Coordinator
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Probability Theory

Exercise Sheet 9

Exercise 9.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $S \leq T$ be two bounded $(\mathcal{F}_n)_{n\geq 0}$-stopping times and let $(X_n)_{n\geq 0}$ be an $(\mathcal{F}_n)_{n\geq 0}$-submartingale. Show that

$$E[X_T|\mathcal{F}_S] \geq X_S,\ P\text{-a.s.}.$$

Exercise 9.2

(a) Let X_n be a supermartingale so that $n \mapsto E[X_n]$ is constant. Show that X_n is a martingale.

(b) Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration and (X_n) $(\mathcal{F}_n)_{n\in\mathbb{N}}$-adapted with $X_n \in L^1$ for all $n \in \mathbb{N}$. Show that X_n is an $\mathcal{F}_n$-martingale if and only if $E[X_\tau] = E[X_0]$ for all bounded $\mathcal{F}_n$-stopping times τ.

Exercise 9.3 Consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_n\}_{n\geq 0}$, and let X_n be an $\mathcal{F}_n$-martingale for which $|X_{n+1} - X_n| \leq M$ P-a.s. for some fixed $M < \infty$. Define the events C, D by

$$\begin{aligned} C &:= \{\lim X_n \text{ exists and is finite}\}, \\ D &:= \{\limsup X_n = +\infty