ETH Zurich HS 2019
D-MATH, D-PHYS
Prof. A.-S. Sznitman

Coordinator
Daniel Balint

Probability Theory

Exercise Sheet 12

Exercise 12.1 Let $(X_n)_{n\geq 0}$ be a uniformly integrable family of random variables on $(\Omega, \mathcal{A}, P)$.

(a) Assume that X_n converges to a random variable X in distribution. Show that

$$E[X_n] \xrightarrow{n\to\infty} E[X].$$

Hint: Compare to (3.6.18)–(3.6.20), p. 111 of the lecture notes.

(b) Assume that X_n converges to a random variable X in probability. Show that $X \in L^1$ and that X_n converges to X in L^1.

Exercise 12.2

Definition: Let $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ be a canonical (time-homogenous) Markov chain with a *countable* state space E, a transition kernel K, and canonical coordinates $(X_n)_{n\geq 0}$. The matrix

$$Q = (Q(x,y))_{x,y\in E} := (K(x,\{y\}))_{x,y\in E} = (P_x[X_1 = y])_{x,y\in E}$$

is then called the *transition matrix* of the Markov chain. For the meanings of notation P_x and transition kernel we refer to p. 145 in lecture notes.

Let E be a countable set, $(S, \mathcal{S})$ a measurable space, $(Y_n)_{n\geq 1}$ a sequence of i.i.d. S-valued random variables. We define a sequence $(Z_n)_{n\geq 0}$ through $Z_0 = x \in E$ and $Z_{n+1} = \Phi(Z_n, Y_{n+1})$, where $\Phi : E \times S \to E$ is a measurable map. Find a transition kernel K on E such that the canonical law P_x with transition kernel K has the same law as $(Z_n)_{n\geq 0}$ (hence $(Z_n)_{n\geq 0}$ induces a time-homogenous Markov chain with transition kernel K). Calculate the corresponding transition matrix.

Exercise 12.3 Let E be a countable set, and $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ a canonical time–homogeneous Markov chain with state space E, canonical coordinate process $(X_n)_{n\geq 0}$ and transition matrix $Q = (Q(x,y))_{x,y\in E}$. Let $F \subset E$ and set $\tau_F := \inf\{n \geq 0 \,|\, X_n \in F\}$.

Let $f : E \to \mathbb{R}^+$ be a bounded function such that $f(x) \geq Qf(x)$ (resp. $=$) for all $x \in F^c$, where

$$Qf(x) := \int_\Omega f(X_1(\omega)) P_x(\mathrm{d}\omega) = \sum_{y\in E} f(y) Q(x,y).$$

Show that $(f(X_{n\wedge\tau_F})_{n\geq 0})$ for all $x \in E$ is a positive P_x-supermartingale (resp. P_x-martingale) with respect to the canonical filtration $(\mathcal{F}_n)_{n\geq 0}$.

Submission: until 14:15, Dec 17., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint