Vector Calculus
6.2 Trigonometric Integrals and Substitutions

Fact: Consider

$$\int \sin^m x \cos^n x \, dx$$

for m,n nonnegative integers.

If m is odd, save one power of sine and use $\sin^2 x=1-\cos^2 x$
⇨ Apply the substitution u=cos(x)
If n is odd, save one power of cosine and use
$\cos^2 x=1-\sin^2 x$
⇨ Apply the substitution u=sin(x)
If m,n are both even, use the half-angle identities

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) \quad \text{and} \quad \sin^2 x = \frac{1}{2}(1-\cos 2x)$$

Sometimes you can save time by using the double-angle formula

$$\sin 2x = 2\sin x \cos x$$

Ex: Compute the most general antiderivative of the given f over the largest possible open subset I of R, and give I.

1. $f(x)=\sin^3 x \cos^2 x$

Sol: We compute

$$\int \sin^3 x \cos^2 x \, dx = \int \sin x \sin^2 x \cos^2 x \, dx$$

$$= \int \sin x \, (1-\cos^2 x)\cos^2 x \, dx$$

$$\underset{\substack{u=\cos x \\ du=-\sin x\,dx}}{=} \int (1-u^2)u^2(-1)\,du \Big|_{u=\cos x}$$

$$= \int -u^2+u^4\,du \Big|_{u=\cos x}$$

$$= -\frac{u^3}{3}+\frac{u^5}{5}+c \Big|_{u=\cos x}$$

$$= \boxed{\begin{array}{c} -\dfrac{\cos^3 x}{3}+\dfrac{\cos^5 x}{5}+c \\ \text{over } I=(-\infty,\infty) \end{array}}$$

2. $f(x) = 4\sin^2 x \cos^2 x$

Sol: Using the double-angle formula,

$$\int 4\sin^2 x\cos^2 x\,dx = \int (2\sin x\cos x)^2\,dx$$

$$= \int (\sin 2x)^2\,dx$$

$$= \int \sin^2 2x\,dx$$

$$= \int \frac{1}{2}\left(1-\cos(2\cdot 2x)\right)dx$$

$$= \int \frac{1}{2}(1-\cos 4x)\,dx$$

$u=4x$

$$= \boxed{\frac{1}{2}\left(x-\frac{\sin 4x}{4}\right)+C \quad \text{over } I=(-\infty,\infty)}$$

Similarly, we compute

3. $f(x)=\sin^2 x$
4. $f(x)=\cos^2 x$

Fact: Consider

$$\int \tan^m x \sec^n x\,dx$$

for m,n nonnegative integers...refer to the notes.

Ex: Compute the most general antiderivative of $f(x)=\tan^3 x \sec(x)$ over the largest possible open subset I of R, and give I.

Sol: We compute

$$\int \tan^3 x \sec x\,dx = \int \tan^2 x \underbrace{\tan x \sec x\,dx}_{du}$$

We will use the Pythagorean Thm in the following form

$$\cos^2 x + \sin^2 x = 1 \implies \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\implies 1 + \tan^2 x = \sec^2 x$$

We compute

$$\int \tan^3 x \sec x \, dx = \int (\sec^2 x - 1) \tan x \sec x \, dx$$

$$\underset{\substack{u = \sec x \\ du = \sec x \tan x \, dx}}{=} \int (u^2 - 1) \, du \Big|_{u = \sec x}$$

$$= \frac{u^3}{3} - u + c \Big|_{u = \sec x}$$

$$= \boxed{\begin{array}{c} \dfrac{\sec^3 x}{3} - \sec x + C \\ \text{over } I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{array}}$$

$f(x) = \tan^3 x \sec x$

Since tan(x) is defined and continuous over (-pi/2,pi/2) and sec(x) is defined and continuous over (-pi/2,pi/2)U(pi/2,3pi/2). So, f is defined and continuous over (-pi/2,pi/2). Look at the Week 1 Lecture notes.

Fact: Evaluating some trigonometric integrals using angle addition formulas...refer to the notes.

Ex: Compute the most general antiderivative of f(x)=sin(4x)cos(5x) over the largest possible open subset I of R, and give I.

Sol: We use the angle addition formula

$$\int \sin 4x \cos 5x \, dx = \int \frac{1}{2}\left(\sin((4-5)x) + \sin((4+5)x)\right) dx$$

$$= \int \frac{1}{2}\left(\sin(-x) + \sin(9x)\right) dx$$

$$= \int \frac{1}{2}\left(-\sin x + \sin 9x\right) dx$$

$$= \boxed{\frac{1}{2}\left(\cos x - \frac{\cos 9x}{9}\right) + C \quad \text{over } I = (-\infty, \infty)}$$

Fact: To compute an integrals involving the expression

$$\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$$

use the following trigonometric substitutions and identities.

Expression	Substitution		Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Ex: Compute the following integrals.

1. $\int_0^2 x^3\sqrt{4-x^2}\,dx$

Sol: We compute

$$\int_0^2 x^3\sqrt{4-x^2}\,dx \underset{\substack{x=2\sin\theta\\ dx=2\cos\theta\,d\theta}}{=\!=} \int_0^{\frac{\pi}{2}} \cancel{(2\sin\theta)^3\sqrt{4-(2\sin}}$$

$x=0 \Rightarrow 2\sin\theta=0$
with $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$ $*$

$\Rightarrow \theta=0$

$x=2 \Rightarrow 2\sin\theta=2$

$\Rightarrow \sin\theta=1$

$\Rightarrow \theta=\frac{\pi}{2}$

$$=\!= \int_0^{\frac{\pi}{2}} (2\sin\theta)^3\sqrt{4-(2\sin\theta)^2}\,(2\cos\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2^4\sin^3\theta\cos\theta\sqrt{\underbrace{4-4\sin^2\theta}_{4(1-\sin^2\theta)}}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2^4\sin^3\theta\cos\theta\sqrt{4\cos^2\theta}\,d\theta$$

$$= 2^5\int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta\sqrt{\cos^2\theta}\,d\theta$$

$$\underset{\substack{\cos\theta \geq 0 \\ \text{for } 0 \leq \theta \leq \frac{\pi}{2}}}{=} 2^5 \int_0^{\frac{\pi}{2}} \sin^3\theta \cos^2\theta \, d\theta$$

$$= 2^5 \left(-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5} \right) \Bigg|_{\theta=0}^{\frac{\pi}{2}}$$

$$= \boxed{2^5 \left(0 + 0 - \left(\frac{-1}{3} + \frac{1}{5} \right) \right)}$$

2. $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{(2x^2+8)^{\frac{3}{2}}} \, dx$

Sol: We compute

$$\int_0^{2\sqrt{3}} \frac{x^3}{(2x^2+8)^{\frac{3}{2}}} \, dx = \int_0^{2\sqrt{3}} \frac{x^2}{(2(x^2+4))^{3/2}} \, dx$$

$$= \frac{1}{2^{3/2}} \int_0^{2\sqrt{3}} \frac{x^2}{(x^2+4)^{3/2}} \, dx$$

$$\underset{\substack{x = 2\tan\theta \\ dx = 2\sec^2\theta \, d\theta}}{=}$$

$x=0 \Rightarrow 2\tan\theta = 0$
with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\Rightarrow \theta = 0$

$x = 2\sqrt{3} \Rightarrow 2\tan\theta = 2\sqrt{3}$
$\Rightarrow \tan\theta = \sqrt{3}$
$\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{\pi}{3}$

$$\frac{1}{2^{3/2}} \int_0^{\frac{\pi}{3}} \frac{(2\tan\theta)^3}{\left((2\tan\theta)^2 + 4\right)^{3/2}} 2\sec^2\theta \, d\theta$$

$$= \frac{2^4}{2^{3/2}} \int_0^{\frac{\pi}{3}} \frac{\tan^3\theta \sec^2\theta}{\left(4\tan^2\theta + 4\right)^{3/2}} d\theta$$

$\tan^2\theta + 1 = \sec^2\theta$

$$= 2^{\frac{5}{2}} \int_0^{\frac{\pi}{3}} \frac{\tan^3\theta \sec^2\theta}{\left(4\sec^2\theta\right)^{3/2}} d\theta$$

$$= \frac{2^{\frac{5}{2}}}{4^{3/2}} \int_0^{\frac{\pi}{3}} \frac{\tan^3\theta \sec^2\theta}{\left(\sqrt{\sec^2\theta}\right)^3} d\theta$$

$\sec\theta = \frac{1}{\cos\theta} > 0$
for $\theta \in [0, \frac{\pi}{3}]$

$$= \frac{2^{5/2}}{4^{3/2}} \int_0^{\frac{\pi}{3}} \frac{\tan^3\theta \sec^2\theta}{(\sec\theta)^3} d\theta$$

$$= 2^{-1/2} \int_0^{\frac{\pi}{3}} \frac{\tan^3\theta}{\sec\theta} d\theta$$

close enough

How do we do this integral? We must use a trigonometric substitution, but first, we compute

$$\int \frac{\tan^3\theta}{\sec\theta}\, d\theta = \int \frac{\left(\frac{\sin\theta}{\cos\theta}\right)^3}{\frac{1}{\cos\theta}}\, d\theta$$

$$= \int \frac{\sin^3\theta}{\cos^2\theta}\, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta} \sin\theta\, d\theta$$

$$= \int \left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) \sin\theta\, d\theta$$

$$\underset{u=\cos\theta}{=} \cdots$$

6.3 Partial Fractions

Def: A rational fraction is a function of the form f(x)=P(x)/Q(x) where P,Q are polynomials.

Ex: Compute the most general antiderivative of the given f over the largest possible open subset I of R, and give I.

1. f(x)=1/(x+1)

$P(x)=1$ constant polynomial
$Q(x)=x+1$ linear polynomial

Sol: We compute

$$\int \frac{1}{x+1}\,dx \underset{\substack{u=x+1\\du=dx}}{=\!=} \int \frac{1}{u}\,du \Big|_{u=x+1}$$

$$= \ln|u| + \text{"}C\text{"} \Big|_{u=x+1}$$

$\int \frac{1}{u}\,dx = \begin{cases} \ln|u|+c_1 & \text{for } u>0 \\ \ln|u|+c_2 & \text{for } u<0 \end{cases}$

$$= \begin{cases} \ln|x+1| + c_1 & \text{for } x>-1 \\ \ln|x+1| + c_2 & \text{for } x<-1 \end{cases}$$

$$= \boxed{\begin{cases} \ln(x+1) + c_1 & \text{for } x>-1 \\ \ln(-(x+1)) + c_2 & \text{for } x<-1 \end{cases} \quad \text{over } I=(-\infty,-1)\cup(-1,\infty)}$$

2. f(x)=(2x+3)/(x^2+3x+2)

$P(x) = 2x+3$ linear

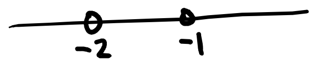
$Q(x) = x^2+3x+2$ quadratic

Sol: To compute this integral, we must first write

$$\frac{2x+3}{x^2+3x+2} = \frac{2x+3}{(x+1)(x+2)} = \frac{(x+1)+(x+2)}{(x+1)(x+2)}$$

$$= \frac{x+1}{(x+1)(x+2)} + \frac{x+2}{(x+1)(x+2)}$$

$$= \frac{1}{x+2} + \frac{1}{x+1} \quad \text{for } x \neq -2, -1$$



This gives

$$\int \frac{2x+3}{x^2+3x+2}\,dx = \int \frac{1}{x+2}\,dx + \int \frac{1}{x+1}\,dx$$

$$= \ln|x+2| + \ln|x+1| + \text{"}c\text{"}$$

$$= \boxed{\begin{cases} \ln|x+2| + \ln|x+1| + c_1 & \text{for } x < -2 \\ \ln|x+2| + \ln|x+1| + c_2 & \text{for } -2 < x < -1 \\ \ln|x+2| + \ln|x+1| + c_3 & \text{for } x > -1 \end{cases} \\ \text{over } I = (-\infty,-2) \cup (-2,-1) \cup (-1,\infty)}$$

Easy Partial Fractions: Suppose P,Q are polynomials, and suppose the degree of P is strictly less than the degree of Q. If we can factor Q into distinct linear factors

$$Q(x) = (a_1x+b_1)\cdot \ldots \cdot (a_kx+b_k)$$

then we can decompose P(x)/Q(x) into partial fractions: there are constants $A_1,\ldots,A_k$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x_1+b_1} + \cdots + \frac{A_k}{a_kx+b_k}$$

Ex: Compute the most general antiderivative of $f(x)= \frac{x^2+2x+2}{x^2-1}$ over the largest possible open subset I of R, and give I.

Sol: First, we must use long division!

$$\begin{array}{r|l} & \quad 1 \\ x^2-1 & \overline{\;x^2+2x+2} \\ & \underline{-x^2 \qquad +1} \\ & \qquad 2x+3 \end{array} \Rightarrow \frac{x^2+2x+2}{x^2-1} = 1 + \underbrace{\frac{2x+3}{x^2-1}}_{\text{easy partial fractions}}$$

Easy Partial Fractions says that there are constants A,B so that

$$\frac{2x+3}{x^2-1} = \frac{2x+3}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

To solve for A,B, we combine the right-hand side again,

$$\frac{2x+3}{x^2-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)} = \frac{(A+B)x+(B-A)}{x^2-1}$$

$$\Rightarrow \quad 2x+3 = (A+B)x + (B-A)$$

$$\Rightarrow \quad \begin{aligned} A+B&=2 \\ -A+B&=3 \end{aligned}$$

Fact: If P,Q are polynomials and P(x)=Q(x) for all x, then the coefficients of P,Q are equal.

Using linear algebra

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{1+1}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2-3 \\ 2+3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 5/2 \end{bmatrix}$$

This gives

$$\frac{2x+3}{x^2-1} = \frac{-1/2}{x+1} + \frac{5/2}{x-1} \quad \text{for } x \neq -1,1$$

Thus

$$\int \frac{x^2+2x+2}{x^2-1}\,dx = \int 1 + \frac{2x+3}{x^2-1}\,dx$$

$$\underset{EPF}{=} \int 1 + \frac{-1/2}{x+1} + \frac{5/2}{x-1} \, dx$$

$$= 1 - \frac{1}{2}\ln|x+1| + \frac{5}{2}\ln|x-1| + \text{"}C\text{"}$$

$$= \boxed{\begin{array}{l} \begin{cases} 1 - \frac{1}{2}\ln|x+1| + \frac{5}{2}\ln|x-1| + c_1 & \text{for } x < -1 \\ 1 - \frac{1}{2}\ln|x+1| + \frac{5}{2}\ln|x-1| + c_2 & \text{for } -1 < x < 1 \\ 1 - \frac{1}{2}\ln|x+1| + \frac{5}{2}\ln|x-1| + c_2 & \text{for } x > 1 \end{cases} \\ \text{over } I = (-\infty,-1) \cup (-1,1) \cup (1,\infty) \end{array}}$$

6.4 Integration with Tables and Computer Algebra Systems

Fact: You can use the Table of Integrals on Reference Page 6-10 in the back of the book for the Homework. But for the MIDTERM and FINAL, you only need to memorize the Basic Table of Integrals (from Week 1).