

Solutions to Michael Spivak's Calculus

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*I thank my employer!

To my mother, friends and all those who influence me.

Preface

This is my own solutions to Michael Spivak's Calculus textbook.

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Part I

Prologue

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Chapter 1

Basic properties of number

Problem 1.1. Prove the following:

- (i) If $ax = a$ for some number $a \neq 0$, then $x = 1$
- (ii) $x^2 - y^2 = (x - y)(x + y)$
- (iii) If $x^2 = y^2$, then $x = y$ or $x = -y$
- (iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
- (v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$
- (vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (There is a particularly easy way to do this using (iv), and it will show you how to find a factorization for $x^n + y^n$ whenever n is odd.)

Solution. (i) By (P7)(Existence of multiplicative inverses), there exists a^{-1} such that,

$$\begin{aligned}(a^{-1} \cdot a)x &= (a^{-1} \cdot a) \\ x &= 1\end{aligned}$$

(ii) By (P9) for 2 times,

$$\begin{aligned}(x - y)(x + y) &\stackrel{1}{=} x \cdot (x + y) + (-y) \cdot (x + y) \\ &\stackrel{2}{=} x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y \\ &= x^2 + x \cdot y + [-(x \cdot y)] + [-(y^2)] \\ &= x^2 - y^2\end{aligned}$$

(iii) From (ii) and since $x^2 = y^2$,

$$x^2 - y^2 = (x - y)(x + y) = 0$$

This means $(x - y) = 0 \vee (x + y) = 0$, which is $x = y \vee x = -y$

(iv) Starting with the right-hand side,

$$\begin{aligned} (x - y)(x^2 + xy + y^2) &= x \cdot (x^2 + xy + y^2) + (-y) \cdot (x^2 + xy + y^2) \\ &= x^3 + x^2y + xy^2 + [-(x^2y)] + [-(xy^2)] + [-(y)^3] \\ &= x^3 - y^3 \end{aligned}$$

(v) I propose two solutions for this problem. The first one is the direct right-hand side manipulation, while the latter is done by induction.

The first solution.

$$\begin{aligned} &(x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1} \\ &\quad + [-(x^{n-1}y)] + [-(x^{n-2}y^2)] + \cdots + [-(xy^{n-1})] + [-(y^n)] \\ &= x^n - y^n \end{aligned}$$

Q.E.D

The second solution. Let $n=1$, then indeed $x - y = x - y$. Suppose the statement holds true for $n = k$ with $k \in \mathbb{N}$, that is

$$x^k - y^k = (x - y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})$$

is true. To finish the proof, we need to prove

$$x^{k+1} - y^{k+1} = (x - y)(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k)$$

That is, the statement holds for $n = k$. Starting from the left hand side,

$$\begin{aligned} &x^{k+1} - y^{k+1} \\ &= x^{k+1} - x^k y + x^k y - y^{k+1} \\ &= x^k(x - y) + y(x^k - y^k) \\ &= x^k(x - y) + y(x - y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1}) \\ &= (x - y)[x^k + y(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})] \\ &= (x - y)(x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1} + y^k) \end{aligned}$$

Q.E.D

(vi) We will use (iv) in our proof,

$$\begin{aligned}
 & x^3 + y^3 \\
 = & x^3 - y^3 + 2y^3 \\
 = & (x - y)(x^2 + xy + y^2) + 2y[(x^2 + xy + y^2) + (-x)(x + y)] \\
 = & (x + y)(x^2 + xy + y^2) + 2[-(xy)](x + y) \\
 = & (x + y)(x^2 - xy + y^2)
 \end{aligned}$$

■

Problem 1.2. What is wrong with the following “proof”? Let $x = y$. Then

$$\begin{aligned}
 x^2 &= xy, \\
 x^2 - y^2 &= xy - y^2, \\
 (x + y)(x - y) &= y(x - y), \\
 x + y &= y, \\
 2y &= y, \\
 2 &= 1.
 \end{aligned}$$

Solution. Note that in the transition from line 3 to line 4, the author “simplifies” $(x - y)$ by dividing $(x - y)$ on both sides. This is wrong since $x - y = 0$, and hence $1/0$ is undefined as implied by (P7) in the textbook. ■

Problem 1.3. Prove the following:

- (i) $\frac{a}{b} = \frac{ac}{bc}$, if $b, c \neq 0$.
- (ii) $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, if $b, d \neq 0$.
- (iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$. (To do this you must remember the defining property of $(ab)^{-1}$.)
- (iv) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{db}$, if $b, d \neq 0$.
- (v) $\frac{a}{b} \bigg/ \frac{c}{d} = \frac{ad}{bc}$, if $b, c, d \neq 0$.
- (vi) If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

Solution. (i) Until (iii) is proved, the solution is to test the equality between two sides.

$$\begin{aligned} a(b)^{-1} &= (ac)(bc)^{-1} \\ a[(b)^{-1}b] &= (ac)(bc)^{-1}b \\ (a^{-1}a) &= (a^{-1}a)c(bc)^{-1}b \\ 1 &= (bc)(bc)^{-1} = 1 \end{aligned}$$

(ii) Similar to the above,

$$\begin{aligned} a(b)^{-1} + c(d)^{-1} &= (ad + bc)(bd)^{-1} \\ a(b)^{-1}bd + c(d)^{-1}bd &= (ad + bc)[(bd)^{-1}(bd)] \\ ad(b^{-1}b) + bc(d^{-1}d) &= (ad + bc) \\ ad + bc &= ad + bc \end{aligned}$$

(iii) Since $a, b \neq 0$, there exists $(ab)^{-1}, a^{-1}, b^{-1}$ such that,

$$\begin{aligned} ab &= ab \\ (ab)^{-1}(ab) &= (ab)^{-1}(ab) = 1 \\ (ab)^{-1}a(bb^{-1}) &= b^{-1} \\ (ab)^{-1}(aa^{-1}) &= b^{-1}a^{-1} \\ (ab)^{-1} &= a^{-1}b^{-1} \end{aligned}$$

(iv) For $b, d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = ab^{-1}cd^{-1} = ac(d^{-1}b^{-1}) = ac(db)^{-1} = \frac{ac}{db}$$

where the next-to-last equality follows from (iii).

(v) I first establish for any number $a \neq 0$,

$$(a^{-1})^{-1} = a$$

Let $t = a^{-1}$, we want to prove $t^{-1} = a$. Observe that

$$\begin{aligned} t &= a^{-1} \\ t \cdot (t)^{-1} &= a^{-1} \cdot (t)^{-1} \\ a \cdot 1 &= (a \cdot a^{-1}) \cdot (t)^{-1} \\ a &= (t)^{-1} \end{aligned}$$

From the left hand side of the statement,

$$\frac{a}{b} \bigg/ \frac{c}{d} = a(b)^{-1}[c(d)^{-1}]^{-1} = a(b)^{-1}(c)^{-1}[(d)^{-1}]^{-1} = (ad)(bc)^{-1} = \frac{ad}{bc}$$

where the second and third equality follows both from (iii) and the proof above.

(vi) Using (ii),

$$\begin{aligned}\frac{a}{b} &= \frac{c}{d} \\ \frac{a}{b} + \left(-\frac{c}{d}\right) &= 0 \\ \frac{ad - bc}{bd} &= 0 \\ ad &= bc\end{aligned}$$

Now, put $c = b \wedge d = a$. It follows that $\frac{a}{b} = \frac{b}{a}$ if and only if $a^2 = b^2$. It follows $(a - b)(a + b) = 0$, or $a = b \vee a = -b$. ■

Problem 1.4. Find all numbers x for which

- (i) $4 - x < 3 - 2x$
- (ii) $5 - x^2 < 8$
- (iii) $5 - x^2 < -2$
- (iv) $(x - 1)(x - 3) > 0$ (When is a product of two numbers positive?)
- (v) $x^2 - 2x + 2 > 0$
- (vi) $x^2 + x + 1 > 2$
- (vii) $x^2 - x + 10 > 16$
- (viii) $x^2 + x + 1 > 0$
- (ix) $(x - \pi)(x + 5)(x - 3) > 0$
- (x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$
- (xi) $2^x < 8$
- (xii) $x + 3^x < 4$
- (xiii) $\frac{1}{x} + \frac{1}{1 - x} > 0$
- (xiv) $\frac{x - 1}{x + 1} > 0$

Solution. (i)

$$\begin{aligned}
 4 - x &< 3 - 2x \\
 4 + (-x + 2x) &< 3 + (-2x + 2x) \\
 (-4 + 4) + x &< -4 + 3 \\
 x &< -1
 \end{aligned}$$

(ii)

$$\begin{aligned}
 5 - x^2 &< 8 \\
 5 - 8 &< x^2 \\
 -3 &< x^2
 \end{aligned}$$

Since $x^2 \geq 0 \forall x \in \mathbb{R}$, the inequality holds $\forall x$.

(iii)

$$\begin{aligned}
 5 - x^2 &< -2 \\
 7 &< x^2 \\
 0 &< x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7})
 \end{aligned}$$

Hence, either $x > \sqrt{7} \wedge x > -\sqrt{7}$ or $x < \sqrt{7} \wedge x < -\sqrt{7}$, which is $x > \sqrt{7} \vee x < -\sqrt{7}$.

(iv)

$$\begin{aligned}
 (x - 1)(x - 3) &> 0 \\
 (x > 1 \wedge x > 3) \vee (x < 1 \wedge x < 3) \\
 x > 3 \vee x < 1
 \end{aligned}$$

(v)

$$\begin{aligned}
 x^2 - 2x + 2 &> 0 \\
 (x^2 - 2x + 1) + 1 &> 0 \\
 (x - 1)^2 + 1 &> 0
 \end{aligned}$$

Hence the inequality is satisfied $\forall x$.

(vi)

$$\begin{aligned}
& x^2 + x + 1 > 2 \\
& x^2 + x - 1 > 0 \\
& x^2 + \left(\frac{1+\sqrt{5}}{2}\right)x + \left(\frac{1-\sqrt{5}}{2}\right)x + \left(\frac{(1-\sqrt{5})(1+\sqrt{5})}{4}\right) > 0 \\
& \left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right) > 0 \\
& x > \left(\frac{\sqrt{5}-1}{2}\right) \vee x < \left(\frac{-(\sqrt{5}+1)}{2}\right)
\end{aligned}$$

(vii)

$$\begin{aligned}
& x^2 - x + 10 > 16 \\
& x^2 - x - 6 > 0 \\
& x^2 - 3x + 2x - 6 > 0 \\
& x(x-3) + 2(x-3) > 0 \\
& (x+2)(x-3) > 0 \\
& x > 3 \vee x < -2
\end{aligned}$$

(viii)

$$\begin{aligned}
& x^2 + x + 1 > 0 \\
& x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 > 0 \\
& \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0
\end{aligned}$$

which is true for all x .

(ix) Divide the problem into two cases: $x > \pi$ and $x < \pi$.

Case 1: $x > \pi$

Then $(x+5)(x-3) > 0$, which is $x > 3 \vee x < -5$.

Case 2: $x < \pi$

Then $(x+5)(x-3) < 0$, which is $-5 < x < 3$.

(x)

$$\begin{aligned}
& (x - \sqrt[3]{2})(x - \sqrt{2}) > 0 \\
& x > \sqrt{2} \vee x < \sqrt[3]{2}
\end{aligned}$$

(xi) (Sometimes, to solve a problem, intuition is a necessity.)

$$2^x < 8$$

$$2^x < 2^3$$

$$x < 3$$

(xii)

$$x + 3^x < 4$$

$$x + 3^x < 1 + 3^1$$

$$x < 1$$

(xiii)

$$\frac{1}{x} + \frac{1}{1-x} > 0$$

$$\frac{1}{x(1-x)} > 0$$

Hence, $x(1-x) > 0$. This means $0 < x < 1$.

(xiv)

$$\frac{x-1}{x+1} > 0$$

Hence, $(x-1)(x+1) > 0$, or $x > 1 \vee x < -1$.

■

Problem 1.5. Prove the following:

- (i) If $a < b$ and $c < d$, then $a + c < b + d$
- (ii) If $a < b$, then $-b < -a$
- (iii) If $a < b$ and $c > d$, then $a - c < b - d$
- (iv) If $a < b$ and $c > 0$, then $ac < bc$
- (v) If $a < b$ and $c < 0$, then $ac > bc$
- (vi) If $a > 1$, then $a^2 > a$
- (vii) If $0 < a < 1$, then $a^2 < a$
- (viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$
- (ix) If $0 \leq a < b$, then $a^2 < b^2$. (Use (viii).)
- (x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$. (Use (ix), backwards.)

Solution. Let P be the set of all positive numbers.

- (i) To prove this, we apply (P11): If $a < b \wedge c < d$, then $(b - a \in P) \wedge (d - c \in P)$. Then $(b - a) + (d - c) = (b + d) - (a + c) \in P$. Therefore, $a + c < b + d$.
- (ii) We provide two solutions: The first one is by Trichotomy Law (P10), and the second one is by adding $[(-a) + (-b)]$ to both sides.

Proof by Trichotomy Law. If $a < b$, then $b - a \in P$. By Trichotomy Law, $a - b \notin P$ and $a - b \neq 0$. Therefore, $a - b < 0$, which is $-b < -a$. Q.E.D

Proof by adding.

$$\begin{aligned} a &< b \\ a + [(-a) + (-b)] &< b + [(-a) + (-b)] \\ [a + (-a)] + (-b) &< [b + (-b)] + (-a) \\ -b &< -a \end{aligned}$$

Q.E.D

- (iii) Using (P11), we have $b - a \in P \wedge c - d \in P$. Then $(b - a) + (c - d) \in P$. Hence, $a - c < b - d$.
- (iv) Using (P12), note that $b - a \in P$. Since $c > 0$, $c(b - a) \in P$, which means $bc - ac > 0$, or $ac < bc$.
- (v) By Trichotomy law(P10), $-c \in P$. Then by (iv), $-(ac) < -(bc)$. By (ii), $ac > bc$.
- (vi) Since $a > 1 > 0$, by (iv), $a^2 > a$.
- (vii) Since $a > 0$, by (iv), $a^2 < a$.
- (viii) Because $0 < b$, $bc < bd$. Furthermore, if $c \geq 0$, $ac \leq bc$ (equality occurs if $c = 0$), by (iv). Therefore, $ac \leq bc < bd$. Hence, $ac < bd$.
- (ix) From (viii), let $c = a$ and $d = b$, then the result follows.
- (x) Suppose $a \geq b$. Then $a \geq b \geq 0$. By (ix) and (P9), $a^2 \geq b^2$. This contradicts $a^2 < b^2$.

■

Problem 1.6. (a) Prove that if $0 \leq x < y$, then $x^n < y^n$, $n = 1, 2, 3, \dots$

- (b) Prove that if $x < y$ and n is odd, then $x^n < y^n$.
- (c) Prove that if $x^n = y^n$ and n is odd, then $x = y$.
- (d) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Solution. (a) Repeatedly apply problem 1.5(viii) for $0 \leq x < y$, we have $x^n < y^n$ with $n = 1, 2, 3, \dots$

- (b) The statement is true for the case $0 \leq x < y$. In the case $x < y \leq 0$, by 1.5(ii), $(-x) > (-y) \geq 0$. By (a), $(-x)^n > (-y)^n$ for all odd n . Since n is odd, $-(x^n) > -(y^n)$. Hence, by 1.5(ii), $x^n < y^n$. In the case $x \leq 0 < y$, since n is odd, $x^n < y^n$.
- (c) Suppose that either $x \neq y$. W.l.o.g, let $x < y$, by (b), $x^n < y^n$ for all odd n , contradicting $x^n = y^n$ for all odd n .
- (d) Suppose that both $x \neq y$ and $x \neq -y$. Then $x^2 - y^2 \neq 0$. W.l.o.g, suppose $x^2 > y^2 \geq 0$. Applying (a), this generalizes to $x^n > y^n$ for all even n , contradicting our assumption. Therefore, $x = y$ or $x = -y$.

The direct proof. In the case $x, y \geq 0$; by (a), if $x^n = y^n$ for all even n , then $x = y$. In the case $x, y \leq 0$; if $x^n = y^n$ for all even n , then $(-x), (-y) \geq 0$ and $(-x)^n = (-y)^n$, so $-x = -y$ and hence $x = y$. In the case of x and y have different signs, then x and $-y$ are either two positive or two negative numbers. In either subcase, if $x^n = y^n$ for all even n , then $x^n = (-y)^n$, and it follows $x = -y$ from the previous case.

■

Problem 1.7. Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b$$

Notice that the inequality $\sqrt{ab} \leq (a+b)/2$ holds for all $a, b \geq 0$. A generalization of this fact occurs in Problem 2.22.

Solution. Let us first establish that $a < \frac{a+b}{2} < b$. Note that,

$$a + a < a + b < b + b$$

and therefore, $a < \frac{a+b}{2} < b$. To finish the proof, we need to prove $a < \sqrt{ab} < \frac{a+b}{2}$. To do this, let us prove that if $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$. Note that since $b - a > 0$,

$$b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) > 0$$

Therefore, $\sqrt{b} > \sqrt{a} > 0$. We rewrite the inequality as follows,

$$\sqrt{a} \cdot (\sqrt{b} - \sqrt{a}) > 0$$

Then

$$a < \sqrt{ab} \quad (1.1)$$

We next notice that since $\sqrt{b} - \sqrt{a} > 0$, it follows that $(\sqrt{b} - \sqrt{a}) \cdot (\sqrt{b} - \sqrt{a}) = (\sqrt{b} - \sqrt{a})^2 > 0$. Expand the left hand side,

$$(\sqrt{b} - \sqrt{a})^2 = a + b - 2\sqrt{ab} > 0$$

which implies,

$$\sqrt{ab} < \frac{a+b}{2} \quad (1.2)$$

From (1.1) and (1.2), we have $a < \sqrt{ab} < \frac{a+b}{2}$. ■

Problem 1.8 (*). Although the basic properties of inequalities were stated in terms of the collection P of all positive numbers, and $<$ was defined in terms of P , this procedure can be reversed. Suppose that P10–P12 are replaced by

(P'10) For any numbers a and b one, and only one, of the following holds:

- (i) $a = b$,
- (ii) $a < b$,
- (iii) $b < a$.

(P'11) For any numbers a , b , and c , if $a < b$ and $b < c$, then $a < c$.

(P'12) For any numbers a , b , and c , if $a < b$, then $a + c < b + c$.

(P'13) For any numbers a , b , and c , if $a < b$ and $0 < c$, then $ac < bc$.

Show that P10–P12 can then be deduced as theorems.

Solution. Let P be the set of all positive numbers.

- To prove P10, let $c = a - b$, from (P'10), P10 follows.
- To prove P11, let $a, b \in P$; it is sufficient to prove that $a + b > 0$. From (P'10), we divide the proof into three subcases:

Case 1: $a = b$

Then $a + b = b + b > 0 + b > 0$, where the first inequality follows from (P'12). By (P'11), $a + b > 0$.

Case 2: $a < b$

Then $a + b > a + a > 0 + a > 0$, where the first and second inequality follow from (P'12). By applying (P'11) twice, $a + b > 0$.

Case 3: $a > b$

Interchanging the role of a and b , we have the result.

- To prove P12, let $a, b \in P$; it is sufficient to prove that $a \cdot b > 0$. From (P'10), we divide the proof into three subcases:

Case 1: $a = b$

Then $a \cdot b = b \cdot b > 0 \cdot b = 0$, where the first inequality follows from (P'13) and the equality after which is from (P9).

Case 2: $a < b$

Then $b \cdot a > a \cdot a > 0 \cdot a = 0$, where the first and second inequality is from (P'13). By (P'11), $a \cdot b > 0$.

Case 3: $a > b$

Interchanging a and b returns us to case 2, which yields the result. ■

Problem 1.9. Express each of the following with at least one less pair of absolute value signs.

- (i) $|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}|$
- (ii) $|(|a + b| - |a| - |b|)|$
- (iii) $|(|a + b| + |c| - |a + b + c|)|$
- (iv) $|(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)|$

Solution. (i) Note $\sqrt{7} - \sqrt{5} > 0$, hence

$$|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}| = \sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$$

- (ii) Since $|a + b| - |a| - |b| \leq 0$,

$$|(|a + b| - |a| - |b|)| = |a| + |b| - |a + b|$$

- (iii) Since $|a + b + c| \leq |a + b| + |c|$,

$$|(|a + b| + |c| - |a + b + c|)| = |a + b| + |c| - |a + b + c|$$

- (iv)

$$|(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)| = |\sqrt{2} + \sqrt{3} - \sqrt{7} + \sqrt{5}|$$

■

Problem 1.10. Express each of the following without absolute value signs, treating various cases separately when necessary.

- (i) $|a + b| - |b|$
- (ii) $|(|x| - 1)|$
- (iii) $|x| - |x^2|$
- (iv) $a - |(a - |a|)|$

Solution. (i) We divide into four cases:

$$a \geq 0 \quad \text{and} \quad b \geq 0 \quad (\text{Case 1})$$

$$a \leq 0 \quad \text{and} \quad b \leq 0 \quad (\text{Case 2})$$

$$a \geq 0 \quad \text{and} \quad b \leq 0 \quad (\text{Case 3})$$

$$a \leq 0 \quad \text{and} \quad b \geq 0 \quad (\text{Case 4})$$

In Case 1 and Case 2, we have $|a + b| - |b| = a$ since $|a + b| \leq |a| + |b|$.

In Case 3, if $a + b \geq 0$, then

$$|a + b| - |b| = (a + b) - (-b) = a + b + b = 2b$$

If $a + b \leq 0$, then

$$|a + b| - |b| = (-a - b) - (-b) = -a + (-b) + b = -a$$

In Case 4, if $a + b \geq 0$, then

$$|a + b| - |b| = (a + b) - (b) = a$$

If $a + b \leq 0$, then

$$|a + b| - |b| = -a + (-b) + (-b) = -a - 2b$$

(ii) We make the problem into 4 cases.

$$x \geq 1 \quad (\text{Case 1})$$

$$0 \leq x \leq 1 \quad (\text{Case 2})$$

$$-1 \leq x \leq 0 \quad (\text{Case 3})$$

$$x \leq -1 \quad (\text{Case 4})$$

In Case 1, $||x| - 1| = x - 1$.

In Case 2, $||x| - 1| = 1 - x$.

In Case 3, $||x| - 1| = x + 1$.

In Case 4, $||x| - 1| = -(x + 1)$.

(iii) Since $x^2 \geq 0$, $|x| - |x^2| = |x| - x^2$.

If $x \geq 0$, then $|x| - x^2 = x(1 - x)$. If $x \leq 0$, then

$$|x| - x^2 = -x + (-x^2) = -x(1 + x).$$

(iv) Note that $|a| \geq a$. Hence,

$$a - |(a - |a|)| = a + a - |a| = 2a - |a|$$

We have two cases,

Case 1: $a \geq 0$

$$2a - |a| = 2a - a = a$$

Case 2: $a \leq 0$

$$2a - |a| = 2a + a = 3a$$

■

Problem 1.11. Find all numbers x for which

- (i) $|x - 3| = 8$
- (ii) $|x - 3| < 8$
- (iii) $|x + 4| < 2$
- (iv) $|x - 1| + |x - 2| > 1$
- (v) $|x - 1| + |x + 1| < 2$
- (vi) $|x - 1| + |x + 1| < 1$
- (vii) $|x - 1| \cdot |x + 1| = 0$
- (viii) $|x - 1| \cdot |x + 2| = 3$

Solution. (i)

$$\begin{aligned} x - 3 &= 8 \vee x - 3 = -8 \\ x &= 11 \vee x = -5 \end{aligned}$$

(ii) Then $-8 < x - 3 < 8$. Hence, $-5 < x < 11$.

(iii) Then $-2 < x + 4 < 2$. Hence, $-6 < x < -2$.

(iv) If $1 \leq x \leq 2$, then the inequality becomes $(x - 1) + (2 - x) = 1$. If $x > 2$, then $2x - 3 > 1$, which is $x > 2$. If $x < 1$, then $-2x + 3 > 1$, which is $x < 1$. Therefore, either $x > 2$ or $x < 1$ satisfies the inequality.

(v) If $-1 \leq x \leq 1$, then $(1 - x) + (x + 1) = 2$. If $x > 1$, then $x < 1$, which is contradictory. If $x < -1$, then $(1 - x) + (-x - 1) = -2x < 2$ only if $x > -1$, which is contradictory. Hence, there is no x to satisfy the inequality.

(vi) It is implied from above that

$$|x - 1| + |x + 1| \geq 2$$

Therefore, there is no x satisfying the inequality.

(vii) Either $x = 1$ or $x = -1$.

- (viii) If $-2 \leq x \leq 1$, then we obtain $x^2 + x + 1 > 0$. Hence, in either $x < -2$ or $x > 1$, we have to solve the equation $x^2 + x - 5 = 0$, whose solution is either $x = \frac{-1 + \sqrt{21}}{2}$ or $x = \frac{-1 - \sqrt{21}}{2}$. ■

Problem 1.12. Prove the following:

- (i) $|xy| = |x| \cdot |y|$
- (ii) $\left| \frac{1}{x} \right| = \frac{1}{|x|}$, if $x \neq 0$. (The best way to do this is to remember what $|x|^{-1}$ is.)
- (iii) $\frac{|x|}{|y|} = \left| \frac{x}{y} \right|$, if $y \neq 0$.
- (iv) $|x - y| \leq |x| + |y|$ (Give a very short proof.)
- (v) $|x| - |y| \leq |x - y|$ (A very short proof is possible, if you write things in the right way.)
- (vi) $||x| - |y|| \leq |x - y|$ (Why does this follow immediately from (v)?)
- (vii) $|x + y + z| \leq |x| + |y| + |z|$. Indicate when equality holds, and prove your statement.

Solution. (i) We have 4 cases,

$$x \geq 0 \quad y \geq 0 \tag{1}$$

$$x \geq 0 \quad y \leq 0 \tag{2}$$

$$x \leq 0 \quad y \geq 0 \tag{3}$$

$$x \leq 0 \quad y \leq 0 \tag{4}$$

In (1), $|x| \cdot |y| = xy = |xy|$

In (4), $|x| \cdot |y| = (-x)(-y) = xy = |xy|$

In (3), $|x| \cdot |y| = (-x)(y) = -(xy) = |xy|$

In (2), interchanging x and y leads to (3).

