REAL ANALYSIS

LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

L^p Spaces

We shall give an introduction for the space L^p space, where $1 \le p < \infty$, given by

$$L^p(E) = \big\{ f : \ f \text{ is measurable on } E, \ |f|^p \in L^1(E) \big\},$$

where E is a measurable set in $\mathbb{R}^n$ (in particular one can take $E = \mathbb{R}^n$). We will see that $L^p(E)$ is a normed space with the norm $\|\cdot\|$ defined by

$$\|f\|_p = \|f\|_{L^p} = \|f\|_{L^p(E)} = \Big(\int_E |f|^p \Big)^{\frac{1}{p}},$$

and is complete (w.r.t. the metric $d_p(f,g) = \|f-g\|_p$) and separable [1]. Also, L^p convergence implies the convergence in measure (hence Riesz theorem can be used).

We will also define the normed space

$$L^\infty(E) = \big\{ f : \ f \text{ is measurable on } E, \ \|f\|_\infty < \infty \big\},$$

where E is a measurable set in $\mathbb{R}^n$ (can be taken as $E = \mathbb{R}^n$), and

$$\|f\|_\infty = \|f\|_{L^\infty} = \|f\|_{L^\infty(E)} = \inf\{M : |f(x)| \le M \ a.e. \ x \in E\}.$$

The norm $\|\cdot\|_\infty$ is also called the essential sup norm. This space is complete (w.r.t. the metric $d_\infty(f,g) = \|f-g\|_\infty$), and L^∞ convergence implies the uniform convergence (outside a zero measure set). But $L^\infty(E)$ is not separable when $m(E) > 0$.

[1] We say a metric space (X,d) is separable, if there exists a countable collection $\{f_k\}$ of elements in X such that their linear combinations are dense in X.

1. Some elementary properties

Lemma 1.1 (Young's inequality)**.** *Let a, b, p, q be positive real numbers, and $1/p+1/q = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.1}$$

The equality holds if and only if $a^p = b^q$. Case $p = q = 2$ is known as Cauchy's inequality.

Proof. Consider the function $\phi(t) = t^{\frac{1}{p}}$, $t \geq 0$. It is obviously concave. Therefore

$$t^{\frac{1}{p}} = \phi(t) \leq \phi'(1)(t-1) + \phi(1) = \frac{1}{p}t + \frac{1}{q}, \quad \forall\, t \geq 0,$$

with equality holding if and only if $t = 1$. Inserting $t = a^p/b^q$ in the inequality and then multiplying b^q at both sides, we infer that

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

The equality holds if and only if $a^p = b^q$.

∎

Lemma 1.2 (Hölder's inequality)**.** *Let E be a measurable set of $\mathbb{R}^n$. Suppose $f \in L^p(E)$ and $g \in L^q(E)$, where $1 < p < \infty$ and $1/p + 1/q = 1$. Then $fg \in L^1(E)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{1.2}$$

The equality holds if and only if there is a constant $\lambda \geq 0$ such that $|f(x)|^p = \lambda |g(x)|^q$ for a.e. $x \in E$. Case $p = q = 2$ is known as Schwarz inequality.

Proof. If $\|f\|_p = 0$ then $f(x) = 0$ for a.e. $x \in E$. Then (1.2) holds. Similar result applies to $\|g\|_q = 0$.

Suppose both $\|f\|_p$ and $\|g\|_q$ are positive. Inserting

$$a = \frac{|f(x)|}{\|f\|_p} \quad \text{and} \quad b = \frac{|g(x)|}{\|g\|_q}$$

in (1.1) gives the following pointwise estimate

$$\frac{|f(x)|}{\|f\|_p}\frac{|g(x)|}{\|g\|_q} \leq \frac{1}{p}\frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(x)|^q}{\|g\|_q^q}.$$

Integrating over E yields

$$\int_E |fg| \leq \|f\|_p \|g\|_q$$

as desired. The equality holds if and only if, by Lemma 1.1,

$$\frac{|f(x)|^p}{\|f\|_p^p} = \frac{|g(x)|^q}{\|g\|_q^q}, \quad \text{for a.e.} x \in E.$$

Namely, $|f(x)|^p = \lambda |g(x)|^q$ for a.e. $x \in E$ and for some constant $\lambda \geq 0$.

□

Lemma 1.3 (Minkowski inequality)**.** *Let E be a measurable set of $\mathbb{R}^n$. Suppose $f, g \in L^p(E)$, $1 \leq p < \infty$. Then $f + g \in L^p(E)$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{1.3}$$

The equality then holds if and only if $|f| = \lambda |g|$ a.e. for some $\lambda \geq 0$.

Proof. It is readily seen that (1.3) holds when $p = 1$, by using the pointwise triangle inequality. The equality then holds if and only if $|f| = \lambda |g|$ a.e.

Consider the case $p > 1$. We have

$$\begin{aligned} \|f + g\|_p^p &\leq \int_E |f||f + g|^{p-1} + \int_E |g||f + g|^{p-1} \\ &\leq \|f\|_p \Big(\int_E |f + g|^p \Big)^{1 - \frac{1}{p}} + \|g\|_p \Big(\int_E |f + g|^p \Big)^{1 - \frac{1}{p}}. \end{aligned}$$

The equality holds if and only if $|f| = \lambda g$ a.e. Dividing $\|f + g\|_p^{p-1}$ at both sides gives the desired result.

□

Lemma 1.4. *Let E be a measurable set in $\mathbb{R}^n$. If $f, g \in L^\infty(E)$, then*

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

The equality holds if and only if $|f| = \lambda |g|$ a.e. for some constant $\lambda \geq 0$.

Proof. It is direct to see this by the definition.

□

We now state the main result of this section.

Theorem 1.1. *Let E be a measurable set of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. Then $L^p(E)$ is a normed linear space.*

Proof. Suppose $a, b \in \mathbb{R}$, $f, g \in L^p(E)$. It is not hard to see $af + bg \in L^p(E)$.

We can also verify

(i) $\|f\|_p = 0$ if and only if $f(x) = 0$ for a.e. $x \in E$;

(ii) $\|\lambda f\|_p = |\lambda| \|f\|_p$;

(iii) $\|f + g\| \leq \|f\|_p + \|g\|_p$.

These are consequences of Lemmas 1.3 and 1.4. $\square$

Remark 1.1. *The L^p space can be defined for complex-valued functions, with some minor but needed changes. We also see that such space is a normed (complex) linear space. In this notes, we consider real-valued functions. But the results also hold for complex-valued functions.*

Some other observations are in order.

One may have occasion to use a generalisation of Hölder's inequality to m functions $f_1, \dots, f_m$, lying respectively in spaces $L^{p_1}, \dots, L^{p_m}$, where

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1.$$

The resulting inequality, obtainable from the case $m = 2$ by an induction argument, is the following.

Proposition 1.1. *Suppose $p_i > 0$, $1 \leq i \leq m$, and $\sum_{i=1}^m p_i^{-1} = 1$. Let E be a measurable set in $\mathbb{R}^n$ and $f_i \in L^{p_i}(E)$, $1 \leq i \leq m$. Then*

$$\int_E f_1 \cdots f_m \leq \|f_1\|_{p_1} \cdots \|f_m\|_{p_m}.$$

As a simple consequence of Hölder inequality, we have the following.

Proposition 1.2. *Let E be measurable set in $\mathbb{R}^n$ and $0 < p \leq \sigma \leq r$. Suppose $\lambda \in [0, 1]$ is such that $1/\sigma = \lambda/p + (1 - \lambda)/r$. Then*

$$\|f\|_\sigma \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}. \tag{1.4}$$

As a consequence, $L^p(E) \cap L^r(E) \subseteq L^\sigma(E)$.

Proof. This is an exercise. $\square$

It is also of interest to study the L^p norm as a function of p. Let E be a measurable in $\mathbb{R}^n$ of finite measure. Given a measurable function f, write

$$\Phi_f(p) = \left(\fint_E |f|^p \right)^{\frac{1}{p}} = [m(E)]^{-\frac{1}{p}} \|f\|_p.$$

Proposition 1.3. *Suppose E is of finite measure in $\mathbb{R}^n$. Let $\Phi_f(p)$ be as above. Then*

(i) *$\Phi_f(p)$ is non-decreasing in p;*
(ii) *$\Phi_f(p)$ is logarithmically convex in p^{-1}.*

As a consequence $L^p(E) \subseteq L^{p'}(E)$ if $p > p' \geq 1$.

Proof. By virtue of Hölder inequality, we have

$$\int_E |f|^{p'} = \int_E 1 \cdot |f|^{p'} \leq [m(E)]^{1-\frac{1}{\lambda}} \Big(\int_E |f|^{p'\lambda} \Big)^{\frac{1}{\lambda}}, \quad \forall \, \lambda > 1.$$

Taking $\lambda = p/p'$, we see that

$$\int_E |f|^{p'} \leq [m(E)]^{1-\frac{p'}{p}} \Big(\int_E |f|^p \Big)^{\frac{p'}{p}}.$$

This shows (i).

Conclusion (ii) is a consequence of (1.4) which implies

$$\log \Phi_f(\sigma) \leq \lambda \log \Phi_f(p) + (1-\lambda) \log \Phi_f(r).$$

By (i), if $f \in L^p(E)$, then $\|f\|_{p'} \leq [m(E)]^{1/p'-1/p} \|f\|_p < \infty$. Hence $f \in L^{p'}(E)$.

□

Proposition 1.4. *Let E be a set of finite measure in $\mathbb{R}^n$. Then*

(i) *$L^\infty(E) \subseteq L^p(E)$ for all $p > 0$, and $\lim_{p\to\infty} \Phi_f(p) = \|f\|_\infty$.*
(ii) *$\lim_{p\to 0} \Phi_f(p) = \exp\Big[⨍_E \log|f| \Big]$, if $\log|f| \in L^1(E)$.*

Proof. We prove (i) for $0 < \|f\|_\infty < \infty$. It is not hard to see that $\Phi_f(p) \leq \|f\|_\infty$. We show the opposite inequality. For a small $\delta > 0$, let

$$S_\delta = \{x \in E : |f(x)| > \|f\|_\infty - \delta\}$$

By the definition of $\|\cdot\|_\infty$, $m(S_\delta) > 0$. We find

$$\Phi_f(p) \geq \Big(\frac{1}{m(E)} \int_{S_\delta} |f|^p \Big)^{\frac{1}{p}} \geq (\|f\|_\infty - \delta)[m(S_\delta)/m(E)]^{\frac{1}{p}} \to \|f\|_\infty - \delta.$$

This shows that $\lim_{p\to\infty} \Phi_f(p) \geq \|f\|_\infty$.

We next show (ii). Firstly, assume $⨍_E \log|f| > -\infty$. Let

$$g(p) = \frac{1}{p} \log ⨍_E |f|^p - ⨍_E \log|f|.$$

Since $t \mapsto \log t$ is concave, we have by Jensen inequality

$$g(p) \geq 0.$$

Using the inequality $\ln(1+t) \leq t$, we see that

$$0 \leq g(p) \leq \frac{1}{p}\Big(\fint_E |f|^p - 1\Big) - \fint_E \log|f| =: h(p).$$

We next show that $\lim_{p\to 0} h(p) = 0$. For this end, we take a sequence p_j which converges to 0 and let

$$f_j(x) = \frac{|f(x)|^{p_j} - 1}{p_j} - \log|f(x)|.$$

Observe that $f_j \to 0$ a.e. If $|f_j|$ is bounded by an integrable function, then we get the conclusion by dominated convergence theorem.

We show it is this case. If $t \geq 1$, $0 < p < 1$, then

$$\Big|\frac{t^p - 1}{p}\Big| = \int_1^t s^{p-1} ds \leq t - 1.$$

Since $s \mapsto s^{p-1}$ is decreasing, and if $0 < t < 1$

$$\Big|\frac{t^p - 1}{p}\Big| = \int_t^1 s^{p-1} ds \leq \int_t^1 s^{-1} ds = -\log t.$$

Now denote $A = \{x \in E : |f(x)| \geq 1\}$. Then

$$|f_j(x)| \leq (|f(x)| - 1)\chi_A(x) - \log|f(x)|\chi_{E\setminus A}(x) \in L^1.$$

$\square$

When E is measurable, not necessarily of finite measure, we have the following.

Proposition 1.5. *Let E be a measurable set in $\mathbb{R}^n$ and f is measurable. If $f \in L^r(E)$ for some $r > 0$, then*

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Moreover $L^\infty(E) \cap L^r(E) \subseteq L^p(E)$ for all $p > 0$.

Proof. Observe that if $\|f\|_\infty = 0$ or respectively $\|f\|_\infty = \infty$, then $\|f\|_p = 0$ or respectively $\|f\|_p = 0$. Hence we only prove the conclusion for $0 < \|f\|_\infty < \infty$.

As in the proof of part (i) in Proposition 1.4, we take $S_{\delta,R} = S_\delta \cap B_R$ where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Then $m(S_{\delta,R}) \in (0, m(B_R))$ for some large R. Hence

$$\|f\|_p \geq (\|f\|_\infty - \delta)[m(S_{\delta,R})]^{\frac{1}{p}} \to \|f\|_\infty - \delta \text{ as } p \to \infty.$$

This yields $\lim_{p\to\infty} \|f\|_p \geq \|f\|_\infty$.

On the contrary, it follows by the Hölder inequality,

$$\|f\|_p = \Big(\int_E |f|^{p-r}|f|^r\Big)^{\frac{1}{p}} \leq \|f\|_\infty^{\frac{p-r}{p}} \|f\|_r^{\frac{r}{p}} \to \|f\|_\infty \text{ as } p \to \infty. \tag{1.5}$$

Hence $\|f\|_\infty \geq \lim_{p\to\infty} \|f\|_p$.

Note that (1.5) implies $L^\infty(E) \cap L^r(E) \subseteq L^p(E)$ for all p. The conclusions is proved.

□

2. Completeness, approximation and separability

Let us recall some notions.

Definition 2.1. *A sequence $\{f_j\}$ in a linear space X that is normed by $\|\cdot\|$ is said to converge to f in X provided*

$$\lim_{j\to\infty} \|f_j - f\| = 0.$$

We write

$$f_j \to f \text{ in } X \text{ or } \lim_{j\to\infty} f_j = f \text{ in } X$$

to mean that each f_j and f belong to X and $\lim_{j\to\infty} \|f - f_j\| = 0$.

Since the essential supremum of a function in $L^\infty(E)$ is an essential upper bound, for a sequence f_j and function f in $L^\infty(E)$, $f_j \to f$ in $L^\infty(E)$ if and only if $f_j \to f$ uniformly on the complement of a set of measure zero.

For a sequence f_j and f in $L^p(E)$, $1 \leq p < \infty$, $f_j \to f$ in $L^p(E)$ if and only if

$$\lim_{j\to\infty} \int_E |f_j - f|^p = 0.$$

Completeness of L^p space for $1 \leq p \leq \infty$

Definition 2.2. *A sequence f_j in a linear space X that is normed by $\|\cdot\|$ is said to be Cauchy in X provided for each $\varepsilon > 0$, there is N such that*

$$\|f_j - f_k\| < \varepsilon \quad \textit{for all } j, k \geq N.$$

A normed linear space X is said to be complete provided every Cauchy sequence in X converges to a function in X. A complete normed linear space is called a Banach space.

Theorem 2.1 (Riesz-Fischer). *Let E be a measurable set of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if f_j converge to f in L^p, then there is a subsequence of f_j converge pointwise a.e. on E to f.*

We first show the following theorem.

Theorem 2.2. *Let $p \in [1, \infty]$, E be a measurable set of $\mathbb{R}^n$, and $f_k, f \in L^p(E)$. Suppose $f_k \to f$ fast in the sense that $\sum_{k \geq 1} \|f_k - f\|_p < \infty$. Then*

$$f_k \to f \text{ a.e. and } \|f_k - f\|_p \to 0.$$

Proof. Case $p = \infty$ is immediate from the definition. We focus on the case $p \in [1, \infty)$. Let us divide the proof into several steps.

Step 1. Write

$$f_k(x) = f_1(x) + \sum_{l=2}^{k} (f_l(x) - f_{l-1}(x)), \tag{2.1}$$

and let

$$\begin{aligned} g_k(x) &= |f_1|(x) + \sum_{l=2}^{k} |f_l(x) - f_{l-1}(x)|, \\ g(x) &= |f_1|(x) + \sum_{l=2}^{\infty} |f_l(x) - f_{l-1}(x)|. \end{aligned} \tag{2.2}$$

Then $g_k \nearrow g$, where possibly $g(x) = \infty$. By the MCT,

$$\int_E g^p = \lim_{k \to \infty} \int_E g_k^p. \tag{2.3}$$

Step 2. By assumption, $K := \sum_{k \geq 1} \int \|f - f_k\|_p < \infty$. The Minkowski inequality implies

$$\|g_k\|_p \leq \|f_1\|_p + \sum_{l=2}^{k} \|f_l - f_{l-1}\|_p \leq \|f_1\|_p + 2K,$$

which is independent of k. This together with (2.3) implies $g \in L^p(E)$. Therefore $g(x)$ is finite a.e. In particular, $\lim_k g_k(x)$ exists for a.e. x.

Step 3. Let x be such that $g(x) < \infty$. Then $\{f_k(x)\}$ is a Cauchy sequence of $\mathbb{R}$. Therefore f_k converges a.e. to

$$h(x) = f_1(x) + \sum_{l=2}^{\infty}(f_l(x) - f_{l-1}(x)).$$

Observe that $|f_k - h|^p \to 0$ a.e. and

$$|f_k(x) - h(x)|^p \leq [2\max\{|f_k(x)|, |h(x)|\}]^p \leq 2^p g^p(x) \in L^1(E).$$

By dominated convergence theorem, $h \in L^p(E)$ and

$$\int_E |f_k - h|^p \to 0 \quad \text{as } k \to \infty.$$

Step 4. It remains to show $f = h$ a.e. This is a consequence the Minkowski inequality

$$\|f - h\|_p \leq \|f - f_k\|_p + \|f_k - h\|_p \to 0.$$

Therefore $f = h$ a.e.

□

It is the position for showing the completeness of L^p space, $p \in [1, \infty]$.

Proof of Theorem 2.1. We divide the proof into several steps.

Step 1. We select a subsequence $\{f_{j_k}\}$ such that $\|f_{j_{k+1}} - f_{j_k}\|_p \leq 2^{-k}$. In particular $\sum_{k\geq 1}\|f_{j_{k+1}} - f_{j_k}\|_p < \infty$. Let

$$f(x) := f_{j_1}(x) + \sum_{k=1}^{\infty}(f_{j_{k+1}}(x) - f_{j_k}(x)),$$

and

$$g(x) := |f_{j_1}(x)| + \sum_{k=1}^{\infty}|f_{j_{k+1}}(x) - f_{j_k}(x)|.$$

By Theorem 2.2, $f_{j_k} \to f$ a.e., and

$$\|f_{j_k} - f\|_p \to 0 \quad \text{as } k \to \infty.$$

Step 2. We next show $f_k \to f$ in the L^p. By Minkowski inequality,

$$\|f_k - f\|_p \leq \|f_k - f_{j_l}\|_p + \|f_{j_l} - f\|_p.$$

Given $\varepsilon > 0$, use the fact that $\{f_k\}$ is L^p-Cauchy to choose N_ε so the first term on RHS is $< \varepsilon/2$ for all $k, j_l > N_\varepsilon$. Then choose j_l so the second term is $< \varepsilon/2$, this being permissible since $f_{j_l} \to f$ in the L^p sense. Then $k > N_\varepsilon$ implies $\|f_k - f\|_p < \varepsilon$, which yields the result.

$\square$

Dense subsets of L^p space

We recall the following notion.

Definition 2.3. *Let X be a normed linear space with norm $\|\cdot\|$. Given two subsets $\mathcal{F}$ and $\mathcal{G}$ of X with $F \subset G$, we say that $\mathcal{F}$ is dense in $\mathcal{G}$, provided for each function g in $\mathcal{G}$ and $\varepsilon > 0$, there is a f in $\mathcal{F}$ for which $\|f - g\| < \varepsilon$.*

The main conclusion of this portion is the following theorem. It says that some subsets of $L^p(E)$ with nice property are indeed dense.

Theorem 2.3. *Let $p \in [1, \infty)$, E be a measurable set of $\mathbb{R}^n$ and $f \in L^p(E)$. Then*

(i) *there exists a sequence of simple functions $\{\phi_k\}$ such that*

$$\|\phi_k - f\|_p \to 0 \text{ and } \phi_k \to f \text{ a.e.}$$

(ii) *there exists a sequence of step functions $\{\psi_k\}$ such that*

$$\|\psi_k - f\|_p \to 0 \text{ and } \psi_k \to f \text{ a.e.}$$

(iii) *there is a sequence of continuous functions with compact support $\{g_k\}$ such that*

$$\|g_k - f\|_p \to 0 \text{ and } g_k \to f \text{ a.e.}$$

(iv) *there is a sequence of smooth functions with compact support $\{g_k\}$ such that*

$$\|g_k - f\|_p \to 0 \text{ and } g_k \to f \text{ a.e.}$$

Proof. It suffices to show the convergence in norm. This is because L^p $(1 \le p < \infty)$ convergence implies convergence in measure, since

$$\|f - g\|_p^p \ge \int_{\{|f-g| \ge \varepsilon\}} |f - g|^p \ge m(\{|f - g| \ge \varepsilon\})\varepsilon^p.$$

Hence a.e. convergence for subsequence is a consequence of Riesz theorem. While by definition L^∞ convergence is uniform convergence outside a set of measure zero.

Conclusions (i)-(iii) are obtained in a similar fashion of those for L^1 case. We give a proof of (i) below as an example. By a zero extension outside E, let us suppose

$f \in L^p(\mathbb{R}^n)$. It is known that there exists a sequence $\{\phi_i\}_{i=1}^{\infty}$ of simple functions such that

(i) $\phi_i \to f$ a.e.

(ii) $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |\phi_k| \leq \cdots \leq |f|$.

Hence $|\phi_k - f| \to 0$ a.e., and

$$|\phi_k - f|^p \leq (|\phi_k| + |f|)^p \leq 2^p |f|^p \in L^1.$$

By dominated convergence theorem, we have $\|\phi_k - f\|_p \to 0$ when $1 \leq p < \infty$.

One can use the regularisation method to show that $C_c^{\infty}(E)$ is dense in $L^p(E)$. The argument is very similar to that for L^1 case, which is left as an exercise.

$\square$

Separability of L^p space

Definition 2.4. *A normed linear space X is said to be separable if there is a countable subset that is dense in X.*

Theorem 2.4. *Let $1 \leq p < \infty$ and E be a measurable set in $\mathbb{R}^n$. Then $L^p(E)$ is separable.*

Proof. This is because step functions with rational values and supported on rectangles with rational vertices, are dense in the L^p norm.

$\square$

We comment that $L^{\infty}(E)$ is not separable if $m(E) > 0$.

Consider e.g. $L^{\infty}([0,1])$. The uncountable functions $\{\chi_{[0,\lambda]}\}_{0<\lambda<1}$ satisfies

$$\|\chi_{[0,\lambda_1]} - \chi_{[0,\lambda_2]}\|_{\infty} = 1$$

for every $\lambda_1 \neq \lambda_2$ (no matter how close λ_1 and λ_2 are). This implies that $\{\chi_{[0,\lambda]}\}_{0<\lambda<1}$ cannot lie in a small neighbourhood of any countable may elements of $L^{\infty}[0,1]$. Note that for any $1 \leq p < \infty$,

$$\|\chi_{[0,\lambda_1]} - \chi_{[0,\lambda_2]}\|_p = |\lambda_1 - \lambda_2|^{1/p} \to 0$$

when λ_2 and λ_2 become very close.