第十七章　多元函数微分学

§1　可微性

1. 求下列函数的偏导数：

(1) $z = x^2 y$；(2) $z = y\cos x$；(3) $z = \dfrac{1}{\sqrt{x^2+y^2}}$；

(4) $z = \ln(x^2+y^2)$；(5) $z = e^{xy}$；(6) $z = \arctan\dfrac{y}{x}$；

(7) $z = xye^{\sin(xy)}$；(8) $u = \dfrac{y}{x} + \dfrac{z}{y} - \dfrac{x}{z}$；

(9) $u = (xy)^z$；(10) $u = x^{y^z}$.

解　(1) $z_x = 2xy, z_y = x^2$. (2) $z_x = -y\sin x, z_y = \cos x$.

(3) $z_x = \dfrac{-\dfrac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \dfrac{-x}{(x^2+y^2)^{3/2}}, z_y = \dfrac{-y}{(x^2+y^2)^{3/2}}$.

(4) $z_x = \dfrac{2x}{x^2+y^2}, z_y = \dfrac{2y}{x+y}$. (5) $z_x = ye^{xy}, z_y = xe^{xy}$.

(6) $z_x = \dfrac{1}{1+\left(\dfrac{y}{x}\right)^2} \cdot \dfrac{-y}{x^2} = \dfrac{-y}{x^2+y^2}, z_y = \dfrac{x}{x^2+y^2}$.

(7) $z_x = ye^{\sin(xy)} + xy^2e^{\sin(xy)}\cos(xy) = [1+xy\cos(xy)]ye^{\sin(xy)}$,

$z_y = [1+xy\cos(xy)]xe^{\sin(xy)}$

(8) $u_x = -\dfrac{y}{x^2} - \dfrac{1}{z}, u_y = \dfrac{1}{x} - \dfrac{z}{y^2}, u_z = \dfrac{1}{y} + \dfrac{x}{z^2}$

(9) $u_x = zy(xy)^{z-1}, u_y = zx(xy)^{z-1}, u_z = (xy)^z\ln(xy)$

(10) $u_x = y^z x^{y^z-1}, u_y = zy^{z-1}x^{y^z}\ln x, u_z = y^z x^{y^z}\ln x \cdot \ln y$

2. 设 $f(x,y) = x + (y-1)\arcsin\sqrt{\dfrac{x}{y}}$；求 $f_x(x,1)$

解　因为 $f(x,1)=x$　所以 $f_x(x,1)=\dfrac{d}{dx}f(x,1)=1$.

3. 设

$$f(x,y)=\begin{cases} y\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

考察函数 f 在原点 $(0,0)$ 的偏导数.

解　由于

$$\lim_{\triangle x\to 0}\frac{f(0+\triangle x,0)-f(0,0)}{\triangle x}=\lim_{\triangle x\to 0}\frac{0-0}{\triangle x}=0$$

$$\lim_{\triangle y\to 0}\frac{f(0,0+\triangle y)-f(0,0)}{\triangle y}=\lim_{\triangle y\to 0}\sin\frac{1}{(\triangle y)^2}$$

所以 $f(x,y)$ 在原点关于 x 的偏导数为 0,关于 y 的偏导数不存在.

4. 证明函数 $z=\sqrt{x^2+y^2}$ 在点 $(0,0)$ 连续但偏导数不存在.

$$\lim_{(x,y)\to(0,0)}\sqrt{x^2+y^2}=0=z(0,0)$$

所以函数 $z=\sqrt{x^2+y^2}$ 在点 $(0,0)$ 连续.

由于当 $\triangle x\to 0$ 时

$$\frac{z(0+\triangle x,0)-z(0,0)}{\triangle x}=\frac{\sqrt{(\triangle x)^2}}{\triangle x}=\frac{|\triangle x|}{\triangle x}$$

极限不存在,因而 $z(x,y)$ 在点 $(0,0)$ 关于 x 的偏导数不存在.

同理可证它关于 y 的偏导数也不存在.

5. 考察函数

$$f(x,y)=\begin{cases} xy\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

在点 $(0,0)$ 处的可微性.

解　由偏导数定义知

$$f_x(0,0)=\lim_{\triangle x\to 0}\frac{f(0+\triangle x,0)-f(0,0)}{\triangle x}=\lim_{\triangle x\to 0}\frac{0-0}{\triangle x}=0$$

同理可得 $f_y(0,0)=0$.

由于

$$\left|\frac{\triangle f-f_x(0,0)\triangle x-f_y(0,0)\triangle y}{\rho}\right|$$

$$=\left|\frac{\triangle x\cdot\triangle y}{\sqrt{(\triangle x)^2+(\triangle y)^2}}\sin\frac{1}{(\triangle x)^2+(\triangle y)^2}\right|$$

$$\leqslant\frac{(\triangle x)^2+(\triangle y)^2}{2\sqrt{(\triangle x)^2+(\triangle y)^2}}$$

$$=\frac{\sqrt{(\triangle x)^2+(\triangle y)^2}}{2}\to 0\left(\sqrt{(\triangle x)^2+(\triangle y)^2}\to 0\right)$$

所以 f 在点 $(0,0)$ 处可微.

6. 证明函数

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0\\ 0; & x^2+y^2=0\end{cases}$$

在点 $(0,0)$ 连续且偏导数存在,但在此点不可微.

证　因为 $\left|\dfrac{x^2y}{x^2+y^2}\right|=\dfrac{|x||xy|}{x^2+y^2}\leqslant\dfrac{|x|}{2}$,从而

$$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^2}=0=f(0,0)$$

所以,$f(x,y)$ 在点 $(0,0)$ 连续.

由偏导数定义知

$$f(0,0)=\lim_{\triangle x\to 0}\frac{f(0+\triangle x,0)-f(0,0)}{\triangle}=\lim_{\triangle x\to 0}\frac{0-0}{\triangle x}=0$$

同理 $f_y(0,0)=0$

所以,$f(x,y)$ 在点 $(0,0)$ 的偏导数存在.

但 $\dfrac{\triangle f-f_x(0,0)\triangle x-f_y(0,0)\triangle y}{\rho}=\dfrac{(\triangle x)^2\cdot\triangle y}{[(\triangle x)^2+(\triangle y)^2]^{3/2}}$

考察 $\dfrac{(\triangle x)^2\cdot\triangle y}{[(\triangle x)^2+(\triangle y)^2]^{3/2}}$,由于当 $\triangle x=\triangle y$ 时其值为 $\dfrac{1}{\sqrt{8}}$,当 $\triangle y=0$ 时其值为 0.

所以，$\lim\limits_{\rho\to0}\dfrac{(\triangle x)^2\cdot\triangle y}{[(\triangle x)^2+(\triangle y)^2]^{3/2}}$ 不存在，故 $f(x,y)$ 在点 $(0,0)$ 不可微.

7. 证明函数

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq0\\ 0, & x^2+y^2=0\end{cases}$$

在点 $(0,0)$ 连续且偏导数存在，但偏导数在点 $(0,0)$ 不连续，而 f 在原点 $(0,0)$ 可微.

证　$\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}=\lim\limits_{\rho\to0}\rho^2\sin\dfrac{1}{\rho^2}=0=f(0,0)$.
因此 f 在点 $(0,0)$ 连续.

当 $x^2+y^2\neq0$ 时

$$f_x(x,y)=2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}$$

当 $x^2+y^2=0$ 时

$$f_x(0,0)=\lim_{\triangle x\to0}\frac{f(0+\triangle x,0)-f(0,0)}{\triangle x}$$

$$=\lim_{\triangle x\to0}\triangle x\sin\frac{1}{\triangle x^2}=0$$

但由于 $\lim\limits_{(x,y)\to(0,0)}2x\sin\dfrac{1}{x^2+y^2}=0$，而

$\lim\limits_{(x,y)\to(0,0)}\dfrac{2x}{x^2+y^2}\cos\dfrac{1}{x^2+y^2}$ 不存在(可考察 $y=x$ 情况)

因此当 $(x,y)\to(0,0)$ 时，$f_x(x,y)$ 的极限不存在，从而 $f_x(x,y)$ 在点 $(0,0)$ 不连续. 同理可证 $f_y(x,y)$ 在点 $(0,0)$ 不连续. 然而

$$\lim_{\rho\to0}\frac{\triangle f-f_x(0,0)\triangle x-f_y(0,0)\triangle y}{\rho}$$

$$=\lim_{(\triangle x,\triangle y)\to(0,0)}\frac{(\triangle x)^2+(\triangle y)^2}{\sqrt{(\triangle x)^2+(\triangle y)^2}}\sin\frac{1}{(\triangle x)^2+(\triangle y)^2}=0,$$

所以 f 在点 $(0,0)$ 可微且 $df\mid_{(0,0)}=0$.

8. 求下列函数在给定点的全微分；

(1) $z = x^4 + y^4 - 4x^2y^2$ 在点$(0,0)$,$(1,1)$；

(2) $z = \dfrac{x}{\sqrt{x^2 + y^2}}$ 在点$(1,0)$,$(0,1)$.

解　(1) 因 $z_x = 4x^3 - 8xy^2, z_y = 4y^3 - 8x^2y$　在$(0,0)$连续，从而 z 在$(0,0)$可微.由 $z_x(0,0) = 0, z_y(0,0) = 0$ 得 $dz\mid_{(0,0)} = 0$.同理 z 在 $(1,1)$ 由 $z_x(1,1) = -4, z_y(1,1) = -4$ 得 $dz\mid_{(1,1)} = -4(dx + dy)$.

(2) 因 $z_x = \dfrac{y^2}{(x^2 + y^2)^{3/2}}, z_y = \dfrac{-xy}{(x^2 + y^2)^{3/2}}$ 在$(1,0)$,$(0,1)$处可微且由 $z_x(1,0) = 0, z_y(1,0) = 0$ 得 $dz\mid_{(1,0)} = 0$.

由 $z_x(0,1) = 1, z_y(0,1) = 0$ 得$dz\mid_{(0,1)} = dx$

9. 求下列函数的全微分；

(1)$z = y\sin(x + y)$;(2)$u = xe^{yz} + e^{-z} + y$

解　(1)$dz = y\cos(x + y)dx + [\sin(x + y) + y\cos(x + y)]dy$

(2)$du = e^{yz}dx + (xze^{yz} + 1)dy + (xye^{yz} - e^{-z})dz$

10. 求曲面 $z = \arctan\dfrac{y}{x}$ 在点$\left(1,1,\dfrac{\pi}{4}\right)$处的切平面方程和法线方程.

解　由于 z 在$(1,1)$处可微,从而切平面存在.因为

$$z_x(1,1) = -\frac{1}{2}, z_y(1,1) = \frac{1}{2},$$

所以切平面方程为

$$-\frac{1}{2}(x - 1) + \frac{1}{2}(y - 1) - \left(z - \frac{\pi}{4}\right) = 0,$$

即 $x - y + 2z = \dfrac{\pi}{2}$.

法线方程为$\dfrac{x-1}{-\frac{1}{2}} = \dfrac{y-1}{\frac{1}{2}} = \dfrac{z-\frac{\pi}{4}}{-1}$

即 $2(1-x)=2(y-1)=\frac{\pi}{4}-z$

11. 求曲面 $3x^2+y^2-z^2=27$ 在点 $(3,1,1)$ 处的切平面方程与法线方程.

解 $z_x=\left.\frac{6x}{2z}\right|_{z=1}^{x=3}=9\quad z_y=\left.\frac{y}{z}\right|_{z=1}^{y=1}=1$

所以切平面方程为 $9(x-3)+(y-1)-(z-1)=0$ 即

$$9x+y-z-27=0.$$

法线方程为 $\frac{x-3}{9}=\frac{y-1}{1}=\frac{z-1}{-1}$

即 $x-3=9(y-1)=9(1-z)$.

12. 在曲面 $z=xy$,上求一点,使这点的切平面平行于平面 $x+3y+z+9=0$,并写出这切平面方程和法线方程.

解 设所求点为 $P(x_0,y_0,x_0y_0)$,点 P 处切平面法向量为 $(z_x(x_0,y_0),z_y(x_0,y_0),-1)=(y_0,x_0,-1)$. 要求切平面与平面 $x+3y+z+9=0$ 平行,故 $\frac{1}{y_0}=\frac{3}{x_0}=-1$,从而 $x_0=-3,y_0=-1$. 得 P 点为 $(-3,-1,3)$ 且点 P 处的切平面方程为 $-(x+3)-3(y+1)-(z-3)=0$ 即 $x+3y+z+3=0$. 法线方程为

$$\frac{x+3}{-1}=\frac{y+1}{-3}=\frac{z-3}{-1}$$

即 $3(x+3)=y+1=3(z-3)$.

13. 计算近似值:

(1) $1.002\times2.003^2\times3.004^3$;(2) $\sin29^\circ\times\tan46^\circ$

解 (1) 设 $u=xy^2z^3,x_0=1,y_0=2,z_0=3,\triangle x=0.002$.

$\triangle y=0.003,\triangle z=0.004$ 根据 $u(x_0+\triangle x,y_0+\triangle y,z_0+\triangle z)\approx u(x_0,y_0,z_0)+u_x(x_0,y_0,z_0)\triangle x+u_y(x_0,y_0,z_0)\triangle y+u_z(x_0,y_0,z_0)\triangle z$. $u(1,2,3)=108,u_x(1,2,3)=108,y_y(1,2,3)=108,u_z(1,2,3)=108$. 知

$1.002\times2.003^2\times3.004^3$

$$\approx 108 + 108 \times 0.002 + 108 \times 0.003 + 108 \times 0.004 = 108.972$$

(2) 设 $u = \sin x \cdot \tan y, x_0 = \frac{\pi}{6}, y_0 = \frac{\pi}{4}, \triangle x = \frac{-\pi}{180}, \triangle y = \frac{\pi}{180}$ 则

$$u\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \frac{1}{2}, u_x\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}, u_y\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = 1$$

因而

$$\sin 29^\circ \cdot \tan 46^\circ \approx \frac{1}{2} - \frac{\sqrt{3}}{2} \times \frac{\pi}{180} + \frac{\pi}{180} \approx 05023$$

14. 设圆台上下底的半径分别为 $R = 30cm, r = 20cm$ 高 $h = 40cm$. 若 R, r, h 分别增加 $3mm, 4mm, 2mm$. 求此圆台体积变化的近似值.

解　圆台体积 $V = \frac{\pi h}{3}(R^2 + Rr + r^2)$　从而

$$\triangle V \approx V_R \cdot \triangle R + V_r \cdot \triangle r + V_h \cdot \triangle h$$

将 $R = 30, r = 20, h = 40$, 及 $\triangle R = 0.3, \triangle r = 0.4, \triangle h = 0.2$ 代入上式得

$$\triangle V \approx \frac{3200\pi}{3} \times 0.3 + \frac{2800\pi}{3} \times 0.4 + \frac{1900\pi}{3} \times 0.2 = 820\pi \approx 2576(cm)^3$$

15. 证明:若二元函数 f 在点 $P(x_0, y_0)$ 的某邻域 $U(P)$ 内的偏导函数 f_x 与 f_y 有界,则 f 在 $U(P)$ 内连续.

证　由 f_x, f_y 在 $U(P)$ 内有界,设此邻域为 $U(P, \delta_1)$,存在 $M > 0$,使 $|f_x| < M, |f_y| < M$ 在 $U(P, \delta_1)$ 内成立,由于

$$\begin{aligned} |\triangle z| &= |f(x + \triangle x, y + \triangle y) - f(x, y)| \\ &= |f_x(x + \theta_1 \triangle x, y + \triangle y)\triangle x + f_y(x, y + \theta_2 \triangle y)\triangle y| \\ &\leqslant M|\triangle x| + M|\triangle y| \end{aligned}$$

所以对任意的正数 ε,存在 $\delta = \min\left\{\delta_1, \frac{\varepsilon}{2(M+1)}\right\}$,当 $|\triangle x| < \delta$, $|\triangle y| < \delta$ 时,有 $|f(x + \triangle x, y + \triangle y) - f(x, y)| < \varepsilon$,故 f 在 $U(P, \delta)$ 内连续.

16. 设二元函数 f 在区域 $D = [a, b] \times [c, d]$ 上连续

(1) 若在 $intD$ 内有 $f_x \equiv 0$，试问 f 在 D 上有何特性？

(2) 若在 $intD$ 内有 $f_x = f_y \equiv 0$，f 又怎样？

(3) 在(1)的讨论中，关于 f 在 D 上的连续性假设可否省略？长方形区域可否改为任意区域？

解　(1) 二元函数 f 在 $D = [a,b] \times [c,d]$ 上连续，若在 $intD$ 内有 $f_x \equiv 0$，则 $f(x,y) = \varphi(y)$.

这是因为对 $intD$ 内任意两点 (x_1,y)，(x_2,y) 由中值定理知

$$f(x_2,y) - f(x_1,y) = f_x(x_1 + \theta(x_2 - x_1),y)(x_2 - x_1) = 0$$

即 $f(x_2,y) = f(x_1,y)$，由 (x_1,y)，(x_2,y) 的任意性知 $f(x,y) = \varphi(y)$.

(2) 若在 $intD$ 内有 $f_x = f_y \equiv 0$，则 $f(x,y) =$ 常数.

事实上，对 $intD$ 内任意两点 (x_1,y_1)，(x_2,y_2) 由中值定理(课本 P_{112} 页)知存在

$$\xi = x_1 + \theta_1(x_2 - x_1), \eta = y_1 + \theta_2(y_2 - y_1) \quad 0 < \theta_1, \theta_2 < 1$$

使得

$$f(x_2,y_2) - f(x_1,y_1) = f_x(\xi,y_2)(x_2 - x_1) + f_y(x_1,\eta)(y_2 - y_1),$$

因为 $f_x = f_y \equiv 0$ 所以 $f(x_2,y_2) \equiv f(x_1,y_1)$. 由 (x_1,y_1)，(x_2,y_2) 的任意性知 $f(x,y) =$ 常数.

(3) 在(1)的讨论中，关于 f 在 D 上的连续性假设不能省略. 否则结论不一定成立. 例如：在矩形区域 $D = \left[-\frac{3}{2},\frac{3}{2}\right] \times [0,2]$ 上二元函数

$$f(x,y) = \begin{cases} y^3, x > 0, y > 0 \\ 0, \quad D \text{ 中其它部分} \end{cases}$$

在 $intD$ 内 $f_x \equiv 0$，可是不连续，$f(1,1) = 1$，$f(-1,1) = 0$，显然 f 与 x 有关，结论不成立.

在(1)的讨论中，长方形区域不能改为任意区域，否则结论不一定成立. 例如：设

$$I = \{(x,y) \mid x = 0, y \geqslant 0\}, D = R^2 - I,\text{二元函数}$$

$$f(x,y)=\begin{cases} y^3, x>0 \quad y>0 \\ 0, \qquad D\text{中其它点}\end{cases}$$

在 D 上连续，且 $f_x\equiv 0$，但 $f(1,1)=1, f(-1,1)\equiv 0$ 即 f 与 x 有关，结论不成立.

17. 试证在原点(0,0)的充分小邻域内有

$$\arctan\frac{x+y}{1+xy}\approx x+y$$

证　设 $f(u,v)=\arctan\dfrac{u+v}{1+uv}, u_0=0, v_0=0, \triangle u=x, \triangle v=y$

则　$\arctan\dfrac{x+y}{1+xy}\approx f(u_0,v_0)+f_u(u_0,v_0)\triangle u+f_v(u_0,v_0)\triangle v$

$f(u_0,v_0)=0, f_u(u_0,v_0)=1 \quad f_v(u_0,v_0)=1$

故 $\arctan\dfrac{x+y}{1+xy}\approx 1\cdot x+1\cdot y=x+y$

18. 求曲面 $z=\dfrac{x^2+y^2}{4}$ 与平面 $y=4$ 的交线在 $x=2$ 处的切线与 OX 轴的交角.

解　设该角为 α，则根据导数的几何意义切线对 OX 轴的斜率为 $z_x(2,4)=\dfrac{x}{2}\Big|_{x=2}=1, \tan\alpha=1, \alpha=\dfrac{\pi}{4}$，所以切线与 OX 轴交角 $\alpha=\dfrac{\pi}{4}$.

19. 试证：

(1) 乘积的相对误差限近似于各因子相对误差限之和；

(2) 商的相对误差限近似于分子和分母相对误差限之和.

证　(1) 设 $u=xy$，则 $du=ydx+xdy$，故

$$\left|\frac{\triangle u}{u}\right|\approx\left|\frac{du}{u}\right|\leqslant\left|\frac{dx}{x}\right|+\left|\frac{dy}{y}\right|$$

(2) 设 $v=\dfrac{x}{y}$ 则 $dv=\dfrac{ydx-xdy}{y^2}, \dfrac{dv}{v}=\dfrac{dx}{x}-\dfrac{dy}{y}$

故 $\left|\dfrac{\triangle v}{v}\right|\approx\left|\dfrac{dv}{v}\right|\leqslant\left|\dfrac{dx}{x}\right|+\left|\dfrac{dy}{y}\right|$

20. 测得一物体的体积 $V = 4.45cm^3$,其绝对误差限为 $0.01cm^3$,又测得重量 $W = 30.80g$,其绝对误差限为 $0.01g$,求由公式 $d = \frac{W}{V}$ 算出的比重 d 的相对误差限和绝对误差限.

解　$|\triangle d| \approx |d_W \cdot \triangle W + d_V \cdot \triangle V| = \left|\frac{\triangle W}{V} - \frac{W}{V^2}\triangle V\right|$

$$|\triangle d| \leqslant \left|\frac{\triangle W}{V}\right| + \left|\frac{W}{V^2}\triangle V\right| = \frac{1}{4.45}\times 0.01 + \frac{30.8}{4.45^2}\times 0.01 \approx 0.017$$

$$\left|\frac{\triangle d}{d}\right| \approx \left|\frac{\triangle W}{W}\right| + \left|\frac{\triangle V}{V}\right| \approx 0.26\%$$

所以 d 的相对误差限为 0.26%,绝对误差限为 0.017.

§2　复合函数微分法

1. 求下列复合函数的偏导数或导数:

(1) 设 $z = \arctan(xy)$, $y = e^x$,求 $\frac{dz}{\alpha x}$;

(2) 设 $z = \frac{x^2 + y^2}{xy}e^{\frac{x^2+y^2}{xy}}$,求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$;

(3) 设 $z = x^2 + xy + y^2$, $x = t^2$, $y = t$,求 $\frac{dz}{dt}$;

(4) 设 $z = x^2\ln y$, $x = \frac{u}{v}$, $y = 3u - 2v$,求 $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$;

(5) 设 $u = f(x + y, xy)$,求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$;

(6) 设 $u = f\left(\frac{x}{y}, \frac{y}{z}\right)$,求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

解　(1) 令 $u = xy$,则 $z = \arctan u$, $y = e^x$　$x = x$

$$\frac{dz}{dx} = \frac{dz}{du}\frac{\partial u}{\partial x} + \frac{dz}{du}\frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{y}{1 + x^2y^2} + \frac{xe^x}{1 + x^2y^2} = \frac{e^x(1 + x)}{1 + x^2y^2}$$

(2) $$\frac{\partial z}{\partial x} = \frac{y(x^2 - y^2)}{x^2y^2}e^{\frac{x^2+y^2}{xy}} + \frac{x^2 + y^2}{xy} \cdot \frac{y(x^2 - y^2)}{x^2y^2}e^{\frac{x^2+y^2}{xy}}$$

$$= \frac{x^2 - y^2}{x^2y}\left(1 + \frac{x^2 + y^2}{xy}\right)e^{\frac{x^2+y^2}{xy}}$$

$$\frac{\partial z}{\partial y}=\frac{x^2-y^2}{xy^2}\left(1+\frac{x^2+y^2}{xy}\right)e^{\frac{x^2+y^2}{xy}}$$

(3) $\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}=(2x+y)2t+(x+2y)\cdot 1$

$$=4t^3+3t^2+2t$$

(4) $\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=2x\ln y\cdot\frac{1}{v}+x^2\cdot\frac{1}{y}\cdot 3$

$$=\frac{u}{v^2}\left[2\ln(3u-2v)+\frac{3u}{3u-2v}\right]$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial v}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=-\frac{2u^2}{v^2}\left[\frac{1}{v}\ln(3u-2v)+\frac{1}{3u-2v}\right]$$

(5) 由于 $du=f_1d(x+y)+f_2d(xy)$

$$=f_1dx+f_1dy+f_2ydx+f_2xdy$$

$$=(f_1+yf_2)dx+(f_1+xf_2)dy$$

所以 $\frac{\partial u}{\partial x}=f_1+yf_2$，$\frac{\partial u}{\partial y}=f_1+xf_2$.

(6) $\frac{\partial u}{\partial x}=\frac{1}{y}f_1,\frac{\partial u}{\partial y}=-\frac{x}{y^2}f_1+\frac{1}{z}f_2,\frac{\partial u}{\partial z}=-\frac{y}{z^2}f_2.$

2. 设 $z=\frac{y}{f(x^2-y^2)}$,其中 f 为可微函数,验证

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$$

证　设 $u=x^2-y^2$,则

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}=-\frac{2xyf'(u)}{f^2(u)},\frac{\partial z}{\partial y}=\frac{f(u)+2y^2f'(u)}{f^2(u)}$$

所以$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{-2yf'(u)+\frac{f(u)}{y}+2yf'(u)}{f^2(u)}=\frac{z}{y^2}$

3. 设 $z=\sin y+f(\sin x-\sin y)$,其中 f 为可微函数,证明:

$$\frac{\partial z}{\partial x}\sec x+\frac{\partial z}{\partial y}\sec y=1$$

证　设 $u=\sin x-\sin y$　则

$$\frac{\partial z}{\partial x} = f'(u)\cos x, \quad \frac{\partial z}{\partial y} = (1 - f'(u))\cos y$$

所以 $\dfrac{\partial z}{\partial x}\sec x + \dfrac{\partial z}{\partial y}\sec y = f'(u) + (1 - f'(u)) = 1$

4. 设 $f(x,y)$ 可微,证明:在坐标旋转变换

$$x = u\cos\theta - v\sin\theta, y = u\sin\theta + v\cos\theta$$

之下.$(f_x)^2 + (f_y)^2$ 是一个形式不变量.即若

$$g(u,v) = f(u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta)$$

则必有$(f_x)^2 + (f_y)^2 = (g_u)^2 + (g_v)^2$.(其中旋转角 θ 是常数)

证 $g_u = f_x\cos\theta + f_y\sin\theta, g_v = f_x(-\sin\theta) + f_y\cos\theta$

$$\begin{aligned}(g_u)^2 + (g_v)^2 &= f_x^2\cos^2\theta + f_y^2\sin^2\theta + 2f_xf_y\sin\theta\cos\theta + f_x^2\sin^2\theta + f_y^2\cos^2\theta \\ &\quad - 2f_xf_y\sin\theta\cos\theta \\ &= f_x^2(\sin^2\theta + \cos^2\theta) + f_y^2(\sin^2\theta + \cos^2\theta) = f_x^2 + f_y^2\end{aligned}$$

故 $(f_x)^2 + (f_y)2 = (g_u)^2 + (g_v)^2$.

5. 设 $f(u)$ 是可微函数.

$$F(x,t) = f(x + 2t) + f(3x - 2t),$$

试求:$F_x(0,0)$ 与 $F_t(0,0)$

解 $F_x = f'(x + 2t) + 3f'(3x - 2t)$,

$F_t = 2f'(x + 2t) - 2f'(3x - 2t)$.

故 $F_x(0,0) = 4f'(0), F_t(0,0) = 0$.

6. 若函数 $u = F(x,y,z)$ 满足恒等式

$$F(tx,ty,tz) = t^kF(x,y,z)(t > 0)$$

则称 $F(x,y,x)$ 为 k 次齐次函数.试证下述关于齐次函数的欧拉定理:可微函数 $F(x,y,z)$ 为 k 次齐次函数的充要条件是:

$$xF_x(x,y,z) + yF_y(x,y,z) + zF_z(x,y,z) = kF(x,y,z)$$

并证明:$z = \dfrac{xy^2}{\sqrt{x^2 + y^2}} - xy$ 为 2 次齐次函数.

证 必要性 由 $F(tx,ty,tz) = t^kF(x,y,z)$. 令 $\xi = tx$, $\eta = ty, \xi = tz$,两边对 t 求导得

$$xF_\xi(\xi,\eta,\zeta)+yF_\eta(\xi,\eta,\zeta)+zF_\zeta(\xi,\eta,\zeta)$$
$$=kt^{t-1}F(x,y,z)$$

令 $t=1$ 则有

$$xF_x(x,y,z)+yF_y(x,y,z)+zF_z(x,y,z)=kF(x,y,z)$$

充分性　设 $\Phi(x,y,z,t)=\frac{1}{t^k}F(tx,ty,tz)(t>0)$

令 $\xi=tx,\eta=ty,\zeta=tz$,求 Φ 关于 t 的偏导数得

$$\frac{\partial\Phi}{\partial t}=\frac{1}{t^{k+1}}\{[xF_\xi(\xi,\eta,\zeta)+yF_\eta(\xi,\eta,\zeta)$$
$$+zF_\zeta(\xi,\eta,\zeta)]t-kF(\xi,\eta,\zeta)\}$$

由已知$\frac{\partial\Phi}{\partial t}=0$,于是 Φ 仅是x,y,z 的函数,记

$\psi(x,y,z)=\Phi(x,y,z,t)$,所以 $t^k\psi(x,y,z)=F(tx,ty,tz)$,令 $t=1$ 时 $\psi(x,y,z)=F(x,y,z)$.因此

$$t^kF(x,y,z)=F(tx,ty,tz).$$

因为 $z(tx,ty)=\frac{(tx)(ty)^2}{\sqrt{(tx)^2+(ty)^2}}-(tx)(ty)$

$$=t^2z(x,y)$$

所以 $z(x,y)$ 为二次齐次函数.

7. 设$f(x,y,z)$具有性质$f(tx,t^ky,t^mz)=t^nf(x,y,z)\ (t>0)$证明:

(1) $f(x,y,z)=x^nf\left(1,\frac{y}{x^k},\frac{z}{x^m}\right)$;

(2) $xf_x(x,y,z)+kyf_y(x,y,z)+mzf_z(x,y,z)=nf(x,y,z)$

证　(1) 由 $f(tx,t^ky,t^mz)=t^nf(x,y,z)$　令 $t=\frac{1}{x}$ 得

$$f\left(1,\frac{y}{x^k},\frac{z}{x^m}\right)=x^{-n}f(x,y,z)$$

即 $f(x,y,z)=x^nf\left(1,\frac{y}{x^k},\frac{z}{x^m}\right)$.

(2) 令 $\xi=tx,\eta=t^ky,\zeta=t^mz$,对 $f(tx,t^ky,t^mz)=t^nf(x,y,z)$ 两

边关于 t 的导数得

$$xf_{\xi}(\xi,\eta,\zeta)+kt^{k-1}yf_{\eta}(\xi,\eta,\zeta)+mt^{m-1}zf_{\zeta}(\xi,\eta,\zeta)$$
$$=nt^{n-1}f(x,y,z)$$

令 $t=1$ 则有

$$xf_x(x,y,z)+kyf_y(x,y,z)+mzf_z(x,y,z)=nf(x,y,z)$$

8. 设由行列式表示的函数

$$D(t)=\begin{vmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{vmatrix}$$

其中 $a_{ij}(t)(i,j=1,2,\cdots,n)$ 的导数都存在，证明

$$\frac{dD(t)}{dt}=\sum_{k=1}^{n}\begin{vmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a'_{k1}(t) & a'_{k2}(t) & \cdots & a'_{kn}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{vmatrix}$$

证　记 $x_{ij}=a_{ij}(t)(i,j=1,2,\cdots,n)$

$$f(x_{11},x_{12},\cdots,x_{ij},\cdots,x_{nn})=\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \tag{1}$$

是由行列式定义知 f 为 n^2 元的可微函数且

$$D(t)=f(a_{11}(t),\cdots,a_{ij}(t),\cdots,a_{nn}(t))$$

于是由复合函数求导法则知

$$D'(t)=\sum_{i,j=1}^{n}\frac{\partial f}{\partial x_{ij}}\cdot\frac{dx_{ij}}{dt}=\sum_{i,j=1}^{n}\frac{\partial f}{\partial x_{ij}}\cdot a'_{ij}(t) \tag{2}$$

记(1)之右边行列式中 x_{ij} 的代数余子式为 A_{ij}，则

$$f(x_{11},\cdots,x_{ij},\cdots,x_{nn})=\sum_{i,j=1}^{n}x_{ij}A_{ij}(i,j=1,2,\cdots,n)$$

从而$\dfrac{\partial f}{\partial x_{ij}} = A_{ij}$

代入(2)得 $D'(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} a'_{ij}(t)A_{ij}(t)$　　(3)

其中 $A_{ij}(t)$ 是将 A_{ij} 的元素 x_{hl} 换为 $a_{hl}(t)$ 后得的 $n-1$ 阶行列式，它恰为行列式

$$\begin{vmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a'_{i1}(t) & a'_{i2}(t) & \cdots & a'_{in}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{vmatrix}$$

中 $a'_{ij}(t)$ 的代数余分式，于是由(3)知

$$D'(t) = \sum_{i=1}^{n}\begin{vmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a'_{i1}(t) & a'_{i2}(t) & \cdots & a'_{in}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{vmatrix}$$

§3　方向导数与梯度

1. 求函数 $u = xy^2 + z^3 - xyz$ 在点(1,1,2)处沿方向 l(其方向角分别为60°,45°,60°)的方向导数.

解　易见 u 在点(1,1,2)处可微，故由

$$u_x(1,1,2) = -1, u_y(1,1,2) = 0, u_z(1,1,2) = 11$$

得

$$u_l(1,1,2) = u_x\cos 60° + u_y\cos 45° + u_z\cos 60° = 5$$

2. 求函数 $u = xyz$ 在点 $A(5,1,2)$ 处沿到点 $B(9,4,14)$ 的方向 $\overrightarrow{AB}$ 上的方向导数.

解　方向导数的方向 $l(4,3,12)$ 方向余弦为 $\left(\dfrac{4}{13},\dfrac{3}{13},\dfrac{12}{13}\right)$. 因为

$u_x(5,1,2)=2, u_y(5,1,2)=10, u_x(5,1,2)=5.$

故有　$u_L(5,1,2)=2\times\frac{4}{13}+10\times\frac{3}{13}+5\times\frac{12}{13}=\frac{98}{13}$

3. 求函数 $u=x^2+2y^2+3z^2+xy-4x+2y-4z$ 在点 $A(0,0,0)$ 及点 $B(5,-3,\frac{2}{3})$ 处的梯度以及它们的模.

解　$u_x(0,0,0)=-4, u_y(0,0,0)=2, u_z(0,0,0)=-4$ 于是

$$\mathrm{grad}u(0,0,0)=(-4,2,-4)$$

$|\mathrm{grad}u(0,0,0)|=\sqrt{(-4)^2+2^2+(-4)^2}=6$

$u_x\left(5,-3,\frac{2}{3}\right)=3\quad u_y\left(5,-3,\frac{2}{3}\right)=-5$

$u_z\left(5,-3,\frac{2}{3}\right)=0$ 于是

$$\mathrm{grad}u\left(5,-3,\frac{2}{3}\right)=(3,-5,0)$$

$\left|\mathrm{grad}u\left(5,-3,\frac{2}{3}\right)\right|=\sqrt{(5)^2+(-3)^2+0^2}=\sqrt{34}$

4. 设函数 $u=\ln\left(\frac{1}{r}\right)$,其中

$$r=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$$

求 u 的梯度;并指出在空间哪些点上成立等式 $|\mathrm{grad}u|=1$.

解　因为

$u_x=u_r\cdot r_x=-\frac{1}{r}\cdot\frac{x-a}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}=\frac{a-x}{r^2},$

$u_y=\frac{b-y}{r^2}, u_z=\frac{c-z}{r^2}$

所以　$\mathrm{grad}u=\left(\frac{a-x}{r^2},\frac{b-y}{r^2},\frac{c-z}{r^2}\right),$

由 $|\mathrm{grad}u|=\frac{1}{r}$,得 $r=1$ 故使 $|\mathrm{grad}u|=1$ 的点是满足方程 $(x-a)^2+(y-b)^2+(z-c)^2=1$ 的点,即在空间以 (a,b,c) 为心,以 1 为半径的球面上都有 $|\mathrm{grad}u|=1$.

5. 设函数 $u=\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}$,求它在点$(a,b,c)$的梯度.

解　因为 $u_x(a,b,c)=-\frac{2}{a},u_y(a,b,c)=-\frac{2}{b}$,

$$u(a,b,c)=-\frac{2}{c},$$

所以　$\text{grad}u=(-\frac{2}{a}-\frac{2}{b},-\frac{2}{c})$.

6. 证明:

(1)$\text{grad}(u+c)=\text{gradu}$($c$ 为常数);

(2)$\text{grad}(\alpha u+\beta v)=\alpha\text{grad}u+\beta\text{grad}v$($\alpha,\beta$ 为常数);

(3)$\text{grad}(uv)=u\text{grad}v+v\text{grad}u$;

(4)$\text{gradf}(u)=f'(u)\text{grad}u$.

证　设 $u=u(x,y,z),v=v(x,y,z)$,则

(1)$\text{grad}(u+c)=(u_x,u_y,u_z)=\text{grad}u$.

(2)$\text{grad}(\alpha u+\beta v)=(\alpha u_x+\beta v_x,\alpha u_y+\beta v_y,\alpha u_z+\beta v_z)$

$$=\alpha(u_x,u_y,u_z)+\beta(v_x,v_y,v_z)$$

$$=\alpha\text{grad}u+\beta\text{grad}v.$$

(3)$\text{grad}(uv)=(uv_x+vu_z,uv_y+vu_y,uv_z+vu_z)$

$$=u(v_x,v_y,v_z)+v(u_x,u_y,u_z)$$

$$=u\text{grad}v+v\text{grad}u$$

(4)$\text{grad}f(u)=(f'(u)u_x,f'(u)u_y,f'(u)u_z)$

$$=f'(u)(u_x,u_y,u_z)=f'(u)\text{grad}u$$

7. 设 $r=\sqrt{x^2+y^2+z^2}$,试求:

(1)$\text{grad}r$;(2)$\text{grad}\frac{1}{r}$.

解　(1) 由 $r_x=\frac{x}{r},r_y=\frac{y}{r},r_z=\frac{z}{r}$　得

$$\text{grad}r=\frac{1}{r}(x,y,z)$$

(2) 设 $u=\frac{1}{r}$,则 $u_x=-\frac{x}{r^3},u_y=-\frac{y}{r^3},u_z=-\frac{z}{r^3}$

$$\operatorname{grad} u=\operatorname{grad}\frac{1}{r}=-\frac{1}{r^3}(x,y,z)$$

8. 设 $u=x^2+y^2+z^2-3xyz$,试问在怎样的点集上 $\operatorname{grad}u$ 分别满足:(1) 垂直于 x 轴;(2) 平行于 x 轴;(3) 恒为零向量.

解　(1) $u_x=2x-3yz, u_y=2y-3xz, u_z=2z-3xy$

由 $\operatorname{grad}u$ 垂直于 x 轴,而 x 轴的方向向量是 $(1,0,0)$,故

$(2x-3yz,2y-3xz,2z-3xy)(1,0,0)=2x-3yz=0.$

即 $2x=3yz$.

(2) 若 $\operatorname{grad}u$ 平行于 x 轴,则

$$\frac{2x-3yz}{1}=\frac{2y-3xz}{0}=\frac{2z-3xy}{0}=\lambda(\text{常数})$$

即　$2x-3yz=\lambda,2y=3xz,2z=3xy.$

(3) $\operatorname{grad}u$ 恒为零向量,则

$(2x-3yz,2y-3xz,2z-3xy)=(0,0,0)$

即 $2x=3yz,2y=3xz,2z=3xy.$

解得　$x^2=y^2=z^2.$

9. 设 $f(x,y)$ 可微,$\boldsymbol{l}$ 是 $\boldsymbol{R}^2$ 上的一个确定向量,倘若处处有 $f_l(x,y)\equiv 0$,试问此函数 f 有何特征?

解　设 $\boldsymbol{R}^2$ 上确定向量 L 的方向余弦为 $\cos\alpha,\cos\beta$,则

$$f_l(x,y)=f_x\cos\alpha+f_y\cos\beta$$

又 $f_l(x,y)\equiv 0$,所以 $f_x\cos\alpha+f_y\cos\beta=0$

即 $(f_x,f_y)(\cos\alpha,\cos\beta)=0$

说明函数 f 在点 $P(x,y)$ 的梯度向量与向量 $\boldsymbol{l}$ 垂直.

10. 设 $f(x,y)$ 可微,$\boldsymbol{l}_1$ 与 $\boldsymbol{l}_2$ 是 $\boldsymbol{R}^2$ 上的一组线性无关向量,试证明:若 $f_{l_i}(x,y)\equiv 0(i=1,2,)$ 则 $f(x,y)\equiv$ 常数.

证　由已知

$$f_{l_1}(x,y)=f_x(x,y)\cos\alpha_1+f_y(x,y)\cos\alpha_2=0 \tag{1}$$

$$f_{l_2}(x,y)=f_x(x,y)\cos\beta_1+f_y(x,y)\cos\beta_2=0 \tag{2}$$

$\cos\alpha_1,\cos\alpha_2$ 为 l_1 的方向余弦，$\cos\beta_1,\cos\beta_2$ 为 l_2 的方向余弦又 l_1 与 l_2 线性无关，所以

$$\begin{vmatrix} \cos\alpha_1, \cos\alpha_2 \\ \cos\beta_1, \cos\beta_2 \end{vmatrix} \neq 0$$

于是由(1)、(2)可得，$f_x = f_y = 0$，故 $f(x,y) \equiv$ 常数.

§4　泰勒公式与极值问题

1. 求下列函数的高阶偏导数：

(1) $z = x^4 + y^4 - 4x^2y^2$，所有二阶偏导数；

(2) $z = e^x(\cos y + x\sin y)$，所有二阶偏导数；

(3) $z = x\ln(xy)$，$\dfrac{\partial^3 z}{\partial x^2\partial y}$，$\dfrac{\partial^3 z}{\partial x\partial y^2}$；

(4) $u = xyze^{x+y+z}$，$\dfrac{\partial^{p+q+r}u}{\partial x^p\partial y^q\partial z^r}$；

(5) $z = f(xy^2, x^2y)$，所以二阶偏导数；

(6) $u = f(x^2+y^2+z^2)$，所有二阶偏导数；

(7) $z = f\left(x+y, xy, \dfrac{x}{y}\right)$，$z_x, z_{xx}, z_{xy}$.

解　(1) $z_x = 4x^3 - 8xy^2, z_y = 4y^3 - 8x^2y, z_{xx} = 12x^2 - 8y^2$,

$$z_{xy} = z_{yx} = -16xy, z_{yy} = 12y^2 - 8x^2$$

(2) $z_x = e^x(\cos y + x\sin y + \sin y), z_y = e^x(x\cos y - \sin y)$.

$$z_{xx} = e^x(\cos y + x\sin y + 2\sin y), z_{xy} = z_{yx}$$
$$= e^x(x\cos y + \cos y - \sin y),$$
$$z_{yy} = -e^x(x\sin y + \cos y).$$

(3) $z_x = \ln x + \ln y + 1, z_{xx} = \dfrac{1}{x}, z_{xy} = \dfrac{1}{y}, z_{x^2y} = 0, z_{xy^2} = -\dfrac{1}{y^2}$

(4) $u = xyze^{x+y+z} = xe^x \cdot ye^y \cdot ze^z$　由归纳法知

$(xe^x)^{(p)} = (x+p)e^x, (ye^y)^{(q)} = (y+q)e^y, (ze^z)^{(r)} = (z+r)e^z$,

所以　$$\frac{\partial^{p+q+r}u}{\partial x^p\partial y^q\partial z^r} = (x+p)(y+q)(z+r)e^{x+y+z}$$

(5) $z_x = y^2 f_1 + 2xyf_2, z_y = 2xyf_1 + x^2 f_2$

$z_{xy} = 2yf_1 + y^2(f_{11} \cdot 2xy + f_{12} \cdot x^2) + 2xf_2 + 2xy(f_{21} \cdot 2xy + f_{22} \cdot x^2)$

$= 2yf_1 + 2xf_2 + 2xy(x^2 f_{22} + y^2 f_{11}) + 5x^2 y^2 f_{21}$

$z_{xx} = y^4 f_{11} + 4xy^3 f_{12} + 4x^2 y^2 f_{22} + 2yf_2,$

$z_{yy} = 2xf_1 + 4x^2 y^2 f_{11} + 4x^3 y f_{21} + x^4 f_{22}$

(6) 设 $w = x^2 + y^2 + z^2$,则 $u = f(w)$

$u_x = 2xf'(w), u_y = 2yf'(w), u_z = 2zf'(w),$

$y_{xx} = 2f'(w) + 4x^2 f''(w),$

$u_{yy} = 2f'(w) + 4y^2 f''(w),$

$u_{zz} = 2f'(w) + 4z^2 f''(w), u_{xy} = 4xyf''(w),$

$u_{yz} = 4yzf''(w), u_{xz} = 4xzf''(w)$

(7) $z_x = f_1 + yf_2 + \frac{1}{y}f_3$

$$z_{xx} = f_{11} + yf_{12} + \frac{1}{y}f_{13} + y\left(f_{21} + yf_{22} + \frac{1}{y}f_{23}\right) + \frac{1}{y}\left(f_{31} + yf_{32} + \frac{1}{y}f_{33}\right)$$

$$= f_{11} + 2yf_{12} + \frac{2}{y}f_{13} + y^2 f_{22} + 2f_{23} + \frac{1}{y^2}f_{33}$$

$$Z_{xy} = f_{11} + (x+y)f_{12} + \frac{1}{y}\left(1 - \frac{x}{y}\right)f_{13} + xyf_{22} - \frac{x}{y^3}f_{33} + f_2 - \frac{1}{y^2}f_3$$

2. 设 $u = f(x,y), x = r\cos\theta, y = r\sin\theta$,证明:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

证　$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,$

$$\frac{\partial^2 u}{\partial r^2} = \cos^2\theta \frac{\partial^2 u}{\partial x^2} + 2\sin\theta \cdot \cos\theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial \theta} = -r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial\theta^2} = r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} - r\cos\theta\frac{\partial u}{\partial x}$$

$$- r\sin\theta\frac{\partial u}{\partial y}$$

所以 $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

3. $u = f(r), r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$　证明：

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \frac{d^2 u}{dr^2} + \frac{n-1}{r}\frac{du}{dr}$$

证　因为 $\dfrac{\partial u}{\partial x_i} = \dfrac{du}{dr}\cdot\dfrac{\partial r}{\partial x_i} = \dfrac{du}{dr}\cdot\dfrac{x_i}{r}$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{d^2 u}{dr^2}\cdot\frac{x_i^2}{r^2} + \frac{du}{dr}\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

所以 $\displaystyle\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} = \frac{d^2 u}{dr^2} + \frac{n}{r}\frac{du}{dr} - \frac{1}{r}\frac{du}{dr} = \frac{d^2 u}{dr^2} + \frac{n-1}{r}\frac{du}{dr}$.

4. 设 $v = \dfrac{1}{r}g\left(t - \dfrac{r}{c}\right)$，$c$ 为常数，$r = \sqrt{x^2 + y^2 + z^2}$，证明：$v_{xx} + v_{yy} + v_{zz} = \dfrac{1}{c^2}v_{tt}$.

证　$v_x = -\dfrac{1}{r^2}\cdot\dfrac{x}{r}g + \dfrac{1}{r}g'\cdot\left(-\dfrac{x}{cr}\right) = -\dfrac{x}{r^3}g - \dfrac{x}{cr^2}g'$

$$v_{xx} = \frac{3x^2 - r^2}{r^5}g + \frac{3x^2 - r^2}{cr^4}g' + \frac{x^2}{c^2r^3}g''$$

同样 $v_{yy} = \dfrac{3y^2 - r^2}{r^5}g + \dfrac{3y^2 - r^2}{cr^4}g' + \dfrac{y^2}{c^2r^3}g''$

$$v_{zz} = \frac{3z^2 - r^2}{r^5}g + \frac{3z^2 - r^2}{cr^4}g' + \frac{z^2}{c^2r^3}g''$$

$$v_t = \frac{1}{r}g',\quad v_{tt} = \frac{1}{r}g''$$

所以 $v_{xx} + v_{yy} + v_{zz} = \dfrac{3(x^2 + y^2 + z^2) - 3r^2}{r^5}g + \dfrac{3(x^2 + y^2 + z^2) - 3r^2}{cr^4}g' + \dfrac{x^2 + y^2 + z^2}{c^2r^3}g''$

$$= \frac{1}{c^2} \cdot \frac{1}{r} g'' = \frac{1}{c^2} v_{tt}$$

5. 证明定理 17.8 的推论.

若函数 f 在区域 D 上存在偏导数,且 $f_x \equiv f_y \equiv 0$,则 f 在区域 D 上为常量函数.

证　设 P 和 P' 是区域 D 中任意两点,由于 D 为区域,可用一条完全在 D 内的折线连接 PP'(见图 17－1).设 x_1 为折线上第一个折点,在直线段 $\overline{Px_1}$ 上每一点 $P_0(x_0,y_0)$,存在邻域 $\overline{U}(P_0) \subset D$,由中值定理知,在 $\overline{U}(P_0)$ 内任一点 $M(x_m,y_m)$ 有 $f(M) - f(P_0) = f_x(\theta_1)(x_m - x_0) + f_y(\theta_1)(y_m - y_0)$

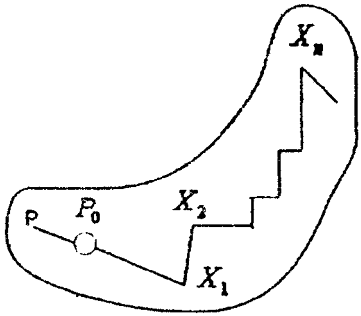

图 17—1

因 $f_x(\theta_1) = f_y(\theta_1) = 0$ 所以 $f(M) - f(P_0) = 0$

即在 $\overline{U}(P_0)$ 内函数 f 是常数.由于在 $\overline{Px_1}$ 上任一点都有这样的邻域 $\overline{U}(P_0)$,使得 $f(x,y)$ = 常数.由有限复盖定理知存在有限个这样的邻域 $\overline{U}(P_1),\cdots,\overline{U}(P_N)$ 复盖 $\overline{Px_1}$,所以 $f(P) = f(x_1)(P \in \overline{Px_1})$.

同理可证　$f(P) = f(x_1) = f(x_2) = \cdots = f(P')$

由 P 和 P' 是区域 D 内任意两点,所以在 D 内,$f(x,y)$ = 常数.

6. 通过对 $F(x,y) = \sin x \cos y$ 施用中值定理,证明对某 $\theta \in (0,1)$,有

$$\frac{3}{4} = \frac{\pi}{3} \cos \frac{\pi\theta}{3} \cos \frac{\pi\theta}{6} - \frac{\pi}{6} \sin \frac{\pi\theta}{3} \sin \frac{\pi\theta}{6}.$$

证　在 $F(x_0+h, y_0+k) = F(x_0,y_0) + F_x(x_0+\theta h, y_0+\theta k)h + F_y(x_0+\theta h, y_0+\theta k)k$ 中,令 $F(x,y) = \sin x \cos y$,$x_0 = 0$,$y_0 = 0$,$h = \frac{\pi}{3}$,$k = \frac{\pi}{6}$,则

$$\sin\frac{\pi}{3}\cos\frac{\pi}{6}=\sin0\cos0+\frac{\pi}{3}\cos\frac{\pi\theta}{6}-\frac{\pi}{6}\sin\frac{\pi\theta}{3}\sin\frac{\pi\theta}{6},$$

即　$\frac{3}{4}=\frac{\pi}{3}\cos\frac{\pi\theta}{3}\cos\frac{\pi\theta}{6}-\frac{\pi}{6}\sin\frac{\pi\theta}{3}\sin\frac{\pi\theta}{6}$

7. 求下列函数在指定点处的泰勒公式：

(1) $f(x,y)=\sin(x^2+y^2)$ 在点(0,0)(到二阶为止)；

(2) $f(x,y)=\frac{x}{y}$ 在点(1,1)(到三阶为止)；

(3) $f(x,y)=\ln(1+x+y)$ 在点(0,0)；

(4) $f(x,y)=2x^2-xy-y^2-6x-3y+5$ 在点(1, -2).

解　(1)$f=\sin(x^2+y^2)$　$f(0,0)=0$

$f_x=2x\cos(x^2+y^2)$, $f_x(0,0)=0$,

$f_y=2y\cos(x^2+y^2)$, $f_y(0,0)=0$,

$f_{x^2}=2\cos(x^2+y^2)-4x^2\sin(x^2+y^2)$　$f_{x^2}(0,0)=2$

$f_{xy}=-4xy\sin(x^2+y^2)$, $f_{xy}(0,0)=0$,

$f_{y^2}=2\cos(x^2+y^2)-4y^2\sin(x^2+y^2)$, $f_{y^2}(0,0)=2$

$f_{x^3}(\theta x,\theta y)=-12\theta x\sin(\theta^2x^2+\theta^2y^2)-8\theta^3x^3\cos(\theta^2x^2+\theta^2y^2)$

$f_{x^2y}(\theta x,\theta y)=-4\theta y\sin(\theta^2x^2+\theta^2y^2)-8\theta^3x^2y\cos(\theta^2x^2+\theta^2y^2)$

$f_{xy^2}(\theta x,\theta y)=-4\theta x\sin(\theta^2x^2+\theta^2y^2)-8\theta^3xy^2\cos(\theta^2x^2+\theta^2y^2)$

$f_{y^3}(\theta,\theta y)=-12\theta y\sin(\theta^2x^2+\theta^2y^2)-8\theta^3y^3\cos(\theta^2x^2+\theta^2y^2)$

$\sin(x^2+y^2)=x^2+y^2+R_2(x,y)$

其中 $R_2(x,y)=-\frac{2}{3}[3\theta(x^2+y^2)^2\sin(\theta^2x^2+\theta^2y^2)-2\theta^3(x^2+y^2)^3\cos(\theta^2x^2+\theta^2y^2)]$.

(2)$f(x,y)=\frac{x}{y}$　$f(1,1,)=1$, $f_x=\frac{1}{y}$, $f_x(1,1,)=1$,

$f_y=-\frac{x}{y^2}$, $f_y(1,1,)=-1$

$f_{x^2}=0$, $f_{x^2}(1,1)=0$, $f_{xy}=\frac{-1}{y^2}$, $f_{xy}(1,1)=-1$,

$f_{y^2}=\dfrac{2x}{y^3}, f_{y^2}(1,1)=2$

$f_{x^3}(1,1)=f_{x^2y}(1,1)=0, f_{xy^2}(1,1)=2$

$f_{y^3}(1,1)=-6, f_{x^4}=f_{x^3y}=f_{x^2y^2}=0$

$f_{xy^3}(1+\theta x,1+\theta y)=\dfrac{6}{(1+\theta y)^4},$

$f_{y^4}(1+\theta x,1+\theta y)=\dfrac{24(1+\theta x)}{(1+\theta y)^5}$

所以$\dfrac{x}{y}=1+(x-1)-(y-1)-(x-1)(y-1)+(y-1)^2+(x-1)(y-1)^2-(y-1)^3+R_3(x,y)$

其中 $R_3(x,y)=-\dfrac{(x-1)(y-1)^3}{[1+\theta(y-1)]^4}+\dfrac{1+\theta(x-1)}{[1+\theta(y-1)]^5}(y-1)^4$

(3) 由于$\dfrac{\partial^k f}{\partial x^k}=\dfrac{(-1)^{k-1}(k-1)!}{(1+x+y)^k}=\dfrac{\partial^k f}{\partial y^k}$

$\dfrac{\partial^k f(0,0)}{\partial x^k}=\dfrac{kf(0,0)}{\partial y^k}=(-1)^{k-1}(k-1)!$

$\dfrac{\partial^n f}{\partial x^p\partial y^{n-p}}=\dfrac{(-1)^n(n-1)!}{(1+x+y)^n}\quad \dfrac{\partial^n f(0,0)}{\partial x^p\partial y^{n-p}}=(-1)^{n-1}(n-1)!$

$$\begin{aligned}\frac{1}{p!}(k\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^p f(0,0)&=\frac{1}{p!}\sum_{i=0}^{p}C_p^i(-1)^{p-1}(p-1)!h^ik^{p-i}\\&=\frac{(-1)^{p-1}}{p}(h+k)^p.\end{aligned}$$

$$\begin{aligned}&\frac{1}{(n+1)!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^{n+1}f(\theta h,\theta k)\\&=\frac{1}{(n+1)!}\sum_{p=0}^{n-1}C_{n+1}^p\frac{(-1)^n n!}{(1+\theta h+\theta k)^{n+1}}h^pk^{n-p}\\&=\frac{(-1)^n}{(n+1)(1+\theta h+\theta k)^{n+1}}(h+k)^{n+1}\end{aligned}$$

$$\begin{aligned}\text{所以}\ \ln(1+x+y)&=\sum_{p=1}^{n}(-1)^{p-1}\frac{(x+y)^p}{p}\\&\quad+(-1)^n\frac{(x+y)^{n+1}}{(n+1)(1+\theta x+\theta y)^{n+1}}(0<\theta<1)\end{aligned}$$

(4) $f(x,y)=2x^2-xy-y^2-6x-3y+5, f(1,-2)=5$

$f_x(1,-2)=0\quad f_y(1,-2)=0, f_{x^2}(1,-2)=4, f_{xy}(1,-2)=-1,$
$f_{y^2}(1,-2)=-2$

所以　$2x^2-xy-6x-y^2-3y+5$

$=5+2(x-1)^2-(x-1)(y+2)-(y+2)^2$

8. 求下列函数的极值点：

(1) $z=3axy-x^3-y^3(a>0)$;

(2) $z=x^2+5y^2-6x+10y+6$;

(3) $z=e^{2x}(x+y^2+2y)$

解　(1) 解方程组 $\begin{cases} z_x=3ay-3x^2=0 \\ z_y=3ax-3y^2=0 \end{cases}$ 得稳定点

$(a,a),(0,0)$，由于 $z_{xx}(a,a)=-6a<0, z_{yy}(a,a)=3a,$

$$z_{yy}(a,a)=-6a,$$

$$z_{xx}(a,a)z_{yy}(a,a)-z_{xy}^2(a,a)=27a^2>0$$

所以 (a,a) 为极大值点

$$z_{yx}(0,0)=0, z_{xy}(0,0)=3a, z_{yy}(0,0)=0,$$

$$z_{xx}(0,0)z_{yy}(0,0)-z_{xy}^2(0,0)=-9a^2<0$$

所以 $(0,0)$ 不是极值点.

(2) 同课本 P_{138} 页例 6.

(3) 解方程组 $\begin{cases} z_x=e^{2x}(2x+2y^2+4y+1)=0 \\ z_y=e^{2x}(2y+2)=0 \end{cases}$ 得稳定点
$\left(\frac{1}{2},-1\right)$

由于 $z_{xx}\left(\frac{1}{2},-1\right)=2e, z_{xy}\left(\frac{1}{2},-1\right)=0, z_{yy}\left(\frac{1}{2},-1\right)=2e,$
$z_{xx}\left(\frac{1}{2},-1\right)\cdot z_{yy}\left(\frac{1}{2},-1\right)-z_{xy}^2\left(\frac{1}{2},-1\right)=4e^2>0$

所以 $\left(\frac{1}{2},-1\right)$ 为极小值点.

9. 求下列函数在指定范围内的最大值与最小值.

(1) $z = x^2 - y^2, \{(x,y) \mid x^2 + y^2 \leqslant 4\}$;

(2) $z = x^2 - xy + y^2, \{(x,y) \mid |x| + |y| \leqslant 1\}$;

(3) $z = \sin x + \sin y - \sin(x + y), \{(x,y) \mid (x,y) \mid x \geqslant 0, y \geqslant 0, x + y \leqslant 2\pi\}$

解　(1) 解方程组 $\begin{cases} z_x = 2x = 0 \\ z_y = -2y = 0 \end{cases}$，得稳定点 $(0,0)$.

由于 $z_{xx} = 2, z_{yy} = -2, z_{xy} = 0, z_{xx}z_{yy} - z_{xy}^2 = -4 < 0$，所以 $(0,0)$ 不是极值点. 在边界 $x^2 + y^2 = 4$ 上，$z = 2x^2 - 4$. 由 $z_x = 4x = 0$ 得稳定点 $x = 0$，这时 $y = \pm 2$，在点 $(0,2)$ 和 $(0,-2)$ 上 $z(0,2) = z(0,-2) = -4$，边界点 $(2,0)$ 和 $(-2,0)$　$z(2,0) = z(-2,0) = 4$，比较各点的函数值知在点 $(2,0)$，$(-2,0)$ 函数取最大值 4，在点 $(0,2)$，$(0,-2)$ 函数取最值 -4.

(2) 解方程组 $\begin{cases} z_x = 2x - y = 0 \\ z_y = -x + 2y = 0 \end{cases}$　得稳定点 $(0,0)$，函数值 $z(0,0) = 0$

考察边界上相应一元函数的稳定点及其函数值有：

$z|_{x+y=1} = 1 - 3x(1 - x), z_x = -3(1 - 2x) = 0$，得 $x = \frac{1}{2}$，$y = \frac{1}{2}, z\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}$.

$z|_{x-y=1} = 1 + x(x - 1), z_x = 2x - 1 = 0$，得 $x = \frac{1}{2}, y = -\frac{1}{2}$，$z\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{3}{4}$

$z|_{x+y=-1} = 1 + 3x(x + 1), z_x = 3(2x + 1) = 0$ 得 $x = -\frac{1}{2}$，$y = -\frac{1}{2}, z\left(-\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4}$

$z|_{y-x=1} = 1 + x(x + 1), z_x = 2x + 1 = 0$ 得 $x = -\frac{1}{2}, y = \frac{1}{2}$，

$z=\left(-\frac{1}{2},\frac{1}{2}\right)=\frac{3}{4}$

而边界点(1,0)(0,1),(-1,0),(0,-1)的函数值都等于1.所以函数的最大值点为(1,0),(0,1),(-1,0),(0,-1),最大值为1,函数的最小值点为(0,0),最小值为0.

(3) 解方程组$\begin{cases} z_x=\cos x-\cos(x+y)=0 \\ z_y=\cos y-\cos(x+y)=0 \end{cases}$ 得 $\cos x=\cos y$,因此稳定点在 $x=y$ 或 $x+y=2\pi$ 上,在区域内部仅$\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)$为稳定点,$z\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)=\frac{3\sqrt{3}}{2}$.而在边界 $x=0,0\leqslant y\leqslant 2\pi$;$y=0,0\leqslant x\leqslant 2\pi$,$x+y=2\pi$ 上函数值均为零.所以函数在点$\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)$取得最大值$\frac{3\sqrt{3}}{2}$.在边界上取得最小值零.

10. 在已知周长为 $2p$ 的一切三角形中,求出面积为最大的三角形.

解　设三角形的三边分别为 x,y,z.则面积

$$S=\sqrt{p(p-x)(p-y)(p-z)},x+y+z=2p.$$

因此 $S=\sqrt{p(p-x)(p-y)(x+y-p)},(x,y)\in D$.其中 $D=\{(x,y)\mid 0\leqslant x\leqslant p,0\leqslant y\leqslant p,x+y\geqslant p\}$,因 S 与$\frac{S^2}{p}$有相同的稳定点,考虑

$$\psi=\frac{\varphi^2}{p}=(p-x)(p-y)(x+y-p)$$

解方程组$\begin{cases} \psi_x=(p-y)(2p-2x-y)=0 \\ \psi_y=(p-x)(2p-2y-x)=0 \end{cases}$ 得 $x=\frac{2}{3}p,y=\frac{2}{3}p$

从而 $z=\frac{2}{3}p$,又在 D 的边界上 $S\equiv 0$,从而 S 在$\left(\frac{2}{3}p,\frac{2}{3}p\right)$处取得最大值,因而面积最大的三角形为边长为$\frac{2}{3}p$ 的等边三角形.面积

$S = \frac{\sqrt{3}}{9} p^2$.

11. 在 xy 平面上求一点，使它到三直线 $x = 0, y = 0$，及 $x + 2y - 16 = 0$ 的距离平方和最小.

解　设所求的点为(x,y)，它到 $x = 0$ 的距离为 $|y|$，它到 $y = 0$ 的距离为 $|x|$，到 $x + 2y - 16 = 0$ 的距离为 $\left|\frac{x + 2y - 16}{\sqrt{5}}\right|$，它到三直线的距离平方和为

$$z = x^2 + y^2 + \frac{(x + 2y - 16)^2}{5}$$

$$由\begin{cases} z_x = 2x + \dfrac{2(x + 2y - 16)}{5} = 0 \\ z_y = 2y + \dfrac{4(x + 2y - 16)}{5} = 0 \end{cases} \quad 得\left(\frac{8}{5}, \frac{16}{5}\right)$$

因为 $z_{xx}\left(\frac{8}{5}, \frac{16}{5}\right) = \frac{12}{5} > 0, z_{xy}\left(\frac{8}{5}, \frac{16}{5}\right) = \frac{4}{5}, z_{yy}\left(\frac{8}{5}, \frac{16}{5}\right) = \frac{18}{5}$, $z_{xx}z_{yy} - z_{xy}^2 = 8 > 0$，因此$\left(\frac{8}{5}, \frac{16}{5}\right)$为 z 的极小值点.

12. 已知平面上 n 个点的坐标分别是

$$A_1(x_1, y_1), A_2(x_2, y_2), \cdots, A_n(x_n, y_n).$$

试求一点，使它与这 n 个点距离的平方和最小.

解　设所求的点为(x,y)，它与各点距离平方和为

$S = \sum_{i=1}^{n} [(x - x_i)^2 + (y - y_i)^2]$. 由

$$\begin{cases} S_x = 2\sum_{i=1}^{n}(x - x_i) = 2nx - 2\sum_{i=1}^{n} x_i = 0 \\ S_r = 2\sum_{i=1}^{n}(y - y_i) = 2ny - 2\sum_{i=1}^{n} y_i = 0 \end{cases}$$

得 $x = \frac{1}{n}\sum_{i=1}^{n} x_i, y = \frac{1}{n}\sum_{i=1}^{n} y_i$

因 $S_{xx} = 2n > 0, S_{xy} = 0, S_{yy} = 2n, S_{xx}S_{yy} - S_{xy}^2 = 4n^2 > 0$.

所以 $\left(\frac{1}{n}\sum_{i=1}^{n}x_i,\frac{1}{n}\sum_{i=1}^{n}y_i\right)$为所求的点.

13. 证明:函数

$$u=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{(x-b)^2}{4a^2t}}\ (a,b\text{ 为常数})$$

满足热传导方程:$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$

证 因为

$$\frac{\partial u}{\partial t}=-\frac{1}{4a\pi^{1/2}t^{3/2}}e^{-\frac{(x-b)^2}{4a^2t}}+\frac{(x-b)^2}{8a^3\pi^{1/2}t^{5/2}}e^{-\frac{(x-b)^2}{4a^2t}}$$

$$=\left(\frac{-1}{4a\pi^{1/2}t^{3/2}}+\frac{1}{2a\pi^{1/2}t^{1/2}}\cdot\frac{(x-b)^2}{4a^2t^2}\right)e^{-\frac{(x-b)^2}{4a^2t}}$$

$$\frac{\partial u}{\partial x}=\frac{-(x-b)}{4a^3\pi^{1/2}t^{3/2}}e^{-\frac{(x-b)^2}{4a^2t}}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{-1}{4a^3\pi^{3/2}t^{3/2}}e^{-\frac{(x-b)^2}{4a^2t}}+\frac{(x-b)^2}{8a^5\pi^{3/2}t^{5/2}}e^{-\frac{(x-b)^2}{4a^2t}}$$

所以 $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$

14. 证明:函数 $u=\ln\sqrt{(x-a)^2+(y-b)^2}$($a,b$ 为常数)满足拉普拉斯方程:$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

证 因为

$$\frac{\partial u}{\partial x}=\frac{x-a}{(x-a)^2+(y-b)^2},$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{(y-b)^2-(x-a)^2}{[(x-a)^2+(y-b)^2]^2}$$

$$\frac{\partial u}{\partial y}=\frac{y-b}{(x-a)^2+(y-b)^2},$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{(x-a)^2-(y-b)^2}{[(x-a)^2+(y-b)^2]^2}$$

所以 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

15. 证明:若函数 $u=f(x,y)$ 满足拉普拉斯方程:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

则函数 $v=f\left(\dfrac{x}{x^2+y^2},\dfrac{y}{x^2+y^2}\right)$ 也满足此方程.

证　令 $s=\dfrac{x}{x^2+y^2}, t=\dfrac{y}{x^2+y^2}$,则有

$$\frac{\partial s}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}=-\frac{\partial t}{\partial y},\frac{\partial t}{\partial x}=\frac{-2xy}{(x^2+y^2)^2}=\frac{\partial s}{\partial y},$$

$$\frac{\partial v}{\partial x}=\frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 f}{\partial s^2}\left(\frac{\partial s}{\partial x}\right)^2+2\frac{\partial^2 f}{\partial s\partial t}\frac{\partial s}{\partial x}\cdot\frac{\partial t}{\partial x}+\frac{\partial^2 f}{\partial t^2}\left(\frac{\partial t}{\partial x}\right)^2+\frac{\partial f}{\partial s}\frac{\partial^2 s}{\partial x^2}+\frac{\partial f}{\partial t}\frac{\partial^2 t}{\partial x^2} \tag{1}$$

同理 $$\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 f}{\partial s^2}\left(\frac{\partial s}{\partial y}\right)^2+2\frac{\partial^2 f}{\partial s\partial t}\frac{\partial s}{\partial y}\cdot\frac{\partial t}{\partial y}+\frac{\partial^2 f}{\partial t^2}\left(\frac{\partial t}{\partial y}\right)^2+\frac{\partial f}{\partial s}\frac{\partial^2 s}{\partial y^2}+\frac{\partial f}{\partial t}\frac{\partial^2 t}{\partial y^2} \tag{2}$$

由于 $\left(\dfrac{\partial s}{\partial x}\right)^2=\left(\dfrac{\partial t}{\partial y}\right)^2,\left(\dfrac{\partial t}{\partial x}\right)^2=\left(\dfrac{\partial s}{\partial y}\right)^2,\dfrac{\partial^2 f}{\partial s^2}+\dfrac{\partial^2 f}{\partial t^2}=0$

$\dfrac{\partial t}{\partial x}\dfrac{\partial s}{\partial x}=\dfrac{\partial t}{\partial y}\cdot\dfrac{\partial s}{\partial y}$

$\dfrac{\partial^2 s}{\partial x^2}=-\dfrac{\partial^2 t}{\partial x\partial y},\dfrac{\partial^2 s}{\partial y^2}=\dfrac{\partial^2 t}{\partial x\partial y}$,故有

$\dfrac{\partial^2 s}{\partial x^2}+\dfrac{\partial^2 s}{\partial y^2}=0$,同理$\dfrac{\partial^2 t}{\partial x^2}+\dfrac{\partial^2 t}{\partial y^2}=0$

将(1)与(2)两式相加,并把上述结果代入整理后得

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial f}{\partial s}\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}\right)+\frac{\partial f}{\partial t}\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}\right)=0$$

16. 设函数 $u=\varphi(x+\psi(y))$,证明:

$$\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial y}\cdot\frac{\partial^2 u}{\partial x^2}$$

证　令 $s=x+\psi(y)$，则

$$\frac{\partial u}{\partial x}=\frac{du}{ds},\frac{\partial^2 u}{\partial x^2}=\frac{d^2u}{ds^2},\frac{\partial^2 u}{\partial x\partial y}=\frac{d^2u}{ds^2}\cdot\psi'(y),$$

$$\frac{\partial u}{\partial y}=\frac{du}{ds}\cdot\psi'(y)$$

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}=\frac{du}{ds}\frac{d^2u}{ds^2}\psi'(y),\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}=\frac{du}{ds}\cdot\psi'(y)\frac{d^2u}{ds^2}$$

故　$\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial^2 u}{\partial x^2}$

17. 设 f_x,f_y 和 f_{yx} 在点 (x_0,y_0) 的某领域内存在，f_{yx} 在点 (x_0,y_0) 连续，证明 $f_{xy}(x_0,y_0)$ 也存在，且 $f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$.

证　由已知条件及中值定理得

$F(\triangle x,\triangle y)=f(x_0+\triangle x,y_0+\triangle y)-f(x_0+\triangle x,y_0)-f(x_0,y_0+\triangle y)+f(x_0,y_0)$

$=f_{yx}(x_0+\theta_1\triangle x,y_0+\theta_2\triangle y)\triangle x\triangle y$

$f_{yx}(x_0+\theta_1\triangle x,y_0+\theta_2\triangle y)$

$$=\left[\frac{f(x_0+\triangle x,y_0+\triangle y)-f(x_0,y_0+\triangle y)}{\triangle x}-\frac{f(x_0,+\triangle x,y_0)-f(x_0,y_0)}{\triangle x}\right]\frac{1}{\triangle y}$$

由于 $f_{yx}(x,y)$ 在 (x_0,y_0) 处连续，故对上式两边取 $\triangle x\to 0$ 得

$$f_{yx}(x_0,y_0+\theta_2\triangle y)=\frac{f_x(x_0,y_0+\triangle y)-f_x(x_0,y_0)}{\triangle y}$$

再让 $\triangle y\to 0$ 时，由 f_{yx} 在 (x_0,y_0) 连续及 f_{xy} 的定义便得

$$f_{yx}(x_0,y_0)=f_{xy}(x_0,y_0)$$

18. 设 f_x,f_y 在点 (x_0,y_0) 的某邻域内存在且在点 (x_0,y_0) 可微，则有

$$f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$$

证　应用中值有(对 $\varphi(x)=f(x,y_0+\triangle y)-f(x,y_0)$)

$F(\triangle x,\triangle y)=f(x_0,+\triangle x,y_0+\triangle y)-f(x_0,+\triangle x,y_0)-$

$f(x, y_0 + \triangle y) + f(x_0, y_0)$

$= \varphi(x_0 + \triangle x) - \varphi(x_0) = \varphi'(x_0 + \theta_1 \triangle x)\triangle x$

$= [f_x(x_0 + \theta_1 \triangle x, y_0 + \triangle y) - f_x(x_0 + \theta_1 \triangle x, y_0)]\triangle x$ $(0 < \theta_1 < 1)$

由 f_x 在(x_0, y_0)处可微知

$F(\triangle x, \triangle y) = [f_x(x_0 + \theta_1 \triangle x, y_0 + \triangle y) - f_x(x_0, y_0)]\triangle x$

$- [f_x(x_0 + \theta_1 \triangle x, y_0) - f_x(x_0, y_0)]\triangle x$

$= [f_{xx}(x_0, y_0)\theta_1 \triangle x + f_{xy}(x_0, y_0)\triangle y + o(\rho)$

$- f_{xx}(x_0, y_0)\theta_1 \triangle x - o(\rho)]\triangle x$

$= f_{xy}(x_0, y_0)\triangle x \triangle y + o(\rho)\triangle x$

所以 $\lim\limits_{(\triangle x, \triangle y) \to (0,0)} \dfrac{F(\triangle x, \triangle y)}{\triangle x \cdot \triangle y} = f_{xy}(x_0, y_0)$

同理由 f_y 在(x_0, y_0)处可微得

$$\lim_{(\triangle x, \triangle y) \to (0,0)} \frac{F(\triangle x, \triangle y)}{\triangle x \triangle y} = f_{yx}(x_0, y_0)$$

从而 $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

19. 设 $u = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

求(1)$u_x + u_y + u_z$;(2)$xu_x + yu_y + zu_z$;(3)$u_{xx} + u_{yy} + u_{zz}$

解 $u_x = \begin{vmatrix} 0 & 1 & 1 \\ 1 & y & z \\ 2x & y^2 & z^2 \end{vmatrix} = (y - z)(-2x + y + z)$

同理 $u_y = (z - x)(x - 2y + z)$ $u_z = (x - y)(x + y - 2z)$

所以(1)$u_x + u_y + u_z = 0$,

(2)$xu_x + uy_y + zu_z = 3(z - y)(x - y)(x - z)$

(3) 由于 $u_{xx} = 2(z - y), u_{yy} = 2(x - z), u_{zz} = 2(y - x)$

所以 $u_{xx} + u_{yy} + u_{zz} = 0$

20. 设 $f(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx$,试按

h,k,l 的正整数幂展开 $f(x+h,y+k,z+l)$.

解　$f_x = 2Ax + Dy + Fz, f_y = 2By + Dx + Ez,$

$f_z = 2Cz + Ey + Fx$

$f_{xx} = 2A, f_{yy} = 2B, f_{zz} = 2C, f_{xy} = f_{yx} = D, f_{yz} = E, f_{xz} = F$

$f(x+h,y+k,z+l) = f(x,y,z) + (2Ax+Dy+Fz)h + (2By+Dx+Ez)k + (2Cz+Ey+Fx)l + Ah^2 + Bk^2 + Cl^2 + Dhk + Ekl + Fhl$

$= f(x,y,z) + (2Ax+Dy+Fz)h + (2By+Dx+Ez)k + (2cz+Ey+Fx)l + f(h,k,l)$

总练习题

1. 设 $f(x,y,z) = x^2y + y^2z + z^2x$,证明

$$f_x + f_y + f_z = (x+y+z)^2$$

证　由 $f_x = 2xy + z^2, f_y = 2yz + x^2, f_z = 2zx + y^2$ 得

$$f_x + f_y + f_z = (x+y+z)^2$$

2. 求函数

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, x^2+y^2 \neq 0 \\ 0, \qquad x^2+y^2 = 0 \end{cases}$$ 在原点的偏导数 $f_x(0,0)$ 与 $f_y(0,0)$,并考察 $f(x,y)$ 在 $(0,0)$ 的可微性.

