Vector Calculus
11.3 Partial Derivatives

Def: Suppose f=f(x,y) is a real-valued function defined near (a,b).

We say the partial derivative of f with respect to x at (a,b) is the limit

curly

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(a,b)} = f_x(a,b) = D_1 f(a,b) = D_x f(a,b)$$

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$$= \lim_{x \to a} \underbrace{\frac{f(x,b)-f(a,b)}{x-a}}_{\text{single-variable function of } x}$$

assuming this limit exists (is finite).

We say the partial derivative of f with respect to y at (a,b) is the limit

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(a,b)} = f_y(a,b) = D_2 f(a,b) = D_y f(a,b) = \lim_{y \to b} \underbrace{\frac{f(a,y)-f(a,b)}{y-b}}_{\text{single-variable function of } y}$$

assuming this limit exists (is finite).

Fact: Suppose f=f(x,y) is a real-valued function defined near (a,b).

Define g(x)=f(x,b), then

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(a,b)} = \frac{d}{dx} g(x) \Big|_{x=a} = g'(a)$$

Define g(y)=g(a,y), then

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(a,b)} = \frac{d}{dy} g(y) \Big|_{y=a} = g'(b).$$

In other words, to compute a partial derivative with respect to x, we can pretend that y is constant and take the derivative with respect to x.

Ex: Let $f(x,y)=x^3+x^2y^2-y$.

1. Compute $f_x(2,3)$ and $f_y(2,3)$.

Sol: First, we compute the partial derivative with respect to x. Consider

$$g(x) \underset{y=3}{=} f(x,3) = x^3 + x^2 \cdot 3^2 - 3$$

$$\Rightarrow \quad g(x) = x^3 + 9x^2 - 3$$

We conclude that

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(2,3)} = \frac{d}{dx} g(x) \Big|_{x=2} = \frac{d}{dx} x^3 + 9x^2 - 3 \Big|_{x=2}$$

$$= 3x^2 + 18x \Big|_{x=2} = \boxed{3 \cdot 2^2 + 18 \cdot 2}$$

We can also more briefly just compute

$$\frac{\partial f}{\partial x}\Big|_{(x,y)=(2,3)} = \frac{\partial}{\partial x}\left(f(x,y)\big|_{y=3}\right)\Big|_{x=2} \quad \checkmark$$

$$= \frac{\partial}{\partial x}\left(f(x,3)\right)\Big|_{x=2} \quad \checkmark$$

You can also first take the derivative with respect to x, where you treat y like a constant, and then plug in x=2 and y=3.

$$\frac{\partial f}{\partial x}\Big|_{(x,y)=(2,3)} = \frac{\partial}{\partial x}\left(x^3 + x^2y^2 - y\right)\Big|_{(x,y)=(2,3)}$$

$$= 3x^2 + 2xy^2 - 0\ \Big|_{(x,y)=(2,3)}$$

$$= 3\cdot 2^2 + 2\cdot 2\cdot 3^2$$

$$= \boxed{3\cdot 2^2 + 18\cdot 2} \quad \checkmark$$

Similarly, we can compute

$$f_y(2,3) = \frac{\partial}{\partial y}\left(x^3+x^2y^2-y\right)\Big|_{(x,y)=(2,3)}$$

$$= 0 + (x^2)2y - 1\ \Big|_{(x,y)=(2,3)}$$

$$= 2^2\cdot 2\cdot 3 - 1 = \boxed{8\cdot 3 - 1}$$

Or we can compute

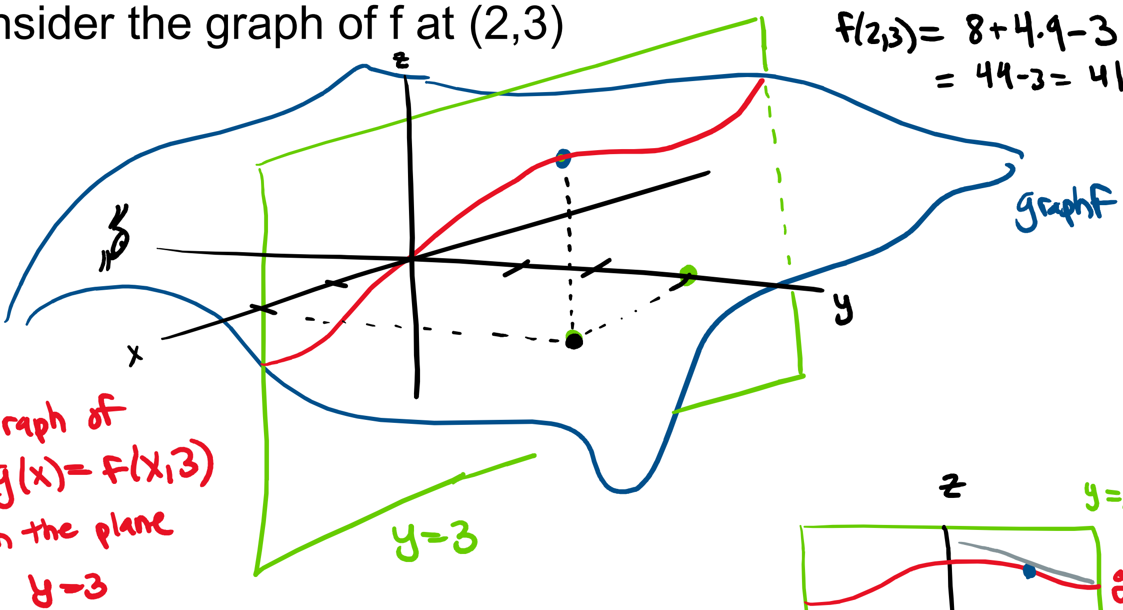
$$f_y(2,3) = \frac{\partial}{\partial y}\left(f(2,y)\right)\Big|_{y=3} = \frac{\partial}{\partial y}\left(8+4y^2-y\right)\Big|_{y=3}$$

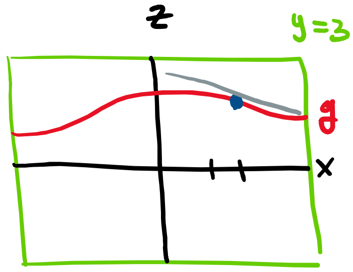
$$= 0 + 8y - 1 \Big|_{y=3} = \boxed{8 \cdot 3 - 1}$$

2. Find a line in the plane y=3 which is tangent to the graph of f at (2,3).

Sol: Consider the graph of f at (2,3)

$$f(2,3) = 8 + 4 \cdot 9 - 3 = 44 - 3 = 41$$





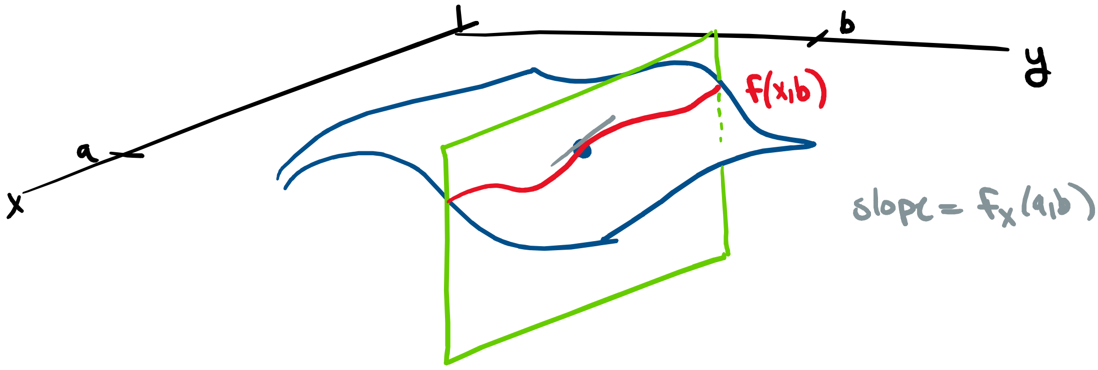
We conclude that a line in the plane y=3 which is tangent to the graph of f at (2,3) is the tangent line of g=g(x) at x=2. This line is given by the equations

$$\begin{cases} z = g'(2)(x-2) + g(2) \\ y = 3 \end{cases}$$

$$\Rightarrow \begin{cases} z = f_x(2,3)(x-2) + f(2,3) \\ y = 3 \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} z = (3\cdot 2^2 + 18\cdot 2)(x-2) + 41 \\ y = 3 \end{cases}}$$

$f_x(a,b)$ gives you the slope of the line in the plane y=b which is tangent to the graph of f at (a,b)



Def: We denote the <u>second partial derivatives of f</u> as follows.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx} = f_{11}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy} = f_{12}$$

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$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx} = f_{21}$$

closer

mixed partial derivatives of f

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy} = f_{22}$$

Thm: Suppose f=f(x,y) is a real-valued function defined near (a,b). If f_{xy} ,f_{yx} exist and are continuous near (a,b), then

$$f_{xy} = f_{yx} \quad \text{near} \quad (a,b)$$

Ex: For $f(x,y)=x^2+xy+e^{x^2y}$, verify $f_{xy}=f_{yx}$ for all (x,y).

Sol: First, we compute

$$\underset{\substack{\uparrow \\ \text{closer} \\ \text{"} \\ \text{First}}}{f_{xy}}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(x^2+xy+e^{x^2y}\right)\right)$$

$$= \frac{\partial}{\partial y}\left(2x+y+2xye^{x^2y}\right)$$

$$= 0+1+2xe^{x^2y}+2xy\cdot x^2e^{x^2y}$$

$$= \boxed{1+2xe^{x^2y}+2x^3ye^{x^2y}}$$

Second, we compute

$$\underset{\substack{\uparrow \\ \text{closer} \\ \text{"} \\ \text{First}}}{f_{yx}}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(x^2+xy+e^{x^2y}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(0+x+x^2e^{x^2y}\right)$$

$$= 1+2xe^{x^2y}+x^2\cdot 2xye^{x^2y}$$

$$= \boxed{1+2xe^{x^2y}+2x^3ye^{x^2y}}$$

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Fact: Similar definitions and results are true for real-valued functions f=f(x,y,z).

Ex: For f(x,y,z)=cos(xy+z), verify that $f_{xyz}=f_{zxy}$ for all (x,y,z).

Sol: We compute

$$f_{xyz}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\cos(xy+z)\right)\right)$$

$$= \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(-y\sin(xy+z)\right)\right)$$

$$= \frac{\partial}{\partial z}\left(-\sin(xy+z) - y\cdot x\cos(xy+z)\right)$$

$$= -\overset{\cos}{\cancel{\sin}}(xy+z) + yx\sin(xy+z)$$

$$= \boxed{-\overset{\cos}{\cancel{\sin}}(xy+z) + xy\sin(xy+z)}$$

$$f_{zxy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\cos(xy+z)\right)\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(-\sin(xy+z)\right)\right)$$

$$= \frac{\partial}{\partial y}\left(-y\cos(xy+z)\right)$$

$$= -\cos(xy+z) - y\left(-x\sin(xy+z)\right)$$

$$= \boxed{-\cos(xy+z) + xy\sin(xy+z)}$$

Ex: Let
$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

then f_{xy}, f_{yx} for all (x,y) near (0,0), but $f_{xy}(0,0)=-1$ while

$f_{yx}(0,0)=1$. The problem is that f_{xy}, f_{yx} are not continuous at (0,0).

Sol: Let's compute $f_{xy}(0,0)$. To do this, we need to compute f_x. First, for (x,y)=/(0,0), we can simply compute

$$f_x = \frac{\partial}{\partial x}\frac{x^3y - xy^3}{x^2+y^2} = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2}$$

At (0,0), we compute

$$f_x(0,0) = \lim_{x\to 0}\frac{f(x,0) - f(0,0)}{x-0} = \lim_{x\to 0}\frac{f(x,0) - 0}{x} = \lim_{x\to 0}\frac{f(x,0)}{x}$$

$$= \lim_{x\to 0}\frac{\frac{x^3\cdot 0 - x\cdot 0^3}{x^2+0^2}}{x} = \lim_{x\to 0}\frac{\frac{0}{x^2}}{x} = \lim_{x\to 0} 0 = 0$$

This gives

$$f_x(x,y) = \begin{cases} \dfrac{(3x^2y-y^3)(x^2+y^2)-(x^3y-xy^3)(2x)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

Now we can compute $f_{xy}(0,0)$, from the definition.

$$f_{xy}(0,0) = (f_x)_y \Big|_{(x,y)=(0,0)} = \lim_{y\to 0} \frac{f_x(0,y) - f_x(0,0)}{y-0}$$

$$= \lim_{y\to 0} \left(\frac{\dfrac{(3\cdot 0^2 y - y^3)(0^2+y^2)-(0^3\cdot y - 0\cdot y^3)(2\cdot 0)}{(0^2+y^2)^2} - 0}{y} \right)$$

$$= \lim_{y\to 0} \left(\frac{\dfrac{(0-y^3)(0+y^2)-0}{y^4}}{y} \right)$$

$$= \lim_{y\to 0} \frac{\frac{-y^5}{y^4}}{y} = \lim_{y\to 0} \frac{-y^5}{y^5} = -1$$

A similar computation shows that

$$f_{yx}(0,0) = 1.$$

Again, the issue is that f_{xy}, f_{yx} are not continuous at (0,0).