REAL ANALYSIS
LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

1. The space L^1 of integrable functions

The set of integrable functions has both algebraic and analytic properties:

- integrable functions form a vector space;
- this vector space is complete in the appropriate norm.

Definition 1.1. *The space $L^1 = L^1(\mathbb{R}^n)$ is the set of integrable functions defined on $\mathbb{R}^n$. The norm is defined by*

$$\|f\| = \|f\|_{L^1} = \|f\|_{L^1(\mathbb{R}^n)} = \int |f|.$$

L^1 is a vector space because it is closed under addition and scalar multiplication. It is normed space because $\|\cdot\|$ satisfies the properties

(a) $\|f+g\| \le \|f\| + \|g\|$,
(b) $\|af\| = |a|\|f\|$ for all $a \in \mathbb{C}$,
(c) $\|f\| = 0$ if and only if $f = 0$ a.e.

So we see it is not really a norm, unless we identify functions that agree almost everywhere. More precisely we could work with equivalence classes of functions, where two functions are equivalent if they are equal a.e. However, it is convenient to retain the (imprecise) terminology that an element $f \in L^1(\mathbb{R}^n)$ is an integrable function, even though it is only an equivalence class of such functions [1].

[1]Note that by the above, the norm $\|f\|$ of an element $f \in L^1(\mathbb{R}^n)$ is well-defined by the choice of any integrable function in its equivalent class.

As in any normed space we can define a corresponding metric. Here

$$d(f,g) = \|f-g\| = \int |f-g|.$$

We say d is a metric if it satisfies: (i) $d(f,g) \geq 0$, (ii) $d(f,g) = d(g,f)$, (iii) $d(f,g) \leq d(f,h) + d(h,g)$, and (iv) $d(f,g) = 0$ if $f = g$ (a.e.).

Definition 1.2. *We say $f_k \to f$ in L^1 sense if $f_k \to f$ in the sense of the metric, namely if $\|f_k - f\| \to 0$ as $k \to \infty$.*

A metric space (V,d) is said to be complete if for every Cauchy sequence $\{x_k\}$ in V (that is, $d(x_k, x_l) \to 0$ as $k, l \to \infty$) there exists $x \in V$ such that $\lim_{k\to\infty} x_k = x$ in the sense that

$$d(x_k, x) \to 0, \quad \text{as } k \to \infty.$$

We will show later that $L^1(\mathbb{R}^n)$ is complete.

<u>L^1 convergence v.s. almost everywhere convergence</u>. Let f, f_k be integrable functions.

- $f_k \to f$ a.e. $\not\Longrightarrow \|f_k - f\| \to 0$.
  E.g. consider $f_k = k^2\chi_{(0,1/k)}$. Then $f_k \to 0$ for all x. But $\|f_k\| = k \to \infty$.
  This example also shows that "$f_k \to f$ a.e." cannot imply "$|f_k| \leq f$ a.e.".
- $\|f_k - f\| \to 0 \not\Longrightarrow f_k \to f$ a.e.
  E.g. consider the sequence of f_k on $[0,1]$ of moving blips

$$\chi_{[0,1]},\ \chi_{[0,\frac{1}{2}]},\ \chi_{[\frac{1}{2},1]},\ \chi_{[\frac{1}{3},1]},\ \chi_{[\frac{1}{3},\frac{2}{3}]},\ \chi_{[\frac{2}{3},1]},\ \chi_{[0,\frac{1}{4}]},\ \chi_{[\frac{1}{4},\frac{2}{4}]},$$
$$\chi_{[\frac{2}{4},\frac{3}{4}]},\ \chi_{[\frac{3}{4},1]},\ \chi_{[0,\frac{1}{5}]},\ \cdots,\ \chi_{[\frac{4}{5},1]},\ \chi_{[0,\frac{1}{6}]},\cdots,\ \chi_{[\frac{5}{6},1]},\ \cdots$$

  Then $\|f_k - 0\| \to 0$. But $f_k \not\to 0$ anywhere on $[0,1]$.

However, the dominated convergence theorem finds an additional assumption under which a.e. convergence does imply L^1 convergence:

$$\text{if } |f_k| \leq g \text{ for some } g \in L^1, \text{ then } f_k \to f \text{ a.e.} \Longrightarrow \|f_k - f\| \to 0.$$

The following proposition applies if $f_k \to f$ geometrically fast in the L^1 sense, that is $\|f_k - f\| \leq Cr^k$ for some $C > 0$ and $r \in [0,1)$.

Theorem 1.1. *Let $f_k, f \in L^1$. Suppose $f_k \to f$ fast in the sense that $\sum_{k\geq 1} \|f_k - f\| < \infty$. Then $f_k \to f$ a.e.*

Proof. We divide the proof into several steps.

Step 1. Write

$$f_k(x) = f_1(x) + \sum_{l=2}^{k} (f_l(x) - f_{l-1}(x)), \tag{1.1}$$

and let

$$\begin{aligned} g_k(x) &= |f_1|(x) + \sum_{l=2}^{k} |f_l(x) - f_{l-1}(x)|, \\ g(x) &= |f_1|(x) + \sum_{l=2}^{\infty} |f_l(x) - f_{l-1}(x)|. \end{aligned} \tag{1.2}$$

Then $g_k \nearrow g$, where possibly $g(x) = \infty$ By the MCT,

$$\int g = \lim_{k\to\infty} \int g_k. \tag{1.3}$$

Step 2. By assumption, $K := \sum_{k\geq 1} \int \|f - f_k\| < \infty$. Using the triangle inequality,

$$\begin{aligned} \int g_k &= \int |f_1| + \sum_{l=2}^{k} \int |f_l - f_{l-1}| \\ &\leq \int |f_1| + \sum_{l=2}^{k} \int |f_{l-1} - f| + \sum_{l=2}^{k} \int |f_l - f| \\ &\leq \|f_1\| + 2K, \end{aligned}$$

which is independent of k. This together with (1.3) implies $g \in L^1$. Therefore $g(x)$ is finite a.e. In particular, $\lim_k g_k(x)$ exists for a.e. x.

Step 3. Because the series (1.2) converges a.e., the series (1.1) converges absolutely a.e., and in particular converges a.e. to a function $h \in L^1$.

It remains to show $f = h$ a.e. Since $|f_k| \leq g_k \leq g \in L^1$ and $f_k \to h$ a.e., we conclude by the DCT that

$$\int |f_k - h| \to 0.$$

Therefore, by the triangle inequality,

$$\|f - h\| \leq \|f - f_k\| + \|f_k - h\| \to 0 \Longrightarrow f = h \text{ } a.e.$$

□

L^1 convergence v.s. convergence in measure.

Proposition 1.1. *If a sequence f_k of integrable functions converges to f in L^1, then f_k converges to f in measure.*

Proof. For any $\varepsilon > 0$, we see that

$$\|f_k - f\| \geq \int_{\{|f_k - f| \geq \varepsilon\}} |f_k - f| \Longrightarrow m(\{|f_k - f| \geq \varepsilon\}) \leq \frac{1}{\varepsilon}\|f_k - f\| \to 0.$$

□

But convergence in measure does not imply L^1 convergence in general. For example let f_k be as follows:

$$\chi_{[0,1]},\ 2\chi_{[0,\frac{1}{2}]},\ 2\chi_{[\frac{1}{2},1]},\ 3\chi_{[\frac{1}{3},1]},\ 3\chi_{[\frac{1}{3},\frac{2}{3}]},\ 3\chi_{[\frac{2}{3},1]},\ 4\chi_{[0,\frac{1}{4}]},\ 4\chi_{[\frac{1}{4},\frac{2}{4}]},$$
$$4\chi_{[\frac{2}{4},\frac{3}{4}]},\ 4\chi_{[\frac{3}{4},1]},\ 5\chi_{[0,\frac{1}{5}]},\ \cdots,\ 5\chi_{[\frac{4}{5},1]},\ 6\chi_{[0,\frac{1}{6}]},\cdots,\ 6\chi_{[\frac{5}{6},1]},\ \cdots$$

Then $f_k \to 0$ in measure, but $\|f_k - 0\| = 1$ for all k.

Theorem 1.2. *Suppose $f_k \in L^1(\mathbb{R}^n)$, and $f_k \to f$ in measure. If there exists $g \in L^1(\mathbb{R}^n)$ such that $|f_k(x)| \leq g(x)$ for all k and a.e. x, then $f \in L^1(\mathbb{R}^n)$, and*

$$\lim_{k\to\infty} \int |f_k - f| = 0.$$

Consequently

$$\lim_{k\to\infty} \int f_k = \int f.$$

Proof. By Riesz Theorem, there is a subsequence $\{f_{k_j}\}$ such that

$$f_{k_j} \to f \quad a.e.$$

It follows that

$$|f| \leq g \quad a.e., \tag{1.4}$$

and hence $f \in L^1$. [2]

Denote $E_{k,\varepsilon} = \{|f_k - f| \geq \varepsilon\}$ and $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. By (1.4),

$$\begin{aligned} \int |f_k - f| &= \int_{E_{k,\varepsilon}} |f_k - f| + \int_{E^c_{k,\varepsilon}\cap B_R} |f_k - f| + \int_{E^c_{k,\varepsilon}\setminus B_R} |f_k - f| \\ &\leq 2\int_{E_{k,\varepsilon}} g + \varepsilon m(B_R) + 2\int_{B^c_R} g. \qquad (1.5) \end{aligned}$$

[2]By the dominated convergence theorem, $\|f_{k_j} - f\| \to 0$.

For any ε', we take $R = R_{\varepsilon',g}$ so that

$$\int_{B_R^c} g < \varepsilon'/6.$$

Then we choose $\varepsilon = \varepsilon_{\varepsilon',R}$ so that $\varepsilon m(B_R) < \varepsilon'/3$. By the absolute continuity of integrals,

$$\int_{E_{k,\varepsilon}} g \le \varepsilon'/6, \quad \text{provided } m(E_{k,\varepsilon}) < \delta = \delta_{\varepsilon',g}.$$

Since $f_k \to f$ in measure, there exists $N_{\varepsilon,\delta}$ (hence depending on n, ε', g only) so that if $k \ge N_{\varepsilon,\delta}$ then $m(E_{k,\varepsilon}) < \delta$. Therefore (1.5) can be further estimated as

$$\int |f_k - f| \le \varepsilon' \ \text{ whenever } k \ge N_{\varepsilon,\delta} = N_{n,\varepsilon',g}.$$

This completes the proof.

□

Corollary 1.1. *Suppose f_k, f are non-negative and $f_k \to f$ in measure. Then*

$$\int f \le \liminf_{k\to\infty} \int f_k.$$

Proof. Suppose $g \in \mathcal{F}(f) = \{h : 0 \le h \le f,\ h \text{ is bounded and } m(\text{supp}(h)) < \infty\}$. Let $g_k = \min\{f_k, g\}$. Observe that

$$\{g - g_k \ge \varepsilon\} \subset \{|f - f_k| \ge \varepsilon\}.$$

Hence $g \to g_k$ in measure. Therefore $\|g - g_k\| \to 0$. Since $|g_k| \le g \in L^1$, it follows by Theorem 1.2 that $\|g_k - g\| \to 0$. Consequently

$$\int g = \lim_{k\to\infty} \int g_k \le \liminf_{k\to\infty} \int f_k, \ \forall\ g \in \mathcal{F}(f)$$

The result then follows by taking supremum on LHS for all $g \in \mathcal{F}(f)$.

□

1.1. **Completeness of L^1 space.**

Theorem 1.3 (Riesz-Fischer)**.** *The vector space L^1 is complete in its metric.*

Proof. Suppose $\{f_k\}$ is a Cauchy sequence in the norm, i.e., $\|f_k - f_j\| \to 0$ as $k, j \to \infty$. The plan of the proof is to extract a subsequence of $\{f_k\}$ that converges to f, both pointwise a.e. and in the norm.

Almost everywhere convergence does not hold for general Cauchy sequences [3]. The main point is that if the convergence in the norm is rapid enough, then a.e. convergence is a consequence, and this can be achieved by dealing with an appropriate subsequence of the original sequence.

Step 1. We select a subsequence $\{f_{j_k}\}$ such that $\|f_{j_{k+1}} - f_{j_k}\| \leq 2^{-k}$. In particular $\sum_{k\geq 1} \|f_{j_{k+1}} - f_{j_k}\| < \infty$. Let

$$f(x) := f_{j_1}(x) + \sum_{k=1}^{\infty} (f_{j_{k+1}}(x) - f_{j_k}(x)),$$

and

$$g(x) := |f_{j_1}(x)| + \sum_{k=1}^{\infty} |f_{j_{k+1}}(x) - f_{j_k}(x)|.$$

By Theorem 1.1, $f_{j_k} \to f$ a.e., and $|f| \leq g \in L^1$. Hence, by the DCT,

$$\|f_{j_k} - f\| \to 0 \quad \text{as } k \to \infty.$$

Step 2. We next show $f_k \to f$ in the L^1 sense. Write

$$\|f_k - f\| \leq \|f_k - f_{j_l}\| + \|f_{j_l} - f\|.$$

Given $\varepsilon > 0$, use the fact that $\{f_k\}$ is L^1-Cauchy to choose N_ε so the first term on RHS is $< \varepsilon/2$ for all $k, j_l > N_\varepsilon$. Then choose j_l so the second term is $< \varepsilon/2$, this being permissible since $f_{j_l} \to f$ in the L^1 sense. Then $k > N_\varepsilon$ implies $\|f_k - f\| > \varepsilon$, which yields the result.

$\square$

Since every sequence that converges in norm is a Cauchy sequence in that norm, the argument in the proof of Theorem 1.3 yields the following.

Theorem 1.4. *If $\{f_k\}_{k\geq 1}$ converges to f in L^1, then there exists a subsequence $\{f_{k_j}\}_{j\geq 1}$ such that*

$$f_{k_j}(x) \to f(x) \quad \textit{for a.e.x.}$$

[3] For example let f_k be

$$\chi_{[0,1]},\ \chi_{[0,\frac{1}{2}]},\ \chi_{[\frac{1}{2},1]},\ \chi_{[\frac{1}{3},1]},\ \chi_{[\frac{1}{3},\frac{2}{3}]},\ \chi_{[\frac{2}{3},1]},\ \chi_{[0,\frac{1}{4}]},\ \chi_{[\frac{1}{4},\frac{2}{4}]},$$
$$\chi_{[\frac{2}{4},\frac{3}{4}]},\ \chi_{[\frac{3}{4},1]},\ \chi_{[0,\frac{1}{5}]},\ \cdots,\ \chi_{[\frac{4}{5},1]},\ \chi_{[0,\frac{1}{6}]},\cdots,\ \chi_{[\frac{5}{6},1]},\ \cdots$$

Then $\{f_k\}$ is a Cauchy sequence in $L^1(\mathbb{R})$, but diverges everywhere on $[0,1]$.

Proof. One sees that

$$\|f_k - f_j\| \le \|f_k - f\| + \|f_j - f\| \to 0 \quad \text{as } k, j \to \infty.$$

Hence $\{f_k\}$ is L^1-Cauchy. The result then follows by *Step 1* in the proof of Theorem 1.3.

□

Other L^1 spaces.

If $E \subset \mathbb{R}^n$ is measurable and $m(E) > 0$ then define

$$L^1(E) = \big\{ f : E \to \mathbb{R} \big| \int_E f \text{ exists and is finite} \big\}.$$

Equivalently, for any $f : E \to \mathbb{R}$ let $\widetilde{f}$ be the zero extension of f to $\mathbb{R}^n$. Then define

$$L^1(E) = \big\{ f : E \to \mathbb{R} \big| \widetilde{f} \in L^1(\mathbb{R}^n) \big\}, \quad \|f\|_{L^1(E)} = \|\widetilde{f}\|_{L^1(\mathbb{R}^n)}.$$

It follows almost immediately that $L^1(E)$ is a complete metric space.

Dense subsets of $L^1(\mathbb{R}^n)$.

Definition 1.3. *A family $\mathcal{G}$ of integrable functions is dense in L^1 if for any $f \in L^1$ and $\varepsilon > 0$, there exists $g \in \mathcal{G}$ so that $\|f - g\| < \varepsilon$.* [4]

Fortunately we are familiar with many families that are dense in L^1. These are useful when one is faced with the problem of proving some fact or identity involving integrable easier to prove for a more restrictive class of functions (like the ones in the theorem below), and then a density (or limiting) argument gives the result in general.

Theorem 1.5. *The following are dense in $L^1(\mathbb{R}^n)$:*

(i) *Simple functions;*
(ii) *Step functions;*
(iii) *Continuous functions with compact support.*

Proof. Let $f \in L^1(\mathbb{R}^n)$ be an integrable function.

Result (i). By setting $f = f^+ - f^-$, we see it is sufficient to approximate positive functions f.

[4] Equivalently, there is a sequence of simple functions $g_k \in \mathcal{G}$ such that $g_k \to f$ in the L^1 sense.

If f is positive then there is a sequence of simple functions $0 \le \phi_k \nearrow f$. By the DCT,

$$\|\phi_k - f\| \to 0.$$

Result (ii). Using result (i), it is sufficient to show any simple function $\phi = \sum_{i=1}^N a_i \chi_{E_i}$ can be approximated by a step function.

For this, it is sufficient to show any characteristic function χ_E can be approximated by a step function. Given $\varepsilon > 0$, any measurable set E with finite measure can be approximated by a finite union of almost disjoint closed cubes Q_j such that $m(E \Delta \bigcup_{j=1}^M Q_j) < \varepsilon$. It follows

$$\|\chi_E - \sum_{j=1}^M \chi_{Q_j}\| < \varepsilon.$$

Result (iii). It is sufficient to show that any step function $\sum_{i=1}^N a_i \chi_{R_i}$, where R_i are closed rectangles can be approximated by a continuous compactly supported function.

Hence it is sufficient to show χ_R where R is a closed rectangle can be approximated by a continuous compactly supported function h. This is done by taking $h = 1$ on R and dying off rapidly to 0 in a narrow strip around R. $\square$

As a consequence of Theorems 1.4 & 1.5, we have the following.

Corollary 1.2. *Let $f \in L^1(\mathbb{R}^n)$ be an integrable function. Then*

(i) *there exists a sequence of simple functions $\{\phi_k\}$ such that*

$$\|\phi_k - f\| \to 0 \text{ and } \phi_k \to f \text{ a.e.}$$

(ii) *there exists a sequence of step functions $\{\psi_k\}$ such that*

$$\|\psi_k - f\| \to 0 \text{ and } \psi_k \to f \text{ a.e.}$$

(iii) *there is a sequence of continuous functions with compact support $\{g_k\}$ such that*

$$\|g_k - f\| \to 0 \text{ and } g_k \to f \text{ a.e.}$$

Exercise 1.1. *Suppose $f \in L^1(\mathbb{R}^n)$. If for any continuous function g with compact support, we have*

$$\int f(x)g(x)dx = 0,$$

then $f = 0$ a.e.

