Vector Calculus
Practice Problems II

1) Determine whether each improper integral is convergent or divergent.

a) $\int_0^1 x \ln x \, dx$

Sol: You must compute $\lim_{a \to 0^+} \int_a^1 x \ln x \, dx$

integrations by parts ...

b) $\int_1^\infty \frac{1}{\sqrt{x} + \sqrt[3]{x}} \, dx$

Sol: Use the Comparison Thm,

$$x \geq 1 \Rightarrow \quad 0 < \sqrt{x} + \sqrt[3]{x} \leq \sqrt{x} + \sqrt{x} = 2\sqrt{x}$$

$x^{\frac{1}{3}} \leq x^{\frac{1}{2}}$

$x \leq x^{\frac{3}{2}}$ ✓

for $x \geq 1$

$$\Rightarrow \quad \frac{1}{2} x^{-\frac{1}{2}} \leq \frac{1}{\sqrt{x} + \sqrt[3]{x}}$$

2) Let $f(x) = 1 - e^{-x}$. Set up, but do not evaluate, an integral for each of the following.

a) The arc length L of the curve $y = f(x)$ for $1 \leq x \leq 3$.

Sol: $\Rightarrow$ $$L = \int_1^3 \sqrt{1 + (f'(x))^2} \, dx$$

compute

b) The surface area S of the surface of revolution formed by rotating the graph of f over [1,3] around the x-axis.

Sol:

$$\Rightarrow \quad S = \int_1^3 2\pi F(x) \sqrt{1 + (f'(x))^2}\, dx$$

$\uparrow$ compute

3) Consider the parametric plane curve

$$C(t) = (\tan^2 t, \sec t) \quad \text{for} \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

a) Eliminate the paraneter to find a Cartesian equation for the curve.

Sol: Use $\left.\begin{array}{l} x = \tan^2 t \\ y = \sec t \end{array}\right\}$

$$\cos^2 t + \sin^2 t = 1$$

$$\Rightarrow \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

$$\Rightarrow 1 + \tan^2 t = \sec^2 t$$

b) Find all the points where the image of C intersects the curve $y=2\sqrt{x}$.

Sol: Set $y=2\sqrt{x}$ into the Cartesian equation for C found in a), and then solve for x. However, you should keep in mind that we need $x \geqslant 0$. You can also consider

$$\sec t = y = 2\sqrt{x} = 2\sqrt{\tan^2 t}$$

$$\Rightarrow \underset{\displaystyle 1+\tan^2 t}{\overset{\parallel}{\sec^2 t}} = 4\tan^2 t$$

$$\Rightarrow 1 + \tan^2 t = 4\tan^2 t \ \cdots$$

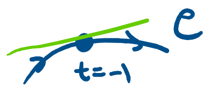
$$t = ?$$

4) Consider the parametric plane curve

$$C(t) = \left(\frac{t^3}{3} - \frac{t^2}{2}, \frac{t^4}{4} - t^3\right) \quad \text{for } t \in \mathbb{R}$$

a) Compute the tangent line of C at t=-1.

Sol: First, we compute



$$x'(-1) = t^2 - t \Big|_{t=-1} = 1 - (-1) = 2 \neq 0$$

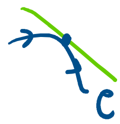
This implies the tangent line of C at t=-1 is

$$y = \underset{\overset{\shortparallel}{F'(x(1))}}{\frac{y'(-1)}{x'(-1)}} (x - x(-1)) + y(-1)$$

We compute

$$C(-1) = \left(\frac{-1}{3} - \frac{1}{2}, \frac{1}{4} + 1\right) = (x(-1), y(-1))$$

$$y'(-1) = t^3 - 3t^2 \Big|_{t=-1} = -1 - 3 = -4$$



$$\Rightarrow \boxed{y = \frac{-4}{2}\left(x - \left(-\frac{1}{3} - \frac{1}{2}\right)\right) + \left(\frac{1}{4} + 1\right)}$$

b) Compute all t so that x'(t)=0, and compute the slope of the tangent line of C at all such t.

Sol: First, we compute

$$0 = x'(t) = t^2 - t = t(t-1) \Rightarrow \boxed{t = 0, 1}$$

Consider t=0,

$$y'(0) = t^3 - 3t^2 \Big|_{t=0} = 0 \qquad \text{oops...}$$

Since x’(0)=y’(0)=0, we need to compute

$$\lim_{t\to 0} \frac{y'(t)}{x'(t)} = \lim_{t\to 0} \frac{t^2-t}{t^3-3t^2} = \lim_{t\to 0} \frac{t\,(t-1)}{t^2\,(t-3)}$$

$$= \lim_{t\to 0} \frac{1}{t}\cdot\frac{t-1}{t-3} = \lim_{t\to 0} \frac{1}{t}\cdot\frac{1-t}{3-t}$$

In this case, we must compute the one-sided limits

$$\lim_{t\to 0^+} \underbrace{\frac{1}{t}}_{\to\infty}\cdot\underbrace{\frac{1-t}{3-t}}_{1/3} = \infty\cdot\frac{1}{3} = \infty$$

$$\lim_{t\to 0^-} \underbrace{\frac{1}{t}}_{\to-\infty}\cdot\underbrace{\frac{1-t}{3-t}}_{\frac{1}{3}} = -\infty\cdot\frac{1}{3} = -\infty$$

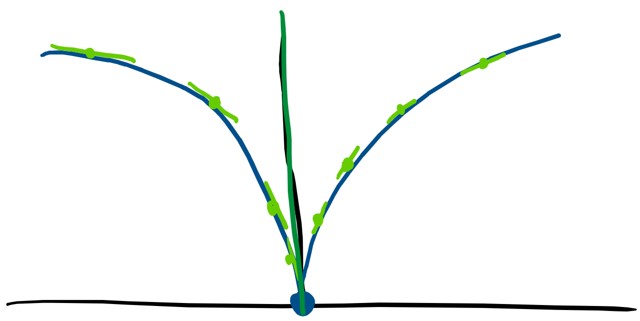
$$\lim_{t\to 0^+} \frac{y'(t)}{x'(t)} = \infty$$

$$\lim_{t\to 0^-} \frac{y'(t)}{x'(t)} = -\infty$$

Since both one-sided limits are a type of infinity, then the tangent line of C at t=0 is a vertical line, and so the slope is undefined.

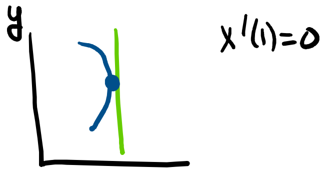
Actually, the image of C has a cusp at t=0.

$C(0)=(0,0)$



Consider t=1,

$$y'(1) = t^3 - 3t^2 \Big|_{t=1} = 1-3 \neq 0$$



y'(1)=0 implies the image of C near t=1 is the graph of a function x=g(y), and x'(1)=0 implies the tangent line is horizontal *with respect to the y-axis*. This means the tangent line is vertical. The slope is undefined.

$$\boxed{t=0,1 \text{ with undefined slope}}$$

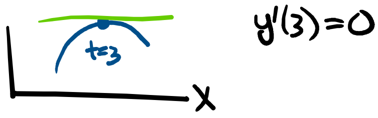
c) Find all t so that the tangent line of C is horizontal.

Sol: First, we compute

$$0 = y'(t) = t^3 - 3t^2 = t^2(t-3) \Rightarrow t = 0, 3$$

We already computed that the tangent line of C at t=0 is a vertical line. Consider t=3,

$$x'(3) = t^2 - t \Big|_{t=3} = 9-3 \neq 0$$



We conclude that the tangent line of C is horizontal only at $\boxed{t=3.}$

5) Consider the parametric plane curve

$$C(t) = \left((t-1)^2,\ e^{t^2} - t^2\right) \text{ for } 0 \le t \le 2$$

a) Show that the image of C is the graph of a function x=g(y) defined over [1,e⁴-4].

Sol: Consider $y(t)=e^{t^2}-t^2$, then

$$y'(t) = 2te^{t^2} - 2t = \underbrace{2t\left(e^{t^2}-1\right)}$$

$$2t > 0 \quad \text{for} \quad t > 0$$

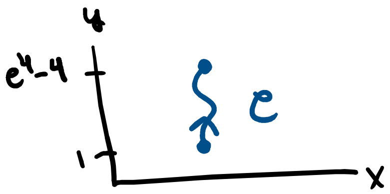
$$e^{t^2} > 1 \quad \text{for} \quad t \neq 0$$

this implies that

$$y'(t) > 0 \quad \text{for} \quad 0 < t < 2$$

This means that y(t) is increasing for $0 \leqslant t \leqslant 2$. We conclude that the image of C is the graph of a function x=g(y) defined over

$$[y(0), y(2)] = [e^0 - 0, e^4 - 4] = [1, e^4 - 4] \checkmark$$



b) Set, but do not evaluate, an integral for the arc length L of the image of C.

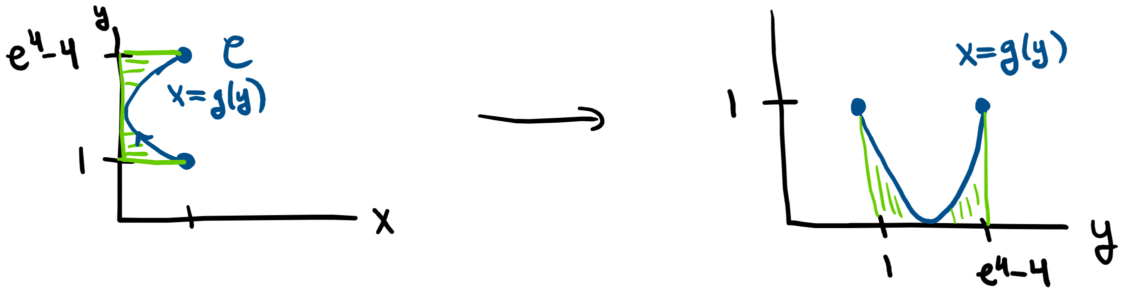
Sol: We give

$$L = \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \ \ldots$$

compute!

c) Set up, but do not evaluate, an integral for the area A bounded by the image of C, the horizontal lines $y=1, y=e^4-4$, and the y-axis.

Sol: Note that we showed the image of C is the graph of a function x=g(y) defined over [1,e^4-4]. We want an integral for the area



This implies that

$$A = \int_1^{e^4-4} g(y)\,dy \underset{y=y(t)}{=\!=} \int_0^2 \overset{g(y(t))}{\overset{\parallel}{x(t)}}\; y'(t)\,dt$$

$y(0)=1$
$y(2)=e^4-4$

Note: 

$$A=\int_a^b y(t)\,x'(t)\,dt$$

We conclude that

$$A=\int_0^2 (t-1)^2 \Big(\underbrace{\tfrac{d}{dt}\, e^{t^2}-t^2}_{\text{compute!}}\Big)\,dt$$

6) Consider the polar parametric plane curve C(theta)=(x(theta),y(theta)) given by the polar parametric equation r=1+cos(theta).

$r = 1+\cos\theta$

a) Compute the tangent line of C at theta=pi/2.

Sol: We compute

$$x'\left(\frac{\pi}{2}\right) = \frac{\partial}{\partial\theta} (1+\cos\theta)\cos\theta \Big|_{\theta=\frac{\pi}{2}}$$

$$= \frac{\partial}{\partial\theta} \cos\theta + \cos^2\theta \Big|_{\theta=\frac{\pi}{2}}$$

$$= -\sin\theta - 2\cos\theta\sin\theta \Big|_{\theta=\frac{\pi}{2}} \quad *$$

$$= -1 - 0 = -1 \neq 0$$

$$y'\left(\frac{\pi}{2}\right) = \frac{\partial}{\partial\theta} (1+\cos\theta)\sin\theta \Big|_{\theta=\frac{\pi}{2}}$$

$$= \frac{\partial}{\partial\theta} \sin\theta + \cos\theta\sin\theta \Big|_{\theta=\frac{\pi}{2}}$$

$$= \cos\theta - \sin^2\theta + \cos^2\theta \Big|_{\theta=\frac{\pi}{2}} \quad *$$

$$= 0 - 1 + 0 = -1 \qquad \Rightarrow \text{slope} = \frac{-1}{-1} = 1$$

$$C\left(\frac{\pi}{2}\right) = \left((1+\cos\theta)\cos\theta, (1+\cos\theta)\sin\theta\right) \Big|_{\theta=\frac{\pi}{2}}$$

$$= (0, 1)$$

We conclude the tangent line of C at theta=pi/2 is

$$\boxed{y = 1\cdot(x-0)+1}$$

b) Compute all theta in [0,2pi) so that x'(theta)=0, and compute the slope of the tangent line of C at all such theta.

Sol: We compute

$$0 = x'(\theta) = -\sin\theta - 2\cos\theta\sin\theta$$

$$\Rightarrow \quad 0 = -\sin\theta\,(1+2\cos\theta)$$

$$\Rightarrow \quad \begin{array}{l}\sin\theta = 0 \\ \cos\theta = -\frac{1}{2}\end{array} \quad \text{for } \theta \in [0, 2\pi)$$

$$\Rightarrow \quad \boxed{\theta = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}}$$

Consider theta=0,

$$y'(0) = \cos\theta - \sin^2\theta + \cos^2\theta \,\Big|_{\theta=0} = 1 - 0 + 1 \neq 0 \Rightarrow \text{undefined}$$

$$y'\left(\tfrac{2\pi}{3}\right) \neq 0, \quad y'\left(\tfrac{4\pi}{3}\right) \neq 0 \Rightarrow \text{undefined}$$

Consider theta=pi,

$$y'(\pi) = \cos\theta - \sin^2\theta + \cos^2\theta \,\Big|_{\theta=\pi} = -1 + 0 + 1 = 0$$

We must consider

$$\lim_{\theta\to\pi} \frac{y'(\theta)}{x'(\theta)} = \lim_{\theta\to\pi} \frac{\cos\theta - \sin^2\theta + \cos^2\theta}{-\sin\theta\,(1+2\cos\theta)}$$

$$= \lim_{\theta\to\pi} \frac{\cos\theta - (1-\cos^2\theta) + \cos^2\theta}{-\sin\theta\,(1+2\cos\theta)}$$

$$= \lim_{\theta \to \pi} \frac{2\cos^2\theta + \cos\theta - 1}{-\sin\theta\,(1+2\cos\theta)}$$

$2x^2+x-1$
$\shortparallel$
$(x+1)(2x-1)$

$$= \lim_{\theta \to \pi} \frac{(\cos\theta+1)(2\cos\theta-1)}{-\sin\theta\,(1+2\cos\theta)}$$

$$= \lim_{\theta \to \pi} \frac{\cos\theta+1}{-\sin\theta} \cdot \lim_{\theta \to \pi} \frac{2\cos\theta-1}{1+2\cos\theta}$$

$\Downarrow$ LHR

$$= \left(\lim_{\theta \to \pi} \frac{-\sin\theta}{-\cos\theta}\right)\left(\frac{-2-1}{1-2}\right)$$

$$= 0 \cdot \frac{-3}{-1} = 0$$

We conclude that x'(theta)=0 at

$\theta = 0$	undefined
* $\theta = \frac{2\pi}{3}$	undefined
$\theta = \pi$	0
* $\theta = \frac{4\pi}{3}$	undefined

c) Find all theta in [0,2pi) so that the tangent line of C at theta is horizontal.

Sol: First we compute

$$0 = y'(\theta) = (\cos\theta+1)(2\cos\theta-1)$$

$$\Rightarrow \quad \begin{matrix} \cos\theta = -1 \\ \cos\theta = \frac{1}{2} \end{matrix} \quad \text{for } \theta \in [0,2\pi)$$

$$\Rightarrow \quad \theta = \overset{\checkmark}{\pi}, \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\pi \longrightarrow \text{slope}=0$$

We already computed that the the tangent line of C at theta=pi is horizontal ($\lim_{\theta\to\pi}\frac{y'(\theta)}{x'(\theta)}=0$).

Consider theta=pi/3,

$$x'\left(\frac{\pi}{3}\right) = -\sin\theta\left(1+2\cos\theta\right)\Big|_{\theta=\frac{\pi}{3}}$$

$$= -\frac{\sqrt{3}}{2}\left(1+2\cdot\frac{1}{2}\right) \neq 0$$

Similarly, x’(5pi/3)=/0. This means the tangent line of C at theta=pi/3,5pi/3 is horizontal.

$$\Rightarrow \quad \boxed{\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}}$$