VECTOR CALCULUS, Week 3

3.7 Antiderivatives; 4.2 The Definite Integral; 4.3 Evaluating Definite Integrals; 4.4 The Fundamental Theorem of Calculus; 4.5 The Substitution Rule; 5.6 Inverse Trigonometric Functions; 6.1 Integration by Parts; 6.6 Improper Integrals

3.7 Antiderivatives

Def: Suppose $I \subseteq \mathbf{R}$.

- An **open interval** is an interval of the form $(a, b), (a, \infty), (-\infty, a)$.
- We say I **is an open subset of R** if and only if for each $x \in I$ there is an open interval (a, b) containing x with $(a, b) \subseteq I$.

Def: Suppose f is defined over an open subset I of $\mathbf{R}$.

- If F is a function so that $F'(x) = f(x)$ for all $x \in I$, then we say F **is an antiderivative of** f **over** I.
- The **general antiderivative of** f **over** I is the collection of all antiderivatives of f over I.

$\int f(x)\, dx$ means to find the most general antiderivative of f over the largest possible open subset I of $\mathbf{R}$, and is called the **indefinite integral of f**.

Ex: Suppose $f(x) = x^{-3}$.

1. Show that $F(x) = \begin{cases} \frac{x^{-2}}{-2} + 1 & \text{for } x < 0 \\ \frac{x^{-2}}{-2} + 2 & \text{for } x > 0 \end{cases}$ is an antiderivative of f over $I = (-\infty, 0) \cup (0, \infty)$.
2. Compute the most general antiderivative of f over the largest possible open subset I of $\mathbf{R}$, and give I.

4.3 Evaluating Definite Integrals (Table of Basic Antiderivatives)

Thm (Table of Basic Antiderivatives):

- $\int\, dx = \int 1\ dx = x + c$ over $I = (-\infty, \infty)$
- If $r \neq -1$, then
$$\int x^r\ dx = \frac{x^{r+1}}{r+1} + \text{“}c\text{”}$$
  - This is over $I = (-\infty, \infty)$, or $I = (0, \infty)$, or $I = (-\infty, 0) \cup (0, \infty)$, depending on the value of r.
  - If $I = (-\infty, 0) \cup (0, \infty)$, then you must compute
$$\int x^r\ dx = \begin{cases} \frac{x^{r+1}}{r+1} + c_1 & \text{for } x < 0 \\ \frac{x^{r+1}}{r+1} + c_2 & \text{for } x > 0 \end{cases}$$

    For example, simply writing $\int x^{-3}\ dx = \frac{x^{-2}}{-2} + c$ is wrong.
- Trigonometric functions: $\begin{cases} \displaystyle\int \cos x\ dx = \sin x + c \\ \displaystyle\int \sin x\ dx = -\cos x + c \\ \displaystyle\int \frac{1}{1+x^2}\ dx = \arctan x + c \end{cases}$
over $I = (-\infty, \infty)$
- $\int e^x\ dx = e^x + c$ over $I = (-\infty, \infty)$
- $\int x^{-1}\ dx = \begin{cases} \ln|x| + c_1 & \text{for } x < 0 \\ \ln|x| + c_2 & \text{for } x > 0 \end{cases}$
over $I = (-\infty, 0) \cup (0, \infty)$

Def: In the Table of Basic Antiderivatives, we say c is the **constant of integration**.

Thm (Basic Antiderivative Rules): Suppose I is an open interval, and suppose F, G are respectively antiderivatives of f, g over I.

- **Simplification Rule:** If $f(x) = g(x)$ for all $x \in I$, then $\int f(x)\ dx = \int g(x)\ dx = G(x) + c$ over I.
- **Addition Rule:** $\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx = F(x) + G(x) + c$ over I.
- **Height Rule:** If $a \in \mathbf{R}$, then $\int af(x)\ dx = a \int f(x)\ dx = aF(x) + c$ over I.

Ex: Compute the most general antiderivative of the given function f over the largest possible open subset I of $\mathbf{R}$, and give I.

1. $f(x) = (x+4)(2x+1)$
2. $f(x) = \frac{x^2+2}{x^2+1}$

4.2 The Definite Integral

Def: Suppose f is continuous over $[a, b]$.

- We define the **area under the graph of f from a to b** to be the *signed* area of the region bounded by the x-axis, the graph of f, and the lines $x = a$ and $x = b$: we add the area of any regions above the x-axis, while *subtract* the area of any regions below the x-axis.

