Solutions to Michael Spivak's Calculus

Son To
<son.trung.to@gmail.com>

Ravintola Kiltakellari *

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*I thank my employer!

To my mother, friends and all those who influence me.

Preface

This is my own solutions to Michael Spivak's Calculus textbook.

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Part I

Prologue

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Chapter 1

Basic properties of number

Problem 1.1. Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$

(ii) $x^2 - y^2 = (x - y)(x + y)$

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$

(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

(v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

(vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (There is a particularly easy way to do this using (iv), and it will show you how to find a factorization for $x^n + y^n$ whenever n is odd.)

Solution. (i) By (P7)(Existence of multiplicative inverses), there exists a^{-1} such that,

$$\begin{aligned}(a^{-1} \cdot a)x &= (a^{-1} \cdot a)\\ x &= 1\end{aligned}$$

(ii) By (P9) for 2 times,

$$\begin{aligned}(x - y)(x + y) &\overset{1}{=} x \cdot (x + y) + (-y) \cdot (x + y)\\ &\overset{2}{=} x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y\\ &= x^2 + x \cdot y + [-(x \cdot y)] + [-(y^2)]\\ &= x^2 - y^2\end{aligned}$$

(iii) From (ii) and since $x^2 = y^2$,

$$x^2 - y^2 = (x-y)(x+y) = 0$$

This means $(x-y) = 0 \vee (x+y) = 0$, which is $x = y \vee x = -y$

(iv) Starting with the right-hand side,

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x \cdot (x^2+xy+y^2) + (-y) \cdot (x^2+xy+y^2) \\ &= x^3 + x^2y + xy^2 + [-(x^2y)] + [-(xy^2)] + [-(y)^3] \\ &= x^3 - y^3\end{aligned}$$

(v) I propose two solutions for this problem. The first one is the direct right-hand side manipulation, while the latter is done by induction.

The first solution.

$$\begin{aligned}&(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1} \\ &+[-(x^{n-1}y)] + [-(x^{n-2}y^2)] + \cdots + [-(xy^{n-1})] + [-(y^n)] \\ &= x^n - y^n\end{aligned}$$

Q.E.D

The second solution. Let n=1, then indeed $x - y = x - y$. Suppose the statement holds true for $n = k$ with $k \in \mathbb{N}$, that is

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})$$

is true. To finish the proof, we need to prove

$$x^{k+1} - y^{k+1} = (x-y)(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k)$$

That is, the statement holds for $n = k$. Starting from the left hand side,

$$\begin{aligned}& x^{k+1} - y^{k+1} \\ =\ & x^{k+1} - x^ky + x^ky - y^{k+1} \\ =\ & x^k(x-y) + y(x^k - y^k) \\ =\ & x^k(x-y) + y(x-y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1}) \\ =\ & (x-y)[x^k + y(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})] \\ =\ & (x-y)(x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1} + y^k)\end{aligned}$$

Q.E.D

(vi) We will use (iv) in our proof,

$$\begin{aligned} & x^3+y^3 \\ = \; & x^3-y^3+2y^3 \\ = \; & (x-y)(x^2+xy+y^2)+2y[(x^2+xy+y^2)+(-x)(x+y)] \\ = \; & (x+y)(x^2+xy+y^2)+2[-(xy)](x+y) \\ = \; & (x+y)(x^2-xy+y^2) \end{aligned}$$

■

Problem 1.2. What is wrong with the following "proof"? Let $x = y$. Then

$$\begin{aligned} x^2 &= xy, \\ x^2-y^2 &= xy-y^2, \\ (x+y)(x-y) &= y(x-y), \\ x+y &= y, \\ 2y &= y, \\ 2 &= 1. \end{aligned}$$

Solution. Note that in the transition from line 3 to line 4, the author "simplifies" $(x-y)$ by dividing $(x-y)$ on both sides. This is wrong since $x-y=0$, and hence $1/0$ is undefined as implied by (P7) in the textbook. ■

Problem 1.3. Prove the following:

(i) $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

(ii) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$, if $b, d \neq 0$.

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$. (To do this you must remember the defining property of $(ab)^{-1}$.)

(iv) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{db}$, if $b, d \neq 0$.

(v) $\dfrac{a}{b} \bigg/ \dfrac{c}{d} = \dfrac{ad}{bc}$, if $b, c, d \neq 0$.

(vi) If $b, d \neq 0$, then $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$. Also determine when $\dfrac{a}{b} = \dfrac{b}{a}$.

