# INSTRUCTOR'S MANUAL

to accompany

*Linear Algebra:*
*4th Edition*

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# 1

# Vector Spaces

## 1.1 INTRODUCTION

**2.** **(b)** $x = (2,4,0) + t(-5,-10,0)$ **(d)** $x = (-2,-1,5) + t(5,10,2)$

**3.** **(b)** $x = (3,-6,7) + s(-5,6,-11) + t(2,-3,-9)$
**(d)** $x = (1,1,1) + s(4,4,4) + t(-7,3,1)$

**4.** $(0,0)$

## 1.2 VECTOR SPACES

**2.** $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

**4.** **(b)** $\begin{pmatrix} 1 & -1 \\ 3 & -5 \\ 3 & 8 \end{pmatrix}$ **(d)** $\begin{pmatrix} 30 & -20 \\ -15 & 10 \\ -5 & -40 \end{pmatrix}$

**(f)** $-x^3 + 7x^2 + 4$ **(h)** $3x^5 - 6x^3 + 12x + 6$

**5.** $\begin{pmatrix} 8 & 3 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 9 & 1 & 4 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 17 & 4 & 5 \\ 6 & 0 & 0 \\ 4 & 1 & 0 \end{pmatrix}$

**16.** Yes **18.** No, (VS 1) fails. **19.** No, (VS 8) fails.

## 1.3 SUBSPACES

**2.** **(b)** $\begin{pmatrix} 0 & 3 \\ 8 & 4 \\ -6 & 7 \end{pmatrix}$ **(d)** $\begin{pmatrix} 10 & 2 & -5 \\ 0 & -4 & 7 \\ -8 & 3 & 6 \end{pmatrix}$ **(f)** $\begin{pmatrix} -2 & 7 \\ 5 & 0 \\ 1 & 1 \\ 4 & -6 \end{pmatrix}$

The trace is 12.

**(h)** $\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}$

The trace is 2.

**8.** **(b)** No **(d)** Yes **(f)** No

**9.** $\mathsf{W}_1 \cap \mathsf{W}_3 = \{0\}$, $\mathsf{W}_1 \cap \mathsf{W}_4 = \mathsf{W}_1$,
$\mathsf{W}_3 \cap \mathsf{W}_4 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 \colon a_1 = -11a_3 \text{ and } a_2 = -3a_3\}$

## 1.4 LINEAR COMBINATIONS AND SYSTEMS OF LINEAR EQUATIONS

**2.** **(b)** $(-2,-4,-3)$
**(d)** $\{x_3(-8,3,1,0)+(-16,9,0,2)\colon\ x_3 \in R\}$
**(f)** $(3,4,-2)$

**3.** **(a)** $(-2,0,3)=4(1,3,0)-3(2,4,-1)$
**(b)** $(1,2,-3)=5(-3,2,1)+8(2,-1,-1)$
**(d)** No
**(f)** $(-2,2,2)=4(1,2,-1)+2(-3,-3,3)$

**4.** **(a)** $x^3-3x+5=3(x^3+2x^2-x+1)-2(x^3+3x^2-1)$
**(b)** No
**(c)** $-2x^3-11x^2+3x+2=4(x^3-2x^2+3x-1)-3(2x^3+x^2+3x-2)$
**(d)** $x^3+x^2+2x+13=-2(2x^3-3x^2+4x+1)+5(x^3-x^2+2x+3)$
**(f)** No

**5.** **(b)** No **(d)** Yes **(f)** No **(h)** No

**11.** The span of $\{x\}$ is $\{0\}$ if $x=0$ and is the line through the origin of $\mathsf{R}^3$ in the direction of $x$ if $x \neq 0$.

**17.** if W is finite

## 1.5 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

**2.** **(b)** Linearly independent **(d)** Linearly dependent
**(f)** Linearly independent **(h)** Linearly independent
**(j)** Linearly dependent

**10.** $(1,0,0)$, $(0,1,0)$, $(1,1,0)$

## 1.6 BASES AND DIMENSION

**2.** **(b)** Not a basis **(d)** Basis

**3.** **(b)** Basis **(d)** Basis

**4.** No, $\dim(\mathsf{P}_3(R))=4$. **5.** No, $\dim(\mathsf{R}^3)=3$.

**8.** $\{u_1,u_3,u_5,u_7\}$

**10.** **(b)** $12-3x$ **(d)** $-x^3+2x^2+4x-5$

**14.** $\{(0,1,0,0,0),(0,0,0,0,1),(1,0,1,0,0),(1,0,0,1,0)\}$ and $\{(-1,0,0,0,1),(0,1,1,1,0)\}$; $\dim(\mathsf{W}_1)=4$ and $\dim(\mathsf{W}_2)=2$.

**16.** $\dim(\mathsf{W})=\frac{1}{2}n(n+1)$

**18.** Let $\sigma_j$ be the sequence such that

$$\sigma_j(i) = \begin{cases} 0 & i = j \\ 1 & i \neq j. \end{cases}$$

Then $\{\sigma_j \colon \ j = 1, 2, \ldots\}$ is a basis for the vector space in Example 5 of Section 1.2.

