Coordinator
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Probability Theory

Exercise Sheet 5

Exercise 5.1

(a) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables converging in probability to a random variable X. Show that $(X_n)_{n\in\mathbb{N}}$ converges to X in distribution.

(b) The converse does not hold in general. Instead, show that if the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution to a *constant* random variable $X = c$, then $(X_n)_{n\in\mathbb{N}}$ converges in probability to c.

Exercise 5.2 Compute the characteristic functions of the following distributions:

(a) The triangular distribution $(1-|x|)1_{[-1,1]}(x)\mathrm{d}x$.

(b) The Cauchy distribution $\frac{\alpha}{\pi}\frac{1}{x^2+\alpha^2}\mathrm{d}x$ with parameter $\alpha > 0$.
Hint: Use a contour integral.

Exercise 5.3 Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of probability measures with

$$\int_{\mathbb{R}} x^k \, P_n(dx) \overset{n\to\infty}{\longrightarrow} \alpha_k \in \mathbb{R} \qquad \text{for all } k \in \mathbb{N}.$$

Assume that there exists *exactly one* probability measure P with the k-th moment $\alpha_k = \int_{\mathbb{R}} x^k \, P(dx)$. Show that $(P_n)_{n\in\mathbb{N}}$ converges weakly towards P.

Hint: First show that $(P_n)_{n\in\mathbb{N}}$ is tight. Then show that each subsequence of $(P_n)_{n\in\mathbb{N}}$ has a sub-subsequence that converges weakly towards P.

Remark: Note that in general, there is no uniqueness of such P. A counter-example would be $Y = e^X$ with X a standard normal random variable.

Submission: until 14:15, Oct 29., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint

Solution 5.1

(a) We denote the distribution functions of X_n and X by F_n and F respectively. We have to show that $\lim_{n\to\infty} F_n(y) = F(y)$ for continuity points y of F.

So, let $y \in \mathbb{R}$ be a continuity point of F and let $\varepsilon > 0$. By the continuity of F in y there is a $\delta > 0$ such that

$$F(y) - \varepsilon \le F(x) \le F(y) + \varepsilon, \quad x \in [y-\delta, y+\delta]. \tag{1}$$

Since the X_n converge to X in probability, there is a $N \in \mathbb{N}$ such that

$$P\left[|X_n - X| > \delta\right] \le \varepsilon, \quad n \ge N. \tag{2}$$

Now, for $n \ge N$,

$$\begin{aligned} F_n(y) = P[X_n \le y] &\le P[\{X \le y+\delta\} \cup \{|X - X_n| > \delta\}] \\ &\le P[X \le y+\delta] + P[|X - X_n| > \delta] \\ &\overset{(2)}{\le} F(y+\delta) + \varepsilon \overset{(1)}{\le} F(y) + 2\varepsilon \end{aligned}$$

and

$$\begin{aligned} F_n(y) = P[X_n \le y] &\ge P[\{X \le y-\delta\} \setminus \{|X - X_n| > \delta\}] \\ &\ge P[X \le y-\delta] - P[|X - X_n| > \delta] \\ &\overset{(2)}{\ge} F(y-\delta) - \varepsilon \overset{(1)}{\ge} F(y) - 2\varepsilon, \end{aligned}$$

so that

$$F(y) - 2\varepsilon \le F_n(y) \le F(y) + 2\varepsilon.$$

Thus,

$$F(y) - 2\varepsilon \le \liminf_{n\to\infty} F_n(y) \le \limsup_{n\to\infty} F_n(y) \le F(y) + 2\varepsilon.$$

But this holds for all $\varepsilon > 0$, so we are done.

(b) We assume that for a $c \in \mathbb{R}$

$$X_n \to c \text{ in distribution.} \tag{3}$$

The constant c has distribution function

$$F(x) = 1_{[c,\infty)},$$

which is continuous except in c. So we know from (3)

$$F_n(z) = P[X_n \le z] \to \begin{cases} 0 & \text{if } z < c, \\ 1 & \text{if } z > c. \end{cases} \tag{4}$$

We want to show that for all $\varepsilon > 0$

$$\lim_{n\to\infty} P[|X_n - c| \ge \varepsilon] = 0.$$

Now,

$$P[|X_n - c| \geq \varepsilon] = P[X_n \leq c - \varepsilon] + P[X_n \geq c + \varepsilon],$$

so that

$$\lim_{n\to\infty} P[|X_n - c| \geq \varepsilon] \leq \lim_{n\to\infty} P[X_n \leq c - \varepsilon] + \lim_{n\to\infty} P[X_n \geq c + \varepsilon].$$

By (4), we have $\lim_{n\to\infty} P[X_n \leq c - \varepsilon] = 0$ for all $\varepsilon > 0$. Furthermore

$$P[X_n \geq c + \varepsilon] = 1 - P[X_n < c + \varepsilon] \leq 1 - P\left[X_n \leq c + \frac{\varepsilon}{2}\right].$$

Thus,

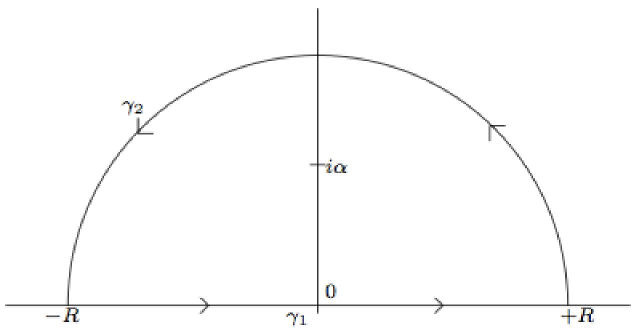
$$\lim_{n\to\infty} P[X_n \geq c + \varepsilon] \leq 1 - \lim_{n\to\infty} P\left[X_n \leq c + \frac{\varepsilon}{2}\right] \overset{(4)}{=} 0.$$

Solution 5.2

(a) We calculate

$$\begin{aligned}
E[e^{itX}] &= \int_{\mathbb{R}} e^{itx}(1-|x|)1_{[-1,1]}(x)\mathrm{d}x = \int_{-1}^{1}(1-|x|)e^{itx}\mathrm{d}x \\
&= \int_{-1}^{1} e^{itx}\mathrm{d}x + \int_{-1}^{0} xe^{itx}\mathrm{d}x - \int_{0}^{1} xe^{itx}\mathrm{d}x \\
&= \frac{1}{it}\left(e^{it} - e^{-it}\right) + \left(\frac{e^{-it}}{it} + \frac{1}{t^2} - \frac{e^{-it}}{t^2}\right) - \left(\frac{e^{it}}{it} + \frac{e^{it}}{t^2} - \frac{1}{t^2}\right) \\
&= \frac{2}{t}\sin(t) - \frac{2}{t}\sin(t) - \frac{2}{t^2}\cos(t) + \frac{2}{t^2} = \frac{2}{t^2}(1-\cos(t)).
\end{aligned}$$

(b) We use the Residue Theorem (for more details on Residue theorem and contour integrals we refer you to the book "Real and complex analysis" by Rudin) for $f(x) = \frac{e^{itx}}{x^2+\alpha^2}$ to calculate $E\left[e^{itX}\right] = \frac{\alpha}{\pi}\int_{-\infty}^{+\infty} \frac{e^{itx}}{x^2+\alpha^2}\mathrm{d}x$. Extend f to $\mathbb{C}$ by $f(z) = \frac{e^{itz}}{z^2+\alpha^2}$. Let us consider the curve $\gamma = \gamma_1 \cup \gamma_2$ as follows:



Thus, we calculate for all $R > \alpha$,

$$\oint_\gamma f(z)\mathrm{d}z = 2\pi i\cdot \underbrace{n(\gamma, i\alpha)}_{=1}\cdot \mathrm{Res}_{i\alpha}(f) = 2\pi i\cdot \lim_{z\to i\alpha}(z-i\alpha)f(z) = 2\pi i\cdot \lim_{z\to i\alpha}\frac{e^{itz}}{z+i\alpha} = \frac{\pi}{\alpha}e^{-\alpha t}.$$

Since we have, for $t \geq 0$,

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| &= \left|\int_0^\pi f\Big(Re^{i\theta}\Big)\, iRe^{i\theta}\mathrm{d}\theta\right| = \left|\int_0^\pi \frac{e^{itRe^{i\theta}}iRe^{i\theta}}{R^2e^{2i\theta}+\alpha^2}\mathrm{d}\theta\right| \\ &\leq \int_0^\pi \left|\frac{e^{itRe^{i\theta}}iRe^{i\theta}}{R^2e^{2i\theta}+\alpha^2}\right|\mathrm{d}\theta = \int_0^\pi \frac{Re^{-Rt\sin(\theta)}}{|R^2e^{2i\theta}+\alpha^2|}\mathrm{d}\theta \\ &\leq \int_0^\pi \frac{R}{|R^2e^{2i\theta}+\alpha^2|}\mathrm{d}\theta \leq \int_0^\pi \frac{R}{R^2-\alpha^2}\mathrm{d}\theta = \frac{\pi R}{R^2-\alpha^2} \underset{R\to\infty}{\to} 0,\end{aligned}$$

we obtain that

$$E\left[e^{itX}\right] = \lim_{R\to\infty}\frac{\alpha}{\pi}\int_{\gamma_1} f(z)\mathrm{d}z = \lim_{R\to\infty}\frac{\alpha}{\pi}\int_{\gamma} f(z)\mathrm{d}z = e^{-\alpha t}.$$

