Identification of Preferences with Unobserved Budget Using Vertical Differentiation within Brands

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Abstract

A lot of discrete-choice analyses routinely skip over the affordability constraints that affect consumers' consideration sets, thereby crediting taste heterogeneity for behavior that is, in truth, driven by limited opportunity. This paper develops a framework for identifying preferences and budget distribution in discrete choice models where unobserved budget constraints determine consumers' consideration sets. Building on the vertical differentiation within brands—where products are ordered by quality and price—we exploit the single-crossing property to recover preference parameters and the distribution of unobserved budgets. Our key insight leverages the asymmetric consumer response to price and quality changes for products that are "Unaffordable First-Best" for marginal consumers. When a product's quality attributes do not affect consumers' choices but its price does, this asymmetry reveals information about consumer budget constraints. We show that preference parameters can be identified without knowing the budget distribution, and once these parameters are identified, the budget distribution can be nonparametrically recovered. We extend our analysis to settings with subsistence levels and show that both preference parameters and subsistence levels can be jointly identified using panel data. Our approach contributes to the literature on nonseparable models and offers practical insights for markets with large-ticket items where affordability constraints significantly influence consumer choices.

Keywords: discrete choice model, budget constraint, consideration set, single-crossing property, nonseparable model, random coefficient

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1 Introduction

Through seminal work of Berry et al. [1995][3], discrete choice models has been widely used in empirical industrial organization literature and further with abundance of competitive structure coming from random coefficients. As the distribution of random coefficients can apparently be interpreted as preference distribution of consumers with respect to the variable the random coefficient is multiplied, the flexibility of the model has a ground of explaining consumer behavior that does not follow logit preference structure as preference heterogeneity in product variables. In this regard, consumers' *opportunity* of buying a product has been swept under the rug for a lot of applied works in discrete choice.

As mentioned in the start of the first chapter on Deaton and Muellbauer [1980][6], such an emphasis on the role of preferences may result in erasing the limits of consumer choice. The limitation of opportunity is rather a global property for all consumers than preferences. Any behavior can be interpreted from the perspective of explained idiosyncratic preference of particular consumer, while the affordability will be the same for all consumers when their budget is the same. Moreover, some part of the opportunity can actually be observable such as average income, subsistence level of the market district of interest, or even individual income which is related to the consumer's budget. They lie as resources for researchers exploit them. In the spirit of the universality, our paper investigates a model with unobservable budget constraint playing a role in consideration set level, which is a natural extension of the discrete choice model for aggregate demand.

We construct an identification strategy that allows us to obtain preference parameters that determines the range of budget that makes consumers to choose certain product. Then, we identify the budget distribution of the consumers nonparametrically. In an effort to obtain the budget distribution, we leverage on particular product relationship—*vertical differentiation*—that is, the relationship between products in a brand that is not affected by the idiosyncratic preferences of consumers. The vertical differentiation within a brand is a prevalent phenomenon for many markets, such as automobile market, smart phone, subscription options in OTT markets, PC markets, and etc., where the products are differentiated by quality and price. Thus, the consumers' choice of product within a brand is determined by their budget allowance.

The vertical differentiation within a brand allows us to exploit the preference ordering across products predetermined by certain budget level (cutoff budget between product pairs), where what properties should these cutoff budgets should have are dictated by Single-Crossing Property (SCP). Such method in empirical Industrial Organization (IO) seems to appear first in Bresnahan [1987][4] by taking consumers' willingness to pay as their type and also assume the heterogeneity of it follows uniform distribution. The major difference of our paper and Bresnahan [1987][4] is in that we deal with the possibilities of some products being unaffordable due to budget constraint, so that different consumers may have different consideration set to start with. Also, we allow for any distribution, even nonparametric one since our identification for budget distribution is nonparametric.

SCP we are using here is extensively aligned with Barseghyan et al. [2021][2]. By assuming the vertical differentiation across products, we have less burden for proving that SCP holds compared to their paper. However, we have an additional complication that the cutoff budget levels are unobservable compared to their setting due to unknown preference parameters that enters into the cutoff budget level formula. Moreover, there will be another layer to consider about the possible orderings SCP regulates by the fact that prices of products can also interact with the cutoff budgets to exactly pin down which budget groups will choose certain products. These are major hindrances to import the framework Barseghyan et al. [2021][2] have devised for insurance choice problem with random consideration set mechanism to typical empirical IO discrete choice setting but with budget constraints.

To resolve the issues, the best way is to find ways to identify the preference parameters separately which will give both cutoff budget levels and cutoff budget orderings. We do this by detecting products that are Unaffordable-First-Best (UFB) for a (marginal) group of consumers. These group of consumers will have a behavioral pattern that the product quality attributes will not affect their choice. They only respond to price change since what they were lacking was not the desirability of the product of interest, but the opportunity itself. These gives us insight of our identification method. By the marginal change in price of these UFB products, the change in market share of the right lower-lineup product will be coming from consumers who were choosing the product as suboptimal because of the limits in affordability of the first-best. These marginal change is special, since we actually know where this marginal group of consumers' budget lies, the price of the UFB. By having the variation in other brands with same-priced UFBs, we can divide the variation to cancel out the density value at the budget, and single-out the idiosyncratic preference shock distribution, which degenerates the identification problem to simple discrete choice without consideration set problem.

Typically in traditional theory, budget is thought of as the set of price of alternatives and the income/wealth/outlay consumers have when they face the choice problem. The concept of budget becomes largely ambiguous[1] when we step outside of the theory and moves towards empirical applications. This poses the question of which budget measure to use even though we have some data on it. In the original BLP paper, for example, extrapolates Current Population Survey (CPS) to estimate the parameters for the

[1] For this reason, we use the terminology "budget" as a replacing word of "income" or "total expenditure" although typically budget is jointly invoking price and income/outlay. Anyway the prices are already given as observables, and thus budget will be unknown only for the upper bound of total expenditure.

distribution of household income using log-normal income distribution, although it is not their model does not involve budget constraint. As automobile market in BLP is large-ticket product and thus the budget constraint is not a trivial issue, it is serious concern that their model is not exactly containing consumer opportunity aspects and, moreover, that general income distribution can easily defy the budget distribution of the market.

In our model, we assume the budget of individual consumers is unknown to the econometrician. This way we can circumvent the issue of which data to use for the budget and rather data tell us how the distribution looks like no matter how researcher perceives it. The budget modelled in our setting will be representing total purchase power of a consumer that can be liquidated for the decision problem. It will be sensitive to the purchase period which will lead to the interpretation of budget for the time lapse. Additionally, the identification of the budget distribution will be generating the data that is not readily available to the researcher, although the identified distribution will depend on our choice of utility function. Despite the pitfall, we can compare the data with the existing data of income or wealth distribution to see how the purchase power in the market district relates to the income or wealth data of the population in the district.

In discrete choice in particular, the effect of budget seems to manifest itself in two roles: (1) it determines the consideration set of consumers, and (2) it affects the indirect utility derived from the choice problem conditional on a discrete alternative. The former role has been insinuated already in McFadden [1981][11] by introducing the choice set of a consumer as a budget set. However, it seems less attention has been paid to the former role. In contrast, standard BLP model has the latter role as a core of the model by situating the budget as a source of affecting the individual price sensitivity.

An interesting point is that, as pointed out in Berry et al. [1995][3] and also recently by Pesendorfer et al. [2023][12], it is possible to capture both roles of budget through specifying the utility for a composite good c as $\alpha \ln(c)$ with α being th homogenous price sensitivity parameter. The indirect utility conditional on a discrete product j, as an example, that is derived from the choice problem will contain $\alpha \ln(y - p_j)$, where y is the budget and p_j being the price of product j. The issue here is that since budget constraints are already internalized in indirect utility function of consumers, there cannot be UFB products, i.e., consumers move on to other products as their budget approaches the price of certain product. This can be translated to saying the price of a product cannot be larger than the relevant cutoff budget level, so that there is no one who desires something that is not affordable as their first best, disabling our identification strategy.

Even in such utility function, we can show that, by deviating a little from standard linear budget constraints, our identification strategy works. To be specific, if linear budget constraint as additional requirements such as there is a minimum composite good consumption level, there occurs disparity between desirability and affordability of the product, that is, UFB can exist. With this specification, we will also show that it is less toxic to use BLP indirect utility to large-ticket product markets with adjustments. The minimum composite good could be understood as subsistence level for consumers which may be observed in public data. We can import this data for the minimum consumption and augment the identification strategy used here. Moreover, if we assume panel data and also assume that this minimum consumption is not changing for a given amount of time, then the value of it can also be treated unknown and be identified. For this paper, we assume the minimum composite good consumption to be homogenous across consumers.

In this regard, our paper contributes on the literature on identification in nonseparable models. We can see in section 2 that the simple direct utility that embeds the budget constraint is nonseparable in unobserved random component, budget. In section 4, although our model is parametric, we can see that this nonseparable utility does not satisfy the conditions for nonparametric identification in relatively new nonseparable discrete choice model literature. Matzkin [2019][10], for example, make use of the observable outside option utility values or assume differentiability which is also the case in Chernozhukov et al. [2019][5]. In our simple utility, value of a product hits the negative infinity when the product is unaffordable, the utility of outside option is not observable due to unobserved budget, and also different people will have different non-differentiable points which complicates the application of identification strategy above. Our strategy is using more structures that allows us the identification of these nonseparable models.

The following content will be organized as follows. In section 2, we will introduce the model assumptions and budget constraints that we will use in the identification strategy. We will discuss the direct utilities that allows budget constraints to manifest in consideration set, and also the SCP. In section 3, we will introduce the main identification strategy of preference parameters and then of the budget distribution. In section 4, we will discuss the identification of the same object on top of the subsistence level, with the indirect utility functions that internalizes budget constraints. In section 5, we will conclude with a discussion on the implications of our findings.

2 Model Assumptions and Budget Constraints

Consider there are T-periods of observations for a market of interest. Each period consists of a continuum of consumers and denote each continuum $\mathcal{I}_t$, where the heterogeneity in this model comes from the additive brand -level idiosyncratic preference shock and the different budget constraint they face, i.e., $i \in \mathcal{I}_t$ are iid drawn from the budget distribution $F_Y(\cdot)$ of potential consumers in the market, assumed to be *fixed* across given period of time[2] and unknown the researcher. The idiosyncratic preference distribution, $F_{\{\varepsilon_\ell\}_{\ell\in\mathcal{L}}}(\cdot)$, will be independent of the budget distribution.

Assumption MA1 (Independence)**.** $F_{\{\varepsilon_\ell\}_{\ell\in\mathcal{L}},Y} = F_{\{\varepsilon_\ell\}_{\ell\in\mathcal{L}}} \cdot F_Y$.

Moreover, assume that $F_Y(\cdot)$ has the density, denoted $f_Y(\cdot)$ which is continuous and strictly positive on $[\underline{y}, \overline{y}] \subseteq \mathbb{R}_+$ and 0 elsewhere. These assumptions are refined as follows:

Assumption MA2 (Budget Distribution)**.** Budget of individual $i \in I_t$ at time $t = 1, \ldots, T$, Y_{it}, satisfy $\{Y_{it}\} \overset{iid}{\sim} F_Y$ for all $i \in \mathcal{I}_t$ and $t = 1, \ldots, T$, where $F_Y(\cdot)$ has density $f_Y(\cdot)$ s.t. $Supp(Y_{it}) = [\underline{y}, \overline{y}] \subseteq \mathbb{R}_+$.

For each period, suppose that we can observe the aggregate quantity of sales and, thus, also observe market shares s_{jt} for each alternative[3] $j = 0, 1, \ldots, J$ and let $\mathcal{J} = \{0, 1, \ldots, J\}$ be the set of all alternatives inside the market. We perceive product 0 as a choice of not buying anything from the market of interest, a.k.a. the outside option. Let $\mathcal{L} = \{\ell_0, \ell_1, \ldots, \ell_{|\mathcal{L}|}\}$ be the partition of $\mathcal{J}$ that represents product lineup for each brand. ℓ_0 consist of the sole constituent, product 0.

In our model, the outside option will be a sensitive object. The amount of people we believe to have considered the choice problem but have chosen not to buy will affect from whom the budget distribution is made out of. Some applied literature make use of the population of the designated market area (DMA) as the total participants which, in turn, make us view the budget distribution is of the total population in the DMA through our setting. If we garner other methods to set the consumers who have chosen outside option, the population the budget distribution is about will also change accordingly.

Let c be a composite good consumption outside of the market of interest, $x_{jt} \in \mathcal{X}_j \subseteq \mathbb{R}^K$ observed characteristics vector of size K for product j and $\xi_j \in \Xi_j \subseteq \mathbb{R}$: time-fixed utility component of j unobserved to the econometrician and cannot be captured by x_{jt}. Denote $\mathcal{X} := \underset{j\in\mathcal{J}}{\times} \mathcal{X}_j$ and $\Xi := \underset{j\in\mathcal{J}}{\times} \Xi_j$. For the outside option, $\mathcal{X}_0 = \Xi_0 = \{0\}$. Let $\theta_1 = (\alpha', \beta')'$. α denote the preference parameter associated with the former and β the latter. $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are parameter space for α and β, each, and thus $\Theta_1 := \mathcal{A} \times \mathcal{B}$. Assume that the idiosyncratic preferences are separable from the systematic part. The consumers in our model have the same systematic direct utility function for each $j \in \mathcal{J}$, $u(\cdot; \alpha) : \mathbb{R} \to \mathbb{R}$ is the parametric direct utility function for the composite good consumption that is monotonic and continuous, and $h(\cdot, \cdot; \beta) : \mathcal{X} \times \Xi \to \mathbb{R}$ is the identical parametric direct utility function for all discrete products.

[2] This is a strong assumption for a durable good, since when there is a price drop on certain product that draws certain budget group to the market outside of our data, we can expect for the next several periods, these people would not show up in the market which undermine the identification of the actual budget distribution.

[3] We use "alternatives", "products", and "goods" interchangeably.

In our case, the fact that the domain of $u(\cdot;\alpha)$ is $\mathbb{R}$ is important, since this is related to the possibility of using the utility function to capture the affordability of the composite good consumption. Even when negative consumption is not allowed for the composite good, the utility function will still be defined for negative consumption so that it is used to compare the desirability of products. However, if we assume that the domain is $\mathbb{R}_{++}$, then the comparison above is not going to be possible by construction, which means utility itself is internalizing the budget constraint.

Let $\ell(j) \in \mathcal{L}$ denote the brand where product j lies in. For each period $t = 1, \ldots, T$, consumer $i \in \mathcal{I}_t$ in the market with a realized budget y_{it} are drawn to a following decision problem with the linear budget constraint:

$$
\begin{aligned}
&\max_{c_t \geq 0, j \in \mathcal{J}} u(c_t;\alpha) + h(x_{jt}, \xi_{\ell(j)};\beta) + \varepsilon_{i\ell(j)t} \\
&\text{s.t. } c_t + \sum_{j \in \mathcal{J}} p_{jt}\mathbf{1}_{jt} \leq y_{it}, \ \sum_{j \in \mathcal{J}} \mathbf{1}_{jt} = 1 \\
\Rightarrow &\max_{j \in \mathcal{J} \text{ s.t. } y_{it} \geq p_{jt}} \left\{ u(y_{it} - p_{jt};\alpha) + h(x_{jt}, \xi_{\ell(j)};\beta) + \varepsilon_{i\ell(j)t} \right\}, \qquad (1)
\end{aligned}
$$

where $\varepsilon_{i\ell t}$ is and the consumer i's idiosyncratic preference for brand $\ell \in \mathcal{L}$, $p_{jt} \in \mathcal{P}_j \subseteq \mathbb{R}_{++}$ is the price of product j at time t with $\mathcal{P}_0 = \{0\}$.

