## VECTOR CALCULUS, Week 13

### 11.6 Directional Derivatives and the Gradient Vector; 11.7 Maximum and Minimum Values

### 11.6 Directional Derivatives and the Gradient Vector

**Def:** Suppose $f = f(x, y)$ is differentiable at $(a, b)$ with $\nabla f(a, b) \neq \vec{0}$, and suppose $k = f(a, b)$.

- We say the **tangent line at** $(a, b)$ **of the level curve of** $f$ **at** $k$ is the line through $(a, b)$ in the direction of $< -f_y(a, b), f_x(a, b) >$, given by the point-direction parameterization
$$\ell(t) =< a, b > +t < -f_y(a, b), f_x(a, b) > \text{ for } t \in \mathbf{R}.$$
- We say the **normal line at** $(a, b)$ **of the level curve** $f$ **at** $k$ is the line through $(a, b)$ in the direction of $\nabla f(a, b)$, given by the point-direction parameterization
$$n(t) =< a, b > +t\nabla f(a, b) \text{ for } t \in \mathbf{R}.$$

**Ex:** Compute the tangent and normal line at $(a, b) = (2, 0)$ of the level curve of $f(x, y) = xe^{xy}$ at $k = 2$.

**Def:** Suppose $f = f(x, y, z)$ is differentiable at $(a, b, c)$ with $\nabla f(a, b, c) \neq \vec{0}$, and suppose $k = f(a, b, c)$.

- We say the **tangent plane at** $(a, b, c)$ **of the level surface of** $f$ **at** $k$ is the plane through $(a, b, c)$ with normal in the direction of $\nabla f(a, b, c)$, given by the scalar equation
$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0.$$
- We say the **normal line at** $(a, b, c)$ **of the level surface of** $f$ **at** $k$ is the line through $(a, b, c)$ in the direction of $\nabla f(a, b, c)$, given by the point-direction parameterization
$$n(t) =< a, b, c > +t\nabla f(a, b, c) \text{ for } t \in \mathbf{R}.$$

**Ex:** Compute the tangent plane and normal line at $(a, b, c) = (-1, 1, 3)$ of the level surface of $f(x, y, z) = z - x^2 - y^2$ at $k = 1$.

## 11.7 Maximum and Minimum Values

**Def:** Suppose $f = f(x, y)$ is a real-valued function defined near $(a, b)$.

- If $f(a, b) \geq f(x, y)$ for all $(x, y)$ near $(a, b)$, then we say

  $(a, b)$ is a **local maximum point of** $f$ and
  $f(a, b)$ is a **local maximum value of** $f$.

- If $f(a, b) \leq f(x, y)$ for all $(x, y)$ near $(a, b)$, then we say

  $(a, b)$ is a **local minimum point of** $f$ and
  $f(a, b)$ is a **local minimum value of** $f$.

- If $(a, b)$ is either a local maximum or a local minimum point of $f$, then we say

  $(a, b)$ is a **local extremum point of** $f$ and
  $f(a, b)$ is a **local extremum value of** $f$.

- If $\begin{cases} f_x(a,b) \text{ DNE, or} \\ f_y(a,b) \text{ DNE, or} \\ \nabla f(a,b) =< 0, 0 > \end{cases}$, then we say $(a, b)$ is a **criticial point** of $f$.

Suppose $\Omega \subseteq \mathbf{R}^2$, and suppose $f$ is defined for all $(x, y) \in \Omega$.

- If $(a, b) \in \Omega$ and $f(a, b) \geq f(x, y)$ for each $(x, y) \in \Omega$, then we say

  $(a, b)$ is an **absolute maximum point of** $f$ **over** $\Omega$ and
  $f(a, b)$ is the **absolute maximum value of** $f$ **over** $\Omega$.

- If $(a, b) \in \Omega$ and $f(x, y) \leq f(a, b)$ for each $(x, y) \in \Omega$, then we say

  $(a, b)$ is an **absolute minimum point of** $f$ **over** $\Omega$ and
  $f(a, b)$ is the **absolute minimum value of** $f$ **over** $\Omega$.

- If $(a, b) \in \Omega$ is either an absolute maximum or an absolute minimum point of $f$ over $\Omega$, then we say

  $(a, b)$ is an **absolute extremum point of** $f$ **over** $\Omega$ and
  $f(a, b)$ is an **absolute extremum value of** $f$ **over** $\Omega$.

We make similar definitions for real-valued functions $f = f(x, y, z)$.

**Def:** Basic definitions in point-set topology. Suppose $\Omega \subseteq \mathbf{R}^2$.

- If for each $(a,b) \in \Omega$, there is a disk of radius $r$

  $$D_{(a,b)} = \{(x,y) : (x-a)^2 + (y-b)^2 < r\}$$

  centered at $(a,b)$ so that $D_{(a,b)} \subseteq \Omega$, then we say $\Omega$ is an **open set**.

- If the complement $\mathbf{R}^2 \setminus \Omega$ of $\Omega$ is an open set, then we say $\Omega$ is a **closed set**.

- Suppose $(a,b) \in \Omega$. If for each disk $D_{(a,b)}$ centered at $(a,b)$, there are $(x_1,y_1) \in D_{(a,b)}$ with $(x_1,y_1) \in \Omega$ and there $(x_2,y_2) \in D_{(a,b)}$ with $(x_2,y_2) \in \mathbf{R}^2 \setminus \Omega$, then we say $(a,b) \in \Omega$ **is in the boundary of** $\Omega$.

  We say the set $\partial\Omega = \{(x,y) \in \Omega : (x,y) \text{ is in the boundary of } \Omega\}$ is the **boundary of** $\Omega$.

  We define the **interior** of $\Omega$ to be $\Omega \setminus \partial\Omega$, the set of points in $\Omega$ which are not in the boundary of $\Omega$.

- If there is an $r > 0$ so that $\Omega \subset \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < r\}$, then we say $\Omega$ is a **bounded set**.

For the same definitions over $\mathbf{R}^3$, use the interior of spheres

$$\{(x,y,z) \in \mathbf{R}^3 : (x-a)^2 + (y-b)^2 + (z-c)^2 < r\}.$$

**Thm:** Suppose $\Omega \subset \mathbf{R}^2$ is a closed and bounded set, and suppose $f = f(x, y)$ is a real-valued function continuous over $\Omega$.

- There is an absolute minimum point $(a_{min}, b_{min}) \in \Omega$ of $f$ over $\Omega$, and an absolute maximum point $(a_{max}, b_{max}) \in \Omega$ of $f$ over $\Omega$.
- If $(a, b) \in \Omega$ is an absolute extremum point of $f$ over $\Omega$, then either

$$\begin{aligned}&(a, b) \in \partial\Omega \text{ or}\\ &(a, b) \text{ is a critical point of } f \text{ in the interior of } \Omega.\end{aligned}$$

The same is true for real-valued continuous functions $f = f(x, y, z)$ over closed bounded sets $\Omega \subset \mathbf{R}^3$.

**Ex:** Find the absolute extremum points and values of the given function $f$ over the given region $\Omega$.

1. $f(x, y) = x^2 + y^2$ over $\Omega = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$.
2. $f(x, y) = x^2 - 2xy + 2y$ over the rectangle

$$\Omega = \{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq 3,\ 0 \leq y \leq 2\}.$$

**Ex:** Find the absolute extremum values of the given function $f$ over the given solid region $E$.

1. $f(x, y, z) = x^2 + y^2 + z^2 - z$ over

$$E = \{(x, y, z) \in \mathbf{R}^3 : 0 \leq z \leq \sqrt{1 - x^2 - y^2}\}$$

2. $f(x, y, z) = xy + z^2$ over $E = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 \leq 1\}$

**Def:** Suppose $f = f(x, y)$ is a real-valued function defined near $(a, b)$, and suppose the second partial derivatives of $f$ exist at $(a, b)$.

- We define the **discriminant of $f$ at** $(a, b)$ to be

$$\Delta = \Delta(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2.$$

- If $(a, b)$ is a critical point of $f$ but not a local extremum point of $f$, then we say $(a, b)$ **is a saddle point of** $f$.

**Thm (Second Derivative Test):** Suppose $f = f(x, y)$ is a real-valued function defined near $(a, b)$, suppose the second partial derivatives of $f$ exist near $(a, b)$ and are continuous at $(a, b)$, and suppose $(a, b)$ is a critical point of $f$.

- If $\Delta > 0$ and $f_{xx}(a, b) > 0$, then $(a, b)$ is a local minimum point of $f$.
- If $\Delta > 0$ and $f_{xx}(a, b) < 0$, then $(a, b)$ is a local maximum point of $f$.
- If $\Delta < 0$, then $(a, b)$ is a saddle point of $f$.
- If $\Delta > 0$ and $f_{xx}(a, b) = 0$ or if $\Delta = 0$, then no conclusion can be made about $(a, b)$.

**Proof:** Use CalcPlot3D.

**Ex:** Find the critical points of the given function $f$, and determine whether the critical points are local minimum, local maximum, or saddle points.

1. $f(x, y) = x^4 + y^4 - 4xy + 1$
2. $f(x, y) = x^4 - 2x^2 + y^3 - 3y$