

# A Note of Calculus-Michael Spivak

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23rd June, 2017

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# Preface

This is the note for the book Calculus writtten by Michael Spivak, citing what I think the most interesting and important subjects mentioned in the book.

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# Part I

## Prologue





# Chapter 1

## Basic properties of number

(P1) If  $a$ ,  $b$ , and  $c$  are any numbers, then

$$a + (b + c) = (a + b) + c$$

See *problem 24* for the generalization of  $a_1 + a_2 + a_3 + \cdots + a_n$  for (P1).

The number 0 has important properties.

(P2) If  $a$  is any number, then

$$a + 0 = 0 + a = a$$

(P3) For every number  $a$ , there is also a number  $-a$  such that

$$a + (-a) = (-a) + a = 0$$

We now prove Lemma 1.

**Lemma 1.** *If  $a + x = a$ , then  $x = 0$*

*Proof.*

If	$a + x = a$	
then	$(-a) + (a + x) = (-a) + a = 0$	(by (P3))
hence	$((-a) + a) + x = 0$	(by (P1))
hence	$0 + x = 0$	(by (P3) again)
therefore,	$x = 0$	(by (P2))

□

Also, remember that the order of addition does not matter.

(P4) If  $a$  and  $b$  are any numbers, then

$$a + b = b + a$$

However, with only (P1)-(P4), we are powerless to figure out what conditions needed to have  $a - b = b - a$ . Therefore, we need to set new properties, and, oddly, they involve multiplication.

(P5) If  $a, b$  and  $c$  are any numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(P6) If  $a$  is any number, then

$$a \cdot 1 = 1 \cdot a = a$$

Moreover,  $1 \neq 0$  (This cannot be proved by other properties listed!)

(P7) For every number  $a \neq 0$ , there is a number  $a^{-1}$  such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1 (\Leftrightarrow 0 \cdot b = 0 \ \forall b)$$

*This is why  $1/0$  is meaningless!*

(P8) If  $a$  and  $b$  are any numbers, then

$$a \cdot b = b \cdot a$$

From (P5), (P6) and (P7), we have two lemmas:

**Lemma 2.** *If  $a \cdot b = a \cdot c$  then  $a = 0 \vee b = c$*

*Proof.* If  $a = 0$  then the lemma is trivial. Suppose now  $a \neq 0$ ,

Multiply  $a^{-1}$  to both sides,  $(a^{-1}) \cdot (a \cdot b) = (a^{-1}) \cdot (a \cdot c)$

By (P5),  $(a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c$

By (P7),  $1 \cdot b = 1 \cdot c$

By (P6),  $b = c$

□

**Lemma 3.** *If  $a \cdot b = 0$  then  $a = 0 \vee b = 0$*

*Proof.* If  $a = 0$ , there is nothing to prove. Suppose now  $a \neq 0$ , follow the proof of Lemma 2 by consecutively applying (P5), (P7) and (P6) in that order to finish the proof. □

We, however, will not be able to prove anything without a relationship between multiplication and addition. Therefore, the next property is definitely necessary.

(P9) If  $a$ ,  $b$  and  $c$  are any numbers, then

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

By (P8), it is also true that  $(b + c) \cdot a = b \cdot a + c \cdot a$

We will see in the next remark and lemmas that properties are not built in a straight line. Rather, it is a result of necessities, of fixes and starts that somehow fits the pieces of a puzzle perfectly.

*Remark.* When  $a - b = b - a$ ?

*Solution.*

Add  $b$  at both sides,  $(a - b) + b = (b - a) + b = b + (b - a)$  by (P4)

By (P1),  $a + (-b + b) = (b + b) + (-a)$

By (P3),  $a + 0 = b + b - a$

By (P2),  $a = b + b - a$

Add both sides to  $a$ ,  $a + a = (b + b - a) + a$

By (P1),  $a + a = b + (b + (-a + a)) = b + b$  by (P2) and (P3)

By (P9),  $a \cdot (1 + 1) = b \cdot (1 + 1)$

By Lemma 2,  $a = b$

■

Note that the proof above based on the presumption that we know  $1 + 1 \neq 0$ . How do we prove it?

