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A systematic review of complex fuzzy sets and logic

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Abstract

Complex fuzzy sets and logic are an extension of type-1 fuzzy sets wherein memberships may be complex-valued. This has been an area of growing research focus in the fuzzy systems community for over a decade, with successful applications in time series forecasting and other areas. We conduct a systematic review of this topic to provide a framework to position new research in the field, consolidate the available theoretical results, catalogue the current applications of complex fuzzy sets and logic, identify the key open questions facing researchers in this area, and suggest possible future directions for research in this field.

Keywords: Complex fuzzy sets; Complex fuzzy logic; Systematic review; Fuzzy connectives and aggregation operators; Fuzzy relations; Neuro-fuzzy systems

1. Introduction

In 2002, Ramot et al. [68] defined a Complex Fuzzy Set (CFS) as an extension of type-1 fuzzy sets in which the codomain of the membership function was the unit disc of the complex plane (the set of complex numbers with modulus ≤ 1). CFS are distinct from Buckley's fuzzy complex numbers, which are type-1 fuzzy subsets of the complex numbers [13]. While complex-valued memberships had been explored in a few earlier works [38,62–64], it was Ramot's paper and the companion piece on Complex Fuzzy Logic (CFL) [67] that became the seminal works in the topic of *Complex Fuzzy Sets and Logic* (CFS&L). Additional early theoretical results appeared in [20,97], and the first applications of CFS&L began appearing in 2007 [49,59]. Since then, the field has been gathering momentum; special sessions on CFS&L were organized at IFSA/NAFIPS 2013, WCCI 2014, NAFIPS 2015, and WCCI 2016, and [67,68] have drawn 116 and 99 citations as of March 2016, respectively.

As a fairly recent research topic, CFS&L has to date advanced unevenly. There is a small body of theoretical results, which have influenced applications of CFS&L (which are largely neuro-fuzzy systems incorporating complex-valued memberships). As the literature grows, it becomes more difficult to track and integrate these disparate results into a cohesive whole. We believe that the CFS&L topic has now grown to the point that a consolidation is needed, in order

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to aid new researchers in this area, focus the field's attention on key open questions, and identify potentially fruitful future directions for the topic.

We undertake a systematic review of the CFS&L topic, following Kitchenham's suggested methodology [39]. Our key research questions focus first on the utility of CFS&L, reviewing the viewpoints of various authors on what they may be useful for. We then examine the truth valuation sets used, functional forms of complex fuzzy sets, complex fuzzy set operations, complex fuzzy relations, complex fuzzy logic, and the applications thereof. We have systematically searched the literature to identify the papers that propose some new advance specifically in CFS&L, or an application thereof, and we then review each included paper through the lenses of our research questions.

The remainder of this paper is organized as follows. In Section 2, we formally construct our research questions, and describe our procedures for identifying relevant primary sources. In Section 3, we examine the arguments for the utility of CFS&L, as well as the truth valuation sets and complex fuzzy membership functions that have appeared in the literature. In Section 4, we review the existing results on complex fuzzy set operations and complex fuzzy relations. In Section 5, we review complex fuzzy logic, and in Section 6 we catalogue the existing applications of CFS&L. We discuss our findings and offer our conclusions in Section 7.

2. Methodology

The overall goal of this review is to consolidate the field of CFS&L research. To accomplish this goal, in this section we formulate more specific research questions, a search strategy, and inclusion/exclusion criteria for primary sources.

2.1. Research questions

CFS&L is a recently-established subtopic within the general field of fuzzy systems research, and more specifically *extensions to type-1 fuzzy sets and logic*. To begin formulating our research questions, we first consider the existential problem of "*what is it good for?*" In our conversations with members of the fuzzy systems community, we invariably find that this is the first question we are asked after defining CFS&L. In other words, we are being asked what unique advantages CFS&L provides. To frame this as a research question, we must first inquire what kind of entity CFS&L actually is. We take the position that, like type-1 fuzzy sets and logic, CFS&L is a mathematical framework (specifically, a family of multi-valued logics and their isomorphic set theories) for reasoning under uncertainty [40, 73]. This then points us to a straightforward formalization of our research question as:

Q1: In what way and for what phenomena is CSF&L a more effective framework for reasoning under uncertainty than existing fuzzy logics?

A type-1 fuzzy set is a set of ordered pairs $(x, \mu(x))$ where $x \in X$ is an element of some universal set X and the membership function $\mu(x)$ defines the degree to which element x belongs to the fuzzy set. In 1965, Zadeh defined the codomain of μ to be the interval $[0, 1]$, a straightforward generalization of the codomain of the characteristic function from traditional (crisp) set theory. While $[0, 1]$ has continued to be very frequently used as the codomain of μ (equivalently, as the truth valuation set of the isomorphic fuzzy logic), there is no reason why other codomains could not be used instead. Indeed, in 1967 Gougen proposed that the codomain of μ could be any complete lattice-ordered semigroup (a complete lattice with an additional operation forming a semigroup over the same set) [23]. These general forms are now more commonly referred to as L-fuzzy sets. The same argument plainly applies to the unit disc of the complex plane; there is no rationale why this must be the sole possible codomain for a CFS membership function, or the sole truth valuation set for any CFL. This leads us to our second research question:

Q2: What are the truth valuation sets currently employed in CFS&L research, and what is known about their properties?

Much of the theoretical research, and most applied work, in type-1 fuzzy systems focuses on the case of parameterized membership functions. The assumption in this case is that a smooth mathematical function can be used to describe the degree to which elements of X belong to a fuzzy set. The prototypical example is relating the measured

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height of a person to the concept "tall," which is a linguistic term that plainly relates to measured heights, but is imprecise. Typical practice in fuzzy systems would be to model "tall" as a unimodal membership function over a subset of the real numbers. Numerous functional forms are available; triangular, trapezoidal, Gaussian, and sigmoidal functions are all possible. Again, no one functional form is universally employed, although properties such as unimodality and normality are often seen as desirable [40]. CFS functional forms are also likely to be very diverse, giving us our next research question:

Q3: What functional forms have been employed for complex fuzzy membership functions, and what are their properties?

Zadeh's seminal paper proposed the operations of fuzzy set union, intersection, and complement, implemented by the MAX, MIN and $1 - x$ functions, respectively. Subsequent research has showed that the MAX and MIN functions do have the unique property of being the smallest (largest) fuzzy union (intersection), but they are only one example each of the infinite classes of functions that meet the axioms for a fuzzy union or intersection. These axioms were first studied by Bellman and Giertz in [12] and later generalized by Voxman and Goetschel [86]; from these works, it became clear that fuzzy intersections were equivalent to the triangular norms, and fuzzy unions to triangular conorms [40]. Thus, "fuzzy set theory" actually refers to an infinite family of related set theories, differentiated by the particular choices of union, intersection and complement operators in that theory. Amongst these, fuzzy set theories that form "DeMorgan triples" (union, intersection and complement operations that together satisfy the DeMorgan laws) have pride of place. There seems no reason to expect that CFS&L will be any different. This leads to our fourth research question:

Q4: What functions and/or classes of functions have been proposed to implement complex fuzzy unions, intersections and complements?

A fuzzy relation between universal sets X and Y is defined as a fuzzy set over the Cartesian product $X \times Y$. The union, intersection and complement of fuzzy relations are defined as for any other fuzzy set. The composition of two relations is defined as [40]:

$$\mu_{R \circ S}(x, z) = \sup_{y \in Y} \left[\mu_R(x, y) \star \mu_S(y, z) \right] \tag{1}$$

where R and S are fuzzy relations over $X \times Y$ and $Y \times Z$, respectively, and $\star$ refers to a t-norm.

We examined fuzzy unions, intersections and complements in research question (Q3), and those results also bear on fuzzy relations. Likewise, complex fuzzy relational compositions and projections also likely form infinite families, which may or may not relate to the operations studied in (Q3); e.g. it is not currently known if the operation "$\star$" in Eq. (1) should still be a t-norm in complex fuzzy relations. This leads to our fifth research question:

Q5: What functions have been proposed to implement complex fuzzy relational compositions, and what are their properties?

"Fuzzy logic" refers to a family of multivalued (actually, the *uncountably infinite* valued) logics, differentiated by the conjunction, disjunction and negation operations, and the truth valuation set, in a particular fuzzy logic [40]. As with fuzzy set theory, fuzzy logics wherein the DeMorgan laws hold are the most commonly studied; e.g. a survey of 30 different fuzzy logics is offered in [84]. As argued in [20], it seems likely that CFL will also be an infinite family of logics, giving us our sixth research question:

Q6: What complex fuzzy logics have been proposed, and what are their properties?

Finally, we inquire how CFS&L has been employed to solve practical, real-world problems. Type-1 fuzzy logic is generally applied to real-world problems through fuzzy inferential systems developed from expert knowledge, or through inductive machine learning. The former includes fuzzy control systems (which rely on the controller model for expert knowledge, e.g. [44,66]) and general inferential systems developed through knowledge elicitation as in [33], while the latter induces fuzzy sets and/or inferential rules from data e.g. [16,31]. Hybrids of the two are also possible, as in [28]. Intuitively, we would expect the same to hold true of CFS&L models for practical problems, leading to our seventh and final research question:

Q7: How are CFS&L operationalized (i.e. models designed and parameterized) for practical applications, and what problems have they been applied to?

2.2. Methodology

The two key methodological steps in a systematic review are to determine a search strategy to locate candidate primary sources, and to establish inclusion and exclusion criteria to reliably identify truly relevant primary sources. We examine both in this section.

CFS&L, as a relatively new topic, can still largely be defined by identifying the primary sources citing Ramot's original papers. Many citation databases exist, some curated (e.g. Scopus) and some based on search algorithms (e.g. Google Scholar, Microsoft Academic Search). As we are interested in maximum coverage of the literature, and will be examining each article for quality, it seems reasonable to turn to the algorithm-based databases. Comparing Google Scholar and Bing Academic on the citations for [68], we find that Google locates 99 citations while Bing Academic finds 45. Similarly, Google finds 116 citations for [67], while Bing finds 39 (as of March 2016). It seems plain that Google Scholar finds a substantially larger pool of candidate primary sources, and so the two sets of citations for [67,68] will be first group of primary sources. We also perform a Google Scholar search for the exact strings (i.e. in quotes) "complex fuzzy sets" and "complex fuzzy logic," which locate 290 and 381 references, respectively (these groups overlap heavily with the first). Finally, we examine the reference lists from each paper to identify any further candidates. Our pool of candidate primary sources is thus the union of these four groups.

Our inclusion criteria are simple: we will include any primary source that adds to the body of knowledge on CFS (fuzzy sets where the codomain Y of the membership function is a subset of the complex numbers; $Y \subseteq \mathbf{C} \wedge Y \not\subset \mathbf{R}$, for $\mathbf{C}$ the complex numbers and $\mathbf{R}$ the reals) or CFL (multivalued logics where the truth valuation set is a subset of the complex numbers as with CFS memberships). This includes both theoretical developments and applications. We exclude articles that merely contain the terms "complex fuzzy sets" or "complex fuzzy logic," or articles that cite [68] or [67], without contributing to the CFS&L body of knowledge. Clearly, these criteria require an exhaustive review of the candidate primary sources; given the size of our candidate pool, this is feasible.

Once we have identified all the primary sources that will be included in this review, we examine each paper to determine what contribution it makes to the CFS&L body of knowledge with respect to each of our research questions. Plainly, this is a qualitative review, having more in common with a taxonomic classification than a quantitative meta-analysis. Since primary sources may include multiple theoretical developments and applications thereof, we expect that many primary sources will appear in the discussion of multiple research questions in this review.

3. Representing uncertainty with complex fuzzy sets

In this section, we first examine research question (Q1), exploring the advantages that various authors have suggested for CFS&L. We then examine research questions (Q2) and (Q3), which focus on the values and functional forms of complex fuzzy membership functions, respectively.

3.1. Complex fuzzy sets and logic for reasoning under uncertainty

Research question (Q1) is fundamental to any new proposed logic; there must be some way in which that new proposal is superior to existing logics. Several authors have suggested possible advantages of CFS&L, beginning with Ramot's seminal papers. Ramot et al. suggested that problems dealing with periodic or recurring phenomena, such as representing solar activity, the effect of financial indicators on each other, and signal processing, can be modeled more faithfully by leveraging the phase component of CFS memberships [67,68]. Dick [20] continued this line of reasoning, suggesting that complex fuzzy sets might provide a general framework for reasoning about phenomena with approximately periodic behavior (which he terms *regularity*, following some of Zadeh's concepts in a 2003 keynote [95]). These are recurring phenomena that never repeat themselves exactly, such as traffic congestion in a

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big city [20]. Reflecting on these sources, the proposed advantage of CFS&L seems to be an efficient representation of recurring behaviors – what the time series forecasting community would call *seasonality*. Some recent work on defining a type-2 complex fuzzy set also asserts that CFS&L is an efficient way to represent seasonality [25,26].

Another line of argument is that complex fuzzy sets and complex-valued truth arise naturally from the mathematics of type-1 fuzzy sets and their extensions. For example, Kosheleva et al. [42] discuss the computational tractability of integration and global optimization (known to be NP-hard in general). Recent results indicate, however, that integration and global optimization can be solved efficiently for a subset of the class of general functions, specifically the analytic functions. Thus, smooth complex fuzzy membership functions represent a limit case for efficient integration and optimization – key operations in defuzzification and alternative selection in fuzzy inferential systems and fuzzy decision support systems, respectively. A second analysis by Kosheleva et al. [41] concerns expert evaluations of a fuzzy proposition (especially compound propositions using t-norms and t-conorms as logical connectives). They discuss the situation where an expert only evaluates a subset of the possible compound propositions (not the individual predicates themselves). One might attempt to deduce the expert's ratings of the individual predicates by treating the available observations as a system of equations to be solved. Complex-valued memberships then arise because, for many choices of t-norm and t-conorm, such a system of equations yields polynomial expressions to be solved, and many of their roots may be complex. Separately, Whalen [87] showed that selecting a negative value for p in the Schweizer–Sklar implication operator, $\mu_{A\to B}(x, y \mid p) = 1 - (\mu_A(x)^{-p} + (1-\mu_B(y))^{-p} - 1)^{-\frac{1}{p}}$, leads to complex truth values. This also makes it possible to have different ranges of selectivity and consensus in implication operators.

Finally, two more analyses have found that CFL possesses desirable mathematical-logic properties. Servin et al. [71] showed that the only distributive extensions of type-1 fuzzy logic to a two-dimensional construct are interval-valued fuzzy logic and complex-valued fuzzy logic. Greenfield and Chiclana [24] compared Ramot's pure complex fuzzy sets [68], Tamir et al.'s pure complex fuzzy sets [77] and type-2 fuzzy sets in terms of rationale, applications, definitions, structures and operations. One point that was raised is that the phase term in a CFS represents additional information (i.e. representing context in a linguistic variable [68]), but the secondary membership in a type-2 fuzzy sets represents a *deficit* of information (uncertainty).

Beyond the philosophical and mathematical arguments in favor of CFS, there is now a growing literature detailing useful applications of them. We will review this literature in detail in Section 6, but a brief summary also seems to be in order. Several papers have now demonstrated the utility of CFS in time series prediction [1,15,92]. However, other papers demonstrate the power of complex fuzzy sets in time series forecasting from a different point of view; they take advantage of the two-dimensional membership degree in the complex fuzzy sets to obtain more information about a system in order to forecast it better [48,51–53]. Moreover, function approximation, image restoration and knowledge discovery have all been efficiently accomplished using CFS-based algorithms [46,47,49,54]. CFS have also been employed as a very natural mechanism for modeling bivariate time series; this is known as the dual-output property [50,53]. Tamir et al. showed that events with fuzzy cyclic behavior, such as the stock market, can be captured by pure complex fuzzy sets [77]. Yager demonstrated an application of Pythagorean membership grades in multi-criteria decision making [88,89]. Alkouri et al. showed that their CAIFS are also effective in multi-criteria decision making [6]. From this summary, we find that there are indeed practical problems where CFS-based approaches have outperformed existing models.

3.2. Codomains of complex fuzzy membership functions

Perhaps the earliest form of CFS was proposed by Nguyen et al., in order to be able to use fuzzy logic in fields with true paradoxes such as philosophy and the humanities [65]. Their codomain was an arbitrary subset of the complex plane. Moses et al., then, introduced a two-dimensional membership grade for fuzzy subsets of the complex numbers, $\mu : C \to [0,1] \times [0,1]$ (the unit square) [62,63].

Ramot et al. proposed a complex fuzzy membership degree as [68]:

$$\mu_s(x) = r_s(x) \cdot e^{(j\omega_s(x))}, \quad j = \sqrt{-1} \tag{2}$$

where $r_s(x)$ is the magnitude and $\omega_s(x)$ the phase of the complex fuzzy set s and $x \in U$ is drawn from a universe of discourse U. Memberships are drawn from the unit disc in the complex plane ($r_s(x) \in [0,1]$), while the phase may take any real value. They suggested that the CFS can be an effective model for problems whose semantics change

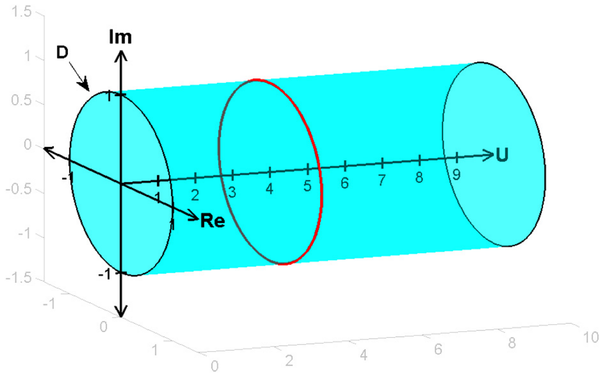

Fig. 1. Complex fuzzy membership function [20].

over time (i.e. the phase represents a changing context). Moreover, this is a straightforward generalization of type-1 fuzzy sets; when $\omega_s(x) = 0$, the CFS and all operations in Ramot's paper reduce to their type-1 counterparts, with the membership magnitude playing the role of the type-1 codomain.

Fig. 1 presents the membership codomain of a CFS from. We begin as usual with the universe of discourse U plotted on the x-axis. Then, the real and imaginary membership axes are placed at right angles (the z- and y-axis, respectively). Since memberships are restricted to the unit disc, we project it along the x-axis, forming a cylinder that includes all possible memberships for every point $x \in U$ [20].

Tamir et al. defined complex fuzzy sets with membership degrees drawn from the unit square $[0, 1] \times [0, 1]$. The memberships are given in Cartesian form, allowing both components of the membership to express type-1 fuzzy information [77], unlike Ramot's formulation. The sets are termed "pure" complex fuzzy sets and the membership grades are defined as [77]:

$$\mu(V, z) = \mu_r(V) + j\mu_i(z) \quad \mu_r, \mu_i \in [0, 1] \tag{3}$$

where V is a fuzzy set over a universe of discourse U, $z \in U$, $\mu_r(V)$ and $\mu_i(z)$ are the real and imaginary part of the complex fuzzy membership grade, $\mu(V, z)$ and can each take any values from the interval $[0, 1]$. $\mu_r(V)$ represents the degree to which the fuzzy set V belongs to the pure complex fuzzy set, and $\mu_i(z)$ represents the membership of z in V. Thus, a pure complex fuzzy set is actually a fuzzy set whose objects are fuzzy sets (i.e. a form of level 2 fuzzy set [40]). Tamir et al. interpreted pure complex fuzzy grades as a way to define fuzzy classes [77]. In general, fuzzy classes are fuzzy sets that can contain both fuzzy sets and objects. A *pure* fuzzy class of order M is a collection of pure fuzzy classes of order $M - 1$; a fuzzy set is thus considered a pure fuzzy class of order 0, and a pure fuzzy class of order 1 contains only fuzzy sets. Tamir defines a *complex fuzzy class* as a pure fuzzy class of order 1, defined by its pure complex membership grade (Eq. (3)). Assume that Γ is a complex fuzzy class, $\Gamma = \{V_i\}_{i=1}^{\infty}$, where V_i is a fuzzy set having objects on the universe of discourse U, $z \in U$; a pure complex fuzzy grade, $\mu_\Gamma(V, z) = \mu_r(V) + j\mu_i(z)$, assigns the membership of an object, z, to a class, Γ via its membership in $V \in \Gamma$. The complex fuzzy class can also be expressed as [77]:

$$\Gamma = \left\{ \left(V, z, \mu_\Gamma(V, z)\right) \mid V \in \Im(U), z \in U \right\} \tag{4}$$

for $\Im(U)$ the set of all type-1 fuzzy subsets of U. Moreover, the polar representation of a pure complex fuzzy grade is defined as [77]:

$$\mu(V, z) = r(V)e^{j\sigma\phi(z)} \tag{5}$$

where $r(V)$ and $\phi(z)$ are respectively the amplitude and phase of the pure complex fuzzy grade, which take values in $[0, 1]$, and σ is a scaling factor to keep the phase within the interval $(0, 2\pi]$. Transformation between the two proposed forms of pure complex fuzzy grade is defined as [77]:

$$\mu_\Gamma(V, z) = T_r\left(\mu_r(V) + j\mu_i(z)\right) = \mu_r(V)e^{j\theta\phi(z)} \tag{6}$$

$$\mu_\Gamma(V, z) = T_r\left(r(V)e^{j\theta\phi(z)}\right) = r(V) + j\phi(z) \tag{7}$$

where T_r is a coordinate transformation function. Tamir notes that maintaining the semantics of the fuzzy class Γ and the fuzzy set V under these transformations might be challenging, and hence the usual transforms of $r\sin\theta$ and $r\cos\theta$ might be inadequate; this is an area of future investigation.

Hata and Murase [27] also define complex fuzzy sets with a codomain of $[0, 1] \times [0, 1]$, as part of the creation of a complex-valued neuro-fuzzy system. Both the real and imaginary components of a complex number are treated as type-1 fuzzy sets, with the goal being to develop a more compact neural network. However, they also explicitly expect the universe of discourse for this CFS to be itself the complex numbers.

Yager defined Pythagorean membership grades as an ordered pair, $(A_Y(x), A_N(x))$, assigning membership and non-membership degree of each $x \in U$ to the fuzzy set A [88].

$$A_Y(x) = r(x)\cos\big(\theta(x)\big) \tag{8}$$

$$A_N(x) = r(x)\sin\big(\theta(x)\big) \tag{9}$$

where $r(x)$ and $\theta(x)$ are respectively the magnitude and phase of the Pythagorean membership vector. This new definition is an extension of intuitionistic fuzzy sets [10], with the main difference being in how the membership and non-membership components are constrained. Intuitionistic fuzzy sets must satisfy $A_Y(x) + A_N(x) \le 1$, while Pythagorean fuzzy sets are constrained by $A_Y(x)^2 + A_N(x)^2 \le 1$ [88]. Thus, Pythagorean fuzzy sets have a strictly larger membership codomain (the unit positive quarter-circle) than intuitionistic fuzzy sets. This is also a subset of Ramot's complex fuzzy grades, with phases limited to $\theta \in [0, \frac{\pi}{2}]$; Yager terms these the $\Pi - i$ numbers [89].

Salleh et al. defined the Complex Atanassov's Intuitionistic Fuzzy Set (CAIFS) which is an extension of the intuitionistic fuzzy sets [10]. In CAIFS, the membership and non-membership grades ($\mu_A(x)$ and $\gamma_A(x)$, respectively) are each complex numbers (drawn from the unit disc in the complex plane), under the constraint $|\mu_A(x) + \gamma_A(x)| \le 1$ [7].

In summation, five different codomains – the unit square, the unit circle, the unit positive quarter-circle, a subset of the cross-product of the unit disc with itself, and an arbitrary subset of the complex plane – have been proposed in the CFS literature to date. There is currently very limited evidence on the properties of these codomains, and no more general conclusions about what other codomains might be useful can currently be drawn.

3.3. Forms of complex fuzzy membership functions

Two classes of complex fuzzy membership functions have been proposed: sinusoidal and Gaussian membership function. Sinusoidal CFS membership functions over the unit disc codomain were introduced by Dick and his collaborators in [15,20] as:

$$\mu(x) = r\big(\omega(x)\big) \cdot e^{(j\omega(x))} \tag{10}$$

where

$$\begin{aligned} r\big(\omega(x)\big) &= d\sin\big(a\big(\omega(x)\big) + b\big) + c \\ \omega(x) &= x \end{aligned} \tag{11}$$

where $r(\theta)$ is amplitude and $\omega(x)$ is the phase of the membership grade of object $x \in U$, drawn from the universe of discourse U. The parameter a changes the frequency of the sine wave, b creates a phase shift, c shifts the wave vertically, and d changes the amplitude of the wave. Since the amplitude of complex fuzzy memberships is limited to [0, 1], the parameters must satisfy the following conditions:

$$0 \le d + c \le 1, \qquad 1 \ge c \ge d \ge 0 \tag{12}$$

Four different forms of Gaussian membership functions have been proposed [27,47,48,75], for either the unit square or unit disc codomains. The first (using the unit square codomain) is [50]:

$$c\,\text{Gaussian}(x, m, \sigma) = \text{Re}\big(c\,\text{Gaussian}(x, m, \sigma)\big) + j\,\text{Im}\big(c\,\text{Gaussian}(x, m, \sigma)\big) \tag{13}$$

where $j = \sqrt{-1}$, and Re(.) and Im(.) are the real and imaginary parts of the membership grade, defined as:

$$\text{Re}\big(c\,\text{Gaussian}(x, m, \sigma)\big) = \exp\left[-0.5\left(\frac{x-m}{\sigma}\right)^2\right] \tag{14}$$

$$\text{Im}\big(c\,\text{Gaussian}(x, m, \sigma)\big) = -\exp\left[-0.5\left(\frac{x-m}{\sigma}\right)^2\right] \times \left(\frac{x-m}{\sigma^2}\right) \tag{15}$$

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where $\{m, \sigma\}$ are the mean and spread of the Gaussian function, and $x \in U$ is an element of the universe of discourse U. The second form is defined for the unit disc codomain [48]:

$$c\,\text{Gaussian}(x, m, \sigma, \lambda) = r_s(x, m, \sigma)\exp\big(j\omega_s(x, m, \sigma, \lambda)\big) \tag{16}$$

where r_s and w_s are the amplitude and phase of the complex fuzzy grade, defined as:

$$r_s(x, m, \sigma) = \text{Gaussian}(x, m, \sigma) = \exp\left[-0.5\left(\frac{x-m}{\sigma}\right)^2\right] \tag{17}$$

$$\omega_s(x, m, \sigma, \lambda) = -\exp\left[-0.5\left(\frac{x-m}{\sigma}\right)^2\right] \times \left(\frac{x-m}{\sigma^2}\right) \times \lambda \tag{18}$$

where $\{m, \sigma, \lambda\}$ are mean and spread and phase frequency factor for the complex fuzzy set, $x \in U$ as above. The phase frequency factor provides an additional degree of freedom in fitting the membership function to data. Shoorangiz and Marhaban [75] also proposed a Gaussian membership function for the unit disc codomain as:

$$\mu(x) = A(x) \cdot e^{jP(x)} \tag{19}$$

where $A(x)$ and $P(x)$ are the amplitude and phase of the complex fuzzy membership function and defined as:

$$A(x) = \exp\left(-\left(\frac{x-c_A}{a_A}\right)^2\right) \tag{20}$$

$$P(x) = 2\pi\exp\left(-\left(\frac{x-c_P}{a_P}\right)^2\right) \tag{21}$$

Finally, Hata and Murase [27] proposed type-1 Gaussian membership functions for the real and imaginary components of a CFS (plainly implying the unit square codomain):

$$\mu_A^R(x_r) = \exp\left[-\frac{(x_r-a_r)^2}{b_r}\right] \tag{22}$$

$$\mu_A^I(x_i) = \exp\left[-\frac{(x_i-a_i)^2}{b_i}\right] \tag{23}$$

where a_r, b_r are the center and width of the Gaussian function for the real-valued component x_r, a_i, b_i are the center and width of the Gaussian for the imaginary component x_i, and $x_r + jx_i$ is the complex-valued input. These membership functions are used as the transfer functions of nodes in a complex-valued neural network. The centers and widths of all Gaussian membership functions are iteratively updated via complex-valued backpropagation.

CFS membership functions and their properties plainly remain a very young topic. Basic properties and manipulations (e.g. what are the core and support of a CFS? What is an α-cut?) have not yet been defined in general. Other functional forms have not yet been proposed (e.g. what are the equivalents to triangular or trapezoidal membership functions?). Furthermore, there has been almost no work in how we *interpret* a CFS. Sinusoids and Gaussian-type membership functions clearly have very different semantics. A Gaussian membership by its nature focuses on one segment of a universe of discourse; one region of feature space, one moment in time, etc. Sinusoids, by contrast, focus on recurring patterns. However, these general observations offer little guidance in associating an intuitive meaning to a specific CFS, or even a collection of them.

Linguistic variables – the association of a linguistic term with a type-1 fuzzy set – are a key part of the success of fuzzy logic, and we expect that the same will hold true for CFS. However, the two-dimensional nature of CFS membership functions has so far made interpreting them extremely difficult. We are aware of only three partial suggestions: first, Alkouri's definition of linguistic variables and hedges for CFS [8], which characterizes complex linguistic variables as a sextuple in which different linguistic values are associated with the magnitude and phase of the CFS, and used to represent uncertainty and periodicity, respectively. Linguistic hedges (as introduced by Zadeh [94]) were also extended to complex fuzzy sets. Six hedges are defined for CFS: *very A, very-very A, indeed A, a little A, slightly A, more or less A and extremely A*. Second, Tamir et al. suggest that a complex fuzzy class can be interpreted as a main term and a modifier [78], yielding propositions of the form "$x \ldots A \ldots B$" where A and B are linguistic values; e.g. "x is a volatile stock in a strong portfolio". Finally, Dick et al.'s suggestion that non-membership and anti-membership (respectively the imaginary and negative real axes of a CFS) could be used to model negations and antonyms from natural language [21].

In summation, the space of complex membership functions remains largely unexplored except for one particular form of sinusoids, and a few variations on the idea of a complex-valued Gaussian function. Likewise, there have only been a few explorations of the idea of complex linguistic variables. The vector-valued nature of CFS memberships currently renders interpretation very difficult.

Finally, while complex-valued extensions to fuzzy soft sets are technically outside the scope of this review, we briefly mention them here for convenience. Kumar and Bajaj [43] introduced complex intuitionistic fuzzy soft sets, which are an extension of intuitionistic fuzzy soft sets defined in [58]; different measurements of distance between these sets, and a definition of entropy for them were developed based on [32]. Rengarajulu [69] developed the idea of parameterized fuzzy soft sets introduced in [14] based on complex fuzzy sets.

4. Complex fuzzy set operations and relations

In this section we focus on research questions (Q4) and (Q5). Complex fuzzy set operations are reviewed first, as they apply equally to individual fuzzy sets and relations between fuzzy sets; we then take up the topic of compositions and projections of complex fuzzy relations.

4.1. Complex fuzzy set operations

Ramot proposed union and intersection operations for his CFS, along with three unique operations manipulating the phase of the CFS membership grade [68]. Consider two complex fuzzy sets, A and B, with membership degrees of $\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{j\omega_B(x)}$, respectively. The union and intersection of these two complex fuzzy sets, are defined as [68]:

$$\mu_{A\cup B}(x) = \left[r_A(x) \oplus r_B(x)\right] e^{j(\omega(x)_{A\cup B})} \tag{24}$$

$$\mu_{A\cap B}(x) = \left[r_A(x) \star r_B(x)\right] e^{j(\omega(x)_{A\cap B})} \tag{25}$$

where $\mu_{A\cup B}(x)$ represents union and $\mu_{A\cap B}(x)$ intersection of the complex fuzzy sets A and B, respectively, $\oplus$ can be any t-conorm, $\star$ is any t-norm, and $x \in U$ is an element of the universe of discourse U. $\omega_{A\cup B}$ and $\omega_{A\cap B}$ are application-dependent; Eqs. (26)–(32) give possible forms for $\omega_{A\cup B}$ (the same functions are suggested for $\omega_{A\cap B}$) [68]:

$$\omega_{A\cup B} = \omega_A + \omega_B \tag{26}$$

$$\omega_{A\cup B} = \max(\omega_A, \omega_B) \tag{27}$$

$$\omega_{A\cup B} = \min(\omega_A, \omega_B) \tag{28}$$

$$\omega_{A\cup B} = \begin{cases} \omega_A & r_A > r_B \\ \omega_B & r_B < r_A \end{cases} \tag{29}$$

$$\omega_{A\cup B} = \frac{r_A \cdot \omega_A + r_B \cdot \omega_B}{r_A + r_B} \tag{30}$$

$$\omega_{A\cup B} = \frac{\omega_A + \omega_B}{2} \tag{31}$$

$$\omega_{A\cup B} = \omega_A - \omega_B \tag{32}$$

The three novel operations are reflection, rotation and directional complex (DC) fuzzy complement. Reflection is defined as [68]:

$$Ref\left(\mu_A(x)\right) = r_A(x) \cdot e^{-j\omega_A(x)} \tag{33}$$

for μ_A as above. Rotation by θ radians is defined as [68]:

$$Rot_\theta\left(\mu_A(x)\right) = r_A(x) \cdot e^{j(\omega_A(x)+\theta)} \tag{34}$$

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Based on the rotation and traditional complement operations, Ramot et al. proposed the directional complex (DC) fuzzy complement as [68]:

$$\mu_{\overline{A}^\theta}(x) = c\big(r_A(x)\big) \cdot e^{j(\omega_A(x)+\theta)} \tag{35}$$

where $c(.)$ is any type-1 fuzzy complement, and θ is again the rotation in radians.

Zhang et al. [97] studied different operations on Ramot's CFS [68] when the membership phase is restricted to $[0, 2\pi]$. They extend the definition of a t-norm to the unit disc by taking a quasi-triangular norm over the phases, and a t-norm over the magnitudes, of both arguments. S-norms are likewise extended as a quasi-triangular norm over the phases, and an s-norm over the magnitudes of the arguments. Then, the union of two CFS A and B over the universe of discourse U with $\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{j\omega_B(x)}$, $x \in U$, is defined in Eq. (36) and proved to be (in their terms) an s-norm [97]:

$$\mu_{A\cup B}(x) = r_{A\cup B}(x) \cdot e^{j\omega_{A\cup B}(x)} = \max\big(r_A(x), r_B(x)\big) \cdot e^{j\max(\omega_A(x),\omega_B(x))}. \tag{36}$$

Likewise, the intersection between these two CFS was defined and proved to be a t-norm [97]:

$$\mu_{A\cap B}(x) = r_{A\cap B}(x) \cdot e^{j\omega_{A\cap B}(x)} = \min\big(r_A(x), r_B(x)\big) \cdot e^{j\min(\omega_A(x),\omega_B(x))}. \tag{37}$$

The complement of a CFS, C, with $\mu_C(x) = r_C(x) \cdot e^{j\omega_C(x)}$ was defined as [97]:

$$\mu_{\overline{C}}(x) = r_{\overline{C}}(x) \cdot e^{j\omega_{\overline{C}}(x)} = \big(1 - r_C(x)\big) \cdot e^{j(2\pi-\omega_C(x))} \tag{38}$$

It was shown that the involutive property for the complement function is also satisfied, $\overline{\overline{C}} = C$. Based on the definitions of union, intersection and complement proposed above, the following properties were obtained [97]:

$$\overline{A\cap B} = \overline{A} \cup \overline{B} \tag{39}$$

$$\begin{aligned}(A\cup B)\cap C &= (A\cap C)\cup(B\cap C)\\ (A\cap B)\cup C &= (A\cup C)\cap(B\cup C)\end{aligned} \tag{40}$$

$$\begin{aligned}(A\cup B)\cap A &= A\\ (A\cap B)\cup A &= A\end{aligned} \tag{41}$$

The following operations were also proposed [97]:

Complex fuzzy product (t-norm):

$$\mu_{A\circ B}(x) = r_{A\circ B} \cdot e^{j\omega_{A\circ B}(x)} = \big(r_A(x) \cdot r_B(x)\big) \cdot e^{j2\pi(\frac{\omega_A(x)}{2\pi}\cdot\frac{\omega_B(x)}{2\pi})} \tag{42}$$

Complex fuzzy Cartesian product:

$$\begin{aligned}\mu_{A_1\times A_2\times\cdots\times A_N}(x) &= r_{A_1\times A_2\times\cdots\times A_N}(x) \cdot e^{j\omega_{A_1\times A_2\times\cdots\times A_N}(x)}\\ &= \min\big(r_{A_1}(x_1), r_{A_2}(x_2), \cdots, r_{A_N}(x_N)\big) \cdot e^{j\min(\omega_{A_1}(x_1),\omega_{A_2}(x_2),\cdots,\omega_{A_N}(x_N))},\\ \text{where } x &= (x_1, x_2, \cdots, x_N) \in \underbrace{U\times U\times\cdots\times U}_{N}.\end{aligned} \tag{43}$$

Complex fuzzy probabilistic sum (s-norm):

$$\begin{aligned}\mu_{A\hat{+}B}(x) &= r_{A\hat{+}B}(x) \cdot e^{j\omega_{A\hat{+}B}(x)}\\ &= \big(r_A(x) + r_B(x) - r_A(x)\cdot r_B(x)\big) \cdot e^{j2\pi(\frac{\omega_A(x)}{2\pi}+\frac{\omega_B(x)}{2\pi}-\frac{\omega_A(x)}{2\pi}\cdot\frac{\omega_B(x)}{2\pi})}\end{aligned} \tag{44}$$

Complex fuzzy bold sum (s-norm):

$$\mu_{A\dot{\cup}B}(x) = r_{A\dot{\cup}B}(x) \cdot e^{j\omega_{A\dot{\cup}B}(x)} = \min\big(1, r_A(x) + r_B(x)\big) \cdot e^{j\min(2\pi,\omega_A(x)+\omega_B(x))} \tag{45}$$

Complex fuzzy bold intersection (t-norm):

$$\mu_{A\dot{\cap}B}(x) = r_{A\dot{\cap}B}(x) \cdot e^{j\omega_{A\dot{\cap}B}(x)} = \max\big(0, r_A(x) + r_B(x) - 1\big) \cdot e^{j\max(0,\omega_A(x)+\omega_B(x)-2\pi)} \tag{46}$$

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Complex fuzzy bounded difference:

$$\mu_{A|-|B}(x) = r_{A|-|B}(x) \cdot e^{j\omega_{A|-|B}(x)} = \max\left[0, r_A(x) - r_B(x) \cdot e^{j\max(0,\omega_A(x)-\omega_B(x))}\right] \tag{47}$$

Complex fuzzy symmetrical difference:

$$\mu_{A\nabla B}(x) = r_{A\nabla B}(x) \cdot e^{j\omega_{A\nabla B}(x)} = \left|r_A(x) - r_B(x)\right| \cdot e^{j|\omega_A(x)-\omega_B(x)|} \tag{48}$$

Complex fuzzy convex linear sum of min and max:

$$\begin{aligned} \mu_{A\|_\lambda B}(x) &= r_{A\|_\lambda B}(x) \cdot e^{j\omega_{A\|_\lambda B}(x)} \\ &= \left[\lambda \min\left(r_A(x), r_B(x)\right) + (1-\lambda)\max\left(r_A(x), r_B(x)\right)\right] \cdot e^{j[\lambda \min(\omega_A(x),\omega_B(x)) + (1-\lambda)\max(\omega_A(x),\omega_B(x))]}, \\ &(0 \le \lambda \le 1) \end{aligned} \tag{49}$$

Moreover, a new definition for distances between two complex fuzzy set was introduced as [97]:

$$d(A, B) = \max\left(\sup_{X\in U}\left|r_A(x) - r_B(x)\right|, \frac{1}{2\pi}\sup_{X\in U}\left|\omega_A(x) - \omega_B(x)\right|\right) \tag{50}$$

where $d(A, B)$ is distance of two complex fuzzy sets A and B. Then, based on this definition, δ-equalities of complex fuzzy sets were proposed; two complex fuzzy sets, A and B, are δ-equal, $A = (\delta)B$, if and only if $d(A, B) \le 1 - \delta$, $0 \le \delta \le 1$. Properties of the δ-equality of complex fuzzy sets were also discussed in the paper. Alternatively, Hamming, Euclidean, Normalized Hamming, and Normalized Euclidean distances and their boundaries were also obtained for complex fuzzy sets in [8].

Two aggregation operations have been proposed for CFS having the unit disc codomain. Ramot et al. proposed a vector aggregation defined as [67]:

$$v : \left\{a \mid a \in \mathbf{C}, |a| \le 1\right\}^n \to \left\{b \mid b \in \mathbf{C}, |b| \le 1\right\} \tag{51}$$

$$\mu_A(x) = v\left(\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x)\right) = \sum_{i=1}^{n} w_i \mu_{A_i}(x) \tag{52}$$

where $w_i \in \{a \mid a \in \mathbf{C}, |a| \le 1\}$ for all i, and $\sum_{i=1}^n |w_i| = 1$. The operation is, in fact, a complex-weighted vector sum. This allows the phase term to influence the aggregation. Ma et al. introduced the product-sum aggregation operation for complex fuzzy sets [57], which is an extension of the aggregation operation proposed by Ramot et al. Let $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ be complex fuzzy sets defined on a universe of discourse, U, with $\mu_{\tilde{A}_i}(u) = x_i(u) + jy_i(u)$ and $c = (c_1, c_2, \ldots, c_n)$ be a complex-valued vector with components drawn from the unit disc of the complex plane with $c_i = x_i' + jy_i'$; the product-sum aggregation operation is defined as [57]:

$$p_c(u) = r\left(p_c(u)\right)e^{j\arg(p_c(u))} \tag{53}$$

where

$$r\left(p_c(u)\right) = \min\left\{1, \sqrt[2]{\left(\sum_{i=1}^{n}\left(x_i(u)x_i' - y_i(u)y_i'\right)\right)^2 + \left(\sum_{i=1}^{n}\left(x_i(u)y_i' + y_i(u)x_i'\right)\right)^2}\right\} \tag{54}$$

$$\arg\left(p_c(u)\right) = \arctan\left(\frac{\sum_{i=1}^n (x_i(u)y_i' + y_i(u)x_i')}{\sum_{i=1}^n (x_i(u)x_i' - y_i(u)y_i')}\right) \tag{55}$$

Tamir et al. proposed union, intersection and complement operations for the pure complex fuzzy classes (unit square codomain) [77]. Assume a complex fuzzy class over the universe of discourse U, $\Gamma = \{V, z, \mu_\Gamma(V, z) \mid V \in \Im(U), z \in U\}$, is defined by a pure complex fuzzy grade $\mu(V, z) = \mu_r(V) + j\mu_i(z)$; the complement of this class is defined as [77]:

$$c\left(\mu_\Gamma(V, z)\right) = c\left(\mu_r(V)\right) + jc\left(\mu_i(z)\right) \tag{56}$$

where $c(.)$ is the complement function, defined as the standard fuzzy complement, $c(\mu_x(y)) = 1 - \mu_x(y)$. The union and intersection of two complex fuzzy classes defined as $\Gamma = \{V, z, \mu_\Gamma(V, z) \mid V \in \Im(U), z \in U\}$ and $\Psi = \{T, z, \mu_\Psi(T, z) \mid T \in \Im(U), z \in U\}$ are given by [77]:

$$\mu_{\Gamma\cup\Psi}(W,z)=\left(\mu_{\Gamma_r}(V)\oplus\mu_{\Psi_r}(T)\right)+j\left(\mu_{\Gamma_i}(z)\oplus\mu_{\Psi_i}(z)\right) \tag{57}$$

$$\mu_{\Gamma\cap\Psi}(W,z)=\left(\mu_{\Gamma_r}(V)\odot\mu_{\Psi_r}(T)\right)+j\left(\mu_{\Gamma_i}(z)\odot\mu_{\Psi_i}(z)\right) \tag{58}$$

where $\mu_{\Gamma\cup\Psi}(W,z)$ and $\mu_{\Gamma\cap\Psi}(W,z)$ are union and intersection functions, respectively; $W\in 2^U$ and $\oplus$ is any t-conorm and $\odot$ is any t-norm operation.

Yager et al. proposed union, intersection and complement operations for two Pythagorean fuzzy sets $A(x)=(A_Y(x),A_N(x))$ and $B(x)=(B_Y(x),B_N(x))$ as follows [89]:

$$A(x)\cup B(x)=\left(\max\left(A_Y(x),B_Y(x)\right),\min\left(A_N(x),B_N(x)\right)\right) \tag{59}$$

$$A(x)\cap B(x)=\left(\min\left(A_Y(x),B_Y(x)\right),\max\left(A_N(x),B_N(x)\right)\right) \tag{60}$$

$$\overline{A}(x)=\left(A_N(x),A_Y(x)\right) \tag{61}$$

He also proposed the following function for mapping a $\Pi-i$ number to the unit interval [0, 1] as a way of ordering Pythagorean fuzzy values [89].

$$F(A)=\frac{1}{2}+r(x)\left(\frac{1}{2}-\frac{2\theta(x)}{\pi}\right) \tag{62}$$

where $r=\sqrt{A_Y^2+A_N^2}$ and $\theta=\tan^{-1}\frac{A_N}{A_Y}$. [89] also proposed an aggregation operation based on the geometric mean for q different criteria that have Pythagorean fuzzy degrees, C_j, $j=1,\dots,q$, as:

$$C(x)=\prod_{k=1}^{q}C_k(x)^{w_k}=\prod_{k=1}^{q}\left(r_k(x)e^{j\theta_k(x)}\right)^{w_k}=\prod_{k=1}^{q}\left(r_k(x)\right)^{w_k}e^{j\sum w_k\theta_k(x)} \tag{63}$$

where $\sum_{k=1}^{q}w_k=1$.

Dick et al. proposed two new complex fuzzy conjunction and disjunction operators which form a DeMorgan triple with the complement of Eq. (61) [21]:

$$x\wedge y=\left(\text{absmin}(x_1,y_1),\text{absmax}(x_2,y_2)\right) \tag{64}$$

$$x\vee y=\left(\text{absmax}(x_1,y_1),\text{absmin}(x_2,y_2)\right) \tag{65}$$

where $x=(x_1+jy_1)$, $y=(x_2+jy_2)$, $x,y\in\mathbf{D}$ (the unit disc of the complex plane), and absmax and absmin are reflexive functions defined as [21]:

$$\text{absmax}(x,y)=\begin{cases}x & \text{if } |x|>|y|\\ y & \text{if } |x|<|y|\\ |x| & \text{if } |x|=|y|\wedge x\neq y\end{cases} \tag{66}$$

$$\text{absmin}(x,y)=\begin{cases}x & \text{if } |x|<|y|\\ y & \text{if } |x|>|y|\\ -|x| & \text{if } |x|=|y|\wedge x\neq y\end{cases} \tag{67}$$

Alkouri and Salleh [7] extended the complements, unions and intersections defined in [68,97] for complex Atanassov's intuitionistic fuzzy sets. They then extended the intuitionistic possibility and necessity measures [10] for CAIFS [9]. The related topic of complex vague relations was investigated by Al-Husban and Salleh in [5].

It is fair to say that there has been a growing volume of work in identifying and evaluating new complex fuzzy set operations. Much like the field of fuzzy logic, the choice of set operators defines different complex fuzzy set theories, having different properties. A significant focus of these efforts has been in finding DeMorgan triples – combinations of unions, intersections and complements that together satisfy the DeMorgan laws of Boolean algebra. These triples are expected to have pride of place in complex fuzzy set theory, much as they do in type-1 fuzzy set theory [40], and the search for them is ongoing. At a deeper level, two further questions have yet to receive attention. Firstly, are there *classes* of functions that are equivalent to the set of complex fuzzy unions or intersections (analogous to the t-norms and t-conorms for fuzzy intersection and union)? Might these classes be specific to specific membership codomains, or could they be common to all forms of CFS&L? Secondly, do generators exist for DeMorgan triples in CFS&L (analogous to the Dombi, Frank, Schweizer, etc. generators defined for type-1 fuzzy logic [40])? Again, would such generators be common to all membership codomains, or codomain-specific?

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4.2. Complex fuzzy relations

Ramot et al. defines complex fuzzy relations as the degree and phase of association, interaction and interconnectedness between elements of different universe of discourses [68]. Let U and V be two universes of discourse; the complex fuzzy relation between them is a complex fuzzy subset of the product space $U \times V$, characterized by a complex-valued membership function in the unit disc, $\mu_R(x, y)$, and defined as [68]:

$$R(U, V) = \{((x, y), \mu_R(x, y)) \mid (x, y) \in U \times V\} \tag{68}$$

Compositions of complex fuzzy relations either defined on the same or different product spaces were proposed by Ramot et al. in [67]. When the product spaces are the same, the complex fuzzy union and intersection operations from the previous section are applicable; otherwise they define the composition as follows. Consider universes of discourse U, V, W, and the relations $R(U, V)$ with $\mu_R(x, y) = r_R(x, y) \cdot e^{j\omega_R(x,y)}$ and $S(V, W)$ with $\mu_S(y, z) = r_S(y, z) \cdot e^{j\omega_S(y,z)}$; the composition is [67]:

$$\mu_{R\circ S}(x, z) = r_{R\circ S}(x, z) \cdot e^{j\omega_{R\circ S}(x,z)} \tag{69}$$

where

$$r_{R\circ S}(x, z) = \sup_{y\in V}\left[r_R(x, y) \star r_S(y, z)\right] \tag{70}$$

$$\omega_{R\circ S}(x, z) = f\left[g\left(\omega_R(x, y), \omega_S(y, z)\right)\right] \tag{71}$$

where $\star$ is any t-norm, and g and f are the membership phase equivalents of $\star$ and sup(.), respectively; Ramot et al. proposed four possible choices for f [67]:

$$\omega_{R\circ S}(x, z) = \sup_{y\in V}\left[g\left(\omega_R(x, y), \omega_S(y, z)\right)\right] \tag{72}$$

$$\omega_{R\circ S}(x, z) = \inf_{y\in V}\left[g\left(\omega_R(x, y), \omega_S(y, z)\right)\right] \tag{73}$$

$$\begin{aligned} &\omega_{R\circ S}(x, z) = g\left(\omega_R(x, y'), \omega_S(y', z)\right) \\ &y' = \left\{y \mid \sup_{y\in V}\left[r_R(x, y) \star r_S(y, z)\right]\right\} \end{aligned} \tag{74}$$

where inf() is the infimum operator.

Zhang et al. [96] proposed operations for complex fuzzy relations for Ramot's complex fuzzy relations, again limiting membership phase to $[0, 2\pi]$. For two complex fuzzy relations defined on the same product space, $\mu_A(x, y) = r_A(x, y) \cdot e^{j\omega_A(x,y)}$ and $\mu_B(x, y) = r_B(x, y) \cdot e^{j\omega_B(x,y)}$, the complex fuzzy union and intersection of the relations are defined as [96]:

Union:

$$\begin{aligned} \mu_{A\cup B}(x, y) &= r_{A\cup B}(x, y) \cdot e^{j\omega_{A\cup B}(x,y)} \\ &= \max\left(r_A(x, y), r_B(x, y)\right) \cdot e^{j\max(\omega_A(x,y),\omega_B(x,y))} \end{aligned} \tag{75}$$

Intersection:

$$\begin{aligned} \mu_{A\cap B}(x, y) &= r_{A\cap B}(x, y) \cdot e^{j\omega_{A\cap B}(x,y)} \\ &= \min\left(r_A(x, y), r_B(x, y)\right) \cdot e^{j\min(\omega_A(x,y),\omega_B(x,y))} \end{aligned} \tag{76}$$

Moreover, the distance of two complex fuzzy relations defined on the same product space was defined as [96]:

$$d(A, B) = \max\left(\sup_{(x,y)\in U\times V}\left|r_A(x, y) - r_B(x, y)\right|, \frac{1}{2\pi}\sup_{(x,y)\in U\times V}\left|\omega_A(x, y) - \omega_B(x, y)\right|\right) \tag{77}$$

Compositions of the complex fuzzy relations, $R(U, V)$ with $\mu_R(x, y) = r_R(x, y) \cdot e^{j\omega_R(x,y)}$ and $S(V, W)$ with $\mu_S(y, z) = r_S(y, z) \cdot e^{j\omega_S(y,z)}$, $x \in U$, $y \in V$, $z \in W$, are defined as [96]:

Sup–min composition:

$$\begin{aligned}\mu_{R\circ S}(x,z) &= r_{R\circ S}(x,z)\cdot e^{j\omega_{R\circ S}(x,z)}\\ &= \sup_{y\in V}\min\big(r_R(x,y), r_S(y,z)\big)\cdot e^{j\,\sup\min_{y\in V}(\omega_R(x,y),\omega_S(y,z))}\end{aligned} \tag{78}$$

Sup-product composition:

$$\mu_{R\circ S}(x,z) = \big(r_R(x,y)\cdot r_S(y,z)\big)\cdot e^{j2\pi((\frac{\omega_R(x,y)}{2\pi}\cdot\frac{\omega_S(y,z)}{2\pi}))} \tag{79}$$

Sup-Lukasiewicz composition

$$\begin{aligned}&\mu_{R\circ S}(x,z) = \big(r_R(x,y)*r_S(y,z)\big)\cdot e^{j(\omega_R(x,y)*\omega_S(y,z))}\\ &\omega_R(x,y)*\omega_S(y,z) = 2\pi\max\left(0, \frac{\omega_R(x,y)}{2\pi}+\frac{\omega_S(y,z)}{2\pi}-1\right)\end{aligned} \tag{80}$$

Das et al. in [17] introduced complement and projection operations for Ramot's complex fuzzy relations $\mu_R(x,y) = r_R(x,y)e^{-j\omega_R(x,y)}$, $x\in U$, $y\in V$, as:
Complement

$$c\big(\mu_R(x,y)\big) = \big[1-r_R(x,y)\cos\big(\omega_R(x,y)\big)\big]+i\big[1-r_R(x,y)\sin\big(\omega_R(x,y)\big)\big] \tag{81}$$

1st projection

$$\begin{aligned}R^{(1)} &= \big\{\big(x,\mu_R^{(1)}(x,y)\big)\big\} = \left\{\left(x,\max_y \mu_R(x,y)\right) \mid (x,y)\in U\times V\right\}\\ &= \left(x,\max_y r_R(x,y)\cdot e^{i\max_y \omega_R(x,y)}\right)\end{aligned}$$

2nd projection

$$\begin{aligned}R^{(2)} &= \big\{\big(x,\mu_R^{(2)}(x,y)\big)\big\} = \left\{\left(y,\max_x \mu_R(x,y)\right) \mid (x,y)\in U\times V\right\}\\ &= \left(y,\max_x r_R(x,y)\cdot e^{i\max_x \omega_R(x,y)}\right)\end{aligned}$$

Total projection

$$R^{(T)} = \max_x\max_y\big\{\mu_R(x,y)\mid (x,y)\in U\times V\big\} \tag{82}$$

Much as with complex fuzzy operations, complex fuzzy relations and operations thereon have been studied for a few instances, but no unifying theories have yet been put forward. For instance, while *sup-star* composition has been studied, inf-ω composition has yet to be examined for complex fuzzy relations. Likewise, the study of projections, extensions and closures of complex fuzzy relations are in their infancy.

5. Complex fuzzy logic

Research question (Q6) focuses on the complex-valued logical systems that have been developed. Complex Fuzzy Logic (CFL) was first proposed by Ramot et al. in [67], using the same unit-disc codomain as his CFS in [68]. This yields an infinite-valued logic wherein the truth values are vectors from the unit disc. This makes CFL one of the very few vector-valued logics in existence; vector logic, matrix logic and quantum logic also have multi-dimensional truth values [61,74,22]. Karacay [35] studied harmonic analysis on fuzzy sets by extending fuzzy logic to the unit circle $\{z : z\in\mathbf{C}, |z|=1\}$.

Dick in 2005 interpreted Ramot's treatment of phase as a relative quantity [67,68] as rotational invariance, meaning that if membership grades of two complex fuzzy sets are rotated with φ radians about the origin, the union, intersection and complement of the complex fuzzy sets are also rotated with the same phase, φ. He showed that the algebraic product and vector negation, $f(x) = -x$, cannot be the conjunction and negation operations in a rotationally invariant logic [20]. He then studied the amplitude and phase of complex fuzzy sets simultaneously, using vector logic [60,61]

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Table 1
Basic complex propositional fuzzy logic connectives.

Operation	Interpretation
Negation	$I('P) = 1 + j - I(p)$
Implication	$I(P \to Q) = \min(1, 1 - I(p_r) + I(p_r)) + j \cdot \min(1, 1 - I(p_i) + I(q_i))$
Conjunction	$I(P \otimes Q) = \min(I(p_r), I(p_r)) + j \cdot \min(I(p_i), I(q_i))$
Disjunction	$I(P \oplus Q) = \max(I(p_r), I(p_r)) + j \cdot \max(I(p_i), I(q_i))$

as framework. It was shown that algebraic product is a possible candidate for the conjunction operator under these assumptions, and the existence of a dual disjunction operator was proved.

Tamir et al. [79] introduced a generalized propositional complex fuzzy logic based on the Łukasiewicz logic system. Modus ponens was adopted as the rule of inference and the following connectives are defined, as in Table 1.

Tamir et al. [82] proposed an extended complex Post logical system based on the extended Post system of order $p > 2$ by Di Zenzo [19]. One of the possible applications of the proposed system is in discrete processes such as digit signal processing (DSP), real time applications and embedded systems. Tamir et al. [80] reviewed the axiom-based complex fuzzy logic and sets.

Dick et al. developed two complete and distributive lattices for Pythagorean fuzzy sets (PFS) based on Pythagorean fuzzy union and intersection (Eqs. (64)–(65)), and on the ranking function (Eq. (62)) proposed in [89]. He then developed two new complete and distributive lattices for complex fuzzy sets based on extending PFS to the whole unit disc. This extension needs a definition for negations of the two axes in PFS ($\mu, \neg\mu$: the membership and non-membership degrees in PFS) to convert it to a complete unit disk; the negations are called anti-membership and anti-non-membership ($\overline{\mu}, \neg\overline{\mu}$). The first lattice is based on the assumption that $\mu = \neg\overline{\mu}$ and $\neg\mu = \overline{\mu}$. There is, thus, a mapping from unit disk to the PFS. The map is used to define a partial ordering over the unit disc. The second lattice is based on the assumption that $\mu \neq \neg\overline{\mu}$ and $\neg\mu \neq \overline{\mu}$; two new complex fuzzy union and intersection operators (Eqs. (64)–(65)) are defined and proven to form the lattice. Moreover, it is shown that Pythagorean negation (Eq. (61)) forms a DeMorgan triple with the new union and intersection operators. Dick suggests that this new logic can be used to represent antonym and negation from natural language [21].

To date, applications of complex fuzzy logic follow a similar approach as type-1 fuzzy logic: a fuzzy rulebase is generated (whether by induction or expert knowledge), and then used to infer conclusions for specific input conditions. While our review of these applications will be discussed in Section 6, the theoretical basis of these applications will be reviewed in the remainder of this section.

Ramot proposed a complex fuzzy implication for the unit-disc codomain in [67]. Consider the rule: *If x is A then y is B*, where x and y are variables taken from two different universe of discourses, U and V, respectively, and A and B are complex fuzzy sets defined on the corresponding universe of discourses; a complex fuzzy implication is characterized by the membership function $\mu_{A\to B}(x, y) = r_{A\to B}(x, y) \cdot e^{j\omega_{A\to B}(x,y)}$. The implication function is the algebraic product:

$$\mu_{A\to B}(x, y) = \mu_A(x) \cdot \mu_B(y) = r_A(x) \cdot r_B(y) \cdot e^{j(\omega_A(x)+\omega_B(y))} \tag{83}$$

Zhang et al. also studied more implication operations for CFS with the unit disk codomain, and with membership phase limited to $[0, 2\pi]$, as follows [96]:

Dienes–Rescher implication operator:

$$\mu_{A\to B}(x, y) = \max\big(1 - r_A(x), r_B(y)\big) \cdot e^{j \max(2\pi - \omega_A(x), \omega_B(y))} \tag{84}$$

Lukasiewicz implication operator:

$$\begin{aligned} &\mu_{A\to B}(x, y) = \big(r_A(x) \cdot r_B(y)\big) \cdot e^{j(\omega_A(x)*\omega_B(y))} \\ &\omega_A(x) * \omega_B(y) = 2\pi \max\left(0, \frac{\omega_A(x)}{2\pi} + \frac{\omega_B(y)}{2\pi} - 1\right) \end{aligned} \tag{85}$$

Zadeh implication operator:

$$\mu_{A\to B}(x, y) = \max\big(1 - r_A(x), \min\big(r_A(x), r_B(y)\big)\big) \cdot e^{j \max(2\pi - \omega_A(x), \min(\omega_A(x), \omega_B(y)))} \tag{86}$$

Table 2
Basic ŁΠ CFL connectives.

Operation	Interpretation
Ł-Implication	$I(P \to_L Q) = \min(1, 1 - I(p_r) + I(q_r)) + j \cdot \min(1, 1 - I(p_i) + I(q_i))$
Π-Implication	$I(P \to_\Pi Q) = \min(0, I(p_r)/I(q_r)) + j \cdot \min(0, I(p_i)/I(q_i))$
Π-Conjunction	$I(P \otimes Q) = I(p_r) \cdot I(q_r) + j \cdot (I(p_i) \cdot I(q_i))$

Godel implication operator:

$$\begin{aligned}
&\mu_{A\to B}(x,y) = r_{A\to B}(x,y) \cdot e^{j\omega_{A\to B}(x,y)} \\
&r_{A\to B}(x,y) = \begin{cases} 1 & \text{if } r_A(x) \le r_B(y) \\ r_B(y) & \text{otherwise} \end{cases} \\
&\omega_{A\to B}(x,y) = \begin{cases} 2\pi & \text{if } \omega_A(x) \le \omega_B(y) \\ \omega_B(y) & \text{otherwise} \end{cases}
\end{aligned} \tag{87}$$

Mamdani implication operator:

$$\mu_{A\to B}(x,y) = \min\bigl(r_A(x), r_B(y)\bigr) \cdot e^{j\min(\omega_A(x),\omega_B(y))} \tag{88}$$

Mamdani product implication operator:

$$\mu_{A\to B}(x,y) = \bigl(r_A(x) \cdot r_B(y)\bigr) \cdot e^{j2\pi(\frac{\omega_A(x)}{2\pi} \cdot \frac{\omega_B(y)}{2\pi})} \tag{89}$$

Fodor implication operator:

$$\begin{aligned}
&\mu_{A\to B}(x,y) = r_{A\to B}(x,y) \cdot e^{j\omega_{A\to B}(x,y)} \\
&r_{A\to B}(x,y) = \begin{cases} 1 & \text{if } r_A(x) \le r_B(y) \\ \max(1 - r_A(x), r_B(y)) & \text{otherwise} \end{cases} \\
&\omega_{A\to B}(x,y) = \begin{cases} 2\pi & \text{if } \omega_A(x) \le \omega_B(y) \\ \max(2\pi - \omega_A(x), \omega_B(y)) & \text{otherwise} \end{cases}
\end{aligned} \tag{90}$$

Reichenbach product implication:

$$\mu_{A\to B}(x,y) = \bigl(1 - r_A(x) + r_A(x) \cdot r_B(y)\bigr) \cdot e^{j(2\pi - \omega_A(x) + 2\pi\frac{\omega_A(x)}{2\pi} \cdot \frac{\omega_B(y)}{2\pi})} \tag{91}$$

Ramot et al. also used the generalized Modus Ponens as the rule of inference in complex fuzzy logic. In the case of multiple complex fuzzy rules, they proposed using the vector aggregation operation (Eq. (52)) to combine the rule consequents into a final output. As Eq. (52) is a vector sum of the outputs, it incorporates membership phase. If the phase terms are aligned, the amplitude of the final output increases; otherwise it can decrease. (This concept has been referred to as *rule interference* in [1,15,59,90–92], and is a key element of the Adaptive Neuro-Complex Fuzzy Inferential System (ANCFIS) discussed in depth in Section 6.) In the defuzzification step, the phase term of the final output is ignored and any of the traditional defuzzification methods can be applied on the amplitude in order to obtain a crisp output [67].

Tamir et al. studied the axiomatization of complex fuzzy logic for the pure complex fuzzy sets defined in [77]. He developed propositional, first order predicate, and generalized propositional complex fuzzy logic, as well as a logical system for discrete complex fuzzy sets [78,79,82]. Similarly to fuzzy propositions, complex fuzzy propositions are assigned a truth value, in this case from the unit square $[0,1] \times [0,i]$. A complex fuzzy proposition P can be defined as "$x \cdots A \cdots B$" where A and B are linguistic values, and ... shows natural language constants; and it can be interpreted as $I(P) = I(p_r) + j \cdot I(p_i)$ or $I(P) = I(r(p)) \cdot e^{j\sigma I(\theta(p))}$ (σ is a scaling factor in the interval $(0, 2\pi]$) where $I(p_r)$ or $I(r(p))$ denotes an interpretation of term A, and $I(p_i)$ or $I(\theta(p))$ is an interpretation of term B.

Propositional and first order predicate complex fuzzy logic are defined in [78], and are denoted as ŁΠ CFL and ŁΠ∀ CFL, respectively. Propositional complex fuzzy logic uses a set of basic and derived connectives along with a set of axioms from [11] to combine complex fuzzy propositions; for two complex fuzzy propositions P and Q, Tables 2 and 3 denote the connectives:

Table 3
Derived ŁΠ CFL connectives.

Operation	Interpretation
Ł-Negation	$I(\neg P) = 1 + j - I(p)$
Π-Delta	$\Delta(I(p)) = 1$ if $I(p) = 1 + j$; else $\Delta(I(p)) = 0 + j0$
Equivalence	$I(P \leftrightarrow Q) = I(P_r \to_L Q_r) \otimes I(Q_r \to_L P_r) + j \cdot (I(P_i \to_L Q_i) \otimes I(Q_i \to_L P_i))$
$P \ominus Q$	$I(P \ominus Q) = \max(0, I(p_r) - I(q_r)) + j \cdot \max(0, I(p_i) - I(q_i))$

Modus ponens and product necessitation are the inference rules of ŁΠ CFL. The first order predicate complex fuzzy logic, ŁΠ∀ CFL, extends propositional complex fuzzy logic by including predicates, constant, variables, the quantifier ∀, and the identity sign =. It follows axioms defined in by Běhounek et al. [11], and also adds generalization as one of the rule inferences. Tamir then extended the approach used in [11] and proposed a formal definition for complex fuzzy class theory (CFCT) based on ŁΠ∀ CFL [78].

Once again, we find that the study of CFL is in its infancy, with a few instances having been studied and their properties examined in varying degrees of detail. Some of the issues we can identify include identifying possible truth valuation sets – and then determining what useful lattices can be formed over them; what propositional and predicate logics can be formed by applying inference rules besides *modus ponens*; how the classes of S-implications and R-implications translate to CFL, and what other classes of implications might exist; and once again, how to meaningfully relate CFL propositions to natural-language expressions. We should also acknowledge the complex-valued propositional logic (S-logic) introduced Sgurev in [72], a six-valued logic containing imaginary logic variables $\{i, \neg i\}$, real logic variables $\{T, F\}$ and two basic states of S-logic $\{T \vee i, F \wedge \neg i\}$.

