一、填空题

1. $\dfrac{361}{400}$;　　2. $\dfrac{1}{4}$;　3. $e^{-2}-e^{-3}$;　4. 2;　　5. 1;　　6. $\dfrac{1}{3}$

二、解答题

1. 设 $A=\{$考试及格$\}$，$B=\{$按时交作业$\}$

(1) 由全概率公式得

$$P(B)=P(A)P(B|A)+P(\overline{A})P(B\mid\overline{A})=0.9\times0.85+0.1\times0.35=0.8$$

(2) 由贝叶斯公式得

$$P(A\mid B)=\frac{P(A)P(B\mid A)}{P(B)}=\frac{0.9\times0.85}{0.8}=\frac{153}{160}=0.95625$$

2. (1) $P(X=k)=\dfrac{C_n^k p^k(1-p)^{n-k}}{1-(1-p)^n},\ k=1,2,\ldots,n.$

(2) $$E(X)=\sum_{k=1}^{n}k\frac{C_n^k p^k(1-p)^{n-k}}{1-(1-p)^n}=\frac{1}{1-(1-p)^n}\sum_{k=1}^{n}kC_n^k p^k(1-p)^{n-k}$$

$$=\frac{1}{1-(1-p)^n}\sum_{k=0}^{n}kC_n^k p^k(1-p)^{n-k}=\frac{np}{1-(1-p)^n}$$

3. 设 X_i 为第 i 位经手人加入的游戏币数目, $i=1,2,\ldots,200$. X 为最终箱子中游戏币增加的数目. 则:

$P(X_i=k)=\dfrac{1}{5}$, $k=-2,-1,0,1,2$; 且 $X_1,X_2,\ldots,X_{200}$ 相互独立,

$$E(X_i)=0,\ D(X_i)=E(X_i^2)=\frac{1}{5}\times[(-2)^2+(-1)^2+0+1^2+2^2]=2$$

$$X=\sum_{i=1}^{200}X_i,\ E(X)=0,\ D(X)=400$$

由中心极限定理, X 近似服从 $N(0,400)$, 于是

$$P(|X|\le10)=2\Phi\left(\frac{10}{\sqrt{400}}\right)-1=2\Phi(0.5)-1=2\times0.6915-1=0.383$$

4. (1) $$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)\mathrm{d}y=\begin{cases}\int_0^x 15xy^2\mathrm{d}y, & 0<x<1\\ 0, & \text{其他}\end{cases}=\begin{cases}5x^4, & 0<x<1\\ 0, & \text{其他}\end{cases};$$

(2) 当 $0<x<1$ 时

$$f_{Y|X}(y\mid x)=\frac{f(x,y)}{f_X(x)}=\begin{cases}\dfrac{15xy^2}{5x^4}, & 0<y<x\\ 0, & \text{其他}\end{cases}=\begin{cases}\dfrac{3y^2}{x^3}, & 0<y<x\\ 0, & \text{其他}\end{cases}$$

$$f_{Y|X}\left(y\left|\frac{1}{2}\right.\right)=\begin{cases}24y^2, & 0<y<\dfrac{1}{2} \\ 0, & 其他\end{cases},\quad P\left(Y<\frac{1}{4}\left|X=\frac{1}{2}\right.\right)=\int_0^{\frac{1}{4}}24y^2\mathrm{d}y=\frac{1}{8}$$

(3) Z 的取值范围为(0, 1). 当 $z<0$ 时, $F_Z(z)=0$; 当 $z\geq 1$ 时, $F_Z(z)=1$. 当 $0\leq z<1$ 时,

$$F_Z(z)=P(Z\leq z)=P\left(\frac{Y}{X}\leq z\right)=\int_0^1\mathrm{d}x\int_0^{zx}15xy^2\mathrm{d}y=\int_0^1 5z^3x^4\mathrm{d}x=z^3$$

