Solutions to Michael Spivak's Calculus

Son To
<son.trung.to@gmail.com>

Ravintola Kiltakellari

25^{th} June, 2017

On maailmalle

Preface

This is my own solutions to Michael Spivak's Calculus textbook.

All the problems are numbered after the chapter; otherwise, the order of problems are the same as the one in the textbook. Chapter 1 has 25 problems.

Vantaa, Finland
25^{th} June, 2017.

Contents

Part I

Prologue

This page is intentionally left blank.

Chapter 1

Basic properties of number

Problem 1.1. Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$

(ii) $x^2 - y^2 = (x-y)(x+y)$

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$

(iv) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

(v) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

(vi) $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$ (There is a particularly easy way to do this using (iv), and it will show you how to find a factorization for $x^n + y^n$ whenever n is odd.)

Solution. (i) By (P7)(Existence of multiplicative inverses), there exists a^{-1} such that,

$$\begin{gathered}(a^{-1} \cdot a)x = (a^{-1} \cdot a)\\ x = 1\end{gathered}$$

(ii) By (P9) for 2 times,

$$\begin{aligned}(x-y)(x+y) &\overset{1}{=} x \cdot (x+y) + (-y) \cdot (x+y)\\ &\overset{2}{=} x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y\\ &= x^2 + x \cdot y + [-(x \cdot y)] + [-(y^2)]\\ &= x^2 - y^2\end{aligned}$$

(iii) From (ii) and since $x^2 = y^2$,

$$x^2 - y^2 = (x-y)(x+y) = 0$$

This means $(x-y) = 0 \lor (x+y) = 0$, which is $x = y \lor x = -y$

(iv) Starting with the right-hand side,

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x \cdot (x^2+xy+y^2) + (-y) \cdot (x^2+xy+y^2) \\ &= x^3 + x^2y + xy^2 + [-(x^2y)] + [-(xy^2)] + [-(y)^3] \\ &= x^3 - y^3\end{aligned}$$

(v) I propose two solutions for this problem. The first one is the direct right-hand side manipulation, while the latter is done by induction.

The first solution.

$$\begin{aligned}&(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1} \\ &+[-(x^{n-1}y)] + [-(x^{n-2}y^2)] + \cdots + [-(xy^{n-1})] + [-(y^n)] \\ &= x^n - y^n\end{aligned}$$

Q.E.D

The second solution. Let n=1, then indeed $x - y = x - y$. Suppose the statement holds true for $n = k$ with $k \in \mathbb{N}$, that is

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})$$

is true. To finish the proof, we need to prove

$$x^{k+1} - y^{k+1} = (x-y)(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k)$$

That is, the statement holds for $n = k$. Starting from the left hand side,

$$\begin{aligned}& x^{k+1} - y^{k+1} \\ = \;& x^{k+1} - x^ky + x^ky - y^{k+1} \\ = \;& x^k(x-y) + y(x^k - y^k) \\ = \;& x^k(x-y) + y(x-y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1}) \\ = \;& (x-y)[x^k + y(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})] \\ = \;& (x-y)(x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1} + y^k)\end{aligned}$$

Q.E.D

(vi) We will use (iv) in our proof,

$$\begin{aligned}
& x^3 + y^3 \\
= \quad & x^3 - y^3 + 2y^3 \\
= \quad & (x-y)(x^2+xy+y^2) + 2y[(x^2+xy+y^2) + (-x)(x+y)] \\
= \quad & (x+y)(x^2+xy+y^2) + 2[-(xy)](x+y) \\
= \quad & (x+y)(x^2-xy+y^2)
\end{aligned}$$

■

Problem 1.2. What is wrong with the following "proof"? Let $x = y$. Then

$$\begin{aligned}
x^2 &= xy, \\
x^2 - y^2 &= xy - y^2, \\
(x+y)(x-y) &= y(x-y), \\
x + y &= y, \\
2y &= y, \\
2 &= 1.
\end{aligned}$$

Solution. Note that in the transition from line 3 to line 4, the author "simplifies" $(x-y)$ by dividing $(x-y)$ on both sides. This is wrong since $x - y = 0$, and hence $1/0$ is undefined as implied by (P7) in the textbook. ■

Problem 1.3. Prove the following:

(i) $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

(ii) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$, if $b, d \neq 0$.

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$. (To do this you must remember the defining property of $(ab)^{-1}$.)

(iv) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{db}$, if $b, d \neq 0$.

(v) $\dfrac{a}{b} \Big/ \dfrac{c}{d} = \dfrac{ad}{bc}$, if $b, c, d \neq 0$.

(vi) If $b, d \neq 0$, then $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$. Also determine when $\dfrac{a}{b} = \dfrac{b}{a}$.

Solution. (i) Until (iii) is proved, the solution is to test the equality between two sides.

$$\begin{aligned} a(b)^{-1} &= (ac)(bc)^{-1} \\ a[(b)^{-1}b] &= (ac)(bc)^{-1}b \\ (a^{-1}a) &= (a^{-1}a)c(bc)^{-1}b \\ 1 &= (bc)(bc)^{-1} = 1 \end{aligned}$$

(ii) Similar to the above,

$$\begin{aligned} a(b)^{-1} + c(d)^{-1} &= (ad+bc)(bd)^{-1} \\ a(b)^{-1}bd + c(d)^{-1}bd &= (ad+bc)[(bd)^{-1}(bd)] \\ ad(b^{-1}b) + bc(d^{-1}d) &= (ad+bc) \\ ad+bc &= ad+bc \end{aligned}$$

(iii) Since $a, b \neq 0$, there exists $(ab)^{-1}, a^{-1}, b^{-1}$ such that,

$$\begin{aligned} ab &= ab \\ (ab)^{-1}(ab) &= (ab)^{-1}(ab) = 1 \\ (ab)^{-1}a(bb^{-1}) &= b^{-1} \\ (ab)^{-1}(aa^{-1}) &= b^{-1}a^{-1} \\ (ab)^{-1} &= a^{-1}b^{-1} \end{aligned}$$

(iv) For $b, d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = ab^{-1}cd^{-1} = ac(d^{-1}b^{-1}) = ac(db)^{-1} = \frac{ac}{db}$$

where the next-to-last equality follows from (iii).

(v) I first establish for any number $a \neq 0$,

$$(a^{-1})^{-1} = a$$

Let $t = a^{-1}$, we want to prove $t^{-1} = a$. Observe that

$$\begin{aligned} t &= a^{-1} \\ t \cdot (t)^{-1} &= a^{-1} \cdot (t)^{-1} \\ a \cdot 1 &= (a \cdot a^{-1}) \cdot (t)^{-1} \\ a &= (t)^{-1} \end{aligned}$$

From the left hand side of the statement,

$$\frac{a}{b}\bigg/\frac{c}{d} = a(b)^{-1}[c(d)^{-1}]^{-1} = a(b)^{-1}(c)^{-1}[(d)^{-1}]^{-1} = (ad)(bc)^{-1} = \frac{ad}{bc}$$

where the second and third equality follows both from (iii) and the proof above.

(vi) Using (ii),

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ \frac{a}{b} + (-\frac{c}{d}) &= 0 \\ \frac{ad - bc}{bd} &= 0 \\ ad &= bc \end{aligned}$$

Now, put $c = b \wedge d = a$. It follows that $\frac{a}{b} = \frac{b}{a}$ if and only if $a^2 = b^2$. It follows $(a-b)(a+b) = 0$, or $a = b \vee a = -b$.

■

Problem 1.4. Find all numbers x for which

(i) $4 - x < 3 - 2x$

(ii) $5 - x^2 < 8$

(iii) $5 - x^2 < -2$

(iv) $(x-1)(x-3) > 0$ (When is a product of two numbers positive?)

(v) $x^2 - 2x + 2 > 0$

(vi) $x^2 + x + 1 > 2$

(vii) $x^2 - x + 10 > 16$

(viii) $x^2 + x + 1 > 0$

(ix) $(x - \pi)(x + 5)(x - 3) > 0$

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$

(xi) $2^x < 8$

(xii) $x + 3^x < 4$

(xiii) $\frac{1}{x} + \frac{1}{1-x} > 0$

(xiv) $\frac{x-1}{x+1} > 0$

Solution. (i)

$$\begin{aligned} 4 - x &< 3 - 2x \\ 4 + (-x + 2x) &< 3 + (-2x + 2x) \\ (-4 + 4) + x &< -4 + 3 \\ x &< -1 \end{aligned}$$

(ii)

$$\begin{aligned} 5 - x^2 &< 8 \\ 5 - 8 &< x^2 \\ -3 &< x^2 \end{aligned}$$

Since $x^2 \geq 0 \ \forall x \in \mathbb{R}$, the inequality holds $\forall x$.

(iii)

$$\begin{aligned} 5 - x^2 &< -2 \\ 7 &< x^2 \\ 0 &< x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7}) \end{aligned}$$

Hence, either $x > \sqrt{7} \ \wedge \ x > -\sqrt{7}$ or $x < \sqrt{7} \ \wedge \ x < -\sqrt{7}$, which is $x > \sqrt{7} \ \vee \ x < -\sqrt{7}$.

(iv)

$$\begin{aligned} (x - 1)(x - 3) &> 0 \\ (x > 1 \ \wedge \ x > 3) \vee (x < 1 \ \wedge \ x < 3) \\ x > 3 \vee x < 1 \end{aligned}$$

(v)

$$\begin{aligned} x^2 - 2x + 2 &> 0 \\ (x^2 - 2x + 1) + 1 &> 0 \\ (x - 1)^2 + 1 &> 0 \end{aligned}$$

Hence the inequality is satisfied $\forall \ x$.

(vi)

$$\begin{aligned}
x^2 + x + 1 &> 2 \\
x^2 + x - 1 &> 0 \\
x^2 + \left(\frac{1+\sqrt{5}}{2}\right)x + \left(\frac{1-\sqrt{5}}{2}\right)x + \left(\frac{(1-\sqrt{5})(1+\sqrt{5})}{4}\right) &> 0 \\
\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right) &> 0 \\
x > \left(\frac{\sqrt{5}-1}{2}\right) \vee x < \left(\frac{-(\sqrt{5}+1)}{2}\right)
\end{aligned}$$

(vii)

$$\begin{aligned}
x^2 - x + 10 &> 16 \\
x^2 - x - 6 &> 0 \\
x^2 - 3x + 2x - 6 &> 0 \\
x(x-3) + 2(x-3) &> 0 \\
(x+2)(x-3) &> 0 \\
x > 3 &\vee x < -2
\end{aligned}$$

(viii)

$$\begin{aligned}
x^2 + x + 1 &> 0 \\
x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 &> 0 \\
(x + \frac{1}{2})^2 + \frac{3}{4} &> 0
\end{aligned}$$

which is true for all x.

(ix) Divide the problem into two cases: $x > \pi$ and $x < \pi$.

Case 1: $x > \pi$
Then $(x+5)(x-3) > 0$, which is $x > 3 \ \vee \ x < -5$.

Case 2: $x < \pi$
Then $(x+5)(x-3) < 0$, which is $-5 < x < 3$.

(x)

$$\begin{aligned}
(x - \sqrt[3]{2})(x - \sqrt{2}) &> 0 \\
x > \sqrt{2} &\vee x < \sqrt[3]{2}
\end{aligned}$$

(xi) (Sometimes, to solve a problem, intuition is a necessity.)

$$
\begin{aligned}
2^x &< 8 \\
2^x &< 2^3 \\
x &< 3
\end{aligned}
$$

(xii)

$$
\begin{aligned}
x + 3^x &< 4 \\
x + 3^x &< 1 + 3^1 \\
x &< 1
\end{aligned}
$$

(xiii)

$$
\begin{aligned}
\frac{1}{x} + \frac{1}{1-x} &> 0 \\
\frac{1}{x(1-x)} &> 0
\end{aligned}
$$

Hence, $x(1-x) > 0$. This means $0 < x < 1$.

(xiv)

$$
\frac{x-1}{x+1} > 0
$$

Hence, $(x-1)(x+1) > 0$, or $x > 1 \ \vee \ x < -1$.