For $t<0$, one can use a similar argument to show $E\left[e^{itX}\right] = e^{\alpha t}$. Thus, we get, for $t \in \mathbb{R}$,

$$E\left[e^{itX}\right] = e^{-\alpha|t|}.$$

Solution 5.3 We first show that the claim in the hint implies the main claim. (cf. (2.3.25) in lecture notes). We assume that $P_n \overset{w}{\not\to} P$. Then there exists a point of continuity of $F(\cdot) := P((-\infty,\cdot])$ and a subsequence $n(k)$ with

$$|F_{n(k)}(y) - F(y)| \geq \epsilon \quad \forall\ k\in\mathbb{N},\ \text{where } F_n(\cdot) := P_n((-\infty,\cdot]).$$

Hence this subsequence has no sub-subsequence converging weakly to P. This contradicts obviously the claim in the hint.
To show the claim in the hint, we need multiple steps. We first show that $(P_n)_{n\in\mathbb{N}}$ is tight. Take $\epsilon>0$ arbitrary. We obtain that

$$\begin{aligned}P_n\Big([-M,M]^c\Big) = \int_{-\infty}^{-M} P_n(dx) + \int_M^\infty P_n(dx) &\leq \int_{-\infty}^{-M}\frac{x^2}{M^2}P_n(dx) + \int_M^\infty \frac{x^2}{M^2}P_n(dx) \\ &\leq \frac{1}{M^2}\int_{-\infty}^\infty x^2 P_n(dx) \overset{n\to\infty}{\longrightarrow} \frac{\alpha_2}{M^2}.\end{aligned}$$

Because $\alpha_2 = \int_{\mathbb{R}} x^2 P(dx) < \infty$, we can choose an $M>0$ such that $\frac{\alpha_2}{M^2} \leq \frac{\epsilon}{2}$. We can therefore find an $N\in\mathbb{N}$ with $P_n\Big([-M,M]^c\Big) \leq \epsilon\ \forall\ n>N$. On the other side it is clear that for each $k\in\{1,\ldots,N\}$, an $M_k>0$ exists with $P_k\Big([-M_k,M_k]^c\Big) \leq \epsilon$. For $M^* := \max\{M,M_1,\ldots,M_N\}$ it follows then

$$\sup_{n\in\mathbb{N}} P_n\Big([-M^*,M^*]^c\Big) \leq \epsilon.$$

The tightness implies by (2.2.26) that each subsequence of $(P_n)_{n\in\mathbb{N}}$ has a sub-subsequence $(P_{n_{k(l)}})_{l\in\mathbb{N}}$ that converges weakly towards a probability measure $\widetilde{P}$.

We now show that $\widetilde{P} = P$. By Proposition 2.7 in the lecture notes, there exist random variables $Y_{n_{k(l)}}$, $l \in \mathbb{N}$ and $\widetilde{Y}$ in a probability space $(\Omega, \mathcal{A}, Q)$ such that

$$Y_{n_{k(l)}} \sim P_{n_{k(l)}},\ l \in \mathbb{N},\ \widetilde{Y} \sim \widetilde{P},\ \text{and } Y_{n_{k(l)}} \xrightarrow{l\to\infty} \widetilde{Y}\ Q\text{-a.s.}$$

We show that the moments of $Y_{n_{k(l)}}$ for $l \to \infty$ converge towards those of $\widetilde{Y}$ respectively. Let $m \in \mathbb{N}$ and $M > 0$ arbitrary. Hence:

$$\begin{aligned}\left|E^Q\left[Y^m_{n_{k(l)}}\right] - E^Q\left[\widetilde{Y}^m|\right]\right| &\le E^Q\left[|Y^m_{n_{k(l)}} - \widetilde{Y}^m|\right] \\ &\le E^Q\left[|Y^m_{n_{k(l)}} - \widetilde{Y}^m|\,1_{\{|Y^m_{n_{k(l)}}|\le M,\,|\widetilde{Y}^m|\le M\}}\right] \\ &+ 3E^Q\left[|Y^m_{n_{k(l)}}|\,1_{\{|Y^m_{n_{k(l)}}|>M\}}\right] + 3E^Q\left[|\widetilde{Y}^m|\,1_{\{|\widetilde{Y}^m|>M\}}\right].\end{aligned} \tag{5}$$

