

A Note of Calculus-Michael Spivak

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23rd June, 2017

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Preface

This is the note for the book Calculus writtten by Michael Spivak, citing what I think the most interesting and important subjects mentioned in the book.

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Part I

Prologue

Chapter 1

Basic properties of number

(P1) If a , b , and c are any numbers, then

$$a + (b + c) = (a + b) + c$$

See *problem 24* for the generalization of $a_1 + a_2 + a_3 + \cdots + a_n$ for (P1).

The number 0 has important properties.

(P2) If a is any number, then

$$a + 0 = 0 + a = a$$

(P3) For every number a , there is also a number $-a$ such that

$$a + (-a) = (-a) + a = 0$$

We now prove Lemma 1.

Lemma 1. *If $a + x = a$, then $x = 0$*

Proof.

If	$a + x = a$	
then	$(-a) + (a + x) = (-a) + a = 0$	(by (P3))
hence	$((-a) + a) + x = 0$	(by (P1))
hence	$0 + x = 0$	(by (P3) again)
therefore,	$x = 0$	(by (P2))

□

Also, remember that the order of addition does not matter.

(P4) If a and b are any numbers, then

$$a + b = b + a$$

However, with only (P1)-(P4), we are powerless to figure out what conditions needed to have $a - b = b - a$. Therefore, we need to set new properties, and, oddly, they involve multiplication.

(P5) If a, b and c are any numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(P6) If a is any number, then

$$a \cdot 1 = 1 \cdot a = a$$

Moreover, $1 \neq 0$ (This cannot be proved by other properties listed!)

(P7) For every number $a \neq 0$, there is a number a^{-1} such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1 (\Leftrightarrow 0 \cdot b = 0 \ \forall b)$$

This is why $1/0$ is meaningless!

(P8) If a and b are any numbers, then

$$a \cdot b = b \cdot a$$

From (P5), (P6) and (P7), we have two lemmas:

Lemma 2. *If $a \cdot b = a \cdot c$ then $a = 0 \vee b = c$*

Proof. If $a = 0$ then the lemma is trivial. Suppose now $a \neq 0$,

Multiply a^{-1} to both sides, $(a^{-1}) \cdot (a \cdot b) = (a^{-1}) \cdot (a \cdot c)$

By (P5), $(a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c$

By (P7), $1 \cdot b = 1 \cdot c$

By (P6), $b = c$

□

Lemma 3. *If $a \cdot b = 0$ then $a = 0 \vee b = 0$*

Proof. If $a = 0$, there is nothing to prove. Suppose now $a \neq 0$, follow the proof of Lemma 2 by consecutively applying (P5), (P7) and (P6) in that order to finish the proof. □

We, however, will not be able to prove anything without a relationship between multiplication and addition. Therefore, the next property is definitely necessary.

(P9) If a , b and c are any numbers, then

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

By (P8), it is also true that $(b + c) \cdot a = b \cdot a + c \cdot a$

We will see in the next remark and lemmas that properties are not built in a straight line. Rather, it is a result of necessities, of fixes and starts that somehow fits the pieces of a puzzle perfectly.

Remark. When $a - b = b - a$?

Solution.

Add b at both sides, $(a - b) + b = (b - a) + b = b + (b - a)$ by (P4)

By (P1), $a + (-b + b) = (b + b) + (-a)$

By (P3), $a + 0 = b + b - a$

By (P2), $a = b + b - a$

Add both sides to a , $a + a = (b + b - a) + a$

By (P1), $a + a = b + (b + (-a + a)) = b + b$ by (P2) and (P3)

By (P9), $a \cdot (1 + 1) = b \cdot (1 + 1)$

By Lemma 2, $a = b$

■

Note that the proof above based on the presumption that we know $1 + 1 \neq 0$. How do we prove it?

Lemma 4. $a \cdot 0 = 0$

Proof.

We have $a \cdot 0 + a \cdot 0 = a \cdot (0 + 0)$ by (P9)

By (P2), $= a \cdot 0$

Add $-a \cdot 0$, $a \cdot 0 = 0$

□

Lemma 5. *The product of two negative numbers is positive*

Proof. We first prove that $(-a) \cdot b = -(a \cdot b)$,

We have by (P9), $(-a) \cdot b + (a \cdot b) = (-a + a) \cdot b = 0$

Adding $-(a \cdot b)$ to both sides, $(-a) \cdot b = -(a \cdot b)$

Now, let's prove the main statement.

$$\begin{aligned} \text{From above,} \quad (-a) \cdot (-b) + [-(a \cdot b)] &= (-a) \cdot (-b) + [(-a) \cdot b] \\ &= (-a) \cdot (-b + b) \\ &= 0 \end{aligned}$$

$$\text{Adding } (a \cdot b) \text{ to both sides,} \quad (-a) \cdot (-b) = (a \cdot b)$$

□

We say that Lemma 5 is a direct consequence of (P1)-(P9).

Also, (P9) has important consequences: Justifying the algebraic manipulations (e.g, $x^2 - 3x + 2 = (x - 1)(x - 2)$) and the way one multiplies arabic numerals,

$$\begin{array}{r} 13 \\ \times 24 \\ \hline 52 \\ 26 \\ \hline 312 \end{array}$$

Denote the set of all positive numbers by P .

(P10) (Trichotomy law) For every number a , one and only one of the following holds:

- (i) $a = 0$
- (ii) $a \in P$
- (iii) $-a \in P$

(P11) (Closure under addition) If $a \in P \wedge b \in P$ then $a + b \in P$

(P12) (Closure under multiplication) If $a \in P \wedge b \in P$ then $a \cdot b \in P$

These properties should be complemented by the following definitions:

$$\begin{aligned} a > b &\text{ if } a - b \in P \\ a < b &\text{ if } b > a \\ a \geq b &\text{ if } a = b \text{ or } a > b \\ a \leq b &\text{ if } a = b \text{ or } a < b \end{aligned}$$

The following lemmas are easy to prove...

Lemma 6. If $a < b$ then $a + c < b + c$

Proof. If $a < b$, then $b - a \in P$, which is surely $(b + c) - (a + c) \in P$ □

Lemma 7. If $a < b \wedge b < c$ then $a < c$

Proof. Then $b - a \in P$ and $c - b \in P$. By (P11), $(b - a) + (c - b) = c - a \in P$ □

Lemma 8. *If $a < 0 \wedge b < 0$ then $a \cdot b > 0$*

Proof. Then $-a > 0 \wedge -b > 0$. By (P12), $(-a) \cdot (-b) = a \cdot b > 0$, by Lemma 5. \square

Lemma 9. *If $a \neq 0$, then $a^2 \neq 0$*

Proof. Because if $a > 0 \wedge b > 0$ and $a < 0 \wedge b < 0$ then $a \cdot b > 0$, let $b = a$ \square

This implies that $1 > 0$ (since $1^2 = 1$).

We now prove a basic theorem relating to the absolute value.

Theorem 1.1. $\forall a \wedge b,$

$$|a + b| \leq |a| + |b|$$

Proof. We apply the straightforward proof. A more elegant proof appears in the exercises. We will consider 4 cases:

$$a \geq 0 \quad \text{and} \quad b \geq 0 \tag{1}$$

$$a \geq 0 \quad \text{and} \quad b \leq 0 \tag{2}$$

$$a \leq 0 \quad \text{and} \quad b \geq 0 \tag{3}$$

$$a \leq 0 \quad \text{and} \quad b \leq 0 \tag{4}$$

For (1), the statement occurs with equality; that is,

$$|a + b| = a + b = |a| + |b|$$

For (4), the same is true by observing,

$$|a + b| = -(a + b) = (-a) + (-b) = |a| + |b|$$

For (2), the job is dumped down to proving that $|a + b| \leq a - b$. This divides the case into two subcases.

Subcase 1: $a + b \geq 0$

Then note that $b \leq (-b)$, which is true since $b \leq 0$.

Subcase 2: $a + b \leq 0$

Then we have $(-a) \leq a$, which is true since $a \geq 0$.

For (3), the case is proved by interchanging the role of a and b . \square

Note from the proof above that equality happens if a and b have the same sign, or one of the two is zero.

Remark. It is crucial to understand that (P1)-(P12) are not enough to account for *all* properties of numbers. The deficiency is profound and subtle; and, hopefully, will be discovered in the rest of the note.

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Chapter 2

Number of various sorts

\mathbb{N} is the basic set and has many deficiencies. ((P2) and (P3)).

Mathematical induction principle is the basic property of \mathbb{N} ; however, even though proof by induction is quite straightforward, *the method by which the formula was discovered remains a mystery.*

Theorem 2.1 (Well-ordering principle). *If A is a nonnull set of natural numbers, then A has a least member.*

Proof. Suppose A has no least member. Let B be the collection of n natural numbers $1, \dots, n$ that are not *all* in A . Clearly, 1 is in B (if not, 1 would be the least member in A). Moreover, if $1, \dots, k$ are not in A , surely $k + 1$ is not in A either (else, $k + 1$ would be the least member in A). This shows that if $k \in B$, then $k + 1 \in B$. Hence, B is the set of all natural numbers, and $A = \emptyset$. \square

\mathbb{N} can be defined either by the well-ordering principle or by mathematical induction since they are equivalent.

Principle of complete induction:

(1) 1 is in A ,

(2) $k + 1$ is in A if $1, \dots, k$ are in A ,

then A is the set of all natural numbers.

Complete induction is the consequence of induction.

Recursive definition.

Deficiencies of \mathbb{N} can be partially remedied by \mathbb{Z} ((P7) fails.), which is remedied by \mathbb{Q} , which is smaller than \mathbb{Z} .

Every natural number n is either odd or even.

if n is odd, n^2 is odd; if n is even, n^2 is even. Hence, if n^2 is even, n is even; if n^2 is odd, n is odd.

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Chapter 3

Functions

Domain of the function := the set of numbers to which the function is defined!

$$f(x) = x^2 \quad \forall x \quad (3.1)$$

$$g(y) = \frac{y^3 + 3y + 5}{y^2 + 1} \quad \forall y \quad (3.2)$$

$$h(c) = \frac{c^3 + 3c + 5}{c^2 - 1} \quad \forall c \neq \pm 1 \quad (3.3)$$

$$r(x) = x^2, \quad \{x : -17 \leq x \leq \frac{\pi}{3}\} \quad (3.4)$$

$$s(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational} \end{cases} \quad (3.5)$$

$$\phi(x) = \begin{cases} 5, & x = 2 \\ \frac{36}{\pi}, & x = 17 \\ 28, & x = \frac{\pi^2}{17} \\ 28, & x = \frac{36}{\pi} \\ 16, & x \neq 2, 17, \frac{\pi^2}{17}, \frac{36}{\pi}, \text{ and } x = a + b\sqrt{2} \text{ for } a, b \text{ in } \mathbb{Q}. \end{cases} \quad (3.6)$$

$$\alpha_x(t) = t^3 + x \text{ for all numbers } t. \quad (3.7)$$

$$y(x) = \begin{cases} n, & \text{exactly } n \text{ 7's appear in the decimal expansion of } x \\ -\pi, & \text{infinitely many 7's appear in the decimal expansion of } x. \end{cases} \quad (3.8)$$

A function f is a **polynomial function** if there are real numbers a_0, \dots, a_n such that

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0, \quad \text{for all } x$$

(Assume $a_n \neq 0$). The highest power of x with a nonzero coefficient is called the **degree** of f .

(3.2) and (3.3) are **rational functions**: Functions of the form p/q where

p and q are polynomials with q is not always 0.

$$f(x) = \frac{x + x^2 + x \sin^2 x}{x \sin x + x \sin^2 x} \quad (3.9)$$

$$f(x) = \sin(x^2) \quad (3.10)$$

$$f(x) = \sin(\sin(x^2)) \quad (3.11)$$

$$f(x) = \sin^2(\sin(\sin^2(x \sin^2 x))) \cdot \sin\left(\frac{x + \sin(x \sin x)}{x + \sin x}\right) \quad (3.12)$$

Let $I(x) = x$, then the concepts of **sum**, **product**, **quotient**, **composite** of a function are necessary.

$$x \rightarrow \sin(x^2)$$

To formalize the definition of a function, one does not ask, “What is a rule?”, “What is an association?”, but “What does one need to know about the function in order to know all about it?”

Definition 3.1. A **function** is a collection of pairs of numbers with the following property: if (a, b) and (a, c) are both in the collection, then $b = c$.

Definition 3.2. If f is a function, the **domain** of f is the set of all a for which there is some b such that (a, b) is in f . If a is in the domain of f , it follows from the definition of a function that there is, in fact, a *unique* number b such that (a, b) is in f . This unique b is denoted by $f(a)$.