Vector Calculus
10.7 Vector Functions and Space Curves

Def: A <u>parametric vector-valued function</u> is a function of the form

$\vec{r}: [a,b] \to \mathbb{R}^2$ or $\vec{r}: [a,b] \to \mathbb{R}^3$

$\vec{r}(t) = \langle x(t), y(t) \rangle$ $\quad$ $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

parametric plane curve $\quad$ parametric space curve

We say t is the <u>parameter</u>, and we say the real-valued functions x,y,z:[a,b]->R are the <u>components</u> of $\vec{r}$. We say the set

$$\{\vec{r}(t) : t \in [a,b]\} \subset \mathbb{R}^2 \text{ or } \mathbb{R}^3$$

is the <u>image</u> of $\vec{r}$.

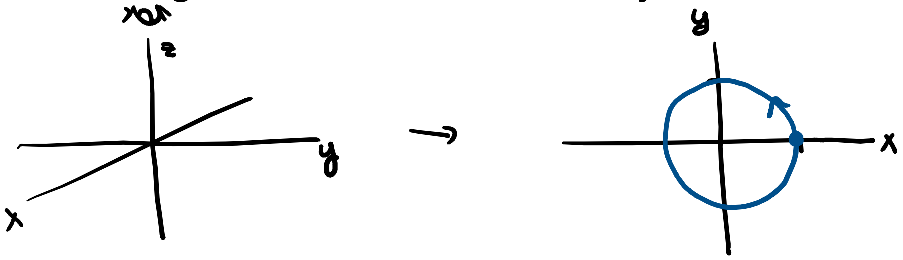
Ex: Sketch the image of the following parametric space curves.

1. $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $0 \le t \le 2\pi$ the helix

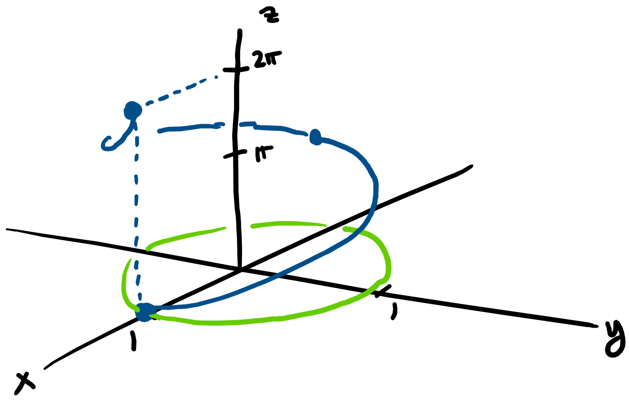
Sol: We compute $\vec{r}(0) = \langle 1,0,0 \rangle$ and $\vec{r}(t) = \langle 1,0,2\pi \rangle$. Also

$$\vec{r}(t) = \langle \underset{x}{\underset{\shortparallel}{\cos t}}, \underset{y}{\underset{\shortparallel}{\sin t}}, t \rangle \implies x^2 + y^2 = 1$$

This means that the image of $\vec{r}$ lies in the unit cylinder $x^2 + y^2 = 1$. In fact,



The image of $\vec{r}$ is

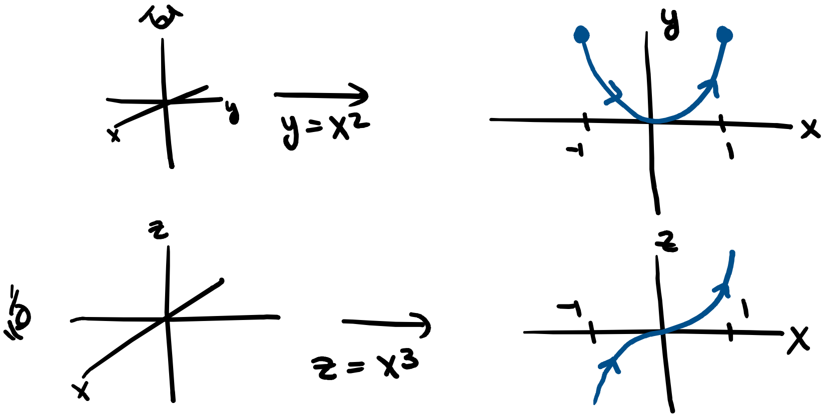


Use [CalcPlot3D (libretexts.org)]

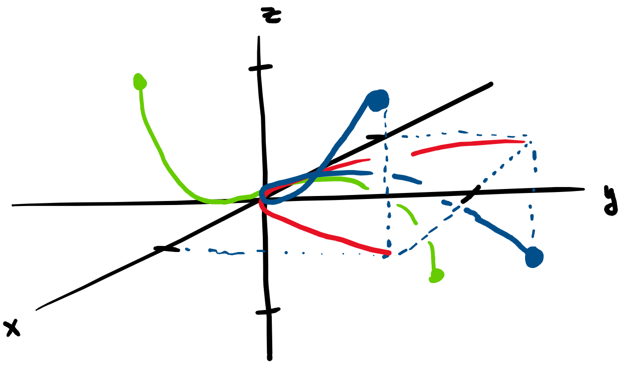
2. r(t)=<t,t^2,t^3> for -1≤t≤1 the twisted cubic

Sol: We compute $\vec{r}$(-1)=<-1,1,-1> and $\vec{r}$(1)=<1,1,1>. Note that

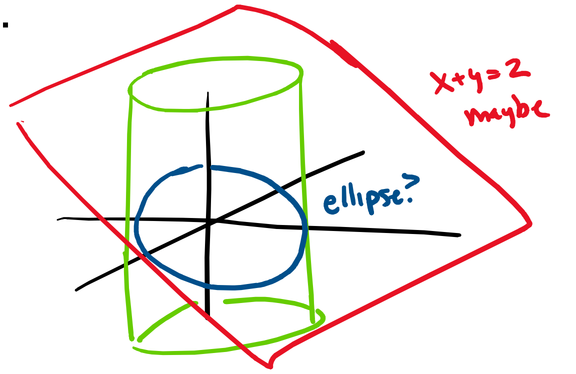
$\vec{r}(t)=\langle \underset{x}{t}, \underset{y}{t^2}, \underset{z}{t^3} \rangle$



"twisted cubic"



Ex: Find a parametric space curve $\vec{r}$ over an interval [a,b] so that the image of $\vec{r}$ is the intersection between the unit cylinder $x^2+y^2=1$ and the plane y+z=2.



Sol: We want the image of $\vec{r}$ to be

Consider

$$x^2+y^2=1 \Rightarrow \begin{array}{l} x=\cos t \\ y=\sin t \end{array}$$

Then

$$y+z=2 \Rightarrow \sin t + z = 2 \Rightarrow z = 2-\sin t$$

We conclude that the image of r(t)=<cos(t),sin(t),2-sin(t)> for $0 \leq t \leq 2\pi$ is the intersection.

Def: Consider a parametric vector-valued function $\vec{r}$ defined for t near a.

We say $\vec{r}$ is differentiable at t=a if and only if the components functions of $\vec{r}$ are differentiable at t=a.
x,y or x,y,z

This occurs if and only if the following limit exists:

$$\left.\frac{d\vec{r}}{dt}\right|_{t=a} = \vec{r}\,'(a) = \lim_{t\to a} \frac{\vec{r}(t)-\vec{r}(a)}{t-a} = \langle x'(a), y'(a), z'(a)\rangle$$

We say $\vec{r}\,'(a)$ is the <u>tangent vector</u> of $\vec{r}$ at t=a.

We say $|\vec{r}\,'(a)|$ is the <u>speed</u> of $\vec{r}$ at t=a.

If $\vec{r}'(a) \neq \vec{0}$, then we say the tangent line of $\vec{r}$ at t=a is the line through $\vec{r}(a)$ in the direction of $\vec{r}'(a)$.

$$\Rightarrow \quad \ell(t) = \vec{r}(a) + t\,\vec{r}'(a) \quad \text{for } t \in \mathbb{R}$$

If $\vec{r}'(t)$ exists for all t near a and is differentiable at t=a, then we let

$$\vec{r}''(a) = \frac{\partial}{\partial t}\vec{r}'(t)\Big|_{t=a}$$

denote the second derivative of $\vec{r}$ at t=a.

Ex: Consider the helix $\vec{r}(t)=\langle\cos(t),\sin(t),t\rangle$ for t in R.

1. Compute the tangent vector and speed of r at t=pi/2.

Sol: We compute

$$\vec{r}'\left(\tfrac{\pi}{2}\right) = \frac{\partial \vec{r}}{\partial t}\Big|_{t=\frac{\pi}{2}} = \frac{\partial}{\partial t}\langle \cos t, \sin t, t\rangle \Big|_{t=\frac{\pi}{2}}$$

$$= \langle -\sin t, \cos t, 1\rangle \Big|_{t=\frac{\pi}{2}}$$

$$\Rightarrow \boxed{\vec{r}'\left(\tfrac{\pi}{2}\right) = \langle -1, 0, 1\rangle} \quad \text{tangent vector}$$

and

$$\left|\vec{r}'\left(\tfrac{\pi}{2}\right)\right| = |\langle -1,0,1\rangle| = \sqrt{1+0+1} = \boxed{\sqrt{2}} \quad \text{speed}$$

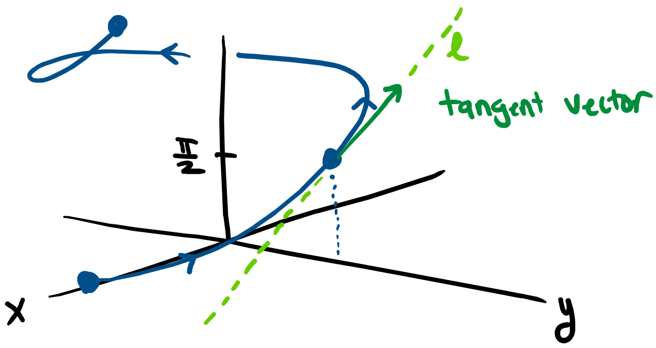
2. Compute the tangent line of $\vec{r}$ at t=pi/2.

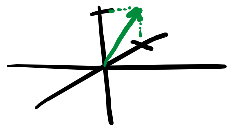
Sol: This is the line

$$\ell(t) = \vec{r}\left(\tfrac{\pi}{2}\right) + t\,\vec{r}'\left(\tfrac{\pi}{2}\right) \quad \text{for } t \in \mathbb{R}$$

$$\Rightarrow \boxed{\ell(t) = \langle 0, 1, \tfrac{\pi}{2}\rangle + t\langle -1,0,1\rangle \text{ for } t \in \mathbb{R}}$$

Check:





The tangent vector, and so the tangent line, are *tangent* to the image of $\vec{r}$ at $\vec{r}(a)$.

Ex: Consider $\vec{r}(t)=\langle t^2/2, t^3/3\rangle$ for t in R.

1. Compute the tangent vector and speed of $\vec{r}$ at t=1.

Sol: We compute

$$\vec{r}\,'(1) = \frac{d}{dt}\left\langle \frac{t^2}{2}, \frac{t^3}{3}\right\rangle\Big|_{t=1} = \langle t, t^2\rangle\Big|_{t=1} = \boxed{\langle 1,1\rangle}$$

with $$|\vec{r}\,'(t)| = \sqrt{1+1} = \boxed{\sqrt{2}}$$

2. Compute the tangent line of $\vec{r}$ at t=1.

Sol: We give

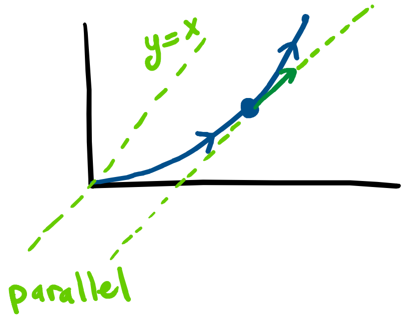
$$\ell(t) = \vec{r}(1) + t\,\vec{r}\,'(1) \quad \text{for } t\in\mathbb{R}$$

$$\Rightarrow \boxed{\ell(t) = \left\langle \frac{1}{2}, \frac{1}{3}\right\rangle + t\langle 1,1\rangle \quad \text{for } t\in\mathbb{R}}$$

Check: for t near 1,

$$y = \frac{t^3}{3} = \frac{1}{3}\left(\sqrt{2}\left(\frac{t^2}{2}\right)^{\frac{1}{2}}\right)^3 = \frac{\sqrt{2}^3}{3}\,x^{\frac{3}{2}}$$

$$\Rightarrow y = \frac{\sqrt{2}^3}{3}\,x^{\frac{3}{2}}$$



Fact: Suppose $\vec{r}$ is a vector-valued function defined over [a,b], and suppose f:[alpha,beta]->[a,b] is continuous.
α β

Suppose f is increasing with

$$f(\alpha)=a \quad \text{and} \quad f(\beta)=b$$

and define the parametric vector-valued function

$$\vec{r}_f(s)=\vec{r}(f(s)) \quad \text{for} \quad \alpha\le s\le\beta$$

Then $\vec{r},\vec{r}_f$ have the same images.

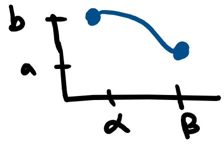
$\vec{r}_f(\alpha)=\vec{r}(f(\alpha))=\vec{r}(a)$ ✓
$\vec{r}_f(\beta)=\vec{r}(f(\beta))=\vec{r}(b)$

However, $\vec{r}_f$ traces the image of $\vec{r}$ with different speed. In fact,

$$|\vec{r}_f(s)|=|f'(s)\,\vec{r}\,'(f(s))|=f'(s)\,|\vec{r}\,'(f(s))|$$

assuming f is differentiable.

Suppose f is decreasing with



$$f(\alpha)=b \quad \text{and} \quad f(\beta)=a$$

and define the parametric vector-valued function

$$\vec{r}_f(s)=\vec{r}(f(s)) \quad \text{for} \quad \alpha\le s\le\beta$$

Then $\vec{r}, \vec{r}_f$ have the same images. However, $\vec{r}_f$ traces the image of $\vec{r}$ in the opposite direction and with different speed. In fact,

Check $\vec{r}_f(\alpha) = \vec{r}(f(\alpha)) = \vec{r}(b)$

$\vec{r}_f(\beta) = \vec{r}(f(\beta)) = \vec{r}(a)$

$$|\vec{r}_f(s)| = |f'(s)\,\vec{r}\,'(f(s))|$$

$$\underset{f \text{ is decreasing}}{=} \underbrace{-f'(s)}_{\geq 0}\,|\vec{r}\,'(f(s))|$$

Def: We say $\vec{r}_f$ is a <u>reparameterization of r.</u>

Ex: Consider the helix

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle \quad \text{for } t \in \mathbb{R}$$

Recall that

$$\vec{r}\left(\tfrac{\pi}{2}\right) = \langle 0, 1, \tfrac{\pi}{2} \rangle, \quad \underbrace{\vec{r}\,'\left(\tfrac{\pi}{2}\right) = \langle -1, 0, 1 \rangle}_{\text{tangent vector}}, \quad \text{and} \quad \underbrace{|\vec{r}\,'(t)|}_{\text{speed}} = \sqrt{2}$$

1. Suppose f(s)=2s, and consider $\vec{r}_f = \vec{r}_f(s)$. Compute $\vec{r}_f$(pi/4), and compute the tangent vector and speed of $\vec{r}_f$ at s=pi/4.

Sol: We compute

$$\vec{r}_f\left(\tfrac{\pi}{4}\right) = \vec{r}_f(s)\Big|_{s=\frac{\pi}{4}} = \vec{r}(f(s))\Big|_{\frac{\pi}{4}}$$

$$= \underbrace{\langle \cos(2s), \sin(2s), 2s \rangle}_{\vec{r}(f(s))}\Big|_{s=\frac{\pi}{4}}$$

$$= \langle \cos(2\cdot\tfrac{\pi}{4}), \sin(2\cdot\tfrac{\pi}{4}), 2\cdot\tfrac{\pi}{4} \rangle$$

$$= \langle \cos(\tfrac{\pi}{2}), \sin(\tfrac{\pi}{2}), \tfrac{\pi}{2} \rangle$$

$$= \boxed{\langle 0, 1, \tfrac{\pi}{2} \rangle} = \vec{r}(\tfrac{\pi}{2})$$

We also compute

$$\vec{r}_f{}'(\tfrac{\pi}{4}) = \frac{d}{ds}\vec{r}_f(s)\Big|_{s=\frac{\pi}{4}} = \frac{d}{ds}\langle \cos(2s), \sin(2s), 2s \rangle\Big|_{s=\frac{\pi}{4}}$$

$$= \langle -2\sin 2s, 2\cos 2s, 2 \rangle \Big|_{s=\frac{\pi}{4}}$$

$$= \langle -2\sin\tfrac{\pi}{2}, 2\cos\tfrac{\pi}{2}, 2 \rangle$$

$$= \boxed{\langle -2, 0, 2 \rangle}$$

$$\text{Speed} = |\vec{r}_f(\tfrac{\pi}{4})| = \sqrt{4+4} = \boxed{2\sqrt{2}}$$

check: Note that $\vec{r}_f$(pi/4)=$\vec{r}$(pi/2)=<0,1,pi/2>. It takes half the time for $\vec{r}_f$ to get to <0,1,pi/2>. This means the speed of $\vec{r}_f$ should be double...$2\sqrt{2}$. ✓

2. Suppose f(s)=pi-s, and consider $\vec{r}_f$=$\vec{r}_f$(s). Compute $\vec{r}_f$(pi/2), and compute the tangent vector and speed of $\vec{r}_f$ at s=pi/2.

   Sol: We compute

$$\vec{r}_F\left(\tfrac{\pi}{2}\right) = \vec{r}(F(s))\Big|_{s=\frac{\pi}{2}} = \langle \cos(\pi - s), \sin(\pi - s), \pi - s \rangle\Big|_{s=\frac{\pi}{2}}$$

$$= \left\langle \cos\left(\tfrac{\pi}{2}\right), \sin\left(\tfrac{\pi}{2}\right), \tfrac{\pi}{2} \right\rangle$$

$$= \boxed{\left\langle 0, 1, \tfrac{\pi}{2} \right\rangle} = r\left(\tfrac{\pi}{2}\right)$$

We also compute

$$\vec{r}_F{}'\left(\tfrac{\pi}{2}\right) = \frac{\partial}{\partial s} \langle \cos(\pi - s), \sin(\pi - s), \pi - s \rangle \Big|_{s=\frac{\pi}{2}}$$

$$= \langle +\sin(\pi - s), -\cos(\pi - s), -1 \rangle \Big|_{s=\frac{\pi}{2}}$$

$$= \left\langle \sin\tfrac{\pi}{2}, -\cos\tfrac{\pi}{2}, -1 \right\rangle$$

$$= \boxed{\langle 1, 0, -1 \rangle} = -\vec{r}\,'\left(\tfrac{\pi}{2}\right)$$

$$\left|\vec{r}_F{}'\left(\tfrac{\pi}{2}\right)\right| = \sqrt{1+1} = \boxed{\sqrt{2}} \quad \text{same speed}$$