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Lab: 03
Monte Carlo Simulation - MA323

Problem 1

1. Implement the acceptance rejection method to generate samples from a distribution with PDF

$$f(x) = 20x(1-x)^3 \quad \text{for} \quad 0 < x < 1.$$

Please use the smallest value of c such that $f(x) \leq cg(x)$ for your choice of g.

(a) What is the average of number of iterations needed to generate a random number and why?

(b) Generate 10000 random numbers from the distribution. Compute the sample mean and compare it with expectation of the PDF f.

(c) What is the approximate value of $P(0.25 \leq X \leq 0.75)$ based on the generated sample in the part (b)? What is the exact value of the probability? Compare them.

(d) Keep a count of number of iterations needed to generate each of the random numbers in part (b). Compute the average of all these values and compare it with the value obtained in part (a).

(e) Draw the histogram of the sample obtained in part (b). Also, draw the PDF f on the same plot. Compare them.

(f) Repeat parts (a)–(d) above with two values of c higher than the smallest value that you have chosen. What are your observations ?

For calculation of smallest c using differential calculus (written calculation attached) :

Calc. for min. value of c such that

$f(x) \leq c\,g(x)$

$g(x) = 1 \qquad 0 \leq x \leq 1.$

So $f(x) \leq c \times 1 \Rightarrow f(x) \leq c$.

So, max value of $f(x) = 20x(1-x)^3$ in $x \in [0,1]$ is needed

$f'(x) = 20\left((1-x)^3 - 3x(1-x)^2\right)$

$f'(x) = 0 \Rightarrow 20(1-x)^2(1 - x - 3x)$

$\Rightarrow x = 1, \ 1/4.$

at $x = 1 \quad f(1) = 0.$

$x = 1/4 \quad f(1/4) = 20(1/4)(1-1/4)^3$

$= \dfrac{135}{64}$

$= 2.109375$ ✓

$\boxed{\max f(x) = \min c = 2.109375}$

(a)Since the probability of a number from the sample being accepted is 1/c on average (as shown below) the total number of iterations required would be c*(size of randomised sample) . So, the average number of iterations required would be approximately equal to c.That is 2.109375 here.

$$\sum_y P(\text{accept}|y)P(Y=y) = \sum_y \frac{f(y)}{cg(y)}g(y) = \frac{1}{c}.$$

(b)Theoretically the value of expectation of PDF f is 1/3=0.333... which approximately matches with the empirically calculated value as follows:

Calculating $E(x)$ for $f(x) = 20x(1-x)^3$

$$E(f(x)) = \int_0^1 x f(x) = \int_0^1 20x^2(1-x)^3\,dx$$

$$= \int_0^1 20x^2(1 - 3x + 3x^2 - x^3)$$

$$= \int_0^1 (20x^2 - 60x^3 + 60x^4 - 20x^5)\,dx$$

$$= \frac{20}{3} - \frac{60}{4} + \frac{60}{5} - \frac{20}{6} = \frac{10}{3} - 3 = \boxed{\frac{1}{3}}$$

classmate

```
C = 2.109375
X = []
Y = generator(50000,53)
U = generator(50000,47)
ctr=0 # in order to keep track of number of iterations

for i in range(50000):
    ctr+=1
    if U[i] <= fx(Y[i])/C :
        X.append(Y[i])
    if len(X)==10000 :
        break

print("The arithmetic mean for the generated random sample: ",np.mean(X)) #0.33300028925032005
```

```
The arithmetic mean for the generated random sample:  0.33300028925032005
```

(c) Theoretically the value of P(0.25X0.75) is 0.617188 which approximately matches with the empirically calculated value as follows:

```
countprob=0
for x in X:
    if x>=0.25 and x<=0.75:
        countprob+=1
probability=countprob/len(X)
print("The probabilty P(0.25≤X≤0.75) for the generated random sample: ",probability)
```

```
The probabilty P(0.25≤X≤0.75) for the generated random sample:  0.6175
```

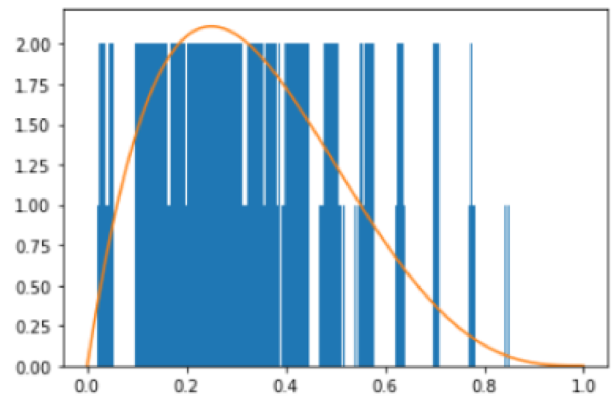
(d) Theoretically the value is c=2.109375 which approximately matches with the empirically calculated value as follows:

```
avgiter=ctr/len(X)
print("The average number of iterations required for generation of each random number:",avgiter)
```

```
The average number of iterations required for generation of each random number: 2.1089
```

