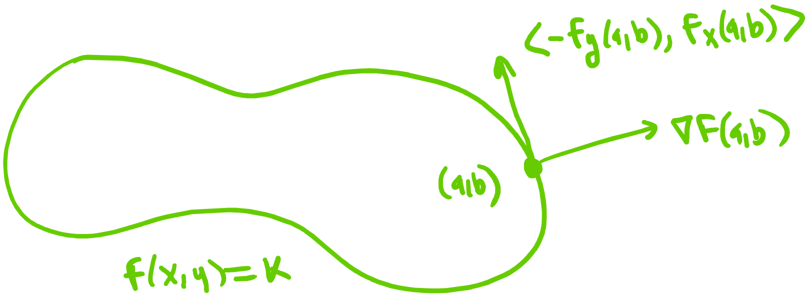
Vector Calculus
11.6 Directional Derivatives and the Gradient Vector

Last time we showed using the Chain Rule that the gradient of $f=f(x,y)$ is perpendicular to the level set



Def: Suppose $f=f(x,y)$ is differentiable at (a,b) with $\nabla f(a,b) \neq \vec{0}$, and suppose $f(a,b)=k$.

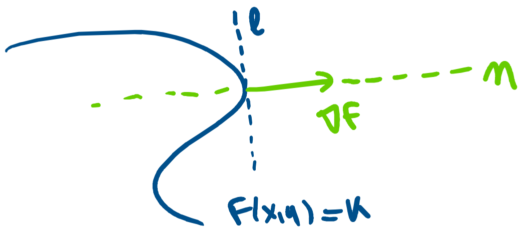
We say the <u>tangent line at (a,b) of the level curve of f at k</u> is the line through (a,b) in the direction of $\langle -f_y(a,b), f_x(a,b)\rangle$, given by the point-direction parameterization

$$\ell(t) = \langle a,b\rangle + t\langle -f_y(a,b), f_x(a,b)\rangle \quad \text{for } t\in\mathbb{R}$$

We say the <u>normal line at (a,b) of the level curve of f at k</u> is the line through (a,b) in the direction of ∇f, given by the point-direction parameterization

$$n(t) = \langle a,b\rangle + t\,\nabla F(a,b) \quad \text{for } t\in\mathbb{R}$$

the gradient gives us the normal direction!



Ex: Compute the tangent and normal line at $(a,b)=\underline{(2,0)}$ of the level curve of $f(x,y)=xe^{xy}$ at $\underline{k=2}$.

$$2\cdot e^{2\cdot 0}=2 \checkmark$$

Sol: First, we compute

$$\nabla f(2,0)=\left\langle \frac{\partial}{\partial x} xe^{xy}, \frac{\partial}{\partial y} xe^{xy}\right\rangle \Big|_{(2,0)}$$

$$=\left\langle e^{xy}+xye^{xy}, x^2e^{xy}\right\rangle \Big|_{(2,0)}$$

$$=\left\langle e^{2\cdot 0}+2\cdot 0e^{2\cdot 0}, 2^2e^{2\cdot 0}\right\rangle = \langle 1,4\rangle$$

We conclude that the tangent line ell and the normal line n at (2,0) of the level curve of f at k=2 is

$$\boxed{\begin{aligned} \ell(t)&=\langle 2,0\rangle + t\langle -4,1\rangle \quad \text{for } t\in\mathbb{R} \\ n(t)&=\langle 2,0\rangle + t\langle 1,4\rangle \quad \text{for } t\in\mathbb{R}\end{aligned}}$$

Def: Suppose $f=f(x,y,z)$ is differentiable at (a,b,c) with $\nabla f(a,b,c) \neq 0$, and suppose $f(a,b,c)=k$.

We say the <u>tangent plane at (a,b,c) of the level surface of f at k</u> is the plane through (a,b,c) with normal in the direction of $\nabla f(a,b,c)$, given by the <u>scalar equation</u>

