VECTOR CALCULUS, Week 4

6.2 Trigonometric Integrals and Substitutions; 6.3 Partial Fractions; 6.4 Integration with Tables and Computer Algebra Systems; 9.1 Parametric Curves

6.2 Trigonometric Integrals and Substitutions

Fact: Consider $\int \sin^m x \cos^n x\, dx$ for m, n nonnegative integers.

- If m is odd, save one power of sine and use $\sin^2 x = 1 - \cos^2 x$.
  $\Rightarrow$ Apply the substitution $u = \cos x$.
- If n is odd, save one power of cosine and use $\cos^2 x = 1 - \sin^2 x$.
  $\Rightarrow$ Apply the substitution $u = \sin x$.
- If m, n are both even, use the **half-angle identities**
  $$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \text{ and } \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$
  Sometimes you can save time by using the **double-angle formula**
  $$\sin 2x = 2 \sin x \cos x.$$

Ex: Compute the most general antiderivative of the given f over the largest possible open subset I of $\mathbf{R}$, and give I.

1. $f(x) = \sin^3 x \cos^2 x$
2. $f(x) = 4 \sin^2 x \cos^2 x$, use the double-angle formula.
3. $f(x) = \sin^2 x$
4. $f(x) = \cos^2 x$

Fact: Consider $\int \tan^m x \sec^n x \; dx$ for m, n non-negative integers.

- If m is odd and $n \geq 1$, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$.
  $\Rightarrow$ Apply the substitution $u = \sec x$.

- If $n \geq 4$ is even, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$.
  $\Rightarrow$ Apply the substitution $u = \tan x$.

- In other cases
  $$\begin{aligned} &m \text{ odd and } n = 0 \\ &m \text{ even and } n = 0, 2 \\ &m \text{ even and } n \text{ odd} \end{aligned}$$
  there is no general strategy.

Ex: Compute the most general antiderivative of $f(x) = \tan^3 x \sec x$ over the largest possible open subset I of $\mathbf{R}$, and give I.

Fact: Evaluating some trigonometric integrals using angle addition formulas.

- To evaulate $\int \sin mx \cos nx \; dx$, use
  $$\sin mx \cos nx = \frac{1}{2}\Big(\sin((m-n)x) + \sin((m+n)x)\Big).$$

- To evaulate $\int \sin mx \sin nx \; dx$, use
  $$\sin mx \sin nx = \frac{1}{2}\Big(\cos((m-n)x) - \cos((m+n)x)\Big).$$

- To evaulate $\int \cos mx \cos nx \; dx$, use
  $$\cos mx \cos nx = \frac{1}{2}\Big(\cos((m-n)x) + \cos((m+n)x)\Big).$$

Ex: Compute the most general antiderivative of $f(x) = \sin 4x \cos 5x$ over the largest possible open subset I of $\mathbf{R}$, and give I.

Fact: To compute an integral involving the expression $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$, use the following trigonometric substitutions and identities.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta \le \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Ex: Compute the following integrals.

1. $\int_0^2 x^3\sqrt{4 - x^2}\ dx$
2. $\int_0^{2\sqrt{3}} \frac{x^3}{(2x^2+8)^{\frac{3}{2}}}\ dx$

6.3 Partial Fractions

Def: A **rational fraction** is a function of the form $f(x) = \frac{P(x)}{Q(x)}$ where P, Q are polynomials.

Ex: Compute the most general antiderivative of the given f over the largest possible open subset I of $\mathbf{R}$, and give I.

1. $f(x) = \frac{1}{x+1}$
2. $f(x) = \frac{2x+3}{x^2+3x+2}$

Easy Partial Fractions: Suppose P, Q are polynomials, and suppose the degree of P is strictly less than the degree of Q. If we can factor Q into distinct linear factors

$$Q(x) = (a_1x + b_1) \cdots (a_kx + b_k),$$

then we can decompose $\frac{P(x)}{Q(x)}$ into **partial fractions**: there are constants $A_1, \ldots, A_k$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \ldots + \frac{A_1}{a_kx + b_k}.$$

Ex: Compute the most general antiderivative of $f(x) = \frac{x^2+2x+2}{x^2-1}$ over the largest possible open subset I of $\mathbf{R}$, and give I.

Fact: If P, Q are polynomials and $P(x) = Q(x)$ for all x, then the coefficients of P, Q are equal.

6.4 Integration with Tables and Computer Algebra Systems

Fact: You can use the Table of Integrals on Reference Page 6-10 in the back of the book for the Homework. But for the MIDTERM and FINAL, you only need to memorize the Basic Table of Integrals.

9.1 Parametric Curves

Def: A **parametric plane curve** is a function $C : [a, b] \to \mathbf{R}^2$, written

$$C(t) = (x(t), y(t)) \text{ for } t \in [a, b]$$

where $x = x(t), y = y(t)$ are functions $x, y : [a, b] \to \mathbf{R}$. This means that for each $t \in [a, b]$, the function $C(t)$ gives you a point $(x(t), y(t))$ in the plane. We say the set

$$C = \{(x(t), y(t)) : t \in [a, b]\}$$

is the **image** of C. We call the variable t the **parameter**, and the equations

$$x = x(t) \text{ and } y = y(t) \text{ for } t \in [a, b]$$

we call **parametric equations** for C. We say $(x(a), y(a))$ is the **initial point**, while $(x(b), y(b))$ is the **terminal point**.

Ex:

1. The image of the curve $C(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$ is the unit circle.

2. The image of the curve $C(t) = (t, f(t))$ for $t \in [a, b]$ is the graph of the function $y = f(x)$ over $[a, b]$.

Ex: For each given parametric plane curve C, do the following.

(a) Eliminate the parameter to find a Cartesian equation for C.

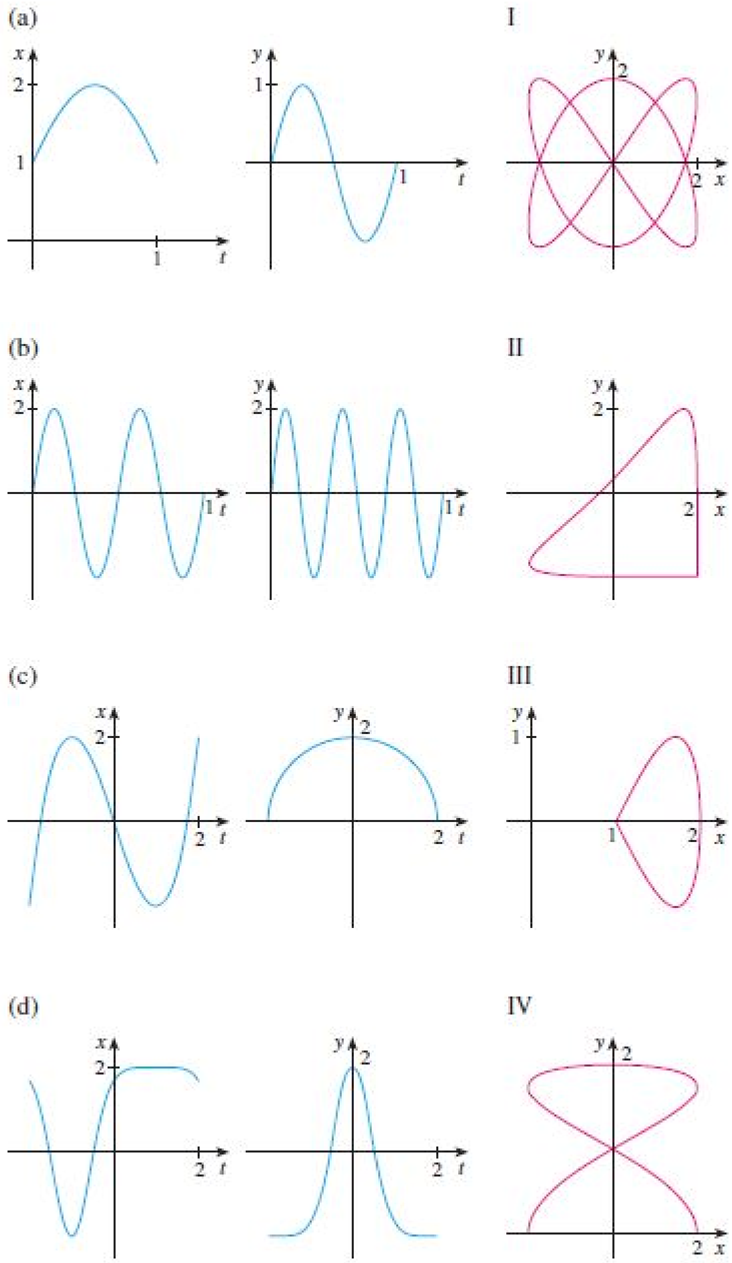
(b) Roughly sketch the image of C. Indicate with an arrow the direction in which the image is traced as t increases.

1. $C(t) = (t^2, \ln t^2)$ for $t \in (0, \infty)$

2. $C(t) = (2 \sin t, 3 \cos t)$ for $t \in [0, 2\pi]$

3. $C(t) = (\sqrt{t}, 1 - t)$ for $t \in [0, \infty)$

Ex: Match the graphs of the parametric equations $x = x(t)$ and $y = y(t)$ with the images of the parametric curves in red.



Ex: Use the graphs of $x = x(t)$ and $y = y(t)$ to sketch the image of the parametric plane curve. Indicate with arrows the direction in which the curve is traced as t increases.