Solution. (i) Until (iii) is proved, the solution is to test the equality between two sides.

$$\begin{aligned}
a(b)^{-1} &= (ac)(bc)^{-1} \\
a[(b)^{-1}b] &= (ac)(bc)^{-1}b \\
(a^{-1}a) &= (a^{-1}a)c(bc)^{-1}b \\
1 &= (bc)(bc)^{-1} = 1
\end{aligned}$$

(ii) Similar to the above,

$$\begin{aligned}
a(b)^{-1} + c(d)^{-1} &= (ad+bc)(bd)^{-1} \\
a(b)^{-1}bd + c(d)^{-1}bd &= (ad+bc)[(bd)^{-1}(bd)] \\
ad(b^{-1}b) + bc(d^{-1}d) &= (ad+bc) \\
ad+bc &= ad+bc
\end{aligned}$$

(iii) Since $a, b \neq 0$, there exists $(ab)^{-1}, a^{-1}, b^{-1}$ such that,

$$\begin{aligned}
ab &= ab \\
(ab)^{-1}(ab) &= (ab)^{-1}(ab) = 1 \\
(ab)^{-1}a(bb^{-1}) &= b^{-1} \\
(ab)^{-1}(aa^{-1}) &= b^{-1}a^{-1} \\
(ab)^{-1} &= a^{-1}b^{-1}
\end{aligned}$$

(iv) For $b, d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = ab^{-1}cd^{-1} = ac(d^{-1}b^{-1}) = ac(db)^{-1} = \frac{ac}{db}$$

where the next-to-last equality follows from (iii).

(v) I first establish for any number $a \neq 0$,

$$(a^{-1})^{-1} = a$$

Let $t = a^{-1}$, we want to prove $t^{-1} = a$. Observe that

$$\begin{aligned}
t &= a^{-1} \\
t \cdot (t)^{-1} &= a^{-1} \cdot (t)^{-1} \\
a \cdot 1 &= (a \cdot a^{-1}) \cdot (t)^{-1} \\
a &= (t)^{-1}
\end{aligned}$$

From the left hand side of the statement,

$$\frac{a}{b}\Big/\frac{c}{d} = a(b)^{-1}[c(d)^{-1}]^{-1} = a(b)^{-1}(c)^{-1}[(d)^{-1}]^{-1} = (ad)(bc)^{-1} = \frac{ad}{bc}$$

where the second and third equality follows both from (iii) and the proof above.

(vi) Using (ii),

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ \frac{a}{b} + (-\frac{c}{d}) &= 0 \\ \frac{ad - bc}{bd} &= 0 \\ ad &= bc \end{aligned}$$

Now, put $c = b \wedge d = a$. It follows that $\frac{a}{b} = \frac{b}{a}$ if and only if $a^2 = b^2$. It follows $(a - b)(a + b) = 0$, or $a = b \vee a = -b$.

■

Problem 1.4. Find all numbers x for which

(i) $4 - x < 3 - 2x$

(ii) $5 - x^2 < 8$

(iii) $5 - x^2 < -2$

(iv) $(x - 1)(x - 3) > 0$ (When is a product of two numbers positive?)

(v) $x^2 - 2x + 2 > 0$

(vi) $x^2 + x + 1 > 2$

(vii) $x^2 - x + 10 > 16$

(viii) $x^2 + x + 1 > 0$

(ix) $(x - \pi)(x + 5)(x - 3) > 0$

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$

(xi) $2^x < 8$

(xii) $x + 3^x < 4$

(xiii) $\frac{1}{x} + \frac{1}{1 - x} > 0$

(xiv) $\frac{x - 1}{x + 1} > 0$

Solution. (i)

$$\begin{aligned} 4 - x &< 3 - 2x \\ 4 + (-x + 2x) &< 3 + (-2x + 2x) \\ (-4 + 4) + x &< -4 + 3 \\ x &< -1 \end{aligned}$$

(ii)

$$\begin{aligned} 5 - x^2 &< 8 \\ 5 - 8 &< x^2 \\ -3 &< x^2 \end{aligned}$$

Since $x^2 \geq 0 \ \forall x \in \mathbb{R}$, the inequality holds $\forall x$.

(iii)

$$\begin{aligned} 5 - x^2 &< -2 \\ 7 &< x^2 \\ 0 &< x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7}) \end{aligned}$$

Hence, either $x > \sqrt{7} \ \wedge \ x > -\sqrt{7}$ or $x < \sqrt{7} \ \wedge \ x < -\sqrt{7}$, which is $x > \sqrt{7} \ \vee \ x < -\sqrt{7}$.

(iv)

$$\begin{aligned} (x - 1)(x - 3) &> 0 \\ (x > 1 \ \wedge \ x > 3) \vee (x < 1 \ \wedge \ x < 3) \\ x > 3 \vee x < 1 \end{aligned}$$

(v)

$$\begin{aligned} x^2 - 2x + 2 &> 0 \\ (x^2 - 2x + 1) + 1 &> 0 \\ (x - 1)^2 + 1 &> 0 \end{aligned}$$

Hence the inequality is satisfied $\forall \ x$.

