

A Note of Calculus-Michael Spivak

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Preface

This is the note for the book Calculus written by Michael Spivak, citing what I think the most interesting and important subjects mentioned in the book.

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Part I

Prologue

Chapter 1

Basic properties of number

(P1) If a , b , and c are any numbers, then

$$a + (b + c) = (a + b) + c$$

See *problem 24* for the generalization of $a_1 + a_2 + a_3 + \cdots + a_n$ for (P1).

The number 0 has important properties.

(P2) If a is any number, then

$$a + 0 = 0 + a = a$$

(P3) For every number a , there is also a number $-a$ such that

$$a + (-a) = (-a) + a = 0$$

We now prove Lemma 1.

Lemma 1. *If $a + x = a$, then $x = 0$*

Proof.

If	$a + x = a$	
then	$(-a) + (a + x) = (-a) + a = 0$	(by (P3))
hence	$((-a) + a) + x = 0$	(by (P1))
hence	$0 + x = 0$	(by (P3) again)
therefore,	$x = 0$	(by (P2))

□

Also, remember that the order of addition does not matter.

(P4) If a and b are any numbers, then

$$a + b = b + a$$

However, with only (P1)-(P4), we are powerless to figure out what conditions needed to have $a - b = b - a$. Therefore, we need to set new properties, and, oddly, they involve multiplication.

(P5) If a, b and c are any numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(P6) If a is any number, then

$$a \cdot 1 = 1 \cdot a = a$$

Moreover, $1 \neq 0$ (This cannot be proved by other properties listed!)

(P7) For every number $a \neq 0$, there is a number a^{-1} such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1 (\Leftrightarrow 0 \cdot b = 0 \ \forall b)$$

This is why $1/0$ is meaningless!

(P8) If a and b are any numbers, then

$$a \cdot b = b \cdot a$$

From (P5), (P6) and (P7), we have two lemmas:

Lemma 2. *If $a \cdot b = a \cdot c$ then $a = 0 \vee b = c$*

Proof. If $a = 0$ then the lemma is trivial. Suppose now $a \neq 0$,

Multiply a^{-1} to both sides, $(a^{-1}) \cdot (a \cdot b) = (a^{-1}) \cdot (a \cdot c)$

By (P5), $(a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c$

By (P7), $1 \cdot b = 1 \cdot c$

By (P6), $b = c$

□

Lemma 3. *If $a \cdot b = 0$ then $a = 0 \vee b = 0$*

Proof. If $a = 0$, there is nothing to prove. Suppose now $a \neq 0$, follow the proof of Lemma 2 by consecutively applying (P5), (P7) and (P6) in that order to finish the proof. □

We, however, will not be able to prove anything without a relationship between multiplication and addition. Therefore, the next property is definitely necessary.

(P9) If a , b and c are any numbers, then

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

By (P8), it is also true that $(b + c) \cdot a = b \cdot a + c \cdot a$

We will see in the next remark and lemmas that properties are not built in a straight line. Rather, it is a result of necessities, of fixes and starts that somehow fits the pieces of a puzzle perfectly.

Remark. When $a - b = b - a$?

Solution.

Add b at both sides, $(a - b) + b = (b - a) + b = b + (b - a)$ by (P4)

By (P1), $a + (-b + b) = (b + b) + (-a)$

By (P3), $a + 0 = b + b - a$

By (P2), $a = b + b - a$

Add both sides to a , $a + a = (b + b - a) + a$

By (P1), $a + a = b + (b + (-a + a)) = b + b$ by (P2) and (P3)

By (P9), $a \cdot (1 + 1) = b \cdot (1 + 1)$

By Lemma 2, $a = b$

■

Lemma 4. $a \cdot 0 = 0$

Proof.

We have $a \cdot 0 + a \cdot 0 = a \cdot (0 + 0)$ by (P9)

By (P2), $= a \cdot 0$

Add $-a \cdot 0$, $a \cdot 0 = 0$

□

Lemma 5. *The product of two negative numbers is positive*

Proof. We first prove that $(-a) \cdot b = -(a \cdot b)$,

We have by (P9), $(-a) \cdot b + (a \cdot b) = (-a + a) \cdot b = 0$

Adding $-(a \cdot b)$ to both sides, $(-a) \cdot b = -(a \cdot b)$

Now, let's prove the main statement.

From above, $(-a) \cdot (-b) + [-(a \cdot b)] = (-a) \cdot (-b) + [(-a) \cdot b]$
 $= (-a) \cdot (-b + b)$
 $= 0$

Adding $(a \cdot b)$ to both sides, $(-a) \cdot (-b) = (a \cdot b)$

□

We say that Lemma 5 is a direct consequence of (P1)-(P9).

Also, (P9) has important consequences: Justifying the algebraic manipulations (e.g, $x^2 - 3x + 2 = (x - 1)(x - 2)$) and the way one multiplies arabic numerals,

$$\begin{array}{r} 13 \\ \times 24 \\ \hline 52 \\ 26 \\ \hline 312 \end{array}$$

Denote the set of all positive numbers by P .

(P10) (Trichotomy law) For every number a , one and only one of the following holds:

- (i) $a = 0$
- (ii) $a \in P$
- (iii) $-a \in P$

(P11) (Closure under addition) If $a \in P \wedge b \in P$ then $a + b \in P$

(P12) (Closure under multiplication) If $a \in P \wedge b \in P$ then $a \cdot b \in P$

These properties should be complemented by the following definitions:

$$\begin{array}{ll} a > b & \text{if } a - b \in P \\ a < b & \text{if } b > a \end{array}$$