《微分几何》第一次课堂练习

1. (15%)

证 (1)(5′) 曲线是弧长参数, $\mathbf{T}=(x'(s),y'(s))$, $|\mathbf{T}|=1$. 若设$\mathbf{T}=(\cos\theta,\sin\theta)=(x'(s),y'(s))$, 则$\mathbf{N}_r=(\cos(\theta+\frac{\pi}{2}),\sin(\theta+\frac{\pi}{2}))=(-\sin\theta,\cos\theta)=(-y'(s),x'(s))$. 3′

由Frenet公式, $\mathbf{r}''(s)=\mathbf{T}'(s)=k_r(s)\mathbf{N}_r(s)$, $\mathbf{r}''(s)=k_r(s)(-y'(s),x'(s))$. 2′

(2)(5′) $k_r(s)=\mathbf{r}''(s)\cdot\mathbf{N}_r(s)=(x'',y'')\cdot(-y',x')=-x''y'+y''x'$.

(3)(5′) 取一般参数$\mathbf{r}=(x(t),y(t))$.

$$\mathbf{T}=\dot{\mathbf{r}}=r'\frac{dt}{ds},\ \left|\frac{dt}{ds}\right|=|\mathbf{r}'|^{-1}=\frac{1}{\sqrt{(x')^2+(y')^2}}.\ 2'$$

$$\dot{x}=x'\frac{dt}{ds},\ \ddot{x}=x''\Big(\frac{dt}{ds}\Big)^2+x'\frac{d^2t}{ds^2}.\quad \dot{y}=x'\frac{dt}{ds},\ \ddot{y}=y''\Big(\frac{dt}{ds}\Big)^2+y'\frac{d^2t}{ds^2}.$$

$$\begin{aligned}k_r(t)&=-\ddot{x}\dot{y}+\ddot{y}\dot{x}=-\Big(x''\Big(\frac{dt}{ds}\Big)^2+x'\frac{d^2t}{ds^2}\Big)y'\frac{dt}{ds}+\Big(y''\Big(\frac{dt}{ds}\Big)^2+y'\frac{d^2t}{ds^2}\Big)x'\frac{dt}{ds}\\&=-y'x''\Big(\frac{dt}{ds}\Big)^3+y''x'\Big(\frac{dt}{ds}\Big)^3=\frac{x'y''-y'x''}{[(x')^2+(y')^2]^{3/2}}.\ 3'\end{aligned}$$

2. (15%) **证**(1).设曲线方程为$x(s)$,其中s为弧长参数，P_0的弧长参数为$s=0$,P的弧长参数为s.则$d=|x(s)-x(0)|$,$\rho=(x(s)-x(0))\cdot x\dot{(0)}$2′ 则

$$\lim_{s\to0}\frac{\rho}{d}=\lim_{s\to0}\frac{(x(s)-x(0))^2}{(s-s_0)|x(s)-x(0)|}=|x\dot{(0)}|=1.2'$$

则$\lim\limits_{s\to0}\frac{h}{d}=\lim\limits_{s\to0}\frac{\sqrt{d^2-\rho^2}}{d}=\lim\limits_{s\to0}\sqrt{1-(\frac{\rho}{d})^2}=0.$2'

证(2).在P_0点进行泰勒展开，有

$$x(s)=x(0)+(s-\frac{k^2s^3}{6})T(0)+(\frac{ks^2}{2}+\frac{\dot{k}s^3}{6})N(0)+\frac{k\tau s^3}{6}B(0)+o(s^3))).3'$$

所以$\rho=s-\frac{k^2s^3}{6}+o(s^3)$, 又因为$\lim\limits_{s\to0}\frac{\rho}{s}=\lim\limits_{s\to0}\frac{(x(s)-x(0))\cdot x\dot{(0)}}{s}=\lim\limits_{s\to0}|x\dot{(0)}|^2=1$,以及

$$h=|x(s)-x(0)-\rho x\dot{(0)}|=|(\frac{ks^2}{2}+\frac{\dot{k}s^3}{6})N(0)|+O(s^3).3'$$

所以

$$\begin{aligned}\lim_{s\to0}\frac{2h}{\rho^2}&=\lim_{\rho\to0}\frac{2h}{\rho^2}\\&=\lim_{\rho\to0}\frac{2h}{s^2}\cdot\frac{s^2}{\rho^2}\\&=\lim_{\rho\to0}\frac{2|(\frac{ks^2}{2}+\frac{\dot{k}s^3}{6})N(0)|+O(s^3)}{s^2}\\&=k3'\end{aligned}$$

3. (15%)

解 (1)

$$\frac{d\tilde{\mathbf{x}}(s)}{ds} = \mathbf{B}(s), \ \left|\frac{d\tilde{\mathbf{x}}(s)}{ds}\right| = |\mathbf{B}(s)| = 1.$$

故s是$\tilde{\mathbf{x}}(s)$的弧长参数. $2'$ 从而$\tilde{\mathbf{T}}(s) = \mathbf{B}(s)$. $3'$

由$\tilde{\mathbf{T}}'(s) = \dot{\mathbf{B}}(s) = -\tau(s)\mathbf{N}(s) = \tilde{k}(s)\tilde{\mathbf{N}}(s)$, 得$\tilde{k}(s) = |\tau(s)| = \tau(s)$(因$\tau(s) > 0$), 这样, $\tilde{\mathbf{N}}(s) = -\mathbf{N}(s)$. $3'$

$\tilde{\mathbf{B}}(s) = \tilde{\mathbf{T}}(s) \times \tilde{\mathbf{N}}(s) = \mathbf{B}(s) \times (-\mathbf{N}(s)) = -\mathbf{B} \times \mathbf{N} = \mathbf{N} \times \mathbf{B} = \mathbf{T}(s)$. $3'$

所以$\tilde{C}$的Frenet标架为$\tilde{\mathbf{T}} = \mathbf{B}(s), \tilde{\mathbf{N}} = -\mathbf{N}(s), \tilde{\mathbf{B}} = \mathbf{T}(s)$.

(2) 由上计算, 知$\tilde{k}(s) = \tau(s)$. $2'$ $\tilde{\mathbf{B}}'(s) = \mathbf{T}'(s) = k(s)\mathbf{N}(s) = -\tilde{\tau}\tilde{\mathbf{N}}(s) = \tilde{\tau}\mathbf{N}$, 于是$\tilde{\tau}(s) = k(s)$. $2'$

4. (20%)

证 由已知，可计算M的第一基本形式为5’

$$I_1 = (v^2 + a^2)(du)^2 + (dv)^2$$

而M的单位法向量为5’

$$n = (-\frac{a}{\sqrt{a^2+v^2}}sinu, -\frac{a}{\sqrt{a^2+v^2}}cosu, -\frac{v}{\sqrt{a^2+v^2}})$$

设M上的每一点借助该点的法向量映到球面的映射是5’

$$\phi(u,v) = (-\frac{a}{\sqrt{a^2+v^2}}sinu, -\frac{a}{\sqrt{a^2+v^2}}cosu, -\frac{v}{\sqrt{a^2+v^2}})$$

求得其第一基本形式为5’

$$I_2 = \frac{a^2}{(a^2+v^2)^2}[(a^2+v^2)(du)^2 + (dv)^2]$$

所以$I_1 == \frac{a^2}{(a^2+v^2)^2} I_2$, 此映射为共性映射。

5. (15%)

证 设曲线$C_1 : u = u(s), v = v(s)$满足$\varphi(u,v) =$常数, $C_2 : u = u(\bar{s}), v = v(\bar{s})$满足$\psi(u,v) =$常数.

$$\varphi_u\frac{du}{ds} + \varphi_v\frac{dv}{ds} = 0 \Rightarrow \frac{du}{dv} = -\frac{\varphi_v}{\varphi_u} \ 3' \Rightarrow \frac{d\mathbf{r}}{ds} = \mathbf{r}_u\frac{du}{ds} + \mathbf{r}_v\frac{dv}{ds} = \frac{dv}{ds}\Big(-\mathbf{r}_u\frac{\varphi_v}{\varphi_u} + \mathbf{r}_v\Big), \ 2'$$

$$\psi_u\frac{du}{d\bar{s}} + \psi_v\frac{dv}{d\bar{s}} = 0 \Rightarrow \frac{du}{dv} = -\frac{\psi_v}{\psi_u} \ 3' \Rightarrow \frac{d\mathbf{r}}{d\bar{s}} = \mathbf{r}_u\frac{du}{d\bar{s}} + \mathbf{r}_v\frac{dv}{d\bar{s}} = \frac{dv}{d\bar{s}}\Big(-\mathbf{r}_u\frac{\psi_v}{\psi_u} + \mathbf{r}_v\Big). \ 2'$$

两方向正交 $\iff$ $0 = \frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{r}}{d\bar{s}}$ $3'$ $= \frac{dv}{ds}\frac{dv}{d\bar{s}}(E\frac{\varphi_v\psi_v}{\varphi_u\psi_u} - F(\frac{\varphi_v}{\varphi_u} + \frac{\psi_v}{\psi_u}) + G)$. 即$E\varphi_v\psi_v - F(\varphi_v\psi_u + \psi_v\varphi_u) + G\varphi_u\psi_u = 0$. $2'$

6. (20%)

解 (1)(10′) 直纹面为$\mathbf{X}(s,v) = \mathbf{x}(s) + v\mathbf{l}(s)$. $2'$

直纹面为可展曲面 $\Longleftrightarrow$ $(\mathbf{x}'(s), \mathbf{l}(s), \mathbf{l}'(s)) = 0$, $2'$ 即$(\mathbf{T}, \mathbf{T} + \lambda\mathbf{B}, k\mathbf{N} + \lambda'\mathbf{B} - \lambda\tau\mathbf{N}) = 0$. $(\mathbf{T}, \lambda\mathbf{B}, (k - \lambda\tau)\mathbf{N}) = 0 \Rightarrow \lambda(k - \lambda\tau) = 0 \Rightarrow \lambda = 0$或$\lambda = \frac{k}{\tau}$. $2'$

$\lambda \equiv 0$时, 直纹面$\mathbf{X} = \mathbf{x}(s) + v\mathbf{T}(s)$, 切线面; $2'$ $\lambda = \frac{k}{\tau}$时, 直纹面$\mathbf{X} = \mathbf{x}(s) + v(\mathbf{T}(s) + \frac{k}{\tau}\mathbf{B}(s))$. $2'$

(2)**($10'$)** ($\Rightarrow$)

如果直纹面是柱面. 在$\lambda \equiv 0$时, $\mathbf{X} = \mathbf{x}(s) + v\mathbf{T}(s)$. $\mathbf{T} =$常向量$\Rightarrow k \equiv 0$, 不合题意, 舍去. $2'$

$\lambda = \frac{k}{\tau}$时, $\mathbf{X} = \mathbf{x}(s) + v(\mathbf{T} + \frac{k}{\tau}\mathbf{B})$, $\mathbf{T} + \frac{k}{\tau}\mathbf{B} =$常向量. $2'$

于是$0 = \dot{\mathbf{T}} + (\frac{k}{\tau})'\mathbf{B} + (\frac{k}{\tau})\dot{\mathbf{B}} = k\mathbf{N} + (\frac{k}{\tau})'\mathbf{B} + \frac{k}{\tau}(-\tau\mathbf{N}) = (\frac{k}{\tau})'\mathbf{B} \Rightarrow (\frac{k}{\tau})' = 0$, $\frac{k}{\tau} =$常数. $2'$

($\Leftarrow$)

$(\frac{k}{\tau}) =$常数, 则$(\mathbf{T} + \frac{k}{\tau}\mathbf{B})' = 0$. $2'$ $\mathbf{T} + \frac{k}{\tau}\mathbf{B} =$常向量. 故$\mathbf{X} = \mathbf{x}(s) + v(\mathbf{T} + \frac{k}{\tau}\mathbf{B})$是可展的柱面. $2'$