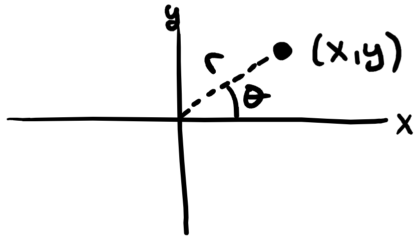
Vector Calculus
9.3 Polar Coordinates

Fact: If (x,y) is in R^2, then there is an r≥0 and theta in [0,2pi) so that (x,y)=(r*cos(theta),r*sin(theta)).



- r^2=x^2+y^2

- If x=/0, then tan(theta)= $\frac{y}{x}$

If x=0 and y>0, then theta=pi/2. If x=0 and y<0, then theta=3pi/2.

Def: Suppose (x,y) is in R^2, and suppose r,theta are in R. If (x,y)=(r*cos(theta),r*sin(theta)), then we say (r,theta)$_p$ are polar coordinates for (x,y). ↖ "polar"

We call the plane with points represented by polar coordinates the polar coordinate system.

We call (0,0)=(0,theta)$_p$ the pole.
any $\theta \in \mathbb{R}$

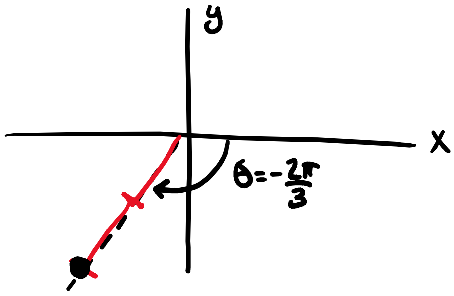
We call the positive x-axis the polar axis.

Note that we allow negative values of r,theta.

Ex: Plot the points given in polar coordinates.

1. (2,-2pi/3)

Sol: r=2 and theta=-2pi/3 gives

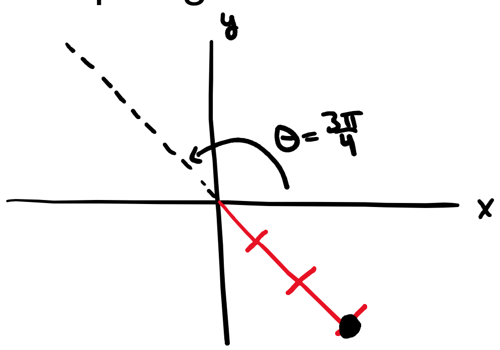


In Cartesian coordinates, this is the point

$$(x,y)=\left(2\cos\left(-\frac{2\pi}{3}\right), 2\sin\left(-\frac{2\pi}{3}\right)\right)$$
$$=\left(2\cdot\left(-\frac{1}{2}\right), 2\left(-\frac{\sqrt{3}}{2}\right)\right)=(-1,-\sqrt{3})$$

2. (-3,3pi/4)

Sol: r=-3 and theta=3pi/4 gives



In Cartesian coordinates, this is the point

$$(x,y)=\left(-3\cos\frac{3\pi}{4}, -3\sin\frac{3\pi}{4}\right)=\left(-3\left(-\frac{\sqrt{2}}{2}\right), -3\left(\frac{\sqrt{2}}{2}\right)\right)=\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$$

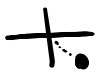
Ex: Give polar coordinates for the following points given in Cartesian coordinates.

1. (1,-1)

Sol: We compute

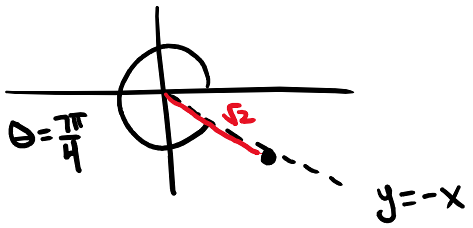
$$r = \sqrt{(1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\tan\theta = \frac{-1}{1} = -1 \underset{\theta = [0, 2\pi)}{\Longrightarrow} \theta = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4} \Longrightarrow \theta = \frac{7\pi}{4}$$



We conclude that

$$(1,-1) = \boxed{\left(\sqrt{2}, \frac{7\pi}{4}\right)_p}$$

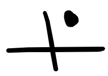


2. (2,2√3)

Sol: We compute

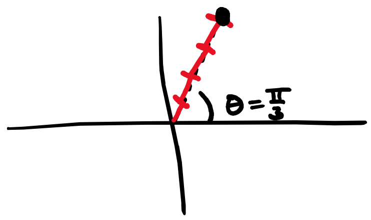
$$r = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4 + 4\cdot 3} = \sqrt{4\cdot 4} = 4$$

$$\tan\theta = \frac{2\sqrt{3}}{2} = \sqrt{3} \underset{\theta \in [0, 2\pi)}{\Longrightarrow} \theta = \frac{\pi}{3}, \frac{4\pi}{3} \Longrightarrow \theta = \frac{\pi}{3}$$



We conclude that

$$(2, 2\sqrt{3}) = \boxed{\left(4, \frac{\pi}{3}\right)_p}$$



Def: A polar parametric plane curve is a parametric plane curve of the form

$$C(\theta)=(x(\theta),y(\theta))=(r(\theta)\cos\theta, r(\theta)\sin\theta)=(r(\theta),\theta)_p \quad \text{for } a\le\theta\le b$$

We say the equation

$$r=r(\theta) \quad \text{for } a\le\theta\le b$$

is a polar parametric equation for C.

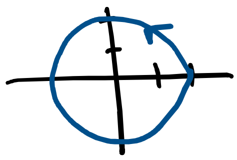
Ex: Identify the images of the polar parametric plane curves given by the following polar parametric equations.

1. r=2

Sol: C is the parametric plane curve

$$C(\theta)=(r(\theta)\cos\theta, r(\theta)\sin\theta)=(2\cos\theta, 2\sin\theta) \quad \text{for all } \theta$$

We conclude that the image of C is the circle of radius =2 centered at the origin.



2. r=2cos(theta), by finding a Cartesian equation for the curve.

Sol: We compute

$$r=2\cos\theta \implies r^2=2r\cos\theta$$

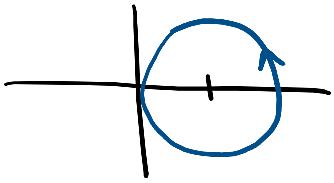
$$\Rightarrow \quad x^2+y^2 = 2x$$

$$\Rightarrow \quad x^2-2x+y^2 = 0$$

$$\Rightarrow \quad x^2-2x+1+y^2 = 1$$

$$\Rightarrow \quad (x-1)^2+(y-0)^2 = 1$$

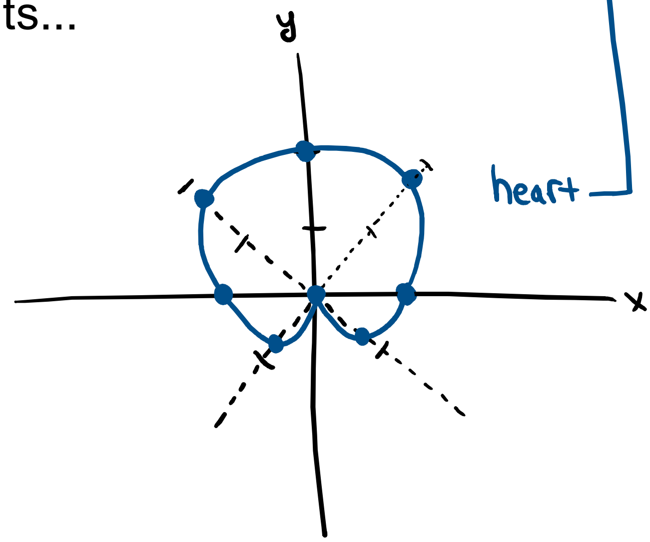
We conclude that the image of C is the unit circle centered at (1,0).



3. r=1+sin(theta), the cardioid

Sol: Let's plot in some points...

θ	r
0	$1 \to (1,0)$
$\frac{\pi}{4}$	$1+\frac{\sqrt{2}}{2} < 2$
$\frac{\pi}{2}$	$2 \to (0,2)$
$\frac{3\pi}{4}$	$1+\frac{\sqrt{2}}{2}$
π	$1 \to (-1,0)$
$\frac{5\pi}{4}$	$1-\frac{\sqrt{2}}{2} < 1$
$\frac{3\pi}{2}$	$1-1 = 0 \to (0,0)$
$\frac{7\pi}{4}$	$1-\frac{\sqrt{2}}{2}$



heart

Ex: Find a polar parametric equation for the curve given by the Cartesian equation

$$(x+1)^2 + (y-2)^2 = 5$$

Sol: We compute

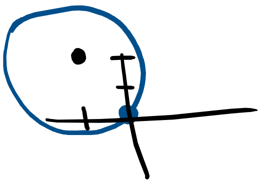
$r = r(\theta)$

$$(x+1)^2 + (y-2)^2 = 5 \Rightarrow x^2 + 2x + 1 + y^2 - 4y + 4 = 5$$

$$\Rightarrow x^2 + y^2 + 2x - 4y = 0$$

$$\Rightarrow r^2 + 2r\cos\theta - 4r\sin\theta = 0$$

$$\Rightarrow r^2 = -2r\cos\theta + 4r\sin\theta$$



$$\Rightarrow \boxed{r = -2\cos\theta + 4\sin\theta \text{ for all } \theta}$$

$(0+1)^2 + (0-2)^2 \overset{\checkmark}{=} 5$

$1 + 4 = 5$

Ex: Consider the cardioid r=1+sin(theta)

1. Find a Cartesian equation for the curve.

Sol: We compute

$$r = 1 + \sin\theta \Rightarrow r^2 = r(1 + \sin\theta)$$

$$\Rightarrow r^2 = r + r\sin\theta$$

$$\Rightarrow \boxed{x^2 + y^2 = \sqrt{x^2 + y^2} + y}$$

2. Compute the slope of the tangent line of C at theta=pi/3,5pi/6.

Sol: Recall that

$$C(\theta) = \Big(\underset{x(\theta)}{\underset{\shortparallel}{(1+\sin\theta)\cos\theta}},\ \underset{y(\theta)}{\underset{\shortparallel}{(1+\sin\theta)\sin\theta}}\Big)$$

First, we compute the slope of the tangent line of C at theta=pi/3. To do this, we compute

$$
\begin{aligned}
x'\left(\tfrac{\pi}{3}\right) &= \frac{\partial}{\partial\theta}(1+\sin\theta)\cos\theta \Big|_{\theta=\frac{\pi}{3}} \\
&= \cos\theta\cos\theta + (1+\sin\theta)(-\sin\theta) \Big|_{\theta=\frac{\pi}{3}} \\
&= \tfrac{1}{2}\cdot\tfrac{1}{2} + \left(1+\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{\sqrt{3}}{2}\right) \\
&= \tfrac{1}{4} - \tfrac{\sqrt{3}}{2} - \tfrac{3}{4} = \frac{-1-\sqrt{3}}{2} \neq 0
\end{aligned}
$$

We conclude that the slope of the tangent line of C at theta=pi/3 is given by

$$\frac{y'\left(\frac{\pi}{3}\right)}{x'\left(\frac{\pi}{3}\right)} = \frac{y'\left(\frac{\pi}{3}\right)}{\left(\frac{-1-\sqrt{3}}{2}\right)}$$

We compute

$$
\begin{aligned}
y'\left(\tfrac{\pi}{3}\right) &= \frac{\partial}{\partial\theta}(1+\sin\theta)\sin\theta \Big|_{\theta=\frac{\pi}{3}} \\
&= \frac{\partial}{\partial\theta}\sin\theta + \sin^2\theta \Big|_{\theta=\frac{\pi}{3}} \\
&= \cos\theta + 2\sin\theta\cos\theta \Big|_{\theta=\frac{\pi}{3}} \\
&= \tfrac{1}{2} + 2\cdot\tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{2} = \frac{1+\sqrt{3}}{2}
\end{aligned}
$$

$$\Rightarrow \quad \text{slope} = \frac{\frac{1+\sqrt{3}}{2}}{\frac{-1-\sqrt{3}}{2}} = \boxed{-1}$$

Second, for theta=5pi/6 we compute

$$x'\left(\tfrac{5\pi}{6}\right) = \frac{\partial}{\partial\theta}(1+\sin\theta)\cos\theta \,\Big|_{\theta=\frac{5\pi}{6}}$$

$$= \cos\theta \cdot \cos\theta + (1+\sin\theta)(-\sin\theta) \,\Big|_{\theta=\frac{5\pi}{6}}$$

$$= -\tfrac{\sqrt{3}}{2} \cdot -\tfrac{\sqrt{3}}{2} + \left(1+\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)$$

$$= \tfrac{3}{4} - \tfrac{3}{2} \cdot \tfrac{1}{2} = 0$$

This means we must compute

$$y'\left(\tfrac{5\pi}{6}\right) = \cos\theta + 2\sin\theta\cos\theta \,\Big|_{\theta=\frac{5\pi}{6}}$$

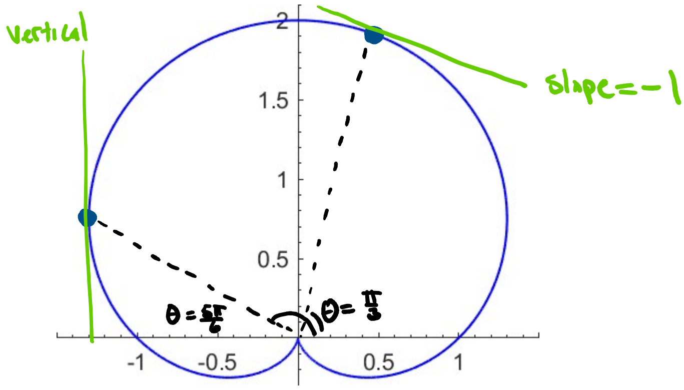
$$= -\tfrac{\sqrt{3}}{2} + 2 \cdot \tfrac{1}{2} \cdot \left(-\tfrac{\sqrt{3}}{2}\right)$$

$$= -\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{2} \neq 0$$

$$\Rightarrow \boxed{\text{slope} = \text{undefined}}$$

In other words, the tangent line of C at theta=5pi/6 is a vertical line.

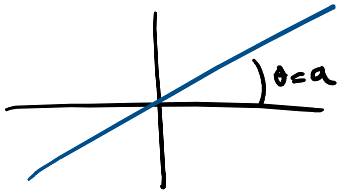
Let's check these two calculations. The cardioid looks like:



$r = 1 + \sin\theta$

9.4 Areas and Lengths in Polar Coordinates

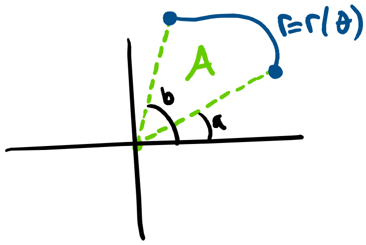
Fact: The set $\{(r,\text{theta})_p : \text{theta}=a\}$ is the line through the origin with angle =a counterclockwise from the positive x-axis.



Ex $\theta = \frac{\pi}{4}$ is the line $y=x$

Fact: Suppose C is the polar parametric plane curve given by the polar parametric equation r=r(theta) for a≤theta≤b, and suppose r is continuous. The (unsigned) area A of the region bounded by C and the lines theta=a and theta=b is given by

$$A = \int_a^b \frac{1}{2}(r(\theta))^2\, d\theta$$



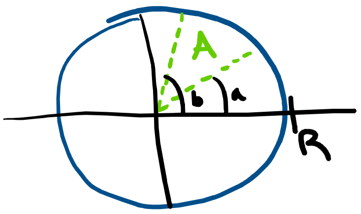
Ex: Give an integral for the area A of the following regions.

1. The region bounded by the polar parametric plane curve C given by the polar parametric equation r=R and the lines theta=a and theta=b.

Sol: This is given by the integral

$$\boxed{A = \int_a^b \frac{1}{2}(R)^2\, d\theta}$$

Note that C is the circle of radius =R centered at the origin. And so A is given by



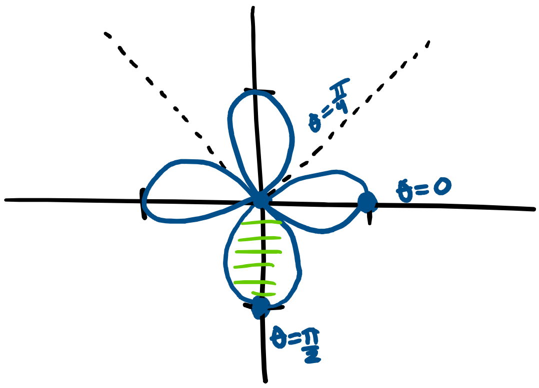
$$\Rightarrow A = \frac{b-a}{2} R^2 = \int_a^b \frac{1}{2} (R)^2 \, d\theta$$

