# REAL ANALYSIS
## LECTURE NOTES

ABSTRACT. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

## 1. FUBINI'S THEOREM

**Notions**: set and function slices.

We work in $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

(i) If $E \subset \mathbb{R}^n$, we write $E^y = \{x \in \mathbb{R}^{n_1} : (x, y) \in E\}$ for the "horizontal" $y$-slice of $E$ where $y \in \mathbb{R}^{n_2}$. Wirte $E_x = \{y \in \mathbb{R}^{n_2} : (x, y) \in \mathbb{R}^n\}$ for the "vertical" $x$-slice where $x \in \mathbb{R}^{n_1}$.

(ii) If $f(x, y)$ is a function in $\mathbb{R}^n$, we write $f^y(x) = f(x, y)$ for the function of the $x \in \mathbb{R}^{n_1}$ variable. Similarly, the slice of $f$ for a fixed $x \in \mathbb{R}^{n_1}$ is $f_x(y) = f(x, y)$.

With the assumption that $f$ is measurable on $\mathbb{R}^n$, it is not necessarily true that the slice $f^y$ is measurable on $\mathbb{R}^{n_1}$ for each $y$; nor does the corresponding assertion necessarily hold for a measurable set: the slice $E^y$ may not be measurable for each $y$.

For example, consider

$$f(x, y) = g(x)g(x + y)\chi_{[0,1]^2}, \quad \text{with } g(t) = \frac{1}{\sqrt{t}}.$$

Then $f^y(x) \in L^1$ for $y \neq 0$, but $f^0(x)$ is not integrable.

Another example arises in $\mathbb{R}^2$ by placing a one-dimensional non-measurable set on the $x$-axis; the set $E$ in $\mathbb{R}^2$ has measure zero, but $E^y$ is not measurable for $y = 0$.

**Theorem 1.1** (Fubini). *Suppose $f(x, y)$ is integrable on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then for almost every $y \in \mathbb{R}^{n_2}$:*

(i) *The slice $f^y(x)$ is measurable in $x$ and integrable on $\mathbb{R}^{n_1}$.*

(ii) *The function defined by $\int_{\mathbb{R}^{n_1}} f^y(x)dx$ is measurable in $y$ and integrable on $\mathbb{R}^{n_2}$.*

*Moreover:*

(iii) $\int_{\mathbb{R}^{n_2}} \left( \int_{\mathbb{R}^{n_1}} f(x,y)dx \right) dy = \int_{\mathbb{R}^n} f$.

*The conclusion is symmetric in $x$ and $y$:*

(i) *The slice $f_x(y)$ is measurable in $y$ and integrable on $\mathbb{R}^{n_2}$.*
(ii) *The function defined by $\int_{\mathbb{R}^{n_2}} f_x(y)dy$ is measurable in $x$ and integrable on $\mathbb{R}^{n_1}$.*
(iii) $\int_{\mathbb{R}^{n_1}} \left( \int_{\mathbb{R}^{n_2}} f(x,y)dy \right) dx = \int_{\mathbb{R}^n} f$.

We assume that $f$ is real-valued. The theorem then clearly applies to the real and imaginary parts of a complex-valued function.

*Proof.* Denote $\mathcal{F} = \{f \in L^1(\mathbb{R}^n) : f$ satisfies all the three conclusions (i)-(iii) in the theorem$\}$. We show that $L^1(\mathbb{R}^n) \subset \mathcal{F}$.

*Step 1.* Any finite linear combination of functions in $\mathcal{F}$ also belongs to $\mathcal{F}$.

Easy to check.

*Step 2.* Suppose $\{f_k\} \subset \mathcal{F}$ so that $f_k \nearrow f$ or $f_k \searrow f$, where $f \in L^1(\mathbb{R}^n)$. Then $f \in \mathcal{F}$.

It suffices to consider the case of an increasing sequence, as we can taking $-f_k$ instead of $f_k$. Also we may replace $f_k$ by $f_k - f_1$ and assume that $f_k$'s are non-negative. It follows by the monotone convergence theorem

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k = \int_{\mathbb{R}^n} f. \tag{1.1}$$

There is a subset $Y \subset \mathbb{R}^{n_2}$ of zero $n_2$-dimensional Lebesgue measure such that

$$f_k^y(x) \in L^1(\mathbb{R}^{n_1}) \ \ \forall\, y \notin Y \text{ and } \forall k, \quad \text{and } g_k(y) := \int_{\mathbb{R}^{n_1}} f_k^y(x)dx \in L^1(\mathbb{R}^{n_2}) \ \forall\, k.$$

Applying the monotone convergence theorem to $f_k^y(x) \nearrow f^y(x)$ for fixed $y \in \mathbb{R}^{n_2} \setminus Y$ (so $f^y(x)$ is measurable in $x$ being a limit of measurable functions), we deduce that

$$g_k(y) = \int_{\mathbb{R}^{n_1}} f_k^y(x)dx \ \text{ increases to a limit } \ g(y) := \int_{\mathbb{R}^{n_1}} f^y(x)dx, \quad \text{for a.e. } y \in \mathbb{R}^{n_2}.$$

Thus $g(y)$ is measurable, as $g(y)$ is a limit of measurable functions $g_k(y)$. Another application of monotone convergence theorem to $g_k(y) \nearrow g(y)$ yields

$$\lim_{k\to\infty} \int_{\mathbb{R}^{n_2}} g_k(y)dy = \int_{\mathbb{R}^{n_2}} g(y)dy. \tag{1.2}$$

It follows (1.1), (1.2) and by $f_k \in \mathcal{F}$ that

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k = \lim_{k\to\infty} \int_{\mathbb{R}^{n_2}} g_k(y)dy = \int_{\mathbb{R}^{n_2}} g(y)dy = \int_{\mathbb{R}^{n_2}} \Big( \int_{\mathbb{R}^{n_1}} f^y(x)dx \Big) dy.$$

This shows that $f$ satisfies (iii).

As $f \in L^1(\mathbb{R}^n)$, $g(y)$ is finite for a.e. $y$, hence $f^y(x) \in L^1(\mathbb{R}^{n_1})$ for a.e. $y$, which implies $f$ satisfies (ii). Recall that, for a.e. $y$, $f^y(x)$ as the limit of $f_k^y(x)$ is measurable on $\mathbb{R}^{n_1}$ and so $f$ satisfies (i).

In summary, $f \in \mathcal{F}$.

*Step 3.* If $E$ is a $G_\delta$ set with finite measure, then $\chi_E \in \mathcal{F}$.

(a) If $E$ is a bouned open cube, it is obvious that $\chi_E \in \mathcal{F}$.

(b) Suppose $E$ is a subset of the boundary of some closed cube. Observe that $m_{\mathbb{R}^n}(E) = 0$. It is direct to check $\chi_E \in \mathcal{F}$.

(c) Suppose $E = \bigcup_{k=1}^N Q_k$ is a finite union of closed cubes whose interiors are disjoint. Then $\chi_E = \sum_{k=1}^N (\chi_{\mathrm{Int}\, Q_k} + \chi_{\partial Q_k})$. So $\chi_E \in \mathcal{F}$ by Step 3 (a), (b) and Step 1.

(d) Suppose $E$ is open and of finite measure. Then $E = \bigcup_{k=1}^\infty Q_k$ with $Q_k$ being almost disjoint closed cubes. Clearly $\chi_{\cup_{j=1}^k Q_k} \nearrow \chi_E \in L^1(\mathbb{R}^n)$. Hence $\chi_E \in \mathcal{F}$ by using Step 3 (c) and Step 2.

(e) Finally, let $E$ be a $G_\delta$ of finite measure. Then $E = \cap_{j\geq 1} \mathcal{U}_j$ with open sets $\mathcal{U}_j$. Since $m(E) < \infty$, there is an open set $\mathcal{O}_0 \supset E$. Let $\mathcal{O}_k = \mathcal{O}_0 \cap \cap_{j=1}^k \mathcal{U}_j$. Then $\mathcal{O}_1 \supset \mathcal{O}_2 \supset \cdots$ with

$$E = \bigcap_{k=1}^\infty \mathcal{O}_k.$$

Obviously $\chi_{\mathcal{O}_k} \searrow \chi_E \in L^1(\mathbb{R}^n)$. Then $\chi_E \in \mathcal{F}$, by Step 3 (d) above and Step 2.

*Step 4.* If $E \subset \mathbb{R}^n$ with $m_{\mathbb{R}^n}(Z) = 0$, then $\chi_Z \in \mathcal{F}$.

There is a $G_\delta$ set $G \supset Z$ with $m(G) = 0$. Step 3 tells us $\chi_G \in \mathcal{F}$. Hence

$$\int_{\mathbb{R}^{n_2}} \Big( \int_{\mathbb{R}^{n_1}} \chi_{G^y}(x)dx \Big) dy = \int_{\mathbb{R}^n} \chi_G = 0.$$

This means

$$m_{\mathbb{R}^{n_1}}(G^y) = \int_{\mathbb{R}^{n_1}} \chi_{G^y}(x)dx = 0 \text{ for a.e. } y,$$

and so $G^y$ is of zero measure. Since $Z^y \subset G^y$, we see that $Z^y$ is of zero measure for a.e. $y$. This shows that $\chi_Z$ satisfies (i) and (ii). Also,

$$\int_{\mathbb{R}^{n_2}} \Big( \int_{\mathbb{R}^{n_1}} \chi_Z(x,y)dx \Big) dy = 0 = \int_{\mathbb{R}^n} \chi_Z.$$

Thus $\chi_Z$ satisfies (iii) and consequently belongs to $\mathcal{F}$.

*Step 5.* If $E$ is measurable and $m(E) < \infty$, then $\chi_E \in \mathcal{F}$.

Note that $E = G \setminus Z$ where $G$ is a $G_\delta$ and $Z$ is of zero measure. The conclusion follows by Step 1, 3 and 4.

*Step 6.* If $f \in L^1(\mathbb{R}^)$, then $f \in \mathcal{F}$.

Since $f = f^+ - f^-$, by Step 1 it suffices to assume $f$ itself is non-negative.

Recall that non-negative $f$ is an increasing limit of simple functions $\phi_k$. It follows from Step 1 and 5, $\phi_k \in \mathcal{F}$. Hence $f \in \mathcal{F}$ by virtue of Step 2.

$\square$

### 1.1. **Tonelli's Theorem.**

Tonelli's Theorem differs from Fubini's theorem in that it applies to any non-negative function $f$, but without the integrability restriction that $\int f < \infty$.

In practice one often wants to apply Fubini's theorem to $f \in \mathbb{R}^n \to \mathbb{R}$ but does not know $f \in L^1(\mathbb{R})$. In this case one can often first use Tonelli's theorem to $|f|$ to show $\int |f| < \infty$. Then one is justified in applying Fubini's theorem to $f$.

**Theorem 1.2** (Tonelli)**.** *Suppose $f(x,y)$ is a non-negative measurable function on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then for almost every $y \in \mathbb{R}^{n_2}$:*

(i) *The slice $f^y(x)$ is measurable in $x$ on $\mathbb{R}^{n_1}$.*
(ii) *The function defined by $\int_{\mathbb{R}^{n_1}} f^y(x)dx$ is measurable in $y$ on $\mathbb{R}^{n_2}$.*
(iii) *$\int_{\mathbb{R}^{n_2}} \Big( \int_{\mathbb{R}^{n_1}} f(x,y)dx \Big) dy = \int_{\mathbb{R}^n} f$ in the extended sense (may take value $\infty$).*

*The conclusion is symmetric in $x$ and $y$:*

(i) *The slice $f_x(y)$ is measurable in $y$ on $\mathbb{R}^{n_2}$.*
(ii) *The function defined by $\int_{\mathbb{R}^{n_2}} f_x(y)dy$ is measurable in $x$ on $\mathbb{R}^{n_1}$.*
(iii) *$\int_{\mathbb{R}^{n_1}} \Big( \int_{\mathbb{R}^{n_2}} f(x,y)dy \Big) dx = \int_{\mathbb{R}^n} f$ in the extended sense (may take value $\infty$).*

