第八章　不定积分

§1　不定积分概念与基本积分公式

1. 验证下列等式,并与(3)(4)两式相比照:

(1)$\int f'(x)dx = f(x) + c$　　(2)$\int df(x) = f(x) + c$

证:(1) $\because$ f(x) 是 f'(x) 的一个原函数.

$$\therefore \int f'(x)dx = f(x) + c$$

(2) $\because \int du = u + c$

$$\therefore \int df(x) = f(x) + c$$

2. 求一曲线 $y = f(x)$,使得曲线上每一点(x,y)处的切线斜率为2x,且通过点(2.5).

解　设所求曲线为 $y = f(x)$,则有 $f'(x) = 2x$,所以 $f(x) = \int 2xdx = x^2 + c$,又曲线过点(2,5),从而 $5 = 2^2 + c$,得 $c = 1$,于是所求的曲线为 $y = x^2 + 1$.

3. 证明 $y = \frac{x^2}{2} sgn x$ 是 $|x|$ 在 $(-\infty, -\infty)$ 上的原函数.

证　当 $x > 0$ 时,$y = \frac{x^2}{2}, y' = \frac{2x}{2} = x$;

当 $x < 0$ 时,$y = -\frac{x^2}{2}, y' = -\frac{2x}{2} = -x$;

当 $x = 0$ 时,$g'|_{x=0}$ 存在,且等于 0;

所以 $y' = |x| (-\infty < x < +\infty)$,故 $y = \frac{x^2}{2} sgn x$ 是 $|x|$ 在 $(-\infty, +\infty)$ 上的原函数.

4. 据理说明为什么每一个含有第一类间断点的函数都没有原函数？

证：一般地，设 x_0 是 $g(x)$ 的第一类间断点，若 $G(x)$ 是 $g(x)$ 在 $U(x_0)$ 上的原函数，则 $G'(x)=g(x), x\in U(x_0)$.

从而 $\lim\limits_{x\to x_0^-} g(x)=\lim\limits_{x\to x_0^-} G'(x)=G_-'(x_0)=G'(x_0)=g(x_0)$.

同理 $\lim\limits_{x\to x_0^+} g(x)=g(x_0)$，可见 $g(x)$ 在 x_0 连续，矛盾.

5. 求下列不定积分：

(1) $\int(1-x+x^3-\frac{1}{\sqrt[3]{x^2}})dx$;

(2) $\int(x-\frac{1}{\sqrt{x}})^2dx$;

(3) $\int\frac{dx}{\sqrt{2gx}}$（g 为正常数）；(4) $\int(2^x+3^x)^2dx$;

(5) $\int(\frac{3}{\sqrt{4-4x^2}}+\sin x)dx$; (6) $\int\frac{x^2}{3(1+x^2)}dx$;

(7) $\int\tan^2xdx$;　　(8) $\sin^2xdx$;

(9) $\int\frac{\cos 2x}{\cos x-\sin x}dx$; (10) $\int\frac{\cos 2x}{\cos^2x\sin^2x}dx$;

(11) $\int 10^t\cdot 3^{2t}dt$; (12) $\int\sqrt{x\sqrt{x\sqrt{x}}}dx$;

(13) $\int(\sqrt{\frac{1+x}{1-x}}+\sqrt{\frac{1-x}{1+x}})dx$;

(14) $\int(\cos x+\sin x)^2dx$;

(15) $\int\cos x\cos 2xdx$; (16) $\int(e^x-e^{-x})^3dx$

解　(1) 原式 $=\int(1-x+x^3-x^{-\frac{2}{3}})dx$

$$=x-\frac{1}{2}x^2+\frac{1}{4}x^4-3x^{\frac{1}{3}}+c$$

(2) 原式 $=\int(x^2-2x^{\frac{1}{2}}+\frac{1}{x})dx$

$=\frac{1}{3}x^3-\frac{4}{3}x^{\frac{3}{2}}+ln|x|+c$

(3) 原式 $=\frac{1}{\sqrt{2g}}\int x^{-\frac{1}{2}}dx=\sqrt{\frac{2}{g}}x^{\frac{1}{2}}+c=\sqrt{\frac{2x}{g}}+c$

(4) 原式 $=\int(4^x+2.6^x+9^x)dx=\frac{4^x}{ln4}+2\times\frac{6^x}{ln6}+\frac{9^x}{ln9}+c$

$=\frac{2^{2x}}{2ln2}+\frac{2^{2x}}{2ln3}+2(\frac{6^x}{ln6})+c$

(5) 原式 $=\int(\frac{3}{2}\cdot\frac{1}{\sqrt{1-x^2}}+sinx)dx$

$=\frac{3}{2}arcsinx-cosx+c$

(6) 原式 $=\int\frac{x^2+1-1}{3(1+x^2)}dx=\int(\frac{1}{3}-\frac{1}{3(1+x^2)})dx$

$=\frac{1}{3}x-\frac{1}{3}arctanx+c$

(7) 原式 $=\int(sec^2x-1)dx=tanx-x+c$

(8) 原式 $=\int\frac{1-cos2x}{2}dx=\frac{1}{2}(x-\frac{1}{2}sin2x)+c$

(9) 原式 $=\int\frac{cos^2x-sin^2x}{cosx-sinx}dx=\int(cosx+sinx)dx$

$=sinx-cosx+c$

(10) 原式 $=\int\frac{cos^2x-sin^2x}{cos^2xsin^2x}dx=\int(\frac{1}{sin^2x}-\frac{1}{cos^2x})dx$

$=-ctanx-tanx+c$

(11) 原式 $=\int 90^tdt=\frac{90^t}{ln90}+c$

(12) 原式 $=\int x^{\frac{7}{8}}dx=\frac{8}{15}x^{\frac{15}{8}}+c$

(13) 原式 $=\int\frac{2}{\sqrt{1-x^2}}dx=2arcsinx+c$

(14) 原式 $= \int(1+2\sin x\cos x)dx = \int(1+\sin 2x)dx$

$$= x - \frac{1}{2}\cos 2x + c$$

(15) $\int \cos x\ \cos 2x dx = \frac{1}{2}\int \cos 3x + \cos x dx$

$$= -\frac{1}{6}\sin 3x - \frac{1}{2}\sin x + c$$

(16) $\int (e^x - e^{-x})^3 dx = \int e^{3x} - e^{-3x} - 3e^x + 3e^{-x} dx$

$$= \frac{1}{3}e^{3x} + \frac{1}{3}e^{-3x} - 3e^x - 3e^{-x} + c$$

§2　换元积分法与分部积分法

1. 应用换元积分法求下列不定积分.

(1) $\int \cos(3x+4)dx$　　(2) $\int xe^{2x^2}dx$;

(3) $\int \frac{dx}{2x+1}$　　(4) $\int (1+x)^n dx$;

(5) $\int (\frac{1}{\sqrt{3-x^2}} + \frac{1}{\sqrt{1-3x^2}})dx$　　(6) $\int 2^{2x+3}dx$

(7) $\int \sqrt{8-3x}dx$　　(8) $\int \frac{dx}{\sqrt[3]{7-5x}}$

(9) $\int x\sin x^2 dx$　　(10) $\int \frac{dx}{\sin^2(2x+\frac{\pi}{4})}$

(11) $\int \frac{dx}{1+\cos x}$　　(12) $\int \frac{dx}{1+\sin x}$

(13) $\int \csc x dx$　　(14) $\int \frac{x}{\sqrt{1-x^2}}dx$

(15) $\int \frac{x}{4+x^4}dx$　　(16) $\int \frac{dx}{x\ln x}$

(17) $\int \frac{x^4}{(1-x^5)^3}dx$　　(18) $\int \frac{x^3}{x^8-2}dx$

(19) $\int \frac{dx}{x(1+x)}$ (20) $\int \cot x dx$

(21) $\int \cos^5 x dx$ (22) $\int \frac{dx}{\sin x \cos x}$

(23) $\int \frac{dx}{e^x + e^{-x}}$ (24) $\int \frac{2x-3}{x^2-3x+8}dx$

(25) $\int \frac{x^2+2}{(x+1)^3}dx$ (26) $\int \frac{dx}{\sqrt{x^2+a^2}}(a>0)$

(27) $\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}}(a>0)$ (28) $\int \frac{x^5}{\sqrt{1-x^2}}dx$

(29) $\int \frac{\sqrt{x}}{1-\sqrt[3]{x}}dx$ (30) $\int \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}dx$

解　(1) $\int \cos(3x+4)dx = \frac{1}{3}\int \cos(3x+4)d(3x+4)$

$$= \frac{1}{3}\sin(3x+4) + c$$

(2) $\int xe^{2x^2}dx = \frac{1}{4}\int e^{2x^2}d2x^2 = \frac{1}{4}e^{2x^2} + c$

(3) $\int \frac{dx}{2x+1} = \frac{1}{2}\int \frac{d(2x+1)}{2x+1} = \frac{1}{2}ln|2x+1| + c$

(4) $\int (1+x)^n dx = \int (1+x)^n d(1+x)$

$$= \begin{cases} \frac{(1+x)^{n+1}}{n+1} + c & n \neq -1; \\ ln|1+x| + c & n = -1 \end{cases}$$

(5) $\int (\frac{1}{\sqrt{3-x^2}} + \frac{1}{\sqrt{1-3x^2}})dx$

$$= \int \frac{1}{\sqrt{1-(\frac{x}{\sqrt{3}})^2}}d(\frac{x}{\sqrt{3}}) + \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{1-(\sqrt{3}x)^2}}d\sqrt{3}x$$

$$= \arcsin\frac{x}{\sqrt{3}} + \frac{1}{\sqrt{3}}\arcsin\sqrt{3}x + c$$

(6) $\int 2^{2x+3}dx = \int 2^{2x+3}d(2x+3) = \frac{2^{2x+3}}{ln2} + c$

(7) $\int \sqrt{8-3x}dx = -\frac{1}{3}\int \sqrt{8-3x}d(8-3x) = -\frac{2}{9}(8-3x)^{\frac{3}{2}} + c$

(8) $\int \frac{dx}{\sqrt[3]{7-5x}} = -\frac{1}{5}\int \frac{d(7-5x)}{\sqrt[3]{7-5x}} = -\frac{3}{10}(7-5x)^{\frac{2}{3}} + c$

(9) $\int sin2xsin3xdx = -\frac{1}{2}\int [cos5x - cos(-x)]dx$

$$= \frac{1}{2}sinx - \frac{1}{10}sin5x + c$$

(10) $\int \frac{dx}{sin^2(2x+\frac{\pi}{4})} = -\frac{1}{2}ctan(2x+\frac{\pi}{4}) + c$

(11) $\int \frac{dx}{1+cosx} = \int \frac{dx}{2cos^2\frac{x}{2}} = tan\frac{x}{2} + c$

(12) $\int \frac{dx}{1+sinx} = \int \frac{dx}{(sin\frac{x}{2}+cos\frac{x}{2})^2} = \frac{1}{2}\int \frac{dx}{cos^2(\frac{\pi}{4}-\frac{x}{2})}$

$$= -tan(\frac{\pi}{4}-\frac{x}{2}) + c$$

(13) $\int \frac{1}{sinx}dx = \int \frac{dx}{2sin\frac{x}{2}cos\frac{x}{2}} = \int \frac{1}{tan\frac{x}{2}cos^2\frac{x}{2}}d(\frac{x}{2})$

$$= \int \frac{1}{tan\frac{x}{2}}dtan\frac{x}{2} = ln|tan\frac{x}{2}| + c$$

(14) $\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} = -(1-x^2)^{\frac{1}{2}} + c$

(15) $\int \frac{x}{4+x^4}dx = \frac{1}{4}\int \frac{d(\frac{x^2}{2})}{1+(\frac{x^2}{2})^2} = \frac{1}{4}arctan\frac{x^2}{2} + c$

(16) $\int \frac{dx}{xlnx} = \int \frac{dlnx}{lnx} = ln|lnx| + c$

(17) $\int \frac{x^4}{(1-x^5)^3}dx = -\frac{1}{5}\int \frac{d(1-x^5)}{(1-x^5)^3} = \frac{1}{10(1-x^5)^2} + c$

$$(18)\int \frac{x^3}{x^8-2}dx = \frac{1}{4}\int \frac{dx^4}{(x^4)^2-(\sqrt{2})^2}$$
$$= \frac{1}{8\sqrt{2}}\left[\int \frac{d(x^4-\sqrt{2})}{x^4-\sqrt{2}} - \int \frac{d(x^4+\sqrt{2})}{x^4+\sqrt{2}}\right]$$
$$= \frac{1}{8\sqrt{2}} ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right| + c$$

$$(19)\int \frac{dx}{x(1-x)} = \int \left(\frac{1}{x} - \frac{1}{1+x}\right)dx = ln\left|\frac{x}{1+x}\right| + c$$

$$(20)\int tan\,x dx = \int \frac{sin\,x}{cos\,x}dx = -\int \frac{d cos\,x}{cos\,x} = -ln|cos\,x| + c$$

$$(21)\int cos^5 x dx = \int (1-sin^2x)^2 d sin x = \int (1-2sin^2x+sin^4x)d sin x$$
$$= sin\,x - \frac{2}{3}sin^3x + \frac{1}{5}sin^5x + c.$$

$$(22)\int \frac{dx}{sin\,x cos\,x} = \int \frac{sin^2x+cos^2x}{sin\,x cos\,x}dx = \int \frac{sin\,x}{cos\,x}dx + \int \frac{cos\,x}{sin\,x}dx$$
$$= -ln|cos\,x| + ln|sin\,x| + c = ln|tan\,x| + c$$

$$(23)\int \frac{dx}{e^x+e^{-x}} = \int \frac{de^x}{1+(e^x)^2} = arctan e^x + c$$

$$(24)\int \frac{2x-3}{x^2-3x+8}dx = \int \frac{d(x^2-3x+8)}{x^2-3x+8}$$
$$= ln|x^2-3x+8| + c$$

(25) 令 $x+1=t$,则 $x=t-1$,$dx=dt$

$$\int \frac{x^2+2}{(x+1)^3}dx = \int \frac{(t-1)^2+2}{t^3}dt$$
$$= \int \frac{t^2-2t+3}{t^3}dt = ln|t| + \frac{2}{t} - \frac{3}{2t^2} + c$$
$$= ln|x+1| + \frac{2}{x+1} - \frac{3}{2(x+1)^2} + c$$

26. 令 $x = a\,tan\,t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$　$dx = a\,sec^2t dt$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\,sec^2t}{a\,sec\,t}dt = \int sec\,t dt$$

$$= \ln|\sec t + \tan t| + c = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + c$$

$$= \ln\left|\sqrt{x^2+a^2} + x\right| + c_1$$

(27) 令 $x = a\tan t$

$$\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}} = \int \frac{a\sec^2 t dt}{(a^2\tan^2 t + a^2)^{\frac{3}{2}}} = \frac{1}{a^2}\int \cos t dt$$

$$= \frac{1}{a^2}\sin t + c = \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}} + c.$$

(28) 令 $x = \sin t$, $|t| < \frac{\pi}{2}$ 则 $dx = \cos t dt$

$$\int \frac{x^5}{\sqrt{1-x^2}}dx = \int \frac{\sin^5 t \cdot \cos t}{\cos t}dt = \int \sin^5 t dt$$

$$= -\int (1-\cos^2 t)^2 d\cos t$$

$$= -\int (1 - 2\cos^2 t + \cos^4 t) d\cos t$$

$$= -\cos t + \frac{2}{3}\cos^3 t - \frac{1}{5}\cos^5 t + c$$

$$= -\sqrt{1-x^2} + \frac{2}{3}(1-x^2)^{\frac{3}{2}} - \frac{1}{5}(1-x^2)^{\frac{5}{2}} + c.$$

(29) 令 $x = t^6$　$dx = 6t^5 dt$

$$\int \frac{\sqrt{x}}{1-\sqrt[3]{x}}dx = \int \frac{6t^8}{1-t^2}dt$$

$$= -6\int (t^6 + t^4 + t^2 + 1 + \frac{1}{t^2-1})dt$$

$$= -6\int \{\frac{1}{7}t^7 + \frac{1}{5}t^5 + \frac{1}{3}t^3 + t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c$$

$$= -\frac{6}{7}x^{\frac{7}{6}} - \frac{6}{5}x^{\frac{5}{6}} - 2x^{\frac{1}{2}} - 6x^{\frac{1}{5}} - 3\ln\left|\frac{\sqrt[6]{x}-1}{\sqrt[6]{x}+1}\right| + c$$

(30) 令 $\sqrt{x+1} = t, x = t^2 - 1, dx = 2tdt$

$$\int \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}dx = \int \frac{t-1}{t+1}\cdot 2tdt = \int 2tdt - 4\int \frac{t}{t+1}dt$$

$$= t^2 - 4t + 4\ln|t+1| + c$$
$$= x + 1 - 4\sqrt{x+1} + 4\ln|\sqrt{x+1}+1| + c$$
$$= x - 4\sqrt{x+1} + 4\ln|\sqrt{x+1}+1| + c_1$$

2. 应用分部积分法求下列不定积分：

(1) $\int \arcsin x\,dx$；(2) $\int \ln x\,dx$；

(3) $\int x^2\cos x\,dx$；(4) $\int \frac{\ln x}{x^3}dx$；

(5) $\int (\ln x)^2 dx$；(6) $\int x\arctan x\,dx$；

(7) $\int [\ln(\ln x) + \frac{1}{\ln x}]dx$；(8) $\int (\arcsin x)^2 dx$；

(9) $\int \sec^3 x\,dx$；(10) $\int \sqrt{x^2 \pm a^2}\,dx\ (a > 0)$；

解　(1) $\int \arcsin x\,dx = x\arcsin x - \int x\,d\arcsin x$

$$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx$$
$$= x\arcsin x + \frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}}$$
$$= x\arcsin x + \sqrt{1-x^2} + c$$

(2) $\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + c$

(3) $\int x^2\cos x\,dx = \int x^2 d\sin x = x^2\sin x - 2\int x\sin x\,dx$

$$= x^2\sin x + 2\int x\,d\cos x$$
$$= x^2\sin x + 2x\cos x - 2\int \cos x\,dx$$
$$= x^2\sin x + 2x\cos x - 2\sin x + c$$

(4) $\int \frac{\ln x}{x^3}dx = -\frac{1}{2}\int \ln x\,d\frac{1}{x^2} = -\frac{1}{2}\frac{\ln x}{x^2} + \frac{1}{2}\int \frac{1}{x^3}dx$

$$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c.$$

$$(5)\int(\ln x)^2dx = x(\ln x)^2 - 2\int\ln xdx$$

$$= x(\ln x)^2 - 2x\ln x + 2\int dx$$

$$= x(\ln x)^2 - 2x\ln x + 2x + c.$$

$$(6)\int x\arctan xdx = \frac{1}{2}\int\arctan xd(x^2+1)$$

$$= \frac{1}{2}(x^2+1)acrtan x - \frac{1}{2}\int dx$$

$$= \frac{1}{2}(x^2+1)acrtan x - \frac{1}{2}x + c$$

$$(7)\int[\ln(\ln x) + \frac{1}{\ln x}]dx = \int\ln(\ln x)dx + \int\frac{dx}{\ln x}$$

$$= x\ln(\ln x) - \int\frac{dx}{\ln x} + \int\frac{dx}{\ln x}$$

$$= x\ln(\ln x) + c.$$

$$(8)\int(\arcsin x)^2dx = x(\arcsin x)^2 - \int\frac{2x\arcsin x}{\sqrt{1-x^2}}dx$$

$$= x(\arcsin x)^2 + 2\int\arcsin xd\sqrt{1-x^2}$$

$$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2\int dx$$

$$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + c.$$

$$(9)\int\sec^3xdx = \int\sec xd\tan x = \sec x\tan x - \int\tan^2x\sec xdx$$

$$= \sec x\tan x + \int(1-\sec^2x)\sec xdx$$

$$= \sec x\tan x + \int\sec xdx - \int\sec^3xdx$$

$$= \sec x\tan x + \ln|\sec x + \tan x| - \int\sec^3xdx$$

故$\int\sec^3xdx = \frac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|) + c$

$$(10)\int\sqrt{x^2\pm a^2}dx = x\sqrt{x^2\pm a^2} - \int\frac{x^2}{\sqrt{x^2\pm a^2}}dx$$

$$= x\sqrt{x^2 \pm a^2} - \int \frac{x^2 \pm a^2}{\sqrt{x^2 \pm a^2}}dx + \int \frac{\pm a^2}{\sqrt{x^2 \pm a^2}}dx$$

$$= x\sqrt{x^2 \pm a^2} - \int \sqrt{x^2 \pm a^2}dx \pm a^2 ln \mid x + \sqrt{x^2 \pm a^2} \mid + c$$

移项并除以 2 可得：

$$\int \sqrt{x^2 \pm a^2}dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} ln \mid x + \sqrt{x^2 \pm a^2} \mid + c_1$$

3. 求下列不定积分：

(1)$\int [f(x)]^\alpha f'(x)dx(\alpha \neq -1)$；(2)$\int \frac{f'(x)}{1 + [f(x)]^2}dx$；

(3)$\int \frac{f'(x)}{f(x)}dx$；(4)$\int e^{f(x)}f'(x)dx$.

解：(1)$\int [f(x)]^\alpha f'(x)dx = \int [f(x)]^\alpha df(x)$

$$= \frac{1}{\alpha + 1}[f(x)]^{\alpha+1} + c$$

(2)$\int \frac{f'(x)}{1 + [f(x)]^2}dx = \int \frac{df(x)}{1 + [f(x)]^2} = arctanf(x) + c$

(3)$\int \frac{f'(x)}{f(x)}dx = \int \frac{df(x)}{f(x)} = ln \mid f(x) \mid + c$

(4)$\int e^{f(x)}f'(x)dx = \int e^{f(x)}df(x) = e^{f(x)} + c$

4. 证明：(1) 若 $I_n = \int tan^n x dx \quad n = 2,3,\cdots$，则

$$I_n = \frac{1}{n-1}tan^{n-1}x - I_{n-2}$$

(2) 若 $I_{(m,n)} = \int cos^m x sin^n x dx$ 则当 $m + n \neq 0$ 时，有

$$I_{m,n} = \frac{cos^{m-1}x sin^{n+1}x}{m+n} + \frac{m-1}{m+n}I_{(m-2,n)}$$

$$= -\frac{sin^{n-1}x cos^{m+1}x}{m+n} + \frac{n-1}{m+n}I_{(m,n-2)}$$

证　(1)$I_n = \int tan^{n-2}x(sec^2x - 1)dx$

$$= \int tan^{n-2}x d(tan x) - \int tan^{n-2}x dx$$

$$= \frac{1}{n-1} tan^{n-1}x - I_{n-2}$$

$$(2) I_{m,n} = \frac{1}{n+1}\int cos^{m-1}x d(sin^{n+1}x)$$

$$= \frac{1}{n+1}[cos^{m-1}x sin^{n+1}x + (m-1)\int sin^{n+2}x cos^{m-2}x dx]$$

$$= \frac{1}{n+1}[cos^{m-1}x sin^{n+1}x + (m-1)\int cos^{m+1}x(1-cos^2x) sin^n x dx]$$

$$= \frac{1}{n+1}[cos^{m-1}x sin^{n+1}x + (m-1)\int cos^{m-2}x sin^n x - (m-1)I_{m,n}]$$

$$= \frac{1}{n+1}[cos^{m-1}x sin^{n+1}x + (m-1)I_{m,2,n} - (m-1)I_{m,n}]$$

移项,合并得:

$$I_{(m,n)} = \frac{cos^{m-1}x sin^{n+1}x}{m+n} + \frac{m-1}{m+n} I_{m,2,n}$$

同理可证:

$$I_{(m,n)} = -\frac{sin^{n-1}x cos^{m+1}x}{m+n} + \frac{n-1}{m+n} I_{(m,n,2)}$$

5. 利用上题的递推公式计算:

$(1)\int tan^3x\, dx \quad (2)\int tan^2x dx \quad (3)\int cos^2x sin^4x dx$

解:$(1)\int tan^3x dx = \frac{1}{3-1} tan^{3-1}x - \int (tan x)^{3-2} dx$

$$= \frac{1}{2} tan^2x + ln\mid cos x\mid + c$$

$(2)\int tan^4x dx = \frac{1}{4-1} lan^{4-1}x - \int (tan x)^{4-2} dx$

$$= \frac{1}{3} tan^3x - \int tan^2x dx = \frac{1}{3} tan^3x + x - tan x + c$$

$(3)\int cos^2x sin^4x dx = \frac{cos x sin^5x}{6} + \frac{1}{6}\int sin^4x dx$

$$= \frac{cos x sin^5x}{6} + \frac{1}{6}\int (\frac{cos 2x - 1}{2})^2 dx$$

$$= \frac{\cos x \sin^5 x}{6} + \frac{1}{6}\int \frac{\cos^2 2x - 2\cos 2x + 1}{4})dx$$

$$= \frac{\cos x \sin^5 x}{6} + \frac{1}{24}[x - \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x] + c$$

6. 导出下列不定积分的递推公式,其中 n 为自然数:

(1)$I_n = \int x^n e^{kx}dx$;(2)$I_n = \int (\ln x)^n dx$;

(3)$I_n = \int (\arcsin x)^n dx$;(4)$I_n = \int e^{ax}\sin^n x dx$

解　(1)$I_n = \frac{1}{k}\int x^n de^{kx} = \frac{1}{k}x^n e^{kx} - \frac{n}{k}\int x^{n-1}e^{kx}dx$

$$= \frac{1}{k}x^n e^{kx} - \frac{n}{k}I_{n-1}$$

(2)$I_n = x(\ln x)^n - n\int x(\ln x)^{n-1} \cdot \frac{1}{x}dx$

$$= x(\ln x)^n - nI_{n-1}$$

(3)$I_n = x(\arcsin x)^n - n\int \frac{x}{\sqrt{1-x^2}}(\arcsin x)^{n-1}dx$

$$= x(\arcsin x)^n + n\int (\arcsin x)^{n-1}d\sqrt{1-x^2}$$

$$= x(\arcsin x)^n + n\sqrt{1-x^2}(\arcsin x)^{n-1} - n(n-1)\int (\arcsin x)^{n-2}dx$$

