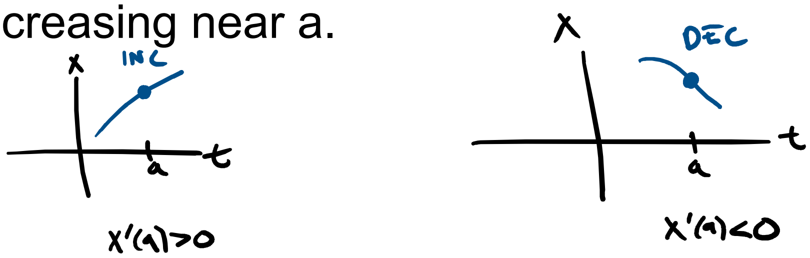
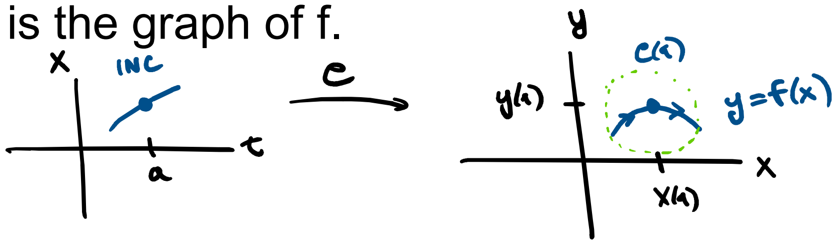
Vector Calculus
9.2 Calculus with Parametric Curves

Fact: If x'(a)=/0, then x=x(t) is either (strictly) increasing or (strictly) decreasing near a.



Fact: Suppose C(t)=(x(t),y(t)) is a parametric plane curve defined for t near a.

If x=x(t) is increasing or decreasing near a, then there is a function y=f(x) defined near x(a) so that the image of C near t=a is the graph of f.

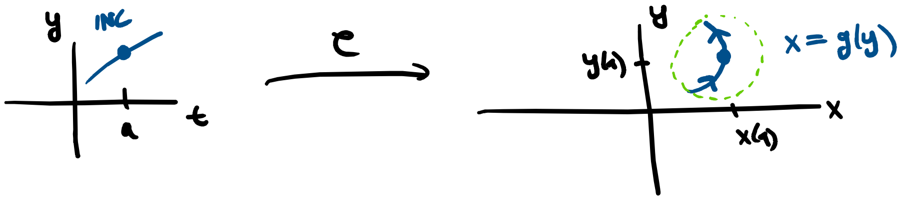


If x,y are differentiable at a with x'(a)=/0, then

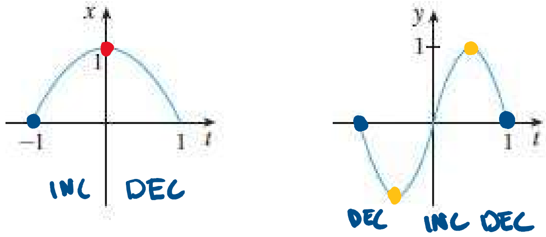
$$f'(x(a)) = \frac{y'(a)}{x'(a)}$$

In other words $$\frac{\partial y}{\partial x} = \frac{\frac{\partial y}{\partial t}}{\frac{\partial x}{\partial t}} \quad \text{if} \quad \frac{\partial x}{\partial t} \neq 0$$

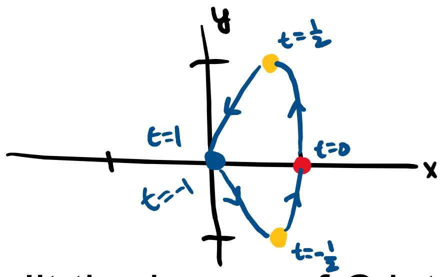
If y=y(t) is increasing or decreasing near a, then there is a function x=g(y) defined near y(a) so that the image of C near t=a is the graph of a function x=g(y).



