

# Solutions to Michael Spivak's Calculus

Son To  
<son.trung.to@gmail.com>

*Ravintola Kiltakellari* \*

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\*I thank my employer!

*To my mother, friends and all those who influence me.*

# Preface

This is my own solutions to Michael Spivak's Calculus textbook.

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# Part I

## Prologue

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# Chapter 1

## Basic properties of number

**Problem 1.1.** Prove the following:

- (i) If  $ax = a$  for some number  $a \neq 0$ , then  $x = 1$
- (ii)  $x^2 - y^2 = (x - y)(x + y)$
- (iii) If  $x^2 = y^2$ , then  $x = y$  or  $x = -y$
- (iv)  $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
- (v)  $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$
- (vi)  $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$  (There is a particularly easy way to do this using (iv), and it will show you how to find a factorization for  $x^n + y^n$  whenever  $n$  is odd.)

*Solution.* (i) By (P7)(Existence of multiplicative inverses), there exists  $a^{-1}$  such that,

$$\begin{aligned}(a^{-1} \cdot a)x &= (a^{-1} \cdot a) \\ x &= 1\end{aligned}$$

(ii) By (P9) for 2 times,

$$\begin{aligned}(x - y)(x + y) &\stackrel{1}{=} x \cdot (x + y) + (-y) \cdot (x + y) \\ &\stackrel{2}{=} x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y \\ &= x^2 + x \cdot y + [-(x \cdot y)] + [-(y^2)] \\ &= x^2 - y^2\end{aligned}$$

(iii) From (ii) and since  $x^2 = y^2$ ,

$$x^2 - y^2 = (x - y)(x + y) = 0$$

This means  $(x - y) = 0 \vee (x + y) = 0$ , which is  $x = y \vee x = -y$

(iv) Starting with the right-hand side,

$$\begin{aligned} (x - y)(x^2 + xy + y^2) &= x \cdot (x^2 + xy + y^2) + (-y) \cdot (x^2 + xy + y^2) \\ &= x^3 + x^2y + xy^2 + [-(x^2y)] + [-(xy^2)] + [-(y)^3] \\ &= x^3 - y^3 \end{aligned}$$

(v) I propose two solutions for this problem. The first one is the direct right-hand side manipulation, while the latter is done by induction.

*The first solution.*

$$\begin{aligned} &(x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1} \\ &\quad + [-(x^{n-1}y)] + [-(x^{n-2}y^2)] + \cdots + [-(xy^{n-1})] + [-(y^n)] \\ &= x^n - y^n \end{aligned}$$

Q.E.D

*The second solution.* Let  $n=1$ , then indeed  $x - y = x - y$ . Suppose the statement holds true for  $n = k$  with  $k \in \mathbb{N}$ , that is

$$x^k - y^k = (x - y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})$$

is true. To finish the proof, we need to prove

$$x^{k+1} - y^{k+1} = (x - y)(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k)$$

That is, the statement holds for  $n = k$ . Starting from the left hand side,

$$\begin{aligned} &x^{k+1} - y^{k+1} \\ &= x^{k+1} - x^k y + x^k y - y^{k+1} \\ &= x^k(x - y) + y(x^k - y^k) \\ &= x^k(x - y) + y(x - y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1}) \\ &= (x - y)[x^k + y(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})] \\ &= (x - y)(x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1} + y^k) \end{aligned}$$

Q.E.D



(vi) We will use (iv) in our proof,

$$\begin{aligned}
 & x^3 + y^3 \\
 = & x^3 - y^3 + 2y^3 \\
 = & (x - y)(x^2 + xy + y^2) + 2y[(x^2 + xy + y^2) + (-x)(x + y)] \\
 = & (x + y)(x^2 + xy + y^2) + 2[-(xy)](x + y) \\
 = & (x + y)(x^2 - xy + y^2)
 \end{aligned}$$

■

**Problem 1.2.** What is wrong with the following “proof”? Let  $x = y$ . Then

$$\begin{aligned}
 x^2 &= xy, \\
 x^2 - y^2 &= xy - y^2, \\
 (x + y)(x - y) &= y(x - y), \\
 x + y &= y, \\
 2y &= y, \\
 2 &= 1.
 \end{aligned}$$

*Solution.* Note that in the transition from line 3 to line 4, the author “simplifies”  $(x - y)$  by dividing  $(x - y)$  on both sides. This is wrong since  $x - y = 0$ , and hence  $1/0$  is undefined as implied by (P7) in the textbook. ■

**Problem 1.3.** Prove the following:

- (i)  $\frac{a}{b} = \frac{ac}{bc}$ , if  $b, c \neq 0$ .
- (ii)  $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ , if  $b, d \neq 0$ .
- (iii)  $(ab)^{-1} = a^{-1}b^{-1}$ , if  $a, b \neq 0$ . (To do this you must remember the defining property of  $(ab)^{-1}$ .)
- (iv)  $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{db}$ , if  $b, d \neq 0$ .
- (v)  $\frac{a}{b} \bigg/ \frac{c}{d} = \frac{ad}{bc}$ , if  $b, c, d \neq 0$ .
- (vi) If  $b, d \neq 0$ , then  $\frac{a}{b} = \frac{c}{d}$  if and only if  $ad = bc$ . Also determine when  $\frac{a}{b} = \frac{b}{a}$ .

*Solution.* (i) Until (iii) is proved, the solution is to test the equality between two sides.

$$\begin{aligned} a(b)^{-1} &= (ac)(bc)^{-1} \\ a[(b)^{-1}b] &= (ac)(bc)^{-1}b \\ (a^{-1}a) &= (a^{-1}a)c(bc)^{-1}b \\ 1 &= (bc)(bc)^{-1} = 1 \end{aligned}$$

(ii) Similar to the above,

$$\begin{aligned} a(b)^{-1} + c(d)^{-1} &= (ad + bc)(bd)^{-1} \\ a(b)^{-1}bd + c(d)^{-1}bd &= (ad + bc)[(bd)^{-1}(bd)] \\ ad(b^{-1}b) + bc(d^{-1}d) &= (ad + bc) \\ ad + bc &= ad + bc \end{aligned}$$

(iii) Since  $a, b \neq 0$ , there exists  $(ab)^{-1}, a^{-1}, b^{-1}$  such that,

$$\begin{aligned} ab &= ab \\ (ab)^{-1}(ab) &= (ab)^{-1}(ab) = 1 \\ (ab)^{-1}a(bb^{-1}) &= b^{-1} \\ (ab)^{-1}(aa^{-1}) &= b^{-1}a^{-1} \\ (ab)^{-1} &= a^{-1}b^{-1} \end{aligned}$$

(iv) For  $b, d \neq 0$ ,

$$\frac{a}{b} \cdot \frac{c}{d} = ab^{-1}cd^{-1} = ac(d^{-1}b^{-1}) = ac(db)^{-1} = \frac{ac}{db}$$

where the next-to-last equality follows from (iii).

(v) I first establish for any number  $a \neq 0$ ,

$$(a^{-1})^{-1} = a$$

Let  $t = a^{-1}$ , we want to prove  $t^{-1} = a$ . Observe that

$$\begin{aligned} t &= a^{-1} \\ t \cdot (t)^{-1} &= a^{-1} \cdot (t)^{-1} \\ a \cdot 1 &= (a \cdot a^{-1}) \cdot (t)^{-1} \\ a &= (t)^{-1} \end{aligned}$$

From the left hand side of the statement,

$$\frac{a}{b} \bigg/ \frac{c}{d} = a(b)^{-1}[c(d)^{-1}]^{-1} = a(b)^{-1}(c)^{-1}[(d)^{-1}]^{-1} = (ad)(bc)^{-1} = \frac{ad}{bc}$$

where the second and third equality follows both from (iii) and the proof above.

(vi) Using (ii),

$$\begin{aligned}\frac{a}{b} &= \frac{c}{d} \\ \frac{a}{b} + \left(-\frac{c}{d}\right) &= 0 \\ \frac{ad - bc}{bd} &= 0 \\ ad &= bc\end{aligned}$$

Now, put  $c = b \wedge d = a$ . It follows that  $\frac{a}{b} = \frac{b}{a}$  if and only if  $a^2 = b^2$ . It follows  $(a - b)(a + b) = 0$ , or  $a = b \vee a = -b$ . ■

**Problem 1.4.** Find all numbers  $x$  for which

- (i)  $4 - x < 3 - 2x$
- (ii)  $5 - x^2 < 8$
- (iii)  $5 - x^2 < -2$
- (iv)  $(x - 1)(x - 3) > 0$  (When is a product of two numbers positive?)
- (v)  $x^2 - 2x + 2 > 0$
- (vi)  $x^2 + x + 1 > 2$
- (vii)  $x^2 - x + 10 > 16$
- (viii)  $x^2 + x + 1 > 0$
- (ix)  $(x - \pi)(x + 5)(x - 3) > 0$
- (x)  $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$
- (xi)  $2^x < 8$
- (xii)  $x + 3^x < 4$
- (xiii)  $\frac{1}{x} + \frac{1}{1 - x} > 0$
- (xiv)  $\frac{x - 1}{x + 1} > 0$

*Solution.* (i)

$$\begin{aligned}
 4 - x &< 3 - 2x \\
 4 + (-x + 2x) &< 3 + (-2x + 2x) \\
 (-4 + 4) + x &< -4 + 3 \\
 x &< -1
 \end{aligned}$$

(ii)

$$\begin{aligned}
 5 - x^2 &< 8 \\
 5 - 8 &< x^2 \\
 -3 &< x^2
 \end{aligned}$$

Since  $x^2 \geq 0 \forall x \in \mathbb{R}$ , the inequality holds  $\forall x$ .

(iii)

$$\begin{aligned}
 5 - x^2 &< -2 \\
 7 &< x^2 \\
 0 &< x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7})
 \end{aligned}$$

Hence, either  $x > \sqrt{7} \wedge x > -\sqrt{7}$  or  $x < \sqrt{7} \wedge x < -\sqrt{7}$ , which is  $x > \sqrt{7} \vee x < -\sqrt{7}$ .