**22.** $\mathsf{W}_1 \subseteq \mathsf{W}_2$

**23.** **(a)** $v \in \mathsf{W}_1$ **(b)** $\dim(\mathsf{W}_2) = \dim(\mathsf{W}_1) + 1$

**25.** $mn$

**27.** If $n$ is even, then $\dim(\mathsf{W}_1) = \dim(\mathsf{W}_2) = \dfrac{n}{2}$; and if $n$ is odd, then $\dim(\mathsf{W}_1) = \dfrac{n+1}{2}$ and $\dim(\mathsf{W}_2) = \dfrac{n-1}{2}$.

**32.** **(a)** Take $\mathsf{W}_1 = \mathsf{R}^3$ and $\mathsf{W}_2 = \text{span}(\{e_1\})$.
**(b)** Take $\mathsf{W}_1 = \text{span}(\{e_1, e_2\})$ and $\mathsf{W}_2 = \text{span}(\{e_3\})$.
**(c)** Take $\mathsf{W}_1 = \text{span}(\{e_1, e_2\})$ and $\mathsf{W}_2 = \text{span}(\{e_2, e_3\})$.

**35.** **(b)** $\dim(\mathsf{V}) = \dim(\mathsf{W}) + \dim(\mathsf{V}/\mathsf{W})$

# 2

# Linear Transformations and Matrices

## 2.1 LINEAR TRANSFORMATIONS, NULL SPACES, AND RANGES

**3.** The nullity is 0, and the rank is 2. Thus $\mathsf{T}$ is one-to-one, but not onto.

**6.** The nullity is $n-1$, and the rank is 1. Thus $\mathsf{T}$ is not one-to-one unless $n=1$, and $\mathsf{T}$ is not onto unless $n=1$.

**11.** $\mathsf{T}(8,11)=(5,-3,16)$

**18.** $\mathsf{T}(a,b)=(b,0)$. $\mathsf{N}(\mathsf{T})=\text{span}\{(1,0)\}=\mathsf{R}(\mathsf{T})$.

**19.** Define $\mathsf{T}=\mathsf{I}$ and $\mathsf{U}=2\mathsf{I}$.

**23.** All of $\mathsf{R}^3$ or a plane in $\mathsf{R}^3$ through the origin

**24.** **(a)** $\mathsf{T}(a,b)=(0,b)$ **(b)** $\mathsf{T}(a,b)=(0,b-a)$

**25.** **(b)** $\mathsf{T}(a,b,c)=(0,0,c)$

**26.** **(a)** $\mathsf{T}=\mathsf{I}_\mathsf{V}$ **(d)** $\mathsf{T}=\mathsf{T}_0$

**27.** **(b)** See (a) and (b) of Exercise 22.

**31.** **(c)** Let $\mathsf{V}=\mathsf{P}(F)$ and $\mathsf{V}=\text{span}(\{1\})$. Define $\mathsf{T}$ first on the standard basis of $\mathsf{V}$ by $\mathsf{T}(1)=\mathsf{T}(x)=\mathit{0}$, and $\mathsf{T}(x^k)=x^{k-1}$ for $k\geq 2$. Now extend $\mathsf{T}$ to a linear transformation from $\mathsf{V}$ to $\mathsf{V}$. Then $\mathsf{N}(\mathsf{T})=\text{span}(\{1,x\})$, and $\mathsf{R}(\mathsf{T})=\text{span}(\{x^k\colon\ k\geq 1\})$. So $\mathsf{V}=\mathsf{R}(\mathsf{T})\oplus\mathsf{W}$, but $\mathsf{W}\neq\mathsf{N}(\mathsf{T})$.

## 2.2 THE MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

**2.** **(b)** $\begin{pmatrix}2 & 3 & -1\\ 1 & 0 & 3\end{pmatrix}$ **(e)** $\begin{pmatrix}1 & 0 & \dots & 0\\ 1 & 0 & \dots & 0\\ \vdots & \vdots & & \vdots\\ 1 & 0 & \dots & 0\end{pmatrix}$

**4.** $\begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\end{pmatrix}$

**5.** **(c)** $(1\ 0\ 0\ 1)$ **(d)** $(1\ 2\ 4)$ **(f)** $\begin{pmatrix}3\\ -6\\ 1\end{pmatrix}$ **(g)** (a)

## 2.3 COMPOSITION OF LINEAR TRANSFORMATIONS AND MATRIX MULTIPLICATION

**2.** **(a)** $(AB)D = \begin{pmatrix} 29 \\ -26 \end{pmatrix}$

**(b)** $A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$, $BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$, $CA = \begin{pmatrix} 20 & 26 \end{pmatrix}$

**3.** **(b)** $[h]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $[\mathsf{U}(h)]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

**4.** **(b)** $\begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$ **(d)** $(12)$

**9.** $\mathsf{T}(a_1, a_2) = (0, a_1 + a_2)$, $\mathsf{U}(a_1, a_2) = (0, a_1)$,

$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$, $CA = \begin{pmatrix} 20 & 26 \end{pmatrix}$

**20.** **(a)** $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $B^3 = \begin{pmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{pmatrix}$ There are no cliques.

