Vector Calculus
11.7 Maximum and Minimum Values

Def: Suppose f=f(x,y) is a real-valued function defined near (a,b), and suppose the second partial derivatives of f exist at (a,b).

We define the discriminant of f at (a,b) to be

$$\underset{\text{"Delta" Greek "D"}}{\Delta} = \Delta(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

If (a,b) is a critical point of f but not a local extremum point of f, then we say (a,b) is a saddle point of f.

Thm (Second Derivative Test): Suppose f=f(x,y) is a real-valued function defined near (a,b), suppose the second partial derivatives of f exist near (a,b) and are continuous at (a,b), and suppose (a,b) is a critical point of f. $\Rightarrow \nabla f(a,b) = \langle 0,0 \rangle$

If $\Delta(a,b)>0$ and $f_{xx}(a,b)>0$, then (a,b) is a local minimum point of f.

Ex $f(x,y) = x^2+y^2$

$\boxed{\text{local minimum}}$

$$\nabla f(0,0) = \langle 2x, 2y \rangle \Big|_{(0,0)} = \langle 0,0 \rangle$$

$$\Delta(0,0) = f_{xx}(0,0)\,f_{yy}(0,0) - (f_{xy}(0,0))^2$$
$$= 2\cdot 2 - (0)^2 = 4 > 0$$
$$f_{xx}(0,0) = 2 > 0$$

If $\Delta(a,b)>0$ and $f_{xx}(a,b)<0$, then (a,b) is a local maximum point of f.

Ex $f(x,y) = -x^2 - y^2$

$\boxed{\text{local maximum}}$

$\nabla f(0,0) = \langle -2x, -2y \rangle \big|_{(0,0)} = \langle 0,0 \rangle$

$\Delta f(0,0) = (-2)(-2) - (0)^2 = 4 > 0$

$f_{xx}(0,0) = -2 < 0$

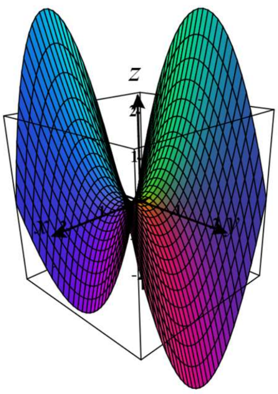
If $\Delta(a,b)<0$, then (a,b) is a saddle point of f.

Ex $f(x,y) = x^2 - y^2$

$\nabla f(0,0) = \langle 2x, -2y \rangle \big|_{(0,0)} = \langle 0,0 \rangle$

$\Delta f(0,0) = 2(-2) - 0^2 = -4 < 0$

$\Rightarrow$ (0,0) is a saddle point.



$\leftarrow$ the graph of f looks like a saddle at (0,0)

Ex: Find the critical points of the given function f, and determine whether the critical points are local minimum, local maximum, or saddle points.

1. $f(x,y)=x^4+y^4-4xy+1$

Sol: We first compute

$f_x = 4x^3 - 4y$ $\quad$ $f_{xx} = 12x^2$

$f_y = 4y^3 - 4x$ $\quad$ $f_{xy} = -4$ $\quad$ $\Delta = 144x^2y^2 - (-4)^2$

$f_{yy} = 12y^2$ $\quad$ $\Delta = 144x^2y^2 - 16$

The critical points are given by

$$\vec{0} = \nabla f = \langle 4x^3 - 4y, 4y^3 - 4x \rangle \Rightarrow \begin{array}{l} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{array}$$

$$\Rightarrow \begin{array}{l} ① \; y = x^3 \\ ② \; y^3 = x \end{array}$$

$$\underset{② \to ①}{\Rightarrow} \; y = x^3 = (y^3)^3$$

$$\Rightarrow \; y = y^9 \Rightarrow \begin{array}{l} y = 0 \\ y^8 = 1 \end{array}$$

$$\Rightarrow \; y = 0, \pm 1$$

$$\underset{②}{\Rightarrow} \; (0^3, 0), (1^3, 1), ((-1)^3, -1)$$

$$\Rightarrow \; \boxed{(0,0), (1,1), (-1,-1)}$$

To determine the type, we compute

$$\Delta(0,0) = 144x^2y^2 - 16 \Big|_{(0,0)} = -16 < 0 \quad \text{saddle}$$

$g(x,y) = x^2 - y^2$
$\Delta = -4 < 0$

$$\Delta(1,1) = 144 - 16 > 0 \quad \underset{g(x,y) = x^2 + y^2}{\Longrightarrow} \quad \text{local minimum}$$

$$f_{xx}(1,1) = 12 > 0$$

$$\Delta(-1,-1) = 144\cdot(-1)^2\cdot(-1)^2 - 16$$
$$= 144 - 16 > 0 \implies \text{local minimum}$$
$$f_{xx}(-1,-1) = 12\cdot(-1)^2 > 0$$

We conclude that

$$\boxed{\begin{array}{l} (0,0) \text{ saddle point} \\ (1,1), (-1,-1) \text{ local minimum points} \end{array}}$$

2. $f(x,y)=x^4-2x^2+y^3-3y$

Sol: We compute

$f_x = 4x^3 - 4x$ $\quad f_{xx} = 12x^2 - 4$

$f_y = 3y^2 - 3$ $\quad f_{xy} = 0$ $\quad \Delta = (12x^2-4)\,6y$

$f_{yy} = 6y$

The critical points are given by

$$\vec{0} = \nabla f = \langle 4x^3-4x, 3y^2-3 \rangle \implies \begin{array}{l} 4x^3-4x=0 \\ 3y^2-3=0 \end{array}$$

$$\implies \begin{array}{l} x(x^2-1)=0 \\ y^2-1=0 \end{array}$$

$$\implies \begin{array}{l} x=0,\pm 1 \\ y=\pm 1 \end{array}$$

$$\implies \boxed{\begin{array}{l} (0,-1),(0,1) \\ (1,-1),(1,1) \\ (-1,-1),(-1,1) \end{array}}$$

To classify these points, we consider

$$\Delta = (12x^2-4)\,6y$$

$$f_{xx} = (12x^2-4)$$

$$\Delta(0,-1) = (-4)(-6) = 24 > 0$$

$$f_{xx}(0,-1) = -4 < 0 \quad \Rightarrow \text{ local maximum}$$

$$\Rightarrow g(x,y) = -x^2-y^2$$

$$\Delta(0,1) = (-4)6 < 0 \Rightarrow \text{ saddle}$$

$$\Delta(\underset{x}{\pm 1},-1) = (12\cdot 1-4)(-6) < 0 \Rightarrow \text{ saddle}$$

$$\Delta(\pm 1,1) = (12\cdot 1-4)(6) > 0 \Rightarrow \text{ local minimum}$$

$$f_{xx}(\pm 1,1) = 12\cdot 1-4 > 0$$

$$\Rightarrow g(x,y) = x^2+y^2$$

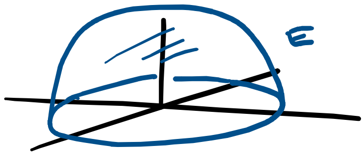
$$\boxed{\begin{array}{ll} (0,1),\ (1,-1),\ (-1,-1) & \text{saddle points} \\ (1,1),\ (-1,1) & \text{local minimum points} \\ (0,-1) & \text{local maximum point} \end{array}}$$

Ex: Find the absolute extremum values of the given function f over the given solid region E.

1. $f(x,y,z)=x^2+y^2+z^2-z$ over

$$E = \{(x,y,z):\ 0 \le z \le \sqrt{1-x^2-y^2}\}$$

Note that the surface $z=\sqrt{1-x^2-y^2}$ is the upper unit sphere, so E is

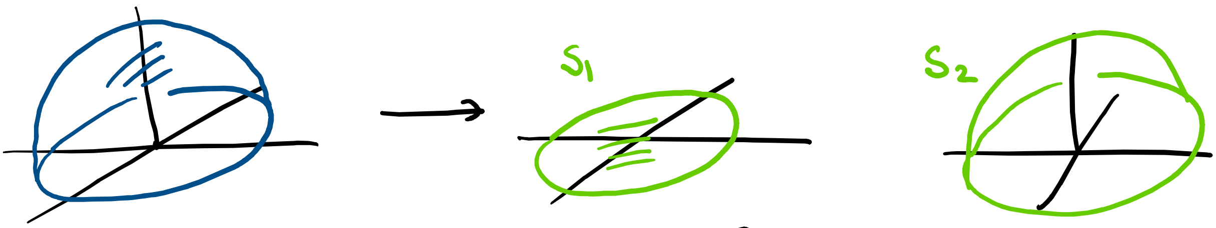


Sol: We must compare the values of f at the interior critical points, and along the boundary of E. First, we compute

$$\vec{0} = \nabla = \langle 2x, 2y, 2z-1 \rangle \Rightarrow x=0, y=0, z=\tfrac{1}{2} \Rightarrow (0,0,\tfrac{1}{2}) \in E \checkmark$$

$$\Rightarrow f(0,0,\tfrac{1}{2}) = 0^2+0^2+(\tfrac{1}{2})^2-\tfrac{1}{2} = \tfrac{1}{4}-\tfrac{1}{2} = \boxed{-\tfrac{1}{4}}$$

Now let's find the absolute extremum values of f over the boundary of E. The boundary of E is given by two pieces,



$\{(x,y,z): x^2+y^2 \le 1, z=0\}$ $\{(x,y,z): x^2+y^2 \le 1, z=\sqrt{1-x^2-y^2}\}$

