Vector Calculus
Review
Practice Problems I

1) Let $\vec{r}(t)=<\frac{t^2}{2},\frac{t^2}{2},\frac{t^3}{3}>$

a) Compute the unit tangent vector and speed of r at t=1.

Sol: We compute

$$\vec{r}'(1)=\langle t,t,t^2\rangle\Big|_{t=1}=\langle 1,1,1\rangle$$

$$\Rightarrow \quad |\vec{r}'(1)|=\boxed{\sqrt{3}} \quad \text{and} \quad \vec{T}(1)=\boxed{\left\langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle}$$

b) Compute the curvature kappa(1).

Sol: We need to compute $\quad \kappa(1)=\dfrac{|\vec{r}'(1)\times\vec{r}''(1)|}{|\vec{r}'(1)|^3}$

We compute

$$\vec{r}'(1)=\langle t,t,t^2\rangle\Big|_{t=1}=\langle 1,1,1\rangle$$

$$|\vec{r}'(1)|=\sqrt{3}$$

$$\vec{r}''(1)=\langle 1,1,2t\rangle\Big|_{t=1}=\langle 1,1,2\rangle$$

$$\vec{r}'(1)\times\vec{r}''(1)=\langle 2-1,-(2-1),0\rangle=\langle 1,-1,0\rangle$$

We conclude that

$$\kappa(1)=\frac{|\langle 1,-1,0\rangle|}{(\sqrt{3})^3}=\boxed{\frac{\sqrt{1+1}}{(\sqrt{3})^3}}$$

c) Compute the arc length function s(t) of r over [0,1].

Sol: We compute

$$s(t)=\int_0^t |\vec{r}\,'(u)|\,du = \int_0^t |\langle u,u,u^2\rangle|\,du$$

$$=\int_0^t \sqrt{u^2+u^2+u^4}\,du = \int_0^t \sqrt{2u^2+u^4}\,du$$

$$\underset{\substack{t\in[0,1]\\ \Rightarrow u\geq 0}}{=} \int_0^t u\sqrt{2+u^2}\,du \underset{\substack{v=2+u^2\\ dv=2u\,du\\ \Rightarrow \frac{1}{2}dv=u\,du\\ u=0\to v=2\\ u=t\to v=2+t^2}}{=} \int_2^{2+t^2} \frac{1}{2} v^{\frac{1}{2}}\,dv$$

$$\Rightarrow \quad s(t) = \frac{1}{2}\left(\frac{v^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_{v=2}^{2+t^2}$$

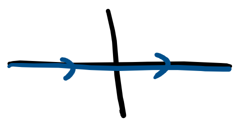
$$= \boxed{\frac{(2+t^2)^{\frac{3}{2}}}{3} - \frac{2^{\frac{3}{2}}}{3}}$$

2) Show that the limit of the given function f at (0,0) exists or does not exist in the extended sense. If the limit exists, given the value.

Guess $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}=1$ $\quad\quad \lim_{x\to 0}\frac{1}{x}$ DNE

a) $f(x,y)=\dfrac{x^2-y}{\sin(x^2+y^2)}$ $\quad\quad \dfrac{0^2-0}{\sin 0^2} = \dfrac{0^2-0}{0^2} = 1-\dfrac{1}{0}$ DNE

Sol: Consider $\vec{r}_1(t)=\langle t,0\rangle$



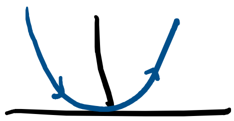
We compute

$$\lim_{t\to 0} f(\vec{r}_1(t)) = \lim_{t\to 0} \frac{x^2-y}{\sin(x^2+y^2)}\Big|_{\substack{x=t\\ y=0}} = \lim_{t\to 0} \frac{t^2-0}{\sin(t^2+0)}$$

$$= \lim_{t\to 0} \frac{t^2}{\sin(t^2)} = 1$$

On the other hand, consider

$$\vec{r}_2(t) = \langle t, t^2 \rangle$$



We compute

$$\lim_{t\to 0} f(\vec{r}_2(t)) = \lim_{t\to 0} \frac{x^2-y}{\sin(x^2+y^2)}\Big|_{\substack{x=t\\ y=t^2}} = \lim_{t\to 0} \frac{t^2-t^2}{\sin(t^2+t^4)}$$

$$= \lim_{t\to 0} 0 = 0$$

Since $$\lim_{t\to 0} f(\vec{r}_1(t)) \neq \lim_{t\to 0} f(\vec{r}_2(t))$$

then we conclude that $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

b) $f(x,y) = \dfrac{x^4+x^5+x^2y^2}{2x^2+y^2}$ Guess $\dfrac{0^4+0^5+0^4}{0^2} = 0^2+0^3 = 0?$

Sol: Let's use the Squeeze Thm. We compute

$$|f(x,y)| = \frac{|x^4+x^5+x^2y^2|}{2x^2+y^2}$$ since $|a+b| \leq |a|+|b|$

$$\leq \frac{x^4+|x|^5+x^2y^2}{2x^2+y^2}$$

$$= \frac{x^4+x^2y^2}{2x^2+y^2} + \frac{|x|^5}{2x^2+y^2}$$

$$= x^2\left(\frac{x^2+y^2}{2x^2+y^2}\right) + |x|^3\left(\frac{x^2}{2x^2+y^2}\right)$$

$$\leq x^2\left(\frac{2x^2+y^2}{2x^2+y^2}\right) + |x|^3\left(\frac{2x^2+y^2}{2x^2+y^2}\right) = x^2+|x|^3$$

$|A| \leq B$
$\Rightarrow -B \leq A \leq B$

This means that

$$-\left(x^2+|x|^3\right) \leq f(x,y) \leq x^2+|x|^3$$

Since

$$\lim_{x\to 0} -\left(x^2+|x|^3\right) = 0 = \lim_{x\to 0} x^2+|x|^3$$

then we conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

by the Squeeze Theorem.

3) Let $f(x,y)=x^2+y+e^{xy}$, and let $x(t)=2t$ and $y(t)=t^2$.

a) Verify by computation that $f_{xy} = f_{yx}$.

Sol: We compute

$$f_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\left(x^2+y+e^{xy}\right) = \frac{\partial}{\partial y}\left(2x+0+ye^{xy}\right)$$

$$= 0 + e^{xy} + xye^{xy}$$

$$= e^{xy} + xye^{xy}$$

$$f_{yx} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(x^2+y+e^{xy}\right) = \frac{\partial}{\partial x}\left(0+1+xe^{xy}\right)$$

$$= 0 + e^{xy} + xye^{xy}$$

$$= e^{xy} + xye^{xy}$$

b) Verify by computation that

$$\frac{\partial}{\partial t} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Sol: We compute

$$\frac{\partial}{\partial t} f(x(t), y(t)) = \frac{\partial}{\partial t}\left(x^2+y+e^{xy} \Big|_{\substack{x=2t\\ y=t^2}} \right)$$

$$= \frac{\partial}{\partial t}\left((2t)^2 + t^2 + e^{2t\cdot t^2}\right)$$

$$= \frac{\partial}{\partial t}\left(4t^2 + t^2 + e^{2t^3}\right)$$

$$= 8t + 2t + 6t^2e^{2t^3} = 10t + 6t^2e^{2t^3}$$

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(x^2+y+e^{xy}\right)\Big|_{(2t,t^2)} \frac{\partial}{\partial t} 2t$$

$$+ \frac{\partial}{\partial y}\left(x^2+y+e^{xy}\right)\Big|_{(2t,t^2)} \frac{\partial}{\partial t} t^2$$

$$= \left(2x + ye^{xy}\right)\Big|_{(2t,t^2)} \cdot 2$$

$$+ \left(1 + xe^{xy}\right)\Big|_{(2t,t^2)} \cdot 2t$$

$$= 2\left(2(2t) + t^2 e^{2t \cdot t^2}\right)$$

$$+ 2t\left(1 + 2t e^{2t \cdot t^2}\right)$$

$$= 8t + 2t^2 e^{2t^3} + 2t + 4t^2 e^{2t^3}$$

$$= 10t + 2t^2 e^{2t^3} + 4t^2 e^{2t^3}$$

$$= 10t + 6t^2 e^{2t^3}$$

✓

4) Suppose f(x,y,z)=x^2+yz.

a) Compute the directional derivative of f in the direction of $\vec{v}$=<2,1,3> at (1,-1,2).

Sol: First, we normalize $\vec{v}$,

$$\frac{\vec{v}}{|\vec{v}|} = \frac{\langle 2,1,3 \rangle}{\sqrt{4+1+9}} = \left\langle \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$$

We also compute

$$\nabla f(1,-1,2) = \langle 2x, z, y \rangle \big|_{(1,-1,2)} = \langle 2, 2, -1 \rangle$$

We conclude that the direction derivative of f in the direciton of $\vec{v}$ at (1,-1,2) is

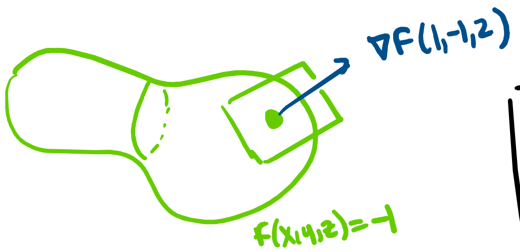
$$D_{\frac{\vec{v}}{|\vec{v}|}} f(1,-1,2) = \frac{\vec{v}}{|\vec{v}|} \cdot \nabla f(1,-1,2) = \left\langle \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle \cdot \langle 2,2,-1 \rangle$$

$$= \boxed{\frac{4}{\sqrt{14}} + \frac{2}{\sqrt{14}} - \frac{3}{\sqrt{14}}}$$

b) Compute the tangent plane and normal line at (1,-1,2) of the level surface of f at k=-1.

Sol: The tangent plane is the plane through (1,-1,2) with normal in the direction of ∇f(1,-1,2).

$\nabla f \perp$ level surface of f at K



$$F(1,-1,2) = 1^2 + (-1)\cdot 2 = -1$$

$$\boxed{2(x-1) + 2(y-(-1)) + (-1)(z-2) = 0}$$

$= 0$

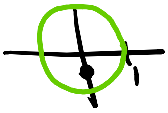
The normal line is the line through (1,-1,2) in the direction of ∇f(1,-1,2).

$$\boxed{\ell(t) = \langle 1,-1,2 \rangle + t\langle 2,2,-1 \rangle \quad \text{for } t \in \mathbb{R}}$$

5) Local and absolute extremum points and values.

a) Find the absolute extremum points and values of $f(x,y)=x^2+y^2+y$ over Omega=$\{(x,y):x^2+y^2\leqslant 1\}$.

unit disk



Sol: First, we find the interior critical points.

$$\vec{0}=\nabla f=\langle 2x, 2y+1\rangle \Rightarrow \begin{matrix} 2x=0 \\ 2y+1=0 \end{matrix} \Rightarrow \begin{matrix} x=0 \\ y=-\frac{1}{2} \end{matrix}$$

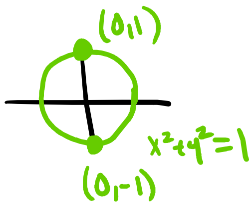
$$\Rightarrow \left(0,-\tfrac{1}{2}\right)\in\Omega$$

We must consider

$$f\left(0,-\tfrac{1}{2}\right)=0+\left(-\tfrac{1}{2}\right)^2-\tfrac{1}{2}$$
$$=\tfrac{1}{4}-\tfrac{1}{2}=\boxed{-\tfrac{1}{4}}$$

We must find the absolute extremum points of $f(x,y)=x^2+y^2+y$ over $x^2+y^2=1$. Note that

$$x^2+y^2=1 \Rightarrow f(x,y)=1+y$$



We must consider

$$f(0,1)=1+1=\boxed{2}$$
$$f(0,-1)=1-1=\boxed{0}$$

Using Lagrange multipliers, we compute

$g(x,y)=x^2+y^2$

① $2x=2\lambda x$

② $2y+1=2\lambda y$

③ $x^2+y^2=1$

$\Longrightarrow$ $\boxed{x=0}$ $\overset{③}{\Downarrow}$ $y=\pm 1$ $\Downarrow$ $\boxed{(0,\pm 1)}$

or

$\boxed{x\neq 0}$ $\overset{①}{\Downarrow}$ $\lambda=1$ $\overset{②}{\Downarrow}$ $2y+1=2y$

$$\Downarrow$$
$$0=1$$
$$\cancel{\emptyset}$$

We conclude that f has absolute maximum point (0,1) over Omega, with absolute maximum value =2, and f has absolute minimum point (0,-1/2) over Omega, with absolute minimum value =-1/4.

b) Find all the critical points of f(x,y)=x^2+y^4+2xy, and determine whether they are local minimum, local maximum, or saddle points.

Sol: First, we find the critical points.

$$\vec{0} = \nabla f = \langle 2x+2y,\ 4y^3+2x \rangle$$

$$\Rightarrow \begin{array}{l} 2x+2y=0 \\ 4y^3+2x=0 \end{array} \Rightarrow \begin{array}{l} 2y=-2x \Rightarrow x=-y \\ 4y^3=-2x \end{array}$$

$$\Rightarrow 2y = 4y^3$$

$$\Rightarrow 4y^3-2y=0$$

$$\Rightarrow 2y(2y^2-1)=0$$

$$\Rightarrow y=0, \pm\frac{\sqrt{2}}{2}$$

$$x=-y \Rightarrow \boxed{(0,0),\ \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),\ \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)}$$

To classify these points, we compute

$$\nabla f = \langle 2x + 2y,\ 4y^3 + 2x \rangle$$

$$\Rightarrow \quad f_{xx} = 2 \qquad f_{xy} = 2$$

$$f_{yy} = 12y^2$$

$$\Rightarrow \quad \Delta = f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 12y^2 - 2^2$$

$$\Rightarrow \quad \Delta = 24y^2 - 4$$

At $(0,0), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ we compute

$$\Delta(0,0) = -4 \qquad g(x,y) = x^2 - y^2 \Rightarrow 2 \cdot (-2) - 0^2 < 0$$

$$\Rightarrow \boxed{(0,0) \text{ is a saddle point}}$$

$$\Delta\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 24\left(-\frac{\sqrt{2}}{2}\right)^2 - 4 = 24 \cdot \frac{2}{4} - 4 = 12 - 4 = 8 > 0$$

$$f_{xx}\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 2 > 0 \qquad g(x,y) = x^2 + y^2$$

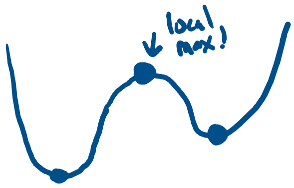
$$\Rightarrow \boxed{\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \text{ is a local minimum}}$$

$$\Delta\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 24\left(\frac{\sqrt{2}}{2}\right)^2 - 4 = 8 > 0$$

$$f_{xx}\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 2 > 0$$

$$\Rightarrow \boxed{\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ is a local minimum}}$$

Note that a single variable function cannot have two local minimum points without a local maximum.



However, this is possible for f=f(x,y).