VECTOR CALCULUS, Week 10

10.8 Arc Length and Curvature; 10.9 Motion in Space: Velocity and Acceleration; 11.1 Functions of Several Variables; 11.2 Limits and Continuity

10.8 Arc Length and Curvature; 10.9 Motion in Space: Velocity and Acceleration

Def: Suppose $\vec{r} = \vec{r}(t)$ is a parametric vector-valued function. Physicists call $\vec{r}(t)$ the **position vector**, $\vec{v}(t) = \vec{r}\,'(t)$ the **velocity vector**, $|\vec{v}(t)| = |\vec{r}\,'(t)|$ the **speed**, and $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$ the **acceleration vector**.

Ex: Compute the position vector, velocity vector, speed, and acceleration vector of $\vec{r}(t) =< 3t, -t^2, t^3 >$ at $t = 2$.

Def: Suppose $\vec{r} = \vec{r}(t)$ is a regular parametric space curve defined for t near a, and suppose $\vec{r}\,''(t) \neq \vec{0}$ exists for t near a.

- We say $\vec{T}(a) = \frac{\vec{r}\,'(a)}{|\vec{r}\,'(a)|}$ is the **unit tangent vector** of $\vec{r}$ at $t = a$.
- We say $\vec{N}(a) = \frac{\frac{d}{dt}\vec{T}(t)|_{t=a}}{|\frac{d}{dt}\vec{T}(t)|_{t=a}|}$ is the **unit normal vector** of $\vec{r}$ at $t = a$.
- We say $\vec{B}(a) = \vec{T}(a) \times \vec{N}(a)$ is the **binormal vector** of $\vec{r}$ at $t = a$.
- We say the vectors $\vec{T}(a), \vec{N}(a), \vec{B}(a)$ are the **TNB frame** or the **orthonormal frame** of $\vec{r}$ at $t = a$.

Ex: Compute the TNB frame of $\vec{r}(t) =< \cos t, \sin t, t >$ at all t.

Fact: Suppose $\vec{r} = \vec{r}(t)$ is a regular parametric space curve defined for t near a, and suppose $\vec{r}\,''(t) \neq \vec{0}$ exists for t near a.

- $\vec{T}(a), \vec{N}(a), \vec{B}(A)$ are **orthonormal**: $|\vec{T}(a)| = |\vec{N}(a)| = |\vec{B}(a)| = 1$ and $\vec{T}(a) \perp \vec{N}(a), \vec{T}(a) \perp \vec{B}(a), \vec{N}(a) \perp \vec{B}(a)$.
- $\vec{T}(a)$ is tangent to the image of $\vec{r}$ at $t = a$.
- $\vec{N}(a)$ points towards the center of the osculating circle of $\vec{r}$ at $t = a$.
- $\vec{B}(a)$ is a normal vector of the plane in space containing the osculating circle of $\vec{r}$ at $t = a$.

Use CalcPlot3D.

11.1 Functions of Several Variables

Def: A **real-valued function of two variables** is a function $f : \mathbf{R}^2 \to \mathbf{R}$ of the variables x, y, and denoted $f = f(x, y)$. The **graph** of f is the surface in space given by the equation $z = f(x, y)$, which is the set

$$\{(x, y, f(x, y)) : x, y \in \mathbf{R}\}.$$

Ex: Some examples.

1. Linear functions $f(x, y) = a + bx + cy$, with graph the non-vertical plane through (0,0,a) with normal in the direction of $\vec{n} =< b, c, -1 >$.
2. $f(x, y) = \sqrt{a^2 - x^2 - y^2}$ for $a > 0$, with graph the upper hemisphere centered at the origin with radius $= a$.

Def: Suppose $f = f(x, y)$ is a real-valued function. The **level curve of** f **at** $k \in \mathbf{R}$ is the curve in the plane $z = k$ given by the equation $f(x, y) = k$.

Ex: Sketch the level curves of $f(x, y) = 4x^2 + y^2$ at $k = 1, 4$.

Def: A **real-valued function of three variables** is a function $f : \mathbf{R}^3 \to \mathbf{R}$ of the variables x, y, z, and denoted $f = f(x, y, z)$. The **graph** of f is the **hypersurface** in $\mathbf{R}^4$ given by the set

$$\{(x, y, z, f(x, y, z)) : x, y, z \in \mathbf{R}\}.$$

The **level surface of** f **at** $k \in \mathbf{R}$ is the surface in space given by the equation $f(x, y, z) = k$.

Ex: Identify the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$ at $k = 1, 4$.

11.2 Limits and Continuity

Fact: Suppose $f = f(x)$ is defined near a, but perhaps not at a. The limit of f at a exists and is L if and only if the left- and right-hand limits of f at a exist and are both L.

Def: The phrase **for all** (x, y) **near** (a, b) means for all (x, y) inside of a circle centered at (a, b).

Def: Suppose $f = f(x, y)$ is a real-valued function defined near (a, b), but perhaps not at (a, b), and suppose $L \in \mathbf{R}$.

- We say $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if and only if $\lim_{t\to 0} f(\vec{r}(t)) = L$ for all continuous parametric plane curves $\vec{r} = \vec{r}(t)$ defined near $t = 0$ with $\vec{r}(0) =< a, b >$.
- If f is defined at (a, b) and $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$, then we say f **is continuous at** (a, b).
- We similarly define $\lim_{(x,y)\to(a,b)} f(x, y)$ **exists in the extended sense**, and write $\lim_{(x,y)\to(a,b)} f(x, y) = \pm\infty$.

Fact: Suppose $f = f(x, y)$ is a real-valued function defined near (a, b), but perhaps not at (a, b). If there are continuous parametric plane curves $\vec{r}_1, \vec{r}_2$ defined near $t = 0$ with $\vec{r}_1(0) = \vec{r}_2(0) =< a, b >$, but so that

$$\lim_{t\to 0} f(\vec{r}_1(t)) \neq \lim_{t\to 0} f(\vec{r}_2(t))$$

(including the possibility that one of the limits does not exist), then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Ex: Show that the limit of the given function f at $(0, 0)$ does not exist (even in the extended sense).

1. $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$
2. $f(x, y) = \frac{xy^2}{x^2+y^4}$...it is not enough to check along $y = mx$.
3. $f(x, y) = \sin\left(\frac{x}{x^2+y^2}\right)$

Fact: Suppose $f = f(x, y)$ is a real-valued function defined near $(0, 0)$, but perhaps not at $(0, 0)$, and suppose $g = g(x)$ is a defined near 0, but perhaps not at 0.

- If $f(x, y) = g(x)$ for all (x, y) near $(0, 0)$, but perhaps not at $(0, 0)$, then $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to 0} g(x)$.
- If $f(x, y) = g(y)$ for all (x, y) near $(0, 0)$, but perhaps not at $(0, 0)$, then $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{y\to 0} g(y)$.
- If $f(x, y) = g(\sqrt{x^2 + y^2})$ for all (x, y) near $(0, 0)$, but perhaps not at $(0, 0)$, then $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} g(r)$.

Similar is true for limits at (a, b).

Ex: Show that the limit of the given function f at the given (a, b) exists in the extended sense.

1. $f(x, y) = e^{2\ln|x|}(x^2 - y^4) + x^2y^4$ at $(a, b) = (1, 1)$
2. $f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ at $(a, b) = (0, 0)$
3. $f(x, y) = \frac{1}{(x^2+y^2)^3}$ at $(a, b) = (0, 0)$

Squeeze Thm: Suppose $g = g(x, y)$ is a real-valued function defined near $(0, 0)$, but perhaps not at $(0, 0)$, suppose $f = f(x), h = h(x)$ are defined near 0, but perhaps not at 0, and suppose $L \in \mathbf{R}$. Also suppose that

$$f(x) \leq g(x, y) \leq h(x)$$

for all (x, y) near $(0, 0)$, but perhaps not at $(0, 0)$.

- If $\lim_{x\to 0} f(x) = L = \lim_{x\to 0} h(x)$, then $\lim_{(x,y)\to(0,0)} g(x, y) = L$.
- If $\lim_{x\to 0} f(x) = \infty$, then $\lim_{(x,y)\to(0,0)} g(x, y) = \infty$.
- If $\lim_{x\to 0} h(x) = -\infty$, then $\lim_{(x,y)\to(0,0)} g(x, y) = -\infty$.

The Squeeze Theorem is also true in case

$$\left.\begin{array}{c} f(x) \\ f(y) \\ f(\sqrt{x^2+y^2}) \end{array}\right\} \leq g(x, y) \leq \left\{\begin{array}{c} h(x) \\ h(y) \\ h(\sqrt{x^2+y^2}) \end{array}\right.$$

(nine possible combinations). Similar is true for limits at (a, b).

Ex: Show that the limit of the given function g exists at $(0,0)$ in the extended sense.

1. $g(x,y) = \frac{|x|e^{-x^2}}{1+y^2}$
2. $g(x,y) = \frac{x^2 y}{x^2+y^2}$
3. $g(x,y) = \frac{e^{x^2}}{x^2+y^2}$
4. $g(x,y) = \frac{y^4}{x^2+y^2} - x^2$

Fact: Suppose $L, M \in \mathbf{R}$.

1. **Simplification Rule:** Suppose $f(x,y) = g(x,y)$ for all (x,y) near (a,b), but perhaps not at (a,b). If $\lim_{(x,y)\to(a,b)} g(x,y) = L$, then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.
2. If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$, then

   Addition Rule: $\lim_{(x,y)\to(a,b)} f(x,y) + g(x,y) = L + M$.

   Multiplication Rule: $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = LM$.

   Division Rule: If $M \neq 0$, then $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$.
3. **t-Substitution Rule:** If $h : \mathbf{R} \to \mathbf{R}$ is defined near $f(a,b)$, but perhaps not at $f(a,b)$, and if $\lim_{t\to f(a,b)} h(t)$ exists, then

$$\lim_{(x,y)\to(a,b)} h(f(x,y)) =_{t=f(x,y)} \lim_{t\to f(a,b)} h(t).$$

Mostly, we have to worry about limits $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)}$ where $\lim_{(x,y)\to(a,b)} g(x,y) = 0$.

Ex: Compute $\lim_{(x,y)\to(0,0)} \left(\frac{xy+2}{x^2+y+3}\right)^3$.

Fact: Similar definitions and facts are true for real-valued functions $f = f(x, y, z)$. In space, the phrase **for all** (x, y, z) **near** (a, b, c) means for all (x, y, z) inside a sphere centered at (a, b, c).

Ex:

1. Show that $\lim_{(x,y,z)\to(0,0,0)} \left(\frac{z^4}{x^2+y^2+z^2} + xyz \right)^4$ exists.

2. Show that the limit of $f(x, y, z) = \frac{xy+z^2}{x^2+y^2+z^2}$ at $(0, 0, 0)$ does not exist, even in the extended sense.