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Synchronization of unified chaotic system by sliding mode/mixed H_2/H_∞ control

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Abstract This paper deals with the synchronization of uncertain unified chaotic system in the presence of two kinds of disturbances, white noise and bounded power signal. A sliding mode controller (SMC) is established to guarantee the sliding motion. Moreover, a proportional-integral (PI) switching surface is used to determine the performance of the system in the sliding motion. Also, by using a mixed H_2/H_∞ approach, the effect of external disturbances on the sliding motion is reduced. The necessary parameters of constructing controller and switching surface are found via semidefinite programming (SDP) which can be solved effectively by a standard software. Finally, a numerical simulation is presented to show the effectiveness of the proposed method.

Keywords Unified chaotic system · Synchronization · Sliding mode control · Mixed H_2/H_∞ approach

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1 Introduction

Chaotic system is a very complex nonlinear system which is excessively sensitive to initial condition. This property implies that two trajectories emerging from two different close initial conditions separate exponentially in the course of time. It appears, at first, it would be very difficult to synchronize two chaotic systems. But Pecoral and Carroll in 1990 showed that it was indeed possible [1]. From then on, chaos synchronization has been extensively investigated due to the potential applications in various fields such as power converters, biological systems, chemical reactors, information processing, and so on [2–4]. Based on various control theories, a number of different methods have been applied theoretically and experimentally to synchronize the chaotic systems [5–8].

Uncertainties and external disturbances always exist in real physical systems that may cause instability and poor performance. Robust control is a common approach to analyze and synthesize such systems. A robust control method that have been used for uncertain chaotic systems is sliding mode control which is preferred because of its inherent advantages such as easy realization, fast response, good transient response, and insensitivity to variations in system parameters [9–11]. Moreover, to reduce the effect of external disturbances on the available output to within a prescribed level, the H_∞ concept was proposed [12–14]. Liao et al. developed a procedure which used the sliding mode method to control a chaotic system in the presence of white noise and reduced the effect of noise via the H_∞ method [15]. The suppression of white noise, however, is more generally guaranteed by the H_2 method. On the other hand, there may be two different kinds of disturbance, white noise and a bounded power signal, which affect the system simultaneously. In this case, a robust approach which can effectively solve the problem is mixed H_2/H_∞.

Mixed H_2/H_∞ is a robust control method which minimizes the H_2 norm of a closed-loop system subject to an H_∞ norm constraint on another closed-loop system. One way to solve the mixed problem is to replace the H_2 norm by a suitable upper bound. The maximum entropy method [16], the auxiliary cost method [17], and the method proposed by Doyle et al. in [18] used this approach. Khargonekar and Rotea [19] gave an efficient convex optimization formulation using the auxiliary cost function of [17]. However, this method has two significant drawbacks: (i) Numerical results show that the mixed controllers designed based on auxiliary cost function may yield an H_2 performance which is worse than that of central H_∞ controllers [20] and (ii) this approach cannot be extended to a more general system, e.g. 2 input-2 output systems. To solve these problems, Halder et al. used the actual H_2 norm instead of an upper bound in [21]. In this technique, the original nonconvex problem was broken up into a series of convex subproblems and each of them can be converted into a semidefinite programming problem that can be easily solved.

In this paper, the synchronization of an uncertain unified chaotic system in the presence of two kinds of disturbance, bounded power signals, and white noise, is examined. To achieve this goal, a sliding mode controller (SMC) is designed which guarantees the occurrence of sliding motion and a proportional-integral (PI) switching surface is used to determine the performance of the system in the sliding motion. The necessary parameter of constructing a PI switching surface and SMC is designed such that the effect of disturbances on the sliding mode is reduced by using the mixed H_2/H_∞ approach adopted in [21].

The organization of this paper is as follows. In Sect. 2, the synchronization problem of uncertain unified chaotic systems in the presence of the disturbances is described. The PI switching surface and the sliding mode controller are designed in Sect. 3. In Sect. 4, a numerical simulation is given to demonstrate the effectiveness of the proposed method. Finally, some conclusion remarks are given in Sect. 5.

2 Synchronization of unified chaotic system

Consider the unified chaotic system which is described by

$$\begin{aligned}
\dot{x} &= (25\alpha + 10)(y - x)\\
\dot{y} &= (28 - 35\alpha)x - xz + (29\alpha - 1)y\\
\dot{z} &= xy - \left(\frac{8+\alpha}{3}\right)z\\
\mathbf{X} &= [x, y, z]'\\
\mathbf{Y} &= \mathbf{E}\mathbf{X}
\end{aligned} \tag{1}$$

where x, y, z are state variables, $\mathbf{Y}$ is the output of the system, $\mathbf{E}$ is an known matrix, and $\alpha \in [0, 1]$. Obviously, system (1) becomes the original Lorenz system for $\alpha = 0$; while it becomes the original Chen system for $\alpha = 1$. When $\alpha = 0.8$, this system becomes the so-called critical system. In particular, system (1) bridges the gap between Lorenz and Chen systems. Moreover, this system is always chaotic in the whole interval $\alpha \in [0, 1]$.

