ETH Zurich HS 2019
D-MATH, D-PHYS
Prof. A.-S. Sznitman

Coordinator
Daniel Balint

Probability Theory

Exercise Sheet 10

Exercise 10.1 (The generalized Borel-Cantelli lemma)
Consider $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_n\}_{n\geq 0}$, and let $A_n \in \mathcal{F}_n$, $n \geq 1$, be a sequence of events. Show that, up to a P-nullset,

$$\limsup_{n\to\infty} A_n = \{\sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty\}.$$

Hint: Use Exercise 9.3.

Exercise 10.2 Consider a Galton-Watson process (see p. 97 of the lecture notes) Z_n, $n \geq 0$, with offspring distribution $\nu = \mathrm{Bin}(2,p)$, $p \in [0,1]$. We are interested in the probability $\vartheta(p) = P[Z_n > 0,\ \forall n \geq 0]$ that the population does not go extinct. Show that

$$\vartheta(p) = \begin{cases} 0 & \text{if } 0 \leq p \leq 1/2; \\ \frac{2p-1}{p^2} & \text{if } 1/2 < p \leq 1. \end{cases}$$

Hint: One way to prove this is to use the results for the various cases (subcritical, critical, supercritical) from Section 3.5 A), pp. 97-101 of the lecture notes.

Exercise 10.3 (Probabilistic solution to the discrete Dirichlet problem)
Let $A \subseteq \mathbb{Z}^d$ be finite, $f : \mathbb{Z}^d \setminus A \to \mathbb{R}$ any function, and $(S_n)_{n\in\mathbb{N}}$ a simple random walk on $\mathbb{Z}^d$ with starting point $S_0 = 0$. For $x \in \mathbb{Z}^d$ let $T_x := \inf\{n \geq 0;\, |\ x + S_n \notin A\}$. Finally, let $\mathcal{F}_n := \sigma(S_0, \dots, S_n)$ and $g(x) := E[f(x + S_{T_x})]$.

(a) Show that $T_x < \infty$ P-a.s. Thus $f(x + S_{T_x})$ exists a.s.
Hint: Use Exercise 9.3.

(b) Show that g solves the discrete Dirichlet problem on A with boundary condition f, i.e.,

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{Z}^d \setminus A \\ \dfrac{1}{2d} \displaystyle\sum_{\substack{\|y-x\|=1 \\ y\in\mathbb{Z}^d}} g(y) & \text{if } x \in A\ . \end{cases}$$

(c) Show that $E\left[f(x + S_{T_x})|\mathcal{F}_1\right] = g(x + S_{T_x\wedge 1})$ P-a.s.

Submission: until 14:15, Dec 03., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint

Solution 10.1 We define $X_0 := 0$, $X_n := \sum_{m=1}^n \left(1_{A_m} - P[A_m|\mathcal{F}_{m-1}]\right)$, $n \geq 1$. Then X_n is an $\mathcal{F}_n$-martingale, since

$$E[X_{n+1} - X_n|\mathcal{F}_n] = E\left[1_{A_{n+1}} - P[A_{n+1}|\mathcal{F}_n]\middle|\mathcal{F}_n\right] = 0.$$

Furthermore $|X_{n+1} - X_n| \leq 2$. We apply the result of Exercise 9.3 to obtain $P[C \cup D] = 1$. Note that:

- $\sum_{n\geq 1} 1_{A_n} = \infty \iff \sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty$ on C.
- $\sum_{n\geq 1} 1_{A_n} = \infty$ and $\sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty$ on D.

Using that $P[C \cup D] = 1$, we get that for an event N with $P[N] = 0$,

$$\left\{\sum_{n\geq 1} 1_{A_n} = \infty\right\} \cap N^c = \left\{\sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty\right\} \cap N^c.$$

Finally, the claim follows since

$$\limsup_{n\to\infty} A_n = \left\{\sum_{n\geq 1} 1_{A_n} = \infty\right\}.$$

Solution 10.2 From (3.5.3), p. 98 of the lecture notes we calculate m:

$$m = 2p \begin{cases} < 1 & \text{if } p \in [0, \frac{1}{2}), \\ = 1 & \text{if } p = \frac{1}{2}, \\ > 1 & \text{if } p \in (\frac{1}{2}, 1]. \end{cases}$$

Thus, if $p \in [0, \frac{1}{2})$, our Galton-Watson process is subcritical, if $p = \frac{1}{2}$ it is critical, and if $p \in (\frac{1}{2}, 1]$ it is supercritical.

For a subcritical Galton-Watson process, we have $P[Z_n = 0 \text{ eventually}] = 1$ by (3.5.7), p. 99, and by (3.5.10), p. 100 of the lecture notes also for a critical process. Hence,

$$\vartheta\,(p) = 0 \quad \forall p \in [0, 1/2].$$

In the supercritical case, we have, by (3.5.13), p. 101 of the lecture notes,

$$P[Z_n = 0 \text{ eventually}] = \varrho \in [0, 1),$$

where ϱ is the unique solution to $\varrho = \varphi(\varrho)$ in $[0, 1)$, and let X be a random variable with distribution ν, we have

$$\varphi(z) = E[z^X] = \sum_{k=0}^{2} P[X = k]z^k = (1-p)^2 + 2p(1-p)z + p^2z^2,$$

(see (3.5.11), p. 100 of the lecture notes, and the explanations right below it). Solving the quadratic equation

$$\varphi(z) = \big((1-p) + pz\big)^2 = z$$

for z, we obtain the solutions $z = 1$ and $z = \frac{(1-p)^2}{p^2}$. Thus, the unique solution to $\varphi(\varrho) = \varrho$ in $[0, 1)$ is

$$\varrho = \frac{(1-p)^2}{p^2},$$

from which it follows that

$$\vartheta(p) = 1 - P[Z_n = 0 \text{ eventually}] = 1 - \varrho = 1 - \frac{(1-p)^2}{p^2} = \frac{2p-1}{p^2},$$

for $p \in (1/2, 1]$.

Solution 10.3 Let $S_n = \sum_{m=1}^n X_m$ ($S_0 = 0$) be the simple random walk on $\mathbb{Z}^d$ such that X_n, $n \geq 1$ are i.i.d. $\mathbb{Z}^d$ valued random variable with $P[X_1 = e] = 1/(2d)$ for any $e \in \mathbb{R}^d$, $\|e\| = 1$ (i.e., $e = e_i$ or $e = -e_i$ for $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the ith entry, $i = 1, \ldots, d$). We can easily show that S_n is an $\mathcal{F}_n$ martingale taking values in $\mathbb{Z}^d$ and S_n^j, the jth coordinate of S_n, is also an $\mathcal{F}_n$ real–valued martingale for each $j = 1, \ldots, d$.

(a) Consider the martingale

$$M_n = \sum_{i=1}^d S_n^i,$$

with bounded increments

$$M_n - M_{n-1} = \sum_{i=1}^d (S_n^i - S_{n-1}^i) \in \{-1, 1\}.$$

By Exercise 9.3, it thus holds that

$$P\left[\{\lim M_n \in \mathbb{Z} \text{ exists}\} \cup \{\liminf M_n = -\infty \text{ and } \limsup M_n = \infty\}\right] = 1.$$

However, since $|M_n - M_{n-1}| = 1$, M_n cannot converge, from which it follows that $\limsup M_n = \infty$ P-a.s. and $\liminf M_n = -\infty$ P-a.s. This, in turn, implies that S_n must exit the finite set A, since $\max_{(a_1,\ldots,a_d)\in A} \sum_{i=1}^d |a_i| < \infty$.

(b) If $x \notin A$, then $T_x = 0 \Rightarrow f(x + S_{T_x}) = f(x)$.

If $x \in A$, then

$$\begin{aligned} E[f(x + S_{T_x})] &= \sum_{\|e\|=1} E\left[f\left(x + S_{T_x}\right) \mathbf{1}_{S_1 = e}\right] \\ &= \frac{1}{2d} \sum_{\|e\|=1} E\left[f\left(x + e + S_{T_{x+e}}\right)\right]. \end{aligned}$$

To see the last equality, we note that if $x + e \notin A$, then in view of the definition of T_{x+e} (resp. T_x) we have $T_{x+e} = 0$ (resp. $T_x = 1$) and in this case it holds that $E\left[f\left(x + S_{T_x}\right) \mathbf{1}_{S_1 = e}\right] = f(x+e)P[S_1 = e] = \frac{f(x+e)}{2d} = E\left[f\left(x + e + S_{T_{x+e}}\right)\right]$. On the other hand, if $x + e \in A$, then we must have $T_x \geq 2$ and $T_{x+e} \geq 1$. In this case we have

$$\begin{aligned} E\left[f\left(x + S_{T_x}\right)\mathbf{1}_{S_1=e}\right] &= \sum_{n\geq 2} E\left[f\left(x+e+S_n-S_1\right)\mathbf{1}_{\{S_1=e\}\cap\{T_x=n\}}\right] \\ &= \sum_{n\geq 2} E\left[f\left(x+e+S_n-S_1\right)\mathbf{1}_{\{S_1=e\}\cap B_n}\right], \end{aligned}$$

where $B_n = \{\forall 2 \leq m < n, x+e+(S_m - S_1) \in A; x+e+S_n-S_1 \notin A\}$. But since $S_m - S_1 = \sum_{j=2}^m X_j$ is independent of $S_1 = X_1$ for all $m \geq 2$, the event B_n is independent of $\{S_1 = e\}$. Moreover, it is easy to see that $x+e+S_m-S_1 = x+e+\sum_{j=2}^m X_j$, $m \geq 1$ is the simple random walk started from $x+e$, which implies that $B_n = \{T_{x+e} = n-1\}$ for all $n \geq 2$ and $x+e+S_n-S_1 = x+e+S_{T_{x+e}}$ on B_n. Hence, we can deduce that

$$\begin{aligned} \sum_{n\geq 2} E\left[f\left(x+e+S_n-S_1\right)\mathbf{1}_{\{S_1=e\}\cap B_n}\right] &= \sum_{n\geq 2} E\left[f\left(x+e+S_n-S_1\right)\mathbf{1}_{B_n}\right] P[S_1=e] \\ &= \frac{1}{2d} E\left[f\left(x+e+S_{T_{x+e}}\right)\mathbf{1}_{\{T_{x+e}\geq 1\}}\right]. \end{aligned}$$

Now we complete our proof by writing $y = x + e$ for some e with $\|e\| = 1$.

(c) In case $x \notin A$ the statement is trivial.

If $x \in A$, then $S_{T_x \wedge 1} = S_1$, whence it follows that

$$E[f(x+S_{T_x})\mathbf{1}_{S_1=e}] = \frac{1}{2d}E[f(x+e+S_{T_{x+e}})] = \frac{1}{2d}g(x+e)$$

and

$$E[g(x+S_1)\mathbf{1}_{S_1=e}] = \frac{1}{2d}E[g(x+e)] = \frac{1}{2d}g(x+e).$$

This concludes the proof.