A Measure of Random Utility Model Violations and Econometric Test

Roy Allen
West Ontario University

Changkuk Im
The Ohio State University

John Rehbeck
The Ohio State University

Woohun Son
The Ohio State University

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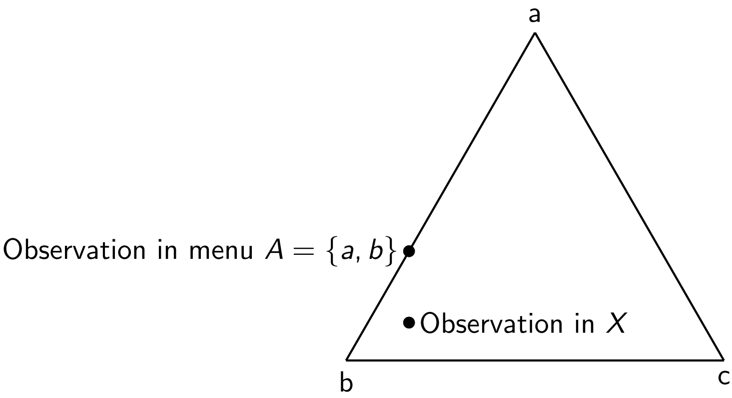
Motivation

- Random Utility Model (RUM) is popular model of individual/aggregate choice
- Few methods to evaluate violation of model

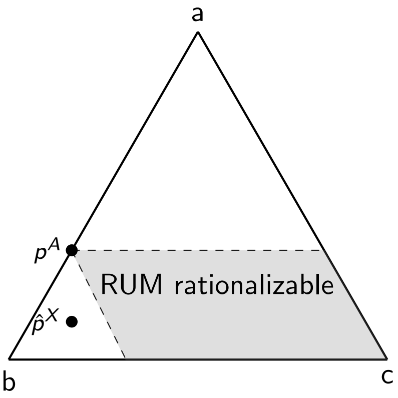
Motivation

- Consider a set of alternatives $X = \{a, b, c\}$



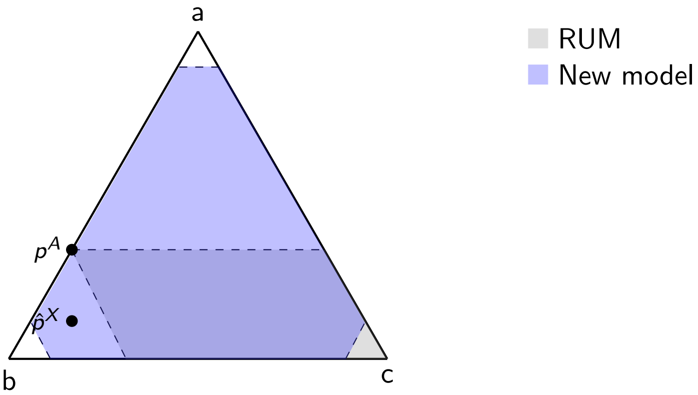
Motivation

- Choice probabilities must not increase as the menu enlarges
  $\Rightarrow$ Data *violates* RUM (Regularity Axiom)



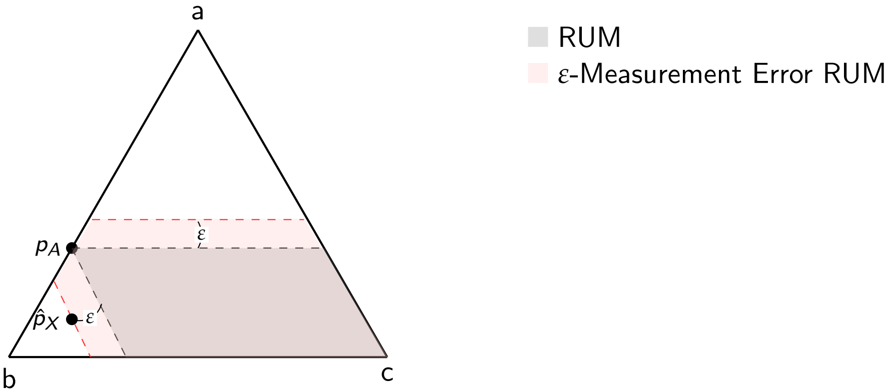
Motivation

- Could rationalize data with a larger model
- E.g. Rational Inattention, Consideration Set models, etc.



Motivation

- What if the data has **measurement error**?
- Extend RUM to **accommodate** measurement error
  ε-Measurement Error RUM (ε-MERUM)



Contribution

- Generate method to quantify measurement error in RUM (MERUM)
- Minimum distance (ε^*) gives ε^*-MERUM rationalization
- ε^*-MERUM gives statistical test via multinomial distributions

Stochastic choice primitives

- X: a finite nonempty set of alternatives
  - e.g. $X = \{a, b, c\}$
- $\mathcal{A} \subseteq 2^X \backslash \{\emptyset\}$: a collection of menus
  - e.g. $\mathcal{A} = \{\{a, b\}, X\}$
- $\mathcal{O} = \{(a, A) \in X \times \mathcal{A} | A \in \mathcal{A},\ a \in A\}$: the collection of all pairs of a menu and an alternative in the menu
  - e.g. $\mathcal{O} = \{(a, \{a, b\}), (b, \{a, b\}), (a, X), (b, X), (c, X)\}$
- $p : \mathcal{O} \to [0, 1]$ is a stochastic choice function
  - $\forall A \in \mathcal{A}, \sum_{a \in A} p(a, A) = 1$

Rationalization by RUM

- $\mathcal{R}$: the set of all strict preference ordering on X
  - e.g. $\mathcal{R} = \{\succ_{abc}, \succ_{acb}, \succ_{bac}, \succ_{bca}, \succ_{cab}, \succ_{cba}\}$
- Random Utility Model (RUM): probability distribution $\mu \in \Delta(\mathcal{R})$
  - Assigns frequency of each preference ordering occurrence
- e.g. p: **RUM rationalizable** if we find μ s.t.

$$\begin{aligned}
p(a,\{a,b\}) &= \mu(\succ_{abc}) + \mu(\succ_{acb}) + \mu(\succ_{cab}) \\
p(b,\{a,b\}) &= \mu(\succ_{bac}) + \mu(\succ_{bca}) + \mu(\succ_{cba}) \\
p(a,X) &= \mu(\succ_{abc}) + \mu(\succ_{acb}) \\
p(b,X) &= \mu(\succ_{bac}) + \mu(\succ_{bca}) \\
p(c,X) &= \mu(\succ_{cab}) + \mu(\succ_{cba})
\end{aligned}$$

Rationalization by RUM

$$\underbrace{\begin{bmatrix} p(a,\{a,b\}) \\ p(b,\{a,b\}) \\ p(a,\{a,b,c\}) \\ p(b,\{a,b,c\}) \\ p(c,\{a,b,c\}) \end{bmatrix}}_{=:p_{(|\mathcal{O}|\times 1)}} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{=:M_{(|\mathcal{O}|\times|\mathcal{R}|)}} \underbrace{\begin{bmatrix} \mu(\succ_{abc}) \\ \mu(\succ_{acb}) \\ \mu(\succ_{bac}) \\ \mu(\succ_{bca}) \\ \mu(\succ_{cab}) \\ \mu(\succ_{cba}) \end{bmatrix}}_{=:\mu_{(|\mathcal{R}|\times 1)}}$$

Definition

A stochastic choice function p is **RUM rationalizable** *if there exists a $\mu \in \Delta(\mathcal{R})$ s.t. $p = M\mu$.*

Extension of RUM with errors

To accommodate the measurement error,

$$\underbrace{\hat{p}}_{\text{observed stochastic choice function}} \qquad \longleftrightarrow \qquad \underbrace{p}_{\text{actual stochastic choice function}}$$

- $\hat{p}$ can be described as p containing error: for each $(a, A) \in \mathcal{O}$,

$$p(a, A) - \varepsilon \leq \hat{p}(a, A) \leq p(a, A) + \varepsilon$$

where $\varepsilon \in [0, 1]$ is an error uniform across $(a, A) \in \mathcal{O}$

Extension of RUM with errors

- Assume that true data is RUM rationalizable: $p = M\mu$

$$
\begin{aligned}
& p(a, A) - \varepsilon \leq \hat{p}(a, A) \leq p(a, A) + \varepsilon \\
\text{plug-in:} \quad & (M\mu)_{(a,A)} - \varepsilon \leq \hat{p}(a, A) \leq (M\mu)_{(a,A)} + \varepsilon \\
\text{rearrange:} \quad & \hat{p}(a, A) - \varepsilon \leq (M\mu)_{(a,A)} \leq \hat{p}(a, A) + \varepsilon \\
\text{stack:} \quad & \underbrace{\begin{bmatrix} \hat{p}(a_1, A_1) \\ \vdots \\ \hat{p}(a_{|\mathcal{O}|}, A_{|\mathcal{O}|}) \end{bmatrix}}_{=:\hat{p}_{(|\mathcal{O}| \times 1)}} - \varepsilon \cdot \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{=:\mathbf{1}_{|\mathcal{O}|,1}} \leq M\mu \leq \underbrace{\begin{bmatrix} \hat{p}(a_1, A_1) \\ \vdots \\ \hat{p}(a_{|\mathcal{O}|}, A_{|\mathcal{O}|}) \end{bmatrix}}_{=:\hat{p}_{(|\mathcal{O}| \times 1)}} + \varepsilon \cdot \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{=:\mathbf{1}_{|\mathcal{O}|,1}} \\
\Leftrightarrow \quad & \hat{p} - \varepsilon \cdot \mathbf{1}_{|\mathcal{O}|,1} \leq M\mu \leq \hat{p} + \varepsilon \cdot \mathbf{1}_{|\mathcal{O}|,1} \qquad (1)
\end{aligned}
$$

- $\exists(\varepsilon, \mu)$ that satisfy (1): ε-**MERUM rationalization** of $\hat{p}$!

Main Proposition

Let $\hat{p}$: an observed s.c.f. and ε^*, where

$$(\varepsilon^*, \mu^*) = \arg \min_{\mu \in \Delta(\mathcal{R}), \varepsilon \in [0,1]} \varepsilon$$

$$\text{s.t. } \hat{p} - \varepsilon \cdot \mathbf{1}_{|\mathcal{O}|,1} \leq M\mu \leq \hat{p} + \varepsilon \cdot \mathbf{1}_{|\mathcal{O}|,1}$$

Then,

1. ε^* always exists for any observed s.c.f. $\hat{p}$ with $\varepsilon^* \in [0, 1]$, and
2. $\varepsilon^* = 0 \Leftrightarrow \hat{p}$: RUM rationalizable

Building the statistical test

- Suppose $\hat{p}$ is ε^*-MERUM rationalized with optimizer (ε^*, μ^*)
- Then, consider the following hypothesis test:

$$H_0 : \; p = M\mu^* \qquad\qquad H_1 : \; p \neq M\mu^*$$

- Observe that $p = M\mu^*$ is a collection of **multinomial** distribution
  - Focus on *certain menu* $A \in \mathcal{A}$ where $|A| = K$
  - The choice probability of k-th alternative in A will be $p_k := p(a_k, A)$

Building the statistical test

$$H_0: \; p = M\mu^* \qquad\qquad H_1: \; p \neq M\mu^*$$

- Construct a confidence interval (CI) under the null (H_0):

$$CI(\hat{p}_k^-(\alpha), \hat{p}_k^+(\alpha)) = P(\hat{p}_k^-(\alpha) \leq p_k \leq \hat{p}_k^+(\alpha), \forall k = 1, \ldots, K)$$

- We have the representation (Sison and Glaz [1995]) of

$$\hat{p}_k^- = \hat{p}_k - \frac{c(\alpha)}{n} \qquad\qquad \hat{p}_k^+ = \hat{p}_k + \frac{c(\alpha)}{n},$$

we can **directly compare** ε^* with $\frac{c(\alpha)}{n}$, since

$$\begin{aligned} \hat{p}_k - \frac{c(\alpha)}{n} &\leq p_k \leq \hat{p}_k + \frac{c(\alpha)}{n} \\ \hat{p}_k - \varepsilon^* &\leq p_k \leq \hat{p}_k + \varepsilon^* \end{aligned}$$

Rationalization problem for simulation

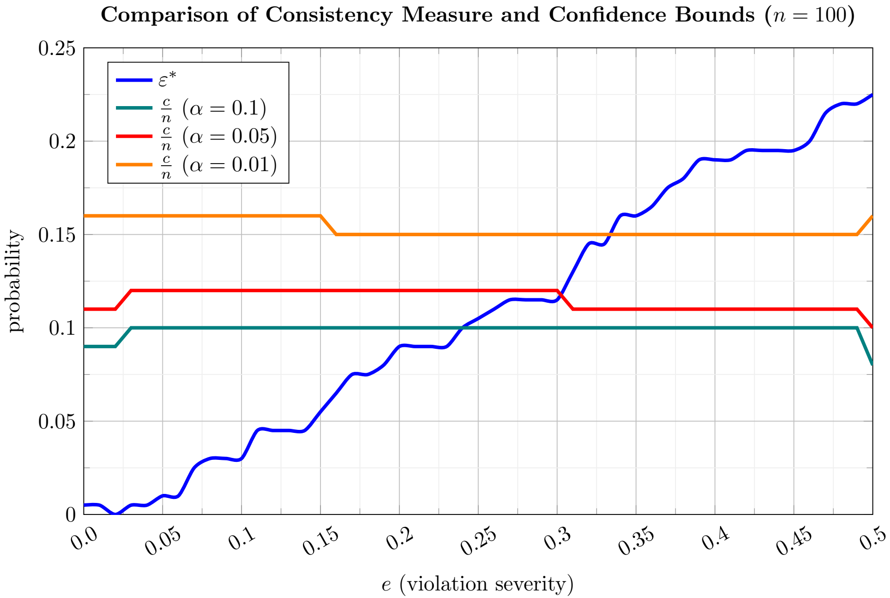
- We consider the following DGP with marginal regularity:

	a	b	c
$\{a, b\}$	$0.5 - e$	$0.5 + e$	$\cdot$
$\{a, b, c\}$	0.5	0.5	0

Table: True data generating process (DGP), p

- Estimate $\hat{p}$ through counting numbers: $\hat{p}(a, A) = \frac{n_a}{n}$
  - n: the number of choice drawn in total
  - n_a: the number of alternative a is chosen

Simulation result of the comparison



Simulation result of the comparison

e \ n	20	30	50	100	200	500
0.0	0	0	0	0	0	0
0.1	0	0	0	0	0.006	0.14
0.2	0.002	0	0.002	0.056	0.422	0.994
0.3	0	0.008	0.052	0.474	0.966	1
0.4	0.008	0.056	0.328	0.97	1	1
0.5	0.028	0.304	0.742	1	1	1

Table: Frequency of ε^* rejection with $\alpha = 0.01$ and 500 simulations

- Measure falls into the rejection region with the following properties
  - The more *severe* the violation (e), the more *frequent* the rejection
  - The *larger* the sample size (n), the more *frequent* the rejection (for $e \neq 0$)

Conclusion

Our results can be summarized as follows:

1. The consistency measure achieved
   - *rationalization with error* as **linear programming** problem
2. The measure captures **severity** of RUM violation of data
   - *stochastic* version of CCEI
3. The measure enters to the **rejection region** with nice properties
   - measure falls into rejection region as e and/or n increases

Thank you!