Vector Calculus

10.1 Three-Dimensional Coordinate Systems; 10.2 Vectors; 10.3 The Dot Product; 10.4 The Cross Product

Def: We use the following notation.

$\cancel{x_1 x_2 x_3}$

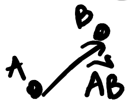
- We use x, y, z-coordinates in $\mathbf{R}^3$, or space.
- Given a point P in space, $P(x, y, z)$ means that P has coordinates (x, y, z).

  $P = (1,2,3) \longrightarrow P(1,2,3)$
- Given two points P_1, P_2 in space, we let $|P_1P_2|$ denote the distance between P_1, P_2.
- The book uses bold letters to denote vectors. We will continue to use arrows, such as $\vec{v}$.

  $\vec{0}$ will denote the zero vector. $\vec{0}$
- If $\vec{v} \in \mathbf{R}^2$ has components v_1, v_2, then we write $\vec{v} =< v_1, v_2 >$. Similarly for $\vec{v} \in \mathbf{R}^3$, we write $\vec{v} =< v_1, v_2, v_3 >$.

  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \langle v_1, v_2 \rangle$

  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \langle v_1, v_2, v_3 \rangle$
- Given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, we denote the vector



$$\vec{AB} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 > .$$

$$= \vec{A} - \vec{B}$$

- If $\vec{v} \in \mathbf{R}^2$ or $\vec{v} \in \mathbf{R}^3$, then we denote the length of $\vec{v}$ by $|\vec{v}|$.

  $\|\vec{v}\| = |\vec{v}|$
- The **standard basis vectors** in $\mathbf{R}^3$ shall be denoted by

$$\vec{i} =< 1, 0, 0 > \quad \vec{j} =< 0, 1, 0 > \quad \vec{k} =< 0, 0, 1 >$$

$$\vec{e}_1 \qquad\qquad \vec{e}_2 \qquad\qquad \vec{e}_3$$

Def: The <u>dot product</u> between $\vec{v},\vec{w}$ in R^3 is

$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3$$

We similarly define the dot product between $\vec{v},\vec{w}$ in R^2.

Thm: If theta is the angle between $\vec{v},\vec{w}$ (in R^2 or R^3), then

$$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta$$

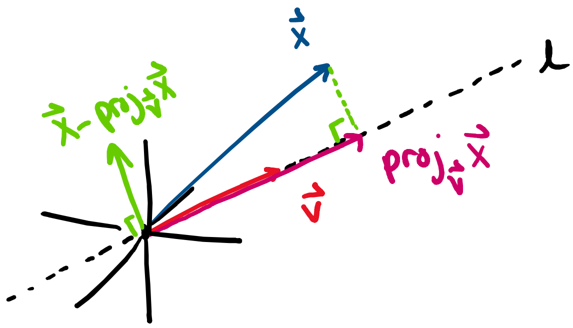
In particular, $\vec{v}\perp\vec{w}$ if and only if $\vec{v}\cdot\vec{w}=0$.

Def: Suppose $\vec{v}$ is a nonzero vector in R^2 or R^3. The <u>vector/orthogonal projection of $\vec{x}$ onto $\vec{v}$</u> is

$$\mathrm{proj}_{\vec{v}}\vec{x} = \frac{\vec{x}\cdot\vec{v}}{|\vec{v}|}\,\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{x}\cdot\vec{v}}{|\vec{v}|^2}\,\vec{v}$$

Thm: $\vec{p}=\mathrm{proj}_{\vec{v}}\vec{x}$ is the unique vector so that

$\vec{p},\vec{v}$ are co-linear
$(\vec{x}-\vec{p})\perp\vec{v}$



Def: Suppose $\vec{v},\vec{w}$ are in R^3. We define the <u>cross product</u> between $\vec{v},\vec{w}$ *in that order* to be the vector given by computing the symbolic determinant