- (ii) Since $x \neq 0$, there exists $|x|^{-1}$ such that

$$|x||x|^{-1} = 1 = |x| \left| \frac{1}{x} \right|$$

where the second equality is by (i). Dividing both sides by $|x|$, we have the result.

(iii) Since $y \neq 0$, from (ii), we immediately have

$$\left| \frac{1}{y} \right| = \frac{1}{|y|}$$

Hence, applying (ii) once more,

$$\left| \frac{x}{y} \right| = |x| \left| \frac{1}{y} \right| = \frac{|x|}{|y|}$$

(iv) Note that,

$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$$

where the last equality follows from (i).

(v) Note that,

$$|x - y + y| \leq |x - y| + |y|$$

Therefore, $|x| - |y| \leq |x - y|$.

(vi) Let the first term be y and the second term be $y - x$. Applying (v), we have

$$|y| - |y - x| \leq |x|$$

Hence, $-|x - y| \leq |x| - |y|$. Combining with (v) gives $||x| - |y|| \leq |x - y|$.

(vii) Notice the pattern,

$$|x + y + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$$

the equality holds only if either x, y, z have the same sign or at least two of them must be equal to 0. It is easy to verify this.

Suppose not, then both x, y, z have different signs and at most one of them is 0. If the latter is true, then, w.l.o.g, suppose $z = 0$, then x, y have different sign, and we are done. If none of them is 0, then, w.l.o.g, suppose $z < 0$ and pick z such that $x + y < -z$. Then,

$$|x + y + z| = -(x + y + z) = -x - y - z < |x| + |y| + |z|$$

where inequality must follow since $x, y \neq 0$. ■

Problem 1.13. The maximum of two numbers x and y is denoted by $\max(x, y)$. Thus $\max(-1, 3) = \max(3, 3) = 3$ and $\max(-1, -4) = \max(-4, -1) = -1$. The minimum of x and y is denoted by $\min(x, y)$. Prove that

$$\max(x, y) = \frac{x + y + |y - x|}{2},$$

$$\min(x, y) = \frac{x + y - |y - x|}{2}.$$

Derive the formula for $\max(x, y, z)$ and $\min(x, y, z)$, using, for example

$$\max(x, y, z) = \max(x, \max(y, z)).$$

Solution. Assume that $x \geq y$, we want to prove that $\max(x, y) = x$.

$$\max(x, y) = \frac{x + y + |y - x|}{2} = \frac{x + y + x - y}{2} = \frac{2x}{2} = x$$

Similarly, we need $\min(x, y) = y$.

$$\min(x, y) = \frac{x + y - |y - x|}{2} = \frac{x + y - (x - y)}{2} = \frac{x + y - x + y}{2} = \frac{2y}{2} = y$$

Let $\max(x, y, z) = \max(x, \max(y, z))$. Then

$$\begin{aligned} \max(x, y, z) &= \frac{x + \max(y, z) + |\max(y, z) - x|}{2} \\ &= \frac{x + \frac{y + z + |z - y|}{2} + \left| \frac{y + z + |z - y|}{2} - x \right|}{2} \\ &= \frac{2x + y + z + |z - y| + |y + z + |z - y| - 2x|}{4} \end{aligned}$$

Similarly,

$$\min(x, y, z) = \frac{2x + y + z - |z - y| - |y + z - |z - y| - 2x|}{4}$$

■

Problem 1.14. (a) Prove that $|a| = |-a|$. (The trick is not to become confused by too many cases. First prove the statement for $a \geq 0$. Why is it then obvious for $a \leq 0$?)

(b) Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$. In particular, it follows that $-|a| \leq a \leq |a|$.

(c) Use this fact to give a new proof that $|a + b| \leq |a| + |b|$.

Solution. (a) Problem 1.12(i) easily tells us that

$$|-a| = |(-1)a| = |-1||a| = 1|a| = |a|$$

(b) If $a \geq 0$, then $a \leq b$. If $a \leq 0$, $-a \leq b$ follows from $a \geq -b$. Therefore, $|a| \leq b$. Conversely, suppose $|a| \leq b$. Then it is certain $a \leq b$ since $a \leq |a| \leq b$. From (a), $|-a| \leq b$, and hence $a \geq -b$. We conclude that $-b \leq a \leq b$. Note that since $|a| \leq |a|$, $-|a| \leq a \leq |a|$.

(c) Because we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, by Problem 1.5(i), we obtain $-(|a| + |b|) \leq a + b \leq |a| + |b|$. From (b), we arrive at the conclusion $|a + b| \leq |a| + |b|$.

■

Problem 1.15 (*). Prove that if x and y are not both 0, then

$$\begin{aligned}x^2 + xy + y^2 &> 0 \\x^4 + x^3y + x^2y^2 + xy^3 + y^4 &> 0\end{aligned}$$

Hint: Use problem 1.

Solution. For the first part, note that

$$x^2 + xy + y^2 = x^2 + 2 \cdot x \cdot \frac{1}{2}y + \frac{1}{4}y^2 - \frac{1}{4}y^2 + y^2 = \left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 > 0$$

For the second part, if $x = y$, then the left-hand side is $5x^4 > 0$. Hence, suppose $x \neq y$. From Problem 1.1(v),

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \neq 0$$

If $x > y$, then $x^5 > y^5$ by Problem 1.6(b). This implies that the second term must be greater than 0. Conversely, $x < y \Rightarrow x^5 < y^5$ implies that it must be greater than 0. ■

Problem 1.16 (*). (a) Show that

$$\begin{aligned}(x + y)^2 &= x^2 + y^2 \text{ only when } x = 0 \text{ or } y = 0, \\(x + y)^3 &= x^3 + y^3 \text{ only when } x = 0 \text{ or } y = 0 \text{ or } x = -y.\end{aligned}$$

(b) Using the fact that

$$x^2 + 2xy + y^2 = (x + y)^2 \geq 0,$$

show that $4x^2 + 6xy + 4y^2 > 0$ unless x and y are both 0.

(c) Use part (b) to find out when $(x + y)^4 = x^4 + y^4$.

(d) Find out when $(x + y)^5 = x^5 + y^5$. Hint: From the assumption $(x + y)^5 = x^5 + y^5$ you should be able to derive the equation $x^3 + 2x^2y + 2xy^2 + y^3 = 0$, if $xy \neq 0$. This implies that $(x + y)^3 = x^2y + xy^2 = xy(x + y)$.

You should know be able to make a good guess as to when $(x + y)^n = x^n + y^n$; the proof is contained in Problem 11.57

Solution. (a) For the first part,

$$(x + y)^2 = x^2 + 2xy + y^2$$

Hence, $(x + y)^2 = x^2 + y^2$ only when $x = 0$ or $y = 0$. For the second part, from Problem 1.1(vi),

$$\begin{aligned}(x + y)^3 - (x + y)(x^2 - xy + y^2) &= 0 \\(x + y)(xy) &= 0\end{aligned}$$

which is true only when $x = 0$ or $y = 0$ or $x = -y$.

(b) Note that $4x^2 + 6xy + 4y^2 = \underbrace{3(x+y)^2}_{\geq 0} + \underbrace{x^2 + y^2}_{> 0} > 0$ unless $x = 0$ and $y = 0$.

(c) Let us expand $(x + y)^4$.

$$\begin{aligned}(x + y)^2(x + y)^2 &= (x^2 + 2xy + y^2)(x^2 + 2xy + y^2) \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\ &= x^4 + y^4 + xy(4x^2 + 6xy + 4y^2)\end{aligned}$$

Hence, $(x + y)^4 = x^4 + y^4$ only when $x = 0$ or $y = 0$, by part (b).

(d) Let us expand $(x + y)^5$.

$$\begin{aligned}(x + y)^4(x + y) &= x^5 + y^5 + xy(x + y)(4x^2 + 6xy + 4y^2) + xy(x^3 + y^3) \\ &= x^5 + y^5 + 5xy(x + y)(x^2 - xy + y^2)\end{aligned}$$

If $xy \neq 0$ and $x + y \neq 0$, let $z = -y$, by Problem 1.6(b), $x^3 \neq z^3$. Hence, $x^2 - xy + y^2 \neq 0$. Therefore, $(x + y)^5 = x^5 + y^5$ only when $x = 0$ or $y = 0$ or $x = -y$.

Remark. Hence, for $(x + y)^n = x^n + y^n$, if n is even, then $x = 0$ or $y = 0$. If n is odd, then $x = 0$ or $y = 0$ or $x = -y$. ■

Problem 1.17. (a) Find the smallest possible value of $2x^2 - 3x + 4$. Hint: “Complete the square”, i.e., write $2x^2 - 3x + 4 = 2(x - 3/4)^2 + ?$

(b) Find the smallest possible value of $x^2 - 3x + 2y^2 + 4y + 2$.

(c) Find the smallest possible value of $x^2 + 4xy + 5y^2 - 4x - 6y + 7$.

Solution. (a) Since $2x^2 - 3x + 4 = 2(x^2 - \frac{3}{2}x + 2)$,

$$2(x^2 - 2 \cdot x \frac{3}{4} + \frac{9}{16} - \frac{9}{16} + 2) = 2(x - \frac{3}{4})^2 + \frac{23}{8}$$

Hence the minimum value is $\frac{23}{8}$ when $x = \frac{3}{4}$.

(b)

$$x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2(y^2 + 2y + 1) = \left(x - \frac{3}{2}\right)^2 + 2(y + 1)^2 - \frac{9}{4}$$

The minimum value is $-\frac{9}{4}$ when $x = \frac{3}{2}$ and $y = -1$.

(c)

$$\begin{aligned}
& \frac{1}{2}x^2 + 4xy + 8y^2 - 3y^2 - 6y + 7 + \frac{1}{2}x^2 - 4x \\
&= \frac{1}{2}(x^2 + 8xy + 16y^2) - 3(y^2 + 2y + 1) + \frac{1}{2}(x^2 - 8x + 16) + 2 \\
&= \frac{1}{2}(x + 4y)^2 - 3(y + 1)^2 + \frac{1}{2}(x - 4)^2 + 2
\end{aligned}$$

Therefore, the minimum value is 2 when $x = 4$ and $y = -1$. ■

Problem 1.18. (a) Suppose that $b^2 - 4c \geq 0$. Show that the numbers

$$\frac{-b + \sqrt{b^2 - 4c}}{2}, \quad \frac{-b - \sqrt{b^2 - 4c}}{2}$$

both satisfy the equation $x^2 + bx + c = 0$.

- (b) Suppose that $b^2 - 4c < 0$. Show that there are no numbers x satisfying $x^2 + bx + c = 0$; in fact, $x^2 + bx + c > 0$ for all x . Hint: Complete the square.
- (c) Use this fact to give another proof that if x and y are not both 0, then $x^2 + xy + y^2 > 0$.
- (d) For which number α is it true that $x^2 + \alpha xy + y^2 > 0$ whenever x and y are not both 0?
- (e) Find the smallest possible value of $x^2 + bx + c$ and of $ax^2 + bx + c$, for $a > 0$.

Solution. (a) Substitution immediately gives the desired result.

(b)

$$x^2 + bx + c = x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c$$

which immediately yields $\left(x + \frac{b}{2}\right)^2 + \frac{[-(b^2 - 4c)]}{4} > 0$ for all x since $b^2 - 4c < 0$.

- (c) If $y = 0$, $x^2 > 0$. Suppose not, using (b), we obtain $-3y^2 < 0$. Hence, $x^2 + xy + y^2 > 0$.
- (d) If $y = 0$, the result follows for all α . Suppose $y \neq 0$, using (b), we obtain $\alpha^2 y^2 - 4y^2 < 0$, which is $y^2(\alpha^2 - 4) < 0$. It follows that $-2 < \alpha < 2$.

- (e) From (b), it follows that the minimum value of $x^2 + bx + c$ is $\frac{[-(b^2 - 4c)]}{4}$ when $x = -b/2$. Since $a > 0$, with the role of b is now b/a and of c is c/a , we easily derive the result.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{[-(b^2 - 4ac)]}{4a^2}$$

So its minimum value is $\frac{[-(b^2 - 4ac)]}{4a^2}$ when $x = -\frac{b}{2a}$. ■

Problem 1.19. The fact that $a^2 \geq 0$ for all numbers a , elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}.$$

(A more general form occurs in Problem 2.21) The three proofs of the Schwarz inequality outlined below have only one thing in common—their reliance on the fact that $a^2 \geq 0$ for all a .

- (a) Prove that if $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for some number λ , then equality holds in Schwarz inequality. Prove the same thing if $y_1 = y_2 = 0$. Now suppose that y_1 and y_2 are not both 0, and that there is no number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. Then

$$\begin{aligned} 0 &< (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 \\ &= \lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2). \end{aligned}$$

Using Problem 1.18, complete the proof of the Schwarz inequality.

- (b) Prove the Schwarz inequality by using $2xy \leq x^2 + y^2$ (how is this derived?) with

$$x = \frac{x_i}{\sqrt{x_1^2 + x_2^2}}, \quad y = \frac{y_i}{\sqrt{y_1^2 + y_2^2}},$$

first for $i = 1$ and then for $i = 2$.

- (c) Prove the Schwarz inequality by first proving that

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

- (d) Deduce, from each of these three proofs, that equality holds only when $y_1 = y_2 = 0$ or when there is a number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

Solution. (a) If $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for every $\lambda \geq 0$,

$$\begin{aligned}\lambda(y_1^2 + y_2^2) &= |\lambda|\sqrt{(y_1^2 + y_2^2)^2} \\ &= \lambda(y_1^2 + y_2^2)\end{aligned}$$

Or if $y_1 = y_2 = 0$, then equality holds since both sides are 0. Otherwise, suppose that y_1 and y_2 are not both 0, and there is no number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$, then

$$\begin{aligned}0 &< \lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2) \\ &= \lambda^2 - 2\lambda \frac{x_1y_1 + x_2y_2}{y_1^2 + y_2^2} + \frac{x_1^2 + x_2^2}{y_1^2 + y_2^2}\end{aligned}$$

This holds only when, by Problem 1.18(b),

$$\frac{4(x_1y_1 + x_2y_2)^2}{(y_1^2 + y_2^2)^2} + \frac{[-4(x_1^2 + x_2^2)(y_1^2 + y_2^2)]}{(y_1^2 + y_2^2)^2} < 0$$

which only holds when

$$x_1y_1 + x_2y_2 < \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

since $a \leq |a|$ for all a .

(b) Note that $(x - y)^2 \geq 0$. For $i = 1$,

$$\frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2} \geq 2 \cdot \frac{x_1y_1}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \quad (1.3)$$

For $i = 2$,

$$\frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2} \geq 2 \cdot \frac{x_2y_2}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \quad (1.4)$$

(1.3)+(1.4), we derive

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

(c)

$$\begin{aligned}& (x_1^2 + x_2^2)(y_1^2 + y_2^2) \\ &= (x_1^2y_1^2 + 2x_1y_1x_2y_2 + x_2^2y_2^2) + (x_1^2y_2^2 - 2x_1y_2x_2y_1 + x_2^2y_1^2) \\ &= (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2\end{aligned}$$

Note that $(x_1y_2 - x_2y_1)^2 \geq 0$. Hence,

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

for $a \leq |a|$ for all a .

- (d) In (a), it is obvious; the proof is based on the separation of two cases, $a^2 = 0$ and $a^2 > 0$. In (b), equality occurs only when $x = y$; by construction, $y_1 = y_2 = 0$ or, if not,

$$\frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{y_1}{\sqrt{y_1^2 + y_2^2}}$$
$$\frac{x_2}{\sqrt{x_1^2 + x_2^2}} = \frac{y_2}{\sqrt{y_1^2 + y_2^2}}$$

implies that for

$$\lambda = \frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{y_1^2 + y_2^2}}$$

$x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

In (c), equality occurs only when $(x_1 y_2 - x_2 y_1)^2 = 0$.

■