2019-2020春夏学期《微分几何》第十五周作业

P_{97}

1. **证明**明显当H达到极大值，K达到极小值时当且仅当k_1达到极大值且k_2达到极小值。由Hilbert引理，这样的点必为脐点，从而矛盾。□

2.(1) **证明**直接计算$K=\frac{h_{11}h_{22}}{g_{11}g_{22}}=-\frac{\varphi''(s)}{\varphi(s)}$.
(2) **证明**$\int_0^1 K'\varphi^2 ds=\int_0^1(\varphi'\varphi''-\varphi\varphi''')ds=(\varphi'^2-\varphi\varphi''\varphi)|_0^1=0$
(3) **证明**若曲率单调递增，则$K'\geq 0$,而$\int_0^1 K'\varphi^2 ds=0$.所以$K'=0$,即$K$为常数，$M$为球面，从而矛盾。□

3.**证明**不妨设$k_1\geq k_2$,由于$\frac{dk_2}{dk_1}\leq 0$, 故存在x_0使得，k_1在x_0处取最大值，k_2在x_0处取最小值，由Hilbert引理，x_0必为脐点,从而$k_1(x_0)\geq k_1(x)\geq k_2(x)\geq k_2(x_0)$,即$k_1\equiv k_2$,所以M必为球面。□

3.**证明** 由于

$$\int\int_{M^2}|K|dA=\int\int_{M^2_+}KdA-\int\int_{M^2_-}KdA\geq 2m\pi,$$

另一方面

$$\int\int_{M^2}KdA=\int\int_{M^2_+}KdA+\int\int_{M^2_-}KdA=2\pi\chi(M^2),$$

于是

$$\int\int_{M^2}|K|dA=2\int\int_{M^2_+}KdA-2\pi\chi(M^2)\geq 2m\pi,$$

从而

$$\int\int_{M^2}H^2dA\geq\int\int_{M^2_+}H^2dA\geq\int\int_{M^2_+}KdA\geq\pi(\chi(M^2)+m).$$

□

4.**证明**环面$T^2=\big((r+a\cos u)\cos v,(r+a\cos u)\sin v,b\sin u\big)$. 计算得

$$g_{11}=b^2\cos^2 u+a^2\sin^2 u,\ g_{12}=0,\ g_{22}=(r+a\cos u)^2,$$

$$h_{11}=\frac{ab}{\sqrt{b^2\cos^2 u+a^2\sin^2 u}},\ h_{12}=0,\ h_{22}=\frac{b\cos u(r+a\cos u)}{\sqrt{b^2\cos^2 u+a^2\sin^2 u}},$$

于是

$$H=\frac{1}{2}(\frac{h_{11}}{g_{11}}+\frac{h_{22}}{g_{22}})=\frac{1}{2}\left(\frac{ab}{(b^2\cos^2 u+a^2\sin^2 u)^{\frac{3}{2}}}+\frac{b\cos u}{(r+a\cos u)\sqrt{b^2\cos^2 u+a^2\sin^2 u}}\right).$$

$$
\begin{aligned}
W &= \int\int_{T^2} H^2 dA \\
&= \frac{\pi}{2}\int_0^{2\pi}\left(\frac{ab(r+a\cos u)+b(b^2\cos^2 u+a^2\sin^2 u)\cos u}{(r+a\cos u)(b^2\cos^2 u+a^2\sin^2 u)^{\frac{3}{2}}}\right)^2 (r+a\cos u)\sqrt{b^2\cos^2 u+a^2\sin^2 u}du \\
&= \frac{b^2\pi}{2}\int_0^{2\pi}\frac{\left(a(r+a\cos u)+(b^2\cos^2 u+a^2\sin^2 u)\cos u\right)^2}{(r+a\cos u)(b^2\cos^2 u+a^2\sin^2 u)^{\frac{5}{2}}}du \\
&= \frac{b^2\pi}{2}\int_0^{2\pi}\frac{2a^2r^3+2r(a^2+b^2\cos^2 u+a^2\sin^2 u)^2\cos^2 u-4a^2r(a^2+b^2\cos^2 u+a^2\sin^2 u)\cos^2 u}{(b^2\cos^2 u+a^2\sin^2 u)^{\frac{5}{2}}(r^2-a^2\cos^2 u)}du
\end{aligned}
$$

当且仅当$2r(a^2+b^2\cos^2 u+a^2\sin^2 u)^2\cos^2 u = 4a^2r(a^2+b^2\cos^2 u+a^2\sin^2 u)\cos^2 u$,
即$a=b$, 此时上式可表示成

$$
W=\frac{b^2\pi}{2}\int_0^{2\pi}\frac{2a^2r^3}{a^5(r^2-a^2\cos^2 u)}du=\frac{\pi r^2}{2a}\int_0^{2\pi}\frac{1}{r+a\cos u}du=\pi^2\frac{1}{\frac{a}{c}\sqrt{1-\frac{a^2}{r^2}}}du
$$

则当$a=b=\frac{1}{\sqrt{2}}c$ 时，W达到极小值且$W=2\pi^2$.□

5.**证明** 由Fary-Milnor定理得：

$$
\int_{M^2} H^2 dA=\pi\int_0^l\frac{|k|ds}{2|Ck|\sqrt{1-C^2k^2}}=\pi\int_0^l |k|ds\geq 4\pi^2.
$$

□