REAL ANALYSIS
LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books for *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

Part 2. Measure Theory

Given an interval $\langle a, b\rangle \subset \mathbb{R}$, where $\langle , \rangle$ means it can be open or closed at the endpoints, the length of the interval is given by $\ell(\langle a, b\rangle) = b - a$. Allowing $a, b = \pm\infty$, we also define the length of unbounded interval to be ∞. The length function gives us a way to measure the size of subsets in $\mathbb{R}$. One may hope to extend the length function ℓ to be defined on $2^{\mathbb{R}}$. You would expect the following natural properties:

(i) $\ell : 2^{\mathbb{R}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$;
(ii) $\ell(\langle a, b\rangle) = b - a$ for all $a, b \in \mathbb{R}$;
(iii) $\ell(E + r) = \ell(E)$ for all $r \in \mathbb{R}$ and $E \subset \mathbb{R}$;
(iv) $\ell(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \ell(E_k)$, provided E_k are disjoint subsets of $\mathbb{R}$.

Surprisingly, such length function ℓ does not exists.

We start by assuming the existence of a function ℓ satisfying properties (i)-(iv), and end up with a contradiction by the axiom of choice.

Firstly we define an equivalence in $[0, 1]$: $x, y \in [0, 1]$ are equivalent if $x - y \in \mathbb{Q}$. We write $x \sim y$. One can verify $x \sim x$; $x \sim y \Longrightarrow y \sim x$; and $x \sim y$ & $y \sim z \Longrightarrow x \sim z$. It follows that two equivalence classes are either identical or disjoint. Hence we can write

$$[0, 1] = \bigcup_{\lambda \in \Lambda} E_\lambda, \tag{0.1}$$

where E_λ are disjoint equivalence classes. Note that each E_λ is countable and so Λ is uncountable.

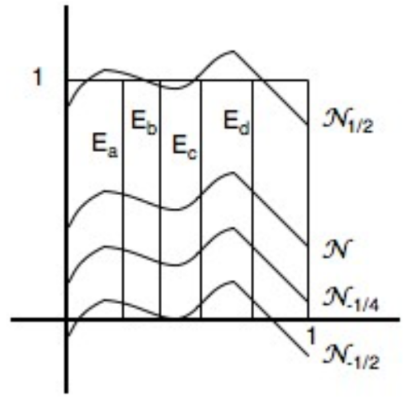

FIGURE 1. Construction of a non measurable set

Let $\mathcal{N}$ be a set that contains exactly one representative from each equivalent class. (This requires the axiom of choice). Given $r \in \mathbb{Q} \cap [-1, 1]$, let $\mathcal{N}_r = \mathcal{N} + r = \{x + r : x \in \mathcal{N}\}$. We claim

(a) $\mathcal{N}_r$ are disjoint;
(b) $[0, 1] \subset \bigcup_{r \in \mathbb{Q} \cap [-1,1]} \mathcal{N}_r \subset [-1, 2]$.

Once this is proved, we deduce by the required properties (ii), (iii) and (iv) that

$$1 = \ell([0, 1]) \leq \sum \ell(\mathcal{N}_r) = \sum \ell(\mathcal{N}) \leq \ell([-1, 2]) = 3,$$

which is impossible. We have used $\ell(E) \leq \ell(F)$ if $E \subset F$. This follows by (iv), as

$$\ell(F) = \ell(E \cup (F \setminus E)) = \ell(E) + \ell(F \setminus E) \geq \ell(E).$$

We show the claim. For (a), suppose $x \in \mathcal{N}_{r_1} \cap \mathcal{N}_{r_2}$. Then $x = \alpha + r_1 = \beta + r_2$ for some $\alpha, \beta \in \mathcal{N}$. Hence $\alpha \sim \beta$. Since we only choose exactly one element from each equivalent class, we obtain $\alpha = \beta$. Therefore $r_1 = r_2$ and so $\mathcal{N}_{r_1}$ coincides with $\mathcal{N}_{r_2}$.

For (b), the second inclusion is obvious. We prove the first inclusion in (b). Given $x \in [0, 1]$, by (0.1) and our construction of $\mathcal{N}$, there is $x' \in \mathcal{N}$ such that $x \sim x'$. Hence $x = x' + r'$ for some $r' \in \mathbb{Q}$. Since $r' = x - x' \in [-1, 1]$, we conclude $x \in \bigcup_{r \in \mathbb{Q} \cap [-1,1]} \mathcal{N}_r$.

The above discussion shows that we cannot expect all the properties (i)-(iv) when we extend the conception of length/area/volume that will be called the measure. We hope the measure satisfies properties (ii)-(iv) which are so natural and needed in the theory, and so can only define this notion for some sets in $\mathbb{R}^n$ rather than all sets.

In next section, we first define the exterior measure for all subsets in $\mathbb{R}^n$, denoted by $m_* : 2^{\mathbb{R}^n} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. We will use the (volume of) rectangles to build up the exterior measure, so that it satisfies $m_*(R) = |R|$ for all rectangles R [1]. The exterior measure of general sets is defined by an approximation procedure via rectangles. One shall see that m_* satisfies the translation invariant as presented by the aforementioned property (iii). However, by our discussion, m_* cannot satisfy the property (iv). The sets for which property (iv) holds are called measurable sets. Let us denote the collection of measurable sets by $\mathscr{M}_{\mathbb{R}^n}$. The measure m is then defined among $\mathscr{M}_{\mathbb{R}^n}$, and for each $E \in \mathscr{M}_{\mathbb{R}^n}$, we assign $m(E) = m_*(E)$.

We present some simple results for rectangles. The volume of the rectangle $R = \prod_{i=1}^n \langle a_i, b_i \rangle$, denoted by $|R|$ is defined to be

$$|R| = \prod_{i=1}^{n} (b_i - a_i).$$

Lemma 0.1. *If a rectangle is the almost disjoint union of finitely many other rectangles, say $R = \bigcup_{k=1}^N R_k$, then*

$$|R| = \sum_{k=1}^{N} |R_k|.$$

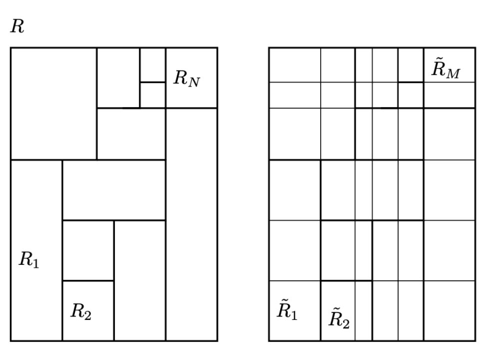

FIGURE 2. The grid formed by the rectangle R_k

Proof. The ingredients of the proof are given below.

[1] Hence m_* satisfies the (high dimensional version of) property (ii) as described above.

(i) The result is direct if the union happens to be obtained by partitioning each side $\langle a_i, b_i \rangle$.

(ii) By extending all edges of all R_k to the edges of R, one obtains a partition $R = \bigcup_j \tilde{R}_k$ for which (i) is true.

(iii) Each R_k is also of the form (i) for a subset of the $\tilde{R}_j$ indexed by $j \in J_k$.

(iv) This gives

$$
\begin{aligned}
|R| &= \sum_j |\tilde{R}_j| && \text{by (ii)} \\
&= \sum_k \sum_{j \in J_k} |\tilde{R}_j| && \text{by rearranging} \\
&= \sum_{k=1}^{N} |R_k|. && \text{by (iii)}
\end{aligned}
\tag{0.2}
$$

□

Lemma 0.2. *If $R, R_1, \ldots, R_N$ are rectangles, and $R \subset \bigcup_{k=1}^N R_k$, then $|R| \leq \sum_{k=1}^N |R_k|$.*

Proof. Extend given rectangles $\{R_k\}_{k=1}^N$ as before, obtaining new rectangles $\tilde{R}_j$. The new "maximal" rectangle contains R, and each $\tilde{R}_j$ may be a subset of more than on R_k. Hence "$=$" should be replaced "$\leq$" in the first and third lines in (0.2).

□

1. Exterior measure

Loosely speaking, the exterior measure m_* assigns to any subset of $\mathbb{R}^n$ a notion of size, which coincides with some of our intuition; however the exterior measure lacks the desirable property of additivity when taking the union of disjoint sets. We remedy this problem later on by restricting our consideration to the measurable sets.

The exterior measure attempts to describe the volume of a set E by approximating it from the outside. The set E is covered by cubes, and if the covering gets finer, with fewer cubes overlapping, the volume of E should be close to the sum of the volumes of the cubes.

Definition 1.1. *For $E \subset \mathbb{R}^n$, the exterior measure of E is*

$$m_*(E) = \inf \left\{ \sum_{j=1}^{\infty} |Q_j| : \; E \subset \bigcup_{j=1}^{\infty} Q_j, \; Q_j \textit{ are closed cubes} \right\}.$$

Remark 1.1. *We make some preliminary remarks.*

(i) *Each cover of E can be finite or countably infinite, by allowing $Q_j = \emptyset$ and defining $|\emptyset| = 0$.*
(ii) *The quantity that would be obtained if one considered only covering of E by finite unions of cubes is in general larger than $m_*(E)$, which would not suffice to our purpose.*
(iii) *One can use (even open) rectangles instead of cubes. This yields the same exterior measure.*
(iv) *If Q is a closed cube then $m_*(Q) = |Q|$.*
(v) *If Q is a cube (not necessarily closed) then $m_*(Q) = |Q|$.*
(vi) *If R is a rectangle (not necessarily closed), then $m_*(R) = |R|$.*
(vii) *$m_*(E) \in [0, \infty]$; $m_*(\emptyset) = 0$; $m_*(\mathbb{R}^n) = \infty$.*

Example 1.1. *If $E \subset \mathbb{R}^n$ is a countable set, then $m_*(E) = 0$.*

Proof. Let $E = \{r_k\}_{k\geq 1}$ and consider the coverings $\left\{ \prod_{i=1}^n [r_{k,i} - \frac{\varepsilon}{2^{(k+1)/n}}, r_{k,i} + \frac{\varepsilon}{2^{(k+1)/n}}] \right\}_{k\geq 1}$, where $r_{k,i}$ are the i-th component of r_k. □

Example 1.2. *The Cantor set $\mathcal{C}$ has exterior measure zero.*

Proof. From the construction of $\mathcal{C}$, we know that $\mathcal{C} \subset C_k$, where each C_k is a disjoint union of 2^k closed intervals, each of length 3^{-k}. Hence $m_*(\mathcal{C}) \leq (2/3)^k$ for all k. Thus $m_*(\mathcal{C}) = 0$. □

1.1. **Properties of the exterior measure.**

We turn to the further study of m_* and prove five properties of exterior measure that are needed in what follows. Basically everything you might expect, except finite and countable additivity for disjoint unions, is true for exterior measure.

Observation 1 (Monotonicity) If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.

This is because any covering of E_2 is a covering of E_1. In particular monotonicity implies that every bounded subset of $\mathbb{R}^n$ has finite exterior measure.

Observation 2 (Countable sub-additivity) If $E = \bigcup_{j=1}^\infty E_j$, then $m_*(E) \leq \sum_{j=1}^\infty m_*(E_j)$.

Proof. Assume $m_*(E_j) < \infty$. Otherwise we are done.

Let $\varepsilon > 0$. For each j we take a covering $E_j \subset \bigcup_{k=1}^{\infty} Q_{j,k}$ by closed cubes with

$$\sum_{k=1}^{\infty} |Q_{j,k}| \leq m_*(E_j) + \frac{\varepsilon}{2^j}.$$

Clearly $\{Q_{j,k}\}_{k,j=1}^{\infty}$ is a covering of E. Hence

$$m_*(E) \leq \sum_{j,k} |Q_{k,j}| = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Q_{j,k}| \leq \sum m_*(E_j) + \varepsilon.$$

This completes the proof by sending $\varepsilon \to 0$.

$\square$

Observation 3 If $E \subset \mathbb{R}^n$, then $m_*(E) = \inf\{m_*(\mathcal{O}) : \ \mathcal{O} \text{ is open and contains } E\}$.

Proof. The direction "$\leq$" is by the monotonicity (Property 1).

We show "$\geq$" below. Let $\varepsilon > 0$. We take a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes such that

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \varepsilon/2. \tag{1.1}$$

Enlarge Q_j a little to an open cube Q_j^o such that $Q_j \subset Q_j^o$ and $|Q_j^o| \leq |Q_j| + \frac{\varepsilon}{2^{j+1}}$. Take $\mathcal{O} = \bigcup_{j=1}^{\infty} Q_j^o$. Then by the countable sub-additivity (Property 2)

$$m_*(\mathcal{O}) \leq \sum_{j=1}^{\infty} |Q_j^o| \leq \frac{\varepsilon}{2} + \sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \varepsilon,$$

where we used (1.1) in the last inequality. This yields the proof by letting $\varepsilon \to 0$.

$\square$

Observation 4 (Additivity for positively separated sets) If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$.

Proof. By the sub-additivity (Property 2), we have "$\leq$".

Given $\varepsilon > 0$, let $\{Q_j\}_{j=1}^{\infty}$ be a family of closed cubes such that $E \subset \bigcup_{j=1}^{\infty} Q_j$ and

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \varepsilon. \tag{1.2}$$

Let $\delta > 0$ such that $d(E_1, E_2) > \delta$. By subdividing the cubes Q_j, we assume without loss of generality diam$Q_j < \delta$ for all j. Hence each Q_j can intersect at most one of the two sets E_1 and E_2. Let $J_i = \{j : Q_j \cap E_j \neq \emptyset\}$, $i = 1, 2$. Then

$$E_1 \subset \bigcup_{j \in J_1} Q_j \quad \text{and} \quad E_1 \subset \bigcup_{j \in J_2} Q_j,$$

and $J_1 \cap J_2 = \emptyset$. Consequently, using (1.2),

$$m_*(E_1) + m_*(E_2) \leq \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| \leq \sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \varepsilon.$$

We complete the proof by sending $\varepsilon \to 0$.

$\square$

Observation 5 (Additivity for almost disjoint cubes) If a set E is the countable union of almost disjoint cubes $E = \bigcup_{j=1}^{\infty} Q_j$, then $m_*(E) = \sum_{j=1}^{\infty} |Q_j|$.

Proof. By sub-additivity (Property 2), we have "$\leq$".

Let $\tilde{Q}_j \subset\subset Q_j$ such that $|Q_j| \leq |\tilde{Q}_j| + \varepsilon/2^j$. Then for all N, $\tilde{Q}_1, \dots, \tilde{Q}_N$ are at a finite positive distance from one another. By virtue of Property 4,

$$m_*(\bigcup_{j=1}^{N} \tilde{Q}_j) = \sum_{j=1}^{N} |\tilde{Q}_j| \geq \sum_{j=1}^{N} |Q_j| - \varepsilon.$$

Since $\bigcup_j \tilde{Q}_j \subset E$, it then follows by the monotonicity (Property 1),

$$m_*(E) \geq m_*(\bigcup_{j=1}^{N} \tilde{Q}_j) \geq \sum_{j=1}^{N} |Q_j| - \varepsilon,$$

which yields the desired result by letting $N \to \infty$ and $\varepsilon \to 0$.

$\square$

Remark 1.2. *By the construction of open sets, the exterior measure of an open set equals the sum of the volumes of the cubes in a decomposition. Moreover, this yields a proof that the sum is independent of the decomposition.*

Despite Observations 4 & 5, one cannot conclude in general that if $E_1 \cup E_2$ is a disjoint union of subsets of $\mathbb{R}^n$, then $m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$.