(vi)

$$\begin{aligned}
x^2+x+1&>2\\
x^2+x-1&>0\\
x^2+\left(\frac{1+\sqrt{5}}{2}\right)x+\left(\frac{1-\sqrt{5}}{2}\right)x+\left(\frac{(1-\sqrt{5})(1+\sqrt{5})}{4}\right)&>0\\
\left(x+\frac{1+\sqrt{5}}{2}\right)\left(x+\frac{1-\sqrt{5}}{2}\right)&>0\\
x>\left(\frac{\sqrt{5}-1}{2}\right)\vee x<\left(\frac{-(\sqrt{5}+1)}{2}\right)
\end{aligned}$$

(vii)

$$\begin{aligned}
x^2-x+10&>16\\
x^2-x-6&>0\\
x^2-3x+2x-6&>0\\
x(x-3)+2(x-3)&>0\\
(x+2)(x-3)&>0\\
x>3\vee x<-2
\end{aligned}$$

(viii)

$$\begin{aligned}
x^2+x+1&>0\\
x^2+x+\frac{1}{4}-\frac{1}{4}+1&>0\\
(x+\frac{1}{2})^2+\frac{3}{4}&>0
\end{aligned}$$

which is true for all x.

(ix) Divide the problem into two cases: $x>\pi$ and $x<\pi$.

Case 1: $x>\pi$
Then $(x+5)(x-3)>0$, which is $x>3 \ \vee\ x<-5$.

Case 2: $x<\pi$
Then $(x+5)(x-3)<0$, which is $-5<x<3$.

(x)

$$\begin{aligned}
(x-\sqrt[3]{2})(x-\sqrt{2})&>0\\
x>\sqrt{2}\vee x<\sqrt[3]{2}
\end{aligned}$$

(xi) (Sometimes, to solve a problem, intuition is a necessity.)

$$\begin{aligned} 2^x &< 8 \\ 2^x &< 2^3 \\ x &< 3 \end{aligned}$$

(xii)

$$\begin{aligned} x + 3^x &< 4 \\ x + 3^x &< 1 + 3^1 \\ x &< 1 \end{aligned}$$

(xiii)

$$\begin{aligned} \frac{1}{x} + \frac{1}{1-x} &> 0 \\ \frac{1}{x(1-x)} &> 0 \end{aligned}$$

Hence, $x(1-x) > 0$. This means $0 < x < 1$.

(xiv)

$$\frac{x-1}{x+1} > 0$$

Hence, $(x-1)(x+1) > 0$, or $x > 1 \ \lor \ x < -1$.

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Problem 1.5. Prove the following:

(i) If $a < b$ and $c < d$, then $a + c < b + d$

(ii) If $a < b$, then $-b < -a$

(iii) If $a < b$ and $c > d$, then $a - c < b - d$

(iv) If $a < b$ and $c > 0$, then $ac < bc$

(v) If $a < b$ and $c < 0$, then $ac > bc$

(vi) If $a > 1$, then $a^2 > a$

(vii) If $0 < a < 1$, then $a^2 < a$

(viii) If $0 \le a < b$ and $0 \le c < d$, then $ac < bd$

(ix) If $0 \le a < b$, then $a^2 < b^2$. (Use (viii).)

(x) If $a, b \ge 0$ and $a^2 < b^2$, then $a < b$.(Use (ix), backwards.)

Solution. Let P be the set of all positive numbers.

(i) To prove this, we apply (P11): If $a < b \ \wedge \ c < d$, then $(b - a \in P) \ \wedge \ (d - c \in P)$. Then $(b - a) + (d - c) = (b + d) - (a + c) \in P$. Therefore, $a + c < b + d$.

(ii) We provide two solutions: The first one is by Trichotomy Law (P10), and the second one is by adding $[(-a) + (-b)]$ to both sides.

Proof by Trichotomy Law. If $a < b$, then $b - a \in P$. By Trichotomy Law, $a - b \notin P$ and $a - b \neq 0$. Therefore, $a - b < 0$, which is $-b < -a$. Q.E.D

Proof by adding.

$$\begin{gathered} a < b \\ a + [(-a) + (-b)] < b + [(-a) + (-b)] \\ [a + (-a)] + (-b) < [b + (-b)] + (-a) \\ -b < -a \end{gathered}$$

Q.E.D

(iii) Using (P11), we have $b - a \in P \ \wedge \ c - d \in P$. Then $(b - a) + (c - d) \in P$. Hence, $a - c < b - d$.

(iv) Using (P12), note that $b - a \in P$. Since $c > 0$, $c(b - a) \in P$, which means $bc - ac > 0$, or $ac < bc$.

(v) By Trichotomy law(P10), $-c \in P$. Then by (iv), $-(ac) < -(bc)$. By (ii), $ac > bc$.

(vi) Since $a > 1 > 0$, by (iv), $a^2 > a$.

(vii) Since $a > 0$, by (iv), $a^2 < a$.

(viii) Because $0 < b$, $bc < bd$. Furthermore, if $c \geq 0$, $ac \leq bc$ (equality occurs if $c = 0$), by (iv). Therefore, $ac \leq bc < bd$. Hence, $ac < bd$.

(ix) From (viii), let $c = a$ and $d = b$, then the result follows.

(x) Suppose $a \geq b$. Then $a \geq b \geq 0$. By (ix) and (P9), $a^2 \geq b^2$. This contradicts $a^2 < b^2$.

■

Problem 1.6. (a) Prove that if $0 \leq x < y$, then $x^n < y^n$, $n = 1, 2, 3, \ldots$.

(b) Prove that if $x < y$ and n is odd, then $x^n < y^n$.

(c) Prove that if $x^n = y^n$ and n is odd, then $x = y$.

(d) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Solution. (a) Repeatedly apply problem 1.5(viii) for $0 \le x < y$, we have $x^n < y^n$ with $n = 1, 2, 3, \ldots$

(b) The statement is true for the case $0 \le x < y$. In the case $x < y \le 0$, by 1.5(ii), $(-x) > (-y) \ge 0$. By (a), $(-x)^n > (-y)^n$ for all odd n. Since n is odd, $-(x^n) > -(y^n)$. Hence, by 1.5(ii), $x^n < y^n$. In the case $x \le 0 < y$, since n is odd, $x^n < y^n$.

(c) Suppose that either $x \neq y$. W.l.o.g, let $x < y$, by (b), $x^n < y^n$ for all odd n, contradicting $x^n = y^n$ for all odd n.

(d) Suppose that both $x \neq y$ and $x \neq -y$. Then $x^2 - y^2 \neq 0$. W.l.o.g, suppose $x^2 > y^2 \ge 0$. Applying (a), this generalizes to $x^n > y^n$ for all even n, contradicting our assumption. Therefore, $x = y$ or $x = -y$.

The direct proof. In the case $x, y \ge 0$; by (a), if $x^n = y^n$ for all even n, then $x = y$. In the case $x, y \le 0$; if $x^n = y^n$ for all even n, then $(-x), (-y) \ge 0$ and $(-x)^n = (-y)^n$, so $-x = -y$ and hence $x = y$. In the case of x and y have different signs, then x and $-y$ are either two positive or two negative numbers. In either subcase, if $x^n = y^n$ for all even n, then $x^n = (-y)^n$, and it follows $x = -y$ from the previous case.

■

Problem 1.7. Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b$$

Notice that the inequality $\sqrt{ab} \le (a+b)/2$ holds for all $a, b \ge 0$. A generalization of this fact occurs in Problem 2.22.

Solution. Let us first establish that $a < \dfrac{a+b}{2} < b$. Note that,

$$a + a < a + b < b + b$$

and therefore, $a < \dfrac{a+b}{2} < b$. To finish the proof, we need to prove $a < \sqrt{ab} < \dfrac{a+b}{2}$. To do this, let us prove that if $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$. Note that since $b - a > 0$,

$$b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) > 0$$

Therefore, $\sqrt{b} > \sqrt{a} > 0$. We rewrite the inequality as follows,

$$\sqrt{a} \cdot (\sqrt{b} - \sqrt{a}) > 0$$

Then

$$a < \sqrt{ab} \tag{1.1}$$

We next notice that since $\sqrt{b} - \sqrt{a} > 0$, it follows that $(\sqrt{b} - \sqrt{a}) \cdot (\sqrt{b} - \sqrt{a}) = (\sqrt{b} - \sqrt{a})^2 > 0$. Expand the left hand side,

$$(\sqrt{b} - \sqrt{a})^2 = a + b - 2\sqrt{ab} > 0$$

which implies,

$$\sqrt{ab} < \frac{a+b}{2} \tag{1.2}$$

From (1.1) and (1.2), we have $a < \sqrt{ab} < \dfrac{a+b}{2}$. ■

Problem 1.8 (*)**.** Although the basic properties of inequalities were stated in terms of the collection P of all positive numbers, and $<$ was defined in terms of P, this procedure can be reversed. Suppose that P10–P12 are replaced by

(P'10) For any numbers a and b one, and only one, of the following holds:

- (i) $a = b$,
- (ii) $a < b$,
- (iii) $b < a$.

(P'11) For any numbers a, b, and c, if $a < b$ and $b < c$, then $a < c$.

(P'12) For any numbers a, b, and c, if $a < b$, then $a + c < b + c$.

(P'13) For any numbers a, b, and c, if $a < b$ and $0 < c$, then $ac < bc$.

Show that P10–P12 can then be deduced as theorems.

Solution. Let P be the set of all positive numbers.

- To prove P10, let $c = a - b$, from (P'10), P10 follows.
- To prove P11, let $a, b \in P$; it is sufficient to prove that $a + b > 0$. From (P'10), we divide the proof into three subscases:

  Case 1: $a = b$
  Then $a + b = b + b > 0 + b > 0$, where the first inequality follows from (P'12). By (P'11), $a + b > 0$.

  Case 2: $a < b$
  Then $a + b > a + a > 0 + a > 0$, where the first and second inequality follow from (P'12). By applying (P'11) twice, $a + b > 0$.

  Case 3: $a > b$
  Interchanging the role of a and b, we have the result.

- To prove P12, let $a, b \in P$; it is sufficient to prove that $a \cdot b > 0$. From (P'10), we divide the proof into three subcases:

  Case 1: $a = b$
  Then $a \cdot b = b \cdot b > 0 \cdot b = 0$, where the first inequality follows from (P'13) and the equality after which is from (P9).

  Case 2: $a < b$
  Then $b \cdot a > a \cdot a > 0 \cdot a = 0$, where the first and second inequality is from (P'13). By (P'11), $a \cdot b > 0$.

  Case 3: $a > b$
  Interchanging a and b returns us to case 2, which yields the result.

■

Problem 1.9. Express each of the following with at least one less pair of absolute value signs.

(i) $|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}|$

(ii) $|(|a + b| - |a| - |b|)|$

(iii) $|(|a + b| + |c| - |a + b + c|)|$

(iv) $|(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)|$

Solution. (i) Note $\sqrt{7} - \sqrt{5} > 0$, hence

$$|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}| = \sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$$

(ii) Since $|a + b| - |a| - |b| \leq 0$,

$$|(|a + b| - |a| - |b|)| = |a| + |b| - |a + b|$$

(iii) Since $|a + b + c| \leq |a + b| + |c|$,

$$|(|a + b| + |c| - |a + b + c|)| = |a + b| + |c| - |a + b + c|$$

(iv)

$$|(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)| = |\sqrt{2} + \sqrt{3} - \sqrt{7} + \sqrt{5}|$$

■

Problem 1.10. Express each of the following without absolute value signs, treating various cases separately when necessary.

(i) $|a + b| - |b|$

(ii) $|(|x| - 1)|$

(iii) $|x| - |x^2|$

(iv) $a - |(a - |a|)|$

Solution. (i) We divide into four cases:

$$a \geq 0 \quad \text{and} \quad b \geq 0 \qquad \text{(Case 1)}$$
$$a \leq 0 \quad \text{and} \quad b \leq 0 \qquad \text{(Case 2)}$$
$$a \geq 0 \quad \text{and} \quad b \leq 0 \qquad \text{(Case 3)}$$
$$a \leq 0 \quad \text{and} \quad b \geq 0 \qquad \text{(Case 4)}$$

In Case 1 and Case 2, we have $|a+b| - |b| = a$ since $|a+b| \leq |a| + |b|$.
In Case 3, if $a + b \geq 0$, then

$$|a+b| - |b| = (a+b) - (-b) = a + b + b = 2b$$

If $a + b \leq 0$, then

$$|a+b| - |b| = (-a-b) - (-b) = -a + (-b) + b = -a$$

In Case 4, if $a + b \geq 0$, then

$$|a+b| - |b| = (a+b) - (b) = a$$

If $a + b \leq 0$, then

$$|a+b| - |b| = -a + (-b) + (-b) = -a - 2b$$

(ii) We make the problem into 4 cases.

$$x \geq 1 \qquad \text{(Case 1)}$$
$$0 \leq x \leq 1 \qquad \text{(Case 2)}$$
$$-1 \leq x \leq 0 \qquad \text{(Case 3)}$$
$$x \leq -1 \qquad \text{(Case 4)}$$

In Case 1, $|(|x| - 1)| = x - 1$.
In Case 2, $|(|x| - 1)| = 1 - x$.
In Case 3, $|(|x| - 1)| = x + 1$.
In Case 4, $|(|x| - 1)| = -(x + 1)$.

(iii) Since $x^2 \geq 0$, $|x| - |x^2| = |x| - x^2$.
If $x \geq 0$, then $|x| - x^2 = x(1-x)$. If $x \leq 0$, then
$|x| - x^2 = -x + (-x^2) = -x(1+x)$.

(iv) Note that $|a| \geq a$. Hence,

$$a - |(a - |a|)| = a + a - |a| = 2a - |a|$$

We have two cases,

Case 1: $a \geq 0$

$$2a - |a| = 2a - a = a$$

Case 2: $a \leq 0$

$$2a - |a| = 2a + a = 3a$$

■

Problem 1.11. Find all numbers x for which

(i) $|x-3| = 8$

(ii) $|x-3| < 8$

(iii) $|x+4| < 2$

(iv) $|x-1| + |x-2| > 1$

(v) $|x-1| + |x+1| < 2$

(vi) $|x-1| + |x+1| < 1$

(vii) $|x-1| \cdot |x+1| = 0$

(viii) $|x-1| \cdot |x+2| = 3$

Solution. (i)

$$x - 3 = 8 \vee x - 3 = -8$$
$$x = 11 \vee x = -5$$

(ii) Then $-8 < x - 3 < 8$. Hence, $-5 < x < 11$.

(iii) Then $-2 < x + 4 < 2$. Hence, $-6 < x < -2$.

(iv) If $1 \leq x \leq 2$, then the inequality becomes $(x-1) + (2-x) = 1$. If $x > 2$, then $2x - 3 > 1$, which is $x > 2$. If $x < 1$, then $-2x + 3 > 1$, which is $x < 1$. Therefore, either $x > 2$ or $x < 1$ satisfies the inequality.

(v) If $-1 \leq x \leq 1$, then $(1-x) + (x+1) = 2$. If $x > 1$, then $x < 1$, which is contradictory. If $x < -1$, then $(1-x) + (-x-1) = -2x < 2$ only if $x > -1$, which is contradictory. Hence, there is no x to satisfy the inequality.

(vi) It is implied from above that

$$|x-1| + |x+1| \geq 2$$

Therefore, there is no x satisfying the inequality.

(vii) Either $x = 1$ or $x = -1$.

(viii) If $-2 \leq x \leq 1$, then we obtain $x^2 + x + 1 > 0$. Hence, in either $x < -2$ or $x > 1$, we have to solve the equation $x^2 + x - 5 = 0$, whose solution is either $x = \dfrac{-1+\sqrt{21}}{2}$ or $x = \dfrac{-1-\sqrt{21}}{2}$.

■

Problem 1.12. Prove the following:

(i) $|xy| = |x| \cdot |y|$

(ii) $\left|\dfrac{1}{x}\right| = \dfrac{1}{|x|}$, if $x \neq 0$. (The best way to do this is to remember what $|x|^{-1}$ is.)

(iii) $\dfrac{|x|}{|y|} = \left|\dfrac{x}{y}\right|$, if $y \neq 0$.

(iv) $|x - y| \leq |x| + |y|$ (Give a very short proof.)

(v) $|x| - |y| \leq |x - y|$ (A very short proof is possible, if you write things in the right way.)

(vi) $|(|x| - |y|)| \leq |x - y|$ (Why does this follow immediately from (v)?)

(vii) $|x + y + z| \leq |x| + |y| + |z|$. Indicate when equality holds, and prove your statement.

Solution. (i) We have 4 cases,

$$x \geq 0 \quad y \geq 0 \tag{1}$$
$$x \geq 0 \quad y \leq 0 \tag{2}$$
$$x \leq 0 \quad y \geq 0 \tag{3}$$
$$x \leq 0 \quad y \leq 0 \tag{4}$$

In (1), $|x| \cdot |y| = xy = |xy|$

In (4), $|x| \cdot |y| = (-x)(-y) = xy = |xy|$

In (3), $|x| \cdot |y| = (-x)(y) = -(xy) = |xy|$

In (2), interchanging x and y leads to (3).

(ii) Since $x \neq 0$, there exists $|x|^{-1}$ such that

$$|x||x|^{-1} = 1 = |x|\left|\frac{1}{x}\right|$$

where the second equality is by (i). Dividing both sides by $|x|$, we have the result.

(iii) Since $y \neq 0$, from (ii), we immediately have

$$\left|\frac{1}{y}\right| = \frac{1}{|y|}$$

Hence, applying (ii) once more,

$$\left|\frac{x}{y}\right| = |x|\left|\frac{1}{y}\right| = \frac{|x|}{|y|}$$

(iv) Note that,

$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$$

where the last equality follows from (i).

(v) Note that,

$$|x - y + y| \leq |x - y| + |y|$$

Therefore, $|x| - |y| \leq |x - y|$.

(vi) Let the first term be y and the second term be $y - x$. Applying (v), we have

$$|y| - |y - x| \leq |x|$$

Hence, $-|x-y| \leq |x|-|y|$. Combining with (v) gives $|(|x|-|y|)| \leq |x-y|$.

(vii) Notice the pattern,

$$|x + y + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$$

the equality holds only if either x, y, z have the same sign or at least two of them must be equal to 0. It is easy to verify this.
Suppose not, then both x, y, z have different signs and at most one of them is 0. If the latter is true, then, w.l.o.g, suppose $z = 0$, then x, y have different sign, and we are done. If none of them is 0, then, w.l.o.g, suppose $z < 0$ and pick z such that $x + y < -z$. Then,

$$|x + y + z| = -(x + y + z) = -x - y - z < |x| + |y| + |z|$$

where inequality must follow since $x, y \neq 0$.

■

Problem 1.13. The maximum of two numbers x and y is denoted by $max(x, y)$. Thus $max(-1, 3) = max(3, 3) = 3$ and $max(-1, -4) = max(-4, -1) = -1$. The minimum of x and y is denoted by $min(x, y)$. Prove that

$$max(x, y) = \frac{x + y + |y - x|}{2},$$

$$min(x, y) = \frac{x + y - |y - x|}{2}.$$

Derive the formula for $max(x, y, z)$ and $min(x, y, z)$, using, for example

$$max(x, y, z) = max(x, max(y, z)).$$

Solution. Assume that $x \geq y$, we want to prove that $max(x, y) = x$.

$$max(x,y) = \frac{x+y+|y-x|}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x$$

Similarly, we need $min(x, y) = y$.

