Solutions to Michael Spivak's Calculus

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*This document would have never been made without the hiring of my employer! Thank you!

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Preface

What is the point of writing a solution manual when one, written by the author, has already existed?

The choice has two personal reasons.

Reason #1: I was first enrolled in Aalto University School of Business back in 2012 as a freshman. Subsequently, I ascertained that the teaching of Aalto is really good for business students, but I am neither a businessman nor an any other type of student: I am a geek who loves his freedom of academic pursuits; therefore, writing the solution manual is simply the way I satisfy myself with one of the legendary books in mathematics.

Reason #2: I have a long history with the book. Since Calculus plays a really important role in Statistics, which eventually plays an important role in Economics(and almost every other fields of science), I searched intensively for a thorough book that *really* teaches Calculus without ommitting the details back when I was a freshman in 2012, and it turned out that Spivak was my choice. However, due to the already intensive teaching of my program, I barely had any time to read the book properly. Moreover, due to my status of not being a native Finnish citizen, I really had so many problems both in term of finance and of legality to be in Finland because of my staunch determination of not wanting to go back to my home country, Vietnam. This further prevents me from reading the book in an edifying way. And now after having acquainted myself with the basic of LaTeX 2_ε, I really want to do as many projects in mathematics as I can, and Spivak's Calculus is obviously my first choice.

Moreover, I really do believe that originality is not something that comes from following a particular way of doing things but rather a strange, personal journey made, written and discovered on your own regardless of how silly and redundant it might be!

Vantaa, Finland
25th June,2017.

Contents

Part I

Prologue

I held every man a debtor
to his profession. . .

Francis Bacon.

Chapter 1

Basic properties of number

Problem 1.1. Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$

(ii) $x^2 - y^2 = (x - y)(x + y)$

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$

(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

(v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

(vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (There is a particularly easy way to do this using (iv), and it will show you how to find a factorization for $x^n + y^n$ whenever n is odd.)

Solution. (i) By (P7)(Existence of multiplicative inverses), there exists a^{-1} such that,

$$\begin{aligned}(a^{-1} \cdot a)x &= (a^{-1} \cdot a)\\ x &= 1\end{aligned}$$

(ii) By (P9) for 2 times,

$$\begin{aligned}(x - y)(x + y) &\overset{1}{=} x \cdot (x + y) + (-y) \cdot (x + y)\\ &\overset{2}{=} x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y\\ &= x^2 + x \cdot y + [-(x \cdot y)] + [-(y^2)]\\ &= x^2 - y^2\end{aligned}$$

(iii) From (ii) and since $x^2 = y^2$,

$$x^2 - y^2 = (x - y)(x + y) = 0$$

This means $(x - y) = 0 \vee (x + y) = 0$, which is $x = y \vee x = -y$

(iv) Starting with the right-hand side,

$$\begin{aligned}(x - y)(x^2 + xy + y^2) &= x \cdot (x^2 + xy + y^2) + (-y) \cdot (x^2 + xy + y^2) \\ &= x^3 + x^2y + xy^2 + [-(x^2y)] + [-(xy^2)] + [-(y)^3] \\ &= x^3 - y^3\end{aligned}$$

(v) I propose two solutions for this problem. The first one is the direct right-hand side manipulation, while the latter is done by induction.

The first solution.

$$\begin{aligned}&(x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1} \\ &+[-(x^{n-1}y)] + [-(x^{n-2}y^2)] + \cdots + [-(xy^{n-1})] + [-(y^n)] \\ &= x^n - y^n\end{aligned}$$

Q.E.D

The second solution. Let n=1, then indeed $x - y = x - y$. Suppose the statement holds true for $n = k$ with $k \in \mathbb{N}$, that is

$$x^k - y^k = (x - y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})$$

is true. To finish the proof, we need to prove

$$x^{k+1} - y^{k+1} = (x - y)(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k)$$

That is, the statement holds for $n = k$. Starting from the left hand side,

$$\begin{aligned}& x^{k+1} - y^{k+1} \\ =\; & x^{k+1} - x^k y + x^k y - y^{k+1} \\ =\; & x^k(x - y) + y(x^k - y^k) \\ =\; & x^k(x - y) + y(x - y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1}) \\ =\; & (x - y)[x^k + y(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})] \\ =\; & (x - y)(x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1} + y^k)\end{aligned}$$

Q.E.D

(vi) We will use (iv) in our proof,

$$
\begin{aligned}
& x^3 + y^3 \\
= \quad & x^3 - y^3 + 2y^3 \\
= \quad & (x-y)(x^2+xy+y^2) + 2y[(x^2+xy+y^2) + (-x)(x+y)] \\
= \quad & (x+y)(x^2+xy+y^2) + 2[-(xy)](x+y) \\
= \quad & (x+y)(x^2-xy+y^2)
\end{aligned}
$$

■

Problem 1.2. What is wrong with the following "proof"? Let $x = y$. Then

$$
\begin{aligned}
x^2 &= xy, \\
x^2 - y^2 &= xy - y^2, \\
(x+y)(x-y) &= y(x-y), \\
x + y &= y, \\
2y &= y, \\
2 &= 1.
\end{aligned}
$$

Solution. Note that in the transition from line 3 to line 4, the author "simplifies" $(x-y)$ by dividing $(x-y)$ on both sides. This is wrong since $x - y = 0$, and hence $1/0$ is undefined as implied by (P7) in the textbook. ■

Problem 1.3. Prove the following:

(i) $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

(ii) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$, if $b, d \neq 0$.

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$. (To do this you must remember the defining property of $(ab)^{-1}$.)

(iv) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{db}$, if $b, d \neq 0$.

(v) $\dfrac{a}{b} \Big/ \dfrac{c}{d} = \dfrac{ad}{bc}$, if $b, c, d \neq 0$.

(vi) If $b, d \neq 0$, then $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$. Also determine when $\dfrac{a}{b} = \dfrac{b}{a}$.

Solution. (i) Until (iii) is proved, the solution is to test the equality between two sides.

$$\begin{aligned} a(b)^{-1} &= (ac)(bc)^{-1} \\ a[(b)^{-1}b] &= (ac)(bc)^{-1}b \\ (a^{-1}a) &= (a^{-1}a)c(bc)^{-1}b \\ 1 &= (bc)(bc)^{-1} = 1 \end{aligned}$$

(ii) Similar to the above,

$$\begin{aligned} a(b)^{-1} + c(d)^{-1} &= (ad+bc)(bd)^{-1} \\ a(b)^{-1}bd + c(d)^{-1}bd &= (ad+bc)[(bd)^{-1}(bd)] \\ ad(b^{-1}b) + bc(d^{-1}d) &= (ad+bc) \\ ad+bc &= ad+bc \end{aligned}$$

(iii) Since $a, b \neq 0$, there exists $(ab)^{-1}, a^{-1}, b^{-1}$ such that,

$$\begin{aligned} ab &= ab \\ (ab)^{-1}(ab) &= (ab)^{-1}(ab) = 1 \\ (ab)^{-1}a(bb^{-1}) &= b^{-1} \\ (ab)^{-1}(aa^{-1}) &= b^{-1}a^{-1} \\ (ab)^{-1} &= a^{-1}b^{-1} \end{aligned}$$

(iv) For $b, d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = ab^{-1}cd^{-1} = ac(d^{-1}b^{-1}) = ac(db)^{-1} = \frac{ac}{db}$$

where the next-to-last equality follows from (iii).

(v) I first establish for any number $a \neq 0$,

$$(a^{-1})^{-1} = a$$

Let $t = a^{-1}$, we want to prove $t^{-1} = a$. Observe that

$$\begin{aligned} t &= a^{-1} \\ t \cdot (t)^{-1} &= a^{-1} \cdot (t)^{-1} \\ a \cdot 1 &= (a \cdot a^{-1}) \cdot (t)^{-1} \\ a &= (t)^{-1} \end{aligned}$$

From the left hand side of the statement,

$$\frac{a}{b}\Big/\frac{c}{d} = a(b)^{-1}[c(d)^{-1}]^{-1} = a(b)^{-1}(c)^{-1}[(d)^{-1}]^{-1} = (ad)(bc)^{-1} = \frac{ad}{bc}$$

where the second and third equality follows both from (iii) and the proof above.

