On Pythagorean and Complex Fuzzy Set Operations

Scott Dick, *Member, IEEE*, Ronald R. Yager, *Fellow, IEEE*, and Omolbanin Yazdanbakhsh, *Member, IEEE*

***Abstract*—Complex fuzzy logic is a new multivalued logic system that has emerged in the last decade. At this time, there are a limited number of known instances of complex fuzzy logic, and only a partial exploration of their properties. There has also been relatively little progress in developing interpretations of complex-valued membership grades. In this paper, we address both problems by examining the recently developed Pythagorean fuzzy sets (a generalization of intuitionistic fuzzy sets). We first characterize two lattices that have been suggested for Pythagorean fuzzy sets and then extend these results to the unit disc of the complex plane. We thereby identify two new complete, distributive lattices over the unit disc, and explore interpretations of them based on fuzzy antonyms and negations.**

***Index Terms*—Complex fuzzy logic (CFL), complex fuzzy sets (CFS), fuzzy intersection, fuzzy union, lattice theory, Pythagorean fuzzy sets (PFS).**

I. Introduction

COMPLEX fuzzy sets (CFS) are a relatively new area of investigation in fuzzy logic. First proposed by Ramot *et al.* in [1] and [2], CFS are subsets of some universal set, which have a membership function whose codomain is the unit disc of the complex plane. Equivalently, a CFS is a set of ordered pairs $(x, \mu(x))$ where $x \in X$ is an element of some universal set, and $\mu(x)$ is the membership of x in the CFS, $\mu(x) \in \{c \in \boldsymbol{C} \mid |c| \leq 1\}$, for $\boldsymbol{C}$ the complex plane. CFS and the isomorphic complex fuzzy logic (CFL) have been shown to create accurate and parsimonious models for time series forecasting [3]–[11], data mining [12]–[14], and image processing [15], [16]. However, this research is still in its infancy, and the architectures that have so far been created are not very diverse. Both the adaptive neuro-complex fuzzy inference system (ANCFIS) architecture [3]–[5], and the complex neuro-fuzzy system family of architecture [6], [7], [9], [10], [12]–[16], are based on Jang's adaptive neuro-fuzzy inference system (ANFIS) [17]. What they all have in common is that they are realizations of a fuzzy inferential system using complex fuzzy sets (CFS) and operations—but the set of available operations is miniscule. The only complex fuzzy conjunction in use is the algebraic product, and inference results are determined using Takagi–Sugeno–Kang inferencing [18]. This lack of diversity is a serious problem because no one learning architecture (or group of architectures) performs optimally on all datasets [19]. In essence, because of the newness of this research area, the design space for creating new complex fuzzy inferential systems (whether declaratively, or inductively via neuro-fuzzy, genetic-fuzzy or other architectures) is artificially constrained. In particular, the very limited theoretical results on what complex fuzzy operations can be combined into lattices obeying the DeMorgan laws is a severe limitation on this design space.

Thus, one of the key research issues surrounding CFS and CFL is to explore and characterize the different possible set-theoretic and logic operators for them. In our view, this is analogous to type-1 fuzzy logic, which is well-known as a family of multi-valued logics with infinite truth-valuation sets [20], [21]. The different fuzzy logics are defined by different choices of the conjunction, disjunction and negation operators. Analyses of fuzzy logic usually employ lattice theory as in [22]; those whose operators together satisfy the DeMorgan laws (known as DeMorgan triples) are the ones of primary interest [23]. A survey of over 30 DeMorgan triples may be found in [24]. As there are several parameterized t-norms that form DeMorgan triples with their dual co-norms, the family of type-1 fuzzy logics is plainly infinite [23]. And while the totally ordered set [0,1] is the iconic truth-valuation set from Zadeh's original work [25], the theory of fuzzy logic was generalized to any complete lattice-ordered semigroup (a complete lattice with an additional operator forming a semigroup over the valuation set, with zero and identity the lattice infimum and supremum, and distributive over the join) by Gougen in 1967 [26]. This of course means that the truth valuation set of a fuzzy logic might only be partially ordered.

It is our contention that CFL is likely also an infinite family of infinite-valued logics, with the signature characteristic that each element of the truth valuation set is a vector rather than a scalar. Ramot *et al.* [1], [2] proposed the first union, intersection and complement operations for CFS, along with their isomorphic counterparts in CFL. These operations focused only on the magnitude of a CFS; membership phase was treated as adding an application-dependent "context." Complex fuzzy implications were to be implemented using the complex product, while fired rules should be aggregated using a weighted vector sum, resulting in *rule interference*. In [22], Dick suggested that membership in a CFS be treated as a vector, and explored one such logic in which the complex product was the intersection, and offered an existence proof for the corresponding union. Zhang *et al.* studied distances and a weakened form of equality between two CFS in [27]. Two CFS are δ-equal if the distance between them is less than or equal to δ. Alkouri and Salleh offered four additional distance metrics, and discussed complex linguistic variables [28]. Alkouri *et al.* extended intuitionistic fuzzy sets [29] to the complex domain; both the membership and non-membership functions of a Complex Atanassov's Intuitionistic

Manuscript received August 25, 2014; revised December 11, 2014, March 27, 2015, June 4, 2015, and August 25, 2015; accepted September 15, 2015. Date of publication November 12, 2015; date of current version October 4, 2016. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under Grant RGPIN 262151, in part by an ARO MURI Grant (Number W911NF-09-1-0392), and in part by the ONR under Grant N00014-13-1-0626.

S. Dick and O. Yazdanbakhsh are with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB T6G 1H9, Canada (e-mail: dick@ece.ualberta.ca; yazdanba@ualberta.ca).

R. R. Yager is with the Department of Information Systems, Iona College, New Rochelle, NY 10801 USA (e-mail: yager@panix.com).

Digital Object Identifier 10.1109/TFUZZ.2015.2500273

Fuzzy Set (CAIFS) are CFS. Axioms for the intersection and union of two CAIFS are discussed, and a complement operation is defined (the latter acts only on phases). Distances between CAIFS are also defined. Yager, on the other hand, developed Pythagorean fuzzy sets (PFS) as a generalization of the original intuitionistic fuzzy sets [30], [31]; this will be discussed further in Section II. Tamir *et al.* offered a different approach to CFS in the so-called "pure" CFS [32]. Memberships in these sets are drawn from the unit square [0,1] × [0,1]. Tamir *et al.* then define a first-order predicate logic for the pure CFS in [33], a propositional logic in [34], [35], and finally an extended Post system of logic in [36]. Moses *et al.* also studied memberships in [0,1] × [0,1], focusing on how transformations from Cartesian coordinates to polar coordinates could be represented in a linguistic variable [37].

Our goal in this article is to explore the relationship between PFS and CFS, thus determining new instances of the isomorphic CFLs and also providing a possible interpretation of complex membership grades. We will begin by analyzing the mathematic-logic properties of PFS (by examining the lattice structure they induce over the set of Pythagorean membership grades). We will then extend this analysis to the entire unit disc, developing new union and intersection operators for CFS. We find that this leads to two different lattices over CFS, each of which is complete and distributive, with lower bound $0 + 1j$ and upper bound $1 + 0j$, $j = \sqrt{-1}$. Finally, we interpret one of these new lattices as a novel, unified means to represent both antonym and negation in a linguistic variable, such as was not possible with type-1 fuzzy sets.

The remainder of this paper is organized as follows. In Section II, we analyze the lattice structures suggested in [30]. In Section III, we extend this analysis to the entire unit disc, with the assumption that the negation of membership is a synonym for non-membership. In Section IV, we remove this assumption, and interpret the resulting lattice in terms of linguistic antonyms and negation. We offer a summary and discussion of future work in Section V.

II. Characterization of Pythagorean Fuzzy Sets

In this section we will characterize two possible PFS theories (and their isomorphic fuzzy logics). Yager [30], [31] proposed Pythagorean fuzzy unions and intersections based on the intuitionistic fuzzy union and intersection. Our analysis in Theorems (2.1) and (2.2) shows that the resulting lattice is complete and distributive, with supremum (1,0) and infimum (0,1). This means that the lattice matches several important characteristics expected of a fuzzy logic. Completeness means we can take the union or intersection of any membership values (not always true in general lattices), and the supremum and infimum match the boundary condition axioms for fuzzy unions and intersections. The usual distributive laws for union and intersection also apply. Yager also discussed a ranking function over Pythagorean memberships; as a lattice is a partially ordered set, we investigate this ranking as an alternative PFS theory. Theorems (2.3) and (2.4), as well as Lemmas (2.1) and (2.2), show that the lattice is complete and distributive with the same infimum and supremum as Theorem (2.2). This means that there is a Pythagorean fuzzy union and intersection based on the ranking function (determined by the join and meet operations in this lattice).

A. Review of Pythagorean Fuzzy Sets

Recall that a PFS A is an ordered triple $(x, \mu_Y(x), \mu_N(x))$, where $x \in U$ is an element of some universe of discourse U, $\mu_Y(x)$ denotes the membership of x in A, and $\mu_N(x)$ denotes non-membership in A. These definitions are the same as in Atanassov's Intuitionistic Fuzzy Sets (IFS) [29]. Where PFS differs from IFS is in the constraints placed upon the member ship and non-membership functions. In an IFS, $0 \le \mu_Y(x) + \mu_N(x) \le 1$ for all x, whereas in a PFS this condition becomes $0 \le (\mu_Y(x))^2 + (\mu_N(x))^2 \le 1$. We follow the convention of plotting membership in an IFS (or PFS) as a point in [0,1] × [0,1], with membership on the horizontal axis and non-membership on the vertical. The membership of an element x in a PFS is thus a point within the subset of [0,1] × [0,1] defined by a quarter circle of radius 1 centered at (0,0) (this region is referred to as the set of $\Pi - i$ numbers in [30]). An alternative formulation in [30] is that a PFS is an ordered triple (x, r_x, θ_x), with $x \in U, r_x$ the *strength of commitment* for x, and θ_x *the direction of commitment*. Strength and direction of commitment relate to membership and non-membership via the well-known relations:

$$\begin{aligned} \mu_Y(x) &= r_x \cdot \cos\theta_x \\ \mu_N(x) &= r_x \cdot \sin\theta_x \end{aligned} \tag{2.1}$$

and the inverse relations:

$$\begin{aligned} r_x &= \sqrt{(\mu_Y(x))^2 + (\mu_N(x))^2} \\ \theta_x &= \tan^{-1}\left(\frac{\mu_N(x)}{\mu_Y(x)}\right). \end{aligned} \tag{2.2}$$

