3.7 Antiderivatives

union of open intervals, including $(1,\infty)$ $(-\infty,1)$

Def: Suppose f is defined over an open subset I of R.

If F is a function so that F'(x)=f(x) for all x in I, then we say F is an antiderivative of f over I.

The general antiderivative of f over I is the collection of all antiderivatives of f over I.

$\int f(x)\,dx$ means to find the most general antiderivative of f over the last largest possible open subset I of R, and is called the indefinite integral of f.

Ex: Suppose f(x)=x.$^{-3}$

1. Show that F(x)= $\begin{cases} \frac{x^{-2}}{-2}+1 & \text{for } x<0 \\ \frac{x^{-2}}{-2}+2 & \text{for } x>0 \end{cases}$ $\leftarrow x\neq 0$

is an antiderivative of f over I=(-infinity,0)U(0,infinity).

Sol: We compute

$$F'(x) = \begin{cases} \frac{d}{dx}\left(\frac{x^{-2}}{-2}+1\right) & \text{for } x<0 \\ \frac{d}{dx}\left(\frac{x^{-2}}{-2}+2\right) & \text{for } x>0 \end{cases}$$

$$= \begin{cases} \frac{(-2)x^{-3}}{-2}+0 & \text{for } x<0 \\ \frac{(-2)x^{-3}}{-2}+0 & \text{for } x>0 \end{cases}$$

$$= \begin{cases} x^{-3} & \text{for } x<0 \\ x^{-3} & \text{for } x>0 \end{cases} = x^{-3} \text{ for } x \neq 0.$$

This means that F'(x)=f(x) for all x=/0. We conclude that F is an antiderivative of f over I=(-infinity,0)U(0,infinity).

2. Compute the most general antiderivative of f over the largest possible open subset I of R, and give I.

Sol: The general antiderivative of f is

$$F(x) = \begin{cases} \dfrac{x^{-2}}{-2} + c_1 & \text{for } x<0 \\ \dfrac{x^{-2}}{-2} + c_2 & \text{for } x>0 \end{cases}$$

where c_1, c_2 are two possibly different constants, over I=(-infinity,0)U(0,infinity).

4.3 Evaluating Definite Integrals
(Table of Basic Antiderivatives)

Thm (Table of Basic Antiderivatives):

$$\int dx = \int 1\, dx = x + \underset{\substack{\uparrow \\ \text{constant}}}{C} \quad \text{over } I = (-\infty, \infty)$$

If r=/-1, then

$$\int x^r dx = \frac{x^{r+1}}{r+1} + \text{"}C\text{"} \quad \leftarrow \text{shorthand}$$

This is over I=(-infinity,infinity), or I=(0,infinity),
or I=(-infinity,0)U(0,infinity), depending on the value of r.
If I=(-infinity,0)U(0,infinity), then you must compute

$$\int x^r dx = \begin{cases} \dfrac{x^{r+1}}{r+1} + C_1 & \text{for } x<0 \\ \dfrac{x^{r+1}}{r+1} + C_2 & \text{for } x>0 \end{cases}$$

For example, simply writing $\int x^{-3} dx = \frac{x^{-2}}{-2} + C$ is wrong. $\emptyset$

Trigonometric functions:

$$\begin{cases} \int \cos x\, dx = \sin x + C \\ \int \sin x\, dx = -\cos x + C \quad \star \\ \int \frac{1}{1+x^2} dx = \arctan x + C \end{cases}$$

over I=(-infinity,infinity).

$$\star \quad \frac{d}{dx}(-\cos x + C) = -(-\sin x) \checkmark$$
$$= \sin x$$

$\int e^x \, dx = e^x + c$ over I=(-infinity,infinity).

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \int \frac{dx}{x} = \begin{cases} \ln|x| + c_1 & \text{for } x<0 \\ \ln|x| + c_2 & \text{for } x>0 \end{cases}$$

over I=(-infinity,0)U(0,infinity).

Def: In the Table of Basic Antiderivatives, we say c is the <u>constant of integration</u>.

Thm (Basic Antiderivative Rules): Suppose I is an open interval, and suppose F,G are respectively antiderivatives of f,g over I. $F' = f$, $G' = g$

<u>Simplification Rule</u>: If f(x)=g(x) for all x in I, then

$$\int f(x) \, dx = \int g(x) \, dx = G(x) + c \quad \text{over } I.$$

⇨ We can simplify inside the integral.

<u>Addition Rule</u>:

$$\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx = F(x) + G(x) + c \quad \text{over } I$$

<u>Height Rule</u>: If a is in R, then

$$\int a f(x) \, dx = a \int f(x) \, dx = a F(x) + c \quad \text{over } I.$$

Ex: Compute the most general antiderivative of the given function f over the largest possible open subset I of R, and give I.

1. f(x)=(x+4)(2x+1)

Sol: We compute

$$\int f(x)\,dx = \int (x+4)(2x+1)\,dx = \int 2x^2+9x+4\,dx$$

$$= \int 2x^2\,dx + \int 9x\,dx + \int 4\,dx$$

$$= 2\int x^2\,dx + 9\int x\,dx + 4\int 1\,dx$$

HEIGHT

$$= \boxed{\begin{array}{l} 2\left(\frac{x^3}{3}\right)+9\left(\frac{x^2}{2}\right)+4x+C \\ \text{over} \quad I=(-\infty,\infty) \end{array}}$$

2. f(x)= $\frac{x^2+2}{x^2+1}$

Sol: We compute

$$\int f(x)\,dx = \int \frac{x^2+2}{x^2+1}\,dx = \int \frac{x^2+1+1}{x^2+1}\,dx$$

$$= \int 1+\frac{1}{1+x^2}\,dx$$

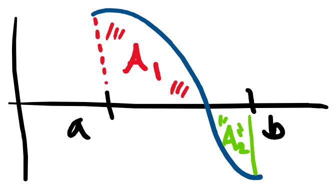
$$= \int dx + \int \frac{1}{1+x^2}\, dx$$

$$= \boxed{\begin{array}{l} x + \arctan x + C \\ \text{over } I = (-\infty, \infty) \end{array}}$$

4.2 The Definite Integral

Def: Suppose f is continuous over [a,b].

We define the area under the graph of f from a to b to be the signed area of the region bounded by the x-axis, the graph of f, and the lines x=a,x=b: we add the area of any regions above the x-axis, while subtract the area of any regions below the x-axis.



area under the graph of F from a to b $=$ $\text{area}(A_1) - \text{area}(A_2)$

We say the definite integral of f from a to b, denoted

$$\int_a^b f(x)\,dx \qquad a<b$$

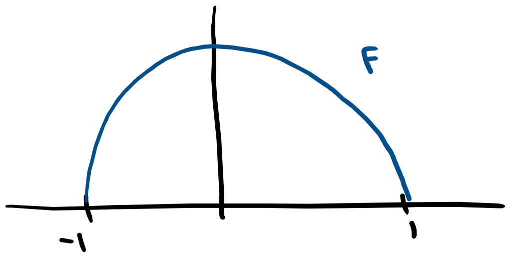
is the area under the graph of f from a to b, and we say a,b are the limits of the integral. We also define

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Ex: Compute $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$

Sol: Let's cheat and use the graph of f(x)=$\sqrt{1-x^2}$.

The graph of f is the upper-half of the unit circle.



By definition,

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \text{area of}$$  $$= \frac{1}{2}\cdot \pi\, 1^2 = \boxed{\frac{\pi}{2}}$$

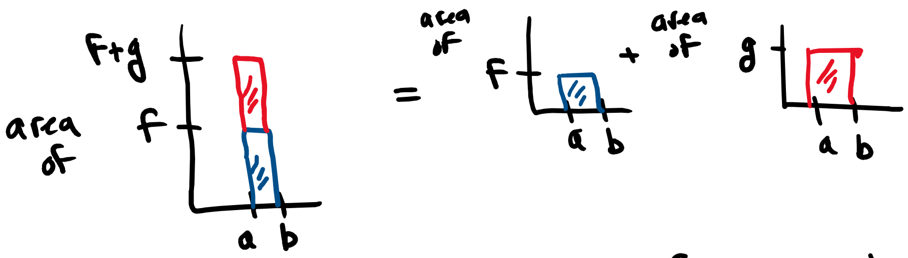
Thm (Basic Geometric Properties): Suppose f,g are continuous over [a,b].

Simplification Rule: if f(x)=g(x) for all x in [a,b], then

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$

⇨ We can simplify inside definite integrals.

Addition Rule: $$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



Base Rule: if c is in [a,b], then $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The same is true for general a,b,c in R, without satisfying $a \leqslant c \leqslant b$, as long as f is continuous between a,b,c so that the definite integrals are defined.

Height Rule: if c is in R, then $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

Height Comparison Rule: if $f(x) \leqslant g(x)$ for x in [a,b], then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Ex: Suppose $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = -7$, compute

$$\int_1^4 2f(x)\,dx.$$

Sol: We compute

$$12 = \int_1^5 f(x)\,dx = \underbrace{\int_1^4 f(x)\,dx}_{?} + \underbrace{\int_4^5 f(x)\,dx}_{\overset{\shortparallel}{-7}}$$

$$\Rightarrow \quad 12 = \int_1^4 f(x)\,dx - 7 \Rightarrow \int_1^4 f(x)\,dx = 19.$$

We conclude that

$$\int_1^4 2f(x)\,dx = 2\int_1^4 f(x)\,dx = \boxed{2 \cdot 19}$$

4.3 Evaluating Definite Integrals (Net Change Theorem)

FToC (Net Change Thm): Suppose f is continuous over [a,b], and suppose F is an antiderivative of f over [a,b]. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{b} = F(b)-F(a)$$

"F evaluated from a to b"

We also have

$$\int_a^a f(x)\,dx = 0 = F(a)-F(a) \quad \text{and}$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx = -\big(F(b)-F(a)\big)$$

$$= F(a)-F(b)$$

Ex: Net Change Thm.

1. Compute $\int_2^5 x^2\,dx$

Sol: We compute

$$\int_2^5 x^2\,dx = \frac{x^3}{3}\Big|_{x=2}^{5} = \boxed{\frac{5^3}{3}-\frac{2^3}{3}}$$

2. WRONG: $\int_{-1}^{2} x^{-3}\,dx = \frac{x^{-2}}{-2}\Big|_{x=-1}^{2}$

$f(x)=x^{-3}$ is not continuous at x=0 in (-1,2).

4.4 The Fundamental Theorem of Calculus
(Area Function Theorem)

Def: Suppose f is continuous over [a,b], and suppose c is in [a,b]. We define the area function F of f over [a,b] centered at c to be the function

$$F(x)=\int_c^x f(t)\,dt \quad \text{for } x\in[a,b]$$

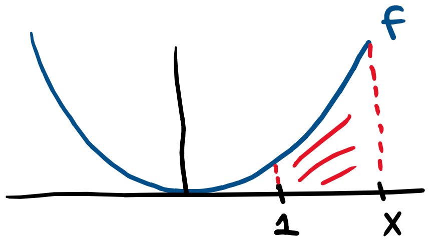
Ex: Compute the area function F of f(x)=x^2 over (-infinity,infinity) centered at c=1.

Sol: We compute

$$F(x)=\int_1^x t^2\,dt = \left.\frac{t^3}{3}\right|_{t=1}^{x}$$

$$\Rightarrow \boxed{F(x)=\frac{x^3}{3}-\frac{1}{3} \quad \text{for all } x}$$

F(x)= the area of



FToC (Area Function Thm): Suppose f is continuous over [a,b], and suppose F is the area function of f over [a,b] centered at c in [a,b]. Then F'(x)=f(x) for each x in (a,b).

$$\Rightarrow \quad \frac{d}{dx}\int_c^x f(t)\,dt = f(x)$$

⇨ Every continuous function has an antiderivative.

Ex: Compute the following derivatives.

1. $\frac{\partial}{\partial x}\int_0^x t+t^3\, \partial t$

Sol: The Area Function Theorem implies that

$$\frac{\partial}{\partial x}\int_{\underset{\text{constant}}{0}}^{x \leftarrow x \text{ by itself}} \underbrace{t+t^3}_{\text{continuous}}\, \partial t = \boxed{x+x^3}$$

We can check this by computing

$$\frac{\partial}{\partial x}\int_0^x t+t^3\, \partial t = \frac{\partial}{\partial x}\left(\frac{t^2}{2}+\frac{t^4}{4} \Big|_{t=0}^{x} \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x^2}{2}+\frac{x^4}{4} - (0+0)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x^2}{2}+\frac{x^4}{4}\right) = x+x^3 \ \checkmark$$

2. $\frac{\partial}{\partial x}\int_0^x e^{t^2}\, \partial t$

Sol: The Area Function Theorem implies that

$$\frac{\partial}{\partial x}\int_0^x e^{t^2}\, \partial t = \boxed{e^{x^2}} \ \checkmark$$

For this example, we cannot check this. The integral

$$\int_0^x e^{t^2}\,dt$$

cannot be computed directly in terms of basic functions.

4.5 The Substitution Rule

Chain Rule/u-Substitution Rule for Integrals: Suppose g is differentiable over [a,b], g' is continuous over [a,b], and f is continuous at each u=g(x) for each x in [a,b].

Indefinite Integral: $\int f(g(x))g'(x)\,dx = \int f(u)\,du \Big|_{u=g(x)}$

over (a,b)

Definite Integral: $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

$u = g(x)$

$du = g'(x)\,dx$

$x = a \to u = g(a)$

$x = b \Rightarrow u = g(b)$

Ex: Compute the most general antiderivative of the given f over the largest possible open subset I of R, and give I.