(vi) Using (ii),

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ \frac{a}{b} + (-\frac{c}{d}) &= 0 \\ \frac{ad-bc}{bd} &= 0 \\ ad &= bc \end{aligned}$$

Now, put $c = b \wedge d = a$. It follows that $\frac{a}{b} = \frac{b}{a}$ if and only if $a^2 = b^2$. It follows $(a-b)(a+b) = 0$, or $a = b \vee a = -b$.

∎

Problem 1.4. Find all numbers x for which

(i) $4 - x < 3 - 2x$

(ii) $5 - x^2 < 8$

(iii) $5 - x^2 < -2$

(iv) $(x-1)(x-3) > 0$ (When is a product of two numbers positive?)

(v) $x^2 - 2x + 2 > 0$

(vi) $x^2 + x + 1 > 2$

(vii) $x^2 - x + 10 > 16$

(viii) $x^2 + x + 1 > 0$

(ix) $(x-\pi)(x+5)(x-3) > 0$

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$

(xi) $2^x < 8$

(xii) $x + 3^x < 4$

(xiii) $\frac{1}{x} + \frac{1}{1-x} > 0$

(xiv) $\dfrac{x-1}{x+1} > 0$

Solution. (i)

$$\begin{aligned} 4 - x &< 3 - 2x \\ 4 + (-x + 2x) &< 3 + (-2x + 2x) \\ (-4 + 4) + x &< -4 + 3 \\ x &< -1 \end{aligned}$$

(ii)

$$\begin{aligned} 5 - x^2 &< 8 \\ 5 - 8 &< x^2 \\ -3 &< x^2 \end{aligned}$$

Since $x^2 \geq 0 \ \forall x \in \mathbb{R}$, the inequality holds $\forall x$.

(iii)

$$\begin{aligned} 5 - x^2 &< -2 \\ 7 &< x^2 \\ 0 &< x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7}) \end{aligned}$$

Hence, either $x > \sqrt{7} \ \wedge \ x > -\sqrt{7}$ or $x < \sqrt{7} \ \wedge \ x < -\sqrt{7}$, which is $x > \sqrt{7} \ \vee \ x < -\sqrt{7}$.

(iv)

$$\begin{aligned} (x-1)(x-3) &> 0 \\ (x > 1 \ \wedge \ x > 3) &\vee (x < 1 \ \wedge \ x < 3) \\ x > 3 &\vee x < 1 \end{aligned}$$

(v)

$$\begin{aligned} x^2 - 2x + 2 &> 0 \\ (x^2 - 2x + 1) + 1 &> 0 \\ (x-1)^2 + 1 &> 0 \end{aligned}$$

Hence the inequality is satisfied $\forall \ x$.

(vi)

$$
\begin{aligned}
x^2 + x + 1 &> 2 \\
x^2 + x - 1 &> 0 \\
x^2 + \left(\frac{1+\sqrt{5}}{2}\right)x + \left(\frac{1-\sqrt{5}}{2}\right)x + \left(\frac{(1-\sqrt{5})(1+\sqrt{5})}{4}\right) &> 0 \\
\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right) &> 0 \\
x > \left(\frac{\sqrt{5}-1}{2}\right) \vee x < \left(\frac{-(\sqrt{5}+1)}{2}\right)
\end{aligned}
$$

(vii)

$$
\begin{aligned}
x^2 - x + 10 &> 16 \\
x^2 - x - 6 &> 0 \\
x^2 - 3x + 2x - 6 &> 0 \\
x(x-3) + 2(x-3) &> 0 \\
(x+2)(x-3) &> 0 \\
x > 3 &\vee x < -2
\end{aligned}
$$

(viii)

$$
\begin{aligned}
x^2 + x + 1 &> 0 \\
x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 &> 0 \\
(x + \frac{1}{2})^2 + \frac{3}{4} &> 0
\end{aligned}
$$

which is true for all x.

(ix) Divide the problem into two cases: $x > \pi$ and $x < \pi$.

Case 1: $x > \pi$
Then $(x+5)(x-3) > 0$, which is $x > 3 \ \vee \ x < -5$.

Case 2: $x < \pi$
Then $(x+5)(x-3) < 0$, which is $-5 < x < 3$.

(x)

$$
\begin{aligned}
(x - \sqrt[3]{2})(x - \sqrt{2}) &> 0 \\
x > \sqrt{2} &\vee x < \sqrt[3]{2}
\end{aligned}
$$

■