**2019-2020春学期《微分几何》第六周作业**

$P_{26}$

1. **证明** 对测地线$C$, 运用Frenet公式,

$$\tau_g = \frac{d\mathbf{N}}{ds}\cdot\mathbf{B} = \frac{d\mathbf{N}}{ds}\cdot(\mathbf{T}\times\mathbf{N}) = (\dot{\mathbf{N}},\mathbf{T},\mathbf{N})$$

由$C$是测地线, $\mathbf{N}=\pm\mathbf{n}$, 故$\tau_g=(\dot{\mathbf{n}},\mathbf{T},\mathbf{n})$. 因

$$\frac{d\mathbf{n}}{ds} = \mathbf{n}_\alpha\frac{du^\alpha}{ds},\ \mathbf{T} = \mathbf{x}_\alpha\frac{du^\alpha}{ds}$$

故

$$\begin{aligned}\tau_g &= (\mathbf{n}_\alpha\frac{du^\alpha}{ds},\mathbf{x}_\beta\frac{du^\alpha}{ds},\mathbf{n}) = (\mathbf{n}_\alpha,\mathbf{x}_\beta,\mathbf{n})\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\\ &= (\mathbf{n}_1,\mathbf{x}_1,\mathbf{n})\Big(\frac{du^1}{ds}\Big)^2 + [(\mathbf{n}_1,\mathbf{x}_2,\mathbf{n})+(\mathbf{n}_2,\mathbf{x}_1,\mathbf{n})]\frac{du^1}{ds}\frac{du^2}{ds} + (\mathbf{n}_2,\mathbf{x}_2,\mathbf{n})\Big(\frac{du^2}{ds}\Big)^2\end{aligned}$$

因

$$\mathbf{n} = \frac{\mathbf{x}_1\times\mathbf{x}_2}{\sqrt{\det g}},$$

$$(\mathbf{n}_1,\mathbf{x}_1,\mathbf{n}) = \frac{1}{\sqrt{\det g}}(\mathbf{n}_1\times\mathbf{x}_1)\cdot(\mathbf{x}_1\times\mathbf{x}_2) = \frac{1}{\sqrt{\det g}}(-h_{11}g_{12}+h_{12}g_{11}).$$

同理,

$$\begin{aligned}(\mathbf{n}_1,\mathbf{x}_2,\mathbf{n}) &= \frac{1}{\sqrt{\det g}}(-h_{11}g_{22}+h_{12}g_{12})\\ (\mathbf{n}_2,\mathbf{x}_1,\mathbf{n}) &= \frac{1}{\sqrt{\det g}}(-h_{12}g_{12}+h_{22}g_{11})\\ (\mathbf{n}_2,\mathbf{x}_2,\mathbf{n}) &= \frac{1}{\sqrt{\det g}}(-h_{12}g_{22}+h_{22}g_{12})\end{aligned}$$

合并得

$$\tau_g = \frac{1}{\sqrt{\det g}}\begin{vmatrix} (\frac{du^2}{ds})^2 & -\frac{du^1}{ds}\frac{du^2}{ds} & (\frac{du^1}{ds})^2\\ g_{11} & g_{12} & g_{22}\\ h_{11} & h_{12} & h_{22}\end{vmatrix}$$

由$\tau_g$的表达式,

$$\tau_g = 0 \iff \begin{vmatrix} (du^2)^2 & -du^1du^2 & (du^1)^2\\ g_{11} & g_{12} & g_{22}\\ h_{11} & h_{12} & h_{22}\end{vmatrix} = 0$$

再由上节习题2, 得曲面上一条曲线为曲率线的充要条件是沿该曲线的测地挠率为零. ▌

3. **证明** 在测地极坐标系$(\rho,\theta)$下, 由Liouville公式, 测地曲率

$$k_g = \frac{d\alpha}{ds} - \frac{1}{2\sqrt{g_{22}}}\frac{\partial\ln g_{11}}{\partial\theta}\cos\alpha + \frac{1}{2\sqrt{g_{11}}}\frac{\partial\ln g_{22}}{\partial\rho}\sin\alpha$$

因$g_{11}=1$, 且测地圆为$\theta$曲线, 与$\rho$曲线切线的夹角$\alpha=\frac{\pi}{2}$, 因此此时

$$k_g=\frac{1}{2}\frac{\partial \ln g_{22}}{\partial \rho}$$

而由上面第2题知, $K$为常数时, $g_{22}$只与$\rho$有关, 因此在测地圆上, $kg=$const. ▌

4. **证明** 已计算得$K=\frac{\det h}{\det g}=\frac{f'f''}{v(1+f'^2)^2}$

$$\frac{f'f''}{v(1+f'^2)^2}=-\frac{1}{a^2}\Rightarrow\frac{f'df'}{(1+f'^2)^2}=-\frac{1}{a^2}vdv$$

积分得

$$\frac{1}{1+f'^2}=\frac{1}{a^2}v^2+c_1$$

取$c_1=0$, 可解出

$$f'=\pm\frac{\sqrt{a^2-v^2}}{v}$$

上式中取负号, 再积分

$$-f=\int\frac{\sqrt{a^2-v^2}}{v}dv$$

令$v=a\cos\varphi$

$$f=\int a\frac{\sin^2\varphi}{\cos\varphi}d\varphi=a[\ln(\sec\varphi+\tan\varphi)-\sin\varphi]+c.$$ ▌

5. **证明** 由测地曲率及法曲率的定义, 有

$$\dot{\mathbf{T}}=k_g\mathbf{Q}+k_n\mathbf{n}$$

因$\mathbf{Q},\mathbf{n}$均为单位向量, 故$\dot{\mathbf{Q}}\cdot\mathbf{Q}=\dot{\mathbf{n}}\cdot\mathbf{n}=0$. 由$\mathbf{Q}=\mathbf{n}\times\mathbf{T}$, 得

$$\dot{\mathbf{Q}}\cdot\mathbf{T}=(\mathbf{n}\times\mathbf{T})'\cdot\mathbf{T}=(\mathbf{n},\dot{\mathbf{T}},\mathbf{T})=(\mathbf{n},k_g\mathbf{Q}+k_n\mathbf{n},\mathbf{T})=-k_g$$

$$\dot{\mathbf{Q}}\cdot\mathbf{n}=(\mathbf{n}\times\mathbf{T})'\cdot\mathbf{n}=(\dot{\mathbf{n}},\mathbf{T},\mathbf{n})=\tau_g$$

于是

$$\dot{\mathbf{Q}}=-k_g\mathbf{T}+\tau_g\mathbf{n}.$$

再由$\mathbf{n}\cdot\mathbf{T}=\mathbf{n}\cdot\mathbf{Q}=0$, 及上两式, 得

$$\dot{\mathbf{n}}\cdot\mathbf{T}=-\mathbf{n}\cdot\dot{\mathbf{T}}=-k_n$$

$$\dot{\mathbf{n}}\cdot\mathbf{Q}=-\mathbf{n}\cdot\dot{\mathbf{Q}}=-\tau_g$$

因此

$$\dot{\mathbf{n}}=-k_n\mathbf{T}-\tau_g\mathbf{n}.$$

三式均得证. ▌

6. **证明**

$$\mathrm{III}-2H\mathrm{II}+K\mathrm{I}=0\Rightarrow\frac{\mathrm{III}}{\mathrm{I}}-2H\frac{\mathrm{II}}{\mathrm{I}}+K=0$$

因

$$\frac{\mathrm{II}}{\mathrm{I}} = k_n,\ \frac{\mathrm{III}}{\mathrm{I}} = \left|\frac{d\mathbf{n}}{ds}\right|^2 = |\dot{\mathbf{n}}|^2 = k_n^2 + \tau_g^2$$

代入即得

$$k_n^2 + \tau_g^2 - 2Hk_n + K = 0 \blacksquare$$