Coordinator
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# Probability Theory

## Exercise Sheet 11

**Exercise 11.1** Let $(X_n)_{n\geq 0}$ be a sequence of random variables with values in $[0,1]$. We set $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$. Suppose that $X_0 = a \in [0,1]$ and

$$P\left[X_{n+1} = \frac{X_n}{2}\Big|\mathcal{F}_n\right] = 1 - X_n, \qquad P\left[X_{n+1} = \frac{1+X_n}{2}\Big|\mathcal{F}_n\right] = X_n.$$

(a) Show that $(X_n)_{n\geq 0}$ is a $\mathcal{F}_n$-martingale that converge to a random variable $X_\infty$ $P$-almost surely and in $L^2$.

(b) Show that $E\left[(X_{n+1} - X_n)^2\right] = \frac{1}{4}E\left[X_n(1-X_n)\right]$.

**Exercise 11.2** Let $Y_n$, $n \geq 0$ be i.i.d. with $P[Y_0 = 1] = p$ and $P[Y_0 = 0] = 1 - p$ for some $p \in (0,1)$. Let $\mathcal{F}_n := \sigma(Y_0, \dots, Y_n)$ for $n \geq 0$ and define

$$T := \inf\{n \geq 0 \mid Y_n = 1\}.$$

Determine the Doob decomposition of $X_n := 1_{\{T \leq n\}}$, $n \geq 0$.

**Hint:** First check that $X_n$ is an $\mathcal{F}_n$-submartingale.

**Exercise 11.3** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration on this space. Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$-martingale such that $M_0 = 0$ and $M_n \in L^2$ for all $n$.

(a) Why is $(M_n^2)_{n\geq 0}$ a submartingale?

(b) Let $(A_n)_{n\geq 0}$ be the non-decreasing and predictable process from the Doob decomposition of $(M_n^2)_{n\geq 0}$. Show that $\tau_a := \inf\{n \geq 0; A_{n+1} > a^2\}$ is a stopping time.

(c) Show that $P\left[\sup_{n\geq 0} |M_{n\wedge\tau_a}| > a\right] \leq \dfrac{E[A_\infty \wedge a^2]}{a^2}$, where $A_\infty$ is the $P$-a.s. limit of $(A_n)_{n\geq 0}$.

**Hint:** First consider $P\left[\sup_{n\leq N} |M_{n\wedge\tau_a}| > a\right]$ for $N \in \mathbb{N}$ and use Doob's inequality.

(d) Show that $P\left[\sup_{n\geq 0} |M_n| > a\right] \leq P[A_\infty > a^2] + P\left[\sup_{n\geq 0} |M_{n\wedge\tau_a}| > a\right]$.

**Submission:** until 14:15, Dec 10., during exercise class or in the tray outside of HG G 53.

**Office hours (Präsenz):** Mon. and Thu., 12:00-13:00 in HG G 32.6.

**Class assignment:**

| Students | Time & Date | Room | Assistant |
|---|---|---|---|
| Afa-Fül | Tue 13-14 | HG F 26.5 | Angelo Abächerli |
| Gan-Math | Tue 13-14 | ML H 41.1 | Zhouyi Tan |
| Meh-Schu | Tue 14-15 | HG F 26.5 | Angelo Abächerli |
| Schü-Zur | Tue 14-15 | ML H 41.1 | Dániel Bálint |

**Solution 11.1**

(a) Since $X_0 = a \in [0,1]$, from the assumption that $P[X_{n+1} = \frac{X_n}{2}$ or $X_{n+1} = \frac{1+X_n}{2}] = 1$ we can use induction argument to conclude that $0 \leq X_n \leq 1$ for all $n$, Hence each $X_n$ is integrable. Moreover, it holds that

$$\begin{aligned} & E[X_{n+1}|\mathcal{F}_n] \\ = \;& \frac{X_n}{2}P[X_{n+1} = \frac{X_n}{2}|\mathcal{F}_n] + \frac{1+X_n}{2}P[X_{n+1} = \frac{1+X_n}{2}|\mathcal{F}_n] \\ = \;& \frac{X_n}{2}(1-X_n) + \frac{1+X_n}{2}X_n = X_n. \end{aligned}$$

Thus $X_n$ is a non-negative martingale, and by the Martingale Convergence Theorem, $X_n$ converge to a random variable $X_\infty$ a.s. Besides, we have that $X_n$ is bounded by 1, then the convergence holds also in $L^p$ for all $p \geq 1$ due to the Dominated Convergence Theorem.

(b) We have that

$$E[(X_{n+1} - X_n)^2] = E[E[(X_{n+1} - X_n)^2|\mathcal{F}_n]] \tag{1}$$

$$= E[E[(X_{n+1}^2 - 2X_{n+1}X_n + X_n^2|\mathcal{F}_n]].$$

It is easy to see that

$$\begin{aligned} E[X_{n+1}^2|\mathcal{F}_n] &= \left(\frac{X_n}{2}\right)^2 P[X_{n+1} = \frac{X_n}{2}|\mathcal{F}_n] + \left(\frac{1+X_n}{2}\right)^2 P[X_{n+1} = \frac{1+X_n}{2}|\mathcal{F}_n] \\ &= \left(\frac{X_n}{2}\right)^2 (1-X_n) + \left(\frac{1+X_n}{2}\right)^2 X_n = \frac{X_n}{4}(1+3X_n). \end{aligned}$$

Plugging this in (1) we have that

$$E[E[(X_{n+1} - X_n)^2|\mathcal{F}_n]] = E[\frac{X_n}{4}(1+3X_n) - 2X_n^2 + X_n^2] = \frac{1}{4}E[X_n(1-X_n)].$$

**Solution 11.2** As in the hint, we first check that $X_n$ is an $\mathcal{F}_n$-submartingale. Clearly, $X_n$ is $\mathcal{F}_n$-adapted. Furthermore, $X_n$ is bounded for all $n$, so it is integrable. Finally, $1_{\{T \leq n+1\}} \geq 1_{\{T \leq n\}}$ for every $n \geq 0$, since $\{T \leq n\} \subseteq \{T \leq n+1\}$. Due to this, and by the monotonicity property of conditional expectation, we obtain

$$E[X_{n+1}|\mathcal{F}_n] = E[1_{\{T \leq n+1\}}|\mathcal{F}_n] \geq E[1_{\{T \leq n\}}|\mathcal{F}_n] = 1_{\{T \leq n\}} = X_n \quad P\text{-a.s.}$$

Hence, $X_n$ is an $\mathcal{F}_n$-submartingale, so the Doob decomposition (unique up to $P$-nullsets) must exist. In other words, there exists a martingale $M_n$, $n \geq 0$, and a predictable, non-decreasing process $A_n$, with $A_0 = 0$, such that

$$X_n = M_n + A_n, \quad n \geq 0.$$

To find $M_n$ and $A_n$, we follow the proof of existence of this decomposition. For our $X_n$, we have for $k \geq 0$:

$$\begin{aligned} E[X_k - X_{k-1}|\mathcal{F}_{k-1}] &= E[1_{\{T\leq k\}} - 1_{\{T\leq k-1\}}|\mathcal{F}_{k-1}] \\ &= E[1_{\{T=k\}}|\mathcal{F}_{k-1}] \\ &= E[1_{\{Y_k=1\}}1_{\{T>k-1\}}|\mathcal{F}_{k-1}] \\ &= 1_{\{T>k-1\}}E[1_{\{Y_k=1\}}|\mathcal{F}_{k-1}] \\ &= 1_{\{T>k-1\}}P[Y_k = 1] \\ &= p1_{\{T>k-1\}} \ (= A_k - A_{k-1}) \quad P\text{-a.s.}, \end{aligned} \tag{2}$$

since $Y$ is independent of $\mathcal{F}_{k-1}$. Thus, we define

$$A_n := \sum_{k=1}^{n} p1_{\{T>k-1\}} = p \cdot (T \wedge n), \quad n \geq 0, \tag{3}$$

and we have that

$$M_n = X_n - A_n = 1_{\{T\leq n\}} - p \cdot (T \wedge n), \quad n \geq 0,$$

is a $\mathcal{F}_n$-martingale, since one can verify with equations (2) and (3) that

$$E[X_{n+1} - A_{n+1}|\mathcal{F}_n] = X_n - A_n \quad P\text{-a.s.}$$

Furthermore, $A_n$ is non-decreasing, $A_0 = 0$ and $A_n$ is predictable, since each indicator function in the sum in equation (3) is $\mathcal{F}_{n-1}$-measurable. Thus, the Doob decomposition is

$$X_n = M_n + A_n = \Big(1_{\{T\leq n\}} - p \cdot (T \wedge n)\Big) + p \cdot (T \wedge n), \quad n \geq 0.$$

**Solution 11.3**

(a) $(M_n^2)_{n\geq 0}$ is a submartinale by Jensen's inequality for the conditional expectation, see (3.2.22) in the lecture notes.

(b) For $n \geq 0$ we obtain,

$$\{\tau_a = n\} = \left( \bigcap_{k=0,\dots,n} \{A_k \leq a^2\} \right) \cap \{A_{n+1} > a^2\} \in \mathcal{F}_n,$$

and thus $\tau_a$ is a stopping time (the events on the right-hand side, even the one involving $A_{n+1}$, are $\mathcal{F}_n$-measurable since $(A_n)_{n\geq 0}$ is predictable).

(c) Note that $A_\infty$ is well-defined $P-$a.s., since the process $(A_n)_{n\geq 0}$ is monotone $P-$a.s. Now let $N \in \mathbb{N}$: Since $(M^2_{n\wedge\tau_a})_{n\geq 0}$ is a non-negative submartingale, Doob's inequality implies that

$$P\Big[\sup_{n\leq N} |M_{n\wedge\tau_a}| > a\Big] = P\Big[\sup_{n\leq N} M^2_{n\wedge\tau_a} > a^2\Big] \leq \frac{E[M^2_{N\wedge\tau_a}]}{a^2} = \frac{E[A_{N\wedge\tau_a}]}{a^2} \leq \frac{E[A_{\tau_a}]}{a^2},$$

where we used that $E[M^2_{N\wedge\tau_a}] = E[A_{N\wedge\tau_a}]$ since $M^2_{n\wedge\tau_a} - A_{n\wedge\tau_a}$ is a martingale (by the optional stopping theorem; recall that $M_n^2 - A_n$ is a martingale and $\tau_a$ a stopping time) with $M^2_{0\wedge\tau_a} - A_{0\wedge\tau_a} = 0$. We also used that $A_{N\wedge\tau_a} \le A_{\tau_a}$, since $A_n$ is non-decreasing. Taking the limit $N \to \infty$ we obtain $P\Big[\sup_{n\ge 0} |M_{n\wedge\tau_a}| > a\Big] \le \frac{E[A_{\tau_a}]}{a^2}$. On the other hand, we have $A_{\tau_a} \le A_\infty$ and

$$
\begin{aligned}
A_{\tau_a} &\le a^2 \quad \text{on } \{\tau_a < +\infty\},\\
A_{\tau_a} &= A_\infty \le a^2 \quad \text{on } \{\tau_a = +\infty\},
\end{aligned}
$$

which implies that $A_{\tau_a} \le A_\infty \wedge a^2$.

(d) Since $\tau_a = +\infty$ on $\{A_\infty \le a^2\}$, we have

$$
\{A_\infty \le a^2\} \cap \Big\{\sup_{n\ge 0} |M_n| > a\Big\} \subseteq \Big\{\sup_{n\ge 0} |M_{n\wedge\tau_a}| > a\Big\}.
$$

From this we obtain

$$
\begin{aligned}
P\Big[\sup_{n\ge 0} |M_n| > a\Big] &= P\Big[A_\infty > a^2, \sup_{n\ge 0} |M_n| > a\Big] + P\Big[A_\infty \le a^2, \sup_{n\ge 0} |M_n| > a\Big]\\
&\le P[A_\infty > a^2] + P\Big[\sup_{n\ge 0} |M_{n\wedge\tau_a}| > a\Big].
\end{aligned}
$$