Vector Calculus
6.1 Integration by Parts

WARNING: $\int_a^b f(x)g(x)\,dx \neq \int_a^b f(x)\,dx \int_a^b g(x)\,dx$

$$\text{Ex} \quad \int_0^1 \underbrace{x \cdot x}_{x^2}\, dx = \frac{1}{3} \neq \underbrace{\int_0^1 x\,dx}_{\frac{1}{2}} \underbrace{\int_0^1 x\,dx}_{\frac{1}{2}} = \frac{1}{4}$$

<u>Integration by Parts:</u> Suppose f,g are differentiable over [a,b], and suppose f',g' are continuous over [a,b].

Indefinite Integral: $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

over (a,b)

Definite Integral: $\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_{x=a}^{b} - \int_a^b f'(x)g(x)\,dx$

Proof: Use the Product Rule,

$$\int_a^b \underbrace{\frac{d}{dx}\big(f(x)g(x)\big)}_{f(x)g'(x) + f'(x)g(x)}\,dx = f(x)g(x)\Big|_{x=a}^{b}$$

$$\Rightarrow \int_a^b f(x)g'(x)\,dx + \int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_{x=a}^{b}$$

Ex: Compute $\int_0^{\frac{\pi}{2}} x\sin x\,dx$

Sol: Using integration by parts,

$$\int_0^{\frac{\pi}{2}} \underbrace{x\sin x}_{f(x)g'(x)}\,dx = x(-\cos x)\Big|_{x=0}^{\frac{\pi}{2}} - \underbrace{\int_0^{\frac{\pi}{2}} 1\cdot(-\cos x)\,dx}_{\text{we hope this integral is easier to compute}}$$

$f(x)=x \leftarrow g'(x)=\sin x$

$f'(x)=1 \leftarrow g(x)=-\cos x$

an antiderivative

$$= -x\cos x\Big|_{x=0}^{\frac{\pi}{2}} + \underbrace{\int_0^{\frac{\pi}{2}} \cos x\,dx}_{\sin x\Big|_{x=0}^{\frac{\pi}{2}}}$$

What if we make the other choice?

$$\int_0^{\frac{\pi}{2}} x\sin x\,dx = (\sin x)\left(\frac{x^2}{2}\right)\Big|_{x=0}^{\frac{\pi}{2}} - \underbrace{\int_0^{\frac{\pi}{2}} (\cos x)\left(\frac{x^2}{2}\right)dx}$$

$f(x)=\sin x \leftarrow g'(x)=x$

$f'(x)=\cos x \leftarrow g(x)=\frac{x^2}{2}$

The problem with this choice is that the integral

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{2}\cos x\,dx$$

is *more* complicated that the original integral.

Ex: Compute the most general antiderivative of the given h over the largest possible open subset I of R, and give I.

1. h(x)=(x^2)e^x

Sol: Using integration by parts,

$$\int x^2 e^x \, dx = x^2 e^x - \underbrace{\int 2x e^x \, dx}_{\text{easier}}$$

$$f(x) = x^2 \quad g'(x) = e^x$$
$$f'(x) = 2x \quad g(x) = e^x$$

We use integration by parts again to compute

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$$

$$f(x) = x \quad g'(x) = e^x$$
$$f'(x) = 1 \quad g(x) = e^x$$

We conclude

$$\boxed{\int x^2 e^x \, dx = x^2 e^x - 2\left(x e^x - e^x\right) + C \quad \text{over } I = (-\infty, \infty)}$$

2. h(x)=ln(x)

Sol: We use integration by parts,

$$\int \ln x \, dx = \int (\ln x) \cdot 1 \, dx$$

$$= x \ln x - \int \frac{1}{x} \cdot x \, dx$$

$$f(x) = \ln x \quad g'(x) = 1$$
$$f'(x) = \frac{1}{x} \quad g(x) = x$$

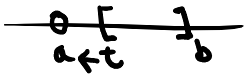
$$= x \ln x - \int dx = \boxed{x \ln x - x + C \quad \text{over } I = (0, \infty)}$$

6.6 Improper Integrals

Def: We define the following <u>improper integrals</u>.

Suppose f is continuous over (a,b]. We define

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$



If this limit exists and is finite, then we say $\int_a^b f(x)\,dx$ is convergent.

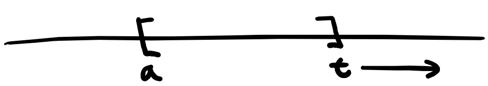
If the limit is $\pm\infty$, then we say $\int_a^b f(x)\,dx$ diverges to $\pm\infty$ and write

$$\int_a^b f(x)\,dx = \pm\infty$$

If the limit does not exist in the extended sense, then we say $\int_a^b f(x)\,dx$ is divergent.

Suppose f is continuous over [a,infinity). We define

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$



If the limit exists and is finite, then we say $\int_a^\infty f(x)\,dx$ is convergent.

If the limit is $\pm\infty$, then we say $\int_a^\infty f(x)\,dx$ diverges to $\pm\infty$

and write

$$\int_a^\infty f(x)\,dx = \pm\infty$$

If the limit does not exist in the extended sense, then we say $\int_a^\infty f(x)\,dx$ is divergent.

We similarly define improper integrals if f is continuous over [a,b),(a,b),(-infinity,a],(a,infinity),(-infinity,a)

we must compute <u>two</u> limits

Ex: Determine whether the following are divergent or convergent.

1. $\int_1^\infty x^{-2}\,dx$

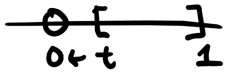
Sol: By definition, we must compute

$$\begin{aligned}
\int_1^\infty x^{-2}\,dx &= \lim_{t\to\infty} \int_1^t x^{-2}\,dx \\
&= \lim_{t\to\infty} \frac{x^{-1}}{-1}\Big|_{x=1}^{t} \\
&= \lim_{t\to\infty} \left(\frac{t^{-1}}{-1} - \frac{1^{-1}}{-1} \right) \\
&= \lim_{t\to\infty} \left(1 - \frac{1}{t} \right) = 1 \quad \boxed{\text{Convergent}}
\end{aligned}$$

2. $\int_0^1 x^{-\frac{1}{2}} dx$

Sol: The problem is at 0. By definition, we compute

$$\int_0^1 x^{-\frac{1}{2}} dx = \lim_{t\to 0^+} \int_t^1 x^{-\frac{1}{2}} dx$$



$$= \lim_{t\to 0^+} \left(\frac{x^{\frac{1}{2}}}{1/2} \Big|_{x=t}^{1} \right)$$

$$= \lim_{t\to 0^+} \left(\frac{1}{1/2} - \frac{t^{1/2}}{1/2} \right)$$

$$= 2 \qquad \boxed{\text{Convergent}}$$

3. $\int_0^1 x^{-2} dx$

Sol: We compute

$$\int_0^1 x^{-2} dx = \lim_{t\to 0^+} \int_t^1 x^{-2} dx$$

$$= \lim_{t\to 0^+} \frac{x^{-1}}{-1} \Big|_{x=t}^{1}$$

$$= \lim_{t\to 0^+} \left(\frac{1^{-1}}{-1} - \frac{t^{-1}}{-1} \right)$$

$$= \lim_{t\to 0^+} \left(\frac{1}{t} - 1 \right) = \infty \qquad \boxed{\text{Diverges to } \infty}$$

Fact: Suppose a>0.

$$\int_0^a x^p\,dx \quad \text{is convergent for } p>-1$$
$$\text{and diverges to } \infty \text{ for } p\le -1$$

$$\int_a^\infty x^p\,dx \quad \text{is convergent for } p<-1$$
$$\text{and diverges to } \infty \text{ for } p\ge -1$$

Similar is true for integrals over [-a,0),(-infinity,-a].

$-a<0$

Comparison Thm: Suppose f,g are continuous over (a,b], and suppose f(x)⩾g(x)⩾0 for all x in (a,b].

$$\text{If } \int_a^b f(x)\,dx \text{ is convergent, then } \int_a^b g(x)\,dx \text{ is convergent}$$

$$\text{If } \int_a^b g(x)\,dx = \infty, \text{ then } \int_a^b f(x)\,dx = \infty .$$

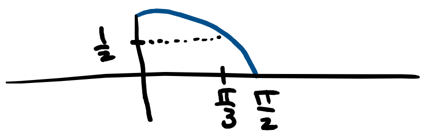
Similar rules hold for other types of improper integrals.

Ex: Use the Comparison Thm to determine whether the following are convergent or divergent.

1. $\int_0^{\pi/3} \frac{\cos x}{x^2}\,dx$

Sol: Note that

$$\frac{1}{2} \le \cos x \quad \text{for} \quad x \in \left[0, \frac{\pi}{3}\right]$$



This implies $\boxed{0 \leq \frac{1/2}{x^2} \leq \frac{\cos x}{x^2} \quad \text{for } x \in (0, \tfrac{\pi}{3}]}$

Since $\underline{\int_0^{\frac{\pi}{3}} x^{-2}\, dx = \infty} \Rightarrow \int_0^{\frac{\pi}{3}} \frac{1/2}{x^2}\, dx = \infty$

$$\underset{\text{Comparison Thm}}{\Longrightarrow} \int_0^{\frac{\pi}{3}} \frac{\cos x}{x^2}\, dx = \infty$$

$\boxed{\text{diverges to } \infty}$

2. $\int_0^{\pi} \frac{\sin x}{\sqrt{x}}\, dx$

Sol: Note that $\int_0^{\pi} \frac{1}{\sqrt{x}}\, dx = \int_0^{\pi} x^{-\frac{1}{2}}\, dx$ converges.

$$0 \leq \frac{\sin x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \text{for all } x \in (0, \pi]$$

Since $\int_0^{\pi} x^{-\frac{1}{2}}\, dx$ converges

then by the Comparison Thm $\int_0^{\pi} \frac{\sin x}{\sqrt{x}}\, dx$ $\boxed{\text{converges}}$

3. $\int_1^{\infty} e^{-x^2}\, dx$

Sol: Note that for $x \geq 1$

$$-x^2 \le -x \qquad\qquad -2^2 \le -2$$

$$\Rightarrow \quad 0 \le e^{-x^2} \le e^{-x} \quad \text{for } x \in [1, \infty).$$

Meanwhile,

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx$$

$$\underset{u=-x}{=} \lim_{t\to\infty} -e^{-x}\Big|_{x=1}^{t}$$

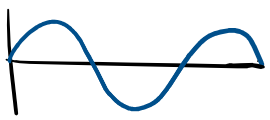
$$= \lim_{t\to\infty} \left(-e^{-t} - (-e^{-1})\right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{e} - \frac{1}{e^t}\right) = \frac{1}{e} \quad \text{converges}$$

We conclude by the Comparison Thm that

$$\int_1^\infty e^{-x^2}\,dx \ \boxed{\text{converges}}$$

Ex: $\int_0^\infty \sin x\,dx$ is divergent.



$$\int_0^{2\pi N} \sin x\,dx = 0 \qquad\qquad N = 1, 2, 3 \ldots$$

$$\int_0^{2\pi N + \pi} \sin x\,dx = 2$$

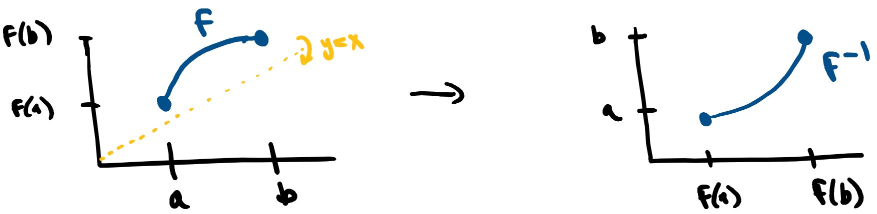
5.6 Inverse Trigonometric Functions

Fact: Defining the inverse of an increasing/decreasing function over an interval.

If f is increasing over [a,b], then we can define

$f(a) < f(b)$

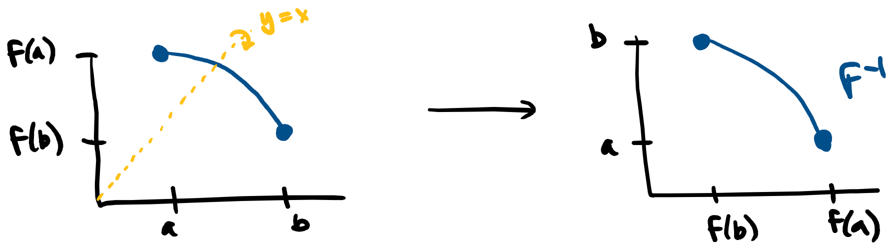
$$f^{-1}: [f(a), f(b)] \to [a,b]$$



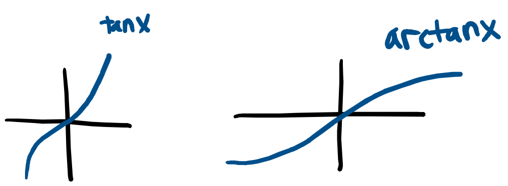
If f is decreasing over [a,b], then we can define

$f(a) > f(b)$

$$f^{-1}: [f(b), f(a)] \to [a,b]$$



Def: Inverse trigonometric functions.



arctangent/inverse tangent

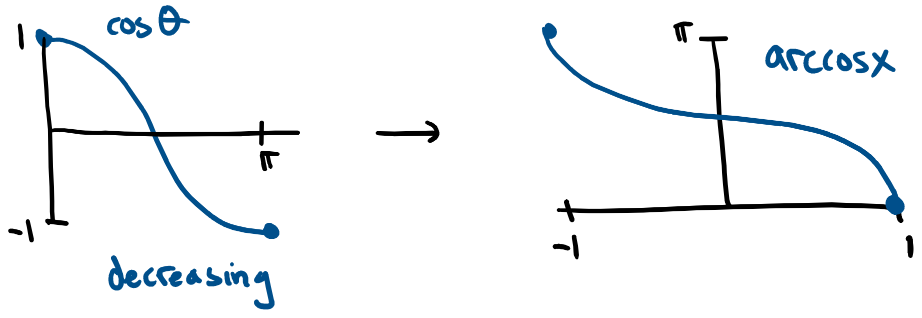
⇨ arctan(x)=$\tan^{-1}$(x) for all x

⇨ tan(arctan(x))=x for all x and arctan(tan(theta))=theta for theta in (-pi/2,pi/2).

arccosine/inverse cosine

⇨ arccos(x)=cos^{-1}(x) for x in [-1,1]

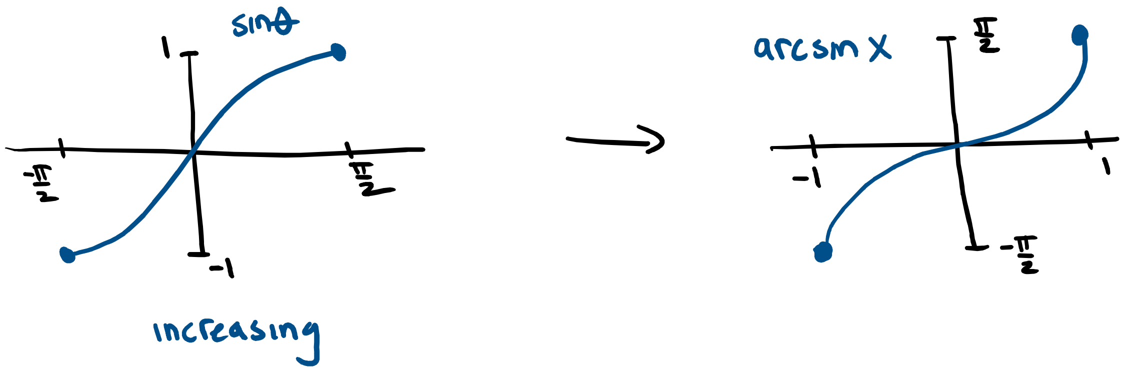
⇨ cos(arccos(x))=x for x in [-1,1] and
arccos(cos(theta))=theta for theta in [0,pi].



arcsine/inverse sine

⇨ arcsin(x)=sin^{-1}(x) for x in [-1,1]

⇨ sin(arcsin(x))=x for x in [-1,1] and
arcsin(sin(theta))=theta for theta in [-pi/2,pi/2].



Fact: Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \quad \text{for } x\in(-1,1)$$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \quad \text{for } x\in(-1,1)$$

Proof: Use the Chain Rule. Note that

$$\cos\left(\cos^{-1}x\right) = x \quad \text{for all } x \in (-1,1)$$

$$\Rightarrow \quad \frac{d}{dx}\cos\left(\cos^{-1}x\right) = \frac{d}{dx}x$$

$$\Rightarrow \quad -\sin\left(\cos^{-1}x\right)\cdot \underbrace{\frac{d}{dx}\cos^{-1}x}_{?} = 1 \quad \text{for all } x\in(-1,1)$$

$$\Rightarrow \quad \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sin\left(\cos^{-1}x\right)}$$

We must show that

$$\sin\left(\cos^{-1}x\right) = \sqrt{1-x^2} \quad \text{for } x\in(-1,1).$$

Using the Pythagorean Thm, we get

$$\left(\cos\left(\cos^{-1}x\right)\right)^2 + \left(\sin\left(\cos^{-1}x\right)\right)^2 = 1$$

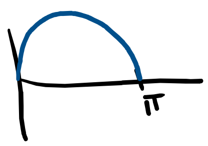
$$\Rightarrow \quad x^2 + \sin^2\left(\cos^{-1}x\right) = 1$$

$$\Rightarrow \quad \sin^2\left(\cos^{-1}x\right) = \underbrace{1-x^2}_{\geq 0} \quad \text{for } x\in(-1,1)$$

$$\Rightarrow \quad \left|\sin\left(\cos^{-1}x\right)\right| = \sqrt{1-x^2} \qquad a^2 = b \Rightarrow |a| = \sqrt{b}$$

By definition, we have

$$\cos^{-1}x \in [0,\pi] \quad \text{for } x\in(-1,1)$$



$$\Rightarrow \quad \sin\left(\cos^{-1}x\right) > 0 \quad \text{for } x\in(-1,1)$$

$$\Rightarrow \quad \underbrace{|\sin(\cos^{-1}x)|}_{\sin(\cos^{-1}x)} = \sqrt{1-x^2}$$

We conclude that

$$\frac{\partial}{\partial x}\cos^{-1}x = \frac{-1}{\sin(\cos^{-1}x)} = \frac{-1}{\sqrt{1-x^2}} \quad \text{for } x \in (-1,1)$$