REAL ANALYSIS

LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books for *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

1. Measurable sets and the Lebesgue measure

The notion of measurability isolates a collection of subsets in $\mathbb{R}^n$ for which the exterior measure satisfies all our desired properties, in particular including (countable) additivity for disjoint unions of sets.

Definition 1.1. *A subset E of $\mathbb{R}^n$ is (Lebesgue) measurable if for any $\varepsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and*

$$m_*(\mathcal{O} - E) \leq \varepsilon.$$

Let $\mathscr{M}_{\mathbb{R}^n}$ be the collection of all measurable sets of $\mathbb{R}^n$. If $E \in \mathscr{M}_{\mathbb{R}^n}$, we define its (Lebesgue) measure $m(E) = m_*(E)$.

Remark 1.1. *We observe for any $E \subset \mathbb{R}^n$, given $\varepsilon > 0$, there is an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m_*(\mathcal{O}) \leq m_*(E) + \varepsilon$. See Observation 3 in previous section. If additivity were available for all open sets and measurable sets, then we should deduce*

$$m_*(\mathcal{O} - E) = m_*(\mathcal{O}) - m_*(E) \leq \varepsilon,$$

which suggests the Definition 1.1.

Theorem 1.1. *A subset E of $\mathbb{R}^n$ is measurable if and only if*

$$m_*(A) = m_*(A \cap E) + m_*(A \setminus E), \quad \forall\ A \subset \mathbb{R}^n. \tag{1.1}$$

Proof. This is an exercise. $\square$

Remark 1.2. *Definition 1.1 relies on the topology as it uses open sets.*

In abstract measure theory, the measure μ is defined on a σ-algebra $\mathscr{M}_X$ [1] of a set X with the requirement $\mu : \mathscr{M}_X \to [0, \infty]$ satisfies

$$\mu\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \mu(E_j),$$

if E_j is a countable family of disjoint sets in $\mathscr{M}_X$.

To build up a measure, one may start with the exterior measure μ_* first. Namely $\mu_* : 2^X \to [0, +\infty]$ satisfies (i) $\mu_*(\emptyset) = 0$; (ii) $\mu_*(E_1) \leq \mu_*(E_2)$ for all $E_1 \subset E_2$; and (iii) $\mu_*(\bigcup_{j\geq 1} E_j) \leq \sum_{j\geq 1} \mu_*(E_j)$.

Given such exterior measure μ_*, the problem that one faces is how to define the corresponding notion of measurable sets. In general X may not be a topological space, and hence the Lebesgue's approach cannot be used. Carathéodory found an ingenious substitute condition, which is a subset $E \subset X$ is Carathéodory measurable if

$$\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \setminus E). \tag{1.2}$$

Setting $\mu(E) = \mu_*(E)$ for all $E \in \mathscr{M}_X$, where $\mathscr{M}_X$ is the collection of measurable sets, then $(X, \mathscr{M}_X, \mu)$ is a measure space. This remarkable fact is summarised as follows

Theorem 1.2. *Given an exterior measure μ_* on a set X, the collection $\mathscr{M}_X$ of Carathéodory measurable sets forms a σ-algebra. Moreover, μ_* restricted to $\mathscr{M}_X$ is a measure.*

1.1. **Some properties.**

In summary, we show the following properties

- (i) $\mathscr{M}_{\mathbb{R}^n}$ contains all open and closed sets.
- (ii) $\mathscr{M}_{\mathbb{R}^n}$ is closed under finite and countably infinite set theoretic operations including union/intersection/complement.
- (iii) Monotonicity, and countable additivity for disjoint unions, hold for measure of measurable sets.
- (iv) The measure of a rectangle is the same as the standard volume.

Property 1 Every open set in $\mathbb{R}^n$ is measurable.

[1] The σ-algebra of X is a collection of subsets of X that is closed under countable unions, countable intersections, and complements.

Proof. This is a direct consequence of Definition 1.1. $\square$

Property 2 If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable and $m_*(F) = 0$.

Proof. Observe 3 of the exterior measure tells us for every ε there is an open set $\mathcal{O} \supset E$ with $m_*(\mathcal{O}) \leq \varepsilon$. Hence $m_*(\mathcal{O} - E) \leq \varepsilon$. This shows E is measurable. $\square$

Example 1.1. *The Cantor set is measurable and has measure zero.*

Property 3 A countable union of measurable sets is measurable.

