Vector Calculus
11.6 Directional Derivatives and the Gradient Vector

Def: Suppose f=f(x,y) is a real-valued function defined near (a,b).

If $\vec{u}=\langle u_1,u_2\rangle$ is a unit-length vector, then we define the directional derivative of f in the direction of $\vec{u}$ at (a,b) to be

$$D_{\vec{u}}f(a,b) = \lim_{t\to 0} \frac{f(a+u_1t, b+u_2t) - f(a,b)}{t}$$

assuming this limit exists.

If $\vec{v}$ is in R^2 with $\vec{v}\neq\vec{0}$, then we define the directional derivative of f in the direction of $\vec{v}$ at (a,b) to be $D_{\frac{\vec{v}}{|\vec{v}|}}f(a,b)$, assuming this limit exists.

If $f_x(a,b), f_y(a,b)$ exist, then we define the gradient of f at (a,b) to be

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle$$

We similar definitions for real-valued functions f=f(x,y,z).

Similar is true for f=f(x,y,z)

Fact: Suppose f=f(x,y) is differentiable at (a,b), and suppose $\vec{u}=\langle u_1,u_2\rangle$ is a unit-length vector.

$$\Rightarrow \quad D_{\vec{u}}f(a,b) = \vec{u}\cdot\nabla f(a,b) = u_1 f_x(a,b) + u_2 f_y(a,b)$$

Suppose ell is the line through (a,b) in the direction of $\vec{u}$.

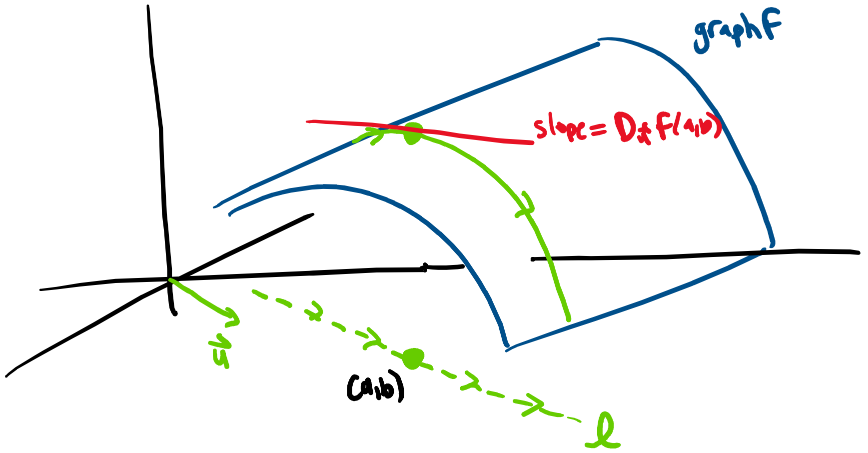
$$\Rightarrow \ell(t) = \langle a,b\rangle + t\vec{u} = \langle a+tu_1, b+tu_2\rangle$$

$D_{\vec{u}}f(a,b)$ measures the instantaneous rate of change of f at (a,b) as we move along ell with increasing values of t, at t=0.

$$\Rightarrow D_{\vec{u}}f(a,b) = \frac{d}{dt} f(\ell(t))\Big|_{t=0} = \frac{d}{dt} f(a+tu_1, b+tu_2)\Big|_{t=0}$$

$$\underset{\text{CHAIN RULE}}{=} \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2$$

$$= \vec{u} \bullet \nabla f(a,b)$$



Ex: Directional derivatives and gradients.

1. Compute the directional derivative of $f(x,y)=x^2+e^{xy}$ in the direction of $\vec{v}=\langle 2,1\rangle$ at (a,b)=(-1,1).

Sol: The directional derivative of f in the direction of $\vec{v}$ at (a,b) is

$$D_{\frac{\vec{v}}{|\vec{v}|}} f(-1,1) = \frac{\vec{v}}{|\vec{v}|} \bullet \nabla f(-1,1)$$

$$= \frac{\langle 2,1\rangle}{\sqrt{4+1}} \cdot \left\langle 2x + ye^{xy}, xe^{xy}\right\rangle\Big|_{(-1,1)}$$

$$= \frac{\langle 2,1\rangle}{\sqrt{5}} \cdot \langle -2+e^{-1}, -e^{-1} \rangle$$

$$= \boxed{\frac{1}{\sqrt{5}}\left(2(-2+e^{-1}) - e^{-1}\right)}$$

2. Compute the directional derivative of f(x,y,z)=xyz^2 in the direction of v=<1,1,2> at (a,b)=(2,3,5).

Sol: We compute

$$D_{\frac{\vec{v}}{|\vec{v}|}} f(2,3,5) = \frac{\vec{v}}{|\vec{v}|} \cdot \nabla f(2,3,5)$$

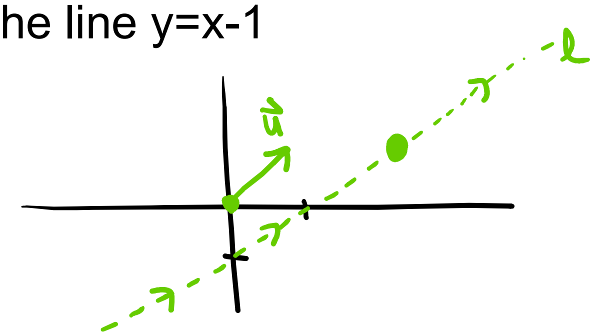
$$= \frac{\langle 1,1,2\rangle}{\sqrt{1+1+4}} \cdot \langle yz^2, xz^2, 2xyz \rangle \Big|_{(2,3,5)}$$

$$= \frac{\langle 1,1,2\rangle}{\sqrt{6}} \langle 3\cdot 25, 2\cdot 25, 2\cdot 2\cdot 3\cdot 5 \rangle$$

$$= \boxed{\frac{2\cdot 35}{\sqrt{6}} + \frac{2\cdot 25}{\sqrt{6}} + \frac{2\cdot 2\cdot 2\cdot 3\cdot 5}{\sqrt{6}}}$$

3. Comute the instantaneous rate of change of $f(x,y)=x^2+xy$ at (a,b)=(2,1) as we move along the line y=x-1 with increasing values of x.

Sol: Consider the line y=x-1



$|\vec{u}|=1$

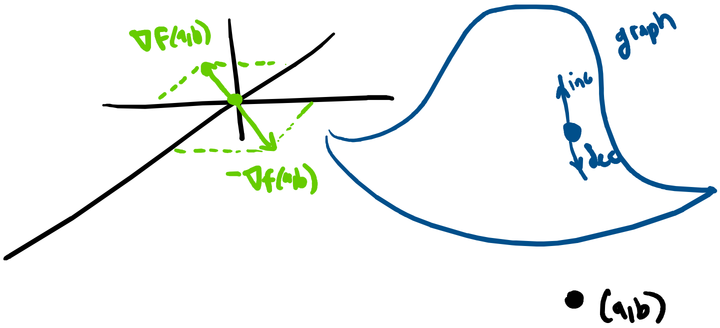
Consider $\vec{u}=\langle\sqrt{2}/2,\sqrt{2}/2\rangle$, then the instantaneous rate of change of f at (a,b)=(2,1) along the line ell is

$F(x,y)=x^2+xy$

$$
\begin{aligned}
D_{\vec{u}}f(2,1) &= \vec{u}\cdot\nabla f(2,1)\\
&= \left\langle \tfrac{\sqrt{2}}{2},\tfrac{\sqrt{2}}{2}\right\rangle\cdot\langle 2x+y, x\rangle\Big|_{(2,1)}\\
&= \left\langle \tfrac{\sqrt{2}}{2},\tfrac{\sqrt{2}}{2}\right\rangle\cdot\langle 5,1\rangle\\
&= \boxed{\frac{5\sqrt{2}}{2}+\frac{1}{2}}
\end{aligned}
$$

"nabla"

Fact: If f=f(x,y) is differentiable at (a,b), then ∇ f(a,b) points in the direction where the values of f increase the most, while $-\nabla$f(a,b) points in the direction where the values of f decrease the most.



The same is true for f=f(x,y,z).

Ex: Compute the direction in which the values of the given function f increase the most and decrease the most at the given point (a,b).

1. $f(x,y)=\sqrt{1-x^2-y^2}$ at (a,b)=(1/2,1/2)

Sol: The values of f increas the most in the direction of

$$\nabla f\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \left\langle \frac{\partial}{\partial x}(1-x^2-y^2)^{\frac{1}{2}}, \frac{\partial}{\partial y}(1-x^2-y^2)^{\frac{1}{2}} \right\rangle \Big|_{\left(\frac{1}{2},\frac{1}{2}\right)}$$

$$= \left\langle \tfrac{1}{2}(1-x^2-y^2)^{-\frac{1}{2}} \cdot -2x, \tfrac{1}{2}(1-x^2-y^2)^{-\frac{1}{2}} \cdot 2y \right\rangle \Big|_{\left(\frac{1}{2},\frac{1}{2}\right)}$$

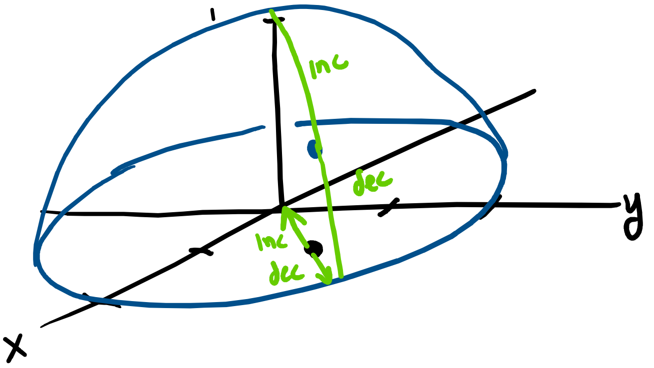
$$= \left\langle \frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}} \right\rangle \Big|_{\left(\frac{1}{2},\frac{1}{2}\right)}$$

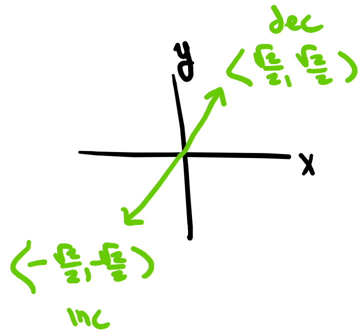
$$= \left\langle \frac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}-\frac{1}{4}}}, \frac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}-\frac{1}{4}}} \right\rangle = \underbrace{\boxed{\left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle}}_{\text{increase}}$$

The values of f decrease the most in the direction of

$$-\nabla f\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \underbrace{\boxed{\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle}}_{\text{decrease}}$$

Let's check this geometrically. The graph of f is the upper hemisphere.





2. $f(x,y,z)=x^2+y^2+z^2$ at $(a,b,c)=(0,2,1)$

Sol: The values of f increase the most in the direction of

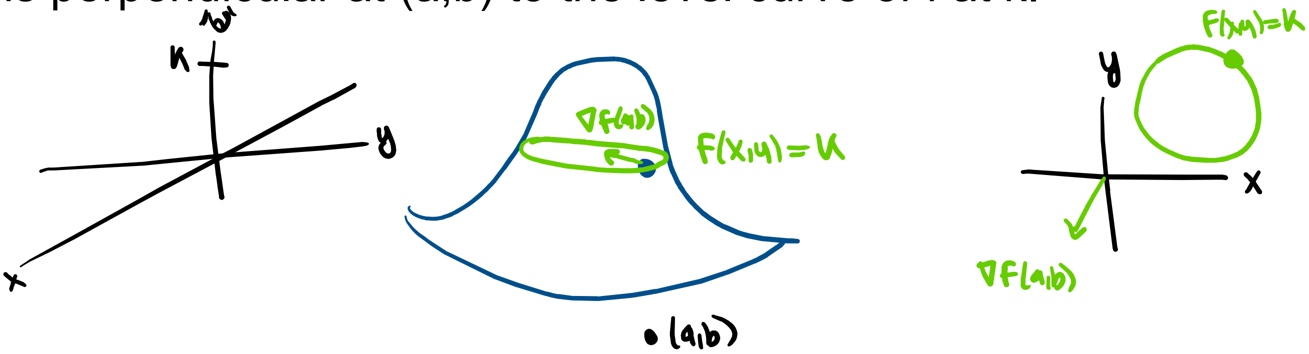
$$\nabla f(0,2,1) = \langle 2x, 2y, 2z \rangle \Big|_{(0,2,1)} = \underbrace{\boxed{\langle 0,4,1 \rangle}}_{\text{increase}}$$

and decrease the most in the direction of

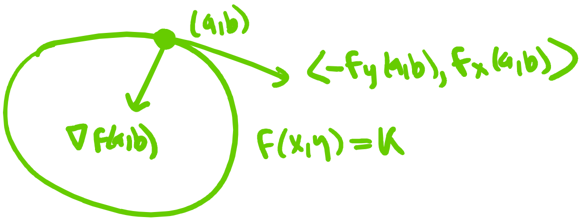
$$-\nabla f(0,2,1) = \underbrace{\boxed{\langle 0,-4,-1 \rangle}}_{\text{decrease}}$$

Fact: Suppose k is in R.

If f=f(x,y) is differentiable at (a,b) and f(a,b)=k, then ∇f(a,b) is perpendicular at (a,b) to the level curve of f at k.



$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$

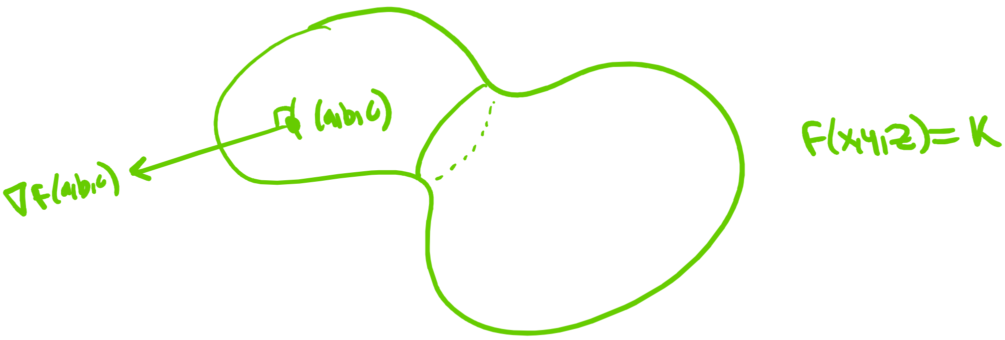


This means $\langle -f_y(a,b), f_x(a,b) \rangle$ is tangent at (a,b) to the level curve of f at k.

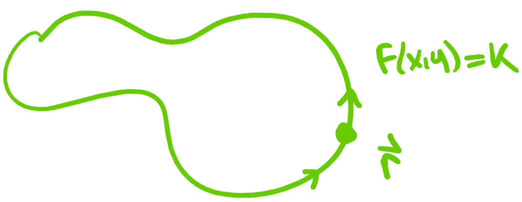
$$\Rightarrow \quad \langle -f_y(a,b), f_x(a,b) \rangle = \mathrm{Rot}_{\frac{\pi}{2}} \nabla f(a,b)$$

90° counterclockwise rotation

If f=f(x,y,z) is differentiable at (a,b) and f(a,b,c)=k, then ∇f(a,b,c) is perpendicular at (a,b,c) to the level surface of f at k.



Proof: Suppose that the level curve f(x,y)=k is given by the parametric plane curve $\vec{r}(t)=\langle x(t),y(t)\rangle$, with $\vec{r}(0)=\langle a,b\rangle$.



Suppose $\vec{r}$ is differentiable. Then

$$K = f(\vec{r}(t)) \quad \text{for all } t \text{ near } 0$$

$$\Rightarrow \quad K = f(x(t), y(t))$$

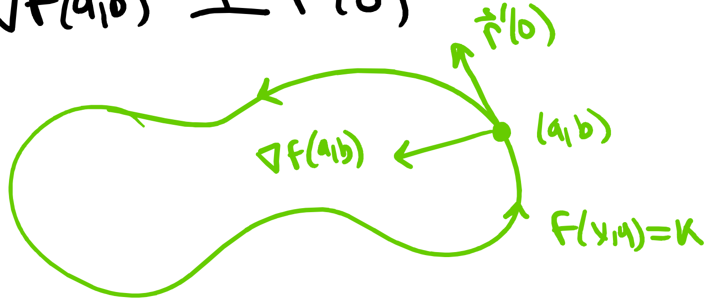
$$\Rightarrow \quad \frac{\partial}{\partial t} k \Big|_{t=0} = \frac{\partial}{\partial t} f(x(t), y(t)) \Big|_{t=0}$$