**(b)** $B = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $B^3 = \begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 3 \\ 3 & 0 & 3 & 2 \end{pmatrix}$

Persons 1, 3, and 4 belong to a clique.

**23.** $\dfrac{n^2 - n}{2}$

## 2.4 INVERTIBILITY AND ISOMORPHISMS

**14.** $\mathsf{T}\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, b, c)$

## 2.5 THE CHANGE OF COORDINATE MATRIX

**2.** **(b)** $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

**3.** **(b)** $\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$ **(f)** $\begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}$

**6.** **(b)** $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $[\mathsf{L}_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

**(d)** $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$, $[\mathsf{L}_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$

**7.** **(b)** $\mathsf{T}(x, y) = \frac{1}{m^2+1}(x + my, mx + m^2y)$

## 2.6 DUAL SPACES

**3.** **(b)** $f_1(a + bx + cx^2) = a$, $f_2(a + bx + cx^2) = b$, $f_3(a + bx + cx^2) = c$

**4.** The basis for V is $\{(.4, -.3, -.1), (.6, .3, .1), (.2, .1, -.3)\}$

**6.** **(a)** $\mathsf{T}^t(f)(x, y) = 7x + 4y$ **(b)** $[\mathsf{T}^t]_{\beta^*} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$

**(c)** $[\mathsf{T}]_\beta = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ and $([\mathsf{T}]_\beta)^t = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$

## 2.7 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

**3.** **(b)** $\{1, e^t\}$ **(d)** $\{e^{-t}, te^{-t}\}$

**4.** $\{t, te^t, t^2e^t\}$

**16.** **(a)** $\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{l}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{l}}t\right)$

**(b)** $\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right)$

**17.** $y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$

**18.** **(a)** *Case 1:* $r^2 = 4km$. $y(t) = e^{-(r/2m)t}[c_1 + c_2t]$

*Case 2:* $r^2 > 4km$. $y(t) = c_1e^{at} + c_2e^{bt}$, where

$$a = \frac{-r}{2m} + \frac{\sqrt{r^2 - 4mk}}{2m}, \quad b = \frac{-r}{2m} - \frac{\sqrt{r^2 - 4mk}}{2m}$$

*Case 3:* $r^2 < 4km$. $y(t) = e^{at}[c_1 \cos bt + c_2 \sin bt]$, where

$$a = \frac{-r}{2m}, \quad b = \frac{\sqrt{4mk - r^2}}{2m}$$

**(b)** Referring to the three cases listed in (a):
*Case 1:* $y(t) = v_0te^{-(r/2m)t}$

*Case 2:* $y(t) = \frac{v_0m}{\sqrt{r^2 - 4mk}}[e^{at} - e^{bt}]$

*Case 3:* $y(t) = \frac{v_0}{b}e^{at} \sin bt$

# 3

# Elementary Matrix Operations and Systems of Linear Equations

## 3.1 ELEMENTARY MATRIX OPERATIONS AND ELEMENTARY MATRICES

**2.** Adding $-1$ times row 1 to row 2 transforms $B$ into $C$.
A sequence of elementary operations that transforms $C$ into $I_2$ is:

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -3 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

**3. (b)** $\begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

## 3.2 THE RANK OF A MATRIX AND MATRIX INVERSES

**2. (b)** 2 **(d)** 1 **(f)** 3

**4. (a)** $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$; the rank is 2.

**5. (b)** The rank is 1; so no inverse exists.

**(d)** The rank is 3, and the inverse is $\begin{pmatrix} -0.5 & 3 & -1 \\ 1.5 & -4 & 2 \\ 1.0 & -2 & 1 \end{pmatrix}$.

**(f)** The rank is 2; so no inverse exists.
**(h)** The rank is 3; so no inverse exists.

**6. (b)** $\mathsf{T}$ is not invertible.
**(d)** $\mathsf{T}^{-1}(ax^2 + bx + c) = (c, 0.5a - 0.5b, 0.5a + 0.5b - c)$
**(f)** $\mathsf{T}$ is not invertible.

**19.** $m$

## 3.3 SYSTEMS OF LINEAR EQUATIONS—THEORETICAL ASPECTS

**2.** **(b)** $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$ **(d)** $\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ **(f)** $\varnothing$

**3.** **(b)** $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}+t\begin{pmatrix}1\\2\\3\end{pmatrix} : t \in R\right\}$ **(d)** $\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\1\end{pmatrix} : t \in R\right\}$

**(f)** $\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$

**4.** **(a)** **(1)** $A^{-1} = \begin{pmatrix}-5 & 3\\ 2 & -1\end{pmatrix}$ **(2)** $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-11\\5\end{pmatrix}$

**5.** $\begin{aligned} x + \ \ y &= 0\\ 2x + 2y &= 0\end{aligned}$

**8.** **(a)** Yes **(b)** Yes

**12.** $\frac{3}{7}$ of the total economic output

**14.** \$300 billion worth of goods and \$200 billion worth of services

## 3.4 SYSTEMS OF LINEAR EQUATIONS—COMPUTATIONAL ASPECTS

**2.** **(b)** $\left\{\begin{pmatrix}9\\4\\0\end{pmatrix}+t\begin{pmatrix}-5\\-3\\1\end{pmatrix} : t \in R\right\}$ **(d)** $\begin{pmatrix}-21\\-16\\14\\-10\end{pmatrix}$

**(f)** $\left\{\begin{pmatrix}-3\\3\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\-2\\0\\1\end{pmatrix} : t \in R\right\}$ **(h)** $\left\{\begin{pmatrix}-3\\-8\\0\\0\\3\end{pmatrix}+t\begin{pmatrix}1\\-2\\1\\0\\0\end{pmatrix} : t \in R\right\}$

**(j)** $\left\{\begin{pmatrix}1\\0\\1\\-1\\0\end{pmatrix}+t\begin{pmatrix}2\\3\\0\\2\\1\end{pmatrix} : t \in R\right\}$

**4.** **(b)** $\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+s\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}1\\0\\1\\2\end{pmatrix} : s, t \in R\right\}$

**6.** $\begin{pmatrix} 1 & -3 & -1 & 1 & 0 & 3 \\ -2 & 6 & 1 & -5 & 1 & -9 \\ -1 & 3 & 2 & 2 & -3 & 2 \\ 3 & -9 & -4 & 0 & 2 & 5 \end{pmatrix}$

**8.** $\{u_1, u_3, u_5, u_7\}$

**9.** $\left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \right\}$

**10.** **(b)** $\{(0,1,1,1,0), (2,1,0,0,0), (-3,0,1,0,0), (-2,0,0,0,1)\}$

**12.** **(b)** $\{(0,-1,0,1,1,0), (1,0,1,1,1,0), (-1,1,0,1,0,0), (-3,-2,0,0,0,1)\}$

# 4

# Determinants

## 4.1 DETERMINANTS OF ORDER 2

**2.** **(b)** $-17$

**3.** **(b)** $36 + 41i$

**4.** **(b)** $10$ **(d)** $14$

## 4.2 DETERMINANTS OF ORDER $n$

**2.** $27$ **4.** $2$ **6.** $-13$

**8.** $-13$ **10.** $4 + 2i$ **12.** $154$

**14.** $-168$ **16.** $36$ **18.** $10$

**20.** $17 - 3i$ **22.** $-100$

**26.** if $n$ is even or $\det(A) = 0$

**28.** $\det(E_1) = -1$ and $\det(E_3) = 1$.

**30.** If $n$ is even, then $\det(B) = (-1)^{\frac{n}{2}} \cdot \det(A)$. If $n$ is odd, then $\det(B) = (-1)^{\frac{n-1}{2}} \cdot \det(A)$.

## 4.3 PROPERTIES OF DETERMINANTS

**2.** $x_1 = \dfrac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$ and $x_2 = \dfrac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$

**4.** $(-1.0, -1.2, -1.4)$ **6.** $(-43, -109, -17)$

**18.** $A_{11}A_{22}\cdots A_{nn}$

**25.** **(b)** $\begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 16 \\ 0 & 0 & 16 \end{pmatrix}$ **(d)** $\begin{pmatrix} 20 & -30 & 20 \\ 0 & 15 & -24 \\ 0 & 0 & 12 \end{pmatrix}$

**(f)** $\begin{pmatrix} 6 & 22 & 12 \\ 12 & -2 & 24 \\ 21 & -38 & -27 \end{pmatrix}$ **(h)** $\begin{pmatrix} -i & -8+i & -1+2i \\ 1-5i & 9-6i & -3i \\ -1+i & -3 & 3-i \end{pmatrix}$

## 4.4 SUMMARY–IMPORTANT FACTS ABOUT DETERMINANTS

**2.** **(b)** $-29$ **(d)** $-24+6i$

**3.** **(b)** $-13$ **(d)** $-13$ **(f)** $4+2i$ **(h)** $154$

**4.** **(b)** $36$ **(d)** $10$ **(f)** $17-3i$ **(h)** $-100$

## 4.5 A CHARACTERIZATION OF THE DETERMINANT

**2.** The 1-linear functions $\delta$: $\mathsf{M}_{1\times 1}(F) \to F$ have the form $\delta(A_{11}) = cA_{11}$ for some scalar $c$.

**4.** No

**6.** No

**8.** No

**10.** Yes

**20.** Define $\delta$: $\mathsf{M}_{3\times 3}(F) \to F$ by $\delta(A) = A_{11}A_{21}A_{31}$. Then $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ has identical rows, but $\delta(B) \neq 0$.

# 5

# Diagonalization

## 5.1 EIGENVALUES AND EIGENVECTORS

**2.** **(b)** $[\mathsf{T}]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$, yes

**(d)** $[\mathsf{T}]_\beta = \begin{pmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ -4 & 0 & 0 \end{pmatrix}$, no

**(f)** $[\mathsf{T}]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, yes

**3.** **(b)** The eigenvalues are 1, 2, and 3. A basis of eigenvectors is

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}. \quad Q = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

**(d)** The eigenvalues are 0 and 1. A basis of eigenvectors is

$$\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}. \quad Q = \begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**4.** **(c)** $\lambda = 2, -1$ $\quad \beta = \{(-1,2,2), (1,1,0), (-2,0,1)\}$
**(d)** $\lambda = -2, -3$ $\quad \beta = \{x+2, 2x+3\}$
**(e)** $\lambda = 4, 2, 0$ $\quad \beta = \{1+x, 3+13x-4x^2, 3-x\}$
**(g)** $\lambda = -1, 1, 2, 3$ $\quad \beta = \{1, 1-x, 2-3x^2, -7+6x+2x^3\}$

**10.** **(b)** $(\lambda - t)^n$, where $n = \dim(\mathsf{V})$

**17.** **(c)** The eigenvectors corresponding to $\lambda = 1$ are the nonzero symmetric matrices.
The eigenvectors corresponding to $\lambda = -1$ are the nonzero skew-symmetric matrices.

**(d)** $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

**(e)** $\beta = \{D_i\colon 1 \le i \le n\} \cup \{E_{ij}\colon 1 \le i < j \le n\} \cup \{F_{ij}\colon 1 \le i < j \le n\}$, where
$D_i$ is the $n \times n$ diagonal matrix with 1 as the $i$th diagonal entry and 0 elsewhere,
$E_{ij}$ is the $n \times n$ matrix with 1 as the $ij$th entry, $-1$ as the $ji$th entry, and 0 elsewhere,
and $F_{ij}$ is the $n \times n$ matrix with 1 as both the $ij$th and $ji$th entries and 0 elsewhere.

**18.** **(b)** Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

## 5.2 DIAGONALIZABILITY

**2.** **(b)** $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ **(d)** $Q = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}$ **(f)** Not diagonalizable

**3.** **(b)** $\beta = \{x, 1 + x^2, -1 + x^2\}$ **(f)** $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

**13.** **(a)** Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. Then $A^t = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$. Notice that the eigenvalues of both $A$ and $A^t$ are 1 and 2. For $\lambda = 1$, $E_\lambda(A)$ is spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $E_\lambda(A^t)$ is spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

**14.** **(a)** $x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

## 5.3 MATRIX LIMITS AND MARKOV CHAINS

**2.** **(b)** $\begin{pmatrix} -0.5 & 0.5 \\ -1.5 & 1.5 \end{pmatrix}$ **(d)** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **(f)** $\begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$

**(h)** $\begin{pmatrix} -2 & -3 & -1 \\ 0 & 0 & 0 \\ 6 & 9 & 3 \end{pmatrix}$ **(j)** No limit exists.

**5.** $A = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

**9.** **(b)** $\begin{pmatrix} 0.50 & 0.50 & 0.50 \\ 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 \end{pmatrix}$ **(d)** $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ **(f)** $\begin{pmatrix} 1 & 0.0 & 0.0 \\ 0 & 0.4 & 0.4 \\ 0 & 0.6 & 0.6 \end{pmatrix}$

**(h)** $\begin{pmatrix} 0.0 & 0.0 & 0 & 0 \\ 0.0 & 0.0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 \\ 0.5 & 0.5 & 0 & 1 \end{pmatrix}$

**10.** **(b)** $\begin{pmatrix} .375 \\ .375 \\ .250 \end{pmatrix}$ after two stages and $\begin{pmatrix} .4 \\ .4 \\ .2 \end{pmatrix}$ eventually

**(d)** $\begin{pmatrix} .252 \\ .334 \\ .414 \end{pmatrix}$ after two stages and $\begin{pmatrix} .25 \\ .35 \\ .40 \end{pmatrix}$ eventually

**(f)** $\begin{pmatrix} .316 \\ .428 \\ .256 \end{pmatrix}$ after two stages and $\begin{pmatrix} .25 \\ .50 \\ .25 \end{pmatrix}$ eventually

**11.** For 1950, the distribution is 19.7% urban, 33.9% unused, and 46.4% agricultural. Eventually, the distribution is 20% urban, 30% unused, and 50% agricultural.

**23.** Here are two examples.

**(a)** Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

**(b)** Take $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

## 5.4 INVARIANT SUBSPACES AND THE CAYLEY-HAMILTON THEOREM

**6.** **(b)** $\{x^3, 6x\}$ **(d)** $\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\right\}$

**9.** **(b)** $t^2$ **(d)** $t(t-3)$

**10.** **(b)** $t^4$ **(d)** $t^2(t-3)^2$

**31.** **(c)** $-(t+1)(t^2-6t+6)$

**41.** $(-t)^{n-2}\left(t^2 - \dfrac{n(n^2+1)}{2}t - \dfrac{n^3(n+1)(n-1)}{12}\right)$

**42.** $(-1)^{n-2}t^{n-1}(t-n)$

# 6

# Inner Product Spaces

## 6.1 INNER PRODUCTS AND NORMS

**4.** **(b)** $\|A\| = 4$, $\|B\| = 2$, $\langle A, B\rangle = -4i$

**5.** $6 - 21i$

**8.** **(a)** Observe that $\langle (1,1),(1,1)\rangle = 0$, which violates (d) of the definition of inner product.

**(b)** Let $A = B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and let $c = 2$. Then $\langle cA, B\rangle = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$, but $c\langle A, B\rangle = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$. Thus (b) of the definition of inner product is violated.