Consider f over S_1, we must find the absolute extremum values of

$$f(x,y,0) = x^2+y^2+0^2-0 = x^2+y^2 \quad \text{for} \quad x^2+y^2 \le 1$$

minimum $\Rightarrow f(0,0)=\boxed{0}$ maximum value $f(x,y)=\boxed{1}$ for all (x,y) with $x^2+y^2=1$

Consider f over S_2, we must find the absolute extremum values of

$$f(x,y,z) = x^2+y^2+\left(\sqrt{1-x^2-y^2}\right)^2 - \sqrt{1-x^2-y^2} \quad \text{with} \quad x^2+y^2 \le 1$$

$$= x^2+y^2+1-x^2-y^2-\sqrt{1-x^2-y^2}$$

$$= 1-\sqrt{1-x^2-y^2}$$

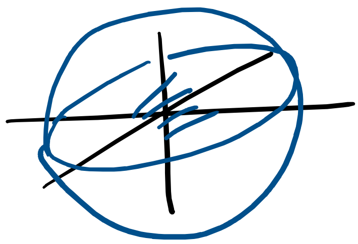
min value $f(0,0) = 1-\sqrt{1} = \boxed{0}$ max value $x^2+y^2=1$ $f(x,y) = 1-\sqrt{1-1} = \boxed{1}$

We conclude that f has absolute minimum value =-1/4 over E, and absolute maximum value =1 over E.

The idea is that to analyze f over the boundary of E, we must consider finding the absolute extremum values of a two-variable function over a region Omega in the plane.

2. $f(x,y,z)=xy+z^2$ over $E=\{(x,y,z): x^2+y^2+z^2\le 1\}$

unit ball

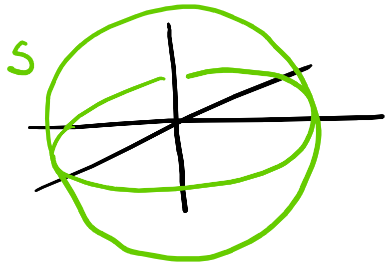


Sol: First, we compute

$$\nabla f = \langle y, x, 2z \rangle \Rightarrow y=0, x=0, z=0$$

$$\langle 0,0,0 \rangle \Rightarrow f(0,0,0) = \boxed{0}$$

Now let's find the absolute extremum values of f over the boundary of E, the unit sphere centered at the origin



$$S = \{(x,y,z) : x^2+y^2+z^2=1\}$$

Consider

$$F(x,y,z) = xy + z^2 \underset{\substack{x^2+y^2+z^2=1 \\ \Rightarrow z^2 = 1-x^2-y^2}}{=} xy + 1 - x^2 - y^2 \quad \text{for } x^2+y^2 \le 1$$

We want to find the absolute extremum values of

$$g(x,y) = xy + 1 - x^2 - y^2 \quad \text{over} \quad \underset{\substack{\text{unit} \\ \text{disk}}}{\Omega} = \{(x,y) : x^2+y^2 \le 1\}$$

First, we compute

$$\vec{0} = \nabla g = \langle y - 2x, x - 2y \rangle \Rightarrow \begin{aligned} -2x + y &= 0 \\ x - 2y &= 0 \end{aligned}$$

$$\Rightarrow \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}}_{\text{invertible}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow g(0,0) = 0 + 1 - 0 - 0 = \boxed{1}$$

Second, we consider the absolute extremum values of g over the unit circle,

$$g(x,y) = xy + 1 - x^2 - y^2 = xy + 1 - 1 = xy$$

$x^2+y^2=1$

We want to find the absolute extremum values of

$h(x,y) = xy$

for $x^2+y^2=1$

max value

$y = x$

$\Rightarrow x = y = \frac{\sqrt{2}}{2}$

$x = y = -\frac{\sqrt{2}}{2}$

$h\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$

$h\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} \cdot -\frac{\sqrt{2}}{2} = \frac{1}{2}$

$\Rightarrow \boxed{\frac{1}{2}}$

min value

$y = -x$

$\Rightarrow \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\Rightarrow \boxed{-\frac{1}{2}}$

The unit circle is given by the parametric plane curve r(t)=<cos(t),sin(t)> for 0≤t≤2pi. We want to find the absolute extremum values of the single-variable function

$$i(t) = h(\vec{r}(t)) = h(\cos t, \sin t) = \cos t \sin t \quad \text{over } [0, 2\pi]$$

We compute $i(0) = 0,\ i(2\pi)$

$0 = i'(t) = -\sin t \cdot \sin t + \cos t \cdot \cos t$

$\Rightarrow \cos^2 t = \sin^2 t$

$\Rightarrow \cos t = \pm \sin t$

$\Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\Rightarrow \vec{r}\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \ldots$

This will give us that i(t) has absolute maximum value =1/2 over [0,2pi], and absolute minimum value =-1/2 over [0,2pi].

We conclude that f has absolute minimum value =-1/2 over E, and absolute maximum value =1 over E.

There must be a better way of doing this, which there is. That is what we will do next time: Lagrange Multipliers.