VECTOR CALCULUS, Week 7

Review

Please read the following instructions for the Midterm.

Name: ________________________ ID: ________________________

- You must show your work.
- You are **not** required to simplify you answers.
  - In particular, you are not required to simplify computations involving basic arithmetic:

addition	subtraction	multiplication
division	roots	powers

    For example, writing $x - 2x$ or $4/2$ as an answer is okay.
  - You are required to evaluate sine and cosine at basic angles.
  - If you are not sure, then ask me before you waste any time simplifying.

Topics

($*$) means I *directly* will test you on this. This outline does not include the first four lectures.

- **6.1 Integration by Parts**
  - Integration by Parts
- **6.2 Trigonometric Integrals and Substitutions**
  - Fact: computing $\int \sin^m x \cos^n x \ dx$
  - Fact: computing $\int \tan^m x \sec^n x \ dx$
  - Fact: evaluating some trigonometric integrals using angle addition formulas.
  - Fact: computing integrals involving $\sqrt{a^2 \pm x^2}$, $\sqrt{x^2 - a^2}$
- **6.3 Partial Fractions**
  - Def: rational fraction, partial fractions.
  - Easy Partial Fractions
- **6.4 Integration with Tables and Computer Algebra**
  - Fact: you only need to know the Basic Table of Integrals.
- **6.6 Improper Integrals.**
  - Def: improper integrals, convergent, diverges to $\pm\infty$, divergent.
  - Fact: $\int_0^a x^p \ dx$ and $\int_a^\infty x^p \ dx$.
  - ($*$) Comparison Thm
- **9.1 Parametric Curves**
  - Def: parametric plane curve, image, parameter, parametric equations, initial point, terminal point.

- **9.2 Calculus with Parametric Curves**
  - (∗) Fact: if $x(t)$ is increasing/decreasing near a, then near $(x(a), y(a))$ the image is the graph of a function $y = f(x)$. Also, $f'(x(a)) = \frac{y'(a)}{x'(a)}$.
  - (∗) Def: the tangent line of C at $t = a$, undefined slope.
  - – Def: area under C.
  - (∗) Fact: computing the area under C.
  - – Def: self-intersections, isolated self-intersections.
  - (∗) Fact: computing the arc length of the image of C.
  - (∗) Fact: computing the arc length of the graph of a function.
  - (∗) Fact: computing the surface area of the surface of revolution formed by the image of C.
  - (∗) Fact: computing the surface area of the surface of revolution formed by the graph of a function.
- **9.3 Polar Coordinates**
  - – Fact: $(x, y) = (r \cos \theta, r \sin \theta)$.
  - – Def: polar coordinates, polar coordinate system, pole, polar axis.
  - (∗) Def: polar parametric plane curve, polar parametric equation.
- **9.4 Areas and Lengths in Polar Coordinates**
  - – Fact: area of a region bounded by a polar parametric plane curve.
  - – Fact: computing the arc length of a polar parametric plane curve.

- **10.1 Three-Dimensional Coordinate Systems; 10.2 Vectors; 10.3 The Dot Product; 10.4 The Cross Product**
  - Def: basic notation for vectors in the plane and in space.
  - Def: dot product.
  - Thm: $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta$.
  - Def: orthogonal/vector projection.
  - Thm: uniqueness of the orthogonal/vector projection.
  - Def: cross product, parallelogram formed by two vectors, right-hand rule.
  - Thm: uniqueness of the cross product.
  - Fact: the cross product is anti-commutative, and gives volumes of parallelpipeds.
- **10.5 Equations of Lines and Planes**
  - Fact: point-direction parameterizations for lines.
  - Fact: normal direction, implicit/vector equation, scalar equation, and linear equations for lines.
- **10.6 Cylinders and Quadric Surfaces**
  - Def: quadric surfaces.
  - Fact: level set method.

Practice Problems I

1) Determine whether each improper integral is convergent or divergent.

(a) $\int_{-\infty}^{0} ze^{2z}\, dz$

(b) $\int_{1}^{\infty} \frac{1}{x^2+x}\, dx$

2) Let $f(x) = e^x + x$. Set up, *but do not evaluate*, an integral for each of the following.

(a) The arc length L of the curve $y = f(x)$ for $1 \leq x \leq 3$.

(b) The surface area S of the surface of revolution formed by rotating the graph of f over $[1, 3]$ around the x-axis.

3) Consider the parametric plane curve

$$C(t) = (\sqrt{t+1}, \sqrt{t-1}) \text{ for } t \geq 1.$$

(a) Eliminate the parameter to find a Cartesian equation for C.

(b) Find all the points where the image of C intersects the line $y = \frac{x}{2}$.

4) Consider the parametric plane curve

$$C(t) = \left(\frac{t^3}{3} - \frac{3t^2}{2} + 2t + 1, \frac{2t^3}{3} - 5t^2 + 12t \right) \text{ for } t \in \mathbf{R}.$$

(a) Compute the tangent line of C at $t = 0$.

(b) Compute all t so that $x'(t) = 0$, and compute the slope of the tangent line of C at all such t.

(c) Find all t so that the tangent line of C at t is horizontal.

5) Consider the parametric plane curve

$$C(t) = (t^2 + 3t + 1, t^3) \text{ for } 0 \le t \le 2.$$

(a) Show that the image of C is the graph of a function $y = f(x)$ defined over $[1, 11]$.

