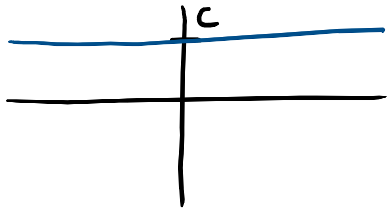
1.2 A Catalog of Essential Functions

Def: We will use the following basic functions and their graphs. The domain is R, unless otherwise specified.

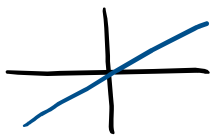
constant functions f(x)=c



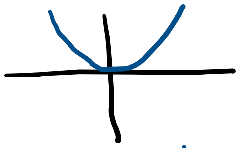
basic power functions

natural power functions $f(x)=x^n$ for n=1,2,3,...

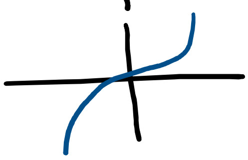
identity function f(x)=x



square function $f(x)=x^2$

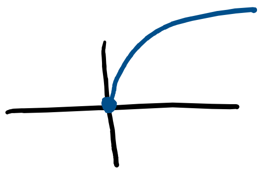


cubic function $f(x)=x^3$

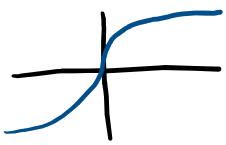


nth root functions $f(x)=x^{\frac{1}{n}}=\sqrt[n]{x}$ for n=2,3,4,...
with domain $\begin{cases} [0,\infty) & \text{if } n \text{ is even} \\ \mathbb{R} & \text{if } n \text{ is odd} \end{cases}$

square root function $f(x)=\sqrt{x}$

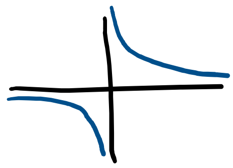


cube root function $f(x)=x^{\frac{1}{3}}$

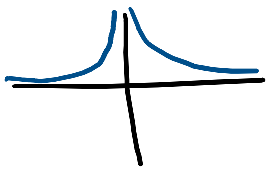


reciprocal functions $f(x)=x^{-n} = \frac{1}{x^n}$ for n=1,2,3,...
with domain x=/0.

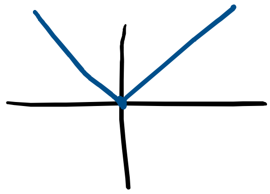
inverse/reciprocal function $f(x)= \frac{1}{x}$



inverse square function $f(x)=\frac{1}{x^2}$

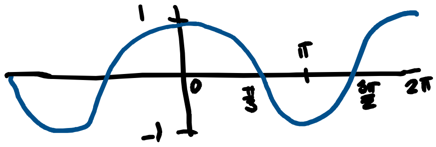


absolute value function f(x)=|x|



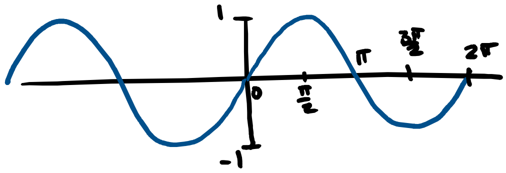
trigonometric functions

cosine f(x)=cos(x)

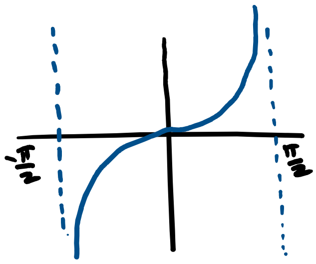


$|\cos x|, |\sin x| \leq 1$

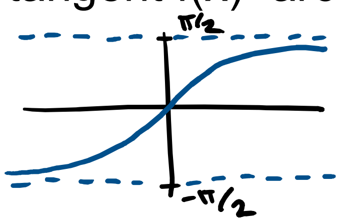
sine f(x)=sin(x)



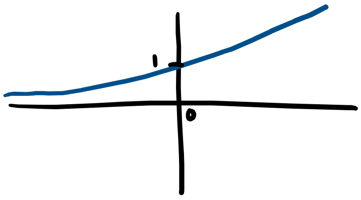
tangent $f(x)=\tan(x)=\frac{\sin x}{\cos x}$ for x in $(-\frac{\pi}{2}, \frac{\pi}{2})$



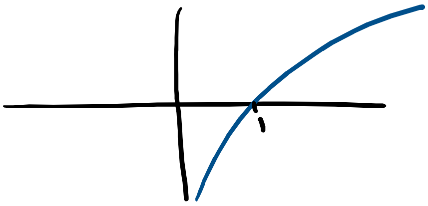
arctangent/inverse tangent $f(x)=\arctan(x)=\tan^{-1}(x)$



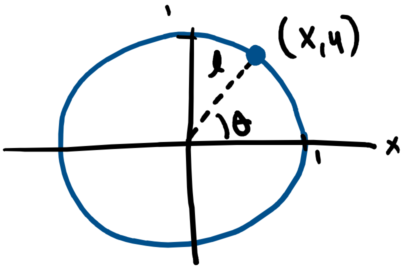
exponential function $f(x)=e^x$



natural logarithm f(x)=ln(x)
with domain (0,infinity)



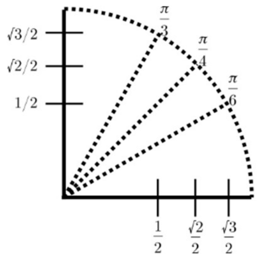
Def: Consider the unit circle, and take an angle theta in R. Draw a unit line segment ell from the origin to the unit circle, such that ell makes angle theta counterclockwise from the positive x-axis;



$$\Rightarrow \quad \begin{aligned} \cos\theta &= x \\ \sin\theta &= y \end{aligned}$$

note that if theta is negative, we make angle |theta| clockwise from the positive x-axis. If (x,y) is the point on the unit circle at the end of ell, then we define cos(theta)=x and sin(theta)=y.

Use the unit circle graph to remember the values of cosine and sine at the basic angles theta=0,pi/6,pi/4,pi/3,pi/2,2pi/3...



Fact: Basic trigonometric formulas:

Pythagorean Thm $\quad \sin^2\theta + \cos^2\theta = 1$ $\qquad x^2+y^2=1 \;\; = \;\; \cos^2\theta + \sin^2\theta$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\cos(-\theta) = \cos\theta \qquad \text{even}$$

$$\sin(-\theta) = -\sin\theta \qquad \text{odd}$$

Def: We define the following additional trigonometric functions.

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad \text{for } 0<\theta<\pi$$

$$\sec\theta = \frac{1}{\cos\theta} \quad \text{for } -\frac{\pi}{2}<\theta<\frac{\pi}{2} \text{ and } \frac{\pi}{2}<\theta<\frac{3\pi}{2}$$

$$\csc\theta = \frac{1}{\sin\theta} \quad \text{for } 0<\theta<\pi \text{ and } \pi<\theta<2\pi$$

These are not basic functions, so you don't need to know their graphs.

Fact: Basic properties of the exponential and natural logarithm.

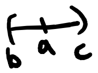
$\ln e^x = x$ for all x, while $e^{\ln x} = x$ for $x > 0$

$e^{a+b} = e^a e^b$ and $(e^a)^b = e^{ab}$

$\ln(ab) = \ln a + \ln b$ and $\ln(a^b) = b \ln a$

1.3 The Limit of a Function

"Def": Suppose a is in R, and suppose f is defined $\underbrace{\text{on an interval containing a}}_{\text{near } a}$, but perhaps not at a itself.



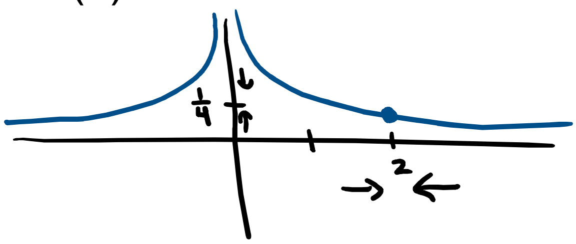
If there is an L in R so that as <u>x approaches a, the value f(x) approaches L</u>, then we say <u>the limit of f at a exists and is L</u>, and we write

$$\lim_{x\to a} f(x) = L$$

If f(x) does not approach any fixed L in R as x approaches a, then we say $\lim_{x\to a} f(x)$ <u>does not exist</u>, or <u>DNE.</u>

Ex: Use the graph of $f(x)=x^{-2}$ to show $\lim_{x\to 2} f(x)=1/4$.

Sol: The graph of f(x) is



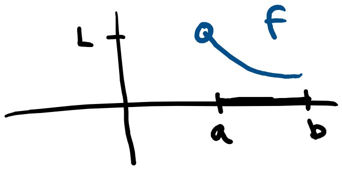
$$\Rightarrow \lim_{x\to 2} \frac{1}{x^2} = \frac{1}{4} \checkmark$$

Fact: Suppose f is a basic function. If f is defined near a and including at a, then $\lim_{x\to a} f(x) = f(a)$.

"Def": We define one-sided limits. Suppose a is in R.

Suppose f is defined on an interval (a,b), where b>a. If there is an L in R so that as x>a approaches a, the value f(x) approaches L, then we say the right-hand limit of f at a exists and is L, and we write

$$\lim_{x\to a^+} f(x) = L$$



We similarly define the left-hand limit of f at a exists and is L, and we write

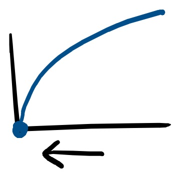
$$\lim_{x\to a^-} f(x) = L$$

We otherwise say $\lim_{x\to a^+} f(x)$ DNE or $\lim_{x\to a^-} f(x)$ DNE.

Fact: All appropriate left- and right-hand limits of basic functions exist at each point of their domains, with limit being the value of the function at the point.

Ex: One-sided limits.

1. Compute $\lim_{x\to 0^+} \sqrt{x}$



Sol:

$$\lim_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0$$

2. However, $\lim_{x\to 0^-} \sqrt{x}$ DNE, because $\sqrt{x}$ is not defined for x<0.

1.4 Calculating Limits

Thm (Basic Limit Laws): Suppose a,b,c,L,M are in R.

1. Simplification Rule: Suppose f(x)=g(x) for all x near a, but perhaps not at a itself. If $\lim_{x\to a} g(x)=L$, then $\lim_{x\to a} f(x)=L$.

   ⇨ You can simplify inside the limit.

2. If $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=M$, then

Addition Rule: $\lim_{x\to a} (f(x)+g(x))=L+M$.

Multiplication Rule: $\lim_{x\to a} f(x)g(x)=LM$.

⇨ $\lim_{x\to a} c*f(x)= c*\lim_{x\to a} f(x)$

Division Rule: If M=/0, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

3. u-Substitution Rule: If

$$\lim_{x\to a} g(x)=b \quad \text{and} \quad \lim_{u\to b} f(u)=L$$

then

$$\lim_{x\to a} f(g(x)) \underset{u=g(x)}{=} \lim_{u\to b} f(u)=L$$

Similar rules hold for one-sided limits.

Ex: Compute the following limits.

1. $\lim\limits_{x\to 0}\sqrt{x+1}$

Sol: We use the u-Subsitution Rule,

$$\lim_{x\to 0}\sqrt{x+1} \underset{\substack{u=x+1 \\ \lim\limits_{x\to 0} x+1 = 0+1=1}}{=} \lim_{u\to 1}\sqrt{u} = \sqrt{1} = \boxed{1}$$

You don't need to do all of this detail on the homework or tests.

2. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}$; multiply by the conjugate.

Sol: We compute

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} \underset{\text{SIMP}}{=} \lim_{x\to 0}\left(\frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

$(a-b)(a+b)=a^2-b^2$

$$\underset{\text{SIMP}}{=} \lim_{x\to 0}\frac{(\sqrt{x+1})^2-1^2}{x(\sqrt{x+1}+1)}$$

$$\underset{\text{SIMP}}{=} \lim_{x\to 0}\frac{x+1-1}{x(\sqrt{x+1}+1)}$$

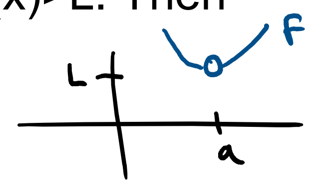
$$\underset{\text{SIMP}}{=} \lim_{x\to 0}\frac{x}{x(\sqrt{x+1}+1)}$$

$$\underset{\text{SIMP}}{=} \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} \underset{\text{DIV}}{=} \boxed{\frac{1}{2}}$$

$$\lim_{x\to 0} \sqrt{x+1}+1 = \sqrt{0+1}+1 = 2 \neq 0$$

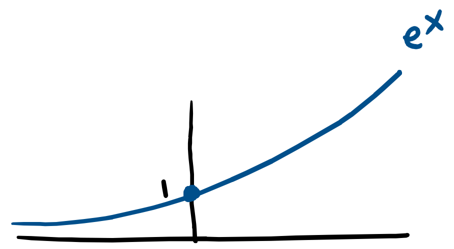
Def: Suppose a,L are in R, and suppose that as x approaches a, the values f(x) approach L with f(x)>L. Then we write

$$\lim_{x\to a} f(x) = L^+$$



We similarly define $\lim_{x\to a, a^\pm} f(x)=L^\pm$ (six combinations).

Ex: Compute $\lim_{x\to 0^+} \sqrt{e^x-1}$.



$\Rightarrow e^x > 1$ for $x>0$

Sol: Note that $e^x-1>0$ for x>0.
So, we compute

$$\lim_{x\to 0^+} \sqrt{e^x-1} \underset{u=e^x-1}{=} \lim_{u\to 0^+} \sqrt{u} = \sqrt{0^+} = \boxed{0^+}$$

$$\lim_{x\to 0^+} (e^x-1) = 0^+$$

1.5 Continuity

Def: Suppose a,b are in R.

We say f is continuous at a if and only if $\lim_{x \to a} f(x)=f(a)$.

We say f is left-continuous at a if and only if $\lim_{x \to a^-} f(x)=f(a)$.

We say f is right-continuous at a if and only if $\lim_{x \to a^+} f(x)=f(a)$.

We say f is continuous over [a,b] if f is continuous at every x in (a,b), f is right-continuous at a, and f is left-continuous at b.

We similarly define f is continuous over

(a,b), $[a,b)$, $(a,b]$, (a,∞), $[a,\infty)$, $(-\infty,b)$, $(-\infty,b]$, or $(-\infty,\infty)$.

Fact: If f is a basic function, then f is appropriately continuous, left-continuous, or right-continuous at every point in its domain.

$\sqrt{x}$ is right- but not left-continuous at $a=0$.

1.6 Limits Involving Infinity

"Def": Suppose a,L are in R.

Suppose f is defined near a, but perhaps not at a. If as x approaches a, the values f(x) get larger in the positive direction, then we say the limit of f at a exists in the extended sense and is infinity, and we write

$$\lim_{x\to a} f(x) = \infty$$

We similarly define $\lim\limits_{x\to a, a^{\pm}} f(x) = \pm\infty$ (six combinations) in the extended sense.

If $\lim\limits_{x\to a, a^{\pm}} f(x) = \pm\infty$, then we say f has a vertical asymptote at x=a.

Suppose f is defined over (a,infinity). If as x gets larger in the positive direction, the values f(x) get closer to L, then we say the limit of f at infinity exists and is L, and we write

$$\lim_{x\to\infty} f(x) = L$$

We similarly define $\lim\limits_{x\to -\infty} f(x) = L$.

If either $\lim\limits_{x\to \pm\infty} f(x) = L$, then we say f has horizontal asymptote y=L.

Suppose is defined over (a,infinity). If as x gets larger in the positive direction, the values f(x) get larger in the positive direction, then we say the limit of f at infinity exists in the extended sense and is infinity, and we write

$$\lim_{x\to\infty} f(x) = \infty .$$

We similarly define $\lim_{x\to\pm\infty} f(x) = \pm\infty$ (four combinations).