1. $f(x)=2xe^{x^2}$

Sol: We compute

$$\int 2xe^{x^2}\,dx = \int \frac{d}{dx}\left(e^{x^2}\right)dx$$

$$= \boxed{e^{x^2} + C \quad \text{over } I = (-\infty, \infty)}$$

$$\int 2xe^{x^2}\,dx = \int (e^{x^2})\,2x\,dx$$

$$\underset{\substack{u=x^2\\ du=2x\,dx}}{=} \int e^u\,du \Big|_{u=x^2}$$

$$= e^u + c \Big|_{u=x^2} = \boxed{\begin{array}{c} e^{x^2}+c \\ \text{over} \\ I=(-\infty,\infty) \end{array}}\ \checkmark$$

2. $f(x)=x^3\sqrt{1+x^2}$

Sol: We compute

$$\underbrace{\int x^3\sqrt{1+x^2}\,dx}_{} \underset{\substack{u=1+x^2\\ \Rightarrow x^2=u-1\\ du=2x\,dx}}{=} \int \tfrac{1}{2}(u-1)\sqrt{u}\,du \Big|_{u=1+x^2}$$

$$\int x^2\sqrt{1+x^2}\;x\,dx$$

$$\int \underbrace{\tfrac{1}{2}x^2}_{\frac{1}{2}(u-1)}\underbrace{\sqrt{1+x^2}}_{\sqrt{u}}\underbrace{(2x)\,dx}_{du}$$

$$= \frac{1}{2}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du \Big|_{u=1+x^2}$$

$$= \frac{1}{2}\left(\frac{u^{\frac{5}{2}}}{(5/2)} - \frac{u^{3/2}}{(3/2)}\right)\Big|_{u=1+x^2} + C$$

$$= \boxed{\begin{array}{c} \dfrac{(1+x^2)^{\frac{5}{2}}}{5} - \dfrac{(1+x^2)^{3/2}}{3} + C \\ \text{over } I = (-\infty, \infty) \end{array}}$$

Ex: Compute the following definite integrals.

1. $\int_1^2 \frac{1}{(1+\sqrt{x})^4}\, dx$

Sol: We compute

$$\int_1^2 \frac{1}{(1+\sqrt{x})^4}\, dx \;\underset{u = 1+\sqrt{x}}{=\!=\!=}\; \int_2^{1+\sqrt{2}} \frac{1}{u^4}\, 2(u-1)\, du$$

$du = \frac{1}{2\sqrt{x}}\, dx$

$\Rightarrow 2\sqrt{x}\, du = dx$

$\Rightarrow 2(u-1)\, du = dx$

$x=1 \Rightarrow u=2$

$x=2 \Rightarrow u=1+\sqrt{2}$

$$= 2\int_2^{1+\sqrt{2}} u^{-3} - u^{-4}\, du$$

$$= 2\left(\frac{u^{-2}}{-2} - \frac{u^{-3}}{(-3)} \Bigg|_{u=2}^{1+\sqrt{2}} \right)$$

$$= \boxed{2\left(\frac{(1+\sqrt{2})^{-2}}{(-2)} - \frac{(2)^{-3}}{(-3)}\right)}$$

2. $\int_{-1}^{1} x^4 \sin x \, dx$

Sol: We compute

$$\int_{-1}^{1} x^4 \sin x \, dx = \int_{-1}^{0} x^4 \sin x \, dx + \int_{0}^{1} x^4 \sin x \, dx$$

$u = -x$
$\Rightarrow x = -u$
$du = -dx$
$\Rightarrow dx = -du$
$x = -1 \Rightarrow u = 1$
$x = 0 \Rightarrow u = 0$

$$= \int_{1}^{0} (-u)^4 \sin(-u)(-1)\,du + \int_{0}^{1} x^4 \sin x \, dx$$

Fact: sin(-x)=-sin(x) and cos(-x)=cos(x) for all x.

$$= \int_{1}^{0} u^4 (-\sin u)(-1)\,du + \int_{0}^{1} x^4 \sin x \, dx$$

$$= \int_{1}^{0} u^4 \sin u \, du + \int_{0}^{1} x^4 \sin x \, dx$$

$$= -\int_0^1 u^4 \sin u \, du + \int_0^1 x^4 \sin x \, dx$$

$$= 0 \qquad \Rightarrow \boxed{\int_{-1}^{1} x^4 \sin x \, dx = 0}$$

Fact: Suppose a>0, and suppose f is continuous over [-a,a] with f(-x)=-f(x) for all x in [-a,a], then

"odd"

$$\int_{-a}^{a} f(x) \, dx = 0.$$