REAL ANALYSIS
LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

1. Egorov's Theorem, Lusin's Theorem, Convergence in measure

1.1. **Littlewood's three principles.**

Littilewood aptly summarised the connections of measurable sets/functions and old concepts in the form of three principles that provide a useful intuition guide in initial study of the theory:

(i) Every measurable set is nearly a finite union of cubes.

See previous lecture.

(ii) Every measurable function is nearly continuous.

See Lusion's Theorem below.

(iii) Every convergent sequence of measurable functions is nearly uniformly convergent.

See Egorov's Theorem below.

Theorem 1.1 (Egorov). *Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}^n$ with $m(E) < \infty$, and assume $f_k \to f$ a.e. on E with f being finite-valued almost everywhere. Given $\varepsilon > 0$, there is a closed set A_ε of $\mathbb{R}^n$ with $m(E \setminus A_\varepsilon) < \varepsilon$ such that $f_k \to f$ uniformly on A_ε.*

Remark 1.1 (Remarks on Egorov's Theorem). *Let us look at some examples.*

(i) *Let* $E = [-1, 1]$*, and*

$$f_k(x) = \begin{cases} 0, & -1 \leq x \leq 0, \\ kx, & 0 \leq x \leq \dfrac{1}{k}, \\ 2 - kx, & \dfrac{1}{k} \leq x \leq \dfrac{2}{k}, \\ 0, & \dfrac{2}{k} \leq x \leq 1. \end{cases}$$

Then $f_k \to f := 0$ *pointwise on* E*, and uniformly on* $[-1, 0] \cup [\varepsilon, 0]$ *for any* $\varepsilon > 0$*.*

(ii) *Condition* $m(E) < \infty$ *is necessary. Consider* $E = \mathbb{R}$ *and* $f_k(x) = x/k$*.*

Proof of Egorov's Theorem 1.1.

Without loss of generality, we assume directly $f_k \to f$ everywhere on E and f is finite-valued.

Step 1. Let $\mathcal{B}_k^i = \bigcup_{j=k}^{\infty} \{x \in E : \ |f_j(x) - f(x)| \geq 1/i\}$, the "bad set". Informally, $x \in \mathcal{B}_k^i$ if x is $1/i$ bad for some f_j with $j \geq k$.

The complement $\mathcal{G}_k^i = E \setminus \mathcal{B}_k^i$ is the "good set", namely $x \in \mathcal{G}_k^i$ if and only if $|f_j(x) - f(x)| < 1/i$ for all $j \geq k$. Informally, $x \in \mathcal{G}_k^i$ if x is $1/i$ good for all f_j with $j \geq k$.

Step 2. Fix i. Then $\mathcal{B}_k^i \searrow \emptyset$ as $k \to \infty$, since $f_j(x) \to f(x)$ for all x in E. Therefore

$$\lim_{k \to \infty} m(\mathcal{B}_k^i) = m(\cap_{k=1}^{\infty} \mathcal{B}_k^i) = 0.$$

Note that $m(E) < \infty$ is used above.

Step 3. Fix $\varepsilon > 0$. By *Step 2*, for each i, there is a $k = k_{i,\varepsilon}$ such that $m(\mathcal{B}_{k_{i,\varepsilon}}^i) < \frac{\varepsilon}{2^{i+1}}$. Let $\mathcal{B}_\varepsilon = \cup_{i=1}^{\infty} \mathcal{B}_{k_{i,\varepsilon}}^i$. Then

(i) $m(\mathcal{B}_\varepsilon) < \varepsilon/2$,

(ii) $x \notin \mathcal{B}_\varepsilon \implies x \in \cap_{i=1}^{\infty} \mathcal{G}_{k_{i,\varepsilon}}^i \implies \forall i, \ |f_j(x) - f(x)| < 1/i$ if $j \geq k_{i,\varepsilon}$.

That is, $f_j \to f$ uniformly on $E \setminus \mathcal{B}_\varepsilon$.

Step 4. Since $E \setminus \mathcal{B}_\varepsilon$ is measurable, there is a closed set A_ε of $\mathbb{R}^n$ such that $A_\varepsilon \subset E \setminus \mathcal{B}_\varepsilon$ and $m((E \setminus \mathcal{B}_\varepsilon) \setminus A_\varepsilon) < \varepsilon/2$.

