中国科学院大学 2014-2015 学年第一学期“数学分析 IA”期末

共九道大题 满分 100 分 时间 180 分钟

1、(16 points) Calculate the following:

(1) $\lim\limits_{x\to 1}\ln x\ln(1-x)$； (2) $\lim\limits_{x\to\infty}(\cos\frac{1}{x})^x$； (3) $\lim\limits_{x\to 0}\dfrac{\cos x-e^{-\frac{x^2}{2}}}{x^4}$；

(4)Choose numbers a and b so that the function $f(x)=e^x-\dfrac{1+ax}{1+bx}$ is an infinitesimal of highest possible order as $x\to 0$.

2、(12 points) Investigate the behavior (convergence or divergence) of $\sum\limits_{n=1}^{\infty}a_n$ if

(1) $a_n=\dfrac{1}{3^n+n}$； (2) $a_n=\dfrac{n^2}{2^n}$； (3) $a_n=(-1)^n\dfrac{\ln n}{n}$.

3、(8 points) Sketch the graph or the function $f(x)=\dfrac{x^2-5x+8}{x-4}$ and discuss its critical points, max./min., monotonicity, convexity, inflection points and asymptotes..

4、(16 points) Calculate the following:

(1) $\int x^2\arctan x\mathrm{d}x$； (2) $\int x^2\sqrt{1-x^2}\mathrm{d}x$； (3) $\int\sqrt{e^x-1}\mathrm{d}x$； (4) $\int \mathrm{e}^x\left(\dfrac{1-x}{1+x^2}\right)^2\mathrm{d}x$.

5、(8 points) Assume that a function f on $\mathbf{R}$ is differentiable and $f'\le 1/2$ .Prove that there exists $x\in\mathbf{R}$ such that $f(x)=x$, and such x is unique.

6、(8 points) Consider a mirror the shape of a parabola $y=3x^2$.

(1)Compute the equation of the line tangent to the parabola at $x=1$；

(2)Find the position of a light source so that the light rays after refection on the mirror are parallel to the y-axis.

7、(8 points) For a function f that is differentiable on [0,1], Assume f' is continuous on (0,1) and $\{x\in(0,1)\,|\,f'(x)>0\}$ is dense in [0,1], then f is increasing in [0,1].

8、(16 points) Let $f:[a,b]\to R$ be a function such that $g(x_0)=\lim_{x\to x_0} f(x)$ exists for every $x_0\in[a,b]$. Prove that：

(1)For any $x_0\in[a,b]$ there exists $\delta>0$ such that $f(x)$ is bounded on $[a,b]\cap U_\delta(x_0)$;

(2) $f(x)$ is bounded on $[a,b]$;
(3)The function $g(x)$ is continuous on $[a,b]$;
(4) $f(x)\neq g(x)$ for at most countably many $x\in[a,b]$.

9、(8 points each only the higher score of the two will be taken)Assume that f is differentiable on [0,1]. Then
(1) f' defined on (0,1) satisfies the intermediate value theorem (i.e. for any $\inf_{x\in(0,1)} f'<c<\sup_{x\in(0,1)} f'$, there exists such that $f'(\xi)=c$).

(2) f' is mean continuous (i.e for any $x_0\in(0,1)$, and closed intervals $I_n=[a_n,b_n]\subset(0,1)$ containing x_0 with $\lim_{n\to\infty}(b_n-a_n)$, the mean value $\frac{f(a_n)-f(b_n)}{a_n-b_n}$ of f' on I_n converges to $f'(x_0)$ as $n\to\infty$).