2019-2020春学期《微分几何》第八周作业

P_{43}

2. **证明** 设$\omega = \sum\limits_{1\le i\le m} \omega^i\sigma_i$, 则

$$\omega\wedge\varphi = \sum_{1\le i\le m} \omega^i\sigma_i\wedge\varphi = \sum_{p+1\le i\le m} \omega^i\sigma_i\wedge\sigma_1\wedge\cdots\wedge\sigma_p = 0$$

而$\sigma_{p+1}\wedge\sigma_1\wedge\cdots\wedge\sigma_p, \cdots, \sigma_m\wedge\sigma_1\wedge\cdots\wedge\sigma_p$线性无关. 于是必有$\omega^{p+1} = \cdots = \omega^m = 0$, 即$\omega$是$\sigma_1, \cdots, \sigma_p$的线性组合. ▌

3. **证明** 由$\omega = \sum\limits_{1\le\alpha<\beta\le m} a_{\alpha\beta}du^\alpha\wedge du^\beta$, 得

$$\begin{aligned} d\omega = & \sum_{1\le\gamma<\alpha<\beta\le m} \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} du^\gamma\wedge du^\alpha\wedge du^\beta \\ & + \sum_{1\le\alpha<\gamma<\beta\le m} \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} du^\gamma\wedge du^\alpha\wedge du^\beta \\ & + \sum_{1\le\alpha<\beta<\gamma\le m} \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} du^\gamma\wedge du^\alpha\wedge du^\beta \end{aligned}$$

对第一项, 作指标替换$\gamma\to\alpha$, $\alpha\to\beta$, $\beta\to\gamma$, 则

$$\sum_{1\le\gamma<\alpha<\beta\le m} \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} du^\gamma\wedge du^\alpha\wedge du^\beta = \sum_{1\le\alpha<\beta<\gamma\le m} \frac{\partial a_{\beta\gamma}}{\partial u^\alpha} du^\alpha\wedge du^\beta\wedge du^\gamma$$

对第二项, 作指标替换$\alpha\to\alpha$, $\beta\to\gamma$, $\gamma\to\beta$, 由$a_{\alpha\beta} = -a_{\beta\alpha}$, 及外积的反对称性, 得

$$\begin{aligned} \sum_{1\le\alpha<\gamma<\beta\le m} \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} du^\gamma\wedge du^\alpha\wedge du^\beta &= \sum_{1\le\alpha<\beta<\gamma\le m} \frac{\partial a_{\alpha\gamma}}{\partial u^\beta} du^\beta\wedge du^\alpha\wedge du^\gamma \\ &= \sum_{1\le\alpha<\beta<\gamma\le m} \frac{\partial a_{\gamma\alpha}}{\partial u^\beta} du^\alpha\wedge du^\beta\wedge du^\gamma \end{aligned}$$

对第三项, 由外积的反对称性得

$$\sum_{1\le\alpha<\beta<\gamma\le m} \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} du^\gamma\wedge du^\alpha\wedge du^\beta = \sum_{1\le\alpha<\beta<\gamma\le m} \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} du^\alpha\wedge du^\beta\wedge du^\gamma$$

合并, 即得

$$d\omega = \sum_{1\le\alpha<\beta<\gamma\le m} \left(\frac{\partial a_{\alpha\beta}}{\partial u^\gamma} + \frac{\partial a_{\beta\gamma}}{\partial u^\alpha} + \frac{\partial a_{\gamma\alpha}}{\partial u^\beta}\right) du^\alpha\wedge du^\beta\wedge du^\gamma. ▌$$

4. **解** (1) 由φ,ψ,η表达式,

$$\begin{aligned}\varphi\wedge\psi &= xyz^2dx\wedge dy+xzdz\wedge dy+\cos ydz\wedge dx,\\ \psi\wedge\eta &= x^2yzdy\wedge dz+xy\cos ydx\wedge dz-\cos y\sin zdx\wedge dy,\\ \eta\wedge\varphi &= xy^2zdz\wedge dx-yz\sin zdy\wedge dx-\sin zdy\wedge dz.\end{aligned}$$

(2)

$$\begin{aligned}d\varphi &= zdy\wedge dx+ydz\wedge dx,\\ d\psi &= zdx\wedge dy+xdz\wedge dy-\sin ydy\wedge dx\\ &= (z+\sin y)dx\wedge dy+xdz\wedge dy,\\ d\eta &= ydx\wedge dz+xdy\wedge dz-\cos zdz\wedge dy\\ &= ydx\wedge dz+(x+\cos z)dy\wedge dz. \blacksquare\end{aligned}$$

5. **证明** 由$x=x(u,v),\ y=y(u,v)$得

$$\begin{aligned}dx\wedge dy &= (x_udu+x_vdv)\wedge(y_udu+y_vdv)\\ &= (x_uy_v-x_vy_u)du\wedge dv=\frac{\partial(x,y)}{\partial(u,v)}du\wedge dv. \blacksquare\end{aligned}$$

6. **证明** 因

$$\begin{aligned}d\omega &= zdy\wedge dx+ydz\wedge dx+zdx\wedge dy+xdz\wedge dy+ydx\wedge dz+xdy\wedge dz\\ &= -zdx\wedge dy+ydz\wedge dx+zdx\wedge dy-xdy\wedge dz-ydz\wedge dx+xdy\wedge dz\\ &= 0\end{aligned}$$

故ω是闭形式.

令$f=xyz$, 容易验证$df=\omega$. $\blacksquare$

7. **证明** 由变换关系知

$$x_u=x_{\tilde u}\tilde u_u+x_{\tilde v}\tilde v_u,\quad x_v=x_{\tilde u}\tilde u_v+x_{\tilde v}\tilde v_v.$$

故

$$\begin{aligned}E &= (x_{\tilde u}\tilde u_u+x_{\tilde v}\tilde v_u)\cdot(x_{\tilde u}\tilde u_u+x_{\tilde v}\tilde v_u)=(\tilde u_u)^2\tilde E+2\tilde u_u\tilde v_u\tilde F+(\tilde v_u)^2\tilde G,\\ F &= (x_{\tilde u}\tilde u_u+x_{\tilde v}\tilde v_u)\cdot(x_{\tilde u}\tilde u_v+x_{\tilde v}\tilde v_v)=\tilde u_u\tilde u_v\tilde E+(\tilde u_u\tilde v_v+\tilde u_v\tilde v_u)\tilde F+\tilde v_u\tilde v_v\tilde G\\ G &= (x_{\tilde u}\tilde u_v+x_{\tilde v}\tilde v_v)\cdot(x_{\tilde u}\tilde u_v+x_{\tilde v}\tilde v_v)=(\tilde u_v)^2\tilde E+2\tilde u_v\tilde v_v\tilde F+(\tilde v_v)^2\tilde G.\end{aligned}$$

计算得

$$EG-F^2=(\tilde u_u\tilde v_v-\tilde u_v\tilde v_u)^2(\tilde E\tilde G-\tilde F^2)=\left(\frac{\partial(\tilde u,\tilde v)}{\partial(u,v)}\right)^2(\tilde E\tilde G-\tilde F^2).$$

联系第5题, 便有

$$\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}d\tilde{u}\wedge d\tilde{v}=\frac{\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}}{\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}}\sqrt{EG-F^2}du\wedge dv=\sqrt{EG-F^2}du\wedge dv. \blacksquare$$