A Note of Calculus-Michael Spivak

Son To [1]

Ravintola Kiltakellari

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[1] Contact me at: son.trung.to@gmail.com

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Preface

This is the note for the book Calculus writtten by Michael Spivak, citing what I think the most interesting and important subjects mentioned in the book.

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Part I

Prologue

Chapter 1

Basic properties of number

(P1) If a, b, and c are any numbers, then

$$a + (b + c) = (a + b) + c$$

See *problem 24* for the generalization of $a_1 + a_2 + a_3 + \cdots + a_n$ for (P1).
The number 0 has important properties.

(P2) If a is any number, then

$$a + 0 = 0 + a = a$$

(P3) For every number a, there is also a number $-a$ such that

$$a + (-a) = (-a) + a = 0$$

We now prove Lemma 1.

Lemma 1. *If $a + x = a$, then $x = 0$*

Proof.

$$\begin{array}{lrl}
\text{If} & a + x = a & \\
\text{then} & (-a) + (a + x) = (-a) + a = 0 & (by\ (P3)) \\
\text{hence} & ((-a) + a) + x = 0 & (by\ (P1)) \\
\text{hence} & 0 + x = 0 & (by\ (P3)\ again) \\
\text{therefore,} & x = 0 & (by\ (P2))
\end{array}$$

□

Also, remember that the order of addition does not matter.

(P4) If a and b are any numbers, then

$$a + b = b + a$$

However, with only (P1)-(P4), we are powerless to figure out what conditions needed to have $a - b = b - a$. Therefore, we need to set new properties, and, oddly, they involve multiplication.

(P5) If a, b and c are any numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(P6) If a is any number, then

$$a \cdot 1 = 1 \cdot a = a$$

Moreover, $1 \neq 0$ (This cannot be proved by other properties listed!)

(P7) For every number $a \neq 0$, there is a number a^{-1} such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1 (\Leftarrow 0 \cdot b = 0 \ \forall b)$$

This is why $1/0$ *is meaningless!*

(P8) If a and b are any numbers, then

$$a \cdot b = b \cdot a$$

From (P5), (P6) and (P7), we have two lemmas:

Lemma 2. *If* $a \cdot b = a \cdot c$ *then* $a = 0 \ \vee \ b = c$

Proof. If $a = 0$ then the lemma is trivial. Suppose now $a \neq 0$,

Multiply a^{-1} to both sides,	$(a^{-1}) \cdot (a \cdot b) = (a^{-1}) \cdot (a \cdot c)$
By (P5),	$(a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c$
By (P7),	$1 \cdot b = 1 \cdot c$
By (P6),	$b = c$

□

Lemma 3. *If* $a \cdot b = 0$ *then* $a = 0 \ \vee \ b = 0$

Proof. If $a = 0$, there is nothing to prove. Suppose now $a \neq 0$, follow the proof of Lemma 2 by consecutively applying (P5), (P7) and (P6) in that order to finish the proof. □

We, however, will not able to prove anything without a relationship between multiplication and addition. Therefore, the next property is definitely necessary.

(P9) If a, b and c are any numbers, then

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

By (P8), it is also true that $(b + c) \cdot a = b \cdot a + c \cdot a$

We will see in the next remark and lemmas that properties are not built in a straight line. Rather, it is a result of necessities, of fixes and starts that somehow fits the pieces of a puzzle perfectly.

Remark. When $a - b = b - a$?

Solution.

Add b at both sides,	$(a - b) + b = (b - a) + b$	$= b + (b - a)$	by (P4)
By (P1),	$a + (-b + b) = (b + b) + (-a)$		
By (P3),	$a + 0 = b + b - a$		
By (P2),	$a = b + b - a$		
Add both sides to a,	$a + a = (b + b - a) + a$		
By (P1),	$a + a = b + (b + (-a + a)) = b + b$		by (P2) and (P3)
By (P9),	$a \cdot (1 + 1) = b \cdot (1 + 1)$		
By Lemma 2,	$a = b$		

■

Lemma 4. $a \cdot 0 = 0$. *Really man?*

Proof.

We have	$a \cdot 0 + a \cdot 0 = a \cdot (0 + 0)$	by (P9)
By (P2),	$= a \cdot 0$	
Add $-a \cdot 0$,	$a \cdot 0 = 0$	

□

Lemma 5. *The product of two negative number is positive*

Proof. □