ETH Zurich HS 2019
D-MATH, D-PHYS
Prof. A.-S. Sznitman

Coordinator
Daniel Balint

Probability Theory

Exercise Sheet 6

Exercise 6.1 Let $(X_i)_{i\geq 1}$ be i.i.d. with symmetric stable distribution of parameter $\alpha \in (0,2)$, see lecture notes p. 63.

(a) Find the distribution of $n^{-1/\alpha}(X_1 + \cdots + X_n)$.

(b) Does $\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$ converge in distribution?

Exercise 6.2 Let $\{X_j\}_{j=1,\dots,n}$, $n \geq 1$ be random variables and let us denote by ϕ_j the characteristic function of X_j. Prove that $\{X_j\}_{j=1,\dots,n}$ are independent if and only if for all $\xi_1, \dots, \xi_n \in \mathbb{R}$.

$$E\left[\exp\left\{i\sum_{j=1}^{n} \xi_j X_j\right\}\right] = \prod_{j=1}^{n} \phi_j(\xi_j).$$

Hint: For $d \geq 1$, and ν a probability measure on $\mathbb{R}^d$, one can define the characteristic function $\phi_\nu : \mathbb{R}^d \to \mathbb{R}$ of ν, as

$$\phi_\nu(\lambda) = \int_{\mathbb{R}^d} \exp(i\lambda \cdot x)\nu(dx),$$

where $\lambda \cdot x$ denotes the scalar product in $\mathbb{R}^d$, and then use (without proof) the following uniqueness property of characteristic functions of $\mathbb{R}^d$-valued random variables: if ν and μ are probability measures on $\mathbb{R}^d$ with the same characteristic function, then $\nu = \mu$, (cf. (2.3.13) the uniqueness property for one-dimensional random variables in the lecture notes).

Exercise 6.3 Let $X_1, X_2, \dots$ be independent random variables for which there exists a constant $M > 0$, such that $|X_n| \leq M$, P-a.s. for $n = 1, 2, \dots$. We write $S_n = X_1 + \ldots + X_n$. Show that, if $\sum \mathrm{Var}(X_n) = \infty$, then there exist constants a_n, b_n such that $(S_n - b_n)/a_n$ converges in distribution towards a standard normal random variable.

Exercise 6.4 (Optional.) Show that when Y_k, $k \geq 1$ are independent uniformly bounded random variables such that $\sum_k Y_k$ converges P-a.s., then $\sum_k \mathrm{Var}(Y_k) < \infty$.
Hint: consider independent copies $\tilde{Y}_k$, $k \geq 1$ of the Y_k, $k \geq 1$ and use Exercise 6.3 with $X_k = Y_k - \tilde{Y}_k$, $k \geq 1$.

Submission: until 14:15, Nov 5., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint