Vector Calculus
11.2 Limits and Continuity

Today, we discuss how to show limits exist.

Fact: Suppose f:R^2->R is defined near (0,0), but perhaps not at (0,0), and suppose g:R->R is defined near 0, but perhaps not at 0.

If f(x,y)=g(x) for all (x,y) near (0,0), but perhaps not at (0,0), then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} g(x)$.

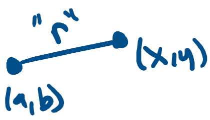
If f(x,y)=g(y) for all (x,y) near (0,0), but perhaps not at (0,0), then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} g(y)$.

If $f(x,y)=g(\underbrace{\sqrt{x^2+y^2}}_{=r})$ for all (x,y) near (0,0), but perhaps not at (0,0),

then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} g(r)$.

Similar is true for limits at (a,b).

$f(x,y) = g\left(\sqrt{(x-a)^2+(y-b)^2}\right)$



Ex: Show that the limit of the given function f at the given (a,b) exists in the extended sense.

1. $f(x,y)=e^{2\ln|x|}(x^2-y^4)+x^2y^4$ at (a,b)=(1,1)

Sol: Note that for all (x,y) near (1,1),

$$f(x,y) = e^{\ln x^2}(x^2-y^4)+x^2y^4 = x^2(x^2-y^4)+x^2y^4$$

$$= x^4 - x^2y^4 + x^2y^4 = x^4 = g(x)$$

Since $\lim_{x\to 1} g(x) = \lim_{x\to 1} x^4 = 1$

then we conclude that $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 1} g(x) = 1$ ✓

2. f(x,y)= $\frac{\sin(x^2+y^2)}{x^2+y^2}$ at (a,b)=(0,0)

Sol: Consider r=$\sqrt{x^2+y^2}$, then r^2= x^2+y^2. This means that for all (x,y) near (0,0) with (x,y)=/(0,0), we have

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2} = \underbrace{\frac{\sin(r^2)}{r^2}}_{g(r)} \Big|_{r=\sqrt{x^2+y^2}}$$

Since

$$\lim_{r\to 0} g(r) = \lim_{r\to 0} \frac{\sin(r^2)}{r^2} \qquad \frac{\to 0}{\to 0}$$

$$\overset{LHR}{=} \lim_{r\to 0} \frac{2r\cos(r^2)}{2r}$$

$$= \lim_{r\to 0} \cos(r^2) = 1$$

then we conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} g(r) = 1$$

3. f(x,y)= $\frac{1}{(x^2+y^2)^3}$ at (a,b)=(0,0)

Sol: Note that for all (x,y) near (0,0) with (x,y)=/(0,0), we have

$$f(x,y) = \frac{1}{(x^2+y^2)^3} = \underbrace{\frac{1}{(r^2)^3}}_{= g(r)} \Big|_{r=\sqrt{x^2+y^2}}$$

Since

$$\lim_{r\to 0} g(r) = \lim_{r\to 0} \frac{1}{r^6} = \infty$$

then we conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} g(r) = \infty$$

in the extended sense.

Squeeze Thm: Suppose g:R^2->R is defined near (0,0), but perhaps not at (0,0), suppose f,h:R->R are defined near 0, but perhaps not at 0, and suppose L is in R. Also suppose that

$$f(x) \leq g(x,y) \leq h(x)$$

for all (x,y) near (0,0), but perhaps not at (0,0).

If $\lim\limits_{x\to 0}$ f(x)=L=$\lim\limits_{x\to 0}$ h(x), then $\lim\limits_{(x,y)\to(0,0)}$ g(x,y)=L.

If $\lim\limits_{x\to 0}$ f(x)=infinity, then $\lim\limits_{(x,y)\to(0,0)}$ g(x,y)=infinity.

If $\lim\limits_{x\to 0}$ h(x)=-infinity, then $\lim\limits_{(x,y)\to(0,0)}$ g(x,y)=-infinity.

The Squeeze Thm is also true in case

$$\left.\begin{matrix} f(x) \\ f(y) \\ f(\sqrt{x^2+y^2}) \end{matrix}\right\} \leq g(x,y) \leq \left\{\begin{matrix} h(x) \\ h(y) \\ h(\sqrt{x^2+y^2}) \end{matrix}\right.$$

(nine possible combinations). Similar is true for limits (a,b).

Ex: Show that the limit of the given function g exists at (0,0) in the extended sense.

1. g(x,y)= $\dfrac{|x|e^{-x^2}}{1+y^2}$ $\qquad$ $\underline{\text{Guess}}$ $\quad \dfrac{0\cdot e^{0^2}}{1+0^2} = 0.$

Sol: Note that for all (x,y) near (0,0), we have

$$0 \leq \frac{|x|e^{-x^2}}{1+y^2} \underset{1+y^2\geq 1}{\leq} \frac{|x|e^{-x^2}}{1} = |x|e^{-x^2}$$

Consider f(x)=0 and h(x)=|x|e^{-x^2}. Since

$$\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} h(x)$$

then we conclude that

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0.$$

2. g(x,y)= $\dfrac{x^2y}{x^2+y^2}$

Sol: We have to be careful, because y can be negative for (x,y) near (0,0). Note that for all (x,y) near (0,0) with (x,y)=/(0,0), we have

$$\left|\frac{x^2y}{x^2+y^2}\right| = \underbrace{\frac{x^2}{x^2+y^2}}_{x^2\le x^2+y^2}\cdot|y| \le |y|$$

FACT
$|A|\le B$
$\Rightarrow -B\le A\le B$

This means that

$$-|y| \le \overset{g(x,y)}{\overset{\shortparallel}{\frac{x^2y}{x^2+y^2}}} \le |y|$$

for all (x,y) near (0,0) with $(x,y)\neq(0,0)$

Consider f(y)=-|y| and h(y)=|y|. Since

$$\lim_{y\to 0} f(y) = 0 = \lim_{y\to 0} h(y)$$

then we conclude by the Squeeze Thm that

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

3. g(x,y)= $\frac{e^{x^2}}{x^2+y^2}$ Guess $\frac{e^0}{\to 0^+} = \infty$

Sol: We must find a f(x or y or r) with f(-)⩽g(x,y) with f(-)->infinity. Note that

$$\frac{e^{x^2}}{x^2+y^2} \underset{x^2\geq 0}{\geq} \frac{e^0}{x^2+y^2} = \frac{1}{x^2+y^2} = \frac{1}{r^2}\Big|_{r=\sqrt{x^2+y^2}}$$

for all $(x,y) \neq (0,0)$

Consider f(r)= $\frac{1}{r^2}$. Since

$$\lim_{r\to 0} f(r) = \infty$$

then we conclude by the Squeeze Thm that

$$\lim_{(x,y)\to(0,0)} g(x,y) = \infty$$

4. g(x,y)= $\frac{y^4}{x^2+y^2} - x^2$ Guess $\frac{0^4}{0^2+0^2} - 0^2 = 0^2 - 0^2 = 0?$

Sol: Note that for all (x,y)=/(0,0)

$$\frac{y^4}{x^2+y^2} - x^2 \geq -x^2 \quad \Rightarrow \quad f(x) = -x^2$$

and

$$\frac{y^4}{x^2+y^2} - x^2 \leq \frac{y^4}{x^2+y^2} = y^2 \cdot \underbrace{\frac{y^2}{x^2+y^2}}_{y^2 \leq x^2+y^2} \leq y^2 \cdot 1 = y^2$$

$$\Rightarrow h(y) = y^2$$

This means that

$$\underset{-x^2}{\overset{}{f(x)}} \le g(x,y) \le \underset{y^2}{h(y)} \quad \text{for all } (x,y) \neq (0,0)$$

Since

$$\lim_{x\to 0} f(x) = 0 = \lim_{y \to 0} h(y)$$

then we conclude by the Squeeze Thm that

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

Fact: Suppose L,M are in R.

1. **Simplification Rule:** Suppose $f(x,y) = g(x,y)$ for all (x,y) near (a,b), but perhaps not at (a,b). If $\lim_{(x,y)\to(a,b)} g(x,y) = L$, then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

2. If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$, then

   Addition Rule: $\lim_{(x,y)\to(a,b)} f(x,y) + g(x,y) = L + M$.

   Multiplication Rule: $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = LM$.

   Division Rule: If $M \neq 0$, then $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$.

3. **t-Substitution Rule:** If $h : \mathbf{R} \to \mathbf{R}$ is defined near $f(a,b)$, but perhaps not at $f(a,b)$, and if $\lim_{t\to f(a,b)} h(t)$ exists, then

$$\lim_{(x,y)\to(a,b)} h(f(x,y)) =_{t=f(x,y)} \lim_{t\to f(a,b)} h(t).$$

Mostly, we have to worry about limits

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)}$$

where $\lim_{(x,y)\to(a,b)} g(x,y)=0$.

Ex: Compute $\lim_{(x,y)\to(0,0)} \left(\dfrac{xy+2}{x^2+y+3}\right)^3$

Sol: Consider

$$f(x,y) = \frac{xy+2}{x^2+y+3}$$

and

$$h(t) = t^3$$

This implies

$$\left(\frac{xy+2}{x^2+y+3}\right)^3 = h(f(x,y))$$

This means that

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy+2}{x^2+y+3}\right)^3 = \lim_{(x,y)\to(0,0)} h(f(x,y))$$

$$\underset{\substack{t=f(x,y)\\ =\frac{xy+2}{x^2+y+3}}}{=} \lim_{t\to \lim_{(x,y)\to(0,0)} f(x,y)} h(t)$$

$$= \lim_{t\to \lim_{(x,y)\to(0,0)} f(x,y)} t^3$$

So, we must compute

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy+2}{x^2+y+3}$$

Note that

$$\lim_{(x,y)\to(0,0)} x^2+y+3 \underset{\text{ADD}}{=} \lim_{(x,y)\to(0,0)} x^2 + \lim_{(x,y)\to(0,0)} y + \lim_{(x,y)\to(0,0)} 3$$

$$= \lim_{x\to 0} x^2 + \lim_{y\to 0} y + \lim_{x\to 0} 3$$

$$= 0+0+3 = 3 \neq 0$$

This means that we can apply the Division Rule to get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{\lim_{(x,y)\to(0,0)}(xy+2)}{3}$$

$$\underset{\text{ADD}}{=} \frac{\lim_{(x,y)\to(0,0)} xy + \lim_{(x,y)\to(0,0)} 2}{3}$$

$$\underset{\text{MULT}}{=} \frac{\left(\lim_{(x,y)\to(0,0)} x\right)\left(\lim_{(x,y)\to(0,0)} y\right) + \lim_{(x,y)\to(0,0)} 2}{3}$$

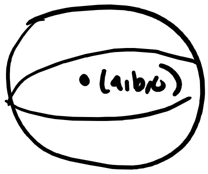
$$= \frac{\lim_{x\to 0} x \lim_{y\to 0} y + \lim_{x\to 0} 2}{3}$$

$$= \frac{0\cdot 0+2}{3} = \frac{2}{3}$$

We conclude that

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy+2}{x^2+y+3}\right)^3 = \lim_{t\to\frac{2}{3}} t^3 = \boxed{\left(\frac{2}{3}\right)^3}$$

Fact: Similar definitions and facts are true for real-valued functions f=f(x,y,z). In space, the phrase <u>for all (x,y,z) near (a,b,c)</u> means for all (x,y,z) inside a sphere centered at (a,b,c).



Ex:

1. Show that $\lim_{(x,y,z)\to(0,0,0)} \left(\frac{z^4}{x^2+y^2+z^2}+xyz\right)^4$ exists.

Sol: Note that

$$\lim_{(x,y,z)\to(0,0,0)} \left(\frac{z^4}{x^2+y^2+z^2}+xyz\right)^4$$

$$\underset{\text{t-SUB}}{=} \left(\lim_{(x,y,z)\to(0,0,0)} \left(\frac{z^4}{x^2+y^2+z^2}+xyz\right)\right)^4$$

$$\underset{\text{ADD}}{=} \left(\lim_{(x,y,z)\to(0,0,0)} \frac{z^4}{x^2+y^2+z^2} + \lim_{(x,y,z)\to(0,0,0)} xyz\right)^4$$

$$\underset{\text{MULT}}{=} \Bigg(\underbrace{\lim_{(x,y,z)\to(0,0,0)} \frac{z^4}{x^2+y^2+z^2}}_{?} + \underbrace{\lim_{x\to 0} x \cdot \lim_{y\to 0} y \cdot \lim_{z\to 0} z}_{\overset{\shortparallel}{0\cdot 0\cdot 0=0}} \Bigg)^4$$

