VECTOR CALCULUS, Week 11

11.3 Partial Derivatives; 11.4 Tangent Planes and Linear Approximations; 11.5 The Chain Rule

11.3 Partial Derivatives

Def: Suppose $f = f(x, y)$ is a real-valued function defined near (a, b).

- We say the **partial derivative of** f **with respect to** x **at** (a, b) is the limit
$$\frac{\partial f}{\partial x}\Big|_{(x,y)=(a,b)} = f_x(a, b) = D_1 f(a, b) = D_x f(a, b) = \lim_{x\to a} \frac{f(x, b) - f(a, b)}{x - a},$$
assuming this limit exists (is finite).

- We say the **partial derivative of** f **with respect to** y **at** (a, b) is the limit
$$\frac{\partial f}{\partial y}\Big|_{(x,y)=(a,b)} = f_y(a, b) = D_2 f(a, b) = D_y f(a, b) = \lim_{y\to b} \frac{f(a, y) - f(a, b)}{y - b},$$
assuming this limit exists (is finite).

Fact: Suppose $f = f(x, y)$ is a real-valued function defined near (a, b).

- Define $g(x) = f(x, b)$, then $\frac{\partial f}{\partial x}|_{(x,y)=(a,b)} = \frac{d}{dx} g(x)|_{x=a} = g'(a)$.
- Define $g(y) = f(a, y)$, then $\frac{\partial f}{\partial y}|_{(x,y)=(a,b)} = \frac{d}{dy} g(y)|_{y=b} = g'(b)$.

In other words, to compute a partial derivative with respect to x, we can pretend that y is a constant and take the derivative with respect to x.

Ex: Let $f(x, y) = x^3 + x^2y^2 - y$.

1. Compute $f_x(2, 3)$ and $f_y(2, 3)$.
2. Find a line in the plane $y = 3$ which is tangent to the graph of f at $(2, 3)$.

Def: We denote the **second partial derivatives of** f as follows.

- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx} = f_{11}$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy} = f_{12}$
- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx} = f_{21}$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy} = f_{22}$

We say $f_{x,y}, f_{yx}$ are the **mixed partial derivatives of** f.

Thm: Suppose $f = f(x, y)$ is a real-valued function defined near (a, b). If f_{xy}, f_{yx} exist and are continuous near (a, b), then $f_{xy} = f_{yx}$ near (a, b).

Ex:

1. For $f(x, y) = x^2 + xy + e^{x^2 y}$, verify $f_{xy} = f_{yx}$ for all (x, y).
2. Let
$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$
then f_{xy}, f_{yx} exist for all (x, y) near $(0, 0)$, but $f_{xy}(0, 0) = -1$ while $f_{x,y}(0, 0) = 1$. The problem is that $f_{x,y}, f_{yx}$ are not continuous at $(0, 0)$.

Fact: Similar definitions and results are true for real-valued functions $f = f(x, y, z)$.

Ex: For $f(x, y, z) = \cos(xy + z)$, verify $f_{xyz} = f_{zxy}$ for all (x, y, z).

11.4 Tangent Planes and Linear Approximations

Def: Suppose $f = f(x, y)$ is a real-valued function defined near (a, b). We say f **is differentiable at** (a, b) if and only if there exists $A, B, C \in \mathbf{R}$ so that

$$\lim_{(x,y)\to(a,b)} \frac{|f(x, y) - (Ax + By + C)|}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$

If f is differentiable at (a, b), then we say the plane $z = Ax + By + C$ is the **tangent plane** or **linear approximation of** f **at** (a, b).

Fact: Suppose $f = f(x, y)$ is a real-valued function defined near (a, b).

- If f is differentiable at (a, b), then the partial derivatives $f_x(a, b), f_y(a, b)$ exist and the tangent plane of f at (a, b) is given by

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

- If f is differentiable at (a, b), then the tangent plane of f at (a, b) gives a good approximation for f near (a, b).

- If f_x, f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

  However, I will give an example of a real-valued function $f = f(x, y)$ so that f_x, f_y exist near $(0, 0)$, but f is not differentiable at $(0, 0)$. The problem with this f is that f_x, f_y are not continuous near $(0, 0)$.

- Similar definitions and results are true for real-valued functions $f = f(x, y, z)$.

  f is differentiable at (a, b, c) if and only if there is a **hyperplane** $w = Ax + By + Cz + D$ in $\mathbf{R}^4$ which gives a good approximation for f near (a, b, c).

Ex: Show that the given function f is differentiable at the given point (a, b), and use the tangent plane to approximate the given value of f.

1. $f(x, y) = xe^{xy}$ at $(a, b) = (1, 0)$, approximate $f(1.1, -0.1)$.
2. $f(x, y) = x^3 + xy$ at $(a, b) = (2, 3)$, approximate $f(1.9, 3.1)$.

Ex: Consider the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

f_x, f_y exist for all (x, y), but f is not differentiable at $(0, 0)$.

11.5 The Chain Rule

Chain Rule: Suppose $f = f(x, y)$ is a real-valued differentiable function, and suppose $g = g(t), h = h(t)$ are differentiable. Then $f(g(t), h(t))$ is differentiable with

$$\frac{d}{dt} f(g(t), h(t)) = \left.\frac{\partial f}{\partial x}\right|_{(g(t),h(t))} g'(t) + \left.\frac{\partial f}{\partial y}\right|_{(g(t),h(t))} h'(t).$$

We write this as

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

For $f = f(x, y, z)$, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Ex: Verify the Chain Rule for $f(x, y) = x^2 + xy$ with $x = t^2$ and $y = \cos t$.

Chain Rule: Suppose $f = f(x, y)$ is a real-valued differentiable function, and suppose $g = g(s, t), h = h(s, t)$ are real-valued differentiable functions, then $f(g(s, t), h(s, t))$ is a real-valued differentiable function with

$$\frac{\partial}{\partial s} f(g(s, t), h(s, t)) = \left.\frac{\partial f}{\partial x}\right|_{(g(s,t),h(s,t))} \frac{\partial g}{\partial s} + \left.\frac{\partial f}{\partial y}\right|_{(g(s,t),h(s,t))} \frac{\partial h}{\partial s}$$

and

$$\frac{\partial}{\partial t} f(g(s, t), h(s, t)) = \left.\frac{\partial f}{\partial x}\right|_{(g(s,t),h(s,t))} \frac{\partial g}{\partial t} + \left.\frac{\partial f}{\partial y}\right|_{(g(s,t),h(s,t))} \frac{\partial h}{\partial t}.$$

In other words,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \text{ and } \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Similar formulas are true for real-valued functions $f = f(x, y, z)$, with x, y, z real-valued functions of three variables.

Ex: Verify the Chain Rule for $f(x, y) = e^x \sin y$ with $x = st^2$ and $y = s^2 t$.