于是

$$f_Z(z)=F_Z'(z)=\begin{cases}3z^2, & 0<z<1 \\ 0, & 其他\end{cases}$$

5. X 的密度函数为 $f(x)=\begin{cases}\mathrm{e}^{-x}, & x>0 \\ 0, & x\leq 0\end{cases}$, $Y=\min\{X, 1\}$.

$$E(Y)=\int_0^1 x\mathrm{e}^{-x}\mathrm{d}x+\int_1^{+\infty}\mathrm{e}^{-x}\mathrm{d}x=1-\mathrm{e}^{-1};$$

$$E(Y^2)=\int_0^1 x^2\mathrm{e}^{-x}\mathrm{d}x+\int_1^{+\infty}\mathrm{e}^{-x}\mathrm{d}x=2-4\mathrm{e}^{-1};$$

$$D(Y)=E(Y^2)-[E(Y)]^2=1-2\mathrm{e}^{-1}-\mathrm{e}^{-2}.$$

6. (1) $Z_1=|X_1-\mu|$的取值范围为$[0, +\infty)$.

当 $z<0$ 时, $F_Z(z)=0$;

当 $z\geq 0$ 时, $F_Z(z)=P(Z_1\leq z)=P(|X_1-\mu|\leq z)=2\Phi\left(\dfrac{z}{\sigma}\right)-1$,

$$f_Z(z)=F_Z'(z)=\frac{2}{\sigma}\varphi\left(\frac{z}{\sigma}\right)=\frac{2}{\sqrt{2\pi}\sigma}e^{-\frac{z^2}{2\sigma^2}}$$

即 $$f_Z(z)=\begin{cases}\dfrac{2}{\sqrt{2\pi}\sigma}e^{-\frac{z^2}{2\sigma^2}}, & z\geq 0 \\ 0, & z<0\end{cases}$$

(2) $$L(\sigma)=\prod_{i=1}^n\frac{2}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{z_i^2}{2\sigma^2}}=\frac{2^n}{(\sqrt{2\pi}\sigma)^n}\mathrm{e}^{-\frac{1}{2\sigma^2}\sum_{i=1}^n z_i^2}$$

$$\ln L(\sigma)=n\ln 2-\frac{n}{2}\ln(2\pi)-n\ln\sigma-\frac{1}{2\sigma^2}\sum_{i=1}^n z_i^2$$

$$\frac{\mathrm{d}\ln L(\sigma)}{\mathrm{d}\sigma}=-\frac{n}{\sigma}+\frac{1}{\sigma^3}\sum_{i=1}^n z_i^2=0,\ 解得\ \hat{\sigma}=\sqrt{\frac{1}{n}\sum_{i=1}^n z_i^2}$$

故σ的极大似然估计量为 $\hat{\sigma}=\sqrt{\dfrac{1}{n}\sum_{i=1}^n Z_i^2}$.

7. (1) $\dfrac{\overline{X}-\mu}{S/\sqrt{n}}=\dfrac{4(\overline{X}-\mu)}{S}\sim t(15)$，$P\left(-t_{0.975}(15)<\dfrac{4(\overline{X}-\mu)}{S}<t_{0.975}(15)\right)=0.95$

即 $P\left(\overline{X}-\dfrac{S}{4}t_{0.975}(15)<\mu<\overline{X}+\dfrac{S}{4}t_{0.975}(15)\right)=0.95$

故μ的置信度为 95%的置信区间为

$$\left(\overline{x}-\frac{s}{4}\times t_{0.975}(15),\overline{x}-\frac{s}{4}\times t_{0.975}(15)\right)=\left(5.2-\frac{1.6}{4}\times 2.1315,5.2+\frac{1.6}{4}\times 2.1315\right)$$

$= (4.3474, 6.0526)$

(2) 检验统计量为 $\chi^2=\dfrac{15S^2}{\sigma_0^2}=\dfrac{15S^2}{2.5}=6S^2$,

检验拒绝域为 $W=\{\chi^2=6S^2>\chi^2_{0.95}(15)\}=\{\chi^2=6S^2>24.996\}$
带入 $s^2=2.56$, 计算得 $6s^2=6\times 2.56=15.36<24.996$
故不能拒绝原假设，不能认为σ^2 显著大于 2.5.