Note that we cannot apply the Division Rule. Instead, we use the Squeeze Thm. For all (x,y,z,)=/(0,0,0), we have

$$0 \leq \frac{z^4}{x^2+y^2+z^2} = z^2 \cdot \underbrace{\frac{z^2}{x^2+y^2+z^2}}_{z^2 \leq x^2+y^2+z^2} \leq z^2 \cdot 1 = z^2$$

Since

$$\lim_{z\to 0} 0 = 0 = \lim_{z\to 0} z^2$$

then we conclude by the Squeeze Thm that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{z^4}{x^2+y^2+z^2} = 0$$

We thus conclude that

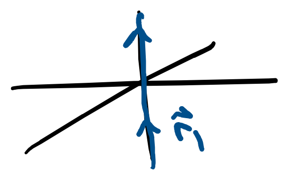
$$\lim_{(x,y,z)\to(0,0,0)} \left(\frac{z^4}{x^2+y^2+z^2} + xyz \right)^4 \underset{\substack{\text{t-SUB}\\ \text{ADD}\\ \text{MULT}\\ \text{SQUEEZE}}}{=} (0+0)^4 = 0 \checkmark$$

2. Show that the limit of f(x,y,z)= $\dfrac{xy+z^2}{x^2+y^2+z^2}$ at (0,0,0) does not exist, even in the extended sense.

Sol: We want to find two continuous *space* curves $\vec{r}_1, \vec{r}_2$ with $\vec{r}_1(0)=\vec{r}_2(0)=\langle 0,0,0\rangle$ so that

$$\lim_{t\to 0} f(\vec{r}_1(t)) \neq \lim_{t\to 0} f(\vec{r}_2(t))$$

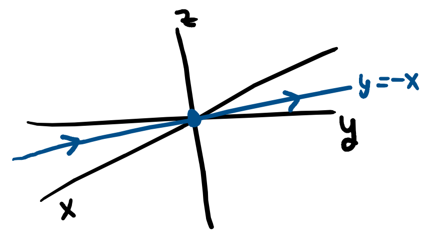
Consider $\vec{r}_1(t)=\langle 0,0,t\rangle$, then



$$\lim_{t\to 0} f(\vec{r}_1(t)) = \lim_{t\to 0} \left.\frac{xy+z^2}{x^2+y^2+z^2}\right|_{\substack{x=0\\ y=0\\ z=t}}$$

$$= \lim_{t\to 0} \frac{0\cdot 0+t^2}{0^2+0^2+t^2} = \lim_{t\to 0} \frac{t^2}{t^2} = \lim_{t\to 0} 1 = 1$$

Consider $\vec{r}_2(t)=\langle t,-t,0\rangle$, then
$y=-x$



$$\lim_{t\to 0} F(\vec{r}_2(t)) = \lim_{t\to 0} \left.\frac{xy+z^2}{x^2+y^2+z^2}\right|_{\substack{x=t\\ y=-t\\ z=0}}$$

$$= \lim_{t\to 0} \frac{t\cdot(-t)+0^2}{t^2+(-t)^2+0^2}$$

$$= \lim_{t \to 0} \frac{-t^2}{2t^2} = \lim_{t\to 0} -\frac{1}{2} = -\frac{1}{2}$$

Since

$$\lim_{t\to 0} f(\vec{r}_1(t)) = 1 \neq -\frac{1}{2} = \lim_{t\to 0} f(\vec{r}_2(t))$$

then we conclude that

$$\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) \text{ DNE.}$$