The first term converges towards 0 for $l \to \infty$ thanks to the Lebesgue dominated convergence theorem. For the second term, one has that

$$E^Q\left[|Y^m_{n_{k(l)}}|\,1_{\{Y^m_{n_{k(l)}}>M\}}\right] = E^Q\left[\frac{Y^{2m}_{n_{k(l)}}}{|Y^m_{n_{k(l)}}|}\,1_{\{|Y^m_{n_{k(l)}}|>M\}}\right] \le \frac{1}{M}E^Q\left[Y^{2m}_{n_{k(l)}}\right] \xrightarrow{l\to\infty} \frac{\alpha_{2m}}{M}.$$

Because we can take $M > 0$ arbitrarily big, we obtain that $E^Q\left[|Y^m_{n_{k(l)}}|\,1_{\{|Y^m_{n_{k(l)}}|>M\}}\right] \xrightarrow{l,M\to\infty} 0$. We estimate the third term in (5) with the help of Fatou's lemma:

$$E^Q\left[\widetilde{Y}^{2m}\right] \le \liminf_{l\to\infty} E^Q\left[Y^{2m}_{n_{k(l)}}\right] = \alpha_{2m}.$$

Note that each $Y^{2m}_{n_{k(l)}}$ is non-negative thanks to the even exponent $2m$ so that Fatou's lemma is valid in this case. This implies that $\widetilde{Y}^{2m} \in L^1$ and therefore, by Jensen's inequality, $\widetilde{Y}^m \in L^1$. Consequently, $E^Q\left[|\widetilde{Y}^m|\,1_{\{|\widetilde{Y}^m|>M\}}\right] \xrightarrow{M\to\infty} 0$. By (5) the convergence of the moments follows, i.e. $E^Q\left[Y^m_{n_{k(l)}}\right] \to E^Q\left[\widetilde{Y}^m\right]$ as $l \to \infty$. Since, by assumption, $E^Q\left[Y^m_{n_{k(l)}}\right] \to \alpha_m$ as $l \to \infty$, we have that the mth moment of $\widetilde{P}$ is α_m for all $m \ge 1$. Since P is the unique measure that satisfies the latter equality for all $m \in \mathbb{N}$, we get $P = \widetilde{P}$.

Remark:

Now let us give a more detailed explanation of the remark after Exercise 5.3. For each $a \in [-1, 1]$, we define a measure μ_a on $\mathbb{R}$ via

$$\mu_a(dx) := \frac{1}{\sqrt{2\pi}x}\exp\left(-\frac{(\log x)^2}{2}\right)(1 + a\sin(2\pi\log x))\mathbf{1}_{(0,\infty)}(x)\,dx.$$

First let us verify that μ_a actually defines a probability measure for all $a \in [-1, 1]$. Indeed, using the substitution $y = \log(x)$ we have

$$\begin{aligned}
\int_{\mathbb{R}} \mu_a(dx) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{x} \exp\left(-\frac{(\log x)^2}{2}\right)(1 + a\sin(2\pi \log x))\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{y^2}{2}\right)(1 + a\sin(2\pi y))\, dy \\
&= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{y^2}{2}\right) dy}_{=1} + \underbrace{\frac{a}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{y^2}{2}\right)\sin(2\pi y)\, dy}_{=0,\text{ since } \sin \text{ is an odd function}} = 1.
\end{aligned}$$

(For $a = 0$, the measure μ_0 is called the distribution of *standard log-normal* random variable, that is, μ_0 is the distribution of random variable $Y = \exp(X)$ with X a standard normal random variable.)

We now claim that the kth moment $\int x^k \mu_a(dx)$ does *not* depend on a. Indeed, for any $a \in [-1.1]$, using the substitution $x = e^{k+u}$ we get:

$$\begin{aligned}
\int_{\mathbb{R}} x^k \mu_a(dx) &= \int_0^\infty \frac{x^{k-1}}{\sqrt{2\pi}} \exp\left(-\frac{(\log x)^2}{2}\right)(1 + a\sin(2\pi \log x))\, dx \\
&= \int_{-\infty}^\infty \frac{\exp(k(k+u))}{\sqrt{2\pi}} \exp\left(-\frac{(k+u)^2}{2}\right)(1 + a\sin(2\pi(k+u)))\, du \\
&= \frac{e^{\frac{k^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{-u^2}{2}}(1 + a\sin(2\pi(k+u)))\, du \\
&= \underbrace{\frac{e^{\frac{k^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{-u^2}{2}}\, du}_{=e^{\frac{k^2}{2}}} + \underbrace{\frac{ae^{\frac{k^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{-u^2}{2}} \sin(2\pi(k+u))\, du}_{=0,\text{ since } \sin \text{ is odd}} = e^{\frac{k^2}{2}}.
\end{aligned}$$

Clearly, this counterexample shows that in general a probability measure P can *not* be uniquely determined by its moments. Therefore the hypothesis regarding the uniqueness of P in the statements of Exercise 5.3 can not be removed.