VECTOR CALCULUS, Week 12

11.6 Directional Derivatives and the Gradient Vector

Def: Suppose $f = f(x, y)$ is a real-valued function defined near (a, b).

- If $\vec{u} =< u_1, u_2 >$ is a unit-length vector, then we define the **directional derivative of f in the direction of $\vec{u}$ at** (a, b) to be

$$D_{\vec{u}} f(a,b) = \lim_{t \to 0} \frac{f(a + u_1 t, b + u_2 t) - f(a,b)}{t},$$

  assuming this limit exists.

- If $\vec{v} \in \mathbf{R}^2$ with $\vec{v} \neq \vec{0}$, then we define the **directional derivative of f in the direction of $\vec{v}$ at** (a, b) to be $D_{\frac{\vec{v}}{|\vec{v}|}} f(a, b)$, assuming this directional derivative exists.

- If $f_x(a, b), f_y(a, b)$ exist, then we define the **gradient of f at** (a, b) to be

$$\nabla f(a,b) =< f_x(a,b), f_y(a,b) > .$$

We make similar definitions for real-valued functions $f = f(x, y, z)$.

Fact: Suppose $f = f(x, y)$ is differentiable at (a, b), and suppose $\vec{u} =< u_1, u_2 >$ is a unit-length vector.

- $D_{\vec{u}} f(a, b) = \vec{u} \cdot \nabla f(a, b) = u_1 f_x(a, b) + u_2 f_y(a, b)$.

- Suppose ℓ is the line through (a, b) in the direction of $\vec{u}$, and consider the point-direction parameterization $\ell(t) =< a, b > +t\vec{u}$ for $t \in \mathbf{R}$. $D_{\vec{u}} f(a, b)$ measures the instantaneous rate of change of f at (a, b) as we move along ℓ with increasing values of t. In other words, $D_{\vec{u}} f(a, b) = \frac{d}{dt} f(\ell(t))|_{t=0}$.

Similar facts are true for $f = f(x, y, z)$.

Ex: Directional derivatives and gradients.

1. Compute the directional derivative of $f(x, y) = x^2 + e^{xy}$ in the direction of $\vec{v} =< 2, 1 >$ at $(a, b) = (-1, 1)$.
2. Compute the directional derivative of $f(x, y, z) = xyz^2$ in the direction of $\vec{v} =< 1, 1, 2 >$ at $(a, b) = (2, 3, 5)$.
3. Compute the instantaneous rate of change of $f(x, y) = x^2 + xy$ at $(a, b) = (2, 1)$ as we move along the line $y = x - 1$ with increasing values of x.

Fact: If $f = f(x, y)$ is differentiable at (a, b), then $\nabla f(a, b)$ points in the direction where the values of f increase the most, while $-\nabla f(a, b)$ points in the direction where the values of f decrease the most. The same is true for real-valued functions $f = f(x, y, z)$.

Ex: Compute the direction in which the values of the given function f increase the most and decrease the most at the given point (a, b).

1. $f(x, y) = \sqrt{1 - x^2 - y^2}$ at $(a, b) = (1/2, 1/2)$.
2. $f(x, y, z) = x^2 + y^2 + z^2$ at $(a, b, c) = (0, 2, 1)$.

Fact: Suppose $k \in \mathbf{R}$.

- If $f = f(x, y)$ is differentiable at (a, b) and $f(a, b) = k$, then $\nabla f(a, b)$ is perpendicular at (a, b) to the level curve of f at k.

  This means $< -f_y(a, b), f_x(a, b) >$ is tangent at (a, b) to the level curve of f at k.
- If $f = f(x, y, z)$ is differentiable at (a, b, c) and $f(a, b, c) = k$, then $\nabla f(a, b, c)$ is perpendicular at (a, b, c) to the level surface of f at k.

Ex:

1. For $f(x, y) = \sqrt{1 - x^2 - y^2}$, verify that $\nabla f(1/2, 1/2)$ is perpendicular to the level curve f at $k = f(1/2, 1/2)$.
2. Find a vector which is perpendicular at $(a, b, c) = (1/2, 1/2, 1/2)$ to the surface $z = 1 - x^2 - y^2$.