REAL ANALYSIS
LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:
[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.
[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

Part 1. Integration Theory

1. The Lebesgue Integral: basic properties and convergence theorems

In this part all sets and functions are measurable, unless otherwise stated.

The integral and its properties are developed in a step-by-step fashion:

(1) Simple functions
(2) bounded functions supported on a set of finite measure
(3) Non-negative functions
(4) Integrable functions (the general case)

At each stage we shall prove appropriate convergence theorems that amount to interchanging the integral with limits. At the end of the process we shall have achieved a general theory of integration that will be decisive the the study of further problems.

1.1. **Stage one: simple functions.**

Properties of simple functions.

Recall that simple functions are functions on $\mathbb{R}^n$ of the form $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$ where $m(E_k) < \infty$.

The canonical form is when a_k are all distinct and E_k are disjoint.

To write ϕ in canonical form, let $\{\alpha_1, \cdots, \alpha_M\}$ be the set of values of ϕ and $F_k = \{x : \phi(x) = \alpha_k\}$. Then $\phi = \sum_{k=1}^{M} \alpha_k \chi_{F_k}$.

Integral of simple functions.

If $\phi(x) = \sum_{k=1}^{M} \alpha_k \chi_{F_k}$ is in canonical form then we define

$$\int \phi = \int \phi(x)dx = \sum_{k=1}^{M} \alpha_k m(F_k).$$

If E has finite measure then define

$$\int_E \phi = \int \phi \chi_E.$$

Properties of integral.

Analogues of the following *six* properties will hold in the subsequent more general situations.

Proposition 1.1. *The integral of simple functions satisfies the following properties:*

(i) *Independence of the representation. If $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of ϕ, then*

$$\int \phi = \sum_{k=1}^{N} a_k m(E_k).$$

(ii) *Linearity. If ϕ and ψ are simple, and $a, b \in \mathbb{R}$, then*

$$\int (a\phi + b\psi) = a \int \phi + b \int \psi.$$

(iii) *Additivity. If E and F are disjoint subsets of $\mathbb{R}^n$ with finite measure, then*

$$\int_{E\cup F} \psi = \int_E \psi + \int_F \psi.$$

(iv) *Monotonicity. If $\phi \le \psi$ are simple, then*

$$\int \phi \le \int \psi.$$

(v) *Triangle inequality. If ϕ is simple, then so is $|\phi|$, and*

$$\left| \int \phi \right| \le \int |\phi|.$$

(vi) *Sets of measure zero do not count. If ϕ and ψ are simple and $\phi = \psi$ a.e., then*

$$\int \phi = \int \psi.$$

Proof. Idea of (i). Suppose that $\phi = \sum_{j=1}^{M} \alpha_j \chi_{F_j}$ is the canonical form. Observe that

$$\alpha_j = \sum_{\{k : m(E_k \cap F_j) > 0\}} a_k.$$

Hence

$$\begin{aligned}
\sum_{j=1}^{M} \alpha_j m(F_j) &= \sum_{j=1}^{M} \sum_{\{k : m(E_k \cap F_j) > 0\}} a_k m(F_j) \\
&= \sum_{k=1}^{N} \sum_{\{j : m(E_k \cap F_j) > 0\}} a_k m(F_j) \\
&= \sum_{k=1}^{N} a_k m(E_k).
\end{aligned}$$

Idea of (ii). It follows from (i).

Idea of (iii). It follows from (ii) and using $\chi_{E \cup F} = \chi_E + \chi_F$.

Idea of (iv). It follows by applying (ii) to $\phi - \psi$.

Idea of (v). Use (i).

Idea of (vi). Use (ii).

□

1.2. **Stage two: bounded functions supported on a set of finite measure.**

Support of a function.

Definition 1.1. *The support of f is $\{x : f(x) \neq 0\}$ [1], written as supp(f).*

We say f is supported on F if $\{x : f(x) \neq 0\} \subset F$.

Suppose f is supported on E with $|f(x)| \leq M$ for all x. Then there is a sequence of simples functions $\{\phi_k\}_{k \geq 1}$ supported on E, with $|\phi_k(x)| \leq M$ for all x, such that $f(x) = \lim_{k \to \infty} \phi_k(x)$ for all x. See Theorem **??**.

Important lemma.

[1] Warning: Usually the support of a function is defined as the smallest closed set where the function is non-zero, i.e. the intersection of all closed sets on which the function is non-zero.

The key lemma that follows allows us to define the integral for the class of bounded functions supported on sets of finite measure.

Lemma 1.1. *Let f be a bounded function supported on E of finite measure. If $\{\phi_k\}_{k\geq 1}$ is any sequence of simple functions bounded by M, supported on E, and with $\phi_k(x) \to f(x)$ for a.e. x, then*

(i) *The limit $\lim_{k\to\infty} \int \phi_k$ exists.*
(ii) *If $f = 0$ a.e., then the limit $\lim_{k\to\infty} \int \phi_k = 0$.*
(iii) *If $\{\psi_k\}_{k\geq 1}$ are also simple functions supported on E, bounded in absolute value by M, and $\psi_k(x) \to f(x)$ for a.e. x, then*

$$\lim_{k\to\infty} \int \phi_k = \lim_{k\to\infty} \int \psi_k.$$

Remark 1.1.

(a) *Avoid blow up! It is necessary that $|\phi_k(x)| \leq M$ for all x. See e.g. $E = [0,1]$, $f(x) = 0$ for all x, $\phi_k = k\chi_{(0,1/k]}$.*
(b) *Avoid escape to ∞! It is necessary that ϕ_k supported on E. See e.g. $E = [0,1]$, $f(x) = 0$ for all x, $\phi_k = \frac{1}{k}\chi_{[0,k]}$.*

Proof of Lemma 1.1. Suppose $\varepsilon > 0$. By Egorov's theorem, $\phi_k \to f$ uniformly on a closed set $F_\varepsilon \subset E$ and $m(E \setminus F_\varepsilon) \leq \varepsilon$. Hence, using Proposition 1.1,

$$\begin{aligned} \Big| \int \phi_k - \int \phi_j \Big| &\leq \int_{F_\varepsilon} |\phi_k - \phi_j| + \int_{E\setminus F_\varepsilon} |\phi_k - \phi_j| \\ &\leq m(E) \sup_{F_\varepsilon} |\phi_k - \phi_j| + 2M\varepsilon. \end{aligned}$$

This shows that $\{\int \phi_k\}_{k\geq 1}$ is a Cauchy sequence and so has a limit.

For (ii), $\phi_k \to 0$ uniformly on F_ε, with $m(E \setminus F_\varepsilon) \leq \varepsilon$, and therefore by a similar argument as above

$$\Big| \int \phi_k \Big| \leq m(E) \sup_{F_\varepsilon} |\phi_k| + 2M\varepsilon \to 0.$$

For (iii), we apply part (ii) to $\phi_k - \psi_k$.

$\square$

<u>*Definition.*</u>

If f is a bounded function supported on E with $m(E) < \infty$, we define its Lebesgue integral by

$$\int f = \lim_{k\to\infty} \int \phi_k,$$

for any sequence ϕ_k of simple functions supported on E, bounded in absolute value by some fixed M, with $\phi_k(x) \to f(x)$ for a.e. x.

For any such f and any $F \subset \mathbb{R}^n$, define

$$\int_F f = \int f\chi_F.$$

Note: The first definition makes sense and is independent of the approximating sequence by Lemma 1.1. The second definition makes sense since if f is bounded and supported on a set of finite measure, then so is $f\chi_F$.

Properties of integral.

Proposition 1.2. *Suppose f and g are bounded and supported on a set of finite measure. Then the following properties hold.*

(i) *Linearity. If $a, b \in \mathbb{R}$, then*

$$\int (af + bg) = a\int f + b\int g.$$

(ii) *Additivity. If E and F are disjoint subsets of $\mathbb{R}^n$, then*

$$\int_{E\cup F} f = \int_E f + \int_F f.$$

(iii) *Monotonicity. If $f \le g$, then*

$$\int f \le \int g.$$

(iv) *Triangle inequality. $|f|$ is also bounded, supported on a set of finite measure, and*

$$\left|\int f\right| \le \int |f|.$$

(v) *Sets of measure zero do not count. If $f = g$ a.e., then*

$$\int f = \int g.$$

Proof. Idea of (i). Take sequences of simple functions (uniformly bounded and supported on a fixed set of finite measure) converging a.e. to f and g.

Idea of (ii). It follows from (i).

Idea of (iii). By (i). $\int f - \int g = \int (f-g)$. But $f - g \geq 0$ and so $\int (f-g) \geq 0$.

Idea of (iv). If $\phi_k \to f$ then $|\phi_k| \to |f|$. Now use the corresponding result for simple functions

Idea of (v). By (i), $\int f - \int g = \int (f-g) = 0$. For the second equality one notes that $h = 0$ a.e. implies $\int h = 0$.

$\square$

Bounded convergence theorem.

Theorem 1.1 (Bounded convergence Theorem). *Let $m(E) < \infty$. Suppose $\{f_k\}_{k\geq 1}$ is a sequence of functions supported on E, bounded in absolute value by M, and $f_n(x) \to f(x)$ for a.e. x. Then f is measurable, bounded, supported on E for a.e. x, and*

$$\lim_{k\to\infty} \int |f_k - f| = 0.$$

In particular,

$$\lim_{k\to\infty} \int f_k = \int f.$$

Proof. Suppose $\varepsilon > 0$. By Egorov's theorem, $f_k \to f$ uniformly on a closed set $F_\varepsilon \subset E$ with $m(E \setminus F_\varepsilon) \leq \varepsilon$. In virtue of Proposition 1.2,

$$\begin{aligned}
\int |f_k - f| &= \int_{F_\varepsilon} |f_k - f| + \int_{E\setminus F_\varepsilon} |f_k - f| \\
&\leq m(E) \sup_{F_\varepsilon} |f_k - f| + 2\varepsilon M \\
&\to 0 \quad \text{as } k \to \infty \text{ and } \varepsilon \to 0.
\end{aligned}$$

For the second, just note that

$$\Big| \int f_k - \int f \Big| \leq \int |f_k - f| \to 0.$$

$\square$

Remark 1.2.

(a) *The proof is essentially the same as Lemma 1.1. The difference is that by this point we can use the basic properties of $\int f_k$ for f_k not just a simple function.*

(b) *The bounded convergence theorem (BCT for short) is in fact a special case of the slightly more general dominated convergence theorem (DCT for short). See later.*

(c) *If $f \geq 0$, f bounded, supported on a set of finite measure, and $\int f = 0$, then $f = 0$ a.e. (We later remove the requirements f bounded and supported on a set of finite measure).*

Proof: Suppose $\{f > 0\}$ has positive measure. Since $\{f > 0\} = \cup_{k \geq 1}\{f \geq 1/k\}$, we infers $F_k := \{f \geq 1/k\}$ has positive measure for some k. But then

$$\int f \geq \int \frac{1}{k}\chi_{F_k} = \frac{1}{k}m(F_k) > 0,$$

arriving a contradiction.

1.3. **Stage three: non-negative functions.**

Definition 1.2. *Suppose $f : \mathbb{R}^n \to [0, \infty]$. Define*

$$\int f = \sup_{g \in \mathcal{F}(f)} \int g,$$

where

$$\mathcal{F}(f) = \{g : 0 \leq g \leq f,\ g \text{ is bounded, supported on a set of finite measure}\}.$$

If $E \subset \mathbb{R}^n$ then we define

$$\int_E f = \int f\chi_E.$$

We say f is (Lebesgue) integrable.

Properties of integral.

Proposition 1.3. *Suppose $f \geq 0$ and $g \geq 0$. Then the following properties hold* [2].

(i) *Linearity. If a and b are positive real numbers, then*

$$\int (af + bg) = a\int f + b\int g.$$

[2] We do not require in the following statements that f or g is integrable unless otherwise stated.

(ii) *Additivity. If E and F are disjoint subsets of $\mathbb{R}^n$, then*

$$\int_{E\cup F} f = \int_E f + \int_F f.$$

(iii) *Monotonicity. If $f \leq g$, then*

$$\int f \leq \int g.$$

(iv) *If f is integrable, then $m(\{f = +\infty\}) = 0$.*

(v) *If $f \geq 0$ and $\int f = 0$, then $f = 0$ a.e.*

Proof. Idea of (i). Firstly

$$\int af = \sup_{\eta\in\mathcal{F}(af)} \int \eta = \sup_{\eta\in\mathcal{F}(f)} \int a\eta = a \sup_{\eta\in\mathcal{F}(f)} \int \eta = a \int f.$$

The second last equality is due to Proposition 1.2. This verifies that $\int af = a\int f$. Next we show (i) when $a = b = 1$. For any $\phi \in \mathcal{F}(f)$ and $\psi \in \mathcal{F}(g)$, we have $\phi+\psi \in \mathcal{F}(f+g)$ and so

$$\int (f+g) \geq \int (\phi+\psi) = \int \phi + \int \psi, \ \forall\ \phi \in \mathcal{F}(f) \text{ and } \psi \in \mathcal{F}(g).$$

Consequently

$$\int (f+g) \geq \int f + \int g.$$

On the other hand, for any $\eta \in \mathcal{F}(f+g)$, taking $\eta_1 = \min\{\eta, f\}$ and $\eta_2 = \eta - \eta_1$, we have $\eta_1 \in \mathcal{F}(f)$ and $\eta_2 \in \mathcal{F}(g)$, and hence

$$\int \eta = \int \eta_1 + \int \eta_2 \leq \int f + \int g, \ \forall\ \eta \in \mathcal{F}(f+g).$$

This yields

$$\int (f+g) \leq \int f + \int g.$$

This finishes part (i).

Idea of (ii). It follows by (i) and $f\chi_{E\cup F} = f\chi_E + f\chi_F$.

Idea of (iii). This is because $\eta \in \mathcal{F}(f)$ implies $\eta \in \mathcal{F}(g)$.

Idea of (iv). Note that $\{f = +\infty\} = \cap_{k\geq 1}\{f \geq k\}$. By (iii),

$$\int f \geq \int f\chi_{\{f\geq k\}} \geq km(\{f \geq k\}).$$

It follows that

$$m(\{f = +\infty\}) = \lim_{k\to\infty} m(\{f \geq k\}) = 0.$$

Idea of (v). Note that $\{f > 0\} = \cup_k \{f > 1/k\}$. For each k, we have

$$0 = \int f \geq \int f\chi_{\{f>1/k\}} \geq \frac{1}{k} m(\{f > 1/k\}) \Longrightarrow m(\{f > \frac{1}{k}\}) = 0 \ \forall\ k \geq 1.$$

Therefore $m(\{f > 0\}) = 0$.

□

Fatou's Lemma and Monotone convergence theorem.

We turn our attention to some convergence theorem for the class of non-negative measurable functions. We ask: suppose $f_k \geq 0$ and $f_k \to f$ almost everywhere; Is it true that $\int f_k \to \int f$? Unfortunately, the example that follows provides a negative answer to this: consider

$$f_k(x) = k\chi_{(0,1/k)}(x), \quad x \in \mathbb{R}.$$

Then $f_k(x) \to 0$ for all x, yet $\int f_k = 1$ for all k.

Lemma 1.2 (Fatou). *Suppose $\{f_k\}$ is a sequence of measurable functions with $f_k \geq 0$. If $\lim_{k\to\infty} f_k(x) = f(x)$ for a.e. x, then*

$$\int f \leq \liminf_{k\to\infty} \int f_k.$$

Remark 1.3.

(i) *Fatou's Lemma allows the case $\int f = \infty$, or $\liminf_{k\to\infty} f_k = \infty$.*

(ii) *Let E_k be a sequence of measurable sets, and $E = \liminf_{k\to\infty} E_k = \cup_{k=1}^\infty \cap_{j=k}^\infty E_j$. One sees that*

$$\chi_{E_k}(x) \to \chi_E(x) \ \text{ for a.e. } x.$$

Recall Corollary **??**: *$m(E) \leq \liminf_{k\to\infty} m(E_k)$, which is equivalent to*

$$\int \chi_E = \int \lim_{k\to\infty} \chi_{E_k} \leq \liminf_{k\to\infty} \int \chi_{E_k}.$$

Hence the Fatou's Lemma is a generalisation (integral version) of Corollary **??**.

Proof of Fatou's Lemma. Let $g \in \mathcal{F}(f)$. Consider $g_k = \min\{g, f_k\}$. Then $g_k \to g$ a.e., and g_k, g are bounded, supported on a set of finite measure (namely $\text{supp}(g)$). By the BCT (i.e. Theorem 1.1),

$$\lim_{k\to\infty} \int g_k = \int \lim_{k\to\infty} g_k = \int g.$$

Consequently

$$\int f = \sup_{g\in\mathcal{F}(f)} \int g = \sup_{g\in\mathcal{F}(f)} \lim_{k\to\infty} \int g_k \leq \lim_{k\to\infty} \int f_k,$$

where the last inequality is because $g_k \leq f_k$ for any $g \in \mathcal{F}(f)$.

□

The following is a special case of the Dominated Convergence Theorem (DCT for short) that will be discussed later, except in the case $\int f = \infty$.

Theorem 1.2. *Suppose f, f_k are non-negative measurable functions, $0 \leq f_k \leq f$ a.e., and $f_k \to f$ a.e. Then* [3]

$$\lim_{k\to\infty} \int f_k = \int f.$$

Proof. Since $f_k \leq f$ a.e., we necessarily have $\limsup_{k\to\infty} \int f_k \leq \int f$. The proof is then completed by using the Fatou's Lemma.

□

An important case is:

Theorem 1.3 (Monotone Convergence Theorem)**.** *Suppose $0 \leq f_k \nearrow f$* [4]*. Then* [5]

$$\int f = \lim_{k\to\infty} \int f_k.$$

The following is an immediate and useful corollary, and allows us to exchange summation and integration in many cases.

Theorem 1.4. *Suppose $g_k \geq 0$ for all k. Then*

$$\int \sum_{k=1}^{\infty} g_k = \sum_{k=1}^{\infty} \int g_k.$$

[3] It is allowed that $\int f = \infty$.

[4] We write $f_k \nearrow f$ whenever $\{f_k(x)\}$ is a non-decreasing sequence for a.e. x, and $f_k(x) \to f(x)$ a.e.; write $f_k \searrow f$ whenever $\{f_k(x)\}$ is a non-increasing sequence for a.e. x, and $f_k(x) \to f(x)$ a.e.

[5] It is allowed that $\int f = \infty$.

If the right hand side is finite, then the series $\sum_{k=1}^{\infty} g_k(x)$ converges for a.e. x.

Proof. Let $f_k = \sum_{j=1}^{k} g_j$ and apply the MCT (namely Theorem 1.3).

□

Exercise 1.1. *Suppose $\sum m(E_k) < \infty$, then $m(\limsup E_k) = 0$.*

Proof: Let $g_k = \chi_{E_k}$. Then

$$\sum_{k=1}^{\infty} \int g_k = \sum_{k=1}^{\infty} m(E_k) < \infty.$$

By Theorem 1.4, $\sum_{k=1}^{\infty} \chi_{E_k}(x)$ is finite for a.e. x. This implies the set of points that belong to infinitely many E_k has measure zero, namely $m(\limsup E_k) = 0$.

Exercise 1.2. *Consider the function*

$$f(x) = \frac{1}{|x|^{n+1}} \text{ if } x \neq 0, \text{ and } f(0) = 0.$$

Then f is integrable outside any ball, $|x| \geq \varepsilon$, and moreover

$$\int_{|x| \geq \varepsilon} f \leq C\varepsilon^{-1}, \quad \text{for some constant } C > 0.$$

Proof: Let $A_k = \{x \in \mathbb{R}^n : 2^k\varepsilon < |x| \leq 2^{k+1}\varepsilon\}$, and define

$$g(x) = \sum_{k=0}^{\infty} g_k(x), \text{ where } g_k(x) = \frac{1}{(2^k\varepsilon)^{n+1}} \chi_{E_k}(x).$$

It can be verified that

$$f(x) \leq g(x) \Longrightarrow \int f \leq \int g$$

Denote $A = \{1 < |x| < 2\}$. Then $m(E_k) = (2^k\varepsilon)^n m(A)$. Hence

$$\int f \leq \sum_{k=0}^{\infty} \int g_k = \sum_{k=0}^{\infty} \frac{m(A)}{2^k\varepsilon} = 2m(A)\varepsilon^{-1}.$$

1.4. **Stage four: general case.**

Definition 1.3. *Let f be a real-valued measurable function f on $\mathbb{R}^n$. Define* [6]

$$\int f = \int f^+ - \int f^-.$$

If both terms are finite, we say that f is (Lebesgue) integrable.

Remark 1.4.

(i) *f is integrable if and only if $\int |f| < \infty$.*

(ii) *If f can be decomposed as $f = g - h$ for two non-negative functions g and h, and $\int g, \int h$ are finite, then $\int f = \int g - \int h$* [7].

Proof: Note that $f^+ + h = g + f^-$. It then follows by integrating both sides.

(iii) *The integrability of f and the value of $\int f$ are unaffected by changing f on a set of measure zero.*

Properties.

Simple applications of the definition and Proposition 1.3 yield all the elementary properties of the integral:

Proposition 1.4. *The integral of Lebesgue integrable functions is linear, additive, monotonic, and satisfies the triangle inequality.*

Important Note:

We can now integrate functions that are not necessarily ≥ 0. It is easy to see that the condition $f_k \geq 0$ in Fatou's Lemma and in Theorems 1.2-1.4 can be replaced by

$$f_k \geq h \text{ for some } h \text{ that is } \textit{integrable}.$$

This is because we can apply Fatou's Lemma and the theorems to $\hat{f}_k = f_k - h \geq 0$.

Theorem 1.5. *Suppose f is integrable. Then for each $\varepsilon > 0$ there exists a ball $B_R(0)$ such that*

$$\int_{B_R^c(0)} |f| \leq \varepsilon.$$

[6] Note that $\int f$ makes sense unless both $\int f^+ = \infty$ and $\int f^- = \infty$, where $\int f^+$ and $\int f^-$ are given by Stage three: the case for non-negative functions.

[7] It is not necessary that $g = f^+$ and $h = f^-$.

Proof. Consider $g_k = |f|\chi_{B_k(0)}$. Then $g_k \nearrow |f|$. By the MCT,

$$\int |f| = \lim_{k\to\infty} \int_{B_k(0)} |f| \Longrightarrow \lim_{k\to\infty} \int_{B_k^c(0)} |f| = 0.$$

□

Theorem 1.6 (Absolute continuity of the integral)**.** *Suppose f is integrable. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_E |f| \leq \varepsilon \quad \textit{whenever } m(E) \leq \delta.$$

Proof. By replacing f with $|f|$ we assume without loss of generality $f \geq 0$.

Define $F_k = \int\{f \leq k\}$. For any $X \subset \mathbb{R}^n$,

$$\int_X f = \int_{X\cap F_k} f + \int_{X\setminus F_k} f. \tag{1.1}$$

For the second integral on the RHS of (1.1) we have

$$\int_{X\setminus F_k} f \leq \int_{\mathbb{R}^n\setminus F_k} f = \int \left(f - f\chi_{F_k}\right) \to 0 \quad \text{(by the MCT)},$$

since $0 \leq f\chi_{E_k} \nearrow f$ a.e. Hence there is N_ε such that

$$\int_{X\setminus F_{N_\varepsilon}} f < \varepsilon/2.$$

Note that N_ε depends on ε but not on X.

Now N_ε is fixed. For the first integral on the RHS of (1.1), we have

$$\int_{X\cap F_{N_\varepsilon}} f \leq N_\varepsilon m(X) < \varepsilon/2,$$

provided $m(X) < \varepsilon/(2N_\varepsilon)$.

We then complete the proof by taking $\delta = \varepsilon/(2N_\varepsilon)$.

□

Dominated Convergence Theorem.

We are now ready to prove a cornerstone the the theory of Lebesgue integration, the dominated convergence theorem (DCT for short), which is a general statement about the interplay between limits and integrals.

Theorem 1.7 (Dominated convergence theorem). *Suppose $f_k \to f$ a.e. Suppose $|f_k| \le g$ a.e. where g is integrable. Then*

$$\lim_{k\to\infty} \int |f_k - f| = 0$$

and consequently

$$\lim_{k\to\infty} \int f_k = \int f.$$

Proof. Fix $\varepsilon > 0$.

1. First define

$$E_N = \{x : |x| \le N,\ g(x) \le N\}.$$

Then, by MCT,

$$g\chi_{E_N} \nearrow g \text{ a.e. as } N \to \infty.$$

So for each $\varepsilon > 0$, we can find and fix N_ε such that

$$\int_{E^c_{N_\varepsilon}} g < \varepsilon. \tag{1.2}$$

2. Note that

$$f_k\chi_{E_{N_\varepsilon}} \to f\chi_{E_{N_\varepsilon}} \text{ a.e. as } k \to \infty.$$

Since $|f_k\chi_{E_{N_\varepsilon}}| \le N_\varepsilon$, we can apply the BCT to conclude that

$$\int_{E_{N_\varepsilon}} |f_k - f| \to 0,$$

which implies the existence of a $K_\varepsilon > 0$ such that

$$\int_{E_{N_\varepsilon}} |f_k - f| < \varepsilon, \quad \forall\ k \ge K_\varepsilon. \tag{1.3}$$

3. By virtue of (1.2) and (1.3),

$$\begin{aligned}
\int |f_k - f| &\le \int_{E_{N_\varepsilon}} |f_k - f| + \int_{E^c_{N_\varepsilon}} |f_k - f| \\
&\le \int_{E_{N_\varepsilon}} |f_k - f| + 2\int_{E^c_{N_\varepsilon}} g \\
&\le 3\varepsilon, \quad \forall\ k \ge K_\varepsilon.
\end{aligned}$$

This proves the theorem.

□

Remark 1.5. *We say f_k converges to f in L^1 sense if $|f_k - f|_{L^1} := \int |f_k - f| \to 0$.*

(i) *$f_k \to f$ a.e. does not imply $f_k \to f$ in L^1. Consider the example: $f_k = k\chi_{[0,\frac{1}{k}]}$. Then $f_k \to 0$. But $\int |f_k| \not\to 0$.*

(ii) *$f_k \to f$ in L^1 does not imply $f_k \to f$ a.e. Consider the example:*

$$f_1 = \chi_{[0,1]}, f_2 = \chi_{[0,1/2]}, f_3 = \chi_{[1/2,1]}, f_4 = \chi_{[0,1/3]}, f_5 = \chi_{[1/3,2/3]}, f_6 = \chi_{[2/3,1]}, \cdots$$

Then $\int |f_k| \to 0$. But $f_k \to 0$ a.e.

1.5. **Complex-valued functions.**

If f is a complex-valued function on $\mathbb{R}^n$, we write it as

$$f(x) = u(x) + \sqrt{-1}v(x),$$

where u and v are real-valued functions. The function f is measurable if and only if both u and v are measurable.

We then say f is (Lebesgue) integrable if the function $|f(x)| = (u^2(x) + v^2(x))^{\frac{1}{2}}$ is (Lebesgue) integrable.

Since $|u| \le |f|$, $|v| \le |f|$, and $|f| \le |u| + |v|$, we deduce that f is integrable if and only if both u and v are integrable. We define

$$\int f = \int u + \sqrt{-1}\int v,$$

and if E is a measurable set of $\mathbb{R}^n$ and $f\chi_E$ is integrable then define

$$\int_E f = \int f\chi_E.$$

The "triangle inequality" also holds for the integral of complex-valued functions. Namely

$$\Big|\int f\Big| \le \int |f|.$$

Proof: Choose θ so that $e^{\sqrt{-1}\theta}\int f$ is non-negative real. Let $\hat{f} = e^{\sqrt{-1}\theta}f$. Hence $\int \hat{f}$ is non-negative real. Write $\hat{f} = \hat{u} + \sqrt{-1}\hat{v}$, where $\hat{u}$ and $\hat{v}$ are real-valued functions. Note that $\int \hat{f} = \int \hat{u} \ge 0$. It follows that

$$\Big|\int f\Big| = \int \hat{f} = \int \hat{u} \le \int |\hat{u}| \le \int |\hat{f}| \le \int |f|.$$