Vector Calculus

11.4 Tangent Planes and Linear Approximations

Def: Suppose f=f(x,y) is a real-valued function defined near (a,b). We say f is differentiable at (a,b) if and only if there exists A,B,C in R so that

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y)-(Ax+By+C)|}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

If f is differentiable at (a,b), then we say the plane z=Ax+By+C is the tangent plane or linear approximation of f at (a,b).

Fact: Suppose f=f(x,y) is a real-valued function defined near (a,b).

If f is differentiable at (a,b), then the partial derivatives $f_x(a,b)$,$f_y(a,b)$ exist and the tangent plane of f at (a,b) is given by

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

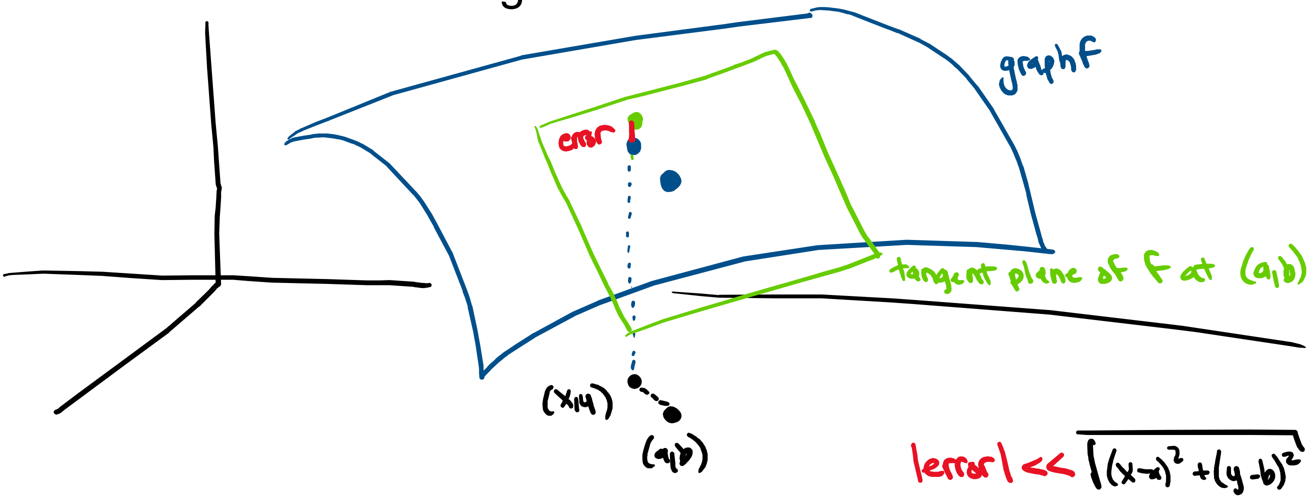
If f is differentiable at (a,b), then the tangent plane of f at (a,b) gives a good approximation for f near (a,b). In fact, the definition says

$$\lim_{(x,y)\to(a,b)} \frac{\left|f(x,y)-\left(\begin{array}{c}\text{tangent plane of } f\\ \text{at } (a,b)\text{, evaluated at } x,y\end{array}\right)\right|}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

Since the denominator is going to zero, then this means that

$$\left| f(x,y) - \left(f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b) \right) \right| \longrightarrow 0$$

faster than the bottom goes to zero.



If f_x, f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

However, I will give an example of a function f=f(x,y) so that f_x, f_y exist near (0,0), but so that f is not differentiable at (0,0). The problem is that f_x, f_y are not continuous at (0,0).

Similar definitions and results are true for real-valued functions f=f(x,y,z).

f is differentiable at (a,b,c) if and only if there is a hyperplane (three-dimensional plane in R^4) w=Ax+By+Cz+D which gives a good approximation for f near (a,b,c).

Ex: Show that the given function f is differentiable at the given point (a,b), and use the tangent plane to approximate the given value of f.

1. $f(x,y)=xe^{xy}$ at $(a,b)=(1,0)$, approximate $f(1.1,-0.1)$.

Sol: Since

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xe^{xy}\right) = e^{xy} + xye^{xy}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xe^{xy}\right) = x^2e^{xy}$$

exist and are continuous for all (x,y), then we conclude that f is differentiable at (a,b)=(1,0). Since (1.1,-0.1) is very close to (a,b)=(1,0), then

$$f(1.1,-0.1) \approx \underbrace{\text{the tangent plane of } f \text{ at } (a,b) \text{ evaluate at } (x,y)=(1.1,-0.1)}$$

$$= \underbrace{f_x(1,0)(x-1)+f_y(1,0)(y-0)+f(1,0)\Big|_{(x,y)=(1.1,-0.1)}}$$

$$= f_x(1,0)(1.1-1)-f_y(1,0)(-0.1-0)+f(1,0)$$

We compute $f(1,0)=1\cdot e^{1\cdot 0}=1$

$$f_x(1,0) = e^{xy} + xye^{xy}\Big|_{(x,y)=(1,0)}$$

$$= e^{1\cdot 0} + 1\cdot 0\, e^{1\cdot 0} = 1$$

$$f_y(1,0) = x^2 e^{xy}\Big|_{(x,y)=(1,0)} = 1^2\cdot e^{1\cdot 0} = 1$$

We conclude that

$$\boxed{f(1.1,-0.1) \approx 1\cdot(1.1-1) + 1\cdot(-0.1-0) + 1}$$

2. f(x,y)=x^3+xy at (a,b)=(2,3), approximate f(1.9,3.1).

Sol: Since

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3+xy) = 3x^2+y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3+xy) = x$$

exist and are continuous for all (x,y), then we conclude that f is differentiable at (a,b)=(2,3). Also, since (1.9,3.1) is very close to (a,b)=(2,3), then

$$f(1.9,3.1) \approx \underbrace{f_x(2,3)(x-2) + f_y(2,3)(y-3) + f(2,3)\Big|_{(x,y)=(1.9,3.1)}}_{\parallel}$$

$$f_x(2,3)(1.9-2) + f_y(2,3)(3.1-3) + f(2,3)$$

We compute $\quad f(2,3) = 2^3 + 2\cdot 3 = 8+6 = 14$

$$f_x(2,3) = 3x^2+y \Big|_{(x,y)=(2,3)} = 3\cdot 2^2+3 = 12+3 = 15$$

$$f_y(2,3) = x \Big|_{(x,y)=(2,3)} = 2$$

We conclude that

$$f(1.9,3.1) \approx -\frac{15}{10} + \frac{2}{10} + 14 = \underbrace{-\frac{13}{10}}_{-1.3} + 14 = \boxed{12.7}$$

$$\parallel$$

$$(1.9)^3 + (1.9)(3.1) = 12.749$$

Ex: Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

f_x, f_y exist for all (x,y), but f is not differentiable at (0,0). The problem is that f_x, f_y are not continuous at (0,0).

Sol: Suppose for contradiction that f is differentiable at (0,0). This implies that f is well approximated at (0,0) by the tangent plane of f at (0,0):

$$z = f_x(0,0)(x-0) + f_y(0,0)(y-0) + \underbrace{f(0,0)}_{= 0}$$

We compute

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{\frac{x\cdot 0}{x^2+0^2} - 0}{x}$$

$$= \lim_{x\to 0} \frac{0}{x} = \lim_{x\to 0} 0 = 0$$

Similarly

$$f_y(0,0) = 0$$

This means that the tangent plane of f at (0,0) is the horizontal plane z=0. Since f is differentiable at (0,0), then we must have

$$\underbrace{\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - (0)|}{\sqrt{(x-0)^2 + (y-0)^2}}}_{\parallel} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{\left|\frac{xy}{x^2+y^2}\right|}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \quad \lim_{(x,y)\to(0,0)} \frac{|xy|}{(x^2+y^2)^{\frac{3}{2}}} = 0 \qquad \cancel{0}$$

However, this limit does not exist! Consider $r_1(t)$=<t,t>, then
$y=x$

$$\lim_{t\to 0} \frac{|t\cdot t|}{(t^2+t^2)^{\frac{3}{2}}} = \lim_{t\to 0} \frac{t^2}{(2t^2)^{\frac{3}{2}}}$$

$$= \lim_{t\to 0} \frac{t^2}{2^{\frac{3}{2}}|t|^3} = \lim_{t\to 0} \frac{1}{2^{\frac{3}{2}}|t|} = \infty$$

Also consider $r_2(t)=<t,0>$, then

$$\lim_{t\to 0} \frac{|t\cdot 0|}{(t^2+0^2)^{\frac{3}{2}}} = \lim_{t\to 0} 0 = 0$$

not equal

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{|xy|}{(x^2+y^2)^{\frac{3}{2}}} \text{ DNE}$$

We conclude that f is not differentiable at (0,0).

