ETH Zurich HS 2019
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 9

Exercise 9.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $S \leq T$ be two bounded $(\mathcal{F}_n)_{n\geq 0}$-stopping times and let $(X_n)_{n\geq 0}$ be an $(\mathcal{F}_n)_{n\geq 0}$-submartingale. Show that

$$E[X_T|\mathcal{F}_S] \geq X_S,\ P\text{-a.s.}.$$

Exercise 9.2

(a) Let X_n be a supermartingale so that $n \mapsto E[X_n]$ is constant. Show that X_n is a martingale.

(b) Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration and (X_n) $(\mathcal{F}_n)_{n\in\mathbb{N}}$-adapted with $X_n \in L^1$ for all $n \in \mathbb{N}$. Show that X_n is an $\mathcal{F}_n$-martingale if and only if $E[X_\tau] = E[X_0]$ for all bounded $\mathcal{F}_n$-stopping times τ.

Exercise 9.3 Consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_n\}_{n\geq 0}$, and let X_n be an $\mathcal{F}_n$-martingale for which $|X_{n+1} - X_n| \leq M$ P-a.s. for some fixed $M < \infty$. Define the events C, D by

$$\begin{aligned} C &:= \{\lim X_n \text{ exists and is finite}\}, \\ D &:= \{\limsup X_n = +\infty \text{ and } \liminf X_n = -\infty\}. \end{aligned}$$

Show that $P[C \cup D] = 1$.
Hint: Show that $P[C^c \cap (\{\sup_{n\in\mathbb{N}} X_n < a\} \cup \{\inf_{n\in\mathbb{N}} X_n > -a\})] = 0$, for all $a > 0$, by considering the processes $\{X_{T_A \wedge n}\}_{n\geq 0}$, for $A = [a, \infty)$ and $A = (-\infty, -a]$, where $T_A = \inf\{n \geq 0 : X_n \in A\}$.

Submission: until 14:15, Nov 26., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint