Vector Calculus
Practice Problems I

1) Determine whether each improper integral is convergent or divergent.

a) $\int_{-\infty}^{0} ze^{2z}\,\delta z$

Sol: We must compute

$$\lim_{a\to-\infty} \underbrace{\int_a^0 ze^{2z}\,\delta z}_{\text{integration by parts}}$$

b) $\int_1^{\infty} \frac{1}{x^2+x}\,\delta x$

Sol: We must compute

$$\lim_{b\to\infty} \int_1^b \underbrace{\frac{1}{x^2+x}}_{\frac{1}{x(x+1)}}\,\delta x \quad \text{Easy Partial Fractions}$$

2) Let f(x)=e^x+x. Set up, *but do not evaluate*, an integral for each of the following.

a) The arc length L of the curve y=f(x) for 1≤x≤3.

Sol: We must give

$$L = \int_1^3 \sqrt{1+(f'(x))^2}\,dx = \int_1^3 \sqrt{1+\Big(\underbrace{\tfrac{d}{dx}(e^x+x)}_{\text{you must compute!}}\Big)^2}\,dx \ \ldots$$

b) The surface area S of the surface of revolution formed by rotating the graph of f over [1,3] around the x-axis.

Sol: Consider C(t)=(x(t),y(t))=(t,f(t)) for 1⩽t⩽3, then

$$S = \int_1^3 2\pi y(t) \sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

$$= \int_1^3 2\pi f(x) \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_1^3 2\pi (e^x+x) \sqrt{1+\Big(\underbrace{\tfrac{d}{dx}(e^x+x)}_{\text{compute!}}\Big)^2}\,dx \ \ldots$$

3) Consider the parametric plane curve

$$C(t) = \left(\sqrt{t+1}, \sqrt{t-1}\right) \text{ for } t \geq 1$$

a) Eliminate the parameter to find a Cartesian equation for C.

Sol: We compute

$$\begin{array}{l} x=\sqrt{t+1} \\ y=\sqrt{t-1} \end{array} \Rightarrow \begin{array}{l} x^2=t+1 \\ y^2=t-1 \end{array}$$

$$\Rightarrow x^2-1 = t = y^2+1$$

$$\Rightarrow \boxed{x^2-1=y^2+1} \quad \text{for } \begin{array}{l} 0 \leq y < \infty \\ \sqrt{2} \leq x < \infty \end{array}$$

b) Find all the points where the image of C intersects the line y=x/2.

Sol: We set $x=\sqrt{t+1}$, $y=\sqrt{t-1}$, and y=x/2 to get

$$\sqrt{t-1} = \frac{\sqrt{t+1}}{2} \quad \text{for } t\geq 1$$

$$t-1 = \frac{t+1}{4}$$

$$\frac{3}{4}t = \frac{5}{4}$$

$$\Rightarrow t=\frac{5}{3}\geq 1$$

$$\Rightarrow \boxed{(x,y) = \left(\sqrt{\tfrac{5}{3}+1}, \sqrt{\tfrac{5}{3}-1}\right)}$$

4) Consider the parametric plane curve

$$C(t) = \left(\frac{t^3}{3} - \frac{3t^2}{2} + 2t + 1, \frac{2t^3}{3} - 5t^2 + 12t\right) \text{ for } t\in\mathbb{R}$$

a) Compute the tangent line of C at t=0.

Sol: First, we compute

$$x'(0) = \frac{\partial}{\partial t} \frac{t^3}{3} - \frac{3t^2}{2} + 2t + 1 \Big|_{t=0} = t^2 - 3t + 2 \Big|_{t=0} = 2 \neq 0$$

x'(0)=/0 implies that the image of C near t=0 is the graph of a function y=f(x) defined for x near x(0). Thus, the tangent line of C at t=0 is given by

$$y = f'(x(0))\,(x-x(0)) + y(0)$$

$$\Rightarrow \quad y = \frac{y'(0)}{x'(0)}(x - x(0)) + y(0)$$

We compute

$$x(0) = 1$$
$$y(0) = 0$$
$$x'(0) = 2$$
$$y'(0) = \frac{d}{dt} \frac{2t^3}{3} - 5t^2 + 12t \Big|_{t=0} = 12$$

We conclude that the tangent line of C at t=0 is

$$\boxed{y = \frac{12}{2}(x-1) + 0}$$

b) Compute all t so that x'(t)=0, and compute the slope of the tangent line of C at all such t.

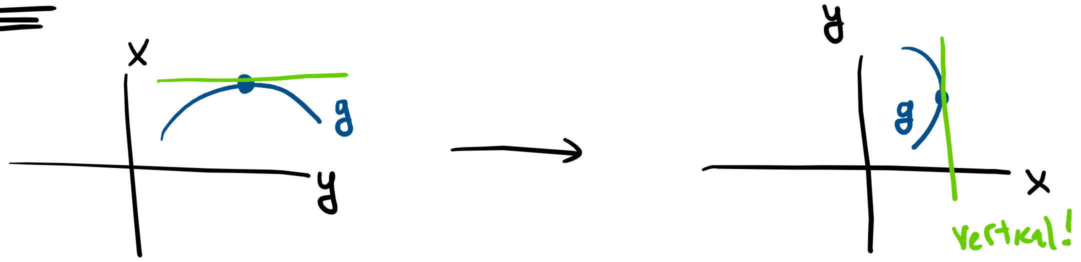
Sol: We compute

$$0 = x'(t) = \frac{d}{dt} \frac{t^3}{3} - \frac{3t^2}{2} + 2t + 1 = t^2 - 3t + 2$$
$$\Rightarrow \quad (t-1)(t-2) = 0$$
$$\Rightarrow \quad \boxed{t = 1, 2}$$

Consider t=1,

$$y'(1) = \frac{d}{dt} \frac{2t^3}{3} - 5t^2 + 12t \Big|_{t=1} = 2t^2 - 10t + 12 \Big|_{t=1}$$
$$= 2 - 10 + 12 = 4$$

y'(1)=/0 implies that the image of C near t=1 is the graph of a function x=g(y). In this case, x'(1)=0 implies that the tangent line of g(y) at y(1) is horizontal *with respect to the y-axis*.



This means that the tangent line of C at t=1 is a *vertical* line. We conclude the slope is $\boxed{\text{undefined.}}$

Consider t=2,

$$y'(2) = 2t^2-10t+12 \Big|_{t=2} = 8-20+12 = 0.$$

Since x'(2)=y'(2)=0, we must compute

$$\lim_{t\to 2} \frac{y'(t)}{x'(t)} = \lim_{t\to 2} \frac{2t^2-10t+12}{t^2-3t+2}$$

$$= \lim_{t\to 2} \frac{2(t^2-5t+6)}{(t-1)(t-2)} = \lim_{t\to 2} \frac{2(t-2)(t-3)}{(t-1)(t-2)}$$

$$= \lim_{t\to 2} \frac{2(t-3)}{t-1} = \frac{2(2-3)}{2-1} = \boxed{-2}$$

5) Consider the parametric plane curve

$$C(t) = (t^2+3t+1, t^3) \text{ for } 0 \le t \le 2$$

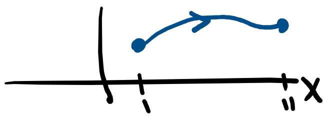
a) Show that the image of C is the graph of a function y=f(x) defined over [1,11].

Sol: Consider x(t)=t^2+3t+1, then

$$x'(t) = 2t+3 > 0+3 = 3 \quad \text{for } 0 \le t \le 2$$

Since x is increasing over [0,2], then we conclude that the image of C is the graph of a function y=f(x) defined over

$$[x(0), x(2)] = [0^2+3\cdot 0+1,\ 2^2+3\cdot 2+1] = [1,11].$$



b) Set up, *but do not evaluate*, an integral for the arc length L of the image of C.

Sol: We give

$$L = \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \qquad \star$$

$$\Rightarrow \quad x(t) = t^2+3t+1 \qquad \dots$$
$$y(t) = t^3$$

c) Set up, *but do not evaluate*, an integral for the area A under C.

Sol: Since x is increasing over [0,2], then

$$A = \int_0^2 y(t)\,x'(t)\, dt$$

$$\Rightarrow \quad x(t) = t^2+3t+1 \quad \dots$$
$$y(t) = t^3$$

Recall, if x(t) is decreasing over [a,b], then

$$A = \int_b^a y(t)x'(t)\,dt.$$

6) Find a Cartesian equation for the polar parametric plane curve given by the polar parametric equation r=5sin(theta).

Sol: We compute

$$r = 5\sin\theta \Rightarrow r^2 = 5r\sin\theta$$
$$\Rightarrow \boxed{x^2+y^2=5y}$$

Note that

$$x^2+y^2-5y=0 \Rightarrow x^2+y^2-5y+\frac{25}{4}=\frac{25}{4}$$
$$\Rightarrow (x-0)^2+\left(y-\frac{5}{2}\right)^2=\frac{25}{4}$$

This is the circle of radius =5/2 centered at (0,5/2).

7) Consider the polar parametric plane curve C(theta)=(x(theta),y(theta)) given by the polar parametric equation r=e$^{\theta}$.

a) Compute the tangent line of C at theta=0.

Sol: C is the parametric plane curve

$$C(\theta) = (\underset{x(\theta)}{\underset{\parallel}{e^\theta \cos\theta}}, \underset{y(\theta)}{\underset{\parallel}{e^\theta \sin\theta}})$$

First, we compute

$$x'(0) = \frac{\partial}{\partial\theta} e^\theta \cos\theta \Big|_{\theta=0} = e^\theta \cos\theta - e^\theta \sin\theta \Big|_{\theta=0}$$

$$= 1\cdot 1 - 1\cdot 0 = 1 \neq 0$$

We also compute

$$y'(0) = \frac{\partial}{\partial\theta} e^\theta \sin\theta \Big|_{\theta=0}$$
$$= e^\theta \sin\theta + e^\theta \cos\theta \Big|_{\theta=0}$$
$$= 0 + 1 = 1$$

$$x(0) = e^0 \cos 0 = 1$$
$$y(0) = e^0 \sin 0 = 0$$

We conclude the tangent line of C at theta=0 is

$$y = \frac{y'(0)}{x'(0)}(x - x(0)) + y(0)$$

$$\Rightarrow \boxed{y = \frac{1}{1}(x-1) + 0}$$

b) Compute all theta in [0,2pi) so that x'(theta)=0, and compute the slope of the tangent line of C at all such theta.

Sol: We compute

$$0 = x'(\theta) = e^\theta \cos\theta - e^\theta \sin\theta$$

$$\Rightarrow \quad \cos\theta - \sin\theta = 0 \quad \text{for } \theta \in [0, 2\pi)$$

$$\Rightarrow \quad \cos\theta = \sin\theta \quad \text{for } \theta \in [0, 2\pi)$$

$$\Rightarrow \quad \boxed{\theta = \frac{\pi}{4}, \frac{5\pi}{4}}$$

Consider theta=pi/4,

$$y'\left(\frac{\pi}{4}\right) = e^{\theta}\sin\theta + e^{\theta}\cos\theta \Big|_{\theta=\frac{\pi}{4}}$$

$$= e^{\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2} + e^{\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2} \neq 0$$

This means the tangent line of C at theta=pi/4 is horizontal *with respect to the y-axis*, which means it is horizontal. We conclude the slope is $\boxed{\text{undefined}}$

Consider theta=5pi/4,

$$y'\left(\frac{5\pi}{4}\right) = e^{\theta}\sin\theta + e^{\theta}\cos\theta \Big|_{\theta=\frac{5\pi}{4}}$$

$$= e^{-\frac{5\pi}{4}}\left(-\frac{\sqrt{2}}{2}\right) + e^{\frac{5\pi}{4}}\left(-\frac{\sqrt{2}}{2}\right) \neq 0$$

similar story

$$\boxed{\text{undefined}}$$

c) Find all theta in [0,2pi) so that the tangent line of C at theta is horizontal.

We compute

$$0 = y'(\theta) = e^{\theta}\sin\theta + e^{\theta}\cos\theta$$

$$\Rightarrow \quad \cos\theta = -\sin\theta \quad \text{for } \theta \in [0, 2\pi)$$

$$\Rightarrow \quad \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Consider theta=3pi/4, then

$$x'\left(\frac{3\pi}{4}\right) = e^{\theta}\cos\theta - e^{\theta}\sin\theta \Big|_{\theta=\frac{3\pi}{4}}$$

$$= e^{\frac{3\pi}{4}}\left(-\frac{\sqrt{2}}{2}\right) - e^{\frac{3\pi}{4}}\left(\frac{\sqrt{2}}{2}\right) \neq 0$$

x'(3pi/4)=/0 implies that the image of C near theta=3pi/4 is the graph of a function y=f(x). Since y'(3pi/4)=0, then tangent line of C is

$$y = \underbrace{f'\left(x\left(\tfrac{3\pi}{4}\right)\right)}_{=\frac{y'\left(\frac{3\pi}{4}\right)}{x'\left(\frac{3\pi}{4}\right)} = 0}\left(x - x\left(\tfrac{3\pi}{4}\right)\right) + y\left(\tfrac{3\pi}{4}\right)$$

$$\Rightarrow \text{horizontal line}$$

We conclude the tangent line of C at theta=3pi/4 is horizontal.

Consider theta=7pi/4, then

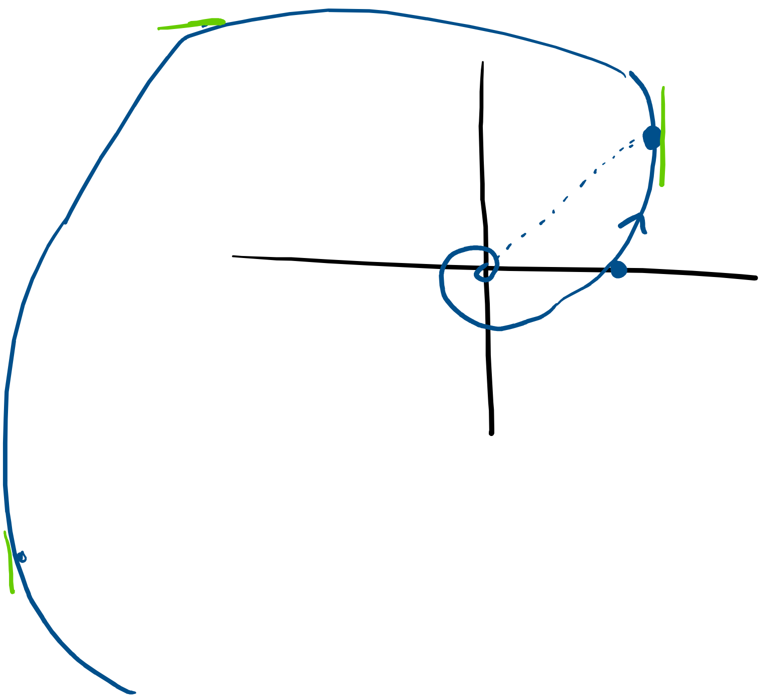
$$x'\left(\frac{7\pi}{4}\right) = e^{\theta}\cos\theta - e^{\theta}\sin\theta \Big|_{\theta=\frac{7\pi}{4}}$$

$$= \underbrace{e^{\frac{7\pi}{4}}\left(\frac{\sqrt{2}}{2}\right) - e^{\frac{7\pi}{4}}\left(-\frac{\sqrt{2}}{2}\right)}_{>0} \neq 0$$

Similarly

We conclude the tangent line of C at theta=7pi/4 is horizontal.

$$\Rightarrow \boxed{\theta = \frac{3\pi}{4}, \frac{7\pi}{4}}$$



$C(\theta) = (e^{\theta}\cos\theta, e^{\theta}\sin\theta)$

spiral

4c) Find all t so that the tangent line of C at t is horizontal.

We compute

$$0 = y'(t) = 2t^2 - 10t + 12 = 2(t^2 - 5t + 6)$$

$$\Rightarrow \quad (t-2)(t-3) = 0 \Rightarrow \quad t = 2, 3$$

Consider t=2. Actually, we previously computed that x’(2)=0, and

$$\lim_{t \to 2} \frac{y'(t)}{x'(t)} = -2.$$

This means the tangent line of C at t=2 is *not* horizontal.
In fact, the slope is -2.

Consider t=3, then

$$x'(3) = t^2 - 3t + 2 \Big|_{t=3} = 9 - 9 + 2 \neq 0$$

We conclude that the tangent line of C at t=3 is horizontal.

$$\Rightarrow \boxed{t=3}$$