Ex $a=0$, $b=2\pi$ ✓

2. One loop of the four-leaved rose r=cos(2theta).

Sol: The image of this curve is given by

θ	$r(\theta)$
0	$1 \to (1,0)$
$\frac{\pi}{4}$	$\cos(\frac{\pi}{2}) = 0$
$\frac{\pi}{2}$	$\cos(\pi) = -1$
$\frac{3\pi}{4}$	$\cos(\frac{3\pi}{2}) = 0$



The area A of one loop is given by

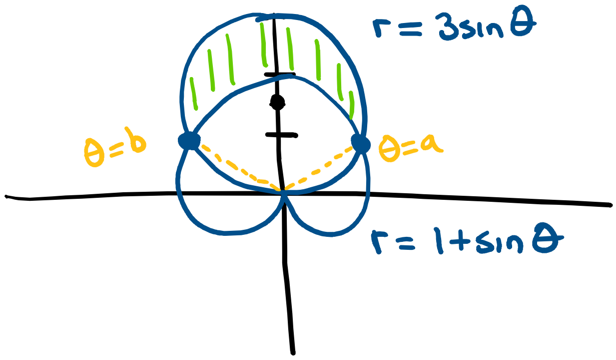
$$A = \boxed{\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2} (\cos 2\theta)^2 \, d\theta}$$

3. The region inside the circle r=3sin(theta) and outside the cardioid r=1+sin(theta).

Sol: To do this, we need to compute which circle we are given. We compute

$$r = 3\sin\theta \Rightarrow r^2 = 3r\sin\theta$$
$$\Rightarrow x^2 + y^2 = 3y$$
$$\Rightarrow x^2 + y^2 - 3y = 0$$
$$\Rightarrow x^2 + y^2 - 3y + \frac{9}{4} = \frac{9}{4}$$
$$\Rightarrow (x-0)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$

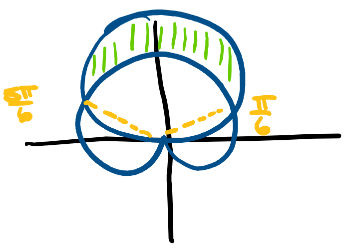
We must find the area of the region



We need to find where these two curves intersect. We need to find theta so that

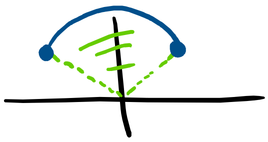
$$3\sin\theta = 1 + \sin\theta$$

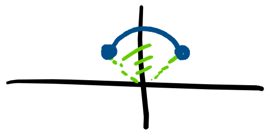
$$\Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}$$
$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$



We conclude that A is given by

$$A = \underbrace{\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \tfrac{1}{2}(3\sin\theta)^2\, d\theta}_{\text{inside the circle}} - \underbrace{\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \tfrac{1}{2}(1+\sin\theta)^2\, d\theta}_{\text{inside the cardioid}}$$





$$= \boxed{\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \tfrac{1}{2}(3\sin\theta)^2 - \tfrac{1}{2}(1+\sin\theta)^2 \; d\theta}$$

Fact: Suppose C is the polar parametric plane curve given by the polar parametric equation r=r(theta) for a≤theta≤b, where r is continuously differentiable over [a,b]. If C has no self-intersections, then the arc length L of the image of C is given by

$$L = \int_a^b \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\; d\theta$$

$$= \int_a^b \sqrt{r(\theta)^2 + (r'(\theta))^2}\; d\theta$$

$x(\theta) = r(\theta)\cos\theta$
$y(\theta) = r(\theta)\sin\theta$
$\cos^2\theta + \sin^2\theta = 1$

The same is true if C only has isolated self-intersections.

Ex: Give an integral for the arc length L of the cardioid r=1+sin(theta) for 0⩽theta⩽2pi.

Sol: Using the simplified formula, we give

$$L = \int_0^{2\pi} \sqrt{\underbrace{(1+\sin\theta)}_{= r(\theta)}^2 + \Big(\underbrace{\tfrac{\partial}{\partial\theta}(1+\sin\theta)}_{= r'(\theta)}\Big)^2}\, \partial\theta$$

$$= \boxed{\int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, \partial\theta}$$