REAL ANALYSIS

LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

1. Some other properties of measurable sets and functions

Proposition 1.1. *Let E_i be measurable set of $\mathbb{R}^{n_i}$, $i = 1, 2$. Then $E = E_1 \times E_2$ are measurable set of $\mathbb{R}^n$ with $n = n_1 + n_2$. Moreover,*

$$m(E) = m(E_1)m(E_2).[1] \tag{1.1}$$

Proof. Since E_1 and E_2 are measurable, we have

$$E_i = \cup_{j\geq 1} E_{i,j}, \quad i = 1, 2,$$

where $E_{i,j}$ is either closed or measure zero. By a cut-off if necessary, we can further require that $m(E_{i,j}) < \infty$ for all i, j. It can be verified that

$$E = E_1 \times E_2 = (\cup_{j\geq 1} E_{1,j}) \times (\cup_{k\geq 1} E_{2,k}) = \cup_{k,j=1}^{\infty} E_{1,j} \times E_{2,k}.$$

Then it suffices to show that

(1) if H and F are both closed, then $H \times F$ is measurable;
(2) if $m(H) = 0$ and $m(F) < \infty$, then $m(H \times F) = 0$.

Note that (1) follows by the closedness of $H \times F$. For (2), given $\varepsilon > 0$, we choose closed cubes $\{Q_j^1\}$ and $\{Q_k^2\}$ such that

$$H \subset \cup_{j\geq 1} Q_j^1,\ F \subset \cup_{j\geq 1} Q_j^2,\ \sum_{j=1}^{\infty} |Q_j^1| \leq \varepsilon \text{ and } \sum_{j=1}^{\infty} |Q_j^2| \leq m(F) + \varepsilon.$$

[1] We understand this equality by the convention $0 \cdot \infty = 0$.

Consequently $H \times F \subset \cup_{j,k\geq 1} Q_j^1 \times Q_k^2$ and

$$\sum_{j,k=1}^{\infty} |Q_j^1 \times Q_k^2| = \sum_{j,k=1}^{\infty} |Q_j^1||Q_k^2| \leq \varepsilon(m(F)+\varepsilon),$$

where we used the fact $Q_j^1 \times Q_k^2$ are rectangles in $\mathbb{R}^n$ and so $|Q_j^1 \times Q_k^2| = |Q_j^1||Q_k^2|$. Therefore $m_*(E) = 0$. Hence (2) follows.

We next show (1.1). Observe the following

(a) If $m(H) = 0$, then $m(H \times F) = 0$.
Proof: This follows by (1) in previous part and the cut-off argument. Namely using $F = \cup_{j\geq 1} F_j$ *with* $F_j = F \cap \{j-1 \leq |x| < j\}$ *we see that* $m(H \times F) = \sum_{j\geq 1} m(H \times F_j) = 0$.

(b) If $\mathcal{O}_i$ are open sets of $\mathbb{R}^{n_i}$, $i = 1, 2$, then $m(\mathcal{O}_1 \times \mathcal{O}_2) = m(\mathcal{O}_1)m(\mathcal{O}_2)$.
Proof: There are almost disjoint closed cubes $\{Q_j^1\}$ *and* $\{Q_j^2\}$ *such that* $\mathcal{O}_i = \cup_{j=1}^{\infty} Q_j^i$, $i = 1, 2$. *Then*

$$\mathcal{O}_1 \times \mathcal{O}_2 = \cup_{j,k\geq 1} Q_j^1 \times Q_k^2.$$

Consequently

$$m(\mathcal{O}_1 \times \mathcal{O}_2) = \sum_{j,k=1}^{\infty} |Q_j^1 \times Q_k^2| = \sum_{j,k=1}^{\infty} |Q_j^1||Q_k^2| = m(\mathcal{O}_1)m(\mathcal{O}_2).$$

