VECTOR CALCULUS, Week 14

14.8 Lagrange Multipliers

Def: Suppose $f = f(x, y, z)$, $g = g(x, y, z)$ are real-valued functions defined near (a, b, c), and suppose $k = g(a, b, c)$.

- If $f(a, b, c) \geq f(x, y, z)$ for all (x, y, z) in the level surface of g at k near (a, b, c), then we say

  (a, b, c) is a **local maximum point of** f **over the level surface of** g **at** k and
  $f(a, b, c)$ is a **local maximum value of** f **over the level surface of** g **at** k.

- If $f(a, b, c) \leq f(x, y, z)$ for all (x, y, z) in the level surface of g at k near (a, b, c), then we say

  (a, b, c) is a **local minimum point of** f **over the level surface of** g **at** k and
  $f(a, b, c)$ is a **local minimum value of** f **over the level surface of** g **at** k.

- If (a, b, c) is either a local maximum or minimum point of f over the level surface of g at k, then we say

  (a, b, c) is a **local extremum point of** f **over the level surface of** g **at** k and
  $f(a, b, c)$ is a **local extremum value of** f **over the level surface of** g **at** k.

Suppose $f = f(x, y, z)$, $g = g(x, y, z)$ are real-valued functions defined over $\mathbf{R}^3$, and suppose $k = g(a, b, c)$.

- If $f(a, b, c) \geq f(x, y, z)$ for all (x, y, z) in the level surface of g at k, then we say

  (a, b, c) is an **absolute maximum point of** f **over the level surface of** g **at** k and
  $f(a, b, c)$ is the **absolute maximum value of** f **over the level surface of** g **at** k.

- If $f(a, b, c) \leq f(x, y, z)$ for all (x, y, z) in the level surface of g at k, then we say

  (a, b, c) is an **absolute minimum point of** f **over the level surface of** g **at** k and
  $f(a, b, c)$ is the **absolute minimum value of** f **over the level surface of** g **at** k.

- If (a, b, c) is either a maximum or minimum point of f over the level surface of g at k, then we say

  (a, b, c) is an **absolute extremum point of** f **over the level surface of** g **at** k and
  $f(a, b, c)$ is an **absolute extremum value of** f **over the level surface of** g **at** k.

We make similar definitions for real-valued functions $f = f(x, y)$, $g = g(x, y)$.

Thm (Lagrange Multipliers): Suppose $f = f(x, y, z), g = g(x, y, z)$ are real-valued functions, and suppose $k \in \mathbf{R}$.

- Suppose $f = f(x, y, z), g = g(x, y, z)$ are differentiable at (a, b, c), and suppose $k = g(a, b, c)$. If (a, b, c) is a local extremum point of f over the level surface of g at k, and if $\nabla g(a, b, c) \neq \vec{0}$, then there is a $\lambda \in \mathbf{R}$ so that
$$\nabla f(a, b, c) = \lambda \nabla g(a, b, c).$$
- Suppose f, g are differentiable functions, and suppose the level surface of g at k is a bounded set. To find the absolute extremum points and values of f over the level surface of g at k, we compare the values of f at all points (x, y, z) in the level surface of g at k so that $\nabla f(a, b, c) = \lambda \nabla g(a, b, c)$ for some $\lambda \in \mathbf{R}$.

A similar theorem is true for real-valued functions $f = f(x, y), g = g(x, y)$.

Proof: Suppose $(a, b, c) = (1, 0, 0)$ is a local extremum point of $f = f(x, y, z)$ over the level surface of $g(x, y, z) = x^2 + y^2 + z^2$ at $k = 1$, prove that $\nabla f(1, 0, 0) = < f_x(1, 0, 0), 0, 0 >$.

Ex: Find the absolute extremum points and values of the given function f over the level curve/surface of the given function g at the given $k \in \mathbf{R}$.

1. $f(x, y) = x^2 + 2y^2$ with $g(x, y) = x^2 + y^2$ at $k = 1$
2. $f(x, y, z) = x^2 + 5y^2 + z^2$ with $g(x, y, z) = 4x + 5y + 2z$ at $k = 1$
3. $f(x, y, z) = xy + z^2$ with $g(x, y, z) = x^2 + y^2 + z^2$ at $k = 1$
4. $f(x, y, z) = xy + \frac{z^3}{3}$ with $g(x, y, z) = x^2 + y^2 + z^2$ at $k = 1$

Ex: Find the absolute extremum values of the given function f over the level curve/surface of the given function g at the given $k \in \mathbf{R}$.

1. $f(x, y, z) = xy^2z$ with $g(x, y, z) = x^2 + y^2 + z^2$ at $k = 4$
2. $f(x, y, z) = x^2 + y^2 + z^2$ with $g(x, y, z) = x^4 + y^4 + z^4$ at $k = 1$

Fact: When solving a Lagrange multiplier problem, consider cases such as $x = 0$ and $x \neq 0$, $y = 0$ and $y \neq 0$, and $z = 0$ and $z \neq 0$. In particular, first consider any equation where the same variable appears on both sides.

Thm (Two Constraint Lagrange Multipliers): Suppose $f = f(x, y, z), g = g(x, y, z), h = h(x, y, z)$ are differentiable at (a, b, c), and suppose $j = g(a, b, c)$ and $k = h(a, b, c)$. Also suppose C is the curve in space given by the intersection between the level surface of g at j and the level surface of h at k. If (a, b, c) is a local extremum point of f over C, and if $\nabla g(a, b, c), \nabla h(a, b, c) \neq \vec{0}$, then there are $\lambda, \mu \in \mathbf{R}$ so that

$$\nabla f(a, b, c) = \lambda \nabla g(a, b, c) + \mu \nabla h(a, b, c).$$

Ex: Find the absolute extremum points and values of $f(x, y, z) = x + 2y + 3z$ over the curve in space C given by the intersection between the plane $x - y + z = 1$ and the unit cylinder $x^2 + y^2 = 1$.