(iv)

$$\begin{aligned}
 (x - 1)(x - 3) &> 0 \\
 (x > 1 \wedge x > 3) \vee (x < 1 \wedge x < 3) \\
 x &> 3 \vee x < 1
 \end{aligned}$$

(v)

$$\begin{aligned}
 x^2 - 2x + 2 &> 0 \\
 (x^2 - 2x + 1) + 1 &> 0 \\
 (x - 1)^2 + 1 &> 0
 \end{aligned}$$

Hence the inequality is satisfied  $\forall x$ .

(vi)

$$\begin{aligned}
& x^2 + x + 1 > 2 \\
& x^2 + x - 1 > 0 \\
& x^2 + \left(\frac{1+\sqrt{5}}{2}\right)x + \left(\frac{1-\sqrt{5}}{2}\right)x + \left(\frac{(1-\sqrt{5})(1+\sqrt{5})}{4}\right) > 0 \\
& \left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right) > 0 \\
& x > \left(\frac{\sqrt{5}-1}{2}\right) \vee x < \left(\frac{-(\sqrt{5}+1)}{2}\right)
\end{aligned}$$

(vii)

$$\begin{aligned}
& x^2 - x + 10 > 16 \\
& x^2 - x - 6 > 0 \\
& x^2 - 3x + 2x - 6 > 0 \\
& x(x-3) + 2(x-3) > 0 \\
& (x+2)(x-3) > 0 \\
& x > 3 \vee x < -2
\end{aligned}$$

(viii)

$$\begin{aligned}
& x^2 + x + 1 > 0 \\
& x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 > 0 \\
& \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0
\end{aligned}$$

which is true for all  $x$ .

(ix) Divide the problem into two cases:  $x > \pi$  and  $x < \pi$ .

*Case 1:*  $x > \pi$

Then  $(x+5)(x-3) > 0$ , which is  $x > 3 \vee x < -5$ .

*Case 2:*  $x < \pi$

Then  $(x+5)(x-3) < 0$ , which is  $-5 < x < 3$ .

(x)

$$\begin{aligned}
& (x - \sqrt[3]{2})(x - \sqrt{2}) > 0 \\
& x > \sqrt{2} \vee x < \sqrt[3]{2}
\end{aligned}$$

(xi) (Sometimes, to solve a problem, intuition is a necessity.)

$$2^x < 8$$

$$2^x < 2^3$$

$$x < 3$$

(xii)

$$x + 3^x < 4$$

$$x + 3^x < 1 + 3^1$$

$$x < 1$$

(xiii)

$$\frac{1}{x} + \frac{1}{1-x} > 0$$

$$\frac{1}{x(1-x)} > 0$$

Hence,  $x(1-x) > 0$ . This means  $0 < x < 1$ .

(xiv)

$$\frac{x-1}{x+1} > 0$$

Hence,  $(x-1)(x+1) > 0$ , or  $x > 1 \vee x < -1$ .

■

**Problem 1.5.** Prove the following:

- (i) If  $a < b$  and  $c < d$ , then  $a + c < b + d$
- (ii) If  $a < b$ , then  $-b < -a$
- (iii) If  $a < b$  and  $c > d$ , then  $a - c < b - d$
- (iv) If  $a < b$  and  $c > 0$ , then  $ac < bc$
- (v) If  $a < b$  and  $c < 0$ , then  $ac > bc$
- (vi) If  $a > 1$ , then  $a^2 > a$
- (vii) If  $0 < a < 1$ , then  $a^2 < a$
- (viii) If  $0 \leq a < b$  and  $0 \leq c < d$ , then  $ac < bd$
- (ix) If  $0 \leq a < b$ , then  $a^2 < b^2$ . (Use (viii).)
- (x) If  $a, b \geq 0$  and  $a^2 < b^2$ , then  $a < b$ . (Use (ix), backwards.)

*Solution.* Let  $P$  be the set of all positive numbers.

- (i) To prove this, we apply (P11): If  $a < b \wedge c < d$ , then  $(b - a \in P) \wedge (d - c \in P)$ . Then  $(b - a) + (d - c) = (b + d) - (a + c) \in P$ . Therefore,  $a + c < b + d$ .
- (ii) We provide two solutions: The first one is by Trichotomy Law (P10), and the second one is by adding  $[(-a) + (-b)]$  to both sides.

*Proof by Trichotomy Law.* If  $a < b$ , then  $b - a \in P$ . By Trichotomy Law,  $a - b \notin P$  and  $a - b \neq 0$ . Therefore,  $a - b < 0$ , which is  $-b < -a$ . Q.E.D

*Proof by adding.*

$$\begin{aligned}
 a &< b \\
 a + [(-a) + (-b)] &< b + [(-a) + (-b)] \\
 [a + (-a)] + (-b) &< [b + (-b)] + (-a) \\
 -b &< -a
 \end{aligned}$$

Q.E.D

- (iii) Using (P11), we have  $b - a \in P \wedge c - d \in P$ . Then  $(b - a) + (c - d) \in P$ . Hence,  $a - c < b - d$ .
- (iv) Using (P12), note that  $b - a \in P$ . Since  $c > 0$ ,  $c(b - a) \in P$ , which means  $bc - ac > 0$ , or  $ac < bc$ .
- (v) By Trichotomy law(P10),  $-c \in P$ . Then by (iv),  $-(ac) < -(bc)$ . By (ii),  $ac > bc$ .
- (vi) Since  $a > 1 > 0$ , by (iv),  $a^2 > a$ .
- (vii) Since  $a > 0$ , by (iv),  $a^2 < a$ .
- (viii) Because  $0 < b$ ,  $bc < bd$ . Furthermore, if  $c \geq 0$ ,  $ac \leq bc$  (equality occurs if  $c = 0$ ), by (iv). Therefore,  $ac \leq bc < bd$ . Hence,  $ac < bd$ .
- (ix) From (viii), let  $c = a$  and  $d = b$ , then the result follows.
- (x) Suppose  $a \geq b$ . Then  $a \geq b \geq 0$ . By (ix) and (P9),  $a^2 \geq b^2$ . This contradicts  $a^2 < b^2$ .

■

**Problem 1.6.** (a) Prove that if  $0 \leq x < y$ , then  $x^n < y^n$ ,  $n = 1, 2, 3, \dots$

- (b) Prove that if  $x < y$  and  $n$  is odd, then  $x^n < y^n$ .
- (c) Prove that if  $x^n = y^n$  and  $n$  is odd, then  $x = y$ .
- (d) Prove that if  $x^n = y^n$  and  $n$  is even, then  $x = y$  or  $x = -y$ .

*Solution.* (a) Repeatedly apply problem 1.5(viii) for  $0 \leq x < y$ , we have  $x^n < y^n$  with  $n = 1, 2, 3, \dots$

- (b) The statement is true for the case  $0 \leq x < y$ . In the case  $x < y \leq 0$ , by 1.5(ii),  $(-x) > (-y) \geq 0$ . By (a),  $(-x)^n > (-y)^n$  for all odd  $n$ . Since  $n$  is odd,  $-(x^n) > -(y^n)$ . Hence, by 1.5(ii),  $x^n < y^n$ . In the case  $x \leq 0 < y$ , since  $n$  is odd,  $x^n < y^n$ .
- (c) Suppose that either  $x \neq y$ . W.l.o.g, let  $x < y$ , by (b),  $x^n < y^n$  for all odd  $n$ , contradicting  $x^n = y^n$  for all odd  $n$ .
- (d) Suppose that both  $x \neq y$  and  $x \neq -y$ . Then  $x^2 - y^2 \neq 0$ . W.l.o.g, suppose  $x^2 > y^2 \geq 0$ . Applying (a), this generalizes to  $x^n > y^n$  for all even  $n$ , contradicting our assumption. Therefore,  $x = y$  or  $x = -y$ .

*The direct proof.* In the case  $x, y \geq 0$ ; by (a), if  $x^n = y^n$  for all even  $n$ , then  $x = y$ . In the case  $x, y \leq 0$ ; if  $x^n = y^n$  for all even  $n$ , then  $(-x), (-y) \geq 0$  and  $(-x)^n = (-y)^n$ , so  $-x = -y$  and hence  $x = y$ . In the case of  $x$  and  $y$  have different signs, then  $x$  and  $-y$  are either two positive or two negative numbers. In either subcase, if  $x^n = y^n$  for all even  $n$ , then  $x^n = (-y)^n$ , and it follows  $x = -y$  from the previous case.

■

**Problem 1.7.** Prove that if  $0 < a < b$ , then

$$a < \sqrt{ab} < \frac{a+b}{2} < b$$

Notice that the inequality  $\sqrt{ab} \leq (a+b)/2$  holds for all  $a, b \geq 0$ . A generalization of this fact occurs in Problem 2.22.

*Solution.* Let us first establish that  $a < \frac{a+b}{2} < b$ . Note that,

$$a + a < a + b < b + b$$

and therefore,  $a < \frac{a+b}{2} < b$ . To finish the proof, we need to prove  $a < \sqrt{ab} < \frac{a+b}{2}$ . To do this, let us prove that if  $0 < a < b$ , then  $0 < \sqrt{a} < \sqrt{b}$ . Note that since  $b - a > 0$ ,

$$b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) > 0$$

Therefore,  $\sqrt{b} > \sqrt{a} > 0$ . We rewrite the inequality as follows,

$$\sqrt{a} \cdot (\sqrt{b} - \sqrt{a}) > 0$$



Then

$$a < \sqrt{ab} \quad (1.1)$$

We next notice that since  $\sqrt{b} - \sqrt{a} > 0$ , it follows that  $(\sqrt{b} - \sqrt{a}) \cdot (\sqrt{b} - \sqrt{a}) = (\sqrt{b} - \sqrt{a})^2 > 0$ . Expand the left hand side,

$$(\sqrt{b} - \sqrt{a})^2 = a + b - 2\sqrt{ab} > 0$$

which implies,

$$\sqrt{ab} < \frac{a+b}{2} \quad (1.2)$$

From (1.1) and (1.2), we have  $a < \sqrt{ab} < \frac{a+b}{2}$ . ■

**Problem 1.8 (\*)**. Although the basic properties of inequalities were stated in terms of the collection  $P$  of all positive numbers, and  $<$  was defined in terms of  $P$ , this procedure can be reversed. Suppose that P10–P12 are replaced by

(P'10) For any numbers  $a$  and  $b$  one, and only one, of the following holds:

- (i)  $a = b$ ,
- (ii)  $a < b$ ,
- (iii)  $b < a$ .

(P'11) For any numbers  $a$ ,  $b$ , and  $c$ , if  $a < b$  and  $b < c$ , then  $a < c$ .

(P'12) For any numbers  $a$ ,  $b$ , and  $c$ , if  $a < b$ , then  $a + c < b + c$ .

(P'13) For any numbers  $a$ ,  $b$ , and  $c$ , if  $a < b$  and  $0 < c$ , then  $ac < bc$ .

Show that P10–P12 can then be deduced as theorems.

*Solution.* Let  $P$  be the set of all positive numbers.

- To prove P10, let  $c = a - b$ , from (P'10), P10 follows.
- To prove P11, let  $a, b \in P$ ; it is sufficient to prove that  $a + b > 0$ . From (P'10), we divide the proof into three subcases:

*Case 1:  $a = b$*

Then  $a + b = b + b > 0 + b > 0$ , where the first inequality follows from (P'12). By (P'11),  $a + b > 0$ .

*Case 2:  $a < b$*

Then  $a + b > a + a > 0 + a > 0$ , where the first and second inequality follow from (P'12). By applying (P'11) twice,  $a + b > 0$ .

*Case 3:  $a > b$*

Interchanging the role of  $a$  and  $b$ , we have the result.

- To prove P12, let  $a, b \in P$ ; it is sufficient to prove that  $a \cdot b > 0$ . From (P'10), we divide the proof into three subcases:

*Case 1:*  $a = b$

Then  $a \cdot b = b \cdot b > 0 \cdot b = 0$ , where the first inequality follows from (P'13) and the equality after which is from (P9).

*Case 2:*  $a < b$

Then  $b \cdot a > a \cdot a > 0 \cdot a = 0$ , where the first and second inequality is from (P'13). By (P'11),  $a \cdot b > 0$ .

*Case 3:*  $a > b$

Interchanging  $a$  and  $b$  returns us to case 2, which yields the result. ■

**Problem 1.9.** Express each of the following with at least one less pair of absolute value signs.

- (i)  $|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}|$
- (ii)  $|(|a + b| - |a| - |b|)|$
- (iii)  $|(|a + b| + |c| - |a + b + c|)|$
- (iv)  $|(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)|$

*Solution.* (i) Note  $\sqrt{7} - \sqrt{5} > 0$ , hence

$$|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}| = \sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$$

- (ii) Since  $|a + b| - |a| - |b| \leq 0$ ,

$$|(|a + b| - |a| - |b|)| = |a| + |b| - |a + b|$$

- (iii) Since  $|a + b + c| \leq |a + b| + |c|$ ,

$$|(|a + b| + |c| - |a + b + c|)| = |a + b| + |c| - |a + b + c|$$

- (iv)

$$|(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)| = |\sqrt{2} + \sqrt{3} - \sqrt{7} + \sqrt{5}|$$
■

**Problem 1.10.** Express each of the following without absolute value signs, treating various cases separately when necessary.

- (i)  $|a + b| - |b|$
- (ii)  $|(|x| - 1)|$
- (iii)  $|x| - |x^2|$
- (iv)  $a - |(a - |a|)|$

*Solution.* (i) We divide into four cases:

$$a \geq 0 \quad \text{and} \quad b \geq 0 \quad (\text{Case 1})$$

$$a \leq 0 \quad \text{and} \quad b \leq 0 \quad (\text{Case 2})$$

$$a \geq 0 \quad \text{and} \quad b \leq 0 \quad (\text{Case 3})$$

$$a \leq 0 \quad \text{and} \quad b \geq 0 \quad (\text{Case 4})$$

In Case 1 and Case 2, we have  $|a + b| - |b| = a$  since  $|a + b| \leq |a| + |b|$ .

In Case 3, if  $a + b \geq 0$ , then

$$|a + b| - |b| = (a + b) - (-b) = a + b + b = 2b$$

If  $a + b \leq 0$ , then

$$|a + b| - |b| = (-a - b) - (-b) = -a + (-b) + b = -a$$

In Case 4, if  $a + b \geq 0$ , then

$$|a + b| - |b| = (a + b) - (b) = a$$

If  $a + b \leq 0$ , then

$$|a + b| - |b| = -a + (-b) + (-b) = -a - 2b$$

(ii) We make the problem into 4 cases.

$$x \geq 1 \quad (\text{Case 1})$$

$$0 \leq x \leq 1 \quad (\text{Case 2})$$

$$-1 \leq x \leq 0 \quad (\text{Case 3})$$

$$x \leq -1 \quad (\text{Case 4})$$

In Case 1,  $||x| - 1| = x - 1$ .

In Case 2,  $||x| - 1| = 1 - x$ .

In Case 3,  $||x| - 1| = x + 1$ .

In Case 4,  $||x| - 1| = -(x + 1)$ .

(iii) Since  $x^2 \geq 0$ ,  $|x| - |x^2| = |x| - x^2$ .

If  $x \geq 0$ , then  $|x| - x^2 = x(1 - x)$ . If  $x \leq 0$ , then

$$|x| - x^2 = -x + (-x^2) = -x(1 + x).$$

(iv) Note that  $|a| \geq a$ . Hence,

$$a - |(a - |a|)| = a + a - |a| = 2a - |a|$$

We have two cases,

Case 1:  $a \geq 0$

$$2a - |a| = 2a - a = a$$

Case 2:  $a \leq 0$

$$2a - |a| = 2a + a = 3a$$

■

**Problem 1.11.** Find all numbers  $x$  for which

- (i)  $|x - 3| = 8$
- (ii)  $|x - 3| < 8$
- (iii)  $|x + 4| < 2$
- (iv)  $|x - 1| + |x - 2| > 1$
- (v)  $|x - 1| + |x + 1| < 2$
- (vi)  $|x - 1| + |x + 1| < 1$
- (vii)  $|x - 1| \cdot |x + 1| = 0$
- (viii)  $|x - 1| \cdot |x + 2| = 3$

*Solution.* (i)

$$\begin{aligned} x - 3 &= 8 \vee x - 3 = -8 \\ x &= 11 \vee x = -5 \end{aligned}$$

(ii) Then  $-8 < x - 3 < 8$ . Hence,  $-5 < x < 11$ .

(iii) Then  $-2 < x + 4 < 2$ . Hence,  $-6 < x < -2$ .

(iv) If  $1 \leq x \leq 2$ , then the inequality becomes  $(x - 1) + (2 - x) = 1$ . If  $x > 2$ , then  $2x - 3 > 1$ , which is  $x > 2$ . If  $x < 1$ , then  $-2x + 3 > 1$ , which is  $x < 1$ . Therefore, either  $x > 2$  or  $x < 1$  satisfies the inequality.

(v) If  $-1 \leq x \leq 1$ , then  $(1 - x) + (x + 1) = 2$ . If  $x > 1$ , then  $x < 1$ , which is contradictory. If  $x < -1$ , then  $(1 - x) + (-x - 1) = -2x < 2$  only if  $x > -1$ , which is contradictory. Hence, there is no  $x$  to satisfy the inequality.

(vi) It is implied from above that

$$|x - 1| + |x + 1| \geq 2$$

Therefore, there is no  $x$  satisfying the inequality.

(vii) Either  $x = 1$  or  $x = -1$ .

- (viii) If  $-2 \leq x \leq 1$ , then we obtain  $x^2 + x + 1 > 0$ . Hence, in either  $x < -2$  or  $x > 1$ , we have to solve the equation  $x^2 + x - 5 = 0$ , whose solution is either  $x = \frac{-1 + \sqrt{21}}{2}$  or  $x = \frac{-1 - \sqrt{21}}{2}$ . ■

**Problem 1.12.** Prove the following:

- (i)  $|xy| = |x| \cdot |y|$
- (ii)  $\left| \frac{1}{x} \right| = \frac{1}{|x|}$ , if  $x \neq 0$ . (The best way to do this is to remember what  $|x|^{-1}$  is.)
- (iii)  $\frac{|x|}{|y|} = \left| \frac{x}{y} \right|$ , if  $y \neq 0$ .
- (iv)  $|x - y| \leq |x| + |y|$  (Give a very short proof.)
- (v)  $|x| - |y| \leq |x - y|$  (A very short proof is possible, if you write things in the right way.)
- (vi)  $||x| - |y|| \leq |x - y|$  (Why does this follow immediately from (v)?)
- (vii)  $|x + y + z| \leq |x| + |y| + |z|$ . Indicate when equality holds, and prove your statement.

*Solution.* (i) We have 4 cases,

$$x \geq 0 \quad y \geq 0 \tag{1}$$

$$x \geq 0 \quad y \leq 0 \tag{2}$$

$$x \leq 0 \quad y \geq 0 \tag{3}$$

$$x \leq 0 \quad y \leq 0 \tag{4}$$

In (1),  $|x| \cdot |y| = xy = |xy|$

In (4),  $|x| \cdot |y| = (-x)(-y) = xy = |xy|$

In (3),  $|x| \cdot |y| = (-x)(y) = -(xy) = |xy|$

In (2), interchanging  $x$  and  $y$  leads to (3).

- (ii) Since  $x \neq 0$ , there exists  $|x|^{-1}$  such that

$$|x||x|^{-1} = 1 = |x| \left| \frac{1}{x} \right|$$

where the second equality is by (i). Dividing both sides by  $|x|$ , we have the result.

(iii) Since  $y \neq 0$ , from (ii), we immediately have

$$\left| \frac{1}{y} \right| = \frac{1}{|y|}$$

Hence, applying (ii) once more,

$$\left| \frac{x}{y} \right| = |x| \left| \frac{1}{y} \right| = \frac{|x|}{|y|}$$

(iv) Note that,

$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$$

where the last equality follows from (i).

(v) Note that,

$$|x - y + y| \leq |x - y| + |y|$$

Therefore,  $|x| - |y| \leq |x - y|$ .

(vi) Let the first term be  $y$  and the second term be  $y - x$ . Applying (v), we have

$$|y| - |y - x| \leq |x|$$

Hence,  $-|x - y| \leq |x| - |y|$ . Combining with (v) gives  $||x| - |y|| \leq |x - y|$ .

(vii) Notice the pattern,

$$|x + y + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$$

the equality holds only if either  $x, y, z$  have the same sign or at least two of them must be equal to 0. It is easy to verify this.

Suppose not, then both  $x, y, z$  have different signs and at most one of them is 0. If the latter is true, then, w.l.o.g, suppose  $z = 0$ , then  $x, y$  have different sign, and we are done. If none of them is 0, then, w.l.o.g, suppose  $z < 0$  and pick  $z$  such that  $x + y < -z$ . Then,

$$|x + y + z| = -(x + y + z) = -x - y - z < |x| + |y| + |z|$$

where inequality must follow since  $x, y \neq 0$ . ■

**Problem 1.13.** The maximum of two numbers  $x$  and  $y$  is denoted by  $\max(x, y)$ . Thus  $\max(-1, 3) = \max(3, 3) = 3$  and  $\max(-1, -4) = \max(-4, -1) = -1$ . The minimum of  $x$  and  $y$  is denoted by  $\min(x, y)$ . Prove that

$$\max(x, y) = \frac{x + y + |y - x|}{2},$$

$$\min(x, y) = \frac{x + y - |y - x|}{2}.$$

Derive the formula for  $\max(x, y, z)$  and  $\min(x, y, z)$ , using, for example

$$\max(x, y, z) = \max(x, \max(y, z)).$$

*Solution.* Assume that  $x \geq y$ , we want to prove that  $\max(x, y) = x$ .

$$\max(x, y) = \frac{x + y + |y - x|}{2} = \frac{x + y + x - y}{2} = \frac{2x}{2} = x$$

Similarly, we need  $\min(x, y) = y$ .

$$\min(x, y) = \frac{x + y - |y - x|}{2} = \frac{x + y - (x - y)}{2} = \frac{x + y - x + y}{2} = \frac{2y}{2} = y$$

Let  $\max(x, y, z) = \max(x, \max(y, z))$ . Then

$$\begin{aligned} \max(x, y, z) &= \frac{x + \max(y, z) + |\max(y, z) - x|}{2} \\ &= \frac{x + \frac{y + z + |z - y|}{2} + \left| \frac{y + z + |z - y|}{2} - x \right|}{2} \\ &= \frac{2x + y + z + |z - y| + |y + z + |z - y| - 2x|}{4} \end{aligned}$$

Similarly,

$$\min(x, y, z) = \frac{2x + y + z - |z - y| - |y + z - |z - y| - 2x|}{4}$$

■

**Problem 1.14.** (a) Prove that  $|a| = |-a|$ . (The trick is not to become confused by too many cases. First prove the statement for  $a \geq 0$ . Why is it then obvious for  $a \leq 0$ ?)

(b) Prove that  $-b \leq a \leq b$  if and only if  $|a| \leq b$ . In particular, it follows that  $-|a| \leq a \leq |a|$ .

(c) Use this fact to give a new proof that  $|a + b| \leq |a| + |b|$ .

*Solution.* (a) Problem 1.12(i) easily tells us that

$$|-a| = |(-1)a| = |-1||a| = 1|a| = |a|$$

(b) If  $a \geq 0$ , then  $a \leq b$ . If  $a \leq 0$ ,  $-a \leq b$  follows from  $a \geq -b$ . Therefore,  $|a| \leq b$ . Conversely, suppose  $|a| \leq b$ . Then it is certain  $a \leq b$  since  $a \leq |a| \leq b$ . From (a),  $|-a| \leq b$ , and hence  $a \geq -b$ . We conclude that  $-b \leq a \leq b$ . Note that since  $|a| \leq |a|$ ,  $-|a| \leq a \leq |a|$ .

(c) Because we have  $-|a| \leq a \leq |a|$  and  $-|b| \leq b \leq |b|$ , by Problem 1.5(i), we obtain  $-(|a| + |b|) \leq a + b \leq |a| + |b|$ . From (b), we arrive at the conclusion  $|a + b| \leq |a| + |b|$ .

■

**Problem 1.15** (\*). Prove that if  $x$  and  $y$  are not both 0, then

$$\begin{aligned}x^2 + xy + y^2 &> 0 \\x^4 + x^3y + x^2y^2 + xy^3 + y^4 &> 0\end{aligned}$$

Hint: Use problem 1.

*Solution.* For the first part, note that

$$x^2 + xy + y^2 = x^2 + 2 \cdot x \cdot \frac{1}{2}y + \frac{1}{4}y^2 - \frac{1}{4}y^2 + y^2 = \left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 > 0$$

For the second part, if  $x = y$ , then the left-hand side is  $5x^4 > 0$ . Hence, suppose  $x \neq y$ . From Problem 1.1(v),

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \neq 0$$

If  $x > y$ , then  $x^5 > y^5$  by Problem 1.6(b). This implies that the second term must be greater than 0. Conversely,  $x < y \Rightarrow x^5 < y^5$  implies that it must be greater than 0. ■

**Problem 1.16** (\*). (a) Show that

$$\begin{aligned}(x + y)^2 &= x^2 + y^2 \quad \text{only when } x = 0 \text{ or } y = 0, \\(x + y)^3 &= x^3 + y^3 \quad \text{only when } x = 0 \text{ or } y = 0 \text{ or } x = -y.\end{aligned}$$

(b) Using the fact that

$$x^2 + 2xy + y^2 = (x + y)^2 \geq 0,$$

show that  $4x^2 + 6xy + 4y^2 > 0$  unless  $x$  and  $y$  are both 0.

(c) Use part (b) to find out when  $(x + y)^4 = x^4 + y^4$ .

(d) Find out when  $(x + y)^5 = x^5 + y^5$ . Hint: From the assumption  $(x + y)^5 = x^5 + y^5$  you should be able to derive the equation  $x^3 + 2x^2y + 2xy^2 + y^3 = 0$ , if  $xy \neq 0$ . This implies that  $(x + y)^3 = x^2y + xy^2 = xy(x + y)$ .

You should know be able to make a good guess as to when  $(x + y)^n = x^n + y^n$ ; the proof is contained in Problem 11.57

*Solution.* (a) For the first part,

$$(x + y)^2 = x^2 + 2xy + y^2$$

Hence,  $(x + y)^2 = x^2 + y^2$  only when  $x = 0$  or  $y = 0$ . For the second part, from Problem 1.1(vi),

$$\begin{aligned}(x + y)^3 - (x + y)(x^2 - xy + y^2) &= 0 \\(x + y)(xy) &= 0\end{aligned}$$

which is true only when  $x = 0$  or  $y = 0$  or  $x = -y$ .



(b) Note that  $4x^2 + 6xy + 4y^2 = \underbrace{3(x+y)^2}_{\geq 0} + \underbrace{x^2 + y^2}_{> 0} > 0$  unless  $x = 0$  and  $y = 0$ .

(c) Let us expand  $(x+y)^4$ .

$$\begin{aligned}(x+y)^2(x+y)^2 &= (x^2 + 2xy + y^2)(x^2 + 2xy + y^2) \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\ &= x^4 + y^4 + xy(4x^2 + 6xy + 4y^2)\end{aligned}$$

Hence,  $(x+y)^4 = x^4 + y^4$  only when  $x = 0$  or  $y = 0$ , by part (b).

(d) Let us expand  $(x+y)^5$ .

$$\begin{aligned}(x+y)^4(x+y) &= x^5 + y^5 + xy(x+y)(4x^2 + 6xy + 4y^2) + xy(x^3 + y^3) \\ &= x^5 + y^5 + 5xy(x+y)(x^2 - xy + y^2)\end{aligned}$$

If  $xy \neq 0$  and  $x+y \neq 0$ , let  $z = -y$ , by Problem 1.6(b),  $x^3 \neq z^3$ . Hence,  $x^2 - xy + y^2 \neq 0$ . Therefore,  $(x+y)^5 = x^5 + y^5$  only when  $x = 0$  or  $y = 0$  or  $x = -y$ .

*Remark.* Hence, for  $(x+y)^n = x^n + y^n$ , if  $n$  is even, then  $x = 0$  or  $y = 0$ . If  $n$  is odd, then  $x = 0$  or  $y = 0$  or  $x = -y$ . ■

**Problem 1.17.** (a) Find the smallest possible value of  $2x^2 - 3x + 4$ . Hint: “Complete the square”, i.e., write  $2x^2 - 3x + 4 = 2(x - 3/4)^2 + ?$

(b) Find the smallest possible value of  $x^2 - 3x + 2y^2 + 4y + 2$ .

(c) Find the smallest possible value of  $x^2 + 4xy + 5y^2 - 4x - 6y + 7$ .

*Solution.* (a) Since  $2x^2 - 3x + 4 = 2(x^2 - \frac{3}{2}x + 2)$ ,

$$2(x^2 - 2 \cdot x \frac{3}{4} + \frac{9}{16} - \frac{9}{16} + 2) = 2(x - \frac{3}{4})^2 + \frac{23}{8}$$

Hence the minimum value is  $\frac{23}{8}$  when  $x = \frac{3}{4}$ . ■