2019-2020春夏学期《微分几何》第十、十一周作业

P_{55}

5. 证明 (1)

$$\mathrm{I}=\frac{(2du^1)^2+(2du^2)^2}{[1-(u^1)^2-(u^2)^2]^2}$$

$$\omega^1=\frac{2}{1-(u^1)^2-(u^2)^2}du^1,\quad \omega^2=\frac{2}{1-(u^1)^2-(u^2)^2}du^2$$

$$\begin{aligned}\omega_1^2 &= \frac{d\omega^1}{\omega^1\wedge\omega^2}\omega^1+\frac{d\omega^2}{\omega^1\wedge\omega^2}\omega^2\\ &= \frac{-4u^2}{4}\frac{2du^1}{1-(u^1)^2-(u^2)^2}+\frac{4u^1}{4}\frac{2du^2}{1-(u^1)^2-(u^2)^2}\\ &= \frac{-2u^2du^1+2u^1du^2}{1-(u^1)^2-(u^2)^2}\end{aligned}$$

$$\begin{aligned}d\omega_1^2 &= \frac{2-2(u^1)^2+2(u^2)^2}{[1-(u^1)^2-(u^2)^2]^2}du^1\wedge du^2+\frac{2+2(u^1)^2-2(u^2)^2}{[1-(u^1)^2-(u^2)^2]^2}du^1\wedge du^2\\ &= \frac{4}{[1-(u^1)^2-(u^2)^2]^2}du^1\wedge du^2\end{aligned}$$

因此

$$K=-\frac{d\omega_1^2}{\omega^1\wedge\omega^2}=-1$$

(2)

$$\omega^1=\frac{du^1}{u^2},\quad \omega^2=\frac{du^2}{u^2}$$

$$\begin{aligned}\omega_1^2 &= \frac{d\omega^1}{\omega^1\wedge\omega^2}\omega^1+\frac{d\omega^2}{\omega^1\wedge\omega^2}\omega^2\\ &= 1\cdot\frac{du^1}{u^2}+0\cdot\frac{du^2}{u^2}=\frac{1}{u^2}du^1\end{aligned}$$

$$d\omega_1^2=\frac{1}{(u^2)^2}du^1\wedge du^2$$

因此

$$K=-\frac{d\omega_1^2}{\omega^1\wedge\omega^2}=-1$$

(3) 曲面的第一基本形式可化为

$$\begin{aligned}\mathrm{I} &= \frac{(du^1-2u^2du^2)^2+(2\sqrt{u^1-(u^2)^2}du^2)^2}{(2\sqrt{u^1-(u^2)^2})^2}\\ &= \left(\frac{du^1-2u^2du^2}{2\sqrt{u^1-(u^2)^2}}\right)^2+(du^2)^2\end{aligned}$$

由此得

$$\omega^1=\frac{du^1-2u^2du^2}{2\sqrt{u^1-(u^2)^2}},\quad \omega^2=du^2$$

因

$$d\omega^1=\frac{-(du^1-2u^2du^2)\wedge(du^1-2u^2du^2)}{2(u^1-(u^2)^2)^{\frac{3}{2}}}=0,\ d\omega^2=0$$

故

$$\omega_1^2=\frac{d\omega^1}{\omega^1\wedge\omega^2}\omega^1+\frac{d\omega^2}{\omega^1\wedge\omega^2}\omega^2=0$$

因此

$$K=-\frac{d\omega_1^2}{\omega^1\wedge\omega^2}=0$$

6. **证明** 在(4.22)中取$\alpha=\beta=1,\ \gamma=\kappa=2$, 则右式为

$$\mathrm{RHS}=h_{11}h_2^2-h_{12}h_1^2=(h_{11}h_{22}-(h_{12})^2)g^{22}=g_{11}K$$

左式为

$$\mathrm{LHS}=(\Gamma_{11}^2)_2-(\Gamma_{12}^2)_1+\Gamma_{11}^1\Gamma_{12}^2+\Gamma_{11}^2\Gamma_{22}^2-\Gamma_{12}^1\Gamma_{11}^2-(\Gamma_{12}^2)^2$$

利用

$$\Gamma_{11}^2=-\frac{1}{2g_{22}}(g_{11})_2,\ \ \Gamma_{12}^2=\frac{1}{2g_{22}}(g_{22})_1,\ \ \Gamma_{11}^1=\frac{1}{2g_{11}}(g_{11})_1$$

$$\Gamma_{22}^2=\frac{1}{2g_{22}}(g_{22})_2,\quad \Gamma_{12}^1=\frac{1}{2g_{11}}(g_{11})_2$$

代入整理得左式为

$$\mathrm{LHS}=-\frac{(g_{11})_{22}}{2g_{22}}-\frac{(g_{22})_{11}}{2g_{22}}+\frac{(g_{11})_1(g_{22})_1+((g_{11})_2)^2}{4g_{11}g_{22}}+\frac{(g_{11})_2(g_{22})_2+((g_{22})_1)^2}{4(g_{22})^2}$$

而由

$$\begin{aligned}
&-\frac{1}{\sqrt{g_{11}g_{22}}}\left[\left(\frac{(\sqrt{g_{11}})_2}{\sqrt{g_{22}}}\right)_2+\left(\frac{\sqrt{g_{22}}_1}{\sqrt{g_{11}}}\right)_1\right]\\
=\quad&-\frac{(g_{11})_{22}}{2g_{11}g_{22}}+\frac{((g_{11})_2)^2g_{22}+(g_{11})_2(g_{22})_2g_{11}}{4(g_{11}g_{22})^2}\\
&-\frac{(g_{22})_{11}}{2g_{11}g_{22}}+\frac{((g_{22})_1)^2g_{11}+(g_{22})_1(g_{11})_1g_{22}}{4(g_{11}g_{22})^2}
\end{aligned}$$

比较得

$$K=\frac{\mathrm{LHS}}{g_{11}}=-\frac{1}{\sqrt{g_{11}g_{22}}}\left[\left(\frac{(\sqrt{g_{11}})_2}{\sqrt{g_{22}}}\right)_2+\left(\frac{\sqrt{g_{22}}_1}{\sqrt{g_{11}}}\right)_1\right]\ \blacksquare$$

7. **证明** 在曲率线网下, 因$h_{12}=0$, Codazzi方程其中两式化为

$$\begin{aligned}
(h_{11})_2-h_{11}\Gamma_{12}^1+h_{22}\Gamma_{11}^2&=0\\
(h_{22})_1-h_{22}\Gamma_{12}^2+h_{11}\Gamma_{22}^1&=0
\end{aligned}$$

因

$$
\begin{aligned}
h_{11}\Gamma_{12}^1 &= \frac{h_{11}(g_{11})_2}{2g_{11}} = \frac{1}{2}k_1(g_{11})_2 \\
-h_{22}\Gamma_{11}^2 &= \frac{h_{22}(g_{11})_2}{2g_{22}} = \frac{1}{2}k_2(g_{11})_2
\end{aligned}
$$

故

$$(h_{11})_2 = H(g_{11})_2$$

同理

$$(h_{22})_1 = H(g_{22})_1$$

当H为常数时, 取曲率线网, 由以上知可设$h_{11} = Hg_{11} + \varphi(u^1)$, $h_{22} = Hg_{22} + \psi(u^2)$. 于是有

$$2H = k_1 + k_2 = \frac{h_{11}}{g_{11}} + \frac{h_{22}}{g_{22}} = 2H + \frac{\varphi}{g_{11}} + \frac{\psi}{g_{22}}$$

因而

$$\frac{\varphi}{g_{11}} = -\frac{\psi}{g_{22}}$$

若其为0, 则有$k_1 = k_2 = 0$, 为脐点, 从而为平面或球面. 除此以外, 可设

$$\frac{\varphi}{g_{11}} = -\frac{\psi}{g_{22}} = \frac{1}{\rho^2}$$

则

$$
\begin{aligned}
\varphi &= \frac{1}{\rho^2}g_{11} = \frac{1}{1+H\rho^2}h_{11} \\
-\psi &= \frac{1}{\rho^2}g_{22} = -\frac{1}{1-H\rho^2}h_{22}
\end{aligned}
$$

从而若作参数变换

$$\tilde{u}^1 = \int \sqrt{\varphi}du^1, \quad \tilde{u}^2 = \int \sqrt{-\psi}du^2$$

则曲面的第一和第二基本形式可化为

$$\mathrm{I} = \rho^2[(d\tilde{u}^1)^2 + (d\tilde{u}^2)^2], \quad \mathrm{II} = (1+H\rho^2)(d\tilde{u}^1)^2 - (1-H\rho^2)(d\tilde{u}^2)^2 \ \blacksquare$$

8. **证明** 由S的方程得

$$g_{11} = a^2(1+u^2), \quad g_{12} = abuv, \quad g_{22} = b^2(1+v^2)$$

$$h_{11} = \frac{a}{\sqrt{1+u^2+v^2}}, \quad h_{12} = 0, \quad h_{22} = \frac{b}{\sqrt{1+u^2+v^2}}$$

从而

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{1}{ab(1+u^2+v^2)^2}$$

同样, 对$\bar{S}$,

$$\bar{K} = \frac{1}{\bar{a}\bar{b}(1+\bar{u}^2+\bar{v}^2)^2}$$

于是, 当$ab = \bar{a}\bar{b}$时, 在点(u,v)与$(\bar{u},\bar{v})$处有相等的Gauss曲率.

设两曲面参数的变换关系为$\bar{u} = \bar{u}(u,v),\ \bar{v} = \bar{v}(u,v)$时, 能等距对应, 由Gauss美妙定理, 曲面$S$和$\bar{S}$在对应点应该有相同的Gauss 曲率, 即

$$\frac{1}{\bar{a}\bar{b}(1+\bar{u}^2+\bar{v}^2)^2} = \frac{1}{ab(1+u^2+v^2)^2}$$

故有

$$\bar{u}^2(u,v) + \bar{v}^2(u,v) = u^2 + v^2$$

因此曲面S上的点$(u,v) = (0,0)$必须对应着曲面$\bar{S}$上的点$(\bar{u},\bar{v}) = (0,0)$, 即

$$\bar{u}(0,0) = 0, \quad \bar{v}(0,0) = 0$$

将前面由Gauss美妙定理得到的恒等式分别对u, v求导得到

$$\bar{u}\frac{\partial\bar{u}}{\partial u} + \bar{v}\frac{\partial\bar{v}}{\partial u} = u, \quad \bar{u}\frac{\partial\bar{u}}{\partial v} + \bar{v}\frac{\partial\bar{v}}{\partial v} = v$$

将上面的式子再次对u, v求导, 并且让$u = 0,\ v = 0$, 则得到

$$\left(\frac{\partial\bar{u}}{\partial u}\right)^2 + \left(\frac{\partial\bar{v}}{\partial u}\right)^2 = 1, \quad \left(\frac{\partial\bar{u}}{\partial v}\right)^2 + \left(\frac{\partial\bar{v}}{\partial v}\right)^2 = 1, \quad \frac{\partial\bar{u}}{\partial u}\frac{\partial\bar{u}}{\partial v} + \frac{\partial\bar{v}}{\partial u}\frac{\partial\bar{v}}{\partial v} = 0$$

命

$$J = \begin{pmatrix} \frac{\partial\bar{u}}{\partial u} & \frac{\partial\bar{v}}{\partial u} \\ \frac{\partial\bar{u}}{\partial v} & \frac{\partial\bar{v}}{\partial v} \end{pmatrix}$$

则前面得到的等式表明$J|_{(u,v)=(0,0)}$是正交矩阵, 不妨设

$$J|_{(u,v)=(0,0)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\epsilon\sin\theta & \epsilon\cos\theta \end{pmatrix}$$

其中$\epsilon = \pm 1$. 因为假定给出的对应是等距对应, 根据第一基本形式相等应该有

$$\begin{pmatrix} a^2(1+u^2) & abuv \\ abuv & b^2(1+v^2) \end{pmatrix} = J \begin{pmatrix} \bar{a}^2(1+\bar{u}^2) & \bar{a}\bar{b}\bar{u}\bar{v} \\ \bar{a}\bar{b}\bar{u}\bar{v} & \bar{b}^2(1+\bar{v}^2) \end{pmatrix} J^T$$

让$(u,v) = (0,0)$, 则根据关于J的假定, 上面的等式成为

$$\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\epsilon\sin\theta & \epsilon\cos\theta \end{pmatrix} \begin{pmatrix} \bar{a}^2 & 0 \\ 0 & \bar{b}^2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\epsilon\sin\theta \\ \sin\theta & \epsilon\cos\theta \end{pmatrix}$$

即

$$\begin{aligned} \bar{a}^2\cos^2\theta + \bar{b}^2\sin^2\theta &= a^2 \\ (\bar{b}^2 - \bar{a}^2)\sin\theta\cos\theta &= 0 \\ \bar{a}^2\sin^2\theta + \bar{b}^2\cos^2\theta &= b^2 \end{aligned}$$

如果$\bar{a}^2 = \bar{b}^2$, 则从上面的第一, 三式得到$\bar{a}^2 = \bar{b}^2 = a^2 = b^2$. 如果$\bar{a}^2 \neq \bar{b}^2$, 则从上面的第二式得到或者$\theta = 0$, 或者$\theta = \frac{\pi}{2}$, 即

$$(a^2, b^2) = (\bar{a}^2, \bar{b}^2) \quad 或 \quad (a^2, b^2) = (\bar{b}^2, \bar{a}^2)$$

因此若曲面S和$\bar{S}$之间存在等距对应, 则必有$(a^2,b^2)=(\bar{a}^2,\bar{b}^2)$或$(a^2,b^2)=(\bar{b}^2,\bar{a}^2)$. 前者时两曲面重合, 后者时两曲面相差一镜面反射. ▌

9. **证明** 由圆环面的方程, 得

$x_1=(-b\sin v\cos v,-b\sin u\sin v,b\cos u),\ x_2=(-(a+b\cos u)\sin v,(a+b\cos u)\cos v,0)$

从而

$$g_{11}=b^2,\quad g_{12}=0,\quad g_{22}=(a+b\cos u)^2$$

为正交参数网, 于是

$$\begin{aligned}
e_1 &= \frac{x_1}{|x_1|}=(-\sin v\cos v,-\sin u\sin v,\cos u)\\
e_2 &= \frac{x_2}{|x_2|}=(-\sin v,\cos v,0)\\
e_3 &= e_1\times e_2=(-\cos u\cos v,-\cos u\sin v,-\sin u)
\end{aligned}$$

及

$$\omega^1=bdu,\quad \omega^2=(a+b\cos u)dv,\quad \omega^3=0$$

因$de_1=\omega_1^3e_3,\ de_2=\omega_2^3e_3$, 故

$$\begin{aligned}
\omega_1^3 &= de_1\cdot e_3=du=\frac{1}{b}\omega^1\\
\omega_2^3 &= de_2\cdot e_3=\cos udv=\frac{\cos u}{a+b\cos v}\omega^2
\end{aligned}$$

得

$$b_{11}=\frac{1}{b},\quad b_{12}=b_{21}=0,\quad b_{22}=\frac{\cos u}{a+b\cos v}$$

因此

$$\begin{aligned}
H &= \frac{1}{2}(b_{11}+b_{22})=\frac{a+2b\cos u}{2b(a+b\cos u)}\\
K &= b_{11}b_{22}-b_{12}^2=\frac{\cos u}{b(a+b\cos u)}
\end{aligned}$$ ▌

10.**解**(u^1,u^2)参数网是曲率线网。如果Gauss方程，Codazzi方程成立，则存在曲面且在相差一个合同变换下是唯一的。

Gauss方程：

$$K=-\frac{1}{\sqrt{g_{11}g_{22}}}\{\Big(\frac{(\sqrt{g_{11}})_2}{\sqrt{g_{22}}}\Big)_2+\Big(\frac{(\sqrt{g_{22}})_1}{\sqrt{g_{11}}}\Big)_1\}=\frac{1}{(1+(u^1)^2)^2}$$

另一方面

$$K=\frac{h_{11}h_{22}}{g_{11}g_{22}}=\frac{1}{(1+(u^1)^2)^2}$$

故Gauss方程成立。

Codazzi方程：

$$(h_{11})_2=H(g_{11})_2,\ (h_{22})_1=H(g_{22})_1$$

因为$(h_{11})_2=0$, $(g_{11})_2=0$, $(h_{22})_1=\frac{2u^1+(u^1)^3}{(1+(u^1)^2)^{\frac{3}{2}}}$, $(g_{22})_1=2u^1$.
而

$$H=\frac{1}{2}\frac{g_{11}h_{22}+g_{22}h_{11}}{g_{11}g_{22}}=\frac{2+(u^1)^2}{2(1+(u^1)^2)^{\frac{3}{2}}}$$

故Codazzi方程成立。
注意到该曲面的第一、第二基本形式只与u^1有关，且$F=M=0$, 这与旋转面的情形相同。设旋转曲面的方程为

$$X(u^1,u^2)=(f(u^1)cosu^2,f(u^1)sinu^2,g(u^1)),\ f(u^1)>0.$$

直接计算可得

$$\widetilde{I}=((f')^2+(g')^2)(du^1)^2+f^2(du^2)^2,$$

$$\widetilde{II}=\frac{f'g''-f''g'}{\sqrt{f'^2+g'^2}}(du^1)^2+\frac{fg'}{\sqrt{f'^2+g'^2}}(du^2)^2$$

