Vector Calculus
Midterm solutions

1) Consider the parametric plane curve

$$C(t) = (t^2+2t+1, t+1) \quad \text{for} \quad t<0$$

a) Find a Cartesian equation for C.

Sol: We compute $\begin{array}{l} x = t^2+2t+1 \Rightarrow x = (t+1)^2 = y^2 \\ y = t+1 \end{array}$

$$\Rightarrow \boxed{x = y^2}$$

b) Find all the points where the image of C intersects the line y=x.

Sol: Set y=x into the Cartesian equation for C, so that

$$x = y^2 = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x = 0, 1$$

Note that the image of C is only given for t<0.

$0 = x = (t+1)^2 \Rightarrow t = -1$ ✓

$1 = x = (t+1)^2 \Rightarrow t = 0$ ∅

We conclude that the image of C intersects the line y=x only at the point C(-1)=$\boxed{(0,0).}$

2) Consider the parametric plane curve

$$C(t) = \left(\frac{t^3}{3} - 5t, e^{2t} + t\right) \quad \text{for} \quad 1 \le t \le 2$$

a) Show that the image of C is the graph of a function y=f(x) defined over [-22/3,-14/3].

Sol: Consider

$$x'(t) = \frac{d}{dt}\,\frac{t^3}{3} - 5t = t^2 - 5$$

$$1 \le t \le 2 \Rightarrow 1^2 \le t^2 \le 2^2$$

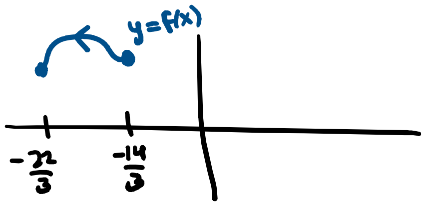
$$\Rightarrow 1-5 \le t^2 - 5 \le 4-5$$

$$\Rightarrow t^2 - 5 \le -1 < 0$$

This means x(t) is decreasing for 1⩽t⩽2. We conclude that the image of C is the graph of a function y=f(x) defined over

$$[x(2), x(1)] = \left[\frac{2^3}{3} - 10, \frac{1}{3} - 5\right] = \left[\frac{8-30}{3}, \frac{1-15}{3}\right] = \left[\frac{-22}{3}, \frac{-14}{3}\right]$$

x decreasing



b) Set up an integral for the arc length L of the image of C.

Sol: We give

$$L = \int_1^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

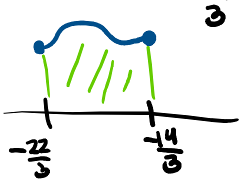
$$= \int_1^2 \sqrt{\left(\frac{d}{dt}\left(\frac{t^3}{3} - 5t\right)\right)^2 + \left(\frac{d}{dt}\left(e^{2t} + t\right)\right)^2}\, dt$$

$$= \boxed{\int_1^2 \sqrt{(t^2-5)^2 + (2e^{2t}+1)^2}\, dt}$$

c) Set up an integral for the area A under C.

Sol: Since the image of C is the graph of y=f(x) defined over [-22/3,-14/3], then

$$A = \int_{-\frac{22}{3}}^{-\frac{14}{3}} f(x)\,dx \underset{\substack{x = x(t) \\ dx = x'(t)dt}}{=} \int_{2}^{1} y(t)\, x'(t)\, dt$$



$x(1) = -\frac{14}{3}$

$x(2) = -\frac{22}{3}$

$$= \boxed{\int_{2}^{1} \left(e^{2t}+t\right)\left(t^2-5\right) dt}$$

$$= \int_{1}^{2} \left(e^{2t}+t\right)\left(5-t^2\right) dt$$

3) Consider the parametric plane curve

$$C(t) = \left(\frac{t^5}{5} - \frac{4t^3}{3},\ \frac{t^4}{4} + \frac{t^3}{3}\right) \text{ for } t \in \mathbb{R}$$

a) Compute x'(t) and y'(t).

Sol: We compute

$$\boxed{x'(t) = t^4 - 4t^2} \quad \text{and} \quad \boxed{y'(t) = t^3 + t^2}$$

b) Compute the tangent line of C at t=1.

Sol: First, we compute

$$x'(1) = 1^4 - 4\cdot 1^2 = 1-4 = -3 \neq 0$$

$$y'(1) = 1^3 + 1^2 = 2$$

$$(x(1), y(1)) = c(1) = \left(\frac{1}{5} - \frac{4}{3}, \frac{1}{4} + \frac{1}{3}\right)$$

Since x'(1)=/0, then the image of C near t=1 is the graph of a function y=f(x) defined near x(1). We conclude that the tangent line of C at t=1 is the tangent line of f at x(1), which is given by

$$y = f'(x(1))(x - x(1)) + f(x(1))$$

$$\Rightarrow \quad y = \frac{y'(1)}{x'(1)}(x - x(1)) + y(1)$$

$$\Rightarrow \quad \boxed{y = \frac{2}{-3}\left(x - \left(\frac{1}{5} - \frac{4}{3}\right)\right) + \left(\frac{1}{4} + \frac{1}{3}\right)}$$

c) Find all t so that x'(t)=0.

Sol: We compute

$$0 = x'(t) = t^4 - 4t^2 = t^2(t^2 - 4) \Rightarrow \boxed{t = 0, \pm 2}$$

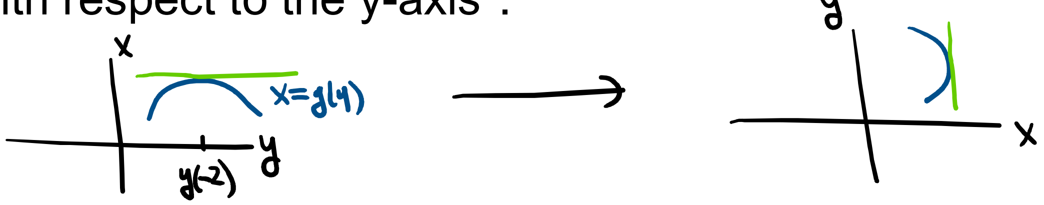
d) Find all t so that the tangent line of C at t is a vertical line.

Sol: Consider t=-2,0,2.

Consider t=-2,

$$y'(-2) = t^3+t^2 \Big|_{t=-2} = -8+4 \neq 0$$

y’(2)=/0 implies that the image of C near t=-2 is the graph of a function x=g(y) defined near y(-2). However, x’(-2)=0 implies that the tangent line of g at y(-2) is a horizontal line *with respect to the y-axis*.



We conclude that the tangent line of C at t=-2 is a vertical line.

Consider t=0,

$$y'(0) = t^3+t^2 \Big|_{t=0} = 0+0 = 0.$$

We must compute

$$\lim_{t\to 0} \frac{y'(t)}{x'(t)} = \lim_{t\to 0} \frac{t^3+t^2}{t^4-4t^2} = \lim_{t\to 0} \frac{t+1}{t^2-4} = -\frac{1}{4}$$

We conclude that the tangent line of C at t=0 is *not* a vertical line.

Consider t=2,

$$y'(2) = t^3+t^2 \Big|_{t=2} = 8+4 \neq 0$$

We conclude that the tangent line of C at t=2 is a vertical line.

$$\Rightarrow \boxed{t = \pm 2}$$

e) Find all t so that y'(t)=0.

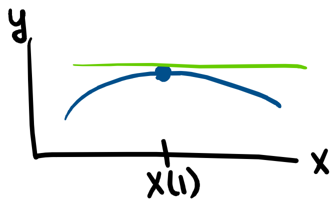
Sol: We compute

$$0 = y'(t) = t^3 + t^2 = t^2(t+1) \Rightarrow t = 0, -1$$

Consider t=-1,

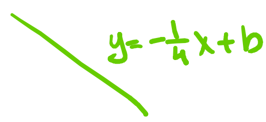
$$x'(-1) = t^4 - 4t^2 \Big|_{t=-1} = 1 - 4 = -3 \neq 0$$

x'(-1)=/0 implies that the image of C near t=-1 is the graph of a function y=f(x) defined near x(-1). In this case, y'(-1)=0 implies that the tangent line of f at x(-1) is a horizontal line. We conclude that the tangent line of C at t=-1 is a horizontal line.



Consider t=0, we already computed

$$\lim_{t \to 0} \frac{y'(t)}{x'(t)} = -\frac{1}{4}$$



We conclude that the tangent line of C at t=0 is *not* a horizontal line.

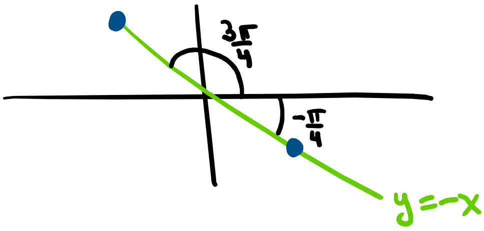
$$\Rightarrow \boxed{t = -1}$$

4) Consider the polar parametric plane curve C given by the polar parametric equation r=theta+pi/2 for -pi⩽theta⩽pi.

$$r = \theta + \frac{\pi}{2}$$

a) Find the points where the image of C intersects the line y=-x.

Sol: Since -pi⩽theta⩽pi, then the image of C intersects the line y=-x at theta=-pi/4,3pi/4.



We conclude that the image of C intersects the line y=-x at

$$C\left(-\frac{\pi}{4}\right) = \left(\left(\theta+\frac{\pi}{2}\right)\cos\theta, \left(\theta+\frac{\pi}{2}\right)\sin\theta\right)\Big|_{\theta=-\frac{\pi}{4}}$$

$$= \boxed{\left(\left(-\frac{\pi}{4}+\frac{\pi}{2}\right)\left(\frac{\sqrt{2}}{2}\right), \left(-\frac{\pi}{4}+\frac{\pi}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)\right)}$$

$$= \left(\frac{\pi\sqrt{2}}{8}, -\frac{\pi\sqrt{2}}{8}\right)$$

$$C\left(\frac{3\pi}{4}\right) = \boxed{\left(\left(\frac{3\pi}{4}+\frac{\pi}{2}\right)\left(-\frac{\sqrt{2}}{2}\right), \left(\frac{3\pi}{4}+\frac{\pi}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right)}$$

b) Compute the tangent line of C at theta=0.

Sol: First, we compute

$$x'(0) = \frac{\partial}{\partial\theta}\left(\theta+\frac{\pi}{2}\right)\cos\theta\,\Big|_{\theta=0}$$

$$= \cos\theta - \left(\theta+\frac{\pi}{2}\right)\sin\theta\,\Big|_{\theta=0} = 1-0 = 1 \neq 0$$

$$y'(0) = \frac{\partial}{\partial\theta}\left(\theta+\frac{\pi}{2}\right)\sin\theta\,\Big|_{\theta=0}$$

$$= \sin\theta + \left(\theta+\frac{\pi}{2}\right)\cos\theta\,\Big|_{\theta=0} = 0+\frac{\pi}{2} = \frac{\pi}{2}$$

$$(x(0), y(0)) = C(0) = \left(\left(0+\frac{\pi}{2}\right)\cos 0, \left(0+\frac{\pi}{2}\right)\sin 0\right)$$

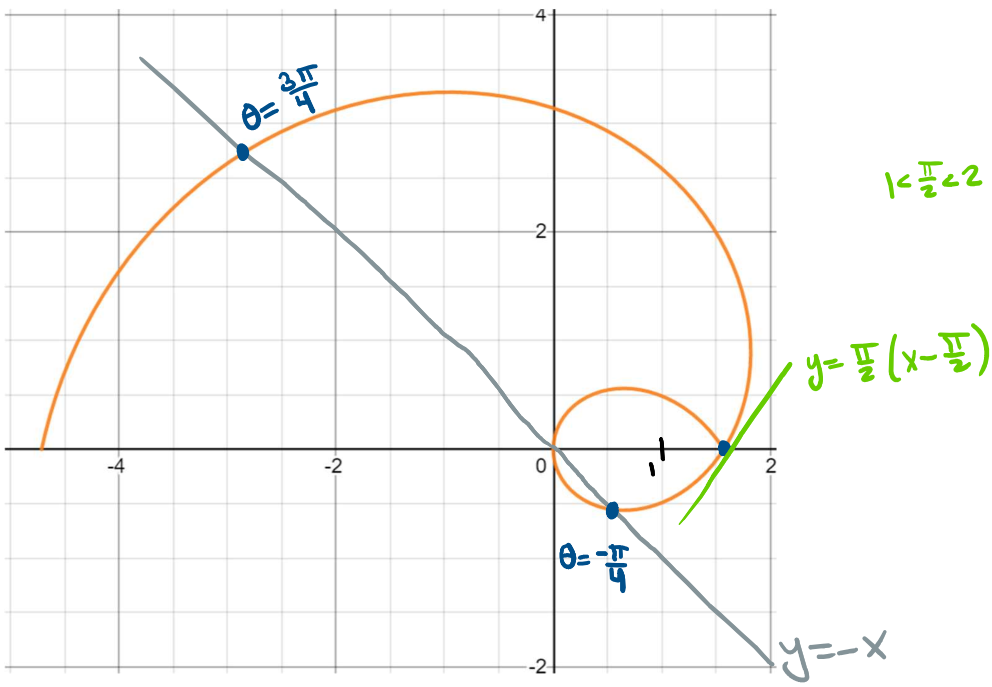
$$= \left(\frac{\pi}{2}, 0\right)$$

Snce x’(0)=/0, then we conclude that the tangent line of C at theta=0 is

$$y = \frac{y'(0)}{x'(0)}(x - x(0)) + y(0)$$

$$\Rightarrow \boxed{y = \frac{\pi/2}{1}\left(x-\frac{\pi}{2}\right)+0}$$

$$\Rightarrow \quad y = \frac{\pi}{2}\left(x-\frac{\pi}{2}\right)$$



4
2
-4
-2
0
2
-2
1
$\theta = \frac{3\pi}{4}$
$\theta = -\frac{\pi}{4}$
$1 < \frac{\pi}{2} < 2$
$y = \frac{\pi}{2}\left(x - \frac{\pi}{2}\right)$
$y = -x$

Common mistakes

1a) You may be tempted to compute

$$x=(t+1)^2 \Rightarrow \sqrt{x}=t+1=y \Rightarrow y=\sqrt{x}$$

This is not good enough. Note that y=t+1 for t<0, so y can be negative. You can try to fix this, by saying

$$x=(t+1)^2 \Rightarrow t+1=\pm\sqrt{x} \Rightarrow y=\pm\sqrt{x}$$

However, the instructions say to give *a* Cartesian equation for C. This is *two* equations for C. The statement "$y=\pm\sqrt{x}$" means

$$y=\sqrt{x} \quad \text{or} \quad y=-\sqrt{x}.$$

1b) Including t=0, which is the point (1,1) is wrong.

2a) x=x(t) is decreasing, so the correct interval is [x(2),x(1)]. Giving [x(1),x(2)]=[-14/3,-22/3] is wrong.

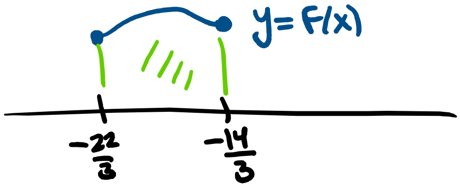
$$-\frac{22}{3}<-\frac{14}{3}$$

2b) You were not required to evaluate the integral.

2c) Note that x=x(t) is decreasing, so we need to give

$$\int_2^1 y(t)\,x'(t)\,dt \quad \text{not} \quad \int_1^2 y(t)\,x'(t)\,dt$$

In fact, note that $y(t)=e^{2t}+t>0$ for $1\leq t\leq 2$. This means that the area under C is a positive number:



$$\Rightarrow \int_{-\frac{22}{3}}^{-\frac{14}{3}} f(x)\,dx > 0$$

and

$$\int_1^2 \underbrace{(e^{2t}+t)}_{>0}\underbrace{(t^2-5)}_{<0}\,dt < 0$$

for $1 \le t \le 2$

The correct integral *is*

$$\int_2^1 (e^{2t}+t)(t^2-5)\,dt = -\int_1^2 (e^{2t}+t)(t^2-5)\,dt$$

$$= \int_1^2 \underbrace{(e^{2t}+t)}_{>0}\underbrace{(5-t^2)}_{>0}\,dt > 0$$

You are not required to compute this integral.

3d) x'(t)=0 is not enough by itself to conclude that the tangent line of C at t is a vertical line. We must careful,

$$x'(0)=0 \quad \text{but} \quad \lim_{t\to 0} \frac{y'(t)}{x'(t)} = -\frac{1}{4}$$

implies the tangent line of C at t=0 is not a vertical line. It is a line with slope =-1/4.

3f) Similarly, y'(t)=0 is not enough by itself to conclude that the tangent line of C at t is a horizontal line.

4a) We only need to find theta in [0,2pi).