$$= x(\arcsin x)^n + n\sqrt{1-x^2}(\arcsin x)^{n-1} - n(n-1)I_{n-2}$$

(4)$I_n = \frac{1}{a}\int \sin^n bx de^{ax}$

$$= \frac{1}{a}\sin^n bx \cdot e^{ax} - \frac{nb}{a}\int e^{ax}\sin^{n-1}bx\cos bx dx$$

$$= \frac{1}{a}e^{ax}\sin^n bx - \frac{nb}{a^2}[e^{ax}\sin^{n-1}bx\cos bx$$

$$- b\int (n-1)\sin^{n-2}bx\cos^2 bx - b\sin^n bx]e^{ax}dx$$

$$= \frac{1}{a}e^{ax}\sin^n bx - \frac{nb}{a^2}e^{ax}\sin^{n-1}bx\cos bx$$

$$+ \frac{nb^2}{a^2}\int e^{ax}[(n-1)\sin^{n-2}bx - n\sin^n bx]dx$$

$$=\frac{1}{a^2}e^{ax}sin^{n-1}bx(asinbx-nbcosbx)+\frac{n(n-1)b^2}{a^2}I_{n-2}-\frac{n^2b^2}{a^2}I_n$$

移项,合并得:

$$I_n=\frac{1}{a^2+b^2n^2}[e^{ax}sin^{n-1}bx(asinx-nbcosx)+n(n-1)b^2I_{n-2}]$$

令 $b=1$ 即得所需证明

7. 利用上题所得递推公式证明:

(1) $\int x^3e^{2x}dx$ (2) $\int(lnx)^3dx$

(3) $\int can(sinx)^3dx$ (4) $\int e^x sin^3xdx$

解 (1) $\int x^3e^{2x}dx=\frac{1}{2}x^3e^{2x}-\frac{3}{2}\int x^2e^{2x}dx$

$$=\frac{1}{2}x^3e^{2x}-\frac{3}{2}[\frac{1}{2}x^2e^{2x}-\frac{2}{2}\int xe^{2x}dx]$$

$$=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{2}[\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}dx]$$

$$=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{4}xe^{2x}-\frac{3}{8}e^{2x}+c$$

(2) $\int(lnx)^3dx=xln^3x-3\int ln^2xdx$

$$=xln^3x-3xln^2x+6\int lnxdx$$

$$=xln^3x-3xln^2x+6xlnx-6x+c$$

(3) $\int(arcsinx)^3dx$

$$=x(arcsinx)^3+3\sqrt{1-x^2}(arcsinx)^2-3x2[\int(arcsinx)^1dx]$$

$$=x(arcsinx)^3+3\sqrt{2-x^2}(arcsinx)^2-6xarcsinx-6\sqrt{1-x^2}+c$$

(4) $\int e^x sin^3xdx$

$$=\frac{1}{1+3^2}[e^x sin^2x(sinx-3cosx)+3(3-1)\int e^x sinxdx]$$

$$=\frac{1}{10}[e^x sin^2x(sinx-3cosx)+3e^x(sinx-cosx)]+c$$

§3　有理函数和可化为有理函数不定积分

一、求下列不定积分：

(1) $\int \frac{x^3}{x-1}dx$　　(2) $\int \frac{x-2}{x^2-7x+12}dx$

(3) $\int \frac{dx}{x^3+1}$　　(4) $\int \frac{dx}{1+x^4}$

(5) $\int \frac{dx}{(x-1)(x^2+1)^2}$　　(6) $\int \frac{x-2}{(2x^2+2x+1)^2}dx$

解：(1) $\int \frac{x^3}{x-1}dx = \int(\frac{x^3-1}{x-1} + \frac{1}{x-1})dx$

$$= \int(x^2+x+1+\frac{1}{x-1})dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + ln(x-1) + c$$

(2) 令 $\frac{x-2}{x^2-7x+12} = \frac{x-2}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$

$$x-2 = A(x-4) + B(x-3)$$

令 $x = 4$ 得 $B = 2$　令 $x = 3$ 得　$A = -1$ 于是

$$\int \frac{x-2}{x^2-7x+12}dx = \int \frac{-1}{x-3}dx + 2\int \frac{1}{x-4}dx$$

$$= ln \mid \frac{(x-4)^2}{x-3} \mid + c$$

(3) 令 $\frac{1}{1+x^3} = \frac{1}{(x+1)(x^2-x+1)}$

$$= \frac{A}{x+1} + \frac{Bx+c}{x^2-x+c}$$ 则

$$1 = A(x^2-x+1) + (Bx+C)(x+1)$$

令 $x = -1$ 得 $A = \frac{1}{3}$ 代入上式有

$$1 = (B+\frac{1}{3})x^2 + (B+C-\frac{1}{3})x + (C+\frac{1}{3})$$

比较两端系数，得 $B = -\frac{1}{3}, C = \frac{2}{3}$ 于是

$$原式 = \int[\frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)}]dx$$

$$= \frac{1}{3}ln\mid x+1\mid - \frac{1}{6}\int\frac{2x-1-3}{x^2-x+1}dx$$

$$= \frac{1}{3}ln\mid x+1\mid - \frac{1}{6}\int\frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int\frac{dx}{x^2-x+1}$$

$$= \frac{1}{3}ln\mid x+1\mid - \frac{1}{6}ln\mid x^2-x+1\mid + \frac{1}{\sqrt{3}}arctan\frac{2x-1}{\sqrt{3}} + c$$

其中

$$\int\frac{dx}{x^2-x+1} = \frac{4}{3}\int\frac{dx}{(\frac{2x-1}{\sqrt{3}})^2+1}$$

(4) $原式 = \frac{1}{2}\int\frac{(1+x^2)-(x^2-1)}{1+x^4}dx$

$$= \frac{1}{2}\int\frac{d(x-\frac{1}{x})}{x^2+\frac{1}{x^2}} - \frac{1}{2}\int\frac{d(x+\frac{1}{x})}{x^2+\frac{1}{x^2}}$$

$$= \frac{1}{2}\int\frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2} - \frac{1}{2}\int\frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2}$$

$$= \frac{\sqrt{2}}{4}arctan\frac{x^2-1}{\sqrt{2}x} + \frac{\sqrt{2}}{8}ln\mid\frac{x^2+\sqrt{2}+1}{x^2-\sqrt{2}+1}\mid + c$$

其中

$$\int\frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2}$$

$$= \int[\frac{1}{x+\frac{1}{x}+\sqrt{2}} - \frac{1}{x+\frac{1}{x}-\sqrt{2}}](-\frac{1}{2\sqrt{2}})d(x+\frac{1}{x})$$

$$= -\frac{1}{2\sqrt{2}}[ln\mid x+\frac{1}{x}+\sqrt{2}\mid - ln\mid x+\frac{1}{x}-\sqrt{2}\mid] + c$$

$$=-\frac{1}{2\sqrt{2}}ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+c$$

(5) 令$\frac{1}{(x-1)(x^2+1)^2}=\frac{A}{x-1}+\frac{Bx+c}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$,则

$$1=A(x^2+1)^2+(Bx+C)(x-1)(x^2+1)+(Dx+E)(x-1)$$

令 $x=1$ 得 $A=\frac{1}{4}$　故

$$1=\frac{1}{4}(x^2+2x^2+1)+Bx^4+(C-B)x^3+(B-C)x^2$$
$$+(C-B)x-c+Dx^2+(E-D)x-E$$
$$=(\frac{1}{4}+B)x^4+(C-B)x^3+(\frac{1}{2}+B-C+D)x^2$$
$$+(C-B+3-D)x+(\frac{1}{4}-C-E)$$

比较两端同次幂项系数,得 $B=-\frac{1}{4}$　$C=-\frac{1}{R}$　$D=-\frac{1}{2}E=-\frac{1}{2}$,故

$$原式=\int\frac{dx}{4(x-1)}-\int\frac{x+1}{4(x^2+1)}dx-\frac{1}{2}\int\frac{x+1}{(x^2+1)^2}dx$$

而

$$\int\frac{x+1}{x^2+1}dx=\frac{1}{2}\int\frac{2x}{x^2+1}dx+\int\frac{dx}{x^2+1}$$
$$=\frac{1}{2}ln(x^2+1)+arctan\,x+C_1$$
$$\int\frac{x+1}{(x^2+1)^2}dx=\frac{1}{2}\int\frac{dx^2}{(x^2+1)^2}+\int\frac{dx}{(x^2+1)^2}$$
$$=\frac{-1}{2(x^2+1)}+\frac{x}{2(1+x^2)}+\frac{1}{2}arctan\,x+c_2$$

所以

$$原式=\frac{1}{4}ln|x-1|-\frac{1}{8}ln(x^2+1)-\frac{1}{4}arctan\,x+\frac{1}{4(x^2+1)}$$
$$-\frac{x}{4(x^2+1)}-\frac{1}{4}arctan\,x+C$$
$$=\frac{1}{4}(ln\frac{|x-1|}{\sqrt{x^2+1}}-2arctan\,x+\frac{1-x}{x^2+1})+C$$

(6) 原式 $= \frac{1}{4}\int \frac{4x+2}{(2x^2+2x+1)^2}dx - \frac{5}{2}\int \frac{dx}{(2x^2+2x+1)^2}$

而

$$\int \frac{dx}{(2x^2+2x+1)^2} = \int \frac{rdx}{[(2x+1)^2+1]^2}$$
$$= \int \frac{2d(2x+1)}{[(2x+1)^2+1]^2}$$
$$= \frac{2x+1}{(2x+1)^2+1} + arctan(2x+1) + C$$

于是

$$原式 = -\frac{1}{4(2x^2+2x+1)} - \frac{5}{4}\cdot\frac{2x+1}{2x^2+2x+1} - \frac{5}{2}arctan(2x+1) + C$$
$$= -\frac{5x+3}{2(2x^2+2x+1)} - \frac{5}{2}arctan(2x+1) + C.$$

2. 求下列不定积分

(1) $\int \frac{dx}{5-3\cos x}$；(2) $\int \frac{dx}{2+\sin^2 x}$；

(3) $\int \frac{dx}{1+\tan x}$；(4) $\int \frac{x^2}{\sqrt{1+x-x^2}}$

(5) $\int \frac{dx}{\sqrt{x^2+x}}$；(6) $\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}dx$

解 (1) 令 $t = \tan\frac{x}{2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$

$$\int \frac{dx}{5-3\cos x} = \int \frac{1}{5-3\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt = \int \frac{1}{1+4t^2}dt$$
$$= \frac{1}{2}\int \frac{1}{1+(2t)^2}d(2t) = \frac{1}{2}arctan 2t + C$$
$$= \frac{1}{2}arctan[2(\tan\frac{x}{2})] + C$$

(2) 原式 $= \int \frac{\sec^2 x dx}{2\sec^2 + \tan^2 x} = \int \frac{d(\tan x)}{3\tan^2 x + 2}$

$$= \frac{\sqrt{6}}{6}\int \frac{d\sqrt{\frac{3}{2}}tan x}{\frac{3}{2}tan^2 x + 1} = \frac{\sqrt{6}}{6} arctan(\sqrt{\frac{3}{2}}tan x) + C$$

(3) 原式 $= \int \frac{cos x}{cos x + sin x}dx$

$$= \frac{1}{2}\int \frac{cos x + sin x - sin x + cos x}{cos x + sin x}dx$$

$$= \frac{1}{2}(\int dx + \int \frac{d(cos x + sin x)}{cos x + sin x})$$

$$= \frac{1}{2}(x + ln\mid cos x + sin x\mid) + C$$

(4) 记 $I = \int \frac{x^2}{\sqrt{1+x-x^2}}dx$，则 $I = -\int \sqrt{1+x-x^2}dx + \int \frac{(x+1)dx}{\sqrt{1+x-x^2}}$

而 $-\int \sqrt{1+x-x^2}dx$

$$= -x\sqrt{1+x-x^2} + \int \frac{-2x+1}{2\sqrt{1+x-x^2}}xdx$$

$$= -x\sqrt{1+x-x^2} - \int \frac{x^2}{\sqrt{1+x-x^2}}dx + \frac{1}{2}\int \frac{xdx}{\sqrt{1+x-x^2}}$$

所以　$I = -x\sqrt{1+x-x^2} - I + \frac{3}{2}\int \frac{x+\frac{3}{2}}{\sqrt{1+x-x^2}}dx$，从而

$$I = -\frac{x}{2}\sqrt{1+x-x^2} + \frac{3}{4}\int \frac{x+\frac{3}{2}}{\sqrt{1+x-x^2}}dx$$

而　$\int \frac{x+\frac{3}{2}}{\sqrt{1+x-x^2}}dx$

$$= -\frac{1}{2}\int \frac{2x+1}{\sqrt{1+x-x^2}}dx + \frac{7}{6}\int \frac{dx}{\sqrt{1+x-x^2}}$$

$$=-\sqrt{1+x-x^2}+\frac{7}{6}\int\frac{d(\frac{2x-1}{\sqrt{5}})}{\sqrt{1-(\frac{2x-1}{\sqrt{5}})^2}}$$

$$=-\sqrt{1+x-x^2}+\frac{7}{6}arcsin(\frac{2x-1}{\sqrt{5}})+C$$

故　原式 $=-\frac{x}{2}\sqrt{1+x-x^2}+\frac{3}{4}-\sqrt{1+x-x^2}$

$$+\frac{7}{6}arcsin(\frac{2x-1}{\sqrt{5}})+C]$$

$$=-\frac{x}{2}\sqrt{1+x-x^2}-\frac{3}{4}\sqrt{1+x-x^2}+\frac{7}{8}arcsin(\frac{2x-1}{\sqrt{5}})+c_1$$

(5) 令 $\sqrt{x^2+x}=t-x$,则 $x=\frac{t}{1+2t}$,$dt=\frac{2(t^2+t)}{(1+2t)^2}dt$

代入被积表达式:

$$\int\frac{dx}{\sqrt{x^2+x}}=\int\frac{2}{1+2t}dt=\int\frac{dt}{t+\frac{1}{2}}$$

$$=ln\mid t+\frac{1}{2}\mid+C$$

$$=ln\mid\sqrt{x^2+x}+x+\frac{1}{2}\mid+C$$

(6) 令 $\frac{1}{x}=t$,则

$$\int\frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}dx=-\int\sqrt{\frac{\frac{1}{x}-1}{\frac{1}{x}+1}}d\frac{1}{x}=-\int\sqrt{\frac{t-1}{t+1}}dt$$

$$=-\int\frac{t-1}{\sqrt{t^2-1}}dt=-\int\frac{t}{\sqrt{t^2-1}}dt+\int\frac{1}{\sqrt{t^2-1}}dt$$

$$=-\sqrt{t^2-1}+ln t+\sqrt{t^2-1}+c$$

$$=-\frac{\sqrt{1-x^2}}{x}+ln\frac{1+\sqrt{1-x^2}}{x}+c$$

总 练 习 题

求下列不定积分

(1)$\int \frac{\sqrt{x}-2\sqrt[3]{x}-1}{\sqrt[4]{x}}dx$

解　原式$=\int(x^{\frac{1}{4}}-2x^{\frac{1}{12}}-x^{-\frac{1}{4}})dx$

$$=\frac{4}{5}x^{\frac{5}{4}}-\frac{24}{13}x^{\frac{13}{12}}-\frac{4}{3}x^{\frac{3}{4}}+C$$

(2)$\int x\arcsin x dx$

解:原式$=-\frac{1}{2}\int \arcsin x d(1-x^2)$

$$=-\frac{1}{2}(1-x^2)\arcsin x+\frac{1}{2}\int\frac{1-x^2}{\sqrt{1-x^2}}dx$$

$$=-\frac{1}{2}(1-x^2)\arcsin x+\frac{1}{2}\int\sqrt{1-x^2}dx$$

令 $x=\sin t$,得

$$\int\sqrt{1-x^2}dx=\int\cos^2 t dt=\int\frac{1+\cos 2t}{2}dt$$

$$=\frac{t}{2}+\frac{1}{4}\sin 2t+c$$

$$=\frac{1}{2}\arcsin x+\frac{1}{2}x\sqrt{1-x^2}+c$$

从而

$$\int x\arcsin x dx=-\frac{1}{2}(1-x^2)\arcsin x+\frac{1}{4}\arcsin x+\frac{1}{4}x\sqrt{1-x^2}+c$$

$$=x^2\arcsin x-\frac{1}{4}\arcsin x+\frac{x}{4}\sqrt{1-x^2}+c$$

(3)$\int\frac{dx}{1+\sqrt{x}}$

解:令$\sqrt{x}=t$,得 $dx=2tdt$,从而

$$\int\frac{dx}{1+\sqrt{x}}=\int\frac{2tdt}{1+t}=2\int(1-\frac{1}{1+t})dt$$

$$= 2(t - \ln|1+t| + c$$
$$= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + c$$

(4) $\int e^{\sin x}\sin 2x dx$

解：原式 $= 2\int e^{\sin x}\sin x d\sin x = 2\int \sin x de^{\sin x}$

$$= 2e^{\sin x}\sin x - 2\int e^{\sin x}d\sin x$$
$$= 2e^{\sin x}\sin x - 2e^{\sin x} + C$$
$$= 2e^{\sin x}(\sin x - 1) + C$$

(5) $\int e^{\sqrt{x}}dx$

解：令 $x = t^2 \quad dx = 2tdt$

原式 $= 2\int te^t dt = 2te^t - 2\int e^t dt + c$

$$= 2e^t(t-1) + c = 2e^{\sqrt{x}}(\sqrt{x} - 1) + c$$

(6) $\int \frac{dx}{x\sqrt{x^2 - 1}}$

解：原式 $= \int \frac{dx}{x^2\sqrt{1 - \frac{1}{x^2}}} = -\int \frac{d\frac{1}{x}}{\sqrt{1-\frac{1}{x^2}}}$

$$= -\arcsin\frac{1}{x} + c = \arccos\frac{1}{x} + c$$

(7) $\int \frac{1 - \tan x}{1 + \tan x}dx$

解：原式 $= \int \frac{(\cos x - \sin x)dx}{\cos x + \sin x} = \int \frac{d(\sin x + \cos x)}{\cos x + \sin x}$

$$= \ln|\cos x + \sin x| + c$$

(8) $\int \frac{x^2 - x}{(x-2)^3}dx$

解：原式 $= \int \frac{x^2 - 4x + 4}{(x-2)^3}dx + 3\int \frac{-x-2}{(x-2)^3}dx + 2\int \frac{1}{(x-2)^3}dx$

$$= \int \frac{dx}{x-2} + 3\int \frac{dx}{(x-2)^2} + 2\int \frac{dx}{(x-2)^3}$$

$$= \ln|x-2| - \frac{3}{(x-2)} - \frac{1}{(x-2)^2} + c$$

(9) $\int \frac{dx}{\cos^4 x}$

解：原式 $= \int \sec^2 x d\tan x = (1 + \tan^2 x) d\tan x$

$$= \tan x + \frac{1}{3}\tan^3 x + c$$

(10) $\int \sin^4 x dx$

解：$\int \sin^4 x dx = \int (\frac{1-\cos 2x}{2})^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) dx$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \frac{1+\cos 4x}{2}) dx$$

$$= \frac{3}{8}x - \frac{1}{4}\sin x + \frac{1}{32}\sin 4x + c$$

(11) $\int \frac{x-5}{x^3 - 3x^2 + 4} dx$

解：令 $\frac{x-5}{x^3-3x^2+4} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$，则

$$x - 5 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

令 $x = -1$　得　$A = -\frac{2}{3}$　令 $x = 2$　得

$C = -1$　代入上式，再令 $x = 0$ 有

$$-5 = -\frac{8}{3} - 2B - 1$$

得 $B = \frac{2}{3}$，于是

原式 $= -\frac{2}{3}\int \frac{dx}{x+1} + \frac{2}{3}\int \frac{dx}{x-2} - \int \frac{dx}{(x-2)^2}$

$$= -\frac{2}{3}\ln|x+1| + \frac{2}{3}\ln|x-2| + \frac{1}{x-2} + c$$

$$= \frac{2}{3} ln \mid \frac{x-2}{x+1} \mid + \frac{1}{x-2} + c$$

(12) $\int arctan(1+\sqrt{x})dx$

解:令 $t = 1+\sqrt{x}$　则 $x = (t-1)^2 dx = 2(t-1)dt$

原式 $= 2\int(t-1)arctan\,tdt = 2\int tarctan\,tdt - 2\int arctan\,tdt$

$$= t^2 arctan\,t - \int \frac{t^2}{1+t^2}dt - 2tarctan\,t + \int \frac{2t}{1+t^2}dt$$

$$= (t^2-2t)arctan\,t - \int dt + \int \frac{1}{t^2+1}dt + ln(t^2+1)$$

$$= (t^2-2t)arctan\,t - t + arctan\,t + ln(t^2+1) + c$$

$$= \sqrt{x}arctan(1+\sqrt{x}) - \sqrt{x} + ln(2+2\sqrt{x}+x) + C$$

(13) $\int \frac{x^7}{x^4+2}dx$

解:原式 $= \int x^3 - \frac{2x^3}{x^4+2}dx$

$$= \int x^3 dx - \frac{1}{2}\int \frac{1}{x^4+2}d(x^4+2)$$

$$= \frac{1}{4}x^4 - \frac{1}{2}ln(x^4+2) + c$$

(14) $\int \frac{tan\,x}{1+tan\,x+tan^2 x}dx$

解:令 $tan\,x = t$,得 $x = arctan\,t, dx = \frac{dt}{1+t^2}$,故

原式 $= \int \frac{t}{1+t+t^2} \cdot \frac{dt}{1+t^2} = \int \frac{dt}{1+t^2} - \int \frac{dt}{1+t+t^2}$

$$= arctan\,t - \frac{2\sqrt{3}}{3}arctan\frac{2t+1}{\sqrt{3}} + c$$

$$= x - \frac{2\sqrt{3}}{3}arctan\frac{2tan\,x+1}{3} + c$$

(15) $\int \frac{x^2}{(1-x)^{100}}dx$

解:原式$=\int \frac{(1-x)^2}{(1-x)^{100}}dx-2\int \frac{1-x}{(1-x)^{100}}dx+\int \frac{1}{(1-x)^{100}}dx$

$$=\frac{1}{97}(1-x)^{-97}-\frac{1}{49}(1-x)^{-98}+\frac{1}{99}(1-x)^{-99}+c$$

(16)$\int \frac{\arcsin x}{x^2}dx$

解:令 $\arcsin x=t$,得 $x=\sin t, dx=\cos t dt$. 故

原式$=\int \frac{t\cos t dt}{\sin^2 t}=-\int t d\frac{1}{\sin t}=-\frac{t}{\sin t}+\int \frac{dt}{\sin t}$

$$=-\frac{t}{\sin t}+\ln\left|\tan\frac{t}{2}\right|+C$$

$$=-\frac{t}{\sin t}+\ln\left|\frac{1-\cos t}{\sin t}\right|+C$$

$$=-\frac{\arcsin x}{x}+\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|+C$$

(17)$\int x\ln(\frac{1+x}{1-x})dx$

解:原式$=\frac{1}{2}\int \ln(\frac{1+x}{1-x})dx^2=\frac{1}{2}x^2\ln(\frac{1+x}{1-x})-\int \frac{x^2}{1-x^2}dx$

$$=\frac{1}{2}x^2\ln(\frac{1+x}{1-x})+\int dx-\int \frac{1}{1-x^2}dx$$

$$=\frac{1}{2}(x^2-1)\ln\left|\frac{1+x}{1-x}\right|+x+c$$

(18)$\int \frac{1}{\sqrt{\sin x\cos^2 x}}dx$

解　令 $t=\tan x$,得 $\sin x=\frac{t}{\sqrt{1+t^2}}$, $\cos x=\frac{1}{\sqrt{1+t^2}}$, $dx=\frac{dt}{1+t^2}$ 故

原式$=\int \frac{1+t^2}{\sqrt{t}}dt-\int t^{\frac{3}{2}}dt+\int t^{\frac{1}{2}}dt$

$$=\frac{2}{5}t^{\frac{5}{2}}+2t^{\frac{1}{2}}+c$$

$$=\frac{2}{5}\tan^{\frac{5}{2}}x+2\tan^{\frac{1}{2}}x+c$$

19.$\int e^x(\frac{1-x}{1+x^2})^2dx$

解：原式 $= \int e^x \frac{1+x^2-2x}{(1+x^2)^2}dx - \int \frac{e^x}{1+x^2}dx + \int e^x d\frac{1}{1+x^2}$

$= \int \frac{e^x}{1+x^2}dx + \frac{e^x}{1+x^2} - \int \frac{e^x}{1+x^2}dx$

$= \frac{e^x}{1+x^2} + C$

(20) $I_n = \int \frac{v^n}{\sqrt{u}}dx$，其中 $u = a_1 + b_1x, v = a_2 + b_2x$，求递推形式解.

解：$I_n = \frac{2}{b_1}\int v^n d\sqrt{u} - \frac{2}{b_1}v^n\sqrt{u} - \frac{2nb}{b_1}\int \sqrt{u}v^{n-1}dx$

而 $\int \sqrt{u}v^{n-1}dx = \int \frac{uv^{n-1}}{\sqrt{u}}dx = \int \frac{(a_1+b_1x)v^{n-1}}{\sqrt{u}}dx$

$= \int \frac{\frac{b_1}{b_2}(a_2+b_2x)v^{n-1}}{\sqrt{u}}dx + \int \frac{(a_1 - \frac{b_1}{b_2}a_2)v^{n-1}}{\sqrt{n}}dx$

$= \frac{b_1}{b_2}\int \frac{v^n}{\sqrt{n}}dx + (a_1 - \frac{b_1}{b_2}a_2)\int \frac{v^{n-1}}{\sqrt{n}}dx$

$= \frac{1}{b_2}[b_1I_n + (a_1b_2 - b_1a_1)I_{n-1}]$

故 $I_n = \frac{2}{b_1}v^n\sqrt{u} - \frac{2n}{b_1}[b_{I_n} + (a_1b_2 - a_2b_1)I_{n-1}]$

移项，合并得：

$I_n = \frac{2}{(2n+1)b_1}[\sqrt{u}v^n + n(a_2b_1 - a_1b_2)I_{n-1}]$