VECTOR CALCULUS, Week 2

2.1 Derivatives and Rates of Change; 2.2 The Derivative as a Function; 2.3 Basic Differentiation Formulas; 2.4 The Product and Quotient Rules; 2.5 The Chain Rule; 2.8 Linear Approximation and Differentials; 3.1 Maximum and Minimum Values; 5.8 Indeterminant Forms and l'Hospital's Rule; The Squeeze Theorem

2.1 Derivatives and Rates of Change

Def: Suppose f is defined near a, including at a itself.

- We say f **is differentiable at** a **with derivative** $f'(a)$ if the following limit exists (is a finite number):

$$f'(a) = \frac{d}{dx} f(x)\Big|_{x=a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

  The notation $f'(a)$ is pronounced f **prime of** a.

- If f is differentiable at a, then we say the line through $(a, f(a))$ with slope $f'(a)$ is the **tangent line of** f **at** a:

$$y = f'(a)(x - a) + f(a).$$

Ex: Definition of the derivative.

1. For $f(x) = x^2$, compute $f'(a)$ for all a.
2. $f(x) = |x|$ is not differentiable at $a = 0$.

2.2 The Derivative as a Function

Def: Suppose f is differentiable at each $x \in (a, b)$

- We say f **is differentiable over** (a, b).
- The function $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for $x \in (a, b)$ is called the **first derivative of** f.
- $\frac{d}{dx} f(x)$ means to find the first derivative of f.
- If f' is continuous over (a, b), then we say f **is continuously differentiable over** (a, b).
- If f' is also differentiable over (a, b), then $f''(x) = \frac{d}{dx} f'(x)$ denotes the **second derivative of** f. The notation $f''(a)$ is pronounced f **double prime of** a. $\frac{d^2}{dx^2} f(x)$ means to find the second derivative of f.

We make similar definitions over $(a, \infty), (-\infty, b), (-\infty, \infty)$.

Ex: For $f(x) = x^2$, we already showed that $f'(x) = \frac{d}{dx} x^2 = 2x$. This means that f is continuously differentiable over $(-\infty, \infty)$.

2.3 Basic Differentiation Formulas

Thm (Table of Basic Derivatives):

- $\frac{d}{dx}c = 0$ for all x.
- $\frac{d}{dx}x^r = rx^{r-1}$ for any real number r, for all x near where x^r is defined.
- trigonometric functions
  $\Rightarrow \frac{d}{dx}\cos x = -\sin x$ for all x.
  $\Rightarrow \frac{d}{d}\sin x = \cos x$ for all x.
  $\Rightarrow \frac{d}{dx}\tan x = \sec^2 x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
  $\Rightarrow \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$ for all x.
- $\frac{d}{dx}e^x = e^x$ for all x.
- $\frac{d}{dx}\ln x = \frac{1}{x}$ for $x > 0$.

2.4 The Product and Quotient Rules

Thm (Basic Derivative Rules): Suppose f, g are differentiable at a.

> **Simplification Rule:** If $f(x) = g(x)$ for all x near a, then $f'(a) = g'(a)$.
>
> **Addition Rule:** $\frac{d}{dx}\big(f(x) + g(x)\big)\Big|_{x=a} = f'(a) + g'(a)$.
>
> **Product Rule:** $\frac{d}{dx}\big(f(x)g(x)\big)\Big|_{x=a} = f'(a)g(a) + f(a)g'(a)$.
>
> $\Rightarrow \frac{d}{dx}cf(x)|_{x=a} = c\frac{d}{dx}f(x)\Big|_{x=a}$ for any $c \in \mathbf{R}$.
>
> **Quotient Rule:** If $g(a) \neq 0$, then $\frac{d}{dx}\frac{f(x)}{g(x)}\Big|_{x=a} = \frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}$.

Ex: Compute the following derivatives.

1. $\frac{d}{dx}x^2$
2. $\frac{d}{dx}\frac{x(x+1)+1}{e^x+1}\Big|_{x=0}$

2.5 The Chain Rule

Chain Rule: Suppose $g(x)$ is differentiable at $x = a$ and $f(u)$ is differentiable at $u = g(a)$, then $f(g(x))$ is differentiable at $x = a$ with derivative

$$\frac{d}{dx} f(g(x))\Big|_{x=a} = \frac{d}{du} f(u)\Big|_{u=g(a)} \cdot \frac{d}{dx} g(x)\Big|_{x=a} = f'(g(a))g'(a).$$

Ex: Compute $\frac{d}{dx}\sqrt{x + \sqrt{x + \sqrt{x}}}$ for $x > 0$.

2.8 Linear Approximation and Differentials

Thm: Suppose f is differentiable at a, then the tangent line $y = f'(a)(x-a) + f(a)$ of f at a is a good approximation for f near a.

Ex: Estimate the value of the following quantities.

1. $\sqrt{\frac{1}{100} + \cos(\frac{1}{100})}$
2. $\sqrt{1.02 + \sin(\frac{1}{100})}$

Fact: Suppose f is differentiable at a. The tangent line of f at a is horizontal if and only if $f'(a) = 0$.

3.1 Maximum And Minimum Values

Def: An **interval** is any subset of $\mathbf{R}$ of the form

$$(a,b),(a,b],[b,a),[a,b],(-\infty,a),(-\infty,a],(a,\infty),[a,\infty),(-\infty,\infty)$$

where $a,b\in\mathbf{R}$.

Def: Suppose I is an interval, and suppose f is defined for all $x\in I$.

- If $c\in I$ and $f(c)\geq f(x)$ for each $x\in I$, then we say

  c is an **absolute maximum point of** f **over** I and
  $f(c)$ is an **absolute maximum value of** f **over** I.

- If $c\in I$ and $f(x)\leq f(c)$ for each $x\in I$, then we say

  c is an **absolute minimum point of** f **over** I and
  $f(c)$ is an **absolute minimum value of** f **over** I.

- If $c\in I$ is an absolute maximum or minimum point of f over I, then we say

  c is an **absolute extremum point of** f **over** I and
  $f(c)$ is an **absolute extremum value of** f **over** I.

Suppose $c\in\mathbf{R}$, and suppose f is defined near c.

- If $f(c)\geq f(x)$ for all x near c, then we say

  c is a **local maximum point of** f and
  $f(c)$ is a **local maximum value of** f.

- If $f(c)\leq f(x)$ for all x near c, then we say

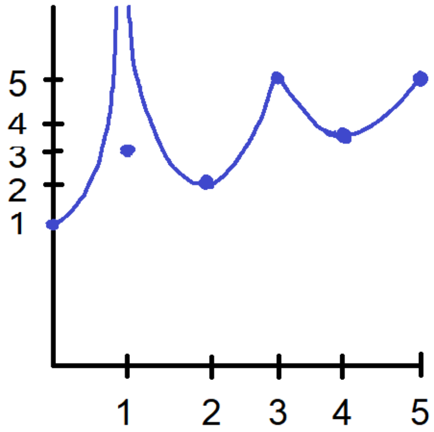
  c is a **local minimum point of** f and
  $f(c)$ is a **local minimum value of** f.

- If c is either a local maximum or a local minimum point of f, then we say c

  c is a **local extremum point of** f and
  $f(c)$ is a **local extremum value of** f.

- If $f'(c)=0$ or $f'(c)$ does not exist, then we say c is a **criticial point of** f.

Ex: Consider the function f defined over $[0, 5]$, given by the following graph.



1. Find the absolute extremum points and values of f over $[2, 5]$.
2. Find the absolute extremum points and values of f over $[0, 5]$.
3. Find the local extremum points and values of f.
4. Find the critical points of f.

Thm: Finding local and absolute extremum points.

- If c is a local extremum point of f, then c is a critical point of f.
- Suppose f is continuous over $[a, b]$.
  - There are $c_{max}, c_{min} \in [a, b]$ so that c_{max} is an absolute maximum point and c_{min} is an absolute minimum point of f over $[a, b]$.
  - If c is an absolute extremum point of f over $[a, b]$, then either $c = a, b$ or c is a critical point of f.

Ex: Find the absolute extremum points and values of $f(x) = 3x^2 - 2x + 1$ over $[0, 2]$.

5.8 Indeterminant Forms and L'Hospital's Rule

Def: Suppose $L, M \in \mathbf{R}$. We define the following extended arithmetic rules, or **determined forms**:

$L + M$, LM, and $\frac{L}{M}$ if $M \neq 0$.

$L + \infty = \infty + L = \infty$ and $L - \infty = -\infty + L = -\infty$.

$\infty + \infty = \infty$ and $-\infty - \infty = -\infty$.

$\frac{L}{\infty} = \frac{L}{-\infty} = 0$.

If $L > 0$, then

$$L \cdot \infty = \infty \cdot L = \frac{\infty}{L} = \infty \text{ and } L \cdot -\infty = -\infty \cdot L = \frac{-\infty}{L} = -\infty.$$

If $L < 0$, then

$$L \cdot \infty = \infty \cdot L = \frac{\infty}{L} = -\infty \text{ and } L \cdot -\infty = -\infty \cdot L = \frac{-\infty}{L} = \infty.$$

$\infty \cdot \infty = -\infty \cdot -\infty = \infty$ and $\infty \cdot -\infty = -\infty \cdot \infty = -\infty$.

The following are **indeterminant forms**, and are undefined:

$$\frac{\infty - \infty}{-\infty + \infty}, \quad \frac{0 \cdot \pm\infty}{\pm\infty \cdot 0}, \quad \frac{\text{anything}}{0}, \quad \frac{\pm\infty}{\pm\infty}$$

Extended Basic Limit Rules: Let $\lim_{x\to\#}$ denote any kind of a limit:

$$\lim_{x\to a},\ \lim_{x\to a^-},\ \lim_{x\to a^+}\ \text{ for } a\in\mathbf{R},\ \text{ or } \lim_{x\to\infty},\ \lim_{x\to-\infty}.$$

1. **Simplification Rule:** If $f(x) = g(x)$ for all x "near #," but perhaps not "at #," and $\lim_{x\to\#} f(x)$ exists in the extended sense, then $\lim_{x\to\#} g(x)$ exists in the extended sense and is equal to $\lim_{x\to\#} f(x)$.

2. Suppose $\lim_{x\to\#} f(x)$ and $\lim_{x\to\#} g(x)$ exist in the extended sense.

   Addition Rule:

$$\text{If } \lim_{x\to\#} f(x) + \lim_{x\to\#} g(x) \text{ is not } \begin{cases} \infty-\infty \\ -\infty+\infty \end{cases},$$
$$\text{then } \lim_{x\to\#}(f(x)+g(x)) = \lim_{x\to\#} f(x) + \lim_{x\to\#} g(x).$$

   Multiplication Rule:

$$\text{If } \lim_{x\to\#} f(x) \cdot \lim_{x\to\#} g(x) \text{ is not } \begin{cases} 0\cdot\pm\infty \\ \pm\infty\cdot 0 \end{cases},$$
$$\text{then } \lim_{x\to\#}(f(x)\cdot g(x)) = \lim_{x\to\#} f(x) \cdot \lim_{x\to\#} g(x).$$

   Division Rule:

$$\text{If } \frac{\lim_{x\to\#} f(x)}{\lim_{x\to\#} g(x)} \text{ is not } \frac{\text{anything}}{0} \text{ or } \frac{\pm\infty}{\pm\infty},$$
$$\text{then } \lim_{x\to\#}\frac{f(x)}{g(x)} = \frac{\lim_{x\to\#} f(x)}{\lim_{x\to\#} g(x)}.$$

3. u-**Substitution Rule**: Suppose

$$\lim_{x\to\#} g(x) \text{ and } \lim_{u\to\lim_{x\to\#} g(x)} f(u)$$

   both exist in the extended sense, then

$$\lim_{x\to\#} f(g(x)) = \lim_{u\to\lim_{x\to\#} g(x)} f(u).$$

Ex: Compute $\lim_{x\to\infty}(x^2 - x)$.

Fact: We need strategies to compute limits of the form $\lim_{x\to\#}\frac{f(x)}{g(x)}$ where $\frac{\lim_{x\to\#} f(x)}{\lim_{x\to\#} g(x)}$ is an indeterminant form of type $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$. To compute limits

$$\lim_{x\to\#}(f(x)+g(x)) \text{ and } \lim_{x\to\#}(f(x)g(x))$$

where the Addition or Multiplication Rules cannot be used, the first step is to simplify to get a limit of the form $\lim_{x\to\#}\frac{f(x)}{g(x)}$.

Degree Analysis Rule: Suppose $p(x), q(x)$ are sums, products, and compositions of constants and basic power functions.

- To compute $\lim_{x\to 0,0^{\pm}}\frac{p(x)}{q(x)}$, factor out the **smallest** power of x from $p(x)$ and $q(x)$.
- To compute $\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}$, factor out the **largest** power of x from $p(x)$ and $q(x)$.

Ex: Compute the following limits.

1. $\lim_{x\to 0^+}\frac{\sqrt{x^7+x^5}+x^2}{x^4-3x^3}$.
2. $\lim_{x\to-\infty}\frac{\sqrt{x^2+1}+(1-x)^{1/4}}{\sqrt[3]{x}+1}$, set $u=-x$.

Fact: To compute $\lim_{x\to 0^-} f(x)$ or $\lim_{x\to-\infty} f(x)$, set $u=-x$.

Easy L'Hospital's Rule: Suppose f, g are differentiable at a with $f(a) = g(a) = 0$ and $g'(a) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.

Ex: Compute $\lim_{x\to 2} \frac{e^{(x-2)}-1}{\arctan x - \arctan 2}$.

L'Hospital's Rule: Let $\lim_{x\to\#}$ denote any kind of limit. Suppose

- $\frac{\lim_{x\to\#} f(x)}{\lim_{x\to\#} g(x)}$ is an indeterminant form of the type $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$.
- $\lim_{x\to\#} \frac{f'(x)}{g'(x)}$ exists in the extended sense.

Then $\lim_{x\to\#} \frac{f(x)}{g(x)} = \lim_{x\to\#} \frac{f'(x)}{g'(x)}$.

Non-Ex: Define the functions

$$f(x) = x\sin(x^{-4})e^{-1/x^2} \text{ and } g(x) = e^{-1/x^2}.$$

Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$ but $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.

Ex: Compute the following limits.

1. $\lim_{x\to 0^+} x\ln x$
2. $\lim_{x\to\infty} \frac{e^x}{x^2}$
3. $\lim_{x\to\infty} \frac{e^x+2x^2+\sqrt{x^4+x+1}}{x^2+\sqrt{x+4x^{1/3}}}$

The Squeeze Theorem

Squeeze Thm: Suppose f, g, h are defined near a, but perhaps not a itself. If

$$f(x) \leq g(x) \leq h(x)$$

for all x near a with $x \neq a$, and if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$. Similar is true for all types of limits, in the extended sense.

Ex: Use the Squeeze Thm to show the following.

1. $\lim_{x \to 0} \frac{x^4}{x^2 + x^4} = 0$
2. $\lim_{x \to 0} x^2 \sin(1/x) = 0$