5.8 Indeterminant Forms and L'Hospital's Rule

Def: Suppose L,M are in R. We define the following extended arithmetic rules, or determined forms:

$$L+M, \quad LM, \quad \text{and} \quad \frac{L}{M} \text{ if } M \neq 0$$

$$L+\infty = \infty + L = \infty \quad \text{and} \quad L-\infty = -\infty + L = -\infty$$

$$\infty + \infty = \infty \quad \text{and} \quad -\infty - \infty = -\infty$$

$$\frac{L}{\infty} = \frac{L}{-\infty} = 0$$

If L>0, then

$$L \cdot \infty = \infty \cdot L = \frac{\infty}{L} = \infty$$

$$\text{and} \quad L \cdot -\infty = -\infty \cdot L = \frac{-\infty}{L} = -\infty$$

If L<0, then

$$L \cdot \infty = \infty \cdot L = \frac{\infty}{L} = -\infty$$

$$L \cdot -\infty = -\infty \cdot L = \frac{-\infty}{L} = \infty$$

$$\infty \cdot \infty = -\infty \cdot -\infty = \infty \quad \text{and} \quad \infty \cdot -\infty = -\infty \cdot \infty = -\infty$$

The following are indeterminant forms, and are undefined:

$$\begin{matrix} \infty - \infty \\ -\infty + \infty \end{matrix}, \quad \begin{matrix} 0 \cdot \pm\infty \\ \pm\infty \cdot 0 \end{matrix}, \quad \frac{\text{anything}}{0}, \quad \frac{\pm\infty}{\pm\infty}$$

Extended Basic Limit Rules: Let $\lim\limits_{x \to \#}$ denote any kind of limit:

$$\lim_{x \to a},\ \lim_{x \to a^+},\ \lim_{x \to a^-} \quad \text{or} \quad \lim_{x \to \infty},\ \lim_{x \to -\infty}$$

for $x \in a$

a

on an interval (a, ∞)

$\lim\limits_{x \to \#} f(x) = \infty$

1. <u>Simplification Rule:</u> If f(x)=g(x) for all x "near #," but perhaps not "at #," and if $\lim\limits_{x \to \#}$ f(x) exists in the extended sense, then $\lim\limits_{x \to \#}$ g(x) exists in the extended sense and is equal to $\lim\limits_{x \to \#}$ f(x).

⇨ You can simplify inside any kind of limit.

Suppose $\lim\limits_{x \to \#}$ f(x) and $\lim\limits_{x \to \#}$ g(x) exist in the extended sense.

same

<u>Addition Rule</u>: If $\lim\limits_{x \to \#} f(x) + \lim\limits_{x \to \#} g(x)$ is NOT $\begin{cases} \infty - \infty \\ -\infty + \infty \end{cases}$

then $$\lim_{x \to \#} (f(x) + g(x)) = \lim_{x \to \#} f(x) + \lim_{x \to \#} g(x).$$

<u>Multiplication Rule</u>: If $(\lim\limits_{x \to \#} f(x))(\lim\limits_{x \to \#} g(x))$ is NOT $\begin{cases} 0 \cdot \pm\infty \\ \pm\infty \cdot 0 \end{cases}$

then $$\lim_{x \to \#} f(x)g(x) = \lim_{x \to \#} f(x) \lim_{x \to \#} g(x)$$

<u>Division Rule</u>: If $\dfrac{\lim_{x \to \#} f(x)}{\lim_{x \to \#} g(x)}$ is NOT $\dfrac{\text{anything}}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$

then $$\lim_{x \to \#} \frac{f(x)}{g(x)} = \frac{\lim_{x \to \#} f(x)}{\lim_{x \to \#} g(x)}$$

2. <u>u-Subsitution Rule</u>: Suppose

$$\lim_{x\to\#} g(x) \quad \text{and} \quad \lim_{u\to \lim_{x\to\#} g(x)} f(u)$$

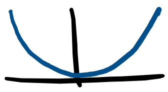
both exist in the extended sense, then

$$\lim_{x\to\#} f(g(x)) = \lim_{u\to \lim_{x\to\#} g(x)} f(u)$$

⇨ If the right-hand side is a determined form, then you can apply the basic limit rule.

