Indian Institute of Technology Jodhpur

Matrix Theory(MAL7051)

Academic Year 2024-25, Semester I

Assignment 1

Due Date: Sep 05, 2024 **Maximum Marks: 100**

Instructions:

- Read each question carefully before answering.
- Plagiarism of any kind will not be tolerated and will result in zero marks.
- Late submissions will incur a penalty of 10% deduction per day.

---

1. Show that the space R^n contains a subspace $W(\neq \{0\})$ such that $W \neq R^n$
2. Prove that the set $\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$ is linearly independent.
3. Find the rank of each of the following matrices:

   (a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

   (b) $B = \begin{pmatrix} 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 10 & 4 & 7 \end{pmatrix}$

   (c) $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

4. Let S be a linearly independent subset of R^n, and let v be a vector in R^n that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.
5. Let

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}.$$

   (a) Find all the eigenvalues of A.

   (b) Find a maximum set of linearly independent eigenvectors of A.

   (c) Is A diagonalizable? If yes, find P such that $D = P^{-1}AP$ is diagonal.

6. Consider the subspace W of R^4 spanned by the vectors

$$v_1 = (1,1,1,1),\ v_2 = (1,1,2,4),\ v_3 = (1,2,-4,-3).$$

   Find (a) an orthogonal basis of W; (b) an orthonormal basis of W.

7. True or false (check addition in each case by an example):

   (a) The symmetric matrices in $\mathbf{M}$ (with $A^T = A$) form a subspace.

(b) The skew-symmetric matrices in $\mathbf{M}$ (with $A^T = -A$) form a subspace.

(c) The unsymmetric matrices in $\mathbf{M}$ (with $A^T \neq A$) form a subspace.

8. Find the singular value decomposition for

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

9. Show that every square real matrix has a polar decomposition.

10. Find the pseudo inverse for the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

11. Find k so that $u = (1, 2, k, 3)$ and $v = (3, k, 7, -5)$ in R^4 are orthogonal.

12. Suppose you do two row operations at once, going from:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} a - Lc & b - Ld \\ c - la & d - lb \end{bmatrix}$$

Find the second determinant. Does it equal $ad - bc$?

13. Consider the rotation of vectors around the x-axis in vector space R^2. Find the corresponding matrix, its eigenvalues, and the corresponding eigenvectors.

14. Let $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$.

(a) Find eigenvalues and corresponding eigenvectors.

(b) Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal.

(c) Find A^8 and $f(A)$ where $f(t) = t^4 - 5t^3 + 7t^2 - 2t + 5$.

(d) Find a matrix B such that $B^2 = A$.

15. Find the eigenvalues for matrix B:

$$B = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \end{bmatrix}$$

16. Find the characteristic roots of the 2-rowed orthogonal matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and verify that they are of unit modulus.

17. Find a real matrix that has no real eigenvalues or eigenvectors.

18. Answer the following :

(a) Given the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, compute its polar decomposition $A = UP$, where U is a unitary matrix and P is a positive semi-definite matrix.

(b) For the matrix $B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, find the matrices U and P in the polar decomposition $B = UP$.

(c) Calculate the polar decomposition of the matrix $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and verify that the decomposition is unique.

19. Which of these transformations are not linear? The input is $\mathbf{v} = (v_1, v_2)$:

(a) $T(\mathbf{v}) = (v_2, v_1)$

(b) $T(\mathbf{v}) = v_1 v_2$

20. Suppose a linear transformation T transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$. Find $T(\mathbf{v})$:

(a) $\mathbf{v} = (-1, 1)$

(b) $\mathbf{v} = (a, b)$

21. Let $T : R^3 \to R^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3, 3x_1 + 4x_2 + x_3, 2x_1 + x_2 - x_3).$$

Find the dimension of the range space of T^2.

22. Let $A = \begin{pmatrix} 3 & -1 \\ -1 & 6 \end{pmatrix}$, a real symmetric matrix. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

23. Find the characteristic polynomial $\Delta(t)$ of each of the following linear operators:

(a) $\mathbf{F} : R^2 \to R^2$ defined by $\mathbf{F}(x, y) = (3x - 7y, \quad 2x + 5y)$.

(b) $\mathbf{D} : V \to V$ defined by $\mathbf{D}(f) = \frac{df}{dt}$, where V is the space of functions with basis $S = \{\sin t, \cos t\}$.

24. Find a linear map $\mathbf{F} : R^3 \to R^4$ whose image is spanned by $(1, 2, 0, -4)$ and $(2, 0, -1, -5)$.

25. Let $G : R^2 \to R^3$ be defined by $G(x, y) = (x + y, x - 2y, 3x + y)$.

(a) Show that G is nonsingular.

(b) Find a formula for G^{-1}.

26. Find the trace and determinant of each of the following linear maps on R^3:

(a) $F(x, y, z) = (x + 3y, \ 3x - 2z, \ x - 4y - 3z)$.

(b) $G(x, y, z) = (y + 3z, \ 2x - 4z, \ 5x + 7y)$.

27. Find a basis (and the dimension) for each of these subspaces of 3×3 matrices:

(a) All diagonal matrices.

(b) All symmetric matrices ($A^T = A$).

(c) All skew-symmetric matrices ($A^T = -A$).

28. Given the matrix

find inverse

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 5 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

Find all the variables of the matrix

$$A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

29. Let T be a linear operator. Show that the following statements are equivalent:

    (a) A scalar λ is an eigenvalue of T.

    (b) The linear operator $\lambda I - T$ is singular.

    (c) The scalar λ is a root of the characteristic polynomial $\Delta(t)$ of T.

30. Consider the following matrix:

$$A = \begin{bmatrix} -3 & 8 & 8 \\ -1 & 5 & +2 \\ -1 & +2 & 9 \end{bmatrix},$$

Find A^{50} (**Think Wisely!!!**).

Best of Luck!!!

MAL 7070: OPTIMIZATION FOR DATA SCIENCE

Full Marks: 50
Deadline: October 8, 2024.

Assignment 3

- Please read the question carefully. Only write necessary and to the point answer.
- Submit a handwritten solution in the class on the 8th of October.

Question 1 (3 marks)

Recall that a hyperplane $H \subseteq \mathbb{R}^n$ is a set of the form

$$H = \{x \, : \, \langle a, x \rangle = b\}.$$

Every hyperplane divide $\mathbb{R}^n$ into two halves

$$H^- = \{x \, : \, \langle a, x \rangle \leq b\}$$

and

$$H^- = \{x \, : \, \langle a, x \rangle \geq b\}$$

that are called half-spaces induced by H.

- Give and example of a hyperplane in $\mathbb{R}^3$.
- Show that the half-spaces induced by it are convex sets.

Question 2 (3 marks)

Let $\{C_t \, : \, t \in T\}$ be a an arbitrary collection of nonempty convex sets. Show that the intersection of these sets, i.e., $\bigcap_{t \in T} C_t$ is convex. Prove that the set $S = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \, : \, |x_1 \cos t + x_2 \cos 2t + x_3 \cos 3t| \leq 1 \text{ for all } t \in [0, 1]\}$ is convex in $\mathbb{R}^3$.

Question 3 (3 marks)

A set $C \subseteq \mathbb{R}^n$ is called cone if for all $x \in C$ and $\lambda \geq 0$, $\lambda x \in C$. Prove that

$$C = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \, : \, x_1 + x_2 t + x_3 t^2 \geq 1 \text{ for all } t \in [0, 1]\}$$

is a cone in $\mathbb{R}^3$.

Question 4 (3 marks)

Let a and b be two distinct points in $\mathbb{R}^2$. Show that the set of all points that are closer to a than to b is a half-space; that is, show that the set $\{x \in \mathbb{R}^2 \, : \, \|x - a\|_2 \leq \|x - b\|_2\}$ is a half-space. For $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ find the half-space.

Question 5 (3 marks)

Consider the set $S = \{(x_1, x_2)^T \in \mathbb{R}^2 \ : \ x_1x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\}$. Is S a convex set? Find a hyperplane that separates $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ with S.

Question 6 (4 marks)

Recall that a polyhedral set is the intersection of a finite number of half-spaces and in general, can be written as $\{x \in \mathbb{R}^n \ : \ Ax \leq b\}$ where $A \in \mathbb{R}^{m \times n}$ is a matrix, $b \in \mathbb{R}^m$ is a vector and the order is component-wise. Consider the set $S = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \ : \ \max\{|x_1|, |x_2|, |x_3|\} \leq 1\}$.. Is S a polyhedral set? Can you write it as $\{x \in \mathbb{R}^3 \ : \ Ax \leq b\}$? If yes, find the A and b. Also, find a hyperplane that separates $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ with S.

Question 7 (4 marks)

Consider the problem

$$\begin{aligned} \text{minimize } & f(x, y) = x^2 + xy + y^2 \\ \text{subject to } & x^2 + y^2 \leq 5 \\ & x^2 - 4y \leq 0. \end{aligned}$$

What is the feasible set here? Can $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ be a local minimizer of this problem? If not, find a descent direction of f at $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ that is also a feasible direction for the feasible set at $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Question 8 (4 marks)

Consider the set $S = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 + x_3^2 \leq 1\}$. Suppose you want to find the minimum distance of the point $\begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$ from this set S. Formulate this as a constrained optimization problem. Then verify whether

- $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a KKT point or not
- $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ is a KKT point or not

Question 9 (3 marks)

Consider the problem

$$\begin{aligned} \text{minimize } & f(x) = \frac{1}{2}x^T Ax + b^T x + c \\ \text{subject to } & -1 \leq x_i \leq 1 \text{ for } i = 1, 2, 3 \end{aligned}$$

where $A = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}$, $b = \begin{pmatrix} -22 \\ 14.5 \\ 13 \end{pmatrix}$ and $c = 1$

Verify whether $x = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}$ is a possible minimizer or not.

Question 10 (4 marks)

Consider the problem

$$\begin{aligned} \text{minimize } & -3x_1^2 + x_2^2 + 2x_3^2 + 2x_1 + 2x_2 + 2x_3 \\ \text{subject to } & x_1^2 + x_2^2 + x_3^2 = 1 \end{aligned}$$

Find the KKT points and the corresponding KKT multipliers.

Question 11 (3 marks)

Consider the problem

$$\begin{aligned} \text{minimize } & x_1^2 + x_2^2 \\ \text{subject to } & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1 \end{aligned}$$

Can you find the global minimizer of this problem? Verify whether the minimizer is Fritz-John point. Is it also a KKT point?

Question 12 (3 marks)

Consider the problem

$$\begin{aligned} \text{minimize } & (x_1 - 3)^2 + (x_2 - 2)^2 \\ \text{subject to } & x_1^2 + x_2^2 \leq 5 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1 + 2x_2 = 4 \end{aligned}$$

Verify whether $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ can be a Fritz-John point. Can it be a KKT point as well?

Question 13 (4 marks)

Consider the problem

$$\begin{aligned} \text{minimize } & x_1^4 + x_2^4 + 12x_1^2 + 6x_2^2 - x_1x_2 - x_1 - x_2 \\ \text{subject to } & x_1 + x_2 \geq 6 \\ & 2x_1 - x_2 \geq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Check whether the following points are KKT or not

- $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$
- $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Problem 14 (6 marks)

A company is planning to assign three jobs to three of its employees: Joseph, Akash, and Rizwan. All three of them are capable of doing all three jobs, but the end result depends on their efficiency, and that incurs some expenses for the company. The following chart explains the expenses of assigning different jobs to different people:

	Job 1	Job 2	Job 3
Joseph	1500	1000	900
Akash	900	1500	1000
Rizwan	1000	1200	800

Only one job must be assigned to one employee and the company wants to get all three jobs done by incurring minimum cost. Formulate a linear programming problem to model the situation. State the assumptions you make. Verify whether the following assignment is optimal or not

$$\begin{aligned} \text{Joseph} &\rightarrow \text{Job 2} \\ \text{Akash} &\rightarrow \text{Job 1} \\ \text{Rizwan} &\rightarrow \text{Job 3} \end{aligned}$$

Full Marks: 30
Deadline: November 8, 2024.

MAL 7070: OPTIMIZATION FOR DATA SCIENCE

Assignment 4

- Please read the question carefully. Only write necessary and to the point answer.
- Submit a handwritten solution on or before the 8th of November (In my office in the Department of Mathematics.

Question 1 (4 marks)

Consider the problem

$$\begin{aligned} &\text{minimize } 2x_1 - x_2 + 2x_3 \\ &\text{subject to } 2x_1 + x_2 + x_3 = 4 \\ &x_1 + 2x_2 + x_4 = 5 \\ &x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Find a basic feasible solution for this problem (that is one extreme point of the feasible set).

Question 2 (4 marks)

Solve the problem using a quadratic penalty function method

$$\begin{aligned} &\text{minimize } x^2 + (y-2)^2 \\ &\text{subject to } y = x + 1 \end{aligned}$$

Question 3 (4 marks)

Solve the problem using a quadratic suitable barrier function method

$$\begin{aligned} &\text{minimize } x + y \\ &\text{subject to } x \geq 0 \\ &y \geq 0 \end{aligned}$$

Question 4 (6 marks)

Write the dual of the following problem.

$$\begin{aligned} &\text{minimize } x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \\ &\text{subject to} \\ &x_1 + 3x_2 + 2x_3 + 3x_4 + 3x_5 = 1 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

4

4-2

Solve the dual and then use complementary slackness to find an optimal solution for the primal. Verify whether the solution you got for the primal problem is a KKT point or not. Can you find all the basic feasible solutions for the primal problem?

Question 5 (4 marks)

Consider the problem

$$\text{minimize } e^{-x_2}$$
$$\text{subject to } \sqrt{x_1^2 + x_2^2} - x_1 \leq 0$$

Write the Lagrange dual. Verify whether this problem has zero duality gap or not.

Question 6 (8 marks)

Use python code to solve the following problem.

$$\begin{aligned} &\text{minimize } x_1^2 + 2x_2^2 \\ &\text{subject to } 2x_1 + 3x_2 - 6 \leq 0 \\ &\quad -x_2 + 1 \leq 0 \end{aligned}$$

- Solve it using a suitable penalty function method. Please use the following instructions to write your code
  - Use $(2,4)^T$ as your starting point and $\gamma = 10$ as your initial penalty parameter.
  - Use any unconstrained optimization algorithm (that is applicable) of your choice at each iteration.
  - Increae the penalty parameter by a factor of 5 after each iteration.
  - Use difference of two consecutive iterations as stopping criteria with tolerance being 10^{-6}
- Solve it using a suitable bariier function method. Please use the following instructions to write your code
  - Use $(0, \frac{3}{2})^T$ as your starting point and $\gamma = 10$ as your initial barrier parameter.
  - Use any unconstrained optimization algorithm (that is applicable) of your choice at each iteration.
  - Decrease the parameter by a factor of $\frac{1}{5}$ after each iteration.
  - Use difference of two consecutive iterations as stopping criteria with tolerance being 10^{-6}

In each case see whether you got a point which is close to a solution of the constrained problem.