7-1 A direct calculation shows

$$\begin{aligned}
& (r_1 + \delta r_1 - r_2 - \delta r_2)^T (r_1 + \delta r_1 - r_2 - \delta r_2) \\
= \ & (r_1 - r_2 + \delta r_1 - \delta r_2)(r_1 - r_2 + \delta r_1 - \delta r_2) \\
= \ & (r_1 - r_2)^T (r_1 - r_2) + 2(r_1 - r_2)^T (\delta r_1 - \delta r_2) + (\delta r_1 - \delta r_2)^T (\delta r_1 - \delta r_2) \\
= \ & (r_1 - r_2)^T (r_1 - r_2) + 2(r_1 - r_2)^T (\delta r_1 - \delta r_2)
\end{aligned}$$

if we neglect the second-order terms in δr_1, δr_2. Therefore, from Equation (7.17) we have

$$\ell^2 = (r_1 - r_2)^T (r_1 - r_2) + 2(r_1 - r_2)^T (\delta r_1 - \delta r_2)$$

Equation (7.18) follows since $(r_1 - r_2)^T (r_1 - r_2) = \ell^2$

7-2 Euler's equation can be expressed as

$$I\dot{\omega} + \omega \times I\omega = 0$$

where

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}; \ \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Computing the cross product yields the three equations

$$\begin{aligned} I_{xx} + (I_{zz} - I_{yy}\omega_y\omega_z) &= 0 \\ I_{yy} + (I_{xx} - I_{zz}\omega_z\omega_x) &= 0 \\ I_{zz} + (I_{yy} - I_{xx}\omega_x\omega_y) &= 0 \end{aligned}$$

7-3 Referring to Figure 7.6 we have

$$\begin{aligned}\int (y^2+z^2)dm &= \int_0^c\int_0^b\int_0^a (y^2+z^2)\rho dxdydz \\ &= \frac{1}{3}\rho abc(b^2+c^2)\end{aligned}$$

Computing the remaining terms similarly we have

$$\begin{aligned} I &= \begin{bmatrix} \int(y^2+z^2)dm & -\int xydm & -\int xzdm \\ -\int xydm & \int(x^2+z^2)dm & -\int yzdm \\ -\int xzdm & -\int yzdm & -\int(x^2+y^2)dm \end{bmatrix} \\ &= \rho\begin{bmatrix} \frac{1}{3}abc(b^2+c^2) & -a^2b^2c/4 & -a^2bc^2/4 \\ -a^2b^2c/4 & \frac{1}{3}abc(a^2+c^2) & -ab^2c^2/4 \\ -a^2bc^2/4 & -ab^2c^2/4 & \frac{1}{3}abc(a^2+b^2) \end{bmatrix}\end{aligned}$$

7-4 From (7.83) we have $\det D = d_{11}d_{22} - d_{12}d_{21}$

$$
\begin{aligned}
&= (m_1\ell c_1^2 + m_2\ell_1^2 + m_2\ell c_2^2 + 2m_2\ell_1\ell c_2\cos q_2 + I_1 + I_2)(m_2\ell c_2^2 + I_2) \\
&\quad (m_2\ell c_2^2 + m_2\ell_1\ell c_2\cos q_2 + I_2)^2 \\
&= m_1 m_2\ell c_2^2 + m_1\ell c_1^2 I_2 + m_2\ell_1^2 I_2 + m_2\ell_1^2 + I_1 I_2 + m_2^2\ell_1^2\ell c_2^2(1 - \cos^2 q_2) + m_2\ell c_2^2 I_1
\end{aligned}
$$

Since $0 \geq 1 - \cos^2 q_2 \geq 1$ we have $\det D > 0$.

7-5 One way to argue that the inertia matrix of an arbitrary n-DOF robot is positive definite is by a consideration of kinetic energy. The kinetic energy of an arbitrary robot is

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

The kinetic energy must be positive for nonzero velocities $\dot{q}$. If $D(q)$ were not sign definite there would be a nonzero velocity vector $\dot{q}$ such that $\dot{q}^T D(q)\dot{q} = 0$ which is a contradiction.

7-6 By symmetry of the inertia matrix we have

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j \quad = \quad \frac{1}{2}\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i}+\frac{\partial d_{jk}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j$$

Since the summation runs over all i, j we can interchange i and j in the second term to obtain the result

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j \quad = \quad \frac{1}{2}\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i}+\frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j$$

7-7 (a) The inertia tensors are

$$J_1 = \begin{bmatrix} \frac{1}{192} & 0 & 0 \\ 0 & \frac{1}{192} & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix}; \quad J_2 = \begin{bmatrix} \frac{1}{192} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{192} \end{bmatrix}; \quad J_3 = \begin{bmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{1}{192} & 0 \\ 0 & 0 & \frac{1}{192} \end{bmatrix}$$

(b) The inertia matrix is

$$D = \begin{bmatrix} m_3 & 0 & 0 \\ 0 & m_2 + m_3 & 0 \\ 0 & 0 & m_1 + m_2 + m_3 \end{bmatrix}$$

(c) Since the inertia matrix is constant, all Christoffel symbols are zero.

(d) From the Euler-Langrange equations, we have

$$\begin{bmatrix} m_3 & 0 & 0 \\ 0 & m_2 + m_3 & 0 \\ 0 & 0 & m_1 + m_2 + m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} g(m_1 + m_2 + m_3) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

For Problems 7-8 to 7-11, download the *Robotica* package from

http://decision.csl.uiuc.edu/ spong/Robotica/.

Use *Robotica* to compute the Euler-Lagrange equations of motion in symbolic form.

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7-12 With the kinetic energy given by

$$K = \frac{1}{2}\sum_{ij}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j$$

we compute

$$p_k = \frac{\partial K}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}\dot{q}_j$$

Now

$$\sum_{k=1}^{n} \dot{q}_k p_k = \sum_{k=1}^{n} \dot{q}_k \sum_{j=1}^{n} d_{kj}\dot{q}_j = d_{kj}\dot{q}_j\dot{q}_k = 2K$$

7-13 (a)

$$H = \sum_{k=1}^{n} \dot{q}_k p_k - L = 2K - (K - V) = K + V$$

(b) From

$$H = \sum_{k=1}^{n} \dot{q}_k p_k - L$$

we have

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$

and

$$\frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial q_k} = u_k - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = u_k - \dot{p}_k$$

The last two inequalities coming from the Euler-Lagrange equations and the definition of p_k, respectively.

7-14 Proof 1) The total derivative dH/dt is given by

$$\frac{dH}{dt} = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k}\dot{q}_k + \frac{\partial H}{\partial p_k}\dot{p}_k$$

From Hamilton's equation this becomes

$$\begin{aligned} \frac{dH}{dt} &= \sum_{k=1}^{n} (\tau_k - \dot{p}_k)\dot{q}_k\dot{p}_k \\ &= \sum_{k=1}^{n} \dot{q}_k\tau_k = \dot{q}^T\tau \end{aligned}$$

Proof 2) From

$$H = K + V = \frac{1}{2}\dot{q}^T D\dot{q} + V(q)$$

we have

$$\frac{dH}{dt} = \dot{q}^T D\ddot{q} + \frac{1}{2}\dot{q}^T\dot{D}\dot{q} + \frac{\partial V}{\partial q}$$

using the Euler-Lagrange equations, this becomes

$$\begin{aligned} \frac{dH}{dt} &= \dot{q}^T\tau + \frac{1}{2}\dot{q}^T(\dot{D} - 2C)\dot{q} \\ &= \dot{q}^T\tau \end{aligned}$$

by the skew-symmetry property. The units of dH/dt are power.