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

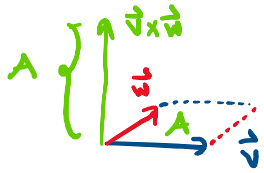
Ex: $\begin{bmatrix}2\\3\\4\end{bmatrix}\times\begin{bmatrix}5\\6\\7\end{bmatrix} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & 4\\ 5 & 6 & 7\end{vmatrix}$

$$= \begin{bmatrix}\begin{vmatrix}3 & 4\\6 & 7\end{vmatrix}\\ -\begin{vmatrix}2 & 4\\5 & 7\end{vmatrix}\\ \begin{vmatrix}2 & 3\\5 & 6\end{vmatrix}\end{bmatrix} = \begin{bmatrix}21-24\\-(14-20)\\12-15\end{bmatrix} = \boxed{\begin{bmatrix}-3\\6\\-3\end{bmatrix}}$$

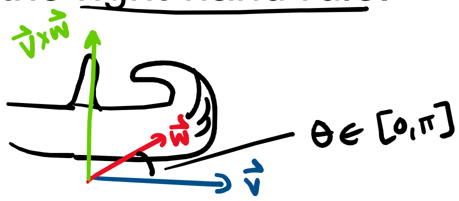
Thm: $\vec{v}\times\vec{w}$ is the unique vector so that

$\vec{v}\times\vec{w}\perp\vec{v},\vec{w}$ $\quad \begin{bmatrix}-3\\6\\-3\end{bmatrix}\cdot\begin{bmatrix}2\\3\\4\end{bmatrix} = -6+18-12=0\checkmark \quad \begin{bmatrix}-3\\6\\-3\end{bmatrix}\cdot\begin{bmatrix}5\\6\\7\end{bmatrix} = -15+36-21 = -36+36=0\checkmark$

$|\vec{v}\times\vec{w}|$ is the area of the parallelogram formed by $\vec{v},\vec{w}$.

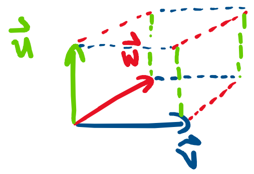


$\vec{v},\vec{w},\vec{v}\times\vec{w}$ in that order satisfy the right-hand rule.



Fact: Suppose $\vec{u},\vec{v},\vec{w}$ are in R^3.

$\vec{w}\times\vec{v}=-(\vec{v}\times\vec{w})$ (anti-commutative law)
$|\vec{u}\cdot(\vec{v}\times\vec{w})|$ is the volume of the parallelpiped formed by $\vec{u},\vec{v},\vec{w}$.


$\vec{u}$
$\vec{w}$
$\vec{v}$

10.5 Equations of Lines and Planes

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Fact: The line ell in space through P(x_0,y_0,z_0) in the direction of $\vec{v}$=<a,b,c>=/$\vec{0}$ is given by the point-direction parameterization

$$\ell(t) = \langle x_0, y_0, z_0\rangle + t\langle a,b,c\rangle \quad \text{for } t\in\mathbb{R}$$

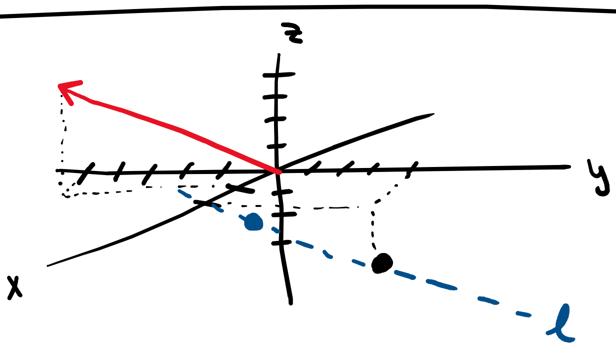
Ex: Suppose ell is the line through A(2,4,-3),B(3,-1,1). Give a point-direction parameterization for ell. At what point does ell pass through the horizontal xy-plane?

