Vector Calculus
Review
Practice Problems II

1) Let $\vec{r}(t)=\langle \frac{t^3}{3}, \frac{2t^{9/2}}{9} \rangle$ for $t \geq 0$.

a) Compute the tangent vector and tangent line of $\vec{r}$ at $t=1$.

Sol: We compute

$$\vec{r}\,'(1) = \langle t^2, t^{7/2} \rangle \Big|_{t=1} = \boxed{\langle 1,1 \rangle}$$

and the tangent line of $\vec{r}$ at t=1 is the line through $\vec{r}(1)$ in the direction of $\vec{r}\,'(1)$.

$$\Rightarrow \boxed{\ell(t) = \langle \tfrac{1}{3}, \tfrac{2}{9} \rangle + t \langle 1,1 \rangle \quad \text{for } t \in \mathbb{R}}$$

b) Compute the radius of the osculating circle of $\vec{r}$ at t=1.

Sol: If $\vec{r}\,''(t) \neq \vec{0}$, then we define kappa(t)= $\frac{1}{\text{radius}}$
and so

$$\text{radius} = \frac{1}{\kappa(t)}$$

Since $\vec{r}$ is a plane curve, then we need to set

$$\vec{r}(t) = \langle \frac{t^3}{3}, \frac{2t^{9/2}}{9}, 0 \rangle$$

and compute

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

We compute

$$\vec{r}'(1) = \langle t^2, t^{7/2}, 0\rangle\big|_{t=1} = \langle 1,1,0\rangle$$

$$\vec{r}''(1) = \langle 2t, \tfrac{7}{2}t^{\frac{5}{2}}, 0\rangle\big|_{t=1} = \langle 2, \tfrac{7}{2}, 0\rangle$$

$$\vec{r}'(1)\times\vec{r}''(1) = \langle 0, 0, \tfrac{7}{2}-2\rangle$$

This means

$$K(1) = \frac{|\langle 0,0,\frac{7}{2}-2\rangle|}{(\sqrt{2})^3} = \frac{\frac{7}{2}-2}{(\sqrt{2})^3}$$

We conclude that the radius of the osculating circle of $\vec{r}$ at t=1 is

$$= \frac{1}{K(1)} = \boxed{\frac{(\sqrt{2})^3}{\frac{7}{2}-2}}$$

c) Compute the arc length function s(t) of $\vec{r}$ over [1,2].

Sol: We compute

$$s(t) = \int_1^t |\vec{r}'(u)|\,du = \int_1^t |\langle u^2, u^{7/2}\rangle|\,du$$

$$= \int_1^t \sqrt{u^4 + (u^{7/2})^2}\,du$$

$1 \le u \le t \le 2$
$\Rightarrow u > 0$

$$= \int_1^t \sqrt{u^4 + u^7}\,du$$

...

2) Show that the limit of the given function f at (0,0) exists or does not exist in the extended sense. If the limit exists, give the value.

a) f(x,y)= $\dfrac{\sin(x^2+y^2)+x^4+2x^2y^2+y^4}{x^2+y^2}$

Sol: We compute

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)+x^4+2x^2y^2+y^4}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)+(x^2+y^2)^2}{x^2+y^2}$$

$$\underset{r=\sqrt{x^2+y^2}}{=} \lim_{r\to 0} \frac{\sin(r^2)+(r^2)^2}{r^2}$$

$$= \lim_{r\to 0}\left(\frac{\sin(r^2)}{r^2}+r^2\right)$$

$$\underset{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}{=} 1+0 = \boxed{1}$$

b) f(x,y)= $\dfrac{1}{x^4-x^6+2x^2y^2+y^4}$

Sol: We use the Squeeze Thm. Note that

$$x^4-x^6+2x^2y^2+y^4 \le x^4+2x^2y^2+y^4$$

We want to say that

$$\left.\frac{1}{(r^2)^2}\right|_{r=\sqrt{x^2+y^2}} = \frac{1}{x^4+2x^2y^2+y^4} \leq \frac{1}{x^4-x^6+2x^2y^2+y^4}$$

We need to make sure that

$$0 < x^4-x^6+2x^2y^2+y^4 \quad \text{for } \underset{\text{near } (0,0)}{(x,y) \neq (0,0)}$$

This is true for (x,y)=/(0,0) inside the unit circle, for $x^2+y^2<1$.
Note that (x,y)=/(0,0) implies that either

$$x \neq 0 \Rightarrow x^4-x^6+2x^2y^2+y^4 \geq x^4(1-x^2) \underset{\substack{x\neq 0 \\ x\neq \pm 1}}{>} 0$$

$|x|<1$ or

$$y \neq 0 \Rightarrow x^4(1-x^2)+2x^2y^2+y^4 \geq 0+0+y^4 > 0$$

Since

$$F(x,y) \geq \left.\frac{1}{(r^2)^2}\right|_{r=\sqrt{x^2+y^2}}$$

for all (x,y) near (0,0), but not at (0,0), and since

$$\lim_{r\to 0} \frac{1}{(r^2)^2} = \infty$$

then we conclude by the Squeeze Thm that

$$\lim_{(x,y)\to(0,0)} F(x,y) = \infty.$$

3) Let $f(x,y)=x^3+y^3$, and let $x(t)=e^t$ and $y(t)=\cos(t)$.

a) Compute the tangent plane of f at (2,1), and use the tangent plane to approximate the value of f(1.9,1.1).

Sol: We compute

$$f(2,1) = 8+1 = 9$$

$$f_x(2,1) = \frac{\partial}{\partial x} f(x,1) \Big|_{x=2} = \frac{\partial}{\partial x} x^3+1 \Big|_{x=2}$$

$$= 3x^2 \Big|_{x=2} = 12$$

$$f_y(2,1) = \frac{\partial}{\partial y} f(2,y) \Big|_{y=1} = \frac{\partial}{\partial y} 8+y^3 \Big|_{y=1}$$

$$= 3y^2 \Big|_{y=1} = 3$$

We conclude that the tangent plane of f at (2,1) is

$$z = f_x(2,1)(x-2) + f_y(2,1)(y-1) + f(2,1)$$

$$\Rightarrow \boxed{\underline{z =} \ 12(x-2)+3(y-1)+9}$$

Do not simply write "$12(x-2)+3(y-1)+9$"

We use this to approximate

$$\underset{(2,1)}{\underset{\approx}{f(1.9,1.1)}} \approx 12(x-2)+3(y-1)+9 \Big|_{(1.9,1.1)}$$

$$= \boxed{12(1.9-2)+3(1.1-1)+9}$$

b) Verify by computation that

$$\frac{\partial}{\partial t} F(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Sol: We need to compute

$$\frac{\partial}{\partial t} F(x(t), y(t)) = \frac{\partial}{\partial t}\left(x^3 + y^3 \Big|_{\substack{x=e^t \\ y=\cos t}} \right)$$

$$= \frac{\partial}{\partial t}\left((e^t)^3 + \cos^3 t \right)$$

$$= \frac{\partial}{\partial t}\left(e^{3t} + \cos^3 t \right)$$

and we need to compute

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(x^3 + y^3\right)\Big|_{\substack{x=e^t \\ y=\cos t}} \frac{\partial}{\partial t} e^t$$

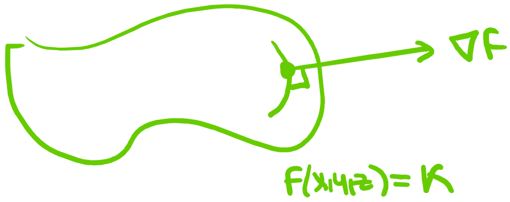
$$+ \frac{\partial}{\partial y}\left(x^3 + y^3\right)\Big|_{\substack{x=e^t \\ y=\cos t}} \frac{\partial}{\partial t} \cos t$$

4) Suppose $f(x,y,z)=e^{xyz}$.

a) Compute the direction in which the values of f increase the most and decrease the most at (2,1,3).

Sol: The values of f increase the most in the direction of the gradient, and decrease the most in the direction of

-gradient.



The values of f are *constant* in the direction perpendicular to the gradient.

We compute

$$\nabla f(2,1,3) = \langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle \Big|_{(2,1,3)}$$

$$= \boxed{\langle 3e^6, 6e^6, 2e^6 \rangle}$$

We conclude that the values of f increase the most in the direction of

$$\langle 3e^6, 6e^6, 2e^6 \rangle$$

and decrease the most in the direction of

$$\langle -3e^6, -6e^6, -2e^6 \rangle.$$

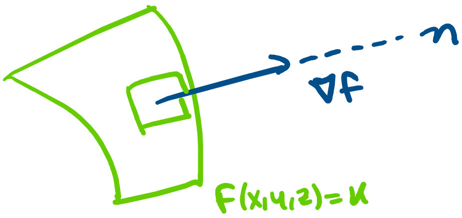
b) Compute the tangent plane and normal line at (2,1,3) of the level surface of f at $k=e^6$.

Sol: Since $f(2,1,3)=e^{2\cdot 1\cdot 3}=e^6$, then the tangent plane at (2,1,3) of the level surface of f at $k=e^6$ is

$$\boxed{3e^6(x-2)+6e^6(y-1)+2e^6(z-3)=0}$$

We also conclude that the normal line at (2,1,3) of the level surface of f at $k=e^6$ is

$$\boxed{n(t) = \langle 2,1,3\rangle + \langle 3e^6, 6e^6, 2e^6\rangle \quad \text{for } t\in\mathbb{R}}$$



5) Find the absolute extremum points and values of the given function f over the level surface of the given function g at the given k in R.

a) $f(x,y,z)=2x+2y+3z$ with $g(x,y,z)=x^2+y^2+z^2$ at $k=1$.

Sol: We must consider the equations

$\nabla f = \lambda \nabla g$

$g(x,y,z)=1$

① $2 = 2\lambda x$

② $2 = 2\lambda y$

③ $3 = 2\lambda z$

④ $x^2+y^2+z^2=1$

$$\xRightarrow{\lambda\neq 0} \quad x = \frac{1}{\lambda}, \quad y = \frac{1}{\lambda}, \quad z = \frac{3}{2\lambda}$$

$$\Rightarrow \left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 1$$

$$\Rightarrow \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} = 1$$

$$\Rightarrow \lambda^2 = 1+1+\frac{9}{4} = \frac{17}{4}$$

$$\Rightarrow \lambda = \pm\frac{\sqrt{17}}{2}$$

$$\Rightarrow \boxed{\left(\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right) \quad \left(-\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}, -\frac{3}{\sqrt{17}}\right)}$$

We must compute

$$f\left(\frac{2}{\sqrt{17}},\frac{2}{\sqrt{17}},\frac{3}{\sqrt{17}}\right)= 2\left(\frac{2}{\sqrt{17}}\right)+2\left(\frac{2}{\sqrt{17}}\right)+3\left(\frac{3}{\sqrt{17}}\right)$$

$$= \frac{4+4+9}{\sqrt{17}} = \sqrt{17}$$

$$f\left(-\frac{2}{\sqrt{17}},-\frac{2}{\sqrt{17}},-\frac{3}{\sqrt{17}}\right) = -\sqrt{17}$$

We conclude that the $\left(\frac{2}{\sqrt{17}},\frac{2}{\sqrt{17}},\frac{3}{\sqrt{17}}\right)$ is the absolute maximum point of f over the level surface of g at k=1, with absolute maximum value $=\sqrt{17}$. We also conclude that $\left(-\frac{2}{\sqrt{17}},-\frac{2}{\sqrt{17}},-\frac{3}{\sqrt{17}}\right)$ is the absolute minimum point of f over the level surface of g at k=1, with absolute minimum value $=-\sqrt{17}$.

b) $f(x,y,z)=x^2+y+z$ with $g(x,y,z)=x^2+y^2+z^2$ at k=1

$\nabla f = \lambda \nabla g$

$g(x,y,z)=k$

Sol: We must compute

① $2x = 2\lambda x$

② $1 = 2\lambda y$

③ $1 = 2\lambda z$

④ $x^2+y^2+z^2=1$

$\Longrightarrow \boxed{x=0}$

② $1=2\lambda y$

③ $1=2\lambda z$

④ $y^2+z^2=1$

$\lambda \neq 0$ $\quad y=\frac{1}{2\lambda}=z$

④ $\Downarrow$

$y=z=\pm\frac{\sqrt{2}}{2}$

$\Downarrow$

$$\boxed{\begin{array}{c}\left(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\\ \left(0,-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)\end{array}}$$

or

$\boxed{x\neq 0}$

① $\Downarrow$

$\lambda=1$

②③ $\Downarrow$

$y=z=\frac{1}{2}$

④ $\Downarrow$

$x^2+\frac{1}{4}+\frac{1}{4}=1$

$\Downarrow$

$x^2=1-\frac{1}{2}=\frac{1}{2}$

$x=\pm\frac{\sqrt{2}}{2}$

Since f(x,y,z)=x^2+y+z, we compute

$\boxed{\left(\pm\frac{\sqrt{2}}{2},\frac{1}{2},\frac{1}{2}\right)}$

$$f\left(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=0+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\sqrt{2}=1.414$$

$$f\left(0,-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)=0-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}=-\sqrt{2}$$

$$f\left(\pm\frac{\sqrt{2}}{2},\frac{1}{2},\frac{1}{2}\right)=\left(\pm\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}+\frac{1}{2}=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}$$

$* \ \sqrt{2}<\frac{3}{2}$

$\Leftrightarrow 2<\frac{9}{4}$ ✓

$\Leftrightarrow 8<9$

We conclude that $\left(\pm\frac{\sqrt{2}}{2},\frac{1}{2},\frac{1}{2}\right)$ are absolute maximum points of f over the level surface of g at k=1, with absolute maximum value =3/2. We also conclude that $\left(0,-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ is the absolute minimum point of f over the level surface of g at k=1, with absolute minimum value =-$\sqrt{2}$.

There is another way to do this. Consider

$$f(x,y,z)=x^2+y+z \quad \text{over} \quad x^2+y^2+z^2=1$$

$$\Rightarrow x^2=1-y^2-z^2$$

We can instead find the absolute extremum points of

$$h(y,z)=1-y^2-z^2+y+z \quad \text{over} \quad \underbrace{y^2+z^2\le 1}_{\Omega}$$

First, we compute

$$\vec{0}=\nabla h=\langle -2y+1,-2z+1\rangle \Rightarrow (y,z)=\left(\frac{1}{2},\frac{1}{2}\right)$$

$$x=\pm\sqrt{1-y^2-z^2} \Rightarrow \left(\pm\sqrt{1-\frac{1}{4}-\frac{1}{4}},\frac{1}{2},\frac{1}{2}\right)$$

$$\Rightarrow \left(\pm\frac{\sqrt{2}}{2},\frac{1}{2},\frac{1}{2}\right)$$

Next, we must find the absolute extremum points of h over the unit circle $y^2+z^2=1$.

$$h(y,z) = 1-y^2-z^2+y+z \qquad \text{over} \quad y^2+z^2=1$$

$$= 1-1+y+z = y+z$$

Do this using Lagrange multipliers, the answer is

$$y=z \Rightarrow \left(\tfrac{\sqrt{2}}{2},\tfrac{\sqrt{2}}{2}\right) \text{ and } \left(-\tfrac{\sqrt{2}}{2},-\tfrac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \left(0,\tfrac{\sqrt{2}}{2},\tfrac{\sqrt{2}}{2}\right) \text{ and } \left(0,-\tfrac{\sqrt{2}}{2},-\tfrac{\sqrt{2}}{2}\right)$$