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作业(10) (P100:2, 4, 5)

2. 解：因为绕原点逆时针旋转 $\frac{\pi}{4}$, 所以此旋 转变换的坐标变换公式得

$$\left\{\begin{array}{l} x' = x\cos\frac{\pi}{4} - y\sin\frac{\pi}{4} \\ y' = x\sin\frac{\pi}{4} + y\cos\frac{\pi}{4} \end{array}\right., \quad 即 \quad \left\{\begin{array}{l} x' = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ y' = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{array}\right. .$$

由此可得

$$\left\{\begin{array}{r} x = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y', \\ y = -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'. \end{array}\right.$$

因此曲线 $2xy = a$ 在此旋转变换下的曲线方程为

$$2\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)\left(-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) = a,$$

即

$$y^2 - x^2 = a.$$

4. 证明： (1). 设 O 是 σ 的不动点，即$\sigma(O) = O$ 以O为原点,建立直角坐标系$\{O, \vec{e_1}, \vec{e_2}\}$,则在上述坐标系下σ的坐标变换公式为

$$\left\{\begin{array}{l} x' = x\cos\theta - y\sin\theta, \\ y' = x\sin\theta + y\cos\theta. \end{array}\right.$$

或者

$$\left\{\begin{array}{l} x' = x\cos\theta + y\sin\theta, \\ y' = x\sin\theta - y\cos\theta. \end{array}\right.$$

由于σ只有一个不动点,则易知$\cos\theta \neq 1$,如果σ的坐标变换为

$$\left\{\begin{array}{l} x' = x\cos\theta + y\sin\theta, \\ y' = x\sin\theta - y\cos\theta. \end{array}\right.$$

记
$$C=\begin{bmatrix} \cos\theta & \sin\theta, \\ \sin\theta & -\cos\theta. \end{bmatrix}$$
则直接计算可得$|C-E|=0$,从而σ还有别的不动点,矛盾. 这样在上述坐标系下σ的坐标变换公式为
$$\begin{cases} x'=x\cos\theta-y\sin\theta, \\ y'=x\sin\theta+y\cos\theta. \end{cases}$$
由于存在旋转变换,其在上面取定坐标系的坐标公式恰为上式，而任一等距变换在取定坐标系的坐标公式是唯一的，即知σ 为旋转变换.

(2). 设 A,B 为 σ 的不动点, 即 $\sigma(A)=A,\sigma(B)=B$. 那么对于 l_{AB} 上任意其它一点 C,

(1): 如果C在线段AB之间,则有
$$d(A,C)+d(C,B)=d(A,\sigma(C))+d(B,\sigma(C))=d(A,B).$$
从而$\sigma(C)$在线段AB 之间, 由于等距变换保持线段的分比, 则$\sigma(C)=C$.

(2): 如果C在线段AB之外, 不妨设$d(B,C)=d(A,C)+d(A,B)$, 则有
$$d(B,\sigma(C))=d(\sigma(C),A)+d(A,B).$$
从而$\sigma(C)$在直线AB上, 再由等距变换保持线段的分比, 则$\sigma(C)=C$.

现以A为原点, AB 为 x 轴建立平面直角坐标系. 利用 σ 至少有两个不动点, 则σ 的坐标变换公式必为
$$\begin{cases} x'=x\cos\theta+y\sin\theta, \\ y'=x\sin\theta-y\cos\theta. \end{cases}$$
再由x 轴上的点均为不动点, 知
$$\cos\theta=1,\quad \sin\theta=0.$$
可得
$$\begin{cases} x'=x, \\ y'=-y. \end{cases}$$
从而σ 为关于 x 轴(也就是直线AB) 的反射.

5. 证明：取平面上的直角坐标系 $\{O;\vec{e}_1,\vec{e}_2\}$, 对$k=1,2$, 设旋转中心 O_k 在此坐标系下的坐标为 (x_k,y_k), 那么 $\vec{r}_k$ 在此直角坐标系下的坐标变换公式为
$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}\begin{pmatrix} x-x_k \\ y-y_k \end{pmatrix}+\begin{pmatrix} x_k \\ y_k \end{pmatrix}.$$

记矩阵

$$C_k=\begin{pmatrix}\cos\theta_k & -\sin\theta_k\\ \sin\theta_k & \cos\theta_k\end{pmatrix},\ k=1,2.$$

那么对任意的点 $P\left(x,y\right)$，有

$$\vec{r}_1\left(P\right)=C_1\begin{pmatrix}x-x_1\\ y-y_1\end{pmatrix}+\begin{pmatrix}x_1\\ y_1\end{pmatrix},$$

$$\begin{aligned}\vec{r}_2\circ\vec{r}_1\left(P\right)&=\vec{r}_2\left(C_1\begin{pmatrix}x-x_1\\ y-y_1\end{pmatrix}+\begin{pmatrix}x_1\\ y_1\end{pmatrix}\right)\\ &=C_2\left(C_1\begin{pmatrix}x-x_1\\ y-y_1\end{pmatrix}+\begin{pmatrix}x_1\\ y_1\end{pmatrix}-\begin{matrix}x_2\\ y_2\end{matrix}\right)+\begin{pmatrix}x_2\\ y_2\end{pmatrix}\\ &=C_2C_1\begin{pmatrix}x-x_1\\ y-y_1\end{pmatrix}+C_2\begin{pmatrix}x_1-x_2\\ y_1-y_2\end{pmatrix}+\begin{pmatrix}x_2\\ y_2\end{pmatrix}.\end{aligned}\tag{5-1}$$

即

$$\begin{aligned}\vec{r}_2\circ\vec{r}_1\left(P\right)=&\begin{pmatrix}\cos\left(\theta_1+\theta_2\right) & -\sin\left(\theta_1+\theta_2\right)\\ \sin\left(\theta_1+\theta_2\right) & \cos\left(\theta_1+\theta_2\right)\end{pmatrix}\begin{pmatrix}x-x_1\\ y-y_1\end{pmatrix}\\ &+\begin{pmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}x_1-x_2\\ y_1-y_2\end{pmatrix}+\begin{pmatrix}x_2\\ y_2\end{pmatrix}.\end{aligned}\tag{5-2}$$

(1). 当 $\theta_1+\theta_2=0$ $\big($或 2π的整数倍$\big)$时，由(5-2)得

$$\begin{aligned}\vec{r}_2\circ\vec{r}_1\left(P\right)&=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x-x_1\\ y-y_1\end{pmatrix}+\begin{pmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}x_1-x_2\\ y_1-y_2\end{pmatrix}+\begin{pmatrix}x_2\\ y_2\end{pmatrix}\\ &=\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}x_1-x_2\\ y_1-y_2\end{pmatrix}-\begin{pmatrix}x_1-x_2\\ y_1-y_2\end{pmatrix}.\end{aligned}$$

注意到

$$\overrightarrow{O_1\vec{r}_2\left(O_1\right)}=\begin{pmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}x_1-x_2\\ y_1-y_2\end{pmatrix}-\begin{pmatrix}x_1-x_2\\ y_1-y_2\end{pmatrix},$$

所有 $\vec{r}_2\circ\vec{r}_1$ 是以 $\overrightarrow{O_1\vec{r}_2\left(O_1\right)}$ 为平移向量的平移变换.

(2). 当 $\theta_1+\theta_2\neq 0$ $\big($不是 2π的整数倍$\big)$时，注意到(5-2)中的坐标变换矩阵$\begin{pmatrix}\cos\left(\theta_1+\theta_2\right) & -\sin\left(\theta_1+\theta_2\right)\\ \sin\left(\theta_1+\theta_2\right) & \cos\left(\theta_1+\theta_2\right)\end{pmatrix}$仍为正交阵, 所以变换$\vec{r}_2\circ\vec{r}_1$至少有一个不动

点. 设 $Q\left(x_0,y_0\right)$ 是 $\vec{r}_2\circ\vec{r}_1$ 的一个不动点，那么再由(5-1)得

$$\begin{aligned}\begin{pmatrix}x_0\\y_0\end{pmatrix} &= C_2C_1\begin{pmatrix}x_0-x_1\\y_0-y_1\end{pmatrix}+C_2\begin{pmatrix}x_1-x_2\\y_1-y_2\end{pmatrix}+\begin{pmatrix}x_2\\y_2\end{pmatrix}\\ &= C_2C_1\begin{pmatrix}x_0\\y_0\end{pmatrix}-C_2C_1\begin{pmatrix}x_1\\y_1\end{pmatrix}+C_2\begin{pmatrix}x_1-x_2\\y_1-y_2\end{pmatrix}+\begin{pmatrix}x_2\\y_2\end{pmatrix}.\end{aligned}$$

上式两边同时左乘 C_2^{-1}, 且移项得

$$\left(C_2-C_1\right)\begin{pmatrix}x_0\\y_0\end{pmatrix}=-C_1\begin{pmatrix}x_1\\y_1\end{pmatrix}+\begin{pmatrix}x_1-x_2\\y_1-y_2\end{pmatrix}+C_2^{-1}\begin{pmatrix}x_2\\y_2\end{pmatrix}, \tag{5-3}$$

即

$$\begin{aligned}&\begin{pmatrix}\cos\theta_2-\cos\theta_1 & \sin\theta_1+\sin\theta_2\\ -\sin\theta_1-\sin\theta_2 & \cos\theta_2-\cos\theta_1\end{pmatrix}\begin{pmatrix}x_0\\y_0\end{pmatrix}\\ &=-\begin{pmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix}+\begin{pmatrix}\cos\theta_2 & \sin\theta_2\\ -\sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}x_2\\y_2\end{pmatrix}+\begin{pmatrix}x_1-x_2\\y_1-y_2\end{pmatrix}.\end{aligned} \tag{5-4}$$

因为

$$\begin{aligned}\det\left(C_2-C_1\right)=\left(\cos\theta_1-\cos\theta_2\right)^2+\left(\sin\theta_1+\sin\theta_2\right)^2=0\\ \Longleftrightarrow\ \theta_1+\theta_2=0\left(\text{或}2\pi\text{的整数倍}\right),\end{aligned}$$

又 $\theta_1+\theta_2\neq 0$, 所以由线性代数的知识知(5-3) 有唯一解. 也就是说 $\vec{r}_2\circ\vec{r}_1$ 有唯一的不动点, 所以是以此点为中心的旋转变换. 且由(5-1)知旋转角为$\theta_1+\theta_2$. 由线性代数中的克莱默法则解得此不动点Q的坐标为

$$\begin{aligned}\begin{pmatrix}x_0\\y_0\end{pmatrix} &= \begin{pmatrix}\frac{1}{2} & \frac{\sin(\theta_1+\theta_2)}{2\cos(\theta_1+\theta_2)-2}\\ -\frac{\sin(\theta_1+\theta_2)}{2\cos(\theta_1+\theta_2)-2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x_2\\y_2\end{pmatrix}\\ &\quad+\begin{pmatrix}\frac{1}{2} & -\frac{\sin(\theta_1+\theta_2)}{2\cos(\theta_1+\theta_2)-2}\\ \frac{\sin(\theta_1+\theta_2)}{2\cos(\theta_1+\theta_2)-2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix}\\ &\quad+\begin{pmatrix}\frac{\cos\theta_1-\cos\theta_2}{2\cos(\theta_1+\theta_2)-2} & \frac{\sin\theta_1+\sin\theta_2}{2\cos(\theta_1+\theta_2)-2}\\ -\frac{\sin\theta_1+\sin\theta_2}{2\cos(\theta_1+\theta_2)-2} & \frac{\cos\theta_1-\cos\theta_2}{2\cos(\theta_1+\theta_2)-2}\end{pmatrix}\begin{pmatrix}x_1-x_2\\y_1-y_2\end{pmatrix}.\end{aligned}$$