REAL ANALYSIS

LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

This part is not required in this course (not included in the final exam).

We attempt to give a very brief introduction as this topic is useful in PDEs.

1. Fourier transform: An introduction

The Fourier transform of an integrable function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\mathscr{F}[f](\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx,$$

and the inverse Fourier transform of $f \in L^1(\mathbb{R}^n)$ is defined by

$$\mathscr{F}[f](x) = f^{\vee}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

Proposition 1.1. *Suppose $f \in L^1(\mathbb{R}^n)$. Then both $\widehat{f}(\xi)$ and $f^{\vee}(x)$ are bounded and continuous.*

Proof. This is an exercise. □

Motivation of Fourier transform

Recall that, for $f \in L^1([-T, T])$, its Fourier series is as follows

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k \cos \frac{\pi}{T} kx + b_k \sin \frac{\pi}{T} kx \Big)$$

This can be also written as (by using $\cos x = (e^{ix} + e^{-ix})/2$ and $\sin x = (e^{ix} - e^{-ix})/2$)

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\frac{\pi}{T}kx}, \text{ where } A_k = \frac{1}{2T} \int_{-T}^{T} f(y) e^{-i\frac{\pi}{T}ky} dy. \tag{1.1}$$

We want to write down a similar triangle series formula for a (not necessarily periodic) function f on all of $\mathbb{R}$. Informally, (1.1) can be written as

$$\begin{aligned} f(x) &= \frac{1}{2T}\sum_{k=-\infty}^{\infty}\Big(\int_{-T}^{T} f(y)e^{-i\frac{\pi}{T}ky}dy\Big)e^{i\frac{\pi}{T}kx} \\ &= \frac{1}{2T}\sum_{k=-\infty}^{\infty}\Big(\int_{-T}^{T} f(y)e^{-2\pi i\xi_k y}dy\Big)e^{2\pi i\xi_k x} \quad (\text{denote } \xi_k = \frac{k}{2T}). \end{aligned}$$

Since $\Delta\xi_k = \xi_{k+1} - \xi_k = \frac{1}{2T}$, sending $T \to \infty$, we expect

$$\int_{-T}^{T} f(y)e^{-2\pi i\xi_k y}dy \to \widehat{f}(\xi_k) = \int_{\mathbb{R}} f(y)e^{-2\pi i\xi_k x} \quad \text{and} \quad \frac{1}{2T}\sum_{k=-\infty}^{\infty} \to \int_{\mathbb{R}} d\xi,$$

and therefore

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)e^{2\pi i\xi x}d\xi = \mathscr{F}^{-1}[\widehat{f}](x). \tag{1.2}$$

The above one is a Fourier inversion formula.

Exercise. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a real-valued function. If f is even, then $\widehat{f}$ is real. If f is odd, then $\widehat{f}$ is imaginary.*

Proof. This is an exercise.

□

Exercise (Important Result). *We have $\mathscr{F}[e^{-\pi|x|^2}](\xi) = e^{-\pi|\xi|^2}$.*

Proof. This is an exercise.

□

1.1. **Some properties of Fourier transform.**

Proposition 1.2. *Suppose $f, g \in L^1(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} f(x)\widehat{g}(x)dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)d\xi.$$

Proof. This is an exercise.

□

Proposition 1.3. *Suppose $f, g \in L^1(\mathbb{R}^n)$. Then $\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.*

Proof. This is an exercise.

□

We next use the notation

$$f(x) \longrightarrow \widehat{f}(\xi)$$

to mean that $\widehat{f}$ denotes the Fourier transform of f.

Proposition 1.4. *Suppose $f \in Sch(\mathbb{R}^n)$. Then*

(i) $\partial_{x^\alpha} f(x) \longrightarrow (2\pi i\xi)^\alpha \widehat{f}(\xi)$,
(ii) $(-2\pi i x)^\alpha f(x) \longrightarrow \partial_{\xi^\alpha} \widehat{f}(\xi)$.

Remark 1.1. *The Schwartz space $Sch(\mathbb{R}^n)$, which consists of all indefinitely differentiable functions f on $\mathbb{R}^n$ such that*

$$\sup_{x\in\mathbb{R}^n} \left| x^\alpha \partial_{x^\beta} f(x) \right| < \infty,$$

for every multi-index α and β. In other words, f and all its derivatives are required to be rapidly decreasing.

Proof. This is an exercise. □

1.2. Applications to differential equations.

We shall prove the Fourier inversion formula in next subsection. Before that, we apply this formula and the properties above to linear, constant-coefficient differential equations in this subsection. We will see that Fourier transform is an especially powerful technique for studying such PDEs.

Ordinary differential equations.

Find a function $u : \mathbb{R} \to \mathbb{R}$ satisfies $\sum_{k=0}^m a_k \frac{d^k}{dx^k} u = f$.

Using Proposition 1.4, after Fourier transform, the ODE reduces to

$$\sum_{k=0}^m a_k (2\pi i\xi)^k \widehat{u}(\xi) = \widehat{f}(\xi).$$

Therefore, the Fourier inversion formula (1.2) shows

$$u(x) = \mathscr{F}^{-1}[\widehat{f}/\sum_{k=0}^m a_k (2\pi i\xi)^k],$$

provided the inverse transform is well-defined. Suppose $f \in C_c(\mathbb{R})$ and we can compute

$$g = \mathscr{F}^{-1}[1/\sum_{k=0}^{m} a_k (2\pi i \xi)^k],$$

and g belongs to a suitable function space. Then we deduce by Proposition 1.3 that

$$u(x) = \mathscr{F}^{-1}[\widehat{f}\widehat{g}] = \mathscr{F}^{-1}[\mathscr{F}[f * g]] = (f * g)(x).$$

Example. Solve $u_{xx} - u = -f$ on $\mathbb{R}$.

Solution. By the method of Fourier transform, we have

$$u(x) \;=\; \int_{\mathbb{R}} f(x-y)\Big[\int_{\mathbb{R}} \frac{e^{2\pi i \xi y}}{4\pi^2\xi^2+1} d\xi\Big] dy = \frac{1}{2}\int_{\mathbb{R}} f(z) e^{-|x-z|} dz,$$

where we use $\mathscr{F}^{-1}[1/(4\pi^2\xi^2+1)](y) = \frac{1}{2}e^{-|y|}$.

For the inverse transform, let us consider the meromorphic function

$$F(z) = \frac{e^{2\pi y i z}}{4\pi^2 z^2 + 1},$$

which has two singularities

$$z_1 = \frac{i}{2\pi} \text{ and } z_2 = -\frac{i}{2\pi}.$$

It is not hard to see that

$$\begin{aligned} F(z) &= \frac{1}{4\pi^2}\frac{e^{2\pi y i z}}{(z+i/(2\pi))(z-i/(2\pi))} \\ &= \frac{e^{-y}}{4\pi i(z-i/(2\pi))} - \frac{e^{y}}{4\pi i)(z+i/(2\pi))} + \text{holomorphic function on } \mathbb{C}\setminus\{z_1,z_2\}, \end{aligned}$$

Observe that

$$\begin{aligned} &\text{when } y>0, \quad \int_{\{Re^{i\theta}:0\le\theta\le\pi\}} F(z)dz \to 0 \text{ as } R\to\infty, \\ &\text{when } y<0, \quad \int_{\{Re^{i\theta}:-\pi\le\theta\le 0\}} F(z)dz \to 0 \text{ as } R\to\infty. \end{aligned}$$

Therefore, using the residue formula, we conclude that

$$\begin{aligned} \int_{\mathbb{R}} \frac{e^{2\pi i\xi y}}{4\pi^2\xi^2+1} d\xi \;&=\; \begin{cases} 2\pi i \mathrm{Res}_{z_1} F & \text{if } y>0, \\ 2\pi i \mathrm{Res}_{z_2} F & \text{if } y<0 \end{cases} \\ &=\; \frac{1}{2}e^{-|y|}. \end{aligned}$$

Exercise. Use the idea above to solve $u_{xx} + u = f$ on $\mathbb{R}$.

The Wave Equation.

The wave equation describes the behaviour of electromagnetic waves in vacuum,

$$\begin{cases} \partial_t^2 u(x,t) = \Delta u(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,\infty), \\ u(x,0) = f(x) \ \text{ and } \ \partial_t u(x,0) = g(x), \end{cases} \tag{1.3}$$

where $f, g \in C_c(\mathbb{R}^n)$.

Let us take the Fourier transform of the equation and of the initial conditions, with respect to the space variables. By Proposition 1.4, this reduces the problem to an ODE in the time variable:

$$\partial_t^2 \widehat{u}(\xi,t) = \mathscr{F}_x[\partial_t^2 u](\xi,t) = \mathscr{F}_x[\Delta u](\xi,t) = -4\pi^2|\xi|^2\widehat{u}(\xi,t)$$

with the initial conditions

$$\widehat{u}(\xi,0) = \widehat{f}(\xi) \text{ and } \partial_t\widehat{u}(\xi,0) = \widehat{g}(\xi).$$

Fix $\xi \in \mathbb{R}^n$. The solution of this ODE in t is given by

$$\widehat{u}(\xi,t) = A(\xi)\cos(2\pi|\xi|t) + B(\xi)\sin(2\pi|\xi|t),$$

where $A(\xi)$ and $B(\xi)$ are unknown constants to be determined by the initial conditions

$$A(\xi) = \widehat{f}(\xi) \quad \text{and} \quad 2\pi|\xi|B(\xi) = \widehat{g}(\xi).$$

Therefore, we find that

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos(2\pi|\xi|t) + \widehat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}. \tag{1.4}$$

The solution u is then given by taking the inverse Fourier transform in the ξ variable

$$u(x,t) = \int_{\mathbb{R}^n} \left[\widehat{f}(\xi)\cos(2\pi|\xi|t) + \widehat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right] e^{2\pi i x\cdot\xi} d\xi.$$

$\boxed{\textit{The wave equation in } \mathbb{R}\times\mathbb{R}}$ In $1D$ case, by the evenness/oddness of triangle functions,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{\mathbb{R}} \widehat{f}(\xi)\left[e^{2\pi i\xi t} + e^{-2\pi i\xi t}\right] e^{2\pi i x\cdot\xi} d\xi + \frac{1}{2}\int_{\mathbb{R}} \widehat{g}(\xi)\left[\frac{e^{2\pi i\xi t} - e^{-2\pi i\xi t}}{2\pi i\xi}\right] e^{2\pi i x\cdot\xi} d\xi \\ &= \frac{1}{2}\int_{\mathbb{R}} \widehat{f}(\xi)\left[e^{2\pi i\xi(x+t)} + e^{2\pi i\xi(x-t)}\right] d\xi + \frac{1}{2}\int_{\mathbb{R}} \widehat{g}(\xi)\left[\frac{e^{2\pi i\xi(x+t)} - e^{2\pi i\xi(x-t)}}{2\pi i\xi}\right] d\xi \\ &= \frac{1}{2}\Big[f(x+t) + f(x-t)\Big] + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds, \end{aligned}$$

where we use in the last equality the inversion formula (1.2) and Proposition 1.4. This is the so-called d'Alembert's formula (in 1747).

$\boxed{\textit{The wave equation in } \mathbb{R}^3\times\mathbb{R}}$ Consider the spherical mean of φ

$$M_t(\varphi)(x) = \fint_{\mathbb{S}^2} \varphi(x-t\gamma)d\sigma(\gamma),$$

where $d\sigma$ is the element of surface area for $\mathbb{S}^2$. Then

$$\begin{aligned}
\widehat{M_t(\varphi)}(\xi) &= \int_{\mathbb{R}^3} e^{-2\pi i\xi\cdot x}\fint_{\mathbb{S}^2} \varphi(x-t\gamma)d\sigma(\gamma)dx \\
&= \fint_{\mathbb{S}^2} e^{-2\pi i\xi\cdot t\gamma}\int_{\mathbb{R}^3} \varphi(x-t\gamma)e^{-2\pi i\xi\cdot(x-t\gamma)}dxd\sigma(\gamma) \\
&= \widehat{\varphi}(\xi)\fint_{\mathbb{S}^2} e^{-2\pi i\xi\cdot t\gamma}d\sigma(\gamma).
\end{aligned}$$

Direct computation shows, by using the spherical coordinate $\gamma=(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$,

$$\begin{aligned}
\fint_{\mathbb{S}^2} e^{-2\pi i\xi\cdot\gamma}d\sigma(\gamma) &= \fint_{\mathbb{S}^2} e^{-2\pi i|\xi|\gamma_3}d\sigma(\gamma) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{-2\pi i|\xi|\cos\theta}\sin\theta d\theta d\varphi \\
&= \frac{1}{2}\int_0^{\pi} e^{-2\pi i|\xi|\cos\theta}\sin\theta d\theta = \frac{1}{2}\int_{-1}^{1} e^{-2\pi i|\xi|t}dt = \frac{\sin(2\pi|\xi|)}{2\pi|\xi|}.
\end{aligned}$$

Therefore

$$\widehat{M_t(\varphi)}(\xi) = \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|t}\widehat{\varphi}(\xi).$$

By (1.4) and the inversion formula, we deduce

$$\begin{aligned}
u(x,t) &= \partial_t\Big(\int_{\mathbb{R}^3}\widehat{f}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}e^{2\pi ix\cdot\xi}d\xi\Big) + t\int_{\mathbb{R}^3}\widehat{M_t(g)}(\xi)e^{2\pi ix\cdot\xi}d\xi \\
&= \partial_t\Big(t\int_{\mathbb{R}^3}\widehat{M_t(f)}(\xi)e^{2\pi ix\cdot\xi}d\xi\Big) + t\int_{\mathbb{R}^3}\widehat{M_t(g)}(\xi)e^{2\pi ix\cdot\xi}d\xi
\end{aligned}$$

Applying the inversion formula again, in $3D$ case, the solution to (1.3) is

$$u(x,t) = \frac{\partial}{\partial t}\fint_{\mathbb{S}^2} tf(x-t\gamma)d\sigma(\gamma) + t\fint_{\mathbb{S}^2} g(x-t\gamma)d\sigma(\gamma).$$

$\boxed{\textit{The wave equation in } \mathbb{R}^2\times\mathbb{R}\textit{: descent}}$

Define the $2D$ means by

$$\widetilde{M}_t(\varphi)(x) = \frac{1}{2\pi}\int_{|y|\le 1}\varphi(x-ty)/\sqrt{1-|y|^2}dy.$$

We find that in $2D$ case, the solution to (1.3) is given by

$$u(x,t) = \frac{\partial}{\partial t}(t\widetilde{M}_t(f)(x)) + t\widetilde{M}_t(g)(x)$$

Let $\tilde{f}(x_1, x_2, x_3) = f(x_1, x_2)$ and $\tilde{g}(x_1, x_2, x_3) = g(x_1, x_2)$. Solving (1.3) with f, g replaced by $\tilde{f}, \tilde{g}$, we obtain $\tilde{u}(x_1, x_2, x_3, t)$. We then expect that $\tilde{u}(x_1, x_2, x_3, t)$ is independent of x_3, which leads to the solution of $2D$ problem $u(x_1, x_2, t) = \tilde{u}(x_1, x_2, 0, t)$.

<u>The Heat Equation</u>.

Consider the heat equation which describes the heat diffusion with the heat source $\varphi \in C_c(\mathbb{R}^n)$ and initial data $f \in C_c(\mathbb{R}^n)$:

$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) + \varphi(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,\infty), \\ u(x,0) = f(x), \quad x \in \mathbb{R}^n. \end{cases}$$

Take the Fourier transform of the equation and the initial condition with respect to the space variable. The heat equation is transferred to the following

$$\partial_t \widehat{u}(\xi,t) = -4\pi^2|\xi|^2 \widehat{u}(\xi,t) + \widehat{\varphi}(\xi,t), \quad \widehat{u}(\xi,0) = \widehat{f}(\xi).$$

The solution to this ODE is

$$\widehat{u}(\xi,t) = e^{-4\pi^2|\xi|^2 t}\widehat{f}(\xi,t) + \int_0^t \widehat{\phi}(\xi,s)e^{-4\pi^2|\xi|^2(t-s)}ds.$$

By Fourier inversion formula and Proposition 1.3,

$$u(x,t) \quad = \quad \big(f * \mathscr{F}^{-1}\big[e^{-4\pi^2 t|\xi|^2}\big]\big)(x) + \int_0^t (h * \mathscr{F}^{-1}[e^{-4\pi^2|\xi|^2(t-s)}])(x,s)ds.$$

Next we compute the inverse transform of $e^{-4\pi^2|\xi|^2 t}$. Direct calculation shows

$$\mathscr{F}^{-1}\big[e^{-4\pi^2|\xi|^2 t}\big](y) \quad = \quad \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{|y|^2}{4t}}.$$

Recall our exercise (marked as "Important Result") at the beginning of the part. This is called the heat kernel.

Consequently the solution to the heat equation is given by

$$u(x,t) = \frac{1}{(2\sqrt{\pi t})^n}\int_{\mathbb{R}^n} f(y)e^{-\frac{|x-y|^2}{4t}}dy + \int_0^t \frac{1}{(2\sqrt{\pi(t-s)})^n}\int_{\mathbb{R}^n} h(y,s)e^{-\frac{|x-y|^2}{4(t-s)}}dyds.$$

<u>The Laplace Equation</u>.

Let us look at a concrete example: the steady-state heat equation in the upper half-plane:

$$\begin{cases} \Delta u = 0 \;\text{ in } H_+^n = \{(x_1, \cdots, x_n) : x_n > 0\}, \\ u(x',0) = f(x'), \quad x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}. \end{cases}$$

By taking Fourier transform of the equation in the x' variable, we obtain

$$-4\pi^2|\xi|^2\widehat{u}(\xi, x_n) + \partial_{x_n}^2\widehat{u}(\xi, x_n) = 0, \quad \widehat{u}(\xi, 0) = \widehat{f}(\xi).$$

Solving this ODE of y gives

$$\widehat{u}(\xi, x_n) = A(\xi)e^{2\pi|\xi|x_n} + B(\xi)e^{-2\pi|\xi|x_n}.$$

We disregard the first term because of its rapid exponential increase. Then

$$\widehat{u}(\xi, y) = \widehat{f}(\xi)e^{-2\pi|\xi|x_n}.$$

It follows that

$$u(x', x_n) = \big(f * \mathscr{F}^{-1}[e^{-2\pi|\xi|x_n}]\big)(x').$$

In two-dimensional case ($n = 2$), we compute

$$\begin{aligned}
\mathscr{F}^{-1}[e^{-2\pi|\xi|x_2}](x_1) &= \int_0^\infty e^{-2\pi\xi(x_2+ix_1)}d\xi + \int_{-\infty}^0 e^{2\pi\xi(x_2-ix_1)}d\xi \\
&= \frac{1}{2\pi(x_2+ix_1)} + \frac{1}{2\pi(x_2-ix_1)} \\
&= \frac{x_2}{\pi(x_1^2+x_2^2)}.
\end{aligned}$$

Therefore Laplace equation in upper half-space has a solution

$$u(x_1, x_2) = \frac{x_2}{\pi}\int_{\mathbb{R}} \frac{f(h)}{(x_1-h)^2+x_2^2}dh.$$

In high dimensions, we have

$$u(x) = \frac{2x_n}{\alpha_n}\int_{\partial H_+^n} \frac{f(y)}{|x-y|^n}dy, \quad \text{where } \alpha_n = |\mathbb{S}^{n-1}|.$$

In the above representation formula,

$$\mathcal{P}_y(x) = \frac{2x_n}{\alpha_n|x-y|^n}$$

is called the Poisson kernel on H_+^n.

1.3. **Fourier inversion formula.**

Theorem 1.1. *Suppose $f \in L^1(\mathbb{R}^n)$ and also $\widehat{f} \in L^1(\mathbb{R}^n)$. Then the inversion formula below holds for almost every x*

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi. \tag{1.5}$$

Consider the modulated Gaussian for given $x \in \mathbb{R}^n$ and $\delta > 0$,

$$G_{\delta,x}(\xi) = e^{-\pi\delta|\xi|^2} e^{2\pi i x\cdot\xi}. \tag{1.6}$$

Straightforward calculation gives

$$\begin{aligned}
\widehat{G}_{\delta,x}(y) &= \int_{\mathbb{R}^n} e^{-\pi\delta|\xi|^2} e^{2\pi i x\cdot\xi} e^{-2\pi i \xi\cdot y} d\xi \\
&= \prod_{k=1}^{n} \int_{\mathbb{R}} e^{-\pi\delta\xi_k^2 + 2\pi i(x_k - y_k)\xi_k} d\xi_k \\
&= \prod_{k=1}^{n} \left[e^{-(x_k-y_k)^2\pi/\delta} \int_{\mathbb{R}} e^{-(\sqrt{\pi\delta}\xi_k - i(x_k-y_k)\sqrt{\pi/\delta})^2} d\xi_k \right] \\
&= \delta^{-\frac{n}{2}} e^{-|x-y|^2\pi/\delta},
\end{aligned}$$

where we employ

$$\int_{\mathbb{R}} e^{-(\sqrt{\pi\delta}t - i(x_k-y_k)\sqrt{\pi/\delta})^2} dt = \int_{\mathbb{R}} e^{-\pi\delta t^2} dt = \frac{1}{\sqrt{\delta}}.$$

Definition 1.1. *We recognise $K_\delta \in L^1(\mathbb{R}^n)$ as good kernels, if*

(i) *$\int_{\mathbb{R}^n} |K_\delta(x)| dx \le A$ for some A independent of A;*
(ii) *$\int_{\mathbb{R}^n} K_\delta(x) dy = 1$;*
(iii) *For each $\eta > 0$, $\int_{|x|>\eta} |K_\delta(x)| dx \to 0$ as $\delta \to 0$.*

Let $K_\delta = \delta^{-n/2} e^{-\pi|x|^2/\delta}$. One may check that K_δ are good kernels. Write $K_{\delta,x}(y) = K_\delta(x-y)$. Then we have

$$K_{\delta,x}(y) = \delta^{-\frac{n}{2}} e^{-\pi|x-y|^2/\delta}. \tag{1.7}$$

which is exactly the Fourier transform of $G_{\delta,x}(\xi)$.

If K_δ are a good kernels, and write $K_{\delta,x}(y) = K_\delta(x-y)$, we usually see that

$$K_{\delta,x}(y) \to \delta_x(y),$$

where $\delta_x(y)$ is a generalised function which enjoys the property $\delta_x * f = f(x)$ [1]. The notation on left-hand side means

$$\delta_x * f := \lim_{\delta_k \to 0} (K_{\delta_k} * f(y))(x)$$

where

$$(K_\delta * f(y))(x) := \int_{\mathbb{R}^n} K_{\delta,x}(y) f(y) dy.$$

The reason of this terminology will become clear in the following.

Lemma 1.1. *Suppose $f \in L^1(\mathbb{R}^n)$. Suppose K_δ are good kernels, namely K_δ satisfy (i), (ii) and (iii). Then $\|K_\delta * f - f\|_{L^1} \to 0$ as $\delta \to 0$. Consequently there is a sequence $\delta_k \to 0$ such that*

$$\lim_{k\to\infty} (K_{\delta_k} * f)(x) = f(x), \ \forall x \in \mathbb{R}^n.$$

Proof. Since K_δ satisfies (ii), we have

$$(K_\delta * f)(x) - f(x) = \int_{\mathbb{R}^n} K_\delta(y) f(x-y) dy - \int_{\mathbb{R}^n} K_\delta(y) f(x) dy.$$

It follows that

$$|(K_\delta * f)(x) - f(x)| \ \ \leq \ \ \int_{\mathbb{R}^n} K_\delta(y) |f(x-y) - f(x)| dy.$$

Note that $f(x)$ and $f(x-y)$ are measurable functions on $\mathbb{R}^n \times \mathbb{R}^n$. We then apply Fubini's theorem to deduce that, by setting $f_y(x) = f(x-y)$,

$$\begin{aligned} \|(K_\delta * f)(x) - f(x)\|_{L^1} &\leq \int_{\mathbb{R}^n} K_\delta(y) \|f_y - f\|_{L^1} dy \\ &\leq \int_{|y|\leq\eta} K_\delta(y) \|f_y - f\|_{L^1} dy + \int_{|y|>\eta} K_\delta(y) \|f_y - f\|_{L^1} dy. \end{aligned}$$

For any $\varepsilon > 0$, there is η_0 such that $\|f_y - f\|_{L^1} < \varepsilon/2$ provided $|y| \leq \eta_0$. Therefore

$$\begin{aligned} \|(K_\delta * f)(x) - f(x)\|_{L^1} &\leq 2^{-1}\varepsilon \int_{|y|\leq\eta} K_\delta(y) dy + 2\|f\|_{L^1} \int_{|y|>\eta} K_\delta(y) dy \\ &\leq \varepsilon/2 + 2\|f\|_{L^1} \int_{|y|>\eta} K_\delta(y) dy, \quad \forall \eta \leq \eta_0. \end{aligned}$$

Fix $\eta = \eta_0$. By property (iii), we can take $\delta < \delta_0(\varepsilon, \eta_0, \|f\|_{L^1})$, such that

$$\|(K_\delta * f)(x) - f(x)\|_{L^1} < \varepsilon.$$

[1] δ_x is also called the Dirac measure.

This implies $K_\delta * f \to f$ in L^1 sense. By Riesz theorem, there is a subsequence that converges to $f(x)$ almost everywhere. $\square$

Proof of Theorem 1.1. In view of Proposition 1.2, for each $x \in \mathbb{R}^n$ and $\delta > 0$,

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) G_{\delta,x}(\xi) d\xi = \int_{\mathbb{R}^n} f(y) K_{\delta,x}(y) dy, \tag{1.8}$$

where $G_{\delta,x}$ and $K_{\delta,x}$ are given by (1.6) and (1.7), which satisfy $\widehat{G}_{\delta,x}(y) = K_{\delta,x}(y)$.

By Lemma 1.1, there is a sequence $\delta_k \to$ such that

$$\int_{\mathbb{R}^n} f(y) K_{\delta_k,x}(y) dy \to f(x). \tag{1.9}$$

On the other hand, observe $|\widehat{f}(\xi) G_{\delta_k,x}(\xi)| \le |\widehat{f}| \in L^1$, and so by the dominated convergence theorem

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) G_{\delta_k,x}(\xi) d\xi \to \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi. \tag{1.10}$$

Now (1.5) follows by combining (1.8)-(1.10).

$\square$

Theorem 1.2 (Plancherel Theorem)**.** *The Fourier transform and its inversion are well-defined on $L^2(\mathbb{R}^n)$ space, and*

$$\|\widehat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in L^2(\mathbb{R}^n). \quad \textit{(Plancherel identity)} \tag{1.11}$$

Moreover the inversion formula (1.5) *holds for all $f \in L^2(\mathbb{R}^n)$.*

Proof. It is quite straightforward to see the Fourier (inverse) transform is well-defined on the Schwartz sapce $\mathrm{Sch}(\mathbb{R}^n)$, and the inversion formula holds in this space.

We verify (1.11) for $f \in \mathrm{Sch}(\mathbb{R}^n)$. Let $f^{\#}(x) = \overline{f(-x)}$. Then

$$\widehat{f^{\#}}(\xi) = \int_{\mathbb{R}^n} \overline{f(-x)} e^{-2\pi i x \cdot \xi} dx = \overline{\widehat{f}(\xi)}.$$

Define $h = f * f^{\#}$. We have

$$\widehat{h}(\xi) = \widehat{f}\widehat{f^{\#}} = |\widehat{f}(\xi)|^2,$$

and

$$h(0) = \int_{\mathbb{R}^n} f(y) f^{\#}(y) dy = \|f\|_{L^2}^2.$$

Apply the inversion formula to h, we have $h(0) = (\widehat{h})^\vee(0) = \|\widehat{f}\|_{L^2}$. That is

$$\|\widehat{f}\|_{L^2} = \|f\|_{L^2}, \quad \forall f \in \mathrm{Sch}(\mathbb{R}^n).$$

The key observation for the extension of Fourier transform and its inverse transform to $L^2(\mathbb{R}^n)$ is that $\mathrm{Sch}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. Given $f \in L^2(\mathbb{R}^n)$ and any $\varepsilon > 0$, if M is large, then $f_M := f\chi_{\{|y| \le M, |f(y)| \le M\}}$ satisfies (by using dominated convergence theorem and $|f_M - f|^2 \le 4f^2 \in L^1(\mathbb{R}^n)$)

$$\|f_M - f\|_{L^2} \le \varepsilon/2.$$

Next we define $g_{M,\delta} = f_M * K_\delta$, where $K_\delta(x,y) = \delta^{-n}\varphi((x-y)/\delta)$. Here

$$\varphi(x) = \begin{cases} c_n e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1, \end{cases}$$

with c_n selected such that $\int_{\mathbb{R}^n} \varphi = 1$. Then φ satisfies

(a) φ is positive and smooth.
(b) φ is supported in the unit ball.
(c) $\varphi \ge 0$.
(d) $\|\varphi\|_{L^1} = 1$.

It is not hard to check that K_δ is a good kernel. By Lemma 1.1, $g_{M,\delta}$ converges to f_M almost everywhere. Note that $|g_{M,\delta}| \le M$ and so $|g_{M,\delta} - f_M|^2 \le 4M^2\chi_{B_M} \in L^1(\mathbb{R}^n)$. Applying the dominated convergence theorem, we have, for small $\delta > 0$,

$$\|g_{M,\delta} - f_M\|_{L^2} < \varepsilon/2.$$

Consequently $\|f - g_{M,\delta}\|_{L^2} \le \varepsilon$. By the smoothness of φ, $g_{M,\delta} \in C_0^\infty(\mathbb{R}^n) \subset \mathrm{Sch}(\mathbb{R}^n)$. This shows the denseness of $\mathrm{Sch}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$.

Now, for each $f \in L^2(\mathbb{R}^n)$, we define $\widehat{f}(\xi) = \widehat{f_k}(\xi)$, where $f_k \in \mathrm{Sch}(\mathbb{R}^n)$ such that $f_k \to f$ in L^2 sense. Since $\|\widehat{f_k} - \widehat{f_j}\|_{L^2} = \|f_k - f_j\|_{L^2} \to 0$ and $L^2(\mathbb{R}^n)$ is complete, we conclude that $\widehat{f_k}(\xi)$ converges to a limit in $L^2(\mathbb{R}^n)$, which is defined as $\widehat{f}(\xi)$.

We show that the limit is independent of the choice of the sequence. Suppose $g_k \in \mathrm{Sch}(\mathbb{R}^n)$ and also $g_k \to f$ in L^2 sense. Suppose the limit of $\widehat{g_k}$ in L^2 sense is φ, while the limit of $\widehat{f_k}$ is ψ. Then

$$\begin{aligned}
\|\psi-\varphi\|_{L^2} &\leq \|\psi-\widehat{f_k}\|_{L^2}+\|\varphi-\widehat{g}_j\|_{L^2}+\|\widehat{f_k}-\widehat{g}_j\|_{L^2} \\
&= \|\psi-\widehat{f_k}\|_{L^2}+\|\varphi-\widehat{g}_j\|_{L^2}+\|f_k-g_j\|_{L^2} \\
&\leq \|\psi-\widehat{f_k}\|_{L^2}+\|\varphi-\widehat{g}_j\|_{L^2}+\|f_k-f\|_{L^2}+\|f-g_j\|_{L^2}\to 0.
\end{aligned}$$

This implies $\psi=\varphi$ a.e.

Similarly, for $f\in L^2(\mathbb{R}^n)$, its Fourier inverse transform $f^\vee(x)$ is defined as the L^2 limit of $f_k^\vee(x)$ where $f_k\in \mathrm{Sch}(\mathbb{R}^n)$ such that $f_k\to f$ in L^2 sense.

It is now easy to see (1.11). For the inversion formula, let $f_k\in \mathrm{Sch}(\mathbb{R}^n)$ such that $f_k\to f$ in L^2, then by the Riesz theorem, we can select a subsequence f_{k_j} such that $f_{k_j}\to f$ a.e. Then

$$f(x)=\lim_{j\to\infty} f_{k_j}(x)=\lim_{j\to\infty}\mathscr{F}^{-1}[\widehat{f}_{k_j}](x)=\mathscr{F}^{-1}[\widehat{f}](x).$$

The last equality follows by the definition of the inverse Fourier transform on $L^2(\mathbb{R}^n)$ space and the fact that

$$\widehat{f}_{k_j}\to\widehat{f}\text{ in }L^2\quad(\text{since }\|\widehat{f}_{k_j}-\widehat{f}\|_{L^2}=\|f_{k_j}-f\|_{L^2}\to 0).$$

□

The argument used to show $\mathrm{Sch}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ in Theorem 1.2 is called the regularisation, or method of mollification. It is very useful in analysis. By this argument and the completeness of $L^p(\mathbb{R}^n)$, one also obtains the following.

Proposition 1.5. *Smooth functions with compact supports are dense in $L^p(\mathbb{R}^n)$, where $1\leq p<\infty$.*

We next use the method of mollification to prove the Weierstrass approximation theorem.

Theorem 1.3. *Let $f\in C([a,b])$. Then for any $\varepsilon>0$, there exists a polynomial P such that*

$$\sup_{x\in[a,b]}|f(x)-P(x)|\leq\varepsilon.$$

Proof. Let $[-L,L]$ be an interval contains $[a,b]$. Let g be a continuous function vanishes outside $[-L,L]$ and $g=f$ on $[a,b]$.

Consider the good kernels $K_\delta(x) = \delta^{-1/2}e^{-\pi x^2/\delta}$. Define $g_\delta(x) = (K_\delta * g)(x)$. Then, for all x,

$$\begin{aligned}
|g_\delta(x) - g(x)| &= \Big| \int_{\mathbb{R}} K_\delta(x-y)g(y)dy - \int_{\mathbb{R}} K_\delta(y)g(x)dy\Big| \\
&\leq \int_{\mathbb{R}} K_\delta(y)|g(x-y) - g(x)|dy \\
&\leq \int_{|y|\leq\eta} K_\delta(y)|g(x-y) - g(x)|dy + 2\sup|g| \int_{|y|>\eta} K_\delta(y)dy \\
&\leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2, \quad \text{provided } \eta, \delta > 0 \text{ are very small.} \qquad (1.12)
\end{aligned}$$

The first $\varepsilon/4$ is due to the uniform continuity of g and the second one is because K_δ are good kernels.

Let us fix $\delta = \delta_0$ such that the above estimate holds.

By the Taylor expansion, there is $N \in \mathbb{N}$ such that

$$\sup_{[-L,L]} \left|\delta_0^{-1/2}e^{-\pi x^2/\delta_0} - Q(x)\right| \leq \frac{\varepsilon}{4L\sup|g|},$$

where $Q(x) = \delta_0^{-1/2}\sum_{k=1}^{N} \frac{1}{k!}(-\pi x^2/\delta_0)^k$. Therefore, for all x,

$$\begin{aligned}
|g_{\delta_0}(x) - (Q*g)(x)| &\leq \int_{\mathbb{R}} |g(x-y)|\left|\delta_0^{-1/2}e^{-\pi y^2/\delta_0} - Q(y)\right|dy \\
&\leq 2L\sup|g| \sup_{[-L,L]} \left|\delta_0^{-1/2}e^{-\pi y^2/\delta_0} - Q(y)\right| \\
&\leq \varepsilon/2.
\end{aligned}$$

This together with (1.12) shows

$$|g(x) - (Q*g)(x)| \leq \varepsilon, \quad \forall\ x.$$

In particular

$$\sup_{x\in[a,b]} |f(x) - (Q*g)(x)| \leq \varepsilon.$$

It is direct to check that $(Q*g)(x)$ is a polynomial. We thus complete the proof. □

Proposition 1.6. *Let $f \in C(\Omega)$ with Ω being a compact subset of $\mathbb{R}^n$. Then there is a sequence functions $f_k \in C^\infty(\mathbb{R}^n)$ such that $\sup_{x\in\Omega}|f_k(x) - f(x)| \to 0$.*

Proof. Consider the good kernels $K_\delta(x) = \delta^{-n/2}e^{-\pi|x|^2/\delta}$. Extend f to be a continuous function $g \in C_0(\mathbb{R}^n)$ such that $g = 0$ outside a neighbourhood of Ω. Define $g_\delta = K_\delta * g$.

Then, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
|g(x) - g_\delta(x)| &\leq \int_{\mathbb{R}^n} K_\delta(y)|g(x-y) - g(x)|dy \\
&\leq \int_{|y|\leq\eta} K_\delta(y)|g(x-y) - g(x)|dy + 2\sup|g| \int_{|y|>\eta} K_\delta(y)dy \\
&\leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2,
\end{aligned}$$

provided $\eta, \delta > 0$ are very small. Hence, there is a sequence $\delta_k \to 0$ such that $f_k := g_{\delta_k}$ satisfies

$$\sup_{x\in\Omega} |f_k(x) - f(x)| \to 0.$$

Obviously $f_k \in C^\infty(\mathbb{R}^n)$, as $|g| \in L^1$ and

$$f_k(x) = \delta_k^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\pi|x-y|^2/\delta_k} g(y)dy.$$

$\square$