■

Problem 1.5. Prove the following:

(i) If $a < b$ and $c < d$, then $a + c < b + d$

(ii) If $a < b$, then $-b < -a$

(iii) If $a < b$ and $c > d$, then $a - c < b - d$

(iv) If $a < b$ and $c > 0$, then $ac < bc$

(v) If $a < b$ and $c < 0$, then $ac > bc$

(vi) If $a > 1$, then $a^2 > a$

(vii) If $0 < a < 1$, then $a^2 < a$

(viii) If $0 \le a < b$ and $0 \le c < d$, then $ac < bd$

(ix) If $0 \le a < b$, then $a^2 < b^2$. (Use (viii).)

(x) If $a, b \ge 0$ and $a^2 < b^2$, then $a < b$.(Use (ix), backwards.)

Solution. Let P be the set of all positive numbers.

(i) To prove this, we apply (P11): If $a < b \wedge c < d$, then $(b - a \in P) \wedge (d - c \in P)$. Then $(b - a) + (d - c) = (b + d) - (a + c) \in P$. Therefore, $a + c < b + d$.

(ii) We provide two solutions: The first one is by Trichotomy Law (P10), and the second one is by adding $[(-a) + (-b)]$ to both sides.

Proof by Trichotomy Law. If $a < b$, then $b - a \in P$. By Trichotomy Law, $a - b \notin P$ and $a - b \neq 0$. Therefore, $a - b < 0$, which is $-b < -a$. Q.E.D

Proof by adding.

$$\begin{aligned} a &< b \\ a + [(-a) + (-b)] &< b + [(-a) + (-b)] \\ [a + (-a)] + (-b) &< [b + (-b)] + (-a) \\ -b &< -a \end{aligned}$$

Q.E.D

(iii) Using (P11), we have $b - a \in P \wedge c - d \in P$. Then $(b - a) + (c - d) \in P$. Hence, $a - c < b - d$.

(iv) Using (P12), note that $b - a \in P$. Since $c > 0$, $c(b - a) \in P$, which means $bc - ac > 0$, or $ac < bc$.

(v) By Trichotomy law(P10), $-c \in P$. Then by (iv), $-(ac) < -(bc)$. By (ii), $ac > bc$.

(vi) Since $a > 1 > 0$, by (iv), $a^2 > a$.

(vii) Since $a > 0$, by (iv), $a^2 < a$.

(viii) Because $0 < b$, $bc < bd$. Furthermore, if $c \geq 0$, $ac \leq bc$ (equality occurs if $c = 0$), by (iv). Therefore, $ac \leq bc < bd$. Hence, $ac < bd$.

(ix) From (viii), let $c = a$ and $d = b$, then the result follows.

(x) Suppose $a \geq b$. Then $a \geq b \geq 0$. By (ix) and (P9), $a^2 \geq b^2$. This contradicts $a^2 < b^2$.

■

Problem 1.6. (a) Prove that if $0 \leq x < y$, then $x^n < y^n$, $n = 1, 2, 3, \ldots$.

(b) Prove that if $x < y$ and n is odd, then $x^n < y^n$.

(c) Prove that if $x^n = y^n$ and n is odd, then $x = y$.

(d) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Solution. (a) Repeatedly apply problem 1.5(viii) for $0 \leq x < y$, we have $x^n < y^n$ with $n = 1, 2, 3, \ldots$

(b) The statement is true for the case $0 \leq x < y$. In the case $x < y \leq 0$, by 1.5(ii), $(-x) > (-y) \geq 0$. By (a), $(-x)^n > (-y)^n$ for all odd n. Since n is odd, $-(x^n) > -(y^n)$. Hence, by 1.5(ii), $x^n < y^n$. In the case $x \leq 0 < y$, since n is odd, $x^n < y^n$.

(c) Suppose that either $x \neq y$. W.l.o.g, let $x < y$, by (b), $x^n < y^n$ for all odd n, contradicting $x^n = y^n$ for all odd n.

(d) Suppose that both $x \neq y$ and $x \neq -y$. Then $x^2 - y^2 \neq 0$. W.l.o.g, suppose $x^2 > y^2 \geq 0$. Applying (a), this generalizes to $x^n > y^n$ for all even n, contradicting our assumption. Therefore, $x = y$ or $x = -y$.

The direct proof. In the case $x, y \geq 0$; by (a), if $x^n = y^n$ for all even n, then $x = y$. In the case $x, y \leq 0$; if $x^n = y^n$ for all even n, then $(-x), (-y) \geq 0$ and $(-x)^n = (-y)^n$, so $-x = -y$ and hence $x = y$. In the case of x and y have different signs, then x and $-y$ are either two positive or two negative numbers. In either subcase, if $x^n = y^n$ for all even n, then $x^n = (-y)^n$, and it follows $x = -y$ from the previous case.

■

Problem 1.7. Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b$$

Notice that the inequality $\sqrt{ab} \leq (a+b)/2$ holds for all $a, b \geq 0$. A generalization of this fact occurs in Problem 2.22.

Solution. Let us first establish that $a < \dfrac{a+b}{2} < b$. Note that,

$$a + a < a + b < b + b$$

and therefore, $a < \dfrac{a+b}{2} < b$. To finish the proof, we need to prove $a < \sqrt{ab} < \dfrac{a+b}{2}$. To do this, let us prove that if $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$. Note that since $b - a > 0$,

$$b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) > 0$$

Therefore, $\sqrt{b} > \sqrt{a} > 0$. We rewrite the inequality as follows,

$$\sqrt{a} \cdot (\sqrt{b} - \sqrt{a}) > 0$$

Then

$$a < \sqrt{ab} \tag{1.1}$$

We next notice that since $\sqrt{b} - \sqrt{a} > 0$, it follows that
$(\sqrt{b} - \sqrt{a}) \cdot (\sqrt{b} - \sqrt{a}) = (\sqrt{b} - \sqrt{a})^2 > 0$. Expand the left hand side,

$$(\sqrt{b} - \sqrt{a})^2 = a + b - 2\sqrt{ab} > 0$$

which implies,

$$\sqrt{ab} < \frac{a+b}{2} \tag{1.2}$$

From (1.1) and (1.2), we have $a < \sqrt{ab} < \dfrac{a+b}{2}$. ∎

Problem 1.8 (*)**.** Although the basic properties of inequalities were stated in terms of the collection P of all positive numbers, and $<$ was defined in terms of P, this procedure can be reversed. Suppose that P10–P12 are replaced by

(P'10) For any numbers a and b one, and only one, of the following holds:

- (i) $a = b$,
- (ii) $a < b$,
- (iii) $b < a$.

(P'11) For any numbers a, b, and c, if $a < b$ and $b < c$, then $a < c$.

(P'12) For any numbers a, b, and c, if $a < b$, then $a + c < b + c$.

(P'13) For any numbers a, b, and c, if $a < b$ and $0 < c$, then $ac < bc$.

Show that P10–P12 can then be deduced as theorems.

Solution. Let P be the set of all positive numbers.

- To prove P10, let $c = a - b$, from (P'10), P10 follows.
- To prove P11, let $a, b \in P$; it is sufficient to prove that $a + b > 0$. From (P'10), we divide the proof into three subscases:

  Case 1: $a = b$
  Then $a + b = b + b > 0 + b > 0$, where the first inequality follows from (P'12). By (P'11), $a + b > 0$.

  Case 2: $a < b$
  Then $a + b > a + a > 0 + a > 0$, where the first and second inequality follow from (P'12). By applying (P'11) twice, $a + b > 0$.

  Case 3: $a > b$
  Interchanging the role of a and b, we have the result.

- To prove P12, let $a, b \in P$; it is sufficient to prove that $a \cdot b > 0$. From (P'10), we divide the proof into three subcases:

  Case 1: $a = b$
  Then $a \cdot b = b \cdot b > 0 \cdot b = 0$, where the first inequality follows from (P'13) and the equality after which is from (P9).

  Case 2: $a < b$
  Then $b \cdot a > a \cdot a > 0 \cdot a = 0$, where the first and second inequality is from (P'13). By (P'11), $a \cdot b > 0$.

  Case 3: $a > b$
  Interchanging a and b returns us to case 2, which yields the result.

■

Problem 1.9. Express each of the following with at least one less pair of absolute value signs.

(i) $|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}|$

(ii) $|(|a + b| - |a| - |b|)|$

(iii) $|(|a + b| + |c| - |a + b + c|)|$

(iv) $|(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)|$

Solution. (i) Note $\sqrt{7} - \sqrt{5} > 0$, hence

$$|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}| = \sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$$

(ii) Since $|a + b| - |a| - |b| \leq 0$,

$$|(|a + b| - |a| - |b|)| = |a| + |b| - |a + b|$$

(iii) Since $|a + b + c| \leq |a + b| + |c|$,

$$|(|a + b| + |c| - |a + b + c|)| = |a + b| + |c| - |a + b + c|$$

(iv)

$$|(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)| = |\sqrt{2} + \sqrt{3} - \sqrt{7} + \sqrt{5}|$$

■

Problem 1.10. Express each of the following without absolute value signs, treating various cases separately when necessary.

(i) $|a + b| - |b|$

(ii) $|(|x| - 1)|$

(iii) $|x| - |x^2|$

(iv) $a - |(a - |a|)|$

Solution. (i) We divide into four cases:

$$a \geq 0 \quad \text{and} \quad b \geq 0 \qquad \text{(Case 1)}$$
$$a \leq 0 \quad \text{and} \quad b \leq 0 \qquad \text{(Case 2)}$$
$$a \geq 0 \quad \text{and} \quad b \leq 0 \qquad \text{(Case 3)}$$
$$a \leq 0 \quad \text{and} \quad b \geq 0 \qquad \text{(Case 4)}$$

In Case 1 and Case 2, we have $|a+b| - |b| = a$ since $|a+b| \leq |a| + |b|$.

In Case 3, if $a+b \geq 0$, then

$$|a+b| - |b| = (a+b) - (-b) = a + b + b = 2b$$

If $a + b \leq 0$, then

$$|a+b| - |b| = (-a-b) - (-b) = -a + (-b) + b = -a$$

In Case 4, if $a+b \geq 0$, then

$$|a+b| - |b| = (a+b) - (b) = a$$

If $a + b \leq 0$, then

$$|a+b| - |b| = -a + (-b) + (-b) = -a - 2b$$

(ii) We make the problem into 4 cases.

$$x \geq 1 \qquad \text{(Case 1)}$$
$$0 \leq x \leq 1 \qquad \text{(Case 2)}$$
$$-1 \leq x \leq 0 \qquad \text{(Case 3)}$$
$$x \leq -1 \qquad \text{(Case 4)}$$

In Case 1, $|(|x|-1)| = x - 1$.

In Case 2, $|(|x|-1)| = 1 - x$.

In Case 3, $|(|x|-1)| = x + 1$.

In Case 4, $|(|x|-1)| = -(x+1)$.

(iii) Since $x^2 \geq 0$, $|x| - |x^2| = |x| - x^2$.

If $x \geq 0$, then $|x| - x^2 = x(1-x)$. If $x \leq 0$, then $|x| - x^2 = -x + (-x^2) = -x(1+x)$.

(iv) Note that $|a| \geq a$. Hence,

$$a - |(a - |a|)| = a + a - |a| = 2a - |a|$$

We have two cases,

Case 1: $a \geq 0$

$$2a - |a| = 2a - a = a$$

Case 2: $a \leq 0$

$$2a - |a| = 2a + a = 3a$$

■

Problem 1.11. Find all numbers x for which

(i) $|x - 3| = 8$

(ii) $|x - 3| < 8$

(iii) $|x + 4| < 2$

(iv) $|x - 1| + |x - 2| > 1$

(v) $|x - 1| + |x + 1| < 2$

(vi) $|x - 1| + |x + 1| < 1$

(vii) $|x - 1| \cdot |x + 1| = 0$

(viii) $|x - 1| \cdot |x + 2| = 3$

Solution. (i)

$$x - 3 = 8 \vee x - 3 = -8$$
$$x = 11 \vee x = -5$$

(ii) Then $-8 < x - 3 < 8$. Hence, $-5 < x < 11$.