对照$f'^2+g'^2=1+(u^1)^2$, $f^2=(u^1)^2$, $\frac{f'g''-f''g'}{\sqrt{f'^2+g'^2}}=\frac{1}{\sqrt{1+(u^1)^2}}$, $\frac{fg'}{\sqrt{f'^2+g'^2}}=\frac{u^2}{\sqrt{1+u^2}}$

$$\Longrightarrow f(u^1)=u^1,\ \ g'(u^1)=u^1,\ \ g(u^1)=\frac{1}{2}(u^1)^2.$$

从而

$$X(u^1,u^2)=(u^1cosu^2,u^1sinu^2,\frac{1}{2}(u^1)^2)$$

▌

8.**证明**三个曲面的Gauss曲率分别>0、$=0$、<0,故不存在等距对应。▌

9.**证明**对于曲面$S:r=(ucosv,usinv,lnu)$, 经计算可得第一、第二基本形式分别为

$$I=(1+\frac{1}{u^2})du^2+u^2dv^2,\quad II=-\frac{1}{u\sqrt{1+u^2}}du^2+\frac{u}{\sqrt{1+u^2}}dv^2.$$

从而

$$K=\frac{h_{11}h_{22}-h_{12}^2}{g_{11}g_{22}-g_{12}^2}=-\frac{1}{(1+u^2)^2}$$

对于曲面$\bar{S}:\bar{r}=(\bar{u}cos\bar{v},\bar{u}sin\bar{v},\bar{v})$, 经计算可得第一、第二基本形式分别为

$$\bar{I}=d\bar{u}^2+(1+\bar{u}^2)d\bar{v}^2,\quad \bar{I}I=-\frac{1}{\sqrt{1+\bar{u}^2}}d\bar{u}d\bar{v}.$$

从而

$$\bar{K}=\frac{h_{11}h_{22}-h_{12}^2}{g_{11}g_{22}-g_{12}^2}=-\frac{1}{(1+\bar{u}^2)^2}$$

当$u^2=\bar{u}^2$时，有$K=\bar{K}$，而这不能保持它们的第一基本形式相同，故S与$\bar{S}$不存在等距对应。▌

11.**证明**对于第一张曲面S_1,$K=\frac{det(h_{\alpha\beta})}{det g_{\alpha\beta}}=-1$, 而$\omega_2^1=0$,故$K=-\frac{d\omega_2^1}{\omega^1\wedge\omega^2}=0$,矛盾，不满足高斯方程，由曲面基本定理知不存在这样的曲面。
对于第二张曲面S_2,在曲率线网下，考率Codazzi方程：

$$(h_{22})_1=(g_{22})_1H$$

而$H=\frac{k_1+k_2}{2}=\frac{cos^2u+\frac{1}{cos^2u}}{2}$, $(h_{22})_1=0$,而$(g_{22})_1H\neq 0$,即Codazzi方程不满足，由曲面论基本定理知不存在这样的曲面。▌

12.**解**(u,v)参数网是正交参数网。如果Gauss方程，Codazzi方程成立，则存在曲面且在相差一个合同变换下是唯一的。
由于Gauss方程：

$$K=-\frac{1}{\sqrt{g_{11}g_{22}}}\{\Big(\frac{(\sqrt{g_{11}})_2}{\sqrt{g_{22}}}\Big)_2+\Big(\frac{(\sqrt{g_{22}})_1}{\sqrt{g_{11}}}\Big)_1\}=1$$

另一方面

$$K=\frac{h_{11}h_{22}}{g_{11}g_{22}}=1$$

故Gauss方程成立。
经过计算可得：$\Gamma_{11}^1=\Gamma_{11}^2=\Gamma_{12}^1=\Gamma_{21}^1=\Gamma_{22}^2=0$, $\Gamma_{12}^2=\Gamma_{21}^2=-\tan u$, $\Gamma_{22}^1=\cos u\sin u$.
则Codazzi方程为：

$$(h_{22})_1-h_{22}\Gamma_{21}^2=-h_{11}\Gamma_{22}^1$$

又因为$(h_{22})_1=-2\cos u\sin u,\quad h_{11}=1,\quad h_{22}=\cos^2 u$，故Codazzi方程成立。
注意到该曲面主曲率为1，且$F=M=0$, 这与单位球面的情形相同。设单位球面的方程为

$$X(u,v)=(\cos u\cos v,\cos u\sin v,\sin u)$$

直接计算可得

$$\widetilde{I}=\widetilde{II}=du^2+\cos^2 udv^2$$

▌