5-1 $\mathcal{Q} = \mathbb{R}^2 \times S^1$.

5-2 $Q = T^3$.

5-3 $\mathcal{Q} = \mathbb{R}^2$.

5-4 $\mathcal{Q} = \mathbb{R} \times S^1$.

5-5 $\mathcal{Q} = \mathbb{R} \times T^2$.

5-6 Assuming all joints are revolute, $\mathcal{Q} = T^6$.

5-7 Let $o_i(q) - o_i(f) = [x(q), y(q), z(q)]^T$. Now,

$$\|o_i(q) - o_i(f)\|^2 = x(q)^2 + y(q)^2 + z(q)^2.$$

and

$$U_{att,i}(q) = \frac{1}{2}\zeta_i \, \|o_i(q) - o_i(f)\|^2 = \frac{1}{2}\zeta_i[x(q)^2 + y(q)^2 + z(q)^2].$$

Therefore,

$$\begin{aligned} F_{att,i}(q) &= -\nabla U_{att,i}(q) \\ &= -\frac{1}{2}\zeta_i[2x(q), 2y(q), 2z(q)]^T \\ &= -\zeta_i[x(q), y(q), z(q)]^T \\ &= -\zeta_i\big(o_i(q) - o_i(f)\big) \end{aligned}$$

5-8 Let A denote the line *segment* passing through a_1 and a_2.
Let P denote the line passing through p that is perpendicular to A.
Define the point $a_\perp$ to be the intersection of A and P.

$$a_\perp = a_1 + t_\perp(a_2 - a_1) \text{ for some } t_\perp \in \mathbb{R}$$

Since line segment A does not extend beyond a_1 or a_2, $t_\perp$ is bounded.

$$t_\perp \in [0, 1]$$

Since $A \perp B$, we know the dot product of the two lines is zero.

$$\begin{aligned} (a_2 - a_1) \cdot (a_\perp - p) &= 0 \\ (a_2 - a_1) \cdot \big((a_1 - p) + t_\perp(a_2 - a_1)\big) &= 0 \end{aligned}$$

We can solve for $t_\perp$ by choosing any of the n components of p, a_1, a_2. For example, using the first component, we have:

$$t_\perp = \frac{(p(1) - a_1(1))(a_2(1) - a_1(1))}{(a_2(1) - a_1(1))^2}.$$

- If $t_\perp \in [0, 1]$, then $a_\perp$ is on line segment A and the minimum distance to point p is $\|a_\perp - p\|$.
- If $t_\perp \notin [0, 1]$, then $a_\perp$ is not on line segment A. Therefore the minimum distance to point p is the smaller of the distances from p to a_1 and from p to a_2. That is,

$$min\{\|a_1 - p\|, \|a_2 - p\|\}.$$

5-9 For a polygon in the plane with vertices $a_i, i = 1 \ldots n$, let $A_i, i = 1 \ldots (n-1)$ be the line segment between vertices a_i and a_{i+1} and let A_n be the line segment between a_n and a_1. Repeat the algorithm given in problem 5-8 to determine the minimum distance d_i between point p and each line segment A_i.

$\Rightarrow$ The shortest distance between point p and the polygon is $min\{d_1, d_2, \ldots, d_n\}$.

5-10 Let G_i denote the ith flat face of the polygon and num_i the number of vertices that define face G_i. Let $a_{ij}, j = 1 \ldots num_i$ denote the n_i vertices defining face G_i.

For each face G_i, at least three vertices are not colinear. We will call these three vertices v_1, v_2, v_3. These points define the plane p_i in which face G_i lies. We can find the equation of this plane by computing four determinants

$$\begin{vmatrix} 1 & & \\ 1 & v_2 & v_3 \\ 1 & & \end{vmatrix} x + \begin{vmatrix} & 1 & \\ v_1 & 1 & v_3 \\ & 1 & \end{vmatrix} x + \begin{vmatrix} & & 1 \\ v_1 & v_2 & 1 \\ & & 1 \end{vmatrix} x - \begin{vmatrix} v_1 & v_2 & v_3 \end{vmatrix} = 0$$

and the vector n_i normal to the plane by taking the cross product

$$n_i = (v_3 - v_1) \times (v_2 - v_1).$$

We will proceed according to the following algorithm.

1. Compute the perpendicular distances from p to each of the faces.
2. Compute the perpendicular distances from p to each of the edges.
3. Compute the distances from p to each of the vertices.

1. For each face G_i, solve for the point of intersection int_i of the plane p_i with the line defined by the normal vector n_i and passing through p. We must now check whether int_i is inside or outside face G_i. To do this, we may construct lines from int_i to each of the vertices a_{ij}.

   – If the sum of the angles of these lines (with a common reference of any line in the plane) is an integer multiple of 2π, the point int_i lies inside the face.
   – If the sum of the angles is not an integer multiple of 2π, the point lies outside the face.

