6-1 From the block diagram of Figure 6.6

$$\frac{\Theta_m}{V} = \left(\frac{1}{s}\right)\frac{\frac{K_m}{(Ls+R)(J_ms+B_m)}}{1+\frac{K_bK_m}{(Ls+R)(J_ms+B_m)}} = \frac{K_m}{s[(Ls+R)(J_ms+B_m)+K_bK_m]}$$

and

$$\frac{\Theta_m}{\tau_\ell} = \frac{\frac{-1/r}{s(J_ms+B_m)}}{1+\frac{K_bK_m}{(Ls+R)(J_ms+B_m)}} = \frac{-(Ls+R)/r}{s[(Ls+r)(J_mS+B_m)+K_bK_m]}$$

6-2 Divide Equations (6.11) and (6.12) by R and set the ratio $\frac{L}{R} = 0$ to get the reduced order system

$$\frac{\Theta_m}{V} = \frac{K_m/R}{s(J_m s + B_m + K_b K_m/R)} \; ; \; \frac{\Theta_m}{\tau_\ell} = \frac{-1/r}{s(J_m s + B_m + K_b K_m/R)}$$

6-3 Compute $\frac{\Theta(s)}{\Theta^d(s)}$ with $D(s) = 0$ and $\frac{\Theta(s)}{D(s)}$ with $\Theta^d(s) = 0$ and combine the resulting transfer functions using the Principle of Superposition.

6-4 The tracking error is computed as

$$\begin{aligned} E(s) &= \Theta^d(s) - \Theta(s) \\ &= \Theta^d(s) - \left[\frac{K_P + K_D s}{\Omega(s)}\Theta^d(s) - \frac{1}{\Omega(s)}D(s)\right] \\ &= \frac{Js^2 + Bs}{\Omega(s)}\Theta^d(s) + \frac{1}{\Omega(s)}D(s) \end{aligned}$$

The Final Value Theorem says that, if $F(s)$ is the Laplace transform of $f(t)$, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} F(s)$$

whenever both limits are well defined. Consult any textbook on control systems for a more detailed statement. The steady state error is defined as

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} (\theta^d(t) - \theta(t))$$

Thus, from the Final Value Theorem, we have $e_{ss} = \lim_{s\to 0} E(s)$ where $E(s)$ is given by (6.20). Substituting (6.21) and (6.22) into (6.20) and computing the limit gives (6.23).

6-5 Equations (6.28) and (6.29) follow exactly as in Problem 6-3 using superposition and block diagram reduction from Figure 6.12.

6-6 The Routh-Hurwitz criterion can be used to derive the following general result. Any linear third-order system with characteristic polynomial $\Omega(s) = s^3 + a_2 s^2 + a_1 s + a_0$ is asymptotically stable if and only if a_0, a_1, a_2 are positive and $a_2 a_1 > a_0$. Applying this result to the characteristic polynomial (6.29) gives

$$\frac{(B + K_D)}{J}\frac{K_P}{J} > \frac{K_I}{J}$$

which reduces to (6.30) after multiplying through by J.

6-7 Use Matlab/Simulink to generate the system of Figure 6.14 and simulate the system with various parameter values.

6-8 From Figure 6.16 we have

$$\Theta = G(s)(H(s)(\Theta^d - \Theta) + F(s)\Theta^d)$$

Solving for Θ gives

$$\Theta = \frac{G(s)H(s) + G(s)F(s)}{1 + G(s)H(s)}$$

Substituting in the expressions for $G(s)$, $H(s)$, and $F(s)$ yields (6.32).

6-9 From Figure 6.17 we have, (suppressing the argument s),

$$\Theta = G\{D + H(\Theta^d - \Theta) + F\Theta^d\}$$

Solving for Θ gives

$$\Theta = \frac{G}{1+GH}D + \frac{GH+GF}{1+GH}\Theta^d$$

Therefore the error $E = \Theta^d - \Theta$ satisfies

$$\begin{aligned} E &= \Theta^d - \Theta \\ &= \Theta^d - \left[\frac{G}{1+GH}D + \frac{GH+GF}{1+GH}\Theta^d\right] \\ &= -\frac{G}{1+GH}D + \frac{1-GF}{1+GH}\Theta^d \\ &= -\frac{G}{1+GH}D \end{aligned}$$

since $1 - GF = 0$. Substituting the expressions for G and H into the above equation gives (6.35).

6-10 From block diagram (Figure 6.21)

$$\frac{\theta_\ell}{u} = \frac{\frac{k}{p_m p_\ell}}{1 - \frac{k^2}{p_m p_\ell}} = \frac{k}{p_m p_\ell - k^2}$$

The open-loop characteristic polynomial is

$$\begin{aligned} p_m p_\ell - k^2 &= (J_\ell s^2 + B_\ell s + k)(J_m s^2 + B_m s + k) - k^2 \\ &= J_\ell J_m s^4 + (J_\ell B_m + J_m B_\ell)s^3 + (k(J_m + J_\ell) + B_m B_\ell)s^2 + k(B_\ell + B_m)s \end{aligned}$$

If $B_m = B_\ell = 0$ the characteristic polynomial reduces to $J_\ell J_m s^4 + k(J_m + J_\ell)s^2$.

6-11 Using A, b, and c given by Equations (6.51) and (6.52), we have

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} adj(sI - A)$$

Carrying out the calculations give,

$$\det(sI - A) = s^4 + \left(\frac{B_m}{J_m} + \frac{B_\ell}{J_\ell}\right) s^3 + \left(\frac{k}{J_m} + \frac{B_\ell B_m}{J_m J_\ell} + \frac{k}{J_\ell}\right) s^2 + \left(\frac{kB_\ell}{J_\ell J_m} + \frac{kB_m}{J_\ell J_m}\right) s$$

and

$$c^T(sI - A)^{-1} b = \frac{k}{J_m J_\ell s^4 + (J_\ell B_m + J_m B_\ell)s^3 + (k(J_m + J_\ell) + B_\ell B_m)s^2 + k(B_\ell + B_m)s}$$

which is identical to (6.45).

There is no published solution for Problem 6.12.

6-13 Both (6.59) and (6.67) are found by direct calculation. It is instructive to write a *Mathematica* function to compute these terms symbolically. In the case of (6.59) we have

$$\det \begin{bmatrix} 0 & 0 & 0 & \frac{k}{J_m J_\ell} \\ 0 & 0 & \frac{k}{J_m J_\ell} & \frac{-B_\ell k}{J_m J_\ell^2} - \frac{B_m k}{J_\ell^2 J_m} \\ 0 & \frac{1}{J_m} & \frac{-B_m}{J_m^2} & \frac{-k}{J_m^2} + \frac{B_m^2}{J_m^3} \\ \frac{1}{J_m} & \frac{-B_m}{J_m^2} & \frac{-k}{J_m^2} + \frac{B_m^2}{J_m^3} & \frac{kB_m}{J_m^3} + \frac{kB_m}{J_m^3} - \frac{B_m^3}{J_m^4} \end{bmatrix} = \left(-\frac{1}{J_m}\right)\left(+\frac{1}{J_m}\right)\left(-\frac{k^2}{J_m^2 J_\ell^2}\right) = \frac{k^2}{J_m^4 J_\ell^2}$$

Equation (6.67) is derived similarly.

6-14 Integrator Wind up – If integral control is used, the integrator can build up large values when the actuator saturates.

Anti-wind up – Turn of the integral control when actuator saturates.

6-15 Adding the first-order dynamics of a permanent-magnet DC motor to the flexible-joint model (6.39)-(6.40) gives

$$\begin{aligned}
J_\ell \ddot{\theta}_\ell + B_\ell \dot{\theta}_\ell + k(\theta_l - \theta_m) &= 0 \\
J_m \ddot{\theta}_m + B_m \dot{\theta}_m - k(\theta_l - \theta_m) &= u \\
u &= k_m I \\
L\dot{I} + rI &= V - k_b \dot{\theta}_m
\end{aligned}$$

where I is the motor current, V is the input voltage, L is the armature inductance, R is the armature resistance and k_m, k_b are the torque and back-emf constants, respectively. The system is thus fifth-order. Defining state variables

$$x_1 = \theta_\ell \; ; \; x_2 = \dot{\theta}_\ell \; ; x_3 = \theta_m \; ; \; x_4 = \dot{\theta}_m \; ; x_5 = I$$

yields the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{J_\ell} & -\frac{B_\ell}{J_\ell} & \frac{k}{J_\ell} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{k}{J_m} & 0 & -\frac{k}{J_m} & -\frac{B_m}{J_m} & \frac{k_m}{J_m} \\ 0 & 0 & 0 & -\frac{k_b}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} V$$

With output $y = q_\ell = x_1$, the output equation is

$$y = [1, 0, 0, 0, 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

It is now straightforward to compute the determinants

$$\det \left[b, Ab, A^2b, A^3b, A^4b \right] \quad \text{and} \quad \det \begin{bmatrix} c \\ cA \\ cA^2 \\ cA^3 \\ cA^4 \end{bmatrix}$$

and show that they are nonzero - hence the system is controllable and observable.

6-16 Choose state variables

$$x = \begin{bmatrix} I_a \\ \theta_m \\ \dot{\theta}_m \end{bmatrix}$$

Then the state equations can be written

$$\dot{x} = \begin{bmatrix} \dot{I}_a \\ \dot{\theta}_m \\ \ddot{\theta}_m \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{K_b}{L} \\ 0 & 0 & 1 \\ \frac{K_m}{J_m} & 0 & -\frac{B_m}{J_m} \end{bmatrix} x + \begin{bmatrix} \frac{V}{L} \\ 0 \\ \frac{-r\tau_\ell}{J_m} \end{bmatrix}$$

In state space, equation is a linear third order system.

6-17 (a) The open loop transfer function is given by Equation (6.45):

$$\frac{\theta_\ell(s)}{U(s)} = \frac{100}{20s^4 + 7s^3 + 1200.5s^2 + 150s}$$

There are 2 real poles at $s = 0$, $s = -0.125$ and a pair of complex poles at $s = -0.1125 \pm 7.7449j$.

6-18 For the system described by

$$
\begin{aligned}
J_1\ddot{q}_1 &= \tau \\
J_2\ddot{q}_2 &= \tau
\end{aligned}
$$

choose state variables

$$x_1 = q_1;\ \ x_2 = \dot{q}_1;\ \ x_3 = q_2;\ \ x_4 = \dot{q}_2$$

Then in state space, we have

$$\dot{x} \;=\; \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tau$$

It is easy to see that the matrix

$$[b, Ab, A^2b, A^3b] \;=\; \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

has rank 2 and, therefore, the system is uncontrollable.

6-19 Controllability follows from the calculation

$$\text{rank } [b\ Ab] = \quad \text{rank} \begin{bmatrix} 1 & 7 \\ -2 & 5 \end{bmatrix} = 2$$

With $k = [k_1, k_2]$ and $u = kx$, the closed loop system matrix is given by

$$A + bk \quad = \quad \begin{bmatrix} 1 + k_1 & -3 + k_2 \\ 1 - 2k_1 & -2 - 2k_2 \end{bmatrix}$$

The characteristic polynomial is

$$\det(sI - A - bk) \quad = \quad s^2 + (1 + 2k_2 - k_1)s + (1 - 8k_1 - 3k_2) = s^2 + 4s + 4$$

Therefore equating coefficients gives

$$\begin{aligned} 1 - k_1 + 2k_2 &= 4 \\ 1 - 8k_1 - 3k_2 &= 4 \end{aligned}$$

Solving for k_1 and k_2 yields

$$k_1 = \frac{-15}{19}, \qquad k_2 = \frac{21}{19}$$

and therefore the state feedback control becomes:

$$u \quad = \quad \frac{-15}{19}x_1 + \frac{21}{19}x_2$$

6-20 In this case the closed loop system matrix is

$$A - bk = \begin{bmatrix} -1 & 0 \\ -k_1 & 2 - k_2 \end{bmatrix}$$

and so the characteristic equation is

$$\det(\lambda I - A + bk) = (\lambda + 1)(\lambda - 2 + k_2) = 0$$

Thus we see that $\lambda = -1$ is a closed loop pole for any choice of feedback gains k_1, k_2. The choice $k_2 = 4$ places one pole at $s = -2$ but it is not possible to place both poles at $s = -2$. However, the closed loop system is stable.

6-21 In this case, a similar calculation as above shows that there is always a pole at $s = +1$ for any choice of gains k_1 and k_2. Therefore, the system cannot be stabilized.

6-22 Choose the feedforward transfer $F(s)$ and PD compensator $C(s)$, respectively, as

$$F(s) \;=\; 2s^2 + s \;;\; C(s) = K_p + K_D s$$

The desired closed-loop characteristic polynomial, with $\omega = 10$ and $\zeta = 0.707$, is

$$s^2 + 2\zeta\omega s + \omega^2 = s^2 + 14.14s + 100$$

With $G(s) = \frac{1}{2s^2+s}$, and PD compensator, the closed loop characteristic polynomial is $2s^2 + (2 + K_D)s + K_P$. Thus, equating coefficients, leads to the PD gains

$$K_P = 200; \;\; K_D = 26.8$$