Vector Calculus
14.8 Lagrange Multipliers

Ex: Find the absolute extremum points and values of $f(x,y,z)=xy+z^2$ over the solid region $E=\{(x,y,z):x^2+y^2+z^2\leq 1\}$.

Sol: This involved finding the absolute extremum points and values of f over the unit sphere $x^2+y^2+z^2=1$.

Def: Suppose $f=f(x,y,z), g=g(x,y,z)$ are real-valued functions defined near (a,b,c), and suppose $k=g(a,b,c)$.

If $f(a,b,c)\geq f(x,y,z)$ for all (x,y,z) in the level surface of g at k near (a,b,c), then we say (a,b,c) is a <u>local maximum point of f over the level surface of g at k</u>, and <u>f(a,b,c) is a local maximum value of f over the level surface of g at k</u>.

We also define <u>local minimum/extremum points and values of f over the level surface of g at k</u>, and we define <u>absolute maximum/minimum/extremum points and values of f over the level surface of g at k</u>.

- ⇨ Ex: Find the absolute extremum points and values of $f(x,y,z)=xy+z^2$ over the level surface of $g(x,y,z)=x^2+y^2+z^2$ at k=1.

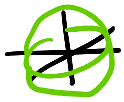
We make similar definitions for real-valued functions $f=f(x,y), g=g(x,y)$. => level curves

Thm (Lagrange Multipliers): Suppose f=f(x,y,z),g=g(x,y,z) are real-valued functions, and suppose k is in R.

Suppose f=f(x,y,z),g=g(x,y,z) are differentiable at (a,b,c), and suppose k=g(a,b,c). If (a,b,c) is a local extremum point of f over the level surface of g at k, and if ∇g(a,b,c)=/$\vec{0}$, then there is a lambda in R so that

$$\nabla f(a,b,c) = \lambda \nabla g(a,b,c)$$

Suppose f,g are differentiable functions, and suppose the level surface of g at k is a bounded set.

<u>Ex</u> $g(x,y,z)=x^2+y^2+z^2 \quad k=1$ 

To find the absolute extremum points and values of f over the level surface of g at k, we compare the values of f at all points (x,y,z) in the level surface of g at k so that

$$\nabla f(x,y,z)=\lambda \nabla g(x,y,z) \quad \text{for some } \lambda \in \mathbb{R}.$$

Similar is true for real-valued functions f=f(x,y),g=g(x,y).

Proof: Suppose (a,b,c)=(1,0,0) is a local extremum point of f=f(x,y,z) over the level surface of g(x,y,z)=$x^2+y^2+z^2$ at k=1, prove that

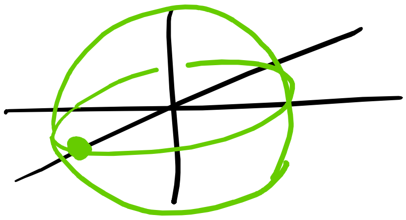
$$\nabla f(1,0,0) = \langle f_x(1,0,0), 0, 0 \rangle$$

$$\nabla g(1,0,0)=\langle 2x,2y,2z \rangle \Big|_{(1,0,0)} = \langle 2,0,0 \rangle$$

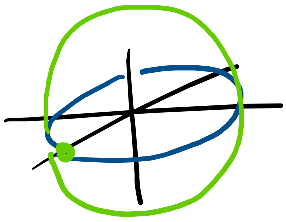
$$\Rightarrow \nabla f(1,0,0) = \left\langle \frac{f_x(1,0,0)}{2} \cdot 2, 0, 0 \right\rangle = \overset{\lambda}{\overset{\shortparallel}{\frac{f_x(1,0,0)}{2}}} \langle 2,0,0 \rangle$$

$$= \lambda \nabla g(1,0,0)$$

Consider the level surface of g at k=1, unit sphere, near (1,0,0).



Consider the parametric space curve



$$\vec{r}_1(t) = \langle \cos t, \sin t, 0 \rangle$$

$$\vec{r}_1(0) = \langle 1, 0, 0 \rangle$$

Consider the single-variable function

$$h_1(t) = f(\vec{r}_1(t)) = f(\cos t, \sin t, 0)$$

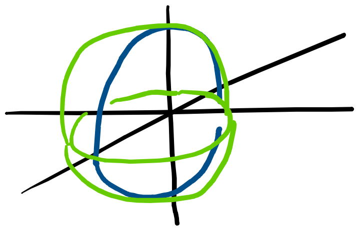
Since the image of $\vec{r}_1$ is contained in the unit sphere, and since (1,0,0) is a local extremum point of f over the unit sphere, then t=0 is a local extremum point of h. This implies

$$0 = h'(0) = \frac{d}{dt} f(\cos t, \sin t, 0) \Big|_{t=0}$$

$$\underset{\text{CHAIN RULE}}{=} \frac{\partial f}{\partial x}\Big|_{(1,0,0)} \left(-\sin t \big|_{t=0}\right) + \frac{\partial f}{\partial y}\Big|_{(1,0,0)} \left(\cos t \big|_{t=0}\right) + 0$$

$$= 0 + f_y(1,0,0) \Rightarrow f_y(1,0,0) = 0$$

Consider the parametric space curve



$$\vec{r}_2(t) = \langle \cos t, 0, \sin t \rangle$$

$$\vec{r}_2(0) = \langle 1, 0, 0 \rangle$$

We also compute that

$$0 = \frac{\partial}{\partial t} f(\cos t, 0, \sin t)\Big|_{t=0}$$

$$= \frac{\partial f}{\partial x}\Big|_{(1,0,0)} \left(-\sin t\big|_{t=0}\right) + 0 + \frac{\partial f}{\partial z}\Big|_{(1,0,0)} \left(\cos t\big|_{t=0}\right)$$

$$= 0 + f_z(1,0,0) \Rightarrow \quad f_z(1,0,0) = 0$$

We conclude that

$$\nabla f(1,0,0) = \langle f_x(1,0,0), 0, 0\rangle = \underbrace{\lambda}_{=\frac{f_x(1,0,0)}{2}} \underbrace{\nabla g(1,0,0)}_{=\langle 2,0,0\rangle}$$

Ex: Find the absolute extremum points and values of the given function f over the level curve/surface of the given function g at the given k in R.

1. f(x,y)=x^2+2y^2 with g(x,y)=x^2+y^2 at k=1.

   $\Rightarrow f(x,y) = x^2+y^2+y^2 = 1+y^2$

⇨ Find the absolute extremum points and values of f over the unit circle $x^2+y^2=1$.

Sol: We find all (x,y) so that

$$\underbrace{\nabla f}_{=\langle 2x, 4y\rangle} = \underbrace{\lambda \nabla g}_{=\lambda\langle 2x, 2y\rangle} \quad \text{with} \quad x^2+y^2=1$$

This means we must solve the equations

① $2x = 2\lambda x$

② $4y = 2\lambda y$

③ $x^2+y^2=1$

Fact: When solving a Lagrange multiplier problem, consider cases such as x=0 and x=/0, y=0 and y=/0, and z=0 and z=/0. In particular, first consider any equation where the same variable appears on both sides.

$x=0$ $\Downarrow$? $\quad$ $x\neq 0$ $\Downarrow$?

① $2x=2\lambda x$

② $4y=2\lambda y$

③ $x^2+y^2=1$

$2x=2\lambda x \Longrightarrow \boxed{x=0}$ or $\boxed{x\neq 0}$

$\boxed{x=0} \overset{③}{\Longrightarrow} y^2=1 \Rightarrow y=\pm 1 \Rightarrow \boxed{(0,\pm 1)}$

$\boxed{x\neq 0} \overset{①}{\Longrightarrow} \lambda=1 \overset{②}{\Longrightarrow} 4y=2y \Rightarrow y=0 \overset{③}{\Longrightarrow} x=\pm 1 \Rightarrow \boxed{(\pm 1,0)}$

Since $f(x,y)=x^2+2y^2$ then

$$f(0,\pm 1) = 0+2 = 2$$

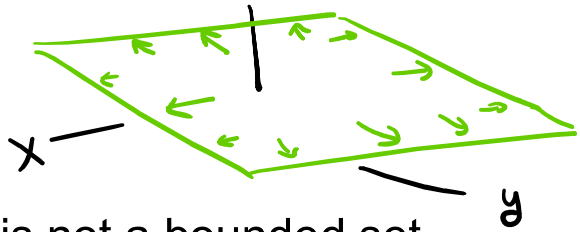
$$f(\pm 1,0) = 1+0 = 1$$

We conclude that f has absolute maximum points (0,±1) over the level surface of g at k=1, with absolute maximum value =2. We also conclude that f has absolute minimum points (±1,0) over the level surface of g at k=1, with absolute minimum value =1.

2. $f(x,y,z)=x^2+5y^2+z^2$ with $g(x,y,z)=4x+5y+2z$ at $k=1$

Sol: Note that the level surface of g at k=1 is the plane

$4x+5y+2z=1$

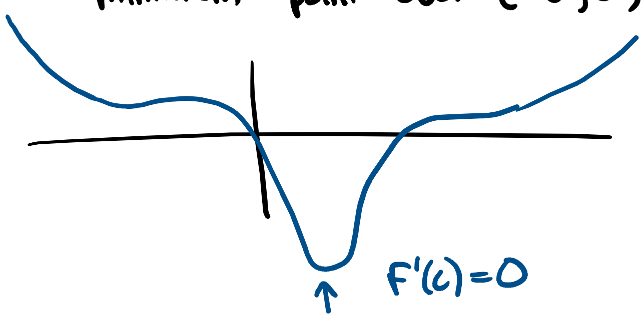


The level surface of g at k=1 is not a bounded set. However, as x->infinity, or y->infinity, or z->infinity, we have that f(x,y,z)->infinity. This means that f must have an absolute minimum point over the level surface of g at k=1.

Recall the following fact for single-variable functions:

FACT IF $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$, f continuous

then f must have an absolute minimum point over $(-\infty,\infty)$



To find the absolute minimum point of f over the level surface of g at k=1, we consider the equations

$\nabla f = \lambda \nabla g$ with $4x+5y+2z=1$

$\nabla f = \langle 2x, 10y, 2z\rangle$, $\lambda\nabla g = \lambda\langle 4,5,2\rangle$

We consider the equations

$$\begin{aligned}
&① \; 2x = 4\lambda \Rightarrow 4x = 8\lambda \\
&② \; 10y = 5\lambda \Rightarrow 5y = \tfrac{5}{2}\lambda \\
&③ \; 2z = 2\lambda \\
&④ \; 4x+5y+2z=1
\end{aligned}$$

$$\Rightarrow 8\lambda + \tfrac{5}{2}\lambda + 2\lambda = 1$$

$$10\lambda + \tfrac{5}{2}\lambda = 1$$

$$\tfrac{25}{2}\lambda = 1$$

$$\Rightarrow \lambda = \frac{2}{25}$$

$$\Rightarrow (x,y,z) = \left(2\left(\tfrac{2}{25}\right), \tfrac{1}{2}\left(\tfrac{2}{25}\right), \tfrac{2}{25}\right) = \left(\tfrac{4}{25}, \tfrac{1}{25}, \tfrac{2}{25}\right)$$

We conclude that f has absolute minimum point (4/25,1/25,2/25) over the level surface of g at k=1, with absolute minimum value

$$f\left(\tfrac{4}{25}, \tfrac{1}{25}, \tfrac{2}{25}\right) = \left(\tfrac{4}{25}\right)^2 + 5\left(\tfrac{1}{25}\right)^2 + \left(\tfrac{2}{25}\right)^2$$

3. $f(x,y,z)=xy+z^2$ with $g(x,y,z)=x^2+y^2+z^2$ at k=1

Sol: We must solve the equations

$$\begin{aligned}
&① \; y = 2\lambda x \\
&② \; x = 2\lambda y \\
\star \; &③ \; 2z = 2\lambda z \\
&④ \; x^2+y^2+z^2=1
\end{aligned}$$

$\boxed{z=0}$ or $\boxed{z\neq 0}$

$\boxed{z=0}$ $\Downarrow$

① $y=2\lambda x$
② $x=2\lambda y$
③ $x^2+y^2=1$

$\boxed{x=0}$ or $\boxed{x\neq 0}$

$\boxed{x=0}$ $\xrightarrow{③}$ $y^2=1$ $\Downarrow$ $\boxed{(0,\pm 1,0)}$

$\boxed{x\neq 0}$: $\boxed{y=0}$ $\xrightarrow{③}$ or $\boxed{y\neq 0}$ $\xrightarrow{①,②}$

$\boxed{z\neq 0}$ $\xrightarrow{③}$ $\lambda = 1$ $\Downarrow$

① $y=2x$
② $x=2y$
④ $x^2+y^2+z^2=1$

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$\boxed{z\neq 0}$

$\Downarrow$

① $x=2y$

② $y=2x \Rightarrow x=\frac{y}{2}$

④ $x^2+y^2+z^2=1$

$\Downarrow$

$2y=x=\frac{y}{2} \Rightarrow \begin{matrix} y=0 \\ x=0 \end{matrix}$

④ $\Downarrow$

$z^2=1$

$\Downarrow$

$\boxed{(0,0,\pm1)}$

$x^2=1$

$\Downarrow$

$\boxed{(\pm1,0,0)}$

$\frac{y}{2x}=\lambda=\frac{x}{2y}$

$x^2+y^2=1$

$\Downarrow$

$y^2=x^2$

$x^2+y^2=1$

$\Downarrow$

$x^2+x^2=1$

$\Downarrow$

$2x^2=1$

$\Downarrow$

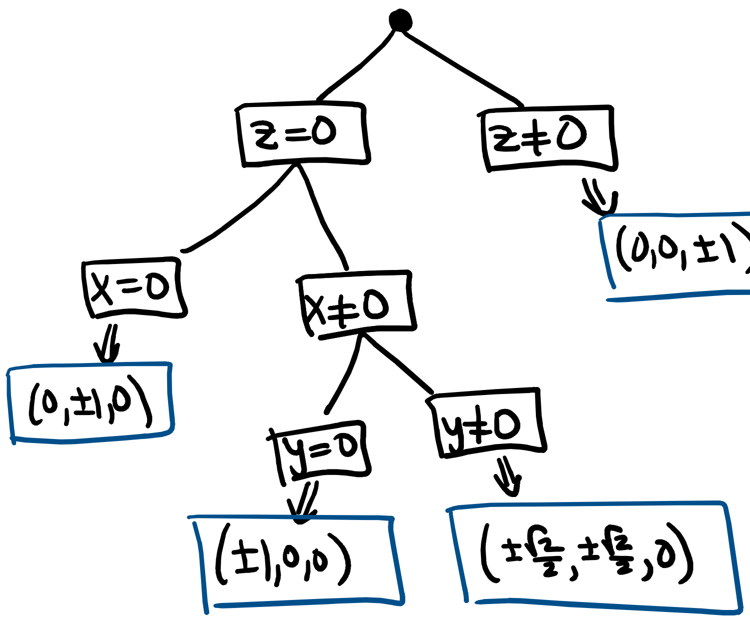
$x^2=\frac{1}{2}$

$\Downarrow$

$x=\pm\frac{\sqrt{2}}{2}$

$\Downarrow$

$y^2=x^2$ $\boxed{\left(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2},0\right)}$



Since $f(x,y,z)=xy+z^2$, then we compute

$$f(0,\pm 1,0) = f(\pm 1,0,0) = 0+0 = 0$$

$$f(0,0,\pm 1) = 0 + (\pm 1)^2 = 1$$

$$f\left(\tfrac{\sqrt2}{2},\tfrac{\sqrt2}{2},0\right) = f\left(-\tfrac{\sqrt2}{2},-\tfrac{\sqrt2}{2},0\right) = \tfrac{\sqrt2}{2}\cdot\tfrac{\sqrt2}{2} + 0 = \tfrac{1}{2}$$

$$f\left(\tfrac{\sqrt2}{2},-\tfrac{\sqrt2}{2},0\right) = f\left(-\tfrac{\sqrt2}{2},\tfrac{\sqrt2}{2},0\right) = -\tfrac{\sqrt2}{2}\cdot\tfrac{\sqrt2}{2} + 0 = -\tfrac{1}{2}$$

We conclude that f has absolute maximum points (0,0,±1) over the level surface of g at k=1, with absolute maximum value =1. We also conclude that f has absolute minimum points $\left(\tfrac{\sqrt2}{2},-\tfrac{\sqrt2}{2},0\right), \left(-\tfrac{\sqrt2}{2},\tfrac{\sqrt2}{2},0\right)$ over the level surface of g at k=1, with absolute minimum value

$$f\left(\tfrac{\sqrt2}{2},-\tfrac{\sqrt2}{2},0\right) = f\left(-\tfrac{\sqrt2}{2},\tfrac{\sqrt2}{2},0\right) = -\tfrac{1}{2}$$

4. $f(x,y,z)=xy+\frac{z^3}{3}$ with $g(x,y,z)=x^2+y^2+z^2$ at k=1

Sol: We must consider the equations

① $y = 2\lambda x$

② $x = 2\lambda y$

* ③ $z^2 = 2\lambda z$

④ $x^2+y^2+z^2=1$

$\boxed{z=0}$ or $\boxed{z\neq 0}$

$\boxed{z=0}$:

① $y = 2\lambda x$

② $x = 2\lambda y$

③ $x^2+y^2=1$

- $\boxed{x=0}$: $\boxed{(0,\pm 1,0)}$
- $\boxed{x\neq 0}$:
  - $\boxed{y=0}$: $\boxed{(\pm 1,0,0)}$
  - $\boxed{y\neq 0}$: $\boxed{\left(\pm\tfrac{\sqrt2}{2},\pm\tfrac{\sqrt2}{2},0\right)}$

$\boxed{z\neq 0}$:

① $y=2\lambda x$

② $x=2\lambda y$

③ $z=2\lambda$

④ $x^2+y^2+z^2=1$

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$\boxed{z\neq 0}$

① $y=2\lambda x$

② $x=2\lambda y$

③ $z=2\lambda$

④ $x^2+y^2+z^2=1$

$\Rightarrow$ $\boxed{x=0} \xrightarrow{①} y=0 \xrightarrow{④} 0+0+z^2=1 \Rightarrow \boxed{(0,0,\pm1)}$

or

$\boxed{x\neq 0} \xrightarrow{②} 2\lambda y\neq 0 \Rightarrow y\neq 0 \xrightarrow{①②} \frac{y}{x}=2\lambda=\frac{x}{y}$, $2\lambda = z$

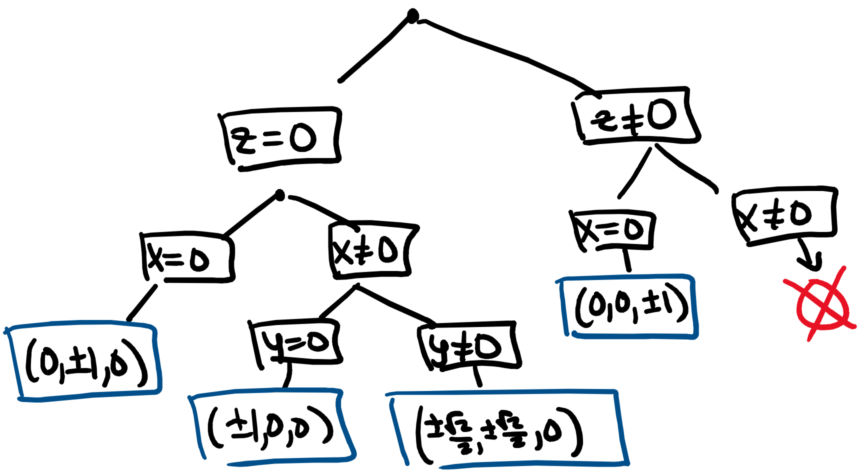
$$\Rightarrow \frac{1}{\left(\frac{x}{y}\right)}=\frac{y}{x}=z=\left(\frac{x}{y}\right)$$

$$\Rightarrow \left(\frac{x}{y}\right)^2=1$$

$$\Rightarrow \underbrace{z^2=1}_{④}$$

$$\Rightarrow x^2+y^2+1=1$$

$$\Rightarrow x=y=0$$



Since $f(x,y,z)=xy+\frac{z^3}{3}$ then we compute

$$f(0,\pm1,0)=f(\pm1,0,0)=0$$

$$f\left(\tfrac{\sqrt2}{2},\tfrac{\sqrt2}{2},0\right)=f\left(-\tfrac{\sqrt2}{2},-\tfrac{\sqrt2}{2},0\right)=\tfrac12$$

$$f\left(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, 0\right) = f\left(-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, 0\right) = -\tfrac{1}{2}$$

$$f(0,0,1) = \tfrac{1}{3}, \quad f(0,0,-1) = -\tfrac{1}{3}$$

We conclude that f has absolute maximum points

$$\left(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, 0\right), \left(-\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, 0\right)$$

over the level surface of g at k=1, with absolute maximum value =1/2. We also conclude that f has absolute minimum points

$$\left(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, 0\right), \left(-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, 0\right)$$

over the level surface of g at k=1, with absolute minimum value k=-1/2.