Proof. Given $\{E_j\}_{j\geq 1} \subset \mathscr{M}_{\mathbb{R}^n}$, let $E = \bigcup_{j\geq 1} E_j$. For any $\varepsilon > 0$, take open sets $\mathcal{O}_j \supset E_j$ with $m_*(\mathcal{O}_j - E_j) \leq \varepsilon/2^j$. Let $\mathcal{O} = \bigcup_{j\geq 1} \mathcal{O}_j$. Then

$$\mathcal{O} - E = \bigcup_{j\geq 1}(\mathcal{O}_j - E) \subset \bigcup_{j\geq 1}(\mathcal{O}_j - E_j).$$

Hence, by the sub-additivity, $m_*(\mathcal{O} - E) = \sum_{j\geq 1} m_*(\mathcal{O}_j - E_j) \leq \varepsilon$. We are done.

$\square$

Property 4 Closed sets are measurable.

Proof. Let F be a closed set. Since $F = \bigcup_{j\geq 1}(F \cap B_j)$, where $\overline{B}_j$ is a closed ball of radius j. By Property 3, it suffices to show each $F_j = F \cap B_j$ is measurable. Note that F_j is compact. Hence we can assume directly F is compact and show it is measurable.

By the previous Observation 3, for any $\varepsilon > 0$, we can select an open set $\mathcal{O} \supset F$ with

$$m_*(\mathcal{O}) \leq m_*(F) + \varepsilon. \tag{1.3}$$

By Theorem **??**, there are almost disjoint closed cubes $\{Q_j\}_{j\geq 1}$ such that

$$\mathcal{O} - F = \bigcup_{j\geq 1} Q_j.$$

For a fixed N, F and $K_N := \bigcup_{j=1}^N Q_j$ are two disjoint compact sets. Hence $d(F, K_N) > 0$ [2]. By Observation 1, 4 and 5 of the exterior measure,

$$m_*(\mathcal{O}) \geq m_*(F \cup K_N) = m_*(F) + m_*(K_N) = m_*(F) + \sum_{j=1}^{N} m_*(Q_j).$$

[2]If F_1 is compact and F_2 is closed, then $d(F_1, F_2) > 0$. This is an exercise.

This together with (1.3) implies

$$\sum_{j=1}^{N} m_*(Q_j) \leq \varepsilon.$$

Letting $N \to \infty$, one concludes by Observation 5 again,

$$m_*(\mathcal{O} - F) = m_*\Big(\bigcup_{j\geq 1} Q_j\Big) = \sum_{j\geq 1} m_*(Q_j) \leq \varepsilon.$$

□

Property 5 The complement of a measurable set is measurable.

Proof. Let E be a measurable set. For each $n \geq 1$, there is an open set $\mathcal{O}_n \supset E$ with $m_*(\mathcal{O}_n - E) \leq \frac{1}{n}$. Define $S = \bigcup_{n\geq 1} \mathcal{O}_n^c$. Then $E^c = (E^c - S) \cup S$.

By Properties 3 & 4, S is measurable. Note that $E^c - S \subset \mathcal{O}_n - E$. By the sub-additivity,

$$m_*(E^c - S) \leq m_*(\mathcal{O}_n - E) \leq \frac{1}{n}.$$

Therefore $E_c - S$ is measurable as it has zero measure. Hence E^c is the union of two measurable sets, and is therefore measurable by Property 3.

□

As an immediate consequence of Properties 3 & 5, we have:

Property 6 A countable intersection of measurable sets is measurable.

Theorem 1.3. *If $E_1, E_2, \dots,$ are disjoint measurable sets, and $E = \bigcup_{j=1}^{\infty} E_j$, then*

$$m(E) = \sum_{j=1}^{\infty} m(E_j).$$

Proof. We only show "$\geq$", as the opposite inequality is due to sub-additivity of exterior measure (Observation 2).

Let us first assume E_j are bounded.

Given $\varepsilon > 0$, we can select a closed set $F_j \subset E_j$ such that $m(E_j - F_j) \leq \varepsilon/2^j$ [3]. Note that F_j are disjoint compact sets, and so $d(F_i, F_j) > 0$ for all $i \neq j$. Fixing N, by

[3]For this, one applies the definition of measurability to E_j^c.

Observation 4 of exterior measure,

$$m(E) \geq m\Big(\bigcup_{j=1}^{N} F_j\Big) = \sum_{j=1}^{N} m(F_j) \geq \sum_{j=1}^{N} \Big(m(E_j) - m(E_j - F_j)\Big) \geq \sum_{j=1}^{N} m(E_j) - \varepsilon.$$

Sending $\varepsilon \to 0$ and $N \to \infty$, we are done.

For possibly unbounded E_j, let $\{S_k\}_{k\geq 1}$ be a partition of $\mathbb{R}^n$ into bounded disjoint measurable sets [4]. Since

$$E = \bigcup_{j,k\geq 1} E_{j,k}, \quad \text{where } E_{j,k} = E_j \cap S_k,$$

and noticing $E_{j,k}$ are disjoint and bounded, we obtain

$$m(E) = m\Big(\bigcup_{j,k\geq 1} E_{j,k}\Big) = \sum_{j\geq 1}\sum_{k\geq 1} m(E_{j,k}) = \sum_{j\geq 1} m(E_j).$$

The second equality is due to the additivity for bounded case.

□

1.2. **Increasing unions and decreasing Intersections.**

Proposition 1.1. *Let $\{E_k\}_{j\geq 1}$ be a sequence of measurable sets in $\mathbb{R}^n$.*

(i) *If $E_k \nearrow E$, then $m(E) = \lim_{N\to\infty} m(E_N)$.*
(ii) *If $E_k \searrow E$ and $m(E_k) < \infty$ for some k, then $m(E) = \lim_{N\to\infty} m(E_N)$.*

Remark 1.3. *Consider $E_k = (k, \infty) \subset \mathbb{R}$. Then $E_k \searrow \emptyset$. Hence $m(E) < \lim_{N\to\infty} E_N$, which shows (ii) in Proposition 1.1 may fail without the assumption $m(E_k) < \infty$.*

Proof. We show (i). Write $E = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \cup \cdots = \bigcup_{k=1}^{\infty} (E_k \setminus E_{k-1})$, where $E_0 = \emptyset$. Since $E_{k+1} \setminus E_k$ are disjoint, by the additivity,

$$m(E) = \sum_{k=1}^{\infty} m(E_k \setminus E_{k-1}) = \lim_{N\to\infty} \sum_{k=1}^{N} m(E_k \setminus E_{k-1}) = \lim_{N\to\infty} m(E_N).$$

For (ii), observe that $E_1 \setminus E_k \searrow E_1 \setminus E$. Without loss of generality, we simply assume $m(E_1) < \infty$. Applying part (i) to $\{E_1 \setminus E_k\}_{k\geq 1}$, we see that

$$m(E_1 \setminus E) = \lim_{N\to\infty} m(E_1 \setminus E_N)$$

[4] For example, let $S_k = \{x : k - 1 \leq |x| < k\}$.

which yields

$$m(E) = m(E_1) - \lim_{N\to\infty} m(E_1 \setminus E_N) = \lim_{N\to\infty} \Big(m(E_1) - m(E_1 \setminus E_N)\Big) = \lim_{N\to\infty} m(E_N).$$

$\square$

Corollary 1.1. *Let $\{E_k\}_{j\geq 1}$ be a sequence of measurable sets in $\mathbb{R}^n$. Then*

$$m(\liminf_{k\to\infty} E_k) = \lim_{k\to\infty} m\big(\bigcap_{j\geq k} E_j\big) \leq \liminf_{k\to\infty} m(E_k),$$

and, if $m(\bigcup_{j\geq k} E_j) < \infty$ for some k,

$$m(\limsup_{k\to\infty} E_k) = \lim_{k\to\infty} m\big(\bigcup_{j\geq k} E_j\big) \geq \limsup_{k\to\infty} m(E_k).$$

The results hold for exterior measure of a sequence of sets (not necessarily measurable).

Proposition 1.2. *Let $\{E_k\}_{j\geq 1}$ be a sequence of sets in $\mathbb{R}^n$.*

(i) *If $E_k \nearrow E$, then $m_*(E) = \lim_{N\to\infty} m_*(E_N)$.*
(ii) *$m_*(\liminf_{k\to\infty} E_k) = \lim_{k\to\infty} m_*\big(\bigcap_{j\geq k} E_j\big) \leq \liminf_{k\to\infty} m_*(E_k)$.*

Proof. By Observation 3 of the exterior measure, for $\varepsilon > 0$, there are open sets $\mathcal{O}_k \supset E_k$ with $m(\mathcal{O}_k) \leq m_*(E_k) + \varepsilon$. Since E_k is increasing, $E_k \subset \bigcap_{j\geq k} \mathcal{O}_j$. By Corollary 1.1,

$$m_*(E) \leq m(\liminf_{k\to\infty} \mathcal{O}_k) \leq \liminf_{k\to\infty} m(\mathcal{O}_k) \leq \liminf_{k\to\infty} m(E_k) + \varepsilon,$$

which gives $m_*(E) \leq \liminf_{k\to\infty} m(E_k)$. On the other hand, by the monotonicity,

$$m_*(E) \geq \limsup_{k\to\infty} m_*(E_k).$$

This proves part (i).

