VECTOR CALCULUS, Week 5

9.2 Calculus with Parametric Curves

Fact: If $x'(a) \neq 0$, then $x = x(t)$ is either (strictly) increasing or (strictly) decreasing near a.

Fact: Suppose $C(t) = (x(t), y(t))$ is a parametric plane curve defined for t near a.

- If $x = x(t)$ is increasing or decreasing near a, then there is a function $y = f(x)$ defined near $x(a)$ so that the image of C near $t = a$ is the graph of f.

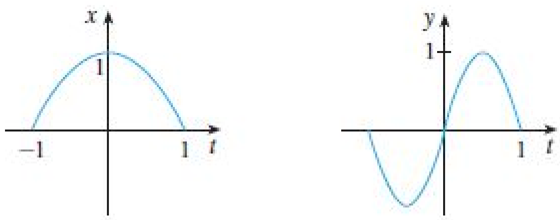
  If x, y are differentiable at a with $x'(a) \neq 0$, then

  $$f'(x(a)) = \frac{y'(a)}{x'(a)}.$$

  In other words, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ if $\frac{dx}{dt} \neq 0$.

- If $y = y(t)$ is increasing or decreasing near a, then there is a function $x = g(y)$ defined near $y(a)$ so that the image of C near $t = a$ is the graph of the function $x = g(y)$.

Proof: Let's consider the example



To find $f'(x(a))$, use $y(t) = f(x(t))$.

Ex: Consider the parametric plane curve $C(t) = (t^2, t^3 - 3t)$.

1. Show that the image of C near $t = 1$ is the graph of a function $y = f(x)$ defined near $x = 1$.
2. Show that the image of C near $t = 0$ is the graph of a function $x = g(y)$ defined near $y = 0$.

Def: Suppose $C(t) = (x(t), y(t))$ is a parametric plane curve defined near a with x, y continuously differentiable near a. We define the **tangent line of C at $t = a$** as follows.

- If $x'(a) \neq 0$, then we say

$$y = \frac{y'(a)}{x'(a)}(x - x(a)) + y(a).$$

  is the tangent line of C at $t = a$.

- If $x'(a) = 0$ and $y'(a) \neq 0$, then we say the tangent line of C at $t = a$ is the vertical line $x = x(a)$, and we say the slope of the tangent line is **undefined**.

- If $x'(a) = y'(a) = 0$, then we need to consider $\lim_{t \to a} \frac{y'(t)}{x'(t)}$.

  - If $\lim_{t \to a} \frac{y'(t)}{x'(t)} = m$, then we say

$$y = m(x - x(a)) + y(a).$$

    is the tangent line of C at $t = a$.

  - If both $\lim_{t \to a^\pm} \frac{y'(t)}{x'(t)}$ are $\pm\infty$, then we say the tangent line of C at $t = a$ is the vertical line $x = x(a)$, and we say the slope of the tangent line is **undefined**.

Otherwise, we say the tangent line of C at $t = a$ does not exist.

Ex: Consider the parametric plane curve $C(t) = (t^2, t^3 - 3t)$.

1. Show that C has two tangent lines at $(3, 0)$, and find their equations.
2. Compute all t so that $x'(t) = 0$, and compute the slope of the tangent line of C at all such t.
3. Find all t so that the tangent line of C at t is horizontal.

Ex: Consider the cycloid $C(t) = (t - \sin t, 1 - \cos t)$.

1. Compute the tangent line of C at $t = \frac{\pi}{3}$.
2. Compute all $t \in [0, 2\pi)$ so that $x'(t) = 0$, and compute the slope of the tangent line of C at all such t.
3. Find all $t \in [0, 2\pi)$ so that the tangent line of C at t is horizontal.

To draw the cycloid, consider the circle of radius $= 1$ with center $(t, 1)$.

Def: Suppose $C(t) = (x(t), y(t))$ for $a \leq t \leq b$ is a parametric plane curve, and suppose x, y are continuous over $[a, b]$. We define the **area under** C to be the *signed* area of the region bounded by the image of C, the x-axis, and the vertical lines $x = x(a), x = x(b)$.

Fact: Suppose $C(t) = (x(t), y(t))$ for $a \leq t \leq b$ is a parametric plane curve, suppose x is continuously differentiable over $[a, b]$, and suppose y is continuous over $[a, b]$. Suppose A is the area under C.

- If x is increasing over $[a, b]$, then the image of C is the graph of a function $y = f(x)$ defined for $x \in [x(a), x(b)]$, and so

$$A = \int_{x(a)}^{x(b)} f(x)\ dx \underset{\substack{x = x(t) \\ dx = x'(t)dt}}{=} \int_a^b y(t)x'(t)\ dt.$$

- If x is decreasing over $[a, b]$, then

$$A = \int_{x(b)}^{x(a)} f(x)\ dx \underset{\substack{x = x(t) \\ dx = x'(t)dt}}{=} \int_b^a y(t)x'(t)\ dt.$$

Ex: Give an integral for the area A under one arc of the cycloid $C(t) = (t - \sin t, 1 - \cos t)$.

Def: Suppose $C(t) = (x(t), y(t))$ for $a \le t \le b$ is a parametric plane curve. If $C(t)$ has the property

$$C(t_1) = C(t_2) \text{ implies } t_1 = t_2 \text{ for all } a \le t_1, t_2 \le b,$$

then we say C **does not self-intersect**.

Fact: Suppose $C(t) = (x(t), y(t))$ for $a \le t \le b$ is a parametric plane curve with x, y continuously differentiable over $[a, b]$. If C does not self-intersect, then the arc length L of the image of C is given by

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

This formula also works if C only has **isolated self-intersections**.

Proof: Use the Pythagorean Thm and approximation.

Ex: Give an integral for the arc length L of one arc of the cycloid $C(t) = (t - \sin t, 1 - \cos t)$.

Fact: Suppose f is continuously differentiable over $[a, b]$. The arc length L of the curve $y = f(x)$ for $a \le x \le b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Ex: Let $f(x) = \sqrt{1 - x^2}$. Give an integral for the arc length L of the curve $y = f(x)$ for $-1 \le x \le 1$.

Fact: Suppose $C(t) = (x(t), y(t))$ for $a \le t \le b$ is a parametric plane curve with x, y continuously differentiable over $[a, b]$. Suppose $y(t) > 0$ for $a \le t \le b$, and suppose C does not self-intersect. The surface area S of the surface of revolution formed by rotating the image of C around the x-axis is

$$S = \int_a^b 2\pi y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\ dt.$$

Ex: Give an integral for the surface area S of the sphere of radius $r > 0$, which is the surface of revolution formed by rotating the image of the parametric plane curve $C(t) = (r\cos t, r\sin t)$ for $0 \le t \le \pi$ around the x-axis.

Fact: Suppose f is continuously differentiable over $[a, b]$ with $f(x) > 0$ for $x \in [a, b]$. The surface area S of the surface of revolution formed by rotating the curve $y = f(x)$ for $a \le x \le b$ around the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\ dx.$$

Ex: Let $f(x) = \sqrt{1 - x^2}$. Give an integral for the surface area S of the surface of revolution formed by rotating the curve $y = f(x)$ for $-1 \le x \le 1$ around the x-axis.