3-1 From Equation (3.13), we know that R has the form

$$R = \begin{bmatrix} c_\theta & r_{12} & r_{13} \\ s_\theta & r_{22} & r_{23} \\ 0 & s_\alpha & c_\alpha \end{bmatrix}$$

Since R is a rotation matrix the column vectors satisfy

$$\begin{aligned} r_{12}^2 + r_{22}^2 &= 1 - s_\alpha^2 = c_\alpha^2 \\ r_{13}^2 + r_{23}^2 &= 1 - c_\alpha^2 = s_\alpha^2 \end{aligned}$$

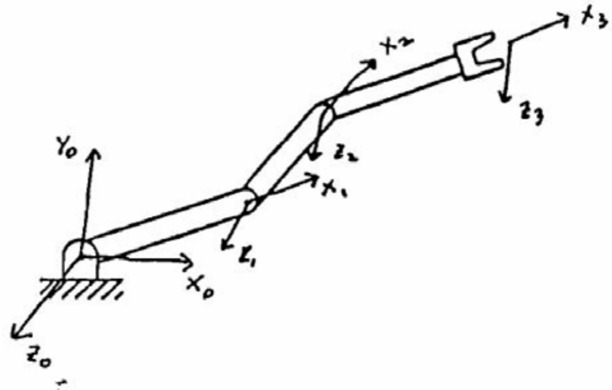
Therefore there is a unique angle θ such that

$$\begin{aligned} r_{12}/c_\alpha = -s_\theta; &\qquad r_{22}/c_\alpha = c_\theta \\ r_{13}/s_\alpha = s_\theta; &\qquad r_{23}/s_\alpha = -c_\theta \end{aligned}$$

and the results follows.

In each of the following problems 3-2 to 3-7, the figure shows the D-H coordinate frames and a table of D-H parameters that is used to generate the A matrices and the T matrix giving the transformation between the base frame and the end-effector frame.

3-2



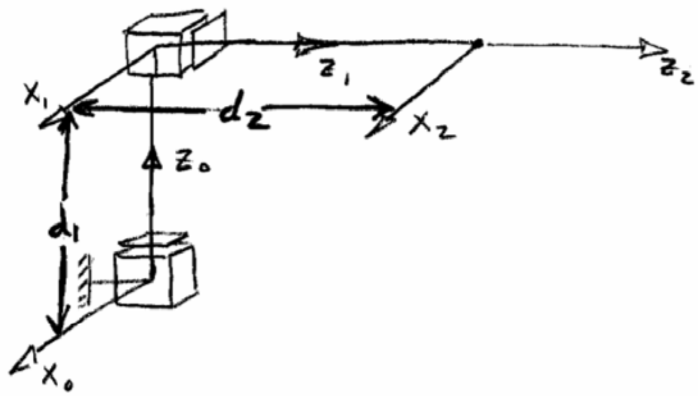
link	a_1	α_i	d_i	θ_i
1	α_1	0	0	θ_1
2	α_2	0	0	θ_2
3	α_3	0	0	θ_3

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; A_2 = \begin{bmatrix} c_2 & -c_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = A_1A_2A_3 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3-3

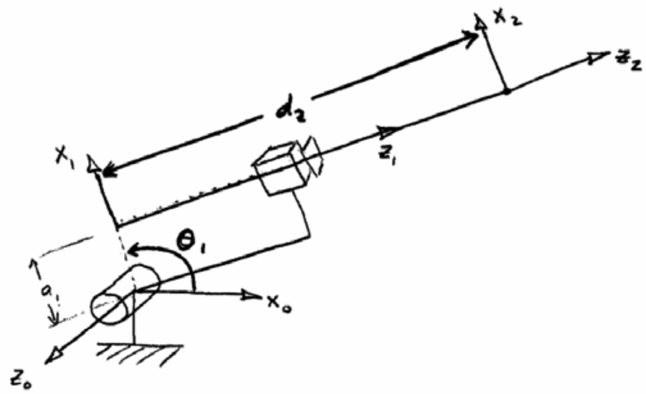


link	a_i	α_i	d_i	θ_i
1	0	-90°	d_1	0
2	0	0	d_2	0

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^2 = A_1 A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3-4

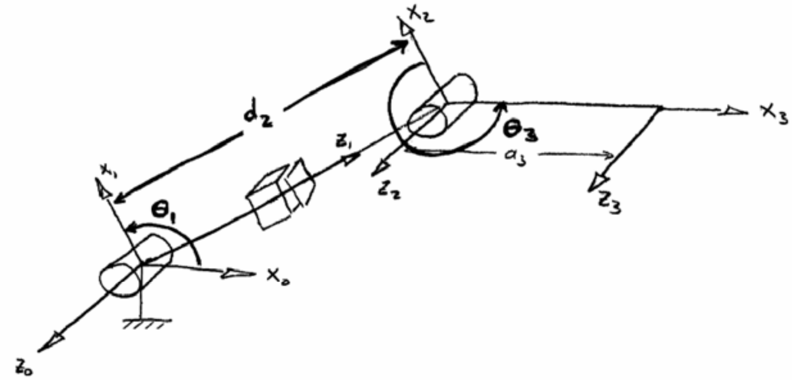


link	a_i	α_i	d_i	θ_i
1	0	90°	0	θ_1
2	0	-90°	d_2	0
3	a_3	0	d_3	θ_3

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^2 = A_1 A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3-5

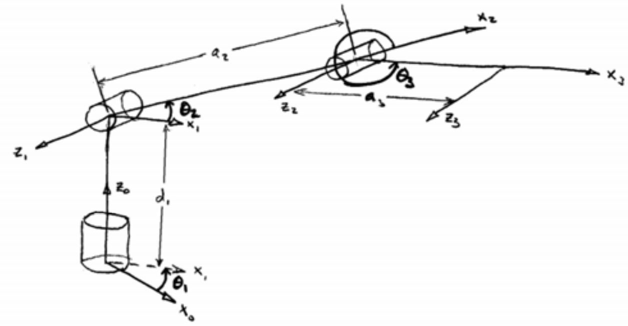


link	a_i	α_i	d_i	θ_i
1	0	90°	0	θ_1
2	0	-90°	d_2	0
3	a_3	0	d_3	θ_3

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = A_1A_2A_3 = \begin{bmatrix} c_{13} & -s_{13} & 0 & s_1d_2 + a_3c_{13} \\ s_{13} & c_{13} & 0 & -c_1d_2 + a_3s_{13} \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



link	a_i	α_i	d_i	θ_i
1	0	90	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

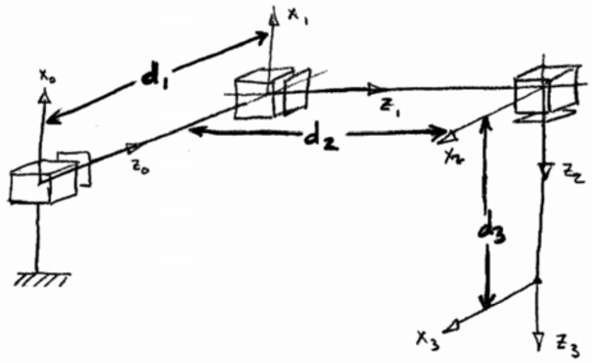
$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$\begin{aligned}
r_{11} &= c_1c_2c_3 - c_1s_2s_3 = c_1c_{23} \\
r_{12} &= -c_1c_2s_3 - c_1c_3c_2 = -c_1s_{23} \\
r_{13} &= s_1 \\
d_x &= a_2a_2c_1c_2 + a_3c_1c_2c_3 - a_3c_1s_2s_3 = a_2c_1c_2 + a_3c_1c_{23} \\
r_{21} &= c_2c_3s_1 - s_1s_2s_3 = x_1c_{23} \\
r_{22} &= -c_2s_1s_3 - c_3s_1s_2 = -s_1s_{23} \\
r_{23} &= -c_1 \\
d_y &= a_2c_2s_1 + a_3c_2c_3s_1 - a_3s_1s_2s_3 = a_2c_2s_1 + a_3s_1c_{23} \\
r_{31} &= c_2s_3 + c_3s_2 = s_{23} \\
r_{32} &= c_2c_3 - s_2s_3 = c_{23} \\
r_{33} &= 0 \\
d_z &= a_2s_2 + a_3c_2s_3 + a_3c_3s_2 = a_2s_2 + a_3s_{23}
\end{aligned}$$

3-7



link	a_i	α_i	d_i	θ_i
1	0	$-90°$	d_1	0
2	0	$90°$	d_2	$90°$
3	0	0	d_3	$-90°$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 \quad = \quad A_1 A_2 A_3 = \begin{bmatrix} 0 & 0 & 1 & d_3 \\ -1 & 0 & 0 & d_2 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

link	a_i	α_i	d_i	θ_i
1	0	90°	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3
4	0	−90°	0	θ_4
5	0	0	0	θ_5
6	0	0	d_6	θ_6

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
r_{11} &= c_1[c_5c_6c_{234} - s_6s_{234}] - s_1s_5c_6 \\
r_{12} &= -c_1[c_5s_6c_{234} + c_6s_{234}] + s_1s_5s_6 \\
r_{13} &= c_1s_5c_{234} + s_1c_5 \\
d_x &= a_2c_1c_2 + a_3c_1c_{23} + d_6[c_1s_5c_{234} + s_1c_5] \\
r_{21} &= c_1s_5s_6 + s_1c_5c_6c_{234} - s_1s_6s_{234} \\
r_{22} &= -c_1s_5s_6 - s_1c_5s_6c_{234} \\
r_{23} &= -c_1c_5 + s_1s_5c_{234} \\
d_y &= a_2s_1c_2 + a_3s_1c_{23} - d_6[c_1c_5 + s_1s_5c_{234}] \\
r_{31} &= s_6c_{234} + c_5s_6s_{234} \\
r_{32} &= c_6s_{234} - c_5s_6s_{234} \\
r_{33} &= s_5s_{234} \\
d_z &= a_2s_2 + a_3c_2s_{23} + d_6s_5s_{234}
\end{aligned}$$

The matrix T_0^3 is given as in Problem 3-7. The matrix T_3^6 is given by Equation (3.15) of the text. Therefore

$$T_0^6 = \begin{bmatrix} -c_6s_5 & s_5s_6 & c_5 & d_3 + d_6c_5 \\ -c_4c_5c_6 + s_4s_6 & c_4c_5s_6 + c_6s_4 & -c_4s_5 & d_2 - d_6c_4s_5 \\ -c_4s_6 - c_5c_6s_4 & -c_4c_6 + c_5s_4s_6 & -s_4s_5 & d_1 - d_6s_4s_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

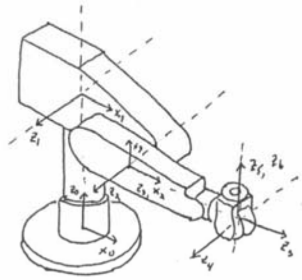
3-9 Attaching a spherical wrist to the robot of Problem 3-7 gives

$$T_0^5 = T_0^3 T_3^6$$

The matrix T_0^3 is given as in Problem 3-7. The matrix T_3^6 is given by Equation (3.15) of the text. Therefore

$$T_0^6 = \begin{bmatrix} -c_6 s_5 & s_5 s_6 & c_5 & d_3 + d_6 c_5 \\ -c_4 c_5 c_6 + s_4 s_6 & c_4 c_5 s_6 + c_6 s_4 & -c_4 s_5 & d_2 - d_6 c_4 s_5 \\ -c_4 s_6 - c_5 c_6 s_4 & -c_4 c_6 + c_5 s_4 s_6 & -s_4 s_5 & d_1 - d_6 s_4 s_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3-10



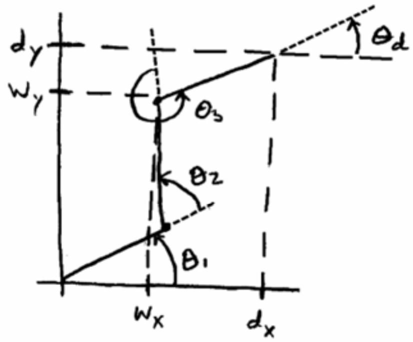
link	a_i	α_i	d_i	θ_i
1	0	90°	13″	θ_1
2	8″	0	d_2	θ_2
3	8″	90°	0	θ_3
4	0	−90°	d_4	θ_4
5	0	90°	0	θ_5
6	0	0	d_6	θ_6

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & 8c_2 \\ s_2 & c_2 & 0 & 8s_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_3 = \begin{bmatrix} c_3 & 0 & s_3 & 8c_3 \\ s_3 & 0 & -c_3 & 8s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where:

$$\begin{aligned}
r_{11} &= c_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_4s_6s_{23}] + s_1[c_4s_6 + s_4c_5c_6] \\
r_{12} &= c_1[-c_{23}(c_4c_5s_6 + s_4c_6) + s_5s_6s_{23}] + s_1[c_4c_6 - s_4c_5s_6] \\
r_{13} &= c_1[c_4s_5c_{23} + c_5s_{23}] - s_1s_4s_5 \\
d_x &= d_2s_1 + d_4c_1s_{23} + d_6[c_1(c_4s_5c_{23} + c_5s_{23}) + s_1s_4 + s_5] + 8c_1[c_{23} + c_2] \\
r_{21} &= -c_1[c_4s_6 + s_4c_5c_6] + s_1[c_{23}(c_4c_5c_6 + s_4s_6) - s_5c_6s_{23}] \\
r_{22} &= c_1[s_4c_5s_6 - c_4c_6] + s_1[-c_{23}(c_4c_5s_6 + s_4c_6) + s_5s_6s_{23}] \\
r_{23} &= -c_1s_4s_5 + s_1[c_4s_5c_{23} + c_5s_{23}] \\
d_y &= -d_2c_1 + d_4s_1s_{23} + d_6[s_1(c_4s_5c_{23} + c_5s_{23}) - c_1s_4s_5] - c_1s_4s_5] + 8s_1[c_{23} + c_2] \\
r_{31} &= s_{23}(c_4c_5c_6 - s_4s_6) + s_5c_6c_{23} \\
r_{32} &= -s_{23}(c_4c_5s_6 + s_4c_6) - s_5s_6c_{23} \\
r_{33} &= -c_5c_{23} + c_4s_5s_{23} \\
d_z &= 13 - d_4c_{23} + d_6[-c_5c_{23} + c_4s_5s_{23}] + 8[s_{23} + s_2]
\end{aligned}$$



1. Given a desired position $d = [d_x, d_y]^T$ of the end-effector only, we can write the coordinates of the end-effector as two equations in three unknowns.

$$
\begin{aligned}
d_x &= a_1\cos(\theta_1) + a_2\cos(\theta_1+\theta_2) + a_3\cos(\theta_1+\theta_2+\theta_3)\\
d_y &= a_1\sin(\theta_1) + a_2\sin(\theta_1+\theta_2) + a_3\sin(\theta_1+\theta_2+\theta_3)
\end{aligned}
$$

Therefore, this problem is underconstrained. In general, there are infinitely many solutions to the inverse kinematics problem. More specifically,

$$
\text{there are} \begin{cases} \infty & \text{solutions if } d \text{ is inside workspace}\\ 1 & \text{solution if } d \text{ is on workspace boundary}\\ 0 & \text{solutions if } d \text{ is outside workspace.} \end{cases}
$$

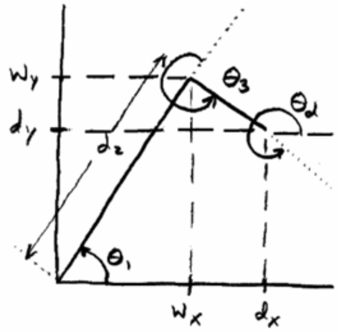
2. Given a desired position $d = [d_x, d_y]^T$ and orientation θ_d of the end-effector, we can write the coordinates of the wrist center $[w_x, w_y]^T$

$$
\begin{aligned}
w_x &= d_x - a_3\cos(\theta_d)\\
w_y &= d_y - a_3\sin(\theta_d).
\end{aligned}
$$

Now we have reduced the problem to finding a solution for the first two links that will reach the wrist center. The solutions for θ_1 and θ_2 are given in Equations (1.7-1.8).

$$\theta_3 = \theta_d - (\theta_1 + \theta_2)$$

$$
\text{There are} \begin{cases} \infty & \text{solutions if the wrist center is the origin}\\ 2 & \text{solutions if the wrist center is inside the 2-link workspace}\\ 1 & \text{solution if the wrist center is on the 2-link workspace boundary}\\ 0 & \text{solutions if the wrist center is outside the 2-link workspace.} \end{cases}
$$



1. Given a desired position $d = [d_x, d_y]^T$ of the end-effector only, we can write the coordinates of the end-effector as two equations in three unknowns.

$$\begin{aligned} d_x &= d_2 \cos(\theta_1) + a_3 \cos(\theta_1 + \theta_3) \\ d_y &= d_2 \sin(\theta_1) + a_3 \sin(\theta_1 + \theta_3) \end{aligned}$$

Therefore, this problem is underconstrained. In general, there are infinitely many solutions to the inverse kinematics problem. More specifically,

$$\text{there are} \begin{cases} \infty & \text{solutions if } d \text{ is inside workspace} \\ 1 & \text{solution if } d \text{ is on workspace boundary} \\ 0 & \text{solutions if } d \text{ is outside workspace.} \end{cases}$$

2. Given a desired position $d = [d_x, d_y]^T$ and orientation θ_d of the end-effector, we can write the coordinates of the wrist center $[w_x, w_y]^T$

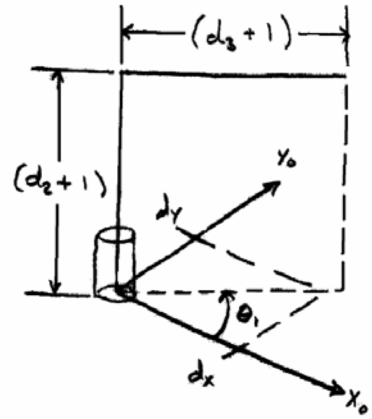
$$\begin{aligned} w_x &= d_x - a_3 \cos(\theta_d) \\ w_y &= d_y - a_3 \sin(\theta_d). \end{aligned}$$

Now we have reduced the problem to finding a solution for the first two links that will reach the wrist center. Solving the geometric problem, we find

$$\begin{aligned} \theta_1 &= \text{Atan2}(w_x, w_y) \\ d_2 &= \sqrt{(w_x^2 + w_y^2)} \\ \theta_3 &= \theta_d - \theta_1 \end{aligned}$$

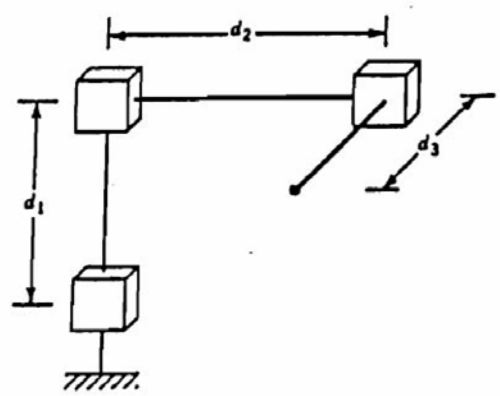
$$\text{There are} \begin{cases} \infty & \text{solutions if the wrist center is the origin} \\ 1 & \text{solution if the wrist center is on or inside the 2-link workspace boundary} \\ 0 & \text{solutions if the wrist center is outside the 2-link workspace.} \end{cases}$$

3-13



Given $d = (d_x, d_y, d_z)^T$, we have

$$
\begin{aligned}
\theta_1 &= \text{Atan2}(d_x, d_y) \\
d_2 &= d_z - 1 \\
d_3 &= -1 + \sqrt{d_x^2 + d_y^2}.
\end{aligned}
$$



3-14 Given $d = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$ can be reached by setting

$$\begin{bmatrix} d_z \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \end{bmatrix}$$

$$R_3^6 \;=\; (R_0^3)^T R = U = \begin{bmatrix} r_{31} & r_{32} & r_{33} \\ r_{11}c_1 + r_{21}s_1 & r_{12}c_1 + r_{22}s_1 & r_{13}c_1 + r_{23}s_1 \\ -r_{11}s_1 + r_{21}c_1 & -r_{21}s_1 + r_{22}c_1 & -r_{13}s_1 + r_{23}c_1 \end{bmatrix}$$

I. If not both $u_{13} + u_{23}$ are zero, then

$$\theta_5 \;=\; A\tan\left(-r_{13}s_1 + r_{23}c_1, \pm\sqrt{1 - (-r_{13}s_1 + r_{23}c_1)^2}\right)$$

a) If the positive square root is chosen

$$\theta_4 \;=\; A\tan\left(r_{33}, r_{13}c_1 + r_{23}s_1\right)$$
$$\theta_6 \;=\; A\tan\left(+r_{11}s_1 - r_{21}c_1, -s_1r_{12} + c_1r_{22}\right)$$

b) If the negative square root is chosen

$$\theta_4 \;=\; A\tan\left(-r_{33}, -r_{13}c_1 + r_{23}s_1\right)$$
$$\theta_6 \;=\; A\tan\left(-r_{11}s_1 + r_{21}c_1, s_1r_{12} - c_1r_{22}\right)$$

II. If $u_{13} = u_{23} = 0$

a) If $u_{33} = 1$

$$0 = r_{33} = r_{13}c_1r_{28}s_1 + r_{28}s_1 = c_4s_5 = s_4s_5 \to s_5 = 0 \quad \theta_5 = 0^\circ$$
$$\theta_4 + \theta_6 \;=\; A\tan\left(r_{31}, r_{11}c_1 + r_{21}s_1\right) = A\tan\left(r_{31}, -r_{32}\right)$$

b) If $u_{33} = 1$ $\theta_4 = 0$; $c_5 = -1$ $s_5 = 0$ $\theta_5 = \pi$

$$\theta_4 - \theta_6 \;=\; A\tan\left(-r_{31}, -r_{32}\right) = A\tan\left(-r_{11}c_1 - r_{21}s_1, -r_{12}c_1 - r_{22}s_1\right)$$

3-16

link	a_i	α_i	d_i	θ_i
1	0	-90	d_1^*	0
2	0	90	d_2^*	90
3	0	0	d_3^*	0
4	0	90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* denotes variable

$$R_0^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Given d and R

$$p_c = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \end{bmatrix}$$

$$r_3^6 = (R_0^6)^T R = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} -r_{31} & -r_{32} & -r_{33} \\ r_{21} & r_{22} & r_{23} \\ r_{11} & r_{12} & r_{13} \end{bmatrix}$$

Equate R_3^6 to matrix (4.4.1). Suppose that r_{33} and r_{23} are unzero, then $r_{13} \neq \pm 1$, so $c_\theta = r_{13}$, $s_\theta = \pm\sqrt{1 - r_{13}^2}$ and $\theta = A\tan\left(r_{13}, \sqrt{1 - r_{13}^2}\right)$, if $s_\theta > 0$, choose $\phi = A\tan(r_{33}, r_{23})$ and $\psi = A\tan(-4_{11}, r_{12})$. However, if $r_{33} = r_{23} = 0$, then $r_{13} = \pm 1$, if

$$r_{13} = +1 \quad - = 0, \phi + \psi = A\tan(-r_{31}, r_{21}) = A\tan(-r_{31}, r_{32})$$

if

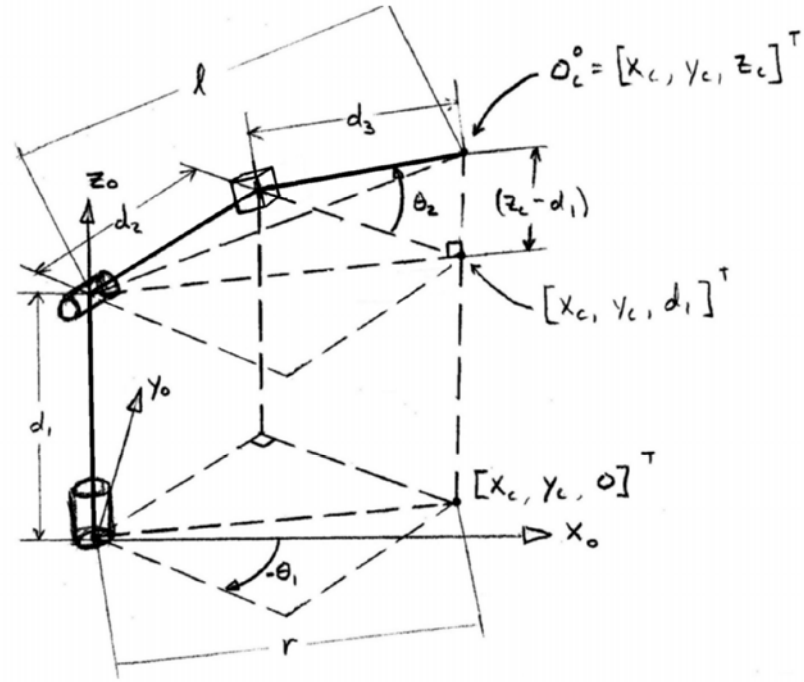
$$r_{13} = -1 \quad \theta = 0, \phi + \psi = A\tan(-r_{31}, r_{21}) = A\tan(-r_{31}, r_{32})$$

if

$$r_{13} = \pm 1,$$

there are an infinite number of solutions.

3-17 machine problem



We are given the desired position d and orientation R of the tool.

1. **desired coordinates of the wrist center**

$$o_c^0 = d - R \begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix}$$

where d_6 is the distance from the wrist center to the origin of the tool frame.

2. **inverse position kinematics**
This problem is difficult to visualize; success will often depend on the quality of the sketch made of the first three links, especially the "cheese wedge" region formed by the upper arm, elbow, and lower arm. Making use of right triangles and the Pythagorean theorem, we have

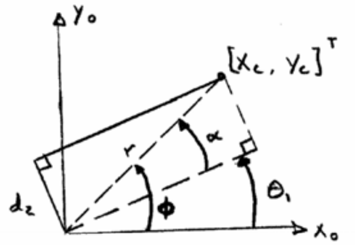
$$\begin{aligned} r^2 &= x_c^2 + y_c^2 \\ \ell^2 &= d_2^2 + d_3^2 \\ \text{and } \ell^2 &= (z_c - d_1)^2 + r^2. \end{aligned}$$

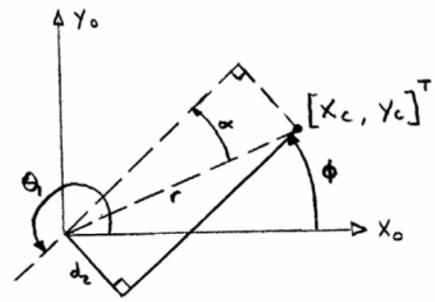
Solving these three equations simultaneously yields a solution for the prismatic joint.

$$d_3 = \sqrt{(z_c - d_1)^2 + x_c^2 + y_c^2 - d_2^2}$$

Again using a right triangle, we find a solution for θ_2.

$$\theta_2 = \begin{cases} \text{Atan2}\big(\sqrt{r^2 - d_2^2}, (z_c - d_1)\big) & \text{left arm} \\ \text{Atan2}\big(-\sqrt{r^2 - d_2^2}, (z_c - d_1)\big) & \text{right arm} \end{cases}$$





This results in a total of *two solutions.* Finally, we project the first three links of the manipulator onto the $x_0 - y_0$ plane to find solutions for waist angle θ_1.

$$\begin{aligned} \phi &= \operatorname{atan2}(x_c, y_c) \\ \alpha &= \operatorname{atan2}(\sqrt{r^2 - d_2^2}, d_2) \end{aligned}$$

$$\theta_1 = \begin{cases} \phi - \alpha & \text{left arm} \\ \phi + \alpha + \pi & \text{right arm} \end{cases}$$

3. **inverse orientation kinematics**

$$R_0^3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 \\ s_1c_2 & c_1 & s_1s_2 \\ -s_2 & 0 & c_2 \end{bmatrix}$$

found by multiplying $A_1A_2A_3$ and extracting first 3 rows and columns

$R_3^6 = (R_0^3)^T R$

$$= \begin{bmatrix} c_1c_2r_{11} + s_1c_2r_{21} - s_2r_{31} & c_1c_2r_{12} + s_1c_2r_{22} - s_2r_{32} & c_1c_2r_{13} + s_1c_2r_{23} - s_2r_{33} \\ -s_1r_{11} + c_1r_{21} & -s_1r_{12} + c_1r_{22} & -s_1r_{13} + c_1r_{23} \\ c_1s_2r_{11} + s_1s_2r_{21} + c_2r_{31} & c_1s_2r_{12} + s_1s_2r_{22}r_{32} & c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33} \end{bmatrix}$$

Assume $r_{13} \neq 0$ and $R_{23} \neq 0$ then

$$c_5 = c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33}$$

and

$$s_5 = \pm\sqrt{1 - (c_2s_2r_{13} + s_1s_2r_{23} + c_2r_{33})^2}$$

if $s_5 > 0$ then

$$\begin{aligned} \theta_5 &= A\tan\left(c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33}, \sqrt{1 - (c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33})^2}\right) \\ \theta_4 &= A\tan(c_1c_2r_{13} + s_1c_2r_{23} - s_2r_{33}, -s_1r_{13} + c_1r_{23}) \\ \theta_6 &= A\tan(c_1s_2r_{11} + s_1s_2r_{21} + c_2r_{31}, -c_1s_2r_{12} - s_1s_2r_{22} - c_2r_{32}) \end{aligned}$$

if $s_5 < 0$ then

$$\begin{aligned}
\theta_5 &= A\tan\left(c_1 s_2 r_{13} + s_1 s_2 r_{23} + c_2 r_{33}, -\sqrt{1-(c_1 s_2 r_{13} + s_1 s_2 r_{23} + c_2 r_{33})^2}\right) \\
\theta_4 &= A\tan(-c_1 c_2 r_{13} - s_1 c_2 r_{23} + s_2 r_{33}, s_1 r_{13} - c_1 r_{23}) \\
\theta_6 &= A\tan(-c_1 s_2 r_{11} - s_1 s_2 r_{21} - c_2 r_{31}, c_1 s_2 r_{12} + s_1 s_2 r_{22} + c_2 r_{32})
\end{aligned}$$

if $r_{13} = r_{23} = 0$ then $r_{33} = \pm 1$
if $r_{33} = +1$ $\theta_5 = \theta_2$ abd $\theta_4 = \pi$
if $s_5 > 0$

$$\theta_6 = A\tan(c_1 s_2 r_{11} + s_1 s_2 r_{21} + c_2 r_{31}, -c_1 s_2 r_{12} - s_1 s_2 r_{22} - c_2 r_{32})$$

if $s_5 < 0$

$$\theta_6 = A\tan(-c_1 s_2 r_{11} - s_1 s_2 r_{21} - c_2 r_{31}, c_1 s_2 r_{12} + s_1 s_2 r_{22} + c_2 r_{32})$$

if $r_{33} = -1$ $\theta_5 = \pi - \theta_2$ and $\theta_4 = 0$
if $s_5 > 0$

$$\theta_6 = A\tan(c_1 s_2 r_{11} + s_1 s_2 r_{21} + c_2 r_{31}, -c_1 s_2 r_{12} - s_1 s_2 r_{22} - c_2 r_{32})$$

if $s_5 < 0$

$$\theta_6 = A\tan(-c_1 s_2 r_{11} - s_1 s_2 r_{21} - c_2 r_{31}, c_1 s_2 r_{12} + s_1 s_2 r_{22} + c_2 r_{32})$$

link	a_1	α_i	d_i	θ_i
1	0	90°	d_1	θ_1^*
2	α_2	0	d_2	θ_2^*
3	α_3	0	0	θ_3^*

* denotes variable

1. **desired coordinates of the wrist center**

$$\begin{aligned} x_c &= x_0 - d_6 c_5 c_1 \\ y_c &= y_0 - d_6 c_5 s_1 \\ z_c &= z_0 - z_{0c} \end{aligned}$$

2. **inverse position kinematics**

$$\begin{aligned} \theta_1 &= \phi - \alpha \\ \phi &= \tan\left(\frac{y_c}{x_c}\right) \\ \alpha &= \tan\left(\frac{a_3 c_{23} + a_2 c_2}{d_2}\right) \end{aligned}$$

Elbow Right

$$\begin{aligned} \theta_1 &= \tan\left(\frac{y_c}{x_c}\right) - \tan\left(\frac{a_3 c_{23} + a_2 c_2}{d_2}\right) \\ \theta_1 &= \phi + \alpha \\ \phi &= \tan\left(\frac{y_c}{x_c}\right) \\ \alpha &= \tan\left(\frac{a_3 c_{23} + a_2 c_2}{d_2}\right) \end{aligned}$$

Elbow Left

$$\theta_1 = \tan\left(\frac{y_c}{x_c}\right) + \tan\left(\frac{a_3 c_{23} + a_2 c_2}{d_2}\right)$$

by the 2-link planar solution

$$\begin{aligned} \theta_3 &= A\tan\left(d, \pm\sqrt{1 - D^2}\right) \text{ where } D = \frac{s_c^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2 a_3} \\ \theta_2 &= A\tan(s_c, z_c - d_1) - A\tan(a_2 + a_3 c_{31} a_3 s_3) \end{aligned}$$

3. **inverse orientation kinematics**

$$R_0^3 = \begin{bmatrix} c_1c_2c_3 - c_1s_2s_3 & -c_1c_2s_3 - c_1s_2c_3 & s_1 \\ s_1c_2c_3 - s_1s_2s_3 & -s_1c_2s_3 - s_1s_2c_3 & -c_1 \\ s_2c_3 + c_2s_3 & -s_2s_3 + c_2c_3 & 0 \end{bmatrix}$$

$$u = (R_0^3)^T R = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}$$

$$\begin{aligned}
u_{11} &= r_{11}(c_1c_2c_3 - c_1s_2s_3) + r_{21}(s_1c_2c_3 - s_1s_2s_3) + r_{31}(s_2c_3 + c_2s_3) \\
u_{21} &= -r_{12}(c_1c_2s_3 + c_1s_2c_3) - r_{21}(s_1c_2s_3 + s_1s_2c_3) + r_{31}(-s_2s_3 + c_2c_3) \\
u_{31} &= r_{11}s_1 - r_{21}c_1 \\
u_{12} &= r_{12}(c_1c_2c_3 - c_1s_2s_3) + r_{22}(s_1c_2c_3 - s_1s_2s_3) + r_{32}(s_2c_3 + c_2s_3) \\
u_{22} &= -r_{12}(c_1c_2s_3 + c_1s_2c_3) - r_{22}(s_1c_2s_3 + s_1s_2c_3) + r_{32}(-s_2s_3 + c_2c_3) \\
u_{32} &= r_{12}s_1 - r_{22}c_1 \\
u_{13} &= r_{13}(c_1c_2c_3 - c_1s_2s_3) + r_{23}(s_1c_2c_3 - s_1s_2s_3) + r_{33}(s_2c_3 + c_2s_3) \\
u_{23} &= -r_{13}(c_1c_2s_3 + c_1s_2c_3) - r_{23}(s_1c_2s_3 + s_1s_2c_3) + r_{33}(-s_2s_3 + c_2c_3) \\
u_{33} &= r_{13}s_1 - r_{23}c_1
\end{aligned}$$

Inverse orientation solutions

I. Suppose not both u_{13}, u_{23} are zero

$$\theta_5 = A\tan\left(u_{33} \pm \sqrt{I - u_{33}^2}\right)$$

a) If the positive square root is chosen

$$\begin{aligned}
\theta_4 &= A\tan(u_{13}, u_{23}) \\
\theta_6 &= A\tan(u_{31}, u_{32})
\end{aligned}$$

b) If the negative square root is chosen

$$\begin{aligned}
\theta_4 &= A\tan(-u_{13}, -u_{23}) \\
\theta_6 &= A\tan(u_{31}, -u_{32})
\end{aligned}$$

II. If $u_{13} = u_{23} = 0$

a) And if $u_{33} = 1$; $\theta_5 = 0$

$$\theta_4 + \theta_6 = A\tan(u_{11}, u_{21})$$

b) Or if $u_{33} = -1$; $\theta_5 = \pi$

$$\theta_4 - \theta_6 = A\tan(-u_{11}, -u_{12})$$

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$$u = R_0^{3T} R$$

$$= \begin{bmatrix} c_1c_{23}r_{11} + s_1c_{23}r_{21} - s_{23}r_{31} & c_1c_{23}r_{12} + s_1c_{23}r_{22} - s_{23}r_{32} & c_1c_{23}r_{13} + s_1c_{23}r_{23} - s_{23}r_{33} \\ -c_1s_{23}r_{11} - s_1s_{23}r_{21} - c_{23}r_{31} & -c_1s_{23}r_{12} - s_1s_{23}r_{22} - c_{23}r_{32} & -c_1s_{23}r_{13} - s_1s_{23}r_{13} - s_1s_{23}r_{23} - c_2r_{33} \\ -s_1r_{11} + c_1r_{21} & -s_1r_{12} + c_1r_{22} & -s_1r_{13} + c_1r_{23} \end{bmatrix}$$

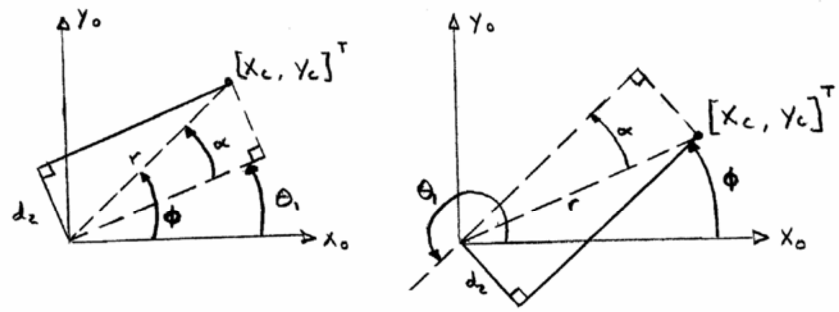
If $u_{13} = u_{23} = 0$ and $u_{33} = 1$

$$\theta_4 + \theta_6 = A\tan(c_1c_{23}r_{11} + s_1c_{23}r_{21} - s_{23}r_{31}, -c_1s_{23}r_{11} - s_1s_{23}r_{21} - c_{23}r_{31})$$

If $u_{33} = -1$

$$\theta_4 - \theta_6 = A\tan(-c_1c_{23}r_{11} - s_1c_{23}r_{21} + s_{23}r_{31}, -c_1c_{23}r_{12} - s_1c_{23}r_{22} + s_{23}r_{32})$$

3-21



Equation (3.47) for θ_1 would become

$$\theta_1 = \begin{cases} \phi - \alpha & \text{left arm} \\ \phi + \alpha + \pi & \text{right arm} \end{cases}$$

where

$$\begin{aligned} \phi &= \text{atan2}(x_c, y_c) \\ \alpha &= \text{atan2}(\sqrt{r^2 - d_2^2}, d_2) \end{aligned}$$

and Equation (3.49) for θ_2 would become

$$\theta_2 = \begin{cases} \text{Atan2}\big(\sqrt{r^2 - d_2^2}, (z_c - d_1)\big) & \text{left arm} \\ \text{Atan2}\big(-\sqrt{r^2 - d_2^2}, (z_c - d_1)\big) & \text{right arm} \end{cases}$$