5. THE TODD-COXETER ALGORITHM

§5.1. Presentations

Presenting a group by generators and relations is by far the most compact way of describing a group and it is often the way it arises in applications. So we need some way of unravelling the structure of the group from its presentation. The Todd-Coxeter Algorithm aims to do just that.

If Γ is a set of symbols, a **group word** on Γ is a string of elements of Γ and their inverses. The inverse of a symbol X is just the formal expression X^{-1}, treated as a single symbol. For convenience in our calculations we will use lower case letters to denote inverses. So, for example, we use the letter a to represent A^{-1}.

Example 1: The group word $AAbABaaa$ represents $A^2B^{-1}ABA^{-3}$. If $A^4 = 1$, the group words $AAAAA$, A and aaa all represent the same element.

In the following description of the algorithm we sometimes use a capital letter, such as X, to represent either a generator or the inverse of a generator. In that case the corresponding lower case letter represents its inverse. So if X is actually the inverse of a generator, x would be the corresponding generator itself.

A **presentation** of a group is a description $\langle\Gamma \mid R\rangle$ where Γ is a set of **generators** and R a set of group words, called **relators**. Every element of the group is a group word on Γ with $\alpha = \beta$ if the equality is a consequence of the relations in R. (In chapter 9 we will define $\langle\Gamma \mid R\rangle$ more precisely as a quotient group of a free group.) Sometimes we write a relator $U^{-1}V$ or UV^{-1} as the **relation** $U = V$.

A **free group** is one with no relations, that is, G is free if $G \cong \langle\Gamma \mid \rangle$ for some Γ. A free group on a single generator is the infinite cyclic group.

A group G is **finitely generated** if it has a presentation $\langle\Gamma \mid R\rangle$ where Γ is finite and it is **finitely presented** if it has a presentation $\langle\Gamma \mid R\rangle$ where both Γ and R are finite. In this chapter we only consider finitely presented groups.

Example 2: $\langle A, B \mid A^4, B^2, BA = A^{-1}B\rangle$ is a finitely presented group.
We will write this as $\langle A, B \mid A^4, B^2, BA = aB\rangle$.
We can also write this as $\langle A, B \mid A^4, B^2, ABAB\rangle$ because we can deduce the relation $(AB)^2 = 1$ from $A^4 = 1$, $B^2 = 1$ and $BA = A^{-1}B$ and conversely we can deduce $BA = A^{-1}B$ from the relations $A^4 = 1$, $B^2 = 1$ and $(AB)^2 = 1$.

An equivalent presentation of this group is $\langle A, B \mid A^4, B^2, B^{-1}AB = A^{-1}\rangle$ which we shall write as $\langle A, B \mid A^4, B^2, bAB = a\rangle$.

The elements of this group are products of powers of A's and B's. Since $BA = A^{-1}B$ we can bring all the A's to the front and so every element can be put in the form A^rB^s. Since $A^4 = 1$ and $B^2 = 1$ we can assume that $r = 0, 1, 2$ or 3 and $s = 0$ or 1. The group thus has 8 elements: $1, A, A^2, A^3, B, AB, A^2B, A^3B$. It is the dihedral group of order 8.

You may feel that you were able to predict that this group has order 8 from the relators A^4 and B^2 (since $4 \times 2 = 8$). But it is not always so simple. For example the following group may appear to have order 4 but in fact it is infinite.

Example 3: $\langle \mathbf{A, B} \mid \mathbf{A^2, B^2} \rangle$ is infinite.
The elements are strings of alternating A's and B's (any successive pair of A's or of B's can be removed) so the distinct elements are:

$$1, A, AB, ABA, ABAB, ..., B, BA, BAB, BABA, ...$$

For example the product of BAB and $BABABA$ is BABBABABA = ABA.
The inverse of $BABA$ is $ABAB$ and the inverse of ABA is ABA itself.

Any string that starts and finishes with the same symbol has infinite order while if the first symbol is different to the first the string has order 2.

So $C = AB$ has infinite order as has $C^{-1} = B^{-1}A^{-1} = BA$. The group can be generated by A and C (since $B = AC$) and since $BC = BAB = C^{-1}B$ we can present the group as $\langle C, B \mid B^2, BC = C^{-1}B \rangle$. Rewriting C as A this becomes $\langle A, B \mid B^2, BA = A^{-1}B \rangle$ which we recognise as belonging to the dihedral family. We call it the **infinite dihedral group** and denote it as D_∞.

On the other hand the group in the next example is much smaller than it might appear.

Example 4: $\langle \mathbf{A, B} \mid \mathbf{A^5, B^3, BA = A^2B} \rangle$ has order 3. This is because $BAB^{-1} = A^2$ so

$$\begin{aligned} B^2AB^{-2} &= B\,(BAB^{-1})B^{-1} \\ &= BA^2B^{-1} \\ &= (BAB^{-1})^2 \\ &= (A^2)^2 \\ &= A^4. \end{aligned}$$

Continuing we get $B^3AB^{-3} = A^8 = A^3$.
Since $B^3 = 1$ it follows that $A = A^3$ and hence $A^2 = 1$.
But $A^5 = 1$ so $A = 1$. So this is simply the cyclic group of order 3 in disguise. A simpler presentation would be $\langle B \mid B^3 \rangle$ or, since we could use any letter for the generator, $\langle A \mid A^3 \rangle$.

§5.2. Chains and Links

Suppose we have a finite presentation $\langle \Gamma \mid R \rangle$ for a group G. An element of G will be represented by many group words. We attempt to assign to each element a unique integer code, with 1 representing the identity and 2, 3, ... , N representing the other elements. If we succeed, the order of the group is N. Of course this will only happen if G is finite.

A **chain** is an expression mWn, representing the equation $mW = n$, where m and n are integer codes and W is a group word.

The **length** of a chain mWn is the length of the word W (that is, the number of symbols, counting each generator or its inverse as a single symbol).

Example 5: 7*AAbbb*2 is a chain of length 5 representing $7 \times A^2B^{-3} = 2$. Here 2, 7 and A^2B^{-3} are all elements of the group. The elements represented by the integer codes 2 and 7 will also be expressible as group words and the group word A^2B^{-3} will also have an integer code.

At the outset the elements of the group being presented will be expressed as words in the generators but, since many different words can represent the same element, this is not altogether a satisfactory notation. Gradually, as the algorithm progresses, we assign a unique integer code to each element.

A chain mWn is a **link** if it has length 1, that is, where the word W is a generator or the inverse of a generator. Once we know all the links we can express the codes in terms of the generators and hence obtain the group table.

Example 6: If B is one of the generators then the chain $6B2$ is a link. It provides the information that $6B = 2$. For example, if we have already expressed 6 as the word BA^3B^{-2} then we can express 2 as BA^3B^{-1}.

The links are used to build up a table, called the **Link Table**:

	A	**B**	**...**
1			
2			
...			

Once we have completed this table we can produce the group table.

Example 7: If we have a group generated by A and B with the following link table, the group has order 6.

	A	**B**
1	2	4
2	3	5
3	1	6
4	6	1
5	4	2
6	5	3

We can use the link table to express each element as a group word. For example, since 1 is the identity it follows that the element 2 is A and the element 4 is B. In the next row we have the fact that $2A = 3$ so $3 = AA = A^2$. Doing this for all the elements we get:

	A	**B**	
1	2	4	1
2	3	5	A
3	1	6	$2A$
4	6	1	B
5	4	2	$2B$
6	5	3	$3B$

The group word associated with each is, of course, not unique. We can now complete the group table. For example, $5.3 = 5.2A = 5AA = 4A = 6$ and $6.5 = 6.2B = 6AB = 5B = 2$.

If G is a generator then the links rGs and sgr, where $g = G^{-1}$, give equivalent information. This is because rGs represents the equation $rG = s$ while sgr represents the equivalent equation $sG^{-1} = r$. We call these links **conjugates** of one another.

Example 8: The conjugate of the link 3*B*5 is the link 5*b*3.

§ 5.3. The Restricted Todd-Coxeter Algorithm

To make it easy to follow the algorithm we shall use a mixture of metaphors. So far we have used the rather mechanical metaphor of "chains" and "links" to describe the ingredients of the algorithm, but to describe the overall behaviour we'll use the metaphor of population dynamics.

A chain is said to **die** when it contracts down to a link. Throughout the course of the algorithm chains are born, contract and eventually die and the success of the algorithm for a particular presentation depends on the balance between the birth and death rates. The birth rate is uniform (with as many new chains born in each generation as there are relators) but the death rate can be unpredictable. At first the set of live chains grows, with more being born than die. If all goes well the time comes when the balance shifts and the population decreases and is finally wiped out. But it can happen that the population grows indefinitely and so the algorithm fails to terminate.

The algorithm proceeds in a cycle with three stages occurring at each generation: BIRTH, CONTRACTION and MARRIAGE. We begin with BIRTH.

BIRTH RULE
Having just created the integer code *m* we add the chain *mRm* to the list of chains for each relator *R*.

Suppose we are processing the presentation ⟨A, B, ... | R, S, ... ⟩. Then the relators *R*, *S*, ... represent the identity. So for every integer code m. the chains *mRm*, *mSm*, ... are valid because they simply state that $m1 = m$. At each generation, after introducing a new integer code, these corresponding chains are born, one for each relator.

This is how the algorithm gets off the ground because we begin with the code 1, representing the identity and so the first chains to be born are 1*R*1, 1*S*1, ...

Example 9: For $\langle A, B \mid A^4, B^2, BA = A^{-1}B\rangle = \langle A, B \mid A^4, B^2, ABAb\rangle$ the process begins with the birth of the chains: 1*AAAA*1, 1*BB*1, 1*ABAb*1.

Next we carry out any contractions that are possible.

CONTRACTION RULE
If *X* is a generator (or the inverse of a generator) and *rXs* is a link we may contract *rX* to *s* at the start of any live chain and contract *Xs* to *r* at the end of any live chain:
$$rX \to s, \quad Xs \to r$$

After the first generation of chains are born there are no links and so no contraction is possible. One way to obtain links is for chains to contract to a link, but clearly that cannot occur until we have some links already. Fortunately there is another way of obtaining links, through so-called "marriage".

The link rXs represents the equation $rX = s$, so clearly rX can be replaced by s.
If αXs is any live chain, ending in Xs then $s = \alpha X = rX$, so $\alpha = r$. Hence $\alpha\ r$ is a valid chain (representing $\alpha = r$).

Example 10: If 3*A*5 is a link (equivalent to the conjugate link 5*a*3) we may perform the following contractions:

3*ABBB*3 → 5*BBB*3,
5*aBBAbA*2 → 3*BBAbA*2,
2*ABAA*5 → 2*ABA*3,
2*ABa*3 → 2*AB*5.

If 5*B*4, 2*B*3 are also links we may make the following further contractions of 5*BBB*3:

5*BBB*3 → 4*BB*3 → 4*B*2.

This chain has now contracted down to a link and so, as a chain, it dies. But it may well be a *new* link, enabling further contractions.

DEATH RULE
Whenever a chain contracts to a link it "dies".
The information it conveys is transferred to the link table.

If the new link has the form rGs, where G is a generator, or the inverse of a generator, we write *s* in row *r*, column *G* of the link table. Then, we write the corresponding information for the equivalent link *sgr*.

When the dust finally settles, and we still have some live chains but can make no more contractions possible, we must create a new integer code.

MARRIAGE RULE
If the set of live chains is non-empty, and no further contractions are possible, assign the next available code, *m*, by creating a new link of the form *mGr* or *rGm* where *G* is a generator or the inverse of a generator for some *r* < *m*.

The creation of the new link can be thought of as a "marriage" where the new code "marries" one of the existing ones. Like marriage in western society there is freedom to choose a partner – the algorithm does not arrange a marriage. But, as with real marriages there are sensible choices and less sensible ones, so it is with the Todd-Coxeter Algorithm. At the time of a marriage it is impossible to know for certain that a given choice is going to be successful!

Unlike human marriage the choice goes beyond that of choosing a partner – the choice of generator comes into it too. And different codes can "marry" the same code, using different generators. Perhaps the analogy is starting to break down, so let us explain in a more mundane fashion how this so-called "marriage rule" operates.

The rule defines the next integer code in terms of the preceding ones. Having done that, a whole new generation of chains is ready to be born. Further possibilities for contraction now arise. There is now a whole new generation of chains to process, but more importantly the new link can be used in further contractions.

Example 11: Suppose we have assigned codes 1, 2, 3 and have generated all the corresponding chains and contracted them as far as possible. At that stage we need to define a new integer code by creating a link to define 4. For example we might create the link $4A2$. This means putting $4A = 2$ or, in other words, defining 4 as $2A^{-1}$. Or we could create $3B4$, which defines 4 as $3B$.

Note that we are not allowed to create a link of the form mAm, defining m in terms of itself. Nor can we define m in terms of a future code. If only codes 1, 2, 3 have been assigned then the code 4 cannot be defined by a link such as $4B7$. In defining a new code m the other code must be less than m.

One very good strategy is to create the new link so as to fill up the next available blank in the table. This could be considered the "default strategy" and it is the one that is mostly used in the following examples. However the unsolvability of the word problem means that there is no strategy that can be guaranteed to always work. So, there will be times when this default strategy will be the wrong thing to do.

Example 12:

Suppose we have assigned codes 1, 2, 3 and we need to assign the code 4 at the stage where our table of links so far is:

	A	**B**
1	2	3
2	1	
3		

We may decide to create the link $2B4$ or $3A4$ or $3B4$. The default strategy would be to create the link $2B4$.

But another consideration is how useful the newly created link would be at the moment. If, in the above situation, we had the chain $3BA3$ we might decide to create the link $3B4$ because this could be used immediately to contract $3BA3$ to $4A3$ thereby giving yet another link.

Finally we must ask, "How does the algorithm begin and how does it terminate?" It begins with the empty set of chains and with the integer code 1 being created, representing the identity. This immediately causes a number of chains to be born, and the process of birth, contraction and death.

STARTING
Begin with an empty set of chains and assign the code 1.

The restricted algorithm that we are describing terminates in one of two ways. For a start there is the possibility that a chain contracts to a link which conflicts with one that we already have. That is, when we come to enter the new link in the link table we find that this cell is already occupied *with a different code*.

For example we may have just contracted the chain 5*AB*2 to the link 5*A*7, but when we go to enter "7" in the "5" row and *A* column we discover that there's already a "3". In other words we have $5A = 3$ and $5A = 7$.

This doesn't necessarily mean that we've made a mistake (though of course this might be the case). It may simply mean that we've inadvertently issued two different integer codes to the same group element. With the full Todd-Coxeter algorithm we then have to carry out an identification process in which we identify the two codes, and continue the process, but with the restricted algorithm we avoid this complication and simply abort the process. Of course this would mean that we don't get a group table for our presentation.

The normal way for the restricted algorithm to terminate, and the only way that gives us a group table, is to arrive at a situation where we have a complete link table, with no blanks. Clearly a completed link table will allow all chains to be contracted to links, so that all chains will now be dead.

FINISHING:
Terminate when either:
(1) All chains have become links and the table of links is complete or
(2) You get a contradiction in the Link Table

If the algorithm terminates under (1) it will have been successful and we will have a group table. If it terminates under (2) the algorithm fails. This will either mean that we could have made better choices (we could begin again and vary our strategy) or it could mean that the group is infinite. A further possibility is that we never reach either state (1) or (2) and the algorithm fails to terminate.

The unrestricted algorithm includes an additional step of identification if and when we reach a contradiction. This involves a considerable amount of work as we follow through the consequences of the identification. In this case either the algorithm will eventually reach a successful outcome or it will never terminate. However we'll only use the Restricted Algorithm and simply abort if we ever get a contradiction.

With the Unrestricted Algorithm it is the case that if we have any presentation for a finite group there will be a sequence of choices that will lead to the algorithm terminating successfully. The problem is that there is no strategy that can be guaranteed to achieve this.

With the Restricted Algorithm it can happen that no sequence of choices will avoid identification and so, no matter what you do, the algorithm will abort. This can occur for certain presentations and often using a different, but equivalent, presentation will lead to success.

TODD-COXETER ALGORITHM (Restricted Version)

Suppose ⟨ $G_1, G_2, \ldots, G_m \mid R_1, R_2, \ldots, R_n$ ⟩ is a presentation for a group *G*.

Initially let *N* = 1, *C* = the empty set of chains, and *L* a table with 2*m* columns with each cell blank. The number of rows is initially 1. The 2*m* columns are labelled by the generators and their inverses.

(1) For each relator R_i adjoin the chain NR_iN to *C*;

(2) Where possible, contract chains using the links in *L*.

Any chain that contracts to a link, together with its conjugate, is transferred to the link table and that chain dies. It is removed from *C*.

If this results in two integers being assigned to the same cell in the Link Table, abort the algorithm.

(3) When no further contraction is possible, and *L* has a blank cell, increase *N* by 1. Then marry *N* to one of the existing codes by inserting a new link, and its conjugate, into the Link Table. Then GO TO (1).

(4) If *L* has no blank cell terminate.

Then the group has order *N* and the group table can be constructed from the information given by the links in *L*.

In performing the Todd-Coxeter Algorithm by hand we set out our working in three tables:

CHAIN TABLE: This describes the successive contractions of the chains. Each of the chains that are born is placed at the start of a new row and their successive contractions are recorded.

LINK TABLE: Once a proper chain becomes a link the information is transferred to the link table. At the right of the table record the link definitions for each code. The code "1" is assigned to the identity so we write 1 in the definition column of the table for the code "1".

	A	**B**	**...**	
1			…	1
2			…	
...	…	…	…	

GROUP TABLE: group table

	1	**2**	**...**
1	1	2	…
2	2		…
...	…	…	…

§ 5.4. Examples of the Todd-Coxeter Algorithm

Perhaps you find all this very confusing. That's not surprising, because this is probably the most complex algorithm you will ever have to carry out. Also, what you might find disquieting, is that the algorithm is not deterministic. At one stage in every generation you have to make a choice and the success depends on whether you make the "right" choices.

We can give advice that can help, but the unsolvability of the word problem guarantees that no choice strategy exists that will always be successful.

Having said this, it is remarkable how often even the restricted algorithm works, provided we start with a "good" presentation and use the couple of simple strategies that we have described. But work through the following examples and you will eventually find that it becomes second nature.

Example 13: $G = \langle A, B \mid A^2, B^2, (AB)^2 \rangle$.

(For clarity we describe the appearance of the tables at each generation and make bold the additional information at each stage. Normally you would carry out your working in the one set of tables.)

We begin with N = 1. For each relator a chain is born.

1AA1
1BB1
1ABAB1

	A	**B**	
1			**1**

Having no links we clearly can't make any contraction so we must create a link to define 2 (the marriage step). Using the default strategy we create the link 1*A*2, thereby defining 2 = *A*. We write 2 in the "1"row and *A* column. We enclose it in parentheses to record the fact that this link was created, rather than found as a result of contracting a chain. This makes it easier to follow the sequence of events when the table is completed.

1AA1
1BB1
1ABAB1

	A	**B**	
1	**(2)**		1
2			**A**

Having defined a new code, a new generation of chains is born, one for each relator.

1AA1
1BB1
1ABAB1
2AA2
2BB2
2ABAB2

	A	**B**	
1	(2)		1
2			**A**

Some contraction is now possible. Any "1*A*" can be replaced by "2" and any "*A*2" can be replaced by "1". We use an arrow to separate a chain from its contraction.

1AA1→**2A1**
1BB1
1ABAB1→**2BAB1**
2AA2→**2A1**
2BB2
2ABAB2

	A	**B**	
1	(2)		1
2			A

Three chains have contracted, and two of them have contracted down to the same link 2*A*1. This is a new link and so is transferred to the link table. Those two chains are considered "dead" and don't enter into the remaining computation so we mark them by the symbol "X".

1AA1→2A1	**X**
1BB1	
1ABAB1→2BAB1	
2AA2→2A1	**X**
2BB2	
2ABAB2	

	A	**B**	
1	(2)		1
2	1		A

The new link 2*A*1 means that chains starting with “2*A*” or ending with “*A*1” can be contracted. This allows just one more contraction: 2*ABAB*2 → 1*BAB*2.

1AA1→2A1	X
1BB1	
1ABAB1→2BAB1	
2AA2→ 2A1	X
2BB2	
2ABAB2**→1BAB2**	

	A	**B**	
1	(2)		1
2	1		A

No further contraction is possible so we must create a new code. The default strategy would create 1*B*3. This seems to be a good idea because it will be useful in contracting the two chains that begin with “1*B*”.

1AA1→2A1	X
1BB1**→3B1**	**X**
1ABAB1→2BAB1	
2AA2→2A1	X
2BB2	
2ABAB2→1BAB2**→3AB2**	
3AA3	
3BB3→3B1	**X**
3ABAB3→3ABA1	

	A	B	
1	(2)	**(3)**	1
2	1		A
3			**B**

We now have (twice) the link 3*B*1. We enter this up in the link table. This link allows some further contraction.

1AA1→2A1	X
1BB1→3B1	X
1ABAB1→2BAB1**→2BA3**	
2AA2→2A1	X
2BB2	
2ABAB2→1BAB2→3AB2	
3AA3	
3BB3→3B1	X
3ABAB3→3ABA1**→3AB2**	

	A	B	
1	(2)	(3)	1
2	1		A
3		**1**	B

No further contraction being possible, we create code “4”.

1AA1→2A1	X
1BB1→3B1	X
1ABAB1→2BAB1→2BA3	
2AA2→2A1	X

	A	**B**	
1	(2)	(3)	1
2	1	**(4)**	A
3		1	B

2BB2		4			**2B**	
2ABAB2→1BAB2→3AB2						
3AA3						
3BB3→3B1	X					
3ABAB3→3ABA1→3AB2						
4AA4						
4BB4						
4ABAB4						

Many contractions are now possible, leading to further links, leading to further contractions until all chains become links and the link table becomes complete.

1AA1→2A1	X	
1BB1→3B1	X	
1ABAB1→2BAB1→2BA3**→4A3**	**X**	
2AA2→2A1	X	
2BB2**→4B2**	**X**	
2ABAB2→1BAB2→3AB2**→4B2**	**X**	
3AA3**→3A4**	**X**	
3BB3→3B1	X	
3ABAB3→3ABA1→3AB2**→4B2**	**X**	
4AA4**→3A4**	**X**	
4BB4**→2B4**	**X**	
4ABAB4→3BAB4→1AB4**→2B4**	**X**	

	A	**B**	
1	(2)	(3)	1
2	1	(4)	A
3	4	1	B
4	3	2	2B

We can now complete the group table. The first row and column are easy, since "1" is the identity.

GROUP TABLE

	1	**2**	**3**	**4**
1	1	2	3	4
2	2			
3	3			
4	4			

Now 2 = *A* so the "2" column in the group table is a copy of the *A* column in the link table. Similarly, as 3 = *B*, the "3" column in the group table is a copy of the *B* column in the link table.

GROUP TABLE

	1	**2**	**3**	**4**
1	1	2	3	4
2	2	1	4	
3	3	4	1	
4	4	3	2	

Finally, since 4 = 2*B* we use the *B* column of the link table to transform the "2" column of the group table into the "4" column. For example, 3.4 = 3(2*B*) = (3.2)*B* = 4*B* (from the "2" column of the group table) = 3 (from the *B* column of the link table).
The entries in the "2" column are 2, 1, 4, 3 respectively and from the link table 2*B* = 4, 1*B* = 3, 4*B* = 2 and 3*B* = 1. So the "4" column of the group table is 4, 3, 2, 1 respectively.

GROUP TABLE

	1	**2**	**3**	**4**
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Example 14: $G = \langle A, B \mid A^3, B^2, BAB = A^{-1}\rangle$

We write the third relator as *BABA*. We show the state of the computation at each generation and, to save space, we omit any rows that contain "dead" chains from the previous step.

***N* = 1, 2**

1AAA1→2AA1	
1BB1→	
1BABA1	
2AAA2→2AA1	=
2BB2	
2BABA2→2BAB1	

	A	B	
1	(2)		1
2			A

The symbol "=" indicates that the chain is a repetition of an earlier one. There is no need to process this further and we can effectively regard it as "dead".

***N* = 3**

1AAA1→2AA1	
1BB1→3B1	X
1BABA1→3ABA1	
2BB2	
2BABA2→2BAB1→2BA3	
3AAA3	
3BB3→1B3	X
3BABA3→1ABA3→2BA3	=

	A	B	
1	(2)	(3)	1
2			A
3		1	B

***N* = 4**

1AAA1→2AA1→4A1	X
1BABA1→3ABA1	
2BB2	
2BABA2→2BAB1→2BA3	
3AAA3	
4AAA4→1AA4→2A4	X
4BB4	
4BABA4→4BAB2	

	A	B	
1	(2)	(3)	1
2	(4)		A
3		1	B
4	1		2A

***N* = 5**

1BABA1→3ABA1	
2BB2→5B2	X
2BABA2→2BAB1→2BA3→5A3	X
3AAA3→3AA5	
4BB4	
4BABA4→4BAB2→4BA5	
5AAA5→3AA5	
5BB5→2B5	X
5BABA5→2ABA5→4BA5	

	A	B	
1	(2)	(3)	1
2	(4)	(5)	A
3		1	B
4	1		2A
5	3	2	2B

***N* = 6**

1BABA1→3ABA1→6BA1→4A1	X
3AAA3→3AA5→6A5	X
4BB4→5B4	X
4BABA4→4BAB2→4BA5→4B6	X
5AAA5→3AA5→6A5	X
5BABA5→2ABA5→4BA5→6A5	X
6AAA6→5AA6→3A6	X
6BB6→4B6	X
6BABA6→4ABA6→1BA6→3A6	X

	A	B	
1	(2)	(3)	1
2	(4)	(5)	A
3	(6)	1	B
4	1	6	2A
5	3	2	2B
6	5	4	3A

The first column of the group table is easy, and since 2 = *A* and 3 = *B* the next two columns can be copied from the link table.

GROUP TABLE

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3			
2	2	4	5			
3	3	6	1			
4	4	1	6			
5	5	3	2			
6	6	5	4			

Now 4 = 2*A* so we take the "2" column and multiply each entry by *A*, using the link table. So 2, 4, 6, 1, 3, 5 becomes 4, 1, 5, 2, 6, 3. The *A* column is a permutation of {1, 2, 3, 4, 5, 6} and we are simply using *A* to permute the "2" column

GROUP TABLE

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4		
2	2	4	5	1		
3	3	6	1	5		
4	4	1	6	2		
5	5	3	2	6		
6	6	5	4	3		

Since 5 = 2*B* we permute the "2" column of the group table by the *B* column of the link table, considered as a permutation. This gives us the "5" column. Similarly 6 = 3*A* so we permute the "3" column by *A*.

GROUP TABLE

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	5	1	6	3
3	3	6	1	5	4	2
4	4	1	6	2	3	5
5	5	3	2	6	1	4
6	6	5	4	3	2	1

Of course we can always complete the last column by writing down the missing symbol in each row.

Example 15: $G = \langle A, B \mid A^4, B^4, A^2 = B^2, B^{-1}AB = A^{-1}\rangle$.
We write the third relator as *AAbb*, and the fourth as *bABA*. This time we present the finished computation, though to help you follow it we enclose in parentheses those links in the link table that came about by creation, rather than the contraction of a chain. Also, since our relators involve the inverse "b" we include a column in the link table for it. Remember that every link that goes in the *B* column will produce an equivalent one in the *b* column.

1AAAA1→2AAA1→4AA1→7A1	X
1BBBB1→3BBB1→4BB1→8B1	X
1AAbb1→2Abb1→2Ab3→4b3	X
1bABA1→1bAB7→1bA5→8A5	X
2AAAA2→2AAA1	=
2BBBB2→5BBB2→5BB6→7B6	X
2AAbb2→4Abb2→4Ab5→7b5	X
2bABA2→2bAB1→6AB1→6A8	X
3AAAA3→3AAA5→6AA5→8A5	X
3BBBB3→3BBB1	=
3AAbb3→3AAb4→6Ab4→8b4	X
3bABA3→1ABA3→2BA3→5A3	X
4AAAA4→4AAA2→4AA1	=
4BBBB4→4BBB3→4BB1	=
4AAbb4→7Abb4→1bb4→8b4	X
4bABA4→3ABA4→3AB2→6B2	X
5AAAA5→3AAA5	=
5BBBB5→5BBB2	=
5AAbb5→3Abb5→6bb5→6b7	X
5bABA5→2ABA5→4BA5→8A5	X
6AAAA6→6AAA3→6AA5	=
6BBBB6→2BBB6→5BB6→7B6	X
6AAbb6→6AAb2→6AA5	=
6bABA6→6bAB3→6bA1→7A1	X
7AAAA7→1AAA7→2AA7→4A7	X
7BBBB7→6BBB7→2BB7→5B7	X
7AAbb7→1Abb7→2bb7→6b7	X

	A	**B**	**b**	
1	(2)	(3)	8	1
2	(4)	(5)	6	A
3	(6)	4	1	B
4	(7)	(8)	3	2A
5	3	7	2	2B
6	8	2	7	3A
7	1	6	5	4A
8	5	1	4	4B

7bABA7→5ABA7→3BA7→4A7	X
8AAAA8→5AAA8→3AA8→6A8	X
8BBBB8→1BBB8→3BB8→4B8	X
8AAbb8→5Abb8→3bb8→1b8	X
8bABA8→4ABA8→7BA8→6A8	X

We can obtain the group table from the link table in the usual way. This group is the quaternion group of order 8.

Example 16: $G = \langle A, B \mid A^3, B^3, (AB)^2 \rangle$

LINK TABLE

1AAA1→2AA1→4A1	X
1BBB1→3BB1→7B1	X
1ABAB1→2BAB1→5AB1→5A7	X
2AAA2→2AA1	=
2BBB2→5BB2→9B2	X
2ABAB2→4BAB2→8AB2→8A9	X
3AAA3→6AA3→10A3	X
3BBB3→3BB1	=
3ABAB3→3ABA1→3AB4→6B4	X
4AAA4→1AA4→2A4	X
4BBB4→4BB6→8B6	X
4ABAB4→1BAB4→3AB4	=
5AAA5→7AA5→11A5	X
5BBB5→5BB2	=
5ABAB5→5ABA2→5AB1	=
6AAA6→6AA3	=
6BBB6→4BB6	=
6ABAB6→6ABA8→10BA8→12A8	X
7AAA7→7AA5	=
7BBB7→1BB7→3B7	X
7ABAB7→7ABA3→7AB10→11B10	X
8AAA8→9AA8→12A8	X
8BBB8→6BB8→4B8	X
8ABAB8→8ABA4→8AB2→9B2	X
9AAA9→9AA8	=
9BBB9→2BB9→5B9	X
9ABAB9→9ABA5→9AB11→12B11	X
10AAA10→3AA10→6A10	X
10BBB10→10BB11→10B12	X
10ABAB10→3BAB10→7AB10	=
11AAA11→5AA11→7A11	X
11BBB11→10BB11	=
11ABAB11→5BAB11→9AB11	=
12AAA12→8AA12→9A12	=
12BBB12→11BB12→10B12	=
12ABAB12→8BAB12→6AB12→10B12	=

	A	**B**	
1	(2)	(3)	1
2	(4)	(5)	A
3	(6)	(7)	B
4	1	(8)	2A
5	7	(9)	2B
6	(10)	4	3A
7	(11)	1	3B
8	9	6	4B
9	(12)	2	5B
10	3	12	6A
11	5	10	7A
12	8	11	9A

GROUP TABLE

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
		=A	=B	=2A	=2B	=3A	=3B	=4B	=5B	=6A	=7A	=9A
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	5	1	8	7	9	3	6	11	12	10
3	3	6	7	10	4	11	1	12	8	15	2	9
4	4	1	8	2	3	9	6	5	7	12	10	11
5	5	7	9	11	1	12	2	10	3	8	4	6
6	6	10	4	3	12	1	8	7	11	2	9	5
7	7	11	1	5	10	2	3	9	12	4	6	8
8	8	9	6	12	2	10	4	11	5	3	1	7
9	9	12	2	8	11	4	5	6	10	1	7	3
10	10	3	12	6	7	8	11	4	1	9	5	2
11	11	5	10	7	9	3	12	1	2	6	8	4
12	12	8	11	9	6	5	10	2	4	7	3	1

It can be shown that $G \cong A_4$

§ 5.5. Why Does It Work?

At this stage it is difficult to give a rigorous proof that the Todd-Coxeter algorithm, when it terminates normally, gives the correct result because we have not yet properly defined what a group presentation really is. But here is an informal proof which will suffice for now.

Suppose we carry out the Todd-Coxeter algorithm on a presentation for a group G and it terminates normally. We end up with a finite number of codes N and a link table which expresses each of the generators as a permutation on $\{1, 2, \ldots, N\}$. These permutations generate a permutation group which will satisfy all the relations given in the presentation. Provided that different codes correspond to different elements this permutation group will be isomorphic to the group presented by the presentation.

Now identifications can occur during the course of the algorithm, forced by contradictions that arise from interactions between the relations. But if at the end we have two different codes for the same group element we would have an unforced identification, that is, one that is not a consequence of the relations.

But the group described by the presentation is one in which the *only* relations that hold are those that are consequences of the ones given in the presentation. An unforced identification would correspond to a relation that is not a consequence of these. It therefore could not arise. Hence the permutation group obtained at the end is indeed isomorphic to the group being presented.

§ 5.6. Marriage Strategies

Probably all of the algorithms you have met so far in your mathematical career are deterministic. This means that the algorithm tells you precisely what to do at each stage and terminates after a finite number of steps. It can therefore be programmed on a computer so that, after feeding in the input data, we simply need to click "GO" and wait for the output.

By contrast the Todd-Coxeter algorithm is a wild beast. It is "non-deterministic". Every so often the algorithm consults you as an external "oracle". "I can't proceed", it says, "please choose a marriage partner for the next code". This you do and then it goes off happily for a while until it has to come back for some more advice.

The following three strategies are useful.

MARRIAGE STRATEGY I Fill up the first blank cell in the Link Table.

For example if our Link Table, at a given stage, is:

	A	**B**
1	(2)	(3)
2	3	
3		

and we are about to create the code "4" we would, following this strategy, create the link 2*B*4 by writing (4) into the table. Our Link Table would now be:

	A	**B**
1	(2)	(3)
2	3	(4)
3		

MARRIAGE STRATEGY II Choose a link that shortens one of the shortest chains.

For example if we have the chains 1*AAAA*3, 2*ABA*3, 1*AABB*2, and we are about to create the code "5" we would, using this principle, create the link 2*A*5, or perhaps 5*A*3. Either way we would now have a chain of length 2 (either 5*BA*3 or 2*AB*5, depending which of the alternatives we chose. Short chains are more likely to give links in the near future and links allow us to shorten other chains, so this seems a sensible strategy.

MARRIAGE STRATEGY III Choose a link that will shorten the most chains.

For example if we have the chains 1*AAAA*3, 2*ABA*3, 1*AABB*2 and we are about to create the code "5" we would, using this principle, create the link 1*A*5 to shorten the first and third chains, or perhaps 5*A*3 to shorten the first and second.

In many cases there will be equally attractive alternatives. We can often choose between them by considering which best satisfies the other strategies. For example using strategy III above we could have created the link 1*A*5 or 5*A*3 but since 5*A*3 also shortens the shortest chain that is the one we should probably choose.

There are many other, less obvious strategies. In fact no finite list of strategies will work in all cases.

§ 5.7. The Unsolvability of the Word Problem

Which of the above strategies is the best? If we could decide that then we could then build this into the algorithm and so make it deterministic. Unfortunately it is not so easy!

The Todd-Coxeter Algorithm (either the restricted or the full version) will, if it terminates normally, give the correct group table and the full version, involving identifications to resolve "contradictions", will only ever terminate normally – if it terminates at all. The intrinsic problem with the Todd-Coxeter Algorithm is that it may never terminate.

Of course, if the presentation is that of an infinite group then quite clearly it can never terminate. We would be forced to keep giving out more and more integer codes.

But what if the group is indeed finite? If the algorithm doesn't terminate there is no way of knowing this. Even if the algorithm continues for an enormous length of time, with integer codes running in to the millions, we cannot be sure that it is destined to go on forever. We may be just about to conclude that we have an infinite group when suddenly all the chains collapse and the algorithm terminates. It might just be a very, very large group, but not an infinite one. Or we could get contradictions and identifications and it might turn out that all the millions of codes represent the same element and we have the trivial group! We can never tell what is around the corner.

On the other hand, even though we have a finite group, because of the choices that we make, the algorithm may go on indefinitely. Certainly it is very easy to make poor choices that would cause this to happen.

What if the group really is finite and we make the "right" choices? Then it is true that the full algorithm will terminate, giving the correct group table. But how can we know what are the right choices? Is there is a strategy, either a combination of the three given above, or perhaps some other fiendishly clever one, that would guarantee that the full algorithm would always terminate? The answer is a resounding "NO". In fact we cannot guarantee that, given a presentation, we can determine whether a given word will reduce to the identity!

It is not simply that nobody has yet come up with such a deterministic algorithm. It has been *proved* that nobody ever will. Like the Halting Problem in computer science, the Word Problem is **unsolvable**.

THE WORD PROBLEM

Construct an algorithm which, given any presentation for a group, and any word in those generators, will determine whether or not that word represents the identity.

Theorem 1: The Word Problem is Unsolvable (i.e. no such algorithm can ever exist).

This has been proved but, as the proof is rather long and technical, we omit it. Basically one assumes that such an algorithm exists and then proceed to obtain a logical contradiction.

What it means is that the Todd-Coxeter algorithm attempts to do something that is intrinsically impossible. Yet it works surprisingly well, for "reasonable" presentations, using "sensible" choice strategies (like the three given above). Needless to say, much research has gone into investigating the effect of different choice strategies for different types of presentation.

Can we avoid identifications by making the "right" choices? This is easily answered. We cannot. For example the presentation $\langle A \mid A^5, A^7\rangle$ represents the trivial group $\{1\}$. This is because $A = A^{15-14} = (A^5)^3(A^{-7})^2 = 1$. Yet we have to begin by introducing the integer code "2", which ultimately must be identified with "1".

In many other cases, however, when a contradiction arises it is possible to avoid identifications by starting again and making better choices.

Example 17: $G = \langle A, B \mid A^7, A^2B^2\rangle$.

If we adopted strategy I at each step we would find that we get a contradiction after having introduced code 21. If you want to recreate this computation the created links would be:
1*A*2, 1*B*3, 2*A*4, 2*B*5, 3*A*6, 3*B*7, 4*A*8, 4*B*9, 5*A*10, 5*B*11, 6*B*12, 7*B*13, 8*A*14, 8*B*15, 9*A*16, 10*B*17, 11*B*18, 12*A*19, 12*B*20, 14*B*21.

If we adopted strategy II at each stage we'd find that we would get a contradiction after having introduced code "20". The created links would be:

1*A*2, 2*A*3, 3*B*4, 3*A*5, 5*B*6, 4*A*7, 7*A*8, 5*A*9, 6*A*10, 10*A*11, 8*A*12, 9*A*13, 12*A*14, 12*B*15, 13*A*16, 1*B*17, 2*B*18, 11*A*19, 14*A*20.

If we adopted strategy III at each stage we would find that we would get a contradiction after having introduced code "17". The created links would be:
1*A*2, 2*A*3, 3*A*4, 4*A*5, 5*A*6, 6*A*7, 1*B*8, 1*b*9, 3*b*10, 5*b*11, 7*b*12, 2*b*13, 4*b*14, 9*a*15, 10*a*16, 10*A*17.

But the following combination of all three basic strategies terminates without recourse to identifications.

We begin by using strategy III at each stage.

1AAAAAAA1→2AAAAAA1→3AAAAA1→4AAAA1 →5AAA1→6AA1→7A1	X
1AABB1→2ABB1→3BB1	
2AAAAAAA2→3AAAAAA2→3AAAAA1	=
2AABB2→3ABB2→4BB2	
3AAAAAAA3→3AAAAAA2→3AAAAA1	=
3AABB3→4ABB3→5BB3	
4AAAAAAA4→4AAAAAA3→4AAAAA2→4AAAA1	=
4AABB4→5ABB4→6BB4	
5AAAAAAA5→5AAAAAA4→5AAAAA3→5AAAA2 →5AAA1	=
5AABB5→6ABB5→7BB5	
6AAAAAAA6→6AAAAAA5→6AAAAA4→6AAAA3 →6AAA2→6AA1	=
6AABB6→7ABB6→1BB6	
7AAAAAAA7→1AAAAAA7→2AAAAA7→3AAAA7 →4AAA7→5AA7→6A7	=
7AABB7→1ABB7→2BB7	

	A	**B**	
1	(2)		1
2	(3)		1A
3	(4)		2A
4	(5)		3A
5	(6)		4A
6	(7)		5A
7	1		6A

We have lots of lovely short chains of the form x*BB*y but without a *B* link we'll have no chance of making links from them. For this reason we now switch to strategy I and fill up the first blank in the *B* column by creating the link 1*B*8.

3BB1	
4BB2	
5BB3	
6BB4	
7BB5	
1BB6→8B6	X
2BB7	
8AAAAAAA8	
8AABB8	

	A	**B**	
1	(2)	(8)	1
2	(3)		1A
3	(4)		2A
4	(5)		3A
5	(6)		4A
6	(7)		5A
7	1		6A
8		6	1B

It would be very natural now to concentrate on *B*s but it turns out that this is not a good idea. Instead we go back to strategy III. It might seem a bad idea to be again filling up the *A* column and neglecting the *B* column, but watch!

3BB1	
4BB2	
5BB3	
6BB4	
7BB5	
2BB7	
8AAAAAAA8→9AAAAAA8→10AAAAA8→11AAAA8 →12AAA8→13AA8	
8AABB8→8AAB1→9AB1→10B1	X
9AAAAAAA9→9AAAAAA8→10AAAAA8	=
9AABB9→10ABB9→11BB9	
10AAAAAAA10→10AAAAAA9→10AAAAA8	=
10AABB10→11ABB10→12BB10	
11AAAAAAA11→11AAAAAA10→11AAAAA9 →11AAAA8	=
11AABB11→12ABB11→13BB11	
12AAAAAAA12→12AAAAAA11→12AAAAA10 →12AAAA9→12AAA8	=
12AABB12→13ABB12	
13AAAAAAA13→13AAAAAA12→13AAAAA11 →13AAAA10→13AAA9→13AA8	=
13AABB13	

	A	**B**	
1	(2)	(8)	1
2	(3)		1A
3	(4)		2A
4	(5)		3A
5	(6)		4A
6	(7)		5A
7	1		6A
8	(9)	6	1B
9	(10)		8A
10	(11)		9A
11	(12)		10A
12	(13)		11A
13			12A

It looks as though we are still a long way off getting rid of all of our chains. But suddenly everything collapses at the very next step!

3BB1→10B1	X
4BB2→11B2	X
5BB3→12B3	X
6BB4→13B4	X
7BB5→14B5	X
2BB7→9B7	X
13AA8→14A8	X
11BB9→2B9	X
12BB10→3B10	X
13BB11→4B11	X
13ABB12→14BB12→5B12	X
13AABB13→14ABB13→8BB13→6B13	X
14AAAAAAA14→8AAAAAA14→9AAAAA14 →10AAAA14→11AAA14→12AA14→13A14	=
14AABB14→8ABB14→9BB14→7B14	X

	A	**B**	
1	(2)	(8)	1
2	(3)	9	1A
3	(4)	10	2A
4	(5)	11	3A
5	(6)	12	4A
6	(7)	13	5A
7	1	14	6A
8	(9)	6	1B
9	(10)	7	8A
10	(11)	1	9A
11	(12)	2	10A
12	(13)	3	11A
13	(14)	4	12A
14	8	5	13A

How did we know that this combination of strategies was going to work? Perhaps intuition. But of course nothing beats good old group *theory*. In many cases we can manipulate the presentation and get our answer much more quickly. In this case we have $A = A^{8-7} = A^8 = (A^2)^4 = (B^{-2})^4 = B^{-8}$ so $G = \langle B \mid B^{-56}, B^{-14} \rangle = \langle B \mid B^{14} \rangle$, the cyclic group of order 14.

§ 5.8. Direct Products

The **direct product** of two groups G, H is the set of ordered pairs where the first component comes from G and the second from H, that is, $\{(g, h) \mid g \in G, h \in H\}$. It is denoted by $\boldsymbol{G \times H}$.

Multiplication in $G \times H$ is component-wise, that is $(g_1, h_1).(g_2, h_2) = (g_1g_2, h_1h_2)$. The identity is $(1, 1)$ and $(g, h)^{-1} = (g^{-1}, h^{-1})$.

Strictly speaking the operation $\times$ is not commutative in that $H \times G$ is usually different to $G \times H$. However they are isomorphic, by matching (g, h) in $G \times H$ with (h, g) in $H \times G$.

If the groups are written additively we call it the **direct sum** and write it as $\boldsymbol{G \oplus H}$. Multiplication is defined by $(g_1, h_1) + (g_2, h_2) = (g_1 + g_2, h_1 + h_2)$. The identity is $(0, 0)$ and the inverse of (g, h) is $(-g, -h)$.

If G, H are finite then the order of $G \times H$ (or $G \oplus H$ if the groups are written additively) is $|G|.|H|$.

Within $G \times H$ there is a subgroup $\{(1, h) \mid h \in H\}$ that is isomorphic to H and another subgroup $\{(g, 1) \mid g \in \mathrm{G}\}$ that is isomorphic to G. Moreover any element of the first subgroup commutes with any element of the second since $(g, 1)(1, h) = (g, h) = (1, h)(g, 1)$.

Example 18: Let $G = S_4$, the symmetric group on $\{1, 2, 3, 4\}$ and $H = \mathrm{Z}_7^{\#}$, the set $\{1, 2, 3, 4, 5, 6\}$ under multiplication modulo 7. Then $|G \otimes H| = 24 \times 6 = 144$.
$((12)(34), 5).((123), 4) = ((12)(34).(123), 5.4) = ((134), 6)$ and
$((1234), 2)^{-1} = ((1432, 4)$.

Example 19: Let G = Z, the group of integers under addition and let $H = \mathbf{R}[\mathrm{x}]$, the set of real polynomials under addition. Then $G \oplus H$ is infinite and $(-3, x - 1) = (3, -x + 1)$.

Example 20: Let $G = \langle \mathrm{A, B} \mid \mathrm{A}^4, \mathrm{B}^2, \mathrm{ABAB} \rangle$ and $\mathrm{H} = \langle \mathrm{A} \mid \mathrm{A}^5 \rangle$. Then
$\mathrm{G} \times \mathrm{H}$ is isomorphic to $\langle \mathrm{A, B, C} \mid \mathrm{A}^4, \mathrm{B}^2, \mathrm{ABAB}_, \mathrm{C}^5\ \mathrm{AC} = \mathrm{CA}, \mathrm{BC} = \mathrm{CB} \rangle$.

If we have a presentation in which the generators can be split into two subsets such that those in one subset commute with those in the other, and if there are no other relations that involve generators from both sets, we can split the group as a direct product.

Example 21: $G = \langle \mathrm{A, B} \mid \mathrm{A}^4, \mathrm{B}^2, \mathrm{AB} = \mathrm{BA} \rangle$.
In this presentation the relation $AB = BA$ states that A commutes with B. Apart from this there's no relation involving both A and B. So the group splits into a direct product:
$G \cong H \times K$ where $H = \langle \mathrm{A} \mid \mathrm{A}^4 \rangle$ and $K = \langle \mathrm{B} \mid \mathrm{B}^2 \rangle$. We don't need to use the Todd-Coxeter algorithm to obtain group tables for these cyclic groups. The group table for H is

	1	**2**	**3**	**4**
1	1	2	3	4
2	2	3	4	1
3	3	4	1	2
4	4	1	2	3

and for K it is

	1	**2**
1	1	2
2	2	1

The elements of $G = H \times K$ are

(1, 1), (2, 1), (3, 1), (4, 1), (1, 2), (2, 2), (3, 2), (4, 2).

Let's code these as follows.

1 = (1, 1)	5 = (1, 2)
2 = (2, 1)	6 = (2, 2)
3 = (3, 1)	7 = (3, 2)
4 = (4, 1)	8 = (4, 2)

In calculating the group table for G we proceed as follows:
$2 \times 2 = (2, 1)(2, 1) = (2 \times 2, 1 \times 1) = (3, 1) = 3$
$2 \times 3 = (2, 1)(3, 1) = (2 \times 3, 1 \times 1) = (4, 1) = 4$

(Remember we look up the table for H to get $2 \times 3 = 4$ and the table for K to get $1 \times 1 = 1$.)

$2 \times 4 = (2, 1)(4, 1) = (2 \times 4, 1 \times 1) = (7, 1) = 7$.
Proceeding in this way we discover that the top 4×4 portion of the group table for G is simply the group table for H. (This is a consequence of the very orderly way in which we ordered the pairs.)

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	2	3	3				
2	2	3	4	1				
3	3	4	1	2				
4	4	1	2	3				

Now

$2 \times 5 = (2, 1)(1, 2) = (2 \times 1, 1 \times 2) = (2, 2) = 6$
$2 \times 6 = (2, 1)(2, 2) = (2 \times 2, 1 \times 2) = (3, 2) = 7$
$2 \times 7 = (2, 1)(3, 2) = (2 \times 3, 1 \times 2) = (4, 2) = 8$
$2 \times 8 = (2, 1)(4, 2) = (2 \times 4, 1 \times 2) = (1, 2) = 5$.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	2	3	3	5	6	7	8
2	2	3	4	1	6	7	8	5
3	3	4	1	2	7	8	5	6
4	4	1	2	3	8	5	6	7

Notice that the next 4×4 block can be obtained by simply adding 4 to each entry in the first block. In fact the entire table is built out of just two 4×4 blocks, X and Y where X is the group table for H and Y is obtained from X by adding 4 to each entry.

X	Y
Y	X

Note too that the pattern of these blocks is that of the group table for K. Noticing these patterns we can make very short work of completing the group table for G. (It's made even easier if we do the whole job in a word processor where we can cut and paste!) Patterns like this always occur whenever we have a direct product.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	2	3	3	5	6	7	8
2	2	3	4	1	6	7	8	5
3	3	4	1	2	7	8	5	6
4	4	1	2	3	8	5	6	7
5	5	6	7	8	1	2	3	3
6	6	7	8	5	2	3	4	1
7	7	8	5	6	3	4	1	2
8	8	5	6	7	4	1	2	3

Example 22: $G = \langle A, B, C \mid A^4, B^3, C^2, AB = BA, ACAC, BC = CB\rangle$.
Here there are two relations that say that B commutes with A and with C. Apart from these there's no relation connecting B with either A or C. So the group splits into a direct product: $G \cong H \times K$ where $H = \langle A, C \mid A^4, C^2, ACAC\rangle$ and $K = \langle B \mid B^3\rangle$.
We don't need to use Todd-Coxeter on the second factor. The group table for $\langle B \mid B^3\rangle$ is:

	1	**2**	**3**
1	1	2	3
2	2	3	1
3	3	1	2

Applying the Todd-Coxeter algorithm to $H = \langle A, C \mid A^4, C^2, ACAC\rangle$ we get:

1AAAA1→2AAA1→4AA1→6A1	X
1CC1→3C1	X
1ACAC1→2CAC1→2CA3→5A3	X
2AAAA2→2AAA1	=
2CC2→5C2	X
2ACAC2→4CAC2→4CA5→7A5	X
3AAAA3→3AAA5→3AA7→8A7	X
3CC3→3C1	=
3ACAC3→3ACA1→3AC6→8C6	X
4AAAA4→4AAA2→4AA1	=
4CC4→7C4	X
4ACAC4→6CAC4→6CA7→6C8	X
5AAAA5→3AAA5	=
5CC5→2C5	=
5ACAC5→3CAC5→1AC5→2C5	=
6AAAA6→1AAA6→2AA6→4A6	=
6CC6→8C6	=
6ACAC6→1CAC6→3AC6	=
7AAAA7→5AAA7→3AA7→8A7	=
7CC7→4C7	=
7ACAC7→5CAC7→2AC7→4C7	=
8AAAA8→7AAA8→5AA8→3A8	=
8CC8→6C8	=
8ACAC8→7CAC8→4AC8→6C8	=

	A	**C**	
1	(2)	(3)	1
2	(4)	(5)	A
3	(8)	1	C
4	(6)	(7)	2A
5	3	2	2C
6	1	8	4A
7	5	4	4C
8	7	6	3A

Notice that we departed from the default choice strategy a little in that we didn't define 6 as 3A. We defined 6 as 4A because we had a short chain that could be contracted to a link by defining it in this way. The group table for H is:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
	1	A	C	2A	2C	4A	4C	3A
1	1	2	3	4	5	6	7	8
2	2	4	5	6	7	1	8	3
3	3	8	1	7	6	5	4	2
4	4	6	7	1	8	2	3	5
5	5	3	2	8	1	7	6	4
6	6	1	8	2	3	4	5	7
7	7	5	4	3	2	8	1	6
8	8	7	6	5	4	3	7	1

The elements of $G = H \times K$ are

$$(1, 1), (2, 1), \ldots , (8, 1), (1, 2), (2, 2), \ldots , (2, 8), (3, 1), \ldots , (3, 8).$$

Let's code these as follows:

1 = (1, 1)	9 = (1, 2)	17 = (1, 3)
2 = (2, 1)	10 = (2, 2)	18 = (2, 3)
3 = (3, 1)	11 = (3, 2)	19 = (3, 3)
4 = (4, 1)	12 = (4, 2)	20 = (4, 3)
5 = (5, 1)	13 = (5, 2)	21 = (5, 3)
6 = (6, 1)	14 = (6, 2)	22 = (6, 3)
7 = (7, 1)	15 = (7, 2)	23 = (7, 3)
8 = (8, 1)	16 = (8, 2)	24 = (8, 3)

When we calculate the group table for G we find that it can be built up as a 24 × 24 table from three 8 × 8 blocks:

X	Y	Z
Y	Z	X
Z	X	Y

The block X is the group table for H, the block Y is obtained from this by adding 8 to each of the entries in X, and Z is obtained from X by adding 16 to each entry. Notice that the pattern for these blocks follows that for the group table for K.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	2	4	5	6	7	1	8	3	10	12	13	14	15	9	16	11	18	20	21	22	23	17	24	19
3	3	8	1	7	6	5	4	2	11	16	9	15	14	13	12	10	19	24	17	23	22	21	20	18
4	4	6	7	1	8	2	3	5	12	14	15	9	16	10	11	13	20	22	23	17	24	18	19	21
5	5	3	2	8	1	7	6	4	13	11	10	16	9	15	14	12	21	19	18	24	17	23	22	20
6	6	1	8	2	3	4	5	7	14	9	16	10	11	12	13	15	22	17	24	18	19	20	21	23
7	7	5	4	3	2	8	1	6	15	13	12	11	10	16	9	14	23	21	20	19	18	24	17	22
8	8	7	6	5	4	3	7	1	16	15	14	13	12	11	15	9	24	23	22	21	20	19	23	17
9	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	1	2	3	4	5	6	7	8
10	10	12	13	14	15	9	16	11	18	20	21	22	23	17	24	19	2	4	5	6	7	1	8	3
11	11	16	9	15	14	13	12	10	19	24	17	23	22	21	20	18	3	8	1	7	6	5	4	2
12	12	14	15	9	16	10	11	13	20	22	23	17	24	18	19	21	4	6	7	1	8	2	3	5
13	13	11	10	16	9	15	14	12	21	19	18	24	17	23	22	20	5	3	2	8	1	7	6	4
14	14	9	16	10	11	12	13	15	22	17	24	18	19	20	21	23	6	1	8	2	3	4	5	7
15	15	13	12	11	10	16	9	14	23	21	20	19	18	24	17	22	7	5	4	3	2	8	1	6
16	16	15	14	13	12	11	15	9	24	23	22	21	20	19	23	17	8	7	6	5	4	3	7	1
17	17	18	19	20	21	22	23	24	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
18	18	20	21	22	23	17	24	19	2	4	5	6	7	1	8	3	10	12	13	14	15	9	16	11
19	19	24	17	23	22	21	20	18	3	8	1	7	6	5	4	2	11	16	9	15	14	13	12	10
20	20	22	23	17	24	18	19	21	4	6	7	1	8	2	3	5	12	14	15	9	16	10	11	13
21	21	19	18	24	17	23	22	20	5	3	2	8	1	7	6	4	13	11	10	16	9	15	14	12
22	22	17	24	18	19	20	21	23	6	1	8	2	3	4	5	7	14	9	16	10	11	12	13	15
23	23	21	20	19	18	24	17	22	7	5	4	3	2	8	1	6	15	13	12	11	10	16	9	14
24	24	23	22	21	20	19	23	17	8	7	6	5	4	3	7	1	16	15	14	13	12	11	15	9

EXERCISES FOR CHAPTER 5

NOTE: When asked to apply the Todd-Coxeter Algorithm, you are only expected to use the Restricted version.

EXERCISE 1: Apply the Todd-Coxeter algorithm to find the order of the group:

$$\langle A, B | A^2, B^3, (BA)^3 \rangle.$$

EXERCISE 2: Apply the Todd-Coxeter algorithm to find the order of the group

$$\langle A, B | A^5 = 1, A^3 = B^2 \rangle.$$

[**HINT:** Use the strategy of example 17.]

EXERCISE 3: Apply the Todd-Coxeter algorithm to construct the group table for

$$G = \langle A, B, C \mid A^2, B^2, C^3, CBCB, AB = BA, AC = CA \rangle.$$

[**HINT:** Write this as the direct product of two groups, one of which is the cyclic group of order 2. Use the Todd-Coxeter only for the other group. Having got the group tables for these two smaller groups put them together in the manner of Example 19.]

EXERCISE 4: Apply the Todd-Coxeter algorithm to the group

$$\langle A, B | A^4, B^2, BABA^2 \rangle.$$

Whatever choices you make you will get contradictions. These can be resolved by identifying codes (the full algorithm) but, using a little group theory you can identify this group.
MORAL: Sometimes using the Todd-Coxeter Algorithm is like using a sledgehammer to crack a nut. Before embarking on it play with the presentation a little to see if it can be simplified.

SOLUTIONS FOR CHAPTER 5

EXERCISE 1:

1AA1→2A1	X
1BBB1→3BB1→6B1	X
1BABABA1→1BABAB2→3ABAB2→5BAB2→5BA8→9A8	X
2AA2→1A2	=
2BBB2→4BB2→8B2	X
2BABABA2→2BABAB1→4ABAB1→4ABA6→7BA6→7B10	X
3AA3→5A3	X
3BBB3→3BB1	=
3BABABA3→3BABAB5→6ABAB5→10BAB5→10BA11 →12A11	X
4AA4→7A4	X
4BBB4→4BB2	=
4BABABA4→4BABAB7→8ABAB7→9BAB7→11AB7→12B7	=
5AA5→3A5	=
5BBB5→9BB5→11B5	X
5BABABA5→5BABAB3→5BABA1→5BAB2	=
6AA6→10A6	X
6BBB6→1BB6→3B6	=
6BABABA6→1ABABA6→2BABA6→4ABA6	=
7AA7→4A7	=
7BBB7→10BB7→12B7	X
7BABABA7→7BABAB4→7BABA2→7BAB1→7BA6	=
8AA8→8A9	X
8BBB8→2BB8→4B8	=
8BABABA8→2ABABA8→1BABA8→3ABA8→5BA8	=
9AA9→8A9	=
9BBB9→9BB5	=
9BABABA9→9BABAB8→9BABA4→9BAB7	=
10AA10→6A10	=
10BBB10→10BB7	=
10BABABA10→10BABAB6→10BABA3→10BAB5→10BA11	=
11AA11→	
11BBB11→5BB11→9B11	=
11BABABA11→5ABABA11→3BABA11→6ABA11→6AB12 →10B12	=
12AA12→11A12	X
12BBB12→7BB12→10B12	=
12BABABA12→7ABABA12→4BABA12→8ABA12→9BA12 →11A12	=

	A	**B**	
1	(2)	(3)	
2	1	(4)	A
3	(5)	(6)	B
4	(7)	(8)	2B
5	3	(9)	3A
6	(10)	1	3B
7	4	10	4A
8	9	2	4B
9	8	(11)	5B
10	6	(12)	6A
11	12	5	9B
12	11	7	10B

EXERCISE 2:

1AAAAA1→2AAAA1→3AAA1→4AA1→5A1	X
1AAAbb1→2AAbb1→3Abb1→4bb1→4b9	X
2AAAAA2→2AAAA1	=
2AAAbb2→3AAbb2→4Abb2→5bb2→5b10	X
3AAAAA3→3AAAA2	=
3AAAbb3→4AAbb3→5Abb3→1bb3→6b3	X
4AAAAA4→4AAAA3→4AAA2→4AA1	=
4AAAbb4→5AAbb4→1Abb4→2bb4→2b7	X
5AAAAA5→1AAAA5→2AAA5→3AA5→4A5	=
5AAAbb5→1AAbb5→2Abb5→3bb5→8b5	X
6AAAAA6→7AAAA6→8AAA6→9AA6→10A6	X
6AAAbb6→6AAAb1→7AAb1→8Ab1→9b1	X
7AAAAA7→7AAAA6→8AAA6→	=
7AAAbb7→8AAbb7→9Abb7→10bb7→10b2	X
8AAAAA8→8AAAA7→8AAA6	=
8AAAbb8→9AAbb8→10Abb8→6bb8→3b8	X
9AAAAA9→9AAAA8→9AAA7→9AA6	=
9AAAbb9→9AAAb4→10AAb4→6Ab4→7b4	X
10AAAAA10→6AAAA10→7AAA10→8AA10→9A10	=
10AAAbb10→6AAbb10→7Abb10→8bb10→5b10	=

	A	**B**	**b**	
1	(2)	9	(6)	
2	(3)	10	7	A
3	(4)	6	8	2A
4	(5)	7	9	3A
5	1	8	10	4A
6	(7)	1	3	B
7	(8)	2	4	6A
8	(9)	3	5	7A
9	(10)	4	1	8A
10	6	5	2	9A

EXERCISE 3: Since A commutes with both B and C and, apart from the commuting relations the other relations involve disjoint sets of generators, we can write the group as:

$$\langle A|A^2\rangle \times \langle B, C \mid B^2, C^3, CBCB\rangle.$$

The first group is simply the cyclic group of order 2. We now use the Todd-Coxeter algorithm on the second group.

1BB1→2B1	X
1CCC1→3CC1→6C1	X
1CBCB1→1CBC2→3BC2→5C2	X
2BB2→1B2	=
2CCC2→4CC2→4C5	X
2CBCB2→2CBC1→4BC1→4B6	X
3BB3→5B3	X
3CCC3→3CC1	=
3CBCB3→3CBC5→6BC5→6B4	X
4BB4→6B4	=
4CCC4→4CC2	=
4CBCB4→5BCB4→3CB4→6B4	=
5BB5→3B5	=
5CCC5→2CC5→4C5	=
5CBCB5→2BCB5→1CB5→3B5	=
6BB6→4B6	=
6CCC6→1CC6→3C6	=
6CBCB6→1BCB6→2CB6→4B6	=

	B	**C**	
1	(2)	(3)	
2	1	(4)	B
3	(5)	(6)	C
4	6	5	2C
5	3	2	3B
6	4	1	3C

The group table for $\langle B, C \mid B^2, C^3, CBCB\rangle$ is:

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	5	6	2	4	1
4	4	6	5	1	3	2
5	5	3	2	6	1	4
6	6	4	1	5	2	3

The group table for $\langle A \mid A^2\rangle$ is:

	1	**2**
1	1	2
2	2	1

Combining them we get:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1	4	3	6	5	8	7	10	9	12	11
3	3	5	6	2	4	1	9	11	12	8	10	7
4	4	6	5	1	3	2	10	12	11	7	9	8
5	5	3	2	6	1	4	11	9	8	12	7	10
6	6	4	1	5	2	3	12	10	7	11	8	9
7	7	8	9	10	11	12	1	2	3	4	5	6
8	8	7	10	9	12	11	2	1	4	3	6	5
9	9	11	12	8	10	7	3	5	6	2	4	1
10	10	12	11	7	9	8	4	6	5	1	3	2
11	11	9	8	12	7	10	5	3	2	6	1	4
12	12	10	7	11	8	9	6	4	1	5	2	3

EXERCISE 4:

1AAAA1→2AAA1→4AA1→7A1	X
1BB1→3B1	X
1BABAA1→3ABAA1→6BAA1→6BA7→6B4	X
2AAAA2→2AAA1	=
2BB2→5B2	X
2BABAA2→2BABA1→5ABA1→5AB7→6B7	4=7!
3AAAA3→6AAA3→3AA3→6A3	=
3BB3→1B3	=
3BABAA3→1ABAA3→2BAA3→5AA3→5A6	X
4AAAA4→4AAA2→4AA1	=
4BB4→4B6	X
4BABAA4→4BABA2→4BAB1→4BA3→6A3	X
5AAAA5→6AAA5→3AA5→6A5	3=5!
5BB5→2B5	=
5BABAA5→2ABAA5→4BAA5→6AA5→3A5	5=6!
6AAAA6→6AAA3	=

	A	**B**	
1	(2)	(3)	
2	(4)	(5)	A
3	(6)	1	B
4	(7)	6	2A
5	6	2	2B
6	3	4	3A
7	1	6	4A

6BB6→4B6	=
6BABAA6→6BABA3→4ABA3→7BA3→7B6	X
7AAAA7→1AAA7→2AA7→4A7	=
7BB7→6B7	4=7!
7BABAA7→7BABA4→7BAB2→7BA5→6A5	3=5!

If we were using the full Todd-Coxeter Algorithm we'd have to identify 7 with 4 and 6 and 5 with 3 and continue. It would eventually terminate (provided we weren't too silly with our choices). But is there an easier way of identifying this group?

Yes indeed. The relations $A^4 = 1$ and $B^2 = 1$ means that we can write $BABAA = 1$ as $B^{-1}AB = A^2$. Then $B^{-1}A^2B = A^4 = 1$ so $A^2 = 1$. But then $B^{-1}AB = 1$ which gives $A = 1$. So the group is simply $\langle B \mid B^2 \rangle$, the cyclic group of order 2.