digit matrix. I observed that digit 0 has significantly higher average rank (~ 8) than digit 1 (~ 5). Explain why this might be happening. (Note: averaged over 2000 samples of handwritten digit images each of size 28×28).**(2 points)**



20. Construct a 2 matrix whose **columns** are orthonormal. **(2 points)**

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $v_1^T v_2 = 0$

$v_i^T v_i = 1$

Linear Algebra

21. Solve the following system of linear equations using **LU decomposition** :

$$\begin{cases} 2x + 3y + z = 9, \\ 4x + 7y + 7z = 31, \\ -2x + 4y + 5z = 3. \end{cases}$$

Hint:

(a) Decompose the coefficient matrix A into $A = LU$, where L is lower-triangular with unit diagonal and U is upper-triangular.

(b) Solve $Ly = b$ for y using forward substitution.

(c) Solve $Ux = y$ for x using backward substitution.

(d) Report the solution vector $x = [x, y, z]^T$.

(5 points)

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 9 \\ 4 & 7 & 7 & 31 \\ -2 & 4 & 5 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 9 \\ 0 & 1 & 5 & 13 \\ 0 & 7 & 6 & 12 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 9 \\ 0 & 1 & 5 & 13 \\ 0 & 0 & -29 & 79 \end{array}\right]$$

L

2

$z = \frac{79}{29}$

$y + 5 \times \frac{79}{29} = 13$

$y = 13 - \frac{395}{29} = -\frac{18}{29}$

$x = \frac{118}{29}$

1

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 7 & 0 \end{bmatrix}$$

U

2

22. **Political Preferences Problem.** Suppose a survey collects the opinions of m users about n political leaders from two political parties. The data is represented as a matrix $A \in \mathbb{R}^{m \times n}$, where each row corresponds to a user, each column corresponds to a leader, and each entry A_{ij} indicates how much user i likes leader j (e.g., on a scale of 0–10).

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(a) Explain qualitatively what you expect the **rank** of the matrix A to be. How does the number of dominant political parties influence the rank? **(2.5 points)**

Rank ~ 2

(b) If a new leader from one of the two parties is added to the survey, how would you expect the rank of A to change? Justify your reasoning. **(2.5 points)**

Rank will ~~increase by~~ ~~f~~ remain Same

Probability and Random Processes

23. Let X and Y be two continuous random variables with finite expectations, and let $a, b \in \mathbb{R}$. Show that the expectation operator is linear, i.e.,

$$\mathbb{E}[\,aX + bY\,] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y].$$

(5 points)

$$E[aX+bY] = \iint_{-\infty}^{\infty} (ax+by) f_{XY}(x,y)\,dx\,dy$$

$$= \iint_{-\infty}^{\infty} ax f_{XY}(x,y) + \iint_{-\infty}^{\infty} by f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} ax \int_{-\infty}^{\infty} f_{XY}\,dy\,dx + \int_{-\infty}^{\infty} by \int f_{X,Y}\,dx\,dy$$

$$= a\int_{-\infty}^{\infty} x f_X\,dx + b\int y f_Y\,dy$$

$$= aE(X) + bE(Y)$$

24. **Airline Overbooking Problem (Note: Final answer is not required, right attempt will be considered.)**

An airline operates a flight with 180 seats. Based on historical data, the airline knows that 8% of passengers with confirmed bookings do not show up for the flight. To maximize revenue, the airline overbooks the flight by selling more tickets than the number of seats.

(a) If the airline sells 190 tickets, what is the probability that every passenger who arrives for the flight gets a seat?

Hint: Let X be the number of passengers who show up, Then the probability that k or fewer passengers show up for the flight is

$$\Pr(X \leq k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}.$$

All passengers can be seated if $X \leq 180$. **(2.5 points)**

$$Pr(X \leq 180) = \sum_{i=0}^{180} \binom{190}{180} p^i (1-p)^{190-i}$$

(b) What is the maximum number of tickets the airline can sell so that there is at least a 95% chance that no passenger is denied boarding? **(2.5 points)**

~~Pr(x~~ find n s.t.

$$Pr(x \leq 180) \geq 0.95$$

(c) The airline makes a profit of 2000 INR for every passenger who is successfully seated, but must pay 8000 INR in compensation to each passenger who is denied boarding due to overbooking. If the airline sells 188 tickets, what is the expected profit or loss? **(5 points)**

$X \le 180$ then Profit $= 2000X$

$X > 180$ then profit $2000 \times 180 - 8000(X-180)$

$E(\text{Profit}) = \cancel{2000 E(X) - 10000 E((X-180)^+)}$

$2000 \sum_{k=0}^{180} k \binom{180}{k} (0.92)^k (0.08)^{188-k}$

$+ \sum_{k=181}^{188} (360{,}000 - 8000(k-180)) \binom{188}{k} (0.92)^k (0.08)^{188-k}]$

25. A company manufactures microchips in two factories: Factory A produces 60% of all chips. Factory B produces 40% of all chips.

The defect rates are: Chips from Factory A are defective with probability 0.02. Chips from Factory B are defective with probability 0.05.

A chip is selected at random from the day's production and is found to be defective.

(a) What is the probability that this defective chip came from Factory B? **(2.5 points)**

$P(A) = 0.6$, $P(B) = 0.4$

$P(D|A) = 0.02$ $\quad P(D|B) = 0.05$

$$P(B|D) = \frac{P(B)\,P(D|B)}{P(B)\,P(D|B) + P(A)\,P(D|A)}$$

$$= \frac{0.4 \times 0.05}{0.4 \times 0.05 + 0.6 \times 0.02}$$

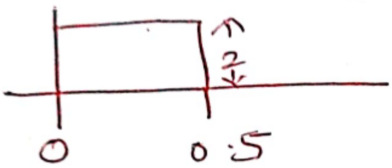
$$= 0.625$$

(b) If a chip is known to come from Factory B, what is the probability that it is non-defective? (2.5 points)

$P(D^c \mid B) = \cancel{\ldots}\ 1 - P(D \mid B)$

$= 0.95$

26. Draw uniform probability density function (PDF) for continuous random variable between that takes values between 0 and 0.5. Compute mean and variance of this PDF. (5 points)



$E(x) = \int_a^b x\, p(x)\, dx = \int_a^b x \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$

$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right] = \frac{a+b}{2}$

$E(x^2) = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{3(b-a)}$

$= \frac{(a^2+ab+b^2)}{3}$

$Var(x) = E(x^2) - (E(x))^2$

$\frac{(a^2+ab+b^2)}{3} - \frac{a^2+b^2-2ab}{4}$

Optimization

$\frac{\cancel{a^2+3b^2+3ab}}{12}$

$= -\frac{\cancel{(a-b)^2}}{4}$

27. Consider the quadratic function

$$f(x) = \frac{1}{2}x^\top Ax - b^\top x, \qquad A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) The gradient of f is given by **(0.5 point)**

$Ax - b$

(b) Using gradient descent with learning rate $\alpha = 0.1$ and initial point

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$\frac{a^2+b^2-2ab}{12}$

$\frac{(b-a)^2}{12}$

compute the value of x after the first iteration, i.e., compute x_1. **(2 points)**

$$x_1 = x_0 - 0.1(Ax_0 - b)$$

$$= x_0 - 0.1(A\underline{x_0} - b)$$

$$= \begin{pmatrix}0\\0\end{pmatrix} - 0.1\left(\begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}\right)$$

$$= \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0.1\\0.1\end{pmatrix}$$

$$= \begin{pmatrix}+0.1\\+0.1\end{pmatrix} \checkmark$$

$$x_{t+1} = x_t - \eta \nabla f$$

A - 1
F - 2

(c) Without using any iterative algorithm, directly solve the linear system $Ax = b$ and find the minimizer x^*. **(2 points)**

Attempt 1
Full 2

$$Ax = b$$

$$3\cancel{x_1} + x_2 = 1$$

$$3\times(\cancel{x_1} + 2x_2 = 1)$$

$$-5x_2 = 1 - 2$$

$$x_2 = \frac{2}{5}$$

$$3x_1 = 1 - \frac{2}{5} = \frac{3}{5}$$

$$x_1 = 1/5$$

$$x^* = \begin{pmatrix}1/5\\2/5\end{pmatrix} = \begin{pmatrix}0.2\\0.4\end{pmatrix} \checkmark$$

(d) Based on your computation in part (1), comment on whether the gradient descent algorithm is *converging* or *diverging* toward x^*. **(0.5 point)**

$x_0 \; x_1 \; x^*$

$$\|x_0 - x^*\| = \sqrt{0.2^2 + 0.4^2} = \sqrt{0.04 + 0.16} = \sqrt{0.20}$$

0.5 $$\|x_1 - x^*\|_2 = \sqrt{(0.1)^2 + (0.3)^2} = \sqrt{0.10}$$

Converging $\|x_1 - x^*\| \leq \|x_0 - x^*\|$

28. Consider the following convex optimization problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & f(x) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1, \\ & x_1 \geq 0, \; x_2 \geq 0. \end{aligned}$$

(a) Rewrite the inequality constraints in the form $g_i(x) \leq 0$. **(1 point)**

$1 - x_1 - x_2 \leq 0$

0.5, 1 $-x_1 \leq 0$

$-x_2 \leq 0$

(b) Form the Lagrangian $\mathcal{L}(x, \lambda, \mu_1, \mu_2)$ for the above problem, where λ corresponds to the constraint $x_1 + x_2 \geq 1$, and μ_1, μ_2 correspond to $x_1 \geq 0$, $x_2 \geq 0$. **(1 point)**

$$\mathcal{L}(x, \lambda, \mu_1, \mu_2) = x_1^2 + x_2^2 + \lambda(1 - x_1 - x_2) + \mu_1(-x_1) + \mu_2(-x_2)$$

1, 0.5

$$= x_1^2 + x_2^2 + \lambda(1 - x_1 - x_2) - \mu_1 x_1 - \mu_2 x_2$$

(c) Compute the dual function **(4 points)**

$$g(\lambda, \mu_1, \mu_2) = \min_{x \in \mathbb{R}^2} \mathcal{L}(x, \lambda, \mu_1, \mu_2).$$

2/4

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda - \mu_1 = 0 \Rightarrow x_1 = \frac{\lambda + \mu_1}{2}$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda - \mu_2 = 0 \quad x_2 = \frac{\lambda + \mu_2}{2}$$

$$g(\lambda, \mu_1, \mu_2) = \left(\frac{\lambda+\mu_1}{2}\right)^2 + \left(\frac{\lambda+\mu_2}{2}\right)^2 + \lambda\left(1 - \frac{\lambda+\mu_1}{2} - \frac{\lambda+\mu_2}{2}\right) - \mu_1\left(\frac{\lambda+\mu_1}{2}\right) - \mu_2\left(\frac{\lambda+\mu_2}{2}\right)$$

$$= \lambda - \frac{1}{2}\lambda^2 - \frac{1}{2}\lambda(\mu_1 + \mu_2) - \frac{1}{4}(\mu_1^2 + \mu_2^2)$$

(d) Write the dual optimization problem: **(4 points)**

$$\max_{\lambda, \mu_1, \mu_2} \; g(\lambda, \mu_1, \mu_2) \quad \text{s.t.} \quad \lambda \geq 0,\; \mu_1 \geq 0,\; \mu_2 \geq 0.$$

2/4

$$\max_{\lambda, \mu_1, \mu_2} \; \lambda - \frac{1}{2}\lambda(\mu_1 + \mu_2) - \frac{1}{4}(\mu_1^2 + \mu_2^2)$$

s.t. : $\lambda \geq 0, \mu_1 \geq 0, \mu_2 \geq 0$

Discrete Mathematics

29. **[Proof]** Prove or Disprove: If a and b are rational numbers then a^b is also a rational number. **[5 Points]**

~~$a = p_1/q_1$, $b = p_2/q_2$~~

~~$p_1 = aq_1$ $p_2 = bq_2$~~

$a^b \neq p/q$

Disprove by counter example

$a = 2$

$b = 1/2$

$\sqrt{2}$ is not rational.

full 5

Partial 2

30. Prove that for every prime number p, either $p = 2$ or p is odd.(5 Points)

~~If $p = 2$ or~~ $\forall p \in \text{Primes} \Rightarrow p = 2$ or $p = 2k+1$

If $p \neq 2$ $p = 2k \Rightarrow p$ is not prime.

5

2

$a^b = \frac{r}{s}$

$\left(\frac{p_1}{q_1}\right)^{(p_2/q_2)} = \frac{r}{s}$

$a = 2$

$b = 1/2$

$2^{1/2}$ $\sqrt{2}$ is not rational