Fuzzy unions, intersections and complements must obey a well-known set of axioms, irrespective of the codomain of the membership function. Unions and intersection must both be continuous, commutative, associative and monotonic. Unions must also be super-additive, with boundary conditions $a \cup 0 = a$ and $a \cup 1 = 1$, while intersections must be sub-additive with boundary conditions $a \cap 0 = 0$ and $a \cap 1 = a$. Complements, on the other hand, must be continuous, monotonic and involutive, with boundary conditions $\neg 1 = 0$ and $\neg 0 = 1$. [30] defines complement, intersection and union for PFS as follows. Consider two PFS over some universe of discourse U, $A(x) = (A_Y(x), A_N(x))$ and $B(x) = (B_Y(x), B_N(x))$, with $A_Y(x)$, $B_Y(x)$ the membership functions and $A_N(x), B_N(x)$ the non-membership functions of A and B, respectively. Complements are defined with respect to the strength of commitment r as:

$$\bar{A}(x) = \left(\sqrt{r^2 - (A_Y(x))^2}, \sqrt{r^2 - (A_N(x))^2}\right). \tag{2.3}$$

It can trivially be shown that $A_N(x) = \sqrt{r^2 - (A_Y(x))^2}$ and $A_Y(x) = \sqrt{r^2 - (A_N(x))^2}$, and so (2.3) can be rewritten as:

$$\bar{A}(x) = (A_N(x), A_Y(x))\,. \tag{2.4}$$

The intersection and union of two PFS are given by:

$$\begin{aligned} A(x) \cap B(x) &= (\min(A_Y(x), B_Y(x)), \\ &\quad \max(A_N(x), B_N(x))) \\ A(x) \cup B(x) &= (\max(A_Y(x), B_Y(x)), \\ &\quad \min(A_N(x), B_N(x))). \end{aligned} \tag{2.5}$$

In addition, [30], [31] examine the usage of PFS in a decision-making scenario. In order to accomplish this, a mapping from the $\Pi - i$ numbers to the unit interval was developed, so that different PFS could be ordered. For an arbitrary $\Pi - i$ number $A = (a, b)$ this mapping is defined as:

$$F(A) = \frac{1}{2} + \sqrt{a^2 + b^2} \cdot \left(\frac{1}{2} - \frac{2 \cdot \tan^{-1}\left(\frac{b}{a}\right)}{\pi}\right). \tag{2.6}$$

Decision alternatives (represented as $\Pi - i$ numbers C_i) can be directly compared by their $F(C_i)$ values. The function F was derived via fuzzy functional modeling from a fuzzy rulebase describing its desired characteristics. Specifically, those rules are:

If r is close to one and d is close to one, then F is 1;
If r is close to one and d is close to zero, then F is 0;
If is r close to zero, then F is 0.5

where r is the magnitude of a Pythagorean membership, and d is related to the phase by $d = 1 - (2\theta/\pi)$ [31]. These rules describe a function that smoothly transitions from the value 1 at the point $(1 + 0i)$ (full membership) to the value 0 at $(0+1i)$ (full non-membership), with all points where $a = b$ having the value 0.5 (no preference for membership or non-membership). Plainly, this is only one of infinitely many mappings from the $\Pi - i$ numbers to [0,1], and other mappings will have different properties. However, as this particular choice was successfully used in a decision-making problem in [31], we will study it here. As we do not study decision-making in the current paper, we direct the reader to [31] for further discussion of the properties of the mapping *F* in that context, as well as an illustrative example.

B. Pythagorean Fuzzy Set Lattice Defined by ∩ and ∪

We first consider the lattice defined by the intersection and union operators in (2.5) (considering ∩ to represent the lattice meet, and ∪ the lattice join). A lattice is an algebraic structure $(L,\wedge,\vee)$ where *L* is a set, $\wedge$ is a binary operation over *L* (the meet) defined as the least upper bound of its arguments, and $\vee$ is a binary operation over *L* (the join) defined as the greatest lower bound of its arguments. Both the join and meet must be commutative, associative, idempotent, and absorptive. (Equivalently, the lattice can be defined as $(L,\leq)$, where $\leq$ is the partial ordering induced by the join and meet. It is equally possible to define the join and meet from a given partial ordering, as in a later section of the current paper [38].)

Theorem 2.1: The operations of ∩ and ∪ given in (2.5) form a lattice over the $\Pi - i$ numbers.

Proof: Commutativity, associativity, absorption, and idempotency of ∩ and ∪ are well-established for intuitionistic fuzzy sets; trivially, these results will also apply to the $\Pi - i$ numbers as well, so long as ∩ and ∪ are closed on that set. This was shown in [30], completing the proof.

Theorem 2.2: The lattice in Theorem (2.1) is (i) bounded, (ii) distributed, and (iii) complete.

Proof Note: throughout the proof, $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ are $\Pi - i$ numbers, which implies that $a_1, a_2, b_1, b_2, c_1, c_2 \in [0,1]$.

1) A lattice is bounded if the lattice infimum ("0") is the identity for the lattice join, and the lattice supremum ("1") is the identity for the lattice meet. Per [30], the infimum and supremum of the $\Pi - i$ numbers are (0,1) and (1,0), respectively

$$\begin{aligned} &A \cup (0,1) = A \\ &(\max(a_1, 0), \min(a_2, 1)) = (a_1, a_2) \\ &(a_1, a_2) = (a_1, a_2) \\ &A \cap (1,0) = A \\ &(\min(a_1, 1), \max(a_2, 0)) = (a_1, a_2) \\ &(a_1, a_2) = (a_1, a_2). \end{aligned}$$

Thus, the lattice is bounded.

2) We need only prove that the join is distributive over the meet (or vice versa)

$$\begin{aligned} &A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ &[\max(a_1, \min(b_1, c_1)), \min(a_2, \max(b_2, c_2))] \\ &\quad = \begin{bmatrix} \min(\max(a_1, b_1), \max(a_1, c_1)), \\ \max(\min(a_2, b_2), \min(a_2, c_2)) \end{bmatrix} \\ &[\max(a_1, \min(b_1, c_1)), \min(a_2, \max(b_2, c_2))] \\ &\quad = [\max(a_1, \min(b_1, c_1)), \min(a_2, \max(b_2, c_2))]\,. \end{aligned}$$

Thus, the lattice is distributive.

3) A lattice is complete if, for any $X \subseteq \Pi - i$ numbers, the lattice join and meet exist for *X*. Observe that, since $A = (a_1,\ a_2)$, $B = (b_1,\ b_2)$ are $\Pi - i$ numbers, $(\max(a_1, b_1), \min(a_2, b_2))$ and $(\min(a_1, a_2),$ $\max(a_2, b_2))$ exist and are elements of [0,1]. Per [30], these meets and joins are themselves $\Pi - i$ numbers. Thus, the join and meet exist for any subset of the $\Pi - i$ numbers, and thus the lattice is complete.

C. Pythagorean Fuzzy Set Lattice Defined by the Ordering Rule F()

We next examine the lattice created by the PFS ordering rule in (2.6). We define this ordering as follows:

Definition 2.1: The ordering $\leq^*$ of two $\Pi - i$ numbers A and B is given by:

$$A \leq *B \equiv F(A) \leq F(B) \tag{2.7}$$

where "$\leq$" denotes the natural ordering of the real numbers and $F()$ is given in (2.6).

Theorem 2.3: The operation of $\leq^*$ in Definition (2.1) forms a lattice over the $\Pi - i$ numbers.

Proof: The real numbers are a totally ordered set; thus, (2.7) clearly implies that $\leq^*$ is a total ordering as well. This necessarily implies the lattice condition (that the lattice join and meet exist for any two $\Pi - i$ numbers). Hence, $\leq^*$ forms a lattice over the $\Pi - i$ numbers. We will henceforth denote this lattice by ($\Pi - i$, $\leq^*$). The next two lemmas identify the infimum and supremum of this lattice.

Lemma 2.1: The infimum of the lattice ($\Pi - i$, $\leq^*$) is (0,1) (in Cartesian coordinates).

Proof: We will use the polar form of a $\Pi - i$ number $A = (r_a, \theta_a)$. By definition [see (2.6)]:

$$F(A) = \frac{1}{2} + r_a \cdot \left(\frac{1}{2} - \frac{2\theta_a}{\pi}\right).$$

The term $(\frac{1}{2} - \frac{2\theta_a}{\pi})$ is monotonic decreasing in the range $[0, \pi/2]$, reaching a minimum of $-1/2$ at the point $\theta_a = \pi/2$. If r_{a} is simultaneously 1, then $F(A) = 0$; in the Cartesian form, this is the point (0,1). Plainly, $F(A) > 0$ for any other $\Pi - i$ number, and so (0,1) is the lattice infimum.

Lemma 2.2: The supremum of the lattice ($\Pi - i$, $\leq^*$) is (1,0) (in Cartesian coordinates).

Proof: As noted in the proof of Lemma 2.1, $(\frac{1}{2} - \frac{2\theta_a}{\pi})$ is monotonic decreasing, with a maximum of 1/2 when $\theta_a = 0$. If r_{a} is simultaneously 1, then $F(A) = 1$; in the Cartesian form, this is the point (1,0). Plainly, $F(A) < 1$ for any other $\Pi - i$ number, and so (1,0) is the lattice supremum.

Theorem 2.4: The lattice ($\Pi - i$, $\leq^*$) is (i) bounded, (ii) distributive, and (iii) complete.

Proof: Note: Throughout the proof, $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ are $\Pi - i$ numbers. Furthermore, the meet is defined as the $\Pi - i$ number having the minimal image under $F()$ (min($F(A)$,F(A))), and the join is the $\Pi - i$ number having the maximal image under $F()$ (max($F(A)$, $F(B)$)), respectively.

i) From Lemma (2.2), the infimum and supremum are (0, 1) and (1, 0), respectively. Plainly, the meet identity for this lattice is (1, 0), the lattice supremum, and the identity for the join is (0, 1), the lattice infimum. Thus, the lattice is bounded.

ii) We will prove distributivity of the join over the meet. From Lemmas (2.1) and (2.2), F(A) $\in$ [0,1]. Thus, from the known properties of max and min on the interval [0,1], we have:

$$\begin{aligned}
&A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\\
&\max(F(A), \min(F(B), F(C))\\
&\quad = \min(\max(F(A), F(B)), \max(F(A), F(C)))\\
&\min(\max(F(A), F(B)), \max(F(A), F(C)))\\
&\quad = \min(\max(F(A), F(B)), \max(F(A), F(C))).
\end{aligned}$$

iii) Observe that any $\Pi - i$ number is mapped to some element of [0,1]. Therefore, any subset of the $\Pi - i$ numbers is mapped to a subset of [0,1] (although perhaps not of the same cardinality). As [0,1] is totally ordered by $\leq$, there must necessarily be a maximal and a minimal element to any subset of [0,1]. Thus, the join and meet of any subset of the $\Pi - i$ numbers exists, and the lattice is therefore complete.