**Lemma 4.**  $a \cdot 0 = 0$

*Proof.*

We have  $a \cdot 0 + a \cdot 0 = a \cdot (0 + 0)$  by (P9)

By (P2),  $= a \cdot 0$

Add  $-a \cdot 0$ ,  $a \cdot 0 = 0$

□

**Lemma 5.** *The product of two negative numbers is positive*

*Proof.* We first prove that  $(-a) \cdot b = -(a \cdot b)$ ,

We have by (P9),  $(-a) \cdot b + (a \cdot b) = (-a + a) \cdot b = 0$

Adding  $-(a \cdot b)$  to both sides,  $(-a) \cdot b = -(a \cdot b)$

Now, let's prove the main statement.

$$\begin{aligned} \text{From above,} \quad (-a) \cdot (-b) + [-(a \cdot b)] &= (-a) \cdot (-b) + [(-a) \cdot b] \\ &= (-a) \cdot (-b + b) \\ &= 0 \end{aligned}$$

$$\text{Adding } (a \cdot b) \text{ to both sides,} \quad (-a) \cdot (-b) = (a \cdot b)$$

□

We say that Lemma 5 is a direct consequence of (P1)-(P9).

Also, (P9) has important consequences: Justifying the algebraic manipulations (e.g,  $x^2 - 3x + 2 = (x - 1)(x - 2)$ ) and the way one multiplies arabic numerals,

$$\begin{array}{r} 13 \\ \times 24 \\ \hline 52 \\ 26 \phantom{0} \\ \hline 312 \end{array}$$

Denote the set of all positive numbers by  $P$ .

(P10) (Trichotomy law) For every number  $a$ , one and only one of the following holds:

- (i)  $a = 0$
- (ii)  $a \in P$
- (iii)  $-a \in P$

(P11) (Closure under addition) If  $a \in P \wedge b \in P$  then  $a + b \in P$

(P12) (Closure under multiplication) If  $a \in P \wedge b \in P$  then  $a \cdot b \in P$

These properties should be complemented by the following definitions:

$$\begin{aligned} a > b &\text{ if } a - b \in P \\ a < b &\text{ if } b > a \\ a \geq b &\text{ if } a = b \text{ or } a > b \\ a \leq b &\text{ if } a = b \text{ or } a < b \end{aligned}$$

The following lemmas are easy to prove...

**Lemma 6.** *If  $a < b$  then  $a + c < b + c$*

*Proof.* If  $a < b$ , then  $b - a \in P$ , which is surely  $(b + c) - (a + c) \in P$  □

**Lemma 7.** *If  $a < b \wedge b < c$  then  $a < c$*

*Proof.* Then  $b - a \in P$  and  $c - b \in P$ . By (P11),  $(b - a) + (c - b) = c - a \in P$  □

**Lemma 8.** *If  $a < 0 \wedge b < 0$  then  $a \cdot b > 0$*

*Proof.* Then  $-a > 0 \wedge -b > 0$ . By (P12),  $(-a) \cdot (-b) = a \cdot b > 0$ , by Lemma 5.  $\square$

**Lemma 9.** *If  $a \neq 0$ , then  $a^2 \neq 0$*

*Proof.* Because if  $a > 0 \wedge b > 0$  and  $a < 0 \wedge b < 0$  then  $a \cdot b > 0$ , let  $b = a$   $\square$

This implies that  $1 > 0$  (since  $1^2 = 1$ ).

We now prove a basic theorem relating to the absolute value.

**Theorem 1.**  $\forall a \wedge b$ ,

$$|a + b| \leq |a| + |b|$$

*Proof.* We apply the straightforward proof. A more elegant proof appears in the exercises. We will consider 4 cases:

$$a \geq 0 \quad \text{and} \quad b \geq 0 \tag{1}$$

$$a \geq 0 \quad \text{and} \quad b \leq 0 \tag{2}$$

$$a \leq 0 \quad \text{and} \quad b \geq 0 \tag{3}$$

$$a \leq 0 \quad \text{and} \quad b \leq 0 \tag{4}$$

For (1), the statement occurs with equality; that is,

$$|a + b| = a + b = |a| + |b|$$

For (4), the same is true by observing,

$$|a + b| = -(a + b) = (-a) + (-b) = |a| + |b|$$

For (2), the job is dumped down to proving that  $|a + b| \leq a - b$ . This divides the case into two subcases.

*Subcase 1:*  $a + b \geq 0$

Then note that  $b \leq (-b)$ , which is true since  $b \leq 0$ .

*Subcase 2:*  $a + b \leq 0$

Then we have  $(-a) \leq a$ , which is true since  $a \geq 0$ .

For (3), the case is proved by interchanging the role of  $a$  and  $b$ .  $\square$

Note from the proof above that equality happens if  $a$  and  $b$  have the same sign, or one of the two is zero.

*Remark.* It is crucial to understand that (P1)-(P12) are not enough to account for *all* properties of numbers. The deficiency is profound and subtle; and, hopefully, will be discovered in the rest of the note.