Problems In Mathematics for Computer Science

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Preface

This is a research project in which I try to read the notes and solve all the problems from [1]

Contents

Part I

Notes

Chapter 1

What is a Proof?

1.1 Propositions

Definition 1.1. A *proposition* is a statement (communication) that is either true or false.

Claim 1.1.1. $\forall\ n \in \mathbb{N},\ p ::= n^2 + n + 41$ *is prime*

Question: Is this claim true or false?

Claim 1.1.2. *No polynomial with integer coefficients can map all nonnegative numbers into primes, unless it's a constant.*

Question: Is this true or false?

Claim 1.1.3 (Euler's Conjecture)**.** $\forall a, b, c, d \in \mathbb{Z}^+.\ a^4 + b^4 + c^4 \neq d^4$

Claim 1.1.4. $313(x^3 + y^3) = z^3$ *has no solution when* $x, y, z \in \mathbb{Z}^+$

Claim 1.1.5 (Four Color Theorem)**.** *Every map can be colored with* 4 *colors so that adjacent regions have different colors.*

Claim 1.1.6 (Fermat's Last Theorem)**.** $\forall a, b, c \in \mathbb{Z}^+\ \forall n > 2, n \in \mathbb{Z}.\ a^n + b^n \neq c^n$

Claim 1.1.7 (Goldbach)**.** *Every even integer greater than* 2 *is the sum of two primes.*

1.2 Predicates

Definition 1.2. A *predicate* is a proposition whose truth depends on the value of one or more variables.

If P is a predicate, then $P(n)$ is either *true* or *false*, depending on the value of n.

1.3 The Axiomatic Method

Definition 1.3. A *proof* is a sequence of logical deductions from a set of axioms and previous proved propositions that concludes with the proposition in question.

- *Theorems*
- *Lemma*
- *Corollary*

$\Rightarrow$ Axiomatic Method

1.4 Our axioms

1.4.1 Logical deductions

Keywords: *Logical deductions(inference rules), antecedents, conclusion, modus ponens*

1.4.2 Patterns of Proof

Many proofs follow specific templates... Many special techniques later on.

1.5 Proving an Implication

Definition 1.4. *Implications* means $P \Rightarrow Q$

1.5.1 Method #1: $P \Rightarrow Q$

1.5.2 Method #2: Contrapositive: $\neg Q \Rightarrow \neg P$

1.6 Proving an "if and only if"

1.6.1 Method #1: Prove each statement implies the other

1.6.2 Method #2: Construct a chain of iffs

1.7 Proof by Cases

Amusing theorem

Theorem 1.7.1. *Every collection of 6 people includes a club of 3 people or a group of 3 strangers.*

Proof. The proof is by case analysis. Let x be one of those 6 people. Among 5 other people, there are two scenarios:

1. At least 3 people have met x
2. At least 3 people have not met x

We argue that these two cases are exhaustive since we are dividing the 5 people into two groups: those who have met x and those who have not.

Case 1: Suppose that at least 3 people have met x
This is divided further more into two subcases:

Case 1.1: No pairs among those people have met each other. In this case, they form a group of at least 3 strangers. Thus, the theorem holds in this subcase.

Case 1.2: At least one pair in those people have met. Adding x to such pair forms a club of at least 3 people. The theorem is proved in this subcase.

This implies that the theorem holds for Case 1.

Case 2: Suppose that at least 3 people have not met x
This again splits the case into two subcases:

Case 2.1: All pairs among those people have met each other. In this case, they form a club of at least 3 people. Thus the theorem holds in this subcase.

Case 2.2: At least one pair in those people have not met. Adding x to such pair forms a group of at least 3 strangers. The theorem holds in this subcase.

This implies that the theorem holds for Case 2.

We have proved the theorem. □

1.8 Proof by Contradiction

Theorem 1.8.1. *$\sqrt{2}$ is irrational*

Proof. We use proof by contradiction. Suppose $\sqrt{2}$ is rational, then $\sqrt{2} = \frac{p}{q}$, where p and q are integers that have no common factors. Then $2 = \frac{p^2}{q^2}$, which means $p^2 = 2q^2$. Since p^2 is even, p must be even (easily proved by contradiction again). W.l.o.g, assume $p = 2k$ for some integer k. Then $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2$, which implies that q is also even. However, this contradicts the fact that p and q have no common factors. Therefore $\sqrt{2}$ is irrational. □

Part II

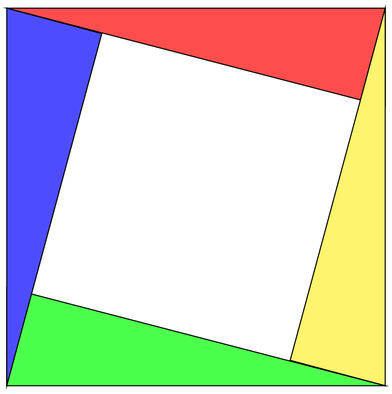
Problems and Exercises

Chapter 1

What is a Proof?

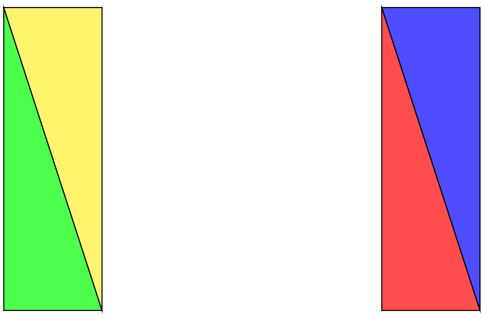
Problem 1.1.

(a) Colors of the triangles are arbitrary since I do not remember the exact ones in the text.



The middle square is a square of $(b - a) \times (b - a)$

(b) [*Possible Errata:* Arrange the same shapes so they form two rectangles, both $a \times b$.]



We prove by construct a chain of iffs.

$$\begin{aligned} & (b-a)^2 = c^2 - 2ab \\ \Leftrightarrow \quad & a^2 + b^2 - 2ab = c^2 - 2ab \\ \Leftrightarrow \quad & a^2 + b^2 = c^2 \end{aligned}$$

(c) The equation would still hold true since $a = b$ is not a requirement for the proof. In fact, note that if $a = b$, the area of the bigger square in **(a)** will now be exactly equal to the sum of area of all triangles inside it, which is equal to the sum of area of two smaller squares in **(b)**. That is, $c^2 = a^2 + b^2$.

(d) Some assumptions about right triangles, squares and lines are,

- 4 identical right triangles.

Bibliography

[1] Eric Lehman, Tom Leighton, and Albert Meyer. *Mathematics for Computer Science*. MIT OCW, 2018.