- We say the **definite integral of f from a to b**, denoted $\int_a^b f(x)\ dx$, is the area under the graph of f from a to b, and we say a, b **are the limits of the integral**. We also define

$$\int_a^a f(x)\ dx = 0 \text{ and } \int_b^a f(x)\ dx = -\int_a^b f(x)\ dx.$$

Ex: Compute $\int_{-1}^1 \sqrt{1 - x^2}\ dx$.

Thm (Basic Geometric Properties): Suppose f, g are continuous over $[a, b]$.

- **Simplification Rule:** if $f(x) = g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\ dx = \int_a^b g(x)\ dx$.

- **Addition Rule:** $\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$.

- **Base Rule:** if $c \in [a, b]$, then $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$.

  The same is true for general $a, b, c \in \mathbf{R}$, as long as f is continuous *between* a, b, c, so that the definite integrals are defined.

- **Height Rule:** if $c \in \mathbf{R}$, then $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$.

- **Height Comparison Rule:** if $f(x) \leq g(x)$ for $x \in [a, b]$, then $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$.

Ex: Suppose f is continuous over $[1, 5]$ with $\int_1^5 f(x)\ dx = 12$ and $\int_4^5 f(x)\ dx = -7$, compute $\int_1^4 2f(x)\ dx$.

4.3 Evaluating Definite Integrals (Net Change Theorem)

FToC (Net Change Thm): Suppose f is continuous over $[a, b]$, and suppose F is an antiderivative of f over $[a, b]$. Then

$$\int_a^b f(x)\ dx = F(x)|_{x=a}^b = F(b) - F(a).$$

We also have

$$\int_a^a f(x)\ dx = 0 = F(a) - F(a) \text{ and}$$
$$\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx = -(F(b) - F(a)) = F(a) - F(b).$$

Ex: Net Change Thm.

1. Compute $\int_2^5 x^2\ dx$.
2. WRONG: $\int_{-1}^2 x^{-3}\ dx = \frac{-1}{2} x^{-2}|_{x=-1}^2$.

4.4 The Fundamental Theorem of Calculus (Area Function Theorem)

Def: Suppose f is continuous over $[a, b]$, and suppose $c \in [a, b]$. We define the **area function** F **of** f **over** $[a, b]$ **centered at** c to be the function

$$F(x) = \int_c^x f(t)\ dt \text{ for } x \in [a, b].$$

Ex: Compute the area function F of $f(x) = x^2$ over $(-\infty, \infty)$ centered at $c = 1$.

FToC (Area Function Thm): Suppose f is continuous over $[a, b]$, and suppose F is the area function of f over $[a, b]$ centered at $c \in [a, b]$. Then $F'(x) = f(x)$ for each $x \in (a, b)$.

$\Rightarrow$ Every continuous function has an antiderivative.

Ex: Compute the following derivatives.

1. $\frac{d}{dx} \int_0^x t + t^3\ dt$
2. $\frac{d}{dx} \int_0^x e^{t^2}\ dt$

4.5 The Substitution Rule

Chain Rule/u-Substitution Rule for Integrals: Suppose $g(x)$ is differentiable over $[a, b]$, $g'(x)$ is continuous over $[a, b]$, and $f(u)$ is continuous at each $u = g(x)$ for $x \in [a, b]$.

Indefinite Integral: $\int f(g(x))g'(x)\ dx = \int f(u)\ du\Big|_{u=g(x)}$ over (a, b)

Definite Integral: $\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$

Ex: Compute the most general antiderivative of the given f over the largest possible open subset I of $\mathbf{R}$, and give I.

1. $f(x) = 2xe^{x^2}$
2. $f(x) = x^3\sqrt{1 + x^2}$

Ex: Compute the following definite integrals.

1. $\int_1^2 \frac{1}{(1+\sqrt{x})^4}\ dx$
2. $\int_{-1}^1 x^4 \sin x\ dx$

Fact: Suppose $a > 0$, and suppose f is continuous over $[-a, a]$ with $f(-x) = -f(x)$ for all $x \in [-a, a]$, then $\int_{-a}^a f(x)\ dx = 0$.

6.1 Integration by Parts

WARNING: $\int_a^b f(x)g(x)\ dx \neq \int_a^b f(x)\ dx \int_a^b g(x)\ dx$ in general.

Integration by Parts: Suppose f, g are differentiable over $[a, b]$, and suppose f', g' are continuous over $[a, b]$.

> Indefinite integral: $\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$ over (a, b).
>
> Definite integral: $\int_a^b f(x)g'(x)\ dx = f(x)g(x)|_{x=a}^{b} - \int_a^b f'(x)g(x)\ dx$

Proof: Use the Product Rule.

Ex: Compute $\int_0^{\frac{\pi}{2}} x \sin x\ dx$.

Ex: Compute the most general antiderivative of the given h over the largest possible open subset I of $\mathbf{R}$, and give I.

1. $h(x) = x^2 e^x$
2. $h(x) = \ln x$

6.6 Improper Integrals

Def: We define the following **improper integrals**.

- Suppose f is continuous over $(a, b]$. We define

$$\int_a^b f(x)\ dx = \lim_{t \to a^+} \int_t^b f(x)\ dx.$$

  - If the limit exists and is finite, then we say $\int_a^b f(x)\ dx$ is **convergent**.
  - If the limit is $\pm\infty$, then we say $\int_a^b f(x)\ dx$ **diverges to** $\pm\infty$ and write $\int_a^b f(x)\ dx = \pm\infty$.
  - If the limit does not exist in the extended sense, then we say $\int_a^b f(x)\ dx$ is **divergent**.

- Suppose f is continuous over $[a, \infty)$. We define

$$\int_a^\infty f(x)\ dx = \lim_{t \to \infty} \int_a^t f(x)\ dx.$$

  - If the limit exists and is finite, then we say $\int_a^\infty f(x)\ dx$ is **convergent**.
  - If the limit is $\pm\infty$, then we say $\int_a^\infty f(x)\ dx$ **diverges to** $\pm\infty$ and write $\int_a^\infty f(x)\ dx = \pm\infty$.
  - If the limit does not exist in the extended sense, then we say $\int_a^\infty f(x)\ dx$ is **divergent**.

We similarly define improper integrals if f is continuous over $[a, b), (a, b), (-\infty, a], (a, \infty), (-\infty, a)$.

Ex: Determine whether the following converge, diverge to $\pm\infty$, or diverge.

1. $\int_1^\infty x^{-2}\ dx$
2. $\int_0^1 x^{-1/2}\ dx$
3. $\int_0^1 x^{-2}\ dx$

Fact: Suppose $a > 0$.

- $\int_0^a x^p \, dx$ is convergent for $p > -1$ and diverges to ∞ for $p \leq -1$.
- $\int_a^\infty x^p \, dx$ is convergent for $p < -1$ and diverges to ∞ for $p \geq -1$.

Similar is true for integrals over $[-a, 0), (-\infty, -a)$.

Comparison Thm: Suppose f, g are continuous over $(a, b]$, and suppose $f(x) \geq g(x) \geq 0$ for all $x \in (a, b]$.

- If $\int_a^b f(x) \, dx$ is convergent, then $\int_a^b g(x)$ is convergent.
- If $\int_a^b g(x) \, dx = \infty$, then $\int_a^b f(x) = \infty$.

Similar rules hold for other types of improper integrals.

Ex: Use the Comparison Thm to determine whether the following converge, diverge to $\pm\infty$, or diverge.

1. $\int_0^{\pi/3} \frac{\cos x}{x^2} \, dx$
2. $\int_0^{\pi} \frac{\sin x}{\sqrt{x}} \, dx$
3. $\int_1^\infty e^{-x^2} \, dx$

Ex: $\int_0^\infty \sin x \, dx$ is divergent.

5.6 Inverse Trigonometric Functions

Fact: Defining the inverse of an increasing/decreasing function over an interval.

- If f is increasing over $[a,b]$, then we can define
$$f^{-1} : [f(a), f(b)] \to [a,b].$$
- If f is decreasing over $[a,b]$, then we can define
$$f^{-1} : [f(b), f(a)] \to [a,b].$$

Def: Inverse trigonometric functions.

- arctangent/inverse tangent
  $\Rightarrow \arctan x = \tan^{-1} x$ for all x
  $\Rightarrow \tan(\arctan x) = x$ for all x and $\arctan(\tan\theta) = \theta$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$
- arccosine/inverse cosine
  $\Rightarrow \arccos x = \cos^{-1} x$ for $x \in [-1,1]$
  $\Rightarrow \cos(\cos^{-1} x) = x$ for $x \in [-1,1]$ and $\cos^{-1}(\cos\theta) = \theta$ for $\theta \in [0,\pi]$
- arcsine/inverse sine
  $\Rightarrow \arcsin x = \sin^{-1} x$ for $x \in [-1,1]$
  $\Rightarrow \sin(\sin^{-1} x) = x$ for $x \in [-1,1]$ and $\sin^{-1}(\sin\theta) = \theta$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Fact: Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \text{ for } x \in (-1,1)$$
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \text{ for } x \in (-1,1)$$

Proof: Use the Chain Rule.