We use the brand-level preference shock instead of product-level to utilize the fact that consumers are clear about what is the best product in terms of quality for each brand. If there is a product-level shock, then consumers disagree with what is the best product although all the quality-specifying characteristics are better for certain product compared to the others in the brand. But consumers will still have some disagreement on which brand is of their best even though systematic part shows that the similar lineup products in different brands have some hierarchy in terms of quality. Likewise, we also assume that the unobserved utility for each product are brand-specific rather than product-specific, i.e., $\xi_j = \xi_\ell$ for $\forall j \in \ell$. This can be relaxed to allow product-specific fixed effects (time-invariant), but the identification requirements will be more demanding with large amount of price variation needed which we will see in section 3.1. Therefore, we may be mixing the usage of ξ_j to express the generality of certain parts.

For practical purpose, as well as expositional one, we will work with $h(x_{jt}, \xi_{\ell t};\beta) = x'_{jt}\beta + \xi_\ell$ and $\{\{\{\varepsilon_\ell\}_{\ell \in \mathcal{L}}\}_{i \in \mathcal{I}_t}\}_{t=1,\ldots,T} \overset{iid}{\sim}$ Type I extreme value distribution with scale parameter 1 (T1EV(1)).

2.1 Consumer Maximization Problem with a Subsistence Level

For some problems we will deal with later, consumer $i \in \mathcal{I}_t$ may face the following maximization problem with a minimum consumption level $c_t^{\min}$ at time t:

$$
\begin{aligned}
&\max_{c_t \geq 0, j \in \mathcal{J}} u(c_t;\alpha) + h(x_{jt}, \xi_{\ell(j)};\beta) + \varepsilon_{i\ell(j)t} \\
&\text{s.t. } c_t + \sum_{j\in\mathcal{J}} p_{jt}\mathbf{1}_{jt} \leq y_{it},\ \sum_{j\in\mathcal{J}} \mathbf{1}_{jt} = 1,\ \&\ c_t \geq c_t^{\min} \\
\Rightarrow &\max_{j\in\mathcal{J} \text{ s.t. } y_{it}-c_t^{\min}\geq p_{jt}} \left\{ u(y_{it}-p_{jt};\alpha) + h(x_{jt}, \xi_{\ell(j)};\beta) + \varepsilon_{i\ell(j)t} \right\},
\end{aligned} \tag{2}
$$

$c_t^{\min}$[4] > 0 is the homogenous minimum consumption level (subsistence level) at time t. The researcher will have a choice to take $c_t^{\min}$ as an observable by letting it be the subsistence level that can be extrapolated outside of the market data, or rather take it as an unobservable which is identified in this paper only when $c_t^{\min} = c^{\min}$, a fixed quantity across the period of interest.

To integrate the affordability into the utility function, we may consider $u(\cdot;\alpha) : \mathbb{R}_{++} \to \mathbb{R}$. If we set $u(c;\alpha) = \alpha \ln c$ as it was introduced in Berry et al. [1995][3], then the following conditional indirect utility function occurs: $\alpha \ln(y_{it} - p_{jt}) + x_{jt}'\beta + \xi_{\ell(j)} + \varepsilon_{ijt}$, where $h(x_{jt}, \xi_{\ell(j)};\beta) = x_{jt}'\beta + \xi_{\ell(j)}$. The functional form is interesting in terms of affordability since it is embedding the exclusion of unaffordable goods by sending the utility of the unaffordable to $-\infty$. Therefore, under the $u(\cdot;\alpha)$ specification given, the consideration set concept is irrelevant with the mere linear budget constraint. That is, the budget constraints are totally regulated through the indirect utility function, not affecting the consideration set level.

The celebrated BLP model can be derived from the extended ARUM form (Allen [2024][1]) above by assuming price is miniscule compared to the budget as pointed out in Pesendorfer et al. [2023][12]. Then,

[4] Eventually, this can also be thought of as transferred consumption level that cannot be liquidated. The interpretation comes from the observation that if $c_t \geq c_t^{\min}$ must hold then, we can change this problem into

$$
\begin{aligned}
&\max_{c_t\geq 0, j\in\mathcal{J}} u(c_t + c_t^{\min};\alpha) + h_j(x_{jt}, \xi_j;\beta) + \varepsilon_{i\ell(j)t} \\
&\text{s.t. } c_t + \sum_{j\in\mathcal{J}} p_{jt}\mathbf{1}_{jt} \leq \underbrace{y_{it} - c_t^{\min}}_{=:y_{it}'},\ \sum_{j\in\mathcal{J}} \mathbf{1}_{jt} = 1.
\end{aligned}
\qquad \Rightarrow \max_{j\in\mathcal{J} \text{ s.t. } y_{it}'\geq p_{jt}} \left\{ u(y_{it}' - p_{jt} + c_t^{\min};\alpha) + h_j(x_{jt}, \xi_j;\beta) + \varepsilon_{i\ell(j)t} \right\}
$$

Viewing $y_{it} - c_t^{\min}$ as just a new budget y_{it}', then this is similar with the original maximization problem, but instead of minimum expenditure on composite good, there is a fixed transferred consumption $c_t^{\min}$. In this regard, it would have been the best to set this heterogeneous across individual ($c_{it}^{\min}$) and also identify the distribution of it, but the author has no knowledge of possibility of identification of the object separately from the budget distribution, and thus put this out of the scope.

by Taylor approximation or, equivalently, using log approximation[5], we get

$$\alpha \ln(y_{it} - p_{jt}) + x'_{jt}\beta + \xi_j + \varepsilon_{ijt} \underset{\text{equivalent to}}{\Longrightarrow} -\alpha \frac{p_{jt}}{y_{it}} + x'_{jt}\beta + \xi_j + \varepsilon_{ijt},$$

which is exactly BLP when $h_j(x_{jt}, \xi_j; \beta) = x'_{jt}\beta + \xi_j$ and also allowing for random coefficients ($\beta \to \beta_i$ & $\alpha_i := \frac{\alpha}{y_{it}}$[6]). This implies that the familiar linear BLP indirect utility function is not meant to be used for the large ticket products which makes the approximation to poor.

Now, with the same indirect utility specification, consider adding the constraint $c_t \geq c_t^{\min}$. We have already checked above that replacement of c_t into $y_{it} - p_{jt}$ is not affected by the additional constraint, and thus we still have $\alpha \ln(y - p_{jt}) + h(x_{jt}, \xi_j; \beta) + \varepsilon_{i\ell(j)t}$. However, we can see that we cannot rely on the term inside $\ln(\cdot)$, since the actual constraint is now $p_{jt} + c_t^{\min} \leq y_{it}$. Therefore, the consideration set will still have a role[7] to catch unaffordable products through the budget constraint with the subsistence level requirement.

We can rewrite the conditional indirect utility function, as already been seen in footnote 4, as $\alpha \ln(y'_{it} + c_t^{\min} - p_{jt}) + h_j(x_{jt}, \xi_j; \beta) + \varepsilon_{i\ell(j)t}$ for j s.t. $y'_{it} \geq p_{jt}$. Then, y'_{it} can be understood as leftover income that can be used for the discrete product. Here, even though p_{jt} could be large compared to y'_{it}, if it is relatively small including $c_t^{\min}$, we can implement the same logic to derive widely used linear BLP indirect utility and obtain

$$-\frac{\alpha}{y_{it} + c_t^{\min}} p_{jt} + x'_{jt}\beta + \xi_{\ell(j)} + \varepsilon_{i\ell(j)t} \text{ for } y_{it} \geq p_{jt}. \tag{3}$$

Equation (3) is similar to the indirect utility function that was used in Song [2015][14]. One salient difference of (3) from the criticism of using linear BLP to the large priced goods is that since the constraint is $p_{jt} \leq y'_{it}$, not $p_{jt} \leq y'_{it} + c_t^{\min}$, it is much less innocuous to large ticket products with consideration set being regulated by the constraint above.

5 $\log(1 + a) \approx a$ when $a \in \mathbb{R}$ is miniscule.

6 Note that since we assumed the distribution of y_{it} is fixed across t, denoting $\frac{\alpha}{y_{it}}$ into α_i, which is also a random variable, is only a loss of realization, not the distribution.

7 We may try to adjust the utility function to have the extended ARUM property by exploiting modified direct utility for composite good: $u(c, c^{\min}; \alpha) = \alpha \ln(c - c^{\min})$. However, our identification cannot apply to such specification. Moreover, sticking with the utility form $u(c; \alpha) = \alpha \ln c$ has a natural interpretation in relationship between two different constraints. We compare the utility value of the same consumer with and without the constraint by using the identical utility function, whereas this would be impossible when it is altered by the above version. If we stick with the $u(c, c^{\min}; \alpha)$, then we would have to set $c^{\min} = 0$ for the linear budget constraint, whereas set $c^{\min} = c_t^{\min}$ for the subsistence level constraint which breaks the uniformity of the two situations.

If we stick with log specification, we can reduce our interest to the following indirect utility function:

$$
\begin{aligned}
V(y_{it}\mathbf{1}_{|\mathcal{J}|} - \boldsymbol{p}_t, \boldsymbol{x}_t, \boldsymbol{\xi}, j; \alpha, \beta) :=& \max_c \left\{ \alpha \ln(c) + x_j'\beta + \xi_{\ell(j)} + \varepsilon_{\ell(j)} \middle| c + p_j \leq y \;\&\; c \geq c^{\min} \right\} \\
& \text{for } j \in \{j' \in \mathcal{J} | p_j + c^{\min} \leq y\} \\
=& \alpha \ln\left(y_{it} - p_{jt}\right) + x_j'\beta + \xi_{\ell(j)} + \varepsilon_{i\ell(j)t} \text{ for } j \in \mathcal{J}(y_{it}),
\end{aligned} \tag{4}
$$

where $\mathcal{J}(y_{it}) = \{j' \in \mathcal{J} | p_{j't} + c_t^{\min} \leq y_{it}\}$, or shortly $\mathcal{J}_{it}$.

2.2 Model Assumptions for Single-Crossing Property

Our goal is to give the point identification of preference parameters α, β and the budget distribution $F_Y(\cdot)$ given the connected support of $[\underline{y}, \overline{y}]$ making use of Single-Crossing Property (SCP) implied by the vertical differentiation within brands. The identification using SCP was investigated heavily on Barseghyan et al. [2021][2]. In our setting, vertical differentiation will imply SCP under mild conditions. We are going to assume vertical differentiation within a brand which is not affected by the idiosyncratic preferences.

Assumption MA3 (Vertical Differentiation)**.** For each brand $\ell \in \mathcal{L}$, label products in an order that satisfies $j_1^\ell < j_2^\ell < \cdots < j_{|\ell|}^\ell$, then $p_{j_1^\ell} < p_{j_2^\ell} < \cdots < p_{j_{|\ell|}^\ell}$. Given the ordering and $\beta \in \mathcal{B}$, $h(x_{j_1^\ell}, \xi_{j_1^\ell}; \beta) < h(x_{j_2^\ell}, \xi_{j_2^\ell}; \beta) < \cdots < h(x_{j_{|\ell|}^\ell}, \xi_{j_{|\ell|}^\ell}; \beta)$ is the case.

Assumption MA3 restricts the relationship between products, specifically between $\boldsymbol{x}, \boldsymbol{\xi}$, and β. For example, if β are all positive and we have linear-in-parameter for h, then $\forall r, s$ s.t. $0 \leq r < s \leq |\ell|$, $x_{j_r} > x_{j_s}$ and $\xi_{j_r} > \xi_{j_s}$ is not allowed. Of course, for some attribute x^q, $x_{j_r}^q > x_{j_s}^q$ or $\xi_{j_r} > \xi_{j_s}$ could be allowed, but in the end, $h_{j_r} < h_{j_s}$ should be maintained to satisfy the assumption. When shedding light on the fact that prices and product qualities (h values) are random variables across time, we can view **Assumption** MA3 as imposing stochastic monotonicity of the price and qualities of products in a brand, each.

The violation of the assumption will occur if some product has higher price but its quality is lower than or equal to the lower-lineup products, then the product will be dominated by such products so that no matter what consumer budget is, it will not be selected. Thus, if there are products that has 0 market shares, then we should rule that product out from the scope. In the follow-up research, if we want to integrate random coefficients on preference parameters of product attributes, then we might want to allow for dominated product to emerge. For example, in smart phone industry, it is typically the case that the most high-end product of each brand is typically the heaviest. Consumers with high sensitivity to the weight of the product will exclude the product from their consideration by making it dominated product, while some consumers with weight endurance will take the product as the highest quality product. However, this is not of our scope in this paper and left to future research.

We can show that this assumption endow us with the SCP defined on Barseghyan et al. [2021][2] along with the following utility assumptions.

Assumption MA4 (Utility Requirements)**.** The utility function for the composite good, $u(c;\alpha)$ is monotonic, continuously differentiable, and strictly concave in c. The utility function for the discrete good, $h(x,\xi;\beta)$ has the identical form for all j, continuously differentiable for (x,ξ).

Definition 1 (Single-Crossing Property)**.** If, for any two $j<k$ in $\mathcal{J}$, $\exists y_{j,k}:\mathcal{P}\times\mathcal{X}\times\Xi\times\mathcal{A}\times\mathcal{B}\to\mathbb{R}_+$ s.t. $V(y\mathbf{1}_{|\mathcal{J}|}-\boldsymbol{p}_t,c_{it},\boldsymbol{x}_t,\boldsymbol{\xi},j;h,\alpha,\beta)\left\{\begin{smallmatrix}>\\=\\<\end{smallmatrix}\right\}V(y\mathbf{1}_{|\mathcal{J}|}-\boldsymbol{p}_t,c_{it},\boldsymbol{x}_t,\boldsymbol{\xi},k;h,\alpha,\beta)$ as $y\left\{\begin{smallmatrix}<\\=\\>\end{smallmatrix}\right\}y_{j,k}$, respectively, then we say the conditional indirect utility V satisfy the Single-Crossing Property (SCP) in $\mathcal{J}$.

Lemma 1. Under **Assumption** MA3 and MA4, the conditional indirect utility V satisfy SCP in each $\ell\in\mathcal{L}$.

For succinctness, denote $h_j:=h(x_j,\xi_j;\beta)$. This can be easily shown by solving $\forall j,k\in\ell$ s.t. $j<k$, $u(y-p_j)+h_j=u(y-p_k)+h_k$ with respect to y, where the solver will be $y_{j,k}$ function. Let $\psi(y,p_j,p_k)=u(y-p_j)-u(y-p_k)$. Since $p_j<p_k$ and $u(\cdot)$ is assumed to be strictly concave, $\psi(\cdot,p_j,p_k)$ will be a decreasing function. Thus, $y_{j,k}(p_j,p_k,x_j,x_k,\xi_j,\xi_k)=\psi^{-1}(h_k-h_j,p_j,p_k)$. This shows that SCP holds for V which is basically the sum of composite good utility with corresponding discrete utility. Also, note that by **Assumption** MA3, $h_k-h_j>0$ and when the gap increases, i.e., when the quality of the higher lineup gets better or lower lineup gets worse, it leads to decrease of $y_{j,k}$. This is an intuitive result considering that even consumers with less budget would also feel k more desirable than j. Note that since $u(\cdot)$ and $h.$ are assumed to be continuously differentiable, so does $y_{j,k}$ with respect to all its arguments.

Note that **Lemma 1** is just a sufficient condition that is useful for indirect utility function with known direct utility such as (4). However, when we use (3), since it was derived from approximation, we cannot find the direct utility function that gives the indirect utility format. However, we can still show that it follows SCP by taking a derivative of it w.r.t. budget y, $\alpha p/(y+c^{\min})^2$ which says larger the price, higher the increase of utility by the increase of budget, which seemingly captures SCP definition. The cutoff budget $y_{j,k}$ from (3) will be

$$-\frac{\alpha}{y+c^{\min}}p_j+h_j+\varepsilon_{\ell(j)}=-\frac{\alpha}{y+c^{\min}}p_k+h_k+\varepsilon_{\ell(k)}\Rightarrow\ y_{j,k}=\frac{\alpha(p_k-p_j)}{h_k-h_j}-c^{\min}. \tag{5}$$

Thus, the cutoff budget of $j,k\in\ell(j)=\ell(k)$ will be re-evaluation of the price difference of the two with price sensitivity and the quality difference of the two, location shifted by transferred consumption $c^{\min}$, thus

$c^{\min}$ does not affect the order. Observe that depending on α value, the cutoff budget level can be negative because of $c^{\min}$, even when **Assumption MA3** holds. If this happens, it violates the SCP definition. To be rigorous, we should take this into account for the usage of cutoff budgets with (3).

On the other hand, if we use (4), we can be sure by **Lemma 1**, that it satisfies SCP and the cutoff budget $y_{j,k}$ will be

$$(y-p_j)^{\alpha} e^{h_j+\varepsilon_{\ell(j)}} = (y-p_k)^{\alpha} e^{h_k+\varepsilon_{\ell(k)}} \Rightarrow y_{j,k} = \frac{p_j - e^{(h_k-h_j)/\alpha} p_k}{1-e^{(h_k-h_j)/\alpha}} = \frac{e^{h_j/\alpha} p_j - e^{h_k/\alpha} p_k}{e^{h_j/\alpha} - e^{h_k/\alpha}}. \tag{6}$$

The interpretation should follow similarly with the above, but here instead of transferred consumption interpretation, $c^{\min}$ is the subsistence level which is latent in budget constraint.

2.2.1 About the Cutoff Orderings

The interesting ordering structure across the cutoff budgets between products arise due to SCP. The results can be condensed into the following fact which is a copy of Barseghyan et al. [2021][2] **Fact 4**.

Fact 1 (Cutoff Relative Order)**.**
Let $j(=1) < k(=2) < l(=3)$[8] $\in \ell \in \mathcal{L}$. Under **Assumption** MA3 and MA4, given $\boldsymbol{p}$, $\boldsymbol{x}$, $\boldsymbol{\xi}$, and θ, if alternatives $1, 2$, and 3 are not dominated[9], then one and only one of the following cases holds:

(i) $y_{1,2} < y_{1,3} < y_{2,3}$: 2 is the first best in $\{1,2,3\}$ if $y \in (y_{1,2}, y_{1,3})$;

(ii) $y_{1,2} > y_{1,3} > y_{2,3}$: 2 is Never-the-First-Best[10] in $\{1,2,3\}$;

(iii) $y_{1,2} = y_{1,3} = y_{2,3}$: 2 is Never-the-First-Best in $\{1,2,3\}$, except for at $y = y_{1,2}$, where ties occur.

The proof is in Barseghyan et al. [2021][2] Online Appendix "*Proof of Fact 4*."

To better understand why such ordering occurs, focus on three products (1, 2, and 3) situation. By SCP, given $\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}, \alpha$ and β, for a small enough y, we have $1 \succ 2 \succ 3$ for a preference ordering $\succ$, while for a large enough y we get $3 \succ 2 \succ 1$. Also, by SCP, as y grows from the region where the former ordering is the case, the preference ordering changes from $1 \succ 2 \succ 3$ to $3 \succ 2 \succ 1$. The rule of the exchange is that first, only the neighbor two product pairs can be exchanged, and second once the switch has occurred, there

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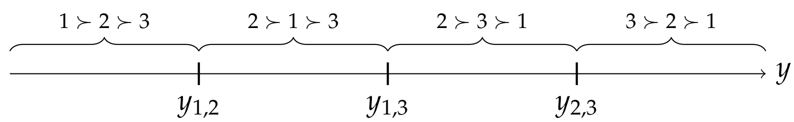
[8] The **Fact 1** holds for all $j < k < l$, but for the visual aid we use 1,2, and 3 instead of j, k, and l.

[9] If, for some good j, $\exists j'$ s.t. for all $y \in [0, +\infty)$, $V(y\mathbf{1}_{|\mathcal{J}|} - \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}, j'; \theta_1) > V(y\mathbf{1}_{|\mathcal{J}|} - \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}, j; \theta_1)$ given $\boldsymbol{x}$, $\boldsymbol{p}$, $\boldsymbol{\xi}$, and θ_1, we call the alternative j is dominated.

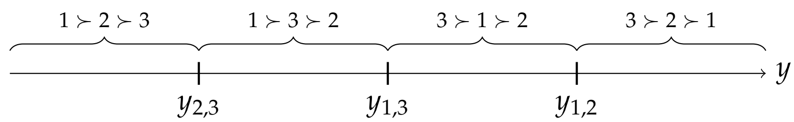
[10] If, for some good j, for each $y \in [0, +\infty)$, $\exists j' \in \mathcal{J}$ s.t. $V(y\mathbf{1}_{|\mathcal{J}|} - \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}, j'; \theta_1) > V(y\mathbf{1}_{|\mathcal{J}|} - \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}, j; \theta_1)$ given $\boldsymbol{x}$, $\boldsymbol{p}$, $\boldsymbol{\xi}$, and θ_1, we call the alternative j is Never-the-First-Best. Basic difference between the dominated good is that Never-the-First-Best doesn't necessarily need the same alternative to be better in all ys unlike the dominated good.

is no reverse. Thus, from $1 \succ 2 \succ 3$, there are only two possibilities: switch 1 and 2, or switch 2 and 3. The former switching point will be $y_{1,2}$ and the latter $y_{2,3}$. Thus, which one moves first will determine the cases **Fact 1**(i) or (ii), or also (iii) if this happens simultaneously at the same point of y.

Observe that, ignoring the tie-case like **Fact 1**(iii), there are two possible ways to change the ordering from $1 \succ 2 \succ 3$ to $3 \succ 2 \succ 1$. These can be visually depicted as follows.



Ordering 1

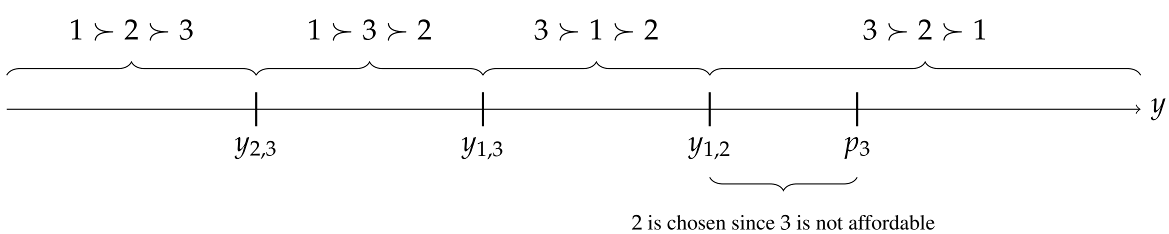


Ordering 2

When there are more than three products, the problem will change to exchange the order in the rule that was explained above but with more products which induces more cases. The number of cases actually explodes in a super-exponential rate. The formula for the number of cases when there are total product of $|\ell|$ is given in Stanley [1984][15] and it is $\frac{\binom{|\ell|}{2}!}{\prod_{l=1}^{|\ell|-1}(2k-1)^{|\ell|-k}}$. This gives 2 for $|\ell| = 3$, 16 for $|\ell| = 4$, and $|\ell| = 5$ is left with 768.

Fact 1 is powerful when it comes to refining the actual case that has occurred. Were affordability not be problematic, by sorting out the products with positive market shares, we can safely say that **Fact 1** (i) is the case for all triplet we have chosen in $\ell \in \mathcal{L}$ for, otherwise, the positive market share for the middle product among the triplet will contradict with the cutoff ordering if the budget distribution does not have a mass at the cutoff-ties.

However, the explosion of cases becomes a real problem when budget constraints kick in, which is a departure of our paper from Barseghyan et al. [2021]. For example, continuing with three products (1, 2, and 3) above, $p_3 > y_{1,2}$ will allow product 2 to have positive market share even when one of **Fact 1** (ii) or (iii). This can be depicted in the following diagram.



Actually, we are not sure if $y_{1,2}$ is the lower bound where people choose 2, since $y_{1,2} < p_2$ will make consumers with budget from p_2 to p_3 to choose 2, but budget below p_2 should still choose 1.

Likewise, affordability issue imposes complication in specifying the cutoff ordering which leads to the obstacle of specifying the likelihood function even assuming the budget distribution to be parametric. The best way to cope with the situation is to identify the determinants of cutoff orderings, which are α and β. This will be the main contribution of our paper, to come up with a strategy to identify these parameters without knowing what the budget distribution of consumers.

2.3 Choice Probability and Integration Regions

We can now construct the choice probability (market share) that is induced from the model discussion above. The basic formula for the market share of product $j \in \mathcal{J}$ can be given as

$$\begin{aligned}
&s_{jt}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta, \theta_2) \\
&= \int_{y_{it}} \int_{\varepsilon_{it}} \mathbf{1}\{j = \arg \max_{j' \in \mathcal{J}(y_{it})} \{V(y_{it}\mathbf{1}_{|\mathcal{J}|} - \boldsymbol{p}_t, c_{it}, \boldsymbol{x}_t, \boldsymbol{\xi}, j'; h, \alpha, \beta)\}\} dF_{\{\varepsilon_\ell\}_{\ell \in \mathcal{L}}}(\boldsymbol{\varepsilon}_{it}; \theta_2) f_Y(y_{it}) dy_{it},
\end{aligned}$$

where s_{jt} is the market share of product $j \in \mathcal{J}$, $\boldsymbol{\varepsilon}_{it} := (\varepsilon_{i1t}, \ldots, \varepsilon_{i|\mathcal{L}|t})$, and θ_2 be the parameters for the idiosyncratic preference joint distribution $F_{\{\varepsilon_\ell\}_{\ell \in \mathcal{L}}}$ with the parameter space Θ_2.

One distinguishing part of this model from others, similar to Song [2015][14], is that once type of a consumer is given, which is budget y_{it} in our model, then the best alternative inside each brand is deterministically given. To make use of this, let $v_\ell^*(y) := \max_{j_l^\ell \in \ell}\{u(y - p_{j_l^\ell}; \alpha) + h(x_{j_l^\ell t}, \xi_{j_l^\ell t}; \beta)\}$. Therefore, if j_1^ℓ is the maximizer at $y = y_{it}$, then $v_\ell^*(y_{it}) = u(y_{it} - p_{j_1^\ell}; \alpha) + h(x_{j_1^\ell t}, \xi_{j_1^\ell t}; \beta)$ suppressing other arguments for succinctness. Additionally, this means that $V(y_{it}\mathbf{1}_{|\mathcal{J}|} - \boldsymbol{p}_t, c_{it}, \boldsymbol{x}_t, \boldsymbol{\xi}, j_1^\ell; h, \alpha, \beta) = v_\ell^*(y_{it}) + \varepsilon_{i\ell t}$.

Making use of these facts, we can refine the market share formula as

$$\begin{aligned}
s_{jt}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta, \theta_2) &= \int_{S_{jt}} \underbrace{\int_{\boldsymbol{\varepsilon_{it}}} \mathbf{1}\{v_{\ell(j)}^*(y_{it}) + \varepsilon_{i\ell(j)t} \geq v_\ell^*(y_{it}) + \varepsilon_{i\ell t}, \forall \ell \in \mathcal{L}\} dF_{\{\varepsilon_\ell\}_{\ell \in \mathcal{L}}}(\boldsymbol{\varepsilon}_{it}; \theta_2)}_{=: P_{\ell(j)}(y_{it}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)} f_Y(y_{it}) dy_{it}, \\
&= \int_{S_{jt}} P_{\ell(j)}(y_{it}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y_{it}) dy_{it} \qquad (7)
\end{aligned}$$

where $S_{jt}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) := \{y \in [\underline{y}, \overline{y}] \big| j = \arg \max_{j' \in \ell(j)} \{u(y - p_{j't}; \alpha) + h_{j't}(x_{j't}, \xi_{j't}; \beta)\} \ \& \ p_{jt} + c^{\min} \leq y\}$ with arguments suppressed to S_{jt}, and $P_{\ell(j)}(y_{it}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)$ is the probability of product in $\ell(j)$ is chosen given y_{it}. The exact purchased product that corresponds to this probability is not determined until we specify where y_{it} lies. It will be the choice probability of j conditional on budget y_{it} if $y_{it} \in S_{jt}$, but not otherwise. Let the choice purchase probability of product j given y_{it} be $P_j(y_{it}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)$. Then, the above observation can be refined as

$$P_{\ell(j)}(y_{it}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)\mathbf{1}\{y_{it} \in S_{jt}\} = P_j(y_{it}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta).$$

Note that S_{jt} is regulating both the best preference region within budget line for product j in brand $\ell(j)$ and also the budget constraints. If the budget constraint doesn't bind solely from the discrete products for any consumers[11], as we have seen in the discussion in the last subsection, then $S_{j_l^\ell} = [y_{j_{l-1}^\ell, j_l^\ell}, y_{j_l^\ell, j_{l+1}^\ell}]$. Once budget constraint restricts the purchase of a product for some consumers that this will not be the case. Finding out what S_j for all $j \in \mathcal{J}$ will be achieved through identification of preference parameters.

Observe that the idiosyncratic preference distribution is of researcher's choice. When we have $\{\varepsilon_\ell\}_{\ell \in \mathcal{L}}$ be independent of each other and follow Type 1 Extreme value distribution, we will have logit formula inside the integration with respect to the budget. Thus,

$$P_{\ell(j)}(y_{it}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) = \frac{e^{v^*_{\ell(j)}(y_{it})}}{\sum_{\ell \in \mathcal{L}} e^{v^*_\ell(y_{it})}} \Rightarrow s_{jt}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta, \theta_2) = \int_{S_{jt}} \frac{e^{v^*_{\ell(j)}(y_{it})}}{\sum_{\ell \in \mathcal{L}} e^{v^*_\ell(y_{it})}} f_Y(y_{it}) dy_{it}.$$

The formula above is the same with Song [2015][14]'s hybrid model except that our model does not have random coefficients and their models does not have budget constraints. The same applies if we use Normal distribution for $\{\varepsilon_\ell\}_{\ell \in \mathcal{L}}$, which will give us Probit model inside the integration with respect to the budget constraint. This paper does not provide identification under nonparametric specification for the idiosyncratic preference distribution.

3 Identification with Linear Budget Constraints

Now, that we have covered the model-induced market share function along with the identification complication that should be resolved, we will first go over the strategy to identify the preference parameters which leads the identification of cutoff ordering that has occurred. The subsistence level can also be identified under strong assumptions, or we can take it as an observable by extrapolating the subsistence level of the relevant market district out side of the given data. Next, we will turn to the identification of budget distribution. The identification of budget distribution will lead to correctly specifying the price elasticities, since it is sensitive to how many consumer's has left certain product not only of their preference but also through budget constraints. The knowledge of the shape of the budget distribution will be important in this sense as pointed out in Pesendorfer et al. [2023][12].

[11] Note that budget constraint cannot be slack in the end because we assume the monotonicity of composite good consumption.

3.1 Identification of Preference Parameters

The core identification problem is that we are ignorant of the budget bounds that bears consumers who choose certain products, i.e., $S_{jt}, \forall j \in \mathcal{J}$ as explained in the previous section. Moreover, the confounding effect comes from discerning whether the variation coming from the budget distribution or from idiosyncratic preference distribution. To take a closer look at the latter issue, consider a product without the affordability problem, that is, if it is product $j_l \in \ell$, $S_{jt} = [y_{j_{l-1},j_l}, y_{j_l,j_{l+1}}]$. Then, the partial derivative of $P_j(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)$ w.r.t. $p_{j_{l+1}t}$[12], for example, will give

$$\frac{\partial}{\partial p_{j_{l+1}}} \int_{\underline{S_{j_l t}}}^{\overline{S_{j_l t}}} P_{j_l}(y_{it}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y) dy = P_{j_l}(\overline{S_{j_l t}}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(\overline{S_{j_l t}}).$$

For the identification of the preference parameters and subsistence level,

Definition 2 (Unaffordable First-Best (UFB) for some consumers)**.** Given product attributes $\boldsymbol{p}, \boldsymbol{x},$ and $\boldsymbol{\xi}$, and a preference parameter vector $\theta_1 \in \Theta_1$, product j is called Unaffordable First-Best for some (UFB) if $p_j = \inf S_j(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \theta_1)$.

The UFB does not arise for every utility function that satisfy SCP. For example, when we try to implement the cutoff budget level function with (6), we will see that $y_{j,k} < p_k$ implies violation of **Assumption MA3**.

$$\frac{p_j - e^{(h_k-h_j)/\alpha} p_k}{1 - e^{(h_k-h_j)/\alpha}} < p_k \Rightarrow p_j - e^{(h_k-h_j)/\alpha} p_k > p_k - e^{(h_k-h_j)/\alpha} p_k (\because \alpha > 0 \,\&\, h_k > h_j) \Rightarrow p_j > p_k. \quad (8)$$

Hence, the identification strategy that rely on the existence of UFB is not globally applicable for all utility functions that satisfy SCP.

The reason why UFB (say, j_{l+1}^{ℓ}) is valuable is because when, ceteris-paribus, the price change of UFB occurs, we can segregate a group of people who move into right lower-lineup product (j_l^{ℓ}) and even know what their budget was, $p_{j_{l+1}^{\ell}}$. We can directly observe this from the partial derivative of the market share of product j_l^{ℓ} with respect to $p_{j_{l+1}^{\ell}}$:

$$\frac{\partial}{\partial p_{j_{l+1}^{\ell}}} s_{j_l^{\ell}}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) = \frac{\partial}{\partial p_{j_{l+1}^{\ell}}} \int_{\underline{S_{j_l^{\ell}}}}^{p_{j_{l+1}^{\ell}}} P_{j_l^{\ell}}(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y) dy = P_{j_l^{\ell}}(p_{j_{l+1}^{\ell}}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(p_{j_{l+1}^{\ell}}).$$

[12] The reason why we take the derivative with respect to the product quality of one-lineup above is because the model we came up with impose a local competition of products within a brand, so that the change of attributes of one product only interact with products neighbor of it, right above and right below.

The partial derivative is only possible when $\underline{S}_{j_l^\ell}$ is not a function of $p_{j_{l+1}^\ell}$. This can be shown in the proof of **Proposition 1** below.

Observe that if there exist another brand that has $p_{j_{l+1}^\ell} = p_{j_{m+1}^{\ell'}}$ $\ell \neq \ell'$, then we have

$$\frac{\partial}{\partial p_{j_{l+1}^\ell}} s_{j_l^\ell} \Bigg/ \frac{\partial}{\partial p_{j_{m+1}^{\ell'}}} s_{j_m^{\ell'}} = \frac{P_{j_l^\ell}(p_{j_{l+1}^\ell}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(p_{j_{l+1}^\ell})}{P_{j_m^{\ell'}}(p_{j_{m+1}^{\ell'}}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(p_{j_{m+1}^{\ell'}})} = \frac{P_{j_l^\ell}(p_{j_{l+1}^\ell}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)}{P_{j_m^{\ell'}}(p_{j_{m+1}^{\ell'}}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)} \tag{9}$$

That is, we can separate out the idiosyncratic preference contribution to the market shares from the budget distribution contribution. Therefore, we can focus on identifying the preference parameters through these ratios of partial derivatives.

Unlike in some situations where partial derivative is not an identifiable object, that is not the case here. That is, the partial derivative in our model setting is possible even when we have $\boldsymbol{\xi}$s as unobservable and correlated with $\boldsymbol{p}$. This is because, the price change of the same product in a market that can be leveraged from our data should be, in most cases, coming time variation of them. The unobservable utilities are time-fixed object that cannot change across time in our setting. Therefore, through investigation of data variation across time, we can identify the partial derivative that is being used in (9) only by checking the observable data structure.

The prerequisite of the analysis above would be to detect which product is actually Unaffordable First-Best (UFB) for some consumers. This can be achieved through the natural asymmetry of the consumer response between quality change and price change, coming from the fact that the product is unaffordable. If a product is unaffordable for a group of consumers although it is the natural first-best according to their preference ordering, then the consumer group will not be responding to the product quality of the unaffordable product, since the reason why they are choosing the suboptimal is not because of the desirability of the optimal product but there is no opportunity of purchasing it. In contrast, the price change is directly affecting this point, by altering the opportunity. The marginal group where budget equals to the their first-best product will lose opportunity by the price change, whereas, in opposite, the marginal unaffordable groups would move onto the newly available first-best if the price marginally goes down.

Whether this is true for all possible cases that can occur in cutoff orderings should be checked and it is indeed the case as the **Proposition 1** states. Also, we can show jointly that if UFB does not exist for a brand, then only one case will be possible as already insinuated in the previous section. The proofs are simple but lengthy, and thus moved to **Appendix 6.1**.

Proposition 1 (Price-Quality Asymmetry)**.** Under **Assumption MA3-MA4**, for a product $j_l \in \ell \in \mathcal{L}$, if

$\frac{\partial P_{j_l}(\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha,\beta)}{\partial x_{j_{l+1}}} = 0$, then j_{l+1} is an Unaffordable First-Best, i.e., $\overline{S_{j_l}} = p_{j_{l+1}}$.

Proposition 2 (No UFB pins down S_js)**.** Under **Assumption MA3-MA4**, if $\nexists$UFB for all products in brand $\ell = \{j_1, \ldots, j_{|\ell|}\}$, then $S_{j_l}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)$ is uniquely determined as $\underline{S_{j_l}} = y_{j_{l-1},j_l}$ and $\overline{S_{j_l}} = y_{j_l,j_{l+1}}$ for $k = 1, \ldots, |\ell|$, $y_{j_0,j_1} = \underline{y}$, and $y_{j_{|\ell|},j_{|\ell|+1}} = \overline{y}$.

According to **Proposition 1**, we can find a product that is Unaffordable First-Best for some consumers. Then, as already explained, using the variation of market share of the right lower-lineup product comes from the marginal change in price of the UFB, we can isolate a group of people whose budget is known to be the price of UFB. Then, if we have groups of consumers who has the same budget, but have chosen different brands, by comparing the product quality-attribute differences between the brands, we can identify the preference parameters α and β through the ratio of partial derivatives of the market shares with respect to the product quality attributes as can be seen in (9).

Now, for the exposition[13], let

$$V(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) = -\alpha \frac{p_j}{y} + x_j'\beta + \xi_{\ell(j)} + \varepsilon_{\ell(j)} \quad \forall j \in \mathcal{J}(y), \ \& \ \varepsilon_{\ell(j)} \sim \text{T1EV}(1). \tag{10}$$

Then, the (9) derived from the above can be expressed as follows:

$$\ln \frac{\partial}{\partial p_{j^\ell_{l+1}t}} s_{j^\ell_l t} \Bigg/ \frac{\partial}{\partial p_{j^{\ell'}_{m+1}t}} s_{j^{\ell'}_m t} = v^*_\ell(y_{it}) - v^*_{\ell'}(y_{it}) = -\alpha \left(\frac{p_{j^\ell_l t} - p_{j^{\ell'}_m t}}{p_{j^\ell_{l+1}t}} \right) + \left(x_{j^\ell_l t} - x_{j^{\ell'}_m t} \right)' \beta + \xi_\ell - \xi_{\ell'}.$$

Observe that p_j/y part containing an unobservable (y) has changed into an observable, $(p_{j^\ell_l} - p_{j^{\ell'}_m})/p_{j^\ell_{l+1}}$, due to knowing $y_{it} = p_{j^\ell_{l+1}}$.

Again, the partial derivative ratio on the left-hand side of (9) is identifiable if the right higher-lineup product is UFB, even when there is a correlation between the price and the unobserved utility. However, now the correlation between the price and the unobserved utility incurs the traditional endogeneity problem which can be solved in two ways. First, by having the same observation that is coming from product pairs distinct with $(j^\ell_l, j^{\ell'}_m)$ in the same brand pair (ℓ, ℓ') at any time t, the unobserved $\xi_\ell - \xi_{\ell'}$ can be eliminated. Or we can either use the time-invariant property of the unobserved utilities and use the differencing between periods. This will give an identification of the preference parameters α and β with the rank condition of observables.

The required exclusion restriction should be $E[\boldsymbol{\xi}|\boldsymbol{z}, \boldsymbol{x}] = \boldsymbol{0}$ instead of $E[\boldsymbol{z}'\boldsymbol{\xi}] = 0$, where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_J)'$ is the vector of unobserved utility components, $\boldsymbol{z} = (z_1, \ldots, z_J)'$ is the vector of instruments,

[13] We have already discussed the toxicity of the BLP expression below. The remedy will be given in the next section.

and $\mathbf{0}$ is a J-dimensional vector of zeros. This means that for each $j = 1, \dots, J$, $E[\xi_j | \boldsymbol{z}, \boldsymbol{x}] = 0$, requiring conditional mean independence rather than just the weaker mean independence implied by $E[\boldsymbol{z}'\boldsymbol{\xi}] = 0$.

To formalize this, denote

$$y_{\ell,\ell',l,m} := \ln\left(\frac{\partial_{p_{j^{\ell}_{l+1}}} s_{j^{\ell}_l}}{\partial_{p_{j^{\ell'}_{m+1}}} s_{j^{\ell'}_m}}\right), \quad w_{\ell,\ell',l,m} := \left(\frac{p_{j^{\ell}_l} - p_{j^{\ell'}_m}}{p_{j^{\ell}_{l+1}}}\right), \quad v_{\ell,\ell',l,m} := x_{j^{\ell}_l} - x_{j^{\ell'}_m}, \quad u_{\ell,\ell'} := \xi_\ell - \xi_{\ell'},$$

$$X_{\ell,\ell',l,m} := \left(w_{\ell,\ell',l,m}, v'_{\ell,\ell',l,m}\right)', \qquad \text{and} \qquad Z_{\ell,\ell',l,m} := \begin{pmatrix} z_{j^{\ell}_l} \\ z_{j^{\ell}_{l+1}} \\ z_{j^{\ell'}_m} \end{pmatrix} \in \mathcal{Z}^3 \subseteq \mathbb{R}^3,$$

where ∂_x denotes the partial derivative with respect to x. The observable equation is

$$y_{\ell,\ell',l,m} = X'_{\ell,\ell',l,m}\theta_1 + u_{\ell,\ell'}.$$

Theorem 1 (Identification of Preference Parameters)**.** Assume the supports of observables and unobservables, $(\mathcal{P}, \mathcal{X}, \Xi)$ satisfy **Assumption MA3**, and $\exists p_{j^{\ell_1}_{l_1}}, \dots, p_{j^{\ell_K}_{l_K}}$ for distinct $\ell_1, \dots, \ell_K \in \mathcal{L}$ that are UFBs and $p_{j^{\ell_1}_{l_1}} = \dots = p_{j^{\ell_K}_{l_K}}$[14]. Given conditional indirect utility

$$V(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha^*, \beta^*) = -\alpha^* \frac{p_j}{y} + x_j'\beta^* + \xi_{\ell(j)} + \varepsilon_{\ell(j)} \quad \forall j \in \mathcal{J}(y), \, \& \, \varepsilon_{\ell(j)} \sim \text{T1EV}(1),$$

where α^* and β^* are the true preference parameters. Then, α^* and β^* (or θ_1^*) can be identified through the ratio of partial derivatives of the market shares in (9) if $\exists \boldsymbol{z} := (z_1, \dots, z_J) \in \mathcal{Z}^{|\mathcal{J}|}$ s.t. $\mathbb{E}[u_{\ell,\ell',l,m} \mid Z_{\ell,\ell',l,m}, X_{\ell,\ell',l,m}] = 0$ a.s.[15] and $\operatorname{rank}\left(\mathbb{E}\left[Z_{\ell,\ell',l,m} X^{\top}_{\ell,\ell',l,m}\right]\right) = 1 + K$, and $\mathbb{E}\|Z_{\ell,\ell',l,m}\|^2 < \infty$ and $\mathbb{E}\|X_{\ell,\ell',l,m}\|^2 < \infty$.

The proof is direct from the above discussion.

Note that identifying α, β can be done without identifying the budget heterogeneity itself. However, to identify the budget distribution itself, we need to recover all the difference between the unobserved utilities itself. However, if we stick with the brand-level unobserved utility, the condition above is the sufficient

[14] The second assumption here may not be needed in identification arguments, since the infinite amount of data is assumed and thus the assumption is absorbed to **Assumption MA3**. However, for clarity and the fact that the products in market cannot be easily assumed to be infinite, I add this line in the assumption.

[15] Equivalently, for every ℓ, l and $\ell' \neq \ell$,

$$\mathbb{E}[\xi_\ell \mid z_{j^{\ell}_l}, z_{j^{\ell}_{l+1}}] = \mathbb{E}[\xi_\ell], \qquad \mathbb{E}[\xi_{\ell'} \mid z_{j^{\ell'}_m}] = \mathbb{E}[\xi_{\ell'}].$$

condition, even when we want to identify the cutoff budget orderings, since brand-level unobserved utility does not affect the cutoff budget orderings that works within the brand. Moreover, we can even allow time varying brand-level unobserved utility $\xi_{\ell t} - \xi_{\ell' t}$, not because we can identify them but since anyway they do not appear in the cutoff budget functions within the same brand.

However, if we allow for product-level unobserved utility, ξ_j, identifying α, β is not enough for identifying the cutoff budget orderings. This will require the unobserved utility to be fixed across the period of interest and the existence of connected graph between all the product, recovering the difference of unobserved utilities across products.

Following the definition in Jackson [2008][7], take $\mathcal{J}\backslash\{0\}$ as a set of nodes and the $J \times J$ adjacency matrix $\mathcal{G}(\{\boldsymbol{p}\}_{t=1}^T, \{\boldsymbol{x}\}_{t=1}^T, \{\boldsymbol{\xi}\}_{t=1}^T; \alpha, \beta)$, where for each $j, k \in \mathcal{J}$ element of $\mathcal{G}$, $\mathcal{G}_{jk}$, is 1 if $\exists t \in \{1, \ldots, T\}$ s.t. $p_{jt} = p_{kt}$ and they are UFB, and $\mathcal{G}_{jk} = 0$ otherwise. Then, the identification of the cutoff budget orderings can be achieved by checking that for any $j, j' \in \ell$ for each ℓ, there is a path in $\mathcal{G}(\{\boldsymbol{p}\}_{t=1}^T, \{\boldsymbol{x}\}_{t=1}^T, \{\boldsymbol{\xi}\}_{t=1}^T; \alpha, \beta)$ that connects the two[16].

Corollary 1.1 (Identification with product-level fixed effects)**.** Assume the supports of observables and unobservables, $(\mathcal{P}, \mathcal{X}, \Xi)$ satisfy **Assumption MA3**, and the graph $\left(\mathcal{J}\backslash\{0\}, \mathcal{G}(\{\boldsymbol{p}\}_{t=1}^T, \{\boldsymbol{x}\}_{t=1}^T, \{\boldsymbol{\xi}\}_{t=1}^T; \alpha, \beta)\right)$ has a path for every $j, j' \in \ell$ for each ℓ. Given conditional indirect utility

$$V(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha^*, \beta^*) = -\alpha^* \frac{p_j}{y} + x_j' \beta^* + \xi_j + \varepsilon_{\ell(j)} \quad \forall j \in \mathcal{J}(y), \ \& \ \varepsilon_{\ell(j)} \sim \text{T1EV}(1).$$

Then, α^* and β^* (or θ_1^*) can be identified through the ratio of partial derivatives of the market shares in (9) if $\exists \boldsymbol{z} := (z_1, \ldots, z_J) \in \mathcal{Z}^{|\mathcal{J}|}$ s.t. $\mathbb{E}\big[u_{\ell,\ell',l,m} \mid Z_{\ell,\ell',l,m}, X_{\ell,\ell',l,m}\big] = 0$ a.s. and $\operatorname{rank}\Big(\mathbb{E}\left[Z_{\ell,\ell',l,m} X_{\ell,\ell',l,m}^\top\right]\Big) = 1 + K$, and $\mathbb{E}\|Z_{\ell,\ell',l,m}\|^2 < \infty$ and $\mathbb{E}\|X_{\ell,\ell',l,m}\|^2 < \infty$.

The proof is direct from the above discussion.

The graph approach is being used to be parsimonious about the price variation. Still, the assumption above could be strong when price variation is not enough. For example, the shortest way j_l^ℓ and j_{l+1}^ℓ having a path is by having $t, t' = 1, \ldots, T$ s.t. $p_{j_l^\ell t} = p_{j_m^{\ell'} t}$ and $p_{j_{l+1}^\ell t'} = p_{j_m^{\ell'} t'}$ holds for some $m \in \ell'$ when all of them were UFBs. This means that price of $j_m^{\ell'}$ has soared its price from $p_{j_l^\ell t}$ to $p_{j_{l+1}^\ell t'}$, which is not a common case when we reflect on the nature of vertically differentiated products. This means that some product's price has changed as if it has become another brands higher-lineup product. Therefore, for the minimum requirement of the identification, we will stick with $\xi_j = \xi_{\ell(j)t}$ instead of $\xi_j = \xi_j$ for each $j \in \mathcal{J}$.

[16] We don't need a full connectedness of graph $(\mathcal{J}\backslash\{0\}, \mathcal{G})$, since for the cutoff budget identification, only the difference between unobserved utilities of products within the same brand matters.

Lastly, if we fully specify the relationship between the unobserved utility and price with known parametric distribution of η_{jt}, then we can identify all the parameters following the well-established identification theorems in the parametric identification literature, Rothenberg [1971][13], in particular. This relaxes the assumption of the idiosyncratic preference distribution from following T1EV to any parametric distribution, such as Normal distribution which means (9) will give us the ratio of two probit market shares. Moreover, we can also allow of other parametric specification for $u(\cdot;\alpha)$ and $h(\cdot,\cdot;\beta)$ if it meets the rank condition for the identification. Again, for expositional purpose, we will be using T1EV distribution for the idiosyncratic preference distribution, but the identification of the preference parameters along with parameters related to specified parametric distributions can be achieved through any parametric distribution of the idiosyncratic preference distribution that satisfies the identification conditions in Rothenberg [1971][13].

Recall that the generalized utility maximization problem with linear budget constraint will give us

$$V(y,\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha^*,\beta^*) = u(y-p_j;\alpha^*) + h(x_j,\zeta(x_j,\xi_j,z_j,\eta_j);\beta^*) + \varepsilon_{\ell(j)},\ \forall j \in \mathcal{J}(y),\ \&\ \varepsilon_{\ell(j)} \overset{iid}{\sim} F_\varepsilon(\cdot;\theta_{21}^*),$$

where α^* and β^* are the true preference parameters, and $\zeta(\cdot;\cdot,\cdot)$ is a known parametric function with known parametric distribution of η_{jt} with the true parameter θ_{22}^*, i.e., $p_{jt} = \zeta(x_{jt},\xi_j,z_{jt},\eta_{jt})$ for each $j \in \mathcal{J}$. Denote $\theta^* := (\theta_1^{*'}\ \theta_{21}^{*'}\ \theta_{22}^{*'})'$. Observe that the generalized version of derivative ratio in (9) can be viewed as the constraint structure of the following:

$$\psi(\theta) := \frac{\partial_{p_{j_l^\ell}} s_{j_l^\ell}}{\partial_{p_{j_m^{\ell'}}} s_{j_m^{\ell'}}} - \frac{P_{j_l^\ell}(p_{j_{l+1}^\ell},\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\theta)}{P_{j_m^{\ell'}}(p_{j_{l+1}^\ell},\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\theta)} = 0.$$

Let the constrained structural parameter space be Θ'.

Our observable distribution of the outcome variable $\mathbf{1}_j$ will depend on $\alpha, \beta,$ and θ_{22} only through $u(\cdot;\alpha)$ and $h(\cdot,\cdot;\beta,\theta_{22})$. $u(\cdot)$, $h(\cdot)$ are called "reduced-form" parameters in Rothenberg [1971][13]. Then, as noted there, the identification possibilities will resort to the identifiability of the original parameters from the reduced-form parameters. Denote the value of the parameters $\phi_1 = \phi_1(\alpha) := u(c;\cdot)$ and $\phi_{2j} = \phi_{2j}(\beta,\theta_{22}) := h(x_j,\xi_j;\cdot,\cdot)$ for $j \in \mathcal{J}$ given $\boldsymbol{x} \in \mathcal{X}$, $\boldsymbol{\xi} \in \Xi$. Define their Jacobian matrices as

$$\Psi(\theta) = \begin{bmatrix} \frac{\partial\psi}{\partial\theta} \end{bmatrix}, \qquad \Phi(\theta) = \begin{bmatrix} \frac{\partial\phi_1}{\partial\theta} \\ \frac{\partial\phi_{21}}{\partial\theta} \\ \vdots \\ \frac{\partial\phi_{2J}}{\partial\theta} \end{bmatrix} \qquad \text{and} \qquad \mathcal{T}(\theta) = \begin{bmatrix} \Psi(\theta) \\ \Phi(\theta) \end{bmatrix}.$$

We say θ is a *regular point* for $\mathcal{T}(\theta)$ if $\exists$open ball around θ, $B(\theta) \subset \mathbb{R}^{\dim(\theta)}$, s.t. $\operatorname{rank} \mathcal{T}(\theta')$ is constant all $\theta' \in B(\theta)$.

Theorem 2 (Local identification under general parametric specification)**.** Assume the supports of observables and unobservables, $(\mathcal{P}, \mathcal{X}, \Xi)$ satisfy **Assumption MA3**, $u(\cdot;\alpha) : \mathbb{R} \to \mathbb{R}$ and $h(\cdot,\cdot;\beta) : \mathbb{R}^2 \to \mathbb{R}$ satisfy **Assumption MA4** and the graph $\left(\mathcal{J}\backslash\{0\}, \mathcal{G}(\{\boldsymbol{p}\}_{t=1}^T, \{\boldsymbol{x}\}_{t=1}^T, \{\boldsymbol{\xi}\}_{t=1}^T; \alpha, \beta)\right)$ has a path for every $j, j' \in \ell$ for each ℓ. Given conditional indirect utility

$$V(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha^*, \beta^*) = u(y - p_j; \alpha^*) + h(x_j, \zeta(x_j, \xi_j, z_j, \eta_j); \beta^*) + \varepsilon_{\ell(j)}, \ \forall j \in \mathcal{J}(y), \ \& \ \varepsilon_{\ell(j)} \overset{iid}{\sim} F_\varepsilon(\cdot; \theta_{21}^*),$$

where θ^* are the true preference parameters, and $\zeta(\cdot;\cdot,\cdot)$ is a known parametric function with known parametric distribution of η_{jt} with the true parameter θ_{22}^*. Then, $\theta^* \in \Theta$ can be locally identified through the ratio of partial derivatives of the market shares in (9), if Θ is an open set, $u(c;\cdot)$ and $h(x,\xi;\cdot)$ are continuously differentiable, ϕ_1 and $\{\phi_{2j}\}_{j\in\mathcal{J}}$ are globally identifiable for $\phi_1 \in \phi_1(\Theta')$ and $\phi_{2_j} \in \phi_2(\Theta')$ for all $j \in \mathcal{J}$, $\exists \boldsymbol{z} = (z_1, \ldots, z_J) \in \mathcal{Z}^{|\mathcal{J}|}$, θ^* is a regular point of $\mathcal{T}(\theta)$, and lastly and mainly, $\mathcal{T}(\theta^*)$ has rank $\dim(\theta)$.

The proof comes from applying **Theorem 5** in Rothenberg [1971][13] to the derivative ratio (9) with replacing the density function there to the density ratio in (9) here.

3.2 Identification of Budget Distribution

Once we have identified the cutoff budget level functions, we are now ready to identify the budget distribution. The identification of the budget distribution is important because it will allow us to estimate the price elasticities of the products, which is sensitive to how many consumers have left certain products not only because of their preference but also through budget constraints. The knowledge of the shape of the budget distribution will be important in this sense as pointed out in Pesendorfer et al. [2023][12].

The idea behind the identification of the budget distribution is that thanks to our identification of the cutoff budget level functions, the cutoff budgets are now working as agents who identify the budget density level at their function values. For example, consider a product $j_l \in \ell$. When the price of product $j_{l+1} \in \ell$ changes, the market share of j_l will vary as below.

$$\frac{\partial}{\partial p_{j_{l+1}}} s_{j_l} = \frac{\partial}{\partial p_{j_{l+1}}} \int_{\underline{S_{j_l}}}^{\overline{S_{j_l}}} P_{j_l}(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y) dy = P_{j_l}(\overline{S_{j_l}}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(\overline{S_{j_l}}) \frac{\partial \overline{S_{j_l}}}{\partial p_{j_{l+1}}}.$$

Since we have identified everything to know $P_{j_l}(\overline{S}_{j_l}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)$ and $\frac{\partial \overline{S_{j_l}}}{\partial p_{j_{l+1}}}$, the only unknown left is $f_Y(\overline{S_{j_l}})$. Thus, the equation above gives us the value of budget density level at $f_Y(\overline{S_{j_l}})$. We can apply the exact same logic to the lower bound of the budget level, $\underline{S_{j_l}}$, and thus we can identify the budget distribution at $f_Y(\underline{S_{j_l}})$. Also, depending on what S_{j_l} is, we can also make use of the variation of attributes of right neighbor products $x_{j_{l+1}}$ and $x_{j_{l-1}}$.

Note that the same cannot be achieved through taking a derivative with respect to the price or quality attributes of j_l itself since there will be an additional confounding effect coming from the fact that consumers who were choosing other brands will also try to buy the product j_l if their budget lies on the budget region that allows that and we have to account for the fact through taking the integral of the derivative of $P_{j_l}(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)$, which gives extra unknown, i.e.,

$$
\begin{aligned}
&\frac{\partial}{\partial p_{j_l}} s_{j_l} = \frac{\partial}{\partial p_{j_l}} \int_{\underline{S_{j_l}}}^{\overline{S_{j_l}}} P_{j_l}(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y) dy \\
=& P_{j_l}(\overline{S_{j_l}}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(\overline{S_{j_l}}) \frac{\partial \overline{S_{j_l}}}{\partial p_{j_l}} + \int_{\underline{S_{j_l}}}^{\overline{S_{j_l}}} \frac{\partial P_{j_l}(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)}{\partial p_{j_l}} f_Y(y) dy - P_{j_l}(\underline{S_{j_l}}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(\underline{S_{j_l}}) \frac{\partial \underline{S_{j_l}}}{\partial p_{j_l}}.
\end{aligned}
$$

Even if we can identify the first term and the last term through the derivative with respect to the price or attributes of right neighbor products the identification of the middle term does not directly tell us information about the budget distribution pointwise. Therefore, we will rather make use of the partial derivative of market shares of product j_l, for example, with respect to the price or attributes of the right neighbor products, j_{l+1} and j_{l-1}, which will not have this confounding effect.

To establish the identification argument for the budget distribution, we should formally define the identification of the budget distribution. We follow Matzkin [2007,2013] [8, 9] for the definition.

Definition 3 (Identification of Budget Distribution)**.** The distribution $F_Y \in \mathcal{F}(\mathbb{R}^+)$ is identified if

$$
\begin{aligned}
\mathcal{M} := \{(u, h, F_{\boldsymbol{s},\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi},Y,\{\varepsilon_\ell\}_{\ell\in\mathcal{L}}) : &\textbf{ Assumption MA1-MA4} \text{ are satisfied, and} \\
&\int F_{\boldsymbol{s},\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi},Y,\{\varepsilon_\ell\}_{\ell\in\mathcal{L}}} dF_{\boldsymbol{\xi},Y,\{\varepsilon_\ell\}_{\ell\in\mathcal{L}}} = F_{\boldsymbol{s},\boldsymbol{p},\boldsymbol{x}}\}
\end{aligned}
$$

is a singleton given $u, h,$ and $F_{\boldsymbol{\xi},\{\varepsilon_\ell\}_{\ell\in\mathcal{L}}}$.

That is, if our model is specified up to the distribution of the budget, then the budget distribution is identified if there is only one distribution that matches the distribution of the observables induced from model with the actual distribution of the observables. As formally stated in Matzkin [2007,2013] [8, 9], if $\mathcal{M}$ has multiple elements, the elements are said to be observationally equivalent, which should not arise to secure the identification.

Now, to specify the agents that identifies the density for each product $j \in \mathcal{J}$, let us introduce the following indexing function. $\sigma : \mathcal{J} \times \mathcal{P} \times \mathcal{X} \times \Xi \to \{1, 2, 3, 4\}$, where

$$\sigma(j_l^\ell, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}) = \begin{cases} 1 & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell - 1, j_l^\ell} \ \& \ \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell, j_l^\ell}, \\ 2 & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell, j_{l+1}^\ell} \ \& \ \underline{S_{j_l^\ell}} = p_{j_l^\ell}, \\ 3 & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell} \ \& \ \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell, j_l^\ell}, \text{ and} \\ 4 & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell} \ \& \ \underline{S_{j_l^\ell}} = p_{j_l^\ell}. \end{cases}$$

Also, let

$$A_1(j_l^\ell) = \{y_{j_l^\ell, j_{l+1}^\ell}, y_{j_{l-1}^\ell, j_l^\ell} : \boldsymbol{x} \in \mathcal{X}, \boldsymbol{p} \in \mathcal{P} \text{ satisfy } \textbf{MA3} \text{ and } \overline{S_{j_l^\ell}} = y_{j_l^\ell - 1, j_l^\ell} \ \& \ \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell, j_l^\ell}\},$$
$$A_2(j_l^\ell) = \{y_{j_l^\ell, j_{l+1}^\ell} : \boldsymbol{x} \in \mathcal{X}, \boldsymbol{p} \in \mathcal{P} \text{ satisfy } \textbf{MA3} \text{ and } \overline{S_{j_l^\ell}} = y_{j_l^\ell, j_{l+1}^\ell} \ \& \ \underline{S_{j_l^\ell}} = p_{j_l^\ell}\},$$
$$A_3(j_l^\ell) = \{p_{j_{l+1}^\ell}, y_{j_{l-1}^\ell, j_l^\ell} : \boldsymbol{x} \in \mathcal{X}, \boldsymbol{p} \in \mathcal{P} \text{ satisfy } \textbf{MA3} \text{ and } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell} \ \& \ \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell, j_l^\ell}\}, \text{ and}$$
$$A_4(j_l^\ell) = \{p_{j_{l+1}^\ell} : \boldsymbol{x} \in \mathcal{X}, \boldsymbol{p} \in \mathcal{P} \text{ satisfy } \textbf{MA3} \text{ and } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell} \ \& \ \underline{S_{j_l^\ell}} = p_{j_l^\ell}\}.$$

Theorem 3. Given α, β, and $\boldsymbol{\xi}$, if $[\underline{y}, \overline{y}] \subseteq \bigcup_{(\boldsymbol{x},\boldsymbol{p},\boldsymbol{\xi}) \in \mathcal{X} \times \mathcal{P} \times \{\boldsymbol{\xi}\} \text{ s.t. } \textbf{MA3} \text{ holds}} \cup_{\ell \in \mathcal{L}} \cup_{j \in \ell} A_{\sigma(j, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi})}(j)$, then F_Y is nonparametrically identified.

The proof is in **Appendix 6.2**.

Another caveat is that the identification of the bounds of the support of the budget, $\underline{y}$ and $\overline{y}$, is possible under rather strong condition even after assuming **MA2**. Because of the assumption, we would directly identify what the bounds are when the upper bounds of the most premium lineup, $j_{|\ell|}$ for ℓ, gives the zero market share for the product, which we didn't allow in the first place. However, even when we allow zero market share to happen, it is not likely that the firms will set the price or attribute values that will give them zero market share back. Therefore, in practice, we will have to assume that the bounds are known or come up with a threshold that takes density value below the threshold to be 0 density, i.e., that is where we will assume where the upper bound lies. The same applies to the lower bound as well.

4 Identification of Preference Parameters and Subsistence Level

4.1 Allowance of Identification for the Budget Constraint Internalized Model

As discussed in section 2, some BLP-like linear-in-parameter models does not in accord with the setting with markets with large price that actually restricts affordability. In section 2.1, this have been somewhat alleviated through the nonlinear budget specification with the existence of minimum composite good consumption, namely, subsistence level. The natural follow-up question is whether identification strategy works under the subsistence level in budget constraint, and if we were to let it be unobserved, how would we identify the subsistence level.

If the subsistence level $c_t^{\min}$ is known for all t, then the identification results above will pass through, since we are just adding another observables to the model. We just change the indirect utility function from (10) to the following using (3):

$$V(y\mathbf{1}_{|\mathcal{J}} - \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}, j; \alpha, \beta) = -\frac{\alpha}{y + c^{\min}} p_j + x_j'\beta + \xi_{\ell(j)} + \varepsilon_{i\ell(j)} \text{ for } j \in \mathcal{J}_{it} = \{j' \in \mathcal{J} : p_{j'} \leq y\}.$$

However, there is newly added utility form that is identifiable, which was mentioned for UFB not arising, without the subsistence level, (4).

$$V(y\mathbf{1}_{|\mathcal{J}} - \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}, j; \alpha, \beta) = \alpha \ln(y - p_j) + x_j'\beta + \xi_{\ell(j)} + \varepsilon_{i\ell(j)} \text{ for } j \in \mathcal{J}_{it} = \{j' \in \mathcal{J} : p_{j'} + c^{\min} \leq y\}. \tag{11}$$

The reason why this is identifiable with the strategies given in this paper is because the subsistence level is lifting the preference from perfectly internalizing the budget constraint unlike standard linear budget constraint. This can be checked when we try to find the counterpart of (8).

$$\frac{p_j - e^{(h_k - h_j)/\alpha} p_k}{1 - e^{(h_k - h_j)/\alpha}} < p_k + c^{\min} \Rightarrow c^{\min} > \frac{p_k}{e^{(h_k - h_j)/\alpha} - 1}. \tag{12}$$

Unlike (8), this inequality does not violate any assumption of the model. (12) says that if the product quality difference is not large enough compared to the price of the higher-lineup product, then the subsistence level can actually exceed the value of the left-hand side, incurring the affordability issue for the group of consumers. That is, the slight deviation from linear budget constraint allows the existence of UFBs and thus the identification results in the previous section.

Now, since the regression equation for (11) is

$$\alpha \ln \frac{p_{j^{\ell}_{l+1}} - p_{j^{\ell}_{l}} + c^{\min}}{p_{j^{\ell'}_{m+1}} - p_{j^{\ell'}_{m}} + c^{\min}} + \left(x_{j^{\ell}_{l} t} - x_{j^{\ell'}_{m} \tau} \right) \beta + \xi_{\ell} - \xi_{\ell'},$$

denote

$$\tilde{w}_{\ell,\ell',l,m} := \ln \left(\frac{p_{j^{\ell}_{l+1}} - p_{j^{\ell}_{l}} + c^{\min}}{p_{j^{\ell'}_{m+1}} - p_{j^{\ell'}_{m}} + c^{\min}} \right) \quad \text{and} \quad \tilde{X}_{\ell,\ell',l,m} := (w_{\ell,\ell',l,m}, v'_{\ell,\ell',l,m})'.$$

Theorem 4 (Identification of Preference Parameters with Subsistence Level)**.** Assume the supports of observables and unobservables, $(\mathcal{P}, \mathcal{X}, \Xi)$ satisfy **Assumption MA3**, and $\exists p_{j^{\ell_1}_{l_1}}, \dots, p_{j^{\ell_M}_{l_M}}$ for $\ell_1, \dots, \ell_M \in \mathcal{L}$ that are distinct UFBs and $p_{j^{\ell_1}_{l_1}} = \cdots = p_{j^{\ell_M}_{l_M}}$. Given conditional indirect utility

$$V(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha^*, \beta^*) = -\frac{\alpha^*}{y + c^{\min}_t} p_j + x'_j \beta^* + \xi_{\ell(j)} + \varepsilon_{\ell(j)} \quad \forall j \in \{j' \in \mathcal{J} : p_{j'} \le y\}, \text{ or}$$
$$V(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha^*, \beta^*) = \alpha^* \ln(y - p_j - c^{\min}) + x'_j \beta^* + \xi_{\ell(j)} + \varepsilon_{\ell(j)} \quad \forall j \in \{j' \in \mathcal{J} : p_{j'} + c^{\min} \le y\},$$

where α^* and β^* are the true preference parameters and $\varepsilon_{\ell(j)} \sim \text{T1EV}(1)$, α^* and β^* can be identified through the ratio of partial derivatives of the market shares in (9) if $\exists \boldsymbol{z} := (z_1, \dots, z_J) \in \mathcal{Z}^{|\mathcal{J}|}$ (or $\tilde{\boldsymbol{z}} := (\tilde{z}_1, \dots, \tilde{z}_J) \in \tilde{\mathcal{Z}}^{|\mathcal{J}|}$) s.t. $\mathbb{E}\big[u_{\ell,\ell',l,m} \mid Z_{\ell,\ell',l,m}, X_{\ell,\ell',l,m}\big] = 0$ ($\mathbb{E}\big[u_{\ell,\ell',l,m} \mid \tilde{Z}_{\ell,\ell',l,m}, \tilde{X}_{\ell,\ell',l,m}\big] = 0$) a.s. and $\operatorname{rank}\Big(\mathbb{E}\big[Z_{\ell,\ell',l,m} X^{\top}_{\ell,\ell',l,m}\big]\Big) = 1 + K$, ($\operatorname{rank}\Big(\mathbb{E}\big[\tilde{Z}_{\ell,\ell',l,m} \tilde{X}^{\top}_{\ell,\ell',l,m}\big]\Big) = 1 + K$) and $\mathbb{E}\|Z_{\ell,\ell',l,m}\|^2 < \infty$ and $\mathbb{E}\|X_{\ell,\ell',l,m}\|^2 < \infty$. ($\mathbb{E}\|\tilde{Z}_{\ell,\ell',l,m}\|^2 < \infty$ and $\mathbb{E}\|\tilde{X}_{\ell,\ell',l,m}\|^2 < \infty$.).

The proof is direct from the above discussion. Note that conditions in the parenthesis is for the second utility function.

Theorem 4 is not trivial repeat of **Theorem 1** because of the additional utility form that is identifiable. Recall that through the discussion with (8), we have shown that the identification strategy didn't hold so that **Theorem 1** couldn't be applied for the counterpart of linear budget constraint of (11). The author is unaware of any other identification strategy that can be applied for the linear budget constraint counterpart. The identification should leverage on identification literature on nonseparable models such as Matzkin [2019][10] or Chernozhukov et al. [2019][5]. None of the conditions in these papers are satisfied in this utility function, since we do not observe outside option utilities (Matzkin [2019][10]) nor do we have continuous differentiability with respect to c (Chernozhukov et al. [2019][5]). Therefore, although our setting is specific, our paper contributes to the identification of nonseparable models with possible novel methods in nonlinear budget constraint setting.

The knowledge of the subsistence level $c_t^{\min}$ is important for the identification of the preference parameters above. The change in bounds of (11) due to the existence of the subsistence level is not an issue, since we will just shift the position of the bounds with known values. We may make use of population-level investigation of the subsistence level as the value that should be used for each period of interest, or any well-founded values that is persuasive for each market can be candidates.

4.2 Identification with Unknown Subsistence Level

Now, if we want to be rather agnostic about $c_t^{\min}$ and detect it from the data, time-varying properties would not be welcomed. Recalling the discussion in the identification without subsistence level, the partial derivative were justified since we could be sure that the unobservable, $\boldsymbol{\xi}$, are not time-varying so that we could be sure if the price change across time is not due to the change in unobserved utility. Same applies to the unobserved time-varying subsistence level, $c_t^{\min}$. If we were to allow the time-varying subsistence level, then we would have to refrain from using the partial derivatives, since we cannot know whether the unobserved subsistence level change is also driving the variation in the market shares. For this reason, we will assume that the subsistence level is not time-varying, i.e., $c_t^{\min} = c^{\min *}$ for all t in the context of identification with unknown subsistence level.

This may be justified if we are using real valued prices as our data. Although inflation rate may change the nominal subsistence level, the real subsistence level may not be as volatile as the nominal one. We can also adjust the length of the period of interest to be short enough so that the subsistence level does not change much. For example, if we are using monthly data, then we can assume that the subsistence level does not change within 24 months (2 years), which is more probable than arguing that it doesn't change for 5 years or so.

Then, the counterpart of (11) will be the following, which is the same as (13) but with the known subsistence level $c^{\min *}$ instead of $c_t^{\min}$.

$$V(y\mathbf{1}_{|\mathcal{J}} - \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}, j; \alpha, \beta) = \alpha \ln(y - p_j) + x_j'\beta + \xi_{\ell(j)} + \varepsilon_{i\ell(j)} \text{ for } j \in \mathcal{J}_{it} = \{j' \in \mathcal{J} : p_{j'} + c^{\min *} \leq y\}. \tag{13}$$

This bears the log derivative ratio of the market shares for $\ell, \ell' \in \mathcal{L}, l \in \ell, m \in \ell', t$, and τ:

$$\ln \frac{\partial_{p_{j_{l+1}^{\ell}t}} s_{j_l^{\ell}t}}{\partial_{p_{j_{m+1}^{\ell'}\tau}} s_{j_m^{\ell'}\tau}} = \alpha^* \ln \left(\frac{p_{j_{l+1}^{\ell}t} - p_{j_l^{\ell}t} + c^{\min *}}{p_{j_{m+1}^{\ell'}\tau} - p_{j_m^{\ell'}\tau} + c^{\min *}} \right) + (x_{j_l^{\ell}t} - x_{j_m^{\ell'}\tau})'\beta^* + \xi_\ell - \xi_{\ell'},$$

where ∂_x denotes the partial derivative with respect to x.

Intuitively, we can directly see that by making use of the panel data structure, we can control for unobserved utility and also the observed utility for product quality. For example, if we can find t', τ' such that product quality attributes doesn't change, we get

$$\ln \frac{\partial_{p_{j^\ell_{l+1}t}} s_{j^\ell_l t}}{\partial_{p_{j^{\ell'}_{m+1}\tau}} s_{j^{\ell'}_m \tau}} - \ln \frac{\partial_{p_{j^\ell_{l+1}t'}} s_{j^\ell_l t'}}{\partial_{p_{j^{\ell'}_{m+1}\tau'}} s_{j^{\ell'}_m \tau'}} = \alpha \ln \left(\frac{p_{j^\ell_{l+1}t} - p_{j^\ell_l t} + c^{\min *}}{p_{j^{\ell'}_{m+1}\tau} - p_{j^{\ell'}_m \tau} + c^{\min *}} \right) - \alpha \ln \left(\frac{p_{j^\ell_{l+1}t'} - p_{j^\ell_l t'} + c^{\min *}}{p_{j^{\ell'}_{m+1}\tau'} - p_{j^{\ell'}_m \tau'} + c^{\min *}} \right)$$

This is one equation, two unknowns, α and $c^{\min *}$. If we can find another equation with the same form, then we can identify α and $c^{\min *}$ through the ratio of the two equations. Formally, denote

$$\pi_{\ell,l,t} := p_{j^\ell_{l+1}t} - p_{j^\ell_l t}, \qquad g_{\ell,\ell',l,m,t,\tau}(c) := \ln \left(\frac{\pi_{\ell,l,t} + c}{\pi_{\ell',m,\tau} + c} \right)$$

and the constraint equation will be

$$\Delta_{t,\tau,t',\tau'} y_{\ell,\ell',l,m} = \alpha \, \Delta_{t,\tau,t',\tau'} g_{\ell,\ell',l,m}(c) + \Delta_{t,\tau,t',\tau'} v'_{\ell,\ell',l,m} \beta, \tag{14}$$

where $\Delta_{t,\tau,t',\tau'} a_{\ell,\ell',l,m} := a_{\ell,\ell',l,m,t,\tau} - a_{\ell,\ell',l,m,t',\tau'}$. First of all, we have to check whether $\alpha \Delta_{t,\tau,t',\tau'} g_{\ell,\ell',l,m}(c)$ is injective function of α and c to check whether we can identify α and c globally from the equation. Because of the functional form that we chose that contains log transformation, the injectivity will be satisfied as can be seen in the proof of the next theorem.

Another point to be aware of is that (14) is not a regression equation, since we do not have the error term in the right-hand side. The identification of the parameters now means the uniqueness of the solution α^*, β^* and $c^{\min *}$. Of course, in reality, if there are more equations than needed, the overidentification will easily result in no solution. Practically, we can instead put the equation above in regression setting, but here we take it as it is, and give conditions the uniqueness of the solution excluding the possibility that there is no solution. That is, if it is overidentified, we will assume that there is a solution.

For ease of notation, let $o := (\ell, \ell', l, m)$ and $s := (t, \tau)$. Define

$$R_{(o,s,s')}(c) := \begin{bmatrix} \Delta_{s,s'} g_o(c) & \partial_c \Delta_{s,s'} g_o(c) & \Delta_{s,s'} v'_o \end{bmatrix} \in \mathbb{R}^{1 \times (2+K)} \text{ for } s, s' \in \mathcal{T}, o \in \mathcal{O},$$

$$R(c) = \begin{bmatrix} R_{(o_1,s_1,s'_1)}(c) \\ \vdots \\ R_{(o_M,s_M,s'_M)}(c) \end{bmatrix} \in \mathbb{R}^{M \times (2+K)} \text{ for } o_1, \ldots, o_M \in \mathcal{O}, s_1, \ldots, s_M \in \mathcal{T}.$$

where $\mathcal{O}$ is the set of all pair of alternatives (i.e., $\mathcal{O} = \mathcal{J} \times \mathcal{J}$) and $\mathcal{T}$ is the set of all time periods.

Theorem 5 (Identification of Preference Parameters and Subsistence Level)**.** Assume the supports of observables and unobservables, $(\mathcal{P}, \mathcal{X}, \Xi)$ satisfy **Assumption MA3**, and $\exists\,(o_1, s_1, s_1'), \ldots, (o_M, s_M, s_M')$ s.t. for each o, s, s', $p_{j^\ell_{l+1}t} = p_{j^{\ell'}_{m+1}\tau}$ and $(p_{j^\ell_{l+1}t'} = p_{j^{\ell'}_{m+1}\tau'}$ are $j^\ell_{l+1}, j^\ell_{m+1}$ are UFBs at t, t' and τ, τ', each, and are distinct. Given conditional indirect utility in (13), the true parameters α^*, β^*, and $c^{\min *}$ can be globally identified if for $M \geq 2 + K$, the matrix $R(c^{\min *})$ has full column rank $(2 + K)$ and if $\exists(o_a, s_a, s_a')$ and (o_b, s_b, s_b') s.t.

$$\frac{\pi_a}{\pi_a'} \neq \frac{\pi_b}{\pi_b'} \quad \text{and} \quad \Delta_{s_a, s_a'} g_{o_a}(c^{\min *}) \neq 0,\ \Delta_{s_b, s_b'} g_{o_b}(c^{\min *}) \neq 0,$$

, where $\pi_a = p_{j^{\ell_a}_{l_a+1}t_a} - p_{j^{\ell_a}_{l_a}t_a}$ and $\pi_a' = p_{j^{\ell_a'}_{m_a+1}\tau_a} - p_{j^{\ell_a'}_{m_a}\tau_a}$ (analogous notation for index b).

Proof of Theorem 5. First, we have to check whether the function $\alpha \Delta_{s,s'} g_o(c)$ is injective in c and α. Define $D_a(c) = \Delta_{s_a, s_a'} g_{o_a}(c)$ and $D_b(c) = \Delta_{s_b, s_b'} g_{o_b}(c)$. Because the price-gap ratios differ, the map $\Phi(c) = D_a(c)/D_b(c)$ is strictly monotone on the admissible c-interval. Recall,

$$\Delta_{s,s'} g_o(c) := \ln\left(\frac{\pi_{\ell,l,t} + c}{\pi_{\ell',m,\tau} + c}\right) - \ln\left(\frac{\pi_{\ell,l,t'} + c}{\pi_{\ell',m,\tau'} + c}\right) \text{ with } \pi_{\ell,l,t} := p_{j^\ell_{l+1}t} - p_{j^\ell_l t}.$$

Assume that there exists an observationally equivalent pairs of parameters (α_1, c_1) and (α_2, c_2). Observe that, by the assumption, we have the variation in $\pi_{\ell,l,t}$, and by the continuous differentiability of $\ln(x)$ for $x > 0$,

$$\begin{aligned}
&\frac{\partial}{\partial \pi_{\ell,l,t}} \alpha \Delta_{s,s'} g_o(c) = \alpha_1 \frac{1}{\pi_{\ell,l,t} + c_1} = \alpha_2 \frac{1}{\pi_{\ell,l,t} + c_2} = \frac{\partial}{\partial \pi_{\ell,l,t}} \alpha_2 \Delta_{s,s'} g_o(c_2) \\
&\Rightarrow \frac{\alpha_1}{\alpha_2} = \frac{\pi_{\ell,l,t} + c_1}{\pi_{\ell,l,t} + c_2}.
\end{aligned}$$

Given that α_1 and α_2 are fixed constants, α_1/α_2 should be a constant function of $\pi_{\ell,l,t}$. However, since

$$\frac{\partial}{\partial \pi_{\ell,l,t}} \frac{\pi_{\ell,l,t} + c_1}{\pi_{\ell,l,t} + c_2} = \frac{c_2 - c_1}{(\pi_{\ell,l,t} + c_2)^2} \neq 0 \text{ for } c_1 \neq c_2,$$

$\frac{\alpha_1}{\alpha_2}$ is not a constant function of $\pi_{\ell,l,t}$ unless $c_1 = c_2$ and hence, $\alpha_1 = \alpha_2$. However, this contradicts the fact that (α_1, c_1) and (α_2, c_2) are distinct. Same logic applies when $\pi_{\ell',m,\tau}, \pi_{\ell,l,t'}$, or $\pi_{\ell',m,\tau'}$ changes instead of $\pi_{\ell,l,t}$. Therefore, the function $\alpha \Delta_{s,s'} g_o(c)$ is injective in (α, c).

After substituting $(\alpha^*, c^{\min *})$, the system is linear in β: $R_3 \beta = \Delta y - \alpha^* \Delta g(c^{\min *})$, where R_3 is the $(M \times K)$ block of Δx's. The rank condition guarantees that R_3 has full column rank K; hence the solution for $\beta = (R_3' R_3)^{-1}(\Delta y - \alpha^* \Delta\, g(c^{\min *}))$ is unique. $\square$

5 Conclusion

In this paper, we have developed an identification strategy for discrete choice models where unobserved budget constraints determine consumers' consideration sets. Our work addresses a significant gap in empirical industrial organization literature, where the emphasis on preference heterogeneity has often overshadowed the role of consumer opportunity (affordability), particularly at the consideration set level.

Our key contributions and findings can be summarized as follows:

First, we introduced a framework that explicitly models how unobserved budget heterogeneity affects discrete choices through consideration set formation. By focusing on the universal nature of affordability constraints (as opposed to idiosyncratic preferences), we provide a more balanced view of consumer decision-making that acknowledges both preference and opportunity.

Second, we established that when products exhibit vertical differentiation within brands, the price-quality asymmetry in consumer responses to unaffordable first-best (UFB) products provides identifying power. Specifically, **Proposition 1** showed that when consumers do not respond to quality changes in a higher-lineup product but do respond to its price changes, this product must be a UFB. This asymmetry allowed us to isolate groups of consumers with known budgets.

Third, we demonstrated in **Theorem 1** that with sufficient price variation and proper instruments, the preference parameters (α and β) can be identified without knowing the budget distribution. The identification relies on comparing the ratios of partial derivatives of market shares with respect to prices across different brands, which effectively differences out the unknown budget density at specific points.

Fourth, once preference parameters are identified, **Theorem 3** showed that the budget distribution can be nonparametrically identified by treating the cutoff budgets as special regressors that reveal the density at those points. This result requires that the range of cutoff budget levels covers the support of the budget distribution. This identification of the budget distribution contributes to correctly specifying price elasticities, which is sensitive to how consumers leave certain products not only because of preference but also through budget constraints, as pointed out in Pesendorfer et al. [2023][12].

Fifth, our analysis extended to settings with subsistence levels (minimum composite good consumption). **Theorem 4** established that preference parameters remain identifiable even with such constraints. More importantly, **Theorem 5** showed that when subsistence levels are unknown but fixed across time, both preference parameters and subsistence levels can be jointly identified using panel data.

Sixth, we contribute to the literature on identification in nonseparable models. Our paper presents a novel identification strategy for nonseparable utility models where standard approaches from the literature (such as those in Matzkin [2019][10] or Chernozhukov et al. [2019][5]) cannot be directly applied due to the lack of observable outside option utilities and the non-differentiability at budget constraint thresholds.

These results offer several methodological advantages. Unlike traditional approaches that rely on parametric assumptions about budget or willingness to pay distributions, our method recovers this distribution directly from choice data. The identification strategy works with multiple utility specifications, including indirect utility functions that partially internalize budget constraints. Our approach also addresses limitations of log-linear BLP specifications for high-priced goods by incorporating realistic modifications to the budget constraint with introduction of the subsistence level.

The primary limitation of our framework is the assumption of homogeneous preference parameters, which could be relaxed in future work to accommodate random coefficients. Other promising extensions include developing estimation methods that efficiently implement these identification results, exploring alternative idiosyncratic error structures beyond the Type 1 Extreme Value distribution, and testing the framework on market-level data for durable goods.

6 Appendix

6.1 Proof of Proposition 1 and 2

Proposition (Price-Quality Asymmetry)**.** Under **Assumption MA3-MA4**, for a product $j_l \in \ell \in \mathcal{L}$, if $\frac{\partial P_{j_l}(\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha,\beta)}{\partial x_{j_{l+1}}} = 0$, then j_{l+1} is an Unaffordable First-Best, i.e., $\overline{S_{j_l}} = p_{j_{l+1}}$.

Proof for Proposition 1. First, note that $P_j(y,\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha,\beta)$ is not a function of attributes of the product $j' \in \ell$ s.t. $j' \neq j$. Once, the budget range chooses the product j as their choice in the brand ℓ, the only competition left for the product j is not the competition within the brand but the competition across brands. Therefore, the partial derivative of $P_j(y,\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha,\beta)$ with respect to the attributes of the product $j' \in \ell$ s.t. $j' \neq j$ is zero. Therefore, the only component that embeds the competition between products within a brand ℓ is the region of budget, S_j for $j \in \ell$.

We are left to show that for each brand $\ell \in \mathcal{L}$, if S_{j_l} for $j_l \in \ell$ are not functions of $x_{j_{l+1}}$, then its upper bound must be $p_{j_{l+1}}$. Let's first think about the possibilities the lower bound can be the function of $x_{j_{l+1}}$. There are two cases that this may be possible: i) $\underline{S_{j_l}} = y_{j_{l-k},j_{l+1}}$ for $l > k \geq 0$, or ii) $\underline{S_{j_l}} = y_{j_{l+1},j_{l+k}}$ for $|\ell| - l \geq k > 1$. To check if this is possible, we can utilize **Fact 1** which can be achieved through **Lemma 1**. Observe that if there is a slight change in $x_{j_{l+1}}$, then case (iii) of **Fact 1** will be absorbed to (i) (if there is an decrease in quality of j_{l+1}) or (ii) above (if there is an increase in quality of j_{l+1}). Thus, it suffices to show the proposition should be true through (i) and (ii).

The triplet (j_{l-k}, j_l, j_{l+1}) from i) must also satisfy one of the cases in **Fact 1**. Under **Fact 1**(i), we have $y_{j_{l-k},j_l} < y_{j_{l-k},j_{l+1}} < y_{j_l,j_{l+1}}$. The only possibility of $y_{j_{l-k},j_{l+1}}$ being the lower bound of budget that buys j_l is when $y_{j_{l-k},j_{l+1}} = p_{j_l}$, since for $y \in (y_{j_{l-k},j_l}, y_{j_{l-k},j_{l+1}})$, $j_l \succ j_{l-k}$ already holds and thus if j_l was affordable in the range, $y_{j_{l-k}}, j_{l+1}$ cannot be the bound. However, even when $y_{j_{l-k},j_{l+1}} = p_{j_l}$, p_{j_l} must be the bound then $y_{j_{l-k},j_{l+1}}$ because when there is a quality change of products that leads to an increase in $y_{j_{l-k},j_{l+1}}$ and thus $p_{j_l} < y_{j_{l-k},j_{l+1}}$, this does not change the market share of product j_l, since the affordability is still a problem for j_l at $y = p_{j_l}$. Therefore, $y_{j-k,j+1}$ cannot be $\underline{S_{j_l}}$ in this case. Note that the same logic applies to impossibility of $y_{j_l,j_{l+1}}$ being the lower bound. Similarly, for **Fact 1**(ii), we have $y_{j_{l-k},j_l} > y_{j_{l-k},j_{l+1}} > y_{j_l,j_{l+1}}$ and, here, j_l is not chosen naturally from the preference orderings. We can directly check that $y_{j_{l-k},j_{l+1}}$ cannot be the bound, since for $y < y_{j_{l-k},j_l}$, $j_{l-k} \succ j_l$ holds which means that for j_l to be chosen in this range $p_{j_{l-k}} > p_{j_l}$, which violates **Assumption MA3**. Following the same logic in **Fact 1**(i) above, for $y_{j_l,j_{l+1}}$ to be the lower bound, $p_{j_l} = y_{j_l,j_{l+1}}$ should be the case but the real bound is p_{j_l} not $y_{j_l,j_{l+1}}$. Thus, $\underline{S_{j_l}} = y_{j_{l-k},j_{l+1}}$ for $l > k \geq 0$ is not possible.

For ii) $\underline{S_{j_l}} = y_{j_{l+1},j_{l+k}}$ with $|\ell| - l \geq k > 1$, similar logic applies with triplets j_l, j_{l+1}, j_{l+k}. If **Fact 1**(i) holds, then we have $y_{j_l,j_{l+1}} < y_{j_l,j_{l+k}} < y_{j_{l+1},j_{l+k}}$. For $y_{j_{l+1},j_{l+k}}$ to be the lower bound, both j_{l+1} and j_{l+k} must be unaffordable and j_l should be affordable at $y_{j_{l+1},j_{l+k}}$, which gives the same problem as in i). Under **Fact 1**(ii), we have $y_{j_l,j_{l+1}} > y_{j_l,j_{l+k}} > y_{j_{l+1},j_{l+k}}$. If $y_{j_{l+1},j_{l+k}}$ is the lower bound, then $p_{j_l} = y_{j_{l+1},j_{l+k}}$, again makes p_{j_l} itself the lower bound. Therefore, $\underline{S_j} = y_{j_{l+1},j_{l+k}}$ cannot be the case.

Now, we are left with showing that if the upper bound is not a function of $x_{j_{l+1}}$, then it must be $p_{j_{l+1}}$. First, note that if $\overline{S_{j_l}}$ cannot be a function that does not contain j_l's attribute other than the situations where the product more preferable is unaffordable, and the product must be j_{l+1}. Generate some triplet by adding j_k, j_m for k, m, l are distinct. For $\overline{S_{j_l}}$ to be y_{j_k,j_m} (WLOG, $k < m$), we must have either $j_k \succ j_m \succ j_l$ or $j_l \succ j_k \succ j_m$ at y s.t. 'relevant switch point'$< y < y_{j_k,j_m}$, since the switch can only happen when it is right next pair. We can directly check that the latter case will not give y_{j_k,j_m} as the upper bound since the rank switch of the non-first-best products is not going to alter what will be chosen if the first-best one is affordable as a matter of fact we are checking the upper bound of j_l being chosen so j_l should definitely be affordable.

Observe that for the former case to be the case, both j_k, j_m must be unaffordable at y with above requirement and $p_{j_k} = y_{j_k,j_m}$ must be the case so that after the switch point $j_m \succ j_k \succ j_l$ is the preference ordering, but since j_m is still not affordable and j_k starts to become affordable, j_l will no longer be chosen. Following the logic in previous proof, in this case p_{j_k} is the upper bound, not y_{j_k,j_m}. However, by **Assumption MA3**, this means that j_{l+1} is affordable in this budget range, but the fact that j_l was being chosen in S_{j_l} implies that if $k \neq l+1$, $j_l \succ j_{l+1}$ was the case. Since once switch happens as y grows, the preference ordering cannot be reversed between j_{l+k} and j_{l+1} and j_{l+k} being affordable, j_{l+1} will be never chosen which will lead to $s_{j_{l+1}} = 0$, a contradiction. Therefore, the only possibility is that $j_k = j_{l+1}$ and thus $\overline{S_{j_l}} = p_{j_{l+1}}$.

The only leftover case is when $\overline{S_{j_l}} = y_{j_l, j_{l+k}}$ or $= y_{j_{l-k}, j_l}$ for $k > 0$. For the latter, we can directly check that this cannot be the case since at $y < y_{j_{l-k}, j_l}$, $j_{l-k} \succ j_l$, where j_{l-k} is affordable for j_l to be affordable. This leads to j_l not being chosen at all, which contradicts the fact that we are checking the upper bound of j_l being chosen. For the former, for $y < y_{j_l, j_{l+k}}$, $j_l \succ j_{l+k}$ holds. If j_l were to be chosen in this range, products that were more preferred than j_l should be all unaffordable, while j_l being affordable, i.e., only j_{l+m} for $m > k$ are allowed to be more preferred and y at the range is smaller than p_m for all $m > k$. For $y_{j_l, j_{l+k}}$ to be the upper bound, j_{l+k} should be affordable at the switch point, then same logic applies as in the previous case, if $k \neq 1$, i.e., the only possible case is $k = 1$. This is the case that we have ruled out as an assumption of the proposition. Assembling all the cases, the only possibility is that $\overline{S_{j_l}} = p_{j_{l+1}}$ for j_{l+1} being the next product in the brand ℓ after j_l.

$\square$

Proposition (No UFB pins down S_js)**.** Under **Assumption MA3-MA4**, if $\nexists$UFB for all products in brand $\ell = \{j_1, \ldots, j_{|\ell|}\}$, then $S_{j_l}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)$ is uniquely determined as $\underline{S_{j_l}} = y_{j_{l-1}, j_l}$ and $\overline{S_{j_l}} = y_{j_l, j_{l+1}}$ for $k = 1, \ldots, |\ell|$, $y_{j_0, j_1} = \underline{y}$, and $y_{j_{|\ell|}, j_{|\ell|+1}} = \overline{y}$.

Proof for Proposition 2. The proof is straightforward from the fact that if there is no UFB, then the only possible case is that the budget range of each product is determined by the right next products in the brand. This can be seen from **Fact 1**(ii) and (iii). If there is no UFB, these two cases will generate Never-the-First-Best products, which will not be chosen in any budget range other than at points. Then, for these products, the market shares will be 0, a contradiction. Therefore, the only possible case is **Fact 1**(i) and thus $\underline{S_{j_l}} = y_{j_{l-1}, j_l}$ and $\overline{S_{j_l}} = y_{j_l, j_{l+1}}$ for $k = 1, \ldots, |\ell|$, $y_{j_0, j_1} = \underline{y}$, and $y_{j_{|\ell|}, j_{|\ell|+1}} = \overline{y}$. $\square$

6.2 Proof of Theorem 3

Theorem. Given α, β, and $\boldsymbol{\xi}$, if $[\underline{y}, \overline{y}] \subseteq \bigcup_{(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{\xi}) \in \mathcal{X} \times \mathcal{P} \times \{\boldsymbol{\xi}\} \text{ s.t. MA3 holds}} \cup_{\ell \in \mathcal{L}} \cup_{j \in \ell} A_{\sigma(j, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi})}(j)$, then F_Y is nonparametrically identified.

*Proof of **Theorem** 3.* From the proof of **Proposition 1,2**, we can infer that the $\overline{S_{j_l^\ell}}$ is either $p_{j_{l+1}^\ell}$ or $y_{j_l^\ell, j_{l+1}^\ell}$ for $l = 1, \ldots, |\ell|$, where $y_{j_{|\ell|}^\ell, j_{|\ell|+1}} = \overline{y}$. Similarly, $\underline{S_{j_l^\ell}}$ is either $p_{j_l^\ell}$ or $y_{j_{l-1}^\ell, j_l^\ell}$ for $l = 1, \ldots, |\ell|$, where $y_{j_0^\ell, j_1^\ell} = \underline{y}$.

For each j_l^ℓ, we can come up with the closed form of $s_{j_l^\ell}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)$ for each case.

$$s_{j_l^\ell}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) = \begin{cases} \int_{y_{j_{l-1}^\ell, j_l^\ell}}^{y_{j_l^\ell, j_{l+1}^\ell}} P_{j_l^\ell}(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y) dy & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell, j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell, j_l^\ell} \\ \int_{p_{j_l^\ell}}^{y_{j_l^\ell, j_{l+1}^\ell}} P_{j_l^\ell}(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y) dy & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell, j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = p_{j_l^\ell} \\ \int_{y_{j_{l-1}^\ell, j_l^\ell}}^{p_{j_{l+1}^\ell}} P_{j_l^\ell}(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y) dy & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell, j_l^\ell} \\ \int_{p_{j_l^\ell}}^{p_{j_{l+1}^\ell}} P_{j_l^\ell}(y, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y) dy & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = p_{j_l^\ell}. \end{cases}$$

Then, by taking a derivative with respect to $p_{j_{l-1}^\ell}$ and $p_{j_{l+1}^\ell}$, we get the following.

$$\frac{\partial s_{j_l^\ell}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)}{\partial p_{j_{l-1}^\ell}} = \begin{cases} -P_{j_l^\ell}(y_{j_{l-1}^\ell, j_l^\ell}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y_{j_{l-1}^\ell, j_l^\ell}) \frac{\partial y_{j_{l-1}^\ell, j_l^\ell}}{\partial p_{j_{l-1}^\ell}} & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell, j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell, j_l^\ell} \\ 0 & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell, j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = p_{j_l^\ell} \\ -P_{j_l^\ell}(y_{j_{l-1}^\ell, j_l^\ell}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y_{j_{l-1}^\ell, j_l^\ell}) \frac{\partial y_{j_{l-1}^\ell, j_l^\ell}}{\partial p_{j_{l-1}^\ell}} & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell, j_l^\ell} \\ 0 & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = p_{j_l^\ell}. \end{cases} \tag{15}$$

$$\frac{\partial s_{j}(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta)}{\partial p_{j_{l+1}^\ell}} = \begin{cases} P_{j_l^\ell}(y_{j_l^\ell, j_{l+1}^\ell}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y_{j_l^\ell, j_{l+1}^\ell}) \frac{\partial y_{j_l^\ell, j_{l+1}^\ell}}{\partial p_{j_{l+1}^\ell}} & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell, j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell, j_l^\ell} \\ P_{j_l^\ell}(y_{j_l^\ell, j_{l+1}^\ell}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(y_{j_l^\ell, j_{l+1}^\ell}) \frac{\partial y_{j_l^\ell, j_{l+1}^\ell}}{\partial p_{j_{l+1}^\ell}} & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell, j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = p_{j_l^\ell} \\ P_{j_l^\ell}(p_{j_{l+1}^\ell}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(p_{j_{l+1}^\ell}) & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell, j_l^\ell} \\ P_{j_l^\ell}(p_{j_{l+1}^\ell}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\xi}; \alpha, \beta) f_Y(p_{j_{l+1}^\ell}) & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = p_{j_l^\ell}. \end{cases} \tag{16}$$

From the derivative with respect to $x_{j_{l-1}^\ell}$ and $x_{j_{l+1}^\ell}$, we can get the following.

$$\frac{\partial s_{j_l^\ell}(\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha,\beta)}{\partial x_{j_{l-1}^\ell}} = \begin{cases} -P_{j_l^\ell}(y_{j_{l-1}^\ell,j_l^\ell},\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha,\beta) f_Y(y_{j_{l-1}^\ell,j_l^\ell}) \frac{\partial y_{j_{l-1}^\ell,j_l^\ell}}{\partial x_{j_{l-1}^\ell}} & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell,j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell,j_l^\ell} \\ 0 & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell,j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = p_{j_l^\ell} \\ -P_{j_l^\ell}(y_{j_{l-1}^\ell,j_l^\ell},\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha,\beta) f_Y(y_{j_{l-1}^\ell,j_l^\ell}) \frac{\partial y_{j_{l-1}^\ell,j_l^\ell}}{\partial x_{j_{l-1}^\ell}} & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell,j_l^\ell} \\ 0 & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = p_{j_l^\ell}. \end{cases} \tag{17}$$

$$\frac{\partial s_{j_l^\ell}(\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha,\beta)}{\partial x_{j_{l+1}^\ell}} = \begin{cases} P_{j_l^\ell}(y_{j_l^\ell,j_{l+1}^\ell},\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha,\beta) f_Y(y_{j_l^\ell,j_{l+1}^\ell}) \frac{\partial y_{j_l^\ell,j_{l+1}^\ell}}{\partial x_{j_{l+1}^\ell}} & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell,j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell,j_l^\ell} \\ P_{j_l^\ell}(y_{j_l^\ell,j_{l+1}^\ell},\boldsymbol{p},\boldsymbol{x},\boldsymbol{\xi};\alpha,\beta) f_Y(y_{j_l^\ell,j_{l+1}^\ell}) \frac{\partial y_{j_l^\ell,j_{l+1}^\ell}}{\partial x_{j_{l+1}^\ell}} & \text{if } \overline{S_{j_l^\ell}} = y_{j_l^\ell,j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = p_{j_l^\ell} \\ 0 & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell,j_l^\ell} \\ 0 & \text{if } \overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}, \underline{S_{j_l^\ell}} = p_{j_l^\ell}. \end{cases} \tag{18}$$

For each $j_l^\ell \in \ell$ for $\forall \ell \in \mathcal{L}$, the derivatives above shows that no matter what the budget range is, if the partial derivatives with respect to $p_{j_{l-1}^\ell}$, $p_{j_{l+1}^\ell}$, $x_{j_{l-1}^\ell}$, and $x_{j_{l+1}^\ell}$ are non-zero, then we can identify at least one point value of the budget density, given the objects that can be identified previously (α, β, and $\boldsymbol{\xi}$). This can be organized as follows:

i) For $\overline{S_{j_l^\ell}} = y_{j_l^\ell,j_{l+1}^\ell}$ and $\underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell,j_l^\ell}$, we identify $f_Y(y_{j_{l-1}^\ell,j_l^\ell})$ and $f_Y(y_{j_l^\ell,j_{l+1}^\ell})$, where we actually have over-identification since we can express both density values from price derivatives and quality attribute derivatives.

ii) For $\overline{S_{j_l^\ell}} = y_{j_l^\ell,j_{l+1}^\ell}$ and $\underline{S_{j_l^\ell}} = p_{j_l^\ell}$, we identify $f_Y(y_{j_l^\ell,j_{l+1}^\ell})$ from both price derivatives quality attribute derivatives with respect to those of right higher lineup product.

iii) For $\overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}$ and $\underline{S_{j_l^\ell}} = y_{j_{l-1}^\ell,j_l^\ell}$, we identify $f_Y(y_{j_{l-1}^\ell,j_l^\ell})$ and $f_Y(p_{j_{l+1}^\ell})$, where the former is from both price derivatives and quality attribute derivatives with respect to those of right lower lineup product, while the latter is solely coming from price derivatives with respect to that of right higher lineup product.

iv) For $\overline{S_{j_l^\ell}} = p_{j_{l+1}^\ell}$ and $\underline{S_{j_l^\ell}} = p_{j_l^\ell}$, we identify $f_Y(p_{j_{l+1}^\ell})$ solely from price derivatives of right higher lineup product.

Observe that $A_s(j)$'s element for $s = 1, \ldots, 4$ indicates the points on the support whose budget density value is identified. If the variation of the observables that satisfy our main assumption **Assumption MA3** is large enough, then we can identify the budget density function f_Y nonparametrically.

□

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