Then $f_j \to f$ uniformly on A_ε and $m(E \setminus A_\varepsilon) < \varepsilon$.

□

Remark 1.2. *In the proof above, we have shown the following result:*

Let f and f_j be measurable functions that are finite-valued almost everywhere. If $f_j \to f$ a.e. on $E \subset \mathbb{R}^n$ with $m(E) < \infty$, then for any $\delta > 0$

$$\lim_{k\to\infty} m(\mathcal{B}_k(\delta)) = 0,$$

where

$$\mathcal{B}_k(\delta) = \bigcup_{j=k}^{\infty} \{x \in E : \ |f_j(x) - f(x)| \geq \delta\}.$$

In fact, this shows that $f_k \to f$ in measure.

Theorem 1.2 (Lusin)**.** *Suppose f is a measurable and finite-valued function on a measurable set E. Then for every $\varepsilon > 0$, there is a closed set F_ε of $\mathbb{R}^n$, with*

$$F_\varepsilon \subset E \quad \textit{and} \quad m(E \setminus F_\varepsilon) \leq \varepsilon.$$

and such that $f|_{F_\varepsilon}$ is continuous.

Moreover there is a continuous function $\tilde{f} : E \to \mathbb{R}$ such that $\tilde{f} = f$ on F_ε.

Remark 1.3 (Remarks on Lusin's Theorem)**.**

(i) *The notation $f|_{F_\varepsilon}$ is the restriction of f on F_ε. The conclusion of the theorem states that if f is viewed as a function defined only on F_ε, then f is continuous. However, the theorem does not make the stronger assertion that the function f defined on E is continuous at the points of F_ε.*

For example, let $f = \chi_{\mathbb{I}}$, where $\mathbb{I}$ is the set of irrationals. Then f is nowhere continuous, but $f|_{\mathbb{I}} \equiv 1$ is continuous everywhere on its domain $\mathbb{I}$. Now choose an $F_\varepsilon \subset \mathbb{I}$, F_ε closed in $\mathbb{R}$, such that $m(\mathbb{I} \setminus F_\varepsilon) < \varepsilon$ [1]. Then $m(\mathbb{R} \setminus F_\varepsilon) < \varepsilon$ and f_{F_ε} is continuous. In this case $\tilde{f} \equiv 1$.

(ii) *It is not necessarily true that f is continuous on some measurable set $F \subset E$ with $m(E \setminus F) = 0$.*

For example, let $\hat{\mathcal{C}} \subset E = [0,1]$ be a Cantor-like set with $m(\hat{\mathcal{C}}) = 1/2$. Consider

$$f(x) = \chi_{\hat{\mathcal{C}}}(x) - \chi_{[0,1]\setminus\hat{\mathcal{C}}}(x), \quad x \in [0,1].$$

It is easy to see that $([0,1] \setminus Z) \cap \hat{\mathcal{C}} \neq \emptyset$, for any zero measure set $Z \subset [0,1]$. Since $\hat{\mathcal{C}}$ has empty interior, $\forall x \in ([0,1] \setminus Z) \cap \hat{\mathcal{C}}$,

$$(x - \delta, x + \delta) \cap \big([0,1] \setminus (Z \cup \hat{\mathcal{C}})\big) \neq \emptyset \ \forall \ \delta > 0.$$

[1] Let $\mathbb{Q} = \{r_k\}_{k=1}^{\infty}$. Consider $F_\varepsilon = \mathbb{R} \setminus G_\varepsilon$, where $G_\varepsilon = \cup_{k=1}^{\infty}(r_k - \frac{\varepsilon}{2^{k+1}}, r_k + \frac{\varepsilon}{2^{k+1}})$.

Consequently, f is discontinuous at $x \in ([0,1] \setminus Z)$.

Proof of Lusin's Theorem 1.2.

First assume $m(E) < \infty$.

Step 1. We select a sequence of step functions f_k such that $f_k \to f$ almost everywhere.

Step 2. For each $f_k = \sum_{j=1}^{N} a_j \chi_{R_j}$, where R_j are disjoint rectangles, we shrink the rectangles a little so that f_k restricted to $E \setminus B_k^{\varepsilon'}$ is continuous, where $m(B_k^{\varepsilon'}) < \varepsilon'/2^k$.

Step 3. Let $B_{\varepsilon'} = \cup_{k \geq 1} B_k^{\varepsilon'}$.

Then $m(B_{\varepsilon'}) < \varepsilon'$, f_k restricted to $E \setminus B_{\varepsilon'}$ is continuous, $f_k \to f$ a.e. on $E \setminus B_{\varepsilon'}$.

Step 4. By Egorov's Theorem, after removing another small set $B'_{\varepsilon'}$ with $m(B'_{\varepsilon'}) < \varepsilon'$, $f_k \to f$ uniformly on $E \setminus (B_{\varepsilon'} \cup B'_{\varepsilon'})$, and f_k restricted to $E \setminus (B_{\varepsilon'} \cup B'_{\varepsilon'})$ are continuous.

Step 5. Hence f restricted to $F'_{\varepsilon'} := E \setminus (B_{\varepsilon'} \cup B'_{\varepsilon'})$ is continuous, being a uniform limit of continuous functions f_k.

Step 6. By taking a slightly smaller closed set F_ε of $\mathbb{R}^n$ with $F_\varepsilon \subset F'_{\varepsilon'}$, and taking $\varepsilon' = \varepsilon/3$, we obtain $f|_{F_\varepsilon}$ is continuous and $m(E \setminus F_\varepsilon) < \varepsilon$.

Note that $m(E) < \infty$ is needed in *Step 4*, as Egorov's Theorem is used there.

For $m(E) = \infty$, we use a cut-off argument as follows.

Step 1. Let $E = \cup_{k \geq 1} E_k$ where $E_k = E \cap \{x : \ 1 - k \leq |x| < k\}$.

Step 2. For each k, take a closed set F_k of $\mathbb{R}^n$ such that $F_k \subset E_k$, $m(E_k \setminus F_k) < \varepsilon/2^k$ and $f|_{F_k^\varepsilon}$ is continuous. Note that F_k^ε are compact and disjoint.

Step 3. Let $F_\varepsilon = \cup_{k \geq 1} F_k^\varepsilon$. Then F_ε is closed [2] and $m(E \setminus F_\varepsilon) < \varepsilon$.

Step 4. Since each $f|_{F_k^\varepsilon}$ is continuous so is $f|_{F_\varepsilon}$. (Consider a sequence $x_j \in F_\varepsilon$, $x_j \to x$. Then eventually $x_j \in F_k^\varepsilon$, say, and now use the continuity of $f|_{F_k^\varepsilon}$.)

For the last part of the theorem: by the Tietze extension theorem, any continuous function defined on a closed subset of $\mathbb{R}^n$ can be extended to a continuous function defined on all of $\mathbb{R}^n$.

□

[2] In general, a countable union of closed sets is not necessarily closed. However any convergent sequence $\{x_j\}_{j \geq 1} \subset F_\varepsilon$ is bounded and so is a subset of the union of finitely many of F_k^ε. The union is closed and so it contains the limit x of the sequence, and so $x \in F_\varepsilon$.

Urysohn's Lemma and Tietze Extension Theorem

Theorem 1.3 (Tietze Extension Theorem)**.** *Suppose $F \subset X$ where X is normal*[3] *and F is closed. Then any continuous function $f : F \to [a, b]$ can be extended to a continuous function $g : X \to [a, b]$.*

The result is also true with $[a, b]$ replaced by $\mathbb{R}$.

We shall use the Urysohn's lemma.

Lemma 1.1 (Urysohn's Lemma)**.** *Suppose A and B are disjoint closed subsets of a normal space X. Then there is a a real-valued continuous function $f : X \to [0, 1]$ such that $f = 0$ on A and $f = 1$ on B.*

Remark 1.4. *If X is a metric space, it suffices to take*

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Proof of Urysohn's Lemma.

Step A. By the definition of normal space, there is an open set $\mathcal{O}_{\frac{1}{2}}$ such that

$$A \subset \mathcal{O}_{\frac{1}{2}} \subset \overline{\mathcal{O}}_{\frac{1}{2}} \subset B^c.$$

Similarly, there are open sets $\mathcal{O}_{\frac{1}{4}}$ and $\mathcal{O}_{\frac{3}{4}}$ such that

$$A \subset \mathcal{O}_{\frac{1}{4}} \subset \overline{\mathcal{O}}_{\frac{1}{4}} \subset \mathcal{O}_{\frac{1}{2}} \subset \overline{\mathcal{O}}_{\frac{1}{2}} \subset \mathcal{O}_{\frac{3}{4}} \subset \overline{\mathcal{O}}_{\frac{3}{4}} \subset B^c.$$

Iterating this, let $\Lambda = \{m/2^n : n = 1, 2, \ldots;\ m = 1, 2, \ldots, 2^n - 1\}$. Then there exist open sets $\mathcal{O}_r$ for each $r \in \Lambda$ such that if $r < s$ then

$$A \subset \mathcal{O}_r \subset \overline{\mathcal{O}}_r \subset \mathcal{O}_s \subset \overline{\mathcal{O}}_s \subset B^c.$$

[3] We recall the following separation properties:

- Tychonoff or T_1 topology: every two points $x, y \in X$ with $x \neq y$, there is a neighbourhood U_x of x which avoids y.
- Hausdorff or T_2 topology: every two points $x, y \in X$ with $x \neq y$, there are disjoint open neighbourhoods of x and y respectively.
- Regular or T_3 topology: Tychonoff (i.e. singletons are closed) + for every point x and closed set A which are disjoint there exist disjoint open neighbourhoods of x and A respectively.
- Normal or T_4 topology: Tychonoff + for every two disjoint closed sets A and B there exist disjoint open neighbourhoods of A and B respectively.

Step B. Define

$$f(x) = \begin{cases} \inf\{r \in \Lambda : x \in O_r\} & \text{if } x \in \cup_{r\in\Lambda}\mathcal{O}_r, \\ 1 & \text{if } x \in X \setminus \cup_{r\in\Lambda}\mathcal{O}_r, \end{cases}$$

Clearly $f : X \to [0,1]$, $f = 0$ on A and $f = 1$ on B.

Step C. We claim that f is continuous.

To do this it is sufficient to show that if $0 < a, b < 1$ then

(i) $f^{-1}[0,b)$ is open;

(ii) $f^{-1}(a,1]$ is open.

Results (i) and (ii) follow by the observation below

$$f(x) < b \iff x \in \cup_{r<b}\mathcal{O}_r,$$
$$f(x) > a \iff x \in \cup_{s>a}\overline{\mathcal{O}}_s^c.$$

$\square$

It is the position to prove the Tietze Extension Theorem.

Proof of Theorem 1.3.

Without loss of generality, we take $[a,b] = [-1,1]$, and so

$$|f(x)| \leq 1, \ \forall\ x \in F.$$

Step A. Let $A = \{x \in F : \ f(x) \leq -1/3\}$ and $B = \{x \in F : \ f(x) \geq 1/3\}$.

Then A and B are disjoint closed subsets of X. By Urysohn's Lemma, there is a continuous function g_1 defined on X such that $g_1 = -1/3$ on A and $g_1 = 1/3$ on B, and

$$|g_1(x)| \leq \frac{1}{3}, \ \forall x \in X.$$

It follows that

$$|f(x) - g_1(x)| \leq \frac{2}{3}, \ \forall\ x \in F.$$

By the same argument as before, now applied to $f - g_1$, there is a continuous function g_2 defined on X such that

$$|g_2(x)| \leq \frac{2}{3}\cdot\frac{1}{3}, \qquad \forall\ x \in X,$$
$$|f(x) - g_1(x) - g_2(x)| \leq \left(\frac{2}{3}\right)^2, \ \forall\ x \in F.$$

Repeating the argument, for each $n \geq 1$ there exists a continuous function g_n defined on X such that

$$\begin{aligned} |g_n(x)| &\leq \left(\frac{2}{3}\right)^{n-1} \cdot \frac{1}{3}, && \forall\, x \in X, \\ |f(x) - \sum_{k=1}^{n} g_k(x)| &\leq \left(\frac{2}{3}\right)^{n}, && \forall\, x \in F. \end{aligned} \tag{1.1}$$

Step B. It can be checked that the sequence of partial sums from $\sum_{n\geq 1} g_n(x)$ is uniformly Cauchy (using the second inequality in (1.1)). Hence $\sum_{n\geq 1} g_n(x)$ converges to a function $g : X \to \mathbb{R}$. Since g_n are continuous, it follows that g is continuous. One easily sees from (1.1) that $f = g$ on F.

Step C. Now suppose that $f : F \to \mathbb{R}$.

Let $\psi : \mathbb{R} \to (-1, 1)$ be the homeomorphism with inverse $\psi^{-1} : (-1, 1) \to \mathbb{R}$, given by

$$\psi(x) = \frac{x}{1 + |x|} \text{ and } \psi^{-1}(y) = \frac{y}{1 - |y|}.$$

Consider the continuous composition $\hat{f} = \psi \circ f : F \to (-1, 1)$.

By the Tietze extension theorem for closed bounded intervals, there is a continuous function $\hat{g} : X \to [-1, 1]$ (not to $(-1, 1)$ unfortunately) such that $\hat{g}(x) = \hat{f}(x)$ for $x \in F$.

By the Urysohn's Lemma there is a continuous function $\phi : X \to [-1, 1]$ such that $\phi(x) = 1$ if $x \in F$ and $\phi(x) = 0$ if $\{x \in X : \hat{g}(x) = \pm 1\}$. The function $\phi\hat{g}$ agrees with $\hat{g}$ on F, and hence agrees with $\hat{f}$ on F. Moreover, it never takes the values ± 1.

It follows that the function

$$\psi^{-1} \circ (\psi\hat{g}) = \frac{\phi\hat{g}}{1 - |\phi\hat{g}|} : X \to \mathbb{R}$$

is always finite and so continuous, and it agrees with $\psi^{-1} \circ \hat{f}$, i.e., with f on F.

$\square$

1.2. **Convergence of measurable functions.**

We are concerned with three types of convergence of measurable functions.

- Almost everywhere convergence.
- Uniform convergence.

- Convergence in measure.

Recall that f_k converges to f almost everywhere on E, write

$$\lim_{k\to\infty} f_k(x) = f(x) \ \ a.e.\ x \in E,$$

if there is a $Z \subset E$ with $m(Z) = 0$ such that $\lim_{k\to\infty} f_j(x) = f(x)$ for all $x \in E \setminus Z$. Note that if f_k are measurable then f is measurable.

The Theorem of Egorov says that the almost everywhere convergence is nearly the uniform convergence, see Theorem 1.1.

We shall also use some other notion of convergence.

Let us consider an example below. Given $n \in \mathbb{N}$, there are unique k and i such that

$$n = 2^k + i, \quad , \ k = 0, 1, 2, \cdots, \text{ and } i = 0, 1, \cdots, 2^k - 1. \tag{1.2}$$

Define a sequence of measurable functions $\{f_n\}_{n\ge 1}$ on $[0,1]$ as follows

$$f_n(x) = \chi_{[i/2^k,(i+1)/2^k]}(x), \ \ x \in [0,1],$$

where k and i are determined by (1.2). For every $x_0 \in [0,1]$, one sees that $f_n(x_0) = 0$ for infinitely many n's and so is $f_n(x_0) = 1$. Hence f_n is not convergent for all $x \in [0,1]$. However f_n is convergent in the sense of measure. For each $\varepsilon > 0$, there is a N such that

$$m\big(\{x \in [0,1] : |f_n(x) - 0| > \varepsilon\}\big) = \frac{1}{2^k} \to 0, \text{ as } n \to \infty.$$

This suggests the following definition.

Definition 1.1. *Let $\{f_k\}$ be a sequence of measurable functions on E, finite-valued almost everywhere. We say $\{f_k\}$ converges in measure to a function f if for every $\varepsilon > 0$,*

$$m\big(\{x \in E : |f_k(x) - f(x)| > \varepsilon\}\big) \to 0 \quad as\ k \to \infty.$$

This notion appears naturally in the proof of Egorov's theorem. See Remark 1.2.

Remark 1.5. *If f_k converges to both f and g in measure, then $f = g$ a.e. on E.*

Proof. Exercise.

Roughly speaking, these notions of convergence have the following connections:

(A) Almost everywhere convergence $\Longrightarrow$ nearly uniform convergence;
See Egorov's Theorem 1.1.

Almost everywhere convergence $\Longrightarrow$ convergence in measure;

See Theorem 1.4

(B) Nearly uniform convergence $\Longrightarrow$ convergence in measure;

Nearly uniform convergence $\Longrightarrow$ almost everywhere convergence;

See Theorem 1.5.

(C) Convergence in measure $\Longrightarrow$ Almost everywhere convergence by a subsequence.

See Riesz's Theorem 1.7.

Theorem 1.4. *Let f and f_k be measurable functions on E that are finite-valued almost everywhere, and $m(E) < \infty$. If $f_k \to f$ a.e. on E, then $f_k \to f$ in measure.*

Proof. See Remark 1.2.

□

Theorem 1.5. *Let f and f_k be measurable functions on E that are finite-valued almost everywhere. Suppose for any $\delta > 0$, there is $E_\delta \subset E$ with $m(E \setminus E_\delta) < \delta$ such that $f_k \to f$ uniformly on E_δ. Then*

(i) *$f_k \to f$ in measure;*

(ii) *$f_k \to f$ a.e. on E, provided $m(E) < \infty$.*

Proof. Given $\varepsilon, \delta > 0$, there is $N_{\varepsilon,\delta} > 0$ such that

$$\bigcup_{k \geq N_{\varepsilon,\delta}} \{x : \ |f_k(x) - f(x)| > \varepsilon\} \subset E \setminus E_\delta, \tag{1.3}$$

which yields

$$m(\{x : \ |f_k(x) - f(x)| > \varepsilon\}) < \delta.$$

Taking limit at the right hand side and then sending $\delta \to 0$, one obtains

$$\lim_{k\to\infty} m(\{x : \ |f_k(x) - f(x)| > \varepsilon\}) = 0.$$

This proves part (i).

Let $G_j(\frac{1}{r}) = \{x : |f_j(x) - f(x)| < 1/r\}$. We have

$$\{x : \ \lim_{k\to\infty} f_k(x) = f(x)\} \ = \ \bigcap_{r=1}^{\infty} \liminf_{j\to\infty} G_j(\frac{1}{r}) = \bigcap_{r=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} \{x : |f_j(x) - f(x)| < \frac{1}{r}\},$$

and so

$$E \setminus \{x : \lim_{k\to\infty} f_k(x) = f(x)\} = \bigcup_{r=1}^{\infty} \limsup_{j\to\infty} D_j(\frac{1}{r}),$$

where $D_j(\frac{1}{r}) = E \setminus G_j(\frac{1}{r})$. Since $m(E) < \infty$, we get

$$m\big(\limsup_{j\to\infty} D_j(\frac{1}{r})\big) = \lim_{j\to\infty} m(\cup_{k\ge j} D_k(\frac{1}{r})) = 0,$$

where the last equality is due to (1.3). Consequently

$$m\big(E \setminus \{x : \lim_{k\to\infty} f_k(x) = f(x)\}\big) = 0.$$

This finishes part (ii).

$\square$

Definition 1.2. *Let f_k be measurable functions on E that are finite-valued almost everywhere. We say $\{f_k\}$ is a Cauchy sequence in measure if for every $\varepsilon > 0$*

$$\lim_{k,j\to\infty} m\big(\{x \in E : |f_k(x) - f_j(x)| > \varepsilon\}\big) = 0.$$

Remark 1.6. *If $f_k \to f$ in measure, then $\{f_k\}$ is a Cauchy sequence in measure.*

Proof. *Exercise.*

Theorem 1.6. *Let $\{f_k\}$ be a Cauchy sequence in measure on E. There is a measurable function f, finite-valued almost everywhere, such that $f_k \to f$ in measure on E.*

Remark 1.7. *Theorem 1.6 together with Remark 1.6 implies that $f_k \to f$ in measure if and only if $\{f_k\}_{k\ge1}$ is a Cauchy sequence in measure.*

Proof of Theorem 1.6. Let us divide the proof into several steps.

Step 1. We first select a subsequence of $\{f_k\}$ that converges to a limit f a.e. on E.

Given $i \ge 1$, there is a N_i such that if $k, j \ge N_i$ then

$$m\big(\{x : |f_k(x) - f_j(x)| > \frac{1}{2^i}\}\big) < \frac{1}{2^i}.$$

This shows that we can extract an increasing sequence k_i such that

$$m(E_i) < \frac{1}{2^i}, \text{ where } E_i = \{x : |f_{k_{i+1}}(x) - f_{k_i}(x)| > \frac{1}{2^i}\}. \tag{1.4}$$

Denote $D = \limsup_{i\to\infty} E_i$. It follows that

$$m(D) = \lim_{k\to\infty} m(\cup_{i\ge k} E_i) \le \lim_{k\to\infty} \frac{1}{2^{k-1}} = 0. \tag{1.5}$$

Consequently $m(E \setminus D) = m(E)$, and for each $x \in E \setminus D$,

$$|f_{k_j} - f_{k_i}| \le \sum_{l=i}^{j-1} |f_{k_{l+1}} - f_{k_l}| \le \sum_{l=i}^{\infty} |f_{k_{l+1}} - f_{k_l}| \le \frac{1}{2^{i-1}}, \ \forall\, j \ge i. \tag{1.6}$$

This implies $\{f_{k_i}(x)\}$ is a Cauchy sequence. Given $x \in E \setminus D$, let

$$f(x) := \lim_{i\to\infty} f_{k_i}(x).$$

By assigning value for f on D, we can assume f is a measurable function on E. Since f_{k_i} is finite-valued almost everywhere, so is f. Obviously $f_{k_i} \to f$ a.e.

Step 2. We next prove $f_{k_i} \to f$ in measure on E.

Let $\mathcal{B}_r = \cup_{i\ge r} E_i$, where E_i is given in (1.4). Recall that $m(\mathcal{B}_r) \to 0$ by (1.5). For $x \in E \setminus \mathcal{B}_r$, we have for any $j \ge r$

$$\begin{aligned} |f_{k_j}(x) - f(x)| &= \Big| f_{k_j}(x) - [f_{k_j}(x) + \sum_{l=j}^{\infty} (f_{k_{l+1}} - f_{k_l})(x)] \Big| \\ &\le \sum_{l=j}^{\infty} |f_{k_{l+1}}(x) - f_{k_l}(x)| \\ &\le \frac{1}{2^{r-1}}. \end{aligned}$$

Therefore $f_{k_i} \to f$ uniformly on $E \setminus \mathcal{B}_r$ [4]. Hence we can apply part (i) in Theorem 1.3 to conclude that $f_{k_i} \to f$ in measure on E. [5]

Step 3. We now show $f_k \to f$ in measure on E.

For any $\varepsilon > 0$, one sees that, by using the triangle inequality,

$$\begin{aligned} \{x : |f_k(x) - f(x)| > \varepsilon\} \quad &\subset \quad \{x : |f_k(x) - f_{k_i}(x)| > \varepsilon/2\} \\ &\qquad \cup \{x : |f_{k_i}(x) - f(x)| > \varepsilon/2\}. \end{aligned}$$

Therefore

$$\begin{aligned} m(\{x : |f_k(x) - f(x)| > \varepsilon\}) \quad &\le \quad m(\{x : |f_k(x) - f_{k_i}(x)| > \varepsilon/2\}) \\ &\qquad + m(\{x : |f_{k_i}(x) - f(x)| > \varepsilon/2\}). \end{aligned}$$

Since

$$\lim_{k,k_i\to\infty} m(\{x : |f_k(x) - f_{k_i}(x)| > \varepsilon/2\}) = 0, \quad (\{f_k\}_{k\ge1} \text{ is Cauchy})$$
$$\lim_{i\to\infty} m(\{x : |f_{k_i}(x) - f(x)| > \varepsilon/2\}) = 0, \quad (f_{k_i} \to f \text{ in measure})$$

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[4] This can be also obtained by letting $j \to \infty$ in (1.6).

[5] If $m(E) < \infty$, we can directly apply Theorem 1.5 to see that $f_{k_i} \to f$ in measure on E.

we conclude that

$$\lim_{k\to\infty} m(\{x : |f_k(x) - f(x)| > \varepsilon\}) = 0.$$

This finishes the proof.

□

Remark 1.8. *We will use an analogue argument as the above again to show the completeness of L^1-space. See the subsequent lectures.*

Theorem 1.7 (Riesz)**.** *Let f and f_k be measurable functions on E that are finite-valued almost everywhere. If $f_k \to f$ in measure on E, then there is a subsequence $\{f_{k_j}\}$ such that $f_{k_j} \to f$ a.e. on E.*

Proof. See *Step 1* in the proof of Theorem 1.6.

□