Probability Theory

Exercise Sheet 11

Exercise 11.1 Let $(X_n)_{n\geq 0}$ be a sequence of random variables with values in $[0,1]$. We set $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Suppose that $X_0 = a \in [0,1]$ and

$$P\left[X_{n+1} = \frac{X_n}{2}\Big|\mathcal{F}_n\right] = 1 - X_n, \qquad P\left[X_{n+1} = \frac{1+X_n}{2}\Big|\mathcal{F}_n\right] = X_n.$$

(a) Show that $(X_n)_{n\geq 0}$ is a $\mathcal{F}_n$-martingale that converge to a random variable X_∞ P-almost surely and in L^2.

(b) Show that $E\left[(X_{n+1} - X_n)^2\right] = \frac{1}{4}E\left[X_n(1 - X_n)\right]$.

Exercise 11.2 Let Y_n, $n \geq 0$ be i.i.d. with $P[Y_0 = 1] = p$ and $P[Y_0 = 0] = 1 - p$ for some $p \in (0,1)$. Let $\mathcal{F}_n := \sigma(Y_0, \ldots, Y_n)$ for $n \geq 0$ and define

$$T := \inf\{n \geq 0 \mid Y_n = 1\}.$$

Determine the Doob decomposition of $X_n := 1_{\{T \leq n\}}$, $n \geq 0$.
  Hint: First check that X_n is an $\mathcal{F}_n$-submartingale.

Exercise 11.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration on this space. Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$-martingale such that $M_0 = 0$ and $M_n \in L^2$ for all n.

(a) Why is $(M_n^2)_{n\geq 0}$ a submartingale?

(b) Let $(A_n)_{n\geq 0}$ be the non-decreasing and predictable process from the Doob decomposition of $(M_n^2)_{n\geq 0}$. Show that $\tau_a := \inf\left\{n \geq 0; A_{n+1} > a^2\right\}$ is a stopping time.

(c) Show that $P\left[\sup_{n\geq 0}|M_{n\wedge\tau_a}| > a\right] \leq \frac{E[A_\infty \wedge a^2]}{a^2}$, where A_∞ is the P-a.s. limit of $(A_n)_{n\geq 0}$.

  Hint: First consider $P\left[\sup_{n\leq N}|M_{n\wedge\tau_a}| > a\right]$ for $N \in \mathbb{N}$ and use Doob's inequality.

(d) Show that $P\left[\sup_{n\geq 0}|M_n| > a\right] \leq P[A_\infty > a^2] + P\left[\sup_{n\geq 0}|M_{n\wedge\tau_a}| > a\right]$.

Submission: until 14:15, Dec 10., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint