2.1 Derivatives and Rates of Change

Def: Suppose f is defined near a, including at a itself.

"f prime of a"

We say that f is differentiable at a with derivative f'(a) if the following limit exists (is a finite number):

$$f'(a) = \frac{d}{dx} f(x)\Big|_{x=a} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

"evaluated at $x=a$"

If f is differentiable at a, then we say the line through (a,f(a)) with slope f'(a) is the tangent line of f at a:

$$y = f'(a)(x-a) + f(a)$$

Ex: Definition of the derivative.

1. For f(x)=x^2, compute f'(a) for all a.

Sol: By definition, we must compute

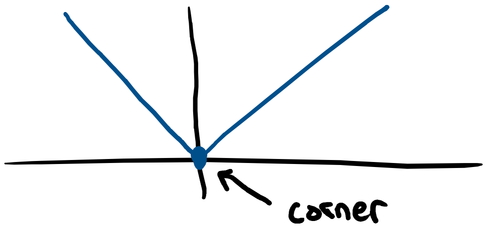
$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{x^2-a^2}{x-a}$$

$$= \lim_{x \to a} \frac{(x-a)(x+a)}{x-a}$$

$$= \lim_{x \to a} (x+a) = a+a = 2a.$$

2. $f(x)=|x|$ is not differentiable at $a=0$.

The problem is that the graph of f has a corner at $a=0$.



2.2 The Derivative as a Function

Def: Suppose f is differentiable at each x in (a,b).

We say f is differentiable over (a,b).

The function f'(x)= $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for each x in (a,b) is called
the first derivative of f.

The notation $\frac{\partial}{\partial x} f(x)$ means to find the first derivative of f.

If f' is continuous over (a,b), then we say f is continuously differentiable over (a,b).

"f double prime of f"

If f' is also differentiable over (a,b), then f''(x)= $\frac{\partial}{\partial x}$ f'(x) denotes the second derivative of f. The notation $\frac{\partial^2}{\partial x^2} f(x)$ means to find the second derivative of f.

We make similar definitions over (a,infinity),(-infinity,b), (-infinity,infinity).

Ex: For f(x)=x^2, we already showed that

$$f'(x) = \frac{\partial}{\partial x} x^2 = 2x \quad \text{for all } x \in \mathbb{R}$$

We conclude that f is continuously differentiable over (-infinity,infinity).

2.3 Basic Differentiation Formulas

Thm (Table of Basic Derivatives):

- $\frac{\partial}{\partial x} c = 0$ for all x
- $\frac{\partial}{\partial x} x^r = r x^{r-1}$ for any real number r, for all x near where x^r is defined.

  ☆ $\frac{\partial}{\partial x} x^{\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}}$ for $x > 0$.

- trigonometric functions

$$\frac{\partial}{\partial x} \cos x = -\sin x \quad \text{for all } x$$

$$\frac{\partial}{\partial x} \sin x = \cos x \quad \text{for all } x$$

$$\frac{\partial}{\partial x} \tan x = \sec^2 x \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\frac{\partial}{\partial x} \arctan x = \frac{1}{1+x^2} \quad \text{for all } x$$

- $\frac{\partial}{\partial x} e^x = e^x$ for all x
- $\frac{\partial}{\partial x} \ln x = \frac{1}{x}$ for $x > 0$

For the Midterm and Final (and the Homework), these are the only derivatives you need to have memorized.

2.4 The Product and Quotient Rules

Thm (Basic Derivative Rules): Suppose f,g are differentiable at a.

Simplification Rule: If f(x)=g(x) for all x near a, then f'(a)=g'(a).

⇨ We can simplify inside of the derivative.

Addition Rule: $\frac{\partial}{\partial x}\left(f(x)+g(x)\right)\Big|_{x=a} = f'(a) + g'(a)$

Product Rule: $\frac{\partial}{\partial x}\left(f(x)g(x)\right)\Big|_{x=a} = f'(a)g(a) + f(a)g'(a)$

$$\Rightarrow \frac{\partial}{\partial x} cf(x)\Big|_{x=a} = c\frac{\partial}{\partial x} f(x)\Big|_{x=a}$$

Quotient Rule: If g(a)=/0, then

$$\frac{\partial}{\partial x}\frac{f(x)}{g(x)}\Big|_{x=a} = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

Ex: Compute the following derivatives.

1. $\frac{\partial}{\partial x}$ x^2

Sol: The idea here is that we will verify the Product Rule,

$$2x = \frac{\partial}{\partial x}x^2 = \frac{\partial}{\partial x}(x \cdot x)$$

$$= \left(\frac{\partial}{\partial x} x\right) \cdot x + x \cdot \frac{\partial}{\partial x} x$$

$$= 1 \cdot x + x \cdot 1 = 2x \quad \checkmark$$

Compare this to

$$\varnothing \quad \left(\frac{\partial}{\partial x} x\right)\left(\frac{\partial}{\partial x} x\right) = 1 \cdot 1 \neq 2x \quad \varnothing$$

We cannot simply multiply the derivatives together. We must use the Product Rule!

2. $\frac{\partial}{\partial x} \frac{x(x+1)+1}{e^x+1} \Big|_{x=0}$

Sol: Using the Quotient Rule,

$$\frac{\partial}{\partial x} \frac{x(x+1)+1}{e^x+1} \Big|_{x=0} = \frac{\left(\frac{\partial}{\partial x}(x(x+1)+1)\right)(e^x+1) - (x(x+1)+1)\frac{\partial}{\partial x}(e^x+1)}{(e^x+1)^2} \Bigg|_{x=0}$$

$$= \frac{\left(\frac{\partial}{\partial x} x(x+1) + \frac{\partial}{\partial x} 1\right)(e^x+1) - (x(x+1)+1)\left(\frac{\partial}{\partial x} e^x + \frac{\partial}{\partial x} 1\right)}{(e^x+1)^2} \Bigg|_{x=0}$$

$$= \frac{\left(\frac{\partial}{\partial x}(x^2+x) + 0\right)(e^x+1) - (x(x+1)+1)(e^x+0)}{(e^x+1)^2} \Bigg|_{x=0}$$

$$= \left. \frac{(2x+1)(e^x+1) - (x(x+1)+1)\,e^x}{(e^x+1)^2} \right|_{x=0}$$

You cannot leave the answer like this, you must plug in x=0.

$$= \boxed{\frac{(0+1)(1+1) - (0+1)\cdot 1}{(1+1)^2}}$$

2.5 The Chain Rule

Chain Rule: Suppose g(x) is differentiable at x=a and f(u) is differentiable at u=g(a), then f(g(x)) is differentiable at x=a with derivative

$$\frac{\partial}{\partial x} f(g(x))\Big|_{x=a} = \frac{\partial}{\partial u} f(u)\Big|_{u=g(a)} \cdot \frac{\partial}{\partial x} g(x)\Big|_{x=a} = f'(g(a))\, g'(a).$$

Ex: Compute $\frac{\partial}{\partial x}\sqrt{x+\sqrt{x+\sqrt{x}}}$ for x>0.

Sol: Using the Chain Rule, we compute

$$\frac{\partial}{\partial x}\sqrt{x+\sqrt{x+\sqrt{x}}} = \frac{\partial}{\partial x}\left(\sqrt{u}\,\Big|_{u=x+\sqrt{x+\sqrt{x}}}\right)$$

$$=\left(\frac{\partial}{\partial u}\sqrt{u}\,\Big|_{u=x+\sqrt{x+\sqrt{x}}}\right)\frac{\partial}{\partial x}\, x+\sqrt{x+\sqrt{x}}$$

$$\frac{\partial}{\partial u}\sqrt{u} = \frac{\partial}{\partial u} u^{\frac{1}{2}} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2u^{\frac{1}{2}}} = \frac{1}{2\sqrt{u}}$$

$$= \left(\frac{1}{2\sqrt{u}}\,\Big|_{u=x+\sqrt{x+\sqrt{x}}}\right)\left(1+\frac{\partial}{\partial x}\sqrt{x+\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left(1+\frac{\partial}{\partial x}\sqrt{x+\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left(1 + \frac{d}{dx}\left(\sqrt{u} \,\Big|_{u=x+\sqrt{x}} \right) \right)$$

$$= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left(1 + \left(\frac{d}{du}\sqrt{u} \,\Big|_{u=x+\sqrt{x}} \right) \frac{d}{dx}\left(x+\sqrt{x}\right) \right)$$

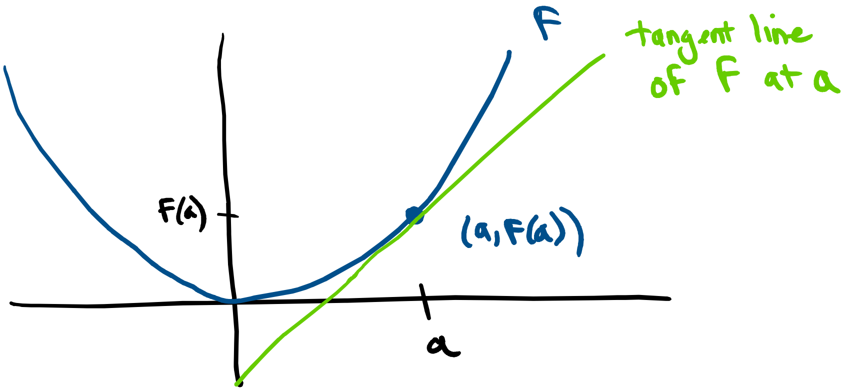
$$= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left(1 + \left(\frac{1}{2\sqrt{u}} \,\Big|_{u=x+\sqrt{x}} \right) \left(1 + \frac{1}{2\sqrt{x}} \right) \right)$$

You must eliminate all of the u's!

$$= \boxed{\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \left(1 + \left(\frac{1}{2\sqrt{x+\sqrt{x}}} \right) \left(1 + \frac{1}{2\sqrt{x}} \right) \right)}$$

2.8 Linear Approximation and Differentials

Thm: Suppose f is differentiable at a, then the tangent line y=f'(a)(x-a)+f(a) of f at a is a good approximation for f near a.



Ex: Estimate the value of the following quantities.

1. $\sqrt{\frac{1}{100}+\cos\left(\frac{1}{100}\right)}$

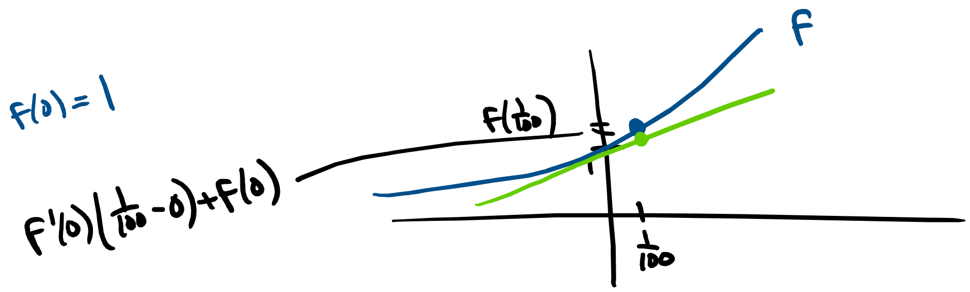
Sol: Consider the function

$$f(x)=\sqrt{x+\cos(x)}$$

We want to approximate the value of f(1/100)=$\sqrt{\frac{1}{100}+\cos\frac{1}{100}}$. Note that f is differentiable at a=0. Since the tangent line of f at a=0 gives a good approximation for f near a=0, then we conclude that

$$f\left(\frac{1}{100}\right) \approx \underbrace{\text{the value of the tangent line of } f \text{ at } a=0, \text{ at } x=\tfrac{1}{100}}_{\shortparallel}$$

$$f'(0)(x-0)+f(0)\Big|_{x=\frac{1}{100}}$$



We must compute

$$F(0) = \sqrt{0+\cos 0} = \sqrt{0+1} = \sqrt{1} = 1$$

$$F'(0) = \frac{d}{dx}\sqrt{x+\cos x}\,\Big|_{x=0} = \frac{1}{2\sqrt{x+\cos x}}\cdot(1-\sin x)\Big|_{x=0}$$

$$= \frac{1-0}{2\sqrt{0+1}} = \frac{1}{2}$$

We conclude that

$$\sqrt{\tfrac{1}{100}+\cos\tfrac{1}{100}} = F\left(\tfrac{1}{100}\right) \approx \underbrace{F'(0)\left(\tfrac{1}{100}-0\right)+F(0)}_{\frac{1}{2}\cdot\frac{1}{100}+1=\boxed{\frac{201}{200}}}$$

My calculator says

$$\sqrt{\tfrac{1}{100}+\cos\left(\tfrac{1}{100}\right)} = 1.00496\ldots$$

$$\frac{201}{200} = 1.005!$$

2. $\sqrt{1.02+\sin\left(\frac{1}{100}\right)}$

Sol: Note that

$$\sqrt{1.02+\sin\left(\tfrac{1}{100}\right)} = \sqrt{1+2\left(\tfrac{1}{100}\right)+\sin\left(\tfrac{1}{100}\right)}.$$

Consider

$$f(x) = \sqrt{1+2x+\sin(x)}$$

We compute

$$f\left(\tfrac{1}{100}\right) \approx \underbrace{\text{the tangent line of } f \text{ at } a=0, \text{ evaluated at } x=\tfrac{1}{100}}_{f'(0)\left(\frac{1}{100}-0\right)+f(0)}$$

We must compute

$$f(0) = \sqrt{1+2\cdot 0+\sin 0} = \sqrt{1+0+0} = 1$$

$$f'(0) = \frac{d}{dx}\sqrt{1+2x+\sin x}\,\Big|_{x=0}$$

$$= \frac{1}{2\sqrt{1+2x+\sin x}}\cdot(0+2+\cos x)\,\Big|_{x=0}$$

$$= \frac{2+\cos 0}{2\sqrt{1+0+0}} = \frac{3}{2}$$

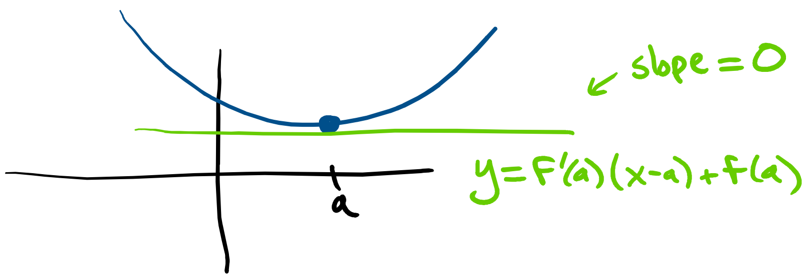
We conclude that

$$f\left(\tfrac{1}{100}\right) \approx \tfrac{3}{2}\left(\tfrac{1}{100}-0\right)+1 = \tfrac{203}{200}.$$

$$\sqrt{1.02+\sin\left(\tfrac{1}{100}\right)} = 1.01488\ldots$$

$$\tfrac{203}{200} = 1.015 \text{ !}$$

Fact: Suppose f is differentiable at a. The tangent line of f at a is horizontal if and only if f'(a)=0.



$\Rightarrow f'(a)=0$

3.1 Maximum and Minimum Values

$(a,b), (a,b], [a,b), [a,b]$
$(-\infty,a), (-\infty,a], (a,\infty), [a,\infty)$
$(-\infty,\infty)$

Def: Suppose I is an interval

and suppose f is defined for all x in I.

If c is in I and f(c)≥f(x) for each x in I, then we say
c is an absolute maximum point of f over I and
f(c) is an absolute maximum value of f over I.

We similarly define:
c is an absolute minimum point of f over I and
f(c) is an absolute minimum value of f over I.

We also define absolute extremum point/value over I.

$\parallel$
either maximum or minimum

Suppose c is in R, and suppose f is defined near c.

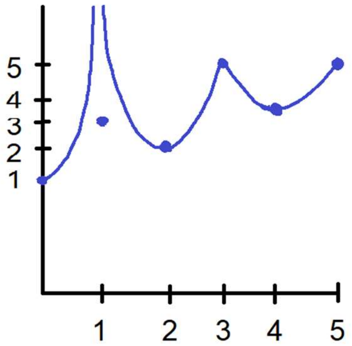
If f(c)≥f(x) for all x near c, then we say
on both sides
c is a local maximum point of f and
f(c) is a local minimum value of f.

We similarly define local minimum point/value and local extremum point/value.

If f'(c)=0 or f'(c) does not exist, then we say c is a critical point of f.

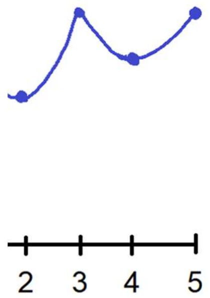
f is defined near c

Ex: Consider the function f defined over [0,5], given by the following graph.



