A Note of Calculus-Michael Spivak

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Preface

This is the note for the book Calculus writtten by Michael Spivak, citing what I think the most interesting and important subjects mentioned in the book.

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Part I

Prologue

Chapter 1

Basic properties of number

(P1) If a, b, and c are any numbers, then

$$a + (b + c) = (a + b) + c$$

See *problem 24* for the generalization of $a_1 + a_2 + a_3 + \cdots + a_n$ for (P1).
The number 0 has important properties.

(P2) If a is any number, then

$$a + 0 = 0 + a = a$$

(P3) For every number a, there is also a number $-a$ such that

$$a + (-a) = (-a) + a = 0$$

We now prove Lemma 1.

Lemma 1. *If $a + x = a$, then $x = 0$*

Proof.

If	$a + x = a$	
then	$(-a) + (a + x) = (-a) + a = 0$	(*by* $(P3)$)
hence	$((-a) + a) + x = 0$	(*by* $(P1)$)
hence	$0 + x = 0$	(*by* $(P3)$ *again*)
therefore,	$x = 0$	(*by* $(P2)$)

□

Also, remember that the order of addition does not matter.

(P4) If a and b are any numbers, then

$$a + b = b + a$$

However, with only (P1)-(P4), we are powerless to figure out what conditions needed to have $a - b = b - a$. Therefore, we need to set new properties, and, oddly, they involve multiplication.

(P5) If a, b and c are any numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(P6) If a is any number, then

$$a \cdot 1 = 1 \cdot a = a$$

Moreover, $1 \neq 0$ (This cannot be proved by other properties listed!)

(P7) For every number $a \neq 0$, there is a number a^{-1} such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1 (\Leftarrow 0 \cdot b = 0 \ \forall b)$$

This is why $1/0$ *is meaningless!*

(P8) If a and b are any numbers, then

$$a \cdot b = b \cdot a$$

From (P5), (P6) and (P7), we have two lemmas:

Lemma 2. *If* $a \cdot b = a \cdot c$ *then* $a = 0 \ \vee \ b = c$

Proof. If $a = 0$ then the lemma is trivial. Suppose now $a \neq 0$,

Multiply a^{-1} to both sides,	$(a^{-1}) \cdot (a \cdot b) = (a^{-1}) \cdot (a \cdot c)$
By (P5),	$(a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c$
By (P7),	$1 \cdot b = 1 \cdot c$
By (P6),	$b = c$

□

Lemma 3. *If* $a \cdot b = 0$ *then* $a = 0 \ \vee \ b = 0$

Proof. If $a = 0$, there is nothing to prove. Suppose now $a \neq 0$, follow the proof of Lemma 2 by consecutively applying (P5), (P7) and (P6) in that order to finish the proof. □

We, however, will not able to prove anything without a relationship between multiplication and addition. Therefore, the next property is definitely necessary.

(P9) If a, b and c are any numbers, then

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

By (P8), it is also true that $(b + c) \cdot a = b \cdot a + c \cdot a$

We will see in the next remark and lemmas that properties are not built in a straight line. Rather, it is a result of necessities, of fixes and starts that somehow fits the pieces of a puzzle perfectly.

Remark. When $a - b = b - a$?

Solution.

Add b at both sides,	$(a - b) + b = (b - a) + b \quad = b + (b - a)$	by (P4)
By (P1),	$a + (-b + b) = (b + b) + (-a)$	
By (P3),	$a + 0 = b + b - a$	
By (P2),	$a = b + b - a$	
Add both sides to a,	$a + a = (b + b - a) + a$	
By (P1),	$a + a = b + (b + (-a + a)) = b + b$	by (P2) and (P3)
By (P9),	$a \cdot (1 + 1) = b \cdot (1 + 1)$	
By Lemma 2,	$a = b$	

■

Note that the proof above based on the presumption that we know $1 + 1 \neq 0$. How do we prove it?

Lemma 4. $a \cdot 0 = 0$

Proof.

We have	$a \cdot 0 + a \cdot 0 = a \cdot (0 + 0)$	by (P9)
By (P2),	$= a \cdot 0$	
Add $-a \cdot 0$,	$a \cdot 0 = 0$	

□

Lemma 5. *The product of two negative numbers is positive*

Proof. We first prove that $(-a) \cdot b = -(a \cdot b)$,

We have by (P9),	$(-a) \cdot b + (a \cdot b) = (-a + a) \cdot b$
	$= 0$
Adding $-(a \cdot b)$ to both sides,	$(-a) \cdot b = -(a \cdot b)$

Now, let's prove the main statement.

From above,
$$\begin{aligned}(-a)\cdot(-b)+[-(a\cdot b)] &= (-a)\cdot(-b)+[(-a)\cdot b]\\ &= (-a)\cdot(-b+b)\\ &= 0\end{aligned}$$
Adding $(a\cdot b)$ to both sides,
$$(-a)\cdot(-b)=(a\cdot b)$$
□

We say that Lemma 5 is a direct consequence of (P1)-(P9).

Also, (P9) has important consequences: Justifying the algebraic manipulations(e.g, $x^2-3x+2=(x-1)(x-2)$) and the way one multiplies arabic numerals,

$$\begin{array}{r} 13\\ \times\, 24\\ \hline 52\\ 26\\ \hline 312\end{array}$$

Denote the set of all positive numbers by P.

(P10) (Trichotomy law) For every number a, one and only one of the following holds:

(i) $a=0$

(ii) $a\in P$

(iii) $-a\in P$

(P11) (Closure under addition) If $a\in P\wedge b\in P$ then $a+b\in P$

(P12) (Closure under multiplication) If $a\in P\wedge b\in P$ then $a\cdot b\in P$

These properties should be complemented by the following definitions:

$$\begin{array}{lll} a>b & \text{if} & a-b\in P\\ a<b & \text{if} & b>a\\ a\geq b & \text{if} & a=b \text{ or } a>b\\ a\leq b & \text{if} & a=b \text{ or } a<b\end{array}$$

The following lemmas are easy to prove...

Lemma 6. *If $a<b$ then $a+c<b+c$*

Proof. If $a<b$, then $b-a\in P$, which is surely $(b+c)-(a+c)\in P$ □

Lemma 7. *If $a<b\wedge b<c$ then $a<c$*

Proof. Then $b-a\in P$ and $c-b\in P$. By (P11), $(b-a)+(c-b)=c-a\in P$ □

Lemma 8. *If $a < 0 \wedge b < 0$ then $a \cdot b > 0$*

Proof. Then $-a > 0 \wedge -b > 0$. By (P12), $(-a) \cdot (-b) = a \cdot b > 0$, by Lemma 5. □

Lemma 9. *If $a \neq 0$, then $a^2 \neq 0$*

Proof. Because if $a > 0 \wedge b > 0$ and $a < 0 \wedge b < 0$ then $a \cdot b > 0$, let $b = a$ □

This implies that $1 > 0$ (since $1^2 = 1$).

We now prove a basic theorem relating to the absolute value.

Theorem 1.1. *$\forall a \wedge b$,*

$$|a + b| \leq |a| + |b|$$

Proof. We apply the straightforward proof. A more elegant proof appears in the exercises. We will consider 4 cases:

$$a \geq 0 \quad \text{and} \quad b \geq 0 \tag{1}$$
$$a \geq 0 \quad \text{and} \quad b \leq 0 \tag{2}$$
$$a \leq 0 \quad \text{and} \quad b \geq 0 \tag{3}$$
$$a \leq 0 \quad \text{and} \quad b \leq 0 \tag{4}$$

For (1), the statement occurs with equality; that is,

$$|a + b| = a + b = |a| + |b|$$

For (4), the same is true by observing,

$$|a + b| = -(a + b) = (-a) + (-b) = |a| + |b|$$

For (2), the job is dumped down to proving that $|a + b| \leq a - b$. This divides the case into two subcases.

Subcase 1: $a + b \geq 0$

Then note that $b \leq (-b)$, which is true since $b \leq 0$.

Subcase 2: $a + b \leq 0$

Then we have $(-a) \leq a$, which is true since $a \geq 0$.

For (3), the case is proved by interchanging the role of a and b. □

Note from the proof above that equality happens if a and b have the same sign, or one of the two is zero.

Remark. It is crucial to understand that (P1)-(P12) are not enough to account for *all* properties of numbers. The deficiency is profound and subtle; and,hopefully, will be discovered in the rest of the note.

Chapter 2

Number of various sorts

$\mathbb{N}$ is the basic set and has many deficiencies. ((P2) and (P3)).

Mathematical induction principle is the basic property of $\mathbb{N}$; however, even though proof by induction is quite straightforward, *the method by which the formula was discovered remains a mystery.*

Theorem 2.1 (Well-ordering principle)**.** *If A is a nonnull set of natural numbers, then A has a least member.*

Proof. Suppose A has no least member. Let B be the collection of n natural numbers $1, \ldots, n$ that are not *all* in A. Clearly, 1 is in B (if not, 1 would be the least member in A). Moreover, if $1, \ldots, k$ are not in A, surely $k+1$ is not in A either (else, $k+1$ would be the least member in A). This shows that if $k \in B$, then $k+1 \in B$. Hence, B is the set of all natural numbers, and $A = \emptyset$. □

$\mathbb{N}$ can be defined either by the well-ordering principle for by mathematical induction since they are equivalent.

Principle of complete induction:

(1) 1 is in A,

(2) $k+1$ is in A if $1, \ldots, k$ are in A,

then A is the set of all natural numbers.

Complete induction is the consequence of induction.

Recursive definition.

Deficiencies of $\mathbb{N}$ can be partially remedied by $\mathbb{Z}$((P7) fails.), which is remedied by $\mathbb{Q}$, which is smaller than $\mathbb{Z}$.

Every natural number n is either odd or even.

if n is odd, n^2 is odd; if n is even, n^2 is even. Hence,
if n^2 is even, n is even; if n^2 is odd, n is odd.

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Chapter 3

Functions

Domain of the function := the set of numbers to which the function is defined!

$$f(x) = x^2 \ \forall x \tag{3.1}$$
$$g(y) = \frac{y^3 + 3y + 5}{y^2 + 1} \ \forall y \tag{3.2}$$
$$h(c) = \frac{c^3 + 3c + 5}{c^2 - 1} \ \forall c \neq \pm 1 \tag{3.3}$$
$$r(x) = x^2, \ \{x : -17 \leq x \leq \frac{\pi}{3}\} \tag{3.4}$$
$$s(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational} \end{cases} \tag{3.5}$$
$$\phi(x) = \begin{cases} 5, & x = 2 \\ \frac{36}{\pi}, & x = 17 \\ 28, & x = \frac{\pi^2}{17} \\ 28, & x = \frac{36}{\pi} \\ 16, & x \neq 2, 17, \frac{\pi^2}{17}, \frac{36}{\pi}, \text{ and } x = a + b\sqrt{2} \text{ for } a, b \text{ in } \mathbb{Q}. \end{cases} \tag{3.6}$$
$$\alpha_x(t) = t^3 + x \text{ for all numbers } t. \tag{3.7}$$
$$y(x) = \begin{cases} n, & \text{exactly n 7's appear in the decimal expansion of } x \\ -\pi, & \text{infinitely many 7's appear in the decimal expansionof } x. \end{cases} \tag{3.8}$$

A function f is a **polynomial function** if there are real numbers $a_0, \ldots, a_n$ such that

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \quad \text{for all } x$$

(Assume $a_n \neq 0$). The highest power of x with a nonzero coefficient is called the **degree** of f.

(3.2) and (3.3) are **rational functions**: Functions of the form p/q where

p and q are polynomials with q is not always 0.

$$f(x) = \frac{x + x^2 + x\sin^2 x}{x\sin x + x\sin^2 x} \tag{3.9}$$

$$f(x) = \sin(x^2) \tag{3.10}$$

$$f(x) = \sin(\sin(x^2)) \tag{3.11}$$

$$f(x) = \sin^2(\sin(\sin^2(x\sin^2 x^2))) \cdot \sin\left(\frac{x + \sin(x\sin x)}{x + \sin x}\right) \tag{3.12}$$

Let $I(x) = x$, then the concepts of **sum, product, quotient, composite** of a function are necessary.

$$x \to \sin(x^2)$$

To formalize the definition of a function, one does not ask, "What is a rule?", "What is an association?", but "What does one need to know about the function in order to know all about it?"

Definition 3.1. A **function** is a collection of pairs of numbers with the following property: if (a, b) and (a, c) are both in the collection, then $b = c$.

Definition 3.2. If f is a function, the **domain** of f is the set of all a for which there is some b such that (a, b) is in f. If a is in the domain of f, it follows from the definition of a function that there is, in fact, a *unique* number b such that (a, b) is in f. This unique b is denoted by $\boldsymbol{f(a)}$.

Appendix: Ordered Pairs

Definition 3.3.

$$(a, b) = \{\{a\}, \{a, b\}\}$$

Theorem 3.1. *If $(a, b) = (c, d)$, then $a = c$ and $b = d$.*

Proof. The hypothesis means that

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

a is the common member of both members of {{a}, {a,b} }. Similarly, c is the common member of both members of {{c}, {c,d} }. Hence $a = c$. To prove $b = d$, we divide the proof into two cases:

Case 1: $b = a$

Then {{a}, {a,d} } contains a unique member a. This implies that $d = a = b$.

Case 2: $b \neq a$

This implies that $\{a, b\} = \{a, d\}$, which means $d \neq a$. This implies that $b = d$. □

Chapter 4

Graphs

Open interval from a to $b := \{x : a < x < b\}$.