Ex: Compute $\lim_{x\to\infty} (x^2-x)$

Sol: Note that $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} -x = \left(\lim_{x\to\infty}(-1)\right)\left(\lim_{x\to\infty} x\right)$



$$-1\cdot\infty \underset{\text{MULT}}{=} -\infty$$

We cannot use the Addition Rule, because

$$\lim_{x\to\infty} (x^2-x) = \lim_{x\to\infty} x^2 + \lim_{x\to\infty} -x$$

$$= \infty - \infty$$

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We must apply the Multiplication Rule instead.

$$\lim_{x\to\infty} x^2-x = \lim_{x\to\infty} x(x-1) \underset{\text{MULT}}{=} \left(\lim_{x\to\infty} x\right)\left(\lim_{x\to\infty} (x-1)\right)$$

$$\underset{\text{ADD}}{=} \infty \cdot (\infty - 1) = \infty \cdot \infty = \boxed{\infty} \checkmark$$

Fact: We need strategies to compute limits of the form

$$\lim_{x \to \#} \frac{f(x)}{g(x)}$$

where

$$\frac{\lim_{x \to \#} f(x)}{\lim_{x \to \#} g(x)}$$

is an indeterminant form of type $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$. To compute limits

$$\lim_{x \to \#} f(x) + g(x) \qquad \text{or} \qquad \lim_{x \to \#} f(x)g(x)$$

where the Addition or Multiplication Rules cannot be used. the first step is to simplify to get a limit of the form $\lim_{x \to \#} \frac{f(x)}{g(x)}$.

Degree Analysis Rule: Suppose p(x),q(x) are sums, products, and compositions of constant and basic power functions.

To compute $\lim_{x \to 0, 0^{\pm}} \frac{p(x)}{q(x)}$, factor out the *smallest* power of x from p(x),q(x).

To compute $\lim_{x \to \pm\infty} \frac{p(x)}{q(x)}$, factor out the *largest* power of x from p(x),q(x).

Ex: Compute the following limits.

1. $\lim\limits_{x\to 0^+} \dfrac{\sqrt{x^7+x^5}+x^2}{x^4-3x^3}$

Sol: Since we are taking the limit at zero, we must factor out the smallest power of x from the top and the bottom.

The powers of x of the bottom are $\underset{\text{smallest}}{\underline{3}}$,4

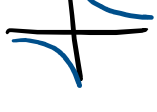
The powers of x of the top are $\underset{\text{smallest}}{\underline{2}}$,5/2,7/2.

We compute

$$\lim_{x\to 0^+} \frac{\sqrt{x^7+x^5}+x^2}{x^4-3x^3} = \lim_{x\to 0^+} \frac{x^2\left(\frac{\sqrt{x^7+x^5}}{x^2}+1\right)}{x^3(x-3)}$$

$$= \lim_{x\to 0^+} \frac{1}{x} \cdot \frac{\sqrt{\frac{x^7+x^5}{x^4}}+1}{x-3}$$

$$= \lim_{x\to 0^+} \underbrace{\frac{1}{x}}_{\to\infty} \cdot \frac{\overbrace{\sqrt{x^3+x}+1}^{\to 1}}{\underbrace{x-3}_{\to -3}}$$



$$= \infty \cdot \frac{1}{-3} = \boxed{-\infty}$$

2. $$\lim_{x\to-\infty} \frac{\sqrt{x^2+1} + (1-x)^{1/4}}{\sqrt[3]{x}+1}$$

Fact: To compute $\lim\limits_{x\to 0^-} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$, set u=-x.

Sol: $$\lim_{x\to-\infty} \frac{\sqrt{x^2+1} + (1-x)^{\frac{1}{4}}}{\sqrt[3]{x}+1} \underset{u=-x}{=} \lim_{u\to\infty} \frac{\sqrt{(-u)^2+1} + (1+u)^{\frac{1}{4}}}{\sqrt[3]{-u}+1}$$

$$= \lim_{u\to\infty} \frac{(u^2+1)^{\frac{1}{2}} + (1+u)^{\frac{1}{4}}}{-u^{\frac{1}{3}}+1}$$

We need to factor out the largest powers of u from the top and the bottom.

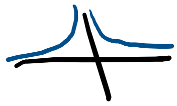
bottom: 0,1/3 largest
top: 0,1/4,1

This gives
$$= \lim_{u\to\infty} \frac{u}{u^{\frac{1}{3}}} \cdot \frac{\left(\frac{(u^2+1)^{\frac{1}{2}}}{u} + \frac{(1+u)^{\frac{1}{4}}}{u}\right)}{\left(-1+u^{-\frac{1}{3}}\right)}$$

$$= \lim_{u\to\infty} u^{\frac{2}{3}} \cdot \frac{\left(\frac{u^2+1}{u^2}\right)^{\frac{1}{2}} + \left(\frac{1+u}{u^4}\right)^{\frac{1}{4}}}{-1+u^{-\frac{1}{3}}}$$

$$= \lim_{u\to\infty} u^{\frac{2}{3}} \quad \frac{\left(1+\frac{1}{u^2}\right)^{\frac{1}{2}} + \left(\frac{1}{u^4}+\frac{1}{u^3}\right)^{\frac{1}{4}}}{-1 + \frac{1}{u^{1/3}}}$$

$$\lim_{u\to\infty} \underbrace{(\underset{\to\infty}{u^2})^{\frac{1}{3}}}_{\to\infty} \qquad \underline{\text{Ex}} \ \lim_{u\to\infty} \frac{1}{u^2} = 0$$



$$= \infty \cdot \frac{1+0}{-1+0}$$

$$= \infty \cdot -1 = \boxed{-\infty}$$

Easy L'Hôspital's Rule: Suppose f,g are differentiable at a with f(a)=g(a) and g'(a)=/0, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Ex: Compute $\lim\limits_{x\to 2} \dfrac{e^{x-2}-1}{\arctan x - \arctan 2}$

Sol: Note that $\lim\limits_{x\to 2}$ e^(x-2)-1=e^0-1=1-1=0 and

$\lim\limits_{x\to 2}$ arctan(x)-arctan(2)=arctan(2)-arctan(2)=0.

Instead, we compute

$$\ell_{x \to 2} \frac{e^{x-2}-1}{\arctan x - \arctan 2} = \ell_{x \to 2} \frac{\left(\frac{e^{x-2}-1}{x-2}\right)}{\left(\frac{\arctan x - \arctan 2}{x-2}\right)}$$

Since $$\ell_{x \to 2} \frac{\arctan x - \arctan 2}{x-2} = \frac{\partial}{\partial x} \arctan x \Big|_{x=2}$$

$$= \frac{1}{1+x^2}\Big|_{x=2} = \frac{1}{5} \neq 0$$

This means we can apply the Division Rule to get

$$= \frac{\ell_{x \to 2} \frac{e^{x-2}-1}{x-2}}{\ell_{x \to 2} \frac{\arctan x - \arctan 2}{x-2}} = \frac{\frac{\partial}{\partial x} e^{x-2}\Big|_{x=2}}{\frac{\partial}{\partial x} \arctan x \Big|_{x=2}}$$

$$= \frac{e^{x-2}\Big|_{x=2}}{1/5} = \boxed{\frac{1}{1/5}}$$

L'Hôspital's Rule: Let $\lim_{x \to \#}$ denote any kind of limit. Suppose

$$\frac{\ell_{x \to \#} f(x)}{\ell_{x \to \#} g(x)}$$ is an indeterminant form of the type $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$

$$\lim_{x \to \#} \frac{f'(x)}{g'(x)} \text{ exists in the extended sense}$$

Then $$\lim_{x \to \#} \frac{f(x)}{g(x)} = \lim_{x \to \#} \frac{f'(x)}{g'(x)}.$$

Non-Ex: Define the functions

$$f(x) = x \sin(x^{-4})\, e^{-\frac{1}{x^2}} \quad \text{and} \quad g(x) = e^{-\frac{1}{x^2}}$$

Then $\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$ but $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ DNE!

Ex: Compute the following limits.

1. $\lim_{x \to 0^+} x \ln(x)$

Sol: Note that $$\left(\lim_{x \to 0^+} x\right)\left(\lim_{x \to 0^+} \ln x\right) = 0 \cdot -\infty \quad \emptyset$$

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We cannot use the Multiplication Rule. Instead, we compute

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1}}$$

$$\lim_{x\to 0^+} \ln x = -\infty$$
$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$
$\checkmark$

$$= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^{-1}}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} \frac{x^2 \cdot \frac{1}{x}}{-x^2 \cdot x^{-2}}$$

$$= \lim_{x\to 0^+} \frac{x}{-1} = 0$$

Since $\lim\limits_{x\to 0^+} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^{-2}}$ exists, then we *can* apply L'Hôspital's Rule.

