ETH Zurich HS 2019
D-MATH, D-PHYS
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Probability Theory

Exercise Sheet 4

Exercise 4.1 Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $E[X_i^+] = \infty$ and $E[X_i^-] < \infty$. Define $S_n = X_1 + \ldots + X_n$. Show that

$$\frac{S_n}{n} \overset{n\to\infty}{\longrightarrow} \infty \qquad \text{a.s.}$$

Exercise 4.2 Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. exponentially distributed random variables with parameter 1, and set for $n \geq 1$

$$M_n = \max_{1\leq i\leq n} X_i.$$

Find a sequence of real numbers a_n, $n \geq 1$, such that $M_n - a_n$ converges in distribution and compute the distribution function of the limiting distribution.

Exercise 4.3

(a) Let f be a (not necessarily Borel-measurable) function from $\mathbb{R}$ to $\mathbb{R}$. Show that the set of discontinuities of f, defined as

$$U_f := \{x \in \mathbb{R} \mid f \text{ is discontinuous in } x\},$$

is Borel-measurable.

(b) Assume that $X_n \to X$ in distribution. Let f be measurable and bounded, such that $P[X \in U_f] = 0$. Use (2.2.13) – (2.2.14) from the lecture notes to show that we have

$$E[f(X_n)] \underset{n\to\infty}{\to} E[f(X)].$$

(c) Let f be measurable and bounded on $[0,1]$, with U_f of Lebesgue measure 0. Show that the corresponding Riemann sums converge to the integral of f, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \underset{n\to\infty}{\to} \int_0^1 f(x)\mathrm{d}x.$$

Submission: until 14:15, Oct 22., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint

Solution 4.1 We define

$$Y_i^M = 1_{\{X_i^+ \le M\}} X_i^+ - X_i^-; \qquad M > 0.$$

We know that $(Y_i^M)_{i\in\mathbb{N}}$ are i.i.d. and $E[|Y_i^M|] < \infty$. Strong law of large numbers implies that

$$\frac{1}{n}\sum_{i=1}^n Y_i^M \overset{P-a.s.}{\longrightarrow} E\left[X_i^+ 1_{\{X_i^+ \le M\}}\right] - E[X_i^-].$$

We take $K > 0$. By monotone convergence theorem, there eixsts $M_0 > 0$, s.t.,

$$E\left[X_i^+ 1_{\{X_i^+ \le M\}}\right] - E[X_i^-] \ge K, \qquad \forall M > M_0.$$

For $P-$a.e. ω, $\exists\, n_0(\omega)$ s.t. $\forall n \ge n_0$, :

$$\frac{1}{n}\sum_{i=1}^n Y_i^M(\omega) \ge \frac{K}{2},$$

and with the observation that

$$Y_i^M \underset{M\to\infty}{\nearrow} X_i,$$

it follows that

$$\frac{1}{n}\sum_{i=1}^n X_i(\omega) \ge \frac{K}{2}, \qquad \forall n \ge n_0(\omega),$$

for $P-$a.e. ω.

Solution 4.2 Take $y \in \mathbb{R}$. Then we have that:

$$\begin{aligned} P\left[M_n - a_n \le y\right] &= P\left[M_n \le y + a_n\right] = P\left[\bigcap_{i=1}^n \{X_i \le y + a_n\}\right] \\ &= \prod_{i=1}^n \underbrace{P\left[X_i \le y + a_n\right]}_{=1-e^{-y-a_n}} = \left(1 - e^{-y-a_n}\right)^n = \left(1 - \frac{e^{-y+(\log n - a_n)}}{n}\right)^n, \end{aligned}$$

which converges whenever $\log n - a_n$ converges to some value $a \in \mathbb{R}$. In such cases, letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} P\left[M_n - a_n \le y\right] = \exp(-e^{-(y-a)}) = \int_{-\infty}^{y-a} e^{-x}\exp(-e^{-x})\,\mathrm{d}x.$$

This is known as the Gumbel distribution.

Solution 4.3

(a) Let $f : \mathbb{R} \to \mathbb{R}$ and

$$V_{\epsilon,\delta} := \left\{x \in \mathbb{R} \middle| \exists y, z \in (x-\delta, x+\delta) \text{ s.t. } |f(y) - f(z)| \ge \epsilon\right\}.$$

(i) Claim: $V_{\epsilon,\delta}$ is open.
Let $x \in V_{\epsilon,\delta}$. Then there are $y, z \in (x-\delta, x+\delta)$ such that $|f(y) - f(z)| \geq \epsilon$. We set $r := \delta - \max\{|y-x|, |z-x|\} > 0$.
$\Rightarrow \forall\, \tilde{x} \in (x-r, x+r)$ it holds that $|y-\tilde{x}| \leq |y-x| + |x-\tilde{x}| < |y-x| + r \leq \delta$, and similarly for z. From this it follows that $y, z \in (\tilde{x}-\delta, \tilde{x}+\delta)$ and $|f(y)-f(z)| \geq \epsilon$, which gives $\tilde{x} \in V_{\epsilon,\delta}$. So $(x-r, x+r) \subset V_{\epsilon,\delta}$, and the claim follows.

