ETH Zurich HS 2019
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 13

Exercise 13.1 Let $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ be a canonical time–homogeneous Markov chain with a countable state space E, canonical coordinate process $(X_n)_{n\geq 0}$ and transition matrix Q. Let $A \subset E$ and τ_A the first entrance time of A, i.e., $\tau_A := \inf\{n \geq 0 \,|\, X_n \in A\}$. Suppose that there exists $n \geq 1$ and $\alpha > 0$ such that for all $x \in A^c$,

$$Q^n(x, A) = \sum_{a\in A} Q^n(x, a) = \sum_{a\in A} P_x[X_n = a] \geq \alpha.$$

Show that for all $x \in E$ we have that $P_x(\tau_A < +\infty) = 1$.

Exercise 13.2 Let $(\Omega, \mathcal{F}, (P_x)_{x\in\mathbb{Z}})$ be a canonical (time-homogeneous) Markov chain with state space $\mathbb{Z}$, transition matrix Q, and canonical coordinate process $(X_n)_{n\geq 0}$. We assume that

$$\sum_{y\in\mathbb{Z}} y^2 Q(x, y) < +\infty \text{ for all } x \in \mathbb{Z},$$

and set $b(x) := E_x[X_1]$, $a(x) := \mathrm{Var}_x(X_1) = E_x[(X_1 - b(x))^2]$.

(a) Represent $b(x)$ and $a(x)$ with the help of the matrix Q.

(b) Show that

$$E_x[X_{n+1}] = E_x[b(X_n)], \quad \mathrm{Var}_x(X_{n+1}) = \mathrm{Var}_x(b(X_n)) + E_x[a(X_n)].$$

Exercise 13.3 With the same notation as p. 145 in lecture notes, consider the canonical Markov chain with state space S and transition kernel K. Let N be an $\mathcal{F}_n$-stopping time, $B \in \mathcal{A}$ and μ be a probability on $(S, \mathcal{S})$. Show that (recall $\mathcal{F}_N$ from (3.3.6), p. 89)

$$E^{P_\mu}\left[1_B \circ \theta_N | \mathcal{F}_N\right] = P_{X_N}\left[B\right] \quad \text{ on } \{N < \infty\} \quad P\text{-a.s.}$$

(Here $1_B \circ \theta_N$ is understood as $1_B(\omega) \circ \theta_{N(\omega)}(\omega)$, if $N(\omega) < \infty$ and 0 otherwise, and $P_{X_N}[B]$ as $P_{X_{N(\omega)}(\omega)}[B]$, if $N(\omega) < \infty$ and 0 otherwise.)

Submission: until 14:15, Dec 19., in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint

Solution 13.1 Let $x \in A$, $\tau_A = 0$ P_x-a.s. For $x \in A^c$, we have that for all $k \geq 0$,

$$P_x(\tau_A > (k+1)n) \leq P_x(\tau_A > kn, X_{(k+1)n} \in A^c) = E_x[E_x[1_{\{\tau_A > kn\}} 1_{\{X_{(k+1)n} \in A^c\}} | \mathcal{F}_{nk}]] \overset{\text{Markov}}{=} E_x[1_{\{\tau_A > kn\}} \underbrace{P_{X_{kn}}[X_n \in A^c]}_{\leq 1-\alpha}] \leq (1-\alpha) P_x(\tau_A > kn).$$

From the last we get by induction that $P_x(\tau_A > kn) \leq (1-\alpha)^k$ and taking the limit as k goes to infinity we get that,

$$P_x(\tau_A = +\infty) = \lim_{k\to\infty} P_x(\tau_A > kn) = 0.$$

Solution 13.2

(a) For $x \in \mathbb{Z}$ one has

$$\begin{aligned} b(x) &= E_x[X_1] = \sum_{y\in\mathbb{Z}} yQ(x,y), \\ a(x) &= E_x[(X_1 - b(x))^2] = \sum_{y\in\mathbb{Z}} (y - b(x))^2 Q(x,y). \end{aligned}$$

(b) Using the Markov property we have (recall that θ_n is the shift operator on Ω, see p. 145)

$$\begin{aligned} E_x[X_{n+1}] &= E_x[X_1 \circ \theta_n] = E_x[E_{X_n}[X_1]] = E_x[b(X_n)], \\ E_x[X_{n+1}^2] &= E_x[X_1^2 \circ \theta_n] \\ &= E_x[E_{X_n}[X_1^2]] = E_x[\mathrm{Var}_{X_n}(X_1) + E_{X_n}[X_1]^2] = E_x[a(X_n) + b(X_n)^2]. \end{aligned}$$

Hence we have

$$\begin{aligned} \mathrm{Var}_x(X_{n+1}) &= E_x[X_{n+1}^2] - E_x[X_{n+1}]^2 \\ &= E_x[a(X_n) + b(X_n)^2] - E_x[b(X_n)]^2 \\ &= E_x[a(X_n) + b(X_n)^2] - \Big(E_x[b(X_n)^2] - \mathrm{Var}_x(b(X_n))\Big) \\ &= \mathrm{Var}_x(b(X_n)) + E_x[a(X_n)]. \end{aligned}$$

Solution 13.3 Observe that on $\{N = n\}$, $P_{X_N}[B] = P_{X_n}[B]$ where the latter is $\mathcal{F}_n$-measurable, and hence X_N and also $P_{X_N}[B]$ are $\mathcal{F}_N$-measurable. Moreover, for $A \in \mathcal{F}_N$, we have

$$\begin{aligned} E^\mu\left[1_B \circ \theta_N | A \cap \{N < \infty\}\right] &= \sum_{n\geq 0} E^\mu\left[1_B \circ \theta_N | A \cap \{N = n\}\right] \\ &= \sum_{n\geq 0} E^\mu\left[P_{X_n}[B] | A \cap \{N = n\}\right] \\ &= E^\mu\left[P_{X_N}[B] | A \cap \{N < \infty\}\right], \end{aligned}$$

where the second equation follows from (4.2.55), p. 145 in lecture notes (weak Markov property). This proves our claim.