Solutions to Michael Spivak's Calculus

Son To
<son.trung.to@gmail.com>

Ravintola Kiltakellari *

25th June, 2017

*I thank my employer!

This page is intentionally left blank

Preface

This is my own solutions to Michael Spivak's Calculus textbook.

To those who have taught me and have had influences on me.

Vantaa, Finland
25th June, 2017.

Contents

Part I

Prologue

I held every man a debtor
to his profession. . .

Francis Bacon.

Chapter 1

Basic properties of number

Problem 1.1. Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$

(ii) $x^2 - y^2 = (x - y)(x + y)$

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$

(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

(v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

(vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (There is a particularly easy way to do this using (iv), and it will show you how to find a factorization for $x^n + y^n$ whenever n is odd.)

Solution. (i) By (P7)(Existence of multiplicative inverses), there exists a^{-1} such that,

$$\begin{aligned} (a^{-1} \cdot a)x &= (a^{-1} \cdot a) \\ x &= 1 \end{aligned}$$

(ii) By (P9) for 2 times,

$$\begin{aligned} (x - y)(x + y) &\overset{1}{=} x \cdot (x + y) + (-y) \cdot (x + y) \\ &\overset{2}{=} x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y \\ &= x^2 + x \cdot y + [-(x \cdot y)] + [-(y^2)] \\ &= x^2 - y^2 \end{aligned}$$

(iii) From (ii) and since $x^2 = y^2$,

$$x^2 - y^2 = (x-y)(x+y) = 0$$

This means $(x-y) = 0 \vee (x+y) = 0$, which is $x = y \vee x = -y$

(iv) Starting with the right-hand side,

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x\cdot(x^2+xy+y^2) + (-y)\cdot(x^2+xy+y^2)\\ &= x^3 + x^2y + xy^2 + [-(x^2y)] + [-(xy^2)] + [-(y)^3]\\ &= x^3 - y^3\end{aligned}$$

(v) I propose two solutions for this problem. The first one is the direct right-hand side manipulation, while the latter is done by induction.

The first solution.

$$\begin{aligned}&(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})\\ &= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1}\\ &+[-(x^{n-1}y)] + [-(x^{n-2}y^2)] + \cdots + [-(xy^{n-1})] + [-(y^n)]\\ &= x^n - y^n\end{aligned}$$

Q.E.D

The second solution. Let n=1, then indeed $x - y = x - y$. Suppose the statement holds true for $n = k$ with $k \in \mathbb{N}$, that is

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})$$

is true. To finish the proof, we need to prove

$$x^{k+1} - y^{k+1} = (x-y)(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k)$$

That is, the statement holds for $n = k$. Starting from the left hand side,

$$\begin{aligned}& x^{k+1} - y^{k+1}\\ =\;& x^{k+1} - x^ky + x^ky - y^{k+1}\\ =\;& x^k(x-y) + y(x^k - y^k)\\ =\;& x^k(x-y) + y(x-y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})\\ =\;& (x-y)[x^k + y(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})]\\ =\;& (x-y)(x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1} + y^k)\end{aligned}$$

Q.E.D

(vi) We will use (iv) in our proof,

$$\begin{aligned} & x^3 + y^3 \\ = \quad & x^3 - y^3 + 2y^3 \\ = \quad & (x-y)(x^2+xy+y^2) + 2y[(x^2+xy+y^2) + (-x)(x+y)] \\ = \quad & (x+y)(x^2+xy+y^2) + 2[-(xy)](x+y) \\ = \quad & (x+y)(x^2-xy+y^2) \end{aligned}$$

■

Problem 1.2. What is wrong with the following "proof"? Let $x = y$. Then

$$\begin{aligned} x^2 &= xy, \\ x^2 - y^2 &= xy - y^2, \\ (x+y)(x-y) &= y(x-y), \\ x + y &= y, \\ 2y &= y, \\ 2 &= 1. \end{aligned}$$

Solution. Note that in the transition from line 3 to line 4, the author "simplifies" $(x - y)$ by dividing $(x - y)$ on both sides. This is wrong since $x - y = 0$, and hence $1/0$ is undefined as implied by (P7) in the textbook. ■

Problem 1.3. Prove the following:

(i) $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

(ii) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$, if $b, d \neq 0$.

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$. (To do this you must remember the defining property of $(ab)^{-1}$.)

(iv) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{db}$, if $b, d \neq 0$.

(v) $\dfrac{a}{b} \bigg/ \dfrac{c}{d} = \dfrac{ad}{bc}$, if $b, c, d \neq 0$.

(vi) If $b, d \neq 0$, then $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$. Also determine when $\dfrac{a}{b} = \dfrac{b}{a}$.

Solution. (i) Until (iii) is proved, the solution is to test the equality between two sides.

$$\begin{aligned} a(b)^{-1} &= (ac)(bc)^{-1} \\ a[(b)^{-1}b] &= (ac)(bc)^{-1}b \\ (a^{-1}a) &= (a^{-1}a)c(bc)^{-1}b \\ 1 &= (bc)(bc)^{-1} = 1 \end{aligned}$$

(ii) Similar to the above,

$$\begin{aligned} a(b)^{-1} + c(d)^{-1} &= (ad+bc)(bd)^{-1} \\ a(b)^{-1}bd + c(d)^{-1}bd &= (ad+bc)[(bd)^{-1}(bd)] \\ ad(b^{-1}b) + bc(d^{-1}d) &= (ad+bc) \\ ad+bc &= ad+bc \end{aligned}$$

(iii) Since $a, b \neq 0$, there exists $(ab)^{-1}, a^{-1}, b^{-1}$ such that,

$$\begin{aligned} ab &= ab \\ (ab)^{-1}(ab) &= (ab)^{-1}(ab) = 1 \\ (ab)^{-1}a(bb^{-1}) &= b^{-1} \\ (ab)^{-1}(aa^{-1}) &= b^{-1}a^{-1} \\ (ab)^{-1} &= a^{-1}b^{-1} \end{aligned}$$

(iv) For $b, d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = ab^{-1}cd^{-1} = ac(d^{-1}b^{-1}) = ac(db)^{-1} = \frac{ac}{db}$$

where the next-to-last equality follows from (iii).

(v) I first establish for any number $a \neq 0$,

$$(a^{-1})^{-1} = a$$

Let $t = a^{-1}$, we want to prove $t^{-1} = a$. Observe that

$$\begin{aligned} t &= a^{-1} \\ t \cdot (t)^{-1} &= a^{-1} \cdot (t)^{-1} \\ a \cdot 1 &= (a \cdot a^{-1}) \cdot (t)^{-1} \\ a &= (t)^{-1} \end{aligned}$$

From the left hand side of the statement,

$$\frac{a}{b}\Big/\frac{c}{d} = a(b)^{-1}[c(d)^{-1}]^{-1} = a(b)^{-1}(c)^{-1}[(d)^{-1}]^{-1} = (ad)(bc)^{-1} = \frac{ad}{bc}$$

where the second and third equality follows both from (iii) and the proof above.

(vi) Using (ii),

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ \frac{a}{b} + (-\frac{c}{d}) &= 0 \\ \frac{ad - bc}{bd} &= 0 \\ ad &= bc \end{aligned}$$

Now, put $c = b \wedge d = a$. It follows that $\frac{a}{b} = \frac{b}{a}$ if and only if $a^2 = b^2$. It follows $(a - b)(a + b) = 0$, or $a = b \vee a = -b$.

■

Problem 1.4. Find all numbers x for which

(i) $4 - x < 3 - 2x$

(ii) $5 - x^2 < 8$

(iii) $5 - x^2 < -2$

(iv) $(x - 1)(x - 3) > 0$ (When is a product of two numbers positive?)

(v) $x^2 - 2x + 2 > 0$

(vi) $x^2 + x + 1 > 2$

(vii) $x^2 - x + 10 > 16$

(viii) $x^2 + x + 1 > 0$

(ix) $(x - \pi)(x + 5)(x - 3) > 0$

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$

(xi) $2^x < 8$

(xii) $x + 3^x < 4$

(xiii) $\frac{1}{x} + \frac{1}{1 - x} > 0$

(xiv) $\frac{x - 1}{x + 1} > 0$

Solution. (i)

$$\begin{aligned}4 - x &< 3 - 2x\\ 4 + (-x + 2x) &< 3 + (-2x + 2x)\\ (-4 + 4) + x &< -4 + 3\\ x &< -1\end{aligned}$$

(ii)

$$\begin{aligned}5 - x^2 &< 8\\ 5 - 8 &< x^2\\ -3 &< x^2\end{aligned}$$

Since $x^2 \geq 0 \; \forall x \in \mathbb{R}$, the inequality holds $\forall x$.

(iii)

$$\begin{aligned}5 - x^2 &< -2\\ 7 &< x^2\\ 0 &< x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7})\end{aligned}$$

Hence, either $x > \sqrt{7} \;\wedge\; x > -\sqrt{7}$ or $x < \sqrt{7} \;\wedge\; x < -\sqrt{7}$, which is $x > \sqrt{7} \;\vee\; x < -\sqrt{7}$.

(iv)

$$\begin{gathered}(x - 1)(x - 3) > 0\\ (x > 1 \;\wedge\; x > 3) \vee (x < 1 \;\wedge\; x < 3)\\ x > 3 \vee x < 1\end{gathered}$$

(v)

$$\begin{aligned}x^2 - 2x + 2 &> 0\\ (x^2 - 2x + 1) + 1 &> 0\\ (x - 1)^2 + 1 &> 0\end{aligned}$$

Hence the inequality is satisfied $\forall\, x$.

(vi)

$$
\begin{aligned}
x^2 + x + 1 &> 2 \\
x^2 + x - 1 &> 0 \\
x^2 + \left(\frac{1+\sqrt{5}}{2}\right)x + \left(\frac{1-\sqrt{5}}{2}\right)x + \left(\frac{(1-\sqrt{5})(1+\sqrt{5})}{4}\right) &> 0 \\
\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right) &> 0 \\
x > \left(\frac{\sqrt{5}-1}{2}\right) \vee x < \left(\frac{-(\sqrt{5}+1)}{2}\right)
\end{aligned}
$$

(vii)

$$
\begin{aligned}
x^2 - x + 10 &> 16 \\
x^2 - x - 6 &> 0 \\
x^2 - 3x + 2x - 6 &> 0 \\
x(x-3) + 2(x-3) &> 0 \\
(x+2)(x-3) &> 0 \\
x > 3 &\vee x < -2
\end{aligned}
$$

(viii)

$$
\begin{aligned}
x^2 + x + 1 &> 0 \\
x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 &> 0 \\
(x + \frac{1}{2})^2 + \frac{3}{4} &> 0
\end{aligned}
$$

which is true for all x.

(ix) Divide the problem into two cases: $x > \pi$ and $x < \pi$.

Case 1: $x > \pi$
Then $(x+5)(x-3) > 0$, which is $x > 3 \ \vee \ x < -5$.

Case 2: $x < \pi$
Then $(x+5)(x-3) < 0$, which is $-5 < x < 3$.

(x)

$$
\begin{aligned}
(x - \sqrt[3]{2})(x - \sqrt{2}) &> 0 \\
x > \sqrt{2} &\vee x < \sqrt[3]{2}
\end{aligned}
$$

(xi) (Sometimes, to solve a problem, intuition is a necessity.)

$$\begin{aligned} 2^x &< 8 \\ 2^x &< 2^3 \\ x &< 3 \end{aligned}$$

(xii)

$$\begin{aligned} x + 3^x &< 4 \\ x + 3^x &< 1 + 3^1 \\ x &< 1 \end{aligned}$$

(xiii)

$$\begin{aligned} \frac{1}{x} + \frac{1}{1-x} &> 0 \\ \frac{1}{x(1-x)} &> 0 \end{aligned}$$

Hence, $x(1-x) > 0$. This means $0 < x < 1$.

(xiv)

$$\frac{x-1}{x+1} > 0$$

Hence, $(x-1)(x+1) > 0$, or $x > 1 \ \vee \ x < -1$.