(e) The histogram plot along with the f(x) curve is as follows :

```
plt.hist(X,bins=10000)
xa = np.arange(0,1,0.003)
ya = xa*20*(1-xa)*(1-xa)*(1-xa)
plt.plot(xa,ya)
plt.show()
```



The histogram plot and the curve are similar and have coinciding behaviour.
(f) Values of c taken are 4 and 7.5

```
print("For c=4, following are ther a-d observations:")
C = 4
X = []
Y = generator(50000,53)
U = generator(50000,47)
ctr=0 # in order to keep track of number of iterations

print("a)The avg number of iterations is equal to c=",C)

for i in range(50000):
    ctr+=1
    if U[i] <= fx(Y[i])/C :
        X.append(Y[i])
    if len(X)==10000 :
        break

print("b)The arithmetic mean for the generated random sample: ",np.mean(X)) #0.33300028925032005

countprob=0
for x in X:
    if x>=0.25 and x<=0.75:
        countprob+=1
probability=countprob/len(X)
print("c)The probabilty P(0.25≤X≤0.75) for the generated random sample: ",probability)

avgiter=ctr/len(X) # the average number of iterations required for generation of each random number
print("d)The average number of iterations required for generation of each random number:",avgiter)
```

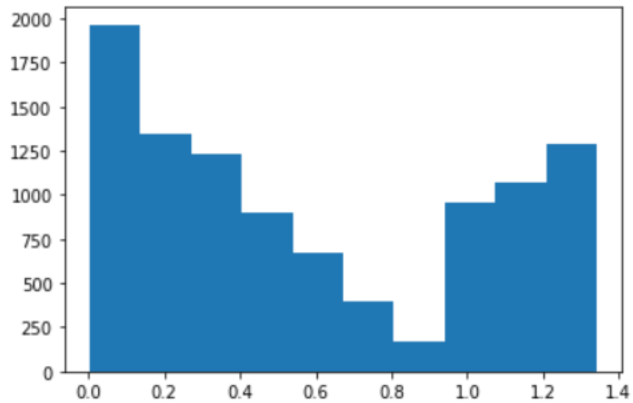
```
For c=4, following are ther a-d observations:
a)The avg number of iterations is equal to c= 4
b)The arithmetic mean for the generated random sample:  0.3330692351462848
c)The probabilty P(0.25≤X≤0.75) for the generated random sample:  0.6192
d)The average number of iterations required for generation of each random number: 3.9997
```

```python
print("For c=7.5, following are ther a-d observations:")
C = 7.5
X = []
Y = generator(50000,53)
U = generator(50000,47)
ctr=0 # in order to keep track of number of iterations

print("a)The avg number of iterations is equal to c=",C)

for i in range(50000):
    ctr+=1
    if U[i] <= fx(Y[i])/C :
        X.append(Y[i])
    if len(X)==10000 :
        break

print("b)The arithmetic mean for the generated random sample: ",np.mean(X)) #0.33300028925032005

countprob=0
for x in X:
    if x>=0.25 and x<=0.75:
        countprob+=1
probability=countprob/len(X)
print("c)The probabilty P(0.25≤X≤0.75) for the generated random sample: ",probability)

avgiter=ctr/len(X) # the average number of iterations required for generation of each random number
print("d)The average number of iterations required for generation of each random number:",avgiter)
```

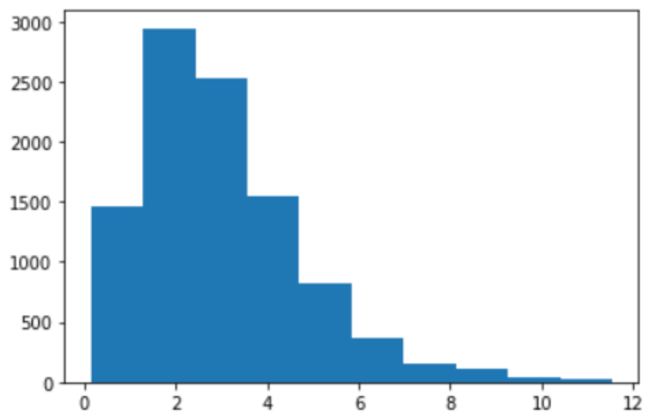
```
For c=7.5, following are ther a-d observations:
a)The avg number of iterations is equal to c= 7.5
b)The arithmetic mean for the generated random sample:  0.3333219846898309
c)The probabilty P(0.25≤X≤0.75) for the generated random sample:  0.6193673530289466
d)The average number of iterations required for generation of each random number: 7.460459564309161
```

Problem 2:

```
Part A

Mean :  0.5946495429085203
Variance :  0.1972118167244329
```




```
Part B

Mean :  2.9937768650513163
Variance :  3.185269821760526
```




```
Part C

Mean :  3.590479674711771
Variance :  3.34335182868699
```