$$min(x,y) = \frac{x+y-|y-x|}{2} = \frac{x+y-(x-y)}{2} = \frac{x+y-x+y}{2} = \frac{2y}{2} = y$$

Let $max(x, y, z) = max(x, max(y, z))$. Then

$$\begin{aligned} max(x,y,z) &= \frac{x+max(y,z)+|max(y,z)-x|}{2} \\ &= \frac{x+\dfrac{y+z+|z-y|}{2}+\left|\dfrac{y+z+|z-y|}{2}-x\right|}{2} \\ &= \frac{2x+y+z+|z-y|+|y+z+|z-y|-2x|}{4} \end{aligned}$$

Similarly,

$$min(x,y,z) = \frac{2x+y+z-|z-y|-|y+z-|z-y|-2x|}{4}$$

■

Problem 1.14. (a) Prove that $|a| = |-a|$. (The trick is not to become confused by too many cases. First prove the statement for $a \geq 0$. Why is it then obvious for $a \leq 0$?)

(b) Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$. In particular, it follows that $-|a| \leq a \leq |a|$.

(c) Use this fact to give a new proof that $|a+b| \leq |a| + |b|$.

Solution. (a) Problem 1.12(i) easily tells us that

$$|-a| = |(-1)a| = |-1||a| = 1|a| = |a|$$

(b) If $a \geq 0$, then $a \leq b$. If $a \leq 0$, $-a \leq b$ follows from $a \geq -b$. Therefore, $|a| \leq b$. Conversely, suppose $|a| \leq b$. Then it is certain $a \leq b$ since $a \leq |a| \leq b$. From (a), $|-a| \leq b$, and hence $a \geq -b$. We conclude that $-b \leq a \leq b$. Note that since $|a| \leq |a|$, $-|a| \leq a \leq |a|$.

(c) Because we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, by Problem 1.5(i), we obtain $-(|a| + |b|) \leq a + b \leq |a| + |b|$. From (b), we arrive at the conclusion $|a + b| \leq |a| + |b|$.

■

Problem 1.15 (*). Prove that if x and y are not both 0, then

$$x^2 + xy + y^2 > 0$$
$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0$$

Hint: Use problem 1.

Solution. For the first part, note that

$$x^2 + xy + y^2 = x^2 + 2 \cdot x \cdot \frac{1}{2}y + \frac{1}{4}y^2 - \frac{1}{4}y^2 + y^2 = \left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 > 0$$

For the second part, if $x = y$, then the left-hand side is $5x^4 > 0$. Hence, suppose $x \neq y$. From Problem 1.1(v),

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \neq 0$$

If $x > y$, then $x^5 > y^5$ by Problem 1.6(b). This implies that the second term must be greater than 0. Conversely, $x < y \Rightarrow x^5 < y^5$ implies that it must be greater than 0. ∎

Problem 1.16 (*). (a) Show that

$$(x + y)^2 = x^2 + y^2 \text{ only when } x = 0 \text{ or } y = 0,$$
$$(x + y)^3 = x^3 + y^3 \text{ only when } x = 0 \text{ or } y = 0 \text{ or } x = -y.$$

(b) Using the fact that

$$x^2 + 2xy + y^2 = (x + y)^2 \geq 0,$$

show that $4x^2 + 6xy + 4y^2 > 0$ unless x and y are both 0.

(c) Use part (b) to find out when $(x + y)^4 = x^4 + y^4$.

(d) Find out when $(x+y)^5 = x^5+y^5$. Hint: From the assumption $(x+y)^5 = x^5+y^5$ you should be able to derive the equation $x^3+2x^2y+2xy^2+y^3 = 0$, if $xy \neq 0$. This implies that $(x + y)^3 = x^2y + xy^2 = xy(x + y)$.

You should know be able to make a good guess as to when $(x + y)^n = x^n + y^n$; the proof is contained in Problem 11.57

Solution. (a) For the first part,

$$(x + y)^2 = x^2 + 2xy + y^2$$

Hence, $(x + y)^2 = x^2 + y^2$ only when $x = 0$ or $y = 0$. For the second part, from Problem 1.1(vi),

$$(x + y)^3 - (x + y)(x^2 - xy + y^2) = 0$$
$$(x + y)(xy) = 0$$

which is true only when $x = 0$ or $y = 0$ or $x = -y$.

(b) Note that $4x^2 + 6xy + 4y^2 = \underbrace{3(x+y)^2}_{\geq 0} + \underbrace{x^2+y^2}_{>0} > 0$ unless $x = 0$ and $y = 0$.

(c) Let us expand $(x+y)^4$.

$$\begin{aligned}(x+y)^2(x+y)^2 &= (x^2+2xy+y^2)(x^2+2xy+y^2)\\ &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\\ &= x^4+y^4+xy(4x^2+6xy+4y^2)\end{aligned}$$

Hence, $(x+y)^4 = x^4+y^4$ only when $x=0$ or $y=0$, by part (b).

(d) Let us expand $(x+y)^5$.

$$\begin{aligned}(x+y)^4(x+y) &= x^5+y^5+xy(x+y)(4x^2+6xy+4y^2)+xy(x^3+y^3)\\ &= x^5+y^5+5xy(x+y)(x^2-xy+y^2)\end{aligned}$$

If $xy \neq 0$ and $x+y\neq 0$, let $z=-y$, by Problem 1.6(b), $x^3 \neq z^3$. Hence, $x^2-xy+y^2 \neq 0$. Therefore, $(x+y)^5 = x^5+y^5$ only when $x=0$ or $y=0$ or $x=-y$.

Remark. Hence, for $(x+y)^n = x^n+y^n$, if n is even, then $x=0$ or $y=0$. If n is odd, then $x=0$ or $y=0$ or $x=-y$.

■

Problem 1.17. (a) Find the smallest possible value of $2x^2-3x+4$. Hint: "Complete the square", i.e., write $2x^2-3x+4 = 2(x-3/4)^2+?$

(b) Find the smallest possible value of $x^2-3x+2y^2+4y+2$.

(c) Find the smallest possible value of $x^2+4xy+5y^2-4x-6y+7$.

Solution. (a) Since $2x^2-3x+4 = 2(x^2-\frac{3}{2}x+2)$,

$$2(x^2-2\cdot x\frac{3}{4}+\frac{9}{16}-\frac{9}{16}+2) = 2(x-\frac{3}{4})^2+\frac{23}{8}$$

Hence the minimum value is $\frac{23}{8}$ when $x=\frac{3}{4}$.

(b)

$$x^2-3x+\frac{9}{4}-\frac{9}{4}+2(y^2+2y+1) = \left(x-\frac{3}{2}\right)^2+2(y+1)^2-\frac{9}{4}$$

The minimum value is $-\frac{9}{4}$ when $x=\frac{3}{2}$ and $y=-1$.

(c)

$$\begin{aligned}
&\frac{1}{2}x^2 + 4xy + 8y^2 - 3y^2 - 6y + 7 + \frac{1}{2}x^2 - 4x \\
=&\frac{1}{2}(x^2 + 8xy + 16y^2) - 3(y^2 + 2y + 1) + \frac{1}{2}(x^2 - 8x + 16) + 2 \\
=&\frac{1}{2}(x + 4y)^2 - 3(y + 1)^2 + \frac{1}{2}(x - 4)^2 + 2
\end{aligned}$$

Therefore, the minimum value is 2 when $x = 4$ and $y = -1$.

■

Problem 1.18. (a) Suppose that $b^2 - 4c \geq 0$. Show that the numbers

$$\frac{-b + \sqrt{b^2 - 4c}}{2}, \quad \frac{-b - \sqrt{b^2 - 4c}}{2}$$

both satisfy the equation $x^2 + bx + c = 0$.

(b) Suppose that $b^2 - 4c < 0$. Show that there are no numbers x satisfying $x^2 + bx + c = 0$; in fact, $x^2 + bx + c > 0$ for all x. Hint: Complete the square.

(c) Use this fact to give another proof that if x and y are not both 0, then $x^2 + xy + y^2 > 0$.

(d) For which number α is it true that $x^2 + \alpha xy + y^2 > 0$ whenever x and y are not both 0?

(e) Find the smallest possible value of $x^2 + bx + c$ and of $ax^2 + bx + c$, for $a > 0$.

Solution. (a) Substitution immediately gives the desired result.

(b)

$$x^2 + bx + c = x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c$$

which immediately yields $\left(x + \frac{b}{2}\right)^2 + \frac{[-(b^2 - 4c)]}{4} > 0$ for all x since $b^2 - 4c < 0$.

(c) If $y = 0$, $x^2 > 0$. Suppose not, using (b), we obtain $-3y^2 < 0$. Hence, $x^2 + xy + y^2 > 0$.

(d) If $y = 0$, the result follows for all α. Suppose $y \neq 0$, using (b), we obtain $\alpha^2 y^2 - 4y^2 < 0$, which is $y^2(\alpha^2 - 4) < 0$. It follows that $-2 < \alpha < 2$.

(e) From (b), it follows that the minimum value of x^2+bx+c is $\dfrac{[-(b^2-4c)]}{4}$ when $x=-b/2$. Since $a>0$, with the role of b is now b/a and of c is c/a, we easily derive the result.

$$x^2+\frac{b}{a}x+\frac{c}{a}=\left(x+\frac{b}{2a}\right)^2+\frac{[-(b^2-4ac)]}{4a^2}$$

So its minimum value is $\dfrac{[-(b^2-4ac)]}{4a^2}$ when $x=-\dfrac{b}{2a}$.

■

Problem 1.19. The fact that $a^2\geq 0$ for all numbers a, elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:

$$x_1y_1+x_2y_2\leq\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}.$$

(A more general form occurs in Problem 2.21) The three proofs of the Schwarz inequality outlined below have only one thing in common–their reliance on the fact that $a^2\geq 0$ for all a.

(a)