■

Problem 1.5. Prove the following:

(i) If $a < b$ and $c < d$, then $a + c < b + d$

(ii) If $a < b$, then $-b < -a$

(iii) If $a < b$ and $c > d$, then $a - c < b - d$

(iv) If $a < b$ and $c > 0$, then $ac < bc$

(v) If $a < b$ and $c < 0$, then $ac > bc$

(vi) If $a > 1$, then $a^2 > a$

(vii) If $0 < a < 1$, then $a^2 < a$

(viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$

(ix) If $0 \leq a < b$, then $a^2 < b^2$. (Use (viii).)

(x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$.(Use (ix), backwards.)

Solution. Let P be the set of all positive numbers.

(i) To prove this, we apply (P11): If $a < b \ \wedge \ c < d$, then $(b - a \in P) \ \wedge \ (d - c \in P)$. Then $(b - a) + (d - c) = (b + d) - (a + c) \in P$. Therefore, $a + c < b + d$.

(ii) We provide two solutions: The first one is by Trichotomy Law (P10), and the second one is by adding $[(-a) + (-b)]$ to both sides.

Proof by Trichotomy Law. If $a < b$, then $b - a \in P$. By Trichotomy Law, $a - b \notin P$ and $a - b \neq 0$. Therefore, $a - b < 0$, which is $-b < -a$. Q.E.D

Proof by adding.

$$\begin{aligned} a &< b \\ a + [(-a) + (-b)] &< b + [(-a) + (-b)] \\ [a + (-a)] + (-b) &< [b + (-b)] + (-a) \\ -b &< -a \end{aligned}$$

Q.E.D

(iii) Using (P11), we have $b - a \in P \ \wedge \ c - d \in P$. Then $(b - a) + (c - d) \in P$. Hence, $a - c < b - d$.

(iv) Using (P12), note that $b - a \in P$. Since $c > 0$, $c(b - a) \in P$, which means $bc - ac > 0$, or $ac < bc$.

(v) By Trichotomy law(P10), $-c \in P$. Then by (iv), $-(ac) < -(bc)$. By (ii), $ac > bc$.

(vi) Since $a > 1 > 0$, by (iv), $a^2 > a$.

(vii) Since $a > 0$, by (iv), $a^2 < a$.

(viii) Because $0 < b$, $bc < bd$. Furthermore, if $c \geq 0$, $ac \leq bc$ (equality occurs if $c = 0$), by (iv). Therefore, $ac \leq bc < bd$. Hence, $ac < bd$.

(ix) From (viii), let $c = a$ and $d = b$, then the result follows.

(x) Suppose $a \geq b$. Then $a \geq b \geq 0$. By (ix) and (P9), $a^2 \geq b^2$. This contradicts $a^2 < b^2$.

■

Problem 1.6. (a) Prove that if $0 \leq x < y$, then $x^n < y^n$, $n = 1, 2, 3, \ldots$.

(b) Prove that if $x < y$ and n is odd, then $x^n < y^n$.

(c) Prove that if $x^n = y^n$ and n is odd, then $x = y$.

(d) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Solution. (a) Repeatedly apply problem 1.5(viii) for $0 \leq x < y$, we have $x^n < y^n$ with $n = 1, 2, 3, \ldots$

(b) The statement is true for the case $0 \leq x < y$. In the case $x < y \leq 0$, by 1.5(ii), $(-x) > (-y) \geq 0$. By (a), $(-x)^n > (-y)^n$ for all odd n. Since n is odd, $-(x^n) > -(y^n)$. Hence, by 1.5(ii), $x^n < y^n$. In the case $x \leq 0 < y$, since n is odd, $x^n < y^n$.

(c) Suppose that either $x \neq y$. W.l.o.g, let $x < y$, by (b), $x^n < y^n$ for all odd n, contradicting $x^n = y^n$ for all odd n.

(d) Suppose that both $x \neq y$ and $x \neq -y$. Then $x^2 - y^2 \neq 0$. W.l.o.g, suppose $x^2 > y^2 \geq 0$. Applying (a), this generalizes to $x^n > y^n$ for all even n, contradicting our assumption. Therefore, $x = y$ or $x = -y$.

The direct proof. In the case $x, y \geq 0$; by (a), if $x^n = y^n$ for all even n, then $x = y$. In the case $x, y \leq 0$; if $x^n = y^n$ for all even n, then $(-x), (-y) \geq 0$ and $(-x)^n = (-y)^n$, so $-x = -y$ and hence $x = y$. In the case of x and y have different signs, then x and $-y$ are either two positive or two negative numbers. In either subcase, if $x^n = y^n$ for all even n, then $x^n = (-y)^n$, and it follows $x = -y$ from the previous case.

■

Problem 1.7. Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b$$

Notice that the inequality $\sqrt{ab} \leq (a+b)/2$ holds for all $a, b \geq 0$. A generalization of this fact occurs in Problem 2.22.

Solution. Let us first establish that $a < \dfrac{a+b}{2} < b$. Note that,

$$a + a < a + b < b + b$$

and therefore, $a < \dfrac{a+b}{2} < b$. To finish the proof, we need to prove $a < \sqrt{ab} < \dfrac{a+b}{2}$. To do this, let us prove that if $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$. Note that since $b - a > 0$,

$$b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) > 0$$

Therefore, $\sqrt{b} > \sqrt{a} > 0$. We rewrite the inequality as follows,

$$\sqrt{a} \cdot (\sqrt{b} - \sqrt{a}) > 0$$

Then

$$a < \sqrt{ab} \tag{1.1}$$

We next notice that since $\sqrt{b} - \sqrt{a} > 0$, it follows that
$(\sqrt{b} - \sqrt{a}) \cdot (\sqrt{b} - \sqrt{a}) = (\sqrt{b} - \sqrt{a})^2 > 0$. Expand the left hand side,

$$(\sqrt{b} - \sqrt{a})^2 = a + b - 2\sqrt{ab} > 0$$

which implies,

$$\sqrt{ab} < \frac{a+b}{2} \tag{1.2}$$

From (1.1) and (1.2), we have $a < \sqrt{ab} < \dfrac{a+b}{2}$. ■

Problem 1.8 (*)**.** Although the basic properties of inequalities were stated in terms of the collection P of all positive numbers, and $<$ was defined in terms of P, this procedure can be reversed. Suppose that P10–P12 are replaced by

(P'10) For any numbers a and b one, and only one, of the following holds:

(i) $a = b$,

(ii) $a < b$,

(iii) $b < a$.

(P'11) For any numbers a, b, and c, if $a < b$ and $b < c$, then $a < c$.

(P'12) For any numbers a, b, and c, if $a < b$, then $a + c < b + c$.

(P'13) For any numbers a, b, and c, if $a < b$ and $0 < c$, then $ac < bc$.

Show that P10–P12 can then be deduced as theorems.

Solution. Let P be the set of all positive numbers.

- To prove P10, let $c = a - b$, from (P'10), P10 follows.
- To prove P11, let $a, b \in P$; it is sufficient to prove that $a + b > 0$. From (P'10), we divide the proof into three subscases:

  Case 1: $a = b$
  Then $a + b = b + b > 0 + b > 0$, where the first inequality follows from (P'12). By (P'11), $a + b > 0$.

  Case 2: $a < b$
  Then $a + b > a + a > 0 + a > 0$, where the first and second inequality follow from (P'12). By applying (P'11) twice, $a + b > 0$.

  Case 3: $a > b$
  Interchanging the role of a and b, we have the result.

- To prove P12, let $a, b \in P$; it is sufficient to prove that $a \cdot b > 0$. From (P'10), we divide the proof into three subcases:

  Case 1: $a = b$
  Then $a \cdot b = b \cdot b > 0 \cdot b = 0$, where the first inequality follows from (P'13) and the equality after which is from (P9).

  Case 2: $a < b$
  Then $b \cdot a > a \cdot a > 0 \cdot a = 0$, where the first and second inequality is from (P'13). By (P'11), $a \cdot b > 0$.

  Case 3: $a > b$
  Interchanging a and b returns us to case 2, which yields the result.

$\blacksquare$