ETH Zurich HS 2019
D-MATH, D-PHYS
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Coordinator
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# Probability Theory

## Exercise Sheet 2

**Exercise 2.1** Take $\Omega = \{a,b,c,d\}$, $\mathcal{A} = \mathcal{P}(\Omega)$ and $\mathcal{C} = \Big\{\{a,b\},\{c,d\},\{a,c\},\{b,d\}\Big\}$. Consider $P$ the equiprobability on $\Omega$ and $Q$ the probability measure $\frac{1}{2}(\delta_a + \delta_d)$ (with $\delta_a$ the point measure at $a$, and $\delta_d$ the point measure at $d$).

(a) Show that $\sigma(\mathcal{C}) = \mathcal{A}$, and $P$ and $Q$ agree on $\mathcal{C}$.

(b) Show that $\{A \in \mathcal{A};\ P(A) = Q(A)\}$ is not a $\sigma$-algebra.

(c) Is $\mathcal{C}$ a $\pi$-system?

**Exercise 2.2** Let $\mathcal{F}$ be a $\sigma$-algebra and $A_i \in \mathcal{F}$ $(i = 1,2,...)$ the event "at time $i$ the phenomena $\Phi$ occurs".
Express with the help of the subsets $A_i$ the following events as subsets $A \in \mathcal{F}$:

(a) "$\Phi$ occurs exactly 17 times"

(b) "$\Phi$ always occurs again"

(c) "$\Phi$ stops occurring at some point"

Describe in words the following event:

(d) $\bigcup_{n\geq 1}\bigcup_{m>n}(A_n \cap A_m)$

Which of these events belong to the asymptotic $\sigma$-algebra $\mathcal{A}^* := \bigcap_{n\geq 1}\sigma\Big(\bigcup_{i\geq n}\{A_i\}\Big)$ ?

**Exercise 2.3** In this exercise, we will construct a countably infinite number of independent random variables, without using a product space with an infinite number of factors.

Consider $\Omega = [0,1)$, equipped with the Borel $\sigma$-algebra and the Lebesgue measure $P$ restricted to $[0,1)$. We define the random variables

$$Y_n : \Omega \to \mathbb{R}\,, \quad n \geq 1\,,$$

by

$$Y_n(\omega) := \begin{cases} 0 & \text{if } \lfloor 2^n\omega \rfloor \text{ is even,} \\ 1 & \text{if } \lfloor 2^n\omega \rfloor \text{ is odd,} \end{cases}$$

where $\lfloor x \rfloor = \max\{z \in \mathbb{Z} \mid z \leq x\}$ denotes the integer part of $x$. Show that $Y_n$, $n \geq 1$, are independent and satisfy $P[Y_n = 0] = P[Y_n = 1] = \frac{1}{2}$.
*Hint:* To gain insight, consider the meaning of $Y_n$ in terms of the binary expansion of $\omega$. You may use the following observation, without proving it:

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $Y_1, Y_2, \ldots$ be random variables on this space, each taking values only in a countable set (that is, for each $i$ there is a countable set $S_i$ such that $P[Y_i \in S_i] = 1$). Assume that

$$P[Y_1 = z_1, Y_2 = z_2, \ldots, Y_n = z_n] = \prod_{i=1}^{n} P[Y_i = z_i] \text{ for all } z_1, \ldots, z_n \in \mathbb{R} \tag{1}$$

holds for all $n \geq 1$. Then, the infinite sequence of random variables $(Y_i)_{i \geq 1}$ is independent.

**Exercise 2.4 (Optional.)** A non-empty family $\mathcal{C}$ of subsets of a non-empty set $\Omega$ is called a $\lambda$-*system*, if

(i) $\Omega \in \mathcal{C}$,

(ii) $A, B \in \mathcal{C}: \; B \subset A \Rightarrow A \setminus B \in \mathcal{C}$,

(iii) $A_n \in \mathcal{C}, \; A_n \subset A_{n+1} \Rightarrow \bigcup_n A_n \in \mathcal{C}$.

Show that the definitions of a Dynkin system and a $\lambda$-system are equivalent.

**Submission:** until 14:15, Oct 8., during exercise class or in the tray outside of HG G 53.

**Office hours (Präsenz):** Mon. and Thu., 12:00-13:00 in HG G 32.6.

**Class assignment:**

| Students | Time & Date | Room | Assistant |
|---|---|---|---|
| Afa-Fül | Tue 13-14 | HG F 26.5 | Angelo Abächerli |
| Gan-Math | Tue 13-14 | ML H 41.1 | Zhouyi Tan |
| Meh-Schu | Tue 14-15 | HG F 26.5 | Angelo Abächerli |
| Schü-Zur | Tue 14-15 | ML H 41.1 | Dániel Bálint |