Probability Theory

Exercise Sheet 10

Exercise 10.1 (The generalized Borel-Cantelli lemma)
Consider $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_n\}_{n \geq 0}$, and let $A_n \in \mathcal{F}_n$, $n \geq 1$, be a sequence of events. Show that, up to a P-nullset,

$$\limsup_{n \to \infty} A_n = \{\sum_{n \geq 1} P[A_n | \mathcal{F}_{n-1}] = \infty\}.$$

Hint: Use Exercise 9.3.

Exercise 10.2 Consider a Galton-Watson process (see p. 97 of the lecture notes) Z_n, $n \geq 0$, with offspring distribution $\nu = \mathrm{Bin}(2, p)$, $p \in [0, 1]$. We are interested in the probability $\vartheta(p) = P[Z_n > 0, \ \forall n \geq 0]$ that the population does not go extinct. Show that

$$\vartheta(p) = \begin{cases} 0 & \text{if } 0 \leq p \leq 1/2; \\ \frac{2p-1}{p^2} & \text{if } 1/2 < p \leq 1. \end{cases}$$

Hint: One way to prove this is to use the results for the various cases (subcritical, critical, supercritical) from Section 3.5 A), pp. 97-101 of the lecture notes.

Exercise 10.3 (Probabilistic solution to the discrete Dirichlet problem)
Let $A \subseteq \mathbb{Z}^d$ be finite, $f : \mathbb{Z}^d \setminus A \to \mathbb{R}$ any function, and $(S_n)_{n \in \mathbb{N}}$ a simple random walk on $\mathbb{Z}^d$ with starting point $S_0 = 0$. For $x \in \mathbb{Z}^d$ let $T_x := \inf\{n \geq 0; \mid x + S_n \notin A\}$. Finally, let $\mathcal{F}_n := \sigma(S_0, \ldots, S_n)$ and $g(x) := E[f(x + S_{T_x})]$.

(a) Show that $T_x < \infty$ P-a.s. Thus $f(x + S_{T_x})$ exists a.s.
   Hint: Use Exercise 9.3.

(b) Show that g solves the discrete Dirichlet problem on A with boundary condition f, i.e.,

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{Z}^d \setminus A \\ \dfrac{1}{2d} \displaystyle\sum_{\substack{\|y-x\|=1 \\ y \in \mathbb{Z}^d}} g(y) & \text{if } x \in A \ . \end{cases}$$

(c) Show that $E\left[f(x + S_{T_x}) | \mathcal{F}_1\right] = g(x + S_{T_x \wedge 1})$ P-a.s.

Submission: until 14:15, Dec 03., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint