Answers to exercises in How To Prove It

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This is to answer all the questions in the books "How to prove it" by Velleman. Comments are appreciated! I benefit greatly from inc for its solutions.

Contents

1 Introduction

Exercise 1.0.1.

(a) $a = 3$, $b = 5 \Rightarrow x = 2^5 - 1 = 31$, $y = 1 + 2^5 + 2^{10} = 1057$

(b) Since $32,767$ is not a prime, $2^{32,767} - 1$ is not a prime either. Therefore, there exists a positive integer $0 < x < 2^{32,767} - 1$ such that $2^{32,767} - 1$ is divisible by x. Hence, by (a), $x = 2^{31} - 1$ satisfies this.

Exercise 1.0.2.

n	$3^n - 1$	$3^n - 2^n$
2, prime	8, not prime	5, prime
3, prime	26, not prime	19, prime
4	80, not prime	65, not prime
5, prime	242, not prime	211, prime
6	728, not prime	665, not prime
7, prime	$2,186$, not prime	$2,059$, prime
8	$6,560$, not prime	$6,305$, not prime
9	$19,682$, not prime	$19,171 = 19 \cdot 1,009$, not prime
10	$59,048$, not prime	$58,025$, not prime

Conjecture 1.1. $3^n - 1$ is even for all n.

Conjecture 1.2. If n is prime, $3^n - 2^n$ is prime.

Conjecture 1.3. If n is not prime, $3^n - 2^n$ is not prime.

Exercise 1.0.3.
We have the following theorem.

Theorem 1.1 (Theorem 3)**.** *There are infinitely many prime numbers.*

Its proof gives a method of finding a prime number n different from the ones in a given list. However, one needs to be careful of prime numbers *not* in the list and smaller than n, since the proof assumes that primes are finite. Example, $n = 3 \cdot 5 + 1 = 15 + 1 = 16$ is not a prime number. Hence,

Remark 1.1 (Method). To find a prime different from the list, check if it is divisible by other primes missing from the list.

(a) Let $n = 2 \cdot 5 + 1 = 11$. n is not divisible by $3, 7$, a new prime number.

(b) Let $n = 2 \cdot 3 + 1 = 7$. n is not divisible by $5, 11$, a new prime number.

Exercise 1.0.4.
$24, 25, 26, 27, 28$

Exercise 1.0.5.
$2^4 \cdot (2^5 - 1) = 496$
$2^6 \cdot (2^7 - 1) = 127.$

Exercise 1.0.6.

Conjecture 1.4. $3, 5, 7$ is the unique triplet prime.

I do not know now to prove it.

2 Chapter 1

2.1 Deductive reasoning and logical connectives

Exercise 2.1.1.

(a) $(R \vee H) \wedge \neg(H \wedge T)$

(b) $S =$ "You go skiing", $N =$ "There is snow".
$\neg S \vee (S \wedge \neg N)$

(c) $\neg((\sqrt{7} = 2) \vee (\sqrt{7} < 2))$

Exercise 2.1.2.

(a) $J =$ John is telling the truth, $B =$ Bill is telling the truth
$(J \wedge B) \vee (\neg J \wedge \neg B)$

(b) $F =$ "I have fish", $C =$ "I have chicken", $M =$ "I have mashed potatoes"
$(F \vee C) \wedge \neg(F \wedge M)$

(c) $(6 \vdots 3) \wedge (9 \vdots 3) \wedge (15 \vdots 3)$

Exercise 2.1.3.
$A =$ Alice is in the room, $B =$ Bob is in the room

(a) $\neg(A \wedge B)$

(b) $\neg A \wedge \neg B$

(c) $\neg A \vee \neg B$

(d) $\neg A \wedge \neg B$

Exercise 2.1.4.
$a)$ and $c)$

Exercise 2.1.5.

(a) I will not buy the pants without the shirt.

(b) I will buy neither the pants nor the shirt.

(c) Either I will not buy the pants or I will not buy the shirt.

Exercise 2.1.6.

(a) At least one of them is happy and at least one of them is not happy.

(b) Either at least one of Steve and George is happy or both are unhappy.

(c) Either Steve is happy or George, not Steve, is happy.

Exercise 2.1.7.

Remark 2.1. An argument is *valid* if the premises cannot all be true without the conclusion being true as well.

(a) Valid.

(b) If Beef and Peas were served, clearly the first two premises were satisfied. The third premise was also satisfied because both Fish and Corn were not served. Hence, the argument is invalid.

(c) We base our argument on Bill,

- If Bill is lying, then John must be telling the truth.
- If Bill is telling the truth, then Sam must be lying.

The conclusion is valid.

Another approach: Suppose John is lying and Sam is telling the truth, then Bill is telling the truth and Sam must be lying: premise is not satisfied. Hence, the conclusion is valid.

(d) Suppose sales and expenses go up. Then the premise is satisfied. Hence, the conclusion is invalid.

2.2 Truth Tables

Exercise 2.2.1.

$\neg P$	Q	$\neg P \vee Q$	$\neg S$	$\neg G$	$S \vee G$	$\neg S \vee \neg G$	$(S \vee G) \wedge (\neg S \vee \neg G)$
TF	T	T	TF	FT	T	T	T
FT	F	F	FT	TF	T	T	T
TF	F	T	TF	TF	F	T	F
FT	T	T	FT	FT	T	F	F

Exercise 2.2.2.
First part is equivalent to $\neg P \vee \neg Q$. The truth table is easy to make.

Exercise 2.2.3.
$P + Q = (P \vee Q) \wedge \neg(P \wedge Q) = (P \vee Q) \wedge (\neg P \vee \neg Q)$. Look at Exercise 2.2.1.

Exercise 2.2.4.
$P \vee Q = \neg(\neg P \wedge \neg Q)$.

Exercise 2.2.5.

(a)

$\neg P$	$\neg Q$	$P \downarrow Q$
FT	TF	F
TF	FT	F
TF	TF	T
FT	FT	F

(b) $P \downarrow Q = \neg P \wedge \neg Q$.

(c) It is easy to see that
$\neg P = P \downarrow P$.
$P \vee Q = \neg(\neg P \wedge \neg Q) = \neg(P \downarrow Q) = (P \downarrow Q) \downarrow (P \downarrow Q)$.
where the second equality is by De Morgan's law; the fourth is by the previous part.
$P \wedge Q = \neg(\neg P \vee \neg Q) = \neg\neg(\neg P \downarrow \neg Q) = (P \downarrow P) \downarrow (Q \downarrow Q)$.

Exercise 2.2.6.

(a)

$\neg P$	$\neg Q$	$P \vert Q$
FT	TF	T
TF	FT	T
TF	TF	T
FT	FT	F

(b) $P|Q = \neg(P \wedge Q) = \neg P \vee \neg Q$.

(c) $\neg P = P|P$.
$P \vee Q = \neg(\neg P) \vee \neg(\neg Q) = \neg P|\neg Q = (P|P)|(Q|Q)$.
$P \wedge Q = \neg(\neg P \vee \neg Q) = \neg(P|Q) = (P|Q)|(P|Q)$.

Exercise 2.2.7.
This exercise is trivial and easily done. Note that if both premises and conclusions are not simultaneously true FOR ALL CASES, then the statement is invalid.

Exercise 2.2.8.
$a) == c)$ and $b) == e)$. $d)$ is not equivalent to any other part.

Exercise 2.2.9.
It's so easy to see that there is no need for truth tables.

(a) Neither.

(b) Contradiction.

(c) Tautology.

(d) A simple derivation yields:
$((P \vee \neg P) \wedge (Q \vee \neg R \vee \neg P)) \vee R = Q \vee (\neg R \vee R) \vee \neg P$,
which equals a tautology.

Exercise 2.2.10.
So obvious.

Exercise 2.2.11.

(a) $P \vee Q$

(b)

$$\begin{aligned} & (P \wedge (P \vee Q)) \wedge ((P \wedge Q) \vee \neg Q) \\ = \quad & P \wedge ((P \vee \neg Q) \wedge (Q \vee \neg Q)) \\ = \quad & P \wedge (P \vee \neg Q) \\ = \quad & P \end{aligned}$$

(c)

$$\begin{aligned} & (\neg P \vee Q) \vee (\neg P \wedge Q) \\ = & Q \vee ((\neg P \vee \neg P) \wedge (\neg P \vee Q)) \\ = \quad & Q \vee (\neg P \wedge (\neg P \vee Q)) \\ = \quad & \neg P \vee Q \end{aligned}$$

Exercise 2.2.12.

(a)

$$\begin{aligned} & (P \wedge \neg Q) \vee (P \wedge \neg R) \\ = \quad & P \wedge (\neg Q \vee \neg R) \end{aligned}$$

(b)

$$\begin{aligned} = & \neg Q \vee [P \vee (P \wedge \neg R)] \\ = \quad & \neg Q \vee P \end{aligned}$$

The first is by De Morgan's law and the associativity. The second is by absorption law.

(c)

$$\begin{aligned} = & [(P \wedge R) \vee (P \wedge \neg R)] \vee (Q \wedge \neg R) \\ = \quad & [P \wedge (R \vee \neg R)] \vee (Q \wedge \neg R) \\ = \quad & P \vee (Q \wedge \neg R) \end{aligned}$$

The first is by distributive and associative law, respectively. The second is by distributive, and the last is by tautology law.

Exercise 2.2.13.

$$\neg(P \vee Q) = \neg(\neg\neg P \vee \neg\neg Q) = \neg[\neg(\neg P \wedge \neg Q)] = (\neg P \wedge \neg Q)$$

The first and the third equality are by double negation law. The second is by De Morgan's first law.

Exercise 2.2.14.

$$[P \wedge (Q \wedge R)] \wedge S = [(P \wedge Q) \wedge R] \wedge S = (P \wedge Q) \wedge (R \wedge S).$$

A general proof can be found in [Spivak, 1994, p. 19, problem 24].

Exercise 2.2.15.
2^n. Using counting techinque.

Exercise 2.2.16.
$P \vee \neg Q$

Exercise 2.2.17.
$P + Q = (P \vee Q) \wedge (\neg P \vee \neg Q)$. Exclusive *or*. See Exercise 2.2.3.

Exercise 2.2.18.
Remember a conclusion is valid when if all the premises are true, then the conclusion is true. So if the conclusion is a tautology. Then there are two cases.

- If the premises are all true. Obviously the argument is valid.
- If at least one of the premises is not true, then the argument is also valid, since the validity check only applies if all the premises are true.

If the conclusion is a contradiction,

- if all the premises are true, then the argument is invalid.
- if at least one of the premises is false, then the argument is valid.

If at least one of the premises is a tautology, it depends on the rest of the premises and the conclusion. If at least one of the premises is a contradiction, the argument is always valid because

- if the conclusion is false, then the argument is valid.
- if the conclusion is true, the validity check does not apply, and hence the argument is valid.

References

inchmeal. https://www.inchmeal.io/.

Michael Spivak. *Calculus*. Publish or Perish, America, third edition, 1994.