VECTOR CALCULUS, Week 15

Review

Please read the following instructions for the final.

Name: ________________________ ID: ________________________

- You must show your work.
- If you are asked to compute a limit, derivative, or integral, then you must compute so that you no longer have any limit, derivative, or integral signs.
- You are **not** required to simplify you answers.
  - In particular, you are not required to simplify computations involving basic arithmetic:

addition	subtraction	multiplication
division	roots	powers

    For example, writing $x - 2x$ or $4/2$ as an answer is okay.
  - You are required to evaluate sine and cosine at basic angles.
  - If you are not sure, then ask me before you waste any time simplifying.

Topics

$(*)$ means I *directly* will test you on this.

- **10.7 Vector Functions and Space Curves**
  - Def: parametric vector-valued function, parameter, real-valued, components, image.
  - $(*)$ Def: differentiable, tangent vector, speed, tangent line, second derivative.
  - Basic Derivative Rules for parametric vector-valued functions.
  - Def: reparameterization.
  - Fact: reparameterizations have the same image, just different speed.
  - Def: integral of a parameteric vector-valued function.
- **10.8 Arc Length and Curvature**
  - $(*)$ Def: regular/smooth, osculating circle, curvature function.
  - Fact: large curvature means small osculating circle, small curvature means large osculating circle.
  - $(*)$ Fact: $\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$ for a space curve, for a plane curve embed $\vec{r}$ into $\mathbf{R}^3$ by setting $\vec{r}(t) =< x(t), y(t), 0 >$.
  - $(*)$ Fact: Arc length of a parameteric vector-valued function.
  - $(*)$ Def: Arc length function.
  - Fact: if $|\vec{r}\,'(t)| = 1$, then the arc length is $L = b - a$ and $\kappa(t) = |\vec{r}\,''(t)|$.
  - Def: parameterization by arc length/unit speed reparameterization.
- **10.9 Motion in Space: Velocity and Acceleration**
  - Def: position vector, velocity vector, speed, acceleration, unit tangent vector, unit normal vector, binormal vector, TNB/orthonormal frame, orthonormal.
- **11.1 Functions of Several Variables**
  - Def: real-valued function of two and three variables, graph, level curve of $f = f(x, y)$ at k, level surface of $f = f(x, y, z)$ at k.

- **11.2 Limits and Continuity**
  - Def: near (a, b) or (a, b, c), limits of two- and three-variable functions, continuity.
  - (∗) Fact: showing a limit does not exist using different paths.
  - (∗) Fact: showing a limit exists using single-variable limits.
  - (∗) Squeeze Theorem
  - Fact: Basic limit laws. Be careful about $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)}$ when $\lim_{(x,y)\to(a,b)} g(x, y) = 0$.
- **11.3 Partial Derivatives**
  - Def: partial derivatives, second partial derivatives, mixed partial derivatives.
  - (∗) Fact: computing partial derivatives.
  - (∗) Thm: for functions with continuous second partial derivatives, the mixed partial derivatives are equal.
- **11.4 Tangent Planes and Linear Approximations**
  - (∗) Def: differentiable, tangent plane/linear approximation.
  - (∗) Fact: if f is differentiable then the partial derivatives exist which give the tangent plane; the tangent plane of f at (a, b) is a good approximation near (a, b); if the partial derivatives are continuous then f is differentiable.
- **11.5 The Chain Rule**
  - (∗) Chain Rule

- **11.6 Directional Derivatives and the Gradient Vector**
  - Def: directional derivative in the direction of a unit vector, directional derivative in the direction of a general vector, gradient.
  - (∗) Fact: computing the directional derivative using the gradient, the directional derivative measures the change of f as we move along the line given by the vector.
  - (∗) Fact: ∇f points in the direction of maximum increase, $-\nabla f$ points in the direction of maximum decrease.
  - Fact: ∇ is perpendicular to the level curve/surface of f at k. For $f = f(x, y)$, the vector $< -f_y(a, b), f_x(a, b) >$ is tangent to the level curve of f at k.
  - (∗) Def: tangent and normal line at (a, b) of the level curve of $f = f(x, y)$ at k, tangent plane and normal line at (a, b, c) of the level surface of $f = f(x, y, z)$ at k.
- **11.7 Maximum and Minimum Values**
  - Def: local and absolute minimum, maximum, and extremum points and values, critical point, open set, closed set, boundary, interior, bounded set.
  - (∗) Thm: computing absolute extremum points and values of a continuous function over a closed and bounded set.
  - Def: discriminant of a function, saddle point.
  - (∗) Second Derivative Test
- **11.8 Lagrange Multipliers**
  - Def: local and absolute maximum, minimum, and extremum points and values of a function over a level curve or surface.
  - (∗) Thm: Lagrange multipliers.
  - (∗) Fact: consider the equations where the same variable appears on both sides.
  - Thm: Two constraint Lagrange multiplies.

Correction

I stated the following fact in class:

Fact: Suppose $f = f(x, y)$ is a real-valued function defined near (a, b). If we define $g(x) = f(x, b)$, then $f_x(a, b) = g'(a)$.

This is true by definition,

$$f_x(a,b) = \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x - a} = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a).$$

This means that to compute $f_x(a, b)$, you can set $y = b$ into $f(x, y)$ and then take the derivative with respect to x. For example, if $f(x, y) = x^{1/3}y$, then

$$f_x(0,0) = \frac{d}{dx}(x^{1/3} \cdot 0)\Big|_{x=0} = \frac{d}{dx} 0\Big|_{x=0} = 0.$$

However, I stated in class that you can also first take the partial derivative of f with respect to x, where you consider y to be a constant, and then input $x = a, y = b$. This is not true! For example, if $f(x, y) = x^{1/3}y$, then

$$\frac{\partial}{\partial x}(x^{1/3}y)\Big|_{(0,0)} = \frac{y}{3x^{2/3}}\Big|_{(0,0)} = \text{ DNE!}$$

Instead, the following fact is true:

Fact: Suppose $f = f(x, y)$ is a real-valued function defined near (a, b). If $f_x(x, y)$ exists near (a, b) and is continuous at (a, b), then

$$f_x(a,b) = \frac{\partial}{\partial x} f(x,y)\Big|_{(a,b)},$$

where the left-hand side means that you take the derivative with respect to x, where you consider y to be a constant, and then input $x = a, y = b$.

This exactly means that we can compute examples such as for $f(x, y) = x^2y$,

$$f_x(2,3) = \frac{\partial}{\partial x} x^2 y\Big|_{(2,3)} = 2xy\Big|_{(2,3)} = 2 \cdot 2 \cdot 3.$$

Practice Problems I

1) Let $\vec{r}(t) =< \frac{t^2}{2}, \frac{t^2}{2}, \frac{t^3}{3} >$.

(a) Compute the unit tangent vector and speed of $\vec{r}$ at $t = 1$.

(b) Compute the curvature $\kappa(1)$.

(c) Compute the arc length function $s(t)$ of $\vec{r}$ over $[0, 1]$.

2) Show that the limit of the given function f at $(0, 0)$ exists or does not exist in the extended sense. If the limit exists, give the value.

(a) $f(x, y) = \frac{x^2 - y}{\sin(x^2+y^2)}$

(b) $f(x, y) = \frac{x^4+x^5+x^2y^2}{2x^2+y^2}$

3) Let $f(x, y) = x^2 + y + e^{xy}$, and let $x(t) = 2t$ and $y(t) = t^2$.

(a) Verify by computation that $f_{xy} = f_{yx}$.

(b) Verify by computation that $\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

4) Suppose $f(x, y, z) = x^2 + yz$.

(a) Compute the directional derivative of f in the direction of $\vec{v} =< 2, 1, 3 >$ at $(1, -1, 2)$.

(b) Compute the tangent plane and normal line at $(1, -1, 2)$ of the level surface of f at $k = -1$.

5) Local and absolute extremum points and values.

(a) Find the absolute extremum points and values of $f(x, y) = x^2 + y^2 + y$ over $\Omega = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$.

(b) Find all the critical points of $f(x, y) = x^2 + y^4 + 2xy$, and determine whether they are local minimum, local maximum, or saddle points.

Practice Problems I Answers

1)

(a) speed is $\sqrt{3}$ and unit tangent vector is $< 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} >$.

(b) $\frac{\sqrt{2}}{3^{3/2}}$

(c) $s(t) = \frac{(2+t^2)^{3/2}}{3} - \frac{(2+0)^{3/2}}{3}$.

