VECTOR CALCULUS, Week 9

10.7 Vector Functions and Space Curves; 10.8 Arc Length and Curvature

10.7 Vector Functions and Space Curves

Def: A **parametric vector-valued function** is a function of the form

$\vec{r} : [a, b] \to \mathbf{R}^2$	or	$\vec{r} : [a, b] \to \mathbf{R}^3$
$\vec{r}(t) =< x(t), y(t) >$		$\vec{r}(t) =< x(t), y(t), z(t) >$
parametric plane curve		parametric space curve.

We say t is the **parameter**, and we say the **real-valued** functions $x, y, z : [a, b] \to \mathbf{R}$ are the **components** of $\vec{r}$. We say the set

$$\{\vec{r}(t) : t \in [a, b]\} \subset \mathbf{R}^2 \text{ or } \mathbf{R}^3$$

is the **image** of $\vec{r}$.

Ex: Sketch the image of the following parametric space curves.

1. $\vec{r}(t) =< \cos t, \sin t, t >$ for $0 \leq t \leq 2\pi$ the helix
2. $\vec{r}(t) =< t, t^2, t^3 >$ for $-1 \leq t \leq 1$ the twisted cubic

Ex: Find a parametric space curve $\vec{r}$ over an interval $[a, b]$ so that the image of $\vec{r}$ is the intersection between the unit cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Def: Consider a parametric vector-valued function $\vec{r}$ defined for t near a.

- We say $\vec{r}$ is differentiable at $t = a$ if and only if the component functions of $\vec{r}$ are differentiable at $t = a$.

  This occurs if and only if the following limit exists:

  $$\left.\frac{d\vec{r}}{dt}\right|_{t=a} = \vec{r}\,'(a) = \lim_{t \to a} \frac{\vec{r}(t) - \vec{r}(a)}{t - a} =< x'(a), y'(a), z'(a) > .$$

- We say $\vec{r}\,'(a)$ is the **tangent vector** of $\vec{r}$ at $t = a$.

- We say $|\vec{r}\,'(a)|$ is the **speed** of $\vec{r}$ at $t = a$.

- If $\vec{r}\,'(a) \neq \vec{0}$, then we say the **tangent line of $\vec{r}$ at** $t = a$ is the line in space through $\vec{r}(a)$ in the direction of $\vec{r}\,'(a)$.

- If $\vec{r}\,'(t)$ exists for all t near a and is differentiable at a, then we let $\vec{r}\,''(a) = \frac{d}{dt}\vec{r}\,'(t)|_{t=a}$ denote the **second derivative of $\vec{r}$ at** $t = a$.

Ex: Consider the helix $\vec{r}(t) =< \cos t, \sin t, t >$ for $t \in \mathbf{R}$.

1. Compute the tangent vector and speed of $\vec{r}$ at $t = \pi/2$.
2. Compute the tangent line of $\vec{r}$ at $t = \pi/2$.

Ex: Consider $\vec{r}(t) =< \frac{t^2}{2}, \frac{t^3}{3} >$ for $t \in \mathbf{R}$.

1. Compute the tangent vector and speed of $\vec{r}$ at $t = 1$.
2. Compute the tangent line of $\vec{r}$ at $t = 1$.

Fact: Suppose $\vec{r}$ is a vector-valued function defined over $[a, b]$, and suppose $f : [\alpha, \beta] \to [a, b]$ is differentiable (and so continuous).

- Suppose f is increasing with

  $$f(\alpha) = a \text{ and } f(\beta) = b,$$

  and define the parametric vector-valued function

  $$\vec{r}_f(s) = \vec{r}(f(s)) \text{ for } \alpha \leq s \leq \beta.$$

  Then $\vec{r}, \vec{r}_f$ have the same images. However, $\vec{r}_f$ traces the image of $\vec{r}$ with different speed. In fact,

  $$|\vec{r}_f(s)| = |f'(s)\vec{r}\,'(f(s))| = f'(s)|\vec{r}\,'(s)|.$$

- Suppose f is decreasing with

  $$f(\alpha) = b \text{ and } f(\beta) = a,$$

  and define the parametric vector-valued function

  $$\vec{r}_f(s) = \vec{r}(f(s)) \text{ for } \alpha \leq s \leq \beta.$$

  Then $\vec{r}, \vec{r}_f$ have the same images. However, $\vec{r}_f$ traces the image of $\vec{r}$ in the opposite direction and with different speed. In fact,

  $$|\vec{r}_f(s)| = |f'(s)\vec{r}\,'(f(s))| = (-f'(s))|\vec{r}\,'(s)|.$$

Def: We say $\vec{r}_f$ is a **reparameterization** of $\vec{r}$.

Ex: Consider the helix

$$\vec{r}(t) =< \cos t, \sin t, t > \text{ for } t \in \mathbf{R}.$$

Recall that

$$\vec{r}(\pi/2) =< 0, 1, \pi/2 >, \vec{r}\,'(\pi/2) =< -1, 0, 1 >, \text{ and } |\vec{r}\,'(t)| = \sqrt{2}.$$

1. Suppose $f(s) = 2s$, and consider $\vec{r}_f = \vec{r}_f(s)$. Compute $\vec{r}_f(\pi/4)$, and compute the tangent vector and speed of $\vec{r}_f$ at $s = \frac{\pi}{4}$.

2. Suppose $f(s) = \pi - s$, and consider $\vec{r}_f = \vec{r}_f(s)$. Compute $\vec{r}_f(\pi/2)$, and compute the tangent vector and speed of $\vec{r}_f$ at $s = \frac{\pi}{2}$.

10.8 Arc Length and Curvature

Let $\vec{0} = < 0, 0 >$ or $= < 0, 0, 0 >$.