(b) Set up, *but do not evaluate*, an integral for the arc length L of the image of C.

(c) Set up, *but do not evaluate*, an integral for the area A under C.

6) Find a Cartesian equation for the polar parametric plane curve given by the polar parametric equation $r = 5\sin\theta$.

7) Consider the polar parametric plane curve $C(\theta) = (x(\theta), y(\theta))$ given by the polar parametric equation $r = e^{\theta}$.

(a) Compute the tangent line of C at $\theta = 0$.

(b) Compute all $\theta \in [0, 2\pi)$ so that $x'(\theta) = 0$, and compute the slope of the tangent line of C at all such θ.

(c) Find all $\theta \in [0, 2\pi)$ so that the tangent line of C at θ is horizontal.

Practice Problems I Answers

1)

(a) convergent

(b) convergent.

2)

(a) $L = \int_1^3 \sqrt{1 + (e^x + 1)^2}\ dx$

(b) $S = \int_1^3 2\pi(e^x + x)\sqrt{1 + (e^x + 1)^2}\ dx$

3)

(a) $x^2 - 1 = y^2 + 1$

(b) $(\sqrt{8/3}, \sqrt{2/3})$

4)

(a) $y = 6(x - 1) + 0$

(b) $t = 1$ with undefined slope, $t = 2$ with slope -2.

(c) $t = 3$

5)

(a) $x'(t) > 0$ for $0 \leq t \leq 2$

(b) $\int_0^2 \sqrt{(2t + 3)^2 + (3t^2)^2}\ dt$

(c) $\int_0^2 t^3(2t + 3)\ dt$

6) $x^2 + y^2 = 5y$, this is a circle!

7)

(a) $y = (x - 1)$

(b) $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ both slopes undefined.

(c) $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

Practice Problems II

1) Determine whether each improper integral is convergent or divergent.

(a) $\int_0^1 x \ln x \; dx$

(b) $\int_1^\infty \frac{1}{\sqrt{x}+\sqrt[3]{x}} \; dx$

2) Let $f(x) = 1 - e^{-x}$. Set up, *but do not evaluate*, an integral for each of the following.

(a) The arc length L of the curve $y = f(x)$ for $1 \leq x \leq 3$.

(b) The surface area S of the surface of revolution formed by rotating the graph of f over $[1, 3]$ around the x-axis.

3) Consider the parametric plane curve

$$C(t) = (\tan^2 t, \sec t) \text{ for } -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

(a) Eliminate the parameter to find a Cartesian equation for the curve

(b) Find all the points where the image of C intersects the curve $y = 2\sqrt{x}$.

4) Consider the parametric plane curve

$$C(t) = \left(\frac{t^3}{3} - \frac{t^2}{2}, \frac{t^4}{4} - t^3 \right) \text{ for } t \in \mathbf{R}.$$

(a) Compute the tangent line of C at $t = -1$.

(b) Compute all t so that $x'(t) = 0$, and compute the slope of the tangent line of C at all such t.

(c) Find all t so that the tangent line of C is horizontal.

5) Consider the parametric plane curve

$$C(t) = ((t-1)^2, e^{t^2} - t^2) \text{ for } 0 \le t \le 2.$$

(a) Show that the image of C is the graph of a function $x = g(y)$ defined over $[1, e^4 - 4]$.

(b) Set up, *but do not evaluate*, an integral for the arc length L of the image of C.

(c) Set up, *but do not evaluate*, an integral for the area A bounded by the image of C, the horizontal lines $y = 1, y = e^4 - 4$, and the y-axis.

6) Consider the polar parametric plane curve $C(\theta) = (x(\theta), y(\theta))$ given by the polar parametric equation $r = 1 + \cos\theta$.

(a) Compute the tangent line of C at $\theta = \frac{\pi}{2}$.

(b) Compute all $\theta \in [0, 2\pi)$ so that $x'(\theta) = 0$, and compute the slope of the tangent line of C at all such θ.

(c) Find all $\theta \in [0, 2\pi)$ so that the tangent line of C at θ is horizontal.

Practice Problems II Answers

1)

(a) convergent

(b) divergent

2)

(a) $L = \int_1^3 \sqrt{1 + (e^{-x})^2}\ dx$

(b) $S = \int_1^3 2\pi(1 - e^{-x})\sqrt{1 + (e^{-x})^2}\ dx$

3)

(a) $x = y^2 - 1$

(b) $(1/3, 2/\sqrt{3})$

4)

(a) $y = -2(x - (-\frac{5}{6})) + \frac{5}{4}$

(b) $t = 0$ with slope $m = 0$, and $t = 1$ with undefined slope.

(c) $t = 0$ and $t = 3$.

5)

(a) Note that $y'(t) > 0$ for $0 < t \leq 2$. This implies $y(t)$ is increasing over $[0, 2]$.

(b) $\int_0^2 \sqrt{(2(t-1))^2 + (2te^{t^2} - 2t)^2}\ dt$

(c) $\int_0^2 (t-1)^2(2te^{t^2} - 2t)\ dt$

6)

(a) $y = 1 \cdot (x - 0) + 1$

(b) θ and slope m

$$\begin{array}{ll} \theta = 0 & m = \text{ undefined} \\ \theta = \frac{2\pi}{3} & m = \text{ undefined} \\ \theta = \pi & \quad\quad m = \ 0 \\ \theta = \frac{4\pi}{3} & m = \text{ undefined} \end{array}$$

(c) $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$