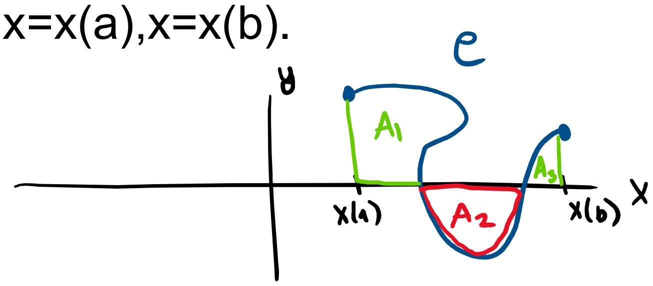
Vector Calculus
9.2 Calculus with Parametric Curves

Def: Suppose $C(t)=(x(t),y(t))$ for $a\leqslant t\leqslant b$ is a parametric plane curve, and suppose x,y are continuous over [a,b]. We define the area under C to be the signed area of the region bounded by the image of C, the x-axis, and the vertical lines $x=x(a), x=x(b)$.



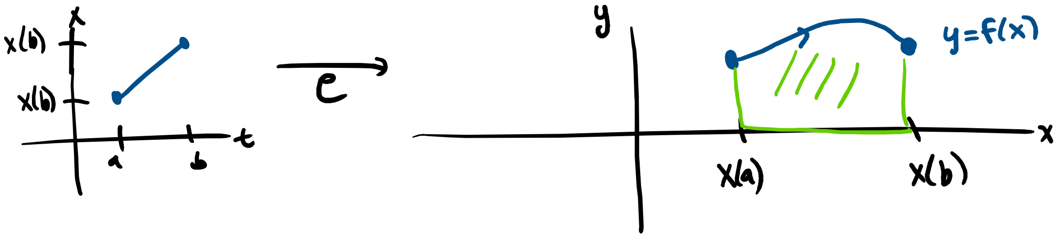
The area under $C=A_1-A_2+A_3$.

Fact: Suppose $C(t)=(x(t),y(t))$ for $a\leqslant t\leqslant b$ is a parametric plane curve, and suppose x is continuously differentiable over [a,b], and suppose y is continuous over [a,b]. Suppose A is the area under C.

If x is increasing over [a,b], then the image of C is the graph of a function $y=f(x)$ defined for x in $[x(a),x(b)]$, and so

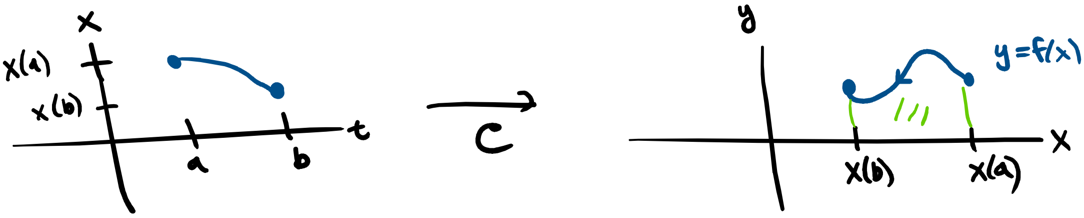
$$A=\int_{x(a)}^{x(b)} f(x)\,dx \underset{\substack{x=x(t)\\ dx=x'(t)dx}}{=} \int_a^b f(x(t))\,x'(t)\,dt$$

$$\Rightarrow \boxed{A=\int_a^b y(t)\,x'(t)\,dt}$$



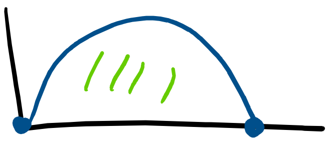
If x is decreasing over [a,b], then

$$A = \int_{x(b)}^{x(a)} f(x)\,dx \underset{\substack{x=x(t)\\ dx=x'(t)}}{=} \int_b^a y(t)\,x'(t)\,dt$$



Ex: Give an integral for the area under one arc of the cycloid C(t)=(t-sin(t),1-cos(t)).

Sol: We must compute the area under C for 0≤t≤2pi.



x(t)=t-sin(t) is increasing over [0,2pi]; we check by computing

$$x'(t) = 1-\cos t > 0 \quad \text{for } t \in (0, 2\pi)$$

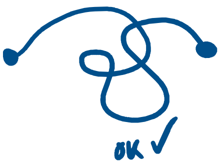
We conclude that the area A under C is given by

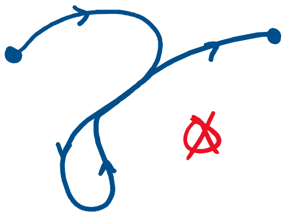
$$A = \int_0^{2\pi} y(t)\,x'(t)\,dt = \boxed{\int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt} > 0$$

Fact: Suppose C(t)=(x(t),y(t)) for a≤t≤b is a parametric plane curve with x,y continuously differentiable over [a,b]. If C does not self-intersect, then the arc length L of the image of C is given by

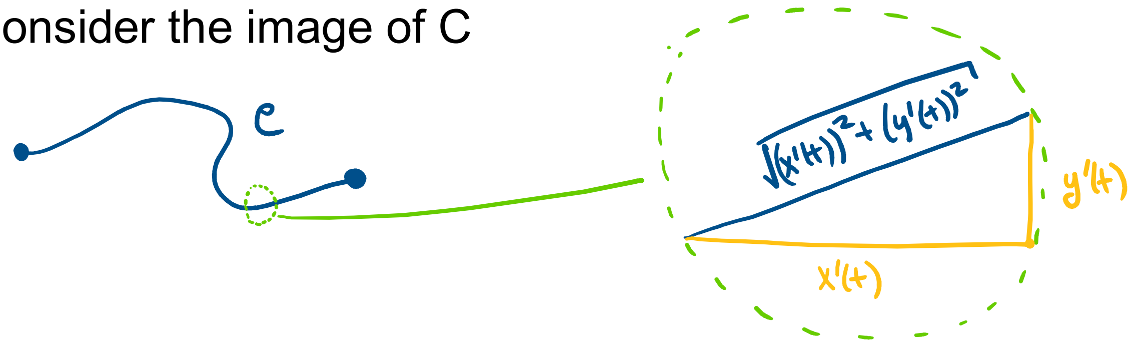
$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

This formula also works if C only has isolated self-intersections.

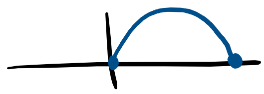




Proof: Use the Pythagorean Thm and approximation. Consider the image of C



Ex: Give an integral for the arc length L of one arc of the cycloid C(t)=(t-sin(t),1-cos(t)).



Sol: We consider the cycloid C(t) for 0≤t≤2pi.
We conclude that

$$L = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \boxed{\int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt}$$

Fact: Suppose f is continuously differentiable over [a,b].
Consider the paremetric plane curve given by

$$C(t) = (t, f(t)) \quad \text{for} \quad a \leq t \leq b.$$

The arc length L of the image of C is given by

$$L = \int_a^b \sqrt{(1)^2 + (f'(t))^2}\, dt$$

We conclude that the arc length L of the curve y=f(x) for a≤x≤b is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Ex: Let $f(x)=\sqrt{1-x^2}$. Give an integral for the arc length L of the curve y=f(x) for -1≤x≤1.

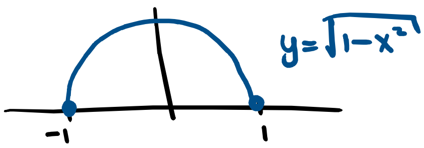
Sol: We compute

$$L = \int_{-1}^{1} \sqrt{1 + \left(\frac{d}{dx}(1-x^2)^{\frac{1}{2}}\right)^2}\, dx$$

← −1 point if you leave it like this

$$= \boxed{\int_{-1}^{1} \sqrt{1 + \left(\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right)^2}\, dx}$$

Note that the graph of f is the upper half unit circle.



This means that L=pi.
We can check that

$$\int_{-1}^{1} \sqrt{1+\left(\tfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right)^2}\, dx$$

$$= \int_{-1}^{1} \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\, dx$$

$$= \int_{-1}^{1} \sqrt{1+\frac{x^2}{1-x^2}}\, dx$$

$$= \int_{-1}^{1} \sqrt{\frac{1-x^2+x^2}{1-x^2}}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$$

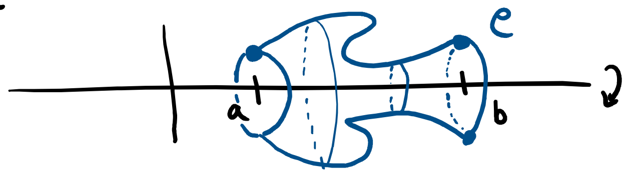
← improper integral

$$\underset{\text{week 3}}{=} \arcsin x \Big|_{x=-1}^{1} = \arcsin 1 - \arcsin -1$$

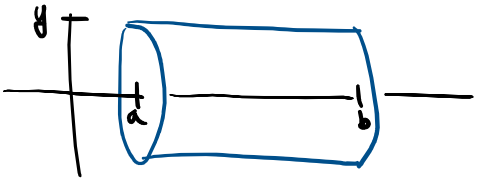
$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \ \checkmark$$

Fact: Suppose C(t)=(x(t),y(t)) for a≤t≤b is a parametric plane curve with x,y continuously differentiable over [a,b]. Suppose y(t)>0 for a≤t≤b, and suppose C does not self-intersect. The surface area S of the surface of revolution formed by rotating the image of C around the x-axis is

$$S = \int_a^b 2\pi y(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$



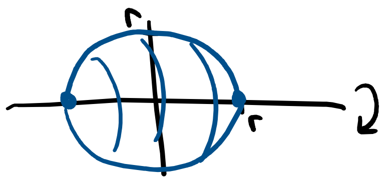
Proof: The surface area of a cylinder is



$$\text{circumference} \bullet \text{height} = 2\pi y\,(b-a)$$

$$S = \int_a^b \underbrace{2\pi y(t)}_{\text{"circumference"}} \underbrace{\sqrt{(x'(t))^2 + (y'(t))^2}}_{\substack{\text{arc length} \\ \| \\ \text{"height"}}}\, dt$$

Ex: Give an integral for the surface area S of the sphere of radius r>0, which is the surface of revolution formed by rotating the image of the parametric plane curve C(t)=(r*cos(t),r*sin(t)) for 0≤t≤pi around the x-axis.



Sol: S is given by

$$S = \int_0^{\pi} 2\pi y(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \boxed{\int_0^{\pi} 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt}$$

$$= \int_0^{\pi} 2\pi r \sin t \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}\, dt$$

$$= \int_0^{\pi} 2\pi r \sin t \sqrt{r^2}\, dt$$

$$= \int_0^{\pi} 2\pi r^2 \sin t\, dt$$

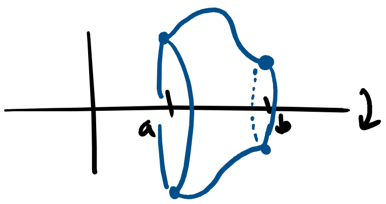
$$= 2\pi r^2 (-\cos t) \Big|_{t=0}^{\pi}$$

$$= 2\pi r^2 \left(-\cos\pi - (-\cos 0)\right)$$

$$= 2\pi r^2 (1+1) = \boxed{4\pi r^2}$$

Fact: Suppose f is continuously differentiable over [a,b] with f(x)>0 for x in [a,b]. The surface area S of the surface of revolution formed by rotating the curve y=f(x) for a≤x≤b around the x-axis is

$$S = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx$$



Proof: Consider C(t)=(t,f(t)) for a≤t≤b.

Ex: Let f(x)=$\sqrt{1-x^2}$. Give an integral for the surface area S of the surface of revolution formed by rotating the curve y=f(x) for -1≤x≤1 around the x-axis.

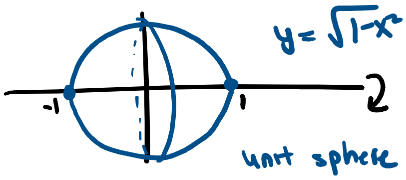
Sol: S is given by

$$S = \int_{-1}^{1} 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$= \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+\left(\frac{d}{dx}(1-x^2)^{\frac{1}{2}}\right)^2}\,dx$$

$$= \boxed{\int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+\left(\tfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right)^2}\,dx}$$

Note that the surface of revolution is the unit sphere.



$S = 4\pi \cdot 1^2 = 4\pi$

Using our computations from the arc length L, we compute

$$S = \int_{-1}^{1} 2\pi\sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

Improper Integral

$$= \int_{-1}^{1} 2\pi\,dx = 2\pi(1-(-1))$$

$$= 4\pi! \checkmark$$