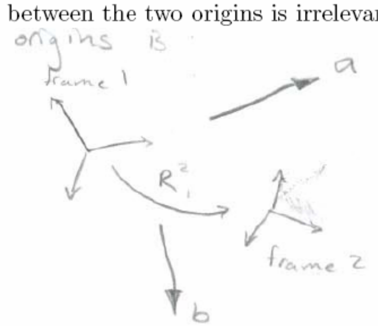
2-1 We are considering free vectors. Consequently, we do not need to know points in space — only direction and magnitude so we only need to know the rotation between the two coordinate frames; the distance between the two origins is irrelevant.



We write $a^2 = R_1^2 a^1$ and $b^2 = R_1^2 b^1$. Now,

$$\begin{aligned} a^2 \cdot b^2 &= (a^2)^T b^2 = (Ra^1)^T (Rb^1) = (a^1)^T R^T R b^1 \\ &= (a^1)^T R^{-1} R b^1 = (a^1)^T b^1 = a^1 \cdot b^1 \end{aligned}$$

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2-2 Notice that $\|v\|^2 = v^T v \Rightarrow \|v\| = +\sqrt{v^T v}$. Therefore,

$$\begin{aligned} \|Rv\| &= +\sqrt{(Rv)^T Rv} = \sqrt{v^T R^T Rv} \\ &= \sqrt{v^T v} = \|v\| \end{aligned}$$

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2-3 This follows from Problem 2-2 with $v = p_1 - p_2$.

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2-4 Let $R = [r_1, r_2, r_3]$ where $r_i = \begin{pmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{pmatrix}$. Then $R^T R = I$ implies

$$\begin{bmatrix} r_1^T r_1 & r_1^T r_2 & r_1^T r_3 \\ r_1^T r_1 & r_2^T r_2 & r_2^T r_3 \\ r_3^T r_1 & r_2^T r_2 & r_3^T r_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Equating entries of the matrices shows that the column vectors of R are of unit length and mutually orthogonal.

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2-5 a) For any matrices A and B, $\det(A^T) = \det(A)$ and $\det(AB) = \det(A)\det(B)$. Thus, if R is orthogonal

$$1 = \det(I) = \det(R^T R) = \det(R^T)\det(R) = (R)^2$$

which implies that

$$\det R \;=\; \pm 1.$$

b) For a right-handed coordinate system, $r_1 \times r_2 = r_3$. This implies that

$$r_{12}r_{23} - r_{13}r_{22} = r_{31}; \quad -r_{11}r_{23} + r_{13}r_{21} = r_{32}; \quad r_{11}r_{22} - r_{12}r_{21} = r_{33}.$$

Therefore, expanding $\det R$ about column 3 gives

$$\begin{aligned}
\det R &= \det\begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \\
&= r_{31}(r_{12}r_{23} - r_{22}r_{13}) - r_{32}(r_{11}r_{23} - r_{21}r_{13}) + r_{33}(r_{11}r_{22} - r_{21}r_{12}) \\
&= r_{31}(r_{31}) + r_{32}(r_{32}) + r_{33}(r_{33}) \\
&= \|r_3\|^2 = 1.
\end{aligned}$$

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2-6 Equation (2.3) is obvious. Equation (2.4) follows from

$$
\begin{aligned}
R_{z,\theta}R_{z,\phi} &= \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\phi - s_\theta s_\phi & -c_\theta s_\phi - c_\phi s_\theta & 0 \\ s_\theta c_\phi + c_\theta s_\phi & -s_\theta s_\phi + c_\theta c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & 0 \\ \sin(\theta+\phi) & \cos(\theta+\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{z,\theta+\phi}.
\end{aligned}
$$

Equation (2.5) follows from (2.3) and (2.4) since

$$R_{z,\theta}R_{z,-\theta} = R_{z,\theta-\theta} = R_{z,0} = I.$$

This can also be shown by noticing that

$$R_{z,\theta}^T = R_{z,-\theta}.$$

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2-7 First, note that $x \in SO(n)$ means that $x^T x = xx^T = I$ and $\det x = 1$.

a) The first property follows from

$$(x_1x_2)^T(x_1x_2) = x_2^T x_1^T x_1 x_2 = x_2^T I x_2 = I$$

so

$$x_1x_2 \in SO(n) \qquad \forall x_1, x_2 \in SO(n)$$

b) By the associative property of matrix multiplication,

$$(x_1, x_2)x_3 \quad = \quad x_1(x_2x_3).$$

for $x_1, x_2, x_3 \in SO(n)$

c) The $n \times n$ identity matrix satisfies the third property.

d) Since $x^T x = xx^T = I$, it follows that $x^T = x^{-1}$

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2-8 For a rotation of θ about the x axis we have

$$\begin{aligned}
x_0 \cdot x_1 &= 1 \\
y_0 \cdot y_1 &= \cos\theta \\
z_0 \cdot z_1 &= \cos\theta \\
z_0 \cdot y_1 &= \sin\theta \\
y_0 \cdot z_1 &= -\sin\theta
\end{aligned}$$

and all other dot products are zero. Substituting into the rotation matrix in Section 2.2.2 gives

$$R_0^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

For a rotation of θ about the y axis we have

$$\begin{aligned}
y_0 \cdot y_1 &= 1 \\
x_0 \cdot x_1 &= \cos\theta \\
z_0 \cdot z_i &= \cos\theta \\
z_0 \cdot x_1 &= -\sin\theta \\
x_0 \cdot z_1 &= \sin\theta
\end{aligned}$$

and all other dot products are zero. Again using the rotation matrix gives

$$R = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

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2-9 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(2).$$

From Cramer's rule and the fact that $A \in SO(3)$ we have

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

which implies that $a = d$ and $b = -c$. Thus

$$A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$$

with $\det A = 1 = a^2 + c^2$. Define $\theta = \tan^{-1}(c/a)$. Then $\cos\theta = a$ and $\sin\theta = c$.

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2-10

$$R = R_{y,\psi} R_{x,\phi} R_{z,\theta}$$

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2-11

$$R = R_{z,\theta} R_{x,\phi} R_{x,\psi}$$

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2-12

$$R = R_{z,\alpha} R_{x,\phi} R_{z,\theta} R_{x,\psi}$$

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2-13

$$R = R_{z,\alpha}R_{z,\theta}R_{x,\phi}R_{x,\psi}$$

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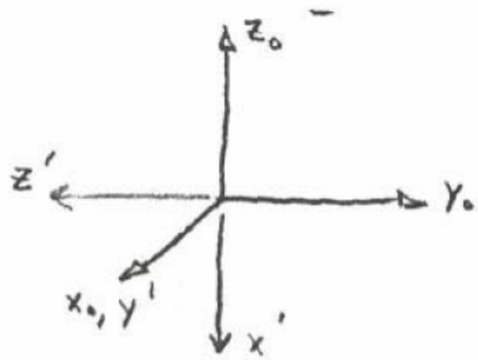
2-14

$$R = R_{y,\frac{\pi}{2}} R_{x,\frac{\pi}{2}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$



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2-15

$$R_3^2 = R_1^2 R_3^1 \quad \text{where} \quad R_1^2 = (R_2^1)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

Therefore,

$$R_3^2 \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 \end{bmatrix}$$

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2-16 If r_{11}, r_{21} are not both zero, then

- $c_\theta \neq 0$ and $r_{31} = -s_\theta \neq \pm 1$
- r_{32}, r_{33} are not both zero.

so, $c_\theta = \pm\sqrt{1 - r_{31}^2}$ and $\theta =$Atan2$\left(\pm \sqrt{1 - r_{31}^2}, r_{31}\right)$.
Follow a development similar to that provided for the Euler angles to find ϕ, θ, and ψ.

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2-17 Straightforward; follow directions given in sentence preceding the equation.

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2-18 Straightforward. Substitute for r_{ij} in Equation (2.45) using the matrix elements given in Equation (2.43).

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2-19 If λ is an eigenvalue of R and k is a unit eigenvector corresponding to λ then, $Rk = \lambda k$. Since R is a rotation $\|Rk\| = \|k\|$. This implies that $|\lambda| = 1$, i.e., the eigenvalues of R are on the unit circle in the complex plane. Since the characteristic polynomial of R is of degree three at least one eigenvalue of R must be real. Hence $+1$ or -1 is an eigenvalue of R. Now, since $+1 = \det R = \lambda_1\lambda_2\lambda_3$ where $\{\lambda_1, \lambda_2, \lambda_3\}$ is the set of eigenvalues of R, it is easy to see that if -1 is an eigenvalue then $\{\lambda_1, \lambda_2, \lambda_3\} = \{-1, -1, +1\}$. In any case $+1$ is always an eigenvalue of R.

The vector k defines the axis of rotation in the angle/axis representation of R.

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2-20

$$R_{k,\theta} \;=\; \begin{bmatrix} \frac{1}{3} & \frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{3}+\frac{1}{\sqrt{3}} \\ \frac{1}{3}+\frac{1}{\sqrt{3}} & \frac{1}{3} & \frac{1}{3}-\frac{1}{\sqrt{3}} \\ \frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{3}+\frac{1}{\sqrt{3}} & \frac{1}{3} \end{bmatrix}$$

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2-21 Straightforward.

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2-22

$R_{x,\theta}R_{y,\phi}R_{z,\pi}R_{y,-\phi}R_{x,-\theta}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} -\cos(2\phi) & -2\cos(\phi)\sin(\phi)\sin(\theta) & \cos(\theta)\sin(2\phi) \\ -2\cos(\phi)\sin(\phi)\sin(\theta) & -\cos(\theta)^2 - \cos(2\phi)\sin(\theta)^2 & -\cos(\phi)^2\sin(2\theta) \\ \cos(\theta)\sin(2\phi) & -\cos(\phi)^2\sin(2\theta) & \cos(\phi)^2\cos(\theta)^2 - \cos(\theta)^2\sin(\phi)^2 - \sin(\theta)^2 \end{bmatrix}$$

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2-23

$$R = R_{y,90}R_{z,45} \quad = \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

$$\theta \quad = \quad \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) = \cos^{1}\left(\frac{\frac{\sqrt{2}}{2}-1}{2}\right) = 98.42^\circ$$

$$k \quad = \quad \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} & - & r_{23} \\ r_{13} & - & r_{31} \\ r_{21} & - & r_{12} \end{bmatrix} = (0.5054481)\begin{bmatrix} 0.7071068 \\ 1.7071068 \\ 0.7071068 \end{bmatrix}$$

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2-24

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

The direction of the x-axis is $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T$.

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2-25 Possible Euler angles:

XYZ	YZX	ZXY
XYX	YZY	ZXZ
XZY	YXZ	ZYX
XZX	YXY	ZYZ

We must be able to rotate about three different axes in order to specify an arbitrary rotation. Therefore, it is not possible to have ZZY Euler angles, since the consecutive Z rotations are rotations about the same axis.

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2-26 For any two complex numbers $c_1,\, c_2 \in \mathbb{C}$,
$c_1 = a + ib = \|c_1\|\,(\cos\theta_1 + i\sin\theta_1)$
$c_2 = e + if = \|c_2\|\,(\cos\theta_2 + i\sin\theta_2)$
where $\theta_1 =$atan2(a, b) and $\theta_2 =$atan2(e, f).

$$\begin{aligned}
c_1c_2 &= \|c_1\|\,\|c_2\|\,(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= \|c_1\|\,\|c_2\|\,\big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2)\big] \\
&= \|c_1\|\,\|c_2\|\,\big[cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big]
\end{aligned}$$

$\Longrightarrow$multiplication of complex numbers corresponds to addition of angles.

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2-27 Group: $\{\mathbb{C}, \cdot\}$
Using complex exponential notation, $c_1 = m_1 e^{j\theta_1}, c_2 = m_2 e^{j\theta_2}, c_3 = m_3 e^{j\theta_3}$.