(c) If G_i are G_δ sets in $\mathbb{R}^{n_i}$ with finite measure, $i = 1, 2$, then

$$m(G_1 \times G_2) = m(G_1)m(G_2).$$

Proof: Suppose $G_i = \cap_{j=1}^{\infty} \mathcal{O}_j^i$. *Then* $(\cap_{j=1}^{N} \mathcal{O}_j^1) \times (\cap_{k=1}^{N} \mathcal{O}_k^2) \searrow G_1 \times G_2$. *Since* G_1 *and* G_2 *are of finite measure, we deduce*

$$\begin{aligned} m(G_1 \times G_2) &= \lim_{N\to\infty} m\big((\cap_{j=1}^{N} \mathcal{O}_j^1) \times (\cap_{k=1}^{N} \mathcal{O}_k^2)\big) \\ &= \lim_{N\to\infty} m\big((\cap_{j=1}^{N} \mathcal{O}_j^1)\big) m\big((\cap_{k=1}^{N} \mathcal{O}_k^2)\big) \quad \textit{[by Observation (b)]} \\ &= m(G_1)m(G_2). \end{aligned}$$

Suppose E_i are of finite measure, $i = 1, 2$. Recall that $E_i = G_i \setminus Z_i$ with G_δ set G_i and measure zero set Z_i. Note that

$$(G_1 \times G_2) \setminus (E_1 \times E_2) = \big((G_1 \setminus E_1) \times G_2\big) \cup \big(G_1 \times (G_2 \setminus E_2)\big) = \big(Z_1 \times G_2\big) \cup \big(G_1 \times Z_2\big).$$

Hence, by virtue of Observation (a),

$$m((G_1 \times G_2) \setminus (E_1 \times E_2)) \le m\big(Z_1 \times G_2\big) + m\big(G_1 \times Z_2\big) = 0.$$

Consequently, by Observation (c), we infer that

$$m(E_1 \times E_2) = m(G_1 \times G_2) = m(G_1)m(G_2) = m(E_1)m(E_2).$$

In general, we consider $E_i = \cup_{k=1}^{\infty} E_{i,k}$ with $E_{i,k} = E_i \cap \{|x| \le k\}$, $i = 1, 2$. Since $E_{i,k} \nearrow E_i$ and $E_{1,k} \times E_{2,k} \nearrow E_1 \times E_2$, one concludes

$$m(E_1 \times E_2) = \lim_{k\to\infty} m(E_{1,k} \times E_{2,k}) = \lim_{k\to\infty} m(E_{1,k})m(E_{2,k}) = m(E_1)m(E_2).$$

□

Proposition 1.2. *Let E be a measurable set of $\mathbb{R}^n$. Then*

$$E^- = \{(x, y) \in \mathbb{R}^{2n} : x - y \in E\}$$

is a measurable set of $\mathbb{R}^{2n}$.

Proof. We have the following observation:

(i) if H is open, then H^- are open;

(ii) if H is closed, then H^- are closed;

(iii) if $R = \prod_{i=1}^{n} \langle a_i, b_i \rangle$, then $R^- = \{(x_1, \cdots, x_n, y_1, \cdots, y_n) \in \mathbb{R}^{2n} : x_i - y_i \in \langle a_i, b_i \rangle\}$. Setting $S_i = \{(x, y) \in \mathbb{R}^2 : x - y \in \langle a_i, b_i \rangle\}$, then

$$R^- \cap [-\ell, \ell]^{2n} = \prod_{i=1}^{n} (S_i \cap [-\ell, \ell]^2).$$

By Proposition 1.1, we obtain

$$m(R^- \cap [-\ell, \ell]^{2n}) = \prod_{i=1}^{n} m(S_i \cap [-\ell, \ell]^2) \le (2\ell)^n \prod_{i=1}^{n} (b_i - a_i) = (2\ell)^n |R|.$$

Recall that there are open sets $\mathcal{O}_j$ and closed sets F_j such that

$$F_j \subset E \subset \mathcal{O}_j, \quad m(E \setminus F_j) \le \frac{1}{j} \text{ and } m(\mathcal{O}_j \setminus E) \le \frac{1}{j}.$$

In view of (i) and (ii), F_j^- and $\mathcal{O}_j^-$ are measurable.

Note that $F_j^- \subset E^- \subset \mathcal{O}_j^-$. Let $G = \cap_{j\ge 1} \mathcal{O}_j^-$. Then $G \setminus E^- \subset \mathcal{O}_j^- \setminus F_j^-$, $\forall j$. We claim, for any $\ell > 0$,

$$m((\mathcal{O}_j^- \setminus F_j^-) \cap [-\ell, \ell]^{2n}) \to 0 \ \text{ as } j \to \infty. \tag{1.2}$$

Once this is done, we conclude that $G \setminus E^-$ is measure zero, and so E^- is measurable.

We shall prove a general result:

$$\text{if } H \subset \mathbb{R}^n \text{ with } m_*(H) < \infty, \text{ then } m_*(H^- \cap [-\ell, \ell]^{2n}) \leq (2\ell)^n m_*(H) \text{ for each } \ell > 0.$$

Then (1.2) is a consequence of the above statement as

$$\mathcal{O}_j^- \setminus F_j^- = (\mathcal{O}_j \setminus F_j)^- \implies m((\mathcal{O}_j^- \setminus F_j^-) \cap [\ell, \ell]^{2n}) \leq (2\ell)^n m(\mathcal{O}_j \setminus F_j) \to 0, \ \forall\, \ell > 0.$$

Take open sets $U_k \supset H$ so that $m_*(H) = m(G)$ where $G = \cap_{k \geq 1} U_k$. Let $G_j = \cap_{k=1}^j U_k$. It is open. Hence $G_j = \cup_{k \geq 1} Q_{j,k}$ for a sequence of almost disjoint closed cubes $\{Q_{j,k}\}_{k \geq 1}$. By Observation (iii),

$$m(Q_{j,k}^- \cap [-\ell, \ell]^{2n}) \leq (2\ell)^n |Q_{j,k}| \implies m(G_j^- \cap [-\ell, \ell]^{2n}) \leq (2\ell)^n m(G_j).$$

Since $G_j^- \searrow G^-$, we see that by sending $j \to \infty$,

$$\begin{aligned} m_*(H^- \cap [-\ell, \ell]^{2n}) &\leq m(G^- \cap [-\ell, \ell]^{2n}) \\ &= \lim_{j \to \infty} m(G_j^- \cap [-\ell, \ell]^{2n}) \\ &\leq (2\ell)^n \lim_{j \to \infty} m(G_j) \\ &= (2\ell)^n m(G) \\ &= (2\ell)^n m_*(H). \end{aligned}$$

□

Proposition 1.3. *Let E_i be measurable set of $\mathbb{R}^{n_i}$ and f_i be (a.e. finite-valued) measurable function on E_i, $i = 1, 2$. Then*

(i) *$E = E_1 \times E_2$ is a measurable set of $\mathbb{R}^n$ with $n = n_1 + n_2$;*
(ii) *$F(x_1, x_2) = f_1(x_1) f_2(x_2)$ is a measurable function on E.*

Proof. For part (i), one sees Proposition 1.1.

We show part (ii). Select two sequences of simple functions $\{f_{1,k}\}_{k \geq 1}$ and $\{f_{2,j}\}_{j \geq 1}$ such that, for $i = 1, 2$,

$$f_i(x) = \lim_{j \to \infty} f_{i,j}(x) \ \text{ for all } x \in E_i. \tag{1.3}$$

By part (i), $F_j(x_1, x_2) := f_{1,j}(x_1) f_{2,j}(x_2)$ are simple functions. It follows by (1.3)

$$F_j(x_1, x_2) \to F(x_1, x_2) \text{ for all } (x_1, x_2) \in E.$$

Consequently $F(x_1, x_2)$ is measurable.

$\square$

Proposition 1.4. *If f is a finite-valued measurable function on E. Then $f(x-y)$ is a measurable function on $E^- = \{(x,y) \in \mathbb{R}^{2n} : x - y \in E\}$.*

Proof. Exercise.

$\square$

Convergence in measure is preserved under some operations.

Proposition 1.5. *Let f_k, g_k, f, g be measurable functions on $E \subset \mathbb{R}^n$. Suppose that $f_k \to f$ in measure and $g_k \to g$ in measure. Then*

(i) $f_k \pm g_k \to f \pm g$ *in measure;*
(ii) $|f_k| \to |f|$ *in measure;*
(iii) $\min\{f_k, g_k\} \to \min\{f, g\}$ *in measure, and* $\max\{f_k, g_k\} \to \max\{f, g\}$ *in measure;*
(iv) *when $m(E) < \infty$, $f_k g_k \to fg$ in measure;*

Proof. Exercise.

$\square$