(ii) Claim: $U_f = \bigcup_n \bigcap_m V_{\frac{1}{n},\frac{1}{m}}$.
"$\subset$" Let $x \in U_f$. Then there is an $n \in \mathbb{N}$, such that

$$\forall\, m \in \mathbb{N}\ \exists\, y \in \left(x - \frac{1}{m}, x + \frac{1}{m}\right) \text{ s.t. } |f(x) - f(y)| \geq \frac{1}{n}.$$

"$\supset$" We assume that for some $n \in \mathbb{N}$, $x \in V_{\frac{1}{n},\frac{1}{m}}$, $\forall\, m$. Then there are $y, z \in (x - \frac{1}{m}, x + \frac{1}{m})$ so that $|f(y) - f(z)| \geq \frac{1}{n}$.
From this it follows that either $|f(y) - f(x)| \geq \frac{1}{2n}$ or $|f(z) - f(x)| \geq \frac{1}{2n}$ must hold. In other words $\exists\, n \in \mathbb{N}$, such that

$$\forall\, m \in \mathbb{N}\ \exists\, y \in \left(x - \frac{1}{m}, x + \frac{1}{m}\right) : |f(y) - f(x)| \geq \frac{1}{2n},$$

which implies that f is discontinuous in x.

Since the $V_{\frac{1}{n},\frac{1}{m}}$ are open, they are Borel measurable. And since any $\sigma-$algebra is closed under countable unions and intersections, $U_f = \bigcup_n \bigcap_m V_{\frac{1}{n},\frac{1}{m}}$ must also be Borel measurable.

(b) By (2.2.13) – (2.2.14) of the lecture notes, there exist $Y_n \overset{d}{=} X_n$, and $Y \overset{d}{=} X$, such that $Y_n \to Y$, P'-almost surely on a probability space $(\Omega', \mathcal{F}', P')$. Of course, we also have $f(Y_n) \overset{d}{=} f(X_n)$, and $f(Y) \overset{d}{=} f(X)$, so that we have $E[f(X_n)] = E'[f(Y_n)]$, and $E[f(X)] = E'[f(Y)]$, where we denote by E' the expectation w.r.t. P'. Thus, it suffices to show that

$$E'[f(Y_n)] \underset{n\to\infty}{\to} E'[f(Y)]. \tag{1}$$

Now, since $Y_n \to Y$, P'-almost surely, we have a set N, with $P'(N) = 0$, such that

$$\left\{\omega' \in \Omega' \mid Y_n(\omega') \underset{n\to\infty}{\to} Y(\omega')\right\} \cup N = \Omega'. \tag{2}$$

On the other hand, we have

$$\begin{aligned}\{\omega' \mid Y_n(\omega') \to Y(\omega')\} &\subseteq \{\omega' \mid f \text{ cont. in } Y(\omega'), Y_n(\omega') \to Y(\omega')\} \cup \{\omega \mid f \text{ discont. in } Y(\omega)\} \\ &\subseteq \{\omega' \mid f(Y_n(\omega')) \to f(Y(\omega'))\} \cup \{\omega' \mid Y(\omega') \in U_f\}.\end{aligned}$$

Consequently, it follows from equation (2) that we have

$$\{\omega' \mid f(Y_n(\omega')) \to f(Y(\omega'))\} \cup \{\omega' \mid Y(\omega') \in U_f\} \cup N = \Omega'.$$

But, by assumption $P'(Y \in U_f) = P(X \in U_f) = 0$ (recall that Y and X have the same distribution). Therefore, we get $f(Y_n) \to f(Y)$, P'-almost surely. Finally f is a bounded function, so by the Dominated Convergence Theorem equation (1) holds.

(c) Let λ be the Lebesgue measure on $[0,1]$, and, for all $a \in [0,1]$, let δ_a denote the Dirac delta measure on $[0,1]$. Let X_n, $n \geq 1$, be random variables with distribution $\frac{1}{n}\sum_{k=1}^{n}\delta_{\frac{k}{n}}$. Note that

$$E[f(X_n)] = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Let X be a uniform random variable on $[0,1]$, hence it has distribution λ, and we note that

$$E[f(X)] = \int_0^1 f(x)\lambda(dx).$$

Thus, it suffices to show that we have

$$E[f(X_n)] \underset{n\to\infty}{\to} E[f(X)]. \tag{3}$$

Since by assumption $P[X \in U_f] = \lambda(U_f) = 0$, part **b)** implies that equation (3) is a consequence of the following:

$$X_n \xrightarrow{d} X. \tag{4}$$

To show equation (4), note that for all $n \in \mathbb{N}$,

$$P\left[X_n \leq a\right] = \begin{cases} 0, & a < 0, \\ \frac{[na]}{n}, & 0 \leq a \leq 1, \\ 1, & 1 < a. \end{cases}$$

Since we have $na - 1 < [na] \leq na$ (i.e. $[na]$ denotes the integer part of na), we get $\frac{[na]}{n} \underset{n\to\infty}{\to} a$, for all $0 \leq a \leq 1$. Thus, we obtain, for $0 \leq a \leq 1$,

$$P\left[X_n \leq a\right] \underset{n\to\infty}{\to} \lambda([0,a]) = P\left[X \leq a\right],$$

which implies equation (4), by definition. (Cases for $a < 0$ and $a > 1$ are trivially verified.)