Vector Calculus
10.8 Arc Length and Curvature

Let $\vec{0}$=<0,0> or =<0,0,0>

Def: Suppose $\vec{r}$ is a parametric vector-valued function defined over [a,b]. We say $\vec{r}$ is <u>regular/smooth</u> if the component functions of $\vec{r}$ are continuously differentiable over [a,b] with $\vec{r}$'(t)=/$\vec{0}$ for each t in [a,b].

x,y,z

Ex: $\vec{r}_1$(t)=<t^3,t^3> is *not* regular at t=0,

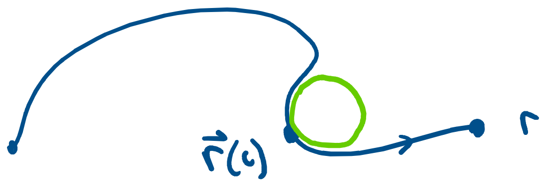
$$\vec{r}_1'(0) = \langle 3t^2, 3t^2 \rangle \Big|_{t=0} = \langle 0,0 \rangle = \vec{0}$$

while $\vec{r}_2$(t)=<t,t> is regular.

$\vec{r}_2'(t)=\langle 1,1\rangle$

The image of both $\vec{r}_1$,$\vec{r}_2$ is the line y=x.

Fact: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over [a,b], and suppose c is in [a,b]. If $\vec{r}$''(c) exists with $\vec{r}$''(c)=/$\vec{0}$, then there is a <u>unique</u> circle which is tangent to the image of $\vec{r}$ at $\vec{r}$(c).

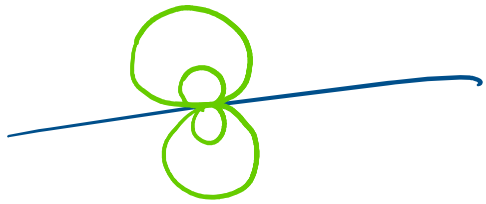


Def: We call this circle the <u>osculating circle of $\vec{r}$ at t=c</u>.

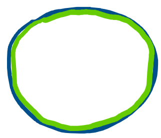
"Kissing"

Ex:

1. Lines do not have *unique* tangent circles.



2. The osculating circle of a circle is itself.



Def: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over [a,b], and suppose $\vec{r}''(t)$ exists for each t in [a,b]. We define the <u>curvative function</u> kappa:[a,b]->[0,infinity) to be

κ "Greek k"

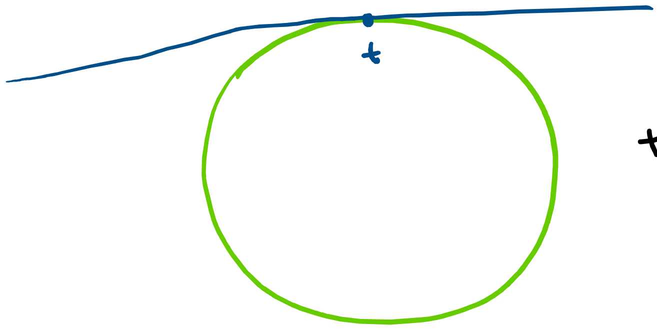
$$\kappa(t) = \begin{cases} 0 & \text{if } \vec{r}''(t) = \vec{0} \\ \dfrac{1}{\left(\text{radius of the osculating circle of } \vec{r} \text{ at } t\right)} & \text{if } \vec{r}''(t) \neq \vec{0} \end{cases}$$

Fact: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over [a,b], and suppose $\vec{r}''(t)$ exists for each t in [a,b].

If kappa(t)>0 is large, then the radius of the osculating circle is small.

$\Rightarrow$ "large curvature"
the image is curving a lot

If kappa(t)>0 is small, then the radius of the osculating circle is big.



"small curvature"
the image curves very little

$$\Rightarrow \kappa(t) = \frac{1}{\text{radius}}$$

If kappa(t)=0, the radius of the osculating circle is infinity, in which case the osculating “circle” is the tangent line.

$\text{radius} = \frac{1}{0} = \infty$

$\Rightarrow \kappa(t) = \frac{1}{\text{radius}} = \frac{1}{\infty} = 0$

We do not define

$$\kappa(t) = \text{radius of the osculating circle of } \vec{r} \text{ at } t$$

because if $\vec{r}''(t)=0$, then the radius of the osculating “circle” is infinity. The actual definition of kappa guarantees that kapp(t) is always a finite value.

We can compute kappa(t) as follows.

If $\vec{r}$:[a,b]->R^3, then

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \quad \text{for } t \in [a,b]$$

$\vec{r}$ regular $\Rightarrow |\vec{r}\,'(t)| \neq 0$

If $\vec{r}$:[a,b]->R^2, then embed $\vec{r}$ into R^2 by setting

$$\vec{r}(t) = \langle x(t), y(t), 0 \rangle \quad \text{for } a \le t \le b$$

We can now compute

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} = \frac{|x'(t)\,y''(t) - x''(t)\,y'(t)|}{\left((x'(t))^2 + (y'(t))^2\right)^{3/2}} \quad \text{for } t \in [a,b]$$

Suppose f:[a,b]->R and suppose

$$\vec{r}(t) = \langle t, f(t) \rangle \quad \text{for } a \le t \le b$$

then

$$\kappa(t) = \frac{|f''(t)|}{\left(1 + (f'(t))^2\right)^{\frac{3}{2}}} \quad \text{for } t \in [a,b].$$

The curvature is like a second derivative, however, we do *not* simply have

$$\kappa(t) = |\vec{r}\,''(t)|$$

in general.

Ex: Compute the curvature function for each of the following.

1. $\vec{r}$(t)=<2cos(3t),2sin(3t)>

Sol: Note that the image of $\vec{r}$ is the circle of radius =2 centered at the origin. This means that the osculating circle of the image of $\vec{r}$ is itself. Thus,

$$K(t) = \frac{1}{\text{radius}} = \boxed{\frac{1}{2}}$$

Let's also compute

$$\vec{r}(t) = \langle 2\cos 3t, 2\sin 3t, 0 \rangle$$

$$\star \quad \vec{r}'(t) = \langle -6\sin 3t, 6\cos 3t, 0 \rangle$$

$$|\vec{r}'(t)| = \sqrt{6^2\sin^2 3t + 6^2\cos^2 3t} = 6$$

$$\star \quad \vec{r}''(t) = \langle -18\cos 3t, -18\sin 3t, 0 \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \langle 0, 0, 3\cdot 6^2\sin^2 3t + 3\cdot 6^2\cos^2 3t \rangle$$
$$= \langle 0, 0, 3\cdot 6^2 \rangle$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = 3\cdot 6^2$$

We conclude that

$$K(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{3\cdot 6^2}{6^3} = \frac{3}{6} = \frac{1}{2}\ !$$

Note that $\quad |\vec{r}''(t)| = 18 \neq K(t)$

2. $\vec{r}(t)$=<cos(t),sin(t),t>

Sol: We compute

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle \qquad |\vec{r}'(t)| = \sqrt{1+1} = \sqrt{2}$$

$$\vec{r}''(t) = \langle -\cos t, -\sin t, 0 \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \langle \sin t, -\cos t, 1 \rangle$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{1+1} = \sqrt{2}$$

We conclude that

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \boxed{\frac{1}{2}}$$

This implies that the radius of the osculating circle is always 1/kappa(t)=1/(1/2)=2.

Let use [CalcPlot3D (libretexts.org)]()

Fact: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over [a,b], and suppose $\vec{r}$ has only <u>isolated self-intersections</u>. The arc length L of the image of $\vec{r}$ is given by

$$L = \int_a^b |\vec{r}'(t)|\,dt = \begin{cases} \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt & \text{or} \\ \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \end{cases}$$

Def: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over [a,b] with only isolated self-intersections, and suppose L is the arc length of the image of $\vec{r}$. We define the <u>arc length function s:[a,b]->[0,L] of $\vec{r}$</u>

to be the function

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du \qquad \text{for } a \le t \le b$$

Note that

$$s(a) = \int_a^a |\vec{r}\,'(u)|\,du = 0 \qquad\qquad 0 < s(t) < L$$
$$s(b) = \int_a^b |\vec{r}\,'(u)|\,du = L \qquad\qquad \text{for all } t \in (a,b)$$

Ex: Compute the arc length L of the image and the arc length function s(t) for each of the following parametric vector-valued functions.

1. $\vec{r}(t)$=<2cos(3t),2sin(3t)> for 0⩽t⩽2pi/3

Sol: First, we already computed

$$|\vec{r}\,'(t)| = |\langle -6\sin 3t, 6\cos 3t\rangle| = 6$$

We conclude the arc length L is

$$L = \int_0^{\frac{2\pi}{3}} |\vec{r}\,'(t)|\,dt = \int_0^{\frac{2\pi}{3}} 6\,dt = 6 \cdot \frac{2\pi}{3} = \boxed{4\pi}$$

and the arc length function s(t) is

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t 6\,du = 6t$$

$$\Rightarrow \boxed{s(t) = 6t \quad \text{for } 0 \le t \le \frac{2\pi}{3}}$$

<u>check</u> $s\left(\frac{2\pi}{3}\right) = 6 \cdot \frac{2\pi}{3} = 4\pi = L \checkmark$

2. $\vec{r}(t)=\langle\cos(t),\sin(t),t\rangle$ for 0<t<2pi

Sol: We already computed

$$|\vec{r}\,'(t)| = |\langle -\sin t, \cos t, t\rangle| = \sqrt{1+1} = \sqrt{2}$$

We conclude that the arc length L is

$$L = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\pi\sqrt{2}}$$

and the arc length function s(t) is

$$s(t) = \int_0^t |\vec{r}\,'(u)|\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\,t$$

$$\Rightarrow \boxed{s(t) = \sqrt{2}\,t \quad \text{for} \quad 0 \le t \le 2\pi}$$

check $s(2\pi) = \sqrt{2}\cdot 2\pi = L$ ✓

Fact: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over [a,b] with only isolated self-intersections, and suppose $|\vec{r}\,'(t)|=1$ for each t in [a,b].

The arc length function s of $\vec{r}$ is s(t)=t-a, and the arc length of $\vec{r}$ is L=b-a.

pf: $s(t) = \int_a^t |\vec{r}\,'(u)|\, du = \int_a^t 1\, du = t-a.$

If $\vec{r}\,''(t)$ exists for each t in [a,b], then kappa(t)=$|\vec{r}\,''(t)|$.

Def: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over [a,b], and suppose that $|\vec{r}\,'(t)|=1$ for each t in [a,b], then we say $\vec{r}$ is a unit-speed parametric vector-valued function, or $\vec{r}$ is parameterized by arc length.

$s(t) = t-a \Rightarrow s = t-a$

Ex: Reparameterize the following parametric vector-valued functions so that they are parameterized by arc length. More precisely, find a real-valued function f=f(s) so that $\vec{r}_f$ is a unit-speed parametric vector valued function, and give $r_f=r_f(s)$.

1. r(t)=<2cos(3t),2sin(3t)> for $0 \leq t \leq 2pi/3$

Sol: Consider the arc length L=4pi and the arc length function

$$s: [0, \tfrac{2\pi}{3}] \to \underbrace{[0, 4\pi]}$$

$$s(t) = 6t$$

Find the inverse of s(t), solve for t in terms of s.

$$s = 6t \Rightarrow t = \frac{s}{6}$$

This is what we will use for f. Define

$$\boxed{f(s) = \frac{s}{6} \quad \text{for} \quad 0 \leq s \leq 4\pi}$$

$$f(0) = 0, \quad f(4\pi) = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$f: [0, 4\pi] \to [0, \tfrac{2\pi}{3}]$$

Consider the reparameterization of $\vec{r}$ given by

$$\vec{r}_f(s) = \vec{r}(f(s)) = \langle 2\cos 3t, 2\sin 3t \rangle \Big|_{t=\frac{s}{6}}$$

$$\Rightarrow \quad \boxed{\vec{r}_F(s) = \left\langle 2\cos\frac{s}{2}, 2\sin\frac{s}{2} \right\rangle \quad \text{for } 0 \le s \le 4\pi}$$

Check: We compute

$$|\vec{r}_F'(s)| = \left|\left\langle -\frac{1}{2}\cdot 2\sin\frac{s}{2}, \frac{1}{2}\cdot 2\cos\frac{s}{2} \right\rangle\right|$$

$$= \left|\left\langle -\sin\frac{s}{2}, \cos\frac{s}{2} \right\rangle\right| = \sqrt{\sin^2\frac{s}{2} + \cos^2\frac{s}{2}} = 1 \quad \checkmark$$

So $\vec{r}_F$ *is* unit-speed parametric vector-valued function. Let's check that kappa(t)=|$\vec{r}$''(t)|. The image of $\vec{r}_F$ is still the circle of radius =2 centered at the origin. This means that that the osculating circle of the image of $\vec{r}$ is itself, and so

$$\kappa(t) = \frac{1}{\text{radius}} = \frac{1}{2}$$

We also have
$$\vec{r}_F'(s) = \left\langle -\sin\frac{s}{2}, \cos\frac{s}{2} \right\rangle$$
$$\vec{r}''_F(s) = \left\langle -\frac{1}{2}\cos\frac{s}{2}, -\frac{1}{2}\sin\frac{s}{2} \right\rangle$$

$$\Rightarrow \quad |\vec{r}''_F(s)| = \frac{1}{2} \quad \checkmark$$

2. $\vec{r}$(t)=<cos(t),sin(t),t> for 0≤t≤2pi

Sol: We compute the arc length is L=2pi$\sqrt{2}$ and the arc length function is

$$s: [0, 2\pi] \to [0, 2\pi\sqrt{2}]$$
$$s(t) = \sqrt{2}\,t$$

Solve for t in terms of s, so that

$$s = \sqrt{2}t \quad \Rightarrow \quad t = \frac{s}{\sqrt{2}}$$

We conclude that we should set

$$\boxed{f(s) = \frac{s}{\sqrt{2}} \quad \text{for} \quad 0 \le s \le 2\pi\sqrt{2}}$$

$$f(0)=0, \; f(2\pi\sqrt{2}) = 2\pi \checkmark$$

and

$$\vec{r}_f(s) = \vec{r}(f(s)) = \langle \cos t, \sin t, t \rangle \Big|_{t=\frac{s}{\sqrt{2}}}$$

$$\Rightarrow \boxed{\vec{r}_f(s) = \left\langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle \quad \text{for} \quad 0 \le s \le 2\pi\sqrt{2}}$$

Check: The curvature should still be $=\frac{1}{2}$, as before, so we check

$$\frac{1}{2} \stackrel{?}{=} |\vec{r}_f''(s)| = \left| \frac{d}{ds} \left\langle -\frac{1}{\sqrt{2}} \sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \right\rangle \right|$$

$$= \left| \left\langle -\frac{1}{2} \cos\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{2} \sin\left(\frac{s}{\sqrt{2}}\right), 0 \right\rangle \right|$$

$$= \frac{1}{2} \checkmark$$