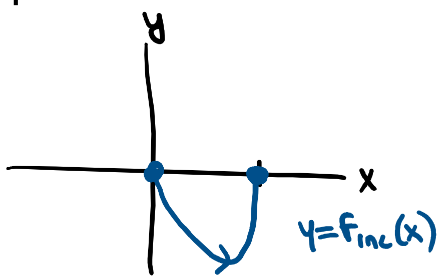
Proof: Let’s consider the example

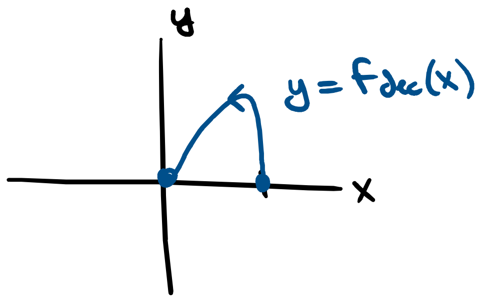


Let’s sketch the image of C.



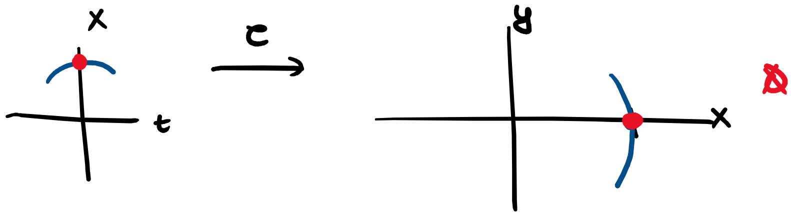
Note that we can split the image of C into the following two graphs of functions of x



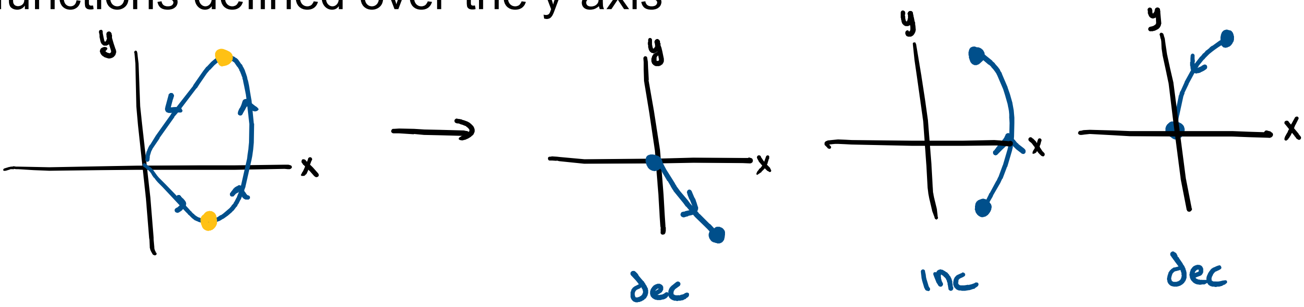


The first part corresponds to the image of C for -1≤t≤0, which is where x is increasing. The second part corresponds to the image of C for 0≤t≤1, which is where x is decreasing.

However, the image of C cannot be given as the graph of *ONE* function y=f(x) near t=0. The reason is that t=0 is precisely where x=x(t) goes from increasing to decreasing.



Similarly, the image of C decomposes into three graphs of functions defined over the y-axis



Let's suppose the image of C near t=a is the graph of a function y=f(x), and suppose x,y are differentiable at t=a with x'(a)=/0. Since the image of C is the graph of y=f(x) then

$$y(t) = F(x(t)) \quad \text{for } t \text{ near } a$$

$$\Rightarrow \quad \frac{d}{dt} y(t)\Big|_{t=a} = \frac{d}{dt} F(x(t))\Big|_{t=a}$$

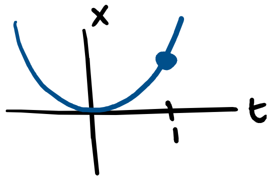
$$\Rightarrow \quad y'(a) = F'(x(a))\,\underset{\neq 0}{x'(a)}$$

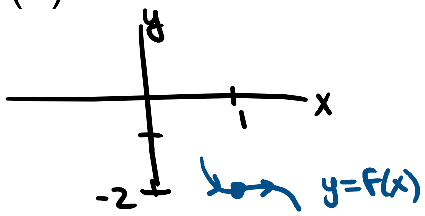
$$\Rightarrow \quad F'(x(a)) = \frac{y'(a)}{x'(a)} \quad \checkmark$$

Ex: Consider the parametric plane curve C(t)=(t^2,t^3-3t).

1. Show that the image of C near t=1 is the graph of a function y=f(x) defined near x=1.

Sol: Consider x(t)=t^2, then x=x(t) is increasing near t=1. We conclude that the image of C near t=1 is the graph of a function y=f(x) defined near x=x(1)=1^2=1.





In this case, we can also do this directly, by setting

$$x = t^2 \longrightarrow t = \sqrt{x} \text{ for } x \approx 1$$
$$y = t^3 - 3t \longrightarrow y = x^{\frac{3}{2}} - 3\sqrt{x} \quad \checkmark$$

2. Show that the image of C near t=0 is the graph of a function x=g(y) defined near y=0.

Sol: First, let's try to solve for x in terms of y, let's try to eliminate the parameter t.

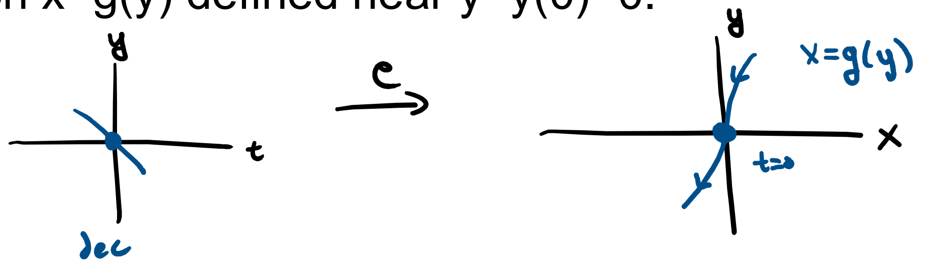
$$x = t^2$$
$$y = t^3 - 3t \xrightarrow[\text{of } y]{\text{solve for } t \text{ in terms}} t = ???$$

Instead, consider y(t)=t^3-3t at t=0. We compute

$$y'(0) = \frac{\partial}{\partial t} t^3 - 3t \Big|_{t=0} = 3t^2 - 3 \Big|_{t=0} = -3$$

This means that y=y(t) is decreasing near t=0. We conclude that the image of C near t=0 is the graph of a function x=g(y) defined near y=y(0)=0.

$x=t^2$



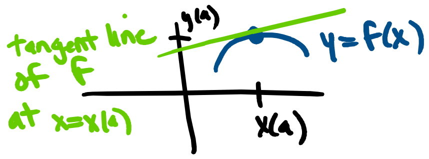
Def: Suppose C(t)=(x(t),y(t)) is a parametric plane curve defined near a with x,y continuously differentiable near a. We define the <u>tangent line of C at t=a</u> as follows.

x', y' continuous

If x'(a)=/0, then we say

$$y=\frac{y'(a)}{x'(a)}\left(x-x(a)\right)+y(a)$$

is the tangent line of C at t=a. Note that x'(a)=/0, then the image of C near t=a is the graph of a function y=f(x) defined near x=x(a).



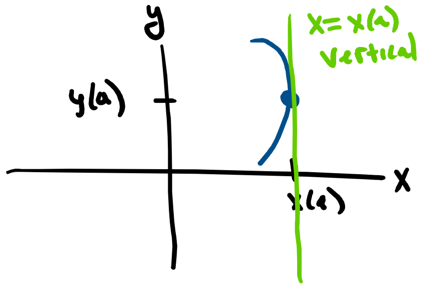
So in this case, we defined the tangent line of C at t=a to be the tangent line of f at x=x(a). This is given by

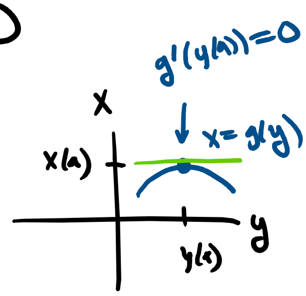
$$y=\underbrace{f'(x(a))}_{=\frac{y'(a)}{x'(a)}}\left(x-x(a)\right)+\underbrace{f(x(a))}_{=y(a)}$$

If x'(a)=0 and y'(a)=/0, then we say the tangent line of C at t=a is the vertical line x=x(a), and we say the slope of the tangent line is <u>undefined</u>. Consider y'(a)=/0, this means that the image of C near t=a is the graph of a function x=g(y) defined near y=y(a). Also, x’(a)=0 in this case implies that g’(y(a))=0.

$$g'(y(a)) = \frac{x'(a)}{y'(a)} = \frac{0}{y'(a)} = 0$$

So, the image of C is





If x’(a)=y’(a)=0, then we need to consider $\lim_{t\to a} \frac{y'(t)}{x'(t)}$

If $\lim_{t\to a} \frac{y'(t)}{x'(t)}$=m, then we say

$$y = m(x - x(a)) + y(a)$$

is the tangent line of C at t=a.

If both one-sided limits $\lim_{t\to a^{\pm}} \frac{y'(t)}{x'(t)}$ are ±infinity, then we say the tangent line of C at t=a is the vertical line x=x(a), and we say the slope of the tangent line is <u>undefined</u>.

Otherwise (if the limit does not exist, or if a one-sided limit is finite while the other infinite), then we say the tangent line of C at t=a does not exist.

Ex: Consider the parametric plane curve C(t)=(t^2,t^3-3t).

1. Show that C has two tangent lines at (3,0), and find their equations.

Sol: Consider $t^2 = 3 \Rightarrow t = \pm\sqrt{3}$

$t^3 - 3t = 0$ check $t(t^2-3) = 0$ at $t = \pm\sqrt{3}$

This means that $C(\sqrt{3}) = C(-\sqrt{3}) = (3,0)$

First, consider t=-$\sqrt{3}$. We compute

$$x'(-\sqrt{3}) = \frac{d}{dt} t^2 \Big|_{t=-\sqrt{3}} = 2t \Big|_{t=-\sqrt{3}} = -2\sqrt{3} \neq 0$$

This means that x=x(t) is decreasing near t=-$\sqrt{3}$, and so the image of C near t=-$\sqrt{3}$ is the graph of a function y=f(x) defined near x=x(-$\sqrt{3}$)=3. Thus, the tangent line of C at t=-$\sqrt{3}$ is the tangent line of f at x=3. This is given by

$$y = \underbrace{f'(x(-\sqrt{3}))}_{= \frac{y'(-\sqrt{3})}{x'(\sqrt{3})} = \frac{y'(-\sqrt{3})}{-2\sqrt{3}}} (x - \underbrace{x(-\sqrt{3})}_{=3}) + \underbrace{f(x(-\sqrt{3}))}_{= y(-\sqrt{3}) = 0}$$

