VECTOR CALCULUS, Week 6

9.3 Polar Coordinates; 9.4 Areas and Lengths in Polar Coordinates; 10.1 Three-Dimensional Coordinate Systems; 10.2 Vectors; 10.3 The Dot Product; 10.4 The Cross Product; 10.5 Equations of Lines and Planes; 10.6 Cylinders and Quadric Surfaces

9.3 Polar Coordinates

Fact: If $(x, y) \in \mathbf{R}^2$, then there is an $r \geq 0$ and $\theta \in [0, 2\pi)$ so that $(x, y) = (r\cos\theta, r\sin\theta)$.

- $r^2 = x^2 + y^2$
- If $x \neq 0$, then $\tan\theta = \frac{y}{x}$.

  If $x = 0$ and $y > 0$, then $\theta = \frac{\pi}{2}$. If $x = 0$ and $y < 0$, then $\theta = \frac{3\pi}{2}$.

Def: Suppose $(x, y) \in \mathbf{R}^2$, and suppose $r, \theta \in \mathbf{R}$. If $(x, y) = (r\cos\theta, r\sin\theta)$, then we say $(r, \theta)_p$ are **polar coordinates** for (x, y).

- We call the plane with points represented by polar coordinates the **polar coordinate system**.
- We call the origin the **pole**.
- We call the positive x-axis the **polar axis**.

Note that we allow negative values of r, θ.

Ex: Plot and give Cartesian coordinates for the following points given in polar coordinates.

1. $(2, -2\pi/3)_p$
2. $(-3, 3\pi/4)_p$

Ex: Give polar coordinates for the following points given in Cartesian coordinates.

1. $(1, -1)$
2. $(2, 2\sqrt{3})$

Def: A **polar parametric plane curve** is a parametric plane curve of the form

$$C(\theta) = (x(\theta), y(\theta)) = (r(\theta)\cos\theta, r(\theta)\sin\theta) = (r(\theta), \theta)_p \text{ for } a \leq \theta \leq b.$$

We say the equation

$$r = r(\theta) \text{ for } a \leq \theta \leq b$$

is a **polar parametric equation** for C.

Ex: Identify the images of the polar parametric plane curves given by the following polar parametric equations.

1. $r = 2$
2. $r = 2\cos\theta$, by finding a Cartesian equation for the curve.
3. $r = 1 + \sin\theta$, the cardioid

Ex: Find a polar parametric equation for the curve given by the Cartesian equation $(x+1)^2 + (y-2)^2 = 5$.

Ex: Consider the cardioid $r = 1 + \sin\theta$.

1. Find a Cartesian equation for the curve.
2. Compute the slope of the tangent line of C at $\theta = \frac{\pi}{3}, \frac{5\pi}{6}$.

9.4 Areas and Lengths in Polar Coordinates

Fact: The set $\{(r,\theta)_p : \theta = a\}$ is the line through the origin with angle $= a$ counterclockwise from the positive x-axis.

Fact: Suppose C is the polar parametric plane curve given by the polar parametric equation $r = r(\theta)$ for $a \leq \theta \leq b$, and suppose r is continuous. The (unsigned) area A of the region bounded by C and the lines $\theta = a$ and $\theta = b$ is given by

$$A = \int_a^b \frac{1}{2}(r(\theta))^2 \, d\theta.$$

Ex: Give an integral for the area A of the following regions.

1. The region bounded by the polar parametric plane curve C given by the polar parametric equation $r = R$, and the lines $\theta = a$ and $\theta = b$.
2. One loop of the four-leaved rose $r = \cos 2\theta$.
3. The region inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Fact: Suppose C is the polar parametric plane curve given by the polar parametric equation $r = r(\theta)$ for $a \leq \theta \leq b$, where r is continously differentiable over $[a,b]$. If C has no self-intersections, then the arc length L of the image of C is given by

$$L = \int_a^b \sqrt{(x'(\theta))^2 + (y'(\theta))^2} \, d\theta = \int_a^b \sqrt{r(\theta)^2 + (r'(\theta))^2} \, d\theta.$$

The same is true if C only has isolated self-intersections.

Ex: Give an integral for the arc length L of the cardioid $r = 1 + \sin\theta$ for $0 \leq \theta \leq 2\pi$.

10.1 Three-Dimensional Coordinate Systems; 10.2 Vectors; 10.3 The Dot Product; 10.4 The Cross Product

Def: We use the following notation.

- We use x, y, z-coordinates in $\mathbf{R}^3$, or space.
- Given a point P in space, $P(x, y, z)$ means that P has coordinates (x, y, z).
- Given two points P_1, P_2 in space, we let $|P_1P_2|$ denote the distance between P_1, P_2.
- The book uses bold letters to denote vectors. We will continue to use arrows, such as $\vec{v}$.

  $\vec{0}$ will denote the zero vector.
- If $\vec{v} \in \mathbf{R}^2$ has components v_1, v_2, then we write $\vec{v} =< v_1, v_2 >$. Similarly for $\vec{v} \in \mathbf{R}^3$, we write $\vec{v} =< v_1, v_2, v_3 >$.
- Given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, we denote the vector

$$\vec{AB} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 > .$$

- If $\vec{v} \in \mathbf{R}^2$ or $\vec{v} \in \mathbf{R}^3$, then we denote the length of $\vec{v}$ by $|\vec{v}|$.
- The **standard basis vectors** in $\mathbf{R}^3$ shall be denoted by

$$\vec{i} =< 1, 0, 0 > \quad \vec{j} =< 0, 1, 0 > \quad \vec{k} =< 0, 0, 1 >$$

Def: The **dot product** between $\vec{v}, \vec{w} \in \mathbf{R}^3$ is

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

We similarly define the dot product between $\vec{v}, \vec{w} \in \mathbf{R}^2$.

Thm: If θ is the angle between $\vec{v}, \vec{w}$ (in $\mathbf{R}^2$ or $\mathbf{R}^3$), then

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta.$$

In particular, $\vec{v} \perp \vec{w}$ if and only if $\vec{v} \cdot \vec{w} = 0$.

Def: Suppose $\vec{v}$ is a nonzero vector in $\mathbf{R}^2$ or $\mathbf{R}^3$. The **vector/orthogonal projection of $\vec{x}$ onto $\vec{v}$** is

$$\mathrm{proj}_{\vec{v}} \vec{x} = \left(\frac{\vec{x} \cdot \vec{v}}{|\vec{v}|} \right) \frac{\vec{v}}{|\vec{v}|} = \left(\frac{\vec{x} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}.$$

Thm: $\vec{p} = \mathrm{proj}_{\vec{v}} \vec{x}$ is the unique vector so that

- $\vec{p}, \vec{v}$ are co-linear.
- $(\vec{x} - \vec{p}) \perp \vec{v}$.

Def: Suppose $\vec{v}, \vec{w} \in \mathbf{R}^3$. We define the **cross product** between $\vec{v}, \vec{w}$ *in that order* to be the vector given by computing the symbolic determinant

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Thm: $\vec{v} \times \vec{w}$ is the unique vector so that

- $\vec{v} \times \vec{w} \perp \vec{v}, \vec{w}$
- $|\vec{v} \times \vec{w}|$ is the area of the **parallelogram formed by** $\vec{v}, \vec{w}$.
- $\vec{v}, \vec{w}, \vec{v} \times \vec{w}$ *in that order* satisfy the **right-hand rule**.

Fact: Suppose $\vec{u}, \vec{v}, \vec{w} \in \mathbf{R}^3$.

- $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$ (anti-commutative law)
- $|\vec{u} \cdot (\vec{v} \times \vec{w})|$ is the volume of the **parallelpiped formed by** $\vec{u}, \vec{v}, \vec{w}$.

10.5 Equations of Lines and Planes

Fact: The line ℓ in space through $P(x_0, y_0, z_0)$ in the direction of $\vec{v} =< a, b, c > \neq \vec{0}$ is given by the point-direction parameterization

$$\ell(t) =< x_0, y_0, z_0 > +t < a, b, c > \text{ for } t \in \mathbf{R}.$$

Ex: Suppose ℓ is the line through $A(2, 4, -3), B(3, -1, 1)$. Give a point-direction parameterization for ℓ. At what point does ℓ pass through the horizontal xy-plane?

Ex: Consider the lines given by the point-direction parameterizations

$$\ell_1(t) =< 1, -2, 4 > +t < 1, 3, -1 > \text{ for } t \in \mathbf{R}$$
$$\ell_2(s) =< 0, 3, -3 > +s < 2, 1, 4 > \text{ for } s \in \mathbf{R}.$$

Show that ℓ_1, ℓ_2 are **skew**: not parallel and do not intersect.

Fact: To describe a plane in space, we need a point in the plane and a **normal direction**. If $P(x_0, y_0, z_0)$ is a point in space and $\vec{n} =< a, b, c > \neq \vec{0}$, then the plane $\mathcal{P}$ through P with normal in the direction of $\vec{n}$ is given by the **implicit/vector equation**

$$\mathcal{P} : \vec{n} \bullet (< x, y, z > - < x_0, y_0, z_0 >) = 0.$$

We also say $\mathcal{P}$ is given by the **scalar equation**

$$\mathcal{P} : a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

or by the **linear equation**

$$\mathcal{P} : ax + by + cz + d = 0.$$

Ex: Find a linear equation for the plane $\mathcal{P}$ through the points $P(1, 3, 2), Q(3, -1, 6), R(5, 2, 0)$.

Ex: Find a point-direction parameterization for the line ℓ of intersection between the planes given by the linear equations

$$\mathcal{P}_1 : x + y + z - 1 = 0 \text{ and } \mathcal{P}_2 : x - 2y + 3z - 1 = 0.$$

10.6 Cylinders and Quadric Surfaces

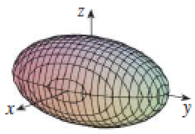
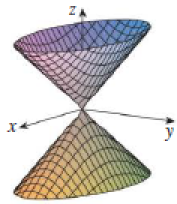
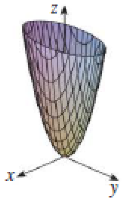
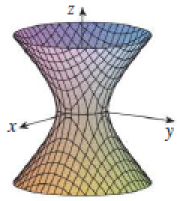
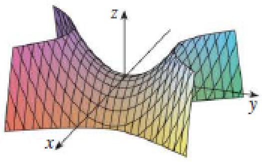
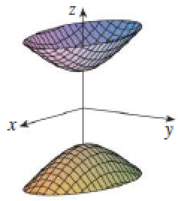
Def: A **quadric surface** is a surface in space given by a second-degree equation in x, y, z of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

Ex: Some familiar quadric surfaces.

1. The unit cylinder centered around the z-axis, given by $x^2 + y^2 - 1 = 0$.
2. The unit sphere centered at the origin, given by $x^2 + y^2 + z^2 - 1 = 0$.

We as well have cylinders and spheres of different radii centered elsewhere.

Surface	Equation	Surface	Equation
Ellipsoid  	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone  	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid  	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet  	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid  	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets  	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Fact: Consider a quadric surface S given by the equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

To determine the type of S, and to sketch S, it is best to use the **level set method**: set a variable equal to a constant, and sketch the resulting curve. For example, set $z = k$ and sketch the curve in the plane $z = k$ given by the equation

$$Ax^2 + By^2 + Dxy + Eky + Fkx + Gx + Hy + Ck^2 + Ik + J = 0.$$

This curve is the intersection between S and the plane $z = k$.

Ex: Use the level set method to determine the type and sketch the quadric surfaces given by the following equations.

1. $x^2 + \frac{y^2}{9} + \frac{z^2}{4} - 1 = 0$
2. $4x^2 + y^2 - z = 0$