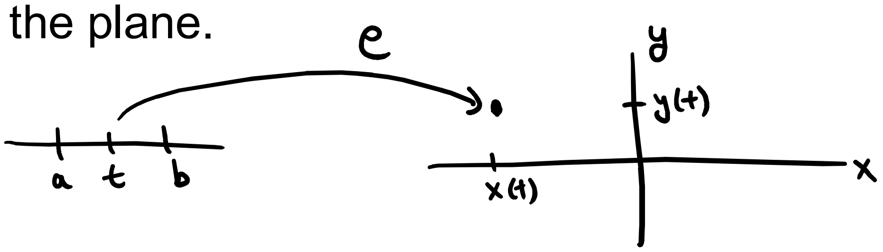
Vector Calculus
9.1 Parametric Curves

Def: A parametric plane curve is a function C:[a,b]->R^2 written

$$C(t) = (x(t), y(t)) \quad \text{for } t \in [a,b]$$

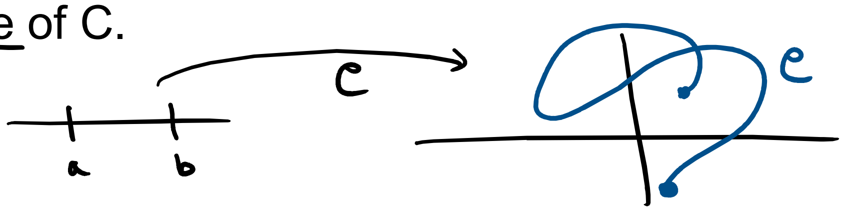
where x=x(t),y=y(t) are functions x,y:[a,b]->R. This means that for each t in [a,b], the function C(t) gives you a point (x(t),y(t)) in the plane.



We say the set

$$C = \{(x(t), y(t)) : t \in [a,b]\}$$

is the image of C.



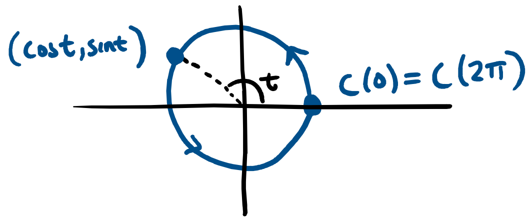
We call the variable t the parameter, and the equations

$$x = x(t) \quad \text{and} \quad y = y(t) \quad \text{for } t \in [a,b]$$

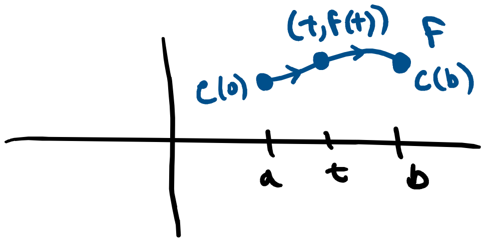
we call parametric equations for C. We say (x(a),y(a)) is the initial point, while (x(b),y(b)) is the terminal point.

Ex:

1. The image of the curve C(t)=(cos(t),sin(t)) for t in [0,2pi] is the unit circle.



2. The image of the curve C(t)=(t,f(t)) for t in [a,b] is the graph of the function y=f(x) over [a,b].



Ex: For each given parametric plane curve, do the following.

(a) Eliminate the parameter to find a Cartesian equation for the curve.

(b) Roughly sketch the image of C. Indicate with an arrow the direction in which the image is traced as t increases.

1. C(t)=(t^2,ln(t^2)) for t in (0,infinity)

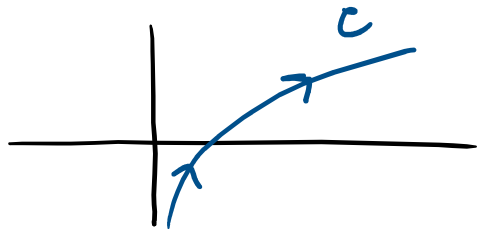
Sol: First, we do (a):

$$\begin{array}{l} x = t^2 \\ y = \ln t^2 \end{array} \Rightarrow \boxed{y = \ln x}$$

To sketch the curve, note that

$$x = t^2 \quad \text{for} \quad t \in (0,\infty) \Rightarrow \quad \begin{array}{l} y = \ln x \\ \text{for} \quad x > 0 \end{array}$$

To this implies the image of C is the graph of y=ln(x).



2. C(t)=(2sin(t),3cos(t)) for t in [0,2pi]

Sol: First, we do (a):

$$\frac{(\overset{x}{\overset{\parallel}{2\sin t}})^2}{4} + \frac{(\overset{y}{\overset{\parallel}{3\cos t}})^2}{9} = \frac{4\sin^2 t}{4} + \frac{9\cos^2 t}{9}$$

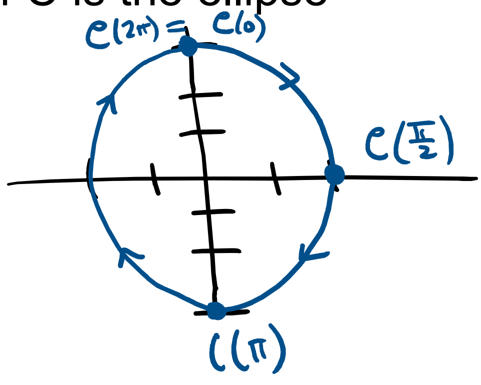
$$= \sin^2 t + \cos^2 t = 1$$

We conclude that a Cartesian equation for C is given by

$$\boxed{\frac{x^2}{4} + \frac{y^2}{9} = 1}$$

ellipse

For (b), the image of C is the ellipse



$[0,2\pi] \ni t$	$C(t) = (2\sin t, 3\cos t)$
0	$(0,3)$
$\pi/2$	$(2,0)$
π	$(0,-3)$
2π	$(0,3)$

3. C(t)=($\sqrt{t}$,1-t) for t in [0,infinity)

Sol: First, we do (a):

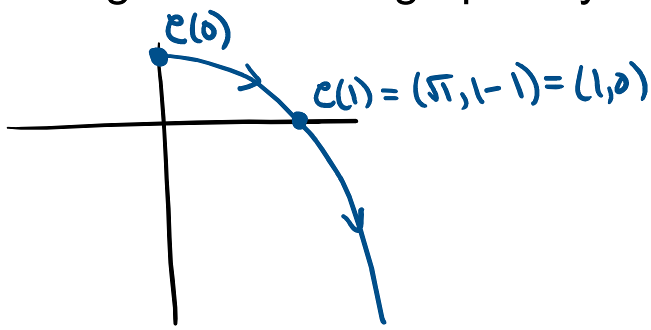
$$\begin{matrix} x = \sqrt{t} \\ y = 1 - t \end{matrix} \quad \underset{t \geq 0}{\Rightarrow} \quad y = 1 - (\sqrt{t})^2 = 1 - x^2$$

$$\Rightarrow \quad \boxed{y = 1 - x^2}$$

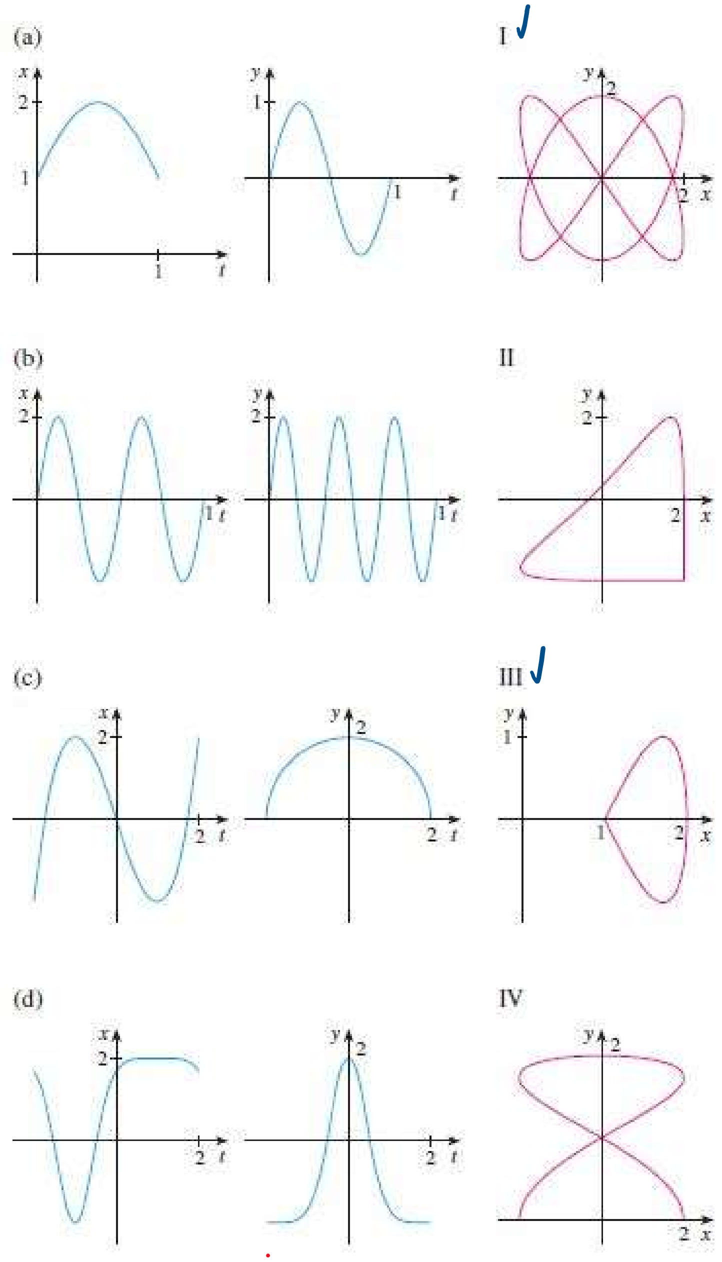
For part (b), we have to be careful. Note that

$$x = \sqrt{t} \quad \text{for} \quad t \geq 0 \quad \Rightarrow \quad x \geq 0$$

This means that the image of C is the graph of y=1-x^2 but *only* for x≥0.



Ex: Match the graphs of the parametric equations x=x(t) and y=y(t) with the images of the parametric curves in red.



(a)

t	c(t)
0	(1,0)
$\frac{1}{2}$	(2,0)
1	(1,0)

"closed" loop

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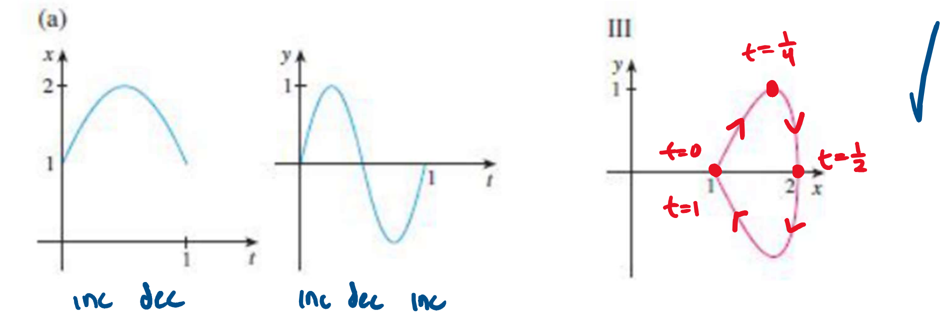
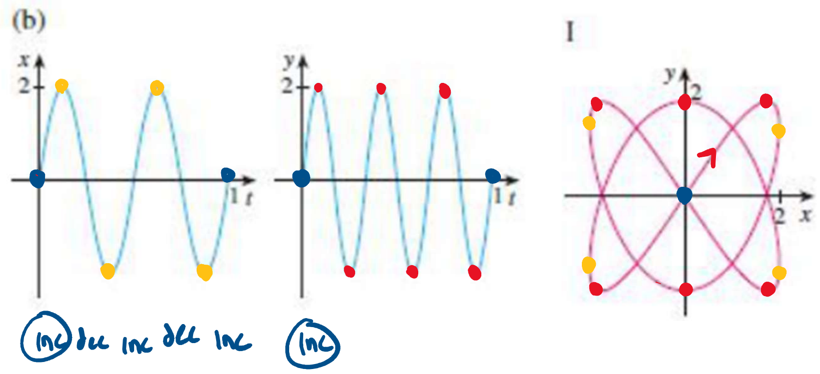
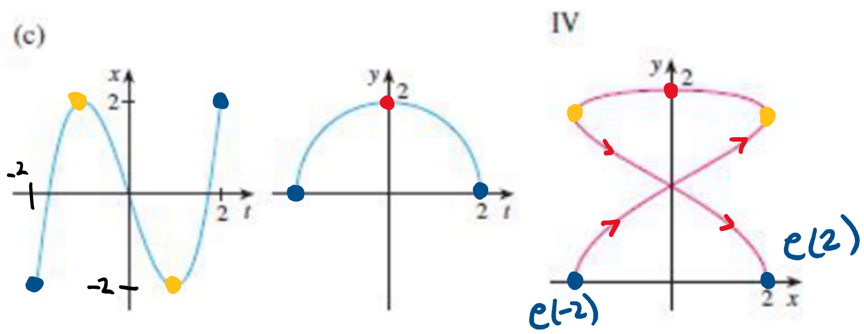


