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Ex. 3.12 (Morten Poulsen). It might help to think of (ii) and (iii) as rotated dictionary orders and drawing a diagram might help as well.

(i). Let $(x_0, y_0) \in \mathbf{Z}_+ \times \mathbf{Z}_+$. Immediate predecessors:

- If $y_0 > 1$ then the immediate predecessor is $(x_0, y_0 - 1)$.
- If $y_0 = 1$ then $(x_0, 1)$ has no immediate predecessor.

The smallest element is $(1, 1)$.

(ii). Let $(x_0, y_0) \in \mathbf{Z}_+ \times \mathbf{Z}_+$. Immediate predecessors:

- If $x_0 = 1$ then $(1, y_0)$ has no immediate predecessor.
- If $x_0 > 1$ and $y_0 = 1$ then the immediate predecessor is $(x_0 - 1, 1)$.
- If $x_0 > 1$ and $y_0 > 1$ then the immediate predecessor is $(x_0 - 1, y_0 - 1)$.

There are no smallest element.

(iii). Let $(x_0, y_0) \in \mathbf{Z}_+ \times \mathbf{Z}_+$. Immediate predecessors:

- If $y_0 > 1$ then the immediate predecessor is $(x_0 + 1, y_0)$.
- If $x_0 > 1$ and $y_0 = 1$ then the immediate predecessor is $(x_0 - 1, y_0)$.
- The element $(1, 1)$ has no immediate predecessor.

The smallest element is $(1, 1)$.

Since (i) has a smallest element, but (ii) hasn't a smallest element, it follows that the order types of (i) and (ii) are different. Similarly are the order types of (ii) and (iii) different. Since (i) has more than one element (actually countably infinite many elements) without an immediate predecessor and (iii) has only one element without an immediate predecessor, it follows that they have different order types.

Ex. 3.13 (Morten Poulsen).

Theorem 1. *If an ordered set A has the least upper bound property, then it has the greatest lower bound property.*

Proof. Assume $A_0 \subset A$ is nonempty and has a lower bound $b \in A$. Let

$$B_0 = \{ a \in A \mid \forall a_0 \in A_0 : a \leq a_0 \},$$

i.e. B_0 is the set of all lower bounds for A_0. [Want to show that B_0 has a largest element].

Now $B_0 \subset A$ is nonempty, since $b \in B_0$, and has an upper bound, e.g. every element in A_0. Since A has the least upper bound property the set C_0 of all upper bounds for B_0, i.e.

$$C_0 = \{ a \in A \mid \forall b_0 \in B_0 : b_0 \leq a \},$$

has a smallest element $c \in C_0$. Since $A_0 \subset C_0$ it follows that c is a lower bound for A_0, hence $c \in B_0$. It follows that c is the largest element in B_0, this means by definition that A_0 has a greatest lower bound, hence A has the greatest lower bound property. □

Ex. 3.15 (Morten Poulsen). Assume that $\mathbf{R}$ has the least upper bound property.

(a). By a argument similar to the one in example 13, it follows that the sets $[0, 1]$ and $[0, 1)$ have the least upper bound property.

(b). The set $X = [0,1] \times [0,1]$ in the dictionary order has the least upper bound property: Suppose $A \subset X$ is nonempty and has an upper bound. Since $[0,1]$ has the least upper bound property the nonempty set

$$X_0 = \{ x \in [0,1] \mid \exists\, y \in [0,1] : (x,y) \in A \} \subset [0,1]$$

has a least upper bound x_0. Let

$$Y_0 = \{ y \in [0,1] \mid (x_0, y) \in A \} \subset [0,1].$$

If Y_0 is empty then $(x_0, 0)$ is clearly the least upper bound of A. If Y_0 is nonempty then Y_0 is bounded above by 1, hence Y_0 has a least upper bound y_0. It follows, by construction, that (x_0, y_0) is the least upper bound of A. Thus X has the least upper bound property.

The set $Y = [0,1] \times [0,1)$ in the dictionary order has not the least upper bound property: Let B be the set $[0, \frac{1}{2}] \times [0,1)$, B is clearly bounded above. But the set of upper bounds for B has no smallest element, since no element of the form $(\frac{1}{2}, y)$, $y \in [0,1)$, is an upper bound for B and given $\varepsilon > 0$ then $(\frac{1+\varepsilon}{2}, 0) < (\frac{1}{2} + \varepsilon, 0)$. Thus Y hasn't the least upper bound property.

The set $Z = [0,1) \times [0,1]$ has the least upper bound property by a argument similar to the one for X.

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Ex. 4.2. We assume that there exists a set $\mathbf{R}$ equipped with two binary operations, $+$ and $\cdot$, and a linear order $<$ such that

(1) $(\mathbf{R}, +, \cdot)$ is a field.
(2) $x < y \Rightarrow x + z < y + z$ and $0 < x, 0 < y \Rightarrow 0 < xy$
(3) $(\mathbf{R}, <)$ is a linear continuum

Using these axioms we can establish all the usual rules of artihmetic.

(c): $\Rightarrow$: Assume that $x > 0$. Adding $-x$ to this gives $0 > -x$. .
$\Leftarrow$: Assume that $-x < 0$. Adding x to this gives $0 < x$.

(g): Since $0 \neq 1$ in a field, we have either $0 < 1$ or $1 < 0$ by Comparability. We rule out the latter possibility. If $1 < 0$, then $-1 > 0$ so also $1 = (-1) \cdot (-1) > 0$, a contradiction. Thus we have $0 < 1$ and then also $-1 < 0$ by point (c).

Ex. 4.3 (Morten Poulsen).

(a). Let $\mathcal{A}$ be a collection of inductive sets. Since $1 \in A$ for all $A \in \mathcal{A}$, it follows that $1 \in \cap_{A \in \mathcal{A}} A$. Let $a \in \cap_{A \in \mathcal{A}} A$. Since A is inductive for all $A \in \mathcal{A}$, it follows that $a + 1 \in A$ for all $A \in \mathcal{A}$, hence $a + 1 \in \cap_{A \in \mathcal{A}} A$. So $\cap_{A \in \mathcal{A}} A$ is inductive.

(b). By definition $\mathbf{Z}_+ = \cap_{A \in \mathcal{A}} A$, where $\mathcal{A}$ is the collection of all inductive subsets of the real numbers.

Proof of (1): The set $\mathbf{Z}_+$ is inductive by (a).

Proof of (2): Suppose $A \subset \mathbf{Z}_+$ is inductive. Since A inductive, it follows by the definition of $\mathbf{Z}_+$ that $\mathbf{Z}_+ \subset A$, i.e. $A = \mathbf{Z}_+$.

Ex. 4.4 (Morten Poulsen).

(a). Let A be the set of $n \in \mathbf{Z}_+$ for which the statement holds.

The set A is inductive: It is clear that $1 \in A$, since the only nonempty subset of $\{1\}$ is $\{1\}$. Suppose $n \in A$. Let B be a nonempty subset of $\{1, \ldots, n+1\}$. If $n + 1 \in B$ then $n + 1$ is the largest element in B. If $n + 1 \notin B$ then the set $B \cap \{1, \ldots, n\}$ contains a largest element, since $n \in A$.

So $A \subset \mathbf{Z}_+$ is inductive, by the principle of induction, it follows that $A = \mathbf{Z}_+$, as desired.

(b). Consider!

Ex. 4.5 (Morten Poulsen).

(a). Let $a \in \mathbf{Z}_+$. Let

$$X = \{ x \in \mathbf{R} \,|\, a + x \in \mathbf{Z}_+ \} .$$

The set X is inductive: $1 \in X$, since $a \in \mathbf{Z}_+$ and $\mathbf{Z}_+$ inductive. Suppose $x \in X$. Since $a + (x + 1) = (a + x) + 1$, $a + x \in \mathbf{Z}_+$ and $\mathbf{Z}_+$ inductive, it follows that $x + 1 \in X$.

By ex. 4.3(a) it follows that $X \cap \mathbf{Z}_+ \subset \mathbf{Z}_+$ is inductive. By the principle of induction, it follows that $X \cap \mathbf{Z}_+ = \mathbf{Z}_+$, which proves (a).

(b). Let $a \in \mathbf{Z}_+$. Let

$$X = \{ x \in \mathbf{R} \,|\, ax \in \mathbf{Z}_+ \} .$$

The set X is inductive: $1 \in X$, since $a1 = a \in \mathbf{Z}_+$. Suppose $x \in X$. Since $a(x + 1) = ax + a$ and $ax, a \in \mathbf{Z}_+$, it follows by (a) that $x + 1 \in X$.

As above, it follows that $X \cap \mathbf{Z}_+ = \mathbf{Z}_+$, which proves (b).

(c). Let

$$X = \{ x \in \mathbf{R} \,|\, x - 1 \in \mathbf{Z}_+ \cup \{0\} \} .$$

The set X is inductive: $1 \in X$, since $1 - 1 = 0 \in \mathbf{Z}_+ \cup \{0\}$. Suppose $x \in X$. Note that $(x + 1) - 1 = (x - 1) + 1$. If $x - 1 = 0$ then $(x - 1) + 1 = 1 \in \mathbf{Z}_+ \cup \{0\}$. If $x - 1 \in \mathbf{Z}_+$ then, since $\mathbf{Z}_+$ is inductive, $(x - 1) + 1 \in Z_+ \subset \mathbf{Z}_+ \cup \{0\}$. So $x + 1 \in X$.

As above, it follows that $X \cap \mathbf{Z}_+ = \mathbf{Z}_+$, which proves (c).

(d). Let $c \in \mathbf{Z} = \mathbf{Z}_- \cup \{0\} \cup \mathbf{Z}_+$, where $\mathbf{Z}_-$ is negatives of the elements of $\mathbf{Z}_+$. First we prove the result for $d = 1$:

(i) $c + 1 \in \mathbf{Z}$: If $c \in \mathbf{Z}_+$ the result follows from (a). It is clear if $c = 0$. If $c \in \mathbf{Z}_-$ then $c + 1 = -(-c - 1)$, since $-c \in \mathbf{Z}_+$, it follows from (c) that $-c - 1 \in \mathbf{Z}_+ \cup \{0\}$, hence $c + 1 \in \mathbf{Z}$.

(ii) $c - 1 \in \mathbf{Z}$: If $c \in \mathbf{Z}_+$ the result follows from (c). It is clear if $c = 0$. If $c \in \mathbf{Z}_-$ then $c - 1 = -(-c + 1)$, since $-c \in \mathbf{Z}_+$, it follows from (a) or by the inductivity of $\mathbf{Z}_+$ that $-c + 1 \in \mathbf{Z}_+$, hence $c - 1 \in \mathbf{Z}$.

Next we prove the result for $d \in \mathbf{Z}_+$: Let

$$X = \{ x \in \mathbf{R} \,|\, c + x \in \mathbf{Z} \}$$

and

$$Y = \{ y \in \mathbf{R} \,|\, c - y \in \mathbf{Z} \} .$$

The set X is inductive: $1 \in X$, c.f. (i). Suppose $x \in X$. Since $c+(x+1) = (c+x)+1$, $c+x \in \mathbf{Z}$ and (i), it follows that $x + 1 \in X$.

The set Y is inductive: $1 \in X$, c.f. (ii). Suppose $y \in Y$. Since $c-(y+1) = (c-y)-1$, $c-y \in \mathbf{Z}$ and (ii), it follows that $y + 1 \in X$.

As above, it follows that $X \cap \mathbf{Z}_+ = \mathbf{Z}_+$ and $Y \cap \mathbf{Z}_+ = \mathbf{Z}_+$. This proves the result for $d \in \mathbf{Z}_+$. The result is clear if $d = 0$. The case $d \in \mathbf{Z}_-$ is now easy: Since $c + d = c - (-d)$ and $-d \in \mathbf{Z}_+$, it follows that $c + d \in \mathbf{Z}$. Since $c - d = c + (-d)$ and $-d \in \mathbf{Z}_+$, it follows that $c - d \in \mathbf{Z}$.

(e). Let $c \in \mathbf{Z}$. Let

$$X = \{ x \in \mathbf{R} \,|\, cx \in \mathbf{Z} \} .$$

The set X is inductive: $1 \in X$, since $c1 = c \in \mathbf{Z}$. Suppose $x \in X$. Since $c(x + 1) = cx + c$ and $cx \in \mathbf{Z}$, it follows, by (d), that $x + 1 \in X$.

As above, it follows that $X \cap \mathbf{Z}_+ = \mathbf{Z}_+$. Thus the result is proved for $d \in \mathbf{Z}_+$ and is clear if $d = 0$. Since $cd = (-c)(-d)$, $-c \in \mathbf{Z}$, hence if $d \in \mathbf{Z}_-$ then $-d \in \mathbf{Z}_+$, this proves the case $d \in \mathbf{Z}_-$.

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Ex. 6.4 (Morten Poulsen). See the proof of Theorem 10.1.

References

Munkres §10

Ex. 10.1. If a subset of a well-ordered set has an upper bound, the smallest upper bound is a least upper bound (supremum) for the set. (This proof is a tautology!)

Ex. 10.2.

(a). The smallest successor x_+ of any element x is the immediate successor. (The iterated successors of x has the order type of a section of $\mathbf{Z}_+$.)

(b). $\mathbf{Z}$.

Ex. 10.4.

(a). Let A be a simply ordered set containing a subset with the order type of $\mathbf{Z}_-$. Then this subset does not have a smallest element so A is not well-ordered. Conversely, let A be simply ordered set containing a nonempty subset B with no smallest element. Let b_1 be any element of B. Since b_1 is not a smallest element of B there is some element b_2 of B such that $b_2 < b_1$. Continuing inductively we obtain an infinite descending chain $\cdots < b_{n+1} < b_n < \cdots < b_2 < b_1$ forming a subset of the same order type as $\mathbf{Z}_-$.

(b). A does not contain a subset with the order type of $\mathbf{Z}_-$.

Ex. 10.6.

(a). For any element α of S_Ω, the set $\{x \in S_\Omega \mid x \leq \alpha\} = S_\alpha \cup \{\alpha\}$ is countable but S_Ω itself is uncountable [Lemma 10.2].

(b). For any element $\alpha \in S_\Omega$, the set $S_\alpha \cup \{\alpha\}$ is countable so its complement, $\{x \in S_\Omega \mid x > \alpha\} = (\alpha, +\infty)$, in the uncountable set S_Ω, is uncountable [Lemma 10.2, Thm 7.5].

(c). We show the stronger statement [Thm 10.3] that X_0 is not bounded from above. We do this by assuming that X_0 has an upper bound α and find a contradiction. The (non-empty) simply ordered set $(\alpha, +\infty)$ is well-ordered [p. 63], it has no largest element by (a), and each element of $(\alpha, +\infty)$, except the smallest element, has an immediate predecessor. Thus $(\alpha, +\infty)$ has the order type of $\mathbf{Z}_+$, in particular $(\alpha, +\infty)$ is countable, contradicting (b). (Let x be any element of $(\alpha, +\infty)$. Since $(\alpha, +\infty)$ does not contain an infinite descending chain [Ex 10.4], α is an iterated immediate predecessor of x and x is an iterated immediate successor of α.)

Ex. 10.7. We show the contrapositive. Let J_0 be any subset of J that is not everything. Let α be the smallest element of the complement $J - J_0$, the smallest element outside J_0. This means that $\alpha \notin J_0$ and that any element smaller than α is in J_0, i.e. $S_\alpha \subset J_0$. Thus J_0 is not inductive.

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Ex. 11.8 (Morten Poulsen). First recall some definitions: Let V be a vector space over a field K. Let A be a (possibly empty) subset of V. The subspace spanned by A is denoted $\operatorname{span}_K A$ and is defined by

$$\operatorname{span}_K A = \{\, k_1a_1 + \cdots + k_na_n \,|\, n \in \mathbf{Z}_+ \cup \{0\},\ k_1, \ldots, k_n \in K,\ a_1, \ldots, a_n \in A\,\}$$

i.e. the set of all finite linear combinations of elements from W. If $n = 0$ then the linear combination is defined to be the zero element in V. If A is the empty set then $\operatorname{span}_K A$ is the subspace $\{0\}$.

The subset A of V is said to be (linearly) independent if $n \in \mathbf{Z}_+ \cup \{0\}, k_1, \ldots, k_n \in K$ and $a_1, \ldots, a_n \in A$ satisfy

$$k_1a_1 + \cdots + k_na_n = 0$$

then

$$k_1 = 0, \ldots, k_n = 0.$$

The empty set is (by definition) independent.

The subset A is said to be a basis if A is independent and $\operatorname{span}_K A = V$. Note that if A is a basis for V then every element has an unique representation as a finite linear combination of elements of A.

If $V = \{0\}$ then the empty set is a basis, thus assume in the following that $V \neq \{0\}$.

(a). Assume $v \notin \operatorname{span}_K A$. Suppose

$$k_1a_1 + \cdots + k_na_n + k_{n+1}v = 0,$$

where $n \in \mathbf{Z}_+ \cup \{0\}$, $k_1, \ldots, k_{n+1} \in K$ and $a_1, \ldots, a_n \in A$.

If $k_{n+1} = 0$ then $k_1 = 0, \ldots, k_n = 0$, since A is independent. If $k_{n+1} \neq 0$ then $v = \frac{1}{k_{n+1}}(k_1a_1 + \cdots + k_na_n)$, contradicting $v \notin \operatorname{span}_K A$. It follows that $A \cup \{v\}$ is independent.

(b). Let W be the set of independent subsets of V. Define a relation $\prec$ on W by

$$\forall\, w_1, w_2 \in W : w_1 \prec w_2 \Leftrightarrow w_1 \subsetneq w_2.$$

The relation $\prec$ is clearly a strict partial order on W.

Let W_0 be a simply ordered subset of W then

$$U = \bigcup_{w_0 \in W_0} w_0$$

is an independent subset of V: Suppose

$$k_1u_1 + \cdots + k_nu_n = 0,$$

where $n \in \mathbf{Z}_+ \cup \{0\}$, $k_1, \ldots, k_n \in K$ and $u_1, \ldots, u_n \in U$. Then there are $w_1, \ldots, w_n \in W_0$, such that $u_i \in w_i$ for $1 \le i \le n$. Since W_0 is simply ordered the subset $\{w_1, \ldots, w_n\} \subset W_0$ has a largest element w, hence $u_1, \ldots, u_n \in w$. Since w is independent it follows that $k_1 = 0, \ldots, k_n = 0$, hence U is independent, i.e. $U \in W$.

The set U is clearly an upper bound for W_0, hence every simply ordered subset of W has an upper bound in W. Zorn's Lemma gives that W has a maximal element B.

(c). The maximal element B in W is basis for V: Suppose $\operatorname{span}_K B$ is a proper subspace of V. Thus there is an element $b \in V - \operatorname{span}_K B$, hence, by (a), $B \cup \{b\} \in W$, contradicting the maximality of B. It follows that B is a basis for V.

Thus we have proved:

Theorem 1. *Every vector space has a basis.*

A few remarks: We know that if a vector space V over the field K has a basis with $n \in \mathbf{Z}_+$ elements then every basis for V has n elements and n is called the dimension of V over K, and is denoted $\dim_K V = n$. The notion of dimension extends to vector spaces with infinite bases. First one proves the following theorem.

Theorem 2. *If A and B are bases for V then A and B have the same cardinality.*

In view of the previous theorem we define the dimension of a vector space V to be the cardinality of some basis A for V, i.e. $\dim_K V = \operatorname{card} A$.

Hamel bases. Regard the real numbers as a vector space over the rationals. Then this vector space has a basis, any basis for this vector space is called a Hamel basis. Furthermore one shows that $\dim_{\mathbf{Q}} \mathbf{R} = \operatorname{card} \mathbf{R}$.

References

Munkres §13

Ex. 13.1 (Morten Poulsen). Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The following are equivalent:

(i) $A \in \mathcal{T}$.
(ii) $\forall x \in A \, \exists U_x \in \mathcal{T} : x \in U_x \subset A$.

Proof. (i) $\Rightarrow$ (ii): If $x \in A$ then $x \in A \subset A$ and $A \in \mathcal{T}$.
(ii) $\Rightarrow$ (i): $A = \bigcup_{x \in A} U_x$, hence $A \in \mathcal{T}$. $\square$

Ex. 13.4 (Morten Poulsen). Note that every collection of topologies on a set X is itself a set: A topologi is a subset of $\mathcal{P}(X)$, i.e. an element of $\mathcal{P}(\mathcal{P}(X))$, hence a collection of topologies is a subset of $\mathcal{P}(\mathcal{P}(X))$, i.e. a set.

Let $\{\mathcal{T}_\alpha\}$ be a nonempty set of topologies on the set X.

(a). Since every $\mathcal{T}_\alpha$ is a topology on X it is clear that the intersection $\bigcap \mathcal{T}_\alpha$ is a topology on X.

The union $\bigcup \mathcal{T}_\alpha$ is in general not a topology on X: Let $X = \{a, b, c\}$. It is straightforward to check that $\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{c\}, \{b, c\}\}$ are topologies on X. But $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology on X, since $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$.

(b). The intersection of all topologies that are finer than all $\mathcal{T}_\alpha$ is clearly the smallest topology containing all $\mathcal{T}_\alpha$.

The intersection of all $\mathcal{T}_\alpha$ is clearly the largest topology that is contained in all $\mathcal{T}_\alpha$.

(c). The topology $\mathcal{T}_3 = \mathcal{T}_1 \cap \mathcal{T}_2 = \{X, \emptyset, \{a\}\}$ is the largest topology on X contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

The topology $\mathcal{T}_4 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ is the smallest topology that contains $\mathcal{T}_1$ and $\mathcal{T}_2$.

Ex. 13.5 (Morten Poulsen). Let $(X, \mathcal{T})$ be a topological space, $\mathcal{A}$ basis for $\mathcal{T}$ and let $\{\mathcal{T}_\alpha\}$ be the set of topologies on X that contains $\mathcal{A}$.

Claim 1. $\mathcal{T} = \bigcap \mathcal{T}_\alpha$.

Proof. "$\subset$": Let $U \in \mathcal{T}$. By lemma 13.1, U is an union of elements of $\mathcal{A}$. Since $\mathcal{T}_\alpha$ is a topology for all α, it follows that $U \in \mathcal{T}_\alpha$ for all α, i.e. $U \in \bigcap \mathcal{T}_\alpha$.

"$\supset$": Clear since $\mathcal{A} \subset \bigcap \mathcal{T}_\alpha \subset \mathcal{T}$. $\square$

Now assume $\mathcal{A}$ is a subbasis.

Claim 2. $\mathcal{T} = \bigcap \mathcal{T}_\alpha$.

Proof. "$\subset$": Let $U \in \mathcal{T}$. By the definition of a subbasis and the remarks at the bottom on page 82, U is an union of finite intersections of elements of $\mathcal{A}$. Since $\mathcal{T}_\alpha$ is a topology for all α, it follows that $U \in \mathcal{T}_\alpha$ for all α, i.e. $U \in \bigcap \mathcal{T}_\alpha$.

"$\supset$": Clear since $\mathcal{A} \subset \bigcap \mathcal{T}_\alpha \subset \mathcal{T}$. $\square$

Ex. 13.6 (Morten Poulsen). The topologies $\mathbf{R}_l$ and $\mathbf{R}_K$ on $\mathbf{R}$ are not comparable:

$\mathbf{R}_l \not\subset \mathbf{R}_K$: Consider $[-1, 0) \in \mathbf{R}_l$. Clearly no basis element $B_K \in \mathbf{R}_K$ satisfy $-1 \in B_K \subset [-1, 0)$, hence $\mathbf{R}_K$ is not finer than $\mathbf{R}_l$, by lemma 13.3.

$\mathbf{R}_K \not\subset \mathbf{R}_l$: Consider $(-1, 1) - K \in \mathbf{R}_K$. Clearly no basis element $B_l \in \mathbf{R}_l$ satisfy $0 \in B_l \subset (-1, 1) - K$, hence $\mathbf{R}_l$ is not finer than $\mathbf{R}_K$, by lemma 13.3.

Ex. 13.7 (Morten Poulsen). We know that $\mathcal{T}_1$ and $\mathcal{T}_2$ are bases for topologies on $\mathbf{R}$. Furthermore $\mathcal{T}_3$ is a topology on $\mathbf{R}$. It is straightforward to check that the last two sets are bases for topologies on $\mathbf{R}$ as well.

The following table show the relationship between the given topologies on $\mathbf{R}$.

	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$
$\mathcal{T}_1$	$=$	$\subset$ (1)	$\not\subset$ (2)	$\subset$ (3)	$\not\subset$ (4)
$\mathcal{T}_2$	$\not\subset$ (5)	$=$	$\not\subset$ (6)	$\subset$ (7)	$\not\subset$ (8)
$\mathcal{T}_3$	$\subset$ (9)	$\subset$ (10)	$=$	$\subset$ (11)	$\not\subset$ (12)
$\mathcal{T}_4$	$\not\subset$ (13)	$\not\subset$ (14)	$\not\subset$ (15)	$=$	$\not\subset$ (16)
$\mathcal{T}_5$	$\subset$ (17)	$\subset$ (18)	$\not\subset$ (19)	$\subset$ (20)	$=$

(1) Lemma 13.3.
(2) Since $\mathbf{R}-(0,1)$ not finite.
(3) Given basis element $(a,b)\in\mathcal{T}_1$ and $x\in(a,b)$ then the basis element $(a,x]\in\mathcal{T}_4$ satisfy $x\in(a,x]\subset(a,b)$, hence $\mathcal{T}_4$ is finer than $\mathcal{T}_1$, by lemma 13.3.
(4) Given a basis element $(a,b)\in\mathcal{T}_1$ and $x\in(a,b)$ then there are clearly no basis element $(-\infty,c)\in\mathcal{T}_5$ such that $x\in(-\infty,c)\subset(a,b)$, hence $\mathcal{T}_5$ is not finer than $\mathcal{T}_1$, by lemma 13.3.
(5) Lemma 13.3.
(6) Since $\mathbf{R}-(0,1)$ not finite.
(7) Given basis element $(a,b)-K\in\mathcal{T}_2$ and $x\in(a,b)-K$. If $x\in(0,1)$ then there exists $m\in\mathbf{Z}_+$ such that $\frac{1}{m}<x<\frac{1}{m-1}$, hence $x\in(\frac{1}{m},x]\subset(a,b)-K$. If $x\notin(0,1)$ then $x\in(a,x]\subset(a,b)-K$. It follows from (4) and lemma 13.3 that $\mathcal{T}_4$ is finer than $\mathcal{T}_2$.
(8) Since $\mathcal{T}_1\not\subset\mathcal{T}_5$ and $\mathcal{T}_1\subset\mathcal{T}_2$.
(9) Let $U\in\mathcal{T}_3$, U nonempty, i.e. $\mathbf{R}-U=\{r_1,\dots,r_n\}$, $r_1<\dots<r_n$. Since

$$U=\left(\bigcup_{i=1}^{\infty}(r_1-i,r_1)\right)\cup\left(\bigcup_{j=1}^{n-1}(r_j,r_{j+1})\right)\cup\left(\bigcup_{k=1}^{\infty}(r_n,r_n+k)\right)$$

it follows that $U\in\mathcal{T}_1$.
(10) Since $\mathcal{T}_3\subset\mathcal{T}_1\subset\mathcal{T}_2$.
(11) Let $U\in\mathcal{T}_3$, U nonempty, i.e. $\mathbf{R}-U=\{r_1,\dots,r_n\}$, $r_1<\dots<r_n$, and let $x\in U$. If $d=\min\{\,|x-r_i|\mid i\in\{1,\dots,n\}\,\}>0$ then $x\in(x-\frac{d}{2},x+\frac{d}{2}]\subset U$. It follows from lemma 13.3 that $\mathcal{T}_4$ is finer than $\mathcal{T}_3$.
(12) Consider $U=\mathbf{R}-\{0\}\in\mathcal{T}_3$. There are no basis element $(-\infty,a)\in\mathcal{T}_5$ such that $1\in(-\infty,a)\subset U$, hence $\mathcal{T}_5$ is not finer than $\mathcal{T}_4$, by lemma 13.3.
(13) Given basis element $(c,x]\in\mathcal{T}_4$ there is clearly no basis element $(a,b)\in\mathcal{T}_1$ such that $x\in(a,b)\subset(c,x]$, hence $\mathcal{T}_1$ is not finer than $\mathcal{T}_4$, by lemma 13.3.
(14) Given basis element $(c,x]\in\mathcal{T}_4$ there is clearly no basis element $B_K\in\mathcal{T}_2$ such that $x\in B_K\subset(c,x]$, hence $\mathcal{T}_2$ is not finer than $\mathcal{T}_4$, by lemma 13.3.
(15) Since $\mathbf{R}-(0,1]$ not finite.
(16) Given basis element $(c,x]\in\mathcal{T}_4$ there is clearly no basis element $(-\infty,a)\in\mathcal{T}_5$ such that $x\in(-\infty,a)\subset(c,x]$, hence $\mathcal{T}_5$ is not finer than $\mathcal{T}_4$, by lemma 13.3.
(17) Since $(-\infty,a)=\bigcup_{i=1}^{\infty}(a-i,a)\in\mathcal{T}_1$ for all $a\in\mathbf{R}$.
(18) Since $\mathcal{T}_5\subset\mathcal{T}_1\subset\mathcal{T}_2$.
(19) Since $\mathbf{R}-(-\infty,0)$ not finite.
(20) Given basis element $(-\infty,a)\in\mathcal{T}_5$ and $x\in(-\infty,a)$ then clearly $x\in(x-|x-a|,x+\frac{|x-a|}{2}]\subset(-\infty,a)$, hence $\mathcal{T}_4$ is finer than $\mathcal{T}_5$, by lemma 13.3.

Ex. 13.8 (Morten Poulsen).

(a). Let

$$\mathcal{B}=\{\,(a,b)\mid a,b\in\mathbf{Q},a<b\,\}.$$

It is straightforward to check that $\mathcal{B}$ is a basis. Let $\mathcal{T}$ be the standard topology on $\mathbf{R}$ generated by the basis:

$$\{\,(r,s)\,|\,r,s\in\mathbf{R}\,\}.$$

Let $U\in\mathcal{T}$ and let $x\in U$. Then (by definition of an open set in a topology generated by a basis) there exists a basis element (r,s), $r,s\in\mathbf{R}$, such that $x\in(r,s)$. Furthermore there exists $a,b\in\mathbf{Q}$ such that $r\le a<x<b\le s$, hence $x\in(a,b)\subset(r,s)$. It follows, by lemma 13.2, that $\mathcal{B}$ is a basis for $\mathcal{T}$.

(b). Let

$$\mathcal{C}=\{\,[a,b)\,|\,a,b\in\mathbf{Q},a<b\,\}.$$

It is straightforward to check that $\mathcal{C}$ is a basis. Let $\mathcal{T}_{\mathcal{C}}$ be the topology on $\mathbf{R}$ generated by $\mathcal{C}$.

Consider $[\sqrt{2},2)\in\mathbf{R}_l$. There are clearly no basis element $[a,b)\in\mathcal{C}$ such that $\sqrt{2}\in[a,b)\subset[\sqrt{2},2)$, hence $\mathcal{T}_{\mathcal{C}}$ is not finer than $\mathbf{R}_l$, by lemma 13.3.

Since $\mathbf{R}_l$ is clearly finer than $\mathcal{T}_{\mathcal{C}}$, it follows that $\mathbf{R}_l$ is strictly finer than $\mathcal{T}_{\mathcal{C}}$.

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Ex. 16.1 (Morten Poulsen). Let $(X, \mathcal{T})$ be a topological space, $(Y, \mathcal{T}_Y)$ be a subspace and let $A \subset Y$.

Let $\mathcal{T}_A^Y$ be the subspace topology on A as a subset of Y and let $\mathcal{T}_A^X$ be the subspace topology on A as a subset of X. Since

$$\begin{aligned} U \in \mathcal{T}_A^Y &\Leftrightarrow \exists U_Y \in \mathcal{T}_Y : U = A \cap U_Y \\ &\Leftrightarrow \exists U_X \in \mathcal{T} : U = A \cap (Y \cap U_X) \\ &\Leftrightarrow \exists U_X \in \mathcal{T} : U = A \cap U_X \\ &\Leftrightarrow U \in \mathcal{T}_A^X \end{aligned}$$

it follows that $\mathcal{T}_A^Y = \mathcal{T}_A^X$.

Ex. 16.3 (Morten Poulsen). Consider $Y = [-1, 1]$ as a subspace of $\mathbf{R}$ with the standard topology. By lemma 16.1 a basis for the subspace topology on Y is sets of the form:

$$Y \cap (a, b) = \begin{cases} (a, b), & a, b \in Y \\ [-1, b), & a \notin Y, b \in Y \\ (a, 1], & a \in Y, b \notin Y \\ Y, \emptyset, & a, b \notin Y. \end{cases}$$

Note that intervals of the form $[a, b)$ are not open in $\mathbf{R}$, since there are no basis element (c, d) such that $a \in (c, d) \subset [a, b)$. Similarly are intervals of the form $(a, b]$ and $[a, b]$ not open in $\mathbf{R}$.

$A = \{\, x \mid \frac{1}{2} < |x| < 1 \,\}$: $A = (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$, hence A open in $\mathbf{R}$. Since $A = Y \cap A$ it follows that A open in Y.

$B = \{\, x \mid \frac{1}{2} < |x| \leq 1 \,\}$: Since $B = Y \cap (-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)$ it follows that B open in Y. Another argument is that $B = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$, i.e. an union of basis elements, hence open in Y. The set B is not open in $\mathbf{R}$, since if B is open in $\mathbf{R}$ then $B \cap (0, 2) = (\frac{1}{2}, 1]$ open in $\mathbf{R}$, contradicting that $(\frac{1}{2}, 1]$ not open in $\mathbf{R}$.

$C = \{\, x \mid \frac{1}{2} \leq |x| < 1 \,\}$: Since there clearly is no basis element U for the subspace topology on Y such that $\frac{1}{2} \in U \subset C$, it follows that C is not open in Y. By an argument similar to the one for the set B, it follows that C not open in $\mathbf{R}$.

$D = \{\, x \mid \frac{1}{2} \leq |x| \leq 1 \,\}$: By arguments similar to the ones above it is easily seen that D is not open in either Y or $\mathbf{R}$.

$E = \{\, x \mid 0 < |x| < 1, \frac{1}{x} \notin \mathbf{Z}_+ \,\}$: Note that $E = (-1, 0) \cup ((0, 1) - K)$, where $K = \{\frac{1}{n} \mid n \in \mathbf{Z}_+ \}$. Since $E = (-1, 0) \cup \bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})$ it follows that E is open in $\mathbf{R}$ and Y.

Ex. 16.4. Let $\pi_k \colon \prod_{j \in J} X_j \to X_k$ be the projection map onto X_k. Observe that π_k maps any basis set, $\prod V_j$ where $V_j \subset X_j$ is open and $V_j = X_j$ for all but finitely many $j \in J$, to an open set in X_k, $\pi_k(\prod V_j) = V_k$. Since maps preserve unions of sets [Ex 2.2], it follows that π_k maps open sets in the product $\prod X_j$ to open sets in X_k.

Ex. 16.6 (Morten Poulsen). The set

$$\mathcal{B} = \{\, (a, b) \times (c, d) \mid a < b, c < d \text{ and } a, b, c, d \in \mathbf{Q} \,\}$$

is a basis for $\mathbf{R}^2$: The set

$$\{\, (r, s) \mid r < s \text{ and } r, s \in \mathbf{Q} \,\}$$

is a basis for $\mathbf{R}$, by Ex. 13.8(a). From Theorem 15.1 it follows that $\mathcal{B}$ is a basis for $\mathbf{R}^2$.

Ex. 16.7. $\mathbf{R}_+ \times \mathbf{R}$ is a convex subset of the linearly ordered set $\mathbf{R} \times \mathbf{R}$ that is not an interval nor a ray.

Ex. 16.9 (Morten Poulsen). Let $\mathbf{R}^2_{dict}$ be $\mathbf{R}^2$ with the dictionary order topology and let $\mathbf{R}_d \times \mathbf{R}$ be the product topology, where $\mathbf{R}_d$ is $\mathbf{R}$ with the discrete topology and $\mathbf{R}$ is $\mathbf{R}$ with the standard topology.

The set
$$\{\, (a \times b, c \times d) \mid a, b \in \mathbf{R} \text{ and } (a < c) \vee (a = c \wedge b < d) \,\}$$
is a basis for $\mathbf{R}^2_{dict}$.

The set
$$\{\, \{a\} \mid a \in \mathbf{R} \,\}$$
is a basis for $\mathbf{R}_d$, c.f. §13 Example 3.

The set
$$\{\, (a, b) \mid a, b \in \mathbf{R} \text{ and } a < b \,\}$$
is a basis for $\mathbf{R}$.

By Theorem 15.1 the set
$$\{\, \{a\} \times (b, c) \mid a, b, c \in \mathbf{R} \text{ and } b < c \,\}$$
is a basis for $\mathbf{R}_d \times \mathbf{R}$.

Claim 1. $\mathbf{R}^2_{dict} = \mathbf{R}_d \times \mathbf{R}$.

Proof. "$\subset$": Given basis element $(a \times b, c \times d) \in \mathbf{R}^2_{dict}$ and $x \times y \in (a \times b, c \times d)$ then $x \times y \in \{x\} \times I \subset (a \times b, c \times d)$, where I is an open interval in $\mathbf{R}$ containing y. By Lemma 13.3 it follows that $\mathbf{R}_d \times \mathbf{R}$ is finer than $\mathbf{R}^2_{dict}$.

"$\supset$": Clear, since every basis element $\{a\} \times (b, c) \in \mathbf{R}_d \times \mathbf{R}$ is a basis element in $\mathbf{R}^2_{dict}$. □

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Ex. 17.3. $A \times B$ is closed because its complement

$$(X \times Y) - (A \times B) = (X - A) \times Y \cup X \times (Y - B)$$

is open in the product topology.

Ex. 17.6.

(a). If $A \subset B$, then all limit points of A are also limit points of B, so [Thm 17.6] $\overline{A} \subset \overline{B}$.

(b). Since $A \cup B \subset \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B}$ is closed [Thm 17.1], we have $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ by (a). Conversely, since $A \subset A \cup B \subset \overline{A \cup B}$, we have $\overline{A} \subset \overline{A \cup B}$ by (a) again. Similarly, $\overline{B} \subset \overline{A \cup B}$. Therefore $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. This shows that closure commutes with *finite* unions.

(c). Since $\bigcup A_\alpha \supset A_\alpha$ we have $\overline{\bigcup A_\alpha} \supset \overline{A_\alpha}$ by (a) for all α and therefore $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$. In general we do not have equality as the example $A_q = \{q\}$, $q \in \mathbf{Q}$, in $\mathbf{R}$ shows.

Ex. 17.8.

(a). By [Ex 17.6.(a)], $\overline{A \cap B} \subset \overline{A}$ and $\overline{A \cap B} \subset \overline{B}$, so $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. It is *not* true in general that $\overline{A \cap B} = \overline{A} \cap \overline{B}$ as the example $A = [0, 1)$, $B = [1, 2]$ in $\mathbf{R}$ shows. (However, if A is open and D is dense then $\overline{A \cap D} = \overline{A}$).

(b). Since $\bigcap A_\alpha \subset A_\alpha$ we have $\overline{\bigcap A_\alpha} \subset \overline{A_\alpha}$ for all α and therefore $\overline{\bigcap A_\alpha} \subset \bigcap \overline{A_\alpha}$. (In fact, (a) is a special case of (b)).

(c). Let $x \in \overline{A} - \overline{B}$. For any neighborhood of x, $U - \overline{B}$ is also a neighborhood of x so

$$U \cap (A - B) = (U - B) \cap A \supset (U - \overline{B}) \cap A \neq \emptyset$$

since x is in the closure of A [Thm 17.5]. So $x \in \overline{A - B}$. This shows that $\overline{A} - \overline{B} \subset \overline{A - B}$. Equality does not hold in general as $\overline{\mathbf{R}} - \overline{\{0\}} = \mathbf{R} - \{0\} \subsetneq \overline{\mathbf{R} - \{0\}} = \mathbf{R}$.

Just to recap we have

(1) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$ $\quad$ ($A \subset B$, B closed $\Rightarrow \overline{A} \subset B$)
(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
(3) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ $\quad$ ($\overline{A \cap D} = \overline{A}$ if D is dense.)
(4) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$
(5) $\overline{\bigcap A_\alpha} \subset \bigcap \overline{A_\alpha}$
(6) $\overline{A} - \overline{B} \subset \overline{A - B}$

Dually,

(1) $A \subset B \Rightarrow \operatorname{Int} A \subset \operatorname{Int} B$ $\quad$ ($A \subset B$, A open $\Rightarrow A \subset \operatorname{Int} B$)
(2) $\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B$
(3) $\operatorname{Int}(A \cup B) \supset \operatorname{Int} A \cup \operatorname{Int} B$

These formulas are really the same because

$$\overline{X - A} = X - \operatorname{Int} A, \quad \operatorname{Int}(X - A) = X - \overline{A}$$

Ex. 17.9. [Thm 19.5] Since $\overline{A} \times \overline{B}$ is closed [Ex 17.3] and contains $A \times B$, it also contains the closure of $A \times B$ [Ex 17.6.(a)], i.e. $\overline{A \times B} \subset \overline{A} \times \overline{B}$.

Conversely, let $(x, y) \in \overline{A} \times \overline{B}$. Any neighborhood of (x, y) contains a product neighborhood of the form $U \times V$ where $U \subset X$ is a neighborhood of x and $V \subset Y$ a neighborhood of y. The intersection of this product neighborhood with $A \times B$

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap Y)$$

is nonempty because $U \cap A \neq \emptyset$ as $x \in \overline{A}$ and $V \cap B \neq \emptyset$ as $y \in \overline{B}$. Since thus any neighborhood of (x, y) intersect $A \times B$ nontrivially, the point (x, y) lies in the closure of $A \times B$ [Thm 17.5]. This shows that $\overline{A} \times \overline{B} \subset \overline{A \times B}$.

Ex. 17.10 (Morten Poulsen).

Theorem 1. *Every order topology is Hausdorff.*

Proof. Let $(X, \leq)$ be a simply ordered set. Let X be equipped with the order topology induced by the simple order. Furthermore let a and b be two distinct points in X, may assume that $a < b$. Let

$$A = \{ x \in X \mid a < x < b \},$$

i.e. the set of elements between a and b.

If A is empty then $a \in (-\infty, b)$, $b \in (a, \infty)$ and $(-\infty, b) \cap (a, \infty) = \emptyset$, hence X is Hausdorff.

If A is nonempty then $a \in (-\infty, x)$, $b \in (x, \infty)$ and $(-\infty, x) \cap (x, \infty) = \emptyset$ for any element x in A, hence X is Hausdorff. $\square$

Ex. 17.11 (Morten Poulsen).

Theorem 2. *The product of two Hausdorff spaces is Hausdorff.*

Proof. Let X and Y be Hausdorff spaces and let $a_1 \times b_1$ and $a_2 \times b_2$ be two distinct points in $X \times Y$. Note that either $a_1 \neq a_2$ or $b_1 \neq b_2$.

If $a_1 \neq a_2$ then, since X is Hausdorff, there exists open sets U_1 and U_2 in X such that $a_1 \in U_1$, $a_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. It follows that $U_1 \times Y$ and $U_2 \times Y$ are open in $X \times Y$. Furthermore $a_1 \times b_1 \in U_1 \times Y$, $a_2 \times b_2 \in U_2 \times Y$ and $(U_1 \times Y) \cap (U_2 \times Y) = (U_1 \cap U_2) \times Y = \emptyset \times Y = \emptyset$, hence $X \times Y$ is Hausdorff.

The case $b_1 \neq b_2$ is similar. $\square$

Ex. 17.12 (Morten Poulsen).

Theorem 3. *Every subspace of a Hausdorff space is Hausdorff.*

Proof. Let A be a subspace of a Haussdorff space X and let a and b be two distinct points in A.

Since X is Hausdorff there exists two open sets U and V in X such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$. Hence $a \in A \cap U$, $b \in A \cap V$ and $(A \cap U) \cap (A \cap V) = (U \cap V) \cap A = \emptyset \cap A = \emptyset$. Since $A \cap U$ and $A \cap V$ are open in A, it follows that A is Hausdorff. $\square$

Ex. 17.13 (Morten Poulsen).

Theorem 4. *A topological space X is Hausdorff if only if the diagonal*

$$\Delta = \{ x \times x \in X \times X \mid x \in X \}$$

is closed in $X \times X$.

Proof. Suppose X is Hausdorff. The diagonal Δ is closed if and only if the complement $\Delta^c = X \times X - \Delta$ is open. Let $a \times b \in \Delta^c$, i.e. a and b are distinct points in X. Since X is Hausdorff there exists open sets U and V in X such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$. Hence $a \times b \in U \times V$ and $U \times V$ open in $X \times X$. Furthermore $(U \times V) \cap \Delta = \emptyset$, since U and V are disjoint. So for every point $a \times b \in \Delta^c$ there exists an open set $U_{a \times b}$ such that $a \times b \in U_{a \times b} \subset \Delta^c$. By Ex. 13.1 it follows that Δ^c open, i.e. Δ closed.

Now suppose Δ is closed. If a and b are two distinct points in X then $a \times b \in \Delta^c$. Since Δ^c is open there exists a basis element $U \times V$, U and V open in X, for the product topology, such that $a \times b \in U \times V \subset \Delta^c$. Since $U \times V \subset \Delta^c$ it follows that $U \cap V = \emptyset$. Hence U and V are open sets such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$, i.e. X is Hausdorff. $\square$

Ex. 17.14 (Morten Poulsen). The sequence converges to every real number, by the following result.

Theorem 5. *Let X be a set equipped with the finite complement topology. If $(x_n)_{n \in \mathbf{Z}_+}$ is an infinite sequence of distinct points in X then (x_n) converges to every x in X.*

Proof. Let U be a neighborhood of $x \in X$, i.e $X - U$ is finite. It follows that $x_n \in U$, for all, but finitely many, $n \in \mathbf{Z}_+$, i.e. (x_n) converges to x. $\square$

Ex. 17.21 (Morten Poulsen). Let X be a topological space. Consider the three operations on $\mathcal{P}(X)$, namely closure $A \mapsto \overline{A}$, complement $A \mapsto X - A$ and interior $A \mapsto A^\circ$. Write A^- instead of $\overline{A}$ and A^c instead of $X - A$, e.g. $X - \overline{X - A} = A^{c-c}$.

Lemma 6. *If $A \subset X$ then $A^\circ = A^{c-c}$.*

Proof. $A^\circ \supset A^{c-c}$: Since $A^c \subset A^{c-}$, $A^{c-c} \subset A$ and A^{c-c} is open.

$A^\circ \subset A^{c-c}$: Since $A^\circ \subset A$, $A^c \subset A^{\circ c}$ and $A^{\circ c}$ is closed, it follows that $A^{c-} \subset A^{\circ c}$, hence $A^\circ \subset A^{c-c}$. □

This lemma shows that the interior operation can be expressed in terms of the closure and complement operations.

(a). The following theorem, also known as Kuratowski's Closure-Complement Problem, was first proved by Kuratowski in 1922.

Theorem 7. *Let X be a topological space and $A \subset X$. Then at most 14 distinct sets can be derived from A by repeated application of closure and complementation.*

Proof. Let $A_1 = A$ and set $B_1 = A_1^c$. Define $A_{2n} = A_{2n-1}^-$ and $A_{2n+1} = A_{2n}^c$ for $n \in \mathbf{Z}_+$. Define $B_{2n} = B_{2n-1}^-$ and $B_{2n+1} = B_{2n}^c$ for $n \in \mathbf{Z}_+$.

Note that every set obtainable from A by repeatedly applying the closure and complement operations is clearly one of the sets A_n or B_n.

Now $A_7 = A_4^{c-c} = A_4^\circ = A_3^{-\circ}$. Since $A_3 = A_1^{-c}$ it follows that A_3 is open, hence $A_3 \subset A_7 \subset A_3^-$, so $A_7^- = A_3^-$, i.e. $A_8 = A_4$, hence $A_{n+4} = A_n$ for $n \geq 4$. Similarly $B_{n+4} = B_n$ for $n \geq 4$.

Thus every A_n or B_n is equal to one of the 14 sets $A_1, \ldots, A_7, B_1, \ldots, B_7$, this proves the result. □

(b). An example:

$$A = ((-\infty, -1) - \{-2\}) \cup ([-1, 1] \cap \mathbf{Q}) \cup \{2\}.$$

The 14 different sets:

$$\begin{aligned}
A_1 &= ((-\infty, -1) - \{-2\}) \cup ([-1, 1] \cap \mathbf{Q}) \cup \{2\} \\
A_2 &= (-\infty, 1] \cup \{2\} \\
A_3 &= (1, \infty) - \{2\} \\
A_4 &= [1, \infty) \\
A_5 &= (-\infty, 1) \\
A_6 &= (-\infty, 1] \\
A_7 &= (1, \infty) \\
B_1 &= \{-2\} \cup ([-1, 1] - \mathbf{Q}) \cup ((1, \infty) - \{2\}) \\
B_2 &= \{-2\} \cup [-1, \infty) \\
B_3 &= (-\infty, -1) - \{-2\} \\
B_4 &= (-\infty, -1] \\
B_5 &= (-1, \infty) \\
B_6 &= [-1, \infty) \\
B_7 &= (-\infty, -1).
\end{aligned}$$

Another example:

$$A = \{\, 1/n \mid n \in \mathbf{Z}_+ \,\} \cup (2, 3) \cup (3, 4) \cup \{9/2\} \cup [5, 6] \cup \{\, x \mid x \in \mathbf{Q}, 7 \leq x < 8 \,\}.$$

References

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Munkres §18

Ex. 18.1 (Morten Poulsen). Recall the ε-δ-definition of continuity: A function $f : \mathbf{R} \to \mathbf{R}$ is said to be continuous if

$$\forall a \in \mathbf{R} \, \forall \varepsilon \in \mathbf{R}_+ \, \exists \delta \in \mathbf{R}_+ \, \forall x \in \mathbf{R} : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Let $\mathcal{T}$ be the standard topology on $\mathbf{R}$ generated by the open intervals.

Theorem 1. *For functions $f : \mathbf{R} \to \mathbf{R}$ the following are equivalent:*

(i) $\forall a \in \mathbf{R} \, \forall \varepsilon \in \mathbf{R}_+ \, \exists \delta \in \mathbf{R}_+ \, \forall x \in \mathbf{R} : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.
(ii) $\forall a \in \mathbf{R} \, \forall \varepsilon \in \mathbf{R}_+ \, \exists \delta \in \mathbf{R}_+ : f((a - \delta, a + \delta)) \subset (f(a) - \varepsilon, f(a) + \varepsilon)$.
(iii) $\forall U \in \mathcal{T} : f^{-1}(U) \in \mathcal{T}$.

Proof. "$(i) \Leftrightarrow (ii)$": Clear.

"$(ii) \Rightarrow (iii)$": Let $U \in \mathcal{T}$. If $a \in f^{-1}(U)$ then $f(a) \in U$. Since U is open there exists $\varepsilon \in \mathbf{R}_+$ such that $(f(a) - \varepsilon, f(a) + \varepsilon) \subset U$. By assumption there exists $\delta_a \in \mathbf{R}_+$ such that $f((a - \delta_a, a + \delta_a)) \subset (f(a) - \varepsilon, f(a) + \varepsilon)$, hence $(a - \delta_a, a + \delta_a) \subset f^{-1}(U)$. It follows that $f^{-1}(U) = \bigcup_{a \in f^{-1}(U)} (a - \delta_a, a + \delta_a)$, i.e. open.

"$(iii) \Rightarrow (ii)$": Let $a \in \mathbf{R}$. Given $\varepsilon \in \mathbf{R}_+$ then $f^{-1}((f(a) - \varepsilon, f(a) + \varepsilon))$ is open and contains a. Hence there exists $\delta \in \mathbf{R}_+$ such that $(a - \delta, a + \delta) \subset f^{-1}((f(a) - \varepsilon, f(a) + \varepsilon))$. It follows that $f((a - \delta, a + \delta)) \subset (f(a) - \varepsilon, f(a) + \varepsilon)$. $\square$

Ex. 18.2. Let $f : \mathbf{R} \to \{0\}$ be the constant map. Then 2004 is a limit point of $\mathbf{R}$ but $f(2004) = 0$ is not a limit point of $f(\mathbf{R}) = \{0\}$. (The question is if $f(A') \subset f(A)'$ in general. The answer is no: In the above example $\mathbf{R}' = \mathbf{R}$ so that $f(\mathbf{R}') = f(\mathbf{R}) = \{0\}$ but $f(\mathbf{R})' = \emptyset$.)

Ex. 18.6 (Morten Poulsen).

Claim 2. *The map $f : \mathbf{R} \to \mathbf{R}$, defined by*

$$f(x) = \begin{cases} x, & x \in \mathbf{R} - \mathbf{Q} \\ 0, & x \in \mathbf{Q}, \end{cases}$$

is continuous only at 0.

Proof. Since $|f(x)| \leq |x|$ for all x it follows that f is continuous at 0.

Let $x_0 \in \mathbf{R} - \{0\}$. Since $\lim_{x \to x_0, x \in \mathbf{Q}} f(x) = 0$ and $\lim_{x \to x_0, x \in \mathbf{R} - \mathbf{Q}} f(x) = x$ it follows that f is not continuous at x_0. $\square$

Ex. 18.7 (Morten Poulsen).

(a). The following lemma describes the continuous maps $\mathbf{R}_\ell \to \mathbf{R}$.

Lemma 3. *Let $f : \mathbf{R} \to \mathbf{R}$. The following are equivalent:*

(i) *$f : \mathbf{R} \to \mathbf{R}$ is right continuous.*
(ii) $\forall x \in \mathbf{R} \, \forall \varepsilon \in \mathbf{R}_+ \, \exists \delta \in \mathbf{R}_+ : f([x, x + \delta)) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.
(iii) *For each $x \in \mathbf{R}$ and each interval (a, b) containing $f(x)$ there exists an interval $[c, d)$ containing x such that $f([c, d)) \subset (a, b)$.*
(iv) *$f : \mathbf{R}_\ell \to \mathbf{R}$ is continuous.*

Proof. "$(i) \Leftrightarrow (ii)$": By definition.

"$(ii) \Rightarrow (iii)$": Let $x \in \mathbf{R}$. Assume $f(x) \in (a, b)$. Set $\varepsilon = \min\{f(x) - a, b - f(x)\} \in \mathbf{R}_+$. Then $(f(x) - \varepsilon, f(x) + \varepsilon) \subset (a, b)$. By (ii) there exists $\delta \in \mathbf{R}_+$ such that $f([x, x + \delta)) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$, hence $[c, d) = [x, x + \delta)$ does the trick.

"$(iii) \Rightarrow (ii)$": Let $x \in \mathbf{R}$ and $\varepsilon \in \mathbf{R}_+$. By (iii) there exists $[c, d)$ such that $x \in [c, d)$ and $f([c, d)) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. Set $\delta = d - x \in \mathbf{R}_+$. Then $[x, x + \delta) = [x, d) \subset [c, d)$, hence $f([x, x + \delta)) \subset (x - \varepsilon, x + \varepsilon)$.

"$(iii) \Leftrightarrow (iv)$": Clear. $\square$

(b). The continuous maps from $\mathbf{R}$ to $\mathbf{R}_\ell$ are the constant maps, c.f. Ex. 25.1.

A map $f\colon \mathbf{R}_\ell \to \mathbf{R}_\ell$ is continuous if and only if for any x and $\varepsilon > 0$ there exists $\delta > 0$ such that $f([x, x+\delta)) \subset [f(x), f(x)+\varepsilon)$. Hence $f\colon \mathbf{R}_\ell \to \mathbf{R}_\ell$ is continuous if and only if $f\colon \mathbf{R} \to \mathbf{R}$ is right continuous and $\forall x \in \mathbf{R} \exists \delta > 0 \forall y \in [x, x+\delta)\colon f(x) \leq f(y)$. (Thanks to Prateek Karandikar for a correction and for this example of a continuous map $\mathbf{R}_\ell \to \mathbf{R}_\ell$.)

Ex. 18.8. Let Y be an ordered set. Give $Y \times Y$ the product topology. Consider the set

$$\Delta^- = \{y_1 \times y_2 \mid y_1 > y_2\}$$

of points below the diagonal. Let $(y_1, y_2) \in \Delta^-$ so that $y_1 > y_2$. If y_2 is the immediate predecessor of y_1 then

$$y_1 \times y_2 \in [y_1, \infty) \times (-\infty, y_2) = (y_2, \infty) \times (-\infty, y_1) \subset \Delta^-$$

and if $y_1 > y > y_2$ for some $y \in Y$ then

$$y_1 \times y_2 \in (y, \infty) \times (-\infty, y) \subset \Delta^-$$

This shows that Δ^- is open.

(a). Since the map $(f, g)\colon X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y$ is continuous, the preimage

$$(f, g)^{-1}(\Delta^-) = \{x \in X \mid f(x) > g(x)\}$$

is open and the complement $\{x \in X \mid f(x) \leq g(x)\}$ is closed.

(b). The map

$$\min\{f, g\}(x) = \begin{cases} f(x) & f(x) \leq g(x) \\ g(x) & f(x) \geq g(x) \end{cases}$$

is continuous according to [1, Thm 18.3].

Ex. 18.10. Let $(f_j\colon X_j \to Y_j)_{j \in J}$ be an indexed family of continuous maps. Define $\prod f_j : \prod X_j \to \prod Y_j$ to be the map that takes $(x_j) \in \prod X_j$ to $(f_j(x_j)) \in \prod Y_j$. The commutative diagram

$$\begin{tikzcd} \prod X_j \arrow[r, "\prod f_j"] \arrow[d, "\pi_k"'] & \prod Y_j \arrow[d, "\pi_k"] \\ X_k \arrow[r, "f_k"'] & Y_k \end{tikzcd}$$

shows that $\pi_k \circ \prod f_j = f_k \circ \pi_k$ is continuous for all $k \in J$. Thus $\prod f_j\colon \prod X_j \to \prod Y_j$ is continuous [1, Thm 18.4, Thm 19.6].

Ex. 18.13. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces where the codomain, Y, is Hausdorff. The equalizer

$$\mathrm{Eq}(f, g) = \{x \in X \mid f(x) = g(x)\} = (f, g)^{-1}(\Delta)$$

is then a closed subset of X for it is the preimage under the continuous map $(f, g)\colon X \to Y \times Y$ of the diagonal $\Delta = \{(y, y) \in Y \times Y \mid y \in Y\}$ which is closed since Y is Hausdorff [1, Ex 17.13]. (This is [1, Ex 31.5].)

It follows that if f and g agree on the subset $A \subset X$ then they also agree on $\overline{A}$ for

$$A \subset \mathrm{Eq}(f, g) \implies \overline{A} \subset \mathrm{Eq}(f, g)$$

In particular, if f and g agree on a dense subset of X, they are equal: Any continuous map into a Hausdorff space is determined by its values on a dense subset.

References

[1] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063

Munkres §19

Ex. 19.7. Any nonempty basis open set in the product topology contains an element from $\mathbf{R}^\infty$, cf. Example 7p. 151. Therefore $\overline{\mathbf{R}^\infty} = \mathbf{R}^\omega$ in the product topology. ($\mathbf{R}^\infty$ is *dense* [Definition p. 191] in $\mathbf{R}^\omega$ with the product topology.)

Let (x_i) be any point in $\mathbf{R}^\omega - \mathbf{R}^\infty$. Put

$$U_i = \begin{cases} \mathbf{R} & \text{if } x_i = 0 \\ \mathbf{R} - \{0\} & \text{if } x_i \neq 0 \end{cases}$$

Then $\prod U_i$ is open in the box topology and $(x_i) \in \prod U_i \subset \mathbf{R}^\omega - \mathbf{R}^\infty$. This shows that $\mathbf{R}^\infty$ is closed so that $\overline{\mathbf{R}^\infty} = \mathbf{R}^\infty$ with the box topology on $\mathbf{R}^\omega$.

See [Ex 20.5] for the closure of $\mathbf{R}^\infty$ in $\mathbf{R}^\omega$ with the uniform topology.

Ex. 19.10.

(a). The topology $\mathcal{T}$ (the *initial topology* for the set maps $\{f_\alpha \mid \alpha \in J\}$) is the intersection [Ex 13.4] of all topologies on A for which all the maps f_α, $\alpha \in J$, are continuous.

(b). Since all the functions $f_\alpha \colon A \to X_\alpha$, $\alpha \in J$, are continuous, $\mathcal{S} = \bigcup \mathcal{S}_\alpha \subset \mathcal{T}$. The topology $\mathcal{T}_\mathcal{S}$ generated by $\mathcal{S}$, which is the coarsest topology containing $\mathcal{S}$ [Ex 13.5], is therefore also contained in $\mathcal{T}$. On the other hand, $\mathcal{T} \subset \mathcal{T}_\mathcal{S}$, for all the functions $f_\alpha \colon A \to X_\alpha$, $\alpha \in J$, are continuous in $\mathcal{T}_\mathcal{S}$ and $\mathcal{T}$ is the coarsest topology with this property. Thus $\mathcal{T} = \mathcal{T}_\mathcal{S}$.

(c). Let $g \colon Y \to A$ be any map. Then

$$\begin{aligned} g \colon Y \to A \text{ is continuous} &\Leftrightarrow \forall U \in \mathcal{S} \colon g^{-1}(U) \in \mathcal{T}_Y \\ &\Leftrightarrow \forall \alpha \in J \; \forall U_\alpha \in \mathcal{T}_\alpha \colon g^{-1}(f_\alpha^{-1} U_\alpha) \in \mathcal{T}_Y \\ &\Leftrightarrow \forall \alpha \in J \; \forall U_\alpha \in \mathcal{T}_\alpha \colon (f_\alpha \circ g)^{-1} U_\alpha \in \mathcal{T}_Y \\ &\Leftrightarrow \forall \alpha \in J \colon f_\alpha \circ g \colon Y \to X_\alpha \text{ is continuous} \\ &\Leftrightarrow f \circ g \colon Y \to \prod X_\alpha \text{ is continuous} \end{aligned}$$

where $\mathcal{T}_Y$ is the topology on Y and $\mathcal{T}_\alpha$ the topology on X_α.

(d). Consider first a single map $f \colon A \to X$, and give A the initial topology so that the open sets in A are the sets of the form $f^{-1}U$ for U open in X. Then $f \colon A \to f(A)$ is always continuous [Thm 18.2] and open because $f(A) \cap U = f(f^{-1}U)$ for all (open) subsets U of X.
Next, note that the initial topology for the set maps $\{f_\alpha \mid \alpha \in J\}$ is the initial topology for the single map $f = (f_\alpha) \colon A \to \prod X_\alpha$. As just observed, $f \colon A \to f(A)$ is continuous and open.

Example: The product topology on $\prod X_\alpha$ is the initial topology for the set of projections $\pi_\alpha \colon \prod X_\alpha \to X_\alpha$.

References

1st December 2004

Munkres §20

Ex. 20.5. Consider $\mathbf{R}^\omega$ with the uniform topology and let d be the uniform metric. Let $C \subset \mathbf{R}^\omega$ be the set of sequences that converge to 0. Then

$$\overline{\mathbf{R}^\infty} = C.$$

$\subset$: Since clearly $\mathbf{R}^\infty \subset C$ it is enough to show that C is closed. Let $(x_n) \in \mathbf{R}^\omega - C$ be a sequence that does not converge to 0. This means that there is some $1 > \varepsilon > 0$ such that $|x_n| > \varepsilon$ for infinitely many n. Then $B_d((x_n), \frac{1}{2}\varepsilon) \subset \mathbf{R}^\omega - C$.

$\supset$: Let $(x_n) \in C$. For any $1 > \varepsilon > 0$ we have $|x_n| < \varepsilon$ for all but finitely many n. Thus $B_d((x_n), 2\varepsilon) \cap \mathbf{R}^\infty \neq \emptyset$.

References

4th January 2005

Munkres §22

Ex. 22.2.

(a). The map $p\colon X \to Y$ is continuous. Let U be a subspace of Y such that $p^{-1}(U) \subset X$ is open. Then

$$f^{-1}(p^{-1}(U)) = (pf)^{-1}(U) = \mathrm{id}_Y^{-1}(U) = U$$

is open because f is continuous. Thus $p\colon X \to Y$ is a quotient map.

(b). The map $r\colon X \to A$ is a quotient map by (a) because it has the inclusion map $A \hookrightarrow X$ as right inverse.

Ex. 22.3. Let $g\colon \mathbf{R} \to A$ be the continuous map $f(x) = x \times 0$. Then $q \circ g$ is a quotient map, even a homeomorphism. If the compostion of two maps is quotient, then the last map is quotient; see [1]. Thus q is quotient.

The map q is not closed for $\{x \times \frac{1}{x} \mid x > 0\}$ is closed but $q(A \cap \{x \times \frac{1}{x} \mid x > 0\} = (0, \infty)$ is not closed.

The map q is not open for $\mathbf{R} \times (-1, \infty)$ is open but $q(A \cap (\mathbf{R} \times (-1, \infty))) = [0, \infty)$ is not open.

Ex. 22.5. Let $U \subset A$ be open in A. Since A is open, U is open in X. Since p is open, $p(U) = q(U) \subset p(A)$ is open in Y and also in $p(A)$ because $p(A)$ is open [Lma 16.2].

SupplEx. 22.3. Let G be a topological group and H a subgroup. Let $\varphi_G\colon G \times G \to G$ be the map $\varphi_G(x, y) = xy^{-1}$ and $\varphi_H\colon H \times H \to H$ the corresponding map for the subgroup H. Since these maps are related by the commutative diagram

$$\begin{tikzcd}
G \times G \arrow[r, "\varphi_G"] & G \\
H \times H \arrow[r, "\varphi_H"] \arrow[u, hook] & H \arrow[u, hook]
\end{tikzcd}$$

and φ_G is continuous, also φ_H is continuous [Thm 18.2]. Moreover, any subspace of a T_1-space is a T_1-space, so H is a T_1-space. Thus H is a topological group by Ex 22.1.

Now, by using [Thm 18.1.(2), Thm 19.5], we get

$$\varphi_G(\overline{H} \times \overline{H}) = \varphi_G(\overline{H \times H}) \subset \overline{\varphi_G(H \times H)} = \overline{H}$$

which shows that $\overline{H}$ is a subgroup of G. But then $\overline{H}$ is a topological group as we have just shown.

SupplEx. 22.5. Let G be a topological group and H a subgroup of G.

(a). Left multiplication with α induces a commutative diagram

$$\begin{tikzcd}
G \arrow[r, "f_\alpha"] \arrow[d, "p"'] & G \arrow[d, "p"] \\
G/H \arrow[r, "f_\alpha/H"'] & G/H
\end{tikzcd}$$

which shows [Thm 22.2] that f_α/H is ¸

(b). The saturation $gH = f_g(H)$ of the point $g \in G$ is closed because H is closed and left multiplication f_g is a homeomorphism [SupplEx 22.4].

(c). The saturation $UH = \bigcup_{h \in H} Uh = \bigcup g_h(U)$ is open because U is open and right multiplication g_h is a homeomorphism [SupplEx 22.4].

(d). The space G/H is T_1 by point (b). The map $G \times G \xrightarrow{\varphi} G \colon (x, y) \mapsto xy^{-1}$ is continuous [SupplEx 22.1] and it induces a commutative diagram

$$\begin{tikzcd} G \times G \arrow[r, "\varphi"] \arrow[d, "p \times p"'] & G \arrow[d, "p"] \\ G/H \times G/H \arrow[r, "\varphi/H"'] & G/H \end{tikzcd}$$

where $p \times p$ is a quotient map since it is open as we have just shown. (It is not true that the product of two quotient maps is a quotient map [Example 7, p. 143] but is is true that a product of two open maps is an open map.) This shows [Thm 22.2] that φ/H is continuous and hence [SupplEx 22.1] G/H is a topological group.

SupplEx. 22.7. Let G be a topological group.

(a). Any neighborhood of U of e contains a symmetric neighborhood $V \subset U$ such that $VV \subset U$.
By continuity of $(x, y) \to xy$, there is a neighborhood W_1 of e such that $W_1W_1 \subset U$. By continuity of $(x, y) \to xy^{-1}$, there is a neighborhood W_2 of e such that $W_2W_2^{-1} \subset W_1$. (Any neighborhood of e contains a symmetric neighborhood of e.) Now $V = W_2W_2^{-1}$ is a symmetric neighborhood of e and $VV \subset W_1W_1 \subset U$.

(b). G is Hausdorff.
Let $x \neq y$. Since the set $\{xy^{-1}\}$ is closed in G, there is a neighborhood U of e not containing xy^{-1}. Find a symmetric neighborhood V of e such that $VV \subset U$. Then $Vx \cap Vy = \emptyset$. (If not, then $gx = hy$ for some $g, h \in V$ and $xy^{-1} = g^{-1}h \in V^{-1}V = VV \subset U$ contradicts $xy^{-1} \notin U$.)

(c). G is regular.
Let $A \subset G$ be a closed subspace and x a point outside A. Since e is outside the closed set xA^{-1}, a whole neighborhood U of e is disjoint from xA^{-1}. Find a symmetric neighborhood V of e such that $VV \subset U$. Then $Vx \cap VA = \emptyset$. (If not, then $gx = ha$ for some $g, h \in V$ and $xA^{-1} \ni xa^{-1} = g^{-1}h \in V^{-1}V = VV \subset U$ contradicts $xA^{-1} \cap U = \emptyset$)

(d). G/H is regular for any closed subgroup $H \subset G$.
The point xH is closed in G/H because the subspace xH is closed in G. Let $x \in G$ be a point ouside a saturated closed subset $A \subset G$. Find a neighborhood V of e such that Vx and VA are disjoint. Then also $VxH \cap VA = \emptyset$ since $A = AH$ is saturated. Since $p \colon G \to G/H$ is open [SupplEx 22.5.(c)], $p(Vx)$ is an open neighborhood of $p(x)$ disjoint from the open neighborhood $p(VA)$ of $p(A)$. This shows that G/H is regular.

References

[1] Jesper M. Møller, *General topology*, http://www.math.ku.dk/ moller/e03/3gt/notes/gtnotes.dvi.

27th January 2005

Munkres §23

Ex. 23.1. Any separation $X = U \cup V$ of $(X, \mathcal{T})$ is also a separation of $(X, \mathcal{T}')$. This means that

$$(X, \mathcal{T}) \text{ is disconnected} \Rightarrow \ (X, \mathcal{T}') \text{ is disconnected}$$

or, equivalently,

$$(X, \mathcal{T}') \text{ is connected} \Rightarrow \ (X, \mathcal{T}) \text{ is disconnected}$$

when $\mathcal{T}' \supset \mathcal{T}$.

Ex. 23.2. Using induction and [1, Thm 23.3] we see that $A(n) = A_1 \cup \cdots \cup A_n$ is connected for all $n \geq 1$. Since the spaces $A(n)$ have a point in common, namely any point of A_1, their union $\bigcup A(n) = \bigcup A_n$ is connected by [1, Thm 23.3] again.

Ex. 23.3. Let $A \cup \bigcup A_\alpha = C \cup D$ be a separation. The connected space A is [Lemma 23.2] entirely contained in C or D, let's say that $A \subset C$. Similarly, for each α, the connected [1, Thm 23.3] space $A \cup A_\alpha$ is contained entirely in C or D. Sine it does have something in common with C, namely A, it is entirely contained in C. We conclude that $A \cup \bigcup A_\alpha = C$ and $D = \emptyset$, contradicting the assumption that $C \cup D$ is a separation

Ex. 23.4 (Morten Poulsen). Suppose $\emptyset \subsetneq A \subsetneq X$ is open and closed. Since A is open it follows that $X - A$ is finite. Since A is closed it follows that $X - A$ open, hence $X - (X - A) = A$ is finite. Now $X = A \cup (X - A)$ is finite, contradicting that X is infinite. Thus X and $\emptyset$ are the only subsets of X that are both open and closed, hence X is connected.

Ex. 23.5. $\mathbf{Q}$ is totally disconnected [1, Example 4, p. 149].
$\mathbf{R}_\ell$ is totally disconnected for $\mathbf{R}_\ell = (-\infty, b) \cup [b, +\infty)$ for any real number b.
Any well-ordered set X is totally disconnected in the order topology for

$$X = (-\infty, \alpha + 1) \cup (\alpha, +\infty) = (-\infty, \alpha] \cup [\alpha + 1, +\infty)$$

for any $\alpha \in X$ and if $A \subset X$ contains $\alpha < \beta$ then $\alpha \in (-\infty, \alpha + 1)$ and $\beta \in (\alpha, +\infty)$.

Ex. 23.6. $X = \text{Int}(A) \cup \text{Bd}(A) \cup \text{Int}(X - A)$ is a partition of X for any subset $A \subset X$ [1, Ex 17.19]. If the subspace $C \subset X$ intersects both A and $X - A$ but not $\text{Bd}(A)$, then C intersects $A - \text{Bd}(A) = \text{Int}(A)$ and $(X - A) - \text{Bd}(X - A) = \text{Int}(X - A)$ and

$$C = (C \cap \text{Int}(A)) \cup (C \cap \text{Int}(X - A))$$

is a separation of C.

Ex. 23.7. $\mathbf{R} = (-\infty, r) \cup [r, +\infty)$ is a separation of $\mathbf{R}_\ell$ for any real number r. It follows [1, Lemma 23.1] that any subspace of $\mathbf{R}_\ell$ containing more than one point is disconnected: $\mathbf{R}_\ell$ is totally disconnected.

Ex. 23.11. Let $X = C \cup D$ be a separation of X. Since fibres are connected, $p^{-1}(p(x)) \subset C$ for any $x \in C$ and $p^{-1}(p(x)) \subset D$ for any $x \in D$ [1, Lemma 23.2]. Thus C and D are saturated open disjoint subspaces of X and therefore $p(C)$ and $p(D)$ are open disjoint subspace of Y. In other words, $Y = p(C) \cup p(D)$ is a separation.

Ex. 23.12. Assume that the subspace Y is connected. Let $X - Y = A \cup B$ be a separation of $X - Y$ and $Y \cup A = C \cup D$ a separation of $Y \cup A$. Then [1, Lemma 23.1]

$$\overline{A} \subset X - B, \quad \overline{B} \subset X - A, \quad \overline{C} \subset X - D, \quad \overline{D} \subset X - C$$

and

$$Y \cup A \cup B = X = B \cup C \cup D$$

are partitions [1, §3] of X.

The connected subspace Y is entirely contained in either C or D [1, Lemma 23.2]; let's say that $Y \subset C$. Then $D = C \cup D - C \subset Y \cup A - Y \subset A$ and $\overline{D} \subset \overline{A} \subset X - B$. From

$$\overline{B \cup C} \overset{[Ex17.6.(b)]}{=} \overline{B} \cup \overline{C} \subset (X - A) \cup (X - D) = (B \cup Y) \cup (B \cup C) \subset B \cup C$$

$$\overline{D} \subset (X - B) \cap (X - C) = D$$

we conclude that $B \cup C = \overline{B \cup C}$ and $D = \overline{D}$ are closed subspaces. Thus $X = (B \cup C) \cup D$ is a separation of X.

References

[1] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063

1st December 2004

Munkres §24

Ex. 24.2 (Morten Poulsen). Let $f: S^1 \to \mathbf{R}$ be a continuous map. Define $g: S^1 \to \mathbf{R}$ by $g(s) = f(s) - f(-s)$. Clearly g is continuous. Furthermore

$$g(s) = f(s) - f(-s) = -(f(-s) - f(s)) = -g(-s),$$

i.e. g is an odd map. By the Intermediate Value Theorem there exists $s_0 \in S^1$ such that $g(s_0) = 0$, i.e. $f(s_0) = f(-s_0)$.

This result is also known as the Borsuk-Ulam theorem in dimension one. Thus there are no injective continuous maps $S^1 \to \mathbf{R}$, hence S^1 is not homeomorphic to a subspace of $\mathbf{R}$, which is no surprise.

Ex. 24.4. [1, §17]. Suppose that X is a linearly ordered set that is not a linear continuum. Then there are nonempty, proper, clopen subsets of X:

- If $(x, y) = \emptyset$ for some points $x < y$ then $(-\infty, x] = (\infty, y)$ is clopen and $\neq \emptyset, X$.
- If $A \subset X$ is a nonempty subset bounded from above which has no least upper bound then the set of upper bounds $B = \bigcap_{a \in A} [a, \infty) = \bigcup_{b \in B} (b, \infty)$ is clopen and $\neq \emptyset, X$.

Therefore X is not connected [2, §23].

Ex. 24.8 (Morten Poulsen).

(a).

Theorem 1. *The product of an arbitrary collection of path connected spaces is path connected.*

Proof. Let $\{A_j\}_{j \in J}$ be a collection of path connected spaces. Let $x = (x_j)_{j \in J}$ and $y = (y_j)_{j \in J}$ be two points in $\prod_{j \in J} A_j$

For each $j \in J$ there exists a path $\gamma_j : [0, 1] \to A_j$ between x_j and y_j, since A_j is path connected for all j. Now the map $\gamma : [0, 1] \to \prod_{j \in J} A_j$ defined by $\gamma(t) = (\gamma_j(t))_{j \in J}$ is a path between x and y, hence the product is path connected. $\square$

(b). This is not true in general: The set $S = \{x \times \sin(x^{-1} \mid 0 < x < \pi^{-1}\}$ is path connected, but $\overline{S} = S \cup (\{0\} \times [-1, 1])$ is not path connected, c.f. example 24.7.

(c).

Theorem 2. *If $f : X \to Y$ is a continuous map and X path connected then $f(X)$ is path connected.*

Proof. Clearly it suffices to consider the case where f is surjective. Let y_1 and y_2 be two points in Y. Then there exists x_1 and x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X path connected there is a path $\gamma : [0, 1] \to X$ between x_1 and x_2. Now $\delta = f \circ g : [0, 1] \to Y$ is a path between $\delta(0) = f(\gamma(0)) = y_1$ and $\delta(1) = f(\gamma(1)) = y_2$, hence Y path connected. $\square$

(d).

Theorem 3. *Let $\{A_j\}_{j \in J}$ be a collection of path connected spaces. If $\bigcap_{j \in J} A_j$ is nonempty then $\bigcup_{j \in J} A_j$ is path connected.*

Proof. Let a and b be two points in $\bigcup_{j \in J} A_j$ and let c be an element of $\bigcap_{j \in J} A_j$. Note that there exists s and t such that $a \in A_s$ and $b \in A_t$. Clearly $c \in A_s \cap A_t$. Since A_s and A_t are path connected there exists a path $f : [0, 1] \to A_s$ between a and c and a path $g : [0, 1] \to A_t$ between c and b.

Now $h : [0, 1] \to \bigcup_{j \in J} A_j$ defined by

$$h(t) = \begin{cases} f(2t), & 0 \leq t \leq 1/2 \\ g(2t - 1), & 1/2 \leq t \leq 1 \end{cases}$$

is a path, by the Pasting lemma, between a and b, hence $\bigcup_{j \in J} A_j$ is path connected. $\square$

Ex. 24.10 (Morten Poulsen). Let U be a nonempty, open and connected subspace of $\mathbf{R}^2$ and let $x_0 \in U$. Furthermore let A be the set of points in U that can be joined to x_0 by a path in U.

A open: Let $a_0 \in A \subset U$. Since U open there is an open rectangle $V(= (a,b) \times (c,d))$ such that $a_0 \in V \subset U$. Since V clearly is path connected it follows that $V \subset A$, hence A open.

A closed: Let $u_0 \in U - A$. If every open rectangle containing u_0 intersects A then clearly $u_0 \in A$, hence u_0 is not a limit point of A. Thus no point of $U - A$ is a limit point of A, hence A is closed.

Since U connected it follows that $A = U$, hence U path connected.

Ex. 24.11 (Morten Poulsen). Let A be a subspace of X.

A connected $\not\Rightarrow$ $\operatorname{Int} A$ and $\operatorname{Bd} A$ connected: If $A = [0,1]$ then $\operatorname{Bd} A = \{0,1\}$ is not connected. If $A = \overline{B(-1 \times 0, 1)} \cup \overline{B(1 \times 0, 1)} \subset \mathbf{R}^2$, then $\operatorname{Int} A = B(-1 \times 0, 1) \cup B(1 \times 0, 1)$ is not connected.

$\operatorname{Int} A$ connected $\not\Rightarrow$ A connected: If $A = (0,1) \cup \{2\}$ then $\operatorname{Int} A = (0,1)$ is connected, but A is not connected.

$\operatorname{Bd} A$ connected $\not\Rightarrow$ A connected: $A = \mathbf{Q}$ is not connected but $\operatorname{Bd} A = \mathbf{R}$ is connected.

$\operatorname{Int} A$ and $\operatorname{Bd} A$ connected $\not\Rightarrow$ A connected: One example is $A = \mathbf{Q}$. An example with nonempty interior is

$$A = ([0,1] \times [0,1]) \cup (\{0,1\} \times [1,2]) \cup (([0,1] \cap \mathbf{Q}) \times \{2\}) \subset \mathbf{R}^2$$

where

$$\operatorname{Bd} A = (\{0,1\} \times [0,2]) \cup (\{0,1,2\} \times [0,1])$$

and

$$\operatorname{Int} A = (0,1) \times (0,1).$$

both are connected but A is not connected.

Ex. 24.12 (The long line). The idea is that the two linear continua $S_\Omega \times [0,1)$ and $\mathbf{Z}_+ \times [0,1) = [1,\infty)$, or rather the long line $L = (S_\Omega \times [0,1)) - \{a_0 \times 0\}$ and the real line $\mathbf{R} = (\mathbf{Z}_+ \times [0,1)) - \{1 \times 0\}$, should have a great deal in common. L satisfies the conditions of a 1-dimensional manifold but 2nd countability. The long line is normal [Ex 32.8] but not metrizable [Ex 50.5].

(a) and (b). Easy.

(c). I do the hint first. Let $a > a_0$ be an element of S_Ω which has no immediate predecessor (there are uncountably many such elements [Ex 10.6]). The set of predecessors $S_a = \{b \in S_\Omega \mid b < a\} = \{b_1, b_2, \dots\}$ is countable and the sets $(b_n, a] = (b_n, a+1)$ is a neighborhood basis at a. Since b_1 is not an immediate predecessor, there is an element $a_1 \in (b_1, a]$. Since $\sup\{a_1, b_2\}$ is not an immediate predecessor, there is an element $a_2 \in (\sup\{a_1, b_2\}, a]$. Proceeding inductively, we find a sequence of elements $a_n < a$ such that $a_n > a_{n-1}$ and $a_n > b_n$ for all n. Then a_n is an increasing sequence and since $a_n > b_n, b_{n-1}, \dots, b_1$ for all n, the sequence a_n converges to a.

Let now J be the set of points a such that $[a_0 \times 0, a \times 1)$ has the order type of $[0,1)$. I claim that J is inductive. Suppose that $S_a \subset J$. If a has an immediate predecessor a_1 then $[a_0 \times 0, a \times 1) = [a_0 \times 0, (a-1) \times 1) \cup [a \times 0, a \times 1)$ has the order type of $[0,1)$ by (a). Otherwise, $[a_0 \times 0, a \times 1)$ has the order type of $[0,1)$ by the hint and (b) (let $x_0, x_1, x_2, \dots$ be the sequence $a_0 \times 0, a_1 \times 1, a_2 \times 1 \dots$).

(d,e). For every point $a \times t$ of $S_\Omega \times [0,1)$, the intervals $[a_0 \times 0, a \times 1)$ and $[a_0 \times 0, a \times t)$ have the order type of $[0,1)$ by (c) and (a). Then $(a_0 \times 0, a \times t)$ has the order type of $(0,1)$. Since intervals are convex, the subspace topology on $(a_0 \times 0, a \times t)$ is the order topology [Thm 16.4] so $(a_0 \times 0, a \times t)$ is homeomorphic to $(0,1)$. From this we see that any two points in L are contained in an interval homeomorphic to $(0,1)$ and therefore there is continuous path between them.

(f). Suppose that L is 2nd countable. Then also $S_\Omega - \{a_0\}$ is 2nd countable since this property is preserved under open continuous maps [Ex 16.4, Ex 30.12]. But $S_\Omega - \{a_0\}$ is not 2nd countable for it does not contain countable dense subsets. (Every countable subset of $S_\Omega - \{a_0\} \subset S_\Omega$ is bounded [Thm 10.3] so that the complement contains an interval of the form (α, Ω) which is non-empty, in fact, uncountable [Ex 10.6], cf. [Ex 30.7].)

References

[1] Jesper M. Møller, *General topology*, `http://www.math.ku.dk/ moller/e03/3gt/notes/gtnotes.dvi`.
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11th December 2004

Munkres §25

Ex. 25.1. $\mathbf{R}_\ell$ is totally disconnected [Ex 23.7]; its components and path components [Thm 25.5] are points. The only continuous maps $f\colon \mathbf{R} \to \mathbf{R}_\ell$ are the constant maps as continuous maps on connected spaces have connected images.

Ex. 25.2.

$\mathbf{R}^\omega$ in product topology: Let X be $\mathbf{R}^\omega$ in the product topology. Then X is is path connected (any product of path connected spaces is path connected [Ex 24.8]) and hence also connected.

$\mathbf{R}^\omega$ in uniform topology: Let X be $\mathbf{R}^\omega$ in the uniform topology. Then X is not connected for $X = B \cup U$ where both B, the set of bounded sequences, and U, the complementary set of unbounded sequences, are open as any sequence within distance $\frac{1}{2}$ of a bounded (unbounded) sequence is bounded (unbounded).

We shall now determine the path components of X. Note first that for any sequence (y_n) we have

$$(0) \text{ and } (y_n) \text{ are in the same path component} \Leftrightarrow (y_n) \text{ is a bounded sequence}$$

$\Rightarrow$: Let $u\colon [0,1] \to X$ be a path from (0) to (y_n). Since $u(0) = (0)$ is bounded, also $u(1) = (y_n)$ is bounded for the connected set $u([0,1])$ can not intersect both subsets in a separation of X.

$\Leftarrow$: The formula $u(t) = (ty_n)$ is a path from (0) to (y_n). To see that u is continuous note that $d(u(t_1), u(t_0)) = \sup\{n \in \mathbf{Z}_+ \mid \min(|(t_1-t_0)y_n|, 1)\} = |t_1-t_0|M$ when $|t_1-t_0| < M^{-1}$ where $M = \sup\{|y_n| \mid n \in \mathbf{Z}_+\}$ and d is the uniform metric.

Next observe that $(y_n) \to (x_n) + (y_n)$ is an isometry or X to itself [Ex 20.7]. It follows that in fact

$$(x_n) \text{ and } (y_n) \text{ are in the same path component} \Leftrightarrow (y_n - x_n) \text{ is a bounded sequence}$$

for any two sequences $(x_n), (y_n) \in \mathbf{R}^\omega$.

This describes the path components of X. It also shows that balls of radius < 1 are path connected. Therefore X is locally path connected so that the path components are the components [Thm 25.5].

$\mathbf{R}^\omega$ in box topology: Let X be $\mathbf{R}^\omega$ in the box topology. Then X is not connected for the box topology is finer than the uniform topology [1, Thm 20.4, Ex 23.1]; in fact, $X = B \cup U$ where both B, the set of bounded sequences, and U, the complementary set of unbounded sequences, are open as they are open in the uniform topology or as any sequence in the neighborhood $\prod(x_n - 1, x_n + 1)$ is bounded (unbounded) if (x_n) is bounded (unbounded), see [1, Example 6, p 151].

The (path) components of X can be described as follows:

$$(x_n) \text{ and } (y_n) \text{ in the same (path) component} \Leftrightarrow x_n = y_n \text{ for all but finitely many } n$$

$\Rightarrow$: Suppose that x_n and y_n are different for infintely many $n \in \mathbf{Z}_+$. For each n, choose a homeomorphism $h_n\colon \mathbf{R} \to \mathbf{R}$ such that $h_n(x_n) = 0$ and $h_n(y_n) = n$ in case $x_n \neq y_n$. Then $h = \prod h_n\colon X \to X$ is a homeomorphism with $h(x_n) = (0)$ and $h(y_n) = n$ for infinitely many n. Since a homeomorphism takes (path) components to (path) components and $h(x_n) = (0) \in B$ and $h(y_n) \in U$ are not in the same (path) component, (x_n) and (y_n) are not in the same (path) component either.

$\Leftarrow$: The map $u(t) = ((1-t)x_n + ty_n)$, $t \in [0,1]$, is constant in all but finitely many coordinates. From this we see that $u\colon [0,1] \to X$ is a continuous path from (x_n) to (y_n). Therefore, (x_n) and (y_n) are in the same (path) component.

X is not locally connected since the components are not open [1, Thm 25.3]. The component of the constant sequence (0) is $\mathbf{R}^\infty$.

$\mathbf{R}^\omega$ in the box topology is an example of a space where the components and the path components are the same even though the space is not locally path connected, cf [1, Thm 25.5].

Ex. 25.3. A connected and not path connected space can not be locally path connected [Thm 25.5]. Any linear continuum is locally connected (the topology basis consists of intervals which are connected in a linear continuum [Thm 24.1]). The subsets $\{x\} \times [0,1] = [x \times 0, x \times 1]$, $x \in [0,1]$, are path connected for they are homeomorphic to $[0,1]$ in the usual order topology [Thm 16.4]. There is no continuous path starting in $[x \times 0, x \times 1]$ and and ending in $[y \times 0, y \times 1]$ when $x \neq y$ for the same reason as there is no path from 0×0 to 1×1 [Example 6, p 156]. Therefore these sets are the path components of I_o^2. Since the path components are not open we see once again that I_o^2 is not locally path connected [Thm 25.4]. (${I_o}^2$ is an example of a space with one component and uncountable many path components.)

Ex. 25.4. Any open subset of a locally path connected space is locally path connected. In a locally path connected space, the components and the path components are the same [Thm 25.5].

Ex. 25.8. Let $p\colon X \to Y$ be a quotient map where X is locally (path-)connected. The claim is that Y is locally (path-)connected.

Let U be an open subspace of Y and C a (path-)component of U. We must show that C is open in Y, ie that that $p^{-1}(C)$ is open in X. But $p^{-1}(C)$ is a union of (path-)components of the open set $p^{-1}(U)$ and in the locally (path-)connected space X open sets have open (path-)components.

References

[1] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063

Munkres §26

Ex. 26.1 (Morten Poulsen).

(a). Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X. Suppose $\mathcal{T}' \supset \mathcal{T}$.

If $(X, \mathcal{T}')$ is compact then $(X, \mathcal{T})$ is compact: Clear, since every open covering if $(X, \mathcal{T})$ is an open covering in $(X, \mathcal{T}')$.

If $(X, \mathcal{T})$ is compact then $(X, \mathcal{T})$ is in general not compact: Consider $[0, 1]$ in the standard topology and the discrete topology.

(b).

Lemma 1. *If $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ are compact Hausdorff spaces then either $\mathcal{T}$ and $\mathcal{T}'$ are equal or not comparable.*

Proof. If $(X, \mathcal{T})$ compact and $\mathcal{T}' \supset \mathcal{T}$ then the identity map $(X, \mathcal{T}') \to (X, \mathcal{T})$ is a bijective continuous map, hence a homeomorphism, by theorem 26.6. This proves the result. □

Finally note that the set of topologies on the set X is partially ordered, c.f. ex. 11.2, under inclusion. From the lemma we conclude that the compact Hausdorff topologies on X are minimal elements in the set of all Hausdorff topologies on X.

Ex. 26.2 (Morten Poulsen).

(a). The result follows from the following lemma.

Lemma 2. *If the set X is equipped with the finite complement topology then every subspace of X is compact.*

Proof. Suppose $A \subset X$ and let $\mathcal{A}$ be an open covering of A. Then any set $A_0 \in \mathcal{A}$ will covering all but a finite number of points. Now choose a finite number of sets from $\mathcal{A}$ covering $A - A_0$. These sets and A_0 is a finite subcovering, hence A compact. □

(b). Lets prove a more general result: Let X be an uncountable set. Let

$$\mathcal{T}_c = \{ A \subset X \mid X - A \text{ countable or equal } X \}.$$

It is straightforward to check that $\mathcal{T}_c$ is a topology on X. This topology is called the countable complement topology.

Lemma 3. *The compact subspaces of X are exactly the finite subspaces.*

Proof. Suppose A is infinite. Let $B = \{b_1, b_2, \ldots\}$ be a countable subset of A. Set

$$A_n = (X - B) \cup \{b_1, \ldots, b_n\}.$$

Note that $\{A_n\}$ is an open covering of A with no finite subcovering. □

The lemma shows that $[0, 1] \subset \mathbf{R}$ in the countable complement topology is not compact.

Finally note that $(X, \mathcal{T}_c)$ is not Hausdorff, since no two nonempty open subsets A and B of X are disjoint: If $A \cap B = \emptyset$ then $X - (A \cap B) = (X - A) \cup (X - B)$, hence X countable, contradicting that X uncountable.

Ex. 26.3 (Morten Poulsen).

Theorem 4. *A finite union of compact subspaces of X is compact.*

Proof. Let $A_1, \ldots, A_n$ be compact subspaces of X. Let $\mathcal{A}$ be an open covering of $\bigcup_{i=1}^n A_i$. Since $A_j \subset \bigcup_{i_1}^n A_i$ is compact, $1 \leq j \leq n$, there is a finite subcovering $\mathcal{A}_j$ of $\mathcal{A}$ covering A_j. Thus $\bigcup_{j=1}^n \mathcal{A}_j$ is a finite subcovering of $\mathcal{A}$, hence $\bigcup_{i=1}^n A_i$ is compact. □

Ex. 26.5. For each $a \in A$, choose [Lemma 26.4] disjoint open sets $U_a \in a$ and $V_a \supset B$. Since A is compact, A is contained in a finite union $U = U_1 \cup \cdots \cup U_n$ of the U_as. Let $V = V_1 \cap \cdots V_n$ be the intersection of the corresponding V_as. Then U is an open set containing A, V is an open set containing B, and U and V are disjoint as $U \cap V = \bigcup U_i \cap V \subset \bigcup U_i \cap V_i = \emptyset$.

Ex. 26.6. Since any closed subset A of the compact space X is compact [Thm 26.2], the image $f(A)$ is a compact [Thm 26.5], hence closed [Thm 26.3], subspace of the Hausdorff space Y.

Ex. 26.7. This is just reformulation of The tube lemma [Lemma 26.8]: Let C be a closed subset of $X \times Y$ and $x \in X$ a point such that the slice $\{x\} \times Y$ is disjoint from C. Then, since Y is compact, there is a neighborhood W of x such that the whole tube $W \times Y$ is disjoint from C.

In other words, if $x \notin \pi_1(C)$ then there is a neighborhood W of x which is disjoint from $\pi_1(C)$. Thus The tube lemma says that $\pi_1 \colon X \times Y \to X$ is closed when Y is compact (so that π_1 is an example of a perfect map [Ex 26.12]). On the other hand, projection maps are always open [Ex 16.4].

Ex. 26.8. Let $G \subset X \times Y$ be the graph of a function $f \colon X \to Y$ where Y is compact Hausdorff. Then

$$G \text{ is closed in } X \times Y \Leftrightarrow f \text{ is continuous}$$

$\Leftarrow$: (For this it suffices that Y be Hausdorff.) Let $(x, y) \in X \times Y$ be a point that is not in the graph of f. Then $y \neq f(x)$ so by the Hausdorff axiom there will be disjoint neighborhoods $V \ni y$ and $W \ni f(x)$. By continuity of f, $f(U) \subset W \subset Y - V$. This means that $(U \times V) \cap G = \emptyset$.
$\Rightarrow$: Let V be a neighborhood of $f(x)$ for some $x \in X$. Then $G \cap (X \times (Y - V))$ is closed in $X \times Y$ so [Ex 26.7] the projection $\pi_1(G \cap (X \times (Y - V)))$ is closed in X and does not contain x. Let U be a neighborhood of X such that $U \times Y$ does not intersect $G \cap (X \times (Y - V))$. Then $f(U)$ does not intersect $Y - V$, or $f(U) \subset V$. This shows that f is continuous at the arbitrary point $x \in X$.

Ex. 26.12. (Any perfect map is *proper*; see the January 2003 exam for more on proper maps.) Let $p \colon X \to Y$ be closed continuous surjective map such that $p^{-1}(y)$ is compact for each $y \in Y$. Then $p^{-1}(C)$ is compact for any compact subspace $C \subset Y$.

For this exercise we shall use the following lemma.

Lemma 5. *Let $p \colon X \to Y$ be a closed map.*

(1) *If $p^{-1}(y) \subset U$ where U is an open subspace of X, then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of y.*
(2) *If $p^{-1}(B) \subset U$ for some subspace B of Y and some open subspace U of X, then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of B.*

Proof. Note that

$$\begin{aligned} p^{-1}(W) \subset U &\Leftrightarrow \big[p(x) \in W \Rightarrow x \in U\big] \Leftrightarrow \big[x \notin U \Rightarrow p(x) \notin W\big] \Leftrightarrow p(X - U) \subset Y - W \\ &\Leftrightarrow p(X - U) \cap W = \emptyset \end{aligned}$$

(1) The point y does not belong to the closed set $p(X - U)$. Therefore a whole neighborhood $W \subset Y$ of y is disjoint from $p(X - U)$, i.e. $p^{-1}(W) \subset U$.
(2) Each point $y \in B$ has a neighborhood W_y such that $p^{-1}(W_y) \subset U$. The union $W = \bigcup W_y$ is then a neighborhood of B with $p^{-1}(W) \subset U$. □

We shall not need point (2) here.

Let $C \subset Y$ be compact. Consider a collection $\{U_\alpha\}_{\alpha \in J}$ of open sets covering of $p^{-1}(C)$. For each $y \in C$, the compact space $p^{-1}(y)$ is contained in a the union of a finite subcollection $\{U_\alpha\}_{\alpha \in J(y)}$. There is neighborhood W_y of y such that $p^{-1}(W_y)$ is contained in this finite union. By compactness of C, finitely many $W_{y_1}, \ldots, W_{y_k}$ cover Y. Then the finite collection $\bigcup_{i=1}^k \{U_\alpha\}_{\alpha \in J(y_i)}$ cover $p^{-1}(C)$. This shows that $p^{-1}(C)$ is compact.

Ex. 26.13. Let G be a topological group and A and B subspaces of G.

(a). A closed and B compact $\Rightarrow AB$ closed
Assume $c \notin AB = \bigcup_{b \in B} Ab$. The regularity axiom for G [Suppl Ex 22.7] implies that there are disjoint open sets $W_b \ni c$ and $U_b \supset Ab$ separating c and Ab for each point $b \in B$. Then $A^{-1}U_b$ is an open neighborhood of b. Since B is compact, it can be covered by finitely many of these open sets $A^{-1}U_b$, say

$$B \subset A^{-1}U_1 \cup \cdots \cup A^{-1}U_k = A^{-1}U$$

where $U = U_1 \cup \cdots \cup U_k$. The corresponding open set $W = W_1 \cap \cdots \cap W_k$ is an open neighborhood of c that is disjoint from AB since $W \cap AB \subset \bigcup W \cap U_i \subset \bigcup W_i \cap U_i = \emptyset$.

(b). H compact subgroup of $G \Rightarrow p\colon G \to G/H$ is a closed map
The saturation AH of any closed subset $A \subset G$ is closed by (a).

(c). H compact subgroup of G and G/H compact $\Rightarrow G$ compact
The quotient map $p\colon G \to G/H$ is a perfect map because it is a closed map by (b) and has compact fibres $p^{-1}(gH) = gH$. Now apply [Ex 26.12].

References

4th January 2005

Munkres §27

Ex. 27.1 (Morten Poulsen). Let $A \subset X$ be bounded from above by $b \in X$. For any $a \in A$ is $[a, b]$ compact.

The set $C = \overline{A} \cap [a, b]$ is closed in $[a, b]$, hence compact, c.f. theorem 26.2. The inclusion map $j : C \to X$ is continuous, c.f. theorem 18.2(b). By the extreme value theorem C has a largest element $c \in C$. Clearly c is an upper bound for A.

If $c \in A$ then clearly c is the least upper bound. Suppose $c \notin A$. If $d < c$ then (d, ∞) is an open set containing c, i.e. $A \cap (d, \infty) \neq \emptyset$, since c is a limit point for A, since $c \in C \subset \overline{A}$. Thus d is not an upper bound for A, hence c is the least upper bound.

Ex. 27.3.

(a). K is an infinite, discrete, closed subspace of $\mathbf{R}_K$, so K can not be contained in any compact subspace of $\mathbf{R}_K$ [Thm 28.1].

(b). The subspaces $(-\infty, 0)$ and $(0, +\infty)$ inherit their standard topologies, so they are connected. Then also their closures, $(-\infty, 0]$ and $[0, +\infty)$ and their union, $\mathbf{R}_K$, are also connected [Thm 23.4, Thm 23.3].

(c). Since the topology $\mathbf{R}_K$ is finer than the standard topology [Lemma 13.4] on $\mathbf{R}$ we have

$$U \text{ is connected in } \mathbf{R}_K \overset{\text{Ex 23.1}}{\Rightarrow} U \text{ is connected in } \mathbf{R} \overset{\text{Thm 24.1}}{\Leftrightarrow} U \text{ is convex}$$

for any subspace U of $\mathbf{R}_K$.

Let now $f\colon [0, 1] \to \mathbf{R}_K$ be a path from $f(0) = 0$ to $f(1) = 1$. The image $f([0, 1])$ is convex since it is connected as a subspace of $\mathbf{R}_K$ [Thm 23.5], and connected subspaces of $\mathbf{R}_K$ are convex as we just noted. Therefore the interval $[0, 1]$ and the its subset K is contained in $f([0, 1])$. The image $f([0, 1])$ is also compact in the subspace topology from $\mathbf{R}_K$ [Thm 26.5]. Thus the image is a compact subspace of $\mathbf{R}_K$ containing K; this is a contradiction (see (a)). We conclude that there can not exist any path in $\mathbf{R}_K$ from 0 to 1.

Ex. 27.5. I first repeat Thm 27.7 in order to emphasize the similarity between the two statements.

Theorem 1 (Thm 27.7)**.** *Let X be a compact Hausdorff space with no isolated points. Then X contains uncountably many points.*

Proof. Let $A = \{a_1, a_2, \dots\}$ be a countable subset of X. We must find a point in X outside A.

We have $X \neq \{a_1\}$ for $\{a_1\}$ is not open. So the open set $X - \{a_1\}$ is nonempty. By regularity [Lemma 26.4, Lemma 31.1], we can find an open nonempty set U_1 such that

$$U_1 \subset \overline{U}_1 \subset X - \{a_1\} \subset X$$

We have $U_1 \neq \{a_2\}$ for $\{a_2\}$ is not open. So the open set $U_1 - \{a_2\}$ is nonempty. By regularity [Lemma 26.4, Lemma 31.1], we can find an open nonempty set U_2 such that

$$U_2 \subset \overline{U}_2 \subset U_1 - \{a_2\} \subset U_1$$

Continuing this way we find a descending sequence of nonempty open sets U_n such that

$$U_n \subset \overline{U}_n \subset U_{n-1} - \{a_n\} \subset U_{n-1}$$

for all n.

Because X is compact, the intersection $\bigcap U_n = \bigcap \overline{U}_n$ is nonempty [p. 170] and contained in $\bigcap (X - \{a_n\}) = X - \bigcup \{a_n\} = X - A$. □

Theorem 2 (Baire category theorem)**.** *Let X be a compact Hausdorff space and $\{A_n\}$ a sequence of closed subspaces. If $\operatorname{Int} A_n = \emptyset$ for all n, then $\operatorname{Int} \bigcup A_n = \emptyset$.*

Proof. (See Thm 48.2.) Let U_0 be any nonempty subspace of X. We must find a point in U_0 outside $\bigcup A_n$.

We have $U_0 \not\subset A_1$ for A_1 has no interior. So the open set $U_0 - A_1$ is nonempty. By regularity [Lemma 26.4, Lemma 31.1], we can find a nonempty open set U_1 such that

$$U_1 \subset \overline{U}_1 \subset U_0 - A_1 \subset U_0$$

We have $U_1 \not\subset A_2$ for A_2 has no interior. So the open set $U_1 - A_2$ is nonempty. By regularity [Lemma 26.4, Lemma 31.1], we can find a nonempty open set U_2 such that

$$U_2 \subset \overline{U}_2 \subset U_1 - A_2 \subset U_1$$

Continuing this way, we find a descending sequence of nonempty open sets U_n such that

$$U_n \subset \overline{U}_n \subset U_{n-1} - A_n \subset U_{n-1}$$

for all n.

Because X is compact, the intersection $\bigcap U_n = \bigcap \overline{U}_n$ is nonempty [p. 170] and contained in $U_0 \cap \bigcap (X - A_n) = U_0 - \bigcup A_n$.

□

Ex. 27.6 (The Cantor set).

(a). The set A_n is a union of 2^n disjoint closed intervals of length $1/3^n$. Let p and q be two points in C. Choose n so that $|p - q| > 1/3^n$. Then there is point r between them that is not in A_n, so not in C. As in [Example 4, p. 149], this shows that any subspace of C containing p and q has a separation.

(b). C is compact because [Thm 26.2] it is closed subspace of the compact space $[0, 1]$.

(c). C is constructed from any of the A_n by removing interior points only. Thus the boundary of A_n is contained in C for all n. Any interval of length $> 1/3^{n+1}$ around any point of A_n contains a boundary point of A_{n+1}, hence a point of C. Thus C has no isolated points.

(d). C is a nonempty compact Hausdorff space with no isolated points, so it contains uncountably many points [Thm 27.7].

References

1st December 2004

Munkres §28

Ex. 28.1 (Morten Poulsen). Let d denote the uniform metric. Choose $c \in (0,1]$. Let $A = \{0,c\}^\omega \subset [0,1]^\omega$. Note that if a and b are distinct points in A then $d(a,b) = c$. For any $x \in X$ the open ball $B_d(x, c/3)$ has diameter less than or equal $2c/3$, hence $B_d(x, c/3)$ cannot contain more than one point of A. It follows that x is not a limit point of A.

Ex. 28.6 (Morten Poulsen).

Theorem 1. *Let (X,d) be a compact metric space. If $f : X \to X$ is an isometry then f is a homeomorphism.*

Proof. Clearly any isometry is continuous and injective. If f surjective then f^{-1} is also an isometry, hence it suffices to show that f is surjective.

Suppose $f(X) \subsetneq X$ and let $a \in X - f(X)$. Note that $f(X)$ is compact, since X compact, hence $f(X)$ closed, since X Hausdorff, i.e. $X - f(X)$ is open. Thus there exists $\varepsilon > 0$ such that $a \in B_d(a, \varepsilon) \subset X - f(X)$.

Define a sequence (x_n) by

$$x_n = \begin{cases} a, & n = 1 \\ f(x_n), & n > 1. \end{cases}$$

If $n \neq m$ then $d(x_n, x_m) \geq \varepsilon$: Induction on $n \geq 1$. If $n = 1$ then clearly $d(a, x_m) \geq \varepsilon$, since $x_m \in f(X)$. Suppose $d(x_n, x_m) \geq \varepsilon$ for all $m \neq n$. If $m = 1$ then $d(x_{n+1}, x_1) = d(f(x_n), a) \geq \varepsilon$. If $m > 1$ then $d(x_{n+1}, x_m) = d(f(x_n), f(x_{m-1})) = d(x_n, x_{m-1}) \geq \varepsilon$.

For any $x \in X$ the open ball $B_d(x, \varepsilon/3)$ has diameter less than or equal to $2\varepsilon/3$, hence $B_d(x, \varepsilon/3)$ cannot contain more than one point of A. It follows that x is not a limit point of A. □

April 21, 2006

Munkres §29

Ex. 29.1. Closed intervals $[a, b] \cap \mathbf{Q}$ in $\mathbf{Q}$ are not compact for they are not even sequentially compact [Thm 28.2]. It follows that all compact subsets of $\mathbf{Q}$ have empty interior (are nowhere dense) so $\mathbf{Q}$ can not be locally compact.

To see that compact subsets of $\mathbf{Q}$ are nowhere dense we may argue as follows: If $C \subset \mathbf{Q}$ is compact and C has an interior point then there is a whole open interval $(a, b) \cap \mathbf{Q} \subset C$ and also $[a, b] \cap \mathbf{Q} \subset C$ for C is closed (as a compact subset of a Hausdorff space [Thm 26.3]). The closed subspace $[a, b] \cap \mathbf{Q}$ of C is compact [Thm 26.2]. This contradicts that no closed intervals of $\mathbf{Q}$ are compact.

Ex. 29.2.

(a). Assume that the product $\prod X_\alpha$ is locally compact. Projections are continuous and open [Ex 16.4], so X_α is locally compact for all α [Ex 29.3]. Furthermore, there are subspaces $U \subset C$ such that U is nonempty and open and C is compact. Since $\pi_\alpha(U) = X_\alpha$ for all but finitely many α, also $\pi_\alpha(C) = X_\alpha$ for all but finitely many α. But C is compact so also $\pi_\alpha(C)$ is compact.

(b). We have $\prod X_\alpha = X_1 \times X_2$ where X_1 is a finite product of locally compact spaces and X_2 is a product of compact spaces. It is clear that finite products of locally compact spaces are locally compact for finite products of open sets are open and all products of compact spaces are compact by Tychonoff. So X_1 is locally compact. X_2 is compact, hence locally compact. Thus the product of X_1 and X_2 is locally compact.

Conclusion: $\prod X_\alpha$ is locally compact if and only if X_α is locally compact for all α and compact for all but finitely many α.

Example: $\mathbf{R}^\omega$ and $\mathbf{Z}_+^\omega$ are not locally compact.

Ex. 29.3. Local compactness is not preserved under continuous maps. For an example, let $S \subset \mathbf{R}^2$ be the graph of $\sin(1/x)$, $x \in (0, 1]$. The space $\{(0, 0)\} \cup S$ is not locally compact at $(0, 0)$: Any neighborhood U of $(0, 0)$ contains an infinite subset without limit points, the intersection of S and a horizontal straight line, so U can not [Thm 28.1] be contained in any compact subset of S. On the other hand, $\{(0, 0)\} \cup S$ is the image of a continuous map defined on the locally compact Hausdorff space $\{-1\} \cup (0, 1]$ [Thm 29.2].

Local compactness is clearly preserved under *open* continuous maps as open continuous maps preserve both compactness and openness.

Ex. 29.4 (Morten Poulsen). Let d denote the uniform metric. Suppose $[0, 1]^\omega$ is locally compact at 0. Then $0 \in U \subset C$, where U is open and C is compact. There exists $\varepsilon > 0$ such that $B_d(0, \varepsilon) \subset U$. Note that $A = \{0, \varepsilon/3\}^\omega \subset B_d(0, \varepsilon)$, hence $A \subset C$. By theorem 28.2 A has a limit point in C, contradicting Ex. 28.1.

Ex. 29.5 (Morten Poulsen).

Lemma 1. *A homeomorphism between locally compact Hausdorff spaces extends to a homeomorphism between the one-point compactifications. In other words, homeomorphic locally compact Hausdorff spaces have homeomorphic one-point compactifications.*

Proof. Let $f : X_1 \to X_2$ be a homeomorphism between locally compact Hausdorff spaces. Furthermore let $\omega X_1 = X_1 \cup \{\omega_1\}$ and $\omega X_2 = X_2 \cup \{\omega_2\}$ denote the one-point compactifications. Define $\widetilde{f} : \omega X_1 \to \omega X_2$ by

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in X_1 \\ \omega_2, & x = \omega_1. \end{cases}$$

Note that $\widetilde{f}$ is bijective. Recall that for a locally compact Hausdorff space X the topology on the one-point compactification, ωX, is the collection

$$\{\, U \,|\, U \subset X \text{ open} \,\} \cup \{\, \omega X - C \,|\, C \subset X \text{ compact} \,\},$$

c.f. the proof of theorem 29.1.

If $U \subset X_2$ is open then $\widetilde{f}^{-1}(U) = f^{-1}(U)$ is open in ωX_1. If $C \subset X_2$ is compact then $\widetilde{f}^{-1}(\omega X_2 - C) = \widetilde{f}^{-1}(\omega X_2) - \widetilde{f}^{-1}(C) = \omega X_1 - f^{-1}(C)$ is open in ωX_1, since $f^{-1}(C) \subset X_1$ is compact. It follows that $\widetilde{f}$ is continuous, hence a homeomorphism, by theorem 26.6. □

Finally note that the converse statement does not hold: If $X_1 = [0, 1/2) \cup (1/2, 1]$ and $X_2 = [0, 1)$ then $\omega X_1 = [0, 1] = \omega X_2$. But X_1 and X_2 are not homeomorphic, since X_1 is not connected and X_2 is connected.

Ex. 29.6 (Morten Poulsen). Let S^n denote the unit sphere in $\mathbf{R}^{n+1}$. Let p denote the point $(0, \dots, 0, 1) \in \mathbf{R}^{n+1}$.

Lemma 2. *The punctured sphere $S^n - p$ is homeomorphic to $\mathbf{R}^n$.*

Proof. Define $f : (S^n - p) \to \mathbf{R}^n$ by

$$f(x) = f(x_1, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n).$$

The map f is also known as stereographic projection. It is straightforward to check that the map $g : \mathbf{R}^n \to (S^n - p)$ defined by

$$g(y) = g(y_1, \dots, y_n) = (t(y)\, y_1, \dots, t(y)\, y_n, 1 - t(y)),$$

where $t(y) = 2/(1 + \|y\|^2)$, is a right and left inverse for f. □

Theorem 3. *The one-point compactification of $\mathbf{R}^n$ is homeomorphic to S^n.*

Proof. By the preceding lemma $\mathbf{R}^n$ is homeomorphic to $S^n - p$. The one-point compactification of $S^n - p$ is clearly S^n. Now the result follows from Ex. 29.5. □

Ex. 29.7. Let X be any linearly ordered space with the least upper bound property. As $[a, b] = [a, b) \cup \{b\}$ is compact Hausdorff [Thm 27.1, Thm 17.11], the right half-open interval $[a, b)$ is locally compact Hausdorff and its Alexandroff compactification is $[a, b]$ [Thm 29.1]. Apply this to $S_\Omega = [1, \Omega) \subset \overline{S}_\Omega = [1, \Omega]$. (Apply also to $\mathbf{Z}_+ = [1, \omega) \subset \mathbf{Z}_+ \times \mathbf{Z}_+$ where $\omega = 2 \times 1$ for (an alternative answer to) [Ex 29.8])

Also X itself is locally compact Hausdorff [Thm 17.11] as all closed and bounded intervals in X are compact [Thm 27.1]. Is the one-point compactification of X a linearly ordered space?

Ex. 29.9. This follows from Ex 29.3 for the quotient map $G \to G/H$ is open [SupplEx 22.5.(c)].

Ex. 29.11. It is not always true that the product of two quotient maps is a quotient map [Example 7, p. 143] but here is a case where it is true.

Lemma 4 (Whitehead Theorem)**.** [1, 3.3.17] *Let $p\colon X \to Y$ be a quotient map and Z a locally compact space. Then*

$$p \times 1\colon X \times Z \to Y \times Z$$

is a quotient map.

Proof. Let $A \subset X \times Z$. We must show: $(p \times 1)^{-1}(A)$ is open $\Rightarrow A$ is open. This means that for any point $(x, y) \in (p \times 1)^{-1}(A)$ we must find a saturated neighborhood U of x and a neighborhood V of y such that $U \times V \subset (p \times 1)^{-1}(A)$.

Since $(p \times 1)^{-1}(A)$ is open in the product topology there is a neighborhood U_1 of x and a neighborhood V of y such that $U_1 \times V \subset (p \times 1)^{-1}(A)$. Since Y is locally compact Hausdorff we may assume [Thm 29.2] that $\overline{V}$ is compact and $U_1 \times \overline{V} \subset (p \times 1)^{-1}(A)$. Note that also $p^{-1}(pU_1) \times \overline{V}$ is contained in $(p \times 1)^{-1}(A)$. The tube lemma [Lemma 26.8] says that each point of $p^{-1}(pU_1)$ has a neighborhood such that the product of this neighborhood with $\overline{V}$ is contained in the open set $(p \times 1)^{-1}(A)$. Let U_2 be the union of these neighborhoods. Then $p^{-1}(pU_1) \subset U_2$ and $U_2 \times \overline{V} \subset (p \times 1)^{-1}(A)$. Continuing inductively we find open sets $U_1 \subset U_2 \subset \cdots \subset U_i \subset U_{i+1} \subset \cdots$ such that $p^{-1}(pU_i) \subset U_{i+1}$ and $U_{i+1} \times \overline{V} \subset (p \times 1)^{-1}(A)$. The open set $U = \bigcup U_i$ is saturated because $U \subset p^{-1}(pU) = \bigcup p^{-1}(pU_i) \subset \bigcup U_{i+1} = U$. Thus also $U \times V$ is saturated and $U \times V \subset \bigcup U_i \times V \subset (p \times 1)^{-1}(A)$. □

Example: If $p\colon X \to Z$ is a quotient map, then also $p \times \mathrm{id}\colon X \times [0,1] \to Z \times [0,1]$ is a quotient map. This fact is important for homotopy theory.

Theorem 5. *Let $p\colon A \to B$ and $q\colon C \to D$ be quotient maps. If B and C are locally compact Hausdorff spaces then $p \times q\colon A \times C \to B \times D$ is a quotient map.*

Proof. The map $p \times q$ is the composition

$$A \times C \xrightarrow{p\times 1} B \times C \xrightarrow{1\times q} B \times D$$

of two quotient maps and therefore itself a quotient map [p. 141]. □

References

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1st December 2004

Munkres §30

Ex. 30.3 (Morten Poulsen). Let X be second-countable and let A be an uncountable subset of X. Suppose only countably many points of A are limit points of A and let $A_0 \subset A$ be the countable set of limit points.

For each $x \in A - A_0$ there exists a basis element U_x such that $x \in U_x$ and $U_x \cap A = \{x\}$. Hence if a and b are distinct points of $A - A_0$ then $U_a \neq U_b$, since $U_a \cap A = \{a\} \neq \{b\} = U_b \cap A$. It follows that there uncountably many basis elements, contradicting that X is second-countable.

Note that it also follows that the set of points of A that are not limit points of A are countable.

Ex. 30.4 (Morten Poulsen).

Theorem 1. *Every compact metrizable space is second-countable.*

Proof. Let X be a compact metrizable space, and let d be a metric on X that induces the topology on X.

For each $n \in \mathbf{Z}_+$ let $\mathcal{A}^n$ be an open covering of X with $1/n$-balls. By compactness of X there exists a finite subcovering $\mathcal{A}_n$.

Now $\mathcal{B} = \bigcup_{n \in \mathbf{Z}_+} \mathcal{A}_n$ is countable, being a countable union of finite sets.

$\mathcal{B}$ is a basis: Let U be an open set in X and $x \in U$. By definition of the metric topology there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$. Choose $N \in \mathbf{Z}_+$ such that $2/N < \varepsilon$. Since $\mathcal{A}_N$ covers X there exists $B_d(y, 1/N)$ containing x. If $z \in B_d(y, 1/N)$ then

$$d(x,z) \leq d(x,y) + d(y,z) \leq 1/N + 1/N = 2/N < \varepsilon,$$

i.e. $z \in B_d(x, \varepsilon)$, hence $B_d(y, 1/N) \subset B_d(x, \varepsilon) \subset U$. It follows that $\mathcal{B}$ is a basis. $\square$

Ex. 30.5. Let X be a metrizable topological space.

Suppose that X has a countable dense subset A. The collection $\{B(a,r) \mid a \in A, r \in \mathbf{Q}_+\}$ of balls centered at points in A and with a rational radius is a countable basis for the topology: It suffices to show that for any $y \in B(x, \varepsilon)$ there are $a \in A$ and $r \in \mathbf{Q}_+$ such that $y \in B(a,r) \subset B(x,\varepsilon)$. Let r be a positive rational number such that $2r < \varepsilon - d(x,y)$ and let $a \in A \cap B(y,r)$. Then $y \in B(a,r)$, of course, and $B(a,r) \subset B(x,\varepsilon)$ for if $d(a,z) < r$ then $d(x,z) \leq d(x,y) + d(y,z) \leq d(x,y) + d(y,a) + d(a,z) < d(x,y) + 2r < \varepsilon$.

Suppose that X is Lindelöf. For each positive rational number r, let A_r be a countable subset of X such that $X = \bigcup_{a \in A_r} B(a,r)$. Then $A = \bigcup_{r \in \mathbf{Q}_+} A_r$ is a dense countable subset: Any open ball $B(x, \varepsilon)$ contains a point of A_r when $0 < r < \varepsilon$, $r \in \mathbf{Q}$.

We now have an extended version of Thm 30.3:

Theorem 2. *Let X be a topological space. Then*

$$\begin{array}{ccccc} X \text{ has a countable dense subset} & \Longleftarrow & X \text{ is 2nd countable} & \Longrightarrow & X \text{ is Lindelöf} \\ & & \Downarrow & & \\ & & X \text{ is 1st countable} & & \end{array}$$

If X is metrizable, the three conditions of the top line equivalent.

Ex. 30.6. $\mathbf{R}_\ell$ has a countable dense subset and is not 2nd countable. According to [Ex 30.5] such a space is not metrizable.

The ordered square I_o^2 is compact and not second countable. Any basis for the topology has uncountably many members because there are uncountably many disjoint open sets $(x \times 0, x \times 1)$, $x \in I$, and each of them contains a basis open set. (Alternatively, note that I_o^2 contains the uncountable discrete subspace $\{x \times \frac{1}{2} \mid x \in I\}$ so it can not be second countable by [Example 2 p 190].) According to [Ex 30.4] or [30.5(b)] a compact space with no countable basis is not metrizable.

Ex. 30.7. (Open ordinal space and closed ordinal space) Sets of the form (α, β), $-\infty \leq \alpha < \beta \leq +\infty$, form bases for the topologies on the *open ordinal space* $S_\Omega = [0, \Omega)$ and the *closed ordinal space* $\overline{S}_\Omega = [0, \Omega]$ [§14, Thm 16.4]. The sets $(\alpha, \beta] = (\alpha, \beta + 1) = [\alpha + 1, \beta]$ are closed and open. Let n denote the nth immediate successor of the first element, 0.

- $[0, \Omega)$ **is first countable:** $\{0\} = [0, 1)$ is open so clearly $[0, \Omega)$ is first countable at the point 0. For any other element, $\alpha > 0$, we can use the collection of neighborhoods of the form $(\beta, \alpha]$ for $\beta < \alpha$.
- $[0, \Omega)$ **does not have a countable dense subset:** The complement of any countable subset contains [Thm 10.3] an interval of the form (α, Ω) (which is nonempty, even uncountable [Lemma 10.2]).
- $[0, \Omega)$ **is not second countable:** If it were, there would be a countable dense subset [Thm 30.3].
- $[0, \Omega)$ **is not Lindelöf:** The open covering consisting of the sets $[0, \alpha)$, $\alpha < \Omega$, does not contain a countable subcovering.
- $[0, \Omega]$ **is not first countable at** Ω**:** This is a consequence of [Lemma 21.2] in that Ω is a limit point of $[0, \Omega)$ but not the limit point of any sequence in $[0, \Omega)$ for all such sequences are bounded [Example 3, p. 181].
- $[0, \Omega]$ **does not have a dense countable subset:** for the same reason as for $[0, \Omega)$.
- $[0, \Omega]$ **is not second countable:** It is not even first countable.
- $[0, \Omega]$ **is Lindelöf:** It is even compact [Thm 27.1].

$S_\Omega = [0, \Omega)$ is limit point compact but not compact [Example 2, p. 179] so it can not be metrizable [Thm 28.2]. S_Ω is first countable and limit point compact so it is also sequentially compact [Thm 28.2].

$\overline{S}_\Omega = [0, \Omega]$ is not metrizable since it is not first countable.

Ex. 30.9. A space X is Lindelöf if and only if any collection of closed subsets of X with empty intersection contains a countable subcollection with empty intersection. Since closed subsets of closed subsets are closed, it follows immediately that closed subspaces of Lindelöf spaces are Lindelöf.

The anti-diagonal $L \subset \mathbf{R}_\ell \times \mathbf{R}_\ell$ is a closed discrete uncountable subspace [Example 4 p 193]. Thus the closed subset L does not have a countable dense subset even though $\mathbf{R}_\ell \times \mathbf{R}_\ell$ has a countable dense subset.

Ex. 30.12. Let $f : X \to Y$ be an open continuous map.

Let $\mathcal{B}$ be a neighborhood basis at the point $x \in X$. Let $f(\mathcal{B})$ be the collection of images $f(B) \subset f(X)$ of members B of the collection $\mathcal{B}$. The sets in $f(\mathcal{B})$ are open in Y, and hence also in $f(X)$, since f is an open map. Let $f(x)$ be a point in $f(X)$. Any neighborhood of $f(x)$ has the form $V \cap f(X)$ for some neighborhood $V \subset Y$ of $f(x)$. Since $p^{-1}(V)$ is a neighborhood of x there is a set B in the collection $\mathcal{B}$ such that $x \in B \subset f^{-1}(V)$. Then $x \in f(B) \subset V \cap f(X)$. This shows that $f(\mathcal{B})$ is a neighborhood basis at $f(x) \in f(X)$.

Let $\mathcal{B}$ be a basis for the topology on X. Let $f(\mathcal{B})$ be the collection of images $f(B) \subset f(X)$ of members B of the collection $\mathcal{B}$. The sets in $f(\mathcal{B})$ are open in Y, and hence also in $f(X)$, since f is an open map. Since $\mathcal{B}$ ia a covering of X, $f(\mathcal{B})$ is a covering of $f(X)$. Suppose that $f(x) \in f(B_1) \cap f(B_2)$ where $x \in X$ and B_1, B_2 are basis sets. Choose a basis set B_3 such that $x \in B_3 \subset f^{-1}(f(B_1) \cap f(B_2))$. Then $f(x) \in f(B_3) \subset f(B_1) \cap f(B_2)$. This shows that $f(\mathcal{B})$ is a basis for a topology $\mathcal{T}_{f(\mathcal{B})}$ on $f(X)$. This topology is coarser than the topology on $f(X)$ since the basis elements are open in $f(X)$. Conversely, let $f(x) \in V \cap f(X)$ where V is open in Y. Choose a basis element B such that $x \in B \subset f^{-1}(V)$. Then $f(x) \in f(B) \subset V \cap f(X)$. This shows that all open subsets of $f(X)$ are in $\mathcal{T}_{f(\mathcal{B})}$. We conclude that $f(\mathcal{B})$ is a basis for the topology on $f(X)$.

We conclude that continuous open maps preserve 1st and 2nd countability.

Ex. 30.13. Let D be a countable dense subset and $\mathcal{U}$ a collection of open disjoint subsets. Pick a member of D inside each of the open open sets in $\mathcal{U}$. This gives an injective map $\mathcal{U} \to D$. Since D is countable also $\mathcal{U}$ is countable.

Ex. 30.16. For each natural number $k \in \mathbf{Z}_+$, let D_k be the set of all finite sequences

$$(I_1, \dots, I_k, x_1, \dots, x_k)$$

where $I_1, \dots, I_k \subset I$ are disjoint closed subintervals of I with rational endpoints and $x_1, \dots, x_k \in \mathbf{Q}$ are rational numbers. Since D_k is a subset of a countable set,

$$D_k \hookrightarrow \overbrace{(\mathbf{Q} \times \mathbf{Q}) \times \dots \times (\mathbf{Q} \times \mathbf{Q})}^{k} \times \overbrace{\mathbf{Q} \times \dots \times \mathbf{Q}}^{k} = \mathbf{Q}^{3k},$$

D_k itself is countable [Cor 7.3]. Put $D = \bigcup_{k \in \mathbf{Z}_+} D_k$. As a countable union of countable sets, D is countable [Thm 7.5].

For each element $(I_1, \dots, I_k, x_1, \dots, x_k) \in D_k$, let $x(I_1, \dots, I_k, x_1, \dots, x_k) \in \mathbf{R}^I$ be the element given by

$$\pi_t x(I_1, \dots, I_k, x_1, \dots, x_k) = \begin{cases} x_j & t \in I_j \text{ for some } j \in \{1, \dots, k\} \\ 0 & t \notin I_1 \cup \dots \cup I_k \end{cases}$$

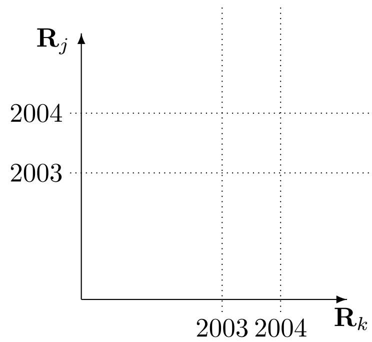
where $\pi_t \colon \mathbf{R}^I \to \mathbf{R}$, $t \in I$, is the projection map. This defines a map $x \colon D \to \mathbf{R}^I$.

(a). The basis open sets in $\mathbf{R}^I$ are finite intersections $\bigcap_{j=1}^k \pi_{i_j}^{-1}(U_{i_j})$ where $i_1, \dots, i_k$ are k distinct points in I and $U_{i_1}, \dots, U_{i_k}$ are k open subsets of $\mathbf{R}$. Choose disjoint closed subintervals I_j such that $i_j \in I_j$ and choose $x_j \in U_{i_j} \cap \mathbf{Q}$, $j = 1, \dots, k$. Then $x(I_1, \dots, I_k, x_1, \dots, x_k) \in \bigcap_{j=1}^k \pi_{i_j}^{-1}(U_{i_j})$ for $\pi_{i_j} x(I_1, \dots, I_k, x_1, \dots, x_k) = x_j \in U_{i_j}$ for all $j = 1, \dots, k$. This shows that any (basis) open set contains an element of $x(D)$, ie that the countable set $x(D)$ is dense in $\mathbf{R}^I$.

(b). Let D be a dense subset of $\mathbf{R}^J$ for some set J. Let $f \colon J \to \mathcal{P}(D)$ be the map from the index set J to the power set $\mathcal{P}(D)$ of D given by $f(j) = D \cap \pi_j^{-1}(2003, 2004)$. Let j and k be two distinct points of J. Then $f(j) \neq f(k)$ for

$$\begin{aligned} f(j) - f(k) &= \left(\pi_j^{-1}(2003, 2004) - \pi_k^{-1}(2003, 2004)\right) \cap D \\ &\supset \left(\pi_j^{-1}(2003, 2004) \cap \pi_k^{-1}(2002, 2003)\right) \cap D \neq \emptyset \end{aligned}$$

since D is dense. This shows that f is injective. Thus card $J \leq$ card $\mathcal{P}(D)$.



References

1st December 2004

Munkres §31

Ex. 31.1 (Morten Poulsen). Let a and b be distinct points of X. Note that X is Hausdorff, since X is regular. Thus there exists disjoint open sets A and B such that $a \in A$ and $b \in B$. By lemma 31.1(a) there exists open sets U and V such that

$$a \in U \subset \overline{U} \subset A \text{ and } b \in V \subset \overline{V} \subset B.$$

Clearly $\overline{U} \cap \overline{V} = \emptyset$.

Ex. 31.2 (Morten Poulsen). Let A and B be disjoint closed subsets of X. Since X normal there exists disjoint open sets U_0 and U_1 such that $A \subset U_0$ and $B \subset U_1$. By lemma 31.1(b) there exists open sets V_0 and V_1 such that

$$A \subset V_0 \subset \overline{V_0} \subset U_0 \text{ and } B \subset V_1 \subset \overline{V_1} \subset U_1$$

Clearly $\overline{U} \cap \overline{V} = \emptyset$.

Ex. 31.3 (Morten Poulsen).

Theorem 1. *Every order topology is regular.*

Proof. Let X be an ordered set. Let $x \in X$ and let U be a neighborhood of x, may assume $U = (a,b)$, $-\infty \leq a < b \leq \infty$. Set $A = (a,x)$ and $B = (x,b)$. Using the criterion for regularity in lemma 31.1(b) there are four cases:

(1) If $u \in A$ and $v \in B$ then $x \in (u,v) \subset \overline{(u,v)} \subset [u,v] \subset (a,b)$.
(2) If $A = B = \emptyset$ then $(a,b) = \{x\}$ is open and closed, since X Hausdorff, c.f. Ex. 17.10.
(3) If $A = \emptyset$ and $v \in B$ then $x \in (a,v) \subset= [x,v) \subset \overline{[x,v)} \subset [x,v] \subset (a,b)$.
(4) If $u \in A$ and $B = \emptyset$ then $x \in (u,b) \subset (u,x] \subset \overline{(u,x]} \subset [u,x] \subset (a,b)$.

Thus X is regular. □

Ex. 31.5. The diagonal $\Delta \subset Y \times Y$ is closed as Y is Hausdorff [Ex 17.13]. The map $(f,g) : X \to Y \times Y$ is continuous [Thm 18.4, Thm 19.6] so

$$\{x \in X \mid f(x) = g(x)\} = (f,g)^{-1}(\Delta)$$

is closed.

Ex. 31.6. Let $p \colon X \to Y$ be closed continuous surjective map. Then X normal $\Rightarrow$ Y normal.
For this exercise and the next we shall use the following lemma from [Ex 26.12].

Lemma 2. *Let $p \colon X \to Y$ be a closed map.*

(1) *If $p^{-1}(y) \subset U$ where U is an open subspace of X, then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of y.*
(2) *If $p^{-1}(B) \subset U$ for some subspace B of Y and some open subspace U of X, then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of B.*

Proof. Note that

$$\begin{aligned} p^{-1}(W) \subset U &\Leftrightarrow \big[p(x) \in W \Rightarrow x \in U\big] \Leftrightarrow \big[x \notin U \Rightarrow p(x) \notin W\big] \Leftrightarrow p(X - U) \subset Y - W \\ &\Leftrightarrow p(X-U) \cap W = \emptyset \end{aligned}$$

(1) The point y does not belong to the closed set $p(X - U)$. Therefore a whole neighborhood $W \subset Y$ of y is disjoint from $p(X-U)$, i.e. $p^{-1}(W) \subset U$.
(2) Each point $y \in B$ has a neighborhood W_y such that $p^{-1}(W_y) \subset U$. The union $W = \bigcup W_y$ is then a neighborhood of B with $p^{-1}(W) \subset U$. □

Since points are closed in X and p is closed, all points in $p(X)$ are closed. All fibres $p^{-1}(y) \subset X$ are therefore also closed. Let y_1 and y_2 be two distinct points in Y. Since X is normal we can separate the disjoint closed sets $p^{-1}(y_1)$ and $p^{-1}(y_1)$ by disjoint neighborhoods U_1 and U_2. Using Lemma 2.(1), choose neighborhoods W_1 of y_1 and W_2 of y_2 such that $p^{-1}(W_1) \subset U_1$ and $p^{-1}(W_2) \subset U_2$. Then W_1 and W_2 are disjoint. Thus Y is Hausdorff.

Essentially the same argument, but now using Lemma 2.(2), shows that we can separate disjoint closed sets in Y by disjoint open sets. Thus Y is normal.

Alternatively, see [Lemma 73.3].

Example: If X is normal and $A \subset X$ is closed, then the quotient space X/A is normal.

Ex. 31.7. Let $p\colon X \to Y$ be closed continuous surjective map such that $p^{-1}(y)$ is compact for each $y \in Y$ (a perfect map).

(a). X Hausdorff $\Rightarrow$ Y Hausdorff.
Let y_1 and y_2 be two distinct points in Y. By an upgraded version [Ex 26.5] of [Lemma 26.4] we can separate the two disjoint compact subspaces $p^{-1}(y_1)$ and $p^{-1}(y_2)$ by disjoint open subspaces $U_1 \supset p^{-1}(y_1)$ and $U_2 \supset p^{-1}(y_2)$ of the Hausdorff space X. Choose (Lemma 2) open sets $W_1 \ni y_1$ and $W_2 \ni y_2$ such that $p^{-1}(W_1) \subset U_1$ and $p^{-1}(W_2) \subset U_2$. Then W_1 and W_2 are disjoint. This shows that Y is Hausdorff as well.

(b). X regular $\Rightarrow$ Y regular.
Y is Hausdorff by (a). Let $C \subset Y$ be a closed subspace and $y \in Y$ a point outside C. It is enough to separate the compact fibre $p^{-1}(y) \subset X$ and the closed set $p^{-1}(C) \subset X$ by disjoint open set. (Lemma 2 will provide open sets in Y separating y and C.) Each $x \in p^{-1}(y)$ can be separated by disjoint open sets from $p^{-1}(C)$ since X is regular. Using compactness of $p^{-1}(y)$ we obtain (as in the proof [Thm 26.3]) disjoint open sets $U \supset p^{-1}(y)$ and $V \supset p^{-1}(C)$ as required.

(c). X locally compact $\Rightarrow$ Y locally compact [1, 3.7.21].
Using compactness of $p^{-1}(y)$ and local compactness of X we construct an open subspace $U \subset X$ and a compact subspace $C \subset X$ such that $p^{-1}(y) \subset U \subset C$. In the process we need to know that a finite union of compact subspaces is compact [Ex 26.3]. By Lemma 2, there is an open set $W \ni y$ such that $p^{-1}(y) \subset p^{-1}(W) \subset U \subset C$. Then $y \in W \subset p(C)$ where $p(C)$ is compact [Thm 26.5]. Thus Y is locally compact.

(d). X 2nd countable $\Rightarrow$ Y 2nd countable.
Let $\{B_j\}_{j \in \mathbf{Z}_+}$ be countable basis for X. For each finite subset $J \subset \mathbf{Z}_+$, let $U_J \subset X$ be the union of all open sets of the form $p^{-1}(W)$ with open $W \subset Y$ and $p^{-1}(W) \subset \bigcup_{j \in J} B_j$. There are countably many open sets U_J. The image $p(U_J)$ is a union of open sets in Y, hence open. Let now $V \subset Y$ be any open subspace. The inverse image $p^{-1}(V) = \bigcup_{y \in V} p^{-1}(y)$ is a union of fibres. Since each fibre $p^{-1}(y)$ is compact, it can be covered by a finite union $\bigcup_{j \in J(y)} B_j$ of basis sets contained in $p^{-1}(V)$. By Lemma 2, there is an open set $W \subset Y$ such that $p^{-1}(y) \subset p^{-1}(W) \subset \bigcup_{j \in J(y)} B_j$. Taking the union of all these open sets W, we get $p^{-1}(y) \subset U_{J(y)} \subset \bigcup_{j \in J(y)} B_j \subset p^{-1}(V)$. We now have $p^{-1}(V) = \bigcup_{y \in V} U_{J(y)}$ so that $V = pp^{-1}(V) = \bigcup_{y \in V} p(U_{J(y)})$ is a union of sets from the countable collection $\{p(U_J)\}$ of open sets. Thus Y is 2nd countable.

Example: If Y is compact, then the projection map $\pi_2\colon X \times Y \to Y$ is perfect. (Show that π_2 is closed!)

Ex. 31.8. It is enough to show that $p\colon X \to G\backslash X$ is a perfect map [Ex 31.6, Ex 31.7]. We show that

(1) The saturation GA of any closed subspace $A \subset X$ is closed. (The map p is closed.)
(2) The orbit Gx of any point $x \in X$ is compact. (The fibres $p^{-1}(Gx) = Gx$ are compact.)

(1) Let $y \in X$ be any point outside $GA = \bigcup_{g \in G} gA$. For any $g \in G$, $g^{-1}y$ is outside the closed set $A \subset X$. By continuity of the action $G \times X \to X$,

$$U_g^{-1} V_g \subset X - A$$

for open sets $G \supset U_g \ni g$ and $X \subset V_g \ni y$. The compact space G can be covered by finitely many of the open sets U_g, say $G = U_1 \cup \cdots \cup U_n$. Let $V = V_1 \cap \cdots \cap V_n$ be the intersection of the corresponding neighborhoods of y. Then

$$G^{-1}V = \bigcup_i U_i^{-1}V \subset \bigcup_i U_i^{-1}V_i \subset X - A$$

so $y \in V \subset G(X - A) = X - GA$.
(2) The orbit Gx of a point $x \in X$ is compact because [Thm 26.5] it is the image of the compact space G under the continuous map $G \to X\colon g \to gx$.

References

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Munkres §32

Ex. 32.1. Let Y be a closed subspace of the normal space X. Then Y is Hausdorff [Thm 17.11]. Let A and B be disjoint closed subspaces of Y. Since A and B are closed also in X, they can be separated in X by disjoint open sets U and V. Then $Y \cap U$ and $V \cap Y$ are open sets in Y separating A and B.

Ex. 32.3. Look at [Thm 29.2] and [Lemma 31.1]. By [Ex 33.7], locally compact Hausdorff spaces are even completely regular.

Ex. 32.4. Let A and B be disjoint closed subsets of a regular Lindelöf space. We proceed as in the proof of [Thm 32.1]. Each point $a \in A$ has an open neighborhood U_a with closure $\overline{U}_a$ disjoint from B. Applying the Lindelöf property to the open covering $\{U_a\}_{a\in A} \cup \{X - A\}$ we get a countable open covering $\{U_i\}_{i\in\mathbf{Z}_+}$ of A such that the closure of each U_i is disjoint from B. Similarly, there is a countable open covering $\{V_i\}_{i\in\mathbf{Z}_+}$ of B such that the closure of each V_i is disjoint from A. Now the open set $\bigcup U_i$ contains A and $\bigcup V_i$ contains B but these two sets are not necessarily disjoint. If we put $U'_1 = U_1 - \overline{V_1}$, $U'_2 = U_2 - \overline{V_1} - \overline{V_2}, \dots,$ $U'_i = U_i - \overline{V_1} - \cdots - \overline{V_i}, \dots$ we subtract no points from A so that the open sets $\{U'_i\}$ still form an open covering of A. Similarly, the open sets $\{V'_i\}$, where $V'_i = V_i - \overline{U_1} - \cdots - \overline{U_i}$, cover B. Moreover, the open sets $\bigcup U'_i$ and $\bigcup V'_i$ are disjoint for U'_i is disjoint from $V_1 \cup \cdots \cup V_i$ and V'_i is disjoint from $U_1 \cup \cdots \cup U_i$.

Ex. 32.5. $\mathbf{R}^\omega$ (in product topology) is metrizable [Thm 20.5], in particular normal [Thm 32.2]. $\mathbf{R}^\omega$ in the uniform topology is, by its very definition [Definition p. 124], metrizable, hence normal.

Ex. 32.6. Let X be completely normal and let A and B be separated subspaces of X; this means that $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. Note that A and B are contained in the open subspace $X - (\overline{A} \cap \overline{B}) = (X - \overline{A}) \cup (X - \overline{B})$ where their closures are disjoint. (The closure of A in $X - (\overline{A} \cap \overline{B})$ is $\overline{A} - \overline{B}$ [Thm 17.4].) The subspace $X - (\overline{A} \cap \overline{B})$ is normal so it contains disjoint open subsets $U \supset A$ and $V \supset B$. Since U and V are open in an open subspace, they are open [Lemma 16.2].

Conversely, suppose that X satisfies the condition (and is a T_1-space). Let Y be any subspace of X and A and B two disjoint closed subspaces of Y. Since $\overline{A} \cap Y$ and $\overline{B} \cap Y$ are disjoint [Thm 17.4], $\overline{A} \cap B = \overline{A} \cap (Y \cap B) = (\overline{A} \cap Y) \cap (B \cap Y) = \emptyset$, and, similarly, $A \cap \overline{B} = \emptyset$. By assumption, A and B can then be separated by disjoint open sets. If we also assume that X is T_1 then it follows that Y is normal.

References

Munkres §33

Ex. 33.1 (Morten Poulsen). Let $r \in [0,1]$. Recall from the proof of the Urysohn lemma that if $p < q$ then $\overline{U}_p \subset U_q$. Furthermore, recall that $U_q = \emptyset$ if $q < 0$ and $U_p = X$ if $p > 1$.

Claim 1. $f^{-1}(\{r\}) = \bigcap_{p>r} U_p - \bigcup_{q<r} U_q$, $p, q \in \mathbf{Q}$.

Proof. By the construction of $f \colon X \to [0,1]$,

$$\bigcap_{p>0} U_p - \bigcup_{q<0} U_q = \bigcap_{p>0} U_p = f^{-1}(\{0\})$$

and

$$\bigcap_{p>1} U_p - \bigcup_{q<1} U_q = X - \bigcup_{q<1} U_q = f^{-1}(\{1\}).$$

Now assume $r \in (0,1)$.

"$\subset$": Let $x \in f^{-1}(\{r\})$, i.e. $f(x) = r = \inf\{p \,|\, x \in U_p\}$. Note that $x \notin \bigcup_{q<r} U_q$, since $f(x) = r$. Suppose there exists $t > r$, $t \in \mathbf{Q}$, such that $x \notin U_t$. Since $f(x) = r$, there exists $s \in \mathbf{Q}$ such that $r \leq s < t$ and $x \in U_s$. Now $x \in U_s \subset \overline{U}_s \subset U_t$, contradiction. It follows that $x \in \bigcap_{p>r} U_p - \bigcup_{q<r} U_q$.

"$\supset$": Let $x \in \bigcap_{p>r} U_p - \bigcup_{q<r} U_q$. Note that $f(x) \leq r$, since $x \in \bigcap_{p>r} U_p$. Suppose $f(x) < r$, i.e. there exists $t < r$ such that $x \in U_t \subset \bigcup_{q<r} U_q$, contradiction. It follows that $x \in f^{-1}(\{r\})$. □

Ex. 33.4 (Morten Poulsen).

Theorem 2. *Let X be normal. There exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) > 0$ for $x \notin A$, if and only if A is a closed G_δ set in X.*

Proof. Suppose $A = f^{-1}(\{0\})$. Since

$$A = f^{-1}(\{0\}) = f^{-1}(\bigcap_{n \in \mathbf{Z}_+} [0, 1/n)) = \bigcap_{n \in \mathbf{Z}_+} f^{-1}([0, 1/n))$$

it follows that A is a closed G_δ set.

Conversely suppose A is a closed G_δ set, i.e. $A = \bigcap_{n \in \mathbf{Z}_+} U_n$, U_n open. Then $X - U_n$ and A are closed and disjoint for all n. By Urysohn's lemma there exists a continuous function $f_n : X \to [0,1]$, such that $f_n(A) = \{0\}$ and $f_n(X - U_n) = \{1\}$.

Now define $f : X \to [0,1]$ by

$$f(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x).$$

Clearly f is well-defined. Furthermore f is continuous, by theorem 21.6, since the sequence of continuous functions $\left(\sum_{i=1}^{n} \frac{1}{2^i} f_i(x)\right)_{n \in \mathbf{Z}_+}$ converges uniformly to f, since

$$\left| \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x) - \sum_{i=1}^{n} \frac{1}{2^i} f_i(x) \right| = \sum_{i=n+1}^{\infty} \frac{1}{2^i} f_i(x) \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} \to 0$$

for $n \to \infty$.

Clearly $f(x) = 0$ for $x \in A$. Furthermore note that if $x \notin A$ then $x \in X - U_n$ for some n, hence $f(x) \geq \frac{1}{2^n} f_n(x) = \frac{1}{2^n} > 0$. □

Ex. 33.5 (Morten Poulsen).

Theorem 3 (Strong form of the Urysohn lemma). *Let X be a normal space. There is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) = 1$ for $x \in B$, and $0 < f(x) < 1$ otherwise, if and only if A and B are disjoint closed G_δ sets in X.*

Proof. Suppose $f : X \to [0,1]$ is a continuous function. Then clearly $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$ are disjoint. Since

$$A = f^{-1}(\{0\}) = f^{-1}(\bigcap_{n\in\mathbf{Z}_+}[0,1/n)) = \bigcap_{n\in\mathbf{Z}_+} f^{-1}([0,1/n))$$

and

$$B = f^{-1}(\{1\}) = f^{-1}(\bigcap_{n\in\mathbf{Z}_+}(1-1/n,1]) = \bigcap_{n\in\mathbf{Z}_+} f^{-1}((1-1/n,1])$$

it follows that A and B are disjoint closed G_δ sets in X.

Conversely suppose A and B are disjoint closed G_δ sets in X. By ex. 33.4 there exists continuous functions $f_A : X \to [0,1]$ and $f_B : X \to [0,1]$, such that $f_A^{-1}(\{0\}) = A$ and $f_B^{-1}(\{0\}) = B$. Now the function $f : X \to [0,1]$ defined by

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}$$

is well-defined and clearly continuous. Furthermore $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$, since

$$f(x) = 0 \Leftrightarrow f_A(x) = 0 \Leftrightarrow x \in A$$

and

$$f(x) = 1 \Leftrightarrow f_A(x) = f_A(x) + f_B(x) \Leftrightarrow f_B(x) = 0 \Leftrightarrow x \in B.$$

□

Ex. 33.7. For any topological space X we have the following implications:

$$\begin{aligned} &X \text{ is locally compact Hausdorff} \\ &\overset{\text{Cor 29.4}}{\Rightarrow} X \text{ is an open subspace of a compact Hausdorff space} \\ &\overset{\text{Thm 32.2}}{\Rightarrow} X \text{ is a subspace of a normal space} \\ &\overset{\text{Thm 33.1}}{\Rightarrow} X \text{ is a subspace of a completely regular space} \\ &\overset{\text{Thm 33.2}}{\Rightarrow} X \text{ is completely regular} \end{aligned}$$

Ex. 33.8. Using complete regularity of X and compactness of A, we see that there is a continuous real-valaued function $g\colon X \to [0,1]$ such that $g(a) < \frac{1}{2}$ for all $a \in A$ and $g(B) = \{1\}$. (There are finitely many continuous functions $g_1, \dots, g_k\colon X \to [0,1]$ such that $A \subset \bigcup\{g_i < \frac{1}{2}\}$ and $g_i(B) = 1$ for all i. Put $g = \frac{1}{k}\sum g_i$.) The continuous [Ex 18.8] function $f = 2\max\{0, g - \frac{1}{2}\}$ maps X into the unit interval, $g(A) = \{0\}$, and $g(B) = \{1\}$.

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Munkres §34

Ex. 34.1. We are looking for a non-regular Hausdorff space. By Example 1 p. 197, $\mathbf{R}_K$ [p. 82] is such a space. Indeed, $\mathbf{R}_K$ is Hausdorff for the topology is finer than the standard topology [Lemma 13.4]. $\mathbf{R}_K$ is 2nd countable for the sets (a,b) and $(a,b)-K$, where the intervals have rational end-points, constitute a countable basis. $\mathbf{R}_K$ is not metrizable for it is not even regular [Example 1, p. 197].
Conclusion: The regularity axiom can not be replaced by the Hausdorff axiom in the Urysohn metrization theorem [Thm 34.1].

Ex. 34.2. We are looking for 1st but not 2nd countable space. By Example 3 p. 192, $\mathbf{R}_\ell$ [p. 82] is such a space. Indeed, the Sorgenfrey right half-open interval topology $\mathbf{R}_\ell$ [p. 82] is completely normal [Ex 32.4], 1st countable, Lindelöf, has a countable dense subset [Example 3, p. 192], but is not metrizable [Ex 30.6].

Ex. 34.3. We characterize the metrizable spaces among the compact Hausdorff spaces.

Theorem 1. *Let X be a compact Hausdorff space. Then*

$$X \text{ is metrizable} \Leftrightarrow X \text{ is 2nd countable}$$

Proof. $\Rightarrow$: Every compact metrizable space is 2nd countable [Ex 30.4].
$\Leftarrow$: Every compact Hausdorff space is normal [Thm 32.3]. Every 2nd countable normal space is metrizable by the Urysohn metrization theorem [Thm 34.1]. □

We may also characterize the metrizable spaces among 2nd countable spaces.

Theorem 2. *Let X be a 2nd countable topological space. Then*

$$X \text{ is metrizable} \overset{\text{Thm 34.1, 32.2}}{\Leftrightarrow} X \text{ is (completely) normal} \overset{\text{Thm 32.1}}{\Leftrightarrow} X \text{ is regular}$$

Ex. 34.4. Let X be a locally compact Hausdorff space. Then

$$X \text{ is metrizable} \Leftarrow X \text{ is 2nd countable}$$

$\not\Rightarrow$: Any discrete uncountable space is metrizable and not 2nd countable.
$\Leftarrow$: Every locally compact Hausdorff space is regular [Ex 32.3] (even completely regular [Ex 33.7]). Every 2nd countable regular space is metrizable by the Urysohn metrization theorem [Thm 34.1].

Ex. 34.5.

Theorem 3. *Let X be a locally compact Hausdorff space and X^+ its one-point-compactification. Then*

$$X^+ \text{ is metrizable} \Leftrightarrow X \text{ is 2nd countable}$$

Proof. $\Rightarrow$: : Every compact metrizable space is 2nd countable [Ex 30.4]. Every subspace of a 2nd countable space is 2nd countable [Thm 30.2].
$\Leftarrow$: Suppose that X has the countable basis $\mathcal{B}$. It suffices to show that also X^+ has a countable basis [Ex 34.3]. Any open subset of X is a union of elements from $\mathcal{B}$. The remaining open sets in X^+ are neighborhoods of ∞. Any neighborhood of ∞ is of the form $X^+ - C$ where C is a compact subspace of X. For each point $x \in C$ there is a basis neighborhood $U_x \in \mathcal{B}$ such that $\overline{U}$ is compact [Thm 29.3]. By compactness, C is covered by finitely many basis open sets $C \subset U_1 \cup \cdots \cup U_k$. Now

$$\infty \in X^+ - (\overline{U}_1 \cup \cdots \overline{U}_k) \subset X^+ - C$$

where $X^+ - (\overline{U}_1 \cup \cdots \overline{U}_k)$ is open in X^+ since $\overline{U}_1 \cup \cdots \overline{U}_k$ is compact in X [Ex 26.3]. This shows that if we supplement $\mathcal{B}$ with all sets of the form $X^+ - (\overline{U}_1 \cup \cdots \overline{U}_k)$, $k \in \mathbf{Z}_+$, $U_i \in \mathcal{B}$, and call the union $\mathcal{B}^+$, then $\mathcal{B}^+$ is a basis for the topology on X^+. Since there are only countable many finite subsets of $\mathcal{B}$ [Ex 7.5.(j)], the enlarged basis $\mathcal{B}^+$ is still countable [Thm 7.5]. □

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Ex. 35.3. Let X be a metrizable topological space.

(i) $\Rightarrow$ (ii): (We prove the contrapositive.) Let d be any metric on X and $\varphi\colon X \to \mathbf{R}$ be an unbounded real-valued function on X. Then $\overline{d}(x,y) = d(x,y) + |\varphi(x) - \varphi(y)|$ is an unbounded metric on X that induces the same topology as d since

$$B_{\overline{d}}(x,\varepsilon) \subset B_d(x,\varepsilon) \subset B_{\overline{d}}(x,\delta)$$

for any $\varepsilon > 0$ and any $\delta > 0$ such that $\delta < \frac{1}{2}\varepsilon$ and $d(x,y) < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \frac{1}{2}\varepsilon$.

(ii) $\Rightarrow$ (iii): (We prove the contrapositive.) Let X be a normal space that is not limit point compact. Then there exists a closed infinite subset $A \subset X$ [Thm 17.6]. Let $f\colon X \to \mathbf{R}$ be the extension [Thm 35.1] of any surjection $A \to \mathbf{Z}_+$. Then f is unbounded.

(iii) $\Rightarrow$ (i): Any limit point compact metrizable space is compact [Thm 28.2]; any metric on X is continuous [Ex 20.3], hence bounded [Thm 26.5].

Ex. 35.4. Let Z be a topological space and $Y \subset Z$ a subspace. Y is a retract of Z if the identity map on Y extends continuously to Z, i.e. if there exists a continuous map $r\colon Z \to Y$ such that

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, hook] & Y \\ Z \arrow[ur, "r"'] & \end{tikzcd}$$

commutes.

(a). $Y = \{z \in Z \mid r(z) = z\}$ is closed if Z is Hausdorff [Ex. 31.5].

(b). Any retract of $\mathbf{R}^2$ is connected [Thm 23.5] but A is not connected.

(c). The continuous map $r(x) = x/|x|$ is a retraction of the punctured plane $\mathbf{R}^2 - \{0\}$ onto the circle $S^1 \subset \mathbf{R}^2 - \{0\}$.

Ex. 35.5. A space Y has the UEP if the diagram

(1)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \\ X \arrow[ur, dotted, "\overline{f}"'] & \end{tikzcd}$$

has a solution for any closed subspace A of a normal space X.

(a). Another way of formulating the Tietze extension theorem [Thm 35.1] is: $[0,1]$, $[0,1)$, and $(0,1) \simeq \mathbf{R}$ have the UEP. By the universal property of product spaces [Thm 19.6], $\mathrm{map}(A, \prod X_\alpha) = \prod \mathrm{map}(A, X_\alpha)$, any product of spaces with the UEP has the UEP.

(b). Any retract Y of a UEP space Z is a UEP space for in the situation

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[r, hook, shift right] & Z \arrow[l, "r"', shift right] \\ X \arrow[urr, "\overline{f}"'] & & \end{tikzcd}$$

the continuous map $r\overline{f}\colon X \to Y$ extends $f\colon A \to Y$.

0.1. **Ex. 35.6.** Let Y be a normal space. We say that Y is an absolute retract if for any imbedding $Y \xhookrightarrow{i} Z$ of Y into a closed subspace of a normal space Z there is a map $r\colon Z \to Y$ such that

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[d, equal] & Z \arrow[dl, dotted, "r"] \\ Y & \end{tikzcd}$$

commutes, i.e. such that ri is the identity on Y.

(a). Any space Y with the UEP is an absolute retract: Apply (1) with input $Z \hookleftarrow Y = Y$.

(b). If Y is compact: Y has the UEP $\Leftrightarrow$ Y is an absolute retract. (Cf. [Ex 35.8])
The compact Hausdorff spaces are precisely the spaces that are homeomorphic to a closed subspace of $[0,1]^J$ for some set J [Thm 34.3, Thm 37.3]. Therefore any compact Hausdorff space that is also an absolute retract is a retract of the UEP space $[0,1]^J$, hence is itself a UEP space [Ex 31.5.(b)].

Ex. 35.7.

(a). The space $C \subset \mathbf{R}^2$ is a closed subspace of the normal space $\mathbf{R}^2$ homeomorphic to $[0,\infty)$. The Tietze theorem (small variation of [Thm 35.1.(b)]) says that $[0,\infty)$ has the UEP. Therefore $[0,\infty)$ is an absolute retract [Ex 35.6] and C is retract of $\mathbf{R}^2$. The continuous map $r\colon \mathbf{R}^2 \to C$ given by

$$r(x) = \begin{cases} |x| \cos\log|x| \times \log|x| \sin\log|x| & x \neq 0 \times 0 \\ 0 \times 0 & x = 0 \times 0 \end{cases}$$

is a retraction of $\mathbf{R}^2$ onto the logarithmic spiral C.

(b). The space $K \subset \mathbf{R}^3$ is a closed subspace of the normal space $\mathbf{R}^3$ homeomorphic to $\mathbf{R}$. The Tietze theorem says that $\mathbf{R}$ has the UEP. Therefore $\mathbf{R}$ is an absolute retract [Ex 35.6] and K is a retract of $\mathbf{R}^3$; I can't find an explicit retraction $\mathbf{R}^3 \to K$, though.

Ex. 35.8. (Adjunction spaces [1, p 93] [2, Chp I, Exercise B, p 56]) Let X and Y be two disjoint topological spaces and $f\colon A \to Y$ a continuous map defined on a closed subspace A of X. Define $X \cup_f Y$ to be the quotient of $X \cup Y$ by the smallest equivalence relation such that $a \in A$ and $f(a) \in Y$ are equivalent for all points $a \in A$. (To picture this, tie an elastic band from each point a of A to its image $f(a)$ in Y and let go!) The equivalence classes, $[y] = f^{-1}(y) \cup \{y\}$ for $y \in Y$ and $[x] = \{x\}$ for $x \in X - A$, are represented by points in Y or in $X - A$. Let $p\colon X \cup Y \to X \cup_f Y$ be the quotient map; p_X the restriction of p to X and p_Y the restriction of p to Y.

The adjunction space $X \cup_f Y$ fits into a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"'] & X \cup_f Y \end{tikzcd}$$

called a *push-out diagram* because of this universal property: If $X \to Z$ and $Y \to Z$ are continuous maps that agree on A then there is [Thm 22.2] a unique continuous map $X \cup_f Y \to Z$ such that the diagram

(2)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, "p_Y"] \arrow[ddr, bend left] & \\ X \arrow[r, "p_X"'] \arrow[drr, bend right] & X \cup_f Y \arrow[dr, dotted, "\exists!"] & \\ & & Z \end{tikzcd}$$

commutes. (This is just the universal property for quotient spaces in this particular situation.)

Here are the main properties of adjunction spaces.

Lemma 1. *Let $p\colon X \cup Y \to X \cup_f Y$ be the quotient map.*

(1) *The quotient map p embeds Y into a closed subspace of $X \cup_f Y$. (We therefore identify Y with its image $p_Y(Y)$ in the adjunction space.)*
(2) *The quotient map p embeds $X - A$ into the open subspace $(X \cup_f Y) - Y$ of the adjunction space.*
(3) *If X and Y are normal, also the adjunction space $X \cup_f Y$ is normal.*
(4) *The projection map $p\colon X \cup Y \to X \cup_f Y$ is closed if (and only if* [1, p 93]*) f is closed.*

Proof. (1) The map $p_Y = p|Y \colon Y \to X \cup_f Y$ is closed for closed sets $B \subset Y \subset X \amalg Y$ have closed saturations $f^{-1}(B) \amalg B$. Since p_Y is also injective it is an embedding.
(2) The map $p_X|X - A\colon X - A \to (X \cup_f Y$ open because the saturation of any (open) subset U of $X - A$ is $U \cup \emptyset \subset X \cup Y$ itself. Since $p_X|X - A$ is also injective it is an embedding.
(3) Points are closed in the quotient space $X \cup_f Y$ because the equivalence classes are closed in $X \cup Y$. Let C and D be two disjoint closed subspaces of $X \cup_f Y$. We will show that there is a continuous map $X \cup_f Y \to [0, 1]$ with value 0 on C and value 1 on D. Since Y is normal, there exists [Thm 33.1] a Urysohn function $g\colon Y \to [0, 1]$ such that $g(Y \cap C) = \{0\}$ and $g(Y \cap D) = \{1\}$. Since X is normal, by the Tietze extension theorem [Thm 35.1], there is a continuous map $X \to [0, 1]$ which is 0 on $p_X^{-1}(C)$, 1 on $p_X^{-1}(D)$, and is $g \circ f$ on A. By the universal property for adjunction spaces (2), there is a map $X \cup_f Y \to [0, 1]$ that is 0 on C and 1 on D. This shows that C and D can be separated by a continuous function and that $X \cup_f Y$ is normal.
(4) Closed subsets of Y always have closed saturations as we saw in item (1). If f is closed then also the saturation, $B \cup f^{-1}f(A \cap B) \cup f(A \cap B) \subset X \cup Y$, of a closed subset $B \subset X$ is closed. (Since closed quotient maps (surjective closed maps) preserve normality [Ex 31.6, Thm 73.3] this gives an easy proof of (3) under the additional assumption that $f\colon A \to Y$ be a closed map.) □

The adjunction space is the disjoint union of a closed subspace homeomorphic to Y and an open subspace homeomorphic to $X - A$.

Theorem 2. [2, Chp I, Exercise C, p 56] *Let Y be a normal space. Then Y has the universal extension property if and only if Y is an absolute retract.*

Proof. One direction was proved already in Ex 35.6. For the other direction, suppose that the normal space Y is an absolute retract. Let X be any normal space, A a closed subspace of X, and $f\colon A \to Y$ a continuous map. Form the adjunction space $Z = X \cup_f Y$. Then Z is normal (as we have just seen) and Y is (homeomorphic) to a closed subspace of Z. Since Y is an absolute retract, there is a retraction $r\colon Z \to Y$ of Z onto Y. These maps are shown in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, "p_Y"] \arrow[dr, equal] & \\ X \arrow[r, "p_X"'] & X \cup_f Y \arrow[r, "r"'] & Y \end{tikzcd}$$

which says that $r \circ p_X \colon X \to Y$ is an extension of $f\colon A \to Y$. This shows that Y has the universal extension property. □

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Ex. 36.1. Any locally euclidean space is locally compact (as open subspaces of euclidean space are locally compact [Cor 29.3]). A manifold is locally euclidean *and* Hausdorff, so it is locally compact Hausdorff, hence regular [Ex 32.3]. A manifold also has a countable basis, so it is normal [Thm 32.1] and metrizable [Thm 34.1].

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Ex. 38.4. Let $X \to \beta X$ be the Stone–Čech compactification and $X \to cX$ an arbitrary compactification of the completely regular space X. By the universal property of the Stone–Čech compactification, the map $X \to cX$ extends uniquely

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & cX \\ & \beta X \arrow[ur] & \end{tikzcd}$$

to a continuous map $\beta X \to cX$. Any continuous map of a compact space to a Hausdorff space is closed. In particular, $\beta X \to cX$ is closed. It is also surjective for it has a dense image since $X \to cX$ has a dense image. Thus $\beta X \to cX$ is a closed quotient map.

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Ex. 38.5.

(a). For any $\varepsilon > 0$ there exists an $\alpha \in S_\Omega = [0, \Omega)$ such that $|f(\beta) - f(\alpha)| < \varepsilon$ for all $\beta > \alpha$. For if no such element existed we could find an increasing sequence of elements $\gamma_n \in (0, \Omega)$ such that $|f(\gamma_n) - f(\gamma_{n-1})| \geq \varepsilon$ for all n. But any increasing sequence in $(0, \Omega)$ converges to its least upper bound whereas the image sequence $f(\gamma_n) \in \mathbf{R}$ does not converge; this contradicts continuity of the function $f\colon (0, \Omega) \to \mathbf{R}$. So in particular, there exist elements α_n such that $|f(\beta) - f(\alpha_n)| < 1/n$ for all $\beta > \alpha_n$. Let α be an upper bound for these elements. Then f is constant on (α, Ω).

(b). Since any real function on $(0, \Omega)$ is eventually constant, any real function, in particular any bounded real function, on $(0, \Omega)$ extends to the one-point-compactification $(0, \Omega]$. But the Stone-Čech compactification is characterized by this property [Thm 38.5] so $(0, \Omega] = \beta(0, \Omega)$.

(c). Use that any compactification of $(0, \Omega)$ is a quotient of $(0, \Omega]$ [Ex 38.4].

Ex. 38.6. ([1, Thm 6.1.14]) Let X be a completely regular space and $\beta(X)$ its Stone-Čech compactification. Then

X is nonconnected $\Leftrightarrow$ There exists a continuous surjective function $X \to \{0, 1\}$

$\overset{\text{[Thm 38.4]}}{\Rightarrow}$ There exists a continuous surjective function $\beta(X) \to \{0, 1\} \Leftrightarrow \beta(X)$ is nonconnected

If X is connected then also βX is connected since it has a connected dense subset [3, Thm 23.4]

Ex. 38.7. ([Exam June 03, Problem 4] [5, 6, 4]) Let X be a discrete space; A a subset of $X \subset \beta(X)$ and U an open subset of $\beta(X)$.

(1) Let $F\colon \beta(X) \to \{0, 1\}$ be the extension [Thm 38.4] of the continuous function $f\colon X \to \{0, 1\}$ given by $f(A) = 0$ and $f(X - A) = 1$. Then $\overline{A} \subset F^{-1}(0)$ and $\overline{X - A} \subset F^{-1}(1)$ so these two subsets are disjoint; in other words $\overline{X - A} \subset \beta(X) - \overline{A}$. The inclusions

$$\beta(X) - \overline{A} \overset{\text{def}}{=} \overline{X} - \overline{A} \overset{[Ex 17.8]}{\subset} \overline{X - A} \subset \beta(X) - \overline{A}$$

tell us that $\beta(X) - \overline{A} = \overline{X - A}$. In particular, $\overline{A}$ is open (and closed).

(2) Since $U \cap X$ is a subset of U, it is clear that $\overline{U \cap X} \subset \overline{U}$ [Ex 17.6.(a)]. Conversely, let x be a point in $\overline{U}$ and V any neighborhood of x. Then $V \cap U \neq \emptyset$ is nonempty for x lies in the closure of U, and hence $(V \cap U) \cap X = V \cap (U \cap X) \neq \emptyset$ is also nonempty as X is dense. Thus every neighborhood V of x intersects $U \cap X$ nontrivially. This means that $x \in \overline{U \cap X}$. We conclude that $\overline{U \cap X} = \overline{U}$. From (1) (with $A = U \cap X$) we see that $\overline{U}$ is open (and closed).

(3) Let Y be any subset of $\beta(X)$ contaning at least two distinct points, x and y. We shall show that Y is not connected. Let $U \subset \beta(X)$ be an open set such that $x \in U$ and $y \notin \overline{U}$; such an open set U exists because $\beta(X)$ is Hausdorff [Definition, p. 237]. Then $Y = (Y \cap \overline{U}) \cup (Y - \overline{U})$ is a separation of Y, so Y is not connected.

A Hausdorff space is said to be extremally disconnected if the closure of every open set is open. A space is totally disconnected if the connected components are one-point sets. Any extremally disconnected space is totally disconnected. We have shown that $\beta(X)$ is extremally disconnected.

Ex. 38.8. The compact Hausdorff space I^I is a compactification of $\mathbf{Z}_+$ since [3, Ex 30.16] it has a countable dense subset (and is not finite). Any compactification of $\mathbf{Z}_+$ is a quotient of the Stone–Čech compactification $\beta\mathbf{Z}_+$ [3, Ex 38.4]. In particular, I^I is a quotient of $\beta\mathbf{Z}_+$ so $\mathrm{card}\beta\mathbf{Z}_+ \geq \mathrm{card} I^I$.

Ex. 38.9. ([Exam June 04, Problem 3])

(a). Suppose that $x_n \in X$ converges to $y \in \beta X - X$. We will show that then y is actually the limit point of two sequences with no points in common. The first step is to find a subsequence where no two points are identical. We recursively define a subsequence x_{n_k} by

$$n_k = \begin{cases} 1 & k = 1 \\ \min\{n > n_{k-1} \mid x_n \notin \{x_{n_1}, \ldots, x_{n_{k-1}}\}\} & k > 1 \end{cases}$$

This definition makes sense since the set we are taking the minimal element of a nonempty set. Since x_n converges to y, the subsequence x_{n_k} also converges to y. Clearly, no two points of the subsequence x_{n_k} are identical. We call this subsequence x_n again.

Let now $A = \{x_1, x_3, \ldots\}$ be the set of odd points and $B = \{x_2, x_4, \ldots\}$ the set of even points in this sequence. We claim that $\overline{A} = A \cup \{y\}$ and $\overline{B} = B \cup \{y\}$.

Any neighborhood of y contains a point from A, so y is in the closure of A. Since $A \subset A \cup \{y\} \subset \overline{A}$, it suffices to show that $A \subset A \cup \{y\}$ is closed, ie that the complement of $A \cup \{y\}$ is open: Let z be a point in the complement. Since z is not the limit of the sequence (x_{2n+1}) (there is just one limit point, namely y, in the Hausdorff space βX) there exists a neighborhood of z, even one that doesn't contain y, containing only finitely many elements from this sequence. Since z is not in A we can remove these finitely many points from the neighborhood to get a neighborhood of z that is disjoint from $A \cup \{y\}$.

This shows that $\overline{A} = A \cup \{y\}$. Similarly, $\overline{B} = B \cup \{y\}$. Therefore the intersection $\overline{A} \cap \overline{B} = \{y\} \neq \emptyset$.

On the other hand, the sets A and B are disjoint since no two points of the sequence x_n are identical. They are closed subsets of X for $\mathrm{Cl}_X A = X \cap \overline{A} = X \cap (A \cup \{y\}) = A$ and similarly for B, of course. By Urysohn's characterization of normal spaces, there exists a continuous function $f \colon X \to [0,1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$. The universal property of the Stone–Čech compactification [2, §27] says that there exists a unique continuous map $\overline{f}$ into the compact Hausdorff space $[0,1]$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & {[0,1]} \\ & \beta X \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes. Since $\overline{A} \subset \overline{f}^{-1}(0)$ and $\overline{B} \subset \overline{f}^{-1}(1)$, $\overline{A}$ and $\overline{B}$ are disjoint.

We have now shown that $\overline{A} \cap \overline{B}$ is both empty an nonempty. This contradiction means that no point in $\beta X - X$ can be the limit of a sequence of points in X.

(b). Assume that X is *normal* and noncompact. X is a proper subspace of βX since βX is compact which X is not. No point in $\beta X - X = \overline{X} - X$ is the limit of a sequence of points in X. Thus βX does not satisfy the Sequence lemma so βX is not first countable, in particular not metrizable.

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