8-1 Since J is a constant, diagonal matrix, $M(q) = D(q) + J$ inherits the properties of $D(q)$. The skew-symmetry and passivity properties both follow from $\dot{M} = \dot{D}$. Likewise the additional term in the dynamic equations (8.6) is $J\ddot{q}$, which is linear in J; hence linearity in the parameters is preserved. M is positive definite since it is the sum of two positive definite matrices.

Write $J = diag\{J_1, \ldots, J_n\}$, where J_i are the positive, elements of the diagonal of J. Let λ_{J_1} and λ_{J_n} be the minimum and maximum values, respectively, of $J_1, \ldots, J_n$. Then

$$\lambda_{J_1} I_{n\times n} \leq J \leq \lambda_{J_n} I_{n\times n}$$

Therefore

$$(\lambda_1 + \lambda_{J_1}) I_{n\times n} \leq D(q) + J \leq (\lambda_n + \lambda_{J_n}) I_{n\times n}$$

8-2 From (8.15) and (8.16) the Lagrangian L is

$$L = \frac{1}{2}\dot{q}_1^T D(q_1)\dot{q}_1 + \frac{1}{2}\dot{q}_2^T J\dot{q}_2 - P(q_1) - \frac{1}{2}(q_1 - q_2)^T K(q_1 - q_2)$$

Thus we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = D(q_1)\ddot{q}_1 + \dot{D}(q_1)\dot{q}_1$$

$$\frac{\partial L}{\partial q_1} = \frac{1}{2}\dot{q}_1^T\frac{\partial D}{\partial q_1} - \frac{\partial V_1}{\partial q_1} - K(q_1 - q_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = J\ddot{q}_2; \qquad \frac{\partial L}{\partial q_2} = K(q_1 - q_2)$$

Therefore, with

$$C(q_1, \dot{q}_1)\dot{q}_1 = \dot{D}\dot{q}_1 - \frac{1}{2}\dot{q}_1^T\frac{\partial D}{\partial q_1}; \qquad g(q_1) = \frac{\partial V}{\partial q_1}$$

the Euler-Lagrange equations for this system are

$$\begin{aligned} D(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + g(q_1) + K(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - K(q_1 - q_2) &= \tau \end{aligned}$$

8-3 Computing $\dot{V}$ from (8.20) using the skew-symmetry property and (8.18) with the gravity term $g(q_1) = 0$, we obtain

$$\dot{V} = -\dot{q}_2^T K_d \dot{q}_2$$

Thus $\dot{V} < 0$ as long as $\dot{q}_2 \neq 0$. If $\dot{q}_2 \equiv 0$, then the second equation in (8.18) implies $K(q_2 - q_1) = K_p \tilde{q}_2$. By taking derivatives on both sides, since q_2^d is constant, we have $\dot{q}_1 \equiv 0$, $\ddot{q} \equiv 0$. Therefore from (8.18) we have $q_1 \equiv q_2$ and, hence, $\tilde{q}_2 = 0$. Asymptotic stability follows from Lasalle's Theorem.

8-4 In the steady state, $q_1 = q_2$ was shown in Problem 8-3. If gravity is present then the steady state equation becomes

$$g(q_1) + K(q_1 - q_2) = 0$$

from (8.18). Given a desired position q_1^d, we can modify the desired set point for the motor angle q_2 to satisfy the above equation as

$$q_2^d = q_1^d + \frac{1}{K} g(q_1^d)$$

8-5 The linear approximation of (8.18) is essentially a multivariable equivalent of the model (6.39)-(6.40) with the damping terms set to zero. As the root locus analysis shows in Figure 6.25, the system with PD-control using the link variables is unstable for all values of the gains.

8-6 Use Matlab/Simulink.

8-7 Use Matlab/Simulink.

8-8 Substituting (8.45) into (8.44) gives

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \hat{M}(q)a_q + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q)$$

Suppressing arguments for simplicity we have

$$\begin{aligned} M\ddot{q} &= \hat{M}a_q + \tilde{C}\dot{q} + \tilde{g} \\ &= \hat{M}a_q - Ma_q + Ma_q + \tilde{C}\dot{q} + \tilde{g} \\ &= Ma_q + \tilde{M}a_q + \tilde{C}\dot{q} + \tilde{g} \end{aligned}$$

Multiplying both sides by M^{-1} gives Equations (8.46) and (8.47).

8-9 Use Matlab/Simulink

8-10 Returning to the expression in Problem 8-8 above we have

$$M\ddot{q} \;=\; \hat{M}a_q + \tilde{C}\dot{q} + \tilde{g}$$

Adding and subtracting $\hat{M}\ddot{q}$ on the left-hand side gives

$$\hat{M}\ddot{q} - \tilde{M}\ddot{q} \;=\; \hat{M}a_q + \tilde{C}\dot{q} + \tilde{g}$$

Rearranging this equation and using linearity in the parameters yields

$$\begin{aligned}\hat{M}(\ddot{q} - a_q) &= \tilde{M}\ddot{q} + \tilde{C}\dot{q} + \tilde{g} \\ &= Y(q, \dot{q}, \ddot{q})\tilde{\theta}\end{aligned}$$

Multiplying both sides by $\hat{M}$ gives Equation (8.77).

8-11 From (8.78) and (8.82)

$$\begin{aligned}
\dot{e} &= Ae + B\Phi\tilde{\theta} \\
V &= e^T P e + \tilde{\theta}^T \Gamma \tilde{\theta}
\end{aligned}$$

we have

$$\begin{aligned}
\dot{V} &= \dot{e}^T P e + e^T P \dot{e} + 2\tilde{\theta}^T \Gamma \dot{\tilde{\theta}} \\
&= (Ae + B\Phi\tilde{\theta})^T P e + e^T P (Ae + B\Phi\tilde{\theta}) + 2\tilde{\theta}^T \Gamma \dot{\tilde{\theta}} \\
&= e^T (A^T P + PA) e + 2\tilde{\theta}^T (\Phi^T B^T P e + \Gamma \dot{\tilde{\theta}}) \\
&= -e^T Q e + 2\tilde{\theta}^T (\Phi^T B^T P e + \Gamma \dot{\tilde{\theta}})
\end{aligned}$$

8-12 (a) The state space is four dimensional.

(b) Choose state and control variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}; \ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Then

$$\begin{aligned} \dot{x} &= \begin{bmatrix} x_2 \\ -3x_1x_3 - x_3^2 \\ x_4 \\ -x_4\cos x_1 - 3(x_1 - x_3) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & x_3 \\ 0 & 0 \\ -3x_3\cos^2 x_1 & \end{bmatrix} u \\ &= f(x) + G(x)u \end{aligned}$$

for obvious definitions of $f(x)$, $G(x)$.

(c) The inverse dynamics control law is then

$$u = G^{-1}(v - F(x)$$

$$= \frac{1}{1 + x_3^2\cos^2 x_1} \begin{bmatrix} 3x_1x_3 + x_3^2 - x_3x_4\cos x_1 - cx_1x_3 + 3x_3^2 + v_1 - x_3v_2 \\ 3x_1x_3^2\cos^2 x_1 + x_3^2\cos^2 x_1 + x_4\cos x_1 + 3(x_1 - x_3) + x_3\cos^2 x_1 v_1 + v_2 \end{bmatrix}$$

where

$$\begin{aligned} v_1 &= -10x_2 - 100x_1 + r_1 \\ v_2 &= -10x_4 - 100x_3 + r_2 \end{aligned}$$

8-13 Mimic the proof of Theorem 3.

8-14 Thanks to Martin Corless for supplying this proof. Let

$$0 < \underline{M} \leq \lambda_{min}(M^{-1}) \qquad \text{and} \qquad \|M^{-1}\| = \lambda_{max}(M^{-1}) \leq \overline{M}\,.$$

where λ_{min} and λ_{max} denote minimum eigenvalue and maximum eigenvalue, respectively. Since

$$E = \frac{2}{\overline{M}+\underline{M}} M^{-1} - I\,,$$

its maximum eigenvalue satisfies

$$\lambda_{max}(E) = \frac{2\lambda_{max}(M^{-1})}{\overline{M}+\underline{M}} - 1 \leq \frac{2\overline{M}}{\overline{M}+\underline{M}} - 1 = \overline{\lambda}$$

where

$$\overline{\lambda} := \frac{\overline{M}-\underline{M}}{\overline{M}+\underline{M}} < 1\,.$$

In a similar fashion one can show that $\lambda_{min}(E) \geq -\overline{\lambda}$. Using the symmetry of E we now obtain that

$$||E||^2 = \lambda_{max}(E^T E) = \lambda_{max}(E^2) \leq \overline{\lambda}^2\,.$$

Hence $||E|| \leq \overline{\lambda} < 1$.

- Note that $\underline{M}$ and $\overline{M}$ can also be obtained from

$$\|M\| = \lambda_{max}(M) \leq 1/\underline{M} \qquad \text{and} \qquad \lambda_{min}(M) \geq 1/\overline{M}\,.$$

So, the above results can also be expressed in terms of bounds on $\lambda_{min}(M)$ and $\lambda_{max}(M)$.