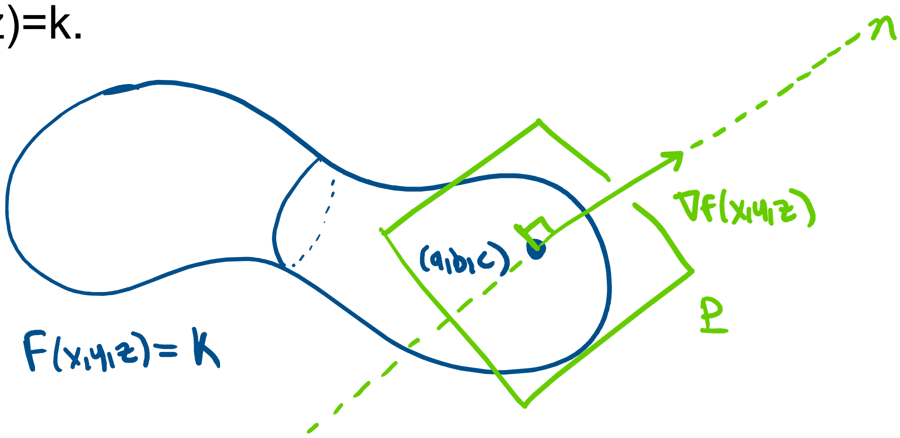
week 6

$$f_x(a,b,c)(x-a)+f_y(a,b,c)(y-b)+f_z(a,b,c)(z-c)=0.$$

We say the <u>normal line at (a,b,c) of the level surface of f at k</u> is the line through (a,b,c) in the direction of $\nabla f(a,b,c)$, given by the point-direction parameterization

$$n(t)=\langle a,b,c\rangle + t\nabla f(a,b,c) \quad \text{for } t\in\mathbb{R}$$

the gradient of f=f(x,y,z) is perpendicular to the level surface f(x,y,z)=k.



Ex: Compute the tangent plane and normal line at (a,b,c)=(-1,1,3) of the level surface of $f(x,y,z)=z-x^2-y^2$ at k=1.

Sol: First, we compute

$$\nabla f(-1,1,3) = \langle -2x, -2y, 1 \rangle \Big|_{(-1,1,3)} = \langle 2,-2,1 \rangle$$

We conclude that the tangent plane and the normal line at (-1,1,3) of the level surface of f at k=1 are given respectively by

$$\boxed{2(x-(-1)) + (-2)(y-1) + (1)(z-3) = 0}$$

$$\boxed{n(t) = \langle -1,1,3 \rangle + t\langle 2,-2,1 \rangle \text{ for } t\in\mathbb{R}}$$

11.7 Maximum and Minimum Values

Def: Suppose f=f(x,y) is a real-valued function defined near (a,b).

If f(a,b)$\geqslant$ f(x,y) for all (x,y) near (a,b), then we say (a,b) is a local maximum point of f and f(a,b) is a local maximum value of f.

We similarly define local minimum point/value of f. $\Rightarrow$ $f(a,b) \leq f(x,y)$

If (a,b) is either a local maximum or a local minimum point of f, then we say (a,b) is a local extremum point of f, and f(a,b) is a local extremum value of f.

⇨ "find the local extremum points" means to find all local maximum *and* minimum points.

If $\begin{cases} f_x(a,b) \text{ DNE or} \\ f_y(a,b) \text{ DNE or} \\ \nabla f(a,b) = \langle 0,0 \rangle \end{cases}$, then we say (a,b) is a critical point of f.

$f'(c)$ DNE

$\Rightarrow$ $f'(c)=0$

Ω Greek "O"

Suppose Omega is a subset of R^2, and suppose f is defined for all (x,y) in Omega.

If (a,b) is in Omega and f(a,b)$\geqslant$f(x,y) for each (x,y) in Omega, then we say (a,b) is an absolute maximum point of f over Omega, and f(a,b) is an absolute maximum value of f over Omega.

We also define absolute minimum/extremum points and values of f over Omega.

⇨ absolute extremum points are always defined with respect to a region Omega. We cannot just compute the absolute extremum points of a function, we must first be given the region Omega.

We make similar definitions for real-valued functions f=f(x,y,z).

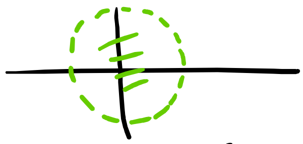
Def: Basic definitions in point-set topoly. Suppose Omega is a subset of R^2.

We say Omega is an open set if Omega does not include its "skin." For example

$\{(x,y): x^2+y^2 \le 1\}$

NOT an open set

$\{(x,y): x^2+y^2 < 1\}$



open unit disk

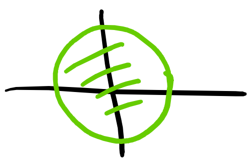
Open ✓

unit circle $x^2+y^2=1$

We say Omega is a closed set if Omega includes its "skin."

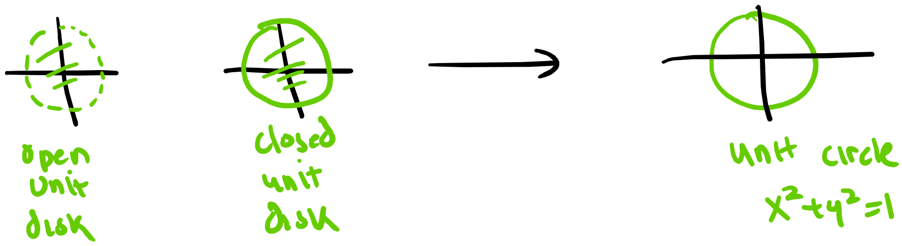
$\{(x,y): x^2+y^2 \le 1\}$

closed set ✓



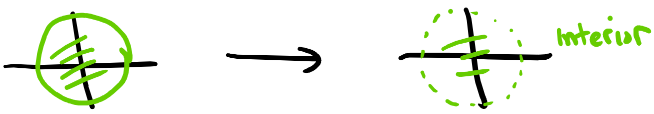
closed unit disk

The boundary of Omega is the “skin” of Omega. For example, the boundary of both the open unit disk and the closed unit disk is the unit circle.



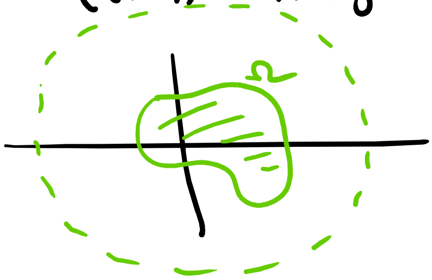
We denote the boundary of Omega by $\partial\Omega$. ∂ curly d

We define the interior of Omega to be Omega without the boundary. For example, the interior of the open unit disk is itself, while the interior of the closed unit disk is the open unit disk.



We say Omega is a bounded set if Omega does not go off to infinity, there is an r>0 so that

$$\Omega \subset \{(x,y) : x^2+y^2<r\}$$



We make similar definitions over R^3.

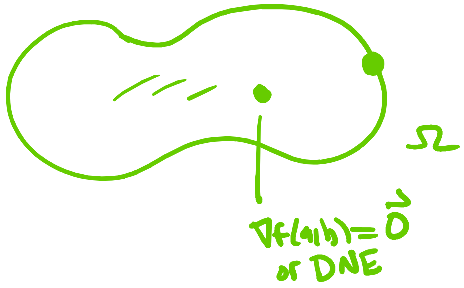
Usually, “<” or “>” means open, while “≤” and “≥” means closed.

Thm: Suppose Omega is a closed and bounded subset of R^2, and suppose f=f(x,y) is a real-valued function continuous over Omega.

There is an absolute minimum point (a_{min},b_{min}) in Omega of f over Omega, and an absolute maximum point (a_{max},b_{max}) in Omega of f over Omega.

- ⇨ f attains a maximum and minimum value over Omega.

If (a,b) in Omega is an absolute extremum point of f over Omega, then either (a,b) is in the boundary of Omega, or (a,b) is a critical point of f in the interior of Omega.



- ⇨ to compute the absolute extremum points of f=f(x) over [a,b], we must compare f(a),f(b) with f(c) at all c in (a,b) so that f'(c) DNE or f'(c)=0.

Ex: Find the absolute extremum points and values of the given function f over the given region Omega.

1. f(x,y)=x^2+y^2 over Omega={(x,y):$x^2+y^2 \leqslant 1$}, the closed unit disk.

Sol: First, let's find all of the interior critical points. We set

$$\vec{0} = \nabla f(x,y) = \langle 2x, 2y \rangle \Rightarrow \begin{array}{l} 2x=0 \\ 2y=0 \end{array} \Rightarrow (x,y)=(0,0)$$

"good critical points"

$$\Rightarrow (0,0) \overset{?}{\in}\checkmark \text{ interior of } \Omega = \{(x,y) : x^2+y^2<1\}$$

open unit disk

$$\Rightarrow f(0,0) = 0^2+0^2 = 0$$

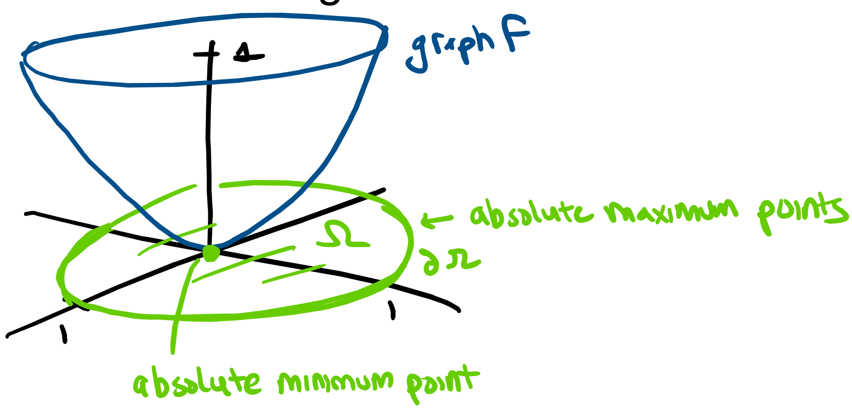
Now we consider the values of f over the boundary of Omega. The boundary of Omega is the unit circle $x^2+y^2=1$.

$$(x,y) \in \partial\Omega \Rightarrow f(x,y) = x^2+y^2 = 1$$

$$\Rightarrow f(x,y)=1 \text{ for } \underline{\text{all}} \ (x,y) \in \partial\Omega$$

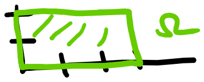
We conclude that (0,0) is the absolute minimum point of f over Omega, with absolute minimum value f(0,0)=0. Meanwhile, every point (x,y) on the boundary of Omega is an absolute maximum point of f over Omega, with absolute maximum value f(x,y)=1.

Note that the graph of f over Omega is



2. $f(x,y)=x^2-2xy+2y$ over the rectangle

$$\Omega = \{(x,y): 0 \le x \le 3,\ 0 \le y \le 2\}$$



Sol: First, we compute the interior critical points.

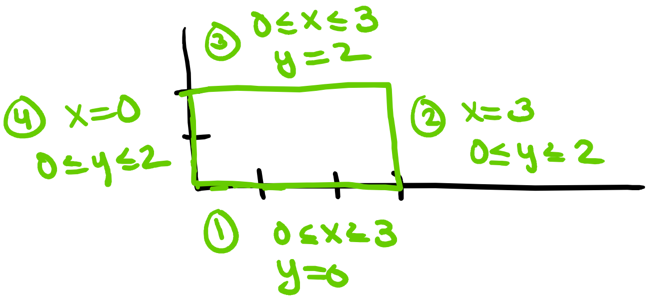
$$\vec{0} = \nabla f = \langle 2x-2y, -2x+2 \rangle$$

$$\Rightarrow \begin{aligned} 2x-2y&=0 \longrightarrow y=1 \\ -2x+2&=0 \Rightarrow x=1 \end{aligned}$$

$$(x,y)=(1,1) \overset{?}{\in} \checkmark \text{ interior of } \Omega = \{(x,y): 0<x<3, 0<y<2\}$$

$$\Rightarrow \boxed{f(1,1) = 1^2 - 2\cdot1\cdot1 + 2\cdot1 = 1-2+2 = 1}$$

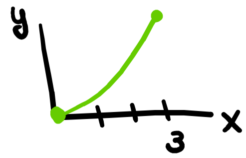
Now we must find the absolute extremum points of f over the boundary of Omega. Consider f over the four pieces of the boundary of Omega, given by



(1) To compute the absolute extremum points of f over the first piece of the boundary of Omega, we find the absolute extremum points of $g(x)=f(x,0)$ over $[0,3]$. This is a single-variable calculus problem.

We must compare the values of g(0),g(3) and g(c) for all interior critical points c in (0,3) of g. We compute

$$g(x) = f(x,0) = x^2 - 2x\cdot 0 + 2\cdot 0 = x^2$$

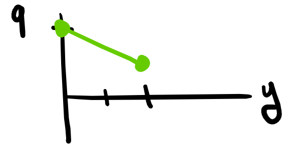


$$\boxed{\begin{aligned} g(0) &= f(0,0) = 0 \\ g(3) &= f(3,0) = 9 \end{aligned}}$$

(2) We must find the absolute extremum points of g(y)=f(3,y) over [0,2].

We compute

$$g(y) = f(3,y) = 3^2 - 2\cdot 3y + 2y = -4y + 9$$



$$\Rightarrow \boxed{\begin{aligned} g(0) &= f(3,0) = 9 \\ g(2) &= f(3,2) = -8+1 = 1 \end{aligned}}$$

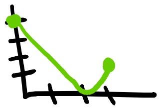
(3) We must find the absolute extremum points of g(x)=f(x,2) over [0,3].

We compute

$$g(x) = f(x,2) = x^2 - 2x\cdot 2 + 2\cdot 2 = x^2 - 4x + 4$$

$$\underset{\text{max}}{g(0)} = 4, \quad g(3) = 9 - 12 + 4 = 1$$

$$0 = g'(c) = 2c - 4 \Rightarrow c = 2 \in (0,3)$$



$$\underset{\min}{g(2)} = 2^2 - 4\cdot 2 + 4 = 0$$

$$\Rightarrow \boxed{\begin{array}{l} g(0) = f(0,2) = 4 \\ g(2) = f(2,2) = 0 \end{array}}$$

(4) We must compute the extremum points of g(y)=f(0,y) over [0,2].

We compute

$$g(y) = f(0,y) = 0 - 2\cdot 0\cdot y + 2y = 2y$$

$$\Rightarrow \boxed{\begin{array}{l} g(0) = f(0,0) = 0 \\ g(2) = f(0,2) = 4 \end{array}}$$

Now we compare the values of f for all of these points, the critical point and the points found on the boundary of Omega.

We conclude that (0,0),(2,2) are absolute minimum points of f over Omega, with absolute minimum value f(0,0)=f(2,2)=0. We also conclude that (3,0) is the absolute maximum point of f over Omega with absolute maximum value f(3,0)=9.