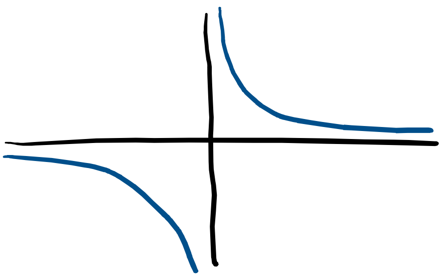
If $\lim_{x\to\pm\infty} f(x) = \pm\infty$, then we appropriately say f blows up/down at $\pm\infty$.

Fact: We can use the graphs of the basic functions to compute their vertical asymptotes, horizontal asymptotes, blow-up, and blow-down behavior.

Ex: Determine the existence of the following limits in the extended sense.

1. $\lim_{x\to 0, 0^\pm} \frac{1}{x}$.

Sol: Recall the graph of the reciprocal function



$$\Rightarrow \left.\begin{aligned} \lim_{x\to 0^+} \frac{1}{x} &= \infty \\ \lim_{x\to 0^-} \frac{1}{x} &= -\infty \end{aligned}\right\} \Rightarrow$$

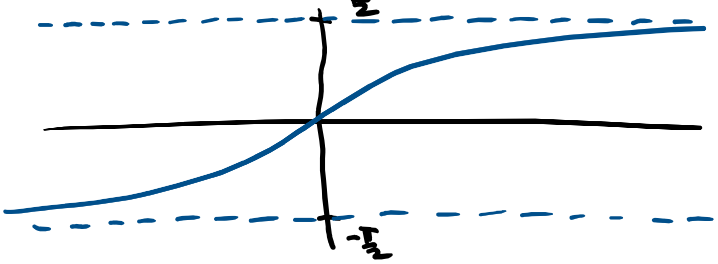
$\frac{1}{x}$ has vertical asymptote at $a=0$

$$\Rightarrow \lim_{x\to 0} \frac{1}{x} \text{ DNE},$$

even in the extended sense

2. $\lim_{x \to \pm\infty} \arctan x$

Sol: Consider the graph of arctangent,

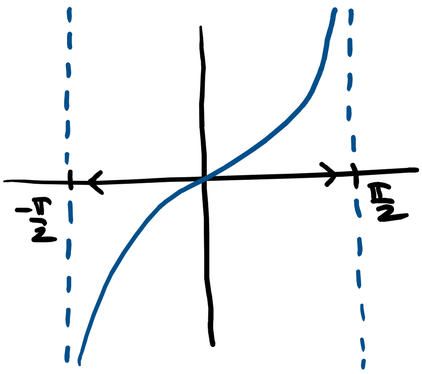


$$\Rightarrow \quad \lim_{x\to\infty} \arctan x = \frac{\pi}{2} \qquad \lim_{x\to-\infty} \arctan x = \frac{-\pi}{2}$$

$\Rightarrow$ arctan x has horizontal asymptotes $y = \frac{\pi}{2}$, $y = \frac{-\pi}{2}$.

3. $\lim_{\theta \to \frac{\pi}{2}^-} \tan(\text{theta})$ and $\lim_{\theta \to -\frac{\pi}{2}^+} \tan(\text{theta})$.

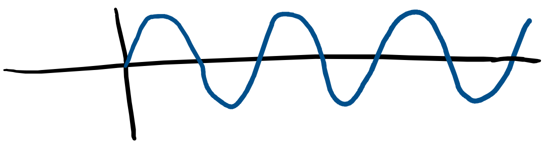
Sol: The graph of tangent is



$$\Rightarrow \quad \lim_{\theta\to\frac{\pi}{2}^-} \tan\theta = \infty \qquad \lim_{\theta\to-\frac{\pi}{2}^+} \tan\theta = -\infty$$

$\Rightarrow$ tangent has vertical asymptotes at $a = \frac{\pi}{2}$ and $a = \frac{-\pi}{2}$.

4. $\lim_{x\to\infty} \sin(x)$ DNE.



Ex: Consider $\lim_{x \to \infty} (x^2-x)$.

Sol: The problem is that $f(x)=x^2-x$ is not a basic function.
We need some Basic Limit Laws involving infinity...