Vector Calculus
10.8 Arc Length and Curvature; 10.9 Motion in Space: Velocity and Acceleration

Def: Suppose $\vec{r}=\vec{r}(t)$ is a parametric vector-valued function. Physicists call $\vec{r}(t)$ the position vector, $\vec{v}(t)=\vec{r}'(t)$ the velocity vector, $|\vec{v}(t)|=|\vec{r}'(t)|$ the speed, and $\vec{a}(t)=\vec{v}'(t)=\vec{r}''(t)$ the acceleration vector.

Ex: Compute the position vector, velocity vector, speed, and acceleration vector of $\vec{r}(t)=<3t,-t^2,t^3>$ at t=2.

Sol: We compute

position

$$\vec{r}(2)=\boxed{\langle 6,-4,8\rangle}$$

Velocity

$$\vec{v}(t)=\langle 3,-2t,3t^2\rangle\Big|_{t=2}=\boxed{\langle 3,-4,12\rangle}$$

$$\vec{a}(t)=\langle 0,-2,6t\rangle\Big|_{t=2}=\boxed{\langle 0,-2,12\rangle}$$

acceleration

$$|\vec{v}(t)|=\boxed{\sqrt{9+16+144}}$$

speed

$\vec{r}'(t)\neq\vec{0}$

Def: Suppose $\vec{r}=\vec{r}(t)$ is a regular parametric space curve defined for t near a, and suppose $\vec{r}''(t)\neq\vec{0}$ exists for t near a.

→ osculating circle exists

We say $\vec{T}(a)=\dfrac{\vec{r}'(a)}{|\vec{r}'(a)|}$ is the unit tangent vector of $\vec{r}$ at t=a.

We say $\vec{N}(a)= \dfrac{\frac{\partial}{\partial t}T(t)\big|_{t=a}}{\left|\frac{\partial}{\partial t}T(t)\big|_{t=a}\right|}$ is the unit normal vector of $\vec{r}$ at t=a.

We say $\vec{B}(a)=\vec{T}(a)\times\vec{N}(a)$ is the binormal vector of $\vec{r}$ at t=a.

We say $\vec{T}(a),\vec{N}(a),\vec{B}(a)$ are the TNB frame or the orthonormal frame of $\vec{r}$ at t=a.

Ex: Compute the TNB frame of $\vec{r}(t)=\langle\cos(t),\sin(t),t\rangle$ at all t.

Sol: We compute

$$\vec{r}\,'(t)=\langle -\sin t, \cos t, 1\rangle$$

$$|\vec{r}\,'(t)|=\sqrt{\sin^2 t+\cos^2 t+1}=\sqrt{2}$$

$$\Rightarrow \quad \vec{T}(t)=\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}=\left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

For $\vec{N}(t)$, we compute

$$\frac{\partial}{\partial t}\vec{T}(t)=\left\langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0\right\rangle$$

$$\left|\frac{\partial}{\partial t}\vec{T}(t)\right|=\sqrt{\frac{\cos^2 t}{2}+\frac{\sin^2 t}{2}}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \vec{N}(t)=\frac{\frac{\partial}{\partial t}\vec{T}(t)}{\left|\frac{\partial}{\partial t}\vec{T}(t)\right|}=\frac{\left\langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0\right\rangle}{\frac{1}{\sqrt{2}}}=\langle -\cos t, -\sin t, 0\rangle$$

Next, we compute

$$\vec{B}(t)=\vec{T}(t)\times\vec{N}(t)=\left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle \times \langle -\cos t, -\sin t, 0\rangle$$

$$= \left\langle \frac{\sin t}{\sqrt{2}}, \frac{-\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

We conclude that

$$\boxed{\begin{aligned} \vec{T}(t) &= \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\ \vec{N}(t) &= \langle -\cos t, -\sin t, 0 \rangle \\ \vec{B}(t) &= \left\langle \frac{\sin t}{\sqrt{2}}, \frac{-\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \end{aligned}}$$

Fact: Suppose $\vec{r}=\vec{r}(t)$ is a regular parametric space curve defined for t near a, and suppose $\vec{r}''(t) \neq \vec{0}$ exists for t near a.

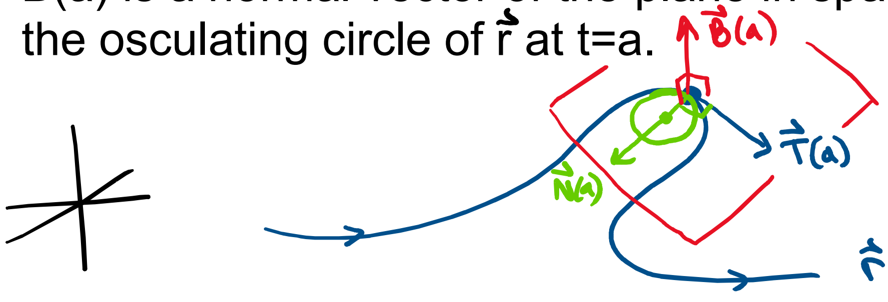
$\vec{T}(a), \vec{N}(a), \vec{B}(a)$ are <u>orthonormal</u>: $|\vec{T}(a)|=|\vec{N}(a)|=|\vec{B}(a)|=1$ and

$$\vec{T}(a) \perp \vec{N}(a), \underbrace{\vec{T}(a) \perp \vec{B}(a), \vec{N}(a) \perp \vec{B}(a)}_{\vec{B}=\vec{T}\times\vec{N}}.$$

$\vec{T}(a)$ is tangent to the image of $\vec{r}$ at t=a.

$\vec{N}(a)$ points towards the center of the osculating circle of $\vec{r}$ at t=a. It does not necessarily point exactly *at* the center, but it does point towards the center.

$\vec{B}(a)$ is a normal vector of the plane in space containing the osculating circle of $\vec{r}$ at t=a.



11.1 Functions of Several Variables

Def: A real-valued function of two variables is a function f:R^2->R of the variables x,y, and denoted f=f(x,y). The graph of f is the surface in space given by the equation z=f(x,y), which is the set

$$\{(x,y,f(x,y)) : x,y \in \mathbb{R}\}$$

Ex: Some examples.

1. Linear functions f(x,y)=a+bx+cy, with graph the non-vertical plane through (0,0,a) with normal in the direction of n=<b,c,-1>.

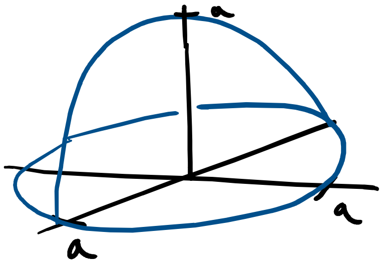
Because, the graph is the surface given by the equation

$$z = a+bx+cy \Rightarrow bx+cy-z+a=0$$
$$\Rightarrow b(x-0)+c(y-0)-(z-a)=0$$
$\Rightarrow$ plane through $(0,0,a)$ with normal in the direction of
$$\vec{n} = \langle b,c,-1\rangle.$$

2. $f(x,y)=\sqrt{a^2-x^2-y^2}$ for a>0, with graph the upper hemisphere centered at the origin with radius =a.

Because the graph of f is the surface given by the equation

$$z=\sqrt{a^2-x^2-y^2} \Rightarrow x^2+y^2+z^2=a^2,\ z\geq 0$$
with $x^2+y^2 \leq a^2$
$$\Downarrow$$

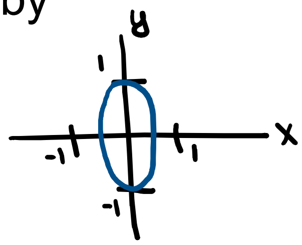


Def: Suppose f=f(x,y) is a real-valued function. The <u>level curve of f at k in R</u> is the curve in the plane z=k given by the equation f(x,y)=k.

Ex: Sketch the level curves of $f(x,y)=4x^2+y^2$ at k=1,4.

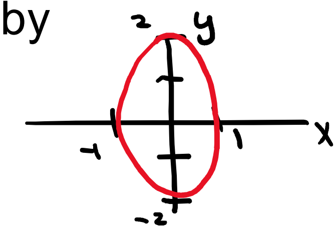
Sol: First, the level curve of f at k=1 is given by

$$4x^2+y^2=1 \Rightarrow \frac{x^2}{1/4}+y^2=1$$

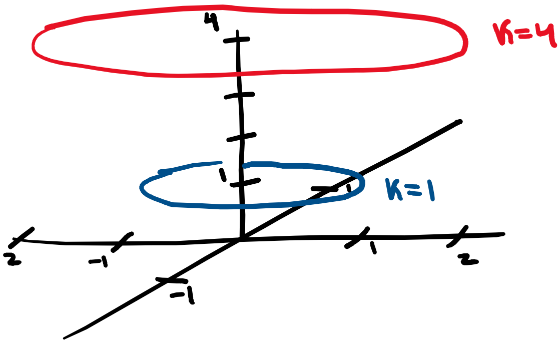


Second, the level curve of f at k=4 is given by

$$4x^2+y^2=4 \Rightarrow x^2+\frac{y^2}{4}=1$$



We conclude that the level curves of f at k=1,4 are



In fact, the graph of f is the quadric surface given by the equation

$$z = 4x^2 + y^2 \qquad \text{elliptic paraboloid.}$$

Def: A real-valued function of three variables is a function f:R^3->R of the variables x,y,z, and denoted f=f(x,y,z). The graph of f is the hypersurface in R^4 given by the set

$$\{ (x,y,z, f(x,y,z)) : x,y,z \in \mathbb{R} \}$$

The level surface of f at k in R is the surface in space given by the equation f(x,y,z)=k.

Ex: Identify the level surfaces of f(x,y,z)=$x^2+y^2+z^2$ at k=1,4.

Sol: We have

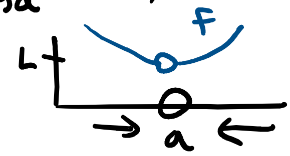
$x^2+y^2+z^2=1 \Rightarrow$ unit sphere centered at the origin.

$x^2+y^2+z^2=4 \Rightarrow$ sphere of radius $=2$ centered at the origin.

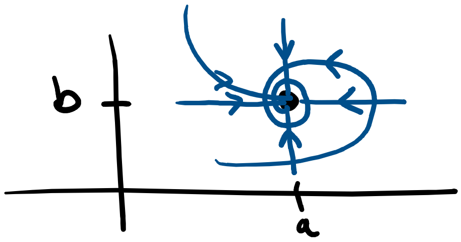
11.2 Limits and Continuity

Fact: Suppose f=f(x) is defined near a, but perhaps not at a. The limit of f at a exists and is L if and only if the left- and right-hand limits of f exist at a and are both L.

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

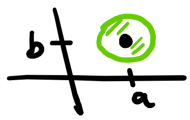


For functions of one variable, you only have to check two directions: from the left and from the right. However, to show the limit of f=f(x,y) at (a,b) exists, we must consider many direction!

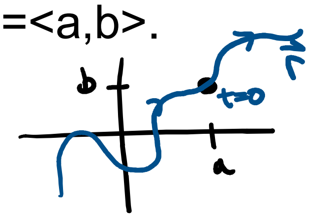


Def: Suppose f=f(x,y) is a real-valued function defined near (a,b), but perhaps not at (a,b), and suppose L is in R.

inside a circle centered at (a,b)



We say $\lim_{(x,y)\to(a,b)} f(x,y)=L$ if and only if $\lim_{t\to 0} f(\vec{r}(t))=L$ for all continuous parametric plane curves $\vec{r}=\vec{r}(t)$ defined near t=0 with $\vec{r}(0)=<a,b>$.



$$\lim_{t\to 0} \underbrace{f(\vec{r}(t))}_{\text{function of one variable}} = L \Rightarrow \lim_{(x,y)\to(a,b)} f(x,y) = L$$

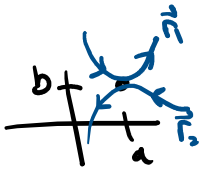
for $\underline{\underline{\text{all}}}$ $\vec{r}$

If f is defined at (a,b) and $\lim\limits_{(x,y)\to(a,b)} f(x,y)=f(a,b)$, then we say <u>f is continuous at (a,b)</u>.

We similarly define $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists in the extended sense, and write

$$\lim_{(x,y)\to(a,b)} f(x,y) = \pm\infty.$$

Fact: Suppose f=f(x,y) is a real-valued function defined near (a,b), but perhaps not at (a,b). If there are continuous parametric plane cures $\vec{r}_1,\vec{r}_2$ defined near t=0 with $\vec{r}_1(0)=\vec{r}_2(0)=$<a,b>, but so that



$$\lim_{t\to 0} f(\vec{r}_1(t)) \neq \lim_{t\to 0} f(\vec{r}_2(t))$$

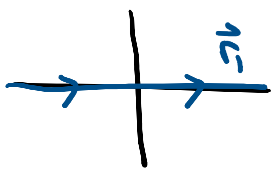
(including the possibility that one of the limits does not exist), then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Ex: Show that the limit of the given function f at (0,0) does not exist (even in the extended sense).

1. $f(x,y)= \dfrac{x^2-y^2}{x^2+y^2}$

Sol: We must find two continuous parametric plane curves $\vec{r}_1,\vec{r}_2$ passing through <0,0> so that $\lim\limits_{t\to 0} f(\vec{r}_1(t)) \neq \lim\limits_{t\to 0} f(\vec{r}_2(t))$.

First consider $\vec{r}_1(t)=$<t,0>

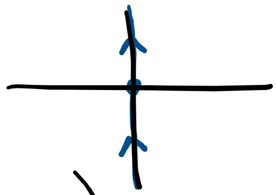


We compute

$$\lim_{t\to 0} f(\vec{r}_1(t)) = \lim_{t\to 0} f(\overset{x}{\overset{\shortparallel}{t}}, \overset{y}{\overset{\shortparallel}{0}}) = \lim_{t\to 0} \left(\frac{x^2-y^2}{x^2+y^2} \Big|_{\substack{x=t\\y=0}} \right)$$

$$= \lim_{t\to 0} \frac{t^2-0^2}{t^2+0^2} = \lim_{t\to 0} \frac{t^2}{t^2} = \lim_{t\to 0} 1 = 1$$

Second, consider $\vec{r}_2(t)=\langle \underset{x}{\underset{\shortparallel}{0}}, \underset{y}{\underset{\shortparallel}{t}} \rangle$.
We compute



$$\lim_{t\to 0} f(\vec{r}_2(t)) = \lim_{t\to 0} \left(\frac{x^2-y^2}{x^2+y^2} \Big|_{\substack{x=0\\y=t}} \right)$$

$$= \lim_{t\to 0} \frac{0^2-t^2}{0^2+t^2} = \lim_{t\to 0} \frac{-t^2}{t^2} = \lim_{t\to 0} -1 = -1$$

Since $$\lim_{t\to 0} f(\vec{r}_1(t)) = 1 \neq -1 = \lim_{t\to 0} f(\vec{r}_2(t))$$

then we conclude that $$\lim_{(x,y)\to(0,0)} f(x,y) \text{ DNE.}$$

2. $f(x,y)= \dfrac{xy^2}{x^2+y^4}$

Sol: Consider $\vec{r}_1(t)=\langle t,0\rangle$, we compute

$$\lim_{t\to 0} f(\vec{r}_1(t)) = \lim_{t\to 0} \frac{xy^2}{x^2+y^4} \Big|_{\substack{x=t\\y=0}}$$

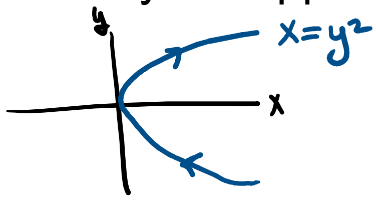
$$= \lim_{t\to 0} \frac{t\cdot 0^2}{t^2+0^4} = \lim_{t\to 0} 0 = 0$$

Consider $\vec{r}_2(t)=\langle 0,t\rangle$, we compute

$$\lim_{t\to 0} F(\vec{r}_2(t)) = \lim_{t\to 0} \left.\frac{xy^2}{x^2+y^4}\right|_{\substack{x=0\\y=t}} = \lim_{t\to 0} \frac{0\cdot t^2}{0^2+t^4}$$

$$= \lim_{t\to 0} 0 = 0$$

WARNING: This does *not* mean that the limit exists. We must consider *all* possible ways of approaching (0,0). Consider $\vec{r}_3(t)=\langle t^2,t\rangle$, we compute



$$\lim_{t\to 0} F(\vec{r}_3(t)) = \lim_{t\to 0} \left.\frac{xy^2}{x^2+y^4}\right|_{\substack{x=t^2\\y=t}} = \lim_{t\to 0} \frac{t^2\cdot t^2}{(t^2)^2+t^4}$$

$$= \lim_{t\to 0} \frac{t^4}{t^4+t^4} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$$

Since

$$\underbrace{\lim_{t\to 0} F(\vec{r}_1(t))}_{x\text{-axis}} = 0 \neq \frac{1}{2} = \underbrace{\lim_{t\to 0} F(\vec{r}_3(t))}_{x=y^2}$$

then we conclude that $\displaystyle\lim_{(x,y)\to(0,0)} F(x,y)$ DNE

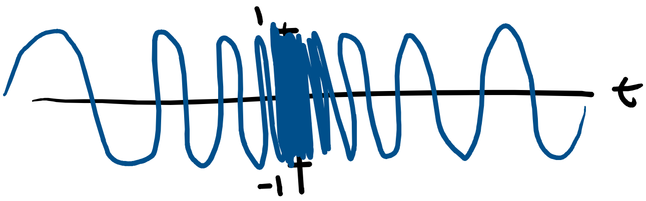
It is not enough to only check along y=mx.

3. $f(x,y)=\sin\left(\frac{x}{x^2+y^2}\right)$

Sol: Consider $\vec{r}(t)=\langle t,0\rangle$, we compute

$$\lim_{t\to 0} f(\vec{r}(t)) = \lim_{t\to 0} \sin\left(\frac{t}{t^2+0^2}\right)$$

$$= \lim_{t\to 0} \sin\left(\frac{1}{t}\right) \text{ DNE}$$



Since $\lim_{t\to 0} \underbrace{f(\vec{r}(t))}_{x\text{-axis}}$ DNE

then we conclude that $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.