$$\Rightarrow \quad 0 = \frac{\partial f}{\partial x}\Big|_{(x(0),y(0))} x'(0) + \frac{\partial f}{\partial y}\Big|_{(x(0),y(0))} y'(0)$$

$$\Rightarrow \quad 0 = \underbrace{\nabla f(x(0), y(0))}_{\vec{r}(0) = \langle a,b \rangle} \bullet \underbrace{\langle x'(0), y'(0) \rangle}_{\vec{r}\,'(0)}$$

$$\Rightarrow \quad 0 = \nabla f(a,b) \bullet \vec{r}\,'(0)$$

$$\Rightarrow \quad \nabla f(a,b) \perp \vec{r}\,'(0)$$



Ex:

1. For $f(x,y)=\sqrt{1-x^2-y^2}$, verify that $\nabla f(1/2,1/2)$ is perpendicular to the level curve of f at k=f(1/2,1/2).

Sol: We already compute that

$$\nabla f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \left\langle -\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2} \right\rangle$$

The level curve of f at k=f(1/2,1/2) is the curve given by the equation

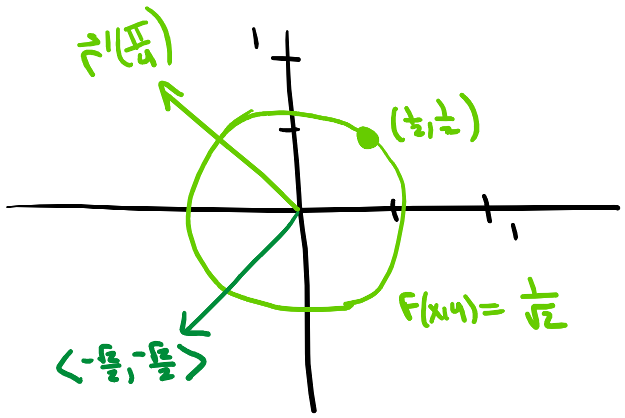
$$f(x,y) = f\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$\sqrt{1-x^2-y^2} = \sqrt{1-\tfrac{1}{4}-\tfrac{1}{4}} = \tfrac{1}{\sqrt{2}}$$

$$\Rightarrow \quad 1-x^2-y^2=\frac{1}{2}$$

$$\Rightarrow \quad x^2+y^2=\frac{1}{2}$$

circle of radius $\frac{1}{\sqrt{2}}$



√

Note that this circle is given by the parametric plane curve

$$\vec{r}(t)=\left\langle \frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\sin t \right\rangle$$

$$\vec{r}(t)=\vec{r}\left(\frac{\pi}{4}\right)=\left\langle \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{2}, \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{2} \right\rangle=\left\langle \frac{1}{2},\frac{1}{2}\right\rangle$$

$$\Rightarrow \quad \vec{r}'\left(\frac{\pi}{4}\right)=\left\langle -\frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\cos t \right\rangle \Big|_{t=\frac{\pi}{4}}$$

$$=\left\langle -\frac{1}{2},\frac{1}{2}\right\rangle \perp \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$$

2. Find a vector which is perpendicular to the surface $z=1-x^2-y^2$ at $(a,b,c)=(1/2,1/2,1/2)$.

$\Downarrow$
level surface
$\Downarrow$
$f(x,y,z)=k$

Sol: Consider $f(x,y,z)=x^2+y^2+z$, then the surface $z=1-x^2-y^2$ is the level surface of f at k=1.

$$f(x,y,z)=1 \Rightarrow x^2+y^2+z=1 \Rightarrow z=1-x^2-y^2 \ ✓$$

Since $f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = \frac{1}{4}+\frac{1}{4}+\frac{1}{2} = 1$

then a vector perpendicular at (a,b,c) to the level surface of f at k is given by

$$\nabla f\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) = \langle 2x, 2y, 1\rangle \Big|_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = \boxed{\langle 1,1,1\rangle}$$

Check

$z = 1 - x^2 - y^2$

elliptic paraboloid