# REAL ANALYSIS
## LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

## 1. Differentiability of functions

We begin with some definitions.

Suppose $f(t)$ is a real-valued function defined on $[a,b]$. Let $P$ be a partition of this interval, i.e., $P = \{t_i\}_{i=0}^N$, $a = t_0 < t_1 < \cdots < t_N = b$. The variation of $F$ on this partition is defined by

$$\mathcal{V}_f(P) := \sum_{j=1}^{N} |f(t_j) - f(t_{j-1})|.$$

Let $P$ and $P'$ be partitions of $[a,b]$. We say $P'$ is a refinement of $P$ if $P \subset P'$. It is not hard to see that

$$\mathcal{V}_f(P) \leq \mathcal{V}_f(P'). \tag{1.1}$$

**Definition 1.1.** *A function $f : [a,b] \to \mathbb{R}$ is said to be of bounded variation if*

$$\sup_P \mathcal{V}_f(P) < \infty.$$

We denote by $BV([a,b])$ the set of all functions of bounded variations on $[a,b]$. This class has a sub-class of functions, called absolutely continuous functions.

**Definition 1.2.** *A function $f : [a,b] \to \mathbb{R}$ is said to be absolutely continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ so that*

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon \quad \textit{whenever} \ \sum_{k=1}^{N} (b_k - a_k) < \delta,$$

*and the intervals $(a_k, b_k)$ with $k = 1, \ldots, N$ are disjoint.*

We denote by $AC([a,b])$ the set of all absolutely continuous functions on $[a,b]$.

From the definition, it is clear that absolutely continuous functions are continuous, and in fact uniformly continuous.

**Lemma 1.1.** $AC([a,b]) \subset BV([a,b])$.

*Proof.* Let $f \in AC([a,b])$. Take $\delta > 0$ such that

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < 1, \quad \text{whenever } \sum_{k=1}^{N} (b_k - a_k) < \delta. \tag{1.2}$$

Let $P = \{t_i\}_{i=0}^{N}$, where $t_0 = a$, $t_i - t_{i-1} = (b-a)/N < \delta$. For each partition $P' = \{s_j\}_{j=0}^{N'}$ of $[a,b]$, we consider the refinement $P'' = P' \cup P$. This partition $P''$ can be written as a union of partitions $P_i$ of $[t_{i-1}, t_i]$. Write $P_i = \{s_{j,i}\}_{j=0}^{\ell_i}$. Applying (1.2) to disjoint intervals $(s_{j-1,i}, s_{j,i})$, for $j = 1, \ldots, \ell_i$ and with fixed $i$, then gives

$$\mathcal{V}_f(P') \leq \mathcal{V}_f(P'') = \sum_{i=1}^{N} \sum_{j=1}^{\ell_i} |f(s_{j,i}) - f(s_{j-1,i})| < N.$$

□

The inclusion in Lemma 1.1 is strict, as Cantor-Lebesgue function (see the end of the notes) is of bounded variation but not absolutely continuous.

Our main task in the following lectures is to prove the following two theorems.

**Theorem 1.1.** *If $f \in BV([a,b])$, then $f'(x)$ exists for a.e. $x$, and $f' \in L^1([a,b])$.*

**Theorem 1.2.** *Suppose $f \in AC([a,b])$. Then $f'$ exists almost everywhere and is integrable. Moreover,*

$$f(x) - f(a) = \int_a^x f'(t)dt, \quad \text{for all } a \leq x \leq b.$$

*By selecting $x = b$ we get $f(b) - f(a) = \int_a^b f'(t)dt$.*

*Conversely, if $f \in L^1([a,b])$, then there exists $F \in AC([a,b])$ such that $F'(x) = f(x)$ almost everywhere, and in fact, we may take $F(x) = \int_a^x f(t)dt$.*

## 1.1. **Functions of bounded variation: Proof of Theorem 1.1.**

1.1.1. *BV functions and their properties.*

We give some examples for functions in $BV([a,b])$.

**Example 1.1.** *If $f$ is real-valued, monotone, and bounded, then $f$ is of bounded variation.*

**Example 1.2.** *If $f$ is differentiable at every point, and $f'$ is bounded, then $f$ is of bounded variation.*

**Example 1.3.** *The function $f$ below is of bounded variation on $[0,1]$ iff $a > b$.*

$$f(x) = \begin{cases} x^a \sin x^{-b}, & 0 < x \le 1, \\ 0, & x = 0. \end{cases}$$

We next define the total variation of $f$ (real-valued) on $[a,x]$ (where $a \le x \le b$) as a function of $x$ by

$$\mathcal{V}_f(a,x) = \sup \sum_{j=1}^{N} |f(t_j) - f(t_{j-1})|,$$

where the sup is over all partitions of $[a,x]$.

**Lemma 1.2.** *Suppose $f \in BV([a,b])$. Then*

(i) $\mathcal{V}_f(a,x) = \mathcal{V}_f(a,y) + \mathcal{V}_f(y,x)$ *for every $y \in [a,x]$.*
(ii) *If $f$ is moreover continuous, then $\mathcal{V}_f(a,x)$ is continuous in $x \in [a,b]$.*

*Proof. Part (i).* Both "$\le$" and "$\ge$" can be checked directly by definition.
*Part (ii).* The total variation $\mathcal{V}_f(a,x)$ is obviously increasing. We first show that for each $\varepsilon > 0$, there is a $x' \in [a,x)$ such that

$$\mathcal{V}_f(a,x') \ge \mathcal{V}_f(a,x) - \varepsilon.$$

Choose a partition $0 = t_0 < \cdots < t_N = x$ such that

$$\mathcal{V}_f(a,x) \le \sum_{j=1}^{N} |f(t_j) - f(t_{j-1})| + \varepsilon/2.$$

By the continuity of $f$, if $t_{N-1} < x' < x$ and $|x - x'|$ is sufficiently small, then $|f(x') - f(x)| \le \varepsilon/2$. Consequently

$$\begin{aligned} \mathcal{V}_f(a,x) &\le \sum_{j=1}^{N-1} |f(t_j) - f(t_{j-1})| + |f(t_{N-1}) - f(x')| + |f(x') - f(x)| + \varepsilon/2 \\ &\le \mathcal{V}_f(a,x') + \varepsilon. \end{aligned}$$

We next show that for each $\varepsilon > 0$, there is a $x' \in (x, b]$ (assuming $x < b$) such that

$$\mathcal{V}_f(a, x') \leq \mathcal{V}_f(a, x) + \varepsilon. \tag{1.3}$$

Fix a $x_1 \in (x, b]$. Let $x = t_0 < t_1 < \cdots < t_N = x_1$ be a partition of $[x, x_1]$ such that

$$\mathcal{V}_f(x, x_1) - \varepsilon/2 \leq \sum_{j=1}^{N} |f(t_j) - f(t_{j-1})|.$$

If $|x_1 - x|$ is very small (so is $|t_1 - x|$), then $|f(t_1) - f(x)| \leq \varepsilon/2$. Therefore

$$\mathcal{V}_f(x, x_1) - \varepsilon/2 \leq \varepsilon/2 + \sum_{j=2}^{N} |f(t_j) - f(t_{j-1})| \leq \varepsilon/2 + \mathcal{V}_f(t_1, x_1).$$

This implies by part (i) that

$$\mathcal{V}_f(x, t_1) \leq \varepsilon.$$

Adding $\mathcal{V}_f(a, x)$ at both sides then yields, by also using part (i) again,

$$\mathcal{V}_f(a, t_1) \leq \mathcal{V}_f(a, x) + \varepsilon.$$

This is exactly (1.2) with $x' = t_1$.

□

**Theorem 1.3.** *A real-valued function $f \in BV([a, b])$ if and only if $f$ is the difference of two increasing bounded functions.*

*Proof.* This is a consequence of Lemma 1.2. “If” part is straightforward. “Only if” part is because we have

$$f(x) = [f(x) + \mathcal{V}_f(a, x)] - \mathcal{V}_f(a, x).$$

Total variation $\mathcal{V}_f(a, x)$ is obviously increasing. If $y \in [a, x]$, then

$$f(y) - f(x) \leq \mathcal{V}_f(y, x) = \mathcal{V}_f(a, x) - \mathcal{V}_f(a, y),$$

which implies the monotonicity $f(y) + \mathcal{V}_f(a, y) \leq f(x) + \mathcal{V}_f(a, x)$.

□

1.1.2. *Differentiability of continuous BV functions.*

We study the differentiability of BV functions. By Theorem 1.3, it suffices to study the differentiability of monotone functions. We shall first assume that $f$ is continuous. This makes the argument simpler. For the general case, it will then be instructive to examine the nature of the possible discontinuities of a BV function, and reduce matters to the case of "jump functions".

We first prove the following result.

**Theorem 1.4.** *If $f \in C([a,b])$ is increasing, then $f'(x)$ exists for a.e. $x$.*

For the proof of the theorem, we define the quantity

$$\delta_h f(x) = \frac{f(x+h) - f(x)}{h}.$$

Consider the four Dini numbers at $x$ given by

$$\begin{aligned} D^+ f(x) &= \limsup_{h\to 0, h>0} \delta_h f(x), \\ D_+ f(x) &= \liminf_{h\to 0, h>0} \delta_h f(x), \\ D^- f(x) &= \limsup_{h\to 0, h<0} \delta_h f(x), \\ D_- f(x) &= \liminf_{h\to 0, h<0} \delta_h f(x). \end{aligned}$$

Clearly $D_+ f(x) \le D^+ f(x)$ and $D_- f(x) \le D^- f(x)$ for all $x$. For the sake of Theorem 1.4, we show

$$D^+ f(x) < \infty \quad \text{for a.e. } x, \tag{1.4}$$

$$D^+ f(x) \le D_- f(x) \quad \text{for a.e. } x. \tag{1.5}$$

Once these results hold, then by applying the result to $-f(-x)$ instead of $f(x)$ we obtain $D^- f(x) \le D_+ f(x)$ for a.e. $x$. Therefore

$$D^+ f(x) \le D_- f(x) \le D^- f(x) \le D_+ f(x) \le D^+ f(x) < \infty \quad \text{for a.e. } x.$$

This means that $f'(x)$ exists for a.e. $x$.

We shall use a technical lemma whose proof is postponed until the end of this section.

**Lemma 1.3** (Rising sun lemma)**.** *Suppose $g \in C(\mathbb{R})$. Let*

$$E = \{x \in \mathbb{R} : g(x+h) > g(x) \ \ \textit{for some } h = h_x > 0\}.$$

*If $E \neq \emptyset$, then it must be open, and hence $E = \bigcup(a_k, b_k)$ for a countable disjoint union of intervals. If $(a_k, b_k)$ is a finite interval in this union, then*

$$g(a_k) = g(b_k).$$

*A slight modification gives:*

*Suppose $g \in C([a, b])$. If $E$ is the set of $x \in (a, b)$ so that $g(x + h) > g(x)$ for some $h > 0$, then $g$ is either empty or open. In the latter case, it is a disjoint union of countably many intervals $(a_k, b_k)$, and $g(a_k) = g(b_k)$, except possibly when $a = a_k$, in which case we only have*

$$g(a_k) \leq g(b_k).$$

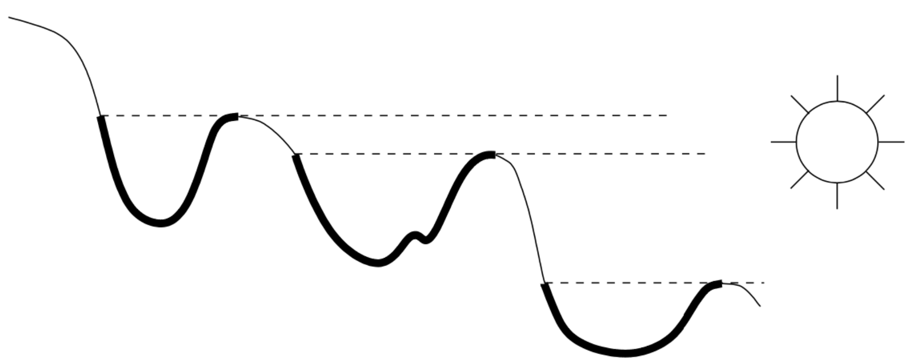

FIGURE 1. Rising sun lemma

*Proof of Theorem 1.4.* For a fixed $\gamma > 0$, let

$$E_\gamma = \{x : D^+ f(x) > \gamma\}.$$

It can be checked that $E_\gamma$ is measurable [1].

Applying Lemma 1.3 to the function

$$g(x) = f(x) - \gamma x,$$

we deduce that $E_\gamma \subset \bigcup_k (a_k, b_k)$, where

$$f(b_k) - f(a_k) \geq \gamma(b_k - a_k).$$

[1]The continuity of $f$ allows one to restrict to countably many $h$ in taking the $\limsup$.

Therefore, we have

$$m(E_\gamma) \le \sum_k (b_k - a_k) \le \frac{1}{\gamma} \sum \big( f(b_k) - f(a_k) \big) \le \frac{1}{\gamma}\big( f(b) - f(a) \big) \to 0,$$

as $\gamma \to \infty$. Since $\{D^+ f(x) = \infty\} \subset E_\gamma$ for all $\gamma$, we obtain (1.4).

Given two real numbers $r, R$ such that $R > r$, we consider

$$E = E_{r,R} = \{x \in (a,b) : D^+ f(x) > R > r > D_- f(x)\}.$$

For the sake of (1.5), we will show that $m(E) = 0$, since it then suffices to let $R$ and $r$ vary over rationals with $R > r$.

Suppose on the contrary $m(E) > 0$. Take an open $\mathcal{O}$ such that $E \subset \mathcal{O} \subset (a,b)$ and

$$m(\mathcal{O}) < \frac{R}{r} m(E). \tag{1.6}$$

Write $\mathcal{O} = \bigcup I_n$, with $I_n$ being disjoint open intervals. Fix $n$ and apply Lemma 1.3 to

$$g(x) = f(-x) + rx \quad \text{on} \quad -I_n,$$

and then reflecting througth the origin again yields an open set

$$\bigcup_k (a_k, b_k) \subset I_n,$$

where the intervals $(a_k, b_k)$ are disjoint, with

$$0 \ge g(-b_k) - g(-a_k) = f(b_k) - f(a_k) - r(b_k - a_k).$$

Next, on each interval $(a_k, b_k)$ we apply Lemma 1.3 this time to

$$g(x) = f(x) - Rx,$$

and so obtain $J_n = \bigcup_{k,j} (a_{k,j}, b_{k,j})$ of disjoint open intervals $(a_{k,j}, b_{k,j})$ with $(a_{k,j}, b_{k,j}) \subset (a_k, b_k)$, for every $j$, and

$$0 \le f(b_{k,j}) - f(a_{k,j}) - R(b_{k,j} - a_{k,j}).$$

Since $f$ is increasing, we find that

$$\begin{aligned} m(J_n) &= \sum_{k,j} (b_{k,j} - a_{k,j}) \le \frac{1}{R} \sum_{k,j} \big( f(b_{k,j}) - f(a_{k,j}) \big) \\ &\le \frac{1}{R} \sum_k \big( f(b_k) - f(a_k) \big) \le \frac{r}{R} \sum_k (b_k - a_k) \le \frac{r}{R} m(I_n). \end{aligned} \tag{1.7}$$

Note that $E \cap I_n \subset J_n$ [2]; and certainly $J_n \subset I_n$. We then sum in $n$ and find

$$m(E) = \sum_n m(E \cap I_n) \leq \sum_n m(J_n) \leq \frac{r}{R} \sum_n m(I_n) = \frac{r}{R} m(\mathcal{O}),$$

where the last second relation is (1.7). It then follows by (1.6) that $m(E) = 0$.

□

We next prove the rising sun lemma.

*Proof of Lemma 1.3.* We only prove the first part of the lemma.

Since $g$ is continuous, $E$ is open if it is non-empty and can therefore be written as a disjoint union of countably many open intervals. If $(a_k, b_k)$ is a finite interval in this decomposition, then $a_k \notin E$; so we cannot have $g(b_k) > g(a_k)$.

Suppose $g(b_k) < g(a_k)$. Then there is $c \in (a_k, b_k)$ such that

$$g(c) = \frac{g(a_k) + g(b_k)}{2}$$

and in fact we can choose $c$ farthest to the right in the interval $(a_k, b_k)$.

Since $c \in E$, there is $d > c$ so that $g(d) > g(c)$. Since $b_k \notin E$, we have $g(x) \leq g(b_k)$ for all $x \geq b_k$; hence $d < b_k$. By continuity, we have $c' \in (d, b_k)$ with property $g(c') = g(c)$, which contradicts with the fact that $c$ was chosen farthest to the right in $(a_k, b_k)$. Hence $g(a_k) = g(b_k)$.

□

The following conclusion is a consequence of Theorem 1.4.

**Corollary 1.1.** *Suppose $f \in C(\mathbb{R})$ is increasing. Then $f'$ exists almost everywhere. Moreover $f'$ is measurable, non-negative, and*

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

*In particular, if $f$ is bounded on $\mathbb{R}$, then $f' \in L^1(\mathbb{R})$.*

*Proof.* For $k \geq 1$, we consider the quotient

$$g_k(x) = \frac{f(x + 1/k) - f(x)}{1/k}.$$

[2]Suppose $x \in E \cap I_n$. Since $D_- f(x) < r$ we find that $x \in (a_k, b_k)$ for some $k$. While, this together with $D^+ f(x) > R$ implies that $x \in (a_{k,j}, b_{k,j})$ for some $j$.

By Theorem 1.4, we have that $g_k(x) \to f'(x)$ for a.e. $x$, which shows in particular that $f'$ is measurable and non-negative.

We now extend $f$ as a continuous function on all of $\mathbb{R}$. By Fatou's lemma,

$$\int_a^b f'(x)dx \leq \liminf_{k\to\infty} \int_a^b g_k(x)dx. \tag{1.8}$$

On the other hand,

$$\begin{aligned}
\int_a^b g_k(x)dx &= \frac{1}{1/k}\int_a^b f(x+1/k)dx - \frac{1}{1/k}\int_a^b f(x)dx \\
&= \frac{1}{1/k}\int_{a+1/k}^{b+1/k} f(x)dx - \frac{1}{1/k}\int_a^b f(x)dx \\
&= \frac{1}{1/k}\int_b^{b+1/k} f(x)dx - \frac{1}{1/k}\int_a^{a+1/k} f(x)dx \\
&\to f(b) - f(a) \quad \text{as } n \to \infty.
\end{aligned}$$

The last step is due to the Lebesgue differentiation theorem. This together with (1.8) completes the proof.

□

1.1.3. *Differentiability of jump functions: completion of Theorem 1.1.*

We next remove the continuity assumption made earlier in the proof of Theorem 1.4. By the decomposition in Theorem 1.3, we consider function $f$ that is increasing and bounded.

**Lemma 1.4.** *A bounded increasing function $f$ on $[a,b]$ has at most countably many discontinuities.*

*Proof.* If $f$ is discontinuous at $x$, then $(f(x^-), f(x^+))$ is an interval which can be associated to a rational number. □

Let $\{x_n\}_{n\geq 1}$ be the points where $f$ is discontinuous. Let $\alpha_n$ denote the jump of $f$ at $x_n$, that is

$$\alpha_n = f(x_n^+) - f(x_n^-),$$

where $f(x_n^+) = \lim_{y>x_n, y\to x_n} f(y)$ and $f(x_n^-) = \lim_{y<x_n, y\to x_n} f(y)$. Set

$$f(x_n) = f(x_n^-) + \theta\alpha_n, \quad \text{for some } \theta_n \in [0,1].$$

If we take

$$j_n(x) = \begin{cases} 0 & \text{if } x < x_n, \\ \theta_n & \text{if } x = x_n, 1 \qquad \text{if } x > x_n, \end{cases}$$

then we define the jump function associated to $f$ by

$$J_f(x) = \sum_{n \geq 1} \alpha_n j_n(x).$$

Obviously if $f$ is bounded then we must have

$$\sum_{n \geq 1} \alpha_n \leq f(b) - f(a) < \infty,$$

and hence the series defining $J_f(x)$ converges absolutely and uniformly.

**Lemma 1.5.** *If $f$ is increasing and bounded on $[a, b]$, then*

(i) *$J_f(x)$ is discontinuous at $\{x_n\}$ and has a jump at $x_n$ equal to that of $f$.*
(ii) *The difference $f(x) - J_f(x)$ is increasing and continuous.*

*Proof. Part (i).* If $x \neq x_n$ for all $n$, each $j_n$ is continuous at $x$, and since the series converges uniformly, $J_f$ is continuous at $x$. If $x = x_N$ for some $N$, then we write

$$J_f(x) = \sum_{n=1}^{N} \alpha_n j_n(x) + \sum_{n=N+1}^{\infty} \alpha_n j_n(x).$$

The series on the right-hand side is continuous at $x_N$. While the finite sum has a jump discontinuity at $x_N$ of size $\alpha_N$. The conclusion follows by the uniform convergence.

*Part (ii).* Continuity of $f(x) - J_f(x)$ follows by part (i). For the monotonicity, if $y > x$, we have

$$J_f(y) - J_f(x) \leq \sum_{x < x_n \leq y} \alpha_n \leq f(y) - f(x),$$

where the last inequality follows since $f$ is increasing. Hence

$$f(x) - J_f(x) \leq f(y) - J_f(y),$$

and the difference $f - J_f$ is increasing, as desired.

$\square$

For each increasing function $f$, we now write $f(x) = [f(x) - J_f(x)] + J_f(x)$, which reduces to the sum of an increasing and continuous function and a jump function. By virtue of Theorem 1.4, for the proof of Theorem 1.1, we show the following.

**Theorem 1.5.** *If $J$ is the jump function considered above, then $J'(x)$ exists and vanishes almost everywhere.*

*Proof.* Given $\varepsilon > 0$, we consider the set

$$E = E_\varepsilon = \Big\{ x : \limsup_{h \to 0} \frac{J(x+h) - J(x)}{h} > \varepsilon \Big\}. \tag{1.9}$$

This is a measurable set [3]. Suppose $\delta = m(E)$. We show that $\delta = 0$.

*Step 1.* For $\eta$ to be chosen later, we take an $N$ large so that $\sum_{n>N} \alpha_n < \eta$. Consider

$$J_0(x) = \sum_{n>N} \alpha_n j_n(x).$$

Because of our choice of $N$, we have

$$J_0(b) - J_0(a) < \eta. \tag{1.10}$$

Observe that $J - J_0$ is a finite sum of terms $\alpha_n j_n(x)$; and the set $E_{0,\varepsilon}$, with $J$ replaced by $J_0$ in (1.9), differs from $E$ by at most a finite set

$$\{x_1, x_2 \cdots, x_N\}.$$

*Step 2.* There is a compact set $K$, with $m(K) \geq \delta/2$ so that $K \subset E_{0,\varepsilon}$. Hence, there are intervals $(a_x, b_x)$ containing $x \in K$, so that

$$J_0(b_x) - J_0(a_x) > \varepsilon(b_x - a_x).$$

We first choose a finite collection of these intervals that covers $K$, and then apply the previous covering Lemma to select intervals $I_1, I_2 \cdots, I_n$ which are disjoint, and satisfy

$$\sum_{j=1}^{n} m(I_j) \geq \frac{1}{3} m(K).$$

Now

$$J_0(b) - J_0(a) \geq \sum_{j=1}^{N} \big[J_0(b_j) - J_0(a_j)\big] > \varepsilon \sum (b_j - a_j) \geq \frac{\varepsilon}{3} m(K) \geq \frac{\varepsilon}{6}\delta.$$

This yields a contradiction if $\eta < \varepsilon\delta/6$. Hence we must have $\delta = 0$ and the theorem is proved.

[3]Given $k > m$, let

$$F_{k,m}^N(x) = \sup_{\frac{1}{k} \leq |h| \leq \frac{1}{m}} \Big| \frac{J_N(x+h) - J_N(x)}{h} \Big|, \quad J_N(x) = \sum_{n=1}^{N} \alpha_n j_n(x).$$

Note that each $F_{k,m}^N(x)$ is a measurable function. Then, successively, let $N \to \infty$, $k \to \infty$, and finally $m \to \infty$.

$\square$

We can now improve Corollary 1.1 by removing the continuity assumption. The proof is the same as that of Corollary 1.1 with the help of Theorem 1.5.

**Theorem 1.6** (Lebesgue)**.** *Let $f$ be an increasing function. Then $f'$ exists almost everywhere, is integrable and*

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

**Remark 1.1.** *The integral inequality cannot improve to be equality in general. See Cantor-Lebesgue function.*

It is now at the position to show Theorem 1.1.

*Proof of Theorem 1.1.* This is a combination of previous conclusions.

By Theorem 1.3, we write $f = f_1 - f_2$ where $f_1$ and $f_2$ are both increasing and bounded.

Denote by $J_{f_1}$ the jump function associated to $f$. Then $f_1 = (f_1 - J_{f_1}) + J_{f_1}$. Since $f_1 - J_{f_1}$ is increasing and continuous, by Theorem 1.4, $f_1 - J_{f_1}$ is a.e. differentiable; On the other hand, it follows by Theorem 1.5 that $J_{f_1}$ is a.e. differentiable. Hence $f_1'(x)$ exists for a.e. $x$.

The same argument yields the a.e. differentiability of $f_2$. Consequently $f'$ exists almost everywhere.

The integrability of $f'$ follows by Theorem 1.6.

$\square$

1.1.4. *The Cantor-Lebesgue function.*

We give the construction of Cantor-Lebesgue function which yields a function $f : [0,1] \to [0,1]$ that is increasing with $f(0) = 0$ and $f(1) = 1$, but $f'(x) = 0$ almost everywhere. Hence $f$ is bounded variation, but

$$\int_a^b f'(x)dx \neq f(b) - f(a).$$

Consider the standard triadic Cantor set $\mathcal{C} \subset [0,1]$, and recall that

$$\mathcal{C} = \bigcap_{k=0}^{\infty} C_k,$$

where each $C_k$ is a disjoint union of $2^k$ closed intervals. Let $F_1(x)$ be the continuous increasing function on $[0,1]$ that satisfies

$$F_1(0) = 0, F_1(x) = 1/2 \text{ if } 1/3 \le x \le 2/3,\ F_1(1) = 1,\ \text{and } F_1 \text{ is linear on } C_1.$$

Similarly, let $F_2(x)$ be continuous and increasing, and such that

$$F_2(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/4 & \text{if } 1/9 \le x \le 2/9, \\ 1/2 & \text{if } 1/3 \le x \le 2/3, \\ 3/4 & \text{if } 7/9 \le x \le 8/9, \\ 1 & \text{if } 7/9 \le x \le 8/9 \end{cases}$$

This process yields a sequence of continuous increasing functions $\{F_n\}_{n=1}^{\infty}$ such that

$$|F_{n+1}(x) - F_n(x)| \le 2^{-n-1}.$$

Hence $\{F_n\}_{n=1}^{\infty}$ converges uniformly to a continuous limit $F$ called the Cantor-Lebesgue function. By construction, $F$ is increasing, $F(0) = 0$, $F(1) = 1$, and $F$ is constant on each interval of the complement of the Cantor set. Since $m(\mathcal{C}) = 0$, we find that $F'(x) = 0$ a.e., as desired.

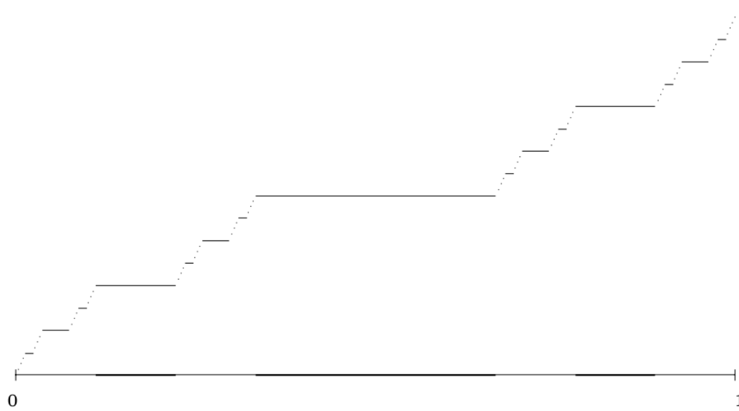


Figure 2. Cantor-Lebesgue function