ETH Zurich HS 2019
D-MATH, D-PHYS
Prof. A.-S. Sznitman

Coordinator
Daniel Balint

Probability Theory

Exercise Sheet 7

Exercise 7.1 Let X and Y be two independent Bernoulli distributed random variables with parameter p. Define $Z = 1_{\{X+Y=0\}}$ and $\mathcal{G} = \sigma(Z)$. Find $E[X|\mathcal{G}]$ and $E[Y|\mathcal{G}]$. Are these random variables also independent?

Exercise 7.2 Let X and Y be random variables whose joint distribution is the uniform distribution on the triangle $\{(x, y) \in \mathbb{R}^2 : \ 0 \le y \le x \le 1\}$.

(a) Compute the distribution of Y/X.

(b) Show that Y/X and X are independent.

(c) Compute the conditional expectation $E[Y|X]$.

Exercise 7.3 Let S be a random variable with $P[S > t] = e^{-t}$, for all $t > 0$. Calculate the following conditional expectations for arbitrary $t > 0$:

(a) $E\Big[S \ \Big| S \wedge t\Big]$, where $S \wedge t := \min(S, t)$;

(b) $E\Big[S \ \Big| S \vee t\Big]$, where $S \vee t := \max(S, t)$.

Exercise 7.4 (Optional.) In this exercise we prove that in Theorem 1.37 (Kolmogorov's Three Series Theorem) (1.4.16) $\Rightarrow$ (1.4.17).

Consider X_k, $k \ge 1$ independent random variables and $A > 0$. Set $Y_k := X_k 1_{|X_k| \le A}$, $k \ge 1$. Assume that $\sum_k X_k$ converges P-a.s.

(a) Show that $P[\liminf_k \{X_k = Y_k\}] = 1$.

(b) Deduce from (a) that $\sum_k P[|X_k| > A] < \infty$ and $\sum_k Y_k$ converges P-a.s.

(c) Show that $\sum_k \mathrm{Var}(Y_k) < \infty$. (**Hint:** use Exercise 6.4.)

(d) Show that $\sum_k E[Y_k]$ converges. (**Hint:** use Theorem 1.34, moreover (c) and (b).)

Submission: until 14:15, Nov 12., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint

Solution 7.1 Since Z is constant on each the sets $A_0 = \{X+Y=0\}$ and $A_1 = \{X+Y \geq 1\}$, we know that $\mathcal{G}$ is generated by this partition. Thus,

$$E[X|\mathcal{G}](\omega) = \alpha_i = \frac{E[X1_{A_i}]}{P(A_i)}, \quad \text{for } \omega \in A_i.$$

On A_0, X is identically 0, so $E[X1_{A_0}] = 0$ and $\alpha_0 = 0$. On the other hand, $X1_{A_1} = 1_{\{X=1\}}1_{\{X+Y\geq 1\}} = 1_{\{X=1\}}$, so it follows that

$$\alpha_1 = \frac{p}{P(A_1)} = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}.$$

Hence, the conditional expectation can be expressed as

$$E[X|\mathcal{G}] = \frac{1}{2-p}1_{\{X+Y\geq 1\}}.$$

By symmetry, $E[Y|\mathcal{G}]$ is given by the same expression, whence we conclude that $E[X|\mathcal{G}] = E[Y|\mathcal{G}]$. Since a non-constant random variable cannot be independent from itself, the two random variables $E[X|\mathcal{G}]$ and $E[Y|\mathcal{G}]$ are not independent.

Solution 7.2

(a) The joint density of X and Y is the function that is constant and equals 2 on the given triangle, and zero outside. Clearly, we have $0 \leq Y/X \leq 1$, P-almost surely. Furthermore, for $t \in [0,1]$, we have

$$P[Y/X \leq t] = P[Y \leq tX] = \int_0^1 \int_0^{tx} 2\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 2tx\,\mathrm{d}x = t.$$

Thus Y/X has the uniform distribution on the interval $[0,1]$.

(b) Let $t_1, t_2 \in (0,1)$. Then we have

$$\begin{aligned} P[Y/X \leq t_1, X \leq t_2] = P[Y \leq t_1 X, X \leq t_2] &= \int_0^{t_2} \int_0^{t_1 x} 2\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{t_2} 2t_1 x\,\mathrm{d}x = t_1 t_2^2 = P[Y/X \leq t_1]P[X \leq t_2]. \end{aligned}$$

This equality holds also trivially for $t_1 \notin (0,1)$ or $t_2 \notin (0,1)$. Thus Y/X and X are independent, since by (1.3.11) of the lecture notes, the distribution of $(Y/X, X)$ equals the product of the distributions of Y/X and X.

(c) Using the properties of the conditional expectation from the lecture, we have

$$E[Y|X] = E[(Y/X)X|X] \overset{(*)}{=} XE[Y/X|X] \overset{(**)}{=} XE[Y/X] \overset{(***)}{=} X/2.$$

($*$) X is $\sigma(X)$-measurable.

($**$) Y/X is independent from $\sigma(X)$ by part (b).

($***$) By (a) Y/X is uniformly distributed on $[0,1]$ and the expectation of the uniform distribution on $[0,1]$ is $1/2$.

Solution 7.3

(a) One has that

$$\begin{aligned} E\Big[S \;\Big|S\wedge t\Big] &= E\Big[S1_{\{S<t\}} \;\Big|S\wedge t\Big] + E\Big[S1_{\{S\geq t\}} \;\Big|S\wedge t\Big] \\ &= E\Big[(S\wedge t)1_{\{S\wedge t<t\}} \;\Big|S\wedge t\Big] + E\Big[S1_{\{S\geq t\}} \;\Big|S\wedge t\Big] \\ &= (S\wedge t)1_{\{S\wedge t<t\}} + E\Big[S1_{\{S\geq t\}} \;\Big|S\wedge t\Big]. \end{aligned} \tag{1}$$

We now compute the second term. Take arbitrary $A \in \mathcal{B}(\mathbb{R})$. Then one has that:

$$\begin{aligned} E\Big[S1_{\{S\geq t\}}1_{\{S\wedge t\in A\}}\Big] &= E\Big[S1_{\{S\geq t\}}1_{\{t\in A\}}\Big] = 1_{\{t\in A\}}\int_t^\infty xe^{-x}dx \\ &= 1_{\{t\in A\}}\left[(-xe^{-x})\Big|_t^\infty + \int_t^\infty e^{-x}dx\right] = 1_{\{t\in A\}}(t+1)e^{-t} \\ &= 1_{\{t\in A\}}(t+1)E\Big[1_{\{S\geq t\}}\Big] = E\Big[(t+1)1_{\{S\geq t\}}1_{\{S\wedge t\in A\}}\Big]. \end{aligned}$$

Because $\{S \geq t\} = \{S\wedge t = t\}$ is $\sigma(S\wedge t)$-measurable, we have that

$$E\Big[S1_{\{S\geq t\}} \;\Big|S\wedge t\Big] = (t+1)1_{\{S\wedge t=t\}}.$$

Then it follows from (1) that

$$E\Big[S \;\Big|S\wedge t\Big] = (S\wedge t)1_{\{S\wedge t<t\}} + (t+1)1_{\{S\wedge t=t\}}.$$

(b) The solution is similar to part **a)**. We know that

$$\begin{aligned} E\Big[S \;\Big|S\vee t\Big] &= E\Big[S1_{\{S\leq t\}} \;\Big|S\vee t\Big] + E\Big[S1_{\{S>t\}} \;\Big|S\vee t\Big] \\ &= E\Big[S1_{\{S\leq t\}} \;\Big|S\vee t\Big] + E\Big[(S\vee t)1_{\{S\vee t>t\}} \;\Big|S\vee t\Big] \\ &= E\Big[S1_{\{S\leq t\}} \;\Big|S\vee t\Big] + (S\vee t)1_{\{S\vee t>t\}}. \end{aligned}$$

Take $A \in \mathcal{B}(\mathbb{R})$. For the first term we have:

$$\begin{aligned} E\Big[S1_{\{S\leq t\}}1_{\{S\vee t\in A\}}\Big] &= E\Big[S1_{\{S\leq t\}}1_{\{t\in A\}}\Big] = 1_{\{t\in A\}}\int_0^t xe^{-x}dx \\ &= 1_{\{t\in A\}}\Big(1-(t+1)e^{-t}\Big) = 1_{\{t\in A\}}\Big(1-(t+1)e^{-t}\Big)\frac{E\Big[1_{\{S\leq t\}}\Big]}{1-e^{-t}} \\ &= E\left[\frac{1-(t+1)e^{-t}}{1-e^{-t}}1_{\{S\leq t\}}1_{\{S\vee t\in A\}}\right]. \end{aligned}$$