(iii) Then $-2 < x + 4 < 2$. Hence, $-6 < x < -2$.

(iv) If $1 \leq x \leq 2$, then the inequality becomes $(x - 1) + (2 - x) = 1$. If $x > 2$, then $2x - 3 > 1$, which is $x > 2$. If $x < 1$, then $-2x + 3 > 1$, which is $x < 1$. Therefore, either $x > 2$ or $x < 1$ satisfies the inequality.

(v) If $-1 \leq x \leq 1$, then $(1 - x) + (x + 1) = 2$. If $x > 1$, then $x < 1$, which is contradictory. If $x < -1$, then $(1 - x) + (-x - 1) = -2x < 2$ only if $x > -1$, which is contradictory. Hence, there is no x to satisfy the inequality.

(vi) It is implied from above that

$$|x - 1| + |x + 1| \geq 2$$

Therefore, there is no x satisfying the inequality.

(vii) Either $x = 1$ or $x = -1$.

(viii) If $-2 \le x \le 1$, then we obtain $x^2 + x + 1 > 0$. Hence, in either $x < -2$ or $x > 1$, we have to solve the equation $x^2 + x - 5 = 0$, whose solution is either $x = \dfrac{-1+\sqrt{21}}{2}$ or $x = \dfrac{-1-\sqrt{21}}{2}$.

$\blacksquare$

Problem 1.12. Prove the following:

(i) $|xy| = |x| \cdot |y|$

(ii) $\left|\dfrac{1}{x}\right| = \dfrac{1}{|x|}$, if $x \neq 0$. (The best way to do this is to remember what $|x|^{-1}$ is.)

(iii) $\dfrac{|x|}{|y|} = \left|\dfrac{x}{y}\right|$, if $y \neq 0$.

(iv) $|x - y| \le |x| + |y|$ (Give a very short proof.)

(v) $|x| - |y| \le |x - y|$ (A very short proof is possible, if you write things in the right way.)

(vi) $|(|x| - |y|)| \le |x - y|$ (Why does this follow immediately from (v)?)

(vii) $|x + y + z| \le |x| + |y| + |z|$. Indicate when equality holds, and prove your statement.

Solution. (i) We have 4 cases,

$$x \ge 0 \quad y \ge 0 \tag{1}$$
$$x \ge 0 \quad y \le 0 \tag{2}$$
$$x \le 0 \quad y \ge 0 \tag{3}$$
$$x \le 0 \quad y \le 0 \tag{4}$$

In (1), $|x| \cdot |y| = xy = |xy|$

In (4), $|x| \cdot |y| = (-x)(-y) = xy = |xy|$

In (3), $|x| \cdot |y| = (-x)(y) = -(xy) = |xy|$

In (2), interchanging x and y leads to (3).

(ii) Since $x \neq 0$, there exists $|x|^{-1}$ such that

$$|x||x|^{-1} = 1 = |x|\left|\frac{1}{x}\right|$$

where the second equality is by (i). Dividing both sides by $|x|$, we have the result.

(iii) Since $y \neq 0$, from (ii), we immediately have

$$\left|\frac{1}{y}\right| = \frac{1}{|y|}$$

Hence, applying (ii) once more,

$$\left|\frac{x}{y}\right| = |x|\left|\frac{1}{y}\right| = \frac{|x|}{|y|}$$

(iv) Note that,

$$|x-y| = |x+(-y)| \leq |x| + |-y| = |x| + |y|$$

where the last equality follows from (i).

(v) Note that,

$$|x-y+y| \leq |x-y| + |y|$$

Therefore, $|x| - |y| \leq |x-y|$.

(vi) Let the first term be y and the second term be $y-x$. Applying (v), we have

$$|y| - |y-x| \leq |x|$$

Hence, $-|x-y| \leq |x|-|y|$. Combining with (v) gives $|(|x|-|y|)| \leq |x-y|$.

(vii) Notice the pattern,

$$|x+y+z| \leq |x+y| + |z| \leq |x| + |y| + |z|$$

the equality holds only if either x, y, z have the same sign or at least two of them must be equal to 0. It is easy to verify this.
Suppose not, then both x, y, z have different signs and at most one of them is 0. If the latter is true, then, w.l.o.g, suppose $z = 0$, then x, y have different sign, and we are done. If none of them is 0, then, w.l.o.g, suppose $z < 0$ and pick z such that $x + y < -z$. Then,

$$|x+y+z| = -(x+y+z) = -x-y-z < |x| + |y| + |z|$$

where inequality must follow since $x, y \neq 0$.

■

Problem 1.13. The maximum of two numbers x and y is denoted by $max(x, y)$. Thus $max(-1, 3) = max(3, 3) = 3$ and $max(-1, -4) = max(-4, -1) = -1$. The minimum of x and y is denoted by $min(x, y)$. Prove that

$$max(x, y) = \frac{x+y+|y-x|}{2},$$
$$min(x, y) = \frac{x+y-|y-x|}{2}.$$

Derive the formula for $max(x, y, z)$ and $min(x, y, z)$, using, for example

$$max(x, y, z) = max(x, max(y, z)).$$

Solution. Assume that $x \geq y$, we want to prove that $max(x, y) = x$.

$$max(x, y) = \frac{x + y + |y - x|}{2} = \frac{x + y + x - y}{2} = \frac{2x}{2} = x$$

Similarly, we need $min(x, y) = y$.

$$min(x, y) = \frac{x + y - |y - x|}{2} = \frac{x + y - (x - y)}{2} = \frac{x + y - x + y}{2} = \frac{2y}{2} = y$$

Let $max(x, y, z) = max(x, max(y, z))$. Then

$$\begin{aligned} max(x, y, z) &= \frac{x + max(y, z) + |max(y, z) - x|}{2} \\ &= \frac{x + \frac{y + z + |z - y|}{2} + \left|\frac{y + z + |z - y|}{2} - x\right|}{2} \\ &= \frac{2x + y + z + |z - y| + |y + z + |z - y| - 2x|}{4} \end{aligned}$$

Similarly,

$$min(x, y, z) = \frac{2x + y + z - |z - y| - |y + z - |z - y| - 2x|}{4}$$

■

Problem 1.14. (a) Prove that $|a| = |-a|$. (The trick is not to become confused by too many cases. First prove the statement for $a \geq 0$. Why is it then obvious for $a \leq 0$?)

(b) Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$. In particular, it follows that $-|a| \leq a \leq |a|$.

(c) Use this fact to give a new proof that $|a + b| \leq |a| + |b|$.

Solution. (a) Problem 1.12(i) easily tells us that

$$|-a| = |(-1)a| = |-1||a| = 1|a| = |a|$$

(b) If $a \geq 0$, then $a \leq b$. If $a \leq 0$, $-a \leq b$ follows from $a \geq -b$. Therefore, $|a| \leq b$. Conversely, suppose $|a| \leq b$. Then it is certain $a \leq b$ since $a \leq |a| \leq b$. From (a), $|-a| \leq b$, and hence $a \geq -b$. We conclude that $-b \leq a \leq b$. Note that since $|a| \leq |a|$, $-|a| \leq a \leq |a|$.

(c) Because we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, by Problem 1.5(i), we obtain $-(|a| + |b|) \leq a + b \leq |a| + |b|$. From (b), we arrive at the conclusion $|a + b| \leq |a| + |b|$.

■

Problem 1.15 (*)**.** Prove that if x and y are not both 0, then

$$\begin{aligned} x^2 + xy + y^2 &> 0 \\ x^4 + x^3y + x^2y^2 + xy^3 + y^4 &> 0 \end{aligned}$$

Hint: Use problem 1.

Solution. For the first part, note that

$$x^2 + xy + y^2 = x^2 + 2 \cdot x \cdot \frac{1}{2}y + \frac{1}{4}y^2 - \frac{1}{4}y^2 + y^2 = \left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 > 0$$

For the second part, if $x = y$, then the left-hand side is $5x^4 > 0$. Hence, suppose $x \neq y$. From Problem 1.1(v),

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \neq 0$$

If $x > y$, then $x^5 > y^5$ by Problem 1.6(b). This implies that the second term must be greater than 0. Conversely, $x < y \Rightarrow x^5 < y^5$ implies that it must be greater than 0. ■

Problem 1.16 (*)**.** (a) Show that

$$\begin{aligned} (x + y)^2 &= x^2 + y^2 \text{ only when } x = 0 \text{ or } y = 0, \\ (x + y)^3 &= x^3 + y^3 \text{ only when } x = 0 \text{ or } y = 0 \text{ or } x = -y. \end{aligned}$$

(b) Using the fact that

$$x^2 + 2xy + y^2 = (x + y)^2 \geq 0,$$

show that $4x^2 + 6xy + 4y^2 > 0$ unless x and y are both 0.

(c) Use part (b) to find out when $(x + y)^4 = x^4 + y^4$.

(d) Find out when $(x+y)^5 = x^5+y^5$. Hint: From the assumption $(x+y)^5 = x^5+y^5$ you should be able to derive the equation $x^3+2x^2y+2xy^2+y^3 = 0$, if $xy \neq 0$. This implies that $(x + y)^3 = x^2y + xy^2 = xy(x + y)$.

You should know be able to make a good guess as to when $(x + y)^n = x^n + y^n$; the proof is contained in Problem 11.57

Solution. (a) For the first part,

$$(x + y)^2 = x^2 + 2xy + y^2$$

Hence, $(x + y)^2 = x^2 + y^2$ only when $x = 0$ or $y = 0$. For the second part, from Problem 1.1(vi),

$$\begin{aligned} (x + y)^3 - (x + y)(x^2 - xy + y^2) &= 0 \\ (x + y)(xy) &= 0 \end{aligned}$$

which is true only when $x = 0$ or $y = 0$ or $x = -y$.

(b) Note that $4x^2 + 6xy + 4y^2 = \underbrace{3(x+y)^2}_{\geq 0} + \underbrace{x^2+y^2}_{>0} > 0$ unless $x = 0$ and $y = 0$.

(c) Let us expand $(x+y)^4$.

$$\begin{aligned}(x+y)^2(x+y)^2 &= (x^2+2xy+y^2)(x^2+2xy+y^2)\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\\ &= x^4 + y^4 + xy(4x^2+6xy+4y^2)\end{aligned}$$

Hence, $(x+y)^4 = x^4 + y^4$ only when $x = 0$ or $y = 0$, by part (b).

(d) Let us expand $(x+y)^5$.

$$\begin{aligned}(x+y)^4(x+y) &= x^5 + y^5 + xy(x+y)(4x^2+6xy+4y^2) + xy(x^3+y^3)\\ &= x^5 + y^5 + 5xy(x+y)(x^2-xy+y^2)\end{aligned}$$

If $xy \neq 0$ and $x + y \neq 0$, let $z = -y$, by Problem 1.6(b), $x^3 \neq z^3$. Hence, $x^2 - xy + y^2 \neq 0$. Therefore, $(x+y)^5 = x^5 + y^5$ only when $x = 0$ or $y = 0$ or $x = -y$.

Remark 1.1. Hence, for $(x+y)^n = x^n + y^n$, if n is even, then $x = 0$ or $y = 0$. If n is odd, then $x = 0$ or $y = 0$ or $x = -y$.

■

Problem 1.17. (a) Find the smallest possible value of $2x^2 - 3x + 4$. Hint: "Complete the square", i.e., write $2x^2 - 3x + 4 = 2(x - 3/4)^2 + ?$

(b) Find the smallest possible value of $x^2 - 3x + 2y^2 + 4y + 2$.

(c) Find the smallest possible value of $x^2 + 4xy + 5y^2 - 4x - 6y + 7$.

Solution. (a) Since $2x^2 - 3x + 4 = 2(x^2 - \frac{3}{2}x + 2)$,

$$2(x^2 - 2 \cdot x\frac{3}{4} + \frac{9}{16} - \frac{9}{16} + 2) = 2(x - \frac{3}{4})^2 + \frac{23}{8}$$

Hence the minimum value is $\frac{23}{8}$ when $x = \frac{3}{4}$.

(b)

$$x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2(y^2+2y+1) = \left(x - \frac{3}{2}\right)^2 + 2(y+1)^2 - \frac{9}{4}$$

The minimum value is $-\frac{9}{4}$ when $x = \frac{3}{2}$ and $y = -1$.

(c)

$$\begin{aligned}
&\frac{1}{2}x^2 + 4xy + 8y^2 - 3y^2 - 6y + 7 + \frac{1}{2}x^2 - 4x \\
=&\frac{1}{2}(x^2 + 8xy + 16y^2) - 3(y^2 + 2y + 1) + \frac{1}{2}(x^2 - 8x + 16) + 2 \\
=&\frac{1}{2}(x + 4y)^2 - 3(y + 1)^2 + \frac{1}{2}(x - 4)^2 + 2
\end{aligned}$$

Therefore, the minimum value is 2 when $x = 4$ and $y = -1$.

■

Problem 1.18. (a) Suppose that $b^2 - 4c \geq 0$. Show that the numbers

$$\frac{-b + \sqrt{b^2 - 4c}}{2}, \quad \frac{-b - \sqrt{b^2 - 4c}}{2}$$

both satisfy the equation $x^2 + bx + c = 0$.

(b) Suppose that $b^2 - 4c < 0$. Show that there are no numbers x satisfying $x^2 + bx + c = 0$; in fact, $x^2 + bx + c > 0$ for all x. Hint: Complete the square.

(c) Use this fact to give another proof that if x and y are not both 0, then $x^2 + xy + y^2 > 0$.

(d) For which number α is it true that $x^2 + \alpha xy + y^2 > 0$ whenever x and y are not both 0?

(e) Find the smallest possible value of $x^2 + bx + c$ and of $ax^2 + bx + c$, for $a > 0$.

Solution. (a) Substitution immediately gives the desired result.

(b)

$$x^2 + bx + c = x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c$$

which immediately yields $\left(x + \frac{b}{2}\right)^2 + \frac{[-(b^2 - 4c)]}{4} > 0$ for all x since $b^2 - 4c < 0$.

(c) If $y = 0$, $x^2 > 0$. Suppose not, using (b), we obtain $-3y^2 < 0$. Hence, $x^2 + xy + y^2 > 0$.

(d) If $y = 0$, the result follows for all α. Suppose $y \neq 0$, using (b), we obtain $\alpha^2 y^2 - 4y^2 < 0$, which is $y^2(\alpha^2 - 4) < 0$. It follows that $-2 < \alpha < 2$.

(e) From (b), it follows that the minimum value of x^2+bx+c is $\dfrac{[-(b^2-4c)]}{4}$ when $x=-b/2$. Since $a>0$, with the role of b is now b/a and of c is c/a, we easily derive the result.

$$x^2+\frac{b}{a}x+\frac{c}{a}=\left(x+\frac{b}{2a}\right)^2+\frac{[-(b^2-4ac)]}{4a^2}$$

So its minimum value is $\dfrac{[-(b^2-4ac)]}{4a^2}$ when $x=-\dfrac{b}{2a}$.

■

Problem 1.19. The fact that $a^2\geq 0$ for all numbers a, elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:

$$x_1y_1+x_2y_2\leq\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}.$$

(A more general form occurs in Problem 2.21) The three proofs of the Schwarz inequality outlined below have only one thing in common–their reliance on the fact that $a^2\geq 0$ for all a.

(a) Prove that if $x_1=\lambda y_1$ and $x_2=\lambda y_2$ for some number λ, then equality holds in Schwarz inequality. Prove the same thing if $y_1=y_2=0$. Now suppose that y_1 and y_2 are not both 0, and that there is no number λ such that $x_1=\lambda y_1$ and $x_2=\lambda y_2$. Then

$$\begin{aligned}0&<(\lambda y_1-x_1)^2+(\lambda y_2-x_2)^2\\&=\lambda^2(y_1^2+y_2^2)-2\lambda(x_1y_1+x_2y_2)+(x_1^2+x_2^2).\end{aligned}$$

Using Problem 1.18, complete the proof of the Schwarz inequality.

(b) Prove the Schwarz inequality by using $2xy\leq x^2+y^2$ (how is this derived?) with

$$x=\frac{x_i}{\sqrt{x_1^2+x_2^2}},\quad y=\frac{y_i}{\sqrt{y_1^2+y_2^2}},$$

first for $i=1$ and then for $i=2$.

(c) Prove the Schwarz inequality by first proving that

$$(x_1^2+x_2^2)(y_1^2+y_2^2)=(x_1y_1+x_2y_2)^2+(x_1y_2-x_2y_1)^2.$$

(d) Deduce, from each of these three proofs, that equality holds only when $y_1=y_2=0$ or when there is a number λ such that $x_1=\lambda y_1$ and $x_2=\lambda y_2$.

Solution. (a) If $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for every $\lambda \geq 0$,

$$\begin{aligned}\lambda(y_1^2 + y_2^2) &= |\lambda|\sqrt{(y_1^2 + y_2^2)^2} \\ &= \lambda(y_1^2 + y_2^2)\end{aligned}$$

Or if $y_1 = y_2 = 0$, then equality holds since both sides are 0. Otherwise, suppose that y_1 and y_2 are not both 0, and there is no number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$, then

$$\begin{aligned}0 &< \lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2) \\ &= \lambda^2 - 2\lambda\frac{x_1y_1 + x_2y_2}{y_1^2 + y_2^2} + \frac{x_1^2 + x_2^2}{y_1^2 + y_2^2}\end{aligned}$$

This holds only when, by Problem 1.18(b),

$$\frac{4(x_1y_1 + x_2y_2)^2}{(y_1^2 + y_2^2)^2} + \frac{[-4(x_1^2 + x_2^2)(y_1^2 + y_2^2)]}{(y_1^2 + y_2^2)^2} < 0$$

which only holds when

$$x_1y_1 + x_2y_2 < \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

since $a \leq |a|$ for all a.

(b) Note that $(x - y)^2 \geq 0$. For $i = 1$,

$$\frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2} \geq 2 \cdot \frac{x_1y_1}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \tag{1.3}$$

For $i = 2$,

$$\frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2} \geq 2 \cdot \frac{x_2y_2}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \tag{1.4}$$

(1.3)+(1.4), we derive

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

(c)

$$\begin{aligned}&(x_1^2 + x_2^2)(y_1^2 + y_2^2) \\ =&(x_1^2y_1^2 + 2x_1y_1x_2y_2 + x_2^2y_2^2) + (x_1^2y_2^2 - 2x_1y_2x_2y_1 + x_2^2y_1^2) \\ =&(x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2\end{aligned}$$

Note that $(x_1y_2 - x_2y_1)^2 \geq 0$. Hence,

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

since $a \leq |a|$ for all a.

(d) In (a), it is obvious; the proof is based on the separation of two cases, $a^2 = 0$ and $a^2 > 0$. In (b), equality occurs only when $x = y$; by construction, $y_1 = y_2 = 0$ or, if not,

$$\frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{y_1}{\sqrt{y_1^2 + y_2^2}}$$
$$\frac{x_2}{\sqrt{x_1^2 + x_2^2}} = \frac{y_2}{\sqrt{y_1^2 + y_2^2}}$$

implies that for

$$\lambda = \frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{y_1^2 + y_2^2}}$$

$x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

In (c), equality occurs only when $(x_1y_2 - x_2y_1)^2 = 0$ and $x_1y_1 + x_2y_2 \geq 0$. These will be satisfied only when $y_1 = y_2 = 0$ or for $\lambda \geq 0$, $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

■

Problem 1.20. Prove that if

$$|x - x_0| < \frac{\epsilon}{2} \quad \text{and} \quad |y - y_0| < \frac{\epsilon}{2},$$

then

$$|(x + y) - (x_0 + y_0)| < \epsilon,$$
$$|(x - y) - (x_0 - y_0)| < \epsilon$$

Solution. This problem mainly uses the results from Problem 1.12. For the first inequality, note that $|(x + y) - (x_0 + y_0)| = |(x - x_0) + (y - y_0)|$, and

$$\begin{aligned} |(x - x_0) + (y - y_0)| &\leq |x - x_0| + |y - y_0| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

For the second inequality, we rewrite $|(x-y)-(x_0-y_0)| = |(x-x_0)-(y-y_0)|$, then

$$\begin{aligned} |(x - x_0) - (y - y_0)| &\leq |x - x_0| + |y - y_0| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

■

Problem 1.21 (*)**.** Prove that if

$$|x - x_0| < \min\left(\frac{\epsilon}{2(|y_0| + 1)}, 1\right) \quad \text{and} \quad |y - y_0| < \frac{\epsilon}{2(|x_0| + 1)},$$

then $|xy - x_0y_0| < \epsilon$.

Solution. We want to utilize the more diverse cases of inequality expression in term of x, therefore we rewrite $|xy - x_0y_0| = |xy - xy_0 + xy_0 - x_0y_0|$. Hence,

$$\begin{aligned} |xy - xy_0 + xy_0 - x_0y_0| &\leq |x||y - y_0| + |y_0||x - x_0| \\ &< (|x_0| + 1)\frac{\epsilon}{2(|x_0| + 1)} + |y_0|\frac{\epsilon}{2(|y_0| + 1)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

where the first strict inequality follows from $|x| - |x_0| \leq |x - x_0|$ (Problem 1.12), and the second strict inequality comes from the fact $\dfrac{|y_0|}{|y_0| + 1} < 1$. ■

Problem 1.22 (*)**.** Prove that if $y_0 \neq 0$ and

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{2}\right),$$

then $y \neq 0$ and

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \epsilon.$$

Solution. Note that the assumption implies $|y| > \dfrac{|y_0|}{2} > 0$, which further implies $\dfrac{1}{|y|} < \dfrac{2}{|y_0|}$; therefore, it must be that $y \neq 0$. Note $\left|\dfrac{1}{y} - \dfrac{1}{y_0}\right| = \left|\dfrac{y - y_0}{yy_0}\right|$, and from that

$$\begin{aligned} \left|\frac{y - y_0}{yy_0}\right| &= |y - y_0|\left|\frac{1}{yy_0}\right| \\ &< \frac{\epsilon|y_0|^2}{2}\frac{2}{|y_0|^2} \\ &= \epsilon \end{aligned}$$

■

Problem 1.23 (*)**.** Replace the question marks in the following statement by expressions involving ϵ, x_0, and y_0 so that the conclusion will be true:

If $y_0 \neq 0$ and

$$|y - y_0| < ? \qquad \text{and} \qquad |x - x_0| < ?$$

then $y \neq 0$ and

$$\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \epsilon.$$

This problem is trivial in the sense that its solution follows from Problem 1.21 and Problem 1.22 with almost no work at all (notice that $x/y = x \cdot 1/y)$). The crucial point is not to become confused; decide which of the two problems should be used first, and don't panic if your answer looks unlikely.

Solution. An observation at both suggested related problems reveals

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{4(|x_0|+1)}\right)$$
$$|x - x_0| < \min\left(\frac{\epsilon|y_0|}{2(|y_0|+1)}, 1\right)$$

Since $y_0 \neq 0$, we easily obtain $y \neq 0$ by Problem 1.22. For the latter part of the proof, notice that

$$\begin{aligned}\left|\frac{x}{y} - \frac{x_0}{y_0}\right| &= \left|x(\frac{1}{y} - \frac{1}{y_0}) + \frac{1}{y_0}(x - x_0)\right| \\ &\leq |x|\left|\frac{1}{y} - \frac{1}{y_0}\right| + \frac{1}{|y_0|}|x - x_0| \\ &< (|x_0|+1)\frac{\epsilon}{2(|x_0|+1)} + \frac{1}{|y_0|}\frac{\epsilon|y_0|}{2(|y_0|+1)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

■

Problem 1.24 (*). This problem shows that the actual placement of parentheses in a sum is irrelevant. The proof involve "mathematical induction"; if you are not familiar with such proofs, but still want to tackle this problem, it can be saved until after Chapter 2, where proofs by induction are explained.

Let us agree, for definiteness, that $a_1 + \cdots + a_n$ will denote

$$a_1 + (a_2 + (a_3 + \cdots + (a_{n-2} + (a_{n-1} + a_n)))\dots)$$

Thus $a_1 + a_2 + a_3$ denotes $a_1 + (a_2 + a_3)$, and $a_1 + a_2 + a_3 + a_4$ denotes $a_1 + (a_2 + (a_3 + a_4))$, etc.

(a) Prove that
$$(a_1 + \cdots + a_k) + a_{k+1} = a_1 + \cdots + a_{k+1}.$$
Hint: Use induction on k.

(b) Prove that if $n \geq k$, then
$$(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) = a_1 + \cdots + a_n.$$
Hint: Use part (a) to give proof by induction on k.

(c) Let $s(a_1, \dots, a_k)$ be some sum formed from $a_1, \dots, a_k$. Show that
$$s(a_1, \dots, a_k) = a_1 + \cdots + a_k$$
Hint: There must be two sums $s'(a_1, \dots, a_l)$ and $s''(a_{l+1}, \dots, a_k)$ such that
$$s(a_1, \dots, a_k) = s'(a_1, \dots, a_l) + s''(a_{l+1}, \dots, a_k).$$

Solution. (a) If $k = 1$, there is nothing to prove. Suppose the equation holds for $k = l$, then for $k = l + 1$,

$$\begin{aligned}
& (a_1 + \cdots + a_{l+1}) + a_{l+2} \\
= & ((a_1 + \cdots + a_l) + a_{l+1}) + a_{l+2} \\
= & a_1 + \cdots + a_l + (a_{l+1} + a_{l+2}) \qquad \text{(since } (a+b)+c = a+(b+c)\text{)} \\
= & a_1 + (a_2 + \cdots + (a_{l-1} + (a_l + (a_{l+1} + a_{l+2}))) \dots) \\
= & a_1 + \cdots + a_{l+1} + a_{l+2}
\end{aligned}$$

The proof is complete.

(b) If $n = k$, there is nothing to prove. Suppose the equation holds for $n \geq k$, then for $n \geq k + 1$,

$$\begin{aligned}
& (a_1 + \cdots + a_{k+1}) + (a_{k+2} + \cdots + a_n) \\
= & ((a_1 + \cdots + a_k) + a_{k+1}) + (a_{k+2} + \cdots + a_n) \quad \text{(by (a))} \\
= & (a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) \\
= & a_1 + \cdots + a_n \qquad \text{(by assumption)}
\end{aligned}$$

The proof is complete.

(c) By (b), there exists a number l such that

$$s'(a_1, \dots, a_l) + s''(a_{l+1}, \dots, a_n) = a_1 + \cdots + a_n$$

Hence, $s(a_1, \dots, a_n) = a_1 + \cdots + a_n$.

■

Problem 1.25. Suppose that we interpret "number" to mean either 0 or 1, and $+$ and $\cdot$ to be the operations defined by the following two tables.

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Check that properties P1-P9 all hold, even though $1 + 1 = 0$.

Solution. It is easy to check! ■

Chapter 2

Numbers of Various Sorts

Problem 2.1. Prove the following formula by induction.

(i) $1^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

(ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$.

Solution. (i) If $n = 1$, the equation holds. Suppose the equation holds for $n = k$, then for $n = k + 1$,

$$\begin{aligned}
& 1^2 + \cdots + k^2 + (k+1)^2 \\
= & \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
= & \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
= & \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
= & \frac{(k+1)(2k^2 + 4k + 3k + 6)}{6} \\
= & \frac{(k+1)[2k(k+2) + 3(k+2)]}{6} \\
= & \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}
\end{aligned}$$

Then the formula holds for every n.

(ii) If $n = 1$, there is nothing to prove. If $n = k$ holds for the equation, then for $n = k + 1$,

$$\begin{aligned}
& [1 + \cdots + k + (k+1)]^2 \\
= & (1 + \cdots + k)^2 + (k+1)^2 + 2(k+1)(1 + \cdots + k) \\
= & 1^3 + \cdots + k^3 + (k+1)[(k+1) + 2\frac{k(k+1)}{2}] \\
= & 1^3 + \cdots + k^3 + (k+1)^3
\end{aligned}$$

This finishes the proof for every n.

■

Problem 2.2. Find a formula for

(i) $\sum_{i=1}^{n}(2i-1) = 1+3+5+\cdots+(2n-1)$.

(ii) $\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2$.

Hint: What do these expressions have to do with $1+2+3+\cdots+2n$ and $1^2+2^2+3^2+\cdots+(2n)^2$?

Solution. (i) Remind that $1+\cdots+2n = n(2n+1)$ and that $2+\cdots+2n = 2\cdot\dfrac{n(n+1)}{2} = n(n+1)$. Hence,

$$\sum_{i=1}^{n}(2i-1) = n(2n+1) - n(n+1) = n^2$$

(ii) Using Problem 2.1(i), we easily derive that $\sum_{i=1}^{2n} i^2 = \dfrac{n(2n+1)(4n+1)}{3}$ and $\sum_{i=1}^{n}(2i)^2 = \dfrac{2n(n+1)(2n+1)}{3}$. Therefore,

$$\sum_{i=1}^{n}(2i-1)^2 = \sum_{i=1}^{2n} i^2 - \sum_{i=1}^{n}(2i)^2 = \frac{n(4n^2-1)}{3}$$

■

Problem 2.3. If $0 \le k \le n$, the "binomial coefficient" $\binom{n}{k}$ is defined by
$\binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{n(n-1)\cdots(n-k+1)}{k!}$, if $k \neq 0, n$
$\binom{n}{0} = \binom{n}{n} = 1$. (This becomes a special case of the first formula if we define $0! = 1$.)

(a) Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

(The proof does not require an induction argument.)

This relation gives rise to the following configuration, known as "Pascal's triangle"—a number not on one of the sides is the sum of two numbers above it; the binomial coefficient $\binom{n}{k}$ is the $(k+1)$st number in the $(n+1)$st row.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

(b) Notice that all numbers in Pascal's triangle are natural numbers. Use part (a) to prove by induction that $\binom{n}{k}$ is always a natural number. (Your entire proof by induction will, in a sense, be summed up in a glance by Pascal's triangle.)

(c) Give another proof that $\binom{n}{k}$ is a natural number by showing that $\binom{n}{k}$ is the number of sets of exactly k integers each chosen from $1, \dots, n$.

(d) Prove the "binomial theorem": If a and b are any numbers and n is a natural number, then

$$
\begin{aligned}
(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n \\
&= \sum_{j=0}^{n} \binom{n}{j} a^{n-j}b^j.
\end{aligned}
$$

(e) Prove that

(i) $\displaystyle\sum_{j=0}^{n} \binom{n}{j} = \binom{n}{0} + \cdots + \binom{n}{n} = 2^n.$

(ii) $\displaystyle\sum_{j=0}^{n} (-1)^j \binom{n}{j} = \binom{n}{0} - \binom{n}{1} + \cdots \pm \binom{n}{n} = 0.$

(iii) $\displaystyle\sum_{l \text{ odd}} \binom{n}{l} = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}$.

(iv) $\displaystyle\sum_{l \text{ even}} \binom{n}{l} = \binom{n}{0} + \binom{n}{2} + \cdots = 2^{n-1}$.

Solution. (a) Starting from the left-hand side,

$$\begin{aligned}\binom{n+1}{k} &= \frac{n!(n-k+1+k)}{k!(n-k+1)!} \\ &= \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!k}{k!(n-k+1)!} \\ &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)![n-(k-1)]!} \\ &= \binom{n}{k} + \binom{n}{k-1}\end{aligned}$$

(b) It is sufficient to prove that $\binom{n}{k}$ is a natural number for all $1 \le k \le (n-1)$. If $n = 1$, then

$$\binom{2}{1} = \binom{1}{0} + \binom{1}{1} = 2$$

Suppose that $\binom{n}{k}$ is natural number for any n and $1 \le k \le n-1$. Then for any $1 \le k \le n$, $\binom{n+1}{k}$ is the sum of two natural numbers, and therefore it must be a natural number.

(c) It is sufficient to prove for the case $0 < k \le n$. If $n = 1$, the claim is trivial. Suppose that $\binom{n}{k}$ is the number of sets of k integers each chosen from $1, \ldots, n$; then $\binom{n+1}{k}$ must include $\binom{n}{k}$ sets of k integers *without* the newly added element and a number of sets of k integers *with* the newly added element. The latter is exactly $\binom{n}{k-1}$ and is a natural number by assumption. Thereby, $\binom{n+1}{k}$ must be a natural number.

(d) We prove by induction on n. If $n = 1$, there is nothing to prove. Suppose the binomial theorem holds for n; then for $n+1$,

$$\begin{aligned}
(a+b)^n(a+b) &= \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j (a+b) \\
&= \sum_{j=0}^{n} \binom{n}{j} a^{n+1-j} b^j + \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^{j+1} \\
&= a^{n+1} + b^{n+1} + \sum_{j=1}^{n} \binom{n}{j} a^{n+1-j} b^j \\
&+ \sum_{j=1}^{n} \binom{n}{j-1} a^{n+1-j} b^j \\
&= \sum_{j=1}^{n} \binom{n+1}{j} a^{n+1-j} b^j + a^{n+1} + b^{n+1} \\
&= \sum_{j=0}^{n+1} \binom{n+1}{j} a^{n+1-j} b^j
\end{aligned}$$

which completes the proof.

(e) This part relies heavily on the binomial theorem from the above.

(i) This is directly from the above: Applying the binomial theorem for $a = b = 1$ yields the result.

(ii) Let $a = 1$ and $b = -1$ yield the result.

(iii) Applying (i) + (ii), we derive that for l even,

$$\sum_{l \text{ even}} \binom{n}{l} = 2^{n-1}$$

Thereby, $\sum_{l \text{ odd}} \binom{n}{l} = 2^n - 2^{n-1} = 2^{n-1}$.

(iv) See the above.

One thing to note: Both of the previous parts do not have a final term expressed in their sum due to the dependence of value n (if n is even or odd).

■

Problem 2.4. (a) Prove that

$$\sum_{k=0}^{l} \binom{n}{k} \binom{m}{l-k} = \binom{n+m}{l}.$$

Hint: Apply the binomial theorem to $(1+x)^n(1+x)^m$.

(b) Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Solution. (a) Remind that $(1+x)^n(1+x)^m = (1+x)^{n+m}$. Hence,

$$\sum_{k=0}^{n} \binom{n}{k} x^k \cdot \sum_{j=0}^{m} \binom{m}{j} x^j = \sum_{l=0}^{n+m} \binom{n+m}{l} x^l$$

Observing that each term of x^l is

$$\sum_{k=0}^{l} \binom{n}{k} \binom{m}{l-k}$$

for every k and j such that $j = l - k$.

(b) From (a), let $m = l = n$ and observe that for all $0 \le k \le n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

since $\dfrac{n!}{k!(n-k)!} = \dfrac{n!}{(n-k)!k!}$. Hence, $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

■

Problem 2.5. (a) Prove by induction on n that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

if $r \neq 1$ (if $r = 1$, evaluating the sum certainly presents no problem).

(b) Derive this result by setting $S = 1 + r + \cdots + r^n$, multiplying this equation by r, and solving the two equations for S.

Solution. (a) If $n = 1$, then the formula immediately holds. Suppose now that it holds for n, then for $n + 1$,

$$\begin{aligned} 1 + r + \cdots + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\ &= \frac{1 - r^{n+2}}{1 - r} \end{aligned}$$

which completes the proof.

(b) Let

$$S = 1 + r + r^2 + \cdots + r^n$$

then

$$rS = r + r^2 + r^3 + \cdots + r^{n+1}$$

Solving for S, we easily obtain

$$S = \frac{1 - r^{n+1}}{1 - r}$$

■

Problem 2.6. The formula for $1^2 + \cdots + n^2$ may be derived as follows. We begin with the formula

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1.$$

Writing this formula for $k = 1, \ldots, n$ and adding, we obtain

$$\begin{aligned} 2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\ 3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1 \\ &\vdots \\ (n+1)^3 - n^3 &= 3 \cdot n^2 + 3 \cdot n + 1 \end{aligned}$$

$$\overline{(n+1)^3 - 1 = 3[1^2 + \cdots + n^2] + 3[1 + \cdots + n] + n.}$$

Thus we can find $\sum_{k=1}^{n} k^2$ if we already know $\sum_{k=1}^{n} k$ (which could have been found in a similar way). Use this method to find

(i) $1^3 + \cdots + n^3$.

(ii) $1^4 + \cdots + n^4$.

(iii) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)}$.

(iv) $\dfrac{3}{1^2 \cdot 2^2} + \dfrac{5}{2^2 \cdot 3^2} + \cdots + \dfrac{2n+1}{n^2(n+1)^2}$.

Solution. (i) We easily derive that

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$$

Hence,

$$\begin{aligned}
2^4 - 1^4 &= 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1 \\
3^4 - 2^4 &= 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1 \\
&\vdots \\
(n+1)^4 - n^4 &= 4 \cdot n^3 + 6 \cdot n^2 + 4 \cdot n + 1
\end{aligned}$$

$$\begin{aligned}
(n+1)^4 - 1 = 4[1^3 + 2^3 + \cdots + n^3] \\
+ 6[1^2 + 2^2 + \cdots + n^2] + 4[1 + 2 + \cdots + n] + n
\end{aligned}$$

It is easily derivable that

$$1^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

(ii) The easiest way to calculate $(k+1)^5 - k^5$ is to use the Pascal triangle from page 31: We derive the result

$$(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1$$

A similar step from above shows that

$$\begin{aligned}
(n+1)^5 - 1 = 5[1^4 + \cdots + n^4] + 10[1^3 + \cdots + n^3] \\
+ 10[1^2 + \cdots + n^2] + 5[1 + \cdots + n] + n
\end{aligned}$$

We derive from the available results,

$$1^4 + 2^4 + \cdots + n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1)$$

(iii) Observe that

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$$

Henceforth,

$$\begin{aligned}
1 - \frac{1}{2} &= \frac{1}{1 \cdot 2} \\
\frac{1}{2} - \frac{1}{3} &= \frac{1}{2 \cdot 3} \\
&\vdots \\
\frac{1}{n} - \frac{1}{n+1} &= \frac{1}{n \cdot (n+1)}
\end{aligned}$$

$$1 - \frac{1}{n+1} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

(iv) See that

$$\frac{1}{k^2} - \frac{1}{(k+1)^2} = \frac{2k+1}{k^2(k+1)^2}$$

Henceforth,

$$\begin{array}{r}
1 - \dfrac{1}{2^2} = \dfrac{3}{1^2 \cdot 2^2} \\
\dfrac{1}{2^2} - \dfrac{1}{3^2} = \dfrac{5}{2^2 \cdot 3^2} \\
\vdots \\
\dfrac{1}{n^2} - \dfrac{1}{(n+1)^2} = \dfrac{2n+1}{n^2(n+1)^2} \\
\hline
\dfrac{3}{1^2 \cdot 2^2} + \dfrac{5}{2^2 \cdot 3^2} + \cdots + \dfrac{2n+1}{n^2(n+1)^2} = \dfrac{n(n+2)}{(n+1)^2}
\end{array}$$

■

Problem 2.7. Use the method of Problem 2.6 to show that $\sum_{k=1}^{n} k^p$ can always be written in the form

$$\frac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + Cn^{p-2} + \cdots.$$

(The first 10 such expressions are

$$\begin{aligned}
\sum_{k=1}^{n} k &= \frac{1}{2}n^2 + \frac{1}{2}n \\
\sum_{k=1}^{n} k^2 &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\
\sum_{k=1}^{n} k^3 &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
\sum_{k=1}^{n} k^4 &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\
\sum_{k=1}^{n} k^5 &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \\
\sum_{k=1}^{n} k^6 &= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n \\
\sum_{k=1}^{n} k^7 &= \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2 \\
\sum_{k=1}^{n} k^8 &= \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n \\
\sum_{k=1}^{n} k^9 &= \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2 \\
\sum_{k=1}^{n} k^{10} &= \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1n^7 + 1n^5 - \frac{1}{2}n^3 + \frac{5}{66}n.
\end{aligned}$$

Notice that the coefficients in the second column are always $\frac{1}{2}$, and that after the third column the powers of n with nonzero coefficients decrease by 2 until n or n^2 is reached. The coefficients in all but the first two columns seem to be rather haphazard, but there is actually is some sort of pattern; finding it may be regarded as a super-perspicacity test. See Problem 27.17 for the complete story.)

Solution. We prove by complete induction on p. Know that

$$(k+1)^{p+2} - k^{p+2} = \sum_{j=2}^{p} \binom{p+2}{j} k^j + (p+2)k^{p+2} + (p+2)k + 1$$

If $p = 1$, it is obvious. Suppose the expression holds for $1, \ldots, p$, then for $p+1$, using the known method presented above, we easily obtain

$$(n+1)^{p+2} - 1 = \sum_{j=2}^{p} \binom{p+2}{j} \sum_{k=1}^{n} k^j + (p+2)\sum_{k=1}^{n} k^{p+1} + (p+2)\sum_{k=1}^{n} k + n$$

Notice that the left-hand side highest power of n is $p+2$ while that on the right is $p+1$ by assumption. Dividing both sides by $p+2$ and solve for $\sum_{k=1}^{n} k^{p+1}$, we have

$$\sum_{k=1}^{n} k^{p+1} = \frac{n^{p+2}}{p+2} + An^{p+1} + Bn^{p} + \cdots$$

for some number A, B. ■

Problem 2.8. Prove that every natural number is either even or odd.

Solution. Suppose that B is the set of all natural numbers that is neither even nor odd. Suppose that $B \neq \emptyset$. Obviously, $1 \notin B$ since 1 is an odd number. If $k \notin B$, then k is either even or odd. Then $k+1$ is either odd or even, and therefore $(k+1) \notin B$. Henceforth, $B = \emptyset$, contradicting assumption. ■

Problem 2.9. Prove that if a set A of natural numbers contains n_0 and contains $k+1$ whenever it contains k, then A contains all natural numbers $\geq n_0$.

Solution. Obivously, $n_0 \in A$. Suppose that $(n_0 + k - 1) \in A$. By assumption, $(n_0 + k) \in A$. Hence, A contains all natural numbers $\geq n_0$. ■

Problem 2.10. Prove the principle of mathematical induction from the well-ordering principle.

Solution. We shall prove the theorem by contradiction,

Theorem 2.1 (Principle of mathematical induction)**.** *If A is the set of natural numbers and*

(1) 1 is in A

(2) $k+1$ is in A wherever k is in A,

then A is the set of all natural numbers.

Let $B \neq \emptyset$ be the set of all natural numbers *not* in A. By the well-ordering principle, B has a least member k. By construction, $k \notin A$. By property (2), $(k-1) \notin A$, which means that $(k-1) \in B$, but this contradicts the fact that k is the least member in B. ■

Problem 2.11. Prove the principle of complete induction from the ordinary principle of induction. Hint: If A contains 1 and A contains $n+1$ whenever it contains $1, \ldots, n$, consider the set B of all k such that $1, \ldots, k$ are all in A.

Solution. We want to prove the following theorem,

Theorem 2.2 (Principle of complete induction)**.** *If A is the set of natural numbers and*

(1) 1 *is in* A

(2) $n+1$ *is in* A *if* $1, \dots, n$ *is in* A

then A *is the set of all natural numbers.*

Let B be the set of all k such that $1, 2, 3, \dots, k$ are all in A. Obviously, $B \subseteq A$. Conversely, $A \subseteq B$ because if not, then $1, \dots, k$ are not all in A which implies $1 \notin A$. Hence, $A = B \neq \emptyset$ since $1 \in A$. We see that B satisfies:

(1) 1 is in B

(2) $k+1$ is in B whenever k is in B. Otherwise, $(k+1) \notin A$, which implies $1, \dots, k$ are not in A, which implies $k \notin B$.

By principle of mathematical induction, B is the set of all natural numbers, which means A is the same. ■

Problem 2.12. (a) If a is rational and b is irrational, is $a+b$ necessarily irrational? What if a and b are both irrational?

(b) If a is rational and b is irrational, is ab necessarily irrational? (Careful!)

(c) Is there a number a such that a^2 is irrational, but a^4 is rational?

(d) Are there two irrational numbers whose sum and product are both rational?

Solution. (a) If $a+b$ were rational then $b = (a+b) - a$ would be rational! Hence, $a+b$ is necessarily irrational. However, if a and b are irrational, then let $b = r - a$ for any rational r, then $a+b$ is rational.

(b) ab is not necessarily irrational for the case when $a = 0$ and $b = \sqrt{2}$. However, if $a \neq 0$, then if ab were rational, then $\dfrac{ab}{a} = b$ would be rational. Hence, ab must be irrational.

(c) Let $a = \sqrt{\sqrt{2}}$: $a^2 = \sqrt{2}$, but $a^4 = 2$.

(d) Let $a = -\sqrt{2}$ and $b = \sqrt{2}$. Then both $a+b$ and ab are rational even though a and b are irrational.

■

Problem 2.13. (a) Prove that $\sqrt{3}, \sqrt{5}$, and $\sqrt{6}$ are irrational. Hint: To treat $\sqrt{3}$, for example, use the fact that every integer is of the form $3n$ or $3n+1$ or $3n+2$. Why does this proof not work for $\sqrt{4}$?

(b) Prove that $\sqrt[3]{2}$ and $\sqrt[3]{3}$ are irrational.

Solution. (a) Since every number is written in the form of either $3n$, $3n+1$ or $3n+2$ for some natural n, then

$$\begin{aligned}(3n+1)^2 &= 9n^2+6n+1 &&= 3(3n^2+2n)+1\\ (3n+2)^2 &= 9n^2+12n+4 &&= 3(3n^2+4n+1)+1\end{aligned}$$

This implies that if a squared integer is divisible by 3, then the integer is divisible by 3. Suppose now that $\sqrt{3}$ were rational. Then there would be a pair of integer p and q with no common divisor such that

$$\sqrt{3} = \frac{p}{q}$$

This implies that $p^2 = 3q^2$. Therefore, there exist a natural k such that $p = 3k$, which implies

$$p^2 = 9k^2 = 3q^2$$

which means that $q^2 = 3k^2$. Henceforth, $q = 3m$ for some natural m, but these imply that p and q have a common divisor: A contradiction. Similar for $\sqrt{5}$, we consider

$$\begin{aligned}(5n+1)^2 &= 25n^2+10n+1 &&= 5(5n^2+2n)+1\\ (5n+2)^2 &= 25n^2+20n+4 &&= 5(5n^2+4n)+4\\ (5n+3)^2 &= 25n^2+30n+9 &&= 5(5n^2+6n+1)+4\\ (5n+4)^2 &= 25n^2+40n+16 &&= 5(5n^2+8n+3)+1\end{aligned}$$

We see that the very same method of proof is applicable for $\sqrt{5}$ and $\sqrt{6}$: If k^2 is divisible by either 5 or 6, then k must be divisible by 5 or 6, respectively. Hence, we easily conclude that $\sqrt{5}$ and $\sqrt{6}$ are irrational.

Now we cannot use this method of proof for $\sqrt{4}$ since the statement *if k^2 is divisible by 4, then k is divisible by 4* is false by letting $k = 2$.

(b) Let us first consider

$$(2n+1)^3 = 8n^3+12n^2+6n+1 = 2(4n^3+6n^2+3n)+1$$

We conclude that if k^3 is divisible by 2, then k is divisible by 2 for any natural k. Suppose $\sqrt[3]{2}$ were rational, then there would be integers p and q with no common divisor such that

$$\sqrt[3]{2} = \frac{p}{q}$$

then $p^3 = 2q^3$: p is divisible by 2, and hence $4k^3 = q^3$ for some k: q is divisible by 2: A contradiction since both have common divisor. We conclude that $\sqrt[3]{2}$ is irrational. Similarly,

$$\begin{aligned}(3n+1)^3 &= 27n^3+27n^2+9n+1 &&= 3(9n^3+9n^2+3n)+1\\ (3n+2)^3 &= 27n^3+54n^2+36n^2+8 &&= 3(9n^3+18n^2+12n^2+2)+2\end{aligned}$$

Hence, for any natural k if k^3 is divisible by 3, then k is divisible by 3. The very same method as above is used to prove that $\sqrt[3]{3}$ is irrational.