We compute $y'(-\sqrt{3}) = \frac{d}{dt} t^3 - 3t \Big|_{t=-\sqrt{3}} = 3t^2 - 3 \Big|_{t=-\sqrt{3}}$

$$= 3 \cdot 3 - 3 = 6$$

The tangent line of C at t=-$\sqrt{3}$ is given by

$$\boxed{y = \frac{6}{-2\sqrt{3}}(x-3)+0}$$

Second, consider t=$\sqrt{3}$. Since

$$x'(\sqrt{3}) = 2t \Big|_{t=\sqrt{3}} = 2\sqrt{3} \neq 0$$

then the tangent line of C at t=$\sqrt{3}$ is given by

$$y = \underbrace{\frac{y'(\sqrt{3})}{x'(\sqrt{3})}}_{=\frac{y'(\sqrt{3})}{2\sqrt{3}}}\big(x - \underbrace{x(\sqrt{3})}_{=3}\big) + \underbrace{y(\sqrt{3})}_{=0}$$

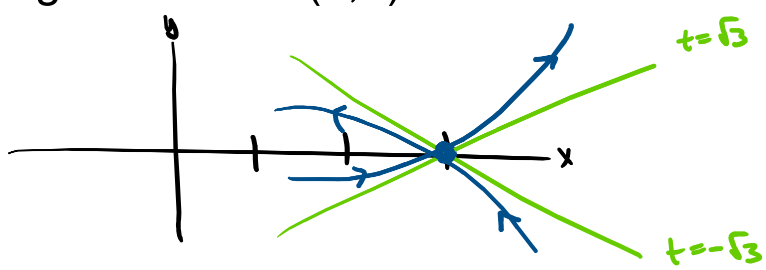
We compute

$$y'(\sqrt{3}) = \frac{d}{dt}\, t^3-3t+1 \Big|_{t=\sqrt{3}} = 3t^2-3 \Big|_{t=\sqrt{3}}$$
$$= 3\cdot 3 - 3 = 6$$

We conclude that the tangent line of C at t=$\sqrt{3}$ is given by

$$\boxed{y = \frac{6}{2\sqrt{3}}(x-3)+0}$$

Roughly, the image of C near (3,0) looks like



2. Compute all t so that x’(t)=0, and compute the slope of the tangent line of C at all such t.

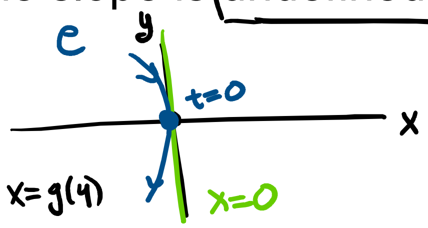
Sol: We compute

$$0 = x'(t) = 2t \quad \Rightarrow \quad \boxed{t=0}$$

We also compute

$$y'(0) = \frac{d}{dt} t^3 - 3t \,\Big|_{t=0} = 3t^2 - 3 \,\Big|_{t=0} = -3 \neq 0$$

Since y’(0)=/0, then the image of C near t=0 is the graph of a function x=g(y). Since x’(0)=0, then the tangent line of g at y=y(0) is horizontal *with respect to the y-axis*. This means that the tangent line of C at t=0 is the *vertical* line x=x(0)=0. Thus, the slope is $\boxed{\text{undefined.}}$



3. Find all t so that the tangent line of C at t is horizontal.

Sol: First, let’s find all t so that y’(t)=0.

$$0 = y'(t) = 3t^2 - 3 \Rightarrow 3(t^2-1) \Rightarrow t = \pm 1$$

Next, we must compute x’(-1),x’(1).

First, $$x'(-1) = 2t \,\Big|_{t=-1} = -2 \neq 0$$

This means that the slope of the tangent line of C at t=-1 is given by

$$\frac{y'(-1)}{x'(-1)} = \frac{0}{-2} = 0$$

We conclude that the tangent line of C at t=-1 is horizontal.

Second, $$x'(1) = 2t\big|_{t=1} = 2 \neq 0$$

This means that the slope of the tangent line of C at t=1 is given by

$$\frac{y'(1)}{x'(1)} = \frac{0}{2} = 0$$

We conclude that the tangent line of C at t=1 is horizontal.

Thus, the tangent line of C at $\boxed{\text{t=-1,1}}$ is horizontal.

Ex: Consider the cycloid C(t)=(t-sin(t),1-cos(t))

1. Compute the tangent line of C at t=pi/3.

Sol: First, we compute

$$x'\left(\tfrac{\pi}{3}\right) = \frac{d}{dt}\,(t-\sin t)\Big|_{t=\frac{\pi}{3}} = \cancel{\sin\tfrac{\pi}{3}}\ 1-\cos\tfrac{\pi}{3} = \cancel{\tfrac{\sqrt{3}}{2}}\ 1-\tfrac{1}{2} = \tfrac{1}{2} \neq 0$$

We conclude that the tangent line of C at t=pi/3 is given by

$$y = \underbrace{\frac{y'\left(\frac{\pi}{3}\right)}{x'\left(\frac{\pi}{3}\right)}}_{=\frac{\sin\left(\frac{\pi}{3}\right)}{1/2} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}}\left(x - \underbrace{x\left(\tfrac{\pi}{3}\right)}_{=\frac{\pi}{3}-\frac{\sqrt{3}}{2}}\right) + \underbrace{y\left(\tfrac{\pi}{3}\right)}_{=1-\frac{1}{2}}$$

$$\Rightarrow \boxed{y = \sqrt{3}\left(x - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right) + \frac{1}{2}}$$

2. Compute all t in [0,2pi) so that x'(t)=0, and compute the slope of the tangent line of C at all such t.

Sol: We compute

$$0 = x'(t) = \frac{d}{dt}\, t - \sin t = 1 - \cos t \underset{t \in [0, 2\pi)}{\Rightarrow} \boxed{t = 0}$$

To compute the tangent line,

$$y'(0) = \frac{d}{dt}\, 1 - \cos t \,\Big|_{t=0} = \sin 0 = 0$$

Oops, this gives x'(0)=y'(0)=0. We must consider

$$\lim_{t \to 0} \frac{y'(t)}{x'(t)} = \lim_{t \to 0} \frac{\sin t}{1 - \cos t} \underset{\text{LHR}}{=} \lim_{t \to 0} \frac{\cos t}{\sin t}$$

Let's consider the one-sided limits:

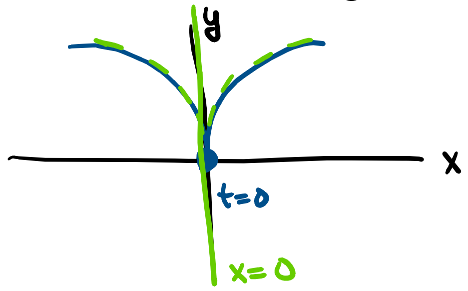
$$\lim_{t \to 0^+} \frac{\cos t}{\sin t} \qquad \frac{\to 1}{\to 0^+} \qquad \infty \qquad \Rightarrow \lim_{t \to 0^+} \frac{y'(t)}{x'(t)} = \infty$$

$$\lim_{t \to 0^-} \frac{\cos t}{\sin t} \qquad \frac{\to 1}{\to 0^-} \qquad -\infty \qquad \Rightarrow \lim_{t \to 0^-} \frac{y'(t)}{x'(t)} = -\infty$$

We conclude that the slope of the tangent line of C at t=0 is undefined.

What this means is that at t=0, the image of C:

$C(0) = (0,0)$



C has a “cusp” at t=0.

3. Find all t in [0,2pi) so that the tangent line of C at t is horizontal.

Sol: We compute

$$0 = y'(t) = \sin t \xrightarrow[t\in[0,2\pi)]{} t = 0, \pi$$

$$\Downarrow$$

vertical ✗

Consider t=pi. We compute

$$x'(\pi) = 1 - \cos t \,\Big|_{t=\pi} = 1 - \cos\pi = 1-(-1) = 2 \neq 0$$

We conclude that the tangent line of C is horizontal at $\boxed{\text{t=pi.}}$

To draw the cylcoid, consider the circle of radius =1 with center (t,1):