解　$f_x(0,0) = \lim\limits_{\triangle x \to 0} \dfrac{f(0+\triangle x,0) - f(0,0)}{\triangle x}$

$$= \lim_{\triangle x \to 0} \frac{(\triangle x)^3}{(\triangle x)^3} = 1$$

$f_y(0,0) = \lim\limits_{\triangle y \to 0} \dfrac{f(0,0+\triangle y) f(0,0)}{\triangle y}$

$$= \lim_{\triangle y \to 0} \frac{-(\triangle y)^3}{(\triangle y)^3} = -1$$

若 $z = f(x,y)$ 在 $(0,0)$ 点可微,则 $dz = \triangle x - \triangle y$ 且

$$\lim_{(\triangle x,\triangle y)\to(0,0)}\frac{\triangle z-dz}{\sqrt{(\triangle x)^2+(\triangle y)^2}}=0$$

而 $\triangle z=f(0+\triangle x,0+\triangle y)-f(0,0)=\dfrac{(\triangle x)^3-(\triangle y)^3}{(\triangle x)^2+(\triangle y)^2}$

当 $\triangle x=-\triangle y$ 时 $\dfrac{\triangle z-dz}{\sqrt{(\triangle x)^2+(\triangle y)^2}}$

$$=\frac{\triangle x\triangle y(\triangle x-\triangle y)}{[(\triangle x)^2+(\triangle y)^2]^{3/2}}=-\frac{\sqrt{2}}{2}$$

从而 $\lim\limits_{(\triangle x,\triangle y)\to(0,0)}\dfrac{\triangle z-dz}{\sqrt{(\triangle x)^2+(\triangle y)^2}}\neq 0$ 所以 $f(x,y)$ 在 $(0,0)$ 不可微.

3. 设 $u=\begin{vmatrix}1 & 1 & \cdots & 1\\ x_1 & x_2 & \cdots & x_n\\ x_1^2 & x_2^2 & \cdots & x_n^2\\ \cdots & \cdots & \cdots & \cdots\\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}\end{vmatrix}$

证明:(1) $\sum\limits_{k=1}^{n}\dfrac{\partial u}{\partial x_k}=0$;(2) $\sum\limits_{k=1}^{n}x_k\dfrac{\partial u}{\partial x_k}=\dfrac{n(n-1)}{2}u$

证 (1) 记 $u=|x_i^j|$, $X_{j+1,i}$ 为 x_i^j 的代数余子式($1\leqslant i\leqslant n,0\leqslant j\leqslant n-1$) 于是 $u=\sum\limits_{j=0}^{n-1}x_i^jX_{j+1,i}$

$$\frac{\partial u}{\partial x_k}=\sum_{j=1}^{n-1}jx_k^{j-1}X_{j+1,k}\quad k=1,2,\cdots,n$$

$$\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}=\sum_{k=1}^{n}\sum_{j=1}^{n-1}jx_k^{j-1}X_{j+1,k}=\sum_{j=1}^{n-1}j\sum_{k=1}^{n}x_k^{j-1}X_{j+1,k}$$

$$\sum_{k=1}^{n} x_k^{j-1} X_{j+1,k} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{j-1} & x_2^{j-1} & \cdots & x_n^{j-1} \\ x_1^{j-1} & x_2^{j-1} & \cdots & x_n^{j-1} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = 0$$

对一切的 $j = 1,2,\cdots,n-1$ 都成立.

所以 $\sum_{k=1}^{n} \frac{\partial u}{\partial x_k} = 0$.

(2) 利用课本 P_{123} 页关于齐次函数的欧拉定理有

$$F(tx_1, tx_2, \cdots, tx_n) = t^n F(x_1, x_2, \cdots, x_n) \Leftrightarrow \sum_{k=1}^{n} x_k F_{x_k} = nF$$

而 u 是 $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$ 次齐次函数.所以

$$\sum_{k=1}^{n} x_k f_{x_k} = \frac{n(n-1)}{2} u$$

4. 设函数 $f(x,y)$ 具有连续的 n 阶偏导数: 试证函数 $g(t) = f(a+ht, b+kt)$ 的 n 阶导数

$$\frac{d^n g(t)}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n (a+ht, b+kt)$$

证　应用数学归纳法证明

当 $n = 1$ 时

$$\frac{dg(t)}{dt} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a+ht, b+kt)$$

且 $$\frac{d^2 g(t)}{dt^2} = \frac{d}{dt}\left(\frac{dg(t)}{dt}\right)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a+ht, b+kt)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(n+ht, b+kt)$$

设$\dfrac{d^{n-1}g(t)}{dt^{n-1}} = \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^{n-1} f(a+ht, b+kt)$ 成立

则$\dfrac{d^n g(t)}{dt^n} = \dfrac{d}{dt}\left(\dfrac{d^{n-1}g(t)}{dt^{n-1}}\right)$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a+ht, b+kt)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n} f(a+ht, b+kt)$$

所以对一切的 n

$$\frac{d^n g(t)}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n} f(a+ht, b+kt)$$

5. 设 $\varphi(x,y,z) = \begin{vmatrix} a+x & b+y & c+z \\ d+z & e+x & f+y \\ g+y & h+z & k+x \end{vmatrix}$ 求$\dfrac{\partial^2 \varphi}{\partial x^2}$

解 $\dfrac{\partial \varphi}{\partial x} = \begin{vmatrix} 1 & b+y & c+z \\ 0 & e+x & f+y \\ 0 & h+z & k+x \end{vmatrix} + \begin{vmatrix} a+x & 0 & c+z \\ d+z & 1 & f+g \\ g+y & 0 & k+x \end{vmatrix} + \begin{vmatrix} a+x & b+y & 0 \\ d+z & e+x & 0 \\ g+y & h+z & 1 \end{vmatrix}$

$$= \begin{vmatrix} e+x & f+y \\ h+z & k+x \end{vmatrix} + \begin{vmatrix} a+x & c+z \\ g+y & k+x \end{vmatrix} + \begin{vmatrix} a+x & b+y \\ d+z & e+x \end{vmatrix}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = k+x+e+x+k+x+a+x+e+x+a+x$$

$$= 6x + 2(a+e+k)$$

6. 设 $\Phi(x,y,z) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(y) & g_2(y) & g_3(y) \\ h_1(z) & h_2(z) & h_3(z) \end{vmatrix}$ 求$\dfrac{\partial^3 \Phi}{\partial x \partial y \partial z}$

解 $\dfrac{\partial \Phi}{\partial x} = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(y) & g_2(y) & g_3(y) \\ h_1(z) & h_2(z) & h_3(z) \end{vmatrix}$

$$\frac{\partial^2 \Phi}{\partial x \partial y} = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1'(y) & g_2'(y) & g_3'(y) \\ h_1(z) & h_2(z) & h_3(z) \end{vmatrix}$$

$$\frac{\partial^3 \Phi}{\partial x \partial y \partial z} = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1'(y) & g_2'(y) & g_3'(y) \\ h_1'(z) & h_2'(z) & h_3'(z) \end{vmatrix}$$

7. 设函数 $u = f(x,y)$ 在 $\boldsymbol{R}^2$ 上有 $u_{xy} = 0$,试求 u 关于 x,y 的函数式.

解　首先证明

若 $f(x,y)$ 在 $\boldsymbol{R}^2$ 上连续,$f_x(x,y) = 0$,则 $f(x,y) = \varphi(y)$.

对 $\boldsymbol{R}^2$ 上任意两点 $(x_1,y),(x_2,y)$,由中值定理

$f(x_2,y) - f(x_1,y) = f_x(x_1 + \theta(x_2 - x_1),y)(x_2 - x_1) = 0$

所以　$f(x_2,y) = f(x_1,y)$

由 $(x_1,y),(x_2,y)$ 对 x 的任意性知 $f(x,y)$ 与 x 无关,即 $f(x,y) = \psi(y)$

再求 u 关于 x,y 的函数式

因 $u_{xy} = 0$,据上述结论知　$u_x = \varphi(x)$.

因而 $\frac{\partial}{\partial x}(u - \int \varphi(x)dx) = 0$　从而 $u - \int \varphi(x)dx = \psi(y)$

所以　$u = \int \varphi(x)dx + \psi(y) = \Phi(x) + \psi(y)$.

8. 设 f 在点 $P_0(x_0,y_0)$ 可微,且在 P_0 给定了 n 个向量 $l_i(i = 1, 2,\cdots,n)$.相邻两个向量之间的夹角为 $\frac{2\pi}{n}$,证明 $\sum_{i=1}^{n} f_{l_i}(P_0) = 0$

证　由于

$$f_{l_1}(P_0) = f_x(P_0)\cos\frac{2\pi}{n} + f_y(P_0)\sin\frac{2\pi}{n}$$

$$f_{l_2}(P_0) = f_x(P_0)\cos\frac{2 \cdot 2\pi}{n} + f_y(P_0)\sin\frac{2 \cdot 2\pi}{n}$$

………

$$f_{l_i}(P_0) = f_x(P_0)\cos\frac{2\pi i}{n} + f_y(P_0)\sin\frac{2\pi i}{n}$$

所以 $$\sum_{i=1}^{n} f_{l_i}(P_0) = f_n(P_0)\sum_{i=1}^{n}\cos\frac{2\pi i}{n} + f_y(P_0)\sum_{i=1}^{n}\sin\frac{2\pi i}{n}$$

而 $$\sum_{i=1}^{n}\cos\frac{2\pi i}{n} = \frac{\sin\left(n+\frac{1}{2}\right)\frac{2\pi}{n}}{2\sin\frac{\pi}{n}} - \frac{1}{2} = 0$$

$$\sum_{i=1}^{n}\sin\frac{2\pi i}{n} = \frac{1}{2} - \frac{\sin\left(n+\frac{1}{2}\right)\frac{2\pi}{n}}{2\sin\frac{\pi}{n}} = 0$$

故 $\sum\limits_{i=1}^{n} f_{l_i}(P_0) = 0.$

9. 设 $f(x,y)$ 为 n 次齐次函数,证明

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^m f = n(n-1)\cdots(n-m+1)f.$$

证 因为 $f(x,y)$ 为 n 次齐次函数,所以 $f(tx,ty) = t^n f(x,y)$

令 $u = tx, v = ty$ 将上式两边对 t 求导得

$$x\frac{\partial f(u,v)}{\partial u} + y\frac{\partial f(u,v)}{\partial v} = nt^{n-1}f(x,y)$$

继续对 t 求导共 m 次得

$$\left(x\frac{\partial f(u,v)}{\partial u} + y\frac{\partial f(u,v)}{\partial v}\right)^m = n(n-1)\cdots(n-m+1)t^{n-m}f(x,y)$$

令 $t = 1$ 则 $u = x$ $v = y$ 得

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^m f = n(n-1)\cdots(n-m+1)f.$$

10. 对于函数 $f(x,y) = \sin\frac{y}{x}$,试证

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^m f = 0.$$

证 因为

$$f(tx,ty) = \sin\frac{ty}{tx} = \sin\frac{y}{x} = f(x,y)$$

所以 $\sin\dfrac{y}{x}$ 为 0 次齐次函数,由上题得

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^m f=0.$$