Probability Theory

Exercise Sheet 1

Exercise 1.1 Let $(\Omega, \mathcal{A}, P) = \Big((0,1), \mathcal{R}, \mu\Big)$, where μ is the Lebesgue measure over $(0,1)$ and $\mathcal{R}$ the Borel σ-algebra on $(0,1)$. Find the distribution function of the random variable

$$X(\omega) := \frac{1}{\lambda} \log \frac{1}{1-\omega}$$

where λ is a given positive parameter.

Exercise 1.2 Let $\mathcal{Z} := (A_i)_{i \in I}$ be a countable decomposition of a set $\Omega \neq \emptyset$ in "atoms" A_i, that is $\Omega = \bigcup_{i \in I} A_i$, where $A_i \cap A_k = \emptyset$ for $i \neq k$, and I countable.

(a) Show that the σ-algebra generated by $\mathcal{Z}$ is of the form

$$\sigma(\mathcal{Z}) = \left\{ \bigcup_{i \in J} A_i \,\middle|\, J \subseteq I \right\}.$$

Hint: Recall the definition of $\sigma(\mathcal{Z})$.

(b) Show that the family of $\sigma(\mathcal{Z})$-measurable random variables is exactly the family of functions on Ω that are constant on "atoms" (that is, all functions f such that for each i, f is constant on A_i).

Exercise 1.3 Let Ω be a non-empty set and let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be two functions. The σ-algebra on Ω generated by X is defined by $\sigma(X) := \Big\{ X^{-1}(B) \mid B \in \mathcal{R} \Big\}$, where $\mathcal{R}$ denotes the Borel σ-algebra on $\mathbb{R}$. In this exercise we will show that:
Claim: Y is $\sigma(X)$-$\mathcal{R}$-measurable $\Longleftrightarrow$ there exists an $\mathcal{R}$-$\mathcal{R}$-measurable function $f : \mathbb{R} \to \mathbb{R}$, such that $Y = f \circ X$.

Hint: For (b)–(e), cf. the proof of (1.2.16) in the lecture notes.

(a) Show the $\Longleftarrow$ direction.

(b) Show the $\Longrightarrow$ direction for any Y of the form $Y = 1_A$, where $A \in \sigma(X)$.

(c) Show the $\Longrightarrow$ direction for any Y that is a linear combination of indicator functions, i.e. for Y of the form $Y = \sum_{i=1}^{n} c_i 1_{A_i}$, where $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \sigma(X)$.

(d) Show the $\Longrightarrow$ direction for any Y such that $Y \geq 0$.

(e) Complete the proof of the claim (i.e. show the $\Longrightarrow$ direction for an arbitrary Y).

Submission: until 14:15, Oct 1., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
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Solution 1.1 Note that $X(\omega) > 0$ for all $\omega \in \Omega = (0,1)$. Thus we obtain that for $y \leq 0$,

$$F_X(y) \overset{\text{Def.}}{=} P[X \leq y] = 0.$$

For $y \geq 0$ we have that

$$\begin{aligned} F_X(y) &\overset{\text{Def.}}{=} P[X \leq y] = P\left[\frac{1}{\lambda} \log \frac{1}{1-\omega} \leq y\right] \\ &= P\left[1 - \omega \geq e^{-\lambda y}\right] \\ &= P\left[\omega \leq 1 - e^{-\lambda y}\right] \\ &= 1 - e^{-\lambda y}, \end{aligned}$$

because $P := \mu$ is the Lebesgue measure over $(0,1)$. Hence X has the Exp(λ)-distribution.

Solution 1.2

(a) By definition, $\sigma(\mathcal{Z})$ is the smallest σ-algebra that contains all A_i, $i \in I$, i.e.,

$$\sigma(\mathcal{Z}) := \bigcap_{\substack{\mathcal{U}\,:\,\mathcal{U} \text{ is a} \\ \sigma\text{-algebra} \\ \text{containing all } A_i}} \mathcal{U}. \tag{1}$$

We now show that $\sigma(\mathcal{Z}) = \left\{ \bigcup_{i \in J} A_i \middle| J \subseteq I \right\}$:

"$\supseteq$" For any σ-algebra $\mathcal{U}$ that contains all A_i it holds that:

$$\bigcup_{i \in J} A_i \in \mathcal{U}, \qquad J \subseteq I,$$

since J, being a subset of I, is countable, and σ-algebras are closed under countable unions by definition. Therefore, we have that

$$\sigma(\mathcal{Z}) \overset{(1)}{=} \bigcap \mathcal{U} \supseteq \left\{ \bigcup_{i \in J} A_i \middle| J \subseteq I \right\}.$$

"$\subseteq$" Since $\mathcal{U}$ contains all A_i, it is sufficient to show that

$$\mathcal{U} = \left\{ \bigcup_{i \in J} A_i \middle| J \subseteq I \right\}$$

is a σ-algebra. We verify the conditions:

- $\bigcup_{i \in J} A_i = \Omega$, by choosing $J = I$, so $\Omega \in \mathcal{U}$,
- for any $J \subset I$, $\left(\bigcup_{i \in J} A_i\right)^c = \bigcup_{i \in I \setminus J} A_i \in \mathcal{U}$,

- if $J_n \subseteq I$, $n \geq 1$, then

$$\bigcup_{n\geq 1}\left(\bigcup_{i\in J_n} A_i\right) = \bigcup_{i\in \underbrace{\bigcup_{n\geq 1} J_n}_{=:J\subseteq I}} A_i \in \mathcal{U}.$$

(b) Let

$$F_1 := \{f : \Omega \to \mathbb{R} \,|\, f \text{ is } \sigma(\mathcal{Z})\text{-measurable}\} \text{ and}$$
$$F_2 := \{f : \Omega \to \mathbb{R} \,|\, f \text{ is constant on } A_i, i \in I\}.$$

We want to show that $F_1 = F_2$:

"$\supseteq$" Let $f \in F_2$. Then we can write

$$f(x) = a_i \text{ for } x \in A_i,$$

for some $a_i \in \mathbb{R}$. To check that f is $\sigma(\mathcal{Z})$-measurable, it suffices to check that $\{x \in \Omega : f(x) \leq a\}$ is a measurable set for all $a \in \mathbb{R}$. So let $a \in \mathbb{R}$ and decompose I in two disjoint sets I_1, I_2 such that

- $a_i \leq a$ for all $i \in I_1$ and
- $a_i > a$ for all $i \in I_2$.

We then have

$$\{f \leq a\} = \bigcup_{i\in I_1} \{f = a_i\} = \bigcup_{i\in I_1} A_i \in \sigma(\mathcal{Z}).$$

"$\subseteq$" Let $f \in F_1$. If f is measurable then the pre-image under f of any Borel measurable subset of $\mathbb{R}$ must be measurable. Therefore $\{x \in \Omega : f(x) = a\} = f^{-1}(\{a\}) \in \sigma(\mathcal{Z})$ for all $a \in \mathbb{R}$. Thus, from part (a) we have $\{x \in \Omega : f(x) = a\} = \bigcup_{i\in J} A_i$ for some $J \subseteq I$. In particular, for all $i \in I$ and $a \in \mathbb{R}$

$$\{f = a\} \cap A_i \in \{\emptyset, A_i\},$$

which implies that f is constant on A_i and $f \in F_2$.

Solution 1.3

(a) If $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{R}$-$\mathcal{R}$-measurable, $Y = f \circ X$ and $B \in \mathcal{R}$ then

$$(f \circ X)^{-1}(B) = X^{-1}(\underbrace{f^{-1}(B)}_{\in\mathcal{R}}) \in \sigma(X).$$

That is Y is $\sigma(X)$-$\mathcal{R}$-measurable.

(b) Since $A \in \sigma(X)$, there is a $B \in \mathcal{R}$ such that $A = X^{-1}(B)$. Therefore

$$Y = 1_A = 1_{X^{-1}(B)} = 1_B \circ X,$$

so the $\implies$ direction holds for indicator functions.

(c) For each i we can apply part (b) to get a $B_i \in \mathcal{R}$ such that $1_{A_i} = 1_{B_i} \circ X$. Then

$$Y = \sum_{i=1}^{n}((c_i 1_{B_i}) \circ X) = (\sum_{i=1}^{n} c_i 1_{B_i}) \circ X = f \circ X,$$

with $f = \sum_{i=1}^{n}(c_i 1_{B_i})$. Furthermore f is $\mathcal{R}$-$\mathcal{R}$-measurable, so $\implies$ direction holds for linear combinations of indicator functions.

(d) Define the "step function approximations"

$$Y_n := \sum_{k=0}^{n2^n-1} \frac{k}{2^n} 1_{\left\{\frac{k}{2^n} \le Y < \frac{k+1}{2^n}\right\}} + n 1_{\{Y \ge n\}}.$$

We then have $Y_n \uparrow Y$. Also Y_n is a linear combination of indicator functions for all n, and since Y is $\sigma(X)$-$\mathcal{R}$-measurable the sets $\left\{\frac{k}{2^n} \le Y < \frac{k+1}{2^n}\right\} \subset \Omega$ are in $\sigma(X)$ (using also that $[k/2^n, (k+1)/2^n)$ and $[n, \infty)$ are in $\mathcal{R}$). Thus, from (c) we know that there are $\mathcal{R}$-$\mathcal{R}$-measurable functions f_n such that $Y_n = f_n \circ X$. We define

$$g(x) := \limsup_{n\to\infty} f_n(x).$$

Since the lim sup of a sequence of measurable functions is measurable, we have that g is a measurable function from $\mathbb{R}$ to $(-\infty, \infty]$. It can happen that $g(x) = \infty$ (but only for x outside the range of X), so to deal with this technicality we set

$$f(x) := 1_{\{g(x)<\infty\}} g(x), x \in \mathbb{R}.$$

Then f is $\mathcal{R}$-$\mathcal{R}$-measurable. Also, since $Y_n \uparrow Y$ we have that $f(x) = \lim_{n\to\infty} f_n(x)$ for x in the range of X, and thus

$$Y = \lim_{n\to\infty} Y_n = \lim_{n\to\infty} f_n \circ X = (\lim_{n\to\infty} f_n) \circ X = f \circ X.$$

This proves the $\implies$ direction for non-negative Y.

(e) Write

$$Y = Y^+ - Y^-,$$

for $Y^+ = 1_{Y\ge 0} Y$ and $Y^- = -1_{Y<0} Y$. Then d) applies to Y^+ and Y^-, so we have functions f and g such that

$$Y^+ = f \circ X \text{ and } Y^- = g \circ X.$$

Clearly

$$Y = (f - g) \circ X,$$

and $f - g$ is $\mathcal{R}$-$\mathcal{R}$-measurable, so the claim follows.