For the unified chaotic system (1), the master and the slave systems can be described by the following equations:

$$\begin{aligned}
\dot{\mathbf{X}}_m &= \mathbf{A}\mathbf{X}_m(t) + \mathbf{B}\mathbf{f}\big(\mathbf{X}_m(t)\big)\\
\mathbf{Y}_m &= \mathbf{E}\mathbf{X}_m(t)\\
\mathbf{X}_m &= [x_m, y_m, z_m]',
\end{aligned} \tag{2}$$

$$\begin{aligned}
\dot{\mathbf{X}}_s &= \mathbf{A}\mathbf{X}_s(t) + \mathbf{B}\mathbf{f}\big(\mathbf{X}_s(t)\big) + \mathbf{B}\Delta\mathbf{f} + \mathbf{B}_2 w_2\\
&\quad + \mathbf{B}_\infty w_\infty + \mathbf{B}\mathbf{u}(t)\\
\mathbf{Y}_s &= \mathbf{E}\mathbf{X}_s(t)\\
\mathbf{X}_s &= [x_s, y_s, z_s]'
\end{aligned} \tag{3}$$

where

$$\mathbf{A} = \begin{bmatrix} -(25\alpha + 10) & (25\alpha + 10) & 0\\ (28 - 35\alpha) & (29\alpha - 1) & 0\\ 0 & 0 & -(\frac{8+\alpha}{3}) \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0\\ 1 & 0\\ 0 & 1 \end{bmatrix},$$

$$\mathbf{f}\big(\mathbf{X}_m(t)\big) = \begin{bmatrix} -x_m z_m\\ x_m y_m \end{bmatrix}, \quad \mathbf{f}\big(\mathbf{X}_s(t)\big) = \begin{bmatrix} -x_s z_s\\ x_s y_s \end{bmatrix},$$

$$\Delta\mathbf{f} = \begin{bmatrix} \Delta f_1\\ \Delta f_2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1\\ u_2 \end{bmatrix}$$

The lower scripts m and s stand for the master system and the slave one, respectively. u_1 and u_2 are the control inputs that should synchronize the two main chaotic systems. w_2 and w_∞ are white noise and bounded power signals, respectively. $\mathbf{B}_2$ and $\mathbf{B}_\infty$ are the noise coefficient matrices which will be specified later. Δf_i $(i = 1, 2)$ denotes the bounded uncertain parameters, and it is assumed there exists a positive number β such that $\|\Delta\mathbf{f}\| \le \beta\|\mathbf{X}_s\|$. Note that the input components appear only in those equations which include uncertainty.

Define the synchronization error as

$$\mathbf{e}(t) = \mathbf{X}_s(t) - \mathbf{X}_m(t) \tag{4}$$

Then the dynamics of synchronization error between master and slave systems given in (2) and (3) can be described by

$$\begin{aligned} \dot{\mathbf{e}}(t) &= \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{f} + \mathbf{B}\Delta\mathbf{f} + \mathbf{B}_2 w_2 + \mathbf{B}_\infty w_\infty + \mathbf{B}\mathbf{u} \\ \mathbf{Y}_e &= \mathbf{E}\mathbf{e} \end{aligned} \tag{5}$$

where $\mathbf{f} = \mathbf{f}(\mathbf{X}_s(t)) - \mathbf{f}(\mathbf{X}_m(t))$.

3 Controller design

This paper aims to use the sliding mode control method to guarantee synchronization between master and slave systems. At the first step, a switching surface with appropriate parameters is selected such that it can provide an asymptotically stable sliding motion. Next, a robust control law is determined which guarantees the error dynamics reach the switching surface and stay on it. Finally, the unknown parameter of the control law and the sliding surface is designed such that the effect of disturbances on the sliding mode is reduced by the mixed H_2/H_∞ approach.

3.1 Switching surface

A proportional-integral (PI) switching surface is chosen as follows [15]:

$$\mathbf{s}(t) = \mathbf{C}\mathbf{e}(t) - \int_0^t (\mathbf{C}\mathbf{A} + \mathbf{C}\mathbf{B}\mathbf{K})\mathbf{e}(s)\,ds \tag{6}$$

where $\mathbf{s}(t) \in R^{2\times 1}$, $\mathbf{C} \in R^{2\times 3}$ and $\mathbf{K} \in R^{2\times 3}$. $\mathbf{C}$ is chosen such that $\mathbf{C}\mathbf{B}$ is nonsingular and $\mathbf{C}\mathbf{B}_2 = 0$, $\mathbf{C}\mathbf{B}_\infty = 0$, and $\mathbf{K}$ is a gain matrix that will be determined later.

The following two conditions must be satisfied on sliding surface [22, 23]

$$\mathbf{s}(t) = \mathbf{C}\mathbf{e}(t) - \int_0^t (\mathbf{C}\mathbf{A} + \mathbf{C}\mathbf{B}\mathbf{K})\mathbf{e}(s)\,ds = 0, \tag{7}$$

$$\dot{\mathbf{s}}(t) = \mathbf{C}\dot{\mathbf{e}}(t) - (\mathbf{C}\mathbf{A} + \mathbf{C}\mathbf{B}\mathbf{K})\mathbf{e}(t) = 0 \tag{8}$$

Hence, the equivalent control $\mathbf{u}_{eq}$ on the switching surface is obtained by substituting (5) into (8), i.e.,

$$\begin{aligned} \dot{\mathbf{s}}(t) &= \mathbf{C}\mathbf{B}(\mathbf{f} + \Delta\mathbf{f} + \mathbf{u} - \mathbf{K}\mathbf{e}) + \mathbf{C}\mathbf{B}_2 w_2 + \mathbf{C}\mathbf{B}_\infty w_\infty \\ &= 0 \end{aligned} \tag{9}$$

Since $\mathbf{C}\mathbf{B}_2 = 0$ and $\mathbf{C}\mathbf{B}_\infty = 0$, the equivalent control $\mathbf{u}_{eq}$ on the sliding mode is given by

$$\mathbf{u}_{eq} = (\mathbf{K}\mathbf{e} - \mathbf{f} - \Delta\mathbf{f}) \tag{10}$$

Substituting $\mathbf{u}_{eq}$ into (5), the sliding mode dynamic is obtained as

$$\begin{aligned} \dot{\mathbf{e}}(t) &= (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{e} + \mathbf{B}_2 w_2 + \mathbf{B}_\infty w_\infty \\ \mathbf{Y}_e &= \mathbf{E}\mathbf{e} \end{aligned} \tag{11}$$

3.2 Sliding mode controller

In the following, a control scheme is presented which ensures the existence of sliding motion.

Theorem 1 *Consider the dynamic system* (5) *and sliding surface* (6). *The reaching condition* $\mathbf{s}^T(t)\dot{\mathbf{s}}(t) < 0$ *of the sliding mode is satisfied if the control input* $\boldsymbol{u}(t)$ *is given by*

$$\begin{aligned} \boldsymbol{u}(t) = \mathbf{K}\mathbf{e} - \mathbf{f} - (\mathbf{C}\mathbf{B})^{-1}\big(&\mu + \eta\|\boldsymbol{s}\|^{\rho-1} \\ &+ \|\mathbf{C}\mathbf{B}\|\beta\|\boldsymbol{X}_s\|\|\boldsymbol{s}\|^{-1}\big)\boldsymbol{s} \end{aligned} \tag{12}$$

where η *and* μ *are arbitrary positive constants. Moreover, for* $0 < \rho < 1$, *the error states reach the sliding surface in a finite time.*

Proof By substituting (5) and (12) into $\mathbf{s}^T(t)\dot{\mathbf{s}}(t)$, we get:

$$\begin{aligned} \mathbf{s}^T\dot{\mathbf{s}} &= \mathbf{s}^T[\mathbf{C}\mathbf{B}\mathbf{f} + \mathbf{C}\mathbf{B}\mathbf{u} + \mathbf{C}\mathbf{B}\Delta\mathbf{f} - \mathbf{C}\mathbf{B}\mathbf{K}\mathbf{e}] \\ &= \mathbf{s}^T\big(\mathbf{C}\mathbf{B}\mathbf{f} + \mathbf{C}\mathbf{B}\mathbf{K}\mathbf{e} - \mathbf{C}\mathbf{B}\mathbf{f} - \big(\mu + \eta\|\mathbf{s}\|^{\rho-1} \\ &\quad + \|\mathbf{C}\mathbf{B}\|\beta\|\mathbf{X}_s\|\|\mathbf{s}\|^{-1}\big)\mathbf{s} + \mathbf{C}\mathbf{B}\Delta\mathbf{f} - \mathbf{C}\mathbf{B}\mathbf{K}\mathbf{e}\big) \\ &\le -\mathbf{s}^T\big(\mu + \eta\|\mathbf{s}\|^{\rho-1}\big)\mathbf{s} - \mathbf{s}^T\big(\|\mathbf{C}\mathbf{B}\|\beta\|\mathbf{X}_s\|\|\mathbf{s}\|^{-1}\big)\mathbf{s} \end{aligned}$$

$$+\,\|\mathbf{s}\|\|\mathbf{CB}\|\beta\|\mathbf{X}_s\| = -\mu\|\mathbf{s}\|^2 - \eta\|\mathbf{s}\|^{\rho+1} \quad (13)$$

Since $\eta, \mu > 0$, the reaching condition $\mathbf{s}^T(t)\dot{\mathbf{s}}(t) < 0$ is always satisfied. To see that the reaching time is finite, refer to [10]. □

3.3 State feedback gain design

In this section, the state feedback gain $\mathbf{K}$ is found such that the effect of disturbances on the sliding surface is reduced.

Consider again the sliding mode error dynamic, i.e.,

$$\dot{\mathbf{e}}(t) = (\mathbf{A} + \mathbf{BK})\mathbf{e} + \mathbf{B}_2 w_2 + \mathbf{B}_\infty w_\infty$$
$$\mathbf{Y}_e = \mathbf{Ee}$$

For simplicity, some new variables are defined as follows:

$$\boldsymbol{\zeta} := \mathbf{e}, \quad \boldsymbol{\nu} := \mathbf{Ke} \tag{14}$$

The variable $\boldsymbol{\nu}$ can be interpreted as a fictitious control input. Therefore, the error dynamic on the switching surface can be thought as a closed-loop system with state feedback controller. Moreover, to generalize the problem and make it compatible with a mixed H_2/H_∞ approach, two outputs are defined as $\mathbf{z}_2$ and $\mathbf{z}_\infty$. Hence,

$$\begin{aligned}
\dot{\boldsymbol{\zeta}} &= \mathbf{A}\boldsymbol{\zeta} + \mathbf{B}\boldsymbol{\nu} + \mathbf{B}_2 w_2 + \mathbf{B}_\infty w_\infty \\
\mathbf{z}_2 &= \mathbf{C}_2\boldsymbol{\zeta} + \mathbf{D}_2\boldsymbol{\nu} \\
\mathbf{z}_\infty &= \mathbf{C}_\infty\boldsymbol{\zeta} + \mathbf{D}_\infty\boldsymbol{\nu} \\
\mathbf{Y}_e &= \mathbf{E}\boldsymbol{\zeta} \\
\boldsymbol{\nu} &= \mathbf{K}\boldsymbol{\zeta}
\end{aligned} \tag{15}$$

It is assumed that w_2 and w_∞ are white noise and a bounded power signal, respectively.

To investigate the effect of these kinds of disturbances, the definition of p space is reviewed. p is the space of all signals with bounded power. A seminorm is defined on p as

$$\begin{aligned}
\|u\|_p &:= \sqrt{\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|u(t)\|^2\, dt} \\
&= \sqrt{Trace(R_{uu}(\cdot))} \quad \forall u \in p
\end{aligned} \tag{16}$$

where

$$R_{uu} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t+\tau)\cdot u^T(t)\, dt \tag{17}$$

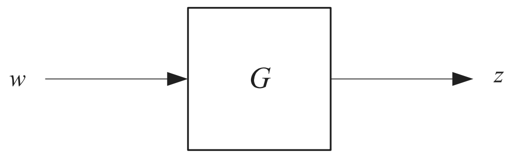

Fig. 1 A stable system

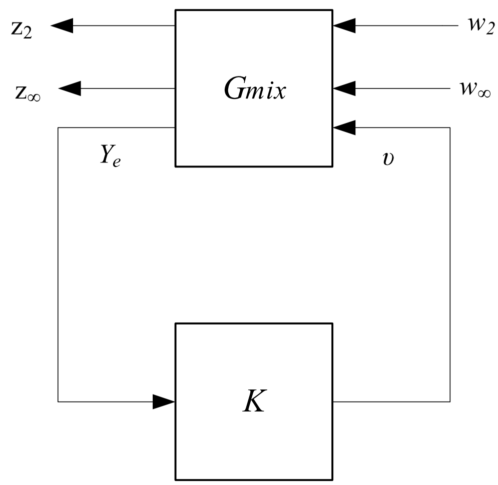

Fig. 2 System model for the mixed H_2/H_∞

Let w be a bounded power signal which excites a stable plant G as in Fig. 1. It can be proved that $\|G\|_\infty = \sup_{w\neq 0} \frac{\|z\|_p}{\|w\|_p}$ [18]. On the other hand, if w is a white noise, then it can be proved that $\|G\|_2^2 = \|z\|_p$. So, by using mixed H_2/H_∞, the effect of these disturbances can be reduced simultaneously.

To reduce the effect of exogenous signals by the H_2/H_∞ method, system (15) is represented by the block diagram given in Fig. 2. Selecting E as identity matrix (I), the system has the following realization:

$$G_{\text{mix}} = \left[\begin{array}{c|ccc}
\mathbf{A} & \mathbf{B}_2 & \mathbf{B}_\infty & \mathbf{B} \\ \hline
\mathbf{C}_2 & 0 & 0 & \mathbf{D}_2 \\
\mathbf{C}_\infty & 0 & 0 & \mathbf{D}_\infty \\
\mathbf{I} & 0 & 0 & 0
\end{array}\right] \tag{18}$$

It is assumed that the following standard assumptions are satisfied:

I. $(\mathbf{A}, \mathbf{B})$ is stabilizable.
II. $\mathbf{D}_\infty^T[\mathbf{C}_\infty\ \mathbf{D}_\infty] = [0\ I]$.
III. $\mathbf{D}_2^T[\mathbf{C}_2\ \mathbf{D}_2] = [0\ I]$.
IV. $\left[\begin{smallmatrix} \mathbf{A}-jw\mathbf{I} & \mathbf{B} \\ \mathbf{C}_2 & \mathbf{D}_2 \end{smallmatrix}\right]$ has full column rank for all $w \in R$.

V. $\left[\begin{smallmatrix} \mathbf{A}-jw\mathbf{I} & \mathbf{B} \\ \mathbf{C}_\infty & \mathbf{D}_\infty \end{smallmatrix}\right]$ has full column rank for all $w \in R$.

For the state feedback controller $\nu(t) = \mathbf{K}\zeta(t)$, the closed-loop system is given by

$$G_{\text{mix}}^{\text{cl}} = \left[\begin{array}{c|cc} \mathbf{A}+\mathbf{B}\mathbf{K} & \mathbf{B}_2 & \mathbf{B}_\infty \\ \hline \mathbf{C}_2+\mathbf{D}_2\mathbf{K} & 0 & 0 \\ \mathbf{C}_\infty+\mathbf{D}_\infty\mathbf{K} & 0 & 0 \end{array}\right] := \left[\begin{array}{c|cc} \bar{\mathbf{A}} & \mathbf{B}_2 & \mathbf{B}_\infty \\ \hline \bar{\mathbf{C}}_2 & 0 & 0 \\ \bar{\mathbf{C}}_\infty & 0 & 0 \end{array}\right] \tag{19}$$

Let $K(G_{\text{mix}})$ denotes the set of admissible controllers which stabilize the closed-loop system. The mixed H_2/H_∞ problem is stated as follows.

Given an achievable H_∞ bound δ, find $K \in K(G_{\text{mix}})$ such that it satisfies

$$\begin{aligned} &\text{minimize} \quad \|T_{z_2 w_2}\|_2 \\ &\text{subject to} \quad \|T_{z_\infty w_\infty}\|_\infty \le \delta \end{aligned} \tag{20}$$

The H_2 cost function is defined as

$$J(\mathbf{K}) := \|T_{z_2 w_2}\|_2^2 = \mathit{Trace}\big(\mathbf{B}_2^T \mathbf{Y}_K \mathbf{B}_2\big) \tag{21}$$

where $\mathbf{Y}_K$ is the observability Gramian of the pair $(\bar{\mathbf{A}}, \bar{\mathbf{C}}_2)$. It is important to note that cost function $J(\mathbf{K})$ is a nonconvex function of the feedback gain matrix $\mathbf{K}$.

According to the bounded real lemma [24], for an admissible $\mathbf{K}$, the closed-loop system $T_{z_\infty w_\infty}$ has H_∞ norm $\le \delta$, if and only if, there exists an $\mathbf{X} \ge 0$ such that

$$\mathbf{X}\bar{\mathbf{A}} + \bar{\mathbf{A}}^T\mathbf{X} + \delta^{-2}\mathbf{X}\mathbf{B}_\infty\mathbf{B}_\infty^T\mathbf{X} + \mathbf{K}^T\mathbf{K} + \mathbf{C}_\infty^T\mathbf{C}_\infty \le 0 \tag{22}$$

The above matrix inequality can be written as the following bilinear matrix inequality (BMI) in the matrices $\mathbf{K}$ and $\mathbf{X}$:

$$\mathbf{B}(\mathbf{X}, \mathbf{K}) := \begin{bmatrix} -\mathbf{X}\bar{\mathbf{A}} - \bar{\mathbf{A}}^T\mathbf{X} - \mathbf{C}_\infty^T\mathbf{C}_\infty & \mathbf{K}^T & \delta^{-1}\mathbf{X}\mathbf{B}_\infty & 0 \\ \mathbf{K} & \mathbf{I} & 0 & 0 \\ \delta^{-1}\mathbf{B}_\infty^T\mathbf{X} & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{X} \end{bmatrix} \ge 0 \tag{23}$$

Moreover, any state feedback gain satisfying the above BMI is also admissible [25]. Therefore, the set of state feedback gain matrices that satisfy H_∞ bound is given as

$$S_K = \big\{\mathbf{K} \in K(G_{\text{mix}}) | \mathbf{B}(\mathbf{X}, \mathbf{K}) \ge 0\big\} \tag{24}$$

Based on the above discussion, the mixed H_2/H_∞ problem can be restated as

$$\min_{\mathbf{K} \in S_K} J(\mathbf{K}) \tag{25}$$

This is a nonconvex minimization problem that is hard to solve. To solve it, as suggested by [21], the nonconvex problem is broken into a series of convex subproblems which can be easily solved. To this end, we first select an admissible feedback gain $\mathbf{K}_0$ from the set S_K which is the central controller for any achievable δ. By using it, a quasiconvex upper bound on $J(\mathbf{K})$ and an internal convex subset of S_K are then constructed, which are represented by $J_{K_0}(\mathbf{K})$ and $\Psi_K(\mathbf{X}_0)$, respectively. These approximations are defined as follows:

$$J_{K_0}(\mathbf{K}) := \mathit{Trace}\big(\mathbf{B}_2^T \Pi_K \mathbf{B}_2\big) \tag{26}$$

where

$$\Pi_K\bar{\mathbf{A}} + \bar{\mathbf{A}}^T\Pi_K + \mathbf{K}^T\mathbf{K} + \mathbf{C}_2^T\mathbf{C}_2 + (\Pi_K - \mathbf{Y}_{K_0})\mathbf{B}\mathbf{B}^T \times (\Pi_K - \mathbf{Y}_{K_0}) = 0 \tag{27}$$

and $\mathbf{Y}_{K_0}$ is the observability Gramian of point $\mathbf{K}_0$, i.e.,

$$\mathbf{Y}_{K_0}(\mathbf{A}+\mathbf{B}\mathbf{K}_0) + (\mathbf{A}+\mathbf{B}\mathbf{K}_0)^T\mathbf{Y}_{K_0} + \mathbf{K}_0^T\mathbf{K}_0 + \mathbf{C}_2^T\mathbf{C}_2 = 0 \tag{28}$$

and

$$\Psi_K(\mathbf{X}_0) := \big\{\mathbf{K} | \mathbf{X} \ge 0, \mathbf{B}_{X_0}(\mathbf{X}, \mathbf{K}) \le 0\big\} \tag{29}$$

where

$$\mathbf{B}_{X_0}(\mathbf{X}, \mathbf{K}) := \mathbf{X}\bar{\mathbf{A}} + \bar{\mathbf{A}}^T\mathbf{X} + \delta^{-2}\mathbf{X}\mathbf{B}_\infty\mathbf{B}_\infty^T\mathbf{X} + \mathbf{K}^T\mathbf{K} + \mathbf{C}_\infty^T\mathbf{C}_\infty + (\mathbf{X}-\mathbf{X}_0)\mathbf{B}\mathbf{B}^T(\mathbf{X}-\mathbf{X}_0) \tag{30}$$

and $\mathbf{X}_0$ is considered as

$$\mathbf{X}_0 := \arg\max_X \det \mathbf{L}_{K_0}(\mathbf{X}) \tag{31}$$

where

$$\mathbf{L}_{K_0}(\mathbf{X}) := \begin{bmatrix} -\mathbf{X}\bar{\mathbf{A}} - \bar{\mathbf{A}}^T\mathbf{X} - \mathbf{K}_0^T\mathbf{K}_0 - \mathbf{C}_\infty^T\mathbf{C}_\infty & \delta^{-1}\mathbf{X}\mathbf{B}_\infty & 0 \\ \delta^{-1}\mathbf{B}_\infty^T\mathbf{X} & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{X} \end{bmatrix} \tag{32}$$

Using these approximations, a convex subproblem can be constructed as

$$\min_{\mathbf{K}\in\Psi_K(\mathbf{X}_0)} J_{K_0}(\mathbf{K}) \tag{33}$$

Note that $J_{K_0}(\mathbf{K})$ and $\Psi_K(\mathbf{X}_0)$ can be expressed in terms of LMIs. Hence, the subproblem can be converted into a semidefinite programming (SDP) problem which can be solved by using the following theorem from [21].

Theorem 2 *The convex subproblem* (33) *is equivalent to the following SDP problem*:

$$\min_{K,\Pi_K,X,\tau} \tau \quad \textit{subject to} \quad \boldsymbol{L}_{X_0}(\boldsymbol{X},\boldsymbol{K}) \geq 0, \quad \textit{and} \quad \boldsymbol{L}_{K_0}(\boldsymbol{K}) \geq 0 \tag{34}$$

where

$$\tau = J_{K_0}(\boldsymbol{K})$$

$$\boldsymbol{L}_{X_0}(\boldsymbol{X},\boldsymbol{K}) := \begin{bmatrix} \boldsymbol{L}_{X_0,11} & \boldsymbol{XB}+\boldsymbol{K}^T & \delta^{-1}\boldsymbol{XB}_\infty & 0 \\ \boldsymbol{B}^T\boldsymbol{X}+\boldsymbol{K} & \boldsymbol{I} & 0 & 0 \\ \delta^{-1}\boldsymbol{B}_\infty^T\boldsymbol{X} & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{X} \end{bmatrix} \tag{35}$$

$$\boldsymbol{L}_{X_0,11} := -\boldsymbol{XA} - \boldsymbol{A}^T\boldsymbol{X} + \boldsymbol{XBB}^T\boldsymbol{X}_0 + \boldsymbol{X}_0\boldsymbol{BB}^T\boldsymbol{X} - \boldsymbol{X}_0\boldsymbol{BB}^T\boldsymbol{X}_0 - \boldsymbol{C}_\infty^T\boldsymbol{C}_\infty$$

and

$$\boldsymbol{L}_{K_0}(\boldsymbol{K}) := \begin{bmatrix} \boldsymbol{L}_{K_0,11} & (\boldsymbol{B}^T\Pi_K+\boldsymbol{K})^T & 0 & 0 \\ (\boldsymbol{B}^T\Pi_K+\boldsymbol{K}) & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & \alpha - Trace(\boldsymbol{B}_2^T\Pi_K\boldsymbol{B}_2) & 0 \\ 0 & 0 & 0 & \Pi_K \end{bmatrix} \tag{36}$$

$$\boldsymbol{L}_{K_0,11} := -\Pi_K\boldsymbol{A} - \boldsymbol{A}^T\Pi_K + \Pi_K\boldsymbol{BB}^T\boldsymbol{Y}_{K_0} + \boldsymbol{Y}_{K_0}\boldsymbol{BB}^T\Pi_K - \boldsymbol{Y}_{K_0}\boldsymbol{BB}^T\boldsymbol{Y}_{K_0} - C_2^TC_2$$

The solution of this SDP problem yields a new controller with a reduced value of function $J_{K_0}(\mathbf{K})$, i.e. τ, unless $\mathbf{K}_0$ is the global minimum of $J_{K_0}(\mathbf{K})$ or the boundary of S_K is reached ($\|T_{z_\infty w_\infty}\| = \delta$). Since $J_{K_0}(\mathbf{K})$ and $\Psi_K(\mathbf{X}_0)$ can be formulated around any point in S_K, a new subproblem can be formulated around the solution of the SDP. This leads to an iterative algorithm that yields the optimal H_2/H_∞ feedback gain $\mathbf{K}$.

4 Numerical simulation

In this section, to verify and demonstrate the effectiveness of the proposed method, we discuss the simulation results for the Lorenz and Chen systems. In the simulation process, the fourth-order Runge–Kutta integration method with the time step of 0.0001 is used to solve the differential equations.

4.1 Lorenz system

When $\alpha = 0$, (2) and (3) correspond to Lorenz system, and the error dynamics are obtained as

$$\dot{\mathbf{e}} = \mathbf{Ae} + \mathbf{Bf} + \mathbf{B}\Delta\mathbf{f} + \mathbf{Bu} + \mathbf{B}_2 w_2 + \mathbf{B}_\infty w_\infty$$

$$\mathbf{Y}_e = \mathbf{Ee}$$

where

$$\mathbf{A} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}, \quad \mathbf{B}_\infty = \begin{bmatrix} 0.2 \\ -0.1 \\ 0 \end{bmatrix},$$

$$\mathbf{C}_2 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}_\infty = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{D}_2 = \mathbf{D}_\infty = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\mathbf{E}$ is taken as an unity matrix. The uncertainties are adopted as

$$\Delta f_1 = \sin t \cdot x_s, \quad \Delta f_2 = \cos t \cdot y_s$$

By choosing $\beta = 1$, $\|\Delta\mathbf{f}\| \le \beta\|\mathbf{X}_s\|$ is satisfied. w_2 is a zero mean Gaussian noise with variance 1, and w_∞ is the sinusoidal signal, $\sin(2t)$, which is in the bounded power signal category. We choose $\mathbf{C} = \left[\begin{smallmatrix} 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ such that $\mathbf{CB} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is nonsingular and $\mathbf{CB}_2 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, $\mathbf{CB}_\infty = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. Moreover, by considering C_2, C_∞, D_2, and D_∞, it can be easily seen that the assumptions $\mathbf{D}_\infty^T[\mathbf{C}_\infty\ \mathbf{D}_\infty] = [0\ I]$, $\mathbf{D}_2^T[\mathbf{C}_2\ \mathbf{D}_2] = [0\ I]$ are satisfied.

The SDP (34) is solved by using the YALMIP interface [26] and the SeDuMi solver [27]. After 3 iterations, with $\delta = 1.5$, we get

$$\mathbf{K} = \begin{bmatrix} -30.3397 & -23.7347 & -0.0630 \\ 0.0106 & -0.0719 & -0.1144 \end{bmatrix}$$

$$\tau = 3.8106$$

The parameters of the controller are $\mu = 12$, $\eta = 10$, $\rho = 0.7$. So, the controller is obtained as

$$\mathbf{u} = \mathbf{Ke} - \mathbf{f} - (\mathbf{CB})^{-1}\big(12 + 10\|\mathbf{s}\|^{-0.3} + \|\mathbf{CB}\|\beta\|\mathbf{X}_s\|\|\mathbf{s}\|^{-1}\big)\mathbf{s}$$

The simulation results with initial conditions $(x_m(0), y_m(0), z_m(0)) = (1, 2, 1)$ and $(x_s(0), y_s(0), z_s(0)) = (-1, -2, -3)$ are shown in Figs. 3, 4, 5. To evaluate the results more precisely, the responses of error states for zero initial conditions are also shown in Fig. 6. As depicted from this figure, the effects of disturbance on the error states are of orders 10^{-2}, 10^{-3}, and 10^{-4}, respectively. Compared with the amplitude of disturbance, based on B_2 and B_∞, it is seen that the controller considerably reduces the effects of disturbance.

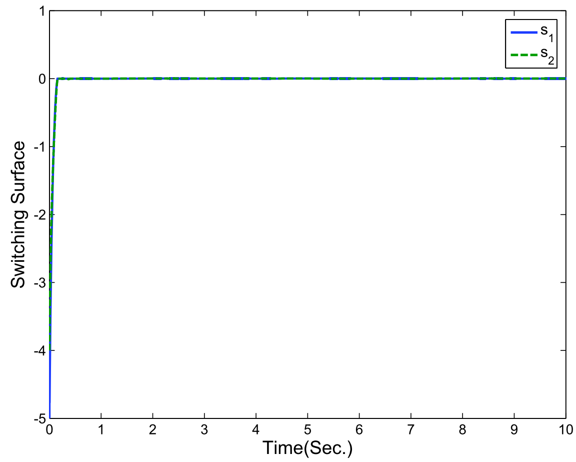

Fig. 3 The time response of switching surface ($\alpha = 0$)

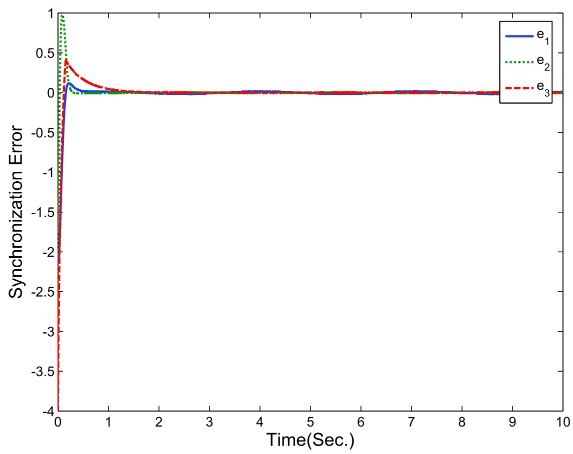

Fig. 4 The time response of synchronization error states ($\alpha = 0$)

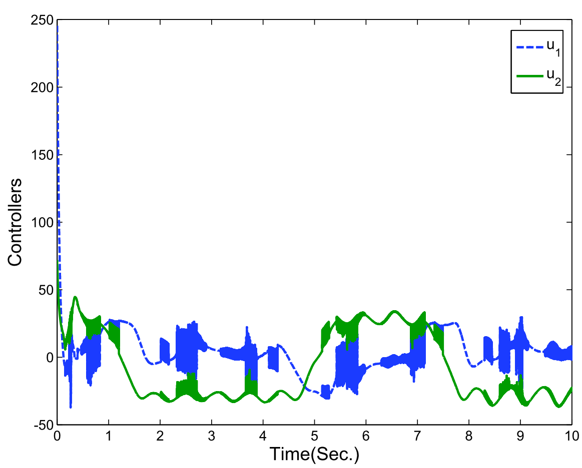

Fig. 5 The time response of controllers ($\alpha = 0$)

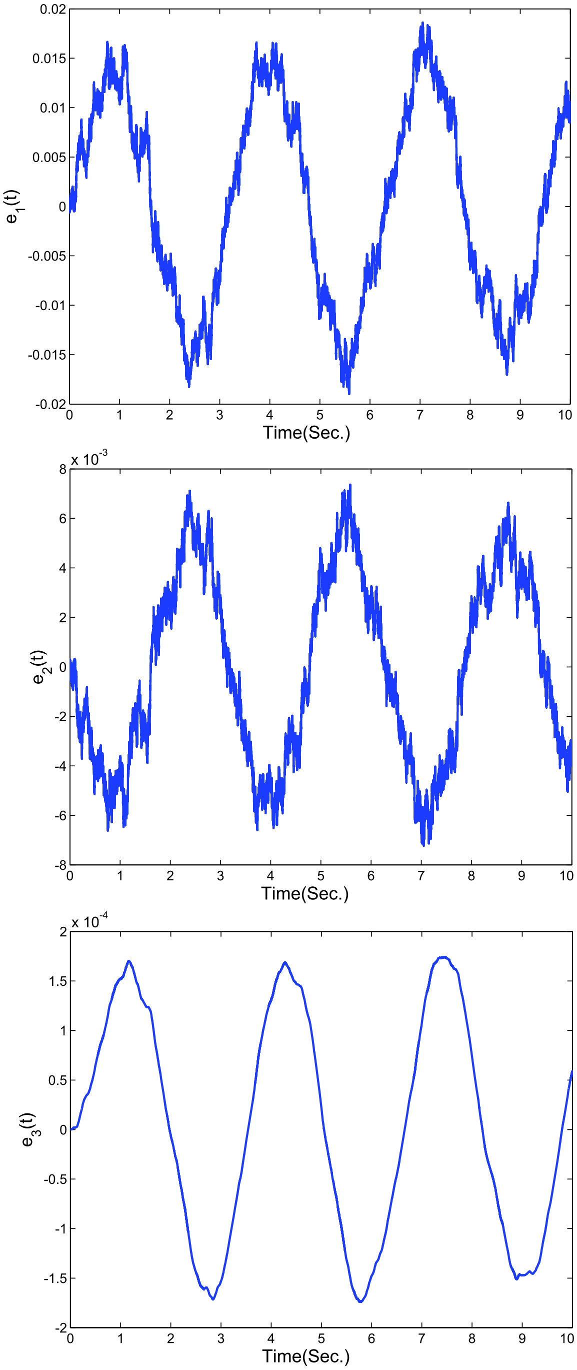

Fig. 6 The time response of synchronization error states with zero initial condition ($\alpha = 0$)

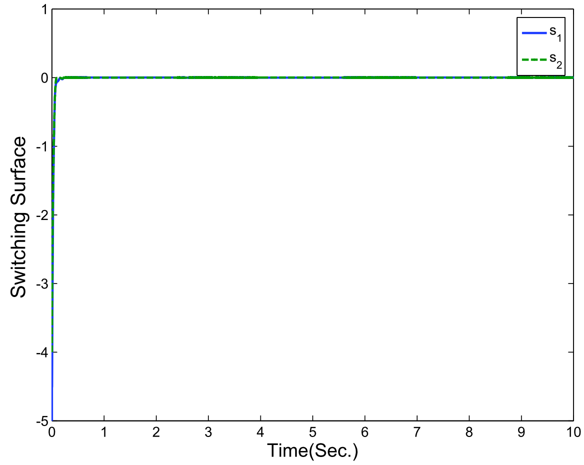

Fig. 7 The time response of switching surface ($\alpha = 1$)

