4-1 straightforward.

$$
\begin{aligned}
S(a)p &= \begin{bmatrix} 0 & -a_x & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} -a_x p_y + a_y p_z \\ a_x p_z - a_x p_x \\ -a_y p_x + a_x p_y \end{bmatrix} \\
a \times p &= \begin{bmatrix} i & j & k \\ a_x & a_y & a_z \\ p_x & p_y & p_z \end{bmatrix} = i(a_y p_z - a_z p_y) - j(a_x p_z - a_z p_x) + k(a_z p_y - a_y p_x)
\end{aligned}
$$

Therefore $S(a)p = a \times p$.

4-3 Let $R = (r_1, r_2, r_3)$, where r_1, r_2, r_3 are the column vectors of R. Let $a = (a_1, a_2, a_3)^T$ and $b = (b_1, b_2, b_3)^T$ be vectors. Then

$$\begin{aligned} Ra &= a_1r_1 + a_2r_2 + a_3r_3 \\ Rb &= b_1r_1 + b_2r_2 + b_3r_3 \end{aligned}$$

Multiplying these together and using the properties of the cross product yields

$$\begin{aligned} Ra \times Rb &= (a_1r_1 + a_2r_2 + a_3r_3) \times (b_1r_1 + b_2r_2 + b_3r_3) \\ &= a_1b_2r_1 \times r_2 + a_1b_3r_1 \times r_3 \\ &+ a_2b_1r_2 \times r_1 + a_2b_3r_2 \times r_3 \\ &+ a_3b_1r_3 \times r_1 + a_3b_2r_3 \times r_2 \\ &= (a_1b_2 - a_2b_1)r_1 \times r_2 + (a_1b_3 - a_3b_1)r_1 \times r_3 + (a_2b_3 - a_3b_2)r_2 \times r_3 \end{aligned}$$

Since R is a rotation matrix, the column vectors satisfy

$$\begin{aligned} r_1 \times r_2 &= r_3 \\ r_1 \times r_3 &= r_2 \\ r_2 \times r_3 &= r_1 \end{aligned}$$

Making these substitutions yields

$$\begin{aligned} Ra \times Rb &= (a_2b_3 - a_3b_2)r_1 + (a_1b_3 - a_3b_1)r_2 + (a_1b_2 - a_2b_1)r_3 \\ &= R(a \times b) \end{aligned}$$

4-4 Set $Y = SX$. By commutativity of the inner product, $X^TY = Y^TX$, or $X^TSX = X^TS^TX$. Since S is skew-symmetric, $S^T + S = 0$. Thus, for any vector X, we have

$$0 = X^T(S + S^T)X = X^TSX + X^TS^TX = 2X^TSX$$

Therefore $X^TSX = 0$.

4-5

$$
\begin{aligned}
\frac{dR_{y,\theta}}{d\theta}R_{y,\theta}^T &= \begin{bmatrix} -s\theta & 0 & c\theta \\ 0 & 0 & 0 \\ -c\theta & 0 & -s\theta \end{bmatrix}\begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = S(j) \\
\frac{dR_{x,\theta}}{d\theta}R_{x,\theta}^T &= \begin{bmatrix} -s\theta & -c\theta & 0 \\ c_\theta & -s\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S(k)
\end{aligned}
$$

4-6

$$
\begin{aligned}
R_{x,90} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}; S(a) = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}; Ra = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \\
S(Ra) &= \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

Then

$$
\begin{aligned}
RS(a)R^T &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = S(Ra)
\end{aligned}
$$

4-7

$$R_1^0 = R_{x,\theta} R_{y,\phi}$$

Then

$$\frac{\partial R_0^1}{\partial \phi} = R_{x,\theta} \frac{\partial R_{y,\phi}}{\partial \phi} = R_{x,\theta} S(j) R_{y,\phi} = \begin{bmatrix} -s\phi & 0 & c\phi \\ s\theta c\phi & 0 & s\theta s\phi \\ -c\theta c\phi & 0 & -s\phi c\theta \end{bmatrix}$$

$$\left.\frac{\partial R_0^1}{\partial \phi}\right|_{\substack{\theta=\phi/2 \\ \phi=\pi/2}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
I + S(k)s_\theta + S^2(k)v_\theta &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -k_z s_\theta & k_y s_\theta \\ k_z s_\theta & 0 & -k_x s_\theta \\ -k_y s_\theta & k_x s_\theta & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} (-k_z^2 - k_y^2)v_\theta & k_x k_y v_\theta & k_x k_z v_\theta \\ k_x k_y v_\theta & (-k_z^2 - k_x^2)v_\theta & k_y k_z v_\theta \\ k_x k_z v_\theta & k_y k_z v_\theta & (-k_y^2 - k_x^2)v_\theta \end{bmatrix}
\end{aligned}
$$

Adding the three matrices and using $k_x^2 + k_y^2 + k_z^2 = 1$ yields (2.2.16).

4-9 $S(k)^3 = -S(k)$ can be verified by direct multiplication. To show (2.5.20), we compute using Problem 2-25

$$\frac{dR}{d\theta} = S(k)\cos\theta + S^2(k)\sin\theta$$

also from Problem 2-25

$$\begin{aligned} S(k)R_{k,\theta} &= S(k) + S^2(k)\sin\theta + S^3(k)(1-\cos\theta) \\ &= S(k)\cos\theta + S^2(k)\sin\theta \end{aligned}$$

Using the fact that $S^3(k) = -S(k)$.

4-10 If $S \in so(3)$ then

$$(e^S)^T = e^{S^T} = e^{-S}$$

which can be verified using the series definition for e^S. Therefore

$$e^S(e^S)^T = e^S e^{-S} = e^{S-S} = e^0 = I$$

Also

$$\det(e^S) = e^{Tr(S)} = e^0 = 1$$

Hence $e^S \in S0(3)$.

4-11

$$
\begin{aligned}
e^{S(k)\theta} &= I + S\theta + \frac{\theta^2}{2!}S^2 + \frac{\theta^3}{3!}S^3 + \cdots \\
&= I + S\theta + \frac{\theta^2}{2!}S^2 + \frac{\theta^3}{4!}(-S^2) + \cdots \\
&= I + S\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) + S^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots\right) \\
&= I + S(k)\sin\theta + S^2(k)(1 - \cos\theta) \\
&= I + S(k)\sin\theta + S^2(k)(\text{vers }(\theta)) = R_{k,\theta}
\end{aligned}
$$

4-12 From 4-11, we know that $R \in SO(3)$ satisfies $\frac{dR}{d\theta} = SR$. Therefore, a matrix S exists that satisfies $R = e^R$. To show $S \in so(3)$, we must show $S^T + S = 0$.

$$S = R^{-1}\frac{dR}{d\theta} = R^T\frac{dR}{d\theta}$$

$$S^T = \left(\frac{dR}{d\theta}\right)^T R = \frac{dR^T}{d\theta}R$$

where the final equality holds because matrix derivative is taken element by element.

$$\begin{aligned} R^T R &= I \\ \frac{d}{d\theta}(R^T R) &= \frac{d}{d\theta}I \\ \frac{dR^T}{d\theta}R + R^T\frac{dR}{d\theta} &= 0 \\ S^T + S &= 0 \end{aligned}$$

So $S \in so(3)$.

4-13 For the Euler angle transformation, we have

$$R = R_{z,\psi} R_{y,\theta} R_{z,\phi}.$$

From Equation (4.18), we know that

$$\frac{dR}{d\theta} = S(k)R.$$

By the chain rule for differentiation, we have

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = S(k)R\dot{\theta}.$$

Applying the product rule for differentiation to the Euler angle transformation, we have

$$\begin{aligned}
\dot{R} &= \dot{R}_z R_y R_z + R_z \dot{R}_y R_z + R_z R_y \dot{R}_z \\
&= [S(\dot{\psi}k)R_{z,\psi}]R_y R_z + R_z[S(\dot{\theta}j)R_{y,\theta}]R_z + R_z R_y[\dot{\phi}k)R_{z,\phi}] \\
&= S(\dot{\psi}k)R_z R_y R_z + S(R_{z,\theta}\dot{\theta}j)R_z R_y R_z + S(R_z R_y \dot{\phi}k)R_z R_y R_z \\
&= [S(\dot{\psi}k) + S(R_z\dot{\theta}j) + S(R_z R_y \dot{\phi}k)]R \\
&= S(\omega)R.
\end{aligned}$$

So

$$\begin{aligned}
\omega &= \dot{\psi}k + R_z\dot{\theta}j + R_z R_y \dot{\phi}k \\
&= (c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta})i + (s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta})j + (\dot{\psi} + c_\theta \dot{\phi})k.
\end{aligned}$$

4-14 For the Euler angle transformation, we have

$$R = R_{z,\phi} R_{y,\theta} R_{x,\psi}.$$

Following the derivation for Problem 4-13 yields

$$\begin{aligned} \dot{R} &= [S(\dot{\phi}k) + S(R_z \dot{\theta} j) + S(R_z R_y \dot{\psi} x)]R \\ &= S(\omega)R. \end{aligned}$$

Therefore,

$$\begin{aligned} \omega &= \dot{\phi}k + R_z \dot{\theta} j + R_z R_y \dot{\psi} x \\ &= (c_\phi c_\theta \dot{\psi} - s_\phi \dot{\theta})i + (c_\phi \dot{\theta} + c_\theta s_\phi \dot{\psi})j + (\dot{\phi} - s_\theta \dot{\psi})k. \end{aligned}$$

4-15

$$
\begin{aligned}
p_0 &= Rp_1 + d \\
\dot{p}_0 &= R\dot{p}_1 \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}
\end{aligned}
$$

4-16 Suppose a_c = distance from joint 2 to o_c, and a_e = length of link 1. Then $o_c = (x_c, y_c, z_c)^T$ where

$$
\begin{aligned}
x_c &= a_1c_1 + a_cc_{12} \\
y_c &= a_1s_1 + a_cc_{12} \\
z_c &= 0
\end{aligned}
$$

Also

$$
\begin{aligned}
z_0 &= z_1 = (0, 0, 1)^T \\
o_0 &= (0, 0, 0)^T \\
o_c &= (a_1c_1 + a_cc_{12}, a_1s_1 + a_cs_{21}, 0)^T \\
o_1 &= (a_1c_1, a_1s_1, 0)^T
\end{aligned}
$$

$$
\begin{aligned}
z_0 \times (o_c - o_0) &= (-a_1s_1 - a_cs_{12}, a_cc_1 + a_cc_{12}, 0)^T \\
z_1 \times (o_c - o_1) &= (-a_cs_{12}, a_cc_{12}, 0)^T
\end{aligned}
$$

Therefore

$$
J = \begin{bmatrix} -a_1s_1 - a_1s_{12} & -a_cs_{12} & 0 \\ a_1c_1 + a_cc_{12} & a_cc_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

4-17 Since all three joins are revolute,

$$J_{11} = [z_0 \times (o_3 - o_0) \quad z_1 \times (o_3 - o_1) \quad z_2 \times (o_3 - o_2)]$$

$$o_0 = o_1 = (0,0,0)^T; \quad o_2 = \begin{bmatrix} a_2c_1c_2 \\ a_2s_1c_2 \\ a_2s_2 \end{bmatrix}; \quad o_3 = \begin{bmatrix} a_2c_1c_2 + a_3c_1c_{23} \\ a_2s_1c_2 + a_3s_1c_{23} \\ a_2s_2 + a_3s_{23} \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad z_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix} \quad z_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

Therefore

$$\begin{aligned} x_0 \times (o_3 - o_0) &= \begin{bmatrix} -a_2s_1c_2 - a_3s_1c_{23} \\ a_2c_1c_2 + a_3c_1c_{23} \\ 0 \end{bmatrix}; \\ z_1 \times (o_3 - o_1) &= \begin{bmatrix} -c_1(a_2s_2 + a_3s_1c_{23} \\ -s_1(a_2s_2 + a_3s_{23}) \\ a_2c_2 + a_3c_{23} \end{bmatrix}; \\ z_2 \times (o_3 - o_2) &= \begin{bmatrix} -a_3c_1s_{23} \\ -a_3s_1s_{23} \\ a_3c_{23} \end{bmatrix} \end{aligned}$$

and hence

$$J_{11} = \begin{bmatrix} -a_2s_1c_2 - a_3s_1c_{23} & -a_2s_2c_1 - a_3s_{23}c_1 & -a_3c_1s_{23} \\ a_2c_1c_2 + a_3c_1c_{23} & -a_2s_1s_2 - a_3s_1s_{23} & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{bmatrix}$$

which agrees with (5.3.14). Next,

$$\begin{aligned} \det J_{11} &= (-a_2s_1c_2 - a_3s_1c_{23})[(a_3c_{23})(-a_2s_1s_2 - a_3s_1s_{23}) + a_3s_1s_{23}(a_2c_2 + a_3c_{23})] \\ &\quad -(a_2c_1c_2 + a_3c_1c_{23})[(a_3c_{23})(-a_2s_2c_1 - a_3s_{23}c_1) + a_3c_1s_{23}(a_2c_2 + a_3c_{23})] \\ &= a_2^2a_3(s_2c_2c_{23} - s_{23}c_2^2) + a_2a_3^2(s_2c_{23}^2 - s_{23}c_2c_{23}) \\ &= -a_2^2a_3c_2s_3 - a_2a_3^2c_{23}s_3 \\ &= -a_2a_3s_3(a_2c_2 + a_3c_{23}) \end{aligned}$$

$$o_0 = 0; \quad o_1 = (0,0,d_1)^T; \quad o_2 = (0,0,d)1)^T; \quad o_3 = (-d_2 s_2 c_1, -d_2 s_2 s_1, d_1 + d_2 c_2)^T$$

$$z_0 = (0,0,1)^T; \quad z_1 = (s_1, -c_1, 0)^T; \quad z_2 = (-s_2 c_1, -s_2 s_1, c_2)^T$$

$$\begin{aligned} z_0 \times (o_3 - o_0) &= (s_1 s_2 d_2, c_1 s_2 d_2, 0)^T \\ z_1 \times (o_3 - o_1) &= (-c_1 c_2 d_2, s_1 c_2 d_2, s_2 d_2)^T \end{aligned}$$

Therefore

$$J = \begin{bmatrix} s_1 s_2 d_2 & -c_1 c_2 d_2 & -s_2 c_1 \\ c_1 s_2 d_2 & s_1 c_2 d_2 & -s_2 s_1 \\ 0 & s_2 d_2 & c_2 \\ 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

4-19 From (5.3.18), the singularities are given by $\alpha_1\alpha_4 - \alpha_2\alpha_3 = 0$. From (5.3.19), we have

$$\begin{aligned}\alpha_1\alpha_4 - \alpha_2\alpha_3 &= (-a_1s_1 - a_2s_{12})(a_1c_{12}) + (a_1s_{12})(a_1c_1 + a_2c_{12}) \\ &= a_1^2 s_2\end{aligned}$$

which agrees with (5.3.20).

4-20 From Figure 3.7,

$$o_0 = (0,0,0)^T; o_3 = (-d_3 s_1, d_3 c_1, 0)^T$$

$$z_0 = (0,0,0)^T; \quad z_1 = (0,0,1)^T; \quad z_2 = (-s_1, c_1, 0)^T$$

$$J \quad = \quad \begin{bmatrix} z_0 \times (o_3 - o_0) & z_1 & z_2 \\ z_0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -c_1 d_3 & 0 & -s_1 \\ -s_1 d_3 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore

$$\det \begin{bmatrix} -c_1 d_3 & 0 & -s_1 \\ -d_3 s_1 & 0 & c_1 \\ 0 & 1 & 0 \end{bmatrix} \quad = \quad c_1^2 d_3 + s_1^2 d_3 = d_3 \neq 0$$

4-21 For cartesian manipulator, all joints are prismatic and hence

$$J = \begin{bmatrix} z_0 & z_1 & z_2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which has rank 3.

$$J = [J_1, \ldots, J_6]$$

where

$$J_1 = \begin{bmatrix} -d_y \\ d_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad J_2 = \begin{bmatrix} c_1 d_z \\ s_1 d_z \\ -s_1 d_y - c_1 d_x \\ -s_1 \\ c_1 \\ 0 \end{bmatrix}; \quad J_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} s_1 s_2 (d_z - o_{3,z}) + c2(d_y - o_{3y}) \\ -c_1 s_1 (d_z - o_{3z}) + c_2(d_x - o_{3x}) \\ -c_1 c_2 s_4 - s_1 c_4 \\ s_2 s_4 \end{bmatrix}$$

$$J_5 = \begin{bmatrix} (-s_1 c_2 s_4 + c_1 c_4)(d_z - o_{3z}) - s_2 s_4 (d_y - o_{3y}) \\ (-c_1 c_2 s_4 + s_1 c_4)(d_z - o_{3z}) + s_2 s_4 (d_x - o_{3x}) \\ (-c_1 c_2 s_4 - s_1 c_4)(d_y - o_{3y}) + (s_1 c_2 s_4 - c_1 c_4)(d_x - o_{3x}) \\ -c_1 c_2 c_4 - s_1 c_4 \\ s_2 s_4 \end{bmatrix}$$

$$J_6 = \begin{bmatrix} (s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5)(d_y - o_{3y}) + (s_2 c_4 s_5 - c_2 c_5)(d_y - o_{3y}) \\ -(c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5)(d_z - o_{3z}) + (s_2 c_4 s_5 - c_2 c_5)(d_x - o_{3x}) \\ c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$

where

$$o_6 = (d_x, d_y, d_z)^T$$

$$o_3 = \begin{bmatrix} o_{3x} \\ o_{3y} \\ o_{3z} \end{bmatrix} = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

4-23

$$\begin{bmatrix} R & SR \\ 0 & R \end{bmatrix} \begin{bmatrix} R^T & -R^T S \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} RR^T & -RR^T S + SRR^T \\ 0 & RR^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

$$\begin{bmatrix} R^T & -R^T S \\ 0 & R^T \end{bmatrix} \begin{bmatrix} R & SR \\ 0 & R \end{bmatrix} = \begin{bmatrix} R^T R & R^T SR - R^T SR \\ 0 & R^T R \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Therefore,

$$\begin{bmatrix} R & SR \\ 0 & R \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T S \\ 0 & R^T \end{bmatrix}.$$

$$det\big(B(\alpha)\big) = c_\psi^2 s_\theta + c_\psi^2 s_\theta = s_\theta$$

Therefore $B(\alpha)$ is invertible whenever $det\big(B(\alpha)\big) = s_\theta$ is nonzero.

4-25 1. Show that $\dot{q} = J^+\xi + (I - J^+J)b$ is a solution to Equation 4.110.

$$\begin{aligned} \xi &= J\dot{q} \\ &= J(J^+\xi + (I - J^+J)b) \\ &= JJ^+\xi + (J - JJ^+J)b \\ &= I\xi + (J - IJ)b \\ &= \xi \end{aligned}$$

2. Show that $b = 0$ minimizes the joint velocities.

$$\|\dot{q}\| = \left\|J^+\xi + (I - J^+J)b\right\|$$

By the triangle inequality, we have

$$\begin{aligned} \|\dot{q}\| &\leq \left\|J^+\xi\right\| + \left\|(I - J^+J)b\right\| \\ &= \left\|J^+\xi\right\| + \left\|(I - J^+J)\right\| \|b\| . \end{aligned}$$

Since $\|(I - J^+J)\| \geq 0$, choosing $b = 0$ minimizes $\|\dot{q}\|$.

4-26 Begin with the singular-value decomposition for J. Following the development in the appendix, we have

$$J = U\Sigma V^T = U\Sigma_m V_m^T.$$

Note that $\Sigma\Sigma^T = \Sigma_m^2$ is symmetric and that U and V are orthogonal matrices.

$$\begin{aligned}
J^+ &= J^T(JJ^T)^{-1} \\
&= (U\Sigma_m V_m^T)^T\Big((U\Sigma_m V_m^T)(U\Sigma_m V_m^T)^T\Big)^{-1} \\
&= V_m\Sigma_m U^T\Big(U\Sigma_m V^T V\Sigma_m^T U^T\Big)^{-1} \\
&= V_m\Sigma_m U^{-1}\Big(U\Sigma_m^2 U^{-1}\Big)^{-1} \\
&= V_m\Sigma_m U^{-1}U\Sigma_m^{-2}U^{-1} \\
&= V_m\Sigma_m^{-1}U^{-1} \\
&= V_m\Sigma_m^{-1}U^T \\
&= \big[V_m|V_{n-m}\big]\big[\Sigma_m^{-1}|0\big]^T U^T \\
&= V\Sigma^+U^T
\end{aligned}$$

4-27 To complete this problem we will use the fact that, for a square matrix A, $(A^T)^{-1} = (A^{-1})^T$.

$$
\begin{aligned}
\|\dot{q}\|^2 &= \dot{q}^T\dot{q} \\
&= (J^+\xi)^T(J^+\xi) \\
&= \left[J^T(JJ^T)^{-1}\xi\right]^T\left[J^T(JJ^T)^{-1}\xi\right] \\
&= \xi^T\left[(JJ^T)^{-1}\right]^T JJ^T(JJ^T)^{-1}\xi \\
&= \xi^T\left[(JJ^T)^{-1}\right]^T\xi \\
&= \xi^T\left[(JJ^T)^T\right]^{-1}\xi \\
&= \xi^T(JJ^T)^{-1}\xi
\end{aligned}
$$

4-28 Note that $\Sigma\Sigma^T = \Sigma_m^2$ is symmetric and that U and V are orthogonal matrices.

$$
\begin{aligned}
\xi^T (JJ^T)^{-1}\xi &= \xi^T \left(U\Sigma V^T)(U\Sigma V^T)^T\right)^{-1}\xi \\
&= \xi^T \left(U\Sigma V^T V\Sigma^T U^T\right)^{-1}\xi \\
&= \xi^T \left(U\Sigma\Sigma^T U^{-1}\right)^{-1}\xi \\
&= \xi^T \left(U\Sigma_m^2 U^{-1}\right)^{-1}\xi \\
&= \xi^T U\Sigma_m^{-2}U^{-1}\xi \\
&= \xi^T U\Sigma_m^{-2}U^T\xi \\
&= (U^T\xi)^T\Sigma_m^{-2}U^T\xi
\end{aligned}
$$