   If any of the normal intersection points int_i lie inside their respective faces, we note the distance $\|p - int_i\|$.
2. For each edge, we follow the algorithm for Problem 5-8. If the perpendicular intersection of a line through p with the line containing the edge occurs within the edge's vertices, we note the distance.
3. Finally, we compute the distances between p and each of the vertices.

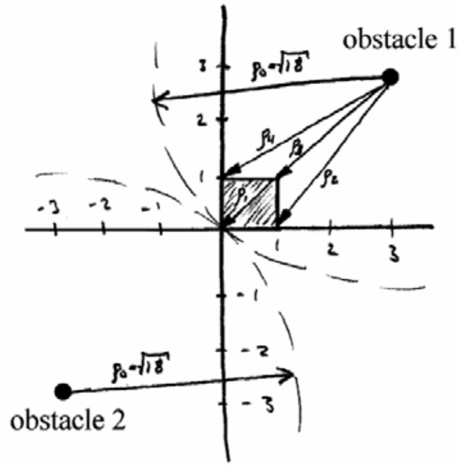
The minimum distance between point p and the polygon is the smallest of these distances.

5-11 Distance $\rho(o_i(q))$ is a function of the location in space $o_i(q)$. Consider writing this as $\rho(x)$, where x is a vector in three dimensions. Then we can treat the gradient ∇ as the partial derivative with respect to the vector x $\frac{\partial}{\partial x}$. The result now follows from the chain rule for differentiation.

$$
\begin{aligned}
F &= -\frac{\partial}{\partial x}\left[\frac{1}{2}\eta_1\left(\rho^{-1}(x) - \frac{1}{\rho_0}\right)^2\right] \\
&= -\frac{1}{2}\eta_1 2\left(\rho^{-1}(x) - \frac{1}{\rho_0}\right)\frac{\partial}{\partial x}\left(\rho^{-1}(x)\right) \\
&= \eta_1\left(\rho^{-1}(x) - \frac{1}{\rho_0}\right)\rho^{-2}(x)\frac{\partial}{\partial x}\rho(x)
\end{aligned}
$$

Therefore,

$$
F_{rep,i}(q) = \eta_i\left(\frac{1}{\rho(o_i(q))} - \frac{1}{\rho_0}\right)\frac{1}{\rho^2(o_i(q))}\nabla\rho(o_i(q)).
$$



– Assume the robot has three degrees of freedom, and thus three "joint variables" $q = \{x, y, \theta\}$. The robot is able to translate $\{x, y\}$ (think of this as two prismatic joints) and rotate $\{\theta\}$ (think of this as a single revolute joint).

1. **artificial workspace forces**
To avoid overlap of the regions of influence, we chose $\rho_0 = \sqrt{18}$ for both obstacles. In its given configuration, the robot is influenced only by obstacle 1. We construct the repulsive potential field and artificial workspace forces according to Equations (5.5) and (5.6).

$$U_{rep,i}(q) = \begin{cases} \frac{1}{2}\eta_i\left(\frac{1}{\rho(a_i(q))} - \frac{1}{\sqrt{18}}\right)^2 & \rho(a_i(q)) \le \sqrt{18} \\ 0 & \rho(a_i(q)) > \sqrt{18} \end{cases}$$
$$F_{rep,i}(q) = \begin{cases} \eta_i\left(\frac{1}{\rho(a_i(q))} - \frac{1}{\sqrt{18}}\right)^2 \frac{1}{\rho^2(a_i(q))}\nabla\rho(a_i(q)) & \rho(a_i(q)) \le \sqrt{18} \\ 0 & \rho(a_i(q)) > \sqrt{18} \end{cases}$$

where

$$\rho(a_i(q)) = \|a_i(q) - b\|$$
$$\nabla\rho(a_i(q)) = \frac{a_i(q) - b}{\|a_i(q) - b\|}.$$

For obstacle 1, $b = [3, 3]^T$ in the equations above; for obstacle 2, $b = [-3, -3]^T$. At time $t = 0$ we have $a_1(q) = [0, 0]^T, a_2(q) = [1, 0]^T, a_3(q) = [1, 1]^T$, and $a_4(q) = [0, 1]^T$. These yield

$$\begin{aligned} \rho(a_1(q)) &= \sqrt{18} & \nabla\rho(a_1(q)) &= \tfrac{[-3,-3]^T}{\sqrt{18}} \\ \rho(a_2(q)) &= \sqrt{13} & \nabla\rho(a_2(q)) &= \tfrac{[-2,-3]^T}{\sqrt{13}} \\ \rho(a_3(q)) &= \sqrt{8} & \nabla\rho(a_3(q)) &= \tfrac{[-2,-2]^T}{\sqrt{8}} \\ \rho(a_4(q)) &= \sqrt{13} & \nabla\rho(a_4(q)) &= \tfrac{[-3,-2]^T}{\sqrt{13}}. \end{aligned}$$

5-13 through 5-16 are machine problems.

5-13 through 5-16 are machine problems.