Exercise 1.2. *Suppose $f \in L^1([a,b])$. If for any differentiable function φ supported in (a,b), we have*

$$\int_{[a,b]} f(x)\varphi'(x)dx = 0,$$

then $f = const.$ for a.e. $x \in [a,b]$.

Exercise 1.3. *Suppose $f(x)$ is bounded measurable on $I = [0,1]$. If*

$$\int_I x^n f(x)dx = 0 \quad \forall\ n \in \mathbb{N},$$

then $f(x) = 0$ a.e. $x \in [0,1]$.

Exercise 1.4. *Suppose $g_k(x)$ are measurable functions on $[a,b]$ with the property*

$$\max_{x\in[a,b]} |g_k| \leq M \quad \forall\ k, \ \ \textit{and} \ \lim_{k\to\infty} \int_{[a,c]} g_k = 0 \quad \forall\ c \in [a,b].$$

Then

$$\lim_{k\to\infty} \int_{[a,b]} f g_k = 0, \quad \forall\ f \in L^1([a,b]).$$

Exercise 1.5. *Suppose $\{\lambda_n\} \subset \mathbb{R}$ and $\lambda_n \to +\infty$. Let*

$$\Lambda = \{x \in \mathbb{R} : \lim_{n\to\infty} \sin(\lambda_n x) \ \textit{exists}\}.$$

Then $m(\Lambda) = 0$.

1.2. **Invariance properties, translations and continuity.**

Translations, dilates and reflections of measurable functions are measurable. Moreover

(i) $\int f(x-h)dx = \int f(x)dx$,
(ii) $r^n \int f(rx)dx = \int f(x)dx$,
(iii) $\int f(-x)dx = \int f(x)dx$.

Proof: Check this by thinking of step by step: the characteristic function χ_E, simple functions, non-negative function f (hence is a limit of $\{\phi_k\}$ which is a sequence of simple functions increasing to f), and the general integrable functions (by the dominated convergence theorem and cut-off argument).

Suppose f and g are measurable functions so that for some fixed $x \in \mathbb{R}^n$ the function

$$y \mapsto f(x-y)g(y) \quad \text{is an integrable function of } y.$$

Then by (i) and (iii) above, the function $f(y)g(x-y)$ is also integrable and we have [5]

$$\int f(x-y)g(y)dy = \int f(y)g(x-y)dy.$$

The RHS is the convolution $f * g$ evaluated at x. The result says that $f * g = g * f$, i.e. convolution is commutative.

We next examine how continuity properties of f are related to the way the translations $f_h(x) := f(x-h)$ vary with h. In general, integrable function f may be discontinuous at every x, even when corrected on a set of measure zero. Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed enumeration $\{r_k\}_{k \geq 1}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} f(x - r_k).$$

Let $F_N(x) = \sum_{k=1}^{N} 2^{-k} f(x - r_k)$. Then

$$\int F_N = \sum_{k=1}^{N} \frac{1}{2^k} \int f(x - r_k) = \sum_{k=1}^{N} \frac{1}{2^k} \int f(x) = \sum_{k=1}^{N} 2^{1-k}.$$

By the monotone convergence theorem,

$$\int F = \lim_{N \to \infty} \int F_N = 2.$$

This implies that the series defining $F(x)$ converges for almost everywhere. Let $\tilde{F}$ be a function such that $\tilde{F}(x) = F(x)$ for a.e. x. We show that $\tilde{F}$ is unbounded in any open interval I. As a by-product, one sees that $\tilde{F}$ is always discontinuous everywhere.

[5] Replace y by $x - y$ and use (i) and (iii).

Fixing a $r_n \in I$, for each $K > 1$, there is sufficiently small δ_K so that, $\forall x \in (r_n, r_n + \delta_K)$, $F(x) \geq 2^{-n} f(x - r_n) > K$. Hence $\tilde{F}$ is unbounded on I.

Nevertheless, there is an overall continuity that an arbitrary $f \in L^1(\mathbb{R}^n)$ enjoys, one that holds in the norm.

Theorem 1.6. *Suppose $f \in L^1(\mathbb{R}^n)$. Then $\|f_h - f\| \to 0$ as $h \to 0$.*

Proof. Given $\varepsilon > 0$, choose a continuous function g such that $\|f - g\| < \varepsilon/3$ (using Theorem 1.5). Then

$$\|f - f_h\| \leq \|f - g\| + \|f_h - g_h\| + \|g - g_h\| \leq 2\|f - g\| + \|g - g_h\| < \varepsilon,$$

provided $|h| < \delta$ so that by the uniform continuity of g and the compactness of $\text{supp}(g)$

$$\int |g(x) - g(x - h)| < \varepsilon/3.$$

Alternative argument: Choose a step function such that $\|f - \psi\| < \varepsilon/3$ (see Theorem 1.5). Argue as above and note that if $|h|$ is small then

$$\|\psi_h - \psi\| < \varepsilon/3.$$

$\square$

1.3. **Interchanging integration with differentiation and limits.**

In this section, $f(x,t)$ is a function of two variables $x \in E \subset \mathbb{R}^n$ and $t \in \mathbb{R}$. Suppose $f(\cdot, t) \in L^1(\mathbb{R}^n)$ for each t. Let

$$F(t) = \int_E f(x,t)dx \tag{1.6}$$

We are concerned with

(1) If $f(x,t)$ is continuous in t for each $x \in E$, does it follow that $F(t)$ is continuous??
(2) If $f(x,t)$ is differentiable in t for each $x \in E$, does it follow that $F'(t)$ exists and

$$F'(t) = \int_E \partial_t f(x,t)dx \ ??$$

The answer is "YES", provided that the functions $f(\cdot, t)$, respectively $\partial_t f(\cdot, t)$ are uniformly integrable over E. That is

$$|f(x,t)| \leq g(x) \in L^1 \text{ for all } t, \text{ respectively } |\partial_t f(x,t)| \leq h(x) \in L^1 \text{ for all } t.$$

The proof is straightforward using (a) the dominated convergence theorem; (b) the observation that the relevant limit exists at t_0 if the limit both exists for any sequence $t_k \to t_0$ and is independent of the sequence.

Theorem 1.7. *Suppose $E \subset \mathbb{R}^n$, $f : E \times (a,b) \to \mathbb{R}$, and $f(x,t)$ is integrable in x for each $t \in (a,b)$. Let $F(t) : (a,b) \to \mathbb{R}$ be as in (1.6).*

- *Suppose $f(x,t)$ is continuous in t for each $x \in E$. Suppose for all $x \in E$ and $t \in (a,b)$ that $|f(x,t)| \le g(x)$ where $g : E \to \mathbb{R}$ is integrable. Then $F(t)$ is continuous in t, namely*

$$\lim_{t\to t_0} \int_E f(x,t)dx = \int_E f(x,t_0)dx, \ \forall\ t_0 \in (a,b). \tag{1.7}$$

- *Suppose $\partial_t f(x,t)$ exists for all t and all x. Suppose for all $x \in E$ and $t \in (a,b)$ that $|\partial_t f(x,t)| \le h(x)$ where $h : E \to \mathbb{R}$ is integrable. Then F is differentiable, and*

$$\frac{\partial}{\partial t}\Big|_{t=t_0} \int_E f(x,t)dx = \int_E \partial_t f(x,t_0)dx, \ \forall\ t_0 \in (a,b). \tag{1.8}$$

Proof. Let t_k be any sequence in (a,b) such that $t_k \to t_0$. Denote $f_k(x) := f(x,t_k)$ and $f_0(x) = f(x,t_0)$ Since $f_k(x) \to f_0(x)$ for all x and $|f_k| \le g \in L^1(E)$, it follows by the dominated convergence theorem that

$$\lim_{k\to\infty} \int_E f(x,t_k)dx = \lim_{k\to\infty} \int_E f_k = \int_E f_0 = \int_E f(x,t_0)dx.$$

Since the RHS is independent of the sequence t_k, (1.7) follows.

Next for $t_k \in (a,b)$, $t_k \to t_0$, $t_k \neq t_0$, we have

$$\delta_k f(x) := \frac{f(x,t_k) - f(x,t_0)}{t_k - t_0} \to \partial_t f(x,t_0).$$

By the mean value theorem, there is a ξ_k between t_0 and t_k such that

$$|\delta_k f(x)| = |\partial_t f(x,\xi_k)| \le g(x) \in L^1(E).$$

Employing the dominated convergence theorem, we deduce that

$$\lim_{k\to\infty} \frac{F(t_k) - F(t_0)}{t_k - t_0} = \lim_{k\to\infty} \int_E \delta_k f(x)dx = \int_E \partial_t f(x,t_0)dx.$$

This proves the differentiability of F and shows (1.8).

□

Exercise 1.6. *Suppose $f(x) \in L^1(\mathbb{R})$ and $xf(x) \in L^1(\mathbb{R})$. Then*
$$\frac{d}{dt}\int_{\mathbb{R}} f(x)\sin(tx)dx = \int_{\mathbb{R}} xf(x)\cos(tx)dx.$$

Proof. Let $g(x,t) = f(x)\sin(tx)$. Then $g(x,t)$ is differentiable in t

$$\partial_t g(x,t) = xf(x)\cos(tx),\ \forall\ (x,t) \in \mathbb{R}^n \times \mathbb{R},\ \text{and}\ |\partial_t g(x,t)| \le |xf(x)| \in L^1(\mathbb{R}).$$

Hence we can apply Theorem 1.7 to conclude the desired result.

□

1.4. **Lebesgue Integral v.s. Riemann Integral.**

This section concerns the Lebesgue/Riemann integrability. Only in this section,

$$\int_E^{\mathcal{L}} f \quad \text{and} \quad \int_E^{\mathcal{R}} f$$

are used to denote the Lebesgue and respectively Riemann integral.

Let f be a bounded real-valued function on $[a,b] \subset \mathbb{R}$. A partition P of $[a,b]$ is a finite sequence of numbers $x_0, x_1, \cdots, x_N$ with

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N \le b.$$

Denote $I_j = [x_{j-1}, x_j]$ and $I_j = |x_j - x_{j-1}|$ the length of I_j. We define the upper and lower sum of f with respect to P by

$$\mathcal{S}_f^+(P) = \sum_{j=1}^N [\sup_{x\in I_j} f(x)]|I_j| \quad \text{and} \quad \mathcal{S}_f^-(P) = \sum_{j=1}^N [\inf_{x\in I_j} f(x)]|I_j|.$$

Obviously $\mathcal{S}_f^+(P) \ge \mathcal{S}_f^-(P)$.

We say f is Riemann integrable if for each $\varepsilon > 0$, there is a partition P so that

$$\mathcal{S}_f^+(P) - \mathcal{S}_f^-(P) \le \varepsilon.$$

We say P' is a refinement of P if P' is obtained from P by adding points. It is easy to check

$$\mathcal{S}_f^+(P) \ge \mathcal{S}_f^+(P') \quad \text{and} \quad \mathcal{S}_f^-(P') \ge \mathcal{S}_f^-(P).$$

It follows that, if P_1 and P_2 are two partitions of $[a,b]$, then

$$\mathcal{S}_f^+(P_1) \ge \mathcal{S}_f^-(P_2),$$

since it is possible to make P' as a common refinement of both P_1 and P_2. By the boundedness of f, we see that both

$$\mathcal{S}_f^+ = \inf_P \mathcal{S}_f^+(P) \quad \text{and} \quad \mathcal{S}_f^- = \sup_P \mathcal{S}_f^-(P)$$

exist and also that $\mathcal{S}_f^+ \geq \mathcal{S}_f^-$. Moreover, f is Riemann integrable if and only if $\mathcal{S}_f^+ = \mathcal{S}_f^-$, and define

$$\int_{[a,b]}^{\mathcal{R}} f = \mathcal{S}_f^+ = \mathcal{S}_f^-.$$

Recall that the oscillation of f over $B_\delta(x)$ [6] is given by,

$$\omega_f(B_\delta(x)) := \sup\big\{|f(x') - f(x'')| : \ x', x'' \in B_\delta(x)\big\},$$

which is decreasing in δ. The oscillation of f at x (a function) is then defined as

$$\omega_f(x) = \lim_{\delta \to 0} \omega_f(B_\delta(x)).$$

Since $\{\omega_f(x) < t\}$ is open [7], we know that $\omega_f(x)$ is a measurable function on $[a, b]$.

We next prove the characterisation of Riemann integrable functions in terms of their discontinuous. The following lemma is crucial.

Lemma 1.1. *Suppose f is a bounded function on $[a, b]$. Then*

$$\int_{[a,b]}^{\mathcal{L}} \omega_f = \mathcal{S}_f^+ - \mathcal{S}_f^-.$$

Proof. Choose P_k^+ and P_k^- such that $\mathcal{S}_f^+(P_k^+) \to \mathcal{S}_f^+$ and $\mathcal{S}_f^-(P_k^-) \to \mathcal{S}_f^-$. Considering the common refinement of P_k^+ and P_k^-, say P_k, we see that

$$\mathcal{S}_f^+(P_k) \to \mathcal{S}_f^+ \quad \text{and} \quad \mathcal{S}_f^-(P_k) \to \mathcal{S}_f^- \quad \text{as } k \to \infty.$$

Suppose $P_k = \{x_i^k\}_{0 \leq i \leq N_k}$. We define a family of simple functions below

$$\omega_{f,P_k}(x) := \sum_{j=1}^{N_k} [\sup_{y \in I_{k,j}} f(y) - \inf_{y \in I_{k,j}} f(y)] \chi_{I_{k,j}^0},$$

where $I_{k,j}^0 = (x_{j-1}^k, x_j^k)$.

[6] In one-dimension, $B_\delta(x) = (x - \delta, x + \delta)$.

[7] See Section 1.5 in the lecture notes.

It is necessary that $\max_{1\le j\le N_k} |I_{k,j}| \to 0$ as $k \to \infty$. Hence $\omega_{f,P_k}(x) \to \omega_f(x)$ for all $x \in [a,b] \setminus Z$, where $Z = \cup_{k=1}^{\infty} \cup_{0\le i\le N_k} \{x_i^k\}$. By the boundedness of f, $\omega_{f,P_k}(x) \le \sup_{[a,b]} f - \inf_{[a,b]} f \in L^1([a,b])$. Employing the bounded convergence theorem,

$$\int_{[a,b]}^{\mathcal{L}} \omega_f = \lim_{k\to\infty} \int_{[a,b]}^{\mathcal{L}} \omega_{f,P_k} = \lim_{k\to\infty} \left(\mathcal{S}_f^+ - \mathcal{S}_f^-\right) = \mathcal{S}_f^+ - \mathcal{S}_f^-.$$

□

Theorem 1.8. *A bounded function f on $[a,b]$ is Riemann integrable if and only if its set of discontinuities has measure zero.*

Proof. It follows by Lemma 1.1 and the fact that f is continuous at x if and only if $\omega_f(x) = 0$. □

Theorem 1.9. *If f is Riemann integrable on $[a,b]$ then f is Lebesgue integrable and*

$$\int_{[a,b]}^{\mathcal{L}} f = \int_{[a,b]}^{\mathcal{R}} f.$$

Proof. In view of Theorem 1.8, f is continuous a.e. on $[a,b]$, hence is measurable. By the boundedness of f, it is Lebesgue integrable. For a $P = \{x_i\}_{i=1}^N$, we

$$\int_{[a,b]}^{\mathcal{L}} f = \sum_{j=1}^{N} \int_{I_j}^{\mathcal{L}} f, \quad \text{where } I_j = [x_{j-1}, x_j].$$

It follows that, for any partition P,

$$\mathcal{S}_f^-(P) = \sum_{j=1}^{N} (\inf_{I_j} f)|I_j| \le \int_{[a,b]}^{\mathcal{L}} f \le \sum_{j=1}^{N} (\sup_{I_j} f)|I_j| = \mathcal{S}_f^+(P)$$

By the arbitrariness of the partition,

$$\mathcal{S}_f^- \le \int_{[a,b]}^{\mathcal{L}} f \le \mathcal{S}_f^+.$$

We the complete the proof by the Riemann integrability of f.

□

Theorem 1.10. *Suppose f_k are bounded functions and are Riemann integrable on $[a,b]$. If $f_k \to f$ uniformly, then f is Riemann integrable on $[a,b]$. Moreover*

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{R}} |f - f_k| = 0.$$

Proof. Exercise. □

Remark 1.1. *The notion of Riemann integration on a rectangle $R \subset \mathbb{R}^n$ is an immediate generalisation of the notion of Riemann integration on an interval $[a, b] \subset \mathbb{R}$.*

Example 1.1 (Lebesgue integrable but not Riemann integrable)**.** *Consider the bounded function f on $[0, 1]$ given by*

$$f(x) = \chi_{\mathbb{Q}\cap[0,1]}(x).$$

It is discontinuous everywhere, and so is not Riemann integrable. But it is Lebesgue integrable and

$$\int_{[0,1]}^{\mathcal{L}} f = 0.$$

Example 1.2 (Limit of Riemann integrable functions may be Lebesgue integrable but not Riemann integrable)**.** *Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. For $n \in \mathbb{N}$, define*

$$f_n(x) = \chi_{\{r_k\}_{k=1}^{n}}(x).$$

Then $f_n \to f := \chi_{\mathbb{Q}\cap[0,1]}$. Clearly f_n is Riemann integrable but f is not.

Theorem 1.9 only holds for integration of bounded function over bounded set. If E is possibly unbounded, the integrability of f on E may follow from the integrability of f on bounded sets E_k which increases to E, with additional conditions as mentioned below.

Proposition 1.2. *Let E_k be measurable sets of $\mathbb{R}^n$ and $E_k \nearrow E$. Suppose $f \in L^1(E_k)$ and $\lim_{k\to\infty} \int_{E_k} |f|$ exists. Then $f \in L^1(E)$ and*

$$\int_E f = \lim_{k\to\infty} \int_{E_k} f.$$

Proof. Note that $0 \le |f|\chi_{E_k} \nearrow |f|\chi_E$. The monotone convergence theorem implies that

$$\int_E |f| = \lim_{k\to\infty} \int_{E_k} |f| < \infty.$$

Therefore $f \in L^1(E)$.

Since $f\chi_{E_k} \to f\chi_E$, and $|f\chi_{E_k}| \le |f|\chi_E \in L^1(E)$, we complete the proof by applying the dominated convergence theorem.

$\square$

Example 1.3. *Let* $f(x) = \sin x/x$. *Then*

$$\lim_{a \to 0+, b \to \infty} \int_{[a,b]} f(x)dx = \frac{\pi}{2}. \tag{1.9}$$

Hence it is Riemann integrable in the above sense. But $f \notin L^1([0,\infty))$ *as*

$$\int_{[0,\infty)} |f(x)|dx = \infty. \tag{1.10}$$