2019-2020春学期《微分几何》第四周作业

1. **解** 由$\mathbf{x}=(a(u^1+u^2),b(u^1-u^2),2u^1u^2)$，得$\mathbf{x}_1=(a,b,2u^2)$，$\mathbf{x}_2=(a,-b,2u^1)$，$\mathbf{x}_{11}=(0,0,0)$，$\mathbf{x}_{12}=(0,0,2)$，$\mathbf{x}_{22}=(0,0,0)$，$\mathbf{n}=\frac{\mathbf{x}_1\times\mathbf{x}_2}{|\mathbf{x}_1\times\mathbf{x}_2|}=\frac{(b(u^1+u^2),a(u^2-u^1),-ab)}{\sqrt{b^2(u^1+u^2)^2+a^2(u^2-u^1)^2+a^2b^2}}$. 从而$g_{11}=\mathbf{x}_1\cdot\mathbf{x}_1=a^2+b^2+4(u^2)^2$，$g_{12}=\mathbf{x}_1\cdot\mathbf{x}_2=a^2-b^2+4u^1u^2$，$g_{22}=\mathbf{x}_2\cdot\mathbf{x}_2=a^2+b^2+4(u^1)^2$，$h_{11}=\mathbf{x}_{11}\cdot\mathbf{n}=0$，$h_{12}=\mathbf{x}_{12}\cdot\mathbf{n}=\frac{-2ab}{\sqrt{b^2(u^1+u^2)^2+a^2(u^2-u^1)^2+a^2b^2}}$，$h_{22}=\mathbf{x}_{22}\cdot\mathbf{n}=0$. 因两个主曲率$k_1,k_2$满足

$$k_1k_2=K=\frac{\det(h_{\alpha\beta})}{\det(g_{\alpha\beta})}=\frac{-h_{12}^2}{\det(g_{\alpha\beta})}$$

$$\frac{1}{2}(k_1+k_2)=H=\frac{1}{2}\frac{h_{11}g_{22}-2h_{12}g_{12}+h_{22}g_{11}}{\det(g_{\alpha\beta})}=-\frac{h_{12}g_{12}}{\det(g_{\alpha\beta})}$$

故主曲率

$$\begin{aligned}k_1,\ k_2 &= H\pm\sqrt{H^2-K}=-\frac{h_{12}g_{12}}{\det(g_{\alpha\beta})}\mp\frac{h_{12}\sqrt{g_{12}^2+\det(g_{\alpha\beta})}}{\det(g_{\alpha\beta})}\\ &= -\frac{h_{12}g_{12}}{\det(g_{\alpha\beta})}\mp\frac{h_{12}\sqrt{g_{11}g_{22}}}{\det(g_{\alpha\beta})}=\frac{h_{12}}{\det(g_{\alpha\beta})}(-g_{12}\mp\sqrt{g_{11}g_{22}})\\ &= \frac{h_{12}}{g_{12}\mp\sqrt{g_{11}g_{22}}},\end{aligned}$$

其中g,h已由上面给出. ▌

2. **解** 记曲面$a=(x,y,f(x,y))$，则

$$a_x=(1,0,f_x),$$

$$a_y=(0,1,f_y),$$

$$n=\frac{a_x\times a_y}{|a_x\times a_y|}=\frac{1}{\sqrt{f_x^2+f_y^2+1}}(-f_x,-f_y,1),$$

$$a_{xx}=(0,0,f_{xx}),$$

$$a_{xy}=(0,0,f_{xy}),$$

$$a_{yy}=(0,0,f_{yy}).$$

则曲面的第一基本形式为$I=(1+f_x^2)dxdx+2f_xf_ydxdy+(1+f_y^2)dydy$，
第二基本形式为$II=\frac{f_{xx}}{\sqrt{f_x^2+f_y^2+1}}dxdx+2\frac{f_{xy}}{\sqrt{f_x^2+f_y^2+1}}dxdy+\frac{f_{yy}}{\sqrt{f_x^2+f_y^2+1}}dydy.$ ▌

