VECTOR CALCULUS, Week 1

1.2 A Catalog of Essential Functions; 1.3 The Limit of a Function; 1.4 Calculating Limits; 1.5 Continuity; 1.6 Limits Involving Infinity

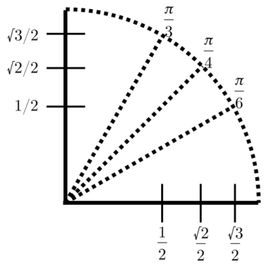
1.2 A Catalog of Essential Functions

Def: We will use the following **basic functions** and their graphs. The domain is $\mathbf{R}$, unless otherwise specified.

- constant functions $f(x) = c$
- basic power functions
  - natural-power functions $f(x) = x^n$ for $n = 1, 2, 3, \ldots$
    $\Rightarrow$ identity function $f(x) = x$
    $\Rightarrow$ square function $f(x) = x^2$
    $\Rightarrow$ cubic function $f(x) = x^3$
  - nth root functions $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ for $n = 2, 3, 4, \ldots$
    with domain $\begin{cases} [0, \infty) & \text{if } n \text{ is even} \\ \mathbf{R} & \text{if } n \text{ is odd.} \end{cases}$
    $\Rightarrow$ square root function $f(x) = \sqrt{x}$
    $\Rightarrow$ cube root function $f(x) = x^{\frac{1}{3}}$.
  - reciprocal functions $f(x) = x^{-n} = \frac{1}{x^n}$ for $n = 1, 2, 3, \ldots$
    with domain $x \neq 0$
    $\Rightarrow$ inverse/reciprocal function $f(x) = \frac{1}{x}$
    $\Rightarrow$ inverse square function $f(x) = \frac{1}{x^2}$
- absolute value function $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
- trigonometric functions
  $\Rightarrow$ cosine $f(x) = \cos x$
  $\Rightarrow$ sine $f(x) = \sin x$
  $\Rightarrow$ tangent $f(x) = \tan x = \frac{\sin x}{\cos x}$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$
  $\Rightarrow$ arctangent/inverse tangent $f(x) = \arctan x = \tan^{-1} x$
- exponential function $f(x) = e^x$
- natural logarithm $f(x) = \ln x$ with domain $(0, \infty)$.

Def: Consider the unit circle, and take an angle $\theta \in \mathbf{R}$. Draw a unit line segment ℓ from the origin to the unit circle, such that ℓ makes angle θ counterclockwise from the positive x-axis; note that if θ is negative, we make angle $|\theta|$ clockwise from the positive x-axis. If (x, y) is the point on the unit circle at the end of ℓ, then we define $\cos\theta = x$ and $\sin\theta = y$.

Use the **unit circle graph** to remember the values of cosine and sine at the **basic angles** $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \ldots$



Fact: Basic trigonometric formulas:

- $\sin^2\theta + \cos^2\theta = 1$
- $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$
- $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$
- $\cos(-\theta) = \cos\theta$
- $\sin(-\theta) = -\sin\theta$

Def: We define the following additional trigonometric functions.

- $\cot\theta = \frac{\cos\theta}{\sin\theta}$ for $0 < \theta < \pi$.
- $\sec\theta = \frac{1}{\cos\theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.
- $\csc\theta = \frac{1}{\sin\theta}$ for $0 < \theta < \pi$ and $\pi < \theta < 2\pi$.

These are not basic functions, so you don't need to know their graphs.

Facts: Basic properties of the exponential and natural logarithm.

- $\ln e^x = x$ for all x, while $e^{\ln x} = x$ for $x > 0$.
- $e^{a+b} = e^a e^b$ and $(e^a)^b = e^{ab}$.
- $\ln(ab) = \ln(a) + \ln(b)$ and $\ln(a^b) = b\ln(a)$.

1.3 The Limit of a Function

"Def": Suppose $a \in \mathbf{R}$, and suppose f is defined on an interval containing a, but perhaps not at a.

- If there is $L \in \mathbf{R}$ so that *as x approaches a, the value $f(x)$ approaches L*, then we say the **limit of f at a exists and is L**, and write $\lim_{x\to a} f(x) = L$.
- If the value $f(x)$ does not approach any fixed $L \in \mathbf{R}$ as x approaches a, then we say $\lim_{x\to a} f(x)$ **does not exist**, or **DNE**.

Def: Suppose $a \in \mathbf{R}$. **Near** a means on an interval containing a.

Ex: Use the graph of $f(x) = x^{-2}$ to show $\lim_{x\to 2} f(x) = \frac{1}{4}$.

Fact: Suppose f is a basic function. If f is defined near $a \in \mathbf{R}$ and including at a, then $\lim_{x\to a} f(x) = f(a)$.

"Def": We define **one-sided limits**. Suppose $a \in \mathbf{R}$

- Suppose f is defined on an interval (a, b), where $b > a$. If there is $L \in \mathbf{R}$ so that *as $x > a$ approaches a, the value $f(x)$ approaches L*, then we say the **right-hand limit of f at a exists and is L**, and write $\lim_{x\to a^+} f(x) = L$.
- Suppose f is defined on an interval (b, a), where $b < a$. If there is $L \in \mathbf{R}$ so that *as $x < a$ approaches a, the value $f(x)$ approaches L*, then we say the **left-hand limit of f at a exists and is L**, and write $\lim_{x\to a^-} f(x) = L$.

We otherwise say $\lim_{x\to a^+} f(x)$ DNE or $\lim_{x\to a^-} f(x)$ DNE.

Fact: All appropriate left- and right-hand limits of basic functions exist at each point of their domains, with limit being the value of the function at the point.

Ex: One-sided limits.

1. Compute $\lim_{x\to 0^+} \sqrt{x}$.
2. $\lim_{x\to 0^-} \sqrt{x}$ DNE.

1.4 Calculating Limits

Thm (Basic Limit Laws): Suppose $a, b, c, L, M \in \mathbf{R}$.

1. **Simplification Rule**: Suppose $f(x) = g(x)$ for all x near a, but perhaps not at a. If $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(x) = L$.

2. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

   Addition Rule: $\lim_{x\to a} f(x) + g(x) = L + M$.

   Multiplication Rule: $\lim_{x\to a} f(x)g(x) = LM$.
   In particular, $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$.

   Division Rule: If $M \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

3. u**-Substitution Rule**:
   If $\lim_{x\to a} g(x) = b$ and $\lim_{u\to b} f(u) = L$, then

$$\lim_{x\to a} f(g(x)) = \lim_{u\to b} f(u) = L.$$

Similar rules hold for one-sided limits.

Ex: Compute the following limits.

1. $\lim_{x\to 0} \sqrt{x+1}$

2. $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$; multiply by the **conjugate**.

Def: Suppose $a, L \in \mathbf{R}$, and suppose that as x approaches a, the values $f(x)$ approach L with $f(x) > L$. Then we write $\lim_{x\to a} f(x) = L^+$.

We similarly define $\lim_{x\to a, a^\pm} f(x) = L^\pm$ (six combinations).

Ex: Compute $\lim_{x\to 0^+} \sqrt{e^x - 1}$.

1.5 Continuity

Def: Suppose $a, b \in \mathbf{R}$.

- We say f **is continuous at** a if and only if $\lim_{x\to a} f(x) = f(a)$.
- We say f is **left-continuous at** a if and only if $\lim_{x\to a^-} f(x) = f(a)$.
- We say f is **right-continuous at** A if and only if $\lim_{x\to a^+} f(x) = f(a)$.
- We say f **is continuous over** $[a, b]$ if f is continuous at every $x \in (a, b)$, f is right-continuous at a, and f is left-continuous at b.

We similarly define f **is continuous over**

$$(a, b), [a, b), (a, b], (a, \infty), [a, \infty), (-\infty, b), (-\infty, b], \text{ or } (-\infty, \infty).$$

Fact: If f is a basic function, then f is appropriately continuous, left-continuous, or right-continuous at every point in its domain.

1.6 Limits Involving Infinity

"Def": Suppose $a, L \in \mathbf{R}$.

- Suppose f is defined near a, but perhaps not at a. If as x approaches a the values of $f(x)$ get larger in the positive direction, then we say **the limit of f at a exists in the extended sense and is infinity**, and write $\lim_{x\to a} f(x) = \infty$.
  - We similarly define $\lim_{x\to a, a^{\pm}} f(x) = \pm\infty$ (six combinations) **in the extended sense**.
  - If $\lim_{x\to a, a^{\pm}} f(x) = \pm\infty$, then we say f has a **vertical asymptote** at $x = a$.
- Suppose f is defined over (a, ∞). If as x gets larger in the positive direction the values of $f(x)$ get closer to L, then we say **the limit of f at ∞ exists and is L**, and write $\lim_{x\to\infty} f(x) = L$.
  - We similarly define $\lim_{x\to-\infty} f(x) = L$.
  - If either $\lim_{x\to\pm\infty} f(x) = L$, then we say f **has horizontal asymptote** $y = L$.
- Suppose f is defined on (a, ∞). If as x gets larger in the positive direction the values of $f(x)$ get larger in the positive direction, then we say **the limit of f at ∞ exists in the extended sense and is ∞**, and write $\lim_{x\to\infty} f(x) = \infty$.
  - We similarly define $\lim_{x\to\pm\infty} f(x) = \pm\infty$ (four combinations).
  - If $\lim_{x\to\pm\infty} f(x) = \pm\infty$, then we appropriately say f **blows up/down at** $\pm\infty$.

Fact: We can use the graphs of the basic functions to compute their vertical asymptotes, horizontal asymptotes, blow-up, and blow-down behavior.

Ex: Determine the existence of the following limits in the extended sense.

1. $\lim_{x\to 0,0^\pm} \frac{1}{x}$
2. $\lim_{x\to\pm\infty} \arctan x$
3. $\lim_{\theta\to\frac{\pi}{2}^-} \tan\theta$ and $\lim_{\theta\to-\frac{\pi}{2}^+} \tan\theta$
4. $\lim_{x\to\infty} \sin x$

Ex: Consider $\lim_{x\to\infty}(x^2 - x)$.