■

Problem 2.14. Prove that

(a) $\sqrt{2} + \sqrt{6}$ is irrational.

(b) $\sqrt{2} + \sqrt{3}$ is irrational.

Solution. (a) Suppose $\sqrt{2}+\sqrt{6}$ is rational. Then $(\sqrt{2}+\sqrt{6})^2 = 8+4\sqrt{3}$ must be rational. Hence, $4\sqrt{3} = 8+4\sqrt{3}-8$ must be rational: A contradiction by since $4 \neq 0$ and $\sqrt{3}$ is irrational.

(b) Suppose that $\sqrt{2} + \sqrt{3}$ is rational. Then $5 + 2\sqrt{6}$ is rational, which implies $2\sqrt{6} = 5 + 2\sqrt{6} - 5$ is rational: A contradiction similar to the above.

■

Problem 2.15. (a) Prove that if $x = p + \sqrt{q}$ where p and q are rational, and m is a natural number, then $x^m = a + b\sqrt{q}$ for some rational a and b.

(b) Prove also that $(p - \sqrt{q})^m = a - b\sqrt{q}$.

Solution. (a) Proof is by induction. Let $m = 1$, the statement holds. Suppose the statement holds for m, then for $m + 1$,

$$\begin{aligned} x^{m+1} &= (a + b\sqrt{q})(p + \sqrt{q}) \\ &= (ap + bq) + (a + bp)\sqrt{q} \end{aligned}$$

The conclusion follows.

(b) The proof is exactly the same as above except now that $x = p - \sqrt{q}$. Hence, for $m + 1$,

$$\begin{aligned} x^{m+1} &= (a - b\sqrt{q})(p - \sqrt{q}) \\ &= (ap + bq) - (a + bp)\sqrt{q} \end{aligned}$$

■

Problem 2.16. (a) Prove that if m and n are natural numbers and $\dfrac{m^2}{n^2} < 2$, then $\dfrac{(m + 2n)^2}{(m + n)^2} > 2$; show, moreover, that

$$\frac{(m + 2n)^2}{(m + n)^2} - 2 < 2 - \frac{m^2}{n^2}$$

(b) Prove the same results with all inequality signs reversed.

(c) Prove that if $\frac{m}{n} < \sqrt{2}$, then there is another rational number $\frac{m'}{n'}$ with $\frac{m}{n} < \frac{m'}{n'} < \sqrt{2}$.

Solution. (a) We first observe from $m^2/n^2 < 2$,

$$(m+n)^2 < 3n^2 + 2nm$$

And from here,

$$\begin{aligned}\frac{(m+2n)^2}{(m+n)^2} &= \frac{(m+n)^2 + n^2 + 2n(m+n)}{(m+n)^2}\\ &= 1 + \frac{3n^2+2nm}{(n+m)^2}\\ &> 1+1\\ &= 2\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{(m+2n)^2}{(m+n)^2} - 2 &= \frac{-(m+n)^2 + n^2 + 2n(m+n)}{(m+n)^2}\\ &= \frac{2n^2-m^2}{(m+n)^2}\\ &< \frac{2n^2-m^2}{n^2}\\ &= 2 - \frac{m^2}{n^2}\end{aligned}$$

(b) Applying the exact method from the above, we see that if $m^2/n^2 > 2$, then $\frac{(m+2n)^2}{(m+n)^2} < 2$. As for the last inequality, note from the third line above: since $2n^2 - m^2 < 0$, the inequality sign is reversed. Therefore, $\frac{(m+2n)^2}{(m+n)^2} - 2 > 2 - \frac{m^2}{n^2}$.

(c) Let $m' = 2n^2 - m^2$ and $n' = (m+n)^2$, we obtain

$$\frac{m^2}{n^2} < \frac{m'^2}{n'^2} < 2$$

Hence, $\frac{m}{n} < \frac{m'}{n'} < \sqrt{2}$.

■

Problem 2.17 (*)**.** It seems likely that $\sqrt{n}$ is irrational whenever the natural number n is not the square of another natural number. Although the method of Problem 2.13 may actually be used to treat any particular case, it is not clear in advance that it will always work, and a proof for the general case requires some extra information. A natural number p is called **prime number** if it is impossible to write $p = ab$ for natural numbers a and b unless one of these is p, and the other 1; for convenience we also agree that 1 is *not* a prime number. The first few prime numbers are $2, 3, 5, 7, 11, 13, 17, 19$. If $n > 1$ is not a prime, then $n = ab$, with a and b both $< n$; if either a or b is not a prime it can be factored similarly; continuing in this way proves that we can write n as a product of primes. For example, $28 = 4 \cdot 7 = 2 \cdot 2 \cdot 7$.

(a) Turn this argument into a rigorous proof by complete induction.

A fundamental theorem about integers, which we will not prove here, states that this factorization is unique, except for the order of the factors. Thus, for example, 28 can never be written as a product of primes one of which is 3, nor can it be written in a way that involves 2 only once (now you should appreciate why 1 is not allowed as a prime).

(b) Using this fact, prove that $\sqrt{n}$ is irrational unless $n = m^2$ for some natural number m.

(c) Prove more generally that $\sqrt[k]{n}$ is irrational unless $n = m^k$

(d) Prove that there cannot be only finitely many prime numbers $p_1, p_2, \dots, p_n$ by considering $p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$.

Solution. (a) Let $n = 2$, then it is a prime itself. Suppose that the argument holds for $n = 1, \dots, k$. Hence, for $n = k + 1$, if $k + 1$ is a prime, we are done. If not, then $k + 1$ can be written as a product of two numbers n_1 and n_2 smaller than $k + 1$. By assumption, n_1 and n_2 can be rewritten as the product of primes: Hence, any number $n > 1$ can be written as a product of primes.

(b) Suppose that $\sqrt{n} = \dfrac{p}{q}$, for any integer p, q. Then $p^2 = nq^2$; because factorization is unique and by (a), nq^2 can be expressed as a product of primes; each of which appears twice, and the same is true for q^2. This implies that $n = m^2$, for some natural m.

(c) Suppose that $\sqrt[k]{n} = \dfrac{p}{q}$, for any integer p, q. Then $p^k = nq^k$; by the same reasoning as above, nq^k and q^k can be expressed as product of primes; each of which appears k times. This implies that $n = m^k$, for some natural m.

(d) The proof is by contradiction. Suppose there were only a finite number of primes $p_1, \dots, p_n$. Then $p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$ could not be a prime, but then it is not evenly divisible by any of $p_1, \dots, p_n$ but 1 and itself: This is then a prime number!

Remark 2.1. BEWARE! This does not say that $p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$ is *definitely* a prime number. What if it is evenly divisible for some primes in $p_{n+1}, p_{n+2}, \dots$? For example:

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \cdot 509$$

Therefore, $p_{n+1} \geq p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$.

■

Problem 2.18 (*). Prove that

(a) If x satisfies

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

for some integers $a_{n-1}, \dots, a_0$, then x is irrational unless x is an integer.(Why is this a generalization of Problem 2.17?)

(b) $\sqrt{6} - \sqrt{2} - \sqrt{3}$ is irrational.

(c) $\sqrt{2} + \sqrt[3]{2}$ is irrational. Hint: Start by working out the first 6 powers of this number.

Solution. (a) We need first to prove the following lemma,

Lemma 2.1. *For any $p > 1$, if p^n is divisible by a prime, so is p.*

Proof. p^n can be written as a product of primes; each of which appears for n times. Since p^n is divisible by a prime, this prime appears as a factor of p^n for n times, which means p is divisible by the prime. □

Suppose x were rational, then $x = \dfrac{p}{q}$ such that p and q are integers that have no common factors. Hence, the equation satisfying x

$$\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_0 = 0$$

is equivalent to

$$p^n + a_{n-1}p^{n-1}q + \cdots + a_0q^n = 0$$

if and only if $q^n = 1$. Suppose $q \neq \pm 1$, then by Problem 2.17, q can be factored as the product of primes. Hence, $a_{n-1}p^{n-1}q + \cdots + a_0q^n$ must be divisible by one of those primes. Since the result is an integer, it is clear that p^n must be also divisible by that prime, and note that $p > 1$. But this implies that p must also be divisible by the prime: A contradiction that p and q have no common factors. Therefore, $q = \pm 1$, and we conclude that x is an integer.

Now let $a_{n-1}, \ldots, a_1$ be 0 and $k = -a_0$, we see that $\sqrt[n]{k} = x$ is irrational unless $k = x^n$ for some integer x since k is an integer.

(b) Let $x = \sqrt{6} - \sqrt{3} - \sqrt{2}$. Then

$$\begin{aligned} x^2 &= 11 + 2\sqrt{6}(1 - \sqrt{2} - \sqrt{3}) \\ (x^2 - 11)^2 &= 24(1 - \sqrt{2} - \sqrt{3})^2 \\ x^4 - 22x^2 + 121 &= 24(6 + 2x) \\ x^4 - 22x^2 - 48x - 23 &= 0 \end{aligned}$$

By (a), x is either an irrational or an integer. We will show that the latter is impossible. Note that since $\sqrt{6}$ and $\sqrt{2} + \sqrt{3}$ are positive and,

$$(\sqrt{2} + \sqrt{3})^2 - 6 = 2\sqrt{6} - 1 > 0$$

Moreover,

$$(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6} < 7 + 2\sqrt{6} = (1 + \sqrt{6})^2$$

Therefore, we conclude $0 < \sqrt{2} + \sqrt{3} - \sqrt{6} < 1$, which means that x must be irrational.

(c) Observe that $\sqrt{2} + \sqrt[3]{2}$ satisfies

$$x^6 - 6x^4 - 4x^3 + 12x^2 - 24x - 4 = 0$$

Therefore, it is either an irrational number or an integer, but $2 < \sqrt{2} + \sqrt[3]{2} < 3$: Hence, the former must be true!

■

Problem 2.19. Prove Bernoulli's inequality: If $h > -1$, then

$$(1 + h)^n \geq 1 + nh.$$

Why is this trivial if $h > 0$?

Solution. The proof is by induction. When $n = 1$, $1 + h = 1 + h$. Suppose the inequality holds for n, then for $n + 1$,

$$\begin{aligned} (1 + h)^{n+1} = (1 + h)^n(1 + h) &\geq (1 + nh)(1 + h) \\ &= 1 + h + nh + nh^2 \\ &\geq 1 + (n + 1)h \end{aligned}$$

Note that without the condition $h > -1$, the first inequality sign will be reversed!
If $h > 0$, expanding the left-hand side using the binomial theorem directly yields the result since $\displaystyle\sum_{k=2}^{n} \binom{n}{k} h^k \geq 0$. ∎

Problem 2.20. The Fibonacci sequence $a_1, a_2, a_3, \dots$ is defined as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-1} + a_{n-2} \quad \text{for } n \geq 3. \end{aligned}$$

Prove that

$$a_n = \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

One way of deriving this astonishing formula is presented in Problem 24.15

Solution. The proof is by complete induction. Let $n = 1$, then $a_1 = 1$. Suppose now that the formula holds for $1, \dots, n$, then for $n+1$,

$$\begin{aligned} a_{n+1} &= a_n + a_{n-1} \\ &= \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} + \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\ &= \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^n \left(\dfrac{3+\sqrt{5}}{1+\sqrt{5}}\right) - \left(\dfrac{1-\sqrt{5}}{2}\right)^n \left(\dfrac{3-\sqrt{5}}{1-\sqrt{5}}\right)}{\sqrt{5}} \\ &= \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} \end{aligned}$$

where in the second equality,

$$\frac{(3+\sqrt{5})(1-\sqrt{5})}{1-5} = \frac{1+\sqrt{5}}{2}$$

and

$$\frac{(3-\sqrt{5})(1+\sqrt{5})}{1-5} = \frac{1-\sqrt{5}}{2}$$

∎

Problem 2.21. The Schwarz inequality (Problem 1.19) actually has a more general form:

$$\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$$

Give three proofs of this, analogous to the three proofs presented in Problem 1.19.

Solution. We will give three proofs just as in Problem 1.19

(a) *The first proof*: Note that the equality occurs when either $y_i = 0 \quad \forall i$ or $x_i = \lambda y_i \quad \forall i$ for $\lambda > 0$. Otherwise,

$$\begin{aligned} 0 &< \sum_{i=1}^{n}(x_i - \lambda y_i)^2 \\ &= \lambda^2(\sum_{i=1}^{n} y_i^2) - 2\lambda(\sum_{i=1}^{n} x_i y_i) + (\sum_{i=1}^{n} x_i^2) \\ &= \lambda^2 - 2\lambda\left(\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^2}\right) + \left(\frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} y_i^2}\right) \end{aligned}$$

This is the case only if

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$$
$$\sum_{i=1}^{n} x_i y_i \leq \left|\sum_{i=1}^{n} x_i y_i\right| \leq \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$$

This concludes our first proof.

(b) *The second proof:* Let

$$x = \frac{x_i}{\sqrt{\sum_{i=1}^{n} x_i^2}} \qquad y = \frac{y_i}{\sqrt{\sum_{i=1}^{n} y_i^2}} \qquad \text{for all } i = 1, \ldots, n$$

and note that $(x - y)^2 \geq 0$. Hence,

$$\frac{x_i^2}{\sum_{i=1}^{n} x_i^2} + \frac{y_i^2}{\sum_{i=1}^{n} y_i^2} \geq 2 \cdot \frac{x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}} \qquad \text{for all } i = 1, \ldots, n$$

Summing up all n inequalities yields the result. Notice also that equality occurs with the same conditions as above.

(c) *The third proof:* Notice that

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) = \left(\sum_{i=1}^{n} x_i y_i\right)^2 + \sum_{\substack{i=1\\ i\neq j}}^{n} (x_i y_j - x_j y_i)^2$$

This essentially delivers the result since $\sum_{\substack{i=1\\ i\neq j}}^{n} (x_i y_j - x_j y_i)^2 \geq 0$ for all terms, with the equality condition as in (a).

■

Problem 2.22. The result in Problem 1.7 has an important generalization: If $a_1, \ldots, a_n \geq 0$, then the "arithmetic mean"

$$A_n = \frac{a_1 + \cdots + a_n}{n}$$

and "geometric mean"

$$G_n = \sqrt[n]{a_1 \cdots a_n}$$

satisfy

$$G_n \leq A_n$$

(a) Suppose that $a_1 < A_n$. Then some a_i satisfies $a_i > A_n$[1]; for convenience, say $a_2 > A_n$. Let $\bar{a}_1 = A_n$ and let $\bar{a}_2 = a_1 + a_2 - \bar{a}_1$. Show that

$$\bar{a}_1 \bar{a}_2 \geq a_1 a_2$$

Why does repeating this process enough times eventually prove that $G_n \leq A_n$? When does this equality hold in the formula $G_n \leq A_n$?

The reasoning in this proof is related to another interesting proof.

(b) Using the fact that $G_n \leq A_n$ when $n = 2$, prove, by induction on k, that $G_n \leq A_n$ for $n = 2^k$.

(c) For a general n, let $2^m > n$. Apply part (b) to the 2^m numbers

$$a_1, \ldots, a_n, \underbrace{A_n, \ldots, A_n}_{2^m - n \text{ times}}$$

to prove that $G_n \leq A_n$

[1] Since $A_n = \dfrac{a_1 + \cdots + a_n}{n} < a_i$ where $a_i = max(a_1, \ldots, a_n)$.

Solution. (a)

$$\begin{aligned}\bar{a}_1\bar{a}_2 - a_1a_2 &= -A_n^2 + (a_1+a_2)A_n - a_1a_2\\ &= (A_n - a_1)(a_2 - A_n) > 0\end{aligned}$$

since $a_1 < A_n < a_2$. Therefore, $\bar{a}_1\bar{a}_2 \geq a_1a_2$.

If we let $\bar{a}_{k+1} = \sum_{i=1}^{k+1} a_i - (\sum_{i=1}^{k} \bar{a}_i) = kA_k + a_{k+1} - kA_k$, we easily prove that $G_n \leq \bar{G}_n$ by induction. Notice further that $\bar{A}_n = A_n$. Hence, it is sufficient to prove that $\bar{G}_n \leq \bar{A}_n = A_n$. This is achieved by complete induction: Let $\bar{a}_1 = A_n$, then there is nothing to prove. Suppose for $\bar{a}_i = A_n$ for $i = 1, \ldots, k$, the inequality holds, then for $\bar{a}_i = A_n$ for $i = 1, \ldots, k, k+1$ the inequality must also hold, which is an equality. The equality holds only for $\bar{a}_i = A_n$ for all $i > 2$ since $\bar{G}_n < G_n$ for some $a_i \neq a_1$.

(b) If $k = 1$, then $n = 2^1 = 2$ and there is nothing to prove. Suppose now that the inequality holds for $n = 2^k$, then for $m = 2^{k+1} = 2n$,

$$\begin{aligned}G_m = \sqrt[2n]{a_1 \cdots a_m} &= \sqrt[n]{\sqrt{a_1a_2} \cdots \sqrt{a_{m-1}a_m}}\\ &\leq \sqrt[n]{\frac{a_1+a_2}{2} \cdot \ldots \cdot \frac{a_{m-1}+a_m}{2}}\\ &\leq \frac{\frac{a_1+a_2}{2} + \cdots + \frac{a_{m-1}+a_m}{2}}{n}\\ &= \frac{a_1 + \cdots + a_m}{2n} = A_m\end{aligned}$$

where in the first inequality, $G_n \leq A_n$ for $n = 2$ is used and in the second inequality the induction assumption is used for $m/2 = n$ terms.

(c) Let $l = 2^m - n$, then

$$\begin{aligned}\sqrt[2^m]{a_1 \cdots a_n \cdot (A_n)^l} &\leq \frac{a_1 + \cdots + a_n + lA_n}{n+l}\\ &= \frac{nA_n + lA_n}{n+l}\\ &= A_n\\ \Rightarrow a_1 \cdots a_n \cdot (A_n)^l &\leq (A_n)^{n+l}\\ a_1 \cdots a_n &\leq (A_n)^n\\ \Rightarrow G_n = \sqrt[n]{a_1 \cdots a_n} &\leq A_n\end{aligned}$$

■

Problem 2.23. The following is a recursive definition of a^n:

$$\begin{aligned}a^1 &= a,\\ a^{n+1} &= a^n \cdot a.\end{aligned}$$

Prove, by induction, that

$$a^{n+m} = a^n \cdot a^m,$$
$$(a^n)^m = a^{nm}.$$

Solution. The proof is by induction on m.
For the first part, let $m = 1$, then $a^{n+1} = a^n \cdot a^1$ by definition. Suppose we have $a^{n+k} = a^n \cdot a^k$, then we want to prove that

$$a^{n+(k+1)} = a^n \cdot a^{k+1}$$

Hence, notice that $n + (k + 1) = (n + k) + 1$, and

$$\begin{aligned} a^{(n+k)+1} &= a^{n+k} \cdot a \\ &= a^n \cdot (a^k \cdot a) \\ &= a^n \cdot a^{k+1} \end{aligned}$$

where the first and third equality are by definition while the second one used the assumption.
For the second part, let $m = 1$, then obviously $(a^n)^1 = a^n$. Assume we have $(a^n)^k = a^{nk}$, then we want to prove,

$$(a^n)^{(k+1)} = a^{n(k+1)}$$

Then

$$\begin{aligned} (a^n)^{(k+1)} &= (a^n)^k \cdot a^n \\ &= a^{nk} \cdot a^n \\ &= a^{n(k+1)} \end{aligned}$$

where the first equality follows from definition; the second follows from assumption, and the third follows from the first part. ■

Problem 2.24. Suppose we know properties P1 and P4 for the natural numbers, but that multiplication has never been mentioned. Then the following can be used as a recursive definition of multiplication:

$$1 \cdot b = b,$$
$$(a + 1) \cdot b = a \cdot b + b.$$

Prove the following (in the order suggested!):

$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$a \cdot 1 = a,$$
$$a \cdot b = b \cdot a$$

Solution. We divide our proofs into three parts.

- $a \cdot (b + c) = a \cdot b + a \cdot c$

  The proof is by induction on a. Let $a = 1$, then by definition $1 \cdot (b + c) = 1 \cdot b + 1 \cdot c$. Assume $k \cdot (b + c) = k \cdot b + k \cdot c$. We want to prove that

  $$(k + 1)(b + c) = (k + 1)b + (k + 1)c$$

  Then

  $$\begin{aligned} (k + 1)(b + c) &= k(b + c) + 1 \cdot (b + c) && \text{by definition} \\ &= (kb + b) + (kc + c) && \text{by assumption, P1 and P4} \\ &= (k + 1)b + (k + 1)c && \text{by P1} \end{aligned}$$

- $a \cdot 1 = a$

  The proof is by induction on a. Let $a = 1$, then $1 \cdot 1 = 1$ by definition. Assume $k \cdot 1 = k$, then

  $$(k + 1) \cdot 1 = k \cdot 1 + 1 \cdot 1 = k + 1$$

  where the first equality is due to definition while the second one is of both assumption and definition.

- $a \cdot b = b \cdot a$

  The proof is by induction on b. Let $b = 1$, then $a \cdot 1 = 1 \cdot a = a$. Assume $a \cdot k = k \cdot a$. Then

  $$\begin{aligned} a \cdot (k + 1) &= a \cdot k + a \cdot 1 && \text{by the first part} \\ &= a \cdot k + a && \text{by the second part} \\ &= k \cdot a + a && \text{by assumption} \\ &= (k + 1) \cdot a && \text{by definition} \end{aligned}$$

■

Problem 2.25. If we accept the real numbers as given, then the natural numbers can be *defined* as the real numbers of the form $1, 1 + 1, 1 + 1 + 1$, etc. The whole point of this problem is to show that there is a rigorous mathematical way of saying "etc."

(a) A set A of real numbers is called **inductive** if

  (1) 1 is in A

  (2) $k + 1$ is in A whenever k is in A.

Prove that

(i) $\mathbb{R}$ is inductive.

(ii) The set of positive real numbers is inductive.

(iii) The set of positive real numbers unequal to $\frac{1}{2}$ is inductive.

(iv) The set of positive real numbers unequal to 5 is not inductive.

(v) If A and B are inductive, then the set C of real numbers which are in both A and B is also inductive.

(b) A real number n will be called a **natural number** if n is in *every* inductive set.

(i) Prove that 1 is a natural number.

(ii) Prove that $k+1$ is a natural number if k is a natural number.

Solution. (a) Let P be the set of positive real numbers and P_n be the set of positive real numbers not equal to n.

(i) By definition, $\mathbb{R}$ is inductive.

(ii) Suppose not. Assume $p \in P$. Then $p+1 \notin P$ whenever $p \in P$. This implies $p \leq -1 < 0$: A contradiction.

(iii) Suppose not. Then there exist some number p such that $p+1 = \frac{1}{2}$, which implies $p = -\frac{1}{2}$: This contradicts $p \in P_{1/2}$.

(iv) 4 is in P_5 but 5 is not in P_5.

(v) Observe first that $1 \in C$ because both A and B are inductive. Suppose $k \in C$, then $k \in A$ and $k \in B$, then $k+1 \in A$ and $k+1 \in B$, and hence $k+1 \in C$. We conclude that C is inductive.

(b) (i) Since every inductive set contains 1, it must be a natural number.

(ii) If k is a natural number, k is in every inductive set; by definition, $k+1$ is in every inductive set, and henceforth $k+1$ is a natural number.

■