ETH Zurich HS 2019
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 8

Exercise 8.1 Let X be a random variable in $L^2(\Omega, \mathcal{A}, P)$ and $\mathcal{F} \subseteq \mathcal{A}$. The *conditional variance* of X given $\mathcal{F}$ is defined as $\mathrm{Var}[X|\mathcal{F}] := E[(X - E[X|\mathcal{F}])^2|\mathcal{F}]$. Prove that

(a) $\mathrm{Var}[X|\mathcal{F}] = E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2$;

(b) $\mathrm{Var}(X) = E[\mathrm{Var}[X|\mathcal{F}]] + \mathrm{Var}[E[X|\mathcal{F}]]$.

(c) Compute $\mathrm{Var}[X|\mathcal{F}]$, where $\mathcal{F} = \sigma(A_1, A_2)$ where $\{A_1, A_2\}$ is a partition of Ω and $P(A_i) > 0$ for $i = 1, 2$.

Exercise 8.2 Let $S, T : \Omega \to \mathbb{N} \cup \{\infty\}$ be $\mathcal{F}_n$-stopping times. Prove or provide a counter example disproving the following statements:

(a) $S - 1$ is a stopping time.

(b) $S + 1$ is a stopping time.

(c) $S \wedge T$ is a stopping time.

(d) $S \vee T$ is a stopping time.

(e) $S + T$ is a stopping time.

Exercise 8.3 (Polya's Urn)
An urn initially contains s black and w white balls. We consider the following process. At each step a random ball is drawn from the urn, and is replaced by t balls of the same colour, for some fixed $t \geq 1$. We define the random variable Y_n as the proportion of black balls in the urn after the n-th iteration. Show that $E[Y_{n+1}|\sigma(Y_1, Y_2, \ldots Y_n)] = Y_n$, for all $n \in \mathbb{N}$, that is, $\{Y_n\}_{n\in\mathbb{N}}$ is a martingale.

Submission: until 14:15, Nov 19., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint