ETH Zurich HS 2019
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 4

Exercise 4.1 Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $E[X_i^+] = \infty$ and $E[X_i^-] < \infty$. Define $S_n = X_1 + \ldots + X_n$. Show that

$$\frac{S_n}{n} \overset{n\to\infty}{\longrightarrow} \infty \qquad \text{a.s.}$$

Exercise 4.2 Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. exponentially distributed random variables with parameter 1, and set for $n \geq 1$

$$M_n = \max_{1\leq i\leq n} X_i.$$

Find a sequence of real numbers a_n, $n \geq 1$, such that $M_n - a_n$ converges in distribution and compute the distribution function of the limiting distribution.

Exercise 4.3

(a) Let f be a (not necessarily Borel-measurable) function from $\mathbb{R}$ to $\mathbb{R}$. Show that the set of discontinuities of f, defined as

$$U_f := \{x \in \mathbb{R} \mid f \text{ is discontinuous in } x\},$$

is Borel-measurable.

(b) Assume that $X_n \to X$ in distribution. Let f be measurable and bounded, such that $P[X \in U_f] = 0$. Use (2.2.13) – (2.2.14) from the lecture notes to show that we have

$$E[f(X_n)] \underset{n\to\infty}{\to} E[f(X)].$$

(c) Let f be measurable and bounded on $[0, 1]$, with U_f of Lebesgue measure 0. Show that the corresponding Riemann sums converge to the integral of f, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \underset{n\to\infty}{\to} \int_0^1 f(x)\mathrm{d}x.$$

Submission: until 14:15, Oct 22., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint