REAL ANALYSIS

LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books for *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

Lecture #2

1. Euclidean Topology

In this note, we shall use the following standard notation. A point $x \in \mathbb{R}^n$ consists of a n-tuple of real numbers

$$x = (x_1, x_2, \cdots, x_n), \quad x_i \in \mathbb{R}, \text{ for } i = 1, \ldots, n.$$

Addition of points is componentwise, and so is multiplication by a real scalar. The norm of x is denoted by $|x|$ and is defined to be the standard Euclidean norm given by

$$|x| = \left(x_1^2 + \cdots x_n^2\right)^{1/2}.$$

The distance between two points x and y is then simply

$$d(x, y) = |x - y|.$$

Let A and B be two subsets of $\mathbb{R}^n$. The distance between these two sets is defined by

$$d(A, B) = \inf |x - y|$$

where the infimum is taken over all $x \in A$ and $y \in B$.

The open ball in $\mathbb{R}^n$ centred at x and of radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^n : \ |y - x| < r\}.$$

Definition 1.1. *Let $G, F \subset \mathbb{R}^n$.*

- *We say G is open if for every $x \in G$ there exists $r > 0$ with $B_r(x) \subset G$.*
- *We say F is closed if $G = \mathbb{R}^n - F$ is open.*
- *$\mathbb{R}^n$ and $\emptyset$ are both open and closed.*

Let $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$. We say E is a neighbourhood of x if there is an open ball $B_r(x) \subset E$.

Theorem 1.1. *Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a collection of open sets. Then*

(i) *Finite intersection of G_λ is open;*
(ii) *$\bigcup_{\lambda\in\Lambda} G_\lambda$ is open.*

Theorem 1.2. *Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a collection of closed sets. Then*

(i) *Finite union of F_λ is closed;*
(ii) *$\bigcap_{\lambda\in\Lambda} F_\lambda$ is closed.*

Theorem 1.3. *A set $F \subset \mathbb{R}^n$ is closed if and only if*

$$\text{for any } \{x_k\} \subset F, \text{ if } x_k \to x \in \mathbb{R}^n \text{ as } k \to \infty, \text{ then } x \in F.$$

Proof. We show "if" part. Let $x \in F^c$. There is a sufficiently large n such that

$$B_{1/n}(x) \subset F^c. \tag{1.1}$$

Otherwise, for any k, $B_{1/k}(x) \cap F \neq \emptyset$. Take $x_k \in B_{1/k}(x) \cap F$. Then $x_k \in F$ and $x_k \to x$. Hence $x \in F$, a contradiction. We conclude that F^c is open by (1.1). Hence F is closed.

We next show "only if" part. Let $\{x_k\} \subset F$ be such that $x_k \to x$. If $x \in F^c$ which is open, then $B_\delta(x) \subset F^c$ and hence $x_k \notin B_\delta(x)$ for all k, arriving a contradiction.

□

Definition 1.2. *Let $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$.*

- *A point x is an interior point of E if $B_r(x) \subset E$ for some $r > 0$. Denote by $\mathrm{Int}\, E$ the set of all interior points of E, and is called the interior of E.*
- *A point x is a limit point of E if for every $r > 0$, $B_r(x) \cap E \neq \emptyset$. Denote by $\overline{E}$ the union of E and all its limit points, and is called the closure of E.*
- *The boundary of E, denoted by ∂E, is defined as $\overline{E} \setminus E$.*
- *A point x is an isolated point of E if $B_r(x) \cap E = \{x\}$ for some $r > 0$.*
- *A closed set E is perfect if E does not have any isolated points.*

Theorem 1.4. *Let $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$. Then*

(i) *$x \in \overline{E}$ if and only if there is a sequence $\{x_k\}$ such that $x_k \to x$;*

(ii) *$Int\,E \subset E$ and $Int\,E$ is the largest* [1] *open subset of E;*
(iii) *$E \subset \overline{E}$, and $\overline{E}$ is the smallest* [2] *closed set containing E;*
(iv) *$E = Int\,E$ if and only if E is open, while $E = \overline{E}$ if and only if E is closed.*

Definition 1.3. *Let $D \subset \mathbb{R}^n$ and $f : D \to \mathbb{R}$ is a function. We say f is continuous at $x \in D$, if*

$$\lim_{y \in D, y \to x} f(y) = f(x).$$

We say f is continuous in D is f is continuous at every $x \in D$.

Theorem 1.5. *A function f on $\mathbb{R}^n$ is continuous if and only if*

$$\{x : \ f(x) > \lambda\} \text{ and } \{x : f(x) < \lambda\} \text{ are open, } \forall\ \lambda \in \mathbb{R}.$$

Proof. We first prove "if" part. Given a $x_0 \in \mathbb{R}^n$, for any $\varepsilon > 0$, we have

$$\begin{aligned} \{y : \ |f(y) - f(x_0)| < \varepsilon\} &= \{y : \ -\varepsilon < f(y) - f(x_0) < \varepsilon\} \\ &= \{y : \ f(y) > f(x_0) - \varepsilon\} \cap \{y : \ f(y) < f(x_0) + \varepsilon\} \\ &=: \mathcal{O}. \end{aligned}$$

Note that $\mathcal{O}$ is open and contains x_0. Hence $B_\delta(x_0) \subset \mathcal{O}$ for some $\delta > 0$. This implies by definition f is continuous at x_0. By the arbitrary of x_0, f is continuous.

Next let us show "only if" part. We aim to prove the openness of $\{x : f(x) > \lambda\}$. Given an $x_0 \in \{x : f(x) > \lambda\}$, i.e. $f(x_0) > \lambda$, by the continuity, $f(y) > \lambda$ for all $y \in B_\delta(x_0)$ provided δ sufficiently small. Hence $B_\delta(x_0) \subset \{x : f(x) > \lambda\}$ and thus $\{x : f(x) > \lambda\}$ is open. Similarly we deduce that $\{x : f(x) < \lambda\}$ is open.

□

Theorem 1.6. *Let $D \subset \mathbb{R}^n$. Then $d(x, D)$ is uniformly continuous.*

Proof. This is a direct consequence of the following inequality

$$|d(x, D) - d(y, D)| \leq d(x, y).$$

□

Theorem 1.7. *If $\{x_k\} \subset \mathbb{R}^n$ is bounded, then $\{x_k\}$ has a subsequence which converges to some point.*

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[1] It means if G is an open subset of E, then $G \subset E$.
[2] It means if F is a closed set and $E \subset F$, then $E \subset F$.

Proof. This is a consequence of the 1-D Bolzano-Weierstrass Theorem. $\square$

Theorem 1.8. *Let $x \in \mathbb{R}^n$ and $F \subset \mathbb{R}^n$ be a closed set. There is a $y \in F$ such that $d(x, y) = d(x, F)$.*

Proof. Take a sequence of points $y_k \in F$ such that

$$d(x, y_k) \to d(x, F).$$

Since $\{y_k\}$ is bounded, we have $y_k \to y$ after passing to a subsequence if necessary. By closedness, $y \in F$. $\square$

Theorem 1.9. *If $F_k \subset \mathbb{R}^n$ are closed, bounded, non-empty and decreasing, then*

$$\bigcap_{k=1}^{\infty} F_k \neq \emptyset.$$

Proof. Take $x_k \in F_k$. Since $\{x_k\}$ is bounded, we have $x_k \to x_\infty$ after passing to a subsequence if necessary. For any fixed k, by the monotonicity, $x_l \in F_k$ whenever $l \geq m_k$ for some m_k. By closedness of F_k, $x_\infty \in F_k$ for all k. This completes the proof. $\square$

Remark 1.1. *Note that $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$; $\bigcap_{n=1}^{\infty}[n, \infty) = \emptyset$; $\bigcap_{n=1}^{\infty}[n, n+1] = \emptyset$.*

Definition 1.4. *We say $E \subset \mathbb{R}^n$ is compact, if for any covering of E by a collection of open sets contains a finite subcovering. Namely given any $\{G_\lambda\}_{\lambda \in \Lambda}$ with G_λ's being open, if $E \subset \bigcup_{\lambda \in \Lambda} G_\lambda$, then there are $G_{\lambda_1}, G_{\lambda_2}, \cdots, G_{\lambda_N}$ such that $E \subset \bigcup_{i=1}^{N} G_{\lambda_i}$.*

Theorem 1.10. *A set $E \subset \mathbb{R}^n$ is compact if and only if E is closed and bounded .*

Proof. We first show the "only if" part. Let E be a compact set. Clearly

$$E \subset \bigcup_{k=1}^{\infty} B_k(0).$$

By the compactness, $E \subset \bigcup_{k=1}^{N} B_k(0)$ for some N. Hence E is bounded. Let $x \in E^c$. We have

$$E \subset \bigcup_{k=1}^{\infty} \{y : d(y, x) > \frac{1}{k}\}$$

This yields an open covering. The compactness then implies

$$E \subset \{y : d(y, x) > \frac{1}{N}\}$$

for some large N. Consequenlty

$$\{y : d(y,x) \le \frac{1}{N}\} \subset E^c,$$

which shows that E^c is open.

Next we show the "if" part. Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be an open covering of E. As E is bounded, there is a closed cube Q with side length $a > 0$, so that $E \subset Q$. Note that Q is the union of 2^n sub-cubes with side length $a/2$. Suppose that $\{G_\lambda\}$ does not have a finite sub-covering for E. There is a closed cube $Q_1 \subset Q$ with side length $a/2$ so that $\{G_\lambda\}$ does not have a finite sub-covering for $E \cap Q_1$. Repeat this procedure. We obtain a sequence of closed cubes Q_k with side length $a/2^k$, $Q_{k+1} \subset Q_k$, such that $\{G_\lambda\}$ does not have a finite sub-covering for $E \cap Q_k$. By Theorem 1.9, there is a $\xi \in \bigcap_{k=1}^\infty (E \cap Q_k)$. As $\{G_\lambda\}$ is a covering, we have $\xi \in G_{\lambda_0}$ for a $G_{\lambda_0} \in \{G_\lambda\}_{\lambda\in\Lambda}$, and so by the openness $B_r(\xi) \subset G_{\lambda_0}$ for some r. Since the side length of Q_k shrinks to zero, we see that $E \cap Q_k \subset B_r(\xi) \subset G_{\lambda_0}$ for large k, thus arriving a contradiction.

□

Theorem 1.11. *Let $F \subset \mathbb{R}^n$ be compact, and f be a continuous function on F. Then*

(i) *f attains its maximum and minimum on F;*

(ii) *f is uniformly continuous on F. That is, for any $\varepsilon > 0$, there is a $\delta > 0$, so that*

$$|f(x) - f(y)| < \varepsilon, \quad \text{whenever } x, y \in F \text{ and } d(x,y) < \delta.$$

Proof. Denote by $K = f(F) \subset \mathbb{R}$, the image of F. Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be an open covering of K. By Theorem 1.5, $f^{-1}(G_\lambda)$ are all open. Obviously

$$F = \bigcup_{\lambda\in\Lambda} f^{-1}(G_\lambda).$$

By the compactness, there is a finite sub-covering of F, say $\{f^{-1}(G_{\lambda_i})\}_{i=1}^N$, where $\lambda_i \in \Lambda$. Hence $K \subset \bigcup_{i=1}^N G_{\lambda_i}$. This shows that K is compact. By Theorem 1.10, K is a bounded and closed. Hence (i) follows.

Since f is continuous, given $\varepsilon > 0$, there are $\delta_x > 0$ (for every $x \in F$) such that

$$|f(x) - f(y)| < \varepsilon/2, \quad \text{for all } y \in B_{\delta_x}(x) \cap F. \tag{1.2}$$

It is trivial that

$$F \subset \bigcup_{x\in F} B_{\delta_x/2}(x).$$

By the compactness, we deduce

$$F \subset \bigcup_{i=1}^{N} B_{\delta_{x_i}/2}(x_i)$$

for some finite $x_i \in F$. Denote $\delta_i' = \delta_{x_i}/2$ and $B_i = B_{\delta_{x_i}/2}(x_i)$. Let $\delta' = \min_{1 \le i \le N}\{\delta_i'\}$. Let $x, y \in F$ be two given points such that $d(x, y) < \delta'$. There is a B_i containing x. Note that $y \in B_{\delta_{x_i}}(x_i)$, as

$$d(y, x_i) \le d(y, x) + d(x, x_i) < \delta' + \delta_{x_i}/2 < \delta_{x_i}.$$

By (1.2), we have

$$|f(x) - f(y)| \le |f(x) - f(x_i)| + |f(x_i) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This finishes the proof.

□

We now proceed to give a description of the structure of open sets in terms of cubes. We begin with the open sets in $\mathbb{R}$.

Theorem 1.12. *Let G be an open set in $\mathbb{R}$. Then G is a countable union of disjoint open intervals.*

Proof. Given $x \in G$, by the openness, we define

$$a_x = \inf\{a \in \mathbb{R} : \ (a, x) \subset G\} \text{ and } b_x = \sup\{b \in \mathbb{R} : \ (x, b) \subset G\}.$$

It is easy to see that

$$G = \bigcup_{x \in G} (a_x, b_x).$$

Clearly the union above is disjoint. As we can assign each interval to a rational number, the disjoint intervals above is countable.

□

A (closed) rectangle R in $\mathbb{R}^n$ is given by the product of n one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

where $a_j \leq b_j$ are real numbers, $j = 1, 2, \ldots, n$. An open rectangle is the product of open intervals. A cube is a rectangle for which $b_1 - a_1 = b_2 - a_2 = \cdots = b_n - a_n$. A union of rectangles is said to be almost disjoint if the interiors of the rectangles are disjoint.

Theorem 1.13. *Every open subset G of $\mathbb{R}^n$ can be written as a countable union of almost disjoint closed cubes.*

Proof. Consider grids of side length 2^{-k} where $k = 1, 2, 3, \ldots$. Consider the corresponding cubes of side length 2^{-k}.

At stage 1, each cube of side length 1 is accepted if it is a subset of G, rejected if a subset of G^c, and held in reserve if it meets both G and G^c.

At stage 2, all reserve cubes from stage 1 are further subdivided and then accepted, rejected or held in reserve according to the above criteria.

At stage 3, all reserve cubes from stage 2 are further subdivided and then accepted, rejected or held in reserve according to the above criteria. Etc.

Let Q_k be the collection of cubes accepted at stage k. Each such cube has side length 2^{-k}.

Claim: $G = \bigcup_{k\geq 1} Q_k$ is the required countable almost disjoint union.

First note it follows from the construction that the union is countable, almost disjoint, and $\bigcup_{k\geq 1} Q_k \subset G$.

Next note that if $x \in G$, then since G is open, for some k there is one or more cubes of size 2^{-k} containing x which is a subset of G. Let k be the smallest such k for which this is true. The corresponding cube will be accepted at this stage and so $x \in \bigcup_{k\geq 1} Q_k$. Hence $G \subset \bigcup_{k\geq 1} Q_k$.

This proves the claim.

□

2. Cantor Set

The Cantor set plays a prominent role set theory and in analysis in general. It and its variants provide a rich source of enlightening examples.

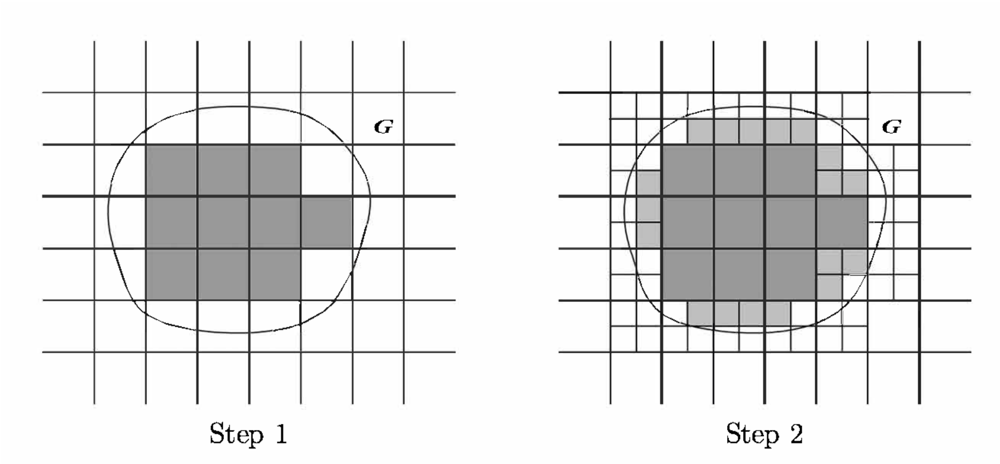

FIGURE 1. Decomposition of G into almost disjoint cubes

We begin with the closed unit interval $C_0 = [0,1]$ and let C_1 denote the set obtained from deleting $I_{1,1} = (\frac{1}{3}, \frac{2}{3})$, the middle third open interval, from $[0,1]$. That is

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1].$$

We also define

$$f(x) = \frac{1}{2}, \quad x \in I_{1,1}.$$

Next we repeat this procedure for each sub-interval of C_1: that is , we delete the middle third open intervals, which are $I_{2,1} = (\frac{1}{9}, \frac{2}{9})$ and $I_{2,2} = (\frac{7}{9}, \frac{8}{9})$. At the second stage we get

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1].$$

We also define at this stage

$$f(x) = \frac{2k-1}{2^2}, \quad x \in I_{2,k}, \ k = 1, 2.$$

We repeat this procedure for each sub-interval of C_2 and so on.

In general, after n step, we have deleted $2^n - 1$ open intervals $\{I_{m,k} : \ 1 \leq m \leq n, 1 \leq k \leq 2^{m-1}\}$, and we obtain

$$C_n = C_{n-1} - \bigcup_{k=1}^{2^{n-1}} I_{n,k} = [0,1] - \bigcup_{m=1}^{n} \bigcup_{k=1}^{2^{m-1}} I_{m,k}.$$

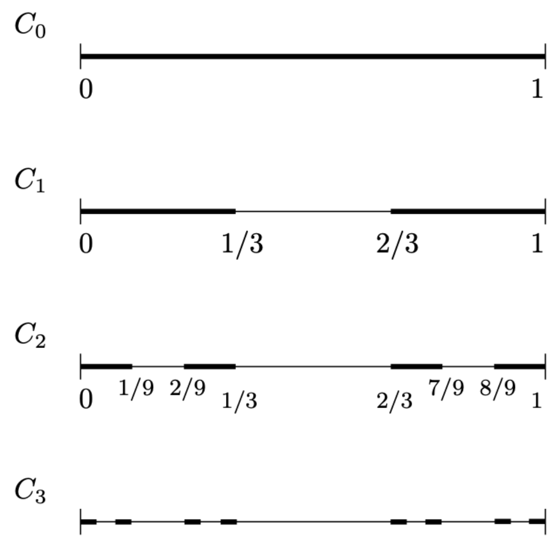

FIGURE 2. Construction of the Cantor set

At the n-th stage, we define

$$f(x) = \frac{2k-1}{2^n}, \;\; x \in I_{n,k}, \;\; 1 \leq k \leq 2^{n-1}.$$

One can check that f is monotone.

This procedure yields a sequence C_n, $n = 0, 1, 2, \ldots$, of compact sets with

$$\cdots \subset C_{n+1} \subset C_n \subset \cdots \subset C_2 \subset C_1 \subset C_0 = [0,1].$$

The Cantor set is by definition the intersection of all C_n's:

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n = [0,1] - G, \;\; \text{where } G = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k}. \tag{2.1}$$

Note that the sum of length of all intervals in G is

$$\sum_{n\geq 1} \sum_{1\leq k\leq 2^{n-1}} |I_{n,k}| = \sum_{n\geq 1} \frac{1}{3^n} \times 2^{n-1} = 1. \tag{2.2}$$

Proposition 2.1. *The Cantor set $\mathcal{C}$ satisfies the following properties:*

(i) *$\mathcal{C}$ is perfect (i.e., it is closed, and has no isolated points);*
(ii) *$[0,1] - \mathcal{C}$ is dense in $[0,1]$;*
(iii) *$\mathcal{C}$ has no interior point;*
(iv) *$\mathcal{C}$ is totally disconnected ($\forall x, y \in \mathcal{C}$, there is $z \notin \mathcal{C}$ that lies between x and y);*

(v) $\mathcal{C}$ *has the cardinality of the continuum.*

Proof. Statement (i) follows by (2.1) immediately. Statement (ii) is equivalent to, for any $(a, b) \subset [0, 1]$, $(a, b) \cap G \neq \emptyset$. It is not hard to be checked by the construction as well. Statement (iii) follows from (ii) immediately. Statement (iv) can be verified directly.

Next we show (iv). For any $x \in (0, 1)$, there is a unique $\mathbf{a} = (a_1, a_2, \cdots, a_n, \cdots) \in \mathcal{A}_3$ such that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}. \tag{2.3}$$

For any $x \in I_{1,1}$, we have $a_1 = 1$. For any $x \in I_{2,1} \cup I_{2,1}$, we have $a_2 = 1$. In general, for any $x \in I_{n,k}$, we have $a_n = 1$. Hence if $x \in G$, then there is an $a_k = 1$. Consequently, if x, written as (2.3), with all a_i being 0 or 2, then $x \in \mathcal{C}$. This yields that

$$\mathrm{Card}(\mathcal{A}_2) \leq \mathrm{Card}(\mathcal{C}).$$

It is direct to see $\mathrm{Card}(\mathcal{C}) \leq \mathrm{Card}([0, 1])$. Recall that $\mathrm{Card}(\mathcal{A}_2) = \mathrm{Card}(\mathcal{C})$. This completes the proof.

□

We turn our attention next to the question of determing the "size" of $\mathcal{C}$. This is a delicate problem, one that may be approached from different angles depending on the notion of size we adopt. For instance, in terms of cardinality the Cantor set is rather large, as large as $[0, 1]$, see (v) in Proposition 2.1. However, from the point of view of "length" the size of $\mathcal{C}$ is small. Roughly speaking, the Cantor set has length zero, by virtue of (2.2). We shall define a notion of measure and make this precise in the following section.

Let $G = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k}$ as above. Let g be a function on $[0, 1]$ such that

$$g(1) = 1,$$
$$g(x) = \inf\{f(y) : \ y > x, y \in G\}, \ 0 \leq x < 1.$$

Clearly g is non-decreasing, $0 \leq g \leq 1$. As $g(G) = f(G)$ is dense in $[0, 1]$, we conclude that $g([0, 1])$ is dense in $[0, 1]$. It follows that g is continuous. Such g is called the Cantor function on $[0, 1]$.