ETH Zurich HS 2019
D-MATH, D-PHYS
Prof. A.-S. Sznitman

Coordinator
Daniel Balint

Probability Theory

Exercise Sheet 2

Exercise 2.1 Take $\Omega = \{a,b,c,d\}$, $\mathcal{A} = \mathcal{P}(\Omega)$ and $\mathcal{C} = \Big\{\{a,b\},\{c,d\},\{a,c\},\{b,d\}\Big\}$. Consider P the equiprobability on Ω and Q the probability measure $\frac{1}{2}(\delta_a + \delta_d)$ (with δ_a the point measure at a, and δ_d the point measure at d).

(a) Show that $\sigma(\mathcal{C}) = \mathcal{A}$, and P and Q agree on $\mathcal{C}$.

(b) Show that $\{A \in \mathcal{A};\ P(A) = Q(A)\}$ is not a σ-algebra.

(c) Is $\mathcal{C}$ a π-system?

Exercise 2.2 Let $\mathcal{F}$ be a σ-algebra and $A_i \in \mathcal{F}$ ($i = 1, 2, \ldots$) the event "at time i the phenomena Φ occurs".
Express with the help of the subsets A_i the following events as subsets $A \in \mathcal{F}$:

(a) "Φ occurs exactly 17 times"

(b) "Φ always occurs again"

(c) "Φ stops occurring at some point"

Describe in words the following event:

(d) $\bigcup_{n\geq 1}\bigcup_{m>n}(A_n \cap A_m)$

Which of these events belong to the asymptotic σ-algebra $\mathcal{A}^* := \bigcap_{n\geq 1}\sigma\Big(\bigcup_{i\geq n}\{A_i\}\Big)$?

Exercise 2.3 In this exercise, we will construct a countably infinite number of independent random variables, without using a product space with an infinite number of factors.

Consider $\Omega = [0,1)$, equipped with the Borel σ-algebra and the Lebesgue measure P restricted to $[0,1)$. We define the random variables

$$Y_n : \Omega \to \mathbb{R}\,, \quad n \geq 1\,,$$

by

$$Y_n(\omega) := \begin{cases} 0 & \text{if } \lfloor 2^n\omega \rfloor \text{ is even,} \\ 1 & \text{if } \lfloor 2^n\omega \rfloor \text{ is odd,} \end{cases}$$

where $\lfloor x \rfloor = \max \{z \in \mathbb{Z} \mid z \leq x\}$ denotes the integer part of x. Show that Y_n , $n \geq 1$, are independent and satisfy $P[Y_n = 0] = P[Y_n = 1] = \frac{1}{2}$.
Hint: To gain insight, consider the meaning of Y_n in terms of the binary expansion of ω. You may use the following observation, without proving it:

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $Y_1, Y_2, \ldots$ be random variables on this space, each taking values only in a countable set (that is, for each i there is a countable set S_i such that $P[Y_i \in S_i] = 1$). Assume that

$$P[Y_1 = z_1, Y_2 = z_2, \ldots, Y_n = z_n] = \prod_{i=1}^{n} P[Y_i = z_i] \text{ for all } z_1, \ldots, z_n \in \mathbb{R} \tag{1}$$

holds for all $n \geq 1$. Then, the infinite sequence of random variables $(Y_i)_{i \geq 1}$ is independent.

Exercise 2.4 (Optional.) A non-empty family $\mathcal{C}$ of subsets of a non-empty set Ω is called a λ-*system*, if

(i) $\Omega \in \mathcal{C}$,

(ii) $A, B \in \mathcal{C}: \; B \subset A \Rightarrow A \setminus B \in \mathcal{C}$,

(iii) $A_n \in \mathcal{C}, \; A_n \subset A_{n+1} \Rightarrow \bigcup_n A_n \in \mathcal{C}$.

Show that the definitions of a Dynkin system and a λ-system are equivalent.

Submission: until 14:15, Oct 8., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint

Solution 2.1

(a) We start with the first claim. Because $\{a\} = \{a,b\} \cap \{a,c\}$, we know that $\{a\} \in \sigma(\mathcal{C})$. By cyclic symmetry we obtain that $\{b\}, \{c\}, \{d\} \in \sigma(\mathcal{C})$ as well. The first claim follows now from Exercise 1.2 (a). For the second claim, we simply observe that $\forall B \in \mathcal{C}$, $P(B) = Q(B) = 1/2$.

(b) Suppose $\{A \in \mathcal{A};\ P(A) = Q(A)\}$ is a σ-algebra. Since this collection contains $\mathcal{C}$, by (a), it would contain also $\mathcal{A}$, by (a). Thus, P and Q would be equal, which is a contradiction.

(c) No. By a direct inspection, we see that

$$\{a,b\} \cap \{a,c\} = \{a\} \notin \mathcal{C}.$$

We can also show this by the following argument: If it were a π-system, by (1.3.11) in the lecture notes, any P and Q agreeing on $\mathcal{C}$ would be equal.

Solution 2.2

(a)

$$\bigcup_{n_1 \geq 1} \bigcup_{n_2 > n_1} \cdots \bigcup_{n_{17} > n_{16}} \left(A_{n_1} \cap A_{n_2} \cap \cdots \cap A_{n_{17}} \cap \left(\bigcap_{k \notin \{n_1, \dots, n_{17}\}} A_k^c \right) \right).$$

This event does not belong in general to the asymptotic σ-algebra since its occurrence is determined by the ocurrence of the event A_1.

(b) $\bigcap_{n\geq 1} \bigcup_{m \geq n} A_m$. This event belongs to the asymptotic σ-algebra, as for each $n_0 \geq 1$

$$\bigcap_{n \geq 1} \bigcup_{m \geq n} A_m = \bigcap_{n \geq n_0} \bigcup_{m \geq n} A_m \in \sigma\left(\bigcup_{i \geq n_0} \{A_i\} \right).$$

(c) $\bigcup_{n\geq 1} \bigcap_{m \geq n} A_m^c$. This event belongs to the asymptotic σ-algebra, as for each $n_0 \geq 1$

$$\bigcup_{n \geq 1} \bigcap_{m \geq n} A_m^c = \bigcup_{n \geq n_0} \bigcap_{m \geq n} A_m^c \in \sigma\left(\bigcup_{i \geq n_0} \{A_i\} \right).$$

(d) Φ occurs at least two times. This event does not belong to the asymptotic σ-algebra.

Solution 2.3 We claim that each $\omega \in \Omega$ can be written as

$$\omega = \sum_{j \geq 1} Y_j(\omega) 2^{-j}. \tag{2}$$

To see this, we write down the binary representation of ω, i.e.

$$\begin{aligned} \omega &= \sum_{j \geq 1} \omega_j 2^{-j}, \quad \omega_j \in \{0,1\}, \\ &= 0.\omega_1 \omega_2 \dots \end{aligned}$$

Technical point: In cases like $\omega = 1/2$, which can be represented as both $0.1000\ldots$ and $0.01111\ldots$, we choose the terminating binary representation, i.e the one which "ends" in an infinite sequence of zeroes, which is the usual convention.

$$\text{For } \omega = 0.\omega_1\omega_2\ldots\omega_j\ldots \Rightarrow 2^j\omega = \omega_1\omega_2\ldots\omega_j.\omega_{j+1}\ldots \in [\omega_1\omega_2\ldots\omega_j,\ \omega_1\omega_2\ldots\omega_j + 1),$$
$$\Rightarrow \lfloor 2^j\omega\rfloor = \omega_1\ldots\omega_j \Rightarrow \begin{cases} \lfloor 2^j\omega\rfloor = \omega_1\ldots\omega_j \text{ is odd}, & \Rightarrow \omega_j = 1, \\ \lfloor 2^j\omega\rfloor = \omega_1\ldots\omega_j \text{ is even}, & \Rightarrow \omega_j = 0. \end{cases}$$

Hence we have $Y_j(\omega) = \omega_j$. From the representation (2), we see that for $n \geq 1$

$$\{Y_n = 0\} = \Omega \cap \bigcup_{j=0}^{2^{n-1}-1} \left[\frac{2j}{2^n}, \frac{2j+1}{2^n}\right).$$

Thus the Y_n are measurable, and $P[Y_n = 0] = 2^{n-1}/(2^n) = 1/2 = P[Y_n = 1]$. To prove independence, we note that for $n \geq 1$ and $z_1, z_2, \ldots, z_n \in \{0,1\}$, we have

$$P\left[\bigcap_{j=1}^{n}\{Y_j = z_j\}\right] = P\left[\left[\sum_{j=1}^{n}\frac{z_j}{2^j}, \sum_{j=1}^{n}\frac{z_j}{2^j} + \frac{1}{2^n}\right)\right] = 2^{-n} = \prod_{j=1}^{n} P[Y_j = z_j].$$

By the observation given in the hint, this implies independence of the infinite sequence $\{Y_n\}$, $n \geq 1$.

Solution 2.4 "$\Leftarrow$" Let $\mathcal{C}$ be a λ-system, then:

- $\Omega \in \mathcal{C}$, because of (i).
- Let A be in $\mathcal{C}$, $A \subset \Omega \overset{\text{(ii)}}{\Rightarrow} A^c = \Omega \setminus A \in \mathcal{C}$.
- Let $A, B \in \mathcal{C}$ disjoint sets. Then we have that $A \subset B^c \in \mathcal{C}$ and due to (ii):
$$B^c \setminus A \in \mathcal{C} \Rightarrow (B^c \setminus A)^c = B \cup A \in \mathcal{C}. \tag{3}$$

Now let $(A_i)_{i\geq 1} \subset \mathcal{C}$ be pairwise disjoint subsets, and set $B_n := \bigcup_{i=1}^n A_i$. By (3), B_n is in $\mathcal{C}$ for every $n \geq 1$, and clearly $B_n \subset B_{n+1}$. Therefore by (iii) we get that,

$$\bigcup_{n\geq 1} B_n = \bigcup_{i\geq 1} A_i \in \mathcal{C}.$$

"$\Rightarrow$" Let $\mathcal{C}$ be a Dynkin-system, then we have:

(i): $\Omega \in \mathcal{C}$.

(ii): Let A, B be in $\mathcal{C}$ with $A \subset B$. Hence $A \cap B^c = \emptyset$, and therefore

$$A \cap B^c = \emptyset \Rightarrow A \cup B^c \in \mathcal{C} \Rightarrow (A \cup B^c)^c = B \setminus A \in \mathcal{C}.$$

(iii): Let $(A_n)_{n\geq 1} \subset \mathcal{C}$ be a sequence satisfying that $A_n \subset A_{n+1}$ for every $n \geq 1$ and set $F_1 = A_1$, $F_n := A_n \setminus A_{n-1} \overset{(ii)}{\in} \mathcal{C}$. Then, $F_n \cap F_k = \emptyset$, for $k \neq n$ and $\bigcup_{n\geq 1} F_n = \bigcup_{k\geq 1} A_k \in \mathcal{C}$.