Def: Suppose $\vec{r}$ is a parametric vector-valued function defined over $[a, b]$. We say $\vec{r}$ is **regular/smooth** if the component functions of $\vec{r}$ are continuously differentiable over $[a, b]$ with $\vec{r}\,'(t) \neq \vec{0}$ for each $t \in [a, b]$.

Ex: $\vec{r_1}(t) = < t^3, t^3 >$ is **not** regular at $t = 0$, while $\vec{r_2}(t) = < t, t >$ is regular. The image of both $\vec{r}_1, \vec{r}_2$ is the line $y = x$.

Fact: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over $[a, b]$, and suppose $c \in [a, b]$. If $\vec{r}\,''(c)$ exists with $\vec{r}\,''(c) \neq \vec{0}$, then there is a unique circle which is tangent to the image of $\vec{r}$ at $\vec{r}(c)$.

Def: We call this circle the **osculating circle of $\vec{r}$ at** $t = c$.

Ex:

1. Lines do not have unique tangent circles.
2. The osculating circle of a circle is itself.

Def: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over $[a, b]$, and suppose $\vec{r}\,''(t)$ exists for each $t \in [a, b]$. We define the **curvature function** $\kappa : [a, b] \to [0, \infty)$ to be

$$\kappa(t) = \begin{cases} 0 & \text{if } \vec{r}\,''(t) = \vec{0} \\ \frac{1}{\text{radius of the osculating circle of } \vec{r} \text{ at } t} & \text{if } \vec{r}\,''(t) \neq \vec{0} \end{cases}$$

Fact: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over $[a, b]$, and suppose $\vec{r}\,''(t)$ exists for each $t \in [a, b]$.

If $\kappa(t) > 0$ is large, then the radius of the osculating circle is small.
If $\kappa(t) > 0$ is small, then the radius of the osculating circle is big.
If $\kappa(t) = 0$, the radius of the osculating circle is infinity,
in which case the osculating "circle" is the tangent line.

We do not define

$$\kappa(t) = \text{ radius of the osculating circle of } \vec{r} \text{ at } t$$

because if $\vec{r}\,''(t) = \vec{0}$, then the radius of the osculating circle is infinity. The actual definition of κ guarantees that $\kappa(t)$ is always a finite value.

We can compute $\kappa(t)$ as follows.

- If $\vec{r} : [a, b] \to \mathbf{R}^3$, then

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \text{ for } t \in [a, b].$$

- If $\vec{r} : [a, b] \to \mathbf{R}^2$, then **embed** $\vec{r}$ into $\mathbf{R}^3$ by setting

$$\vec{r}(t) =< x(t), y(t), 0 > \text{ for } a \leq t \leq b.$$

  We can now compute

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{((x'(t))^2 + (y'(t))^2)^{3/2}} \text{ for } t \in [a, b].$$

- Suppose $f : [a, b] \to \mathbf{R}$ and suppose

$$\vec{r}(t) =< t, f(t) > \text{ for } a \leq t \leq b,$$

  then

$$\kappa(t) = \frac{|f''(t)|}{(1 + (f'(t))^2)^{3/2}} \text{ for } t \in [a, b].$$

Ex: Compute the curvature function for each of the following.

1. $\vec{r}(t) =< 2\cos 3t, 2\sin 3t >$
2. $\vec{r}(t) =< \cos t, \sin t, t >$

Fact: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over $[a,b]$, and suppose $\vec{r}$ has only **isolated self-intersections**. The arc length L of the image of $\vec{r}$ is given by

$$L = \int_a^b |\vec{r}\,'(t)|\;dt = \begin{cases} \displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\;dt \text{ or} \\ \displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\;dt \end{cases}$$

Def: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over $[a,b]$ with only isolated self-intersections, and suppose L is the arc length of the image of $\vec{r}$. We define the **arc length function** $s : [a,b] \to [0,L]$ **of** $\vec{r}$ to be the function

$$s(t) = \int_a^t |\vec{r}\,'(u)|\;du \text{ for } a \le t \le b.$$

Ex: Compute the arc length L of the image and the arc length function $s(t)$ for each of the following parametric vector-valued functions.

1. $\vec{r}(t) =< 2\cos 3t, 2\sin 3t >$ for $0 \le t \le \frac{2\pi}{3}$
2. $\vec{r}(t) =< \cos t, \sin t, t >$ for $0 \le t \le 2\pi$

Fact: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over $[a, b]$ with only isolated self-intersections, and suppose $|\vec{r}\,'(t)| = 1$ for each $t \in [a, b]$.

- The arc length function s of $\vec{r}$ is $s(t) = t - a$, and the arc length of $\vec{r}$ is $L = b - a$.
- If $\vec{r}\,''(t)$ exists for each $t \in [a, b]$, then $\kappa(t) = |\vec{r}\,''(t)|$

Def: Suppose $\vec{r}$ is a regular parametric vector-valued function defined over $[a, b]$, and suppose that $|\vec{r}\,'(t)| = 1$ for each $t \in [a, b]$, then we say $\vec{r}$ is a **unit-speed parametric vector-valued function**, or $\vec{r}$ **is parameterized by arc length**.

Ex: Reparameterize the following parametric vector-valued functions so that they are parameterized by arc length. More precisely, find a real-valued function $f = f(s)$ so that $\vec{r}_f$ is a unit-speed parametric vector-valued function, and give $\vec{r}_f = \vec{r}_f(s)$.

1. $\vec{r}(t) =< 2\cos 3t, 2\sin 3t >$ for $0 \leq t \leq \frac{2\pi}{3}$
2. $\vec{r}(t) =< \cos t, \sin t, t >$ for $0 \leq t \leq 2\pi$