Solutions to Michael Spivak's Calculus

Son To
<son.trung.to@gmail.com>

Ravintola Kiltakellari *

25th June, 2017

*I thank my employer!

To my mother, friends and all those who influence me.

Preface

This is my own solutions to Michael Spivak's Calculus textbook.

Vantaa, Finland
25th June, 2017.

Contents

Part I

Prologue

This page is intentionally left blank.

Chapter 1

Basic properties of number

Problem 1.1. Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$

(ii) $x^2 - y^2 = (x - y)(x + y)$

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$

(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

(v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

(vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (There is a particularly easy way to do this using (iv), and it will show you how to find a factorization for $x^n + y^n$ whenever n is odd.)

Solution. (i) By (P7)(Existence of multiplicative inverses), there exists a^{-1} such that,

$$\begin{aligned} (a^{-1} \cdot a)x &= (a^{-1} \cdot a) \\ x &= 1 \end{aligned}$$

(ii) By (P9) for 2 times,

$$\begin{aligned} (x - y)(x + y) &\overset{1}{=} x \cdot (x + y) + (-y) \cdot (x + y) \\ &\overset{2}{=} x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y \\ &= x^2 + x \cdot y + [-(x \cdot y)] + [-(y^2)] \\ &= x^2 - y^2 \end{aligned}$$

(iii) From (ii) and since $x^2 = y^2$,

$$x^2 - y^2 = (x-y)(x+y) = 0$$

This means $(x-y) = 0 \vee (x+y) = 0$, which is $x = y \vee x = -y$

(iv) Starting with the right-hand side,

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x \cdot (x^2+xy+y^2) + (-y) \cdot (x^2+xy+y^2)\\ &= x^3 + x^2y + xy^2 + [-(x^2y)] + [-(xy^2)] + [-(y)^3]\\ &= x^3 - y^3\end{aligned}$$

(v) I propose two solutions for this problem. The first one is the direct right-hand side manipulation, while the latter is done by induction.

The first solution.

$$\begin{aligned}&(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})\\ &= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1}\\ &+[-(x^{n-1}y)] + [-(x^{n-2}y^2)] + \cdots + [-(xy^{n-1})] + [-(y^n)]\\ &= x^n - y^n\end{aligned}$$

Q.E.D

The second solution. Let n=1, then indeed $x - y = x - y$. Suppose the statement holds true for $n = k$ with $k \in \mathbb{N}$, that is

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})$$

is true. To finish the proof, we need to prove

$$x^{k+1} - y^{k+1} = (x-y)(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k)$$

That is, the statement holds for $n = k$. Starting from the left hand side,

$$\begin{aligned}&x^{k+1} - y^{k+1}\\ =\ &x^{k+1} - x^ky + x^ky - y^{k+1}\\ =\ &x^k(x-y) + y(x^k - y^k)\\ =\ &x^k(x-y) + y(x-y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})\\ =\ &(x-y)[x^k + y(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})]\\ =\ &(x-y)(x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1} + y^k)\end{aligned}$$

Q.E.D

(vi) We will use (iv) in our proof,

$$\begin{aligned}
& x^3+y^3 \\
= \;& x^3-y^3+2y^3 \\
= \;& (x-y)(x^2+xy+y^2)+2y[(x^2+xy+y^2)+(-x)(x+y)] \\
= \;& (x+y)(x^2+xy+y^2)+2[-(xy)](x+y) \\
= \;& (x+y)(x^2-xy+y^2)
\end{aligned}$$

■

Problem 1.2. What is wrong with the following "proof"? Let $x=y$. Then

$$\begin{aligned}
x^2 &= xy, \\
x^2-y^2 &= xy-y^2, \\
(x+y)(x-y) &= y(x-y), \\
x+y &= y, \\
2y &= y, \\
2 &= 1.
\end{aligned}$$

Solution. Note that in the transition from line 3 to line 4, the author "simplifies" $(x-y)$ by dividing $(x-y)$ on both sides. This is wrong since $x-y=0$, and hence $1/0$ is undefined as implied by (P7) in the textbook. ■

Problem 1.3. Prove the following:

(i) $\dfrac{a}{b}=\dfrac{ac}{bc}$, if $b,c\neq 0$.

(ii) $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}$, if $b,d\neq 0$.

(iii) $(ab)^{-1}=a^{-1}b^{-1}$, if $a,b\neq 0$. (To do this you must remember the defining property of $(ab)^{-1}$.)

(iv) $\dfrac{a}{b}\cdot\dfrac{c}{d}=\dfrac{ac}{db}$, if $b,d\neq 0$.

(v) $\dfrac{a}{b}\Big/\dfrac{c}{d}=\dfrac{ad}{bc}$, if $b,c,d\neq 0$.

(vi) If $b,d\neq 0$, then $\dfrac{a}{b}=\dfrac{c}{d}$ if and only if $ad=bc$. Also determine when $\dfrac{a}{b}=\dfrac{b}{a}$.

Solution. (i) Until (iii) is proved, the solution is to test the equality between two sides.

$$\begin{aligned} a(b)^{-1} &= (ac)(bc)^{-1} \\ a[(b)^{-1}b] &= (ac)(bc)^{-1}b \\ (a^{-1}a) &= (a^{-1}a)c(bc)^{-1}b \\ 1 &= (bc)(bc)^{-1} = 1 \end{aligned}$$

(ii) Similar to the above,

$$\begin{aligned} a(b)^{-1} + c(d)^{-1} &= (ad+bc)(bd)^{-1} \\ a(b)^{-1}bd + c(d)^{-1}bd &= (ad+bc)[(bd)^{-1}(bd)] \\ ad(b^{-1}b) + bc(d^{-1}d) &= (ad+bc) \\ ad+bc &= ad+bc \end{aligned}$$

(iii) Since $a, b \neq 0$, there exists $(ab)^{-1}, a^{-1}, b^{-1}$ such that,

$$\begin{aligned} ab &= ab \\ (ab)^{-1}(ab) &= (ab)^{-1}(ab) = 1 \\ (ab)^{-1}a(bb^{-1}) &= b^{-1} \\ (ab)^{-1}(aa^{-1}) &= b^{-1}a^{-1} \\ (ab)^{-1} &= a^{-1}b^{-1} \end{aligned}$$

(iv) For $b, d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = ab^{-1}cd^{-1} = ac(d^{-1}b^{-1}) = ac(db)^{-1} = \frac{ac}{db}$$

where the next-to-last equality follows from (iii).

(v) I first establish for any number $a \neq 0$,

$$(a^{-1})^{-1} = a$$

Let $t = a^{-1}$, we want to prove $t^{-1} = a$. Observe that

$$\begin{aligned} t &= a^{-1} \\ t \cdot (t)^{-1} &= a^{-1} \cdot (t)^{-1} \\ a \cdot 1 &= (a \cdot a^{-1}) \cdot (t)^{-1} \\ a &= (t)^{-1} \end{aligned}$$

From the left hand side of the statement,

$$\frac{a}{b}\Big/\frac{c}{d} = a(b)^{-1}[c(d)^{-1}]^{-1} = a(b)^{-1}(c)^{-1}[(d)^{-1}]^{-1} = (ad)(bc)^{-1} = \frac{ad}{bc}$$

where the second and third equality follows both from (iii) and the proof above.

(vi) Using (ii),

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ \frac{a}{b} + (-\frac{c}{d}) &= 0 \\ \frac{ad-bc}{bd} &= 0 \\ ad &= bc \end{aligned}$$

Now, put $c = b \wedge d = a$. It follows that $\frac{a}{b} = \frac{b}{a}$ if and only if $a^2 = b^2$. It follows $(a-b)(a+b) = 0$, or $a = b \vee a = -b$.

∎

Problem 1.4. Find all numbers x for which

(i) $4 - x < 3 - 2x$

(ii) $5 - x^2 < 8$

(iii) $5 - x^2 < -2$

(iv) $(x-1)(x-3) > 0$ (When is a product of two numbers positive?)

(v) $x^2 - 2x + 2 > 0$

(vi) $x^2 + x + 1 > 2$

(vii) $x^2 - x + 10 > 16$

(viii) $x^2 + x + 1 > 0$

(ix) $(x-\pi)(x+5)(x-3) > 0$

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$

(xi) $2^x < 8$

(xii) $x + 3^x < 4$

(xiii) $\dfrac{1}{x} + \dfrac{1}{1-x} > 0$

(xiv) $\dfrac{x-1}{x+1} > 0$

Solution. (i)

$$\begin{aligned} 4 - x &< 3 - 2x \\ 4 + (-x + 2x) &< 3 + (-2x + 2x) \\ (-4 + 4) + x &< -4 + 3 \\ x &< -1 \end{aligned}$$

(ii)

$$\begin{aligned} 5 - x^2 &< 8 \\ 5 - 8 &< x^2 \\ -3 &< x^2 \end{aligned}$$

Since $x^2 \geq 0 \ \forall x \in \mathbb{R}$, the inequality holds $\forall x$.

(iii)

$$\begin{aligned} 5 - x^2 &< -2 \\ 7 &< x^2 \\ 0 &< x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7}) \end{aligned}$$

Hence, either $x > \sqrt{7} \ \wedge \ x > -\sqrt{7}$ or $x < \sqrt{7} \ \wedge \ x < -\sqrt{7}$, which is $x > \sqrt{7} \ \vee \ x < -\sqrt{7}$.

(iv)

$$\begin{aligned} (x - 1)(x - 3) &> 0 \\ (x > 1 \ \wedge \ x > 3) \vee (x < 1 \ \wedge \ x < 3) \\ x > 3 \vee x &< 1 \end{aligned}$$

(v)

$$\begin{aligned} x^2 - 2x + 2 &> 0 \\ (x^2 - 2x + 1) + 1 &> 0 \\ (x - 1)^2 + 1 &> 0 \end{aligned}$$

Hence the inequality is satisfied $\forall \ x$.

(vi)

$$\begin{aligned}
x^2 + x + 1 &> 2 \\
x^2 + x - 1 &> 0 \\
x^2 + \left(\frac{1+\sqrt{5}}{2}\right)x + \left(\frac{1-\sqrt{5}}{2}\right)x + \left(\frac{(1-\sqrt{5})(1+\sqrt{5})}{4}\right) &> 0 \\
\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right) &> 0 \\
x > \left(\frac{\sqrt{5}-1}{2}\right) \vee x < \left(\frac{-(\sqrt{5}+1)}{2}\right)
\end{aligned}$$

(vii)

$$\begin{aligned}
x^2 - x + 10 &> 16 \\
x^2 - x - 6 &> 0 \\
x^2 - 3x + 2x - 6 &> 0 \\
x(x-3) + 2(x-3) &> 0 \\
(x+2)(x-3) &> 0 \\
x > 3 &\vee x < -2
\end{aligned}$$

(viii)

$$\begin{aligned}
x^2 + x + 1 &> 0 \\
x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 &> 0 \\
(x + \frac{1}{2})^2 + \frac{3}{4} &> 0
\end{aligned}$$

which is true for all x.

(ix) Divide the problem into two cases: $x > \pi$ and $x < \pi$.

Case 1: $x > \pi$
Then $(x+5)(x-3) > 0$, which is $x > 3 \ \vee \ x < -5$.

Case 2: $x < \pi$
Then $(x+5)(x-3) < 0$, which is $-5 < x < 3$.

(x)

$$\begin{aligned}
(x - \sqrt[3]{2})(x - \sqrt{2}) &> 0 \\
x > \sqrt{2} &\vee x < \sqrt[3]{2}
\end{aligned}$$

(xi) (Sometimes, to solve a problem, intuition is a necessity.)

$$\begin{aligned} 2^x &< 8 \\ 2^x &< 2^3 \\ x &< 3 \end{aligned}$$

(xii)

$$\begin{aligned} x + 3^x &< 4 \\ x + 3^x &< 1 + 3^1 \\ x &< 1 \end{aligned}$$

(xiii)

$$\begin{aligned} \frac{1}{x} + \frac{1}{1-x} &> 0 \\ \frac{1}{x(1-x)} &> 0 \end{aligned}$$

Hence, $x(1-x) > 0$. This means $0 < x < 1$.

(xiv)

$$\frac{x-1}{x+1} > 0$$

Hence, $(x-1)(x+1) > 0$, or $x > 1 \ \vee \ x < -1$.

■

Problem 1.5. Prove the following:

(i) If $a < b$ and $c < d$, then $a + c < b + d$

(ii) If $a < b$, then $-b < -a$

(iii) If $a < b$ and $c > d$, then $a - c < b - d$

(iv) If $a < b$ and $c > 0$, then $ac < bc$

(v) If $a < b$ and $c < 0$, then $ac > bc$

(vi) If $a > 1$, then $a^2 > a$

(vii) If $0 < a < 1$, then $a^2 < a$

(viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$

(ix) If $0 \leq a < b$, then $a^2 < b^2$. (Use (viii).)

(x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$.(Use (ix), backwards.)

Solution. Let P be the set of all positive numbers.

(i) To prove this, we apply (P11): If $a < b \ \wedge \ c < d$, then $(b - a \in P) \ \wedge \ (d - c \in P)$. Then $(b - a) + (d - c) = (b + d) - (a + c) \in P$. Therefore, $a + c < b + d$.

(ii) We provide two solutions: The first one is by Trichotomy Law (P10), and the second one is by adding $[(-a) + (-b)]$ to both sides.

Proof by Trichotomy Law. If $a < b$, then $b - a \in P$. By Trichotomy Law, $a - b \notin P$ and $a - b \neq 0$. Therefore, $a - b < 0$, which is $-b < -a$. Q.E.D

Proof by adding.

$$\begin{gathered} a < b \\ a + [(-a) + (-b)] < b + [(-a) + (-b)] \\ [a + (-a)] + (-b) < [b + (-b)] + (-a) \\ -b < -a \end{gathered}$$

Q.E.D

(iii) Using (P11), we have $b - a \in P \ \wedge \ c - d \in P$. Then $(b - a) + (c - d) \in P$. Hence, $a - c < b - d$.

(iv) Using (P12), note that $b - a \in P$. Since $c > 0$, $c(b - a) \in P$, which means $bc - ac > 0$, or $ac < bc$.

(v) By Trichotomy law(P10), $-c \in P$. Then by (iv), $-(ac) < -(bc)$. By (ii), $ac > bc$.

(vi) Since $a > 1 > 0$, by (iv), $a^2 > a$.

(vii) Since $a > 0$, by (iv), $a^2 < a$.

(viii) Because $0 < b$, $bc < bd$. Furthermore, if $c \geq 0$, $ac \leq bc$ (equality occurs if $c = 0$), by (iv). Therefore, $ac \leq bc < bd$. Hence, $ac < bd$.

(ix) From (viii), let $c = a$ and $d = b$, then the result follows.

(x) Suppose $a \geq b$. Then $a \geq b \geq 0$. By (ix) and (P9), $a^2 \geq b^2$. This contradicts $a^2 < b^2$.

■

Problem 1.6. (a) Prove that if $0 \leq x < y$, then $x^n < y^n$, $n = 1, 2, 3, \ldots$.

(b) Prove that if $x < y$ and n is odd, then $x^n < y^n$.

(c) Prove that if $x^n = y^n$ and n is odd, then $x = y$.

(d) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Solution. (a) Repeatedly apply problem 1.5(viii) for $0 \leq x < y$, we have $x^n < y^n$ with $n = 1, 2, 3, \ldots$

(b) The statement is true for the case $0 \leq x < y$. In the case $x < y \leq 0$, by 1.5(ii), $(-x) > (-y) \geq 0$. By (a), $(-x)^n > (-y)^n$ for all odd n. Since n is odd, $-(x^n) > -(y^n)$. Hence, by 1.5(ii), $x^n < y^n$. In the case $x \leq 0 < y$, since n is odd, $x^n < y^n$.

(c) Suppose that either $x \neq y$. W.l.o.g, let $x < y$, by (b), $x^n < y^n$ for all odd n, contradicting $x^n = y^n$ for all odd n.

(d) Suppose that both $x \neq y$ and $x \neq -y$. Then $x^2 - y^2 \neq 0$. W.l.o.g, suppose $x^2 > y^2 \geq 0$. Applying (a), this generalizes to $x^n > y^n$ for all even n, contradicting our assumption. Therefore, $x = y$ or $x = -y$.

The direct proof. In the case $x, y \geq 0$; by (a), if $x^n = y^n$ for all even n, then $x = y$. In the case $x, y \leq 0$; if $x^n = y^n$ for all even n, then $(-x), (-y) \geq 0$ and $(-x)^n = (-y)^n$, so $-x = -y$ and hence $x = y$. In the case of x and y have different signs, then x and $-y$ are either two positive or two negative numbers. In either subcase, if $x^n = y^n$ for all even n, then $x^n = (-y)^n$, and it follows $x = -y$ from the previous case.

■

Problem 1.7. Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b$$

Notice that the inequality $\sqrt{ab} \leq (a+b)/2$ holds for all $a, b \geq 0$. A generalization of this fact occurs in Problem 2.22.

Solution. Let us first establish that $a < \dfrac{a+b}{2} < b$. Note that,

$$a + a < a + b < b + b$$

and therefore, $a < \dfrac{a+b}{2} < b$. To finish the proof, we need to prove $a < \sqrt{ab} < \dfrac{a+b}{2}$. To do this, let us prove that if $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$. Note that since $b - a > 0$,

$$b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) > 0$$

Therefore, $\sqrt{b} > \sqrt{a} > 0$. We rewrite the inequality as follows,

$$\sqrt{a} \cdot (\sqrt{b} - \sqrt{a}) > 0$$

Then

$$a < \sqrt{ab} \tag{1.1}$$

We next notice that since $\sqrt{b} - \sqrt{a} > 0$, it follows that $(\sqrt{b} - \sqrt{a}) \cdot (\sqrt{b} - \sqrt{a}) = (\sqrt{b} - \sqrt{a})^2 > 0$. Expand the left hand side,

$$(\sqrt{b} - \sqrt{a})^2 = a + b - 2\sqrt{ab} > 0$$

which implies,

$$\sqrt{ab} < \frac{a+b}{2} \tag{1.2}$$

From (1.1) and (1.2), we have $a < \sqrt{ab} < \dfrac{a+b}{2}$. ■

Problem 1.8 (*)**.** Although the basic properties of inequalities were stated in terms of the collection P of all positive numbers, and $<$ was defined in terms of P, this procedure can be reversed. Suppose that P10–P12 are replaced by

(P'10) For any numbers a and b one, and only one, of the following holds:

(i) $a = b$,

(ii) $a < b$,

(iii) $b < a$.

(P'11) For any numbers a, b, and c, if $a < b$ and $b < c$, then $a < c$.

(P'12) For any numbers a, b, and c, if $a < b$, then $a + c < b + c$.

(P'13) For any numbers a, b, and c, if $a < b$ and $0 < c$, then $ac < bc$.

Show that P10–P12 can then be deduced as theorems.

Solution. Let P be the set of all positive numbers.

- To prove P10, let $c = a - b$, from (P'10), P10 follows.
- To prove P11, let $a, b \in P$; it is sufficient to prove that $a + b > 0$. From (P'10), we divide the proof into three subscases:

  Case 1: $a = b$
  Then $a + b = b + b > 0 + b > 0$, where the first inequality follows from (P'12). By (P'11), $a + b > 0$.

  Case 2: $a < b$
  Then $a + b > a + a > 0 + a > 0$, where the first and second inequality follow from (P'12). By applying (P'11) twice, $a + b > 0$.

  Case 3: $a > b$
  Interchanging the role of a and b, we have the result.

- To prove P12, let $a, b \in P$; it is sufficient to prove that $a \cdot b > 0$. From (P'10), we divide the proof into three subcases:

  Case 1: $a = b$
  Then $a \cdot b = b \cdot b > 0 \cdot b = 0$, where the first inequality follows from (P'13) and the equality after which is from (P9).

  Case 2: $a < b$
  Then $b \cdot a > a \cdot a > 0 \cdot a = 0$, where the first and second inequality is from (P'13). By (P'11), $a \cdot b > 0$.

  Case 3: $a > b$
  Interchanging a and b returns us to case 2, which yields the result.

■

Problem 1.9. Express each of the following with at least one less pair of absolute value signs.

(i) $|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}|$

(ii) $|(|a + b| - |a| - |b|)|$

(iii) $|(|a + b| + |c| - |a + b + c|)|$

(iv) $|(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)|$

Solution. (i) Note $\sqrt{7} - \sqrt{5} > 0$, hence

$$|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}| = \sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$$

(ii) Since $|a + b| - |a| - |b| \leq 0$,

$$|(|a + b| - |a| - |b|)| = |a| + |b| - |a + b|$$

(iii) Since $|a + b + c| \leq |a + b| + |c|$,

$$|(|a + b| + |c| - |a + b + c|)| = |a + b| + |c| - |a + b + c|$$

(iv)

$$|(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)| = |\sqrt{2} + \sqrt{3} - \sqrt{7} + \sqrt{5}|$$

■

Problem 1.10. Express each of the following without absolute value signs, treating various cases separately when necessary.

(i) $|a + b| - |b|$

(ii) $|(|x| - 1)|$

(iii) $|x| - |x^2|$

(iv) $a - |(a - |a|)|$

Solution. (i) We divide into four cases:

$$a \geq 0 \quad \text{and} \quad b \geq 0 \tag{Case 1}$$
$$a \leq 0 \quad \text{and} \quad b \leq 0 \tag{Case 2}$$
$$a \geq 0 \quad \text{and} \quad b \leq 0 \tag{Case 3}$$
$$a \leq 0 \quad \text{and} \quad b \geq 0 \tag{Case 4}$$

In Case 1 and Case 2, we have $|a+b| - |b| = a$ since $|a+b| \leq |a| + |b|$.
In Case 3, if $a + b \geq 0$, then

$$|a+b| - |b| = (a+b) - (-b) = a + b + b = 2b$$

If $a + b \leq 0$, then

$$|a+b| - |b| = (-a-b) - (-b) = -a + (-b) + b = -a$$

In Case 4, if $a + b \geq 0$, then

$$|a+b| - |b| = (a+b) - (b) = a$$

If $a + b \leq 0$, then

$$|a+b| - |b| = -a + (-b) + (-b) = -a - 2b$$

(ii) We make the problem into 4 cases.

$$x \geq 1 \tag{Case 1}$$
$$0 \leq x \leq 1 \tag{Case 2}$$
$$-1 \leq x \leq 0 \tag{Case 3}$$
$$x \leq -1 \tag{Case 4}$$

In Case 1, $|(|x| - 1)| = x - 1$.
In Case 2, $|(|x| - 1)| = 1 - x$.
In Case 3, $|(|x| - 1)| = x + 1$.
In Case 4, $|(|x| - 1)| = -(x + 1)$.

(iii) Since $x^2 \geq 0$, $|x| - |x^2| = |x| - x^2$.
If $x \geq 0$, then $|x| - x^2 = x(1-x)$. If $x \leq 0$, then
$|x| - x^2 = -x + (-x^2) = -x(1+x)$.

(iv) Note that $|a| \geq a$. Hence,

$$a - |(a - |a|)| = a + a - |a| = 2a - |a|$$

We have two cases,

Case 1: $a \geq 0$

$$2a - |a| = 2a - a = a$$

Case 2: $a \leq 0$

$$2a - |a| = 2a + a = 3a$$

■

Problem 1.11. Find all numbers x for which

(i) $|x - 3| = 8$

(ii) $|x - 3| < 8$

(iii) $|x + 4| < 2$

(iv) $|x - 1| + |x - 2| > 1$

(v) $|x - 1| + |x + 1| < 2$

(vi) $|x - 1| + |x + 1| < 1$

(vii) $|x - 1| \cdot |x + 1| = 0$

(viii) $|x - 1| \cdot |x + 2| = 3$

Solution. (i)

$$x - 3 = 8 \vee x - 3 = -8$$
$$x = 11 \vee x = -5$$

(ii) Then $-8 < x - 3 < 8$. Hence, $-5 < x < 11$.

(iii) Then $-2 < x + 4 < 2$. Hence, $-6 < x < -2$.

(iv) If $1 \leq x \leq 2$, then the inequality becomes $(x - 1) + (2 - x) = 1$. If $x > 2$, then $2x - 3 > 1$, which is $x > 2$. If $x < 1$, then $-2x + 3 > 1$, which is $x < 1$. Therefore, either $x > 2$ or $x < 1$ satisfies the inequality.

(v) If $-1 \leq x \leq 1$, then $(1 - x) + (x + 1) = 2$. If $x > 1$, then $x < 1$, which is contradictory. If $x < -1$, then $(1 - x) + (-x - 1) = -2x < 2$ only if $x > -1$, which is contradictory. Hence, there is no x to satisfy the inequality.

(vi) It is implied from above that

$$|x - 1| + |x + 1| \geq 2$$

Therefore, there is no x satisfying the inequality.

(vii) Either $x = 1$ or $x = -1$.

(viii) If $-2 \leq x \leq 1$, then we obtain $x^2 + x + 1 > 0$. Hence, in either $x < -2$ or $x > 1$, we have to solve the equation $x^2 + x - 5 = 0$, whose solution is either $x = \dfrac{-1+\sqrt{21}}{2}$ or $x = \dfrac{-1-\sqrt{21}}{2}$.

■

Problem 1.12. Prove the following:

(i) $|xy| = |x| \cdot |y|$

(ii) $\left|\dfrac{1}{x}\right| = \dfrac{1}{|x|}$, if $x \neq 0$. (The best way to do this is to remember what $|x|^{-1}$ is.)

(iii) $\dfrac{|x|}{|y|} = \left|\dfrac{x}{y}\right|$, if $y \neq 0$.

(iv) $|x - y| \leq |x| + |y|$ (Give a very short proof.)

(v) $|x| - |y| \leq |x - y|$ (A very short proof is possible, if you write things in the right way.)

(vi) $|(|x| - |y|)| \leq |x - y|$ (Why does this follow immediately from (v)?)

(vii) $|x + y + z| \leq |x| + |y| + |z|$. Indicate when equality holds, and prove your statement.

Solution. (i) We have 4 cases,

$$x \geq 0 \quad y \geq 0 \tag{1}$$
$$x \geq 0 \quad y \leq 0 \tag{2}$$
$$x \leq 0 \quad y \geq 0 \tag{3}$$
$$x \leq 0 \quad y \leq 0 \tag{4}$$

In (1), $|x| \cdot |y| = xy = |xy|$

In (4), $|x| \cdot |y| = (-x)(-y) = xy = |xy|$

In (3), $|x| \cdot |y| = (-x)(y) = -(xy) = |xy|$

In (2), interchanging x and y leads to (3).

(ii) Since $x \neq 0$, there exists $|x|^{-1}$ such that

$$|x||x|^{-1} = 1 = |x|\left|\frac{1}{x}\right|$$

where the second equality is by (i). Dividing both sides by $|x|$, we have the result.

(iii) Since $y \neq 0$, from (ii), we immediately have

$$\left|\frac{1}{y}\right| = \frac{1}{|y|}$$

Hence, applying (ii) once more,

$$\left|\frac{x}{y}\right| = |x|\left|\frac{1}{y}\right| = \frac{|x|}{|y|}$$

(iv) Note that,

$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$$

where the last equality follows from (i).

(v) Note that,

$$|x - y + y| \leq |x - y| + |y|$$

Therefore, $|x| - |y| \leq |x - y|$.

(vi) Let the first term be y and the second term be $y - x$. Applying (v), we have

$$|y| - |y - x| \leq |x|$$

Hence, $-|x-y| \leq |x|-|y|$. Combining with (v) gives $|(|x|-|y|)| \leq |x-y|$.

(vii) Notice the pattern,

$$|x + y + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$$

the equality holds only if either x, y, z have the same sign or at least two of them must be equal to 0. It is easy to verify this.
Suppose not, then both x, y, z have different signs and at most one of them is 0. If the latter is true, then, w.l.o.g, suppose $z = 0$, then x, y have different sign, and we are done. If none of them is 0, then, w.l.o.g, suppose $z < 0$ and pick z such that $x + y < -z$. Then,

$$|x + y + z| = -(x + y + z) = -x - y - z < |x| + |y| + |z|$$

where inequality must follow since $x, y \neq 0$.

■

Problem 1.13. The maximum of two numbers x and y is denoted by $max(x, y)$. Thus $max(-1, 3) = max(3, 3) = 3$ and $max(-1, -4) = max(-4, -1) = -1$. The minimum of x and y is denoted by $min(x, y)$. Prove that

$$max(x, y) = \frac{x + y + |y - x|}{2},$$

$$min(x, y) = \frac{x + y - |y - x|}{2}.$$

Derive the formula for $max(x, y, z)$ and $min(x, y, z)$, using, for example

$$max(x, y, z) = max(x, max(y, z)).$$

Solution. Assume that $x \geq y$, we want to prove that $max(x, y) = x$.

$$max(x, y) = \frac{x + y + |y - x|}{2} = \frac{x + y + x - y}{2} = \frac{2x}{2} = x$$

Similarly, we need $min(x, y) = y$.

$$min(x, y) = \frac{x + y - |y - x|}{2} = \frac{x + y - (x - y)}{2} = \frac{x + y - x + y}{2} = \frac{2y}{2} = y$$

Let $max(x, y, z) = max(x, max(y, z))$. Then

$$\begin{aligned} max(x, y, z) &= \frac{x + max(y, z) + |max(y, z) - x|}{2} \\ &= \frac{x + \frac{y + z + |z - y|}{2} + \left|\frac{y + z + |z - y|}{2} - x\right|}{2} \\ &= \frac{2x + y + z + |z - y| + |y + z + |z - y| - 2x|}{4} \end{aligned}$$

Similarly,

$$min(x, y, z) = \frac{2x + y + z - |z - y| - |y + z - |z - y| - 2x|}{4}$$

■

Problem 1.14. (a) Prove that $|a| = |-a|$. (The trick is not to become confused by too many cases. First prove the statement for $a \geq 0$. Why is it then obvious for $a \leq 0$?)

(b) Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$. In particular, it follows that $-|a| \leq a \leq |a|$.

(c) Use this fact to give a new proof that $|a + b| \leq |a| + |b|$.

Solution. (a) Problem 1.12(i) easily tells us that

$$|-a| = |(-1)a| = |-1||a| = 1|a| = |a|$$

(b) If $a \geq 0$, then $a \leq b$. If $a \leq 0$, $-a \leq b$ follows from $a \geq -b$. Therefore, $|a| \leq b$. Conversely, suppose $|a| \leq b$. Then it is certain $a \leq b$ since $a \leq |a| \leq b$. From (a), $|-a| \leq b$, and hence $a \geq -b$. We conclude that $-b \leq a \leq b$. Note that since $|a| \leq |a|$, $-|a| \leq a \leq |a|$.

(c) Because we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, by Problem 1.5(i), we obtain $-(|a| + |b|) \leq a + b \leq |a| + |b|$. From (b), we arrive at the conclusion $|a + b| \leq |a| + |b|$.

■

Problem 1.15 (*)**.** Prove that if x and y are not both 0, then

$$x^2 + xy + y^2 > 0$$
$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0$$

Hint: Use problem 1.

Solution. For the first part, note that

$$x^2 + xy + y^2 = x^2 + 2 \cdot x \cdot \frac{1}{2}y + \frac{1}{4}y^2 - \frac{1}{4}y^2 + y^2 = \left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 > 0$$

For the second part, if $x = y$, then the left-hand side is $5x^4 > 0$. Hence, suppose $x \neq y$. From Problem 1.1(v),

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \neq 0$$

If $x > y$, then $x^5 > y^5$ by Problem 1.6(b). This implies that the second term must be greater than 0. Conversely, $x < y \Rightarrow x^5 < y^5$ implies that it must be greater than 0. ■

Problem 1.16 (*)**.** (a) Show that

$$(x + y)^2 = x^2 + y^2 \text{ only when } x = 0 \text{ or } y = 0,$$
$$(x + y)^3 = x^3 + y^3 \text{ only when } x = 0 \text{ or } y = 0 \text{ or } x = -y.$$

(b) Using the fact that

$$x^2 + 2xy + y^2 = (x + y)^2 \geq 0,$$

show that $4x^2 + 6xy + 4y^2 > 0$ unless x and y are both 0.

(c) Use part (b) to find out when $(x + y)^4 = x^4 + y^4$.

(d) Find out when $(x+y)^5 = x^5+y^5$. Hint: From the assumption $(x+y)^5 = x^5+y^5$ you should be able to derive the equation $x^3+2x^2y+2xy^2+y^3 = 0$, if $xy \neq 0$. This implies that $(x + y)^3 = x^2y + xy^2 = xy(x + y)$.

You should know be able to make a good guess as to when $(x + y)^n = x^n + y^n$; the proof is contained in Problem 11.57

Solution. (a) For the first part,

$$(x + y)^2 = x^2 + 2xy + y^2$$

Hence, $(x + y)^2 = x^2 + y^2$ only when $x = 0$ or $y = 0$. For the second part, from Problem 1.1(vi),

$$\begin{aligned}(x + y)^3 - (x + y)(x^2 - xy + y^2) &= 0\\ (x + y)(xy) &= 0\end{aligned}$$

which is true only when $x = 0$ or $y = 0$ or $x = -y$.

(b) Note that $4x^2 + 6xy + 4y^2 = \underbrace{3(x+y)^2}_{\geq 0} + \underbrace{x^2 + y^2}_{>0} > 0$ unless $x = 0$ and $y = 0$.

(c) Let us expand $(x+y)^4$.

$$\begin{aligned}(x+y)^2(x+y)^2 &= (x^2 + 2xy + y^2)(x^2 + 2xy + y^2) \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\ &= x^4 + y^4 + xy(4x^2 + 6xy + 4y^2)\end{aligned}$$

Hence, $(x+y)^4 = x^4 + y^4$ only when $x = 0$ or $y = 0$, by part (b).

(d) Let us expand $(x+y)^5$.

$$\begin{aligned}(x+y)^4(x+y) &= x^5 + y^5 + xy(x+y)(4x^2 + 6xy + 4y^2) + xy(x^3 + y^3) \\ &= x^5 + y^5 + 5xy(x+y)(x^2 - xy + y^2)\end{aligned}$$

If $xy \neq 0$ and $x + y \neq 0$, let $z = -y$, by Problem 1.6(b), $x^3 \neq z^3$. Hence, $x^2 - xy + y^2 \neq 0$. Therefore, $(x+y)^5 = x^5 + y^5$ only when $x = 0$ or $y = 0$ or $x = -y$.

Remark. Hence, for $(x+y)^n = x^n + y^n$, if n is even, then $x = 0$ or $y = 0$. If n is odd, then $x = 0$ or $y = 0$ or $x = -y$.

■

Problem 1.17. (a) Find the smallest possible value of $2x^2 - 3x + 4$. Hint: "Complete the square", i.e., write $2x^2 - 3x + 4 = 2(x - 3/4)^2 + ?$

(b) Find the smallest possible value of $x^2 - 3x + 2y^2 + 4y + 2$.

(c) Find the smallest possible value of $x^2 + 4xy + 5y^2 - 4x - 6y + 7$.

Solution. (a) Since $2x^2 - 3x + 4 = 2(x^2 - \frac{3}{2}x + 2)$,

$$2(x^2 - 2 \cdot x\frac{3}{4} + \frac{9}{16} - \frac{9}{16} + 2) = 2(x - \frac{3}{4})^2 + \frac{23}{8}$$

Hence the minimum value is $\frac{23}{8}$ when $x = \frac{3}{4}$.

(b)

$$x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2(y^2 + 2y + 1) = \left(x - \frac{3}{2}\right)^2 + 2(y+1)^2 - \frac{9}{4}$$

The minimum value is $-\frac{9}{4}$ when $x = \frac{3}{2}$ and $y = -1$.

(c)

$$\begin{aligned}
&\frac{1}{2}x^2 + 4xy + 8y^2 - 3y^2 - 6y + 7 + \frac{1}{2}x^2 - 4x \\
=&\frac{1}{2}(x^2 + 8xy + 16y^2) - 3(y^2 + 2y + 1) + \frac{1}{2}(x^2 - 8x + 16) + 2 \\
=&\frac{1}{2}(x + 4y)^2 - 3(y + 1)^2 + \frac{1}{2}(x - 4)^2 + 2
\end{aligned}$$

Therefore, the minimum value is 2 when $x = 4$ and $y = -1$.

■

Problem 1.18. (a) Suppose that $b^2 - 4c \geq 0$. Show that the numbers

$$\frac{-b + \sqrt{b^2 - 4c}}{2}, \quad \frac{-b - \sqrt{b^2 - 4c}}{2}$$

both satisfy the equation $x^2 + bx + c = 0$.

(b) Suppose that $b^2 - 4c < 0$. Show that there are no numbers x satisfying $x^2 + bx + c = 0$; in fact, $x^2 + bx + c > 0$ for all x. Hint: Complete the square.

(c) Use this fact to give another proof that if x and y are not both 0, then $x^2 + xy + y^2 > 0$.

(d) For which number α is it true that $x^2 + \alpha xy + y^2 > 0$ whenever x and y are not both 0?

(e) Find the smallest possible value of $x^2 + bx + c$ and of $ax^2 + bx + c$, for $a > 0$.

Solution. (a) Substitution immediately gives the desired result.

(b)

$$x^2 + bx + c = x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c$$

which immediately yields $\left(x + \frac{b}{2}\right)^2 + \frac{[-(b^2 - 4c)]}{4} > 0$ for all x since $b^2 - 4c < 0$.

(c) If $y = 0$, $x^2 > 0$. Suppose not, using (b), we obtain $-3y^2 < 0$. Hence, $x^2 + xy + y^2 > 0$.

(d) If $y = 0$, the result follows for all α. Suppose $y \neq 0$, using (b), we obtain $\alpha^2 y^2 - 4y^2 < 0$, which is $y^2(\alpha^2 - 4) < 0$. It follows that $-2 < \alpha < 2$.

(e) From (b), it follows that the minimum value of x^2+bx+c is $\dfrac{[-(b^2-4c)]}{4}$ when $x=-b/2$. Since $a>0$, with the role of b is now b/a and of c is c/a, we easily derive the result.

$$x^2+\frac{b}{a}x+\frac{c}{a}=\left(x+\frac{b}{2a}\right)^2+\frac{[-(b^2-4ac)]}{4a^2}$$

So its minimum value is $\dfrac{[-(b^2-4ac)]}{4a^2}$ when $x=-\dfrac{b}{2a}$.

■

Problem 1.19. The fact that $a^2\geq 0$ for all numbers a, elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:

$$x_1y_1+x_2y_2\leq\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}.$$

(A more general form occurs in Problem 2.21) The three proofs of the Schwarz inequality outlined below have only one thing in common–their reliance on the fact that $a^2\geq 0$ for all a.

(a) Prove that if $x_1=\lambda y_1$ and $x_2=\lambda y_2$ for some number λ, then equality holds in Schwarz inequality. Prove the same thing if $y_1=y_2=0$. Now suppose that y_1 and y_2 are not both 0, and that there is no number λ such that $x_1=\lambda y_1$ and $x_2=\lambda y_2$. Then

$$\begin{aligned}0&<(\lambda y_1-x_1)^2+(\lambda y_2-x_2)^2\\&=\lambda^2(y_1^2+y_2^2)-2\lambda(x_1y_1+x_2y_2)+(x_1^2+x_2^2).\end{aligned}$$

Using Problem 1.18, complete the proof of the Schwarz inequality.

(b) Prove the Schwarz inequality by using $2xy\leq x^2+y^2$ (how is this derived?) with

$$x=\frac{x_i}{\sqrt{x_1^2+x_2^2}},\quad y=\frac{y_i}{\sqrt{y_1^2+y_2^2}},$$

first for $i=1$ and then for $i=2$.

(c) Prove the Schwarz inequality by first proving that

$$(x_1^2+x_2^2)(y_1^2+y_2^2)=(x_1y_1+x_2y_2)^2+(x_1y_2-x_2y_1)^2.$$

(d) Deduce, from each of these three proofs, that equality holds only when $y_1=y_2=0$ or when there is a number λ such that $x_1=\lambda y_1$ and $x_2=\lambda y_2$.

Solution. (a) If $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for every $\lambda \geq 0$,

$$\begin{aligned}\lambda(y_1^2 + y_2^2) &= |\lambda|\sqrt{(y_1^2 + y_2^2)^2} \\ &= \lambda(y_1^2 + y_2^2)\end{aligned}$$

Or if $y_1 = y_2 = 0$, then equality holds since both sides are 0. Otherwise, suppose that y_1 and y_2 are not both 0, and there is no number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$, then

$$\begin{aligned}0 &< \lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2) \\ &= \lambda^2 - 2\lambda\frac{x_1y_1 + x_2y_2}{y_1^2 + y_2^2} + \frac{x_1^2 + x_2^2}{y_1^2 + y_2^2}\end{aligned}$$

This holds only when, by Problem 1.18(b),

$$\frac{4(x_1y_1 + x_2y_2)^2}{(y_1^2 + y_2^2)^2} + \frac{[-4(x_1^2 + x_2^2)(y_1^2 + y_2^2)]}{(y_1^2 + y_2^2)^2} < 0$$

which only holds when

$$x_1y_1 + x_2y_2 < \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

since $a \leq |a|$ for all a.

(b) Note that $(x - y)^2 \geq 0$. For $i = 1$,

$$\frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2} \geq 2 \cdot \frac{x_1y_1}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \tag{1.3}$$

For $i = 2$,

$$\frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2} \geq 2 \cdot \frac{x_2y_2}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \tag{1.4}$$

(1.3)+(1.4), we derive

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

(c)

$$\begin{aligned}&(x_1^2 + x_2^2)(y_1^2 + y_2^2) \\ =&(x_1^2y_1^2 + 2x_1y_1x_2y_2 + x_2^2y_2^2) + (x_1^2y_2^2 - 2x_1y_2x_2y_1 + x_2^2y_1^2) \\ =&(x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2\end{aligned}$$

Note that $(x_1y_2 - x_2y_1)^2 \geq 0$. Hence,

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

since $a \leq |a|$ for all a.

(d) In (a), it is obvious; the proof is based on the separation of two cases, $a^2 = 0$ and $a^2 > 0$. In (b), equality occurs only when $x = y$; by construction, $y_1 = y_2 = 0$ or, if not,

$$\frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{y_1}{\sqrt{y_1^2 + y_2^2}}$$
$$\frac{x_2}{\sqrt{x_1^2 + x_2^2}} = \frac{y_2}{\sqrt{y_1^2 + y_2^2}}$$

implies that for

$$\lambda = \frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{y_1^2 + y_2^2}}$$

$x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

In (c), equality occurs only when $(x_1y_2 - x_2y_1)^2 = 0$ and $x_1y_1 + x_2y_2 \geq 0$. These will be satisfied only when $y_1 = y_2 = 0$ or for $\lambda \geq 0$, $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

■