REAL ANALYSIS
LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures. The text is from two books for *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press.

[2] Elias M. Stein & Rami Shakarchi: Real Analysis, Princeton University Press.

Lecture #1

Part 1. Preliminaries

Some basic notions in Set Theory/Euclidean Topology are introduced.

1. Sets and their operations

The union, intersection, difference, and complement of sets are well-known operations in the set theory. The following proposition is straightforward.

Proposition 1.1 (De Morgan Law)**.** *Let A_λ be a family of subsets of X, $\lambda \in \Lambda$. Then*

$$\Big(\bigcup_{\lambda\in\Lambda} A_\lambda\Big)^c = \bigcap_{\lambda\in\Lambda} A_\lambda^c,$$

$$\Big(\bigcap_{\lambda\in\Lambda} A_\lambda\Big)^c = \bigcup_{\lambda\in\Lambda} A_\lambda^c.$$

The notation $A\Delta B$ stands for the symmetric difference between sets A and B, defined by

$$A\Delta B = (A - B) \cup (B - A),$$

which consists of elements that belong to only one of the two sets A or B.

Proposition 1.2. *$A\Delta B = A \cup B - A \cap B$; $A\Delta B = B\Delta A$.*

Let $\{A_k\}_{k\geq 1}$ be a countable collection of subsets of X. We say $A_1, A_2, \ldots$ increases to A (written as $A_n \nearrow A$), if $A_k \subset A_{k+1}$ for all k, and $A = \bigcup_{k=1}^\infty A_k$. Similarly, we say $A_1, A_2, \ldots$ decreases to A (written as $A_n \searrow A$), if $A_{k+1} \subset A_k$ for all k, and $A = \bigcap_{k=1}^\infty A_k$.

Given any countable collection of sets $\{A_k\}_{k\geq 1}$, we define

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k$$

and

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k.$$

We say $\{A_k\}_{k\geq 1}$ has a limit if $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$, and denote

$$\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n.$$

Proposition 1.3. *Let $\{A_n\}_{n\geq 1}$ be a countable collection of sets.*

(i) $x \in \limsup_{n\to\infty} A_n$ *if and only if, for any n, there is a $N = N(n) \geq n$ such that $x \in A_N$. Namely there are infinitely many A_n containing x.*

(ii) $x \in \liminf_{n\to\infty} A_n$ *if and only if there is a n_x such that, for any $N \geq n_x$, $x \in A_N$. Namely there are at most finitely many A_n such that $x \notin A_n$.*

(iii) $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.

(iv) *If $\{A_n\}$ is increasing or decreasing, then $\{A_n\}$ has a limit and*

$$\lim_{n\to\infty} A_n = \begin{cases} \bigcup_{n=1}^{\infty} A_n & \text{if } \{A_n\} \text{ is increasing,} \\ \bigcap_{n=1}^{\infty} A_n & \text{if } \{A_n\} \text{ is decreasing.} \end{cases}$$

Proof. This is a direct consequence of the definitions of $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$.

Let us show (iv).

Suppose $\{A_n\}$ is increasing. Then $\bigcup_{k\geq n} A_k = \bigcup_{k\geq m} A_k$ for any m and n. Hence $\limsup_{n\to\infty} A_n = \bigcup_{n\geq 1} A_n$. Clearly $\limsup_{n\to\infty} A_n = \bigcup_{n\geq 1} \bigcap_{k\geq n} A_k = \bigcup_{n\geq 1} A_n$. Therefore $\lim_{n\to\infty} = \bigcup_{n\geq 1} A_n$.

Suppose $\{A_n\}$ is decreasing. We have $\bigcap_{k\geq n} A_k = \bigcap_{k\geq m} A_k$ for any m and n, and so $\liminf_{n\to\infty} A_n = \bigcap_{n\geq 1} A_n$. On the other hand, $\limsup_{n\to\infty} A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k = \bigcap_{n\geq 1} A_n$. Hence $\lim_{n\to\infty} A_n = \bigcap_{n\geq 1} A_n$.

$\square$

Example 1.1. *For $n \geq 1$, let $A_n = \big\{m/n : \; m \in \mathbb{Z}\big\}$. Find* $\limsup_{n\to\infty} A_n$ *and* $\liminf_{n\to\infty} A_n$.

Solution. We have

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \big\{m/k : \ m \in \mathbb{Z}, \ k \geq n\big\} = \mathbb{Q}.$$

For the second inequality, let $x \in \mathbb{Q}$, thus $x = p/q$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$. Note that $x = np/(nq)$. Since $nq \geq n$, we see that

$$x \in \big\{m/k : \ m \in \mathbb{Z}, \ k \geq n\big\} \ \ \forall\, n \geq 1.$$

Hence $\mathbb{Q} \subseteq \limsup_{n\to\infty} A_n$. The opposite inclusion is obvious.

Next, let $x = p/q$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$, be such that $x \in \mathbb{Q} - \mathbb{Z}$. Without loss of generality, suppose p and q are relatively prime. Clearly $x \notin A_n$ when $n \neq kq$ for some $k \in \mathbb{Z}_+$. Hence

$$x \notin \bigcap_{k\geq n} A_k, \quad \text{for any fixed } n.$$

Hence $\mathbb{Q} - \mathbb{Z}$ and $\liminf_{n\to\infty} A_n$ are disjoint. On the other hand,

$$\mathbb{Z} \subseteq \liminf_{n\to\infty} A_n \subseteq \mathbb{Q}.$$

It then follows that

$$\liminf_{n\to\infty} A_n = \mathbb{Z}.$$

□

2. Cardinality of Sets

We say two sets A and B are equivalent (in the sense of cardinality), written as $A \sim B$, if there is a one-to-one [1] map between A and B.

Theorem 2.1. *Let $\{A_\lambda : \lambda \in \Lambda\}$ and $\{B_\lambda : \lambda \in \Lambda\}$ be two families of disjoint sets. If $A_\lambda \sim B_\lambda$ for all λ, then*

$$\bigcup_{\lambda\in\Lambda} A_\lambda \sim \bigcup_{\lambda\in\Lambda} B_\lambda.$$

We say A is a finite set if A is equivalent to $\{1, 2, \ldots, n\}$ for some n; otherwise A is an infinite set. We say A is a countable set if A is equivalent to $\mathbb{N} = \mathbb{Z}_+$.

Theorem 2.2. *The following statements hold*

[1] This means the map is both injective and surjective.

(i) *any infinite set contains a countable set;*

(ii) *any infinite subset of a countable set is countable;*

(iii) *the union of at most countably many countable sets is countable.*

Example 2.1. *The set of rational numbers is a countable set.*

Example 2.2. *Consider the set $\mathcal{I} = \{I_\lambda\}_{\lambda \in \Lambda}$, where I_λ are disjoint open intervals in $\mathbb{R}$. Then $\mathcal{I}$ is finite or countable.*

Theorem 2.3. *Let A be an infinite set, and B is a countable set. Then $A \sim A \cup B$.*

Proof. Let A_1 be a countable subset of A. By (iii) in Theorem 2.2, we have

$$A_1 \cup B \sim A_1.$$

It follows from Theorem 2.1 that

$$A = (A - A_1) \cup A_1 \sim (A - A_1) \cup (A_1 \cup B) = A \cup B.$$

□

Example 2.3. *The closed interval $[0, 1]$ is not countable.*

Proof. Suppose $[0, 1] = \{a_1, a_2, \cdots, a_n, \cdots\}$. Then we have a sequence of closed intervals, say I_k, such that

$$I_k \subseteq I_{k-1} \quad \text{and} \quad a_k \notin I_k, \quad \text{for all } k = 1, 2, \ldots,$$

where in particular we take $I_0 = [0, 1]$. Clearly $\cap_{k \geq 1} I_k \neq \emptyset$. Take $\xi \in \cap_{k \geq 1} I_k$. But $\xi \neq a_n$ for any n. This shows that $\cap_{k \geq 1} I_k \not\subset [0, 1]$, a contradiction. □

