Probability Theory

Exercise Sheet 12

Exercise 12.1 Let $(X_n)_{n\geq 0}$ be a uniformly integrable family of random variables on $(\Omega, \mathcal{A}, P)$.

(a) Assume that X_n converges to a random variable X in distribution. Show that

$$E[X_n] \xrightarrow{n\to\infty} E[X].$$

Hint: Compare to (3.6.18)–(3.6.20), p. 111 of the lecture notes.

(b) Assume that X_n converges to a random variable X in probability. Show that $X \in L^1$ and that X_n converges to X in L^1.

Exercise 12.2

Definition: Let $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ be a canonical (time-homogenous) Markov chain with a *countable* state space E, a transition kernel K, and canonical coordinates $(X_n)_{n\geq 0}$. The matrix

$$Q = (Q(x,y))_{x,y\in E} := (K(x,\{y\}))_{x,y\in E} = (P_x[X_1 = y])_{x,y\in E}$$

is then called the *transition matrix* of the Markov chain. For the meanings of notation P_x and transition kernel we refer to p. 145 in lecture notes.

Let E be a countable set, $(S, \mathcal{S})$ a measurable space, $(Y_n)_{n\geq 1}$ a sequence of i.i.d. S-valued random variables. We define a sequence $(Z_n)_{n\geq 0}$ through $Z_0 = x \in E$ and $Z_{n+1} = \Phi(Z_n, Y_{n+1})$, where $\Phi : E \times S \to E$ is a measurable map. Find a transition kernel K on E such that the canonical law P_x with transition kernel K has the same law as $(Z_n)_{n\geq 0}$ (hence $(Z_n)_{n\geq 0}$ induces a time-homogenous Markov chain with transition kernel K). Calculate the corresponding transition matrix.

Exercise 12.3 Let E be a countable set, and $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ a canonical time–homogeneous Markov chain with state space E, canonical coordinate process $(X_n)_{n\geq 0}$ and transition matrix $Q = (Q(x,y))_{x,y\in E}$. Let $F \subset E$ and set $\tau_F := \inf\{n \geq 0 \,|\, X_n \in F\}$.

Let $f : E \to \mathbb{R}^+$ be a bounded function such that $f(x) \geq Qf(x)$ (resp. $=$) for all $x \in F^c$, where

$$Qf(x) := \int_\Omega f(X_1(\omega))P_x(\mathrm{d}\omega) = \sum_{y\in E} f(y)Q(x,y).$$

Show that $(f(X_{n\wedge\tau_F})_{n\geq 0})$ for all $x \in E$ is a positive P_x-supermartingale (resp. P_x-martingale) with respect to the canonical filtration $(\mathcal{F}_n)_{n\geq 0}$.

Submission: until 14:15, Dec 17., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

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Solution 12.1

(a) Since $X_n \xrightarrow{n\to\infty} X$ in distribution, we know by Proposition 2.7, p. 50 of the lecture notes that one can construct random variables Y_n, $n \in \mathbb{N}$, and Y on a common probability space $(\Omega', \mathcal{A}', P')$, such that $Y_n \overset{d}{=} X_n$, for all $n \in \mathbb{N}$, $Y \overset{d}{=} X$, and $Y_n \to Y$, P'-almost surely. It is easy to verify that the family $\{Y_n\}_{n\in\mathbb{N}}$ is also uniformly integrable, since

$$\lim_{M\to\infty} \sup_{n\in\mathbb{N}} E_{P'}\left[|Y_n| 1_{\{|Y_n|>M\}}\right] \overset{Y_n \overset{d}{=} X_n}{=} \lim_{M\to\infty} \sup_{n\in\mathbb{N}} E_P\left[|X_n| 1_{\{|X_n|>M\}}\right] = 0.$$

So by (3.6.18)-(3.6.19), p. 111 of the lecture notes, we have the L^1-convergence:

$$E_{P'}[|Y_n - Y|] \xrightarrow{n\to\infty} 0.$$

Moreover, by Jensen's inequality, for each $n \geq 0$, $0 \leq |E[Y_n - Y]| \leq E[|Y_n - Y|]$ and we have $\lim_{n\to\infty} E_{P'}[Y_n] = E_{P'}[Y]$. But then the result follows since $E_P[X_n] = E_{P'}[Y_n]$, for all $n \in \mathbb{N}$, and $E_P[X] = E_{P'}[Y]$.

(b) Since convergence in probability implies convergence in distribution, we can find $(Y_n)_n$ and Y as in 12.1 (a), and by using the uniform integrability property of $(Y_n)_n$ (see 12.1 (a)), we can find M large enough such that for all $n \geq 0$, $E_{P'}[|Y_n| 1_{\{|Y_n|\geq M\}}] \leq \epsilon$ and hence

$$E_P[|X| 1_{\{|X|>M\}}] = E_{P'}[|Y| 1_{\{|Y|>M\}}] \overset{\text{Fatou}}{\leq} \liminf_{n\to\infty} E_{P'}\left[|Y_n| 1_{\{|Y_n|\geq M\}}\right] \leq \epsilon.$$

This shows $X \in L^1$.

Now we prove the convergence in L^1. By convergence in probability, there exists $n_0 \geq 0$ such that, for all $n \geq n_0$, we have for the M as above that

$$P\Big[\underbrace{|X_n - X| \geq \epsilon}_{A_n}\Big] < \frac{\epsilon}{M}.$$

Hence by (3.6.21) in lecture notes it holds that for all $n \geq n_0$,

$$\begin{aligned}
E_P[|X_n - X|] &\leq E_P\left[|X_n - X| 1_{\{|X_n|\leq M, |X|\leq M\}}\right] \\
&\quad + 3\underbrace{E_P\left[|X_n| 1_{\{|X_n|>M\}}\right]}_{\leq \epsilon} + 3\underbrace{E_P\left[|X| 1_{\{|X|>M\}}\right]}_{\leq \epsilon} \\
&\leq E_P\Big[\underbrace{|X_n - X| 1_{\{|X_n|\leq M, |X|\leq M\}}}_{\leq 2M} 1_{A_n}\Big] \\
&\quad + E_P\Big[\underbrace{|X_n - X| 1_{\{|X_n|\leq M, |X|\leq M\}} 1_{A_n^c}}_{\leq \epsilon}\Big] + 6\epsilon \\
&\leq 2MP[A_n] + 7\epsilon \leq 9\epsilon.
\end{aligned}$$

Therefore, X_n converges to X in L^1.

Solution 12.2 Define for each $x, y \in E$, $K(x, \{y\}) := P[\Phi(x, Y_1) = y]$. Consider the probability measure $P_x = P_{\delta_x}$ on $E^{\mathbb{N}}$ as in (4.2.53) on p. 144 in lecture notes with transitional kernel K and initial distribution $\mu := \delta_x$, where δ denotes the Dirac-delta function. (Note that the existence of P_x is provided by Ionescu-Tulcea theorem). We need to show that P_x has the same law as $(Z_n)_{n\geq 0}$.

It is sufficient to show that for any $n \geq 0$ and bounded functions $f_0, f_1, \dots, f_n : E \to \mathbb{R}$,

$$E[f_0(Z_0)f_1(Z_1)\dots f_n(Z_n)] = E^{P_x}[f_0(X_0)f_1(X_1)\dots f_n(X_n)], \tag{1}$$

where $(X_n)_{n\geq 0}$ is the canonical coordinate process on E, i.e. for each $n \geq 0$ and $e = (e_1, e_2, \dots, e_n)$, $X_n(e) = e_n$. We are going to use induction. For $n = 0$, $Z_0 = x$ and $X_0 = x$ P_x-a.s. (cf. (4.2.53)), hence $E[f_0(Z_0)] = f_0(x) = E[f_0(X_0)]$. For the induction step, fix $n > 1$ and assume that (1) holds for n. For any $f : E \to \mathbb{R}$, define $Kf(x) := \sum_{e\in E} K(x, \{e\})f(e)$ and note that by the i.i.d. property of $(Y_n)_{n\geq 1}$, for each $n \geq 0$ we have

$$\begin{aligned}
E\left[f\left(\Phi(Z_{n+1})\right)|\sigma(Z_0, \dots, Z_n)\right] &= E\left[f\left(\Phi(Z_n, Y_{n+1})\right)|\sigma(Z_0, \dots, Z_n)\right] \\
&= \sum_{z\in E}\sum_{e\in E} P[\Phi(z, Y_{n+1}) = e]f(e)1_{\{Z_n=z\}} \\
&= \sum_{z\in E} Kf(z)1_{\{Z_n=z\}} = Kf(Z_n).
\end{aligned}$$

Hence with $f_n' := f_n K f_{n+1}$ it follows for the LHS of (1) that

$$\begin{aligned}
E[f_0(Z_0)f_1(Z_1)\dots f_{n+1}(Z_{n+1})] &= E\left[f_0(Z_0)f_1(Z_1)\dots f_n(Z_n)E\left[f_{n+1}(Z_{n+1})|\sigma(Z_0, \dots, Z_n)\right]\right] \\
&= E\left[f_0(Z_0)f_1(Z_1)\dots f_n(Z_n)Kf_{n+1}(Z_n)\right] \\
&= E\left[f_0(Z_0)f_1(Z_1)\dots f_{n-1}(Z_{n-1})f_n'(Z_n)\right].
\end{aligned}$$

For the RHS of (1), we obtain by (4.2.53)

$$\begin{aligned}
E^{P_x}[f_0(X_0)f_1(X_1)\dots f_{n+1}(X_{n+1})] &= E^{P_x}\left[f_0(X_0)f_1(X_1)\dots f_n(X_n)Kf_{n+1}(X_n)\right] \\
&= E^{P_x}\left[f_0(X_0)f_1(X_1)\dots f_{n-1}(X_{n-1})f_n'(X_n)\right],
\end{aligned}$$

and hence (1) for $n + 1$ follows from the induction hypothesis.

This shows that $(Z_n)_{n\geq 0}$ is a time homogeneous Markov chain and the transition matrix is given through

$$Q(x, y) = P_x[X_1 = y] = E^{P_x}\left[E[1_{\{X_1=y\}}|\mathcal{F}_0]\right] = P[\Phi(x, Y_1) = y].$$

We remark that the time homogeneity of $(X_n)_{n\geq 0}$ follows from the observation that for all $n \geq 0$ and $x, y \in E$,

$$P[X_{n+1} = y|X_n = x] = P[X_1 = y|X_0 = x] = Q(x, y).$$

Solution 12.3 By (4.2.58) in lecture notes, the process

$$M_n := f(X_n) - \sum_{k=0}^{n-1} (Qf - f)(X_k) \tag{2}$$

for $n \geq 1$, $M_0 = 0$ is an $\mathcal{F}_n$-martingale under P_x. Note that here Qf coincides with Kf appeared in (4.2.58) when the state space E is countable.
Furthermore, it is not hard to check that τ_F is an $\mathcal{F}_n$–stopping time (as F is a countable set), and therefore by Corollary 3.24 (optional stopping theorem) the stopped process $M_{n\wedge\tau_F}$ is also a martingale under P_x. Now we consider the stopped process $f(X_{n\wedge\tau_F})$. It holds that (see (2))

$$f(X_{n\wedge\tau_F}) = M_{n\wedge\tau_F} + \sum_{k=0}^{(n\wedge\tau_F)-1} (Qf - f)(X_k). \tag{3}$$

Moreover, since $X_k(\omega) \in F^c$ for all $k < \tau_F(\omega)$ and all $\omega \in \Omega$, the assumption that $f(x) \geq Qf(x)$ for all $x \in F^c$ ensures that the process $\sum_{k=0}^{(n\wedge\tau_F)-1}(Qf - f)(X_k)$, $n \geq 0$ is nonincreasing. In view of (3) we can conclude that $f(X_{n\wedge\tau_F})$ is a supermartingale under P_x, cf. Proposition 3.19. The same argument shows that it is a martingale if $Qf(x) = f(x)$ for all $x \in F^c$,