1. Find the absolute extremum points and values of f over [2,5].

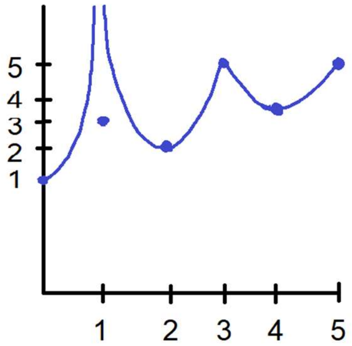
Sol: We must find the absolute maximum and minimum points and values of f over [2,5].



x=2 is ~~an~~ the absolute minimum point of f over [2,5], with absolute minimum value f(2)=2.
x=3,5 are absolute maximum points of f over [2,5], with absolute maximum value f(3)=f(5)=5.

2. Find the absolute extremum points and values of f over [0,5].

Sol: We must find both absolute maximum and minimum points and values of f over [0,5].

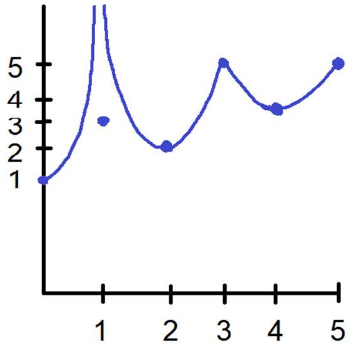


Since $\lim_{x\to 1} f(x)=\infty$, then f does not have an absolute maximum point and value over [0,5].
x=0 is the absolute minimum point of f over [0,5], with absolute minimum value f(0)=1.

3. Find the local extremum points and values of f.

near

Sol: Let's consider the graph

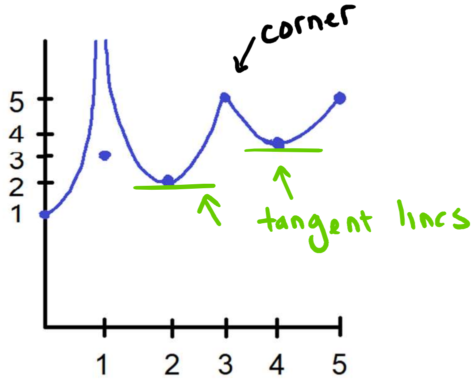


Since f is not defined on *both* sides of x=0, then x=0 is *not* a local minimum.
x=2 is a local minimum with local minimum value f(2)=2.
x=4 is a local minimum with local minimum value f(4)=4.

x=3 is a local maximum with local maximum value f(3)=5.

4. Find the critical points of f.

Sol: We must find all c in (0,5) so that f'(c)=0 or f'(c) does not exist.



Again, we do not count endpoints as critical points.

Since f' does not exist at x=1,3, then x=1,3 are critical points.
At x=2,4 f has horizontal tangent lines. This means the slope of the tangent line of f at x=2,4 is zero, and so f'(2)=f'(4)=0. We conclude that x=2,4 are also critical points of f.

So x=1,2,3,4 are critical points of f.

Thm: Finding local and absolute extremum points.

If c is a local extremum point of f, then c is a critical point of f. $f'(c)$ DNE or $f'(c)=0$

Suppose f is continuous over [a,b].

f has an absolute maximum point and an absolute minimum point over [a,b]. There are c_{max}, c_{min} in [a,b] so that c_{max} is an absolute maximum point of f over [a,b] and so that c_{min} is an absolute minimum point of f over [a,b].

If c is an absolute extremum point of f over [a,b], then either c=a,b or c is a critical point.

Ex: Find the absolute extremum points and values of

$$f(x) = 3x^2 - 2x + 1$$

over [0,2].

Sol: Since f is continuous, and differentiable for all x, then we must compare the values of f(0),f(2) and all c in (0,2) with f'(c)=0. We compute

$$f(0) = 0 - 0 + 1 = 1$$

$$f(2) = 3 \cdot 4 - 4 + 1 = 12 - 3 = 9$$

$$0 = f'(c) = 6c - 2 \Rightarrow 6c = 2 \Rightarrow c = \frac{1}{3} \in (0,2)$$

$$f\left(\tfrac{1}{3}\right) = 3 \cdot \frac{1}{9} - \frac{2}{3} + 1 = \frac{1}{3} - \frac{2}{3} + 1 = \frac{2}{3}$$

x=1/3 is the absolute minimum point of f over [0,2] with absolute minimum value f(1/3)=2/3.
x=2 is the absolute maximum point of f over [0,2] with absolute maximum value f(2)=9.