2. $\lim\limits_{x\to\infty} \frac{e^x}{x^2}$

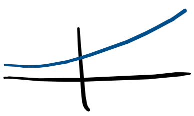
Sol: We compute

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{?}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}x^2}$$

← we must also verify this limit exists

$\checkmark$

$$\lim_{x\to\infty} e^x = \infty$$
$$\lim_{x\to\infty} x^2 = \infty$$



$$\overset{?}{=} \lim_{x\to\infty} \frac{e^x}{2x}$$

$$\lim_{x\to\infty} e^x = \infty \quad \overset{?}{=} \quad \lim_{x\to\infty} \frac{\frac{\partial}{\partial x} e^x}{\frac{\partial}{\partial x} 2x}$$

← does this limit exist?

$$\lim_{x\to\infty} 2x = \infty$$

use L'Hôspitals Rule again

$$= \lim_{x\to\infty} \frac{e^x}{2} = \frac{1}{2}\cdot\infty = \boxed{\infty} \checkmark$$

Technically speaking, the correct order of the logic is

$$\lim_{x\to\infty} \frac{e^x}{2} = \infty \Rightarrow \lim_{x\to\infty} \frac{e^x}{2x} \underset{LHR}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

↑ exists in the extended sense

↓

$$\lim_{x\to\infty} \frac{e^x}{2x} = \infty \Rightarrow \lim_{x\to\infty} \frac{e^x}{x^2} \underset{LHR}{=} \lim_{x\to\infty} \frac{e^x}{2x} = \infty$$

3. $\lim\limits_{x\to\infty} \dfrac{e^x + 2x^2 + \sqrt{x^4+x+1}}{x^2 + \sqrt{x + 4x^{1/3}}}$

Sol: We will use Degree Analysis, since we know

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \infty.$$

We compute

$$\lim_{x\to\infty} \frac{e^x + 2x^2 + \sqrt{x^4+x+1}}{x^2 + \sqrt{x + 4x^{1/3}}}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^2} \cdot \frac{\left(\frac{e^x}{x^2} + 2 + \frac{(x^4+x+1)^{\frac{1}{2}}}{x^2}\right)}{\left(1 + \frac{(x+4x^{1/3})^{1/2}}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{e^x}{x^2} + 2 + \left(\frac{x^4+x+1}{x^4}\right)^{\frac{1}{2}}}{1 + \left(\frac{x+4x^{1/3}}{x^4}\right)^{1/2}}$$

$$= \lim_{x\to\infty} \frac{\frac{e^x}{x^2} + 2 + \left(1 + \frac{1}{x^3} + \frac{1}{x^4}\right)^{\frac{1}{3}}}{1 + \left(\frac{1}{x^3} + \frac{4}{x^{11/3}}\right)^{1/2}}$$

$$= \frac{\infty + 2 + (1+0+0)^{\frac{1}{3}}}{1 + (0+0)^{1/2}} = \frac{\infty+2+1}{1}$$

$$= \boxed{\infty}$$

The Squeeze Theorem

Squeeze Thm: Suppose f,g,h are defined near a, but perhaps not at a itself. If

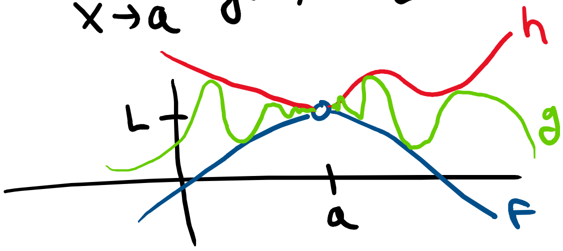
$$f(x) \le g(x) \le h(x)$$

for all x near a, but perhaps not at a itself, and if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$



Similar is ture for all types of limits, in the extended sense.

Ex: Use the Squeeze Thm to show the following.

1. $\lim\limits_{x \to 0} \dfrac{x^4}{x^2+x^4} = 0$

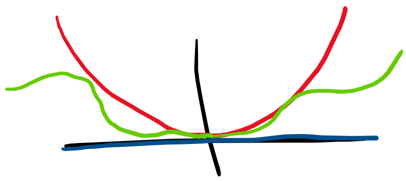
Sol: First, note that

$$0 \le \frac{x^4}{x^2+x^4} = x^2 \underbrace{\left(\frac{x^2}{x^2+x^4}\right)}_{\le 1} \le x^2 \quad \text{for } x \neq 0$$

since $x^2 \le x^2 + x^4$

Since $$\lim_{x\to 0} 0 = \lim_{x\to 0} x^2 = 0$$

then $$\lim_{x\to 0} \frac{x^4}{x^2+x^4} \underset{ST}{=} 0$$



2. $\lim_{x\to 0} x^2\sin(1/x) = 0$

Sol: Recall that

$$\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$$

$$\Rightarrow \quad -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\underset{x^2 \geq 0}{\Rightarrow} \quad -x^2 \leq x^2\sin\left(\frac{1}{x}\right) \leq x^2 \quad \text{for } x \neq 0$$

Since $$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$$

then $$\lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) \underset{ST}{=} 0$$