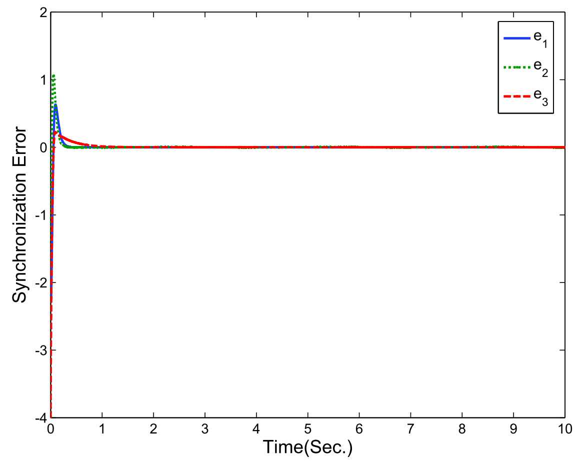

Fig. 8 The time response of synchronization error states ($\alpha = 1$)

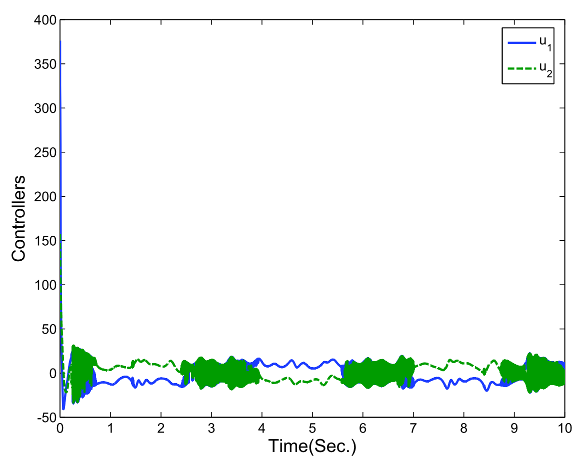

Fig. 9 The time response of controller ($\alpha = 1$)

4.2 Chen system

When $\alpha = 1$, (2) and (3) correspond to the Chen system. The parameters of these equations are chosen

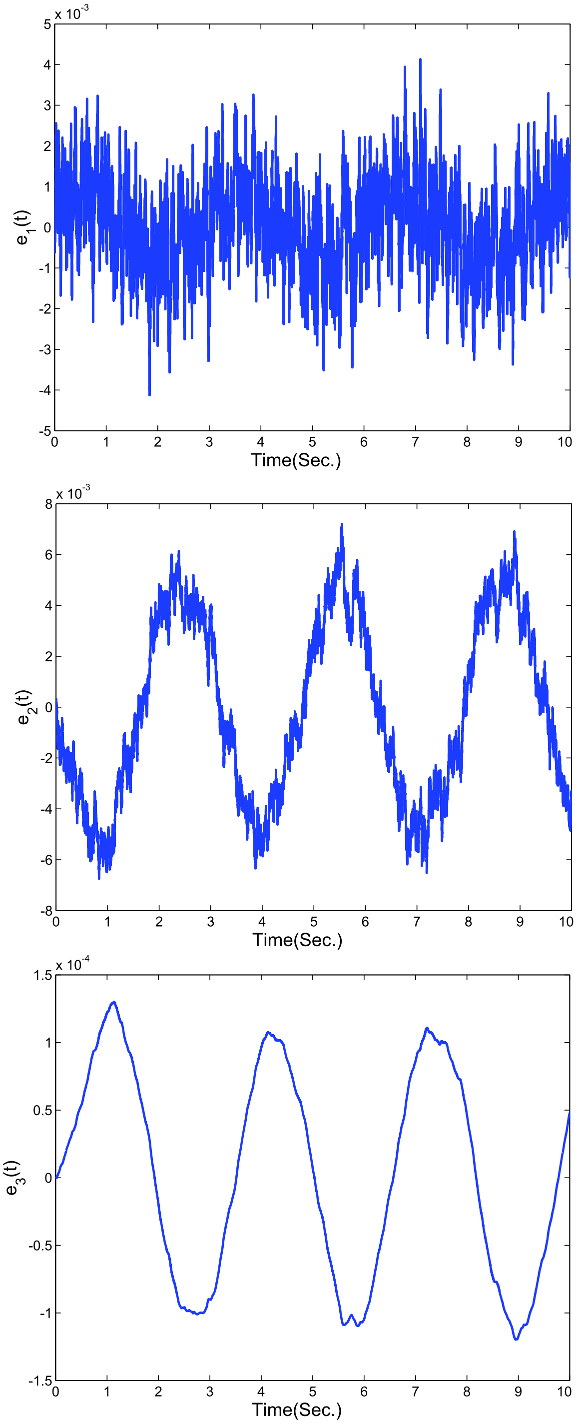

Fig. 10 The time response of synchronization error states with zero initial condition ($\alpha = 1$)

similar to that of Lorenz system except for the uncertainties which are adopted as

$$\Delta f_1 = 0.5 \cos t \cdot y_s, \quad \Delta f_2 = 0.5 \sin t \cdot z_s$$

By choosing $\beta = 0.5$, $\|\Delta \mathbf{f}\| \le \beta \|\mathbf{X}_s\|$ is satisfied.

Moreover, by considering $\delta = 1.2$ in the SDP (34), after 3 iterations, we get

$$\mathbf{K} = \begin{bmatrix} 5.6741 & -47.7688 & -0.0838 \\ 0.0032 & -0.0824 & -0.1070 \end{bmatrix}$$

$$\tau = 4.2768$$

The parameters of the controller are selected as $\mu = 25$, $\eta = 25$, $\rho = 0.7$. Figures 7, 8, 9, 10 show the simulation results for the same initial conditions as the Lorenz system.

5 Conclusion

In this paper, a sliding mode controller using a mixed H_2/H_∞ approach has been designed which can guarantee the synchronization of uncertain unified chaotic systems in the presence of two kinds of disturbance, white noise and a bounded power signal. Furthermore, a numerical simulation is provided to show the effectiveness of the proposed method.

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