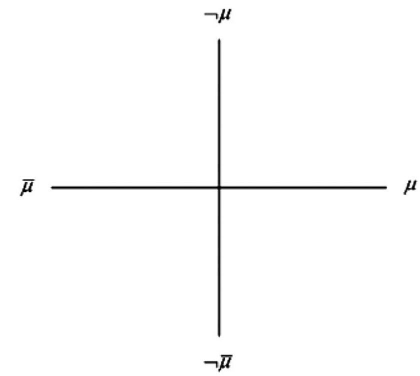


Fig. 1. Nonmembership and negation in the complex plane.

III. Complex Fuzzy Sets: Negation Equalling Non-Membership

We now consider the extension of PFS to the whole unit disc of the complex plane $\boldsymbol{D}$. In Fig. 1, we have labeled the membership and non-membership axes, using μ and $\neg\mu$ to denote membership and non-membership, respectively. In extending PFS to the unit disc, we will also have to incorporate the negation of these axes; they are labeled $\bar{\mu}$ and $\neg\bar{\mu}$. Let us refer to these as anti-membership and anti-non-membership, respectively. As usual, a PFS membership is an ordered pair representing the degree to which an object exhibits membership and non-membership in the PFS. The first question we need to resolve is, do we treat anti-membership and non-membership as being equivalent? In other words, are μ and $\neg\mu$ equivalent to $\neg\bar{\mu}$ and $\bar{\mu}$, respectively? In this section, we consider the lattice structure that results if they are; we will explore the lattice structure that arises if they are not in Section IV.

A. Main Results

We assume that the corollary of assuming anti-membership equals non-membership is that anti-non-membership is equivalent to membership. Plainly, if μ and $\neg\mu$ are equivalent to $\neg\bar{\mu}$ and $\bar{\mu}$, respectively, then the unit disc should in fact map back to the $\Pi - i$ numbers.We begin by formalizing this concept in Definition (3.1), identifying a function that maps the four quadrants of the unit disc of the complex plane into the $\Pi - i$ numbers. Theorem (3.1) discusses an interesting property of the function G, showing that it maps vector negation into the intuitionistic complement. We will see in Section 4 that breaking the equivalence of anti-membership and non-membership also breaks the relation between these two complements. In Theorems (3.2) and (3.3), we show that the mapping G yields a partial ordering of the unit disc, and then that this partial ordering is actually a complete, distributive lattice, with infimum (0,1) and supremum

(1,0). Thus, a complex fuzzy union and intersection arise from the join and meet of this lattice.

Definition 3.1: The function G is a mapping:

$$G : \boldsymbol{D} \rightarrow \ \Pi - i$$

$$G(r,\theta) = \begin{cases} (r,\theta), & \text{if} \quad \theta \in [0,\pi/2] \\ (r,\pi/2), & \text{if} \quad \theta \in [\pi/2,\pi] \\ (r,(-\theta+3\pi/2)), & \text{if} \quad \theta \in [\pi,3\pi/2] \\ (r,0), & \text{if} \quad \theta \in [3\pi/2,2\pi]. \end{cases} \tag{3.1}$$

Plainly, this function acts differently on each quadrant of the unit disc, in order to match the PFS interpretation of the *x* and *y* axes. The upper-right quadrant (which is of course the set Π – *i*) is left unchanged. The upper-left quadrant collapses into the positive imaginary axis; the rationale for this is that, given $\bar{x} = \neg x$, no point in the upper-left quadrant has any support for membership in the set; only non-membership is supported. The lower-left quadrant is the set of points for which anti-membership and anti-non-membership coexist; these points are reflected through the origin (anti-non-membership becomes membership, and anti-membership becomes non-membership). The lower right quadrant is collapsed into the positive real axis, as no point in this quadrant supports non-membership. Naturally, this is only one of infinitely many possible mappings from $\boldsymbol{D}$ to the Π – *i* numbers, but we make the (admittedly arbitrary) choice to investigate this one.

The mapping G also extends the Pythagorean complement defined in [30] to the unit disc. Consider the natural negation of a complex number (*r*, *θ*), given by $(-r, \theta)$. Theorem (3.1) relates this function to the Pythagorean complement.

Theorem 3.1: The function *G* of Definition (3.1) maps the vector negation of a complex number (*r*, *θ*) (given by (–*r*, *θ*)) into the Pythagorean complement defined in [30].

Proof: Throughout the following, $A = (r, \theta) \in \boldsymbol{D}$ (the unit disc of the complex plane).

Case 1–$\theta \in [0, \pi/2]$:

$$\begin{aligned} &A = (r,\theta) \\ &\Rightarrow \bar{A} = (r,\phi), \phi \in [\pi, 3\pi/2] \\ &\Rightarrow G(\bar{A}) = (r, (-\phi + 3\pi/2)) \\ &= (r, (-(\theta+\pi) + 3\pi/2)) \\ &= (r, (\pi/2 - \theta)). \end{aligned}$$

Equivalently, if $A = (a+bi) \in \boldsymbol{D}$ for *a*, *b* non-negative, then $G(\bar{A}) = (b + ai)$, the Pythagorean complement.

Case 2–$\theta \in [\pi/2, \pi]$:

$$\begin{aligned} &A = (r,\theta) \\ &\Rightarrow \bar{A} = (r,\phi), \phi \in [3\pi/2, 2\pi] \\ &\Rightarrow G(\bar{A}) = (r, 0). \end{aligned}$$

Equivalently, if $A = (a+bi) \in \boldsymbol{D}$ for *a* non-positive, b non-negative, then $G(\bar{A}) = (\sqrt{a^2+b^2}, 0)$. This is consistent with the definition of *G*; recall that points in the upper left quadrant were mapped to the positive imaginary axis, as there is no support for membership. The negation of this statement is thus, naturally, that there is no support for non-membership; and so the negation is mapped to the positive real axis. It is consistent with the Pythagorean complement since the negation of $B = (r, \pi/2)$ is defined as $\bar{B} = (r, 0)$ in [30].

Case 3–$\theta \in [\pi, 3\pi/2]$:

$$\begin{aligned} &A = (r,\theta); \\ &\Rightarrow \bar{A} = (r,\phi), \phi \in [0, 2\pi] \\ &\Rightarrow G(\bar{A}) = (r, \phi) \\ &\Rightarrow G(A) = (r, \pi/2 - \phi). \end{aligned}$$

Equivalently, if $A = (a+bi) \in \boldsymbol{D}$ for a, b non-positive, then $G(A) = (b + ai)$, and $G(\bar{A}) = (a + bi)$, giving the Pythagorean complement.

Case 4–$\theta \in [3\pi/2, 2\pi]$:

$$\begin{aligned} &A = (r,\theta) \\ &\Rightarrow \bar{A} = (r,\phi), \phi \in [\pi/2, \pi] \\ &\Rightarrow G(\bar{A}) = (r, \pi/2)\,. \end{aligned}$$

As in Case 2, the function G maps the negation of "no support for non-membership" to exclusive non-membership. This is the complement of *G*(*A*) (mapping the lower-right quadrant to the positive real axis) as defined by [30]. (Equivalently, if $A = (a+bi) \in \boldsymbol{D}$ for a non-negative, b non-positive, then $G(\bar{A}) = (0, \sqrt{a^2+b^2})$.) This completes the proof.

We next consider how a lattice may be formed over the unit disc $\boldsymbol{D}$ using the mapping *G*. In other words, what are the implications for CFL when we interpret the real and imaginary axes as membership and non-membership, and assume $\bar{x} = \neg x$? We first propose an ordering of the unit disc using a composition of the functions *F* and *G*.

Definition 3.2: The ordering $\leq^{**}$ of two complex numbers *A*, *B* from the unit disc $\boldsymbol{D}$ is defined as follows:

$$A \leq^{**} \text{B if } F(G(A)) \leq F(G(B)).$$

This definition is similar to the ordering $\leq^{*}$ investigated in Section II. We now investigate the lattice properties of this new ordering.

Theorem 3.2: The relation $\leq^{**}$ of Definition (3.2) is a partial ordering of the unit disc $\boldsymbol{D}$.

Proof Reflexivity:

Case 1–$\theta \in [0, \pi/2]$: This was proven in Theorem (2.5)

Case 2–$\theta \in [\pi/2, \pi]$:

$$\begin{aligned} &A \leq^{**} A \\ &F(G(A)) \leq F(G(A)) \\ &F(r, \pi/2) \leq F(r, \pi/2) \\ &\frac{1-r}{2} \leq \frac{1-r}{2}. \end{aligned}$$

Plainly, this is true for all $r \in [0,1]$.

Case 3–$\theta \in [\pi, 3\pi/2]$:

$$A \leq^{**} A$$

$$F(G(A)) \le F(G(A))$$
$$F\left(r, (-\theta + 3\pi/2)\right) \le F\left(r, (-\theta + 3\pi/2)\right)$$
$$\frac{1}{2} + r \cdot \left(\frac{1}{2} - \frac{2 \cdot (3\pi/2 - \theta)}{\pi}\right) \le \frac{1}{2}$$
$$+ r \cdot \left(\frac{1}{2} - \frac{2 \cdot (3\pi/2 - \theta)}{\pi}\right)$$
$$\frac{1}{2} + r \cdot \left(\frac{1}{2} - \frac{\pi - 2\theta}{\pi}\right) \le \frac{1}{2} + r \cdot \left(\frac{1}{2} - \frac{\pi - 2\theta}{\pi}\right)$$
$$\frac{1-r}{2} - \frac{2\theta r}{\pi} \le \frac{1-r}{2} - \frac{2\theta r}{\pi}.$$

Again, this is true for all $r \in [0,1]$ and $\theta \in [0,2\pi]$.

Case 4–$\theta \in [3\pi/2, 2\pi]$: The proof is similar to Case 1, and is omitted here. Thus, reflexivity is proven.

Anti-Symmetry:

Case 1–$\theta \in [0, \pi/2]$: This was proven in Theorem (2.5)

Case 2–$\theta \in [\pi/2, \pi]$:

Assume that $A \le^{**} B$ and $B \le^{**} A$ when $A \ne B$, $A = (r_a, \theta_a)$ and $B = (r_b, \theta_b)$, $A, B \in \boldsymbol{D}$. Then,

$$F(G(r_a, \pi/2)) \le F(G(r_b, \pi/2)) \wedge F(G(r_b, \pi/2))$$
$$\le F(G(r_a, \pi/2)) \wedge \neg(F(G(r_b, \pi/2)) = F(G(r_a, \pi/2)))$$
$$\Rightarrow \left(\frac{1}{2} - \frac{r_a}{2}\right) \le \left(\frac{1}{2} - \frac{r_b}{2}\right) \wedge \left(\frac{1}{2} - \frac{r_b}{2}\right)$$
$$\le \left(\frac{1}{2} - \frac{r_a}{2}\right) \wedge \neg\left(\left(\frac{1}{2} - \frac{r_a}{2}\right) = \left(\frac{1}{2} - \frac{r_b}{2}\right)\right)$$
$$\Rightarrow \left(\frac{1}{2} - \frac{r_a}{2}\right) \le \left(\frac{1}{2} - \frac{r_b}{2}\right)$$
$$\le \left(\frac{1}{2} - \frac{r_a}{2}\right) \wedge \neg\left(\left(\frac{1}{2} - \frac{r_a}{2}\right) = \left(\frac{1}{2} - \frac{r_b}{2}\right)\right).$$

This is a contradiction, so Case 2 is proven.

Case 3–$\theta \in [\pi, 3\pi/2]$:

Assume that $A \le^{**} B$ and $B \le^{**} A$ when $A \ne B$, $A = (r_a, \theta_a)$ and $B = (r_b, \theta_b)$, $A, B \in \boldsymbol{D}$. Then,

$$F\left(r_a, (-\theta_a + 3\pi/2)\right) \le F\left(r_b, (-\theta_b + 3\pi/2)\right)$$
$$\wedge F\left(r_b, (-\theta_b + 3\pi/2)\right) \le F\left(r_a, (-\theta_a + 3\pi/2)\right) \wedge$$
$$\neg\left(F\left(r_a, (-\theta_a + 3\pi/2)\right) = F\left(r_b, (-\theta_b + 3\pi/2)\right)\right)$$
$$\Rightarrow \left(\frac{1-r_a}{2} - \frac{2\theta_a r_a}{\pi}\right) \le \left(\frac{1-r_b}{2} - \frac{2\theta_b r_b}{\pi}\right)$$
$$\wedge \left(\frac{1-r_b}{2} - \frac{2\theta_b r_b}{\pi}\right) \le \left(\frac{1-r_a}{2} - \frac{2\theta_a r_a}{\pi}\right) \wedge$$
$$\neg\left(\left(\frac{1-r_a}{2} - \frac{2\theta_a r_a}{\pi}\right) = \left(\frac{1-r_b}{2} - \frac{2\theta_b r_b}{\pi}\right)\right)$$
$$\Rightarrow \left(\frac{1-r_a}{2} - \frac{2\theta_a r_a}{\pi}\right) \le \left(\frac{1-r_b}{2} - \frac{2\theta_b r_b}{\pi}\right)$$
$$\le \left(\frac{1-r_a}{2} - \frac{2\theta_a r_a}{\pi}\right) \wedge \neg\left(\left(\frac{1-r_a}{2} - \frac{2\theta_a r_a}{\pi}\right)\right.$$
$$\left.= \left(\frac{1-r_b}{2} - \frac{2\theta_b r_b}{\pi}\right)\right).$$

This is a contradiction, so Case 3 is proven.

Case 4–$\theta \in [3\pi/2, 2\pi]$:

Assume that $A \le^{**} B$ and $B \le^{**} A$ when $A \ne B, A = (r_a, \theta_a)$ and $B = (r_b, \theta_b)$, $A, B \in \boldsymbol{D}$. Then,

$$F(G(r_a, 0)) \le F(G(r_b, 0)) \wedge F(G(r_b, 0)) \le F(G(r_a, 0)) \wedge$$
$$\neg(F(G(r_b, 0)) = F(G(r_a, 0)))$$
$$\Rightarrow \left(\frac{1}{2} + \frac{r_a}{2}\right) \le \left(\frac{1}{2} + \frac{r_b}{2}\right) \wedge \left(\frac{1}{2} + \frac{r_b}{2}\right)$$
$$\le \left(\frac{1}{2} + \frac{r_a}{2}\right) \wedge \neg\left(\left(\frac{1}{2} + \frac{r_a}{2}\right) = \left(\frac{1}{2} + \frac{r_b}{2}\right)\right)$$
$$\Rightarrow \left(\frac{1}{2} + \frac{r_a}{2}\right) \le \left(\frac{1}{2} + \frac{r_b}{2}\right) \le \left(\frac{1}{2} + \frac{r_a}{2}\right) \wedge$$
$$\neg\left(\left(\frac{1}{2} + \frac{r_a}{2}\right) = \left(\frac{1}{2} + \frac{r_b}{2}\right)\right).$$

This is a contradiction, so Case 4 is proven.

Transitivity.

Given $A, B, C \in \boldsymbol{D}$, $A = (r_a, \theta_a), B = (r_b, \theta_b), C = (r_c, \theta_c)$.

Case 1–$\theta \in [0, \pi/2]$: This was proven in Theorem (2.5)

Case 2–$\theta \in [\pi/2, \pi]$:

$$F(G(r_a, \pi/2)) \le F(G(r_b, \pi/2)) \wedge F(G(r_b, \pi/2))$$
$$\le F(G(r_c, \pi/2))$$
$$\Rightarrow \left(\frac{1}{2} - \frac{r_a}{2}\right) \le \left(\frac{1}{2} - \frac{r_b}{2}\right) \wedge \left(\frac{1}{2} - \frac{r_b}{2}\right)$$
$$\le \left(\frac{1}{2} - \frac{r_c}{2}\right)$$
$$\Rightarrow \left(\frac{1}{2} - \frac{r_a}{2}\right) \le \left(\frac{1}{2} - \frac{r_b}{2}\right) \le \left(\frac{1}{2} - \frac{r_c}{2}\right)$$
$$\Rightarrow \left(\frac{1}{2} - \frac{r_a}{2}\right) \le \left(\frac{1}{2} - \frac{r_c}{2}\right).$$

By transitivity of the real numbers, Case 2 is proven.

Case 3–$\theta \in [\pi, 3\pi/2]$:

$$F\left(r_a, (-\theta_a + 3\pi/2)\right) \le F\left(r_b, (-\theta_b + 3\pi/2)\right)$$
$$\wedge F\left(r_b, (-\theta_b + 3\pi/2)\right) \le F\left(r_c, (-\theta_c + 3\pi/2)\right)$$
$$\Rightarrow \left(\frac{1-r_a}{2} - \frac{2\theta_a r_a}{\pi}\right) \le \left(\frac{1-r_b}{2} - \frac{2\theta_b r_b}{\pi}\right)$$
$$\wedge \left(\frac{1-r_b}{2} - \frac{2\theta_b r_b}{\pi}\right) \le \left(\frac{1-r_c}{2} - \frac{2\theta_c r_c}{\pi}\right)$$
$$\Rightarrow \left(\frac{1-r_a}{2} - \frac{2\theta_a r_a}{\pi}\right) \le \left(\frac{1-r_b}{2} - \frac{2\theta_b r_b}{\pi}\right)$$
$$\le \left(\frac{1-r_c}{2} - \frac{2\theta_c r_c}{\pi}\right)$$

$$\Rightarrow \left(\frac{1-r_a}{2} - \frac{2\theta_a r_a}{\pi}\right) \leq \left(\frac{1-r_c}{2} - \frac{2\theta_c r_c}{\pi}\right).$$

By transitivity of the real numbers, Case 3 is proven.

Case 4–$\theta \in [3\pi/2, 2\pi]$:

$$F(G(r_a,0)) \leq F(G(r_b,0)) \wedge F(G(r_b,0)) \leq F(G(r_c,0))$$
$$\Rightarrow \left(\frac{1}{2}+\frac{r_a}{2}\right) \leq \left(\frac{1}{2}+\frac{r_b}{2}\right) \wedge \left(\frac{1}{2}+\frac{r_b}{2}\right) \leq \left(\frac{1}{2}+\frac{r_c}{2}\right)$$
$$\Rightarrow \left(\frac{1}{2}+\frac{r_a}{2}\right) \leq \left(\frac{1}{2}+\frac{r_b}{2}\right) \leq \left(\frac{1}{2}+\frac{r_c}{2}\right)$$
$$\Rightarrow \left(\frac{1}{2}+\frac{r_a}{2}\right) \leq \left(\frac{1}{2}+\frac{r_c}{2}\right).$$

By transitivity of the real numbers, Case 4 is proven. This completes the proof of Theorem (3.2).

Theorem 3.3: (***D***, $\leq^{**}$) is a complete, distributive lattice.

Proof: 1) *Completeness:* We will show that any subset of **D** has a lower bound and an upper bound under the ordering $\leq^{**}$; this implies both that (**D**, $\leq^{**}$) is a lattice, and that it is complete. For any A, B $\in$ **D**, $A \leq^{**} B$ iff $F(G(A)) \leq F(G(B))$. The codomain of the mapping *F*(*G*()) is the real numbers, which is known to be a complete lattice under the ordering $\leq$. Thus, for any subset of **D**, there is necessarily a least element and a greatest element of the subset under $\leq^{**}$, and so the lattice is complete. *Corollary* The lattice (**D**, $\leq^{**}$) is bounded, with infimum (0+i) and supremum (1+0i).

2) *Distributivity:* Let us denote the meet of the lattice by $\cap$ and the join by $\cup$. For any $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2) \in$ **D**, the mapping *F*(*G*()) is real valued. Using the known properties of the real-valued ordering $\leq$, we have:

$$\begin{aligned} &A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\\ &\max(F(G(A)), \min(F(G(B)), F(G(C)))\\ &\quad = \min(\max(F(G(A)), F(G(B))),\\ &\qquad \max(F(G(A)), F(G(C))))\\ &\min(\max(F(G(A)), F(G(B))),\\ &\qquad \max(F(G(A)), F(G(C))))\\ &\quad = \min(\max(F(G(A)), F(G(B))),\\ &\qquad \max(F(G(A)), F(G(C)))).\end{aligned}$$

This completes the proof.

Theorem 3.3 is the key finding of this section, as it establishes that (***D***, $\leq^{**}$) is a complete, distributive lattice; in other words, the ordering $\leq^{**}$ defined by (2.6) and (3.1) holds over every element of the unit disc, not just the $\Pi - i$ numbers. Thus, when we form a CFL by treating the lattice join as the disjunction, and the lattice meet as the conjunction (as usual), we immediately have the properties of commutativity, associativity, distributivity, idempotency, and the boundary conditions for the lattice infimum (0+*i*) and supremum (1+0*i*) for all elements of the unit disc.

B. Example: Complex Fuzzy Inference

We now demonstrate one usage of the logic developed in this section, by using it in a complex fuzzy inference system. Specifically, we will re-examine an example of time-series forecasting from [4] (Section 3.4 in that paper), in which we carry out a one-step-ahead forecast for one observation of a time series (specifically, the Santa Fe A (Laser) dataset, a well-known chaotic time series often used as a benchmark for time series forecasting algorithms [39]). We first provide a brief overview of the operation of ANCFIS to establish the context of this example. We then modify the inference process of ANCFIS, combining it with the logic formed by (***D***, $\leq^{**}$), and execute the whole process on the example data point from [4].

1) Adaptive Neuro-Complex Fuzzy Inference System: ANCFIS is a neuro-complex-fuzzy system based on the well-known ANFIS architecture [17] (which implements a Takagi–Sugeno–Kang fuzzy inference system as an artificial neural network). In order to use CFL within the basic ANFIS framework, several changes are made in [4]. Firstly, the fuzzy sets representing rule antecedents are replaced with CFS of the form

$$r(\theta) = d \cdot \sin\left(a \cdot (\theta = t) + b\right) + c \tag{3.2}$$

where $t \in X$ is an element of the universal set *X*, while θ is the phase and *r* the magnitude of a complex membership. An input to ANCFIS consists of a window of a time series (several consecutive observations), which is convolved with each CFS. The CFS is first sampled, and the samples converted to Cartesian coordinates:

$$\theta_k = \frac{2\pi}{n}k \tag{3.3}$$
$$x_k = r_k \cos\left(\theta_k\right) \tag{3.4}$$
$$y_k = ir_k \sin\left(\theta_k\right) \tag{3.5}$$

The samples are then convolved with the input window *t*():

$$h\left(k\right) = \sum_j t\left(j\right) g\left(k+1-j\right) \tag{3.6}$$

$$\begin{aligned} &g\left(k+1-j\right) = x_{k+1-j} + iy_{k+1-j} =\\ &r_{k+1-j}\cos(\theta_{k+1-j}) + ir_{k+1-j}\sin(\theta_{k+1-j}) \end{aligned} \tag{3.7}$$

$$\text{conv} = \sum_{k=1}^{2n-1} h(k) = \sum_{k=1}^{2n-1} \sum_{j=\max(1,k+1-n)}^{\min(k,n)} t(j) g\left(k+1-j\right). \tag{3.8}$$

Finally, the convolution sum *conv* is restricted to the unit disc via the Elliot function:

$$E\left(z\right) = \frac{z}{1+|z|} \tag{3.9}$$

for *z* a complex number. This is the output of Layer 1. Layer 2 implements a complex fuzzy conjunction between the antecedents of a single complex fuzzy rule; in [4] this is the algebraic product, determined to be a complex fuzzy conjunction in [22]:

$$O_{2,j} = \prod_i O_{1,i}, \quad i = 1, 2, \ldots, |O_1| \tag{3.10}$$

where $O_{1,i}$ is the output of the i-th node in Layer 1, and $|O_1|$ is the number of nodes in Layer 1. Due to the input window representation we are using, each time series variate is a single input variable to ANCFIS; this means that for a univariate time series such as Santa Fe A, the second layer actually reduces to the identity function. In Layer 3, a second normalization is used to ensure that the firing degree of the complex fuzzy rule remains in the unit circle:

$$O_{3,i} = \bar{w}_i = \frac{O_{2,i}}{\sum_{j=1}^{|O_2|} |O_{2,j}|}, \quad i = 1, 2, \ldots, |O_2|. \tag{3.11}$$

A new layer is now added to implement the concept of *rule interference*; the idea that the firing strengths of different rules may interfere constructively or destructively with each other. In ANCFIS this is accomplished by summing all firing strengths, and then taking the dot product of each rule's firing strength against that sum:

$$O_{4,i} = w_i^{DP} = \bar{w}_i \bullet \sum_{j=1}^{|O_3|} \bar{w}_j, \quad i = 1, 2, \ldots, |O_3|. \tag{3.12}$$

The fifth layer implements the consequent function. In ANCFIS, this is a linear function of the components of the input window:

$$O_{5,i} = w_i^{DP} \left[\sum_{j=1}^{n} p_{i,j} t_j + r_i \right]. \tag{3.13}$$

In the sixth layer, the final network output is computed as the sum of all layer 5 outputs.

The inference process we will examine for the logic of ($\boldsymbol{D}$, $\leq^{**}$) is similar to that of ANCFIS. We again use the convolution sum of (3.6)–(3.8), between an input window and sampled membership functions of the form of (3.2), although we do not normalize it using the Elliot function. The resulting complex fuzzy membership grades are mapped to the $\Pi - i$ numbers using the mapping G of Definition (3.1). We will use the meet of ($\boldsymbol{D}$, $\leq^{**}$) as the conjunction between antecedents in a complex fuzzy rule. We again remove the Layer 3 normalization, and proceed with the rule interference step of Layer 4, the consequent function of Layer 5, and find the overall output via the summation of Layer 6.

2) Numerical Example: The example we will use is the chronologically first input window from the Santa Fe A dataset, which (after normalization during preprocessing of the dataset) is $\boldsymbol{x}$ = (0.3373, 0.5529, 0.3725, 0.1608, 0.0863, 0.0824, 0.1255, 0.2824), with an actual next value of 0.5412. As the window size is eight, the CFS will be sampled at $\theta = \frac{7\pi}{4}, \frac{6\pi}{4}, \ldots, 0$. In this example, there are two fuzzy sets (and hence two rules since the time series is univariate). The parameters of the first CFS are $a_1 = 0.3333$, $b_1 = 0.2313$, $c_1 = 0.6309$, $d_1 = 0.1948$, and for the second node $a_2 = 0.6265$, $b_2 = 0.6145$, $c_2 = 0.1237$, $d_2 = 0.0459$. The node 1 convolution sum is thus (–0.4202 –0.2656i), and in node 2 (–0.0499 + 0.2911i), or $\mu_1 = 0.4971 \angle \pi{+}0.5636$ and $\mu_2 = 0.2659 \angle \pi - 1.4011$, respectively (using the notation $r \angle \theta$ to denote the polar form of a complex number). When we now apply the mapping G, we obtain:

$$G(\mu_1) = 0.4971\angle\frac{\pi}{2} - 0.5636 = (0.2656 + 0.4202i) \tag{3.14}$$

$$G(\mu_2) = 0.2659\angle\frac{\pi}{2} = (0 + 0.2659i). \tag{3.15}$$

We now execute the rule interference step. Summing (3.14) and (3.15) we obtain (0.2656 + 0.6858i). Taking the dot product between this sum and each of (3.14) and (3.15) yields the final firing strengths for each rule:

$$w_1 = G(\mu_1) \cdot (G(\mu_1) + G(\mu_2)) = 0.3587 \tag{3.16}$$

$$w_2 = G(\mu_2) \cdot (G(\mu_1) + G(\mu_2)) = 0.1823. \tag{3.17}$$

The consequent parameters in ANCFIS, as with ANFIS, are determined by least-squares estimation across all the training patterns in the forward pass. In the example from [4], we take these estimated parameters and apply them in the consequent function unchanged; in the current example, we do the same. The parameters of the consequent functions are $\boldsymbol{p}_1$ = (0.6956, 0.7509, –0.3487, 0.0666, –0.7006, 0.5378, –1.5935, 1.4762), $r_1 = 0.4155$ in rule 1, and $\boldsymbol{p}_2$ = (–0.4407, –0.1057, –0.2254, –0.2878, 0.5020, –1.0027, 1.4191, –1.8481), $r_2 = 0.1167$ in rule 2. Thus, the consequents are:

$$\mathrm{rule}_1 = \left[\sum_{j=1}^{n} p_{1,j} t_j + r_1 \right] = 1.1468 \tag{3.18}$$

$$\mathrm{rule}_2 = \left[\sum_{j=1}^{n} p_{2,j} t_j + r_2 \right] = -0.6036. \tag{3.19}$$

Taking the weighted sum of the consequents for the final output, we thus have

$$w_1 \cdot \mathrm{rule}_1 + w_2 \cdot \mathrm{rule}_2 = 0.3013. \tag{3.20}$$

The forecast error for this example is 0.2398. By contrast, the ANCFIS network in [4] obtained an output of 0.2527, giving a forecast error of 0.2885. We thus obtain a substantially lower forecast error on this example using the logic of ($\boldsymbol{D}$, $\leq^{**}$).

IV. Complex Fuzzy Sets: Non-Membership Versus Anti-Membership

We will next examine the case where anti-membership and non-membership have different meanings. We first construct a complete, distributive lattice over the unit disc $\boldsymbol{D}$ that expresses this property, and then find that the intuitionistic complement forms a DeMorgan triple with the join and meet of that lattice. Having established that a logic that distinguishes between anti-membership and non-membership exists, we then turn our attention to what that logic might represent. We suggest that it might be a vehicle for representing linguistic antonym and negation in fuzzy systems. Examining the literature on modern linguistics, we find several properties that a common framework for antonym and negation should have, and show that they are consistent with the logic developed in this section.

A. Lattice Over the Unit Disc

If we now assume that anti-membership and non-membership are not equivalent, then we cannot collapse the unit disc into the positive quarter-circle as in Section III. Instead, both membership and non-membership draw truth values from the whole range [–1,1]. One particular feature of this range is that the zero element of the real line occurs at the center of this interval. The implication of this fact is that a positive linear transform from [0,1] to [–1,1] is not an isomorphism, because the relative location of the *zero* element is altered. Thus, simply mapping t-norms or conorms from [0,1] is not an effective approach in defining meets and joins for CFS. We will first define a pair of functions, absmin and absmax. These functions compare the absolute values of their arguments, with an additional rule for the case of equal absolute values. We prove that associativity and distributivity hold for these functions in Lemmas (4.1) and (4.2). With these properties proven, we define a complex fuzzy union and intersection, and in Theorems (4.1) and (4.2) we show that the lattice they form is complete and distributive, with infimum (0,1) and supremum (1,0). We then investigate two possible complex fuzzy negations: the intuitionistic complement, and vector negation of a complex number. We prove that the intuitionistic complement meets the axioms of a fuzzy negation in Theorem (4.3); it is trivial that vector negation does not (the boundary conditions are not satisfied). In Theorem (4.4) we show that the intuitionistic negation forms a DeMorgan triple with the complex fuzzy union and intersection in Theorem (4.1) and (4.2) (i.e. these three operators satisfy the DeMorgan laws [23]), and we give an counter-example showing that vector negation does not.

Definition 4.1: The function absmax is given by:

$$\operatorname{absmax}(x,y) = \begin{cases} x, & \text{if } |x| > |y| \\ y, & \text{if } |x| < |y| \\ |x|, & \text{if } |x| = |y| \wedge x \neq y. \end{cases} \tag{4.1}$$

Definition 4.2 : The function absmin is given by:

$$\operatorname{absmin}(x,y) = \begin{cases} x, & \text{if } |x| < |y| \\ y, & \text{if } |x| > |y| \\ -|x|, & \text{if } |x| = |y| \wedge x \neq y. \end{cases} \tag{4.2}$$

The functions absmin() and absmax() order the elements $x,y \in [-1,1]$ by their absolute values, with an additional rule that, in the case of equal absolute values, the positive argument is greater than the negative one. absmin() and absmax() are further assumed to be idempotent, as with the *min* and *max* operations, and are clearly commutative. Other authors have investigated this approach of *symmetrizing* functions in the range [0,1], with perhaps the best-known being [40]. That article concentrated on forming operators over ordinal scales that possess as many properties of a ring as possible; the resulting operators were not associative. Our absmin() and absmax() operators, on the other hand, are associative, as we show in Lemma (4.1).

Lemma 4.1 : The functions absmax() and absmin(), defined in Definitions (4.1) and (4.2), respectively, are associative.

Proof: We prove associativity for absmax(); the proof for absmin() is similar and omitted here. By definition, associativity implies

$$\operatorname{absmax}(a, \operatorname{absmax}(b,c)) = \operatorname{absmax}(\operatorname{absmax}(a,b), c) \tag{4.3}$$

for any $a,b,c \in \boldsymbol{R}$. Let us first consider the case where the absolute values of a,b,c are unequal. From Definition (4.1), it is plain that both the left and right hand sides of (4.3) will reduce to whichever of a, b, or c has the greatest absolute value, as the *max*() operator is known to be associative.

Now let us consider the case where the absolute values of a, b, c are equal, but their signs may differ. In this situation, the absolute value of both the LHS and RHS is known; all that remains to be determined is the sign of each side. Consider first the case of a being positive, while b and c are negative. On the left-hand side, absmax(b, c) will be negative, but since a is positive, the LHS evaluates to positive. On the RHS, if a is positive then absmax(a, b) must be positive, and thus the RHS is positive. This illustrates the more general conclusion: when absolute values are equal, the LHS and RHS in (4.3) are negative if and only if a, b, c are all negative.

Finally, consider the case where a subset of a, b, c have equal absolute values. In this situation, the outcome depends on whether the absolute value of the remaining argument (let us arbitrarily select a for the sake of discussion) is greater or lesser than the absolute value of the others. If $|a| < |b|$ and $|c|$, then the LHS and RHS of (4.3) will evaluate to $|b|$ if either b or c is positive, or $-|b|$ if both are negative. On the other hand, if $|a| > |b|$ and $|c|$, then both the LHS and RHS of (4.3) evaluate to a. Clearly, the same argument holds for $|a| = |b| \neq |c|$, and $|a| = |c| \neq |b|$ as well. All cases have now been enumerated, and associativity holds for all; thus, absmax() is associative.

In addition to associativity, in Lemma (4.2) we also show that absmin() and absmax() are distributive over each other.

Lemma 4.2: The operator absmax() distributes over absmin(), and vice versa, where absmax() and absmin() are given in Definitions (4.1) and (4.2), respectively.

Proof: We show that absmin() distributes over absmax(); the dual proof is similar, and is omitted here. Throughout the proof, a, b, $c \in \boldsymbol{R}$. Distributivity of absmin() over absmax() is defined as:

$$\operatorname{absmin}(a, \operatorname{absmax}(b,c)) = \\ \operatorname{absmax}(\operatorname{absmin}(a,b), \operatorname{absmin}(a,c)). \tag{4.4}$$

Consider first the case where the absolute values of a, b, and c are unequal. By Definitions (4.1) and (4.2), (4.4) reduces to:

$$\min(|a|, \max(|b|, |c|)) = \max(\min(|a|, |b|), \min(|a|, |c|).$$

From the known properties of the max and min operators, this equation is true.

Next, consider the case where the absolute values of a, b, and c are equal, but signs may differ. The LHS of (4.4) is always negative if a is negative (irrespective of the sign of b and c); or if a is positive, and b and c are both negative. If a is positive and at least one of b, c are positive, then the LHS is positive. The exact same relationships clearly hold for the RHS of (4.4), and so this case is proven by enumeration.

Finally, consider the case where a subset of a, b, c have equal absolute values. In this situation, the outcome depends on whether the absolute value of the remaining argument (let us arbitrarily select a for the sake of discussion) is greater or lesser than the absolute value of the others. If $|a| < |b|$ and $|c|$, then the LHS of (4.4) is obviously a, while the RHS reduces to absmax$(a, a) = a$. On the other hand, if $|a| > |b|$ and $|c|$, the RHS obviously reduces to absmax(b,c), which equals $|b|$ unless both b and c are negative. Meanwhile, the LHS equals $|b|$ unless both b and c are negative; the sign of a is irrelevant. This completes the proof.

We use these two functions to construct conjunction and disjunction operators for CFL, again interpreting the two components of each vector as membership and non-membership degrees, as follows.

Definition 4.3 : The complex fuzzy conjunction $\wedge$ for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \boldsymbol{D}$ is given by:

$$x \wedge y = (\text{absmin}(x_1, y_1), \text{absmax}(x_2, y_2)). \tag{4.5}$$

Definition 4.4 : The complex fuzzy disjunction $\vee$ for $x = (x_1,x_2)$, $y = (y_1,y_2) \in \boldsymbol{D}$ is given by:

$$x \vee y = (\text{absmax}(x_1, y_1), \text{absmin}(x_2, y_2)). \tag{4.6}$$

Theorem 4.1 : The conjunction and disjunction defined in Definitions (4.3) and (4.4) form a lattice over $\boldsymbol{D}$, with $\wedge$ the meet and $\vee$ the join.

Proof: Commutativity and idempotence are trivial from Definitions (4.1) through (4.4), while associativity follows directly from Lemma 4.1. We thus need only prove absorption. Throughout the proof, $A = (a_1, a_2)$, $B = (b_1, b_2) \in \boldsymbol{D}$. By the definition of absorption,

$$A \vee (A \wedge B) = A$$
$$(\text{absmax}(a_1, \text{absmin}(a_1, b_1)),$$
$$\text{absmin}(a_2, \text{absmax}(a_2, b_2))) = (a_1, a_2).$$

Consider first the case where the absolute values of a_1 and b_1 (and a_2 and b_2; the components of the complex numbers may be considered independent for this proof) are different. If $|a_1| < |b_1|$, then absmin(a_1, b_1) will equal a_1, and then absmax$(a_1, a_1) = a_1$. On the other hand, if $|a_1| > |b_1|$, then absmin$(a_1, b_1) = b_1$, but absmax$(a_1, b_1) = a_1$. A similar argument holds for the second (non-membership) component.

Next, consider the case where the absolute values of a_1 and b_1 are equal but signs may differ. Consider the first (membership) component. If b_1 is positive, then absmin(a_1,b_1) will equal a_1 (it can only be positive if both a_1 and b_1 are positive), and thus the first component would equal a_1. If b_1 is negative then absmin(a_1,b_1) equals b_1—but then absmax$(a_1$,absmin$(a_1,b_1))$ must necessarily equal a_1, so once again the first component is a_1. Again, a similar argument holds for the second (non-membership) component, and so absorption is proven. The proof for $A \wedge (A \vee B)$ is similar, and is omitted here.

Theorem 4.2 : The lattice $(\boldsymbol{D}, \wedge, \vee)$ is (i) bounded, (ii) complete, and (iii) distributive

Proof 1) *Boundedness*

For any $A = (a_1,a_2) \in \mathbf{D}$:

$$A \vee (0, 1) = (\text{absmax}(a_1, 0), \text{absmin}(a_2, 1))$$
$$= (a_1, a_2)$$
$$A \wedge (1, 0) = (\text{absmin}(a_1, 1), \text{absmax}(a_2, 0))$$
$$= (a_1, a_2).$$

2) *Completeness*

Note that for any two-element subset of $\mathbf{D}$, the join and meet are both defined. By idempotence and associativity in Theorem (4.1), this result extends to all subsets of $\mathbf{D}$. Thus, the lattice is complete.

3) *Distributivity*

For any $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2) \in \mathbf{D}$:

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$
$$\Rightarrow (\text{absmin}(a_1, \text{absmax}(b_1, c_1)),$$
$$\text{absmax}(a_2, \text{absmin}(b_2, c_2)))$$
$$= (\text{absmax}(\text{absmin}(a_1, b_1), \text{absmin}(a_1, c_1)),$$
$$\text{absmin}(\text{absmax}(a_2, b_2), \text{absmax}(a_2, c_2))).$$

By Lemma (4.2), the above equation is always true.

B. Complex Fuzzy Negation

Negation of PFS is defined in (2.4) as the exchange of membership and non-membership values, just as in intuitionistic fuzzy sets. CFS, however, also include the inverse of both membership and non-membership grades, and so we must examine 1) whether the Pythagorean negation remains a proper fuzzy negation over $\boldsymbol{D}$; 2) whether it forms a DeMorgan triple with the intersection and union of Definitions (4.3) and (4.4); 3) whether the complex negation $\sim x = -x$ might also form a fuzzy negation and be a DeMorgan triple with (4.3) and (4.4).

Theorem 4.3 : The Pythagorean fuzzy complement $\neg$ of (2.4) is monotonic, involutive, and continuous over $\boldsymbol{D}$, and satisfies the boundary conditions $\neg(0,1) = (1,0)$ and $\neg(1,0) = (0,1)$.

Proof: Involution and continuity are trivial, and so we will only prove monotonicity and the boundary conditions.

1) *Monotonicity:* Consider the ordering of the unit disc $\boldsymbol{D}$ implied by the union and intersection of Definitions (4.3–4.4) Let $a = (a_1 + a_2 i)$, $b = (b_1 + b_2 i)$ be two complex numbers drawn from $\boldsymbol{D}$, such that $a < b$ (i.e. $a \wedge b = a$, and $a \vee b = b$). Monotonicity therefore implies that $\neg a$

$> \neg b$, or equivalently that $\neg a \wedge \neg b = \neg b$, and $\neg a \vee \neg b = \neg a$. For any a, b as above, we find that

$$\neg a = (a_2 + a_1) \text{ and } \neg b = (b_2 + b_1 i)$$
$$a < b \Rightarrow (a_1 < b_1) \text{ and } (a_2 > b_2)$$
$$\Rightarrow (\text{absmax}(a_2, b_2) + i \cdot \text{absmin}(a_1, b_1)) = a_2 + a_1 i$$
$$\Rightarrow \neg a \vee \neg b = \neg a$$
$$\Rightarrow \neg a > \neg b.$$

Thus, monotonicity is proven.