5-13 through 5-16 are machine problems.

5-13 through 5-16 are machine problems.

5-17

$$\det \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} = (t_f - t_0)^4$$

5-18 The problem is somewhat open-ended. Students should discuss issues related to matching the velocity of the conveyor, planning a straight line path, etc.

5-19 Using formulas given, the constants for the cubic polynomial are:

$$
\begin{aligned}
a_0 &= q_0 \\
a_1 &= 0 \\
a_2 &= \frac{(3q_1 - 3q_0 - 2)}{4} \\
a_3 &= \frac{(q_0 - q_1 + 1)}{4}
\end{aligned}
$$

The cubic polynomial for position is:

$$
\begin{aligned}
q_i^d(t) &= a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + a_3(t - t_0)^3 \\
&= q_0 + \frac{(3q_1 - 3q_0 - 2)}{4}(t - t_0)^2 + \frac{(q_0 - q_1 + 1)}{4}(t - t_0)^3
\end{aligned}
$$

5-20 For $t \in [2-t_b, 2]$, the *parabolic* trajectory and speed are given by:

$$\begin{aligned} q(t) &= b_0 + b_1 t + b_2 t^2 \\ \dot{q}(t) &= b_1 + 2b_2 t. \end{aligned}$$

Using the boundary condition $\dot{q}(2) = 1$, we find that

$$b_1 = 1 - 4b_2.$$

For $t \in [t_b, 2-t_b]$, the *linear* trajectory and speed are given by:

$$\begin{aligned} q(t) &= q(t_b) + Vt \\ \dot{q}(t) &= V. \end{aligned}$$

At time $t = 2 - t_b$, the *speeds* of the parabolic and linear segments must match

$$\dot{q}(2-t_b) = (1-4b_2) + 2b_2(t_b) = V$$

which implies

$$b_2 = \frac{V-1}{2t_b - 4}.$$

Using the boundary condition at $q(2) = q_1$ for the *parabolic* trajectory,

$$q(2) = b_0 - 4\frac{V-1}{2t_b-4} + 2 = q_1$$

which implies

$$b_0 = q_1 - 2 + 4\frac{V-1}{2t_b-4}.$$

At time $t = 2 - t_b$, the *trajectories* of the parabolic and linear segments must match

$$\begin{aligned} q(2-t_b) = b_0 + b_2\big[(2-t_b)^2 - 4(2-t_b)\big] + (2-t_b) &= q(t_b) + V(2-t_b) \\ \left(q_1 - 2 + 4\frac{V-1}{2t_b-4}\right) + \left(\frac{V-1}{2t_b-4}\right)\big[(2-t_b)^2 - 4(2-t_b)\big] &= q(t_b) + V(2-t_b) \end{aligned}$$

which may be solved to find slope t_b as a function of blend time V. From the development in the book, we know that

$$t_b = \frac{q_0 - q_1 + 2V}{V}.$$

Setting equal the two expressions for t_b, we can solve for V and then for t_b.

$$q(t) = \begin{cases} q_0 + \frac{V}{2t_b}t^2 & t \in [0, t_b] \\ \frac{q_0+q_1-2V}{2} + Vt & t \in [t_b, 2-t_b] \\ b_0 + b_1 t + b_2 t^2 & t \in [2-t_b, 2] \end{cases}$$

5-21 For $t_f - t_b < t \leq t_f$, the desired *parabolic* trajectory and speed are given by:

$$\begin{aligned} q(t) &= b_0 + b_1 t + b_2 t^2 \\ \dot{q}(t) &= b_1 + 2b_2 t. \end{aligned}$$

Using the boundary condition $\dot{q}(t_f) = 0$ we find that

$$b_1 = -2b_2 t_f$$

For $t \in [t_b, 2 - t_b]$, the *linear* trajectory and speed are given by:

$$\begin{aligned} q(t) &= q(t_b) + Vt \\ \dot{q}(t) &= V. \end{aligned}$$

At time $t = t_f - t_b$, the *speeds* of the parabolic and linear trajectories must match

$$\dot{q}(t_f - t_b) = b_1 + 2b_2(t_f - t_b) = V$$

which gives us

$$\begin{aligned} b_2 &= \frac{-V}{2t_b} \\ b_1 &= \frac{Vt_f}{t_b}. \end{aligned}$$

Using the boundary condition $q(t_f) = q_f$ for the *parabolic* trajectory we find

$$q(t_f) = b_{at} \frac{Vt_f^2}{2t_b} = q_f$$

which implies

$$b_0 = q_f - \frac{Vt_f^2}{2t_b}.$$

Let $\alpha = \frac{V}{t_b}$.
Then for $t_f - t_b < t \leq t_f$ the trajectory is given by

$$q(t) = q_f - \frac{\alpha}{2} t_f^2 + \alpha t_f t - \frac{\alpha}{2t^2}.$$

5-22 and 5-23 are machine problems.

5-22 and 5-23 are machine problems.