Because of the $\sigma(S\wedge t)$-measurability of $1_{\{S\leq t\}} = 1_{\{S\vee t=t\}}$ it follows that

$$E\Big[S1_{\{S\leq t\}} \;\Big|S\vee t\Big] = \frac{1-(t+1)e^{-t}}{1-e^{-t}}1_{\{S\vee t=t\}},$$

and hence

$$E\Big[S \;\Big|S\vee t\Big] = \frac{1-(t+1)e^{-t}}{1-e^{-t}}1_{\{S\vee t=t\}} + (S\vee t)1_{\{S\vee t>t\}}.$$

Solution 7.4

(a) We are going to argue by contraposition. Assume that $P[\liminf_k \{X_k = Y_k\}] < 1$, i.e. $P(A := \Omega \setminus \liminf_k \{X_k = Y_k\}) > 0$. Then by the definition of lim inf, for every $\omega \in A$ and $n \in \mathbb{N}$ there exists a $k > n$ such that $X_k(\omega) \neq Y_k(\omega)$. By the definition of Y this means that $|X_k(\omega)| > A$ for such a k. This implies by Cauchy's convergence test that $\sum_k X_k(\omega)$ does not converge. Since this is true for each $\omega \in A$ and $P(A) > 0$, it follows that $\sum_k X_k$ does not converge P-a.s.

(b) First, note that $\{|X_k| > A\} = \{X_k \neq Y_k\}$. Second, by de Morgan's law we have $\limsup_k \{X_k = Y_k\}^c = (\liminf_k \{X_k = Y_k\})^c$. Putting these together, we obtain due to (a) the following:

$$\begin{aligned} P(\limsup_k \{|X_k| > A\}) &= P(\limsup_k \{X_k \neq Y_k\}) = P(\limsup_k \{X_k = Y_k\}^c) \\ &= P((\liminf_k \{X_k = Y_k\})^c) = 1 - P(\liminf_k \{X_k = Y_k\}) = 0. \end{aligned} \tag{2}$$

Now since X_k are independent, so are the events $\{|X_k| > A\}$, and hence the contraposition of the second lemma of Borel-Cantelli (Lemma 1.26 in lecture notes) implies that $\sum_k P[|X_k| > A] < \infty$.

For the second statement, observe that $\sum_k Y_k = \sum_k X_k - \sum_k X_k 1_{|X_k|>A}$. Due to (2) we have $P(\limsup_k \{|X_k| > A\}) = 0$. This means that there is a set $A \in \mathcal{F}$ with $P(A) = 1$ such that for each $\omega \in A$ there exists a N such that for all $k > N$, $\omega \notin \{|X_k(\omega)| > A\}$, i.e. $1_{|X_k|>A}(\omega) = 0$. This in particular implies that $\sum_k X_k(\omega) 1_{|X_k|>A}(\omega)$ converges as the sum has only finitely many non-zero summands. Sine this is true for all $\omega \in A$ and $P(A) = 1$, it follows that $\sum_k X_k 1_{|X_k|>A}$ converges P-a.s. Therefore $\sum_k Y_k$ is a sum of two P-a.s convergent series and hence it converges P-a.s. itself.

(c) This is a direct consequence of Exercise 6.4. Indeed, Y_k are independent as X_k are, and uniformly bounded by A by construction. Now since in (b) we have shown that $\sum_k Y_k$ converges P-a.s., Exercise 6.4 implies that $\sum_k \mathrm{Var}(Y_k) < \infty$ and hence we are done.

(d) Define $Z_k := Y_k - E[Y_k]$. Then Z_k are independent as X_k and hence Y_k are, $\sum_k \mathrm{Var}(Z_k) = \sum_k \mathrm{Var}(Y_k) < \infty$ due to (c) and $E[Z_k] = 0$ for each k by construction. Hence the conditions of Theorem 1.34 in the lecture notes are satisfied and it follows that $\sum_k Z_k$ converges P-a.s. But since $\sum_k E[Y_k] = \sum_k Y_k - \sum Z_k$ and $\sum_k Y_k$ converges P-a.s. by (b), it follows that $\sum_k E[Y_k]$ also converges P-a.s. Since of course $E[Y_k]$ is deterministic for each k, it follows that $\sum_k E[Y_k]$ converges (for each $\omega \in \Omega$).