**(c)** Let $f(x)$ be the constant polynomial 1. Then $\langle f, f\rangle = 0$, which violates (d) of the definition of inner product.

**11.** The sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the sides.

## 6.2 THE GRAM-SCHMIDT ORTOGONALIZATION PROCESS AND ORTHOGONAL COMPLEMENTS

**2.** **(a)** The orthonormal basis is $\left\{ \frac{\sqrt{2}}{2}(1,0,1),\ \frac{\sqrt{6}}{6}(-1,2,1),\ \frac{\sqrt{3}}{3}(1,1,-1) \right\}$.

The Fourier coefficients are $\frac{3\sqrt{2}}{2}$, $\frac{\sqrt{6}}{2}$, 0.

**(d)** The orthonormal basis is $\left\{ \frac{\sqrt{2}}{2}(1,i,0),\ \frac{\sqrt{17}}{34}(1+i,1-i,8i) \right\}$.

The Fourier coefficients are $\frac{\sqrt{2}}{2}(7+i)$, $\sqrt{17}i$.

**(f)** The orthonormal basis is $\left\{ \frac{1}{\sqrt{15}}(1,-2,-1,3), \frac{1}{\sqrt{10}}(2,2,1,1), \frac{1}{\sqrt{30}}(-4,2,1,3) \right\}$.

The Fourier coefficients are $-\frac{\sqrt{15}}{5}$, $\frac{2\sqrt{10}}{5}$, $\frac{2\sqrt{30}}{5}$.

**(h)** The orthonormal basis is $\left\{ \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \frac{1}{7}\begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix}, \frac{1}{\sqrt{373}}\begin{pmatrix} 8 & -8 \\ 7 & -14 \end{pmatrix} \right\}$.

The Fourier coefficients are $5\sqrt{13}$, $-14$, $\sqrt{373}$.

**(j)** The orthonormal basis is

$$\left\{ \frac{1}{\sqrt{8}}(1,i,2-i,-1), \frac{1}{\sqrt{20}}(1+3i,2i,-1,1+2i), \frac{1}{\sqrt{140}}(-7+i,6+2i,5,5) \right\}.$$

The Fourier coefficients are $6\sqrt{2}$, $4\sqrt{5}$, $2\sqrt{35}$

(l) The orthonormal basis is

$$\left\{\frac{1}{\sqrt{40}}\begin{pmatrix}1-i & -2-3i\\ 2+2i & 4+i\end{pmatrix}, \frac{1}{\sqrt{50}}\begin{pmatrix}6i & -1-i\\ 1-3i & 1+i\end{pmatrix}, \frac{1}{\sqrt{8075}}\begin{pmatrix}-2-43i & 1-21i\\ -68i & 34i\end{pmatrix}\right\}.$$

The Fourier coefficients are $\sqrt{10}(2-6i)$, $10\sqrt{2}$, 0.

**3.** $\dfrac{7}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}$

**9.** An orthonormal basis for W is $\left\{\dfrac{1}{\sqrt{2}}(i,0,1)\right\}$.

An orthonormal basis for $\mathsf{W}^{\perp}$ is $\left\{\dfrac{1}{\sqrt{2}}(1,0,i),(0,1,0)\right\}$.

**19.** **(c)** $x+\dfrac{13}{3}$

**20.** **(a)** $\dfrac{2}{\sqrt{17}}$ **(c)** $\dfrac{5}{\sqrt{15}}$

**21.** The best approximation is $\dfrac{3}{4e}(5e^2-35)t^2+\dfrac{3}{e}t+\dfrac{3}{4e}(11-e^2)$.

**22.** **(a)** $\{\sqrt{3}t, \sqrt{2}(5\sqrt{t}-6t)\}$

**(b)** $\dfrac{45}{28}t-\dfrac{5}{7}\sqrt{t}$

## 6.3 THE ADJOINT OF A LINEAR OPERATOR

**2.** **(b)** $y=(1,-2)$

**3.** **(b)** $\mathsf{T}^*(z_1,z_2)=(5+i,-1-3i)$

**7.** T: $\mathsf{R}^2\to\mathsf{R}^2$ defined by $\mathsf{T}(a_1,a_2)=(a_2,0)$

**11.** Yes

**20.** **(c)** The linear function is $y=-1.8x+0.8$ with $E=0.4$, and the quadratic function is $y=-t^2/7-9t/5+38/35$ with $E\approx 0.11429$.

**22.** **(a)** $x=2$, $y=4$, $z=-2$ **(c)** $x=1$, $y=-\frac{1}{2}$, $z=\frac{1}{2}$

## 6.4 NORMAL AND SELF-ADJOINT OPERATORS

**2.** **(b)** T is neither self-adjoint nor normal. If we let $A=[\mathsf{T}]_\beta$, where $\beta$ is the standard ordered basis, then $AA^*\neq A^*A$.

**(d)** T is not normal.

**(f)** T is self-adjoint. An orthonormal basis of eigenvectors is

$$\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 0\\ 1 & 0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}\right\}$$

with corresponding eigenvalues 1, 1, −1, −1.

## 6.5 UNITARY AND ORTHOGONAL OPERATORS AND THEIR MATRICES

**2.** **(b)** $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

**(c)** $P = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ 1+i & \frac{\sqrt{2}}{2}(1+i) \end{pmatrix}, \quad D = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$

**(e)** $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

**9.** No. Let $\mathsf{U}$ be the linear operator on $\mathsf{C}^2$ defined by $\mathsf{U}(z_1, z_2) = (z_1 + z_2, 0)$, and let $\beta$ be the standard basis for $\mathsf{C}^2$.

**11.** $\frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}$

**16.** In the notation of Example 3 of Section 6.4, let $\mathsf{U} = \mathsf{T}$, and let $\mathsf{W} = \text{span}(\{f_0, f_1, f_2, \ldots\})$.

**27.** **(b)** $x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$, and $y = -\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$
The quadratic form is $(x')^2 + 3(y')^2$.

**(d)** $x = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'$, and $y = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$
The quadratic form is $4(x')^2 + 2(y')^2$.

**(e)** $x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$, and $y = -\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$
The quadratic form is $2(x')^2$.

**28.** $x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{3}}z'$, $y = -\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{3}}z'$, $z = -\frac{2}{\sqrt{6}}y' + \frac{1}{\sqrt{3}}z'$.
The quadratic form is $(x')^2 + (y')^2 + 4(z')^2$.

## 6.6 ORTHOGONAL PROJECTIONS AND THE SPECTRAL THEOREM

**2.** For $\mathsf{W} = \text{span}(\{(1,0,1)\})$, $[\mathsf{T}]_\beta = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

**3.** **(b)** $\mathsf{T}_1(a,b) = \frac{1}{2}(a + ib, -ia + b)$ and $\mathsf{T}_2(a,b) = \frac{1}{2}(a - ib, ia + b)$

**(c)** $\mathsf{T}_1(a,b) = \frac{1}{3}(a + (1+i)b, (1+i)a + 2ib)$ and $\mathsf{T}_2(a,b) = \frac{1}{3}(2a + (1+i)b, (1+i)a + ib)$

**(e)** $\mathsf{T}_1(a,b,c) = \frac{1}{2}(a-b, -a+b, 0)$

$\mathsf{T}_2(a,b,c) = \frac{1}{6}(a+b-2c, a+b-2c, -2a-2b+4c)$, and

$\mathsf{T}_3(a,b,c) = \frac{1}{3}(a+b+c, a+b+c, a+b+c)$

**5. (a)** Let $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^2$ be the projection on the line $y = x$ defined by $\mathsf{T}_1(a,b) = (a,a)$. Then $\|\mathsf{T}(1,0)\| = \|(1,1)\| = \sqrt{2} > 1 = \|(1,0)\|$. In the case of equality, $\mathsf{T}$ is the identity operator.

## 6.7 THE SINGULAR VALUE DECOMPOSITION AND THE PSEUDOINVERSE

**2. (b)** $v_1 = \sqrt{\frac{5}{8}}(3x^2-1)$, $v_2 = \frac{1}{\sqrt{2}}$, $v_3 = \sqrt{\frac{3}{2}}x$, $\quad u_1 = \frac{1}{\sqrt{2}}$, $u_2 = \sqrt{\frac{3}{2}}x \quad \sigma_1 = 3\sqrt{5}$

**(d)** $v_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$, $v_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1-i \\ -1 \end{pmatrix}$, $\quad u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1+i \\ 1 \end{pmatrix}$

$\sigma_1 = 2$, $\sigma_2 = 1$

**3. (b)** $$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^*$$

**(d)** $$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^*$$

**(f)** $$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}^*$$

**4. (b)** $$WP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 20 & 4 & 0 \\ 4 & 20 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**5. (b)** $\mathsf{T}^\dagger(a+bx) = \frac{a}{6}(3x^2-1)$

**(d)** $\mathsf{T}^\dagger(z_1,z_2) = \mathsf{T}^{-1}(z_1,z_2) = \frac{1}{2}(-z_1 + (1-i)z_2, (1+i)z_1)$

**6. (b)** $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}$ **(d)** $\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ **(f)** $\frac{1}{6}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

**7. (b)** $\mathsf{Z}_1 = \mathsf{N}(\mathsf{T})^\perp = \mathsf{P}_1(R)^\perp = \mathrm{span}(\{3x^2-1\})$ and $\mathsf{Z}_2 = \mathsf{R}(\mathsf{T}) = \mathrm{span}(\{1\})$.

**(d)** $\mathsf{Z}_1 = \mathsf{N}(\mathsf{T})^\perp = \mathsf{C}^2$ and $\mathsf{Z}_2 = \mathsf{R}(\mathsf{T}) = \mathsf{C}^2$.

**8.** **(b)** The system is consistent. $\frac{1}{2}\begin{pmatrix}1\\0\\2\\0\end{pmatrix}$

## 6.8 BILINEAR AND QUADRATIC FORMS

**6.** **(b)** $A = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

**9.** **(b)** Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for $\mathsf{V}$. For each $i, j$, $1 \le i, j \le n$, let $H_{ij}$ be the unique bilinear form such that $H_{ij}(v_i, v_j) = 1$ and $H_{ij}(v_p, v_q) = 0$ if $(p, q) \neq (i, j)$. Then $\{H_{ij}\colon 1 \le i, j \le n\}$ is the required basis.

**17.** **(a)** $H(x,y) = -2x_1y_1 + 2x_1y_2 + 2x_2y_1 + x_2y_2$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$

**(b)** $H(x,y) = 7x_1y_1 - 4x_1y_2 - 4x_2y_1 + x_2y_2$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$

**(c)** $H(x,y) = 3x_1y_1 + 3x_2y_2 + 3x_3y_3 - x_1y_3 - x_3y_1$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$

**24.** **(c)** $\mathsf{T}$ is self-adjoint and positive definite.

**(d)** For any scalar $c$, $H(x, cy) = \langle x, \mathsf{T}(cy)\rangle = \bar{c}\,\langle x, \mathsf{T}(y)\rangle = \bar{c}H(x,y)$. So if $c$ is not real and $H(x,y) \neq 0$, then $H(x, cy) \neq cH(x,y)$.

## 6.10 CONDITIONING AND THE RAYLEIGH QUOTIENT

**2.** **(b)** 6

## 6.11 THE GEOMETRY OF ORTHOGONAL OPERATORS

**11.** $\mathsf{T}(x,y,z) = (-x, -z, y)$

# 7

# Canonical Forms

## 7.1 JORDAN CANONICAL FORM I

**2.** **(b)** For $\lambda = -1$, $\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$, For $\lambda = 4$, $\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$ $J = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$

**(d)** for $\lambda = 2$, $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right\}$ For $\lambda = 3$, $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ $J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

**3.** **(b)** For $\lambda = 0$, $\{1, t, \frac{1}{2}t^2\}$, For $\lambda = 1$, $\{e^t, te^t\}$ $J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

**(d)** For $\lambda = 3$, $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$ For $\lambda = 1$, $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$

$$J = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

## 7.2 JORDAN CANONICAL FORM II

**4.** **(b)** $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

**(c)** $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & -3 & 2 \\ -1 & -3 & -1 \\ 1 & 9 & 0 \end{pmatrix}$

**5.** **(b)** $J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\beta = \{1, 12x, 6x^2, x^3\}$

**(e)** $J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$ and $\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ 3 & 0 \end{pmatrix}\right\}$

**(f)** $J = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \cline{4-6} 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ and $\beta = \{2, 2x, x^2, 2x - 2y, x^2 - y^2, x^2 + y^2 - 2xy\}$

**8.** **(c)** For example, since $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda = 2$, we may add this vector to the end vector $\begin{pmatrix}0\\1\\2\\0\end{pmatrix}$ of the first cycle given in Example 2. Thus

$$\beta' = \left\{\begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}$$ is also a Jordan canoncial basis for $\mathsf{L}_A$.

**15.** Define $\mathsf{T}$: $\mathsf{R}^3 \to \mathsf{R}^3$ by $\mathsf{T}(x, y, z) = (0, -z, y)$. In general, $\mathsf{T}$ is such an operator if its characteristic polynomial is of the form $f(t) = t^k g(t)$, where $k \geq 1$ and $g(t)$ is a polynomial of degree greater than 1 with no zeros in the underlying field.

## 7.3 THE MINIMAL POLYNOMIAL

**2.** **(b)** $(t-1)^2$

**3.** **(b)** $(t-1)^3$

## 7.4 RATIONAL CANONICAL FORM

**2.** **(b)** $C = \begin{pmatrix}0 & -1\\1 & -1\end{pmatrix}$ $\quad Q = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

**(d)** $C = \begin{pmatrix}0 & 0 & 0 & -1\\1 & 0 & 0 & 0\\0 & 1 & 0 & -2\\0 & 0 & 1 & 0\end{pmatrix}$ $\quad Q = \begin{pmatrix}1 & 0 & -7 & -4\\0 & 1 & -4 & -3\\0 & 0 & -4 & -4\\0 & 0 & -4 & -8\end{pmatrix}$

**3.** **(b)** $(t^2+1)^2$ $\quad C = \begin{pmatrix}0 & 0 & 0 & -1\\1 & 0 & 0 & 0\\0 & 1 & 0 & -4\\0 & 0 & 1 & 1\end{pmatrix}$

$\beta = \{x\sin x, \sin x + x\cos x, 2\cos x - x\sin x, -3\sin x - x\cos x\}$

**(d)** $t^2$ and $t^2+4$ $\quad C = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & -4\\0 & 0 & 1 & 0\end{pmatrix}$

$\beta = \{\sin x \sin y + \cos x \cos y, \sin x \cos y - \cos x \sin y, \sin x \cos y + \cos x \sin y, 2(\cos x \cos y - \sin x \sin y)\}$