Sol: ell is the line in the direction of

$$\vec{AB} = \langle 3-2, -1-4, 1-(-3)\rangle = \langle 1,-5,4\rangle$$

$$\Rightarrow \boxed{\ell(t) = \langle 2,4,-3\rangle + t\underbrace{\langle 1,-5,4\rangle}_{=\vec{v}} \quad \text{for } t\in\mathbb{R}}$$

$\neq 0$



To see where ell crosses the xy-plane, we set

$$-3+4t=0 \Rightarrow t=\frac{3}{4}$$

$$\Rightarrow \quad \ell\left(\frac{3}{4}\right) = \langle 2,4,-3\rangle + \frac{3}{4}\langle 1,-5,4\rangle$$

$$= \langle 2+\tfrac{3}{4}, 4-\tfrac{15}{4}, 0\rangle$$

ell crosses the xy-plane at $\boxed{P(2+3/4,4-15/4,0).}$

Ex: Consider the lines given by point-direction parameterizations

$$\ell_1(t) = \langle 1,-2,4\rangle + t\langle 1,3,-1\rangle \text{ for } t\in\mathbb{R}$$

$$\ell_2(t) = \langle 0,3,-3\rangle + s\langle 2,1,4\rangle \text{ for } s\in\mathbb{R}$$

Show that ell$_1$,ell$_2$ are <u>skew</u>: not parallel and do not intersect.

Sol: Since the direction vectors of ell$_1$ and ell$_2$ are <1,3,-1> and <2,1,4> respectively, which are not co-linear, then ell$_1$ and ell$_2$ are not parallel.

Now we check that ell$_1$ and ell$_2$ do not intersect. The two lines intersect if and only if there are t,s in R so that

$$\langle 1,-2,4\rangle + t\langle 1,3,-1\rangle = \langle 0,3,-3\rangle + s\langle 2,1,4\rangle$$

$$\Leftrightarrow \quad \langle 1,-5,7\rangle = t\langle -1,-3,1\rangle + s\langle 2,1,4\rangle$$

Consider

$$\langle -1,-3,1\rangle \times \langle 2,1,4\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -3 & 1 \\ 2 & 1 & 4 \end{vmatrix} = \langle -13,6,5\rangle$$

If ell$_1$,ell$_2$ intersect, then there are t,s in R so that

$$\langle -13,6,5\rangle \cdot \langle 1,-5,7\rangle = \langle -13,6,5\rangle \cdot \left(t\langle -1,-3,1\rangle + s\langle 2,1,4\rangle\right)$$

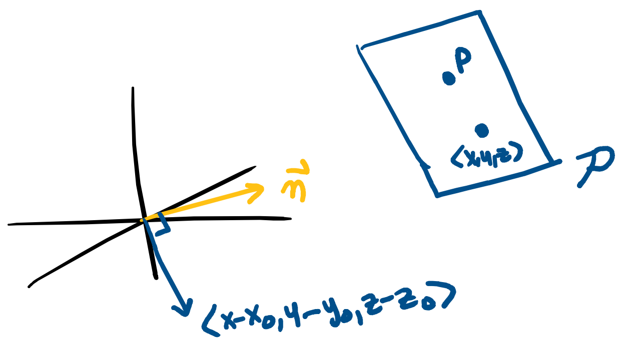
$$\underbrace{-13-30+35}_{\neq 0} \qquad = 0$$

This means there is no t,s in R so that $ell_1(t)=ell_2(s)$, and so the two lines do not intersect.

We conclude that ell_1, ell_2 are skew.

Fact: To describe a plane in space, we need a point in the plane and a normal direction. If $P(x_0,y_0,z_0)$ is a point in space and $\vec{n}=\langle a,b,c\rangle \neq \vec{0}$, then the plane $\mathcal{P}$ through P with normal in the direction of $\vec{n}$ is given by the implicit/vector equation

$$\mathcal{P}: \vec{n}\cdot\left(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle\right)=0$$



We also say that $\mathcal{P}$ is given by the scalar equation

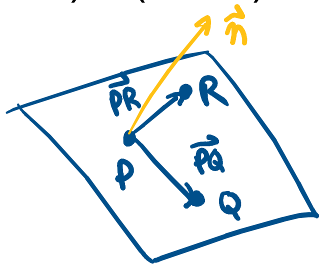
$$\mathcal{P}: a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

or by the linear equation

$$\mathcal{P}: \; ax+by+cz+\underset{\substack{\shortparallel \\ -ax_0-by_0-cz_0}}{d}=0$$

Ex: Find a linear equation for the plane $\mathcal{P}$ through the points P(1,3,2),Q(3,-1,6),R(5,2,0).

Sol: We compute



$$\Rightarrow \quad \vec{n} = \vec{PR} \times \vec{PQ}$$

$$\vec{PR}\times\vec{PQ} = \langle 4,-1,-2\rangle \times \langle 2,-4,4\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & -2 \\ 2 & -4 & 4 \end{vmatrix}$$

$$= \langle -12, -(20), -14\rangle = \langle -12,-20,-14\rangle$$

This gives

$$\mathcal{P}: \langle -12,-20,-14\rangle \cdot \left(\langle x,y,z\rangle - \langle 1,3,2\rangle\right)$$

$$\Rightarrow \boxed{\mathcal{P}: \; -12x-20y-14z+(12+60+28)=0}$$

Ex: Find a point-direction parameterization for the line ell of intersection between the planes given by the linear equations

$\mathcal{P}_1: x+y+z-1=0$ and $\mathcal{P}_2: x-2y+3z-1=0$

$\vec{n}_1=\langle 1,1,1\rangle$ $\vec{n}_2=\langle 1,-2,3\rangle$

Sol: A direction vector for ell is given by

$$\vec{v}=\vec{n}_1\times\vec{n}_2=\langle 1,1,1\rangle\times\langle 1,-2,3\rangle$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 1 & 1\\ 1 & -2 & 3\end{vmatrix}=\langle 5,-(2),-3\rangle$$

$$=\langle 5,-2,-3\rangle$$

Since $\vec{v}$=<5,-2,-3>, then ell crosses the horizontal xy-plane somewhere. $\overset{\#}{0}$

This means we can look for a point P=<p_1,p_2,0> which is on both planes $\mathcal{P}_1,\mathcal{P}_2$. This gives

$$\mathcal{P}_1 \Rightarrow \quad p_1+p_2+0-1=0 \qquad p_1+p_2=1$$

$$\Rightarrow$$

$$\mathcal{P}_2 \Rightarrow \quad p_1-2p_2+0-1=0 \qquad p_1-2p_2=1$$

$$\Rightarrow \begin{bmatrix}p_1\\p_2\end{bmatrix}=\begin{bmatrix}1 & 1\\1 & -2\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{-3}\begin{bmatrix}-2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}p_1\\p_2\end{bmatrix}=-\frac{1}{3}\begin{bmatrix}-3\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow P(1,0,0)$$

We conclude that

$$\boxed{\ell(t)=\langle 1,0,0\rangle+t\langle 5,-2,-3\rangle \text{ for } t\in\mathbb{R}}$$

10.6 Cylinders and Quadric Surfaces

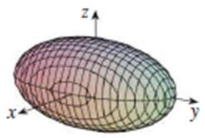
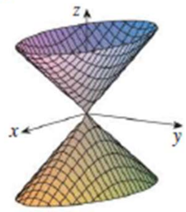
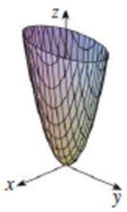
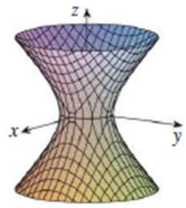
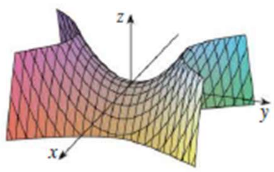
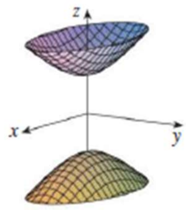
Def: A quadric surface is a surface in space given by a second-degree equation in x,y,z of the form

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz$$
$$+Gx+Hy+Iz+J=0$$

Ex: Some familiar quadric surfaces.

