5.8, The Squeeze Theorem Practice Problems

Ex: Compute the following limits.

1. $\lim_{x \to 0} \dfrac{x^3+2x}{x+1+\frac{1}{x}}$

Sol: We must factor out the smallest power of x from the top and the bottom.

bottom: -1,0,1

$\frac{1}{x} = x^{-1}$

top: 1,3

We compute

$$\lim_{x \to 0} \frac{x^3+2x}{x+1+\frac{1}{x}} = \lim_{x \to 0} \frac{x}{\left(\frac{1}{x}\right)} \cdot \frac{(x^2+2)}{(x^2+x+1)}$$

$$= \lim_{x \to 0} x^2 \cdot \frac{x^2+2}{x^2+x+1}$$

$$= 0 \cdot \frac{0+2}{0+0+1} = \boxed{0}$$

2. $\lim_{x \to \infty} \dfrac{\sqrt[5]{2x^2+x}}{4x^{\frac{2}{3}}+x^{\frac{1}{10}}}$

Sol: We need to factor out the largest power of x from the top and bottom.

bottom: 1/10, 2/5
top: 1/5, 2/5

We compute

$$\lim_{x\to\infty} \frac{\sqrt[5]{2x^2+x}}{4x^{2/5}+x^{1/10}} = \lim_{x\to\infty} \frac{x^{2/5}}{x^{2/5}} \cdot \frac{\frac{(2x^2+x)^{\frac{1}{5}}}{x^{\frac{2}{5}}}}{\left(4+x^{\frac{1}{10}-\frac{2}{5}}\right)}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{2x^2+x}{x^2}\right)^{\frac{1}{5}}}{4+x^{\frac{-3}{10}}}$$

$$= \lim_{x\to\infty} \frac{\left(2+\frac{1}{x}\right)^{\frac{1}{5}}}{4+\frac{1}{x^{\frac{3}{10}}}}$$

$$= \boxed{\frac{(2+0)^{\frac{1}{5}}}{4+0}}$$

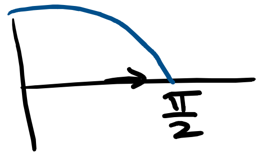
3. $\lim_{x\to\frac{\pi}{2}^-} \sec(x)-\tan(x)$

Sol: First, note that

$$\lim_{x\to\frac{\pi}{2}^-} \sec x = \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cos x}$$

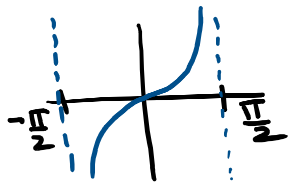
$$=$$

$$u = \cos x \qquad \lim_{u \to 0^+} \frac{1}{u} = \infty$$



$$\lim_{x \to \frac{\pi}{2}^-} \cos x = 0^+$$

$$\lim_{x \to \frac{\pi}{2}^-} -\tan x = -\lim_{x \to \frac{\pi}{2}^-} \tan x = -\infty$$



We cannot use the Addition Rule, because we get the Indeterminant Form

$$\lim_{x \to \frac{\pi}{2}^-} \sec x + \lim_{x \to \frac{\pi}{2}^-} -\tan x = \infty - \infty$$

Instead, we use L'Hospital's Rule. We compute

$$\lim_{x \to \frac{\pi}{2}^-} \sec x - \tan x = \lim_{x \to \frac{\pi}{2}^-} \frac{1}{\cos x} - \frac{\sin x}{\cos x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{1 - \sin x}{\cos x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{\partial}{\partial x}(1-\sin x)}{\frac{\partial}{\partial x}\cos x}$$

$\lim_{x\to\frac{\pi}{2}^-} 1-\sin x = 1-\sin\frac{\pi}{2} = 1-1=0$

$\lim_{x\to\frac{\pi}{2}^-} \cos x = \cos\frac{\pi}{2}=0$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\cos x}{\sin x}$$

$$= \frac{0}{1} = \boxed{0}$$

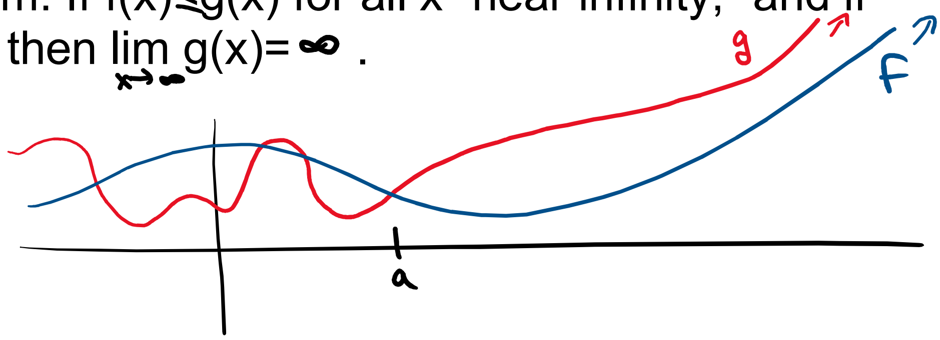
$\lim_{x\to\frac{\pi}{2}^-} \cos x = 0$

$\lim_{x\to\frac{\pi}{2}^-} \sin x = \sin\frac{\pi}{2} = 1 \neq 0$

DIV

Ex: Use the Squeeze Thm to show $\lim_{x\to\infty} x^2+\frac{\sin x}{x^2}=\infty$.

Squeeze Thm: If f(x)≤g(x) for all x "near infinity," and if $\lim_{x\to\infty} f(x)=\infty$, then $\lim_{x\to\infty} g(x)=\infty$.



Sol: Since

$$-1 \leq \sin x$$

then

$$-\frac{1}{x^2} \leq \frac{\sin x}{x^2}$$

and so

$$x^2 - \frac{1}{x^2} \leq x^2 + \frac{\sin x}{x^2} \quad \text{for} \quad x \neq 0$$

Since

$$\lim_{x\to\infty} x^2 - \frac{1}{x^2} = \infty - 0 = \infty$$

then by the Squeeze Thm

$$\lim_{x\to\infty} x^2 + \frac{\sin x}{x^2} = \infty .$$