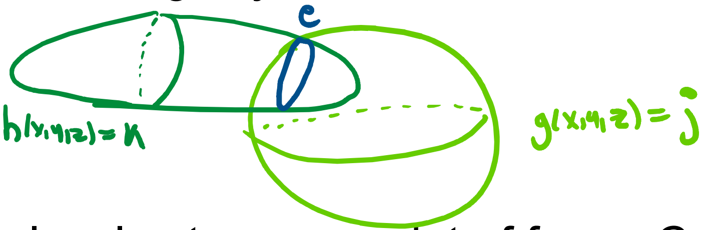
Vector Calculus
14.8 Lagrange Multipliers

Thm (Two Constraint Lagrange Multipliers): Suppose f=f(x,y,z),g=g(x,y,z),h=h(x,y,z) are differentiable at (a,b,c), and suppose j=g(a,b,c) and k=h(a,b,c). Also suppose that C is the curve in space given by the intersection between the level surface of g at j and the level surface of h at k.

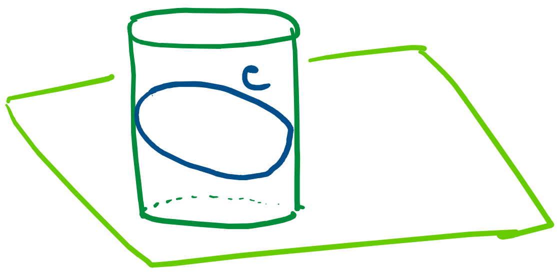



If (a,b,c) is a local extremum point of f over C, and if $\nabla$g(a,b,c), $\nabla$h(a,b,c)=/0, then there is lambda,mu in R so that

$\mu$ Greek "m"

$$\nabla f(a,b,c) = \lambda \nabla g(a,b,c) + \mu \nabla h(a,b,c)$$

Ex: Find the absolute extremum points and values of f(x,y,z)=x+2y+4z over the curve in space C given by the intersection between the plane x-y+z=1 and the unit cylinder $x^2+y^2=1$.




Sol: Consider g(x,y,z)=x-y+z and j=1, and h(x,y,z)=$x^2+y^2$ and k=1. We must consider the equations

$$\nabla f = \lambda \nabla g + \mu \nabla h \qquad g(x,y,z)=1, \quad h(x,y,z)=1$$

We must consider the equations

$$\begin{aligned}
&① \quad 1 = \lambda + 2\mu x \\
&② \quad 2 = -\lambda + 2\mu y \\
&③ \quad 3 = \lambda \\
&④ \quad x-y+z=1 \\
&⑤ \quad x^2+y^2=1
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&① \quad -2 = 2\mu x \Rightarrow x = \frac{-1}{\mu} \\
&② \quad 5 = 2\mu y \Rightarrow y = \frac{5}{2\mu} \\
&④ \quad x-y+z=1 \Rightarrow z = 1-x+y \\
&⑤ \quad x^2+y^2=1
\end{aligned}$$

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

$$\Downarrow$$

$$\mu^2 = \frac{29}{4}$$

$$\mu = \frac{\pm\sqrt{29}}{2}$$

$$\boxed{\mu = \frac{\sqrt{29}}{2}}$$

$$x = -\frac{2}{\sqrt{29}}$$

$$y = \frac{5}{2}\cdot\frac{2}{\sqrt{29}} = \frac{5}{\sqrt{29}}$$

$$z = 1 + \frac{2}{\sqrt{29}} + \frac{5}{\sqrt{29}}$$

$$z = 1 + \frac{7}{\sqrt{29}}$$

$$\boxed{\left(\frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1+\frac{7}{\sqrt{29}}\right)}$$

$$\boxed{\mu = \frac{-\sqrt{29}}{2}}$$

$$x = \frac{2}{\sqrt{29}}$$

$$y = -\frac{5}{\sqrt{29}}$$

$$z = 1 - \frac{2}{\sqrt{29}} - \frac{5}{\sqrt{29}}$$

$$z = 1 - \frac{7}{\sqrt{29}}$$

$$\boxed{\left(\frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}, 1-\frac{7}{\sqrt{29}}\right)}$$

Since f(x,y,z)=x+2y+3z, then we compute

$$f\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1+\frac{7}{\sqrt{29}}\right) = -\frac{2}{\sqrt{29}} + \frac{10}{\sqrt{29}} + 3 + \frac{21}{\sqrt{29}} = 3 + \frac{29}{\sqrt{29}}$$
$$= 3+\sqrt{29}$$

$$f\left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1-\frac{7}{\sqrt{29}}\right) = \frac{2}{\sqrt{29}} - \frac{10}{\sqrt{29}} + 3 - \frac{21}{\sqrt{29}} = 3 - \frac{29}{\sqrt{29}}$$
$$= 3-\sqrt{29}$$

We conclude that f has absolute maximum point

$$\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1+\frac{7}{\sqrt{29}}\right)$$

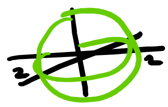
over the curve C with absolute maximum value $=3+\sqrt{29}$, and f has absolute minimum point

$$\left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1-\frac{7}{\sqrt{29}}\right)$$

with absolute minimum value $=3-\sqrt{29}$.

Ex: Find the absolute extremum values of the given function f over the level surface of the given function g at the given k in R.

1. $f(x,y,z)=xy^2z$ with $g(x,y,z)=x^2+y^2+z^2$ at $k=4$



Sol: Consider the equations

① $y^2z = 2\lambda x$

☆ ② $2xyz = 2\lambda y$

③ $xy^2 = 2\lambda z$

④ $x^2+y^2+z^2=4$

$\Rightarrow$ $\boxed{y=0}$ $\downarrow$ $f(x,0,z)=\boxed{0}$

or

$y \neq 0$

① $y^2z=2\lambda x$

② $xz=\lambda$

③ $xy^2=2\lambda z$

④ $x^2+y^2+z^2=4$

$\Leftarrow$

$y \neq 0$ and $x, z \neq 0$

$$\text{①}\ y^2 z = 2\lambda x \Rightarrow \frac{y^2 z}{2x} = \lambda$$

$$\text{②}\ xz = \lambda \Rightarrow xz = \lambda$$

$$\text{③}\ xy^2 = 2\lambda z \Rightarrow \frac{xy^2}{2z} = \lambda$$

$$\text{④}\ x^2 + y^2 + z^2 = 4$$

$$\Downarrow$$

$\boxed{x=0} \Rightarrow f(0,y,z) = \boxed{0}$

$\boxed{x \neq 0} \Rightarrow \boxed{z=0}$ or $\boxed{z \neq 0}$

$\boxed{z=0} \Rightarrow f(x,y,0) = \boxed{0}$

$$\Rightarrow \begin{cases} \dfrac{y^2 z}{2x} = xz = \dfrac{xy^2}{2z} \\ x^2 + y^2 + z^2 = 4 \end{cases}$$

$$\frac{y^2 z}{2x} = xz \Rightarrow y^2 = 2x^2$$

$$xz = \frac{xy^2}{2z} \Rightarrow 2z^2 = y^2 = 2x^2$$

$$\Rightarrow z^2 = x^2$$

$$\text{④} \Rightarrow x^2 + 2x^2 + x^2 = 4$$

$$\Rightarrow 4x^2 = 4$$

$$\Rightarrow x = \pm 1 \Rightarrow \begin{matrix} y^2 = 2 \\ z^2 = 1 \end{matrix}$$

$$\Rightarrow (\pm 1, \pm\sqrt{2}, \pm 1) \quad \text{8 points}$$

Since $f(x,y,z)=xy^2z$, then we compute

$$f(1, \pm\sqrt{2}, 1) = f(-1, \pm\sqrt{2}, -1) = 1 \cdot (\pm\sqrt{2})^2 \cdot 1 = 2$$

$$f(1, \pm\sqrt{2}, -1) = f(-1, \pm\sqrt{2}, 1) = -1 \cdot (\pm\sqrt{2})^2 \cdot 1 = -2$$

We conclude that f has absolute maximum value =2 over the level surface of g at k=4, and f has absolute minimum value =-2 over the level surface of g at k=4.

2. $f(x,y,z)=x^2+y^2+z^2$ with $g(x,y,z)=x^4+y^4+z^4$ at $k=1$

Sol: Consider the equations

$*$ ① $2x=4\lambda x^3$

② $2y=4\lambda y^3$

③ $2z=4\lambda z^3$

④ $x^4+y^4+z^4=1$

$\Rightarrow$ $\boxed{x=0}$ or $\boxed{x\neq 0}$

$\boxed{x\neq 0}$ $\Downarrow$ next page

$\boxed{x=0}$:

② $y=2\lambda y^3$

③ $z=2\lambda z^3$

④ $y^4+z^4=1$

$\boxed{y=0}$ or $\boxed{y\neq 0}$

$\boxed{y=0}$ $\Downarrow$ $z^4=1$ $\Downarrow$ $\boxed{(0,0,\pm1)}$

$\boxed{y\neq 0}$:

② $1=2\lambda y^2$

③ $z=2\lambda z^3$

④ $y^4+z^4=1$

$\boxed{z=0}$ or $\boxed{z\neq 0}$

$\boxed{z=0}$ $\Downarrow$ $y^4=1$ $\Downarrow$ $\boxed{(0,\pm1,0)}$

$\boxed{z\neq 0}$:

① $1=2\lambda y^2$

③ $1=2\lambda z^2$

④ $y^4+z^4=1$

①② $\Rightarrow$ $\lambda\neq 0$

$\Rightarrow$ $y^2=\frac{1}{2\lambda}=z^2$

$\Rightarrow$ $y^4=z^4$

④ $\Rightarrow$ $2y^4=1$

$\Rightarrow$ $y^4=\frac{1}{2}$

$\Rightarrow$ $y=\pm\frac{1}{\sqrt[4]{2}}$

$$\boxed{\left(0,\pm\frac{1}{\sqrt[4]{2}},\pm\frac{1}{\sqrt[4]{2}}\right)}$$

$\boxed{x \neq 0}$

$*$ ① $2x = 4\lambda x^3 \Rightarrow$ ① $1 = 2\lambda x^2$

② $2y = 4\lambda y^3$ ② $y = 2\lambda y^3 \Rightarrow \boxed{y=0}$ or $\boxed{y \neq 0}$

③ $2z = 4\lambda z^3$ ③ $z = 2\lambda z^3$

④ $x^4 + y^4 + z^4 = 1$

$\boxed{y=0}$ $\Downarrow$

① $1 = 2\lambda x^2$

③ $z = 2\lambda z^3$

④ $x^4 + z^4 = 1$

$\boxed{y \neq 0}$

① $1 = 2\lambda x^2$

② $1 = 2\lambda y^2$

③ $z = 2\lambda z^3$

④ $x^4 + y^4 + z^4 = 1$

$\Leftarrow$ $\boxed{z=0}$ or $\boxed{z \neq 0}$

$\boxed{z=0}$ ④ $\Downarrow$ $x^4 = 1 \Downarrow$ $\boxed{(\pm 1, 0, 0)}$

$\boxed{z \neq 0}$

① $1 = 2\lambda x^2$

③ $1 = 2\lambda z^2$

④ $x^4 + z^4 = 1$

$\Downarrow$

$\boxed{\left(\pm \frac{1}{\sqrt[4]{2}}, 0, \pm \frac{1}{\sqrt[4]{2}}\right)}$

$\boxed{x \neq 0}$

$\boxed{y \neq 0}$

① $1 = 2\lambda x^2$

② $1 = 2\lambda y^2$

③ $z = 2\lambda z^3$

④ $x^4 + y^4 + z^4 = 1$

$\Rightarrow$ $\boxed{z=0}$ or $\boxed{z \neq 0}$

$\boxed{z=0}$

① $1 = 2\lambda x^2$

② $1 = 2\lambda y^2$

④ $x^4 + y^4 + z^4 = 1$

$\boxed{\left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}, 0\right)}$

$\boxed{z \neq 0}$

① $1 = 2\lambda x^2$

② $1 = 2\lambda y^2$

③ $1 = 2\lambda z^2$

④ $x^4 + y^4 + z^4 = 1$

①②③ $\Rightarrow x^2 = y^2 = z^2$

④ $\Rightarrow x^4 + x^4 + x^4 = 1$

$$\Rightarrow \quad 3x^4 = 1$$

$$\Rightarrow \quad x = \pm\frac{1}{\sqrt[4]{3}}$$

$$\Rightarrow \quad \boxed{\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[4]{3}}\right)}$$

- $x=0$
  - $y=0$ $\Rightarrow$ $\boxed{(0,0,\pm 1)}$
  - $y\neq 0$
    - $z=0$ $\Rightarrow$ $\boxed{(0,\pm 1,0)}$
    - $z\neq 0$ $\Rightarrow$ $\boxed{\left(0,\frac{\pm 1}{\sqrt[4]{2}},\frac{\pm 1}{\sqrt[4]{2}}\right)}$
- $x\neq 0$
  - $y=0$
    - $z=0$ $\Rightarrow$ $\boxed{(\pm 1,0,0)}$
    - $z\neq 0$ $\Rightarrow$ $\boxed{\left(\frac{\pm 1}{\sqrt[4]{2}},0,\frac{\pm 1}{\sqrt[4]{2}}\right)}$
  - $y\neq 0$
    - $z=0$ $\Rightarrow$ $\boxed{\left(\frac{\pm 1}{\sqrt[4]{2}},\frac{\pm 1}{\sqrt[4]{2}},0\right)}$
    - $z\neq 0$ $\Rightarrow$ $\boxed{\left(\frac{\pm 1}{\sqrt[4]{3}},\frac{\pm 1}{\sqrt[4]{3}},\frac{\pm 1}{\sqrt[4]{3}}\right)}$

Since $f(x,y,z)=x^2+y^2+z^2$, then we compute

$$f(0,0,\pm 1) = f(0,\pm 1,0) = f(\pm 1,0,0) = 1$$

$$f\left(\frac{\pm 1}{\sqrt[4]{2}},\frac{\pm 1}{\sqrt[4]{2}},0\right) = f\left(\frac{\pm 1}{\sqrt[4]{2}},0,\frac{\pm 1}{\sqrt[4]{2}}\right) = f\left(0,\frac{\pm 1}{\sqrt[4]{2}},\frac{\pm 1}{\sqrt[4]{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$f\left(\frac{\pm 1}{\sqrt[4]{3}},\frac{\pm 1}{\sqrt[4]{3}},\frac{\pm 1}{\sqrt[4]{3}}\right) = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

We conclude that f has absolute maximum value $=\sqrt{3}$ over the level surface of g at k=1, and f has absolute minimum value =1 over the level surface of g at k=1.

Ex: For f(x,y,z)=x+2y+3z over C the intersection between x-y+z=1 and $x^2+y^2=1$, parameterize C by setting

$$x=\cos t$$
$$y=\sin t$$
$$z=1-x+y=1-\cos t+\sin t$$

$$\Rightarrow \quad \vec{r}(t)=\langle \cos t, \sin t, 1-\cos t+\sin t\rangle \text{ for } 0\le t\le 2\pi$$

This means that we want to find the absolute extremum values of

$$h(t)=f(\vec{r}(t))=\cos t+2(\sin t)+3(1-\cos t+\sin t)$$
$$\text{for } 0\le t\le 2\pi$$

This is a single-variable problem. Compare the values of h(0),h(2pi) and h(c) for all c in (0,2pi) with h'(c)=0.