2) *Boundary Conditions:* As proven in Theorem (4.2), the supremum and infimum of the lattice ($\boldsymbol{D}$, $\wedge$, $\vee$) are $\mathbf{1} = (1+0i)$ and $\mathbf{0} = (0+1i)$, respectively. Plainly, $\neg\mathbf{1} = \mathbf{0}$ and $\neg\mathbf{0} = \mathbf{1}$, and so the boundary conditions are proven. This completes the proof of Theorem (4.3), and so the mapping $\neg$ satisfies all the properties of a fuzzy complement.

Theorem 4.4 : The operations $\neg$, $\wedge$, $\vee$ given by (2.4), Definitions (4.3), (4.4), respectively, form a De Morgan triple over the unit disc $\boldsymbol{D}$.

Proof: Consider $x = (x_1+x_2 i)$, $y = (y_1+y_2 i) \in \boldsymbol{D}$.

$$\neg x \wedge \neg y = \neg(x \vee y)$$
$$\text{absmin}(x_2, y_2) + i \cdot \text{absmax}(x_1, y_1)$$
$$= \neg(\text{absmax}(x_1, y_1) + i \cdot \text{absmin}(x_2, y_2))$$
$$\text{absmin}(x_2, y_2) + i \cdot \text{absmax}(x_1, y_1)$$
$$= \text{absmin}(x_2, y_2) + i \cdot \text{absmax}(x_1, y_1).$$

The dual proof is similar, and so Theorem (4.4) is proven.

As for complex negation, this operation does not yield a De Morgan triple when combined with Definitions (4.3–4.4). As an example, consider $x = (0.1 + 0.2i)$ and $y = (0.15 + 0.05i)$. We then have:

$$\neg x \wedge \neg y = \neg(x \vee y)$$
$$\text{absmin}(-x_1, -y_1) + i \cdot \text{absmax}(-x_2, -y_2)$$
$$= \neg(\text{absmax}(x_1, y_1) + i \cdot \text{absmin}(x_2, y_2))$$
$$\text{absmin}(-0.1, -0.15) + i \cdot \text{absmax}(-0.2, -0.05)$$
$$= \neg(\text{absmax}(0.1, 0.15) + i \cdot \text{absmin}(0.2, 0.05))$$
$$-0.1 - 0.2i = -0.15 - 0.05i.$$

This is a contradiction, and so complex negation does not form a De Morgan triple with Definitions (4.3) and (4.4).

C. Interpretation of Complex Fuzzy Sets

We now consider the interpretation of a CFS membership when anti-membership and non-membership are different. An obvious place to start is interpreting one as representing the negation of a predicate, and the other as representing the antonym of a predicate. Fuzzy systems researchers have considered the distinction between antonym and negation at length (see for example [41]–[48]), guided by research into antonymy in modern linguistics [49], [50]. The antonyms and negations of interest to fuzzy systems researchers seem to be what Aristotle referred to as Contraries and Contradictories, respectively. Contraries are predicates whose assertion implies the denial of its opposite, but for which both may simultaneously be false. Negating a Contradictory, on the other hand, implies the assertion of its opposite (one or the other must be true) [49]. Commonly, antonyms in a linguistic variable are opposing terms in a defined scale (understood as an ordered set of discrete measurement values, expressed linguistically); [42] calls these "collective nouns," while [49] calls them "gradable antonyms." This intuitive meaning also underlies the notion of "opposite" fuzzy sets [46], and is implicit in [44] and [48]. Negations are a distinct concept as argued by linguists and demonstrated by Turksen in [47]. Furthermore, negation implicitly defines two categories: things that belong to a concept, and things that do not. While antonyms also implicitly define these categories, they also define a third: things neither belong to a concept, nor belong to its opposite. This "zone of indifference" is commonly accepted as a feature of antonyms by researchers in both fields [43], [45], [49]. Furthermore, while negations are generally required to be involutive, antonyms might have the weaker property that "an antonym of an antonym is a synonym," i.e. returning a synonym of the original term rather than the original term itself [51]. However, the two concepts are not wholly separate; [41], [43] asserts that antonym implies negation, but not vice versa, and claims that "the antonym of the negation is the negation of the antonym," while [52] asserts that *coherence with negation* is a necessary property of antonyms.

It seems plain that anti-membership and non-membership can be used to model antonym and negation, in a way that was not previously possible with type-1 fuzzy sets. Mathematically, anti-membership and non-membership are orthogonal concepts, whose composition can be assumed to be commutative. That is, taking the negation of an antonym, or the antonym of a negation (irrespective of whether antonym means anti-membership or non-membership), would in both cases yield the anti-non-membership of the original concept. This also means that negation does not imply antonymity. Antonymity implying negation can then be built into the semantics of a complex linguistic variable, as can coherence between antonym and negation and the "zone of indifference." While these last three properties are not implicit in our interpretation of CFS, we stress that they are *compatible* with it.

To complete our interpretation of CFS, we now have to decide whether non-membership represents negation and anti-membership represents antonym, or vice-versa. Recall that in Section 4.2 we showed that the Pythagorean negation formed a DeMorgan triple with the intersection and union of Definitions (4.3) and (4.4), while the complex negation did not. Given this result, it seems that we must treat non-membership as negation and anti-membership as antonym. This is also consistent with the mathematical analysis in [41]. In that article, antonyms were considered an "inner" operation, while negation is an "external" operation. This means that antonyms map the membership of a predicate onto the complement of the predicate's support, while negation inverts the predicate's membership on its current support. From the definition of the Pythagorean negation, we exchange the membership and non-membership components of the CFS membership, without altering the support of the CFS, matching the semantics of the "external" negation in [41]. As for the "inner" operation of antonymy, there will again need to be additional semantics in a complex linguistic variable to map anti-membership in one predicate to membership in another. Again, we stress that our interpretation is fully compatible with such semantics, even if they are not "baked in."

V. Summary and Future Work

In this article, we have examined the lattice-theoretic properties of PFS, and then extended them to the unit disc of the complex plane. We found that the operations proposed for PFS in [30] yielded two different complete and distributed lattices, and that extending these ideas to the unit disc also produced two additional complete and distributed lattices; these form two new instances of CFS theory (or isomorphically, two new examples of CFL). We also found that the Pythagorean negation forms a DeMorgan triple with one of these lattices. We then offered an interpretation of this latter as a framework for the semantics of antonym and negation. In particular, we found that multiple fundamental linguistic properties of antonym and negation are implicit in this interpretation of CFS, and several others are compatible with it.

Our future work in this area will follow three major strands of research in CFS and logic. First, we will explore how our new complex fuzzy operations perform in creating complex fuzzy models of real-world phenomena, through machine-learning architectures that incorporate CFS and CFL. We will focus on modifying our ANCFIS architecture [4] to use the new operators, and comparing the resulting system to our previous results. Second, we will explore the mathematical-logic implications of these new operations; in particular, we will develop a propositional logic based on anti-membership and non-membership. Thirdly, we will explore the development of complex linguistic variables incorporating the semantics we identified as being necessary in Section 4.3. This will be followed by development of a complex fuzzy inferential system that accommodates antonym and negation in the fuzzy rulebase. Additionally, we will continue to investigate additional complex fuzzy operations; bipolar fuzzy sets [20], in particular, appear to be a promising framework for creating new complex fuzzy operations.

Acknowledgment

The authors would like to thank the anonymous reviewers, whose comments significantly improved the paper.

References

[1] D. Ramot, M. Friedman, G. Langholz, and A. Kandel, "Complex fuzzy logic," *IEEE T. Fuzzy Syst.*, vol. 11, pp. 450–461, 2003.

[2] D. Ramot, R. Milo, M. Friedman, and A. Kandel, "Complex fuzzy sets," *IEEE T. Fuzzy Syst.*, vol. 10, pp. 171–186, 2002.

[3] J. Man, Z. Chen, and S. Dick, "Towards Inductive Learning of Complex Fuzzy Inference Systems," presented at the Proc. NAFIPS Int. C., San Diego, CA, USA, 2007.

[4] Z. Chen, S. Aghakhani, J. Man, and S. Dick, "ANCFIS: A Neuro-Fuzzy Architecture Employing Complex Fuzzy Sets," *IEEE Trans Fuzzy Syst*, vol. 19, pp. 305–322, 2011.

[5] S. Aghakhani and S. Dick, "An on-line learning algorithm for complex fuzzy logic," presented at the Proc. World Conf. Computational Intelligence, Barcelona, Spain, 2010.

[6] C. Li, "Complex Neuro-Fuzzy ARIMA Forecasting. A New Approach Using Complex Fuzzy Sets," *IEEE Trans Fuzzy Syst*, vol. 21, pp. 567–584, 2013.

[7] C. Li and T.-W. Chiang, "Complex Fuzzy Computing to Time Series Prediction A Multi-Swarm PSO Learning Approach" *Lecture Notes in Computer Science*, vol. 6592, 2011, pp. 242–251.

[8] J. Ma, G. Zhang, and J. Lu, "A method for multiple periodic factor prediction problems using complex fuzzy sets," *IEEE T. Fuzzy Syst.*, vol. 20, pp. 32–45, 2012.

[9] C. Li and T.-W. Chiang, "Intelligent financial time series forecasting: A complex neuro-fuzzy approach with multi-swarm intelligence," *International Journal of Applied Mathematics and Computer Science*, vol. 22, pp. 787–800, 2012.

[10] C. Li, T.-W. Chiang, and L.-C. Yeh, "A novel self-organizing complex neuro-fuzzy approach to the problem of time series forecasting," *Neurocomputing*, 2012.

[11] O. Yazdanbakhsh, A. Krahn, and S. Dick, "Predicting solar power output using complex fuzzy logic," presented at the Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, AB, 2013.

[12] C. Li and T.-W. Chiang, "Function Approximation with Complex Neuro-Fuzzy System Using Complex Fuzzy Sets. A New Approach" *New Generation Computing*, vol. 29, 2011, pp. 261–276.

[13] C. Li and T.-W. Chiang, "Complex fuzzy model with PSO-RLSE hybrid learning approach to function approximation," *Int. J. Int. Inf. Database* 926 *Syst.,* vol. 5, pp. 409–430, 2011.

[14] C. Li and T.-W. Chiang, "Complex Neuro-Fuzzy Self-learning Approach to Function Approximation," *Lecture Notes in Computer Science*, vol. 5991, 2010, pp. 289–299.

[15] C. Li and F. Chan, "Complex-Fuzzy Adaptive Image Restoration . An Artificial-Bee-Colony-Based Learning Approach" *Lecture Notes in Computer Science*, vol. 6592, 2011, pp. 90–99.

[16] C. Li, T. Wu, and F.-T. Chan, "Self-learning complex neuro-fuzzy system with complex fuzzy sets and its application to adaptive image noise canceling," *Neurocomputing*, vol. 94, pp. 121–139, 2012.

[17] J.-S. R. Jang, "ANFIS: adaptive-network-based fuzzy inference system," *IEEE T. Syst., Man, Cybern.*, vol. 23, pp. 665–685, 1993.

[18] J.-S. R. Jang, C.-T. Sun, and E. Mizutani, *Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence* Englewood Cliffs, NJ, USA: Prentice-Hall, 1997.

[19] T. Mitchell, *Machine Learning.* New York, NY: McGraw-Hill, 1997.

[20] B. R. Gaines, "Fuzzy reasoning and the logics of uncertainty," presented at the Proc. Int. Symp. Multiple-Valued Logic, Logan, Utah, USA, 1976.

[21] X. Zhiwei, "Multivalued logic and fuzzy logic . their relationship, minimization, and application to fault diagnosis," *IEEE Trans. Comput.,* vol. 33, pp. 679–681, 1984.

[22] S. Dick, "Towards Complex Fuzzy Logic," *IEEE T. Fuzzy Syst.*, vol. 13, pp. 405–414, 2005.

[23] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications.* Upper Saddle River, NJ: Prentice Hall PTR, 1995.

[24] E. Turunen, "Algebraic structures in fuzzy logic," *Fuzzy Sets Syst.,* vol. 52, pp. 181–188, 1992.

[25] L. A. Zadeh, "Fuzzy sets," *Inf. Control,* vol. 8, pp. 338–353, 1965.

[26] J. A. Gougen, "L-fuzzy sets," *J. Math. Anal. Appl.,* vol. 18, pp. 145–174, 1967.

[27] G. Zhang, T. S. Dillon, K.-Y. Cai, J. Ma, and J. Lu, "Operation properties and delta-equalities of complex fuzzy sets," *Int. J. Approx. Reason.,* vol. 50, pp. 1227–1249, 2009.

[28] A. U. M. Alkouri and A. R. Salleh, "Linguistic variables, hedges and several distances on complex fuzzy sets," *J. Intell. Fuzzy Syst.,* In Press.

[29] K. T. Atanassov, "Intuitionistic Fuzzy Sets," *Fuzzy Sets Syst.,* vol. 20, pp. 87–96, 1986.

[30] R. R. Yager and A. M. Abbasov, "Pythagorean Membership Grades, Complex Numbers, and Decision Making," *Int. J. Intelligent Systems*, vol. 28, pp. 436–452, 2013.

[31] R. R. Yager, "Pythagorean membership grades in multi-criteria decision making," *IEEE Trans Fuzzy Syst.*, vol. 22, pp. 958–965, 2014.

[32] D. E. Tamir, L. Jin, and A. Kandel, "A new interpretation of complex membership grade," *Int. J. Intell. Syst.,* vol. 26, pp. 285–312, 2011.

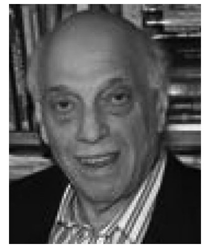
[33] D. E. Tamir and A. Kandel, "Axiomatic theory of complex fuzzy logic and complex fuzzy classes," *Int. J. Comput., Commun. Control,* vol. 4, pp. 562–576, 2011.

[34] D. E. Tamir, M. Last, and A. Kandel, "Generalized complex fuzzy propositional logic," presented at the Proc. World C. Soft Computing, San Francisco, CA, USA, 2011.
[35] D. E. Tamir, M. Last, and A. Kandel, "The Theory and Applications of Generalized Complex Fuzzy Propositional Logic," in *Soft Computing: State of the Art Theory and Novel Applications*, ed: Springer, 2013, pp. 177–192.
[36] D. E. Tamir, H. N. Teodorescu, M. Last, and A. Kandel, "Discrete complex fuzzy logic," in *Fuzzy Information Processing Society (NAFIPS), 2012 Annual Meeting of the North American*, 2012, pp. 1–6.
[37] D. Moses, O. Degani, H.-N. Teodorescu, M. Friedman, and A. Kandel, "Linguistic coordinate transformations for complex fuzzy sets," presented at the Proc. IEEE Int. C. Fuzzy Systems, Seoul, Korea, 1999.
[38] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*. Boston, MA, USA: Cambridge University Press, 2002.
[39] A. Weigend. (2008). *The Santa Fe Time Series Competition Data*. http://www-psych.stanford.edu/~andreas/Time-Series/SantaFe.html
[40] M. Grabisch, "The Mobius transform on symmetric ordered structures and its application to capacities on finite sets," *Discrete Mathematics*, vol. 287, pp. 17–34, 2004.
[41] A. R. de Soto and E. Trillas, "On antonym and negate in fuzzy logic," *Int. J. Intell. Syst.,* vol. 14, pp. 295–303, 1999.
[42] I. Garcia-Honrado and E. Trillas, "An essay on the linguistic roots of fuzzy sets," *Information Sciences*, vol. 181, pp. 4061–4074, 2011.
[43] E. Trillas, C. Moraga, S. Guadarrama, S. Cubillo, and E. Castineira, "Computing with Antonyms," in *Forging New Frontiers: Fuzzy Pioneers I*, M. Nikravesh, *et al.*, Eds., ed Berlin, Germany: Springer, 2007, pp. 133–153.
[44] V. Torra, "Aggregation of linguistic labels when semantics is based on antonyms," *Int. J. Intell. Syst.,* vol. 16, pp. 513–524, 2001.
[45] V. Novak, "Antonyms and linguistic qualifiers in fuzzy logic," *Fuzzy Sets & Systems*, vol. 124, pp. 335–351, 2001.
[46] H. Tizhoosh, "Opposite fuzzy sets with applications in image processing," presented at the Proc. IFSA-EUSFLAT, Lisbon, Portugal, 2009.
[47] A. M. Norwich and I. B. Turksen, "A model for the measurement of membership and the consequences of its empirical implementation," *Fuzzy Sets Syst.,* vol. 12, pp. 1–25, 1984.
[48] R. R. Yager, "Connectives and qualifiers in fuzzy sets," *Fuzzy Sets Syst.,* vol. 40, pp. 39–75, 1991.
[49] A. Lehrer and K. Lehrer, "Antonymy," *Linguistics and Philosophy*, vol. 5, pp. 483–501, 1982.
[50] A. Lehrer, "Markedness and Antonymy," *J. Linguistics,* vol. 21, pp. 397–429, 1985.
[51] S. V. Ovchinnikov, "Representations of synonymy and antonymy by automorphisms in fuzzy set theory" *Sotochastica*, vol. 5, pp. 95–107, 1981.
[52] S. Guadarrama and A. Ruiz-Mayor, "Approximate robotic mapping from sonar data by modeling perceptions with antonyms," *Inf. Sci.,* vol. 180, pp. 4164–4188, 2010.

Scott Dick (M'02) received the B.Sc. degree in 1997, the M.Sc. degree in 1999, and the Ph.D. in 2002, all from the University of South Florida, Tampa, FL, USA.

From 2002 to 2008 he was an Assistant Professor of electrical and computer engineering with the University of Alberta, Edmonton, AB, USA, where he has been an Associate Professor since 2008. His research interests include computational intelligence, machine learning, data mining, and the application of these technologies to real-world problems (e.g., smart Grid, livestock disease management, and antiphishing technologies). His recent focus is on the topic of "complex fuzzy logic," an extension of type-1 fuzzy logic to complex-valued membership grades. His work includes both theoretical analysis of this new concept and constructing machine learning systems based on this new concept. His work has been funded by the Natural Sciences and Engineering Research Council of Canada, the Alberta Science and Research Authority, Hewlett-Packard, PRECARN Inc., and Transport Canada.

Dr. Dick received the USF Outstanding Dissertation Prize for his Ph.D. dissertation in 2003. He is a Member of the IEEE Computational Intelligence Society's Fuzzy Systems Technical Committee, and the Chair of the FSTC Task Force on Complex Fuzzy Sets and Logic. He is an Associate Editor for the IEEE TRANSACTIONS ON FUZZY SYSTEMS, as well as *Evolving Systems*. He is a Member of the ACM and ASEE.



Ronald R. Yager (F'97) received the Undergraduate degree from the City College of New York, New York, NY, USA, and the Ph.D. degree from the Polytechnic Institute of New York University, Brooklyn, NY, USA.

He has worked in the area of computational intelligence for more than 25 years. He is the Director of the Machine Intelligence Institute and Professor of information systems with Iona College, New Rochelle, NY, USA. He has published over 500 papers and edited over 30 books in areas related to fuzzy sets, human behavioral modeling, decision-making under uncertainty and the fusion of information. He is a Visiting Distinguished Scientist with King Saud University, Riyadh, Saudi Arabia. He is an Adjunct Professor with Aalborg University, Aalborg, Denmark.

Dr. Yager is the Editor-in-Chief of the *International Journal of Intelligent Systems*. He is among the world's top 1% most highly cited researchers with more than 45 000 citations in Google Scholar. He has received the IEEE Computational Intelligence Society Pioneer Award in fuzzy systems, the Special Honorary Medal of the 50th Anniversary of the Polish Academy of Sciences, and the Lifetime Outstanding Achievement Award from International the Fuzzy Systems Association. He has recently received honorary doctorate degrees, honoris causa, from the Azerbaijan Technical University and the State University of Information Technologies, Sofia, Bulgaria. He is a Fellow of the New York Academy of Sciences and the Fuzzy Systems Association. He has served with the National Science Foundation as the Program Director in the Information Sciences program. He was a NASA/Stanford Visiting Fellow and a Research Associate with the University of California, Berkeley. He has been a Lecturer with NATO Advanced Study Institutes. He was a Program Director with the National Science Foundation. He is the 2016 recipient of the IEEE Frank Rosenblatt Award, the most prestigious honor given by the IEEE Computational Intelligent Society.

Omolbanin Yazdanbakhsh (M'13) received the B.Sc. and M.Sc. degrees in electrical engineering from Islamic Azad University, Najafabad Branch, Esfahan, Iran, in 2007 and 2010, respectively. Since 2012, she has been working toward the Ph.D. degree in computer engineering with the University of Alberta, Edmonton, AB, Canada.

Her research interests include machine learning and data mining, focusing on time series forecasting.

Ms. Yazdanbakhsh received the Best Student Paper Award at NAFIPS 2015, Queen Elizabeth II Graduate Scholarship in 2015, and University of Alberta Doctoral Recruitment Scholarship in 2012.