Very recently, a few authors have investigated complex fuzzy algebraic structures besides lattices. While this topic is outside of our research questions, we will include a brief discussion of these articles for completeness. Husban and Salleh proposed the idea of a complex fuzzy space, and ring structures within it, in [3]. They further develop these concepts, introducing complex fuzzy hyperrings in [2] and complex fuzzy normal subgroups in [4]. Complex fuzzy matrices (having elements of the form $a + bj$, with $0 \leq a + b \leq 1$, $j = \sqrt{-1}$) are studied by Zhao and Ma in [98].

6. Applications of complex fuzzy sets

Our final research question examines how CFS&L is used in practical applications. We begin by examining how CFS have been operationalized; CFS (or indeed, fuzzy sets in general) are not directly "applied" to a practical problem, but are rather a fundamental concept in constructing an intelligent system to solve a problem (e.g. a fuzzy inferential system, or a neuro-fuzzy network). We then review the specific applications that have been proposed.

Dick and his collaborators [15,59] proposed the Adaptive Neuro-Complex Fuzzy Inferential System (ANCFIS), which realizes the complex fuzzy logic proposed by Dick [20] and Ramot [67]. ANCFIS is based on the well-known ANFIS system proposed by Jang [31], although there are several differences between them. Firstly, ANCFIS uses sinusoidal membership functions as defined in Eqs. (10)–(12). Fuzzification is accomplished by convolving a window of the time series with the sinusoidal memberships. An additional layer implements *rule interference* via the dot product (rather than the vector aggregation proposed by Ramot et al. [67]). The learning algorithm is a hybrid as in ANFIS, with both architectures using least-squares optimization in the forward pass. However gradient descent in the ANCFIS backward pass is combined with a new chaotic simulated annealing technique. ANCFIS has been applied to time series forecasting (both univariate and multivariate), and has been shown to create a very accurate and compact forecasting model. Aghakhani and Dick [1] proposed an online learning algorithm for ANCFIS. The architecture uses recursive least-squares instead of the least-squares algorithm in the forward pass, and replaces chaotic simulated annealing with the downhill-simplex algorithm in the backward pass. Yazdanbakhsh et al. [92] applied ANCFIS to predict solar power output. Yazdanbakhsh and Dick [91] investigated different input formats for ANCFIS, and they studied multivariate time series prediction with ANCFIS in [90].

Li and Chiang [49] proposed a different variation of ANFIS called the Complex Neuro-Fuzzy System (CNFS) for function approximation. The system has a 6-layer network and uses the basic complex fuzzy membership function introduced by Ramot et al. [68]. Consequent and premise parameters are updated by Recursive Least Squares Estimation (RLSE) and Particle Swarm Optimization (PSO), respectively. Li and Chiang [50] extended their work in [49] by using the Gaussian-type complex fuzzy set defined in Eqs. (13)–(15). This paper also proposed the "dual-output property," which refers to treating the real and imaginary components of the complex output (in Cartesian form) as separate variates. Li et al. [54] applied the CNFS proposed in [50] for adaptive image noise cancellation, and Li and Chan [47] used the artificial bee colony (ABC) algorithm instead of PSO, applying it for adaptive image noise cancellation as well. Li and Chan [46], as well as Li [45] used the CNFS from [47] for knowledge discovery and adaptive image restoration, respectively.

Li and Chiang [48] replaced PSO in the CNFS of [50] with the hierarchical multi-swarm particle swarm optimization (HMSPSO) algorithm, combined with the complex Gaussian membership function defined in Eqs. (16)–(18) for time series forecasting. They also applied the system proposed in [48] for financial time series forecasting [51]. Thirunavukarasu et al. [83] devised a new hybrid learning algorithm based on PSO-GA (particle swarm optimization and genetic algorithm) and RLSE for the CNFS proposed in [48]. Li et al. [52] proposed a CNFS using the Gaussian membership function introduced in [48] and PSO-RLSE learning. To minimize the rulebase of CNFS, a clustering algorithm called FCM-Based Splitting Algorithm (FBSA) is employed [76]. Li and Chiang [53] replace the linear consequent function in CNFS with an ARIMA model [52]. This work also employs the "dual-output property" to naturally handle bi-variate time series.

Shoorangiz and Marhaban [75] proposed a five-layer network, also based on ANFIS, called the Adaptive Complex Neuro-Fuzzy Inferential System (ACNFIS). The network updates its premise and consequent parameters with least square estimation (LSE) and the Levenberg–Marquardt algorithm, respectively; the membership functions used in the network are defined in Eqs. (21)–(23). The algorithm was tested on function approximation problems. Hata and Murase [27] also designed their complex-valued neuro-fuzzy system for function-approximation problems. The inputs to their network are notionally complex numbers; however, they are in fact simply pairs of real-valued features, combined together to reduce the size of the neural network. The network is tested on samples from five real-valued analytical functions.

The complex-valued neuro-fuzzy systems above are all plainly relatives of general complex-valued neural networks (CVNNs) [29,30]. In general, the relationship is very similar to that between real-valued neural networks and neuro-fuzzy systems based on type-1 fuzzy sets. That is, the neuro-fuzzy systems employ different activation functions than the generic CVNNs, and the neuron connections through the network typically represent fuzzy rules. Some (e.g. ANCFIS [15]) employ complex-valued backpropagation, while others (e.g. the CNFS architectures [47]) employ derivative-free optimizations for learning. Finally, there are different strategies for determining a final output from the system; ANCFIS converts all network signals to real values in its fourth layer (rule interference), resulting in real-valued outputs. CNFS, on the other hand, simply interprets the complex-valued output as two real-valued outputs (the "dual-output property").

Ma et al. [57] developed a prediction method based on complex fuzzy sets in order to solve multiple periodic factor prediction problems in multisensory data fusion applications containing semantic uncertainty and periodicity. The method first represents observations of each factor by a complex-valued membership grade; the values are assigned based on historical knowledge of the relationship between the factors and the event. Then the complex-valued membership grades are combined by the product sum aggregation proposed in the paper (Eqs. (53)–(55)). In the prediction step, a product-sum aggregation of predicted factors is calculated and compared against the model; the test input is labeled to match the most-similar observation from the training dataset.

Ma et al. [56] proposed a conceptual method for modeling residential utility consumption based on complex fuzzy sets introduced in [68], using complex fuzzy sets for modeling uncertainty and seasonal features of infrastructure utility consumption in data-driven forecasting models of regional infrastructure service demands. Finally, Deshmukh et al. [18] designed a hardware implementation for the CFL proposed in [67].

Applications of pure complex fuzzy classes in disaster mitigation and management, and epidemical crisis prediction were explored in [34,81]; as each disaster unfolds, there are simultaneous problems of uncertainty and severity. Complex fuzzy logic was proposed as a means to express these two characteristics together. Karpenko et al. studied the existence of a solution for the Cauchy problem for fuzzy differential equations, based on pure complex fuzzy sets [36].

Loo et al. [55] applied t-norm and t-conorm operations on complex fuzzy sets to develop a complex-valued version of the simplified fuzzy Adaptive Resonance Theory Map (ARTMAP) [37,85]. Seki and Nakashima [70] extend the single input rule modules (SIRMs) connected fuzzy inference model [93] with complex fuzzy sets; in this model, complex fuzzy sets are applied in the antecedent part. Tamir et al. [80] offer a survey of CFS&L with a focus on axiomatic-based fuzzy logic and applications of CFL.

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The design space of complex fuzzy operationalizations appears to be vast, with only a few examples studied to date. Some of the key issues we observe in this area include the further exploration of this design space (including both algorithmic details, and the more fundamental questions of membership codomain, membership function forms, complex fuzzy operations, etc. from previous sections); development of efficient, next-generation learning systems (e.g. *genetic* complex fuzzy systems and kernel-based algorithms); scaling up CFS&L learning algorithms for large-scale learning and Big Data; and again, the interpretation of these models (i.e. explanatory or knowledge-extraction mechanisms for CFS&L models).

7. Conclusions

In this article, we have conducted a systematic review of the field of complex fuzzy sets and logic, a new area of research focus for the fuzzy systems community. This work is intended to consolidate the topic in order to aid new researchers in the area, identify key open questions, and highlight possible future directions for research. After examining the articles citing Ramot's two seminal papers on complex fuzzy sets and logic [67,68], as well as executing Google Scholar searches for "complex fuzzy sets" and "complex fuzzy logic," and applying our inclusion and exclusion criteria, we have identified a set of primary sources that we believe constitute the current field of CFS&L. We have examined this literature through the lens of seven research questions:

Q1: In what way and for what phenomena is CSF&L a more effective framework for reasoning under uncertainty than existing fuzzy logics?

Q2: What are the truth valuation sets currently employed in CFS&L research, and what is known about their properties?

Q3: What functional forms have been employed for complex fuzzy membership functions, and what are their properties?

Q4: What functions and/or classes of functions have been proposed to implement complex fuzzy unions, intersections and complements?

Q5: What functions have been proposed to implement complex fuzzy relational compositions, and what are their properties?

Q6: What complex fuzzy logics have been proposed, and what are their properties?

Q7: How are CFS&L operationalized (i.e. models designed and parameterized) for practical applications, and what problems have they been applied to?

Overall, we have found that CFS&L is an area of growing interest to the fuzzy systems community, with the number of published articles increasing rapidly in the last few years. There are many important open questions in the field, and rich possibilities for future development. From the basic questions of what membership codomain and membership functions to use in a given situation, to the development of efficient and scalable learning algorithms, to the question of how CFS and models based on them may be interpreted, all of these questions still await answers. We suggest that the question of interpretation is perhaps the most critical facing the field today, as its resolution is critical for forming *intuitive* CFS models (such as a Complex Fuzzy Inferential System developed from expert knowledge). As this is considered one of the major practical advantages of type-1 fuzzy logic, we suspect that this question of intuitive development will again be a critical characteristic for the uptake of CFS&L as a practical engineering methodology in the real world.

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