1. Group is closed under group operation.
   For all c_1,$c_2 \in \mathbb{C}$,
   $$\begin{aligned} c_1 \cdot c_2 &= m_1 e^{j\theta_1} m_2 e^{j\theta_2} \\ &= m_1 m_2 e^{j(\theta_1+\theta_2)} = m_3 e^{j\theta_3} \end{aligned}$$
   where $m_3 = m_1 m_2$ and $\theta_3 = \theta_1 + \theta_2$.
2. Associativity
   For all c_1,c_2,$c_3 \in \mathbb{C}$,
   $$\begin{aligned} (c_1 c_2) c_3 &= (m_1 e^{j\theta_1} m_2 e^{j\theta_2}) m_3 e^{j\theta_3} \\ &= m_1 m_2 m_3 e^{j(\theta_1+\theta_2+\theta_3)} \\ &= m_1 e^{j\theta_1} (m_2 m_3 e^{j(\theta_2+\theta_3)}) \\ &= c_1 (c_2 c_3). \end{aligned}$$
3. Identity element $I = 1 + j0 = 1e^{j0}$
   For all $c \in \mathbb{C}$,
   $$cI = c = Ic.$$
4. Inverse element
   For all $c_1 \in \mathbb{C}$, let inverse $c_2 \in \mathbb{C}$ be defined as $c_2 = \frac{1}{m_1} e^{-j\theta_1}$.
   $$c_1 c_2 = m_1 \frac{1}{m_1} e^{j\theta_1} e^{-j\theta_1} = c_2 c_1 = 1e^{j0} = I$$

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2-28 Quaternion $Q = q_o + iq_1 + jq_2 + kq_3 = (q_0, q_1, q_2, q_3)$
$R_{k,\theta} \rightarrow Q = (\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2})$
Now, $\|k\| = \sqrt{n_x^2 + n_y^2 + n_z^2} = 1$ because $k = [n_x n_y n_z]^T$ is a unit vector.

$$
\begin{aligned}
\|Q\| &= \sqrt{\cos^2\frac{\theta}{2} + (n_x^2 + n_y^2 + n_z^2)\sin^2\frac{\theta}{2}} \\
&= \sqrt{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} \\
&= \sqrt{1} = 1
\end{aligned}
$$

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2-29 $Q = (q_0, q_1, q_2, q_3) = (\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2})$.
Find rotation matrix $R_{k,\theta} \Rightarrow$ find k, θ.

1. $\theta = \cos^{-1}(2q_0)$
2. $k = [n_x, n_y, n_z]^T = \left[\frac{q_1}{\sin\frac{\theta}{2}}, \frac{q_2}{\sin\frac{\theta}{2}}, \frac{q_3}{\sin\frac{\theta}{2}}\right]^T$
3. Substitute values for k, θ into

$$R_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

where $v_\theta = \text{vers}\theta = 1 - c_\theta$.

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2-30 Given R, find $Q = (q_0, q_1, q_2, q_3)$.

$$\theta = \cos^{-1}\left[\frac{Tr(R) - 1}{2}\right]$$

$$k = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

If $\|k\| \neq 1$, then $k' = \frac{k}{\|k\|}$.
$q_0 = \cos\frac{\theta}{2}, q_1 = n_x \sin\frac{\theta}{2}, q_2 = n_y \sin\frac{\theta}{2}, q_3 = n_z \sin\frac{\theta}{2}$

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2-31 $X = x_0 + ix_1 + jx_2 + kx_3 = (x_0, x)$
$Y = y_0 + iy_1 + jy_2 + ky_3 = (y_0, y)$

$$\begin{aligned}
Z = XY &= (x_0 + ix_1 + jx_2 + kx_3)(y_0 + iy_1 + jy_2 + ky_3) \\
&\vdots \\
&= x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 + x_0(iy_1 + jy_2 + ky_3) + y_0(ix_1 + jx_2 + kx_3) \\
&\quad +i(x_2y_3 - x_3y_2) - j(x_1y_3 - y_1x_3) + k(x_1y_2 - y_1x_2) \\
&= x_0y_0 - x^Ty + x_0y + y_0x + x \times y \\
&= \Big(x_0y_0 - x^Ty, (x_0y + y_0x + x \times y)\Big) \\
&= (z_0, z)
\end{aligned}$$

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2-32 Given $Q = (q0, q)$ and $\|q\| = 1$,
show that $Q_I = (1, [0,0,0]^T) = (d_0, d)$ is the identity for unit quaternion multiplication.
We see that $d^T q = q^T d = 0$, and $d \times q = q \times d = [0,0,0]^T$.
Now, applying the result from problem 2-30,

$$\begin{aligned} QQ_I &= \Big(q_0 d_0 - d^T q, (q_0 d + d_0 q + q \times d)\Big) \\ &= (q_0 d_0, d_0 q) \\ &= (q_0, q) = Q. \end{aligned}$$

Similarly, we left-multiply by Q_I and find that $Q_I Q = Q$.

$$\Rightarrow QQ_I \quad = \quad Q_I Q = Q$$

Therefore Q_I is the identity element.

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2-33 $Q^* = (q_0, q^*)$, where $q^* = [-q1, -q2, -q3]^T$.
Recall Q is a unit quaternion, so $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

$$q^T q^* = q^{*T} q = -q_1^2 - q_2^2 - q_3^2$$

$$q_0^2 - 1 = -q_1^2 - q_2^2 - q_3^2 = q^T q^*$$

$$\begin{aligned} q \times q^* &= i(-q_2 q_3 + q_2 q_3) - j(-q_1 q_3 + q_1 q_3) + k(q_1 q_2 + q_1 q_2) \\ &= [0, 0, 0]^T \\ &= q^* \times q \end{aligned}$$

$$\begin{aligned} QQ^* &= \Big(q_0 q_0 + 1 - q_0^2, (q_0 q^* + q_0 q + q \times q^*)\Big) \\ &= \Big(1, (0 + q \times q^*)\Big) \\ &= \Big(1, [0, 0, 0]^T\Big) = Q_I \end{aligned}$$

Similarly, $Q^* Q = \Big(q_0 q_0 + 1 - q_0^2, (q_0 q + q_0 q^* + q^* \times q)\Big) = \Big(1, [0, 0, 0]^T\Big)$.
$\Rightarrow QQ^* = Q^* Q = Q_I$.

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2-34 Consider $\Big(0, [v_x, v_y, v_z]^T\Big)Q^* = X$

$$x_0 = 0 - [v_x, v_y, v_z]\begin{bmatrix} -q1 \\ -q2 \\ -q3 \end{bmatrix} = v_x q_1 + v_y q_2 + v_z q_3$$

$$\begin{aligned} x &= 0 + q_0[v_x, v_y, v_z]^T + [v_x, v_y, v_z]^T \times [-q1, -q2, -q3]^T \\ &= i(q_0 v_x - q_3 v_y + q_2 v_z) + j(q_3 v_x + q_0 v_y - q_1 v_z) + k(-q_2 v_x + q_1 v_y + q_0 v_z) \end{aligned}$$

Now, consider $Q\Big(0, [v_x, v_y, v_z]^T\Big)Q^* = QX = Y$

$$\begin{aligned} y_0 &= q_0(v_x q_1 + v_y q_2 + v_z q_3) - [q1, q2, q3]^T x \\ &= q_0 q_1 v_x + q_0 q_2 v_y + q_0 q_3 v_z - q_0 q_1 v_x + q_1 q_3 v_y - q_1 q_2 v_z \\ &\quad - q_2 q_3 v_x - q_0 q_2 v_y + q_1 q_2 v_z + q_2 q_3 v_x - q_1 q_3 v_y - q_0 q_3 v_z \\ &= 0 \end{aligned}$$

$$\begin{aligned} y &= q_0 x + x_0 q + q \times x \\ &= i(q_0^2 v_x + q_0 q_2 v_y + q_0 q_2 v_z + q_1^2 v_x + q_1 q_2 v_y + q_1 q_3 v_z) \\ &= +j(q_0 q_3 v_x + q_0^2 v_y - q_0 q_1 v_z + q_1 q_2 v_x + q_2^2 v_y + q_2 q_3 v_z) \\ &= +k(-q_0 q_2 v_x + q_0 q_1 v_y + q_0^2 v_x + q_1 q_3 v_x + q_2 q_3 v_y + q_3^2 v_z) + q \times x \end{aligned}$$

$$\begin{aligned} q \times x &= i(-q_2^2 v_x + q_1 q_2 v_y + q_0 q_2 v_z - q_3^2 v_x - q_0 q_3 v_y + q_1 q_3 v_z) \\ &\quad + j(q_0 q_3 v_x + q_3^2 v_y + q_2 q_3 v_z + q_1 q_2 v_x - q_1^2 v_y - q_0 q_1 v_z) \\ &\quad + k(q_1 q_3 v_x + q_0 q_1 v_y - q_1^2 v_z - q_0 q_2 v_x + q_2 q_3 v_y - q_2^2 v_z) \end{aligned}$$

We now separate y by coefficients of i, j, k and v_x, v_y, v_z.

$$\begin{aligned} q_0^2 + q_1^2 - q_2^2 - q_3^2 &= (q_0^2 + q_1^2 + q_2^2 + q_3^2) - q_2^2 - q_3^2 - q_2^2 - q_3^2 \\ &= 1 - 2q_2^2 - 2q_3^2 \end{aligned}$$

Similarly, $q_0^2 + q_2^2 - q_1^2 - q_3^2 = 1 - 2q_1^2 - 2q_3^2$
and $q_0^2 + q_3^2 - q_1^2 - q_2^2 = 1 - q_1^2 - 2q_2^2$.

$$\Rightarrow y = \begin{bmatrix} 1 - 2q_2^2 - 2q_3^2 & 2q_1 q_2 - 2q_0 q_3 & 2q_2 q_3 + 2q_0 q_2 \\ 2q_1 q_2 + 2q_0 q_3 & 1 - 2q_1^2 - 2q_3^2 & 2q_2 q_3 - 2q_0 q_1 \\ 2q_1 q_3 - 2q_0 q_2 & 2q_2 q_3 + 2q_0 q_1 & 1 - 2q_1^2 - 2q_2^2 \end{bmatrix}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = R_v$$

Hence, $Y = Q(0, v_x, v_y, v_z)Q^* = (0, R_v)$ where R_v are the new rotated coordinates of v.

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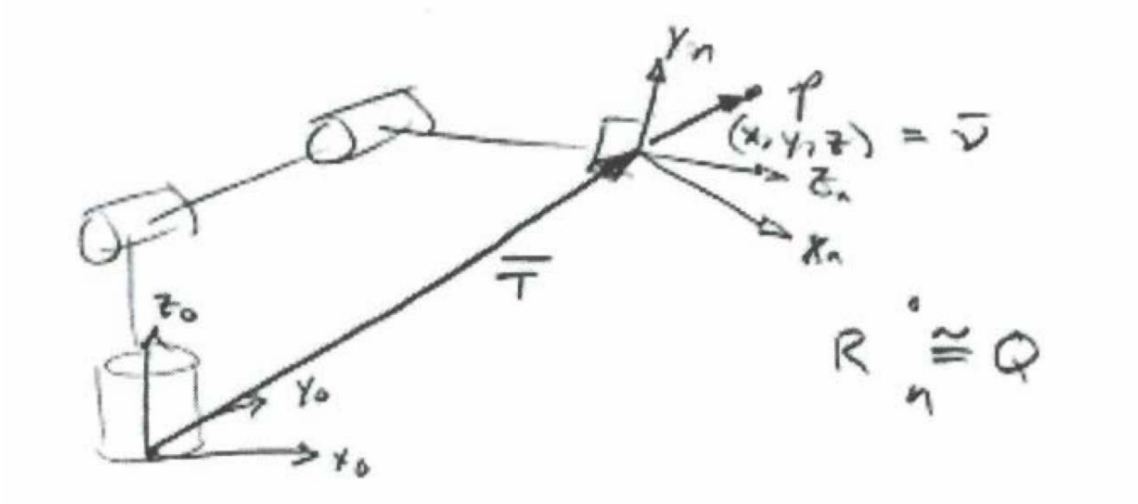

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2-35 Suppose *point* p has been expressed in frame n as $p^n = [x, y, z]^T$. Ignoring quaternions, we know we can write the location of p in base frame coordinates as

$$p^0 \quad = \quad R_n^0 p^n + T.$$

Now, we apply the result from problem 2-33 which gives the following equivalence

$$(0, R_n^0 p^n \quad = \quad Q(0, p^n)Q^*.$$

Since T is just the vector between the two frames, we can now write the expression

$$(0, p^0) \quad = \quad (0, T) + Q(0, p^n)Q^*$$

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2-36

$$H^{-1}H = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T R & R^T d - R^T d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$HH^{-1} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR^T & -RR^T d + d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} = I.$$

So H^{-1} is the inverse of H.

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2-37

$$
\begin{aligned}
T &= T_{y,1}T_{x,3}T_{z,\pi/2} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

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2-38

$$H_1^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_2^0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_2^1 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^0 \quad = \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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2-39

$$H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_2^0 = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_3^0 = \begin{bmatrix} 0 & 1 & 0 & -.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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2-40

$$H_2^1 = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & .5 \\ 0 & 0 & 1 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = H_1^0 H_2^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & .5 & \\ 0 & 0 & 1 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = H_2^0 H_3^2 = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & -1 & 0 & 1.5 \\ 0 & 0 & -1 & 3.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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2-41

$$H_3^2 = \begin{bmatrix} 1 & 0 & 0 & -.3 \\ 0 & -1 & 0 & .4 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The homogeneous transformation from the block frame to the base frame is

$$H_2^0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & .8 \\ 0 & 0 & 1 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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2-42 The earth rotates in the *ecliptic* plane about the sun, at a distance of approximately 150 million km. At the summer solstice ($t = 0$), the earth's axis of rotation z_{earth} is tilted 23.5° toward the sun. Let x_{earth} point in direction of the motion of the earth, always lying in the ecliptic plane and perpendicular to the vector from the sun to the earth. Let the z axis of the sun z_{sun} pass through the center of the sun and be perpendicular to the ecliptic plane. Noting that at $t = 0$ the earth's coordinate frame is coincident with the base frame, we write the homogeneous transformation between the base frame and the sun frame as follows.

$$H_{sun}^{base} = \begin{bmatrix} R_{x,23.5} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \begin{bmatrix} 0 \\ 150 \times 10^6 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Suppose the units of time to be days. Let θ be the angle in degrees between x_{sun} and the ray from the center of the sun to the center of the earth. Since the earth makes a complete revolution about the sun in 365.25 days, we write

$$\theta = \frac{t}{365.25} 360° - 90°$$

where $-90°$ is the offset of θ when $t = 0$. We are now prepared to write the homogeneous transformation from the sun frame to the earth frame at any time t

$$H_{earth}^{sun} = \begin{bmatrix} I & \begin{bmatrix} 150 \times 10^6 \cos\theta \\ 150 \times 10^6 \sin\theta \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{x,-23.5} & 0 \\ 0 & 1 \end{bmatrix}.$$

The homogeneous transformation between the base frame and earth frame is given by

$$H_{earth}^{base} = H_{sun}^{base} H_{earth}^{sun}.$$

The instantaneous orientation of the earth frame w.r.t. the base frame is the product of the rotation matrices given above

$$R_{earth}^{base} = R_{x,23.5} I I R_{x,-23.5} = I.$$

This is as we expect, since the axis of the earth maintains the same tilt as the earth revolves around the sun.

$$H = Rot_{x,\alpha} \text{Trans}_{x,b} \text{Trans}_{z,d} Rot_{z,\theta}$$

Translation and Rotations about the same axis commute because the orientation of the axis is preserved.

Translations commute because the orientation of the reference axes is preserved.

$$H = \left\{ \begin{array}{cccc} R_{x,\alpha} & T_{z,d} & T_{x,b} & R_{z,\theta} \\ T_{x,b} & R_{x,\alpha} & T_{z,d} & R_{z,\theta} \\ T_{x,b} & R_{x,\alpha} & R_{z,\theta} & T_{z,d} \\ R_{x,\alpha} & T_{x,b} & R_{z,\theta} & T_{z,d} \end{array} \right.$$

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