Part(ii) is a consequence of part (i).

$\square$

Exercise 1.1. *Let A_t, $t \in (0,1)$, be sets in $\mathbb{R}^n$, and $A_{t_1} \subset A_{t_2}$ when $t_1 < t_2$. Show*

(i) *$m_*(\bigcup_{t\in(0,1)} A_t) = \lim_{t\to 1-} m_*(A_t)$;*
(ii) *$m_*(\bigcap_{t\in(0,1)} A_t) \leq \lim_{t\to 0+} m_*(A_t)$;*
(iii) *Suppose A_t are measurable. Then $m(\bigcup_{t\in(0,1)} A_t) = \lim_{t\to 1-} m(A_t)$; and*

$$m\big(\bigcap_{t\in(0,1)} A_t\big) = \lim_{t\to 0+} m(A_t),$$

provided $m(A_{t_0}) < \infty$ for some $t_0 \in (0,1)$.

Exercise 1.2. *Let $E \subset \mathbb{R}$, $0 < m_*(E) < \infty$. Show $f(x) = m_*((-\infty, x) \cap E)$ is Lipschitz continuous, and $I = \{m_*(F) : F \subset E\}$ is a bounded closed interval.*

1.3. **Approximating measurable sets.**

An important geometric and analytic insight into the nature of measurable sets is that, in effect, an arbitrary measurable set can be well approximated by the open sets that contain it, and alternatively, by the closed sets it contains.

Theorem 1.4. *Let E be a measurable set in $\mathbb{R}^n$. Then, for every $\varepsilon > 0$:*

(i) *There is an open set $\mathcal{O} \supset E$ with $m(\mathcal{O} - E) \leq \varepsilon$.*
(ii) *There is a closed set $F \subset E$ with $m(E - F) \leq \varepsilon$.*
(iii) *If $m(E) < \infty$, then there is a compact set $K \subset E$ with $m(E - K) \leq \varepsilon$.*
(iv) *If $m(E) < \infty$, then there is a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that*
$$m(E \Delta F) \leq \varepsilon.$$

Proof. Part (i) is directly from the definition of measurability of E.

Part (ii) is from the definition of measurability of E^c and the take complement.

Part (iii) follows by taking $K = F \cap \overline{B}_N$ for sufficiently large N. More precisely, we select F such that $m(E - F) < \varepsilon/2$. Let $K_n := F \cap B_n$. On the other hand, since $E - K_n \searrow E - F$ and $m(E - F) < \infty$, we have $m(E - K_N) \leq m(E - F) + \varepsilon/2$ for some large N. We are done.

For part (iv), we first take almost disjoint closed cubes Q_j such that $E \subset \bigcup_{j \geq 1} Q_j$ and $\sum_{j=1}^{\infty} |Q_j| \leq m(E) + \varepsilon/2$. Since $m(E) < \infty$, we next choose large N so that $\sum_{j=N+1}^{\infty} |Q_j| \leq \varepsilon/2$. Let $F = \bigcup_{j=1}^{N} Q_j$. Then
$$m(E \Delta F) = m(E - F) + m(F - E) \leq m\Big(\bigcup_{j \geq N+1} Q_j\Big) + m\Big(\bigcup_{j \geq 1} Q_j - E\Big) \leq \varepsilon.$$
□

1.4. **Invariance properties of Lebesgue measure.**

One can check that measurability is invariant under the translation, dilation, reflection and linear transformation; and moreover, if $E \subset \mathbb{R}^n$ is measurable, then

(a) translation-invariance: $m(E) = m(E_h)$, where $E_h = \{x + h : x \in E\}$;

(b) dilation-invariance: $m(\lambda E) = |\lambda|^n m(E)$, $\lambda E = \{\lambda x : x \in E\}$;

(c) linear transformation invariance: $m(\mathcal{L}(E)) = |\det \mathcal{L}| m(E)$, where $\mathcal{L}$ is a linear transformation of $\mathbb{R}^n$.

Using Definition 1.1, one can give a quick check for (a) and (b).

Exercise: Show if E is measurable then $\mathcal{L}(E)$ is measure.

The quantitative statement $m((E)) = |\det \mathcal{L}| m(E)$ is a consequence of the Fubini's theorem, which will be discussed later on.

1.5. σ-algebras and Borel sets.

Let X be a set.

Definition 1.2. *A σ-algebra in X is a collection of subsets of X that is closed under countable unions, countable intersections, and complements.*

Remark 1.4. *We collect some simple facts.*

(i) 2^X *is of course a σ-algebra in X.*

(ii) $\mathscr{M}_{\mathbb{R}^n}$*, the collection of measurable sets of* $\mathbb{R}^n$*, is a σ-algebra.*

(iii) *For any collection $\mathscr{C}$ of subsets of X, there is a smallest σ-algebra $\mathscr{P}(\mathscr{C})$ containing $\mathscr{C}$.*

Proof: 2^X *is certainly a σ-algebra containing $\mathscr{C}$. Note that the intersection of any family of σ-algebras is itself a σ-algebra. Hence $\mathscr{P}(\mathscr{C})$ is the intersection of the family of all σ-algebras containing $\mathscr{C}$. Clearly such intersection exists, is a σ-algebra containing $\mathscr{C}$; and moreover if $\mathscr{T}$ is any σ-algebra containing $\mathscr{C}$ then $\mathscr{P}(\mathscr{C}) \subset \mathscr{T}$.*

Definition 1.3. *The Borel σ-algebra in $\mathbb{R}^n$, denoted by $\mathscr{B}_{\mathbb{R}^n}$, is the smallest σ-algebra that contains all open sets. The elements of this σ-algebra are called Borel sets.*

Remark 1.5. *Obviously $\mathscr{B}_{\mathbb{R}^n} \subset \mathscr{M}_{\mathbb{R}^n}$. Is this inclusion is strict? The answer is "yes".*

Claim: If $\phi \in C(\mathbb{R})$ is strictly increasing/deceasing, then $\phi(E) \in \mathscr{B}_{\mathbb{R}}$, $\forall\ E \in \mathscr{B}_{\mathbb{R}}$. [5]

Proof: Let $\mathcal{B}^* = \{E :\ E$ and $\phi(E)$ are both Borel sets$\}$. We have

- *For $E \in \mathcal{B}^*$, we have $\phi(E^c) = \phi(\mathbb{R}) - \phi(E) \in \mathscr{B}_{\mathbb{R}^n}$, and so $E^c \in \mathcal{B}^*$.*

[5]It is not true that the image of a Borel set under a continuous function is necessarily Borel. Such images are called Souslin sets.

- *For a sequence $E_j \in \mathcal{B}^*$, we have $\bigcup_{j\geq 1} E_j$ and $\phi\big(\bigcup_{j\geq 1} E_j\big) = \bigcup_{j\geq 1} \phi(E_j)$ are both Borel sets.*

Hence $\mathcal{B}^*$ is a σ-algebra, and by the continuity of ϕ, containing all open sets. Therefore $\mathcal{B}^* = \mathscr{B}_{\mathbb{R}}$. This shows the Claim.

*Recall the Cantor-Lebesgue function $g : [0,1] \to [0,1]$ at the end of Chapter **??**. Take $\phi(x) = g(x) + x : [0,1] \to [0,2]$. We have*

- *ϕ is continuous and strictly increasing.*
- *$\phi([0,1]) = [0,2]$ and, $m(\phi(G)) = 1$ where $G = [0,1] \setminus \mathcal{C}$ (note $g(G)$ is discrete).*

Since $\phi(\mathcal{C}) = [0,2] \setminus \phi(G)$, one sees that $\phi(\mathcal{C})$ is measurable and $m\big(\phi(\mathcal{C})\big) = 1$. Since it has positive measure, we can construct a non-measurable subset of $\phi(\mathcal{C})$, say $\mathcal{N}_{\phi(\mathcal{C})}$, by using the same approach for constructing $\mathcal{N} \subset [0,1]$ at the beginning of this chapter. Let $E = \phi^{-1}(\mathcal{N}_{\phi(\mathcal{C})})$. Obviously $E \subset \mathcal{C}$, and so it is measurable. On the other hand, since $\phi(E) = \mathcal{N}_{\phi(\mathcal{C})}$ is not measurable, we infer, by the Claim, that $E \notin \mathscr{B}_{\mathbb{R}}$. Hence we obtain a set $E \in \mathscr{M}_{\mathbb{R}} \setminus \mathscr{B}_{\mathbb{R}}$ as desired.

Recall that E is called G_δ if it is a countable intersection of open sets; and is called F_σ if it is a countable union of closed sets.

One sees that G_δ sets and F_σ sets [6] are the simplest Borel sets after the open and closed sets. The Lebesgue measurable sets arise as the completion of the Borel σ-algebra.

Proposition 1.3. *A subset E of $\mathbb{R}^n$ is measurable*

(i) *if and only if E differs from a G_δ by a set of measure zero,*
(ii) *if and only if E differs from an F_σ by a set of measure zero.*

Proof. This is an exercise. □

[6] The terminology G_δ comes from German "Gebiete" and "Durschnitt"; F_σ comes from French "fermé" and "somme".