REAL ANALYSIS

LECTURE NOTES

ABSTRACT. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books for *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

Lecture #3

1. Cantor Set

The Cantor set plays a prominent role set theory and in analysis in general. It and its variants provide a rich source of enlightening examples.

We begin with the closed unit interval $C_0 = [0, 1]$ and let C_1 denote the set obtained from deleting $I_{1,1} = (\frac{1}{3}, \frac{2}{3})$, the middle third open interval, from $[0, 1]$. That is

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1].$$

We also define

$$f(x) = \frac{1}{2}, \quad x \in I_{1,1}.$$

Next we repeat this procedure for each sub-interval of C_1: that is , we delete the middle third open intervals, which are $I_{2,1} = (\frac{1}{9}, \frac{2}{9})$ and $I_{2,2} = (\frac{7}{9}, \frac{8}{9})$. At the second stage we get

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1].$$

We also define at this stage

$$f(x) = \frac{2k-1}{2^2}, \quad x \in I_{2,k}, \ k = 1, 2.$$

We repeat this procedure for each sub-interval of C_2 and so on.

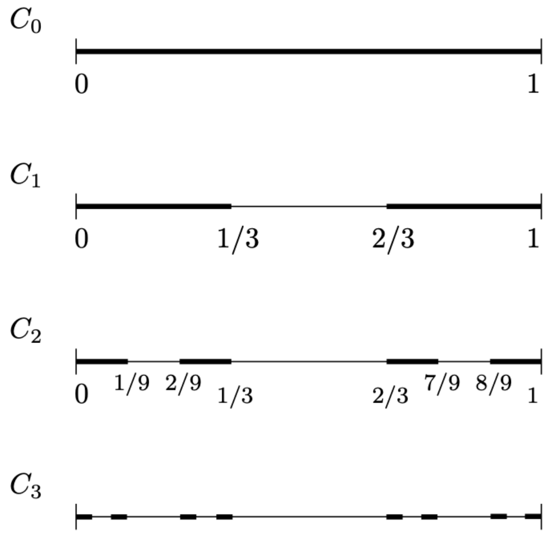

FIGURE 1. Construction of the Cantor set

In general, after n step, we have deleted $2^n - 1$ open intervals $\{I_{m,k} : \ 1 \le m \le n, 1 \le k \le 2^{m-1}\}$, and we obtain

$$C_n = C_{n-1} - \bigcup_{k=1}^{2^{n-1}} I_{n,k} = [0,1] - \bigcup_{m=1}^{n} \bigcup_{k=1}^{2^{m-1}} I_{m,k}.$$

At the n-th stage, we define

$$f(x) = \frac{2k-1}{2^n}, \ x \in I_{n,k}, \ 1 \le k \le 2^{n-1}.$$

One can check that f is monotone.

This procedure yields a sequence C_n, $n = 0, 1, 2, \ldots$, of compact sets with

$$\cdots \subset C_{n+1} \subset C_n \subset \cdots \subset C_2 \subset C_1 \subset C_0 = [0,1].$$

The Cantor set is by definition the intersection of all C_n's:

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n = [0,1] - G, \ \text{where } G = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k}. \tag{1.1}$$

Note that the sum of length of all intervals in G is

$$\sum_{n \ge 1} \sum_{1 \le k \le 2^{n-1}} |I_{n,k}| = \sum_{n \ge 1} \frac{1}{3^n} \times 2^{n-1} = 1. \tag{1.2}$$

Proposition 1.1. *The Cantor set $\mathcal{C}$ satisfies the following properties:*

(i) $\mathcal{C}$ *is perfect (i.e., it is closed, and has no isolated points);*
(ii) $[0,1]-\mathcal{C}$ *is dense in* $[0,1]$*;*
(iii) $\mathcal{C}$ *has no interior point;*
(iv) $\mathcal{C}$ *is totally disconnected (* $\forall x, y \in \mathcal{C}$*, there is* $z \notin \mathcal{C}$ *that lies between* x *and* y*);*
(v) $\mathcal{C}$ *has the cardinality of the continuum.*

Proof. Statement (i) follows by (1.1) immediately. Statement (ii) is equivalent to, for any $(a,b) \subset [0,1]$, $(a,b)\cap G \neq \emptyset$. It is not hard to be checked by the construction as well. Statement (iii) follows from (ii) immediately. Statement (iv) can be verified directly.

Next we show (iv). For any $x \in (0,1)$, there is a unique $\mathbf{a} = (a_1, a_2, \cdots, a_n, \cdots) \in \mathcal{A}_3$ such that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}. \tag{1.3}$$

For any $x \in I_{1,1}$, we have $a_1 = 1$. For any $x \in I_{2,1} \cup I_{2,1}$, we have $a_2 = 1$. In general, for any $x \in I_{n,k}$, we have $a_n = 1$. Hence if $x \in G$, then there is an $a_k = 1$. Consequently, if x, written as (1.3), with all a_i being 0 or 2, then $x \in \mathcal{C}$. This yields that

$$\mathrm{Card}(\mathcal{A}_2) \leq \mathrm{Card}(\mathcal{C}).$$

It is direct to see $\mathrm{Card}(\mathcal{C}) \leq \mathrm{Card}([0,1])$. Since $\mathrm{Card}(\mathcal{A}_2) = \mathrm{Card}([0,1])$, we are done

□

Remark 1.1. *We assert any nonempty perfect set* $E \subset \mathbb{R}$ *has the cardinality of the continuum. Hence (v) in Proposition 1.1 follows by (i) of the proposition.*

The proof of the assertion is essentially the same as that of the above proposition.

We first take a closed interval I *such that* $E \cap I$ *is an infinite set. Since* E *has no isolated points, such* I *exists. Denote by* $|I|$ *the length of* I*. We then take two disjoint closed subintervals* I_1 *and* I_2*, with* $|I_i| < \frac{1}{2}|I|$*, such that* $E \cap I_i$*,* $i = 1, 2$*, are both infinite. The existence of such intervals is again because* E *is perfect. Next take disjoint closed subintervals* $I_{1,1}$ *and* $I_{1,2}$ *of* I_1 *such that* $|I_{1,j}| < \frac{1}{2}|I_1|$ *and* $I_{1,j} \cap E$ *are infinite for* $j = 1, 2$*. Do the same for* I_2 *and obtain* $I_{2,1}$*,* $I_{2,2}$*. After* n *steps, we obtain a collection of disjoint closed intervals* $\mathcal{I}_n = \{I_{i_1,i_2,\cdots,i_n} : \ i_k = 1, 2, \ 1 \leq k \leq n\}$*, where* $I_{i_1,i_2,\cdots,i_n} \subset I_{i_1,i_2,\cdots,i_{n-1}}$*,* $|I_{i_1,i_2,\cdots,i_n}| < \frac{1}{2}|I_{i_1,i_2,\cdots,i_{n-1}}|$*, and* $I' \cap E$ *is infinite for any* $I' \in \mathcal{I}_n$*.*

Let us construct a map $f : \mathcal{A}_2 \to E$*. For any* $\mathbf{i} \in \mathcal{A}_2$*, namely* $\mathbf{i} = (i_1, i_2, \cdots, i_n, \cdots)$ *such that* i_k *take values* 1 *or* 2*, we define* $f(\mathbf{i}) = \bigcap_{n=1}^{\infty} I_{i_1,\cdots,i_n}$*. Since* E *is closed and the*

$|I_{i_1,\cdots,i_n}| \to 0$ *as* $n \to \infty$, f *is well-defined. This shows that*

$$Card(E) \geq Card(\mathcal{A}_2) = Card(\mathbb{R}).$$

On the other hand, $Card(E) \leq Card(\mathbb{R})$. *We hence finish the proof.*

We turn our attention next to the question of determing the "size" of $\mathcal{C}$. This is a delicate problem, one that may be approached from different angles depending on the notion of size we adopt. For instance, in terms of cardinality the Cantor set is rather large, as large as $[0,1]$, see (v) in Proposition 1.1. However, from the point of view of "length" the size of $\mathcal{C}$ is small. Roughly speaking, the Cantor set has length zero, by virtue of (1.2). We shall define a notion of measure and make this precise in the following section.

Let $G = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k}$ as above. Let g be a function on $[0,1]$ such that

$$g(1) = 1,$$
$$g(x) = \inf\{f(y) : \ y > x, y \in G\}, \ 0 \leq x < 1.$$

Clearly g is non-decreasing, $0 \leq g \leq 1$. As $g(G) = f(G)$ is dense in $[0,1]$, we conclude that $g([0,1])$ is dense in $[0,1]$. It follows that g is continuous. Such g is called the Cantor function on $[0,1]$.

2. Baire Category Theorem

Let X be a closed set of $\mathbb{R}^n$. The Baire category theorem says that a countable intersection of dense open subsets of X is not empty, and in fact is dense.

Definition 2.1. *A set* $E \subset X$ *is dense in* X *if* $E \cap U \neq \emptyset$ *for any open set* $U \subset X$. *A set* $E \subset X$ *is nowhere dense in* X *if* $\overline{E}$ *has empty interior.*

Remark 2.1. *Let us look at some examples.*

(i) *The set of rational numbers* $\mathbb{Q}$ *is dense in* $\mathbb{R}$.
(ii) *The Cantor set is nowhere dense in* $[0,1]$. *See (i) & (iii) in Proposition 1.1.*

We have some observations:

(iii) E *is dense in* X *if and only if* $\overline{E} = X$.
(iv) E *is nowhere dense* $\Longrightarrow$ $\overline{E}$ *is nowhere dense* $\Longrightarrow$ $\overline{E}^c$ *is open and dense.*

Definition 2.2. *A set $E \subset X$ is of the first category if it is a countable union of nowhere dense sets of X. A set $E \subset X$ is of the second category if it is not of the first category.*

Remark 2.2. *Look at some examples.*

- *The Cantor set is of the first category.*
- *$\mathbb{Q} \subset \mathbb{R}$ is of the first category (this is obvious).*
- *Any countable set in $\mathbb{R}^n$ is of the first category (this is because $E = \bigcup_{x \in E}\{x\}$).*
- *$\mathbb{I}$, the set of irrational numbers, is of the second category. Because if it were, then so would $\mathbb{R}$ and thus $\mathbb{R} = \bigcup_{i=1}^{\infty} A_i$, where A_i's are closed and nowhere dense (the closedness is due to (iv) in Remark 2.1). Consequently $\emptyset = \mathbb{R}^c = \bigcap_{i=1}^{\infty} A_i^c$ with A_i^c being open and dense. This contradicts with the Baire category theorem.*

Theorem 2.1 (The Baire Category Theorem)**.** *Let X be a closed set of $\mathbb{R}^n$. Then*

(i) *If $\{U_n\}_{n\geq 1}$ is a sequence of dense open sets in X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X, and in particular is non-empty.*
(ii) *X is not a countable union of nowhere dense sets.*

Remark 2.3. *The Baire Category Theorem holds for any complete metric space (X, d).*

Proof. Suppose U is an arbitrary open subset of X and $B_{r_0}(x_0) \subset U$ [1].

By the denseness and openness, U_1 meets $B_{r_0}(x_0)$ and so there is $B_{r_1}(x_1) \subset U_1 \cap B_{r_0}(x_0)$. Decreasing r_1 a little bit if needed, we assume $\overline{B}_{r_1}(x_1) \subset U_1 \cap B_{r_0}(x_0)$.

Similarly U_2 meets $B_{r_1}(x_1)$ and so there exists $\overline{B}_{r_2}(x_2) \subset U_2 \cap B_{r_1}(x_1)$; and also U_3 meets $B_{r_2}(x_2)$ and so there exists $\overline{B}_{r_2}(x_2) \subset U_2 \cap B_{r_1}(x_1)$. Repeat this procedure, we obtain a sequence of balls $B_{r_n}(x_n)$ such that

$$\overline{B}_{r_n}(x_n) \subset U_n \cap B_{r_{n-1}}(x_{n-1}).$$

By decreasing the r_n if necessary (e.g. $r_n < \frac{1}{n}$) we can assume $r_n \to 0$.

If $m \geq n$ then $x_m \in B_{r_n}(x_n)$. It follows the sequence x_n is a Cauchy sequence (in fact $d(x_n, x_m) \leq 2r_N < 2/N$ for all $m, n \geq N$), and so by the completeness of X (this is the only place where we use the closedness of X) we deduce $x_n \to x$ for some $x \in X$.

As for all $m \geq n$ by construction $x_m \in \overline{B}_{r_n}(x_n) \subset U_n$ Fixing n and sending $m \to \infty$, one sees $x \in B_{r_n}(x_n) \subset U_n$, for each n, and so $x \in \bigcap_{n\geq 1} U_n$. Particularly $x \in B_{r_0}(x_0) \subset$

[1] Here $B_r(x) = \{y \in X : d(y, x) < r\}$ is an open ball of X, centred at x of radius r.

U. Therefore $(\bigcap_{n\geq 1} U_n) \cap U \neq \emptyset$. Since U is arbitrary, $\bigcap_{n\geq 1} U_n$ is dense. This proves (i).

For (ii), suppose $X = \bigcup_{n\geq 1} A_n$, where A_n are nowhere dense sets of X. By Remark 2.1, we assume directly A_n are closed. Then A_n^c are open and dense in X^c. By the Baire category theorem, $X^c = \bigcap A_n^c \neq \emptyset$, where X^c is the complement of X in itself and thus must be empty, arriving a contradiction.

$\square$

We point out that the existence of the limit x in the proof of Theorem 2.1 is also a consequence of the (high dimensional) Nested Intervals Theorem.

Definition 2.3. *Countable intersection of open sets are called G_δ sets. Countable union of closed sets are called F_σ sets.*

Theorem 2.2. *Let f be a function on $\mathbb{R}^n$. Denote $E = \{x \in \mathbb{R}^n : \ f \text{ is continuous at } \ x\}$. Then E is a G_δ set and $E^c = \{x \in \mathbb{R}^n : \ f \text{ is discontinuous at } \ x\}$ is a F_σ set.*

Proof. For $x \in E$ and $k \geq 1$, let $r_{x,k} > 0$ be such that $|f(y) - f(x)| < \frac{1}{k}$ for all $y \in B_{r_{x,k}}(x)$, the open ball at x of radius $r_{x,k}$. The existence of such positive $r_{x,k}$ is due to the continuity. We claim $E = \bigcap_{k=1}^{\infty} \bigcup_{x\in E} B_{\frac{1}{2}r_{x,k}}(x)$. This implies that E is G_δ.

For the claim, we only show $\bigcap_{k=1}^{\infty} \bigcup_{x\in E} B_{r_{x,k}}(x) \subset E$, as the opposite inclusion is obvious. Given $z \in \bigcap_{k=1}^{\infty} \bigcup_{x\in E} B_{\frac{1}{2}r_{x,k}}(x)$, then for any k, there is $x = x(k) \in E$ such that $z \in B_{\frac{1}{2}r_{x,k}}(x)$. For any $y \in B_{\frac{1}{2}r_{x,k}}(z)$,

$$d(y, x) \leq d(y, z) + d(z, x) < r_{x,k},$$

thus $y \in B_{r_{x,k}}(x)$. Therefore

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(z) - f(x)| < \frac{2}{k}, \ \forall \ y \in B_{\frac{1}{2}r_{x,k}}(z).$$

Since k is arbitrary, we see that f is continuous at z, and so $z \in E$. The claim is proved.

By the De Morgan law, $E^c = \bigcup_{k=1}^{\infty} \bigcap_{x\in E} B^c_{\frac{1}{2}r_{x,k}}(x)$ is F_σ set.

$\square$

Corollary 2.1. *Let E be a dense set in $\mathbb{R}^n$. If E is countable, then there is no function that is continuous on E and discontinuous on E^c.*

Proof. Let f be a function on $\mathbb{R}^n$. Suppose f is continuous on E. By Theorem 2.2, $E^c = \bigcup_{k=1}^{\infty} F_k$ for some closed sets F_k. We claim that all F_k are nowhere dense. Otherwise, by the closedness, there is a k such that F_k has nonempty interior. Hence $B_r \subset E^c$. This implies $E \cap B_r = \emptyset$, contradicting with the denseness of E.

It follows that E^c is the first category set, so is $\mathbb{R}^n = E \cap E^c$, which contradicts with (ii) in Theorem 2.1.

$\square$

Remark 2.4. *Theorem 2.2 and Corollary 2.1 also hold if* $(\mathbb{R}^n, d_{euclid})$ *is replaced by any complete metric space* (X, d).

Corollary 2.1 says in particular if f is continuous at a countable and dense set $E \subset \mathbb{R}^n$, then it is continuous somewhere in E^c. However there are functions that are continuous at all irrationals, but discontinuous at all rationals. The best-known examples include

$$f(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = p/q \in \mathbb{Q},\ q > 0, \text{ and } p, q \text{ have no common divisors} \\ 0, & \text{if } x \in \mathbb{I}, \end{cases}$$

and the function

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{I}. \end{cases}$$