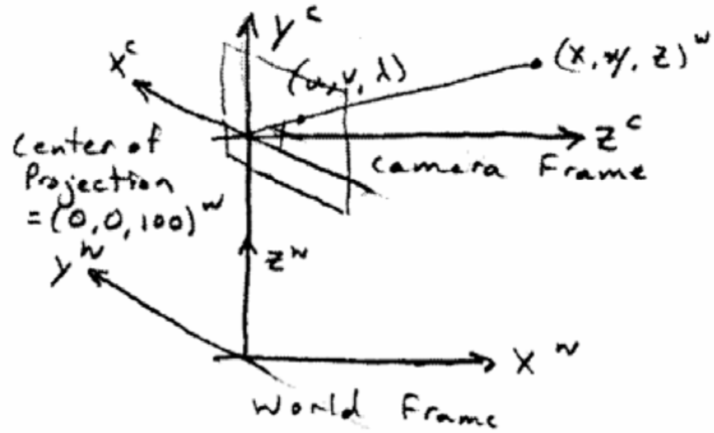
11-1

$$k \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} u \\ v \\ \lambda \end{bmatrix}$$

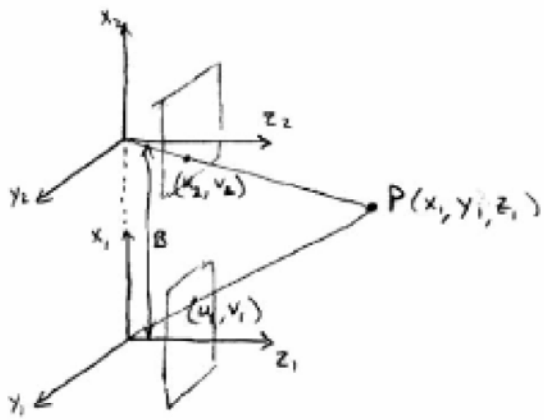
$$\begin{aligned} k &= \frac{\lambda}{z^c} \\ u &= kx^c \\ v &= ky^c \end{aligned}$$

a)	$(25, 25, 50)^c$	$\rightarrow$	$(5, 5) = (u, v)$
b)	$(-25, -25, 50)^c$	$\rightarrow$	$(-5, -5)$
c)	$(20, 5, -50)^c$		invisible (behind the image plane)
d)	$(15, 10, 25)^c$	$\rightarrow$	$(6, 4)$
e)	$(0, 0, 50)^c$	$\rightarrow$	$(0, 0)$
f)	$(0, 0, 100)^c$	$\rightarrow$	$(0, 0)$



$$
\begin{aligned}
x^c &= y^w \\
y^c &= z^2 - 100 \\
z^c &= x^w
\end{aligned}
$$

a)	$(25, 25, 50)^w$	$\rightarrow$	$(25, -50, 25)^c$	$\rightarrow$	$(10, -20) = (u, v)$
b)	$(-25, -25, 50)^w$	$\rightarrow$	$(-25, -30, -25)^c$		invisible (behind image plane)
c)	$(20, 5, -50)^w$	$\rightarrow$	$(5, -150, 20)^c$	$\rightarrow$	$(2.5, -75)$
d)	$(15, 10, 25)^w$	$\rightarrow$	$(10, -75, 15)^c$	$\rightarrow$	$\left(\frac{20}{3}, -50\right)$
e)	$(0, 0, 50)^w$	$\rightarrow$	$(0, -50, 0)^c$		invisible
f)	$(0, 0, 100)^w$	$\rightarrow$	$(0, 0, 0)^c$		invisible



From the transformation, we have

$$(x_2, y_2, z_2) \quad = \quad (x_1 - B, y_1, z_1)$$

We know that u_1, y_1, λ_1 and u_2, v_2, λ_2. We want to find *depth* z_1.

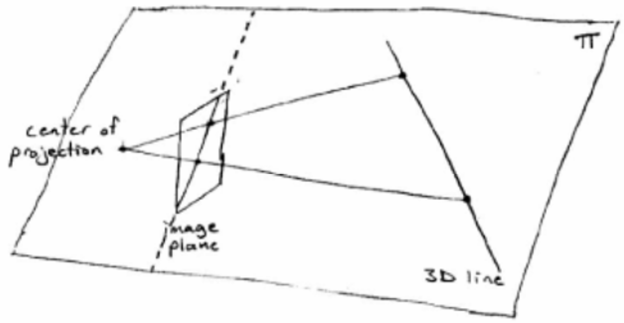
$$K_1 = \frac{\lambda_1}{z_1} \qquad K_2 = \frac{\lambda_2}{z_2} = \frac{\lambda_2}{z_1}$$

$$x_1 = \frac{u_1}{k_1} \qquad x_2 = \frac{u_2}{k_2}$$

$$x_1 = u_1\left(\frac{z_1}{\lambda_1}\right) \qquad x_1 - B = u_2\left(\frac{z_1}{\lambda_2}\right)$$

Setting equal:

$$\begin{aligned} x_1 &= u_1\left(\frac{z_1}{\lambda_1}\right) = B + u_2\left(\frac{z_1}{\lambda_2}\right) \\ B &= z_1\left[\frac{u_1}{\lambda_1} - \frac{u_1}{\lambda_2}\right] \\ \Rightarrow z_1 &= \frac{B}{\left(\frac{u_1}{\lambda_1} - \frac{u_2}{\lambda_2}\right)} \end{aligned}$$



Let Π be a plane defined by any two points on the 3D line and the center of the projection. The intersection of the image plane and plane Π is a straight line.

Degenerate cases:

1. Plane Π is *parallel* to and does not intersect the image plane. For this case to arise, the 3D line must be parallel to the image plane and lie behind the image plane.
2. If the 3D line passes through the center of projection, there are infinite possible planes Π. In this case, the 3D line appears as a single point in the image.

11-5 We are given two lines that are parallel in the camera frame.

$$L_1 : \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \gamma\overline{U}$$
$$L_2 : \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \gamma\overline{U}$$

where $\overline{U} = [U_x U_y U_z]^T$ is a unit vector.

From our equations relating points in the camera frame to coordinate in the image frame

$$k \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} u \\ v \\ \lambda \end{bmatrix}$$

we have

$$u = \frac{x^c}{z^c}\lambda \qquad v = \frac{y^c}{z^c}\lambda.$$

Substitute our lines:

$$u = \frac{(x_i + \gamma U_x)}{(z_i + \gamma U_2)}\lambda$$

Now take the limits as $\gamma \to \infty$.

$$\begin{aligned} u_\infty &= \lim_{\gamma\to\infty} \frac{(x_1 + \gamma U_x)}{(z_i + \gamma U_z)}\lambda = \lim_{\gamma\to\infty} \frac{\left(\frac{x_i}{\gamma} + U_x\right)}{\left(\frac{z_i}{\gamma} + U_z\right)}\lambda \\ &= \frac{U_x}{U_z}\lambda \end{aligned}$$

Similarly, we can find

$$v_\infty = \frac{U_y}{U_z}\lambda$$

The coordinate (u_∞, v_∞) is the vanishing point.

Remarks:

1. (u_∞, v_∞) does *not* depend on x_i, y_i, or z_i! This implies that *any* line with unit direction $\overline{U}$ will pass through (u_∞, v_∞).
2. (u_∞, v_∞) does not exist when $U_z = 0$. Why? When $U_z = 0$ the plane containing the 3D lines is parallel to the image plane. In this case, the two parallel lines remain parallel in the image; they never intersect.

11-6 All horizontal lines will have $U_y = 0$. From Problem 11-5, we substitute into the expressions for the vanishing point to see that all such lines converge to a point with $v_\infty = 0$. Therefore, all horizontal lines vanish at a point along the line $v = 0$ in the image.

11-7 From problem 11-5 we have

$$u_\infty = \lambda \frac{U_x}{U_z} \qquad v_\infty = \lambda . \frac{U_y}{U_z}$$

Since $\overline{U}$ is a unit vector, we also know

$$U_x^2 + U_y^2 + U_z^2 \quad = \quad 1.$$

Now substitute

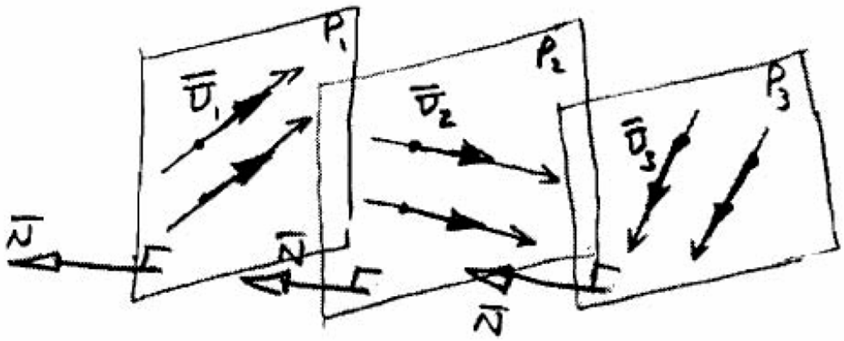
$$\begin{aligned} u_\infty^2 + v_\infty^2 + \lambda^2 &= \frac{\lambda^2}{U_z^2}(U_x^2 + U_y^2) + \lambda^2 \\ &= \frac{\lambda^2}{u_z^2}\underbrace{(U_x^2 + U_y^2 + U_z^2)}_{1} \\ &= \frac{\lambda^2}{U_z^2}. \end{aligned}$$

Rearranging terms yields

$$U_z \quad = \quad \frac{\lambda}{\sqrt{u_\infty^2 + v_\infty^2 + \lambda^2}}$$

Substituting this formula for U_z back into the expressions for u_∞ and v_∞, we get

$$\begin{aligned} U_x &= \frac{u_\infty}{\sqrt{u_\infty^2 + v_\infty^2 + \lambda^2}} \\ U_x &= \frac{v_\infty}{\sqrt{u_\infty^2 + v_\infty^2 + \lambda^2}} \end{aligned}$$



Let two parallel lines with unit vector U_i define plane P_i.

$$U_i = \begin{bmatrix} U_{xi} \\ U_{yi} \\ U_{zi} \end{bmatrix}$$

Consider three such pairs of parallel lines with planes P_1, P_2, P_3 all parallel. Since the planes are parallel, they share a common normal, $\overline{N}$.

$$N = \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}$$

From our formulas for vanishing points, we know

$$\left\{ \begin{array}{rcl} u_{\infty i} & = & \lambda \frac{U_{xi}}{U_{zi}} \\ v_{\infty i} & = & \lambda \frac{U_{yi}}{U_{zi}} \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{rcl} U_{xi} & = & \frac{u_{\infty i} U_{zi}}{\lambda} \\ U_{yi} & = & \frac{v_{\infty i} U_{zi}}{\lambda}. \end{array} \right.$$

Since $\overline{N}$ is normal to any of the lines, we know $U_i \cdot \overline{N} = 0$ for all i.

$$\begin{aligned} U_i \cdot \overline{N} = 0 \Rightarrow U_{xi} N_x + U_{yi} N_y + U_{zi} N_z &= 0 \\ \frac{u_{\infty i} U_{zi}}{\lambda} N_x + \frac{v_{\infty i} U_{zi}}{\lambda} N_y + U_{zi} N_z &= 0 \end{aligned}$$

When $\overline{U}_{zi} \neq 0$, we have

$$\left(\frac{Nx}{\lambda} \right) u_{\infty i} + \left(\frac{Ny}{\lambda} \right) v_{\infty i} + N_z = 0$$

which is the equation of a 2D line of the form $au_\infty + bv_\infty + c = 0$ (in the image plane). Therefore, all vanishing points $(u_{\infty i}, v_{\infty, i})$ lie along this line.
Remark: When $\overline{U}_{zi} = 0$, the plane is parallel to the image plane and the parallel lines do not converge in the image.

11-9

1. To show that $\angle V_aCV_b = \frac{\pi}{2}$, it is sufficient to show $C\vec{V}_a \cdot C\vec{V}_b = 0$. Since vectors $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), andc = (c_1, c_2, c_3)$ define the edges of the cube, we know they are perpendicular, thus $a \cdot b = b \cdot c = a \cdot c = 0$. From our formulas for vanishing points, we have that
   $C\vec{V}_a = \lambda(\frac{a_1}{a_3}, \frac{a_2}{a_3}, 1)$ and $C\vec{V}_b = \lambda(\frac{b_1}{b_3}, \frac{b_2}{b_3}, 1)$.

$$\begin{aligned}
C\vec{V}_a \cdot C\vec{V}_b &= \lambda^2\left(\frac{a_1b_1}{a_3b_3} + \frac{a_2b_2}{a_3b_3} + 1\right) \\
&= \frac{\lambda^2}{a_3b_3}(a_1b_2 + a_2b_2 + a_3b_3) \\
&= \frac{\lambda^2}{a_3b_3}(a \cdot b) \\
&= 0.
\end{aligned}$$

   Therefore, $C\vec{V}_a \perp C\vec{V}_b$ which implies $\angle V_aCV_b = \frac{\pi}{2}$. Similar proofs hold for the other two angles in the problem statement.

2. To show $V_b\vec{V}_c$ is perpendicular to the plane, it is sufficient to show that two distinct vectors in the plane are perpendicular to $V_a\vec{V}_b$. That is, the dot products to two distinct vectors in the plane with $V_b\vec{V}_c$ are both zero.
   One such vector is the altitude h_a, which is perpendicular to $V_b\vec{V}_c$ by definition. Another such vector is $C\vec{V}_a$.
   By vector addition $C\vec{V}_b + V_b\vec{V}_c = C\vec{V}_c$, so $V_b\vec{V}_c = C\vec{V}_c - C\vec{V}_b$. We have

$$\begin{aligned}
C\vec{V}_a \cdot V_b\vec{V}_C &= C\vec{V}_a \cdot \left(C\vec{V}_c - C\vec{V}_b\right) \\
&= C\vec{V}_a \cdot C\vec{V}_c - C\vec{V}_a \cdot C\vec{V}_b \\
&= 0 - 0.
\end{aligned}$$

   Therefore, $C\vec{V}_a$, another vector in the plane, is also perpendicular to $V_b\vec{V}_c$. We conclude that $V_b\vec{V}_c$ is perpendicular to the plane, and hence is the vector normal to the plane.

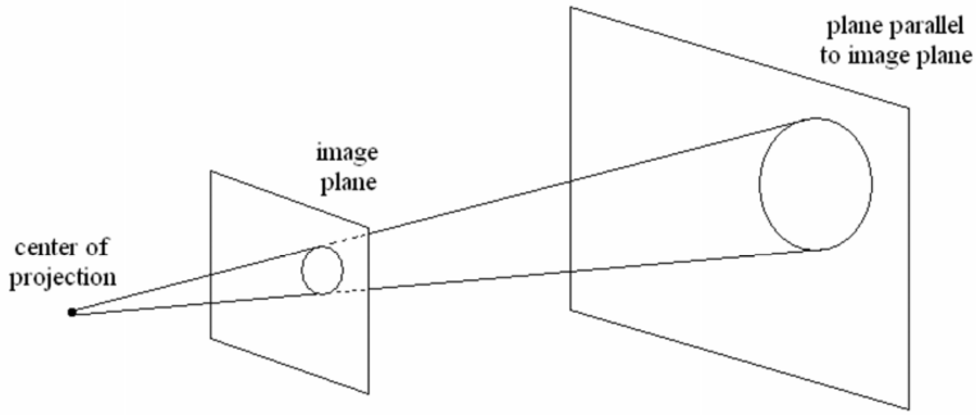
3. From the previous part, we know that $V_b\vec{V}_c$ is normal to plane P_a. The vector $\vec{N} = (0, 0, 1)$ is normal to the image plane. To show that plane P_a is perpendicular to the image plane, it is sufficient to show that the respective normal vectors are perpendicular.

$$\begin{aligned}
V_b\vec{V}_c \cdot \vec{N} &= (u_c - u_b)0 + (v_c - v_b)0 + (0)1 \\
&= 0
\end{aligned}$$

   Therefore, the image plane is perpendicular to plane P_a.

4. The orthocenter of the triangle $V_aV_bV_c$ is $(0, 0, \lambda)$. It lies on the camera's z axis.

11-10 The fun solution.



Consider rays from the center of projection to every point on the circle. Taken together, these rays form a cone. The locus of the intersection of the cone with the image plane is a circle in the image.

$$\begin{aligned}
&\sum_{i=1}^{N}(x_i - \mu)^2 p(x_i) \\
&\quad = \sum (x_i^2 - 2\mu x_i + \mu^2) P(x_i) \\
&\quad = \sum x_i^2 P(x_i) - \sum 2\mu x_i P(x_i) + \sum \mu^2 P(x_i) \\
&\quad = \sum x_i^2 P(x_i) - 2\mu \underbrace{\sum x_i P(x_i)}_{\mu} + \mu^2 \underbrace{\sum P(x_i)}_{1} \\
&\quad = \sum x_i^2 P(x_i) - 2\mu^2 + \mu^2 \\
&\quad = \sum x_i^2 P(x_i) - \mu^2
\end{aligned}$$

11-13 Students simply need to know the formulas for the row and column centroids.

$$\bar{r} = \frac{m_{10}}{m_{00}} = \frac{\sum_{r,c} r\mathcal{I}(r,c)}{\sum_{r,c} \mathcal{I}(r,c)} \qquad \bar{c} = \frac{m_{01}}{m_{00}} = \frac{\sum_{r,c} c\mathcal{I}(r,c)}{\sum_{r,c} \mathcal{I}(r,c)}$$

Equation (11.21) is found by substituting $\bar{r}m_0 0$ for $\sum_{r,c} r\mathcal{I}(r,c)$ and $\bar{c}m_0 0$ for $\sum_{r,c} c\mathcal{I}(r,c)$.

11-14 $P_0(z)$ is the pdf for background pixels, which are low intensity (high z values).
$P_1(z)$ is the pdf for object pixels, which are high intensity (high z values).
Given a chosen threshold value t,

– the probability that a background pixel is misclassified as an object pixel is $\int_t^\infty P_0(z)dz$

– the probability that an object pixel is misclassified as a background pixel is $\int_{-\infty}^t P_1(z)dz$.

11-15 We can write the total probability of error as $E_{total} = \int_{-\infty}^{t} P_1(z)dz + \int_{t}^{\infty} P_0(z)dz$. To minimize E_{total}, so set its derivative with respect to t equal to zero. Using the Fundametnal Theorem of Calculus, we find

$$\begin{aligned}
0 &= \frac{d}{dt}E_{total} \\
&= \frac{d}{dt}\int_{-\infty}^{t} P_1(z)dz + \frac{d}{dt}\int_{t}^{\infty} P_0(z)dz \\
&= P_1(t) + \frac{d}{dt}\int_{\infty}^{t} -P_0(z)dz \\
&= P_1(t) - P_0(t).
\end{aligned}$$

Therefore, the total probability of error is minimized when $P_1(t) = P_0(t)$.
It is a fair assumption that the measured pixel intensity is the sum of a true intensity plus a noise term (which we assume to be additive Gaussian noise). Since all of the pixel values are manufactured by the same process, it's reasonable to assume that the same sort of noise will be introduced for each pixel. Therefore, we get two independent, identically distributed ($\sigma_0^2 = \sigma_1^2$) Gaussian random variables $P_0(t)$ and $P_1(t)$. So the pdfs are of the form

$$P(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}.$$

We know the total probability of error is minimized when $P_1(t) = P_0(t)$, so we substitute the above Gaussian pdf with variances equal. Working through the math leads to the conclusion $(t-\mu_1)^2 = (t-\mu_0)^2$. Due to the even power, this does not imply that $t-\mu_1 = t-\mu_0$. Rather, it implies $|t-\mu_1| = |t-\mu_0|$, so t is equidistant from μ_1 and μ_0. So we choose

$$t^* = \frac{\mu_1 + \mu_0}{2}.$$