2)

(a) DNE

(b) exists and is zero.

3)

(a) Be sure to compute f_{xy} and f_{yx} correctly!

(b) Be careful!

4)

(a) $3/\sqrt{14}$

(b) The tangent plane is

$$2(x-1) + 2(y-(-1)) + (-1)(z-2) = 0$$

and the normal line is

$$\ell(t) =< 1, -1, 2 > +t < 2, 2, -1 > \text{ for } t \in \mathbf{R}.$$

5)

(a) The absolute maximum point of f over Ω is $(0, 1)$, with absolute maximum value $= 2$. The absolute minimum point of f over Ω is $(0, -1/2)$, with absolute minimum value $= -1/4$.

(b) $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ are local minimum points, $(0, 0)$ is a saddle point.

Practice Problems II

1) Let $\vec{r}(t) =< \frac{t^3}{3}, \frac{2t^{9/2}}{9} >$ for $t \geq 0$.

(a) Compute the tangent vector and tangent line of $\vec{r}$ at $t = 1$.

(b) Compute the radius of the osculating circle of $\vec{r}$ at $t = 1$.

(c) Compute the arc length function $s(t)$ of $\vec{r}$ over $[1, 2]$.

2) Show that the limit of the given function f at $(0, 0)$ exists or does not exist in the extended sense. If the limit exists, give the value.

(a) $f(x, y) = \frac{\sin(x^2+y^2)+x^4+2x^2y^2+y^4}{x^2+y^2}$

(b) $f(x, y) = \frac{1}{x^4-x^6+2x^2y^2+y^4}$

3) Let $f(x, y) = x^3 + y^3$, and let $x(t) = e^t$ and $y(t) = \cos t$.

(a) Compute the tangent plane of f at $(2, 1)$, and use the tangent plane to approximate the value of $f(1.9, 1.1)$.

(b) Verify by computation that $\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

4) Suppose $f(x, y, z) = e^{xyz}$.

(a) Compute the direction in which the values of f increase the most and decrease the most at $(2, 1, 3)$.

(b) Compute the tangent plane and normal line at $(2, 1, 3)$ of the level surface of f at $k = e^6$.

5) Find the absolute extremum points and values of the given function f over the level surface of the given function g at the given $k \in \mathbf{R}$.

(a) $f(x, y, z) = 2x + 2y + 3z$ with $g(x, y, z) = x^2 + y^2 + z^2$ at $k = 1$

(b) $f(x, y, z) = x^2 + y + z$ with $g(x, y, z) = x^2 + y^2 + z^2$ at $k = 1$

Practice Problems II Answers

1)

(a) The tangent vector is $< 1, 1 >$, the tangent line is

$$\ell(t) = < 1/3, 2/9 > + t < 1, 1 > \text{ for } t \in \mathbf{R}.$$

(b) The radius of the osculating circle is $\frac{(1+1)^{3/2}}{\frac{7}{2}-2}$.

(c) $s(t) = \frac{(1+t^3)^{3/2}}{9/2} - \frac{(1+1)^{3/2}}{9/2}$ for $t \in [1, 2]$.

2)

(a) Exists with value $= 1$.

(b) Exists with extended value $= \infty$.

3)

(a) The tangent plane is $z = 12(x-2) + 3(y-1) + 9$, and $f(1.9, 1.1) \approx 12(1.9-2) + 3(1.1-1) + 9$.

(b) Be careful!

4)

(a) The direction of maximum increase is $< 3e^6, 6e^6, 2e^6 >$ and the direction of maximum decrease is $< -3e^6, -6e^6, -2e^6 >$.

(b) The tangent plane is given by

$$3e^6(x-2) + 6e^6(y-1) + 2e^6(z-3) = 0,$$

the normal line is given by

$$\ell(t) = < 2, 1, 3 > + t < 3e^6, 6e^6, 2e^6 > \text{ for } t \in \mathbf{R}.$$

5)

(a) The absolute maximum point is $(\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}})$ with absolute maximum value $\sqrt{17}$, the absolute minimum point is $(\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}} \frac{3}{\sqrt{17}})$ which absolute minimum value $-\sqrt{17}$.

(b) The absolute maximum points are $(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2})$ with absolute maximum value $3/2$, the absolute minimum point is $(0, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ with absolute minimum value $-\sqrt{2}$.