2018-2019春学期《微分几何》第二周作业

$\mathrm{P}_6: 10, 12.$; $\mathrm{P}_{13}: 1, 2, 3, 5, 6, 8.$

10. **证明** 设$\mathbf{x}_1(s) = \mathbf{x}(s) + \lambda(s)\mathbf{T}(s)$, s为$\mathbf{x}$的弧长参数, 对s求导得$\mathbf{x}_1' = (1+\lambda')\mathbf{T} + \lambda k\mathbf{N}$. 因$\mathbf{x}_1$正交于$\mathbf{T}$, 故$1+\lambda' = 0$, $\lambda = c_1 - s$, c为常数. 于是可设$\mathbf{x}_1(s) = \mathbf{x}(s) + (c_1 - s)\mathbf{T}(s)$, $\mathbf{x}_2(s) = \mathbf{x}(s) + (c_2 - s)\mathbf{T}(s)$. 对于$i = 1, 2$, 由$\mathbf{x}_i' = (c_i - s)k\mathbf{N}$, 知$\mathbf{T}_i = \pm\mathbf{N}$. 因而$\mathbf{T}_i' = (k_i\mathbf{N}_i)\frac{ds_i}{ds} = \pm\dot{\mathbf{N}} = \pm(-k\mathbf{T} + \tau\mathbf{B})$, 即$\mathbf{N}_i = \pm\frac{1}{\sqrt{k^2+\tau^2}}(-k\mathbf{T} + \tau\mathbf{B})$. 由此, $\mathbf{x}_1, \mathbf{x}_2$为Bertrand曲线对 $\Longleftrightarrow$ 它们具有公共主法线方向$\mathbf{x}_2 - \mathbf{x}_1 = (c_2 - c_1)\mathbf{T} \Longleftrightarrow \mathbf{N}_i = \pm\mathbf{T} \Longleftrightarrow \tau = 0 \Longleftrightarrow \mathbf{x}$为平面曲线. ▌

12. **证明** (1)必要性：若$T(s)$为大圆，则$k = 1, \tau = 0$. 设s 为T 的弧长参数，$\bar{s}$为x的弧长参数，τ 为$T(s)$ 的挠率，$\bar{\tau}$为x的挠率，则

$$\bar{\tau} = (\frac{d\bar{s}}{ds})^3 \frac{(T, \dot{T}, \ddot{T})}{|T \times \dot{T}|^2}.$$

又因为$\dot{T} = N$, $\ddot{T} = -T$, 则代入上式有

$$\bar{\tau} = 0.$$

充分性：若$x(s)$为平面曲线，s为x的弧长参数, $\vec{n}_0$为平面的任意单位法向量，则

$$x(s) - x(0) \cdot \vec{n}_0 = 0,$$

求导得

$$T(s) \cdot \vec{n}_0 = 0.$$

所以$T(s)$为平面曲线且为大圆或大圆的一部分。

(2)若主法线的球面标线为常值曲线，则

$$|\frac{dN(s)}{ds}| = |-kT + \tau B| = \sqrt{k^2 + \tau^2} = 0,$$

则$k = \tau = 0$, 矛盾。▌

P_{13}

1. **解** 设曲面的参数表示为$\mathbf{x}(u, v) = (x^1(u,v), x^2(u,v), x^3(u,v))$, 则$\mathbf{x}_u = (\frac{\partial x^1}{\partial u}, \frac{\partial x^2}{\partial u}, \frac{\partial x^3}{\partial u})$, $\mathbf{x}_v = (\frac{\partial x^1}{\partial v}, \frac{\partial x^2}{\partial v}, \frac{\partial x^3}{\partial v})$. 又因$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x^1}\frac{\partial x^1}{\partial u} + \frac{\partial f}{\partial x^2}\frac{\partial x^2}{\partial u} + \frac{\partial f}{\partial x^3}\frac{\partial x^3}{\partial u} = 0$, 故$\mathbf{m} = (\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3})$与$\mathbf{x}_u$正交, 同理, $\mathbf{m}$也与$\mathbf{x}_v$正交. 因此$\mathbf{m} // \mathbf{x}_u \times \mathbf{x}_v$. 曲面单位法向量$\mathbf{n} = (\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3}) / \sqrt{(\frac{\partial f}{\partial x^1})^2 + (\frac{\partial f}{\partial x^2})^2 + (\frac{\partial f}{\partial x^1})^2}$. 不妨设在曲面上某点局部, $\frac{\partial f}{\partial x^3} \neq 0$. 则在该局部曲面可表示为$\mathbf{x} = \mathbf{x}(x^1, x^2) = (x^1, x^2, x^3(x^1, x^2))$, $\mathbf{x}_1 = (1, 0, \frac{\partial x^3}{\partial x^1})$, $\mathbf{x}_2 = (0, 1, \frac{\partial x^3}{\partial x^2})$. 于是$E = \mathbf{x}_1 \cdot \mathbf{x}_1 = 1 + (\frac{\partial x^3}{\partial x^1})^2$, $F = \mathbf{x}_1 \cdot \mathbf{x}_2 = \frac{\partial x^3}{\partial x^1}\frac{\partial x^3}{\partial x^2}$, $G = \mathbf{x}_2 \cdot \mathbf{x}_2 = 1 + (\frac{\partial x^3}{\partial x^2})^2$. 曲面第一基本形式$I = (1 + (\frac{\partial x^3}{\partial x^1})^2)(dx^1)^2 + 2\frac{\partial x^3}{\partial x^1}\frac{\partial x^3}{\partial x^2}dx^1dx^2 + (1 + (\frac{\partial x^3}{\partial x^2})^2)(dx^2)^2$.

对于曲面$x^1x^2x^3 = c^3$, 知在任意一点(a^1,a^2,a^3)处, 曲面的法向量为$\mathbf{m} = (a^2a^3, a^1a^3, a^1a^2)$, 于是在该点的切平面为$a^2a^3(x^1-a^1)+a^1a^3(x^2-a^2)+a^1a^2(x^3-a^3) = 0$. 与三坐标平面交于$(3a^1,0,0),(0,3a^2,0),(0,0,3a^3)$点. 故四面体体积为$\frac{1}{2}\times\frac{1}{3}\times|3a^1\cdot 3a^2\cdot 3a^3| = \frac{9}{2}|c^3|$是常数. ▌

2. **解** $\mathbf{x}_u = (-\sin u,\cos u,0) + v(\frac{1}{2}\cos\frac{u}{2}\cos u - \sin\frac{u}{2}\sin u, \frac{1}{2}\cos\frac{u}{2}\sin u + \sin\frac{u}{2}\cos u, -\frac{1}{2})\sin\frac{u}{2}$, $\mathbf{x}_v = (\sin\frac{u}{2}\cos u, \sin\frac{u}{2}\sin u, \cos\frac{u}{2})$. $\mathbf{x}_u\times\mathbf{x}_v = (\cos\frac{u}{2}\cos u, \cos\frac{u}{2}\sin u, -\sin\frac{u}{2}) + \frac{v}{2}(\sin u + \sin u\cos u, \sin^2 u - \cos u, \cos u - 1)$, 单位法向量$\mathbf{n} = \frac{\mathbf{x}_u\times\mathbf{x}_v}{|\mathbf{x}_u\times\mathbf{x}_v|}$. ▌

3. **证明** 对任意不在坐标平面上的点(x^1,x^2,x^3), 令$f(\lambda) = \frac{(x^1)^2}{a-\lambda} + \frac{(x^2)^2}{b-\lambda} + \frac{(x^3)^2}{c-\lambda} - 1$. 则$x\to-\infty$时, $f<0$, $x\to c$时, $f(\lambda)>0$. 由f在$(-\infty,c)$的连续性, 知其在$(-\infty,c)$内有根. 同理f在$(c,b),(b,a)$内都有根. 因此, 过(x^1,x^2,x^3), 有三族曲面的一张通过.

要证明曲面的正交性, 只需证明在该点法向量的正交性. 设$\mathbf{n}_i(i=1,2,3)$为对应于λ_i的曲面的法向量, λ_i分别在三个区间段内. 由第一题结论, $\mathbf{n}_i = (\frac{x^1}{a-\lambda_i}, \frac{x^2}{b-\lambda_i}, \frac{x^3}{c-\lambda_i})$. 于是$i\neq j$时, $\mathbf{n}_i\cdot\mathbf{n}_j = \frac{(x^1)^2}{(a-\lambda_i)(a-\lambda_j)} + \frac{(x^2)^2}{(b-\lambda_i)(b-\lambda_j)} + \frac{(x^3)^2}{(c-\lambda_i)(c-\lambda_j)} = \frac{1}{\lambda_i-\lambda_j}(f(\lambda_i) - f(\lambda_j)) = 0$. 因此曲面在交点相互正交. ▌

5. **证明** 设曲面原参数为v^1,v^2, 由已知, 对任一固定的v^2, v^1曲线从参数v_1^1到v_2^1的长$\int_{v_1^1}^{v_2^1}\sqrt{E(v^1,v^2)}dv^1$ 与v^2无关, 从而$E(v^1,v^2)$与v^2无关. 同理$G(v^1,v^2)$与v^1无关. 这时曲面第一基本形式为

$$I = E(v^1)(dv^1)^2 + 2F(v^1,v^2)dv^1dv^2 + G(v^2)(dv^2)^2.$$

引入新参数u^1,u^2, 满足$du^1 = \sqrt{E(v^1)}dv^1$, $du^2 = \sqrt{G(v^2)}dv^2$. 由于$\frac{\partial(u^1,u^2)}{\partial(v^1,v^2)} = \sqrt{E(v^1)G(v^2)} > 0$, 则新参数可作为曲面参数. 又因$\cos\theta = \frac{F(v^1,v^2)}{\sqrt{E(v^1)G(v^2)}}$, 故采用新参数后, 第一基本形式为

$$I = (du^1)^2 + 2\cos\theta du^1du^2 + (du^2)^2.$$

对于平移曲面$\mathbf{x}(u,v) = \mathbf{a}(u) + \mathbf{b}(v)$, 显然$E$与$v$无关, G与u无关, 故它的参数网构成Chebyshev网. ▌

6. **证明** 对单位球面$\mathbf{x} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$, $\mathbf{x}_\theta = (-\sin\theta\cos\varphi, -\sin\theta\sin\varphi, \cos\theta)$, $\mathbf{x}_\varphi = (-\cos\theta\sin\varphi, \cos\theta\cos\varphi, 0)$. 于是$E = \mathbf{x}_\varphi\cdot\mathbf{x}_\varphi = \cos^2\theta$, $F = \mathbf{x}_\varphi\cdot\mathbf{x}_\theta = 0$, $G = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = 1$. 故第一基本形式$I = \cos^2\theta(d\varphi)^2 + (d\theta)^2$.

由$x^1 = \varphi$, $x^2 = \ln|\tan(\frac{\theta}{2}+\frac{\pi}{4})|$, 得$dx^1 = d\varphi$, $dx^2 = \frac{1}{\cos\theta}d\theta$. 于是$\bar{I} = (dx^1)^2 + (dx^2)^2 = (d\varphi)^2 + \frac{1}{\cos^2\theta}(d\theta)^2 = (\frac{1}{\cos\theta})^2 I$, 即共性对应成立. ▌

8. **解** 设球面参数方程$\mathbf{x}(u,v) = (r\cos u\cos v, r\cos u\sin v, r\sin u)$, 则$\mathbf{x}_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$, $\mathbf{x}_v = (-r\cos u\sin v, r\cos u\cos v, 0)$. 于是$E =$

$r^2,\ F=0,\ G=r^2\cos^2 u$. 子午线(即经线)为$u$曲线, 切方向$(\delta u, \delta v)=(1,0)$. 假定球面上切方向为$(du, dv)$的曲线$C$为斜驶线, 它与子午线夹角的余弦

$$\cos\theta = \frac{Edu + Fdv}{\sqrt{E}\sqrt{E(du)^2 + 2Fdudv + G(dv)^2}} = \frac{1}{c}, \quad |c| \geq 1$$

于是曲线C满足微分方程

$$c^2(du)^2 = (du)^2 + \cos^2 u(dv)^2, \quad c^2 \geq 1,$$

即

$$dv = \pm\sqrt{c^2-1}\frac{du}{\cos u}.$$

解得

$$\begin{aligned} v &= \pm\sqrt{c^2-1}\int\frac{du}{\cos u} \\ &= \pm\sqrt{c^2-1}\ln\tan\left(\frac{u}{2}+\frac{\pi}{4}\right) + c_1, \quad c_1\text{为常数} \end{aligned}$$

此即为斜驶线的方程. ▌