10-1 By definition,

$$L_g h = \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} g_j$$

Therefore, we have

$$\begin{aligned} L_f L_g h &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} \frac{\partial h}{\partial x_j} g_j \right) f_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 h}{\partial x_i \partial x_j} g_j + \frac{\partial h}{\partial x_j} \frac{\partial g_j}{\partial x_i} \right) f_i \end{aligned}$$

Likewise, by interchanging f and g above we have

$$L_g L_f h = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 h}{\partial x_i \partial x_j} f_j + \frac{\partial h}{\partial x_j} \frac{\partial f_j}{\partial x_i} \right) g_i$$

Therefore, using the fact that

$$\frac{\partial^2 h}{\partial x_i d x_j} = \frac{\partial^2 h}{\partial x_j \partial x_i}$$

we have

$$L_f L_g h - L_g L_f h = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} \left(\frac{\partial g_i}{\partial x_i} f_i - \frac{\partial f_j}{\partial x_i} g_i \right) = L_{[f,g]} h$$

10-2 If $h = z - \phi(x,y)$, then

$$dh = \left(-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}, 1\right)$$

If X_1, X_2 are given by Equation (10.15) then

$$\begin{aligned} L_{X_1}h &= -\frac{\partial\phi}{\partial x}\cdot 1 - \frac{\partial\phi}{\partial y}\cdot 0 + 1\cdot f(x,y,\phi(x,y)) \\ &= f(x,y,\phi) - \frac{\partial\phi}{\partial x} = 0 \end{aligned}$$

since ϕ satisfies Equation (10.10). Similarly,

$$L_{X_2}h = g(x,y,\phi) - \frac{\partial\phi}{\partial y} = 0$$

10-3 If $h(x, y, z) = 0$ and $\partial h/\partial z \neq 0$ then, by the implicit function theorem we may solve for z as $x = \phi(x, y)$. Furthermore,

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= -\frac{\partial h}{\partial x}\Big/\frac{\partial h}{\partial z} \\ \frac{\partial \phi}{\partial y} &= -\frac{\partial h}{\partial y}\Big/\frac{\partial h}{\partial z} \end{aligned}$$

Now

$$L_{X_1} h = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial z} f = -\frac{\partial h}{\partial z}\frac{\partial \phi}{\partial x} + \frac{\partial h}{\partial z} f = 0$$

which implies

$$\frac{\partial \phi}{\partial x} = f$$

since $\frac{\partial h}{\partial z} \neq 0$. The second equation is shown similarly.

10-4 By repeated application of Lemma 10.1, we have $L_{ad_f^i(g)}T_1 = (-1)^i L_g T_{i+1}$. Thus for $i < n-1$, $L_{ad_f^i(g)}T_1 = 0$ and $L_{ad_f^{n-1}(g)}T_1 = (-1)^{n-1} L_g T_n \neq 0$.

10-5 The vector fields f and g are given by

$$f = \begin{bmatrix} x_1^3 + x_2 \\ x_2^3 \end{bmatrix}; \; g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The dimension of the state space is $n = 2$ and therefore the necessary and sufficient conditions for local feedback linearizability are rank$\{g, [f, g]\} = 2$ and involutivity of the set $\{g\}$. Now, any distribution spanned by a single vector field is involutive and so for second order systems the rank condition alone is necessary and sufficient for local feedback linearizability. Since g is constant, the Lie Bracket $[f, g]$ is given by

$$[f, g] = -\frac{\partial f}{\partial x} g = -\begin{bmatrix} 3x_1^2 & 1 \\ 0 & 3x_2^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3x_2^2 \end{bmatrix}$$

Therefore

$$\{g, [f, g]\} = \begin{bmatrix} 0 & 1 \\ -1 & -3x_2^2 \end{bmatrix}$$

which has rank 2 for all x. Therefore the system is globally feedback linearizable since the rank condition holds globally. To find the change of coordinates we must solve the PDE's

$$L_g T_1 = 0$$

with the additional condition that $L_{[f,g]} \neq 0$. The first equation says, in effect that

$$\frac{\partial T_1}{\partial x_2} = 0$$

while the additional condition implies

$$\frac{\partial T_1}{\partial x_1} \neq 0$$

Thus we may take the simplest solution $T_1 = x_1$ and compute T_2 from

$$T_2 = L_f T_1 = x_1^3 + x_2$$

Therefore, the change of variables is

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_1^3 + x_2 \end{aligned}$$

The feedback linearizing control is found from

$$\begin{aligned} \dot{z}_2 &= 2x_1^2 \dot{x}_1 + \dot{x}_2 \\ &= 2x_1^2(x_1^3 + x_2) + x_2^3 + u = v \end{aligned}$$

Solving for u gives

$$u = v - [2x_1^2(x_1^3 + x_2) + x_2^3]$$

10-6 Choosing q_1 and q_2 as generalized coordinates, the kinetic energy is

$$K = \frac{1}{2}I\dot{q}_1^2 + \frac{1}{2}J\dot{q}_2^2$$

The potential energy is

$$V = MgL(1 - \cos q_1) + \frac{1}{2}k(q_1 - q_2)^2$$

The Lagrangian is

$$L = K - V = \frac{1}{2}I\dot{q}_2^2 + \frac{1}{2}J\dot{q}_2^2 - MgL(1 - \cos q_1) - \frac{1}{2}k(q_1 - q_2)^2$$

Therefore we compute

$$\frac{\partial L}{\partial \dot{q}_1} = I\dot{q}_1; \qquad \frac{\partial L}{\partial \dot{q}_2} = J\dot{q}_2$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = I\ddot{q}_1; \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = J\ddot{q}_2$$

$$\begin{aligned} \frac{\partial L}{\partial q_1} &= -MgL\sin q_1 - k(q_1 - q_2) \\ \frac{\partial L}{\partial q_2} &= k(q_1 - q_2) \end{aligned}$$

Therefore the equations of motion, ignoring damping, are given by

$$\begin{aligned} I\ddot{q}_1 + MgL\sin q_1 + k(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - k(q_1 - q_2) &= u \end{aligned}$$

10-7 If there are damping terms $B_1\dot{q}_1$ and $B_2\dot{q}_2$ on the link and motor, respectively, the equations of motion are

$$
\begin{aligned}
I\ddot{q}_1 + B_1\dot{q}_1 + MgL\sin q_1 + k(q_1 - q_2) &= 0 \\
J\ddot{q}_2 + B_2\dot{q}_2 - k(q_1 - q_2) &= u
\end{aligned}
$$

$$g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}; \qquad f = \begin{bmatrix} x_2 \\ -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix}$$

Therefore, since g is a constant vector field

$$ad_f(g) \quad = \quad [f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = -\frac{\partial f}{\partial x} g$$

Now,

$$\frac{\partial f}{\partial x} \quad = \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-MgL}{I}\cos x_1 - \frac{k}{I} & 0 & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & 0 \end{bmatrix}$$

and so we have

$$ad_f(g) \quad = \quad \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{J} \\ 0 \end{bmatrix}$$

Similarly,

$$ad_f^2(f) \quad = \quad [f, ad_f(g)] = \frac{-\partial f}{\partial x} ad_f(g) = \begin{bmatrix} 0 \\ \frac{k}{IJ} \\ 0 \\ -\frac{k}{J^2} \end{bmatrix}$$

and

$$ad_f^3(g) \quad = \quad [f, ad_f^2(g)] = \frac{\partial f}{\partial x} ad_f^2(g) = \begin{bmatrix} -\frac{k}{IJ} \\ 0 \\ \frac{k}{J^2} \\ 0 \end{bmatrix}$$

$$L_g T_1 \quad = \quad \frac{1}{J}\frac{\partial T_1}{\partial x_4} = 0 \Longrightarrow \frac{\partial T_1}{\partial x_4} = 0$$

$$L_{[f,g]} T_1 \quad = \quad -\frac{1}{J}\frac{\partial T_1}{\partial x_3} = 0 \Longrightarrow \frac{\partial T_1}{\partial x_3} = 0$$

$$L_{ad_f^2(g)} T_1 \quad = \quad \frac{k}{IJ}\frac{\partial T_1}{\partial x_2} - \frac{k}{J_2}\frac{\partial T_1}{\partial x_4} = 0 \Longrightarrow \frac{\partial T_1}{\partial x_2} = 0$$

Since

$$\frac{\partial T_1}{\partial x_4} \quad = \quad 0$$

Finally

$$L_{ad_f^3(g)} T_1 \quad = \quad \frac{-k}{IJ}\frac{\partial T_1}{\partial x_1} + \frac{k}{J^2}\frac{\partial T_1}{\partial x_3} \neq 0 \Longrightarrow \frac{\partial T_1}{\partial x_1} \neq 0$$

since

$$\frac{\partial T_1}{\partial x_3} \quad = \quad 0$$

$$T_1 = x_1$$

$$T_2 = L_f T_1 = (1,0,0,0)f = x_2$$

$$T_3 = L_f T_2 = (0,1,0,0)f = \frac{-mgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3)$$

$$\begin{aligned} T_4 = L_f T_3 &= \left(\frac{-MgL}{I}\cos x_1 \frac{k}{I}, 0, \frac{k}{I}, 0, \frac{k}{I}, 0\right) f \\ &= \left(\frac{-MgL}{I}\cos x_1 - \frac{k}{I}\right) x_2 + \frac{k}{I} x_4 \\ &= \frac{-MgL}{I}\cos x_1 \cdot x_2 - \frac{k}{I}(x_2 - x_4) \end{aligned}$$

$$
\begin{aligned}
L_g T_4 &= \frac{1}{J}\frac{\partial T_4}{\partial x_4} = \frac{k}{IJ} \\
\\
L_f T_4 &= \left(\frac{MgL}{I}\sin x_1 \cdot x_2, \frac{-MgL}{I}\cos x_1 - \frac{k}{I}, 0, \frac{k}{I}\right) f \\
&= \frac{MgL}{I}\sin x_1 \cdot x_2^2 - \left(\frac{MgL}{I}\cos x_1 + \frac{k}{I}\right)\left(\frac{-MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3)\right) \\
&\quad + \frac{k^2}{IJ}(x_1 - x_3) \\
&= \frac{MgL}{I}\sin x_1\left[x_2^2 + \frac{MgL}{I}\cos x_1 + \frac{k}{I}\right] + \frac{k}{I}(x_1 - x_3)\left(\frac{k}{I} + \frac{k}{J} + \frac{MgL}{I}\cos x_1\right)
\end{aligned}
$$

10-12 The coordinate transformation is

$$
\begin{aligned}
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\
y_4 &= \frac{-MgL}{I}\cos x_1 \cdot x_2 - \frac{k}{I}(x_2 - x_4)
\end{aligned}
$$

Therefore the inverse transformation is

$$
\begin{aligned}
x_1 &= y_1 \\
x_2 &= y_2 \\
x_3 &= \frac{I}{k}\left(y_3 + \frac{MgL}{I}\sin x_1 + \frac{k}{I}x_1\right) \\
&= \frac{I}{k}y_3 + \frac{MgL}{k}\sin y_1 + y_1 \\
&= y_1 + \frac{I}{k}\left(y_3 + \frac{MgL}{I}\sin y_1\right) \\
x_4 &= \frac{I}{k}\left(y_4 + \frac{MgL}{I}\cos x_1 \cdot x_2 + \frac{k}{I}x_2\right) \\
&= y_2 + \frac{1}{k}\left(y_4 + \frac{MgL}{I}\cos y_1 \cdot y_2\right)
\end{aligned}
$$

10-13 This is an open-ended design problem. Use Matlab's pole placement or LQR routines to generate linear feedback gains.

10-14 In the case of a single-link rigid robot with a permanent-magnet DC motor we can write the dynamics equations of motion as

$$\begin{aligned} I\ddot{\theta} + Mg\ell\sin(\theta) &= u \\ L\dot{I} + RI &= V - K_b\dot{\theta} \\ u &= KI \end{aligned}$$

Define state and control variables as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ I \end{bmatrix}; \; u = V$$

and write the equations of motion as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{K}{I}x_3 - \frac{Mg\ell}{I}\sin(x_1) \\ \dot{x}_3 &= -\frac{K_b}{L}x_2 - \frac{R}{L}x_3 + \frac{1}{L}u \end{aligned}$$

The transformed state variables under which the above system can be feedback linearized are

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= \frac{K}{I}x_3 - \frac{Mg\ell}{I}\sin(x_1) \end{aligned}$$

The feedback linearizing control law u is found from

$$\begin{aligned} \dot{z}_3 &= \frac{K}{I}\dot{x}_3 - \frac{Mg\ell}{I}\cos(x_1)\dot{x}_1 \\ &= -\frac{K}{I}(\frac{K_b}{L}x_2 - \frac{R}{L}x_3) - \frac{Mg\ell}{I}\cos(x_1)x_2 + \frac{K}{I}\frac{1}{L}u = v \end{aligned}$$

which results in

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= v \end{aligned}$$

10-15 From Equation (10.73) we see that, as $k \to \infty$,

$$x_3 \to y_1 = x_1$$
$$x_4 \to y_2 = x_2$$

The physical interpretation is that, as the joint stiffness tends to infinity, the link and motor position and velocity coincide. This makes sense because the shaft connecting the motor and link becomes a single DOF rigid body. Examining Equation (10.54), if we eliminate the spring torque $k(q_1 - q_2)$ and set $\ddot{q}_2 = \ddot{q}_1$ we recover the equations for a single-link rigid-joint robot

$$(I + J)\ddot{q} + Mg\ell \sin(q) = u$$

10-16 With spring force $F = \phi(q_1 - q_2)$ the equations of motion (10.54) become

$$
\begin{aligned}
I\ddot{q}_1 + Mg\ell\sin(q_1) + \phi(q_1 - q_2) &= 0 \\
J\ddot{q}_2 - \phi(q_1 - q_2) &= u
\end{aligned}
$$

Thus the corresponding vector fields f, and g are

$$
g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}; \qquad f = \begin{bmatrix} x_2 \\ -\frac{MgL}{I}\sin x_1 - \frac{1}{I}\phi(x_1 - x_3) \\ x_4 \\ \frac{1}{J}\phi(x_1 - x_3) \end{bmatrix}
$$

Let ϕ' denote $\frac{\partial\phi}{\partial z}|_{z=x_1-x_3}$. Then a straightforward calculation shows that the distribution $[g, ad_f g, ad_f^2 g, ad_f^3 g]$ is modified as

$$
[g, ad_f(g), ad_f^2(g), ad_f^3(g)] = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{IJ}\phi' \\ 0 & 0 & \frac{1}{IJ}\phi' & 0 \\ 0 & \frac{1}{J} & 0 & -\frac{1}{J^2}\phi' \\ \frac{1}{J} & 0 & -\frac{1}{J^2}\phi' & 0 \end{bmatrix}
$$

which has rank four (and hence is feedback linearizable) provided $\phi' \neq 0$. The coordinate transformation and feedback linearizing control can be obtained by replacing $k(x_1 - x_3)$ in Equations (10.65)-(10.66) by $\phi(x_1 - x_3)$ and replacing k in Equations (10.68) and (10.69) by ϕ'.

10-17 With $y = q_1 = x_1$,we note that the vector field f in Equation (10.56) can be written as

$$
\begin{aligned}
f(x) &= \begin{bmatrix} x_2 \\ -\frac{MgL}{I}\sin(x_1) - \frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{MgL}{I}\sin(x_1) \\ 0 \\ 0 \end{bmatrix} \\
&= Ax + \phi(y)
\end{aligned}
$$

where

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{I} & 0 & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & 0 \end{bmatrix} \quad \phi(y) = \begin{bmatrix} 0 \\ -\frac{MgL}{I}\sin(y) \\ 0 \\ 0 \end{bmatrix}
$$

Therefore, the flexible joint robot model can be written as

$$
\begin{aligned}
\dot{x} &= Ax + \phi(y) + bu \\
y &= Cx
\end{aligned}
$$

Therefore, with the observer equation defined as

$$
\dot{\hat{x}} = A\hat{x} + \phi(y) + bu + L(y - C\hat{x})
$$

the estimation error $e = x - \hat{x}$ is easily seen to satisfy

$$
\dot{e} = (A - LC)e
$$

and, therefore, observability of the pair (C, A) is sufficient to design an observer to estimate $\dot{q}_1, q_2, \dot{q}_2$ given only measurement of q_1.

10-18 With g_1 and g_2 given by

$$g_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad g_2 = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ 0 \\ 0 \end{bmatrix}$$

and with $q = (x, y, \theta, \phi)^T$ we have, since g_1 is constant, that

$$\begin{aligned} [g_1, g_2] &= \frac{\partial g_2}{\partial q} g_1 \\ &= \begin{bmatrix} 0 & 0 & -r\sin(\theta) & 0 \\ 0 & 0 & r\cos(\theta) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -r\sin\theta \\ r\cos(\theta) \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

10-19 A direct calculation shows that g_1 and g_2 are both orthogonal to w_1 and w_2. Since g_1 is constant, the Lie Bracket $[g_1, g_2]$ is

$$\begin{aligned}
[g_1, g_2] &= \frac{\partial g_2}{\partial q} g_1 \\
&= \begin{bmatrix} 0 & 0 & -\sin(\theta) & 0 \\ 0 & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & \frac{1}{d}\sec^2(\phi) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{d}\sec^2(\phi) \\ 0 \end{bmatrix}
\end{aligned}$$

which cannot be expressed as a linear combination of g_1 and g_2.

10-20 This follows simply from the definition of the cross product with some rearranging of terms.

$$\begin{aligned}
\omega \times u &= \begin{bmatrix} \imath & \jmath & k \\ \omega_1 & \omega_2 & \omega_3 \\ u_1 & u_2 & u_3 \end{bmatrix} = \imath(\omega_2 u_3 - \omega_3 u_2) - \jmath(\omega_1 u_3 - \omega_3 u_1) + k(\omega_1 u_2 - \omega_2 u_1) \\
&= \begin{bmatrix} 0 \\ \omega_3 \\ -\omega_2 \end{bmatrix} u_1 + \begin{bmatrix} -\omega_3 \\ 0 \\ \omega_1 \end{bmatrix} u_2 + \begin{bmatrix} \omega_2 \\ -\omega_1 \\ 0 \end{bmatrix} u_3
\end{aligned}$$

10-23 With $v_1 = a\sin(\omega t)$ and $v_2 = b\cos(\omega t)$, the first two equations in (10.119) can be explicitly integrated as

$$\begin{aligned} x_1(t) &= x_1(0) + \int_0^t a\sin(\omega\tau)d\tau = \frac{a}{\omega}(1 - \cos(\omega t)) \\ x_2(t) &= x_2(0) + \int_0^t b\cos(\omega\tau)d\tau = \frac{b}{\omega}\sin(\omega t) \end{aligned}$$

At $t = \frac{2\pi}{\omega}$ we have $x_1(\frac{2\pi}{\omega}) = x_1(0)$ and $x_2(\frac{2\pi}{\omega}) = x_2(0)$.

From the third equation in (10.119) we have

$$\dot{x}_3 = \frac{ab}{\omega}\sin(\omega t)(1 - cos(\omega t))$$

Hence, it is straightforward to compute by direct integration

$$x_3(\frac{2\pi}{\omega}) = x_3(0) + \frac{ab\pi}{\omega^2}$$

and the result follows with $a = \omega = \pi$ and $b = 10$.