Probability Theory

Exercise Sheet 8

Exercise 8.1 Let X be a random variable in $L^2(\Omega, \mathcal{A}, P)$ and $\mathcal{F} \subseteq \mathcal{A}$. The *conditional variance* of X given $\mathcal{F}$ is defined as $\mathrm{Var}[X|\mathcal{F}] := E[(X - E[X|\mathcal{F}])^2|\mathcal{F}]$. Prove that

(a) $\mathrm{Var}[X|\mathcal{F}] = E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2$;

(b) $\mathrm{Var}(X) = E[\mathrm{Var}[X|\mathcal{F}]] + \mathrm{Var}[E[X|\mathcal{F}]]$.

(c) Compute $\mathrm{Var}[X|\mathcal{F}]$, where $\mathcal{F} = \sigma(A_1, A_2)$ where $\{A_1, A_2\}$ is a partition of Ω and $P(A_i) > 0$ for $i = 1, 2$.

Exercise 8.2 Let $S, T : \Omega \to \mathbb{N} \cup \{\infty\}$ be $\mathcal{F}_n$-stopping times. Prove or provide a counter example disproving the following statements:

(a) $S - 1$ is a stopping time.

(b) $S + 1$ is a stopping time.

(c) $S \wedge T$ is a stopping time.

(d) $S \vee T$ is a stopping time.

(e) $S + T$ is a stopping time.

Exercise 8.3 (Polya's Urn)
An urn initially contains s black and w white balls. We consider the following process. At each step a random ball is drawn from the urn, and is replaced by t balls of the same colour, for some fixed $t \geq 1$. We define the random variable Y_n as the proportion of black balls in the urn after the n-th iteration. Show that $E[Y_{n+1}|\sigma(Y_1, Y_2, \dots Y_n)] = Y_n$, for all $n \in \mathbb{N}$, that is, $\{Y_n\}_{n\in\mathbb{N}}$ is a martingale.

Submission: until 14:15, Nov 19., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint

Solution 8.1

(a) Expanding the square in the definition, we obtain

$$\begin{aligned}\mathrm{Var}[X|\mathcal{F}] &:= E[(X - E[X|\mathcal{F}])^2|\mathcal{F}] \\ &= E[X^2 - 2XE[X|\mathcal{F}] + E[X|\mathcal{F}]^2|\mathcal{F}] \\ &= E[X^2|\mathcal{F}] - 2E[X|\mathcal{F}]E[X|\mathcal{F}] + E[X|\mathcal{F}]^2 \\ &= E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2.\end{aligned}$$

(b) Using the tower property and (a),

$$\begin{aligned}E[\mathrm{Var}[X|\mathcal{F}]] + \mathrm{Var}[E[X|\mathcal{F}]] &= E[E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2] + \mathrm{Var}[E[X|\mathcal{F}]] \\ &= E[X^2] - E[E[X|\mathcal{F}]^2] + E[E[X|\mathcal{F}]^2] - E[E[X|\mathcal{F}]]^2 \\ &= E[X^2] - E[X]^2 = \mathrm{Var}(X).\end{aligned}$$

(c) Using (a),

$$\begin{aligned}\mathrm{Var}[X|\mathcal{F}] &= \sum_{i=1}^{2} 1_{A_i} \left(\frac{E[X^2 1_{A_i}]}{P(A_i)} - \frac{E[X 1_{A_i}]^2}{P(A_i)^2} \right) \\ &= \sum_{i=1}^{2} 1_{A_i} \left(E[X^2|A_i] - E[X|A_i]^2 \right).\end{aligned}$$

Solution 8.2 First we show that an $\mathcal{F}_n$-stopping time S can be defined equivalently by the conditions $\{S = n\} \in \mathcal{F}_n$ or $\{S \leq n\} \in \mathcal{F}_n$.

If $\{S = n\} \in \mathcal{F}_n$ for all $n \geq 0$, then for all $0 \leq k \leq n$, $\{S = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$, which implies that $\{S \leq n\} = \bigcup_{k=0}^{n} \{S = k\} \in \mathcal{F}_n$. On the other hand, $\{S \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$ implies that

- $\{S = 0\} \in \mathcal{F}_0$;
- $\{S \leq n-1\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$ for all $n \geq 1$. Hence one knows that for all $n \geq 1$, $\{S = n\} = \{S \leq n\} \setminus \{S \geq n-1\} \in \mathcal{F}_n$.

From now on we will use the one most convenient for our purpose in the following.

(a) In general, $S - 1$ need not be a stopping time. Intuitively, this is because to know whether $S - 1$ has 'happened' by time n, information about time $n + 1$ is needed. A counter example can be constructed as follows:

Let $\Omega = \{0, 1\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \{\emptyset, \{0\}, \{1\}, \Omega\}$ for $n \geq 1$. We define

$$S = 1_{\{1\}} + 2 \cdot 1_{\{0\}}.$$

Then $\{S \leq 0\} = \emptyset \in \mathcal{F}_0$, $\{S \leq 1\} = \{1\} \in \mathcal{F}_1$, and $\{S \leq k\} = \Omega \in \mathcal{F}_k$ for all $k \geq 2$. Thus, S is a $\mathcal{F}_n$-stopping time. However, $\{S - 1 \leq 0\} = \{S \leq 1\} = \{1\} \notin \mathcal{F}_0$, so $S - 1$ is not a stopping time.

(b) $S+1$ is a stopping time, since for any $n \geq 0$,

$$\{S+1 \leq n\} = \{S \leq n-1\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n.$$

(c) $S \wedge T$ is a stopping time, since for any $n \geq 0$,

$$\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\} \in \mathcal{F}_n.$$

(d) $S \vee T$ is a stopping time, since for any $n \geq 0$,

$$\{S \vee T \leq n\} = \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n.$$

(e) $S+T$ is a stopping time, since for any $n \geq 0$,

$$\{S+T = n\} = \bigcup_{k=0}^{n} \underbrace{\{S = k\}}_{\in \mathcal{F}_k \subseteq \mathcal{F}_n} \cap \underbrace{\{T = n-k\}}_{\in \mathcal{F}_{n-k} \subseteq \mathcal{F}_n} \in \mathcal{F}_n.$$

Solution 8.3 The total number of balls after the n-th iteration is given by $K(n) = s + w + n(t-1)$. For $n \geq 1$, let A_n be the event that the n-th ball to be drawn is black. Then the conditional probability of A_{n+1} given $Y_1, \ldots, Y_n$ equals Y_n, that is, for $n \geq 1$,

$$P[A_{n+1} | \sigma(Y_1, \ldots, Y_n)] = Y_n. \tag{1}$$

Note that we have, for $n \geq 1$,

$$Y_{n+1}(\omega) = \begin{cases} \frac{Y_n K(n) + (t-1)}{K(n+1)}, & \text{if } \omega \in \ A_{n+1}, \\ \frac{Y_n K(n)}{K(n+1)}, & \text{if } \omega \in \ A_{n+1}^c. \end{cases} \tag{2}$$

Thus we get, setting $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$,

$$\begin{aligned} E[Y_{n+1} | \mathcal{F}_n] &= E[Y_{n+1} \mathbf{1}_{A_{n+1}} + Y_{n+1} \mathbf{1}_{A_{n+1}^c} | \mathcal{F}_n] \\ &\overset{(2)}{=} E\left[\frac{Y_n K(n) + (t-1)}{K(n+1)} \mathbf{1}_{A_{n+1}} + \frac{Y_n K(n)}{K(n+1)} \mathbf{1}_{A_{n+1}^c} \middle| \mathcal{F}_n \right] \\ &= \frac{Y_n K(n) + (t-1)}{K(n+1)} P[A_{n+1} | \mathcal{F}_n] + \frac{Y_n K(n)}{K(n+1)} P[A_{n+1}^c | \mathcal{F}_n] \\ &\overset{(1)}{=} \frac{Y_n K(n) + (t-1)}{K(n+1)} Y_n + \frac{Y_n K(n)}{K(n+1)} (1 - Y_n) = Y_n, \end{aligned}$$

since $K(n) + (t-1) = K(n+1)$.