《微分几何》第三次课堂练习参考答案

1.(20分)(1)曲面的高斯曲率K,经计算得

$$K=\frac{h_{11}h_{22}-(h_{12})^2}{g_{11}g_{22}-(g_{12})^2}=-1.$$

取$w^1=du,\ w^2=dv,\ w_1^2=0$,由Gauss方程得

$$K=-\frac{dw_1^2}{w^1\wedge w^2}=0.$$

故不满足Gauss方程，因而曲面不存在。(10')

(2)曲面的平均曲率H,

$$H=\frac{h_{11}g_{22}-2h_{12}g_{12}+h_{22}g_{11}}{det(g_{\alpha\beta})}=\frac{1+cos^4u}{cos^2u}.$$

显然不满足Codazzi方程$(h_{11})_2=H(g_{112}),\ (h_{22})_1=H(g_{221})$,因而曲面不存在。(10')□

2.(20分)由题意得$g_{11}=1,\ g_{12}=g_{21}=0,\ g_{22}=G(u,v)$.

(1)$\Gamma_{11}^1=\frac{1}{2}(\log g_{11})_1=0$,

$\Gamma_{22}^2=\frac{1}{2}(\log g_{22})_2=\frac{1}{2}\frac{G_v}{G(u,v)}$,

$\Gamma_{12}^1=\Gamma_{21}^1=\frac{1}{2}(\log g_{11})_2=0$,

$\Gamma_{12}^2=\Gamma_{21}^2=\frac{1}{2}(\log g_{22})_1=\frac{1}{2}\frac{G_u}{G(u,v)}$,

$\Gamma_{22}^1=-\frac{1}{2}g_{11}^{-1}(g_{22})_1=-\frac{1}{2}G_u$,

$\Gamma_{11}^2=-\frac{1}{2}g_{22}^{-1}(g_{11})_2=0.$(6')

(2)对u,v为参数曲线，则$\theta=0$有$g_{11}=1,\ g_{22}=G(u,v)$, 则由Liouville公式

$$k_g=\frac{d\theta}{ds}-\frac{1}{2\sqrt{g_{22}}}\frac{\partial lng_{11}}{\partial u^2}-\frac{1}{2}\frac{\partial lng_{22}}{\partial u^1}tan\theta=0$$

所以u曲线是测地线。(4')

(3)由Gauss曲率为

$$K=-\frac{1}{\sqrt{g_{11}g_{22}}}\left[\left(\frac{(\sqrt{g_{11}})_2}{\sqrt{g_{22}}}\right)_2+\left(\frac{(\sqrt{g_{22}})_1}{\sqrt{g_{11}}}\right)_1\right]=-\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2}.(\mathbf{4'})$$

(4)若一测地线与u线的交角为θ, 由

$$\frac{du}{ds}=\frac{cos\theta}{\sqrt{g_{11}}},\ \frac{dv}{ds}=\frac{sin\theta}{\sqrt{g_{22}}}(\mathbf{2'})$$

且Liouville公式

$$k_g=\frac{d\theta}{ds}-\frac{1}{2\sqrt{g_{22}}}\frac{\partial lng_{11}}{\partial u^2}-\frac{1}{2}\frac{\partial lng_{22}}{\partial u^1}tan\theta=0(\mathbf{2'})$$

则有

$$\frac{d\theta}{dv}=\frac{1}{2}\frac{\partial lng_{11}}{\partial v}cot\theta-\frac{1}{2}\sqrt{\frac{g_{22}}{g_{11}}}\frac{\partial lng_{22}}{\partial u}=-\frac{\partial\sqrt{G}}{\partial u}.(\mathbf{2'})$$

□

3.(20分) 在曲面上取测地极坐标系, 因而曲面的第一基本形式成为

$$\mathrm{I}=(du)^2+G(u,v)(dv)^2$$

其中函数$G(u,v)$满足条件

$$\lim_{u\to 0}\sqrt{G}(u,v)=0,\quad \lim_{u\to 0}(\sqrt{G})_u(u,v)=1 \quad (\mathbf{3'})$$

曲面的Gauss曲率是

$$K=-\frac{1}{\sqrt{G}}(\sqrt{G})_{uu} \quad (\mathbf{2'})$$

现在K是正的常数, 所以$\sqrt{G}(u,v)$关于变量u满足常系数二阶常微分方程

$$(\sqrt{G})_{uu}+K\sqrt{G}=0$$

该方程的通解是

$$\sqrt{G}(u,v)=a(v)\cos(\sqrt{K}u)+b(v)\sin(\sqrt{K}u) \quad (\mathbf{2'})$$

让$u\to 0$, 利用函数$\sqrt{G}(u,v)$所满足的条件得到

$$0=a(v),\quad \sqrt{G}(u,v)=b(v)\sin(\sqrt{K}u)$$

$$(\sqrt{G})_u(u,v)=b(v)\sqrt{K}\cos(\sqrt{K}u)$$

让最后一式中的$u\to 0$得到

$$1=b(v)\sqrt{K},\quad \sqrt{G}(u,v)=\frac{1}{\sqrt{K}}\sin(\sqrt{K}u) \quad (\mathbf{2'})$$

所以曲面的第一基本形式成为

$$\mathrm{I}=(du)^2+\frac{1}{K}\sin^2(\sqrt{K}u)(dv)^2 \quad (\mathbf{1'})$$

此时测地线的微分方程是

$$\frac{du}{ds}=\cos\theta,\quad \frac{dv}{ds}=\frac{\sqrt{K}}{\sin(\sqrt{K}u)}\sin\theta$$

$$\frac{d\theta}{ds}=-\frac{1}{2\sqrt{E}}\frac{\partial\log G}{\partial u}\sin\theta=-\frac{\sqrt{K}\cos(\sqrt{K}u)}{\sin(\sqrt{K}u)}\sin\theta$$

其中θ是测地线和u-曲线的夹角, s是测地线的弧长参数. $(\mathbf{3'})$ 将第一个方程和第三个相除得

$$\frac{du}{d\theta}=-\frac{\sin(\sqrt{K}u)}{\sqrt{K}\cos(\sqrt{K}u)}\frac{\cos\theta}{\sin\theta}$$

积分得

$$\sin(\sqrt{K}u)\sin\theta=c,\quad \cos(\sqrt{K}u)=\sqrt{1-\frac{c^2}{\sin^2\theta}}=\frac{\sqrt{\sin^2\theta-c^2}}{\sin\theta} \quad (\mathbf{2'})$$

将第二个方程与第三个相除得

$$\frac{dv}{d\theta} = -\frac{1}{\cos(\sqrt{K}u)} = -\frac{\sin\theta}{\sqrt{\sin^2\theta - c^2}}, \quad dv = \frac{d\sin\theta}{\sqrt{1-c^2-\cos^2\theta}}$$

积分得

$$v = \arcsin\frac{\cos\theta}{\sqrt{1-c^2}} + v_0, \quad \sin(v-v_0) = \frac{\cos\theta}{\sqrt{1-c^2}} \quad \mathbf{(2')}$$

于是

$$\cos(v-v_0) = \sqrt{1-\frac{\cos^2\theta}{1-c^2}} = \frac{\sqrt{\sin^2\theta - c^2}}{\sqrt{1-c^2}} = \frac{1}{\sqrt{1-c^2}}\sqrt{\frac{c^2}{\sin^2(\sqrt{K}u)} - c^2} = \frac{c}{\sqrt{1-c^2}}\frac{\cos(\sqrt{K}u)}{\sin(\sqrt{K}u)}$$

将最后的式子展开得到

$$\sin(\sqrt{K}u)(\cos v_0 \cos v + \sin v_0 \sin v) = \frac{c}{\sqrt{1-c^2}}\cos(\sqrt{K}u)$$

取$A = \cos v_0$, $B = \sin v_0$, $C = -\frac{c}{\sqrt{1-c^2}}$则得所要关系式. $\mathbf{(3')}$ □

4.(20分) 由曲面方程计算得

$$\mathbf{x}_1 = (\cos v, \sin v, g'), \quad \mathbf{x}_2 = (-u\sin v, u\cos v, 0)$$

从而

$$g_{11} = 1 + g'^2, \quad g_{12} = 0, \quad g_{22} = u^2 \quad \mathbf{(2')}$$

故

$$w^1 = \sqrt{1+g'^2}du,\ w^2 = udv,\ w^3 = 0. \quad \mathbf{(3')}$$

求它们的外微分得到

$$d\omega^1 = 0, \quad d\omega^2 = du \wedge dv$$

并且

$$\omega^1 \wedge \omega^2 = u\sqrt{1+g'^2}du \wedge dv$$

从而

$$\omega_1^2 = -\omega_2^1 = \frac{d\omega^1}{\omega^1\wedge\omega^2}\omega^1 + \frac{d\omega^2}{\omega^1\wedge\omega^2}\omega^2 = \frac{1}{\sqrt{1+g'^2}}dv \quad \mathbf{(3')}$$

假定

$$\begin{aligned}\omega_1^3 &= a\omega^1 + b\omega^2 = a\sqrt{1+g'^2}du + budv \\ \omega_2^3 &= b\omega^1 + c\omega^2 = b\sqrt{1+g'^2}du + cudv \mathbf{(2')}\end{aligned}$$

故

$$\begin{aligned}\mathrm{II} &= \frac{g''}{\sqrt{1+g'^2}}(du)^2 + \frac{ug'}{\sqrt{1+g'^2}}(dv)^2 = \omega^1\omega_1^3 + \omega^2\omega_2^3 \\ &= a(1+g'^2)(du)^2 + 2bu\sqrt{1+g'^2}dudv + cu^2(dv)^2 \mathbf{(4')}\end{aligned}$$

比较上式系数得到

$$a = \frac{g''}{(\sqrt{1+g'^2})^3}, \quad b = 0, \quad c = \frac{g'}{u\sqrt{1+g'^2}}$$

因此

$$\begin{aligned} \omega_1^3 &= -\omega_3^1 = \frac{g''}{1+g'^2}du \\ \omega_2^3 &= -\omega_3^2 = \frac{g'}{\sqrt{1+g'^2}}dv \quad \textbf{(6}'\textbf{)} \end{aligned}$$

□

5.(20分) 由曲面第一基本形式

$$\mathrm{I} = \left(\sqrt{F(u)+G(v)}du\right)^2 + \left(\sqrt{F(u)+G(v)}dv\right)^2$$

得

$$\omega^1 = \sqrt{F(u)+G(v)}du, \quad \omega^2 = \sqrt{F(u)+G(v)}dv \quad \textbf{(5}'\textbf{)}$$

从而

$$\omega_1^2 = \frac{d\omega^1}{\omega^1\wedge\omega^2}\omega^1 + \frac{d\omega^2}{\omega^1\wedge\omega^2}\omega^2 = -\frac{1}{2}\frac{G'}{F+G}du + \frac{1}{2}\frac{F'}{F+G}dv \quad \textbf{(5}'\textbf{)}$$

根据Gauss美妙定理

$$K = -\frac{d\omega_1^2}{\omega^1\wedge\omega^2} = -\frac{G''+F''}{2(G+F)^2} + \frac{G'^2+F'^2}{2(G+F)^3} \quad \textbf{(10}'\textbf{)}$$