*Proof.* Define the truncations of $f$ for $k = 1, 2, \cdots$:

$$f_k(x,y) = \begin{cases} f(x,y) & \text{if } |(x,y)| \le k \text{ and } f(x,y) \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Fubini's theorem to $f_k \in L^1(\mathbb{R}^n)$, we conclude that

(a) for a.e. $y$ the slice $f_k^y(x)$ is measurable for every $k$;
(b) for a.e. $y$, $\int_{\mathbb{R}^{n_1}} f_k(x,y)dx$ is measurable in $y$ and is integrable on $\mathbb{R}^{n_2}$ for every $k$.

Observe for each $y$, $f_k^y(x) \nearrow f^y(x)$. Hence $f^y(x)$ is measurable in $x$ and thus (i) holds. By the monotone convergence theorem,

$$g_k(y) := \int_{\mathbb{R}^{n_1}} f_k(x,y)dx \nearrow g(y) := \int_{\mathbb{R}^{n_1}} f(x,y)dx.$$

Note that $g(y)$, being the limit of measurable functions $g_k(y)$, is measurable. So (ii) follows. Applying the monotone convergence theorem to $\{g_k\}$, one sees that

$$\int_{\mathbb{R}^{n_2}} \Big( \int_{\mathbb{R}^{n_1}} f_k(x,y)dx \Big) dy \to \int_{\mathbb{R}^{n_2}} \Big( \int_{\mathbb{R}^{n_1}} f(x,y)dx \Big) dy. \tag{1.3}$$

Since $f_k \nearrow f$ on $\mathbb{R}^n$, using monotone convergence theorem again,

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k = \lim_{k\to\infty} \int_{\mathbb{R}^{n_2}} \Big( \int_{\mathbb{R}^{n_1}} f_k(x,y)dx \Big) dy = \int_{\mathbb{R}^{n_2}} \Big( \int_{\mathbb{R}^{n_1}} f(x,y)dx \Big) dy,$$

where the second equality is the use of iterated integration of $f_k$ by Fubini's theorem, and the last equality is (1.3). This verifies (iii).

□

## 1.2. Applications of Fubini and Tonelli Theorems.

As an immediate consequence of Tonelli's theorem applied to $\chi_E$, we obtain the following.

**Corollary 1.1.** *If $E$ is a measurable set of $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then for almost every $y \in \mathbb{R}^{n_2}$ the slice $E^y = \{x \in \mathbb{R}^{n_1} : (x,y) \in E\}$ is a measurable subset of $\mathbb{R}^{n_1}$. Moreover $m(E^y)$ is a measurable function of $y$ and*

$$m(E) = \int_{\mathbb{R}^{n_2}} m(E^y)dy.$$

*A symmetric result holds for the $x$-slices $E_x = \{y \in \mathbb{R}^{n_2} : (x,y) \in E\}$ in $\mathbb{R}^{n_2}$.*

One might be tempted to think that the converse assertion holds. To see that this is not the case, note that if we let $\mathcal{N}$ be a non-measurable subset of $\mathbb{R}$, and define

$$E = [0,1] \times \mathcal{N} \subset \mathbb{R} \times \mathbb{R},$$

we see that

$$E^y = \begin{cases} [0,1] & \text{if } y \in \mathcal{N}, \\ \emptyset & \text{if } y \notin \mathcal{N}. \end{cases}$$

Thus $E^y$ is measurable for every $y$. However, if $E$ were measurable, then the corollary would imply that $E_x$ is measurable for almost every $x \in \mathbb{R}$, which is not true since $E_x = \mathcal{N}$ for all $x \in [0,1]$.

There is a weird example in Stein's book page 82-83 where all $y$-slices and all $x$-slices are measurable, but $E$ is not measurable.

**Proposition 1.1.** *If $E = E_1 \times E_2$ is a measurable subset of $\mathbb{R}^n$, and $m_*(E_2) > 0$, then $E_1$ is measurable.*

*Proof.* By Tonelli's theorem, for a.e. $y \in \mathbb{R}^{n_2}$, the slice function

$$\chi^y_{E_1 \times E_2}(x) = \chi_{E_1}(x)\chi_{E_2}(y)$$

is measurable as a function of $x$.

Denote by $F$ the set of $y \in \mathbb{R}^{n_2}$ such that the slice $E^y$ is measurable. Tonelli's theorem asserts that $m(F^c) = 0$. Since $m_*(E_2) > 0$, we have $E_2 \cap F \neq \emptyset$ (otherwise $E_2 = (E_2 \cap F) \cup (E_2 \cap F^c)$ implies $m_*(E) = 0$).

Take $y_0 \in E_2 \cap F$. We infer that $\chi_{E_1}(x) = \chi^{y_0}_{E_1 \times E_2}(x)$ is measurable.

□

The converse of the above result is presented in previous lecture notes, which says $E = E_1 \times E_2$ is a measurable subset of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ if $E_1 \subset \mathbb{R}^{n_1}$ and $E_2 \subset \mathbb{R}^{n_2}$ are both measurable, and

$$m(E) = m(E_1)m(E_2)$$

with the understanding that if one of the sets $E_j$ has measure zero, then $m(E) = 0$. As a consequence of this, we conclude that the measurability of functions is preserved under the trivial extension of variables.

**Proposition 1.2.** *Suppose $f$ is a measurable function on $\mathbb{R}^{n_1}$. Then the function $\tilde{f}$ defined by $\tilde{f}(x,y) = f(x)$ is measurable on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.*

*Proof.* We assume $f$ is real-valued. For any $a \in \mathbb{R}$,

$$\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \tilde{f}(x,y) < a\} = \{x \in \mathbb{R}^{n_1} : f(x) < a\} \times E_2$$

is measurable. Hence by definition $\tilde{f}(x,y)$ is measurable.

□

Integrals of functions and Areas of graphs

**Proposition 1.3.** *Suppose $E$ is a measurable subset of $\mathbb{R}^n$, and $f_1, \cdots, f_d$ are real-valued measurable functions. Let*

$$\mathcal{G} = \{(x, f_1(x), \cdots, f_d(x)) \in \mathbb{R}^{n+d} : x \in E\}.$$

*Then $\mathcal{G}$ is a measurable subset of $\mathbb{R}^{n+d}$ and $m(\mathcal{G}) = 0$.*

*Proof.* Suppose $m(E) < \infty$. Given $\delta > 0$, let $Q_k = \prod_{i=1}^d (a_i^k, b_i^k]$ be disjoint cubes with side length $\delta$ such that $\mathbb{R}^d = \bigcup_{k=1}^\infty Q_k$. Let

$$E_k \ = \ \{x \in E : (f_1(x), \cdots, f_d(x)) \in Q_k\}.$$

Observe $E_k$ is measurable, as it can be written as a intersection of measurable sets,

$$E_k \ = \ \bigcap_{i=1}^d \big\{x \in E : f_i(x) \in (a_i^k, b_i^k]\big\}.$$

Since $\mathcal{G} \subset \bigcup_{k=1}^\infty (E_k \times Q_k)$, we deduce

$$m_*(\mathcal{G}) \le m(\bigcup_{k=1}^\infty (E_k \times Q_k)) = \sum_{k=1}^\infty m(E_k \times Q_k) = \delta^d \sum_{k=1}^\infty m(E_k) = \delta^d m(E).$$

Sending $\delta \to 0$, we find that $m_*(\mathcal{G}) = 0$.

We next deal with the case $m(E) = \infty$. For this end, write $E = \bigcup_{N=1}^\infty E_N$ where $E_N = E \cap \{x \in \mathbb{R}^n : |x| \le N\}$. Set

$$\mathcal{G}_N = \{(x, f_1(x), \cdots, f_d(x)) \in \mathbb{R}^{n+d} : x \in E_N\}.$$

It follows that $m(\mathcal{G}_N) = 0$. As $\mathcal{G} = \cup_{N=1}^\infty \mathcal{G}_N$, we conclude that $\mathcal{G}$ is of zero measure.

□

We next return to an interpretation of the integral that arose first in the calculus.

**Proposition 1.4.** *Suppose $f(x)$ is a non-negative function on $\mathbb{R}^n$, and let*

$$\mathcal{A} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq y \leq f(x)\}.$$

*Then*

(i) *$f$ is measurable on $\mathbb{R}^n$ if and only if $\mathcal{A}$ is measurable in $\mathbb{R}^{n+1}$.*
(ii) *If the conditions in (i) hold, then*

$$\int_{\mathbb{R}^n} f(x)dx = m(\mathcal{A}).$$

*Proof.* If $f$ is measurable, then Proposition 1.2 guarantees that $F(x,y) = y - f(x)$ is measurable on $\mathbb{R}^{n+1}$. So

$$\mathcal{A} = \{(x,y) : y \geq 0\} \cap \{(x,y) : F(x,y) \leq 0\} \subset \mathbb{R}^{n+1}$$

is measurable.

Conversely, suppose that $\mathcal{A}$ is measurable. Note that for each $x \in \mathbb{R}^n$ the slice

$$\mathcal{A}_x = \{y \in \mathbb{R} : (x,y) \in \mathcal{A}\} = [0, f(x)]$$

is a closed segment. Then Tonelli's theorem (or Corollary 1.1) gives the measurability of $m(\mathcal{A}_x) = f(x)$.

Moreover, by Tonelli's theorem

$$m(\mathcal{A}) = \int_{\mathbb{R}^{n+1}} \chi_{\mathcal{A}}(x,y)dxdy = \int_{\mathbb{R}^n} m(\mathcal{A}_x)dx = \int_{\mathbb{R}^n} f(x)dx.$$

□

Convolution of functions

Recall that if $f$ is measurable on $\mathbb{R}^n$ then $f(x-y)$ is measurable on $\mathbb{R}^{2n}$. Let $f, g$ be two integrable functions on $\mathbb{R}^n$. Their convolution is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy.$$

**Theorem 1.3.** *Suppose $f, g \in L^1(\mathbb{R}^n)$. Then $(f * g)(x)$ is well-defined for a.e. $x$,* [1] *and is integrable on $\mathbb{R}^n$. Moreover*

$$\|(f * g)\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},$$

*with equality if $f$ and $g$ are non-negative.*

[1] That is $f(x-y)g(y)$ is integrable on $\mathbb{R}^n$ for a.e. $x$.

*Proof.* Applying the Tonelli's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2n}} |f(x-y)g(y)|dxdy &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|f(x-y)|dx\Big)|g(y)|dy \\ &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|f(x-y)|dx\Big)|g(y)|dy \quad \text{(by translation invariance)} \\ &= \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)} < \infty.\end{aligned}$$

This shows that $f(x-y)g(y) \in L^1(\mathbb{R}^{2n})$. By Fubini's theorem $(f*g)(x)$, as the integral along the $x$-slice, is finite for a.e. $x$ and is integrable on $\mathbb{R}^n$. Hence $(f*g)$ is well-defined.

Observe that

$$\|f*g\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} f(x-y)g(y)dy\Big|dx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(x-y)g(y)|dydx.$$

This together with the previous equality yields

$$\|f*g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)},$$

with equality if $f$ and $g$ are non-negative. □

**Exercise 1.1.** *Let $f, g$ are measurable functions on $\mathbb{R}^n$. Then*

(i) *$f*g$ is uniformly continuous provided $f \in L^1(\mathbb{R}^n)$ and $g$ is bounded;*
(ii) *$(f*g)(x) \to 0$ as $|x| \to \infty$ provided $f, g \in L^1(\mathbb{R}^n)$ and $g$ is bounded.*

*Proof.* Exercise.

□

Fourier transform

The Fourier transform of an integrable function is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi\sqrt{-1}x\cdot\xi}dx.$$

It is direct to see $\widehat{f}$ is bounded and a continuous function of $\xi$. This is because

$$|\widehat{f}| \leq \int_{\mathbb{R}^n}|f|dx = \|f\|_{L^1(\mathbb{R}^n)},$$

and by the dominated convergence theorem,

$$\lim_{|\eta_k|\to 0}\widehat{f}(\xi+\eta_k) = \lim_{|\eta_k|\to 0}\int_{\mathbb{R}^n} f(x)e^{-2\pi\sqrt{-1}x\cdot(\xi+\eta_k)}dx = \widehat{f}(\xi).$$

**Exercise 1.2.** *Suppose $f$ and $g$ are integrable functions. Then $\widehat{f*g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.*

*Proof.* Exercise. □

**Exercise 1.3.** *Suppose* $f \in L^1(\mathbb{R}^n)$. *Then* $\widehat{f}(\xi) \to 0$ *as* $|\xi| \to \infty$.

*Proof.* Exercise.

□

**Exercise 1.4.** *Suppose* $f$ *is integrable on* $[0, 2\pi]$. *Then*

$$\int_{[0,2\pi]} f(x)e^{-\sqrt{-1}nx}dx \to 0 \quad as\ |n| \to \infty.$$

*Consequently if* $E \subset [0, 2\pi]$ *is measurable, then*

$$\int_E \cos^2(nx + t_n) \to \frac{m(E)}{2} \quad as\ n \to \infty.$$

*for any sequence* $t_n$.

*Proof.* Exercise. □

Distribution functions

Let $f$ be a measurable function on $E$. The distribution function of $f$ is given by

$$\mu_f(t) = m(\{x \in E : |f(x)| > t\}).$$

**Theorem 1.4.** *Suppose* $f$ *is a measurable function on* $E$. *Given* $1 \leq p < \infty$,

$$\int_E |f|^p = p \int_{[0,\infty)} t^{p-1}\mu_f(t)dt.$$

*Consequently* $f \in L^p(E)$ *if and only if* $t^{p-1}\mu_f(t) \in L^1([0, \infty))$.

*Proof.* Let $\mathcal{S} = \{(x, t) \in E \times \mathbb{R} : 0 \leq t < |f(x)|\}$ and $F(x, t) = |f(x)| - t$. Clearly $F(x, t)$ is measurable on $\mathbb{R}^{n+1}$, as the measurability is preserved under the trivial extension of variables (see Proposition 1.2). Hence $\mathcal{S}$ is measurable, as

$$\mathcal{S} = \{(x, t) \in E \times \mathbb{R} : t \geq 0\} \cap \{(x, t) \in E \times \mathbb{R} : F(x, t) > 0\}.$$

Applying the Tonelli's theorem to $pt^{p-1}\chi_{\mathcal{S}}(x,t) \geq 0$, we obtain

$$\begin{aligned} p\int_{[0,\infty)} t^{p-1}\mu_f(t)dt &= \int_{\mathbb{R}^{n+1}} pt^{p-1}\chi_{\mathcal{S}}(x,t)dxdt \\ &= \int_{\mathbb{R}^n}\int_{[0,\infty)} pt^{p-1}\chi_{\mathcal{S}}(x,t)dtdx \\ &= \int_E \int_{[0,|f(x)|)} pt^{p-1}dtdx \\ &= \int_E |f|^p. \end{aligned}$$

□

As an immediate consequence, we see from the theorem above that

- if $\mu_f(t)$ behaves like $t^\alpha$ as $t \to \infty$ for some $\alpha \geq -p$, then $f \notin L^p$.
- if $\mu_f(t)$ behaves like $t^\alpha$ as $t \to 0$ for some $\alpha \leq -p$, then $f \notin L^p$.

Hence Theorem 1.4 gives criterion for the $L^p$-integrability of measurable function $f$ through checking the integrability of its distribution function.