Definition 2.1. *We say A has the cardinality of the continuum if $A \sim [0, 1]$.*

Proposition 2.1. *The set of all real numbers $\mathbb{R}$ has the cardinality of the continuum.*

Proof. It is direct to see $\mathbb{R} \sim (0, 1) \sim [0, 1]$. The last step is due to Theorem 2.3. □

We next consider a class of sets whose elements are arrays of infinite length. Given $n \in \mathbb{Z}_+$, let $\mathcal{A}_n$ be the set consisting of elements $\mathbf{a} = \{a_k\}_{k \geq 1}$, where $a_k \in \{0, 1, \cdots, n-1\}$.

Proposition 2.2. *Let $n \geq 2$. The set $\mathcal{A}_n$ has the cardinality of the continuum.*

Proof. We shall show that $\mathcal{A}_n \sim (0,1]$. For any $x \in (0,1]$, there is a unique $k_1 \in [1,n]$ such that

$$\frac{k_1 - 1}{n} < x \le \frac{k_1}{n}.$$

Let $x_1 = x - \frac{k_1-1}{n}$. Then $x_1 \in (0, \frac{1}{n})$. There is a unique $k_2 \in [1,n]$ such that

$$\frac{k_2 - 1}{n^2} < x_1 \le \frac{k_2}{n^2}.$$

Next let $x_2 = x_1 - \frac{k_2-1}{n^2}$. We see that $x_2 \in (0, \frac{1}{n^2})$ and so there is a unique $k_3 \in [1,n]$ such that

$$\frac{k_3 - 1}{n^3} < x_2 \le \frac{k_3}{n^3}.$$

Repeat the above procedure. We obtain a sequence of $k_i \in [1,n]$ (here $i = 1, 2, \ldots$) such that if

$$x_i = x_{i-1} - \frac{k_i - 1}{n^i},$$

where in particular $x_0 = x$, then

$$\frac{k_{i+1} - 1}{n^{i+1}} < x_i \le \frac{k_{i+1}}{n^{i+1}}.$$

It thus follows that

$$\sum_{i=1}^{m} \frac{k_i - 1}{n^i} < x \le \sum_{i=1}^{m-1} \frac{k_i - 1}{n^i} + \frac{k_m}{n^m}, \text{ for any } m \ge 1.$$

Let $a_i = k_i - 1$. Sending $m \to \infty$, we infer that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{n^i}. \tag{2.1}$$

This yields a one-to-one mapping f between $(0,1]$ and $\mathcal{A}_n$, namely

$$f(x) = \{a_1, a_2, \cdots, a_i, \cdots\},$$

where a_i's are such that (2.1) holds. □

Given a set X, we denote by 2^X the set of all subsets of X.

Proposition 2.3. *Set $2^{\mathbb{N}}$ has the cardinality of the continuum.*

Proof. Let $A \in 2^{\mathbb{N}}$. Given any $n \ge 1$, we take

$$a_n = \begin{cases} 1, & n \in A, \\ 0, & n \in \mathbb{N} - A. \end{cases}$$

Then $f(A) = \{a_1, a_2, \cdots, a_n, \cdots\}$ is an one-to-one mapping between $2^{\mathbb{N}}$ and $\mathcal{A}_1$, thus completing the proof by Proposition 2.1.

$\square$

Theorem 2.4. *Let $\{X_i\}_{i\geq 1}$ be a collection of sets with $X_i \sim [0,1]$ for all $i \in \mathbb{N}$. Then $X = \prod_{i=1}^{\infty} X_i$ has continuum.*

Proof. By Proposition 2.2,

$$X \sim \prod_{i=1}^{\infty} \mathcal{A}_1.$$

Given $x = (x_1, x_2, \cdots, x_n, \cdots) \in \prod_{i=1}^{\infty} \mathcal{A}_1$, we write $x_i = (x_i^1, x_i^2, \cdots, x_i^n, \cdots)$. Let us define $y \in \mathcal{A}_1$ by setting

$$y = (x_1^1, x_2^1, x_1^2, x_3^1, x_2^2, x_1^3, \cdots, x_n^1, x_{n-1}^2, \cdots, x_1^n, \cdots).$$

This yields a mapping $f : \prod_{i=1}^{\infty} \mathcal{A}_1 \to \mathcal{A}_1$, by $y = f(x)$. It is not hard to see that f is one-to-one. Hence

$$X \sim \prod_{i=1}^{\infty} \mathcal{A}_1 \sim \mathcal{A}_1 \sim [0,1].$$

The last relation is due to Proposition 2.2.

$\square$

As a corollary, $\mathbb{R}^n$ has the cardinality of the continuum.

The cardinalities of sets can be compared. Theorem below is a tool for this.

Theorem 2.5. *Let A_0, A_1, A_2 be sets such that*

$$A_2 \subset A_1 \subset A_0.$$

If $A_0 \sim A_2$, then $A_0 \sim A_1$.

Proof. Let $h : A_0 \to A_2$ be a one-to-one mapping. Define, for $n = 1, 2, 3, \ldots$,

$$A_{n+2} = h(A_n) = \begin{cases} h^k(A_1), \text{ if } n = 2k-1, \\ h^k(A_2), \text{ if } n = 2k. \end{cases}$$

We thus obtain a sequence of sets $A_3, A_4, A_5, \cdots$, which are subsets of A_2, and

$$A_n \sim A_{n+2}, \ n = 1, 2, 3, \ldots.$$

Since $A_1 \subset A_0$, we have $A_3 = h(A_1) \subset h(A_0) = A_2$. In general, one can check that

$$A_{i+1} \subset A_i, \quad \text{for all } i = 0, 1, 2, \ldots.$$

Namely $\{A_n\}$ is decreasing. We then take

$$A_{-1} = \bigcap_{n=0}^{\infty} A_n,$$

thus

$$A_0 = A_2 \bigcup (A_0 - A_2) = A_4 \bigcup (A_2 - A_4) \bigcup (A_0 - A_2) = \cdots = A_{-1} \bigcup_{n=0}^{\infty} (A_{2n} - A_{2n+2}), \tag{2.2}$$

and similarly

$$A_1 = A_{-1} \bigcup_{n=0}^{\infty} (A_{2n+1} - A_{2n+3}). \tag{2.3}$$

Since $\{A_n\}$ is decreasing, we obtain $A_{2n+2} - A_{2n+3} = h(A_{2n} - A_{2n+1})$, namely

$$A_{2n+2} - A_{2n+3} \sim A_{2n} - A_{2n+1}, \quad n = 0, 1, 2, \ldots.$$

It then follows by Theorem 2.1 that

$$\begin{aligned} A_{2n+1} - A_{2n+3} &= (A_{2n+1} - A_{2n+2}) \cup (A_{2n+2} - A_{2n+3}) \\ &\sim (A_{2n+1} - A_{2n+2}) \cup (A_{2n} - A_{2n+1}) \\ &= A_{2n} - A_{2n+2}. \end{aligned}$$

Then, using Theorem 2.1, we conclude from (2.2) and (2.3) that

$$A_0 = A_{-1} \bigcup_{n=0}^{\infty} (A_{2n} - A_{2n+2}) \sim A_{-1} \bigcup_{n=0}^{\infty} (A_{2n+1} - A_{2n+3}) = A_1.$$

□

Given two sets A and B, we say

- $\mathrm{Card}(A) = \mathrm{Card}(B)$ if $A \sim B$;
- $\mathrm{Card}(A) \leq \mathrm{Card}(B)$ if A is equivalent to a subset of B;
- $\mathrm{Card}(A) < \mathrm{Card}(B)$ if $\mathrm{Card}(A) \leq \mathrm{Card}(B)$ and $\mathrm{Card}(A) \neq \mathrm{Card}(B)$.

This yields an order for sets.

Theorem 2.6. *Let A and B be sets. Then*

(i) *$Card(A) \leq Card(A)$;*

(ii) *if* $Card(A) \le Card(B)$ *and* $Card(B) \le Card(C)$, *then* $Card(A) \le Card(C)$;
(iii) *if* $Card(A) \le Card(B)$ *and* $Card(B) \le Card(A)$, *then* $Card(A) = Card(B)$.

Proof. We only prove (iii). By definition, let $B_1 \subset B$ and $A_1 \subset A$ be such that

$$A \sim B_1 \quad \text{and} \quad B \sim A_1 \tag{2.4}$$

Denote by h a one-to-one map from B to A_1. Using $B_1 \subset B$,

$$B_1 \sim A_2 := h(B_1) \subset A_1.$$

Hence $A_2 \subset A_1 \subset A$ and $A_2 \sim B_1 \sim A$. By Theorem 2.5, $A_1 \sim A$. By (2.4), $B \sim A$.

□

By the above comparison principle, we have the following result.

Example 2.4. *Let* $C([0,1])$ *be the set of all continuous functions on* $[0,1]$. *Then* $C([0,1])$ *has the cardinality of the continuum.*

Proof. Let $f_\lambda : [0,1] \to \mathbb{R}$ be the function such that $f_\lambda(x) = \lambda$. Clearly

$$[0,1] \sim \{f_\lambda\}_{0\le\lambda\le1} \subset C([0,1]) \Longrightarrow \mathrm{Card}([0,1]) \le \mathrm{Card}(C([0,1])). \tag{2.5}$$

On the other hand, given a $f \in C([0,1])$, let

$$X = X(f) = (f(r_1), f(r_2), \cdots, f(r_n), \cdots)$$

where $\{r_k\}_{k\ge1}$ is the set of all rational numbers in $[0,1]$. Thus we define a mapping

$$X : C([0,1]) \to \mathbb{R}^\infty := \prod_{i=1}^{\infty} R_i, \text{ with } R_i = \mathbb{R} \text{ for all } i.$$

By the continuity, if $f, g \in C([0,1])$ and $f \ne g$, then $X(f) \ne X(g)$. Hence X is injective and so

$$\mathrm{Card}(C([0,1])) \le \mathrm{Card}(\mathbb{R}^\infty) = \mathrm{Card}([0,1]). \tag{2.6}$$

The last equality follows from Theorem 2.4.

In view of Theorem 2.6, we deduce that $C([0,1]) \sim [0,1]$ by (2.5) and (2.6).

□

Next we show that the cardinality can be "arbitrarily large".

Theorem 2.7. *For any set* X, *there holds* $Card(X) < Card(2^X)$.

Proof. Obviously $X \sim \big\{\{x\}\big\}_{x\in X} \subset 2^X$. Hence $\mathrm{Card}(X) \leq \mathrm{Card}(2^X)$. For completing the proof, we suppose by contradiction there is a one-to-one map $f : X \to 2^X$. Let

$$X^* = \{x \in X : \; x \notin f(x)\}.$$

Since X^* is a subset of X, there is a x^* such that $f(x^*) = X^*$.

If $x^* \in f(x^*) = X^*$, then by the definition of X^* we have $x^* \notin X^*$; if $x^* \notin f(x^*) = X^*$, then by the definition again we obtain $x^* \in X^*$; arriving contradictions for both cases. $\square$