11.5 The Chain Rule

Chain Rule: Suppose f=f(x,y) is a real-valued differentiable function, and suppose g=g(t),h=h(t) are differentiable. Then f(g(t),h(t)) is differentiable with

$$\frac{\partial}{\partial t} f(g(t),h(t)) = \left.\frac{\partial f}{\partial x}\right|_{(g(t),h(t))} g'(t) + \left.\frac{\partial f}{\partial y}\right|_{(g(t),h(t))} h'(t)$$

We write this as

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

For f=f(x,y,z), we have

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Ex: Verify the Chain Rule is true for

$$f(x,y) = x^2 + xy \quad \text{and} \quad \underset{g(t)=t^2}{x = t^2}, \quad \underset{h(t)=\cos t}{y = \cos t}$$

Sol: First, we compute

$$\frac{\partial}{\partial t} f(x(t),y(t)) = \frac{\partial}{\partial t} f(t^2, \cos t)$$

$$= \frac{\partial}{\partial t}\left(x^2 + xy \Big|_{\substack{x=t^2 \\ y=\cos t}} \right)$$

$$= \frac{\partial}{\partial t}\left((t^2)^2 + t^2 \cos t \right)$$

$$= \frac{\partial}{\partial t}\left(t^4 + t^2\cos t\right)$$

$$\Rightarrow \text{"}\frac{\partial f}{\partial t}\text{"} = \boxed{4t^3 + 2t\cos t - t^2\sin t}$$

Second, we compute

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}(x^2+xy)\Big|_{(t^2,\cos t)} \cdot \frac{\partial}{\partial t} t^2$$

$$+ \frac{\partial}{\partial y}(x^2+xy)\Big|_{(t^2,\cos t)} \cdot \frac{\partial}{\partial t}\cos t$$

$$= (2x+y)\Big|_{(t^2,\cos t)} \cdot 2t$$

$$+ (x)\Big|_{(t^2,\cos t)} \cdot (-\sin t)$$

$$= (2t^2 + \cos t)2t + (t^2)(-\sin t)$$

$$= \boxed{4t^3 + 2t\cos t - t^2\sin t}$$

Chain Rule: Suppose f=f(x,y) is a real-valued differentiable function, and suppose g=g(s,t),h=h(s,t) are real-valued differentiable functions, then f(g(s,t),h(s,t)) is a real-valued differentiable function with

$$\frac{\partial}{\partial s} f(g(s,t),h(s,t)) = \left.\frac{\partial f}{\partial x}\right|_{(g(s,t),h(s,t))} \frac{\partial g}{\partial s} + \left.\frac{\partial f}{\partial y}\right|_{(g(s,t),h(s,t))} \frac{\partial h}{\partial s}$$

and

$$\frac{\partial}{\partial t} f(g(s,t),h(s,t)) = \left.\frac{\partial f}{\partial x}\right|_{(g(s,t),h(s,t))} \frac{\partial g}{\partial t} + \left.\frac{\partial f}{\partial y}\right|_{(g(s,t),h(s,t))} \frac{\partial h}{\partial t}$$

In other words

$$\text{"}\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\text{"} \quad \text{and} \quad \text{"}\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\text{"}$$

Similar formulas are true for real-valued functions f=f(x,y,z) with x,y,z real-valued functions of three variables.

Ex: Verify the Chain Rule for

$$f(x,y) = e^x \sin y \quad \text{with} \quad \underset{g(s,t)=st^2}{x = st^2} \quad \text{and} \quad \underset{h(s,t)=s^2t}{y = s^2 t}$$

Sol: We compute

$$\frac{\partial}{\partial s} f(x(s,t),y(s,t)) = \frac{\partial}{\partial s}\left(e^x \sin y \,\Big|_{(st^2,s^2t)}\right)$$

$$= \frac{\partial}{\partial s}\left(e^{st^2} \sin(s^2 t)\right)$$

$$= \boxed{t^2 e^{st^2} \sin(s^2 t) + 2st\, e^{st^2} \cos(s^2 t)}$$

Second, we compute

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial}{\partial x}\left(e^x \sin y\right)\Big|_{(st^2, s^2t)} \frac{\partial}{\partial s}\left(st^2\right)$$

$$+ \frac{\partial}{\partial y}\left(e^x \sin y\right)\Big|_{(st^2, s^2t)} \frac{\partial}{\partial s}\left(s^2 t\right)$$

$$= e^x \sin y \,\Big|_{(st^2, s^2t)} \cdot t^2$$

$$+ e^x \cos y \,\Big|_{(st^2, s^2t)} \cdot 2st$$

$$= \boxed{t^2 e^{st^2} \sin(s^2 t) + 2st\, e^{st^2} \cos(s^2 t)}$$