

Solutions to Michael Spivak's Calculus

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*I thank my employer!

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Preface

This is my own solutions to Michael Spivak's Calculus textbook.

To those who have taught me and have had influences on me.

Vantaa, Finland
25th June, 2017.

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Part I

Prologue

I held every man a debtor
to his profession. . .

Francis Bacon.

Chapter 1

Basic properties of number

Problem 1.1. Prove the following:

- (i) If $ax = a$ for some number $a \neq 0$, then $x = 1$
- (ii) $x^2 - y^2 = (x - y)(x + y)$
- (iii) If $x^2 = y^2$, then $x = y$ or $x = -y$
- (iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
- (v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$
- (vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (There is a particularly easy way to do this using (iv), and it will show you how to find a factorization for $x^n + y^n$ whenever n is odd.)

Solution. (i) By (P7)(Existence of multiplicative inverses), there exists a^{-1} such that,

$$\begin{aligned}(a^{-1} \cdot a)x &= (a^{-1} \cdot a) \\ x &= 1\end{aligned}$$

(ii) By (P9) for 2 times,

$$\begin{aligned}(x - y)(x + y) &\stackrel{1}{=} x \cdot (x + y) + (-y) \cdot (x + y) \\ &\stackrel{2}{=} x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y \\ &= x^2 + x \cdot y + [-(x \cdot y)] + [-(y^2)] \\ &= x^2 - y^2\end{aligned}$$

(iii) From (ii) and since $x^2 = y^2$,

$$x^2 - y^2 = (x - y)(x + y) = 0$$

This means $(x - y) = 0 \vee (x + y) = 0$, which is $x = y \vee x = -y$

(iv) Starting with the right-hand side,

$$\begin{aligned} (x - y)(x^2 + xy + y^2) &= x \cdot (x^2 + xy + y^2) + (-y) \cdot (x^2 + xy + y^2) \\ &= x^3 + x^2y + xy^2 + [-(x^2y)] + [-(xy^2)] + [-(y)^3] \\ &= x^3 - y^3 \end{aligned}$$

(v) I propose two solutions for this problem. The first one is the direct right-hand side manipulation, while the latter is done by induction.

The first solution.

$$\begin{aligned} &(x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1} \\ &\quad + [-(x^{n-1}y)] + [-(x^{n-2}y^2)] + \cdots + [-(xy^{n-1})] + [-(y^n)] \\ &= x^n - y^n \end{aligned}$$

Q.E.D

The second solution. Let $n=1$, then indeed $x - y = x - y$. Suppose the statement holds true for $n = k$ with $k \in \mathbb{N}$, that is

$$x^k - y^k = (x - y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})$$

is true. To finish the proof, we need to prove

$$x^{k+1} - y^{k+1} = (x - y)(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k)$$

That is, the statement holds for $n = k$. Starting from the left hand side,

$$\begin{aligned} &x^{k+1} - y^{k+1} \\ &= x^{k+1} - x^k y + x^k y - y^{k+1} \\ &= x^k(x - y) + y(x^k - y^k) \\ &= x^k(x - y) + y(x - y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1}) \\ &= (x - y)[x^k + y(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})] \\ &= (x - y)(x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1} + y^k) \end{aligned}$$

Q.E.D

(vi) We will use (iv) in our proof,

$$\begin{aligned}
 & x^3 + y^3 \\
 = & x^3 - y^3 + 2y^3 \\
 = & (x - y)(x^2 + xy + y^2) + 2y[(x^2 + xy + y^2) + (-x)(x + y)] \\
 = & (x + y)(x^2 + xy + y^2) + 2[-(xy)](x + y) \\
 = & (x + y)(x^2 - xy + y^2)
 \end{aligned}$$

■

Problem 1.2. What is wrong with the following “proof”? Let $x = y$. Then

$$\begin{aligned}
 x^2 &= xy, \\
 x^2 - y^2 &= xy - y^2, \\
 (x + y)(x - y) &= y(x - y), \\
 x + y &= y, \\
 2y &= y, \\
 2 &= 1.
 \end{aligned}$$

Solution. Note that in the transition from line 3 to line 4, the author “simplifies” $(x - y)$ by dividing $(x - y)$ on both sides. This is wrong since $x - y = 0$, and hence $1/0$ is undefined as implied by (P7) in the textbook. ■

Problem 1.3. Prove the following:

- (i) $\frac{a}{b} = \frac{ac}{bc}$, if $b, c \neq 0$.
- (ii) $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, if $b, d \neq 0$.
- (iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$. (To do this you must remember the defining property of $(ab)^{-1}$.)
- (iv) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{db}$, if $b, d \neq 0$.
- (v) $\frac{a}{b} \bigg/ \frac{c}{d} = \frac{ad}{bc}$, if $b, c, d \neq 0$.
- (vi) If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

Solution. (i) Until (iii) is proved, the solution is to test the equality between two sides.

$$\begin{aligned} a(b)^{-1} &= (ac)(bc)^{-1} \\ a[(b)^{-1}b] &= (ac)(bc)^{-1}b \\ (a^{-1}a) &= (a^{-1}a)c(bc)^{-1}b \\ 1 &= (bc)(bc)^{-1} = 1 \end{aligned}$$

(ii) Similar to the above,

$$\begin{aligned} a(b)^{-1} + c(d)^{-1} &= (ad + bc)(bd)^{-1} \\ a(b)^{-1}bd + c(d)^{-1}bd &= (ad + bc)[(bd)^{-1}(bd)] \\ ad(b^{-1}b) + bc(d^{-1}d) &= (ad + bc) \\ ad + bc &= ad + bc \end{aligned}$$

(iii) Since $a, b \neq 0$, there exists $(ab)^{-1}, a^{-1}, b^{-1}$ such that,

$$\begin{aligned} ab &= ab \\ (ab)^{-1}(ab) &= (ab)^{-1}(ab) = 1 \\ (ab)^{-1}a(bb^{-1}) &= b^{-1} \\ (ab)^{-1}(aa^{-1}) &= b^{-1}a^{-1} \\ (ab)^{-1} &= a^{-1}b^{-1} \end{aligned}$$

(iv) For $b, d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = ab^{-1}cd^{-1} = ac(d^{-1}b^{-1}) = ac(db)^{-1} = \frac{ac}{db}$$

where the next-to-last equality follows from (iii).

(v) I first establish for any number $a \neq 0$,

$$(a^{-1})^{-1} = a$$

Let $t = a^{-1}$, we want to prove $t^{-1} = a$. Observe that

$$\begin{aligned} t &= a^{-1} \\ t \cdot (t)^{-1} &= a^{-1} \cdot (t)^{-1} \\ a \cdot 1 &= (a \cdot a^{-1}) \cdot (t)^{-1} \\ a &= (t)^{-1} \end{aligned}$$

From the left hand side of the statement,

$$\frac{a}{b} \bigg/ \frac{c}{d} = a(b)^{-1}[c(d)^{-1}]^{-1} = a(b)^{-1}(c)^{-1}[(d)^{-1}]^{-1} = (ad)(bc)^{-1} = \frac{ad}{bc}$$

where the second and third equality follows both from (iii) and the proof above.

(vi) Using (ii),

$$\begin{aligned}\frac{a}{b} &= \frac{c}{d} \\ \frac{a}{b} + \left(-\frac{c}{d}\right) &= 0 \\ \frac{ad - bc}{bd} &= 0 \\ ad &= bc\end{aligned}$$

Now, put $c = b \wedge d = a$. It follows that $\frac{a}{b} = \frac{b}{a}$ if and only if $a^2 = b^2$. It follows $(a - b)(a + b) = 0$, or $a = b \vee a = -b$. ■

Problem 1.4. Find all numbers x for which

- (i) $4 - x < 3 - 2x$
- (ii) $5 - x^2 < 8$
- (iii) $5 - x^2 < -2$
- (iv) $(x - 1)(x - 3) > 0$ (When is a product of two numbers positive?)
- (v) $x^2 - 2x + 2 > 0$
- (vi) $x^2 + x + 1 > 2$
- (vii) $x^2 - x + 10 > 16$
- (viii) $x^2 + x + 1 > 0$
- (ix) $(x - \pi)(x + 5)(x - 3) > 0$
- (x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$
- (xi) $2^x < 8$
- (xii) $x + 3^x < 4$
- (xiii) $\frac{1}{x} + \frac{1}{1 - x} > 0$
- (xiv) $\frac{x - 1}{x + 1} > 0$

Solution. (i)

$$\begin{aligned}
 4 - x &< 3 - 2x \\
 4 + (-x + 2x) &< 3 + (-2x + 2x) \\
 (-4 + 4) + x &< -4 + 3 \\
 x &< -1
 \end{aligned}$$

(ii)

$$\begin{aligned}
 5 - x^2 &< 8 \\
 5 - 8 &< x^2 \\
 -3 &< x^2
 \end{aligned}$$

Since $x^2 \geq 0 \forall x \in \mathbb{R}$, the inequality holds $\forall x$.

(iii)

$$\begin{aligned}
 5 - x^2 &< -2 \\
 7 &< x^2 \\
 0 &< x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7})
 \end{aligned}$$

Hence, either $x > \sqrt{7} \wedge x > -\sqrt{7}$ or $x < \sqrt{7} \wedge x < -\sqrt{7}$, which is $x > \sqrt{7} \vee x < -\sqrt{7}$.

(iv)

$$\begin{aligned}
 (x - 1)(x - 3) &> 0 \\
 (x > 1 \wedge x > 3) \vee (x < 1 \wedge x < 3) \\
 x > 3 \vee x < 1
 \end{aligned}$$

(v)

$$\begin{aligned}
 x^2 - 2x + 2 &> 0 \\
 (x^2 - 2x + 1) + 1 &> 0 \\
 (x - 1)^2 + 1 &> 0
 \end{aligned}$$

Hence the inequality is satisfied $\forall x$.

(vi)

$$\begin{aligned}
& x^2 + x + 1 > 2 \\
& x^2 + x - 1 > 0 \\
& x^2 + \left(\frac{1+\sqrt{5}}{2}\right)x + \left(\frac{1-\sqrt{5}}{2}\right)x + \left(\frac{(1-\sqrt{5})(1+\sqrt{5})}{4}\right) > 0 \\
& \left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right) > 0 \\
& x > \left(\frac{\sqrt{5}-1}{2}\right) \vee x < \left(\frac{-(\sqrt{5}+1)}{2}\right)
\end{aligned}$$

(vii)

$$\begin{aligned}
& x^2 - x + 10 > 16 \\
& x^2 - x - 6 > 0 \\
& x^2 - 3x + 2x - 6 > 0 \\
& x(x-3) + 2(x-3) > 0 \\
& (x+2)(x-3) > 0 \\
& x > 3 \vee x < -2
\end{aligned}$$

(viii)

$$\begin{aligned}
& x^2 + x + 1 > 0 \\
& x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 > 0 \\
& \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0
\end{aligned}$$

which is true for all x .

(ix) Divide the problem into two cases: $x > \pi$ and $x < \pi$.*Case 1:* $x > \pi$ Then $(x+5)(x-3) > 0$, which is $x > 3 \vee x < -5$.*Case 2:* $x < \pi$ Then $(x+5)(x-3) < 0$, which is $-5 < x < 3$.

(x)

$$\begin{aligned}
& (x - \sqrt[3]{2})(x - \sqrt{2}) > 0 \\
& x > \sqrt{2} \vee x < \sqrt[3]{2}
\end{aligned}$$

(xi) (Sometimes, to solve a problem, intuition is a necessity.)

$$2^x < 8$$

$$2^x < 2^3$$

$$x < 3$$

(xii)

$$x + 3^x < 4$$

$$x + 3^x < 1 + 3^1$$

$$x < 1$$

(xiii)

$$\frac{1}{x} + \frac{1}{1-x} > 0$$

$$\frac{1}{x(1-x)} > 0$$

Hence, $x(1-x) > 0$. This means $0 < x < 1$.

(xiv)

$$\frac{x-1}{x+1} > 0$$

Hence, $(x-1)(x+1) > 0$, or $x > 1 \vee x < -1$.

■

Problem 1.5. Prove the following:

- (i) If $a < b$ and $c < d$, then $a + c < b + d$
- (ii) If $a < b$, then $-b < -a$
- (iii) If $a < b$ and $c > d$, then $a - c < b - d$
- (iv) If $a < b$ and $c > 0$, then $ac < bc$
- (v) If $a < b$ and $c < 0$, then $ac > bc$
- (vi) If $a > 1$, then $a^2 > a$
- (vii) If $0 < a < 1$, then $a^2 < a$
- (viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$
- (ix) If $0 \leq a < b$, then $a^2 < b^2$. (Use (viii).)
- (x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$. (Use (ix), backwards.)

Solution. Let P be the set of all positive numbers.

- (i) To prove this, we apply (P11): If $a < b \wedge c < d$, then $(b - a \in P) \wedge (d - c \in P)$. Then $(b - a) + (d - c) = (b + d) - (a + c) \in P$. Therefore, $a + c < b + d$.
- (ii) We provide two solutions: The first one is by Trichotomy Law (P10), and the second one is by adding $[(-a) + (-b)]$ to both sides.

Proof by Trichotomy Law. If $a < b$, then $b - a \in P$. By Trichotomy Law, $a - b \notin P$ and $a - b \neq 0$. Therefore, $a - b < 0$, which is $-b < -a$. Q.E.D

Proof by adding.

$$\begin{aligned}
 a &< b \\
 a + [(-a) + (-b)] &< b + [(-a) + (-b)] \\
 [a + (-a)] + (-b) &< [b + (-b)] + (-a) \\
 -b &< -a
 \end{aligned}$$

Q.E.D

- (iii) Using (P11), we have $b - a \in P \wedge c - d \in P$. Then $(b - a) + (c - d) \in P$. Hence, $a - c < b - d$.
- (iv) Using (P12), note that $b - a \in P$. Since $c > 0$, $c(b - a) \in P$, which means $bc - ac > 0$, or $ac < bc$.
- (v) By Trichotomy law(P10), $-c \in P$. Then by (iv), $-(ac) < -(bc)$. By (ii), $ac > bc$.
- (vi) Since $a > 1 > 0$, by (iv), $a^2 > a$.
- (vii) Since $a > 0$, by (iv), $a^2 < a$.
- (viii) Because $0 < b$, $bc < bd$. Furthermore, if $c \geq 0$, $ac \leq bc$ (equality occurs if $c = 0$), by (iv). Therefore, $ac \leq bc < bd$. Hence, $ac < bd$.
- (ix) From (viii), let $c = a$ and $d = b$, then the result follows.
- (x) Suppose $a \geq b$. Then $a \geq b \geq 0$. By (ix) and (P9), $a^2 \geq b^2$. This contradicts $a^2 < b^2$.

■

Problem 1.6. (a) Prove that if $0 \leq x < y$, then $x^n < y^n$, $n = 1, 2, 3, \dots$

- (b) Prove that if $x < y$ and n is odd, then $x^n < y^n$.
- (c) Prove that if $x^n = y^n$ and n is odd, then $x = y$.
- (d) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Solution. (a) Repeatedly apply problem 1.5(viii) for $0 \leq x < y$, we have $x^n < y^n$ with $n = 1, 2, 3, \dots$

- (b) The statement is true for the case $0 \leq x < y$. In the case $x < y \leq 0$, by 1.5(ii), $(-x) > (-y) \geq 0$. By (a), $(-x)^n > (-y)^n$ for all odd n . Since n is odd, $-(x^n) > -(y^n)$. Hence, by 1.5(ii), $x^n < y^n$. In the case $x \leq 0 < y$, since n is odd, $x^n < y^n$.
- (c) Suppose that either $x \neq y$. W.l.o.g, let $x < y$, by (b), $x^n < y^n$ for all odd n , contradicting $x^n = y^n$ for all odd n .
- (d) Suppose that both $x \neq y$ and $x \neq -y$. Then $x^2 - y^2 \neq 0$. W.l.o.g, suppose $x^2 > y^2 \geq 0$. Applying (a), this generalizes to $x^n > y^n$ for all even n , contradicting our assumption. Therefore, $x = y$ or $x = -y$.

The direct proof. In the case $x, y \geq 0$; by (a), if $x^n = y^n$ for all even n , then $x = y$. In the case $x, y \leq 0$; if $x^n = y^n$ for all even n , then $(-x), (-y) \geq 0$ and $(-x)^n = (-y)^n$, so again $x = y$. In the case of x and y have different signs, then x and $-y$ are either two positive or two negative numbers. In either subcase, if $x^n = y^n$ for all even n , then $x^n = (-y)^n$, and it follows $x = -y$ from the previous case.

■