解 析 几 何

November 30, 2019

6.解： 不失一般性，可设平面的方程为$\pi: \lambda y + z - \mu = 0$. 那么其法向为 $\vec{n} = (0,\lambda,1)$, 且通过点$(0,0,\mu)$. 取 $O^* = (0,0,\mu) \in \pi$, $\vec{e_1^*} = (1,0,0)$, $\vec{e_3^*} = \frac{1}{\sqrt{1+\lambda^2}}(0,\lambda,1)$ 和 $\vec{e_2^*} = \vec{e_3^*} \times \vec{e_1^*} = \left(0, \frac{1}{\sqrt{\lambda^2+1}}, -\frac{\lambda}{\sqrt{\lambda^2+1}}\right)$, 则得直角坐标系 $I^* = \{O^*; \mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$.

记原坐标系为$I = (O, \vec{e_1}, \vec{e_2}, \vec{e_3})$. 则由上面的表达式知$I$到$I^*$的过渡矩阵为

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda^2+1}} & \frac{\lambda}{\sqrt{1+\lambda^2}} \\ 0 & -\frac{\lambda}{\sqrt{\lambda^2+1}} & \frac{1}{\sqrt{1+\lambda^2}} \end{pmatrix}.$$

对任意点P, 记它在I, I^* 下的坐标分别为(x,y,z) 和(x^*,y^*,z^*),则有坐标变换公式为

$$\begin{cases} x = & x^* \\ y = & \frac{1}{\sqrt{\lambda^2+1}}y^* + \frac{\lambda}{\sqrt{1+\lambda^2}}z^* \\ z = & -\frac{\lambda}{\sqrt{\lambda^2+1}}y^* + \frac{1}{\sqrt{1+\lambda^2}}z^* + \mu \end{cases}.$$

设P为平面π和双曲面$\Sigma: \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $(a > b)$ 的交点. 那么由 $P \in \pi$, 知 $z^*(P) = 0$. 所以将z^*和上面的坐标变换公式代入Σ 的方程得

$$\frac{x^{*2}}{a^2} + \frac{c^2 - b^2\lambda^2}{b^2c^2(\lambda^2+1)}y^{*2} + 2\frac{\lambda\mu}{\sqrt{\lambda^2+1}c^2}y^* - \frac{\mu^2}{c^2} = 1. \tag{1}$$

若取

$$\lambda^2 = \frac{(a^2-b^2)c^2}{(a^2+c^2)b^2} > 0,$$

那么上式可化为

$$\frac{x^{*2}}{a^2} + \left(\frac{y^*}{a} - \frac{(a^2-b^2)\mu^2}{(c^2+b^2)c^2}\right)^2 = 1 + \frac{a^2+c^2}{c^2+b^2}\frac{\mu^2}{c^2}.$$

因此取$\lambda^2 = \frac{\left(a^2-b^2\right)c^2}{(a^2+c^2)b^2}$时，交线$\Gamma = \pi \cap \Sigma$是圆.

7. 建立新的直角坐标系$\{O, \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*\}$, 这里$\vec{e}_1^* = (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$, $\vec{e}_2^* = (\frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{-4}{\sqrt{21}})$, $\vec{e}_3^* = (\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}})$, 有$\sqrt{6}x^* = x + 2y + z$, $\sqrt{21}y^* = 2x + y - 4z$, 即

$$f(x + 2y + z, 2x + y - 4z) = f(\sqrt{6}x^*, \sqrt{21}y^*) = 0.$$

为柱面方程，其母线方向为$(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}})$, 准线方程为

$$\left\{\begin{array}{l} f(x + 2y + z, 2x + y - 4z) = 0 \\ 3x - 2y + z = 0 \end{array}\right. \tag{2}$$