t	$c(t)$
0	(1,0)
$\frac{1}{2}$	(2,0)
1	(1,0)
$\frac{1}{4}$	$(\frac{3}{2},1)$



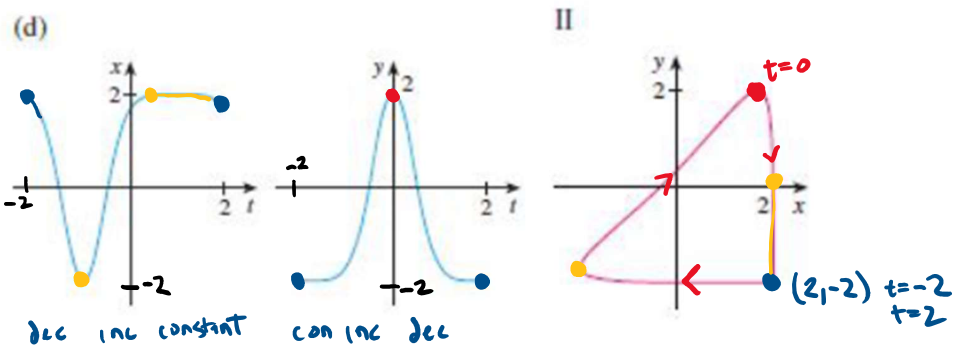
closed loop ✓

t	$c(t)$
0	(0,0)



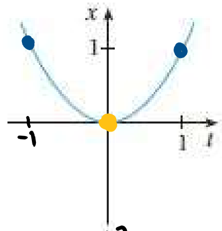
t	$c(t)$
-2	(-2,0)
2	(2,0)

not a closed loop

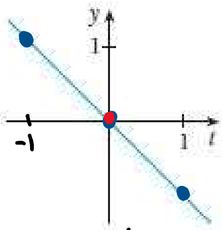


closed loop ✓

Ex: Use the graphs of x=x(t) and y=y(t) to sketch the image of the parameteric plane curve. Indicate with arrows the direction in which the curve is traced as t increases.

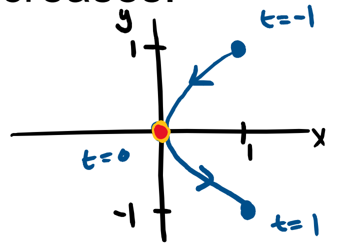


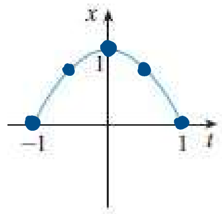
$x=t^2$

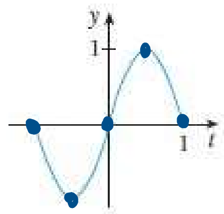


$y=-t \Rightarrow x=(-y)^2=y^2 \Rightarrow x=y^2$

t	c(t)
-1	(1,1)
0	(0,0)
1	(1,-1)

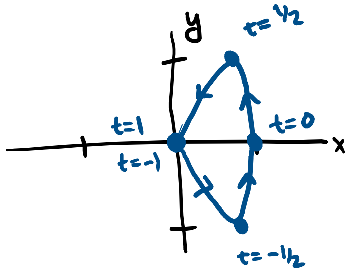


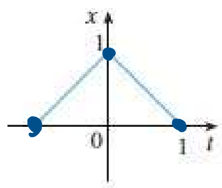




dec inc dec

t	c(t)
-1	(0,0)
-1/2	(3/4,-1)
0	(1,0)
1/2	(3/4,1)
1	(0,0)



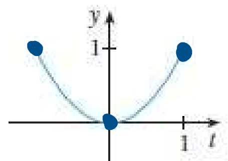


$x=t+1$ $x=1-t$

$\Downarrow$ $\Downarrow$

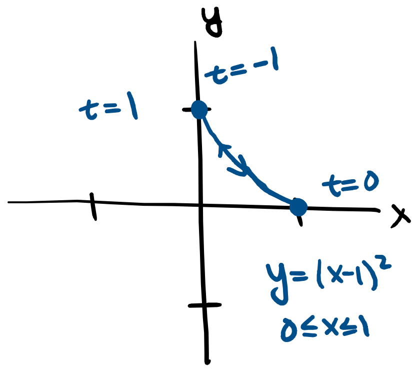
$y=(x-1)^2$ $y=(1-x)^2=(x-1)^2$

$0\le x\le 1$ $0\le x\le 1$



$y=t^2$

t	c(t)
-1	(0,1)
0	(1,0)
1	(0,1)



$y=(x-1)^2$

$0\le x\le 1$