1. The unit cylinder centered around the z-axis, given by x^2+y^2-1=0.
2. The unit sphere centered at the origin, given by x^2+y^2+z^2-1=0.

We as well have cylinders and spheres of different radii centered elsewhere.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Fact: Consider a quadric surface S given by the equation

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0.$$

To determine the type of S, and to sketch S, it is best to use the level set method: set a variable equal to a constant, and sketch the resulting curve. For example, set z=k and sketch the curve in the plane z=k given by the equation

$$Ax^2+Ay^2+Dxy+Eky+Fkx+Gx+Hy+Ck^2+Ik+J=0$$

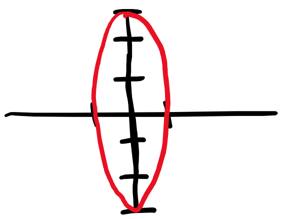
This curve is the intersection between S and the plane z=k.

Ex: Use the level set method to determine the type and sketch the quadric surfaces given by the following equations.

1. $x^2+\frac{y^2}{9}+\frac{z^2}{4}-1=0$

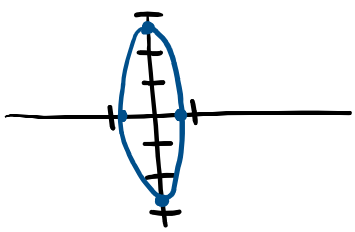
Sol: We set z=0,±1,±2.

$z=0 \Rightarrow x^2+\frac{y^2}{9}=1 \Rightarrow$



$z=\pm 1 \Rightarrow x^2+\frac{y^2}{9}=1-\frac{1}{4}$

$\Rightarrow x^2+\frac{y^2}{9}=\frac{3}{4}$



$y=0 \Rightarrow x=\pm\sqrt{\frac{3}{4}}$

$x=0 \Rightarrow y^2=\frac{27}{4}$

$\Rightarrow y=\pm\sqrt{\frac{27}{4}}$

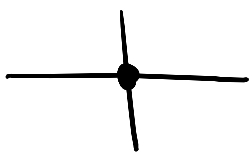
$2<\sqrt{\frac{27}{4}}<3$

$4<\frac{27}{4}<9$

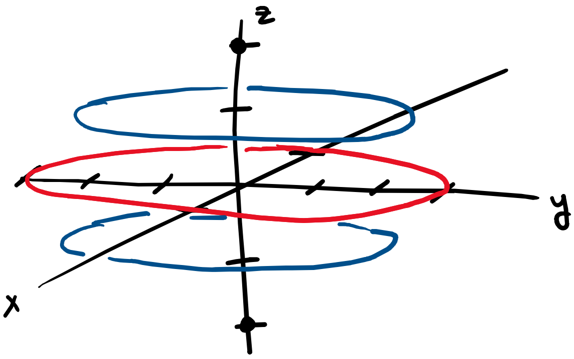
$16<27<36$ ✓

$z=\pm 2 \Rightarrow x^2+\frac{y^2}{9}+1-1=0$

$\Rightarrow x=y=0$



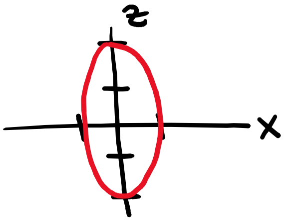
So far, our surface looks like



$$x^2+\frac{y^2}{9}+\frac{z^2}{4}-1=0$$

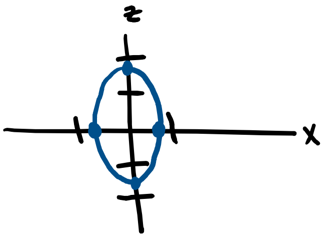
Let's set y=0,±1

$\underline{y=0} \Rightarrow x^2+\frac{z^2}{4}=1$



$y=\pm 1 \Rightarrow x^2+\frac{z^2}{4}=1-\frac{1}{9}$

$\Rightarrow x^2+\frac{z^2}{4}=\frac{8}{9}$

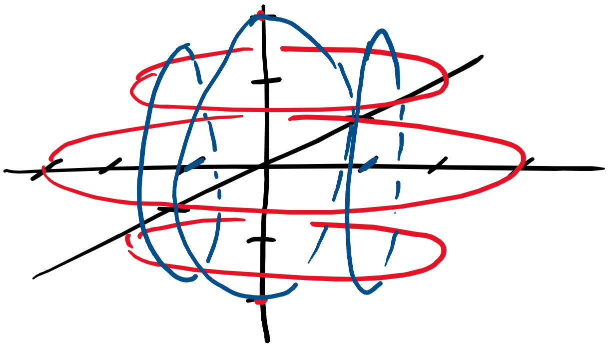


$x=\pm\sqrt{\frac{8}{9}}$

$z=\pm\sqrt{\frac{32}{9}}$

$1<\sqrt{\frac{32}{9}}<2$

We conclude that the surface is the ellipsoid



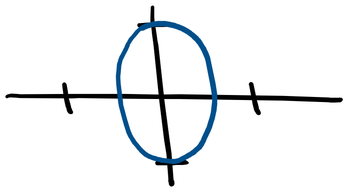
2. 4x^2+y^2-z=0 $\Rightarrow 4x^2+y^2=z$

Sol: First, we set z=0,1,4

$z=0 \Rightarrow x=y=0$

$z=1 \Rightarrow 4x^2+y^2=1$

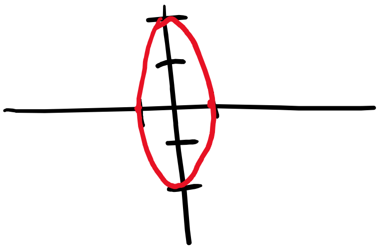
$\Rightarrow \frac{x^2}{1/4}+y^2=1$



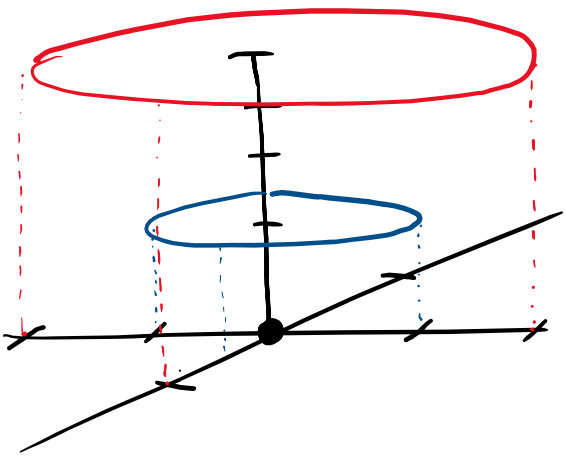
$x=\pm\sqrt{\frac{1}{4}}=\pm\frac{1}{2}$

$z=4 \Rightarrow 4x^2+y^2=4$

$\Rightarrow x^2+\frac{y^2}{4}=1$



So far, we have

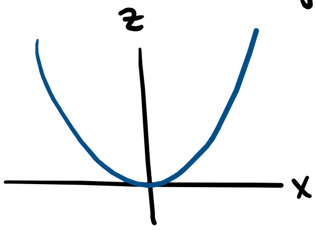


Now, let's set y=0,±1

$4x^2+y^2-z=0$

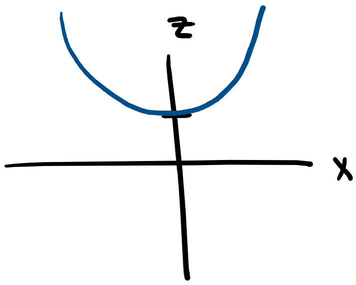
$y=0 \Rightarrow 4x^2-z=0$

$\Rightarrow z=4x^2$



$y = \pm 1 \Rightarrow 4x^2 + 1 - z = 0$

$\Rightarrow z = 4x^2 + 1$



The surface is the elliptic paraboloid