Problems In Mathematics for Computer Science

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Preface

This is a research project in which I try to read the notes and solve all the problems from [1]

Contents

Part I

Notes

Chapter 1

What is a Proof?

1.1 Propositions

Definition 1.1. A *proposition* is a statement (communication) that is either true or false.

Claim 1.1.1. $\forall\ n \in \mathbb{N},\ p ::= n^2 + n + 41$ *is prime*

Question: Is this claim true or false?

Claim 1.1.2. *No polynomial with integer coefficients can map all nonnegative numbers into primes, unless it's a constant.*

Question: Is this true or false?

Claim 1.1.3 (Euler's Conjecture)**.** $\forall a, b, c, d \in \mathbb{Z}^+.\ a^4 + b^4 + c^4 \neq d^4$

Claim 1.1.4. $313(x^3 + y^3) = z^3$ *has no solution when* $x, y, z \in \mathbb{Z}^+$

Claim 1.1.5 (Four Color Theorem)**.** *Every map can be colored with* 4 *colors so that adjacent regions have different colors.*

Claim 1.1.6 (Fermat's Last Theorem)**.** $\forall a, b, c \in \mathbb{Z}^+\ \forall n > 2, n \in \mathbb{Z}.\ a^n + b^n \neq c^n$

Claim 1.1.7 (Goldbach)**.** *Every even integer greater than* 2 *is the sum of two primes.*

1.2 Predicates

Definition 1.2. A *predicate* is a proposition whose truth depends on the value of one or more variables.

If P is a predicate, then $P(n)$ is either *true* or *false*, depending on the value of n.

1.3 The Axiomatic Method

Definition 1.3. A *proof* is a sequence of logical deductions from a set of axioms and previous proved propositions that concludes with the proposition in question.

- *Theorems*
- *Lemma*
- *Corollary*

$\Rightarrow$ Axiomatic Method

1.4 Our axioms

1.4.1 Logical deductions

Keywords: *Logical deductions(inference rules), antecedents, conclusion, modus ponens*

1.4.2 Patterns of Proof

Many proofs follow specific templates... Many special techniques later on.

1.5 Proving an Implication

Definition 1.4. *Implications* means $P \Rightarrow Q$

1.5.1 Method #1: $P \Rightarrow Q$

1.5.2 Method #2: Contrapositive: $\neg Q \Rightarrow \neg P$

1.6 Proving an "if and only if"

1.6.1 Method #1: Prove each statement implies the other

1.6.2 Method #2: Construct a chain of iffs

1.7 Proof by Cases

Amusing theorem

Theorem 1.7.1. *Every collection of 6 people includes a club of 3 people or a group of 3 strangers.*

Proof. The proof is by case analysis. Let x be one of those 6 people. Among 5 other people, there are two scenarios:

1. At least 3 people have met x
2. At least 3 people have not met x

We argue that these two cases are exhaustive since we are dividing the 5 people into two groups: those who have met x and those who have not.

Case 1: Suppose that at least 3 people have met x
This is divided further more into two subcases:

Case 1.1: No pairs among those people have met each other. In this case, they form a group of at least 3 strangers. Thus, the theorem holds in this subcase.

Case 1.2: At least one pair in those people have met. Adding x to such pair forms a club of at least 3 people. The theorem is proved in this subcase.

This implies that the theorem holds for Case 1.

Case 2: Suppose that at least 3 people have not met x
This again splits the case into two subcases:

Case 2.1: All pairs among those people have met each other. In this case, they form a club of at least 3 people. Thus the theorem holds in this subcase.

Case 2.2: At least one pair in those people have not met. Adding x to such pair forms a group of at least 3 strangers. The theorem holds in this subcase.

This implies that the theorem holds for Case 2.

We have proved the theorem. □

1.8 Proof by Contradiction

Theorem 1.8.1. *$\sqrt{2}$ is irrational*

Proof. We use proof by contradiction. Suppose $\sqrt{2}$ is rational, then $\sqrt{2} = \frac{p}{q}$, where p and q are integers that have no common factors. Then $2 = \frac{p^2}{q^2}$, which means $p^2 = 2q^2$. Since p^2 is even, p must be even (easily proved by contradiction again). W.l.o.g, assume $p = 2k$ for some integer k. Then $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2$, which implies that q is also even. However, this contradicts the fact that p and q have no common factors. Therefore $\sqrt{2}$ is irrational. □

Part II

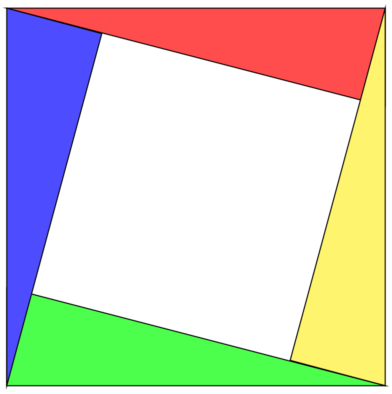
Problems and Exercises

Chapter 1

What is a Proof?

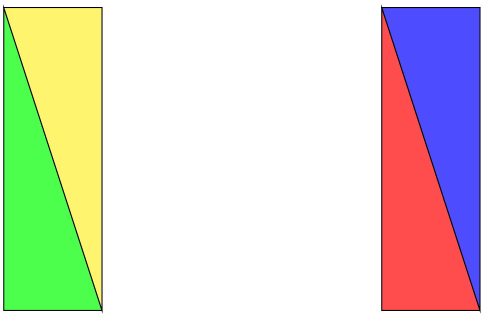
Problem 1.1.

(a) Colors of the triangles are arbitrary since I do not remember the exact ones in the text.



The middle square is a square of $(b-a) \times (b-a)$

(b) [*Possible Errata:* Arrange the same shapes so they form two rectangles, both $a \times b$.]



We prove by construct a chain of iffs.

$$\begin{aligned} & (b-a)^2 = c^2 - 2ab \\ \Leftrightarrow \quad & a^2 + b^2 - 2ab = c^2 - 2ab \\ \Leftrightarrow \quad & a^2 + b^2 = c^2 \end{aligned}$$

(c) The equation would still hold true since $a = b$ is not a requirement for the proof. In fact, note that if $a = b$, the area of the bigger square in (a) will now be exactly equal to the sum of area of all triangles inside it, which is equal to the sum of area of two smaller squares in (b). That is, $c^2 = a^2 + b^2$.

(d) Some assumptions about right triangles, squares and lines are,

- 4 identical right triangles.
- For every 2 points, there is a line.

Problem 1.2.

(a) Mistake:

$$\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$$

The right-hand side of this equation is undefined while the left-hand side is defined.

(b) Suppose $1 = -1$, then

$$\begin{aligned} & 0 = 0 \\ \Rightarrow \quad & 1^2 - 1^2 = 1 + 1 \\ \Rightarrow \quad & (1-1)(1+1) = 1 + 1 \end{aligned}$$

At this stage, we cancel off $1 + 1$ on each side since they are non-zero, the equation then becomes,

$$\begin{aligned} \Rightarrow \quad & 1 - 1 = 1 \\ \Rightarrow \quad & 1 + 1 = 1 \\ \Rightarrow \quad & 2 = 1 \end{aligned}$$

where the second equation is by the antecedent $1 = -1$. Hence, we have proved $2 = 1$.

(c) We shall prove the following lemma,

Lemma 1.2.1. *If $r, s > 0$, then $\sqrt{rs} = \sqrt{r}\sqrt{s}$*

Proof. Assume $r, s > 0$. For every positive integer x, there is one $\sqrt{x} > 0$ such that $x = (\sqrt{x})^2$; therefore,

$$(\sqrt{r})^2(\sqrt{s})^2 = rs \tag{1.1}$$

By commutative and associative property of multiplication,

$$(\sqrt{r})^2(\sqrt{s})^2 = (\sqrt{r}\sqrt{s})^2 \tag{1.2}$$

which leads to

$$(\sqrt{r}\sqrt{s})^2 = rs \tag{1.3}$$

Since $rs > 0$, there is also one $\sqrt{rs} > 0$ such that $(\sqrt{rs})^2 = rs$. Therefore,

$$(\sqrt{r}\sqrt{s})^2 = (\sqrt{rs})^2 \tag{1.4}$$

Since $\sqrt{r}\sqrt{s} > 0$ and $\sqrt{rs} > 0$, we conclude

$$\sqrt{r}\sqrt{s} = \sqrt{rs} \tag{1.5}$$

□

Problem 1.3.

(a) Mistake,

$$\begin{gathered} 3 > 2 \\ 3\log_{10}(1/2) > 2\log_{10}(1/2) \end{gathered}$$

since $\log_{10} n \; \forall 0 < n < 1$ is always negative.

(b) Wrong because all arithmetic operations must be done on numbers with the same currency, so that they will be in the same field.

(c) Since $a = b$, $a - b = 0$, and therefore we cannot cancel $a - b$ on both sides since we cannot divide each side by 0.

Problem 1.4. The questionable step is from step 2 to step 3, namely,

$$\begin{aligned} a + b &\overset{?}{\geq} 2\sqrt{ab} \\ a^2 + 2ab + b^2 &\overset{?}{\geq} 4ab \end{aligned}$$

Assume that $a, b < 0$. Then even though the first inequality is false, the second inequality is true, which will lead to all subsequent inequalities to be true. Therefore, we have just "proved" that arithmetic mean is at least as large as geometric mean for all negative numbers a, b!

To fix this,

$$\begin{aligned}\frac{a+b}{2} &\geq \sqrt{ab}\\ a+b-2\sqrt{ab} &\geq 0\\ (\sqrt{a}-\sqrt{b})^2 &\geq 0\end{aligned}$$

This ensures that $a, b \geq 0$ for the last inequality to be defined.

Bibliography

[1] Eric Lehman, Tom Leighton, and Albert Meyer. *Mathematics for Computer Science.* MIT OCW, 2018.