《微分几何》第二次课堂练习参考答案

1.$(\mathbf{15'})$ 证明：必要性：设球心为P_0 的球面：$X(u,v)-P_0=tn(u,v)$，t是正常数，而每点处的法线为

$$a(s)=X(u,v)+sn(u,v)$$

显然法线都经过固定点$P_0$$(\mathbf{5'})$。

充分性：由于曲面$X(u,v)$上每点处的法线通过一个定点P_0，则有

$$X(u,v)-P_0=\lambda(u,v)n(u,v)(\mathbf{2'})$$

求导可得

$$X_u=\lambda_u n(u,v)+\lambda n_u,$$
$$X_v=\lambda_v n(u,v)+\lambda n_v,(\mathbf{2'})$$

对以上两式点乘n可得

$$X_u\cdot n=\lambda_u n\cdot n+\lambda n_u\cdot n,$$
$$X_v\cdot n=\lambda_v n\cdot n+\lambda n_v\cdot n,(\mathbf{2'})$$

从而

$$\lambda_u=\lambda_v=0\Longrightarrow\lambda\text{是常数。}(\mathbf{2'})$$

于是

$$X(u,v)-P_0=\lambda n(u,v)\Longrightarrow(X(u,v)-P_0)^2=\lambda^2.$$

此时曲面为以P_0为球心，λ为半径的球面（一部分）。$(\mathbf{2'})$

2.证明：(10’)(1)设曲面$X(x,y)=(x,y,f(x,y))$, 于是
$X_1=(1,0,f_1)$, $X_2=(0,1,f_2)$, $n=\frac{X_1\times X_2}{|X_1\times X_2|}=\frac{1}{\sqrt{1+f_1^2+f_2^2}}(-f_1,-f_2,1)$.
$X_{11}=(0,0,f_{11})$, $X_{12}=(0,0,f_{12})$, $X_{22}=(0,0,f_{22})$.
于是

$$g_{11}=1+f_1^2,\ \ g_{12}=f_1f_2,\ \ g_{22}=1+f_2^2.(1'+1'+1')$$

$$h_{11}=\frac{f_{11}}{\sqrt{1+f_1^2+f_2^2}},,\ \ h_{12}=\frac{f_{12}}{\sqrt{1+f_1^2+f_2^2}},\ \ h_{22}=\frac{f_{22}}{\sqrt{1+f_1^2+f_2^2}}(1'+1'+1')$$

$$K=\frac{det(h_{\alpha\beta})}{detg_{\alpha\beta}}=\frac{h_{11}h_{22}-h_{12}^2}{g_{11}g_{22}-g_{12}^2}=\frac{f_{11}f_{22}-f_{12}^2}{(1+f_1^2+f_2^2)^2}(2')$$

$$H=\frac{1}{2}\frac{g_{11}h_{22}-2g_{12}h_{12}+g_{22}h_{11}}{det(g_{\alpha\beta})}=\frac{1}{2}\frac{(1+f_1^2)f_{22}-2f_1f_2f_{12}+(1+f_2^2)f_{11}}{(1+f_1^2+f_2^2)^{\frac{3}{2}}}(2')$$

(10’)（2）证明：必要性：若曲面为平面，则$H=K=0$, 满足$H^2=K$;
若曲面为半径为r的球面，则$H=\frac{1}{r}$,$K=\frac{1}{r^2}$, 满足$H^2=K$，$(4')$ 下证充分性：由$H^2=K$知, $(k_1+k_2)^2=4k_1k_2\Rightarrow(k_1-k_2)^2=0$, $k_1=k_2=k$, 即M上每一点都是脐点. 在M上取正交参数网, 这时$h_{\alpha\beta}=kg_{\alpha\beta}$, 即$h_\alpha^\beta=k\delta_\beta^\alpha$.

由Gauss-Weingarten公式,

$$\mathbf{n}_\alpha = -h_\alpha^\beta \mathbf{e}_\beta = -k\delta_\alpha^\beta \mathbf{e}_\beta = -k\mathbf{e}_\alpha$$

$$\begin{aligned} \mathbf{n}_{\alpha\gamma} &= -k_\gamma \mathbf{e}_\alpha - k_\alpha \mathbf{e}_\gamma \\ &= -k_\gamma \mathbf{e}_\alpha - k(\Gamma_{\alpha\gamma}^\beta \mathbf{e}_\beta + h_{\alpha\gamma}\mathbf{n}) \\ &= -k_\gamma \mathbf{e}_\alpha - k(\Gamma_{\alpha\gamma}^\beta \mathbf{e}_\beta + k g_{\alpha\gamma}\mathbf{n}) \end{aligned}$$

由于$\mathbf{n}_{\alpha\gamma} = \mathbf{n}_{\gamma\alpha}$, $\Gamma_{\alpha\gamma}^\beta = \Gamma_{\gamma\alpha}^\beta \Rightarrow \alpha$与$\gamma$指标可交换, 即$k_\gamma \mathbf{e}_\alpha = k_\alpha \mathbf{e}_\gamma$.
取$\gamma = 1$, $\alpha = 2$, 则$\frac{\partial k}{\partial u^1}\mathbf{e}_2 = \frac{\partial k}{\partial u^2}\mathbf{e}_1$.
由于$\mathbf{e}_1, \mathbf{e}_2$线性无关$\Rightarrow \frac{\partial k}{\partial u^1} = \frac{\partial k}{\partial u^2} = 0 \Rightarrow k =$常数. **(3′)**

① $k = 0$时, M上的点都是平点$\Rightarrow \mathbf{n}_\alpha = 0$, $\mathbf{n}$是常向量

$$\frac{\partial}{\partial u^\alpha}[(\mathbf{x} - \mathbf{x}_0)\cdot \mathbf{n}] = \mathbf{x}_\alpha \cdot \mathbf{n} = 0 \Rightarrow (\mathbf{x} - \mathbf{x}_0)\cdot \mathbf{n} = \text{常数} = C_1.$$

又$\mathbf{x} = \mathbf{x}_0$时, $C_1 = 0 \Rightarrow (\mathbf{x} - \mathbf{x}_0)\cdot \mathbf{n} = 0$是平面的方程.

② $k \neq 0$. 不妨设$k > 0$

$$\frac{\partial}{\partial u^\alpha}\left(\mathbf{x} + \frac{1}{k}\mathbf{n}\right) = \mathbf{x}_\alpha + \frac{1}{k}(-k\mathbf{x}_\alpha) = 0,\ \mathbf{x} + \frac{1}{k}\mathbf{n} = \text{常向量}\mathbf{b}.$$

$|\mathbf{x} - \mathbf{b}| = |\frac{1}{k}\mathbf{n}| = \frac{1}{k}$, 是球面. **(3′)** □

3.**(15′)**由曲面表达式计算可得$g_{11} = r^2$,$g_{12} = 0$,$g_{22} = (a + rcosu)^2$, $h_{11} = r$,$h_{12} = 0$,$h_{22} = (a + rcosu)cosu$,**(3′)** 所以$\Gamma_{11}^1 = 0$,$\Gamma_{11}^2 = 0$, $\Gamma_{12}^1 = 0$,$\Gamma_{11}^2 = \frac{-rsinu}{a+rcosu}$,$\Gamma_{22}^1 = \frac{(a+rcosu)sinu}{r}$, $\Gamma_{22}^2 = 0$,**(4′)**所以Gauss公式为:

$$x_{11} = r\mathbf{n}, \mathbf{x}_{12} = \frac{-rsinu}{a + rcosu}\mathbf{x}_2, \mathbf{x}_{22} = \frac{(a + rcosu)sinu}{r}\mathbf{x}_1 + (a + rcosu)cosu\mathbf{n},\ \textbf{(4′)}$$

Weingarten 公式为

$$\mathbf{n}_1 = -\frac{1}{r}\mathbf{x}_1, \mathbf{n}_2 = -cosua + rcosu\mathbf{x}_2,\ \textbf{(4′)}$$

4. **(20′)** **证明**（1）曲面为$\mathbf{r}(x^1, x^2) = (x^1, x^2, f(x^1) + g(x^2))$. 计算得

$$g_{11} = 1 + f'^2,\ g_{12} = f'g',\ g_{22} = 1 + g'^2.$$

$$h_{11} = \frac{f''}{1 + f'^2 + g'^2},\ h_{12} = 0,\ h_{22} = \frac{g''}{1 + f'^2 + g'^2}.\textbf{(2′)}$$

由其为极小曲面得$g_{11}h_{11} - 2g_{12}h_{12} + g_{22}h_{11} = 0$, 代入, 有

$$-\frac{f''}{1 + f'^2} = \frac{g''}{1 + g'^2}.\textbf{(3′)}$$

因上面左式是关于x^1的函数, 右式是关于x^2的函数, 故必有

$$-\frac{f''}{1+f'^2}=\frac{g''}{1+g'^2}=a=\text{const}.(\mathbf{3'})$$

得$(\arctan(-f'))'=a$, $f'=-\tan(ax^1+c_1)$, $f=-\frac{1}{a}\ln\cos(ax^1+c_1)+c_2$, 同理$g=\frac{1}{a}\ln\cos(ax^2+c_3)+c_4$. 因此除相差一常数外, $ax^3=\ln\frac{\cos ax^2}{\cos ax^1}$.$(\mathbf{2'})$

（2）设旋转曲面$X(u,v)=(f(u)cosv,f(u)sinv,u)$，直接计算得

$X_u=(f'cosv,f'sinv,1)$，$X_v=(-fsinv,fcosv,0)$，$n=\frac{1}{|f|\sqrt{1+f'^2}}(-fcosv,-fsinv,ff')$,

$X_{uu}=(f''cosv,f''sinv,0)$，$X_{uv}=(-f'sinv,f'cosv,0)$，$X_{vv}=(-fcosv,-fsinv,0)$.

从而

$$g_{11}=f'^2+1,\ \ g_{12}=g_{21}=0,\ \ g_{22}=f^2(\mathbf{2'})$$

$$h_{11}=\frac{-ff''}{|f|\sqrt{1+f'^2}},\ \ h_{12}=h_{21}=0,\ \ h_{22}=\frac{|f|}{\sqrt{1+f'^2}}(\mathbf{2'})$$

则Gauss曲率

$$K=\frac{det(h_{\alpha\beta})}{det(g_{\alpha\beta})}=-\frac{f''}{f(1+f'^2)^2}.(\mathbf{2'})$$

若$K=0$，则有$f''=0$，从而$f=au+b$，其中a,b为常数。$(\mathbf{2'})$ 此时，旋转面为平面或者圆锥面。$(\mathbf{2'})$

5. $(\mathbf{15'})$ **证明** 由题意, $\mathrm{III}=\varphi^2\mathrm{I}$, 代入$\mathrm{III}-2H\mathrm{II}+K\mathrm{I}=0$, 得$2H\mathrm{II}=(\varphi^2+K)\mathrm{I}$.$(\mathbf{5'})$ 若$H=0$, 则曲面为极小曲面;$(\mathbf{5'})$ 若$H\neq 0$, 则$\mathrm{II}=\frac{\varphi^2+K}{2H}\mathrm{I}$, 即为脐点, 从而是球面或平面. 又因平面也是极小曲面, 因此曲面必为球面或极小曲面.$(\mathbf{5'})$

6.$(\mathbf{15'})$ **证明**(1)由测地曲率$k_g=(\dot{T},n,T)$即可得

$$k_g=|(\frac{d^2}{ds^2}x,n,\frac{d}{ds}x)|=|(n,\frac{d}{ds}x,\frac{d^2}{ds^2}x)|(\mathbf{5'})$$

(2)设旋转面$X(u^1,u^2)=(f(u^1)cosu^2,f(u^1)sinu^2,u^1)$，则$X_1=(f'(u^1)cosu^2,f'(u^1)sinu^2,1)$，$X_2=(-f(u^1)sinu^2,f(u^1)cosu^2,0)$，$X_{11}=(f''(u^1)cosu^2,f''(u^1)sinu^2,0)$，$X_{22}=(-f(u^1)cosu^2,-f(u^1)sinu^2,0)$，$(\mathbf{2'})$ 从而$n=\frac{1}{|f|\sqrt{f'^2+1}}(-fcosu^2,-fsinu^2,ff')(\mathbf{2'})$

由于u^1曲线称为经线，u^2 曲线称为纬线，$(\mathbf{2'})$ 故

经线的表达式为$a(u^1)=X(u^1,u_0^2)=(f(u^1)cosu_0^2,f(u^1)sinu_0^2,u^1)$

纬线的表达式为$b(u^2)=X(u_0^1,u^2)=(f(u_0^1)cosu^2,f(u_0^1)sinu^2,u_0^1)$

计算得

$T_a=X_1\frac{du^1}{ds}$，$\dot{T}_a=X_{11}(\frac{du^1}{ds})^2+X_1\frac{d^2u^1}{ds^2}$，故

经线的测地曲率$k_g=|(n,T_a,\dot{T}_a)|=|(n,X_1\frac{du^1}{ds},X_{11}(\frac{du^1}{ds})^2+X_1\frac{d^2u^1}{ds^2})|=|(n,X_1\frac{du^1}{ds},X_{11}(\frac{du^1}{ds})^2)|=0$，即经线为测地线。$(\mathbf{2'})$

类似的，纬线的测地曲率$k_g=|(n,X_2\frac{du^2}{ds},X_{22}(\frac{du^2}{ds})^2)|=\frac{f'}{f\sqrt{1+f'^2}}$。$(\mathbf{2'})$