3. **证明** 由行列式乘法法则 $detA = detA^T$, $det(A\cdot B) = detA \cdot detB$, 可得

$$\begin{aligned}
(\mathbf{r_{11}}, \mathbf{r_1}, \mathbf{r_2})(\mathbf{r_{22}}, \mathbf{r_1}, \mathbf{r_2}) &= \det\begin{pmatrix} r_{11}^1 & r_{11}^2 & r_{11}^3 \\ r_1^1 & r_1^2 & r_1^3 \\ r_2^1 & r_2^2 & r_2^3 \end{pmatrix} \det\begin{pmatrix} r_{22}^1 & r_{22}^2 & r_{22}^3 \\ r_1^1 & r_1^2 & r_1^3 \\ r_2^1 & r_2^2 & r_2^3 \end{pmatrix} \\
&= \det\begin{pmatrix} r_{11}^1 & r_{11}^2 & r_{11}^3 \\ r_1^1 & r_1^2 & r_1^3 \\ r_2^1 & r_2^2 & r_2^3 \end{pmatrix} \det\begin{pmatrix} r_{22}^1 & r_{22}^2 & r_{22}^3 \\ r_1^1 & r_1^2 & r_1^3 \\ r_2^1 & r_2^2 & r_2^3 \end{pmatrix}^T \\
&= \left|\begin{pmatrix} r_{11}^1 & r_{11}^2 & r_{11}^3 \\ r_1^1 & r_1^2 & r_1^3 \\ r_2^1 & r_2^2 & r_2^3 \end{pmatrix} \begin{pmatrix} r_{22}^1 & r_1^1 & r_2^1 \\ r_{22}^2 & r_1^2 & r_2^2 \\ r_{22}^3 & r_1^3 & r_2^3 \end{pmatrix}\right| \\
&= \begin{vmatrix} r_{11}\cdot r_{22} & r_{11}\cdot r_1 & r_{11}\cdot r_2 \\ r_1\cdot r_2 2 & E & F \\ r_2\cdot r_{22} & F & G \end{vmatrix}
\end{aligned}$$

同理可得（2）. ▌

4. **证明** 由定义得

$$L = r_{11}\cdot n = r_{11}\cdot\frac{r_1\times r_2}{|r_1\times r_2|} = \frac{(r_{11}, r_1, r_2)}{\sqrt{|r_1|^2|r_2|^2-(r_1\cdot r_2)^2}} = \frac{(r_{11}, r_1, r_2)}{\sqrt{\det(g_{\alpha\beta})}}$$

同理可得

$$N = \frac{(r_{22}, r_1, r_2)}{\sqrt{\det(g_{\alpha\beta})}},$$

$$M = \frac{(r_{12}, r_1, r_2)}{\sqrt{\det(g_{\alpha\beta})}},$$

从而

$$LN - M^2 = \frac{1}{g}[(r_{11}, r_1, r_2)(r_{22}, r_1, r_2) - (r_{12}, r_1, r_2)^2].$$

▌

5. **证明** 由于 $E =< r_1, r_1 >$, $F =< r_1, r_2 >$, $G =< r_2, r_2 >$
$E_1 = 2 < r_{11}, r_1 >$, $F_1 =< r_{11}, r_2 > + < r_1, r_{12} >$, $G_1 = 2 < r_{12}, r_2 >$
$E_2 = 2 < r_{12}, r_1 >$, $F_2 =< r_{12}, r_2 > + < r_1, r_{22} >$, $G_2 = 2 < r_{22}, r_2 >$
可得
$< r_{11}, r_1 >= \frac{E_1}{2}$, $< r_{12}, r_1 >= \frac{E_2}{2}$
$< r_{22}, r_2 >= \frac{G_2}{2}$, $< r_{12}, r_2 >= \frac{G_1}{2}$
$< r_{11}, r_2 >= F_1 - \frac{E_2}{2}$, $< r_{22}, r_1 >= F_2 - \frac{G_1}{2}$ ▌

2. **证明** (1) 若曲线$C:\mathbf{x}(s)=\mathbf{x}(u^\alpha(s))$是曲率线, 则$\mathrm{d}\mathbf{x}$为主方向, 即是Weingarten变换的特征方向, 有

$$W(\mathrm{d}\mathbf{x})=h_\alpha^\beta\mathbf{x}_\beta\mathrm{d}u^\alpha=\lambda\mathrm{d}\mathbf{x}=\lambda\mathbf{x}_\alpha\mathrm{d}u^\alpha$$

其中λ为主曲率. 于是$(h_\alpha^\beta\mathbf{x}_\beta-\lambda\mathbf{x}_\alpha)\mathrm{d}u^\alpha=0$. 两边同点乘$\mathbf{x}_\beta$, 得$(h_{\alpha\beta}-\lambda g_{\alpha\beta})\mathrm{d}u^\alpha=0$. 分别令$\beta=1,2$得

$$\begin{cases}(h_{11}\mathrm{d}u^1+h_{12}\mathrm{d}u^2)-\lambda(g_{11}\mathrm{d}u^1+g_{12}\mathrm{d}u^2)=0\\(h_{12}\mathrm{d}u^1+h_{22}\mathrm{d}u^2)-\lambda(g_{12}\mathrm{d}u^1+g_{22}\mathrm{d}u^2)=0\end{cases}$$

因$(1,-\lambda)$是上述方程组得非零解, 于是有

$$\begin{vmatrix}h_{11}\mathrm{d}u^1+h_{12}\mathrm{d}u^2 & g_{11}\mathrm{d}u^1+g_{12}\mathrm{d}u^2\\h_{12}\mathrm{d}u^1+h_{22}\mathrm{d}u^2 & g_{12}\mathrm{d}u^1+g_{22}\mathrm{d}u^2\end{vmatrix}=0$$

即

$$\begin{vmatrix}(\mathrm{d}u^2)^2 & -\mathrm{d}u^1\mathrm{d}u^2 & (\mathrm{d}u^1)^2\\g_{11} & g_{12} & g_{22}\\h_{11} & h_{12} & h_{22}\end{vmatrix}=0.$$

(2) 必要性

因曲率线网为正交网, 且$W(\mathbf{x}_\alpha)=k_\alpha\mathbf{x}_\alpha$, 故$g_{12}=0$且

$$\mathrm{II}=(W(\mathbf{x}_\alpha\mathrm{d}u^\alpha))(\mathbf{x}_\beta\mathrm{d}u^\beta)=\sum_{\alpha\beta}k_\alpha g_{\alpha\beta}\mathrm{d}u^\alpha\mathrm{d}u^\beta$$

即此时有$g_{12}=h_{12}=0$.

充分性

若$g_{12}=h_{12}=0$, 由上面曲率线的微分方程得$(g_{22}h_{11}-g_{11}h_{22})\mathrm{d}u^1\mathrm{d}u^2=0$. 若$g_{22}h_{11}-g_{11}h_{22}=0$, 则$h_{\alpha\beta}=\lambda g_{\alpha\beta}$, 即曲面有脐点, 矛盾! 故只能是$\mathrm{d}u^1\mathrm{d}u^2=0$, 此即为参数曲线, 因而参数网为曲率线网. ▌

3. **解** 令$\mathbf{x}=(x^1,x^2,x^3)=(x^1,x^2,f(x^1,x^2))$及$f_1=\frac{\partial f}{\partial x^1},f_2=\frac{\partial f}{\partial x^2}$. 则$\mathbf{x}_1=(1,0,f_1)$, $\mathbf{x}_2=(0,1,f_2)$, $\mathbf{x}_{11}=(0,0,f_{11})$, $\mathbf{x}_{12}=(0,0,f_{12})$, $\mathbf{x}_{22}=(0,0,f_{22})$, $\mathbf{n}=\frac{(-f_1,-f_2,1)}{\sqrt{f_1^2+f_2^2+1}}$. 于是$g_{11}=1+f_1^2$, $g_{12}=f_1f_2$, $g_{22}=1+f_2^2$, $h_{11}=\frac{f_{11}}{\sqrt{f_1^2+f_2^2+1}}$, $h_{12}=\frac{f_{12}}{\sqrt{f_1^2+f_2^2+1}}$, $h_{22}=\frac{f_{22}}{\sqrt{f_1^2+f_2^2+1}}$. 由平均曲率表达式, 知$H=0\iff h_{11}g_{22}-2h_{12}g_{12}+h_{22}g_{11}=0$, 代入得$f$所满足的微分方程$f_{11}(1+f_2^2)-2f_{12}f_1f_2+f_{22}(1+f_1^2)=0$.

对于$x^3=a\arctan\frac{x^2}{x^1}$, 知$f(x^1,x^2)=a\arctan\frac{x^2}{x^1}$, 得$f_1=\frac{-ax^2}{(x^1)^2+(x^2)^2}$, $f_2=\frac{ax^1}{(x^1)^2+(x^2)^2}$, $f_{11}=\frac{2ax^1x^2}{((x^1)^2+(x^2)^2)^2}$, $f_{12}=\frac{-a((x^1)^2-(x^2)^2)}{((x^1)^2+(x^2)^2)^2}$, $f_{22}=\frac{-2ax^1x^2}{((x^1)^2+(x^2)^2)^2}$. 将其代入前面所得的微分方程, 验证知等号成立, 因此该曲面为极小曲面. ▌

9. **证明** 曲面为$\mathbf{r}(x^1,x^2)=(x^1,x^2,f(x^1)+g(x^2))$. 计算得

$$g_{11}=1+f'^2,\ g_{12}=f'g',\ g_{22}=1+g'^2.$$

$$h_{11}=\frac{f''}{1+f'^2+g'^2},\ h_{12}=0,\ h_{22}=\frac{g''}{1+f'^2+g'^2}.$$

由其为极小曲面得$g_{11}h_{11}-2g_{12}h_{12}+g_{22}h_{11}=0$, 代入, 有

$$-\frac{f''}{1+f'^2}=\frac{g''}{1+g'^2}.$$

因上面左式是关于x^1的函数, 右式是关于x^2的函数, 故必有

$$-\frac{f''}{1+f'^2}=\frac{g''}{1+g'^2}=a=\mathrm{const}.$$

得$(\arctan(-f'))'=a$, $f'=-\tan(ax^1+c_1)$, $f=-\frac{1}{a}\ln\cos(ax^1+c_1)+c_2$, 同理$g=\frac{1}{a}\ln\cos(ax^2+c_3)+c_4$. 因此除相差一常数外, $ax^3=\ln\frac{\cos ax^2}{\cos ax^1}$. ▌

10. **证明** 由曲面方程得$\mathbf{x}_u=(3(1-u^2+v^2),6uv,6u)$, $\mathbf{x}_v=(6uv,3(1+u^2-v^2),-6v)$, $\mathbf{x}_{uu}=(-6u,6v,6)$, $\mathbf{x}_{uv}=(6v,6u,0)$, $\mathbf{x}_{vv}=(6u,-6v,-6)$, $\mathbf{n}=\frac{1}{1+u^2+v^2}(-2u,2v,1-u^2-v^2)$. 从而$g_{11}=9(1+u^2+v^2)^2$, $g_{12}=0$, $g_{22}=9(1+u^2+v^2)^2$, $h_{11}=6$, $h_{12}=0$, $h_{22}=-6$. 因$h_{11}g_{22}-2h_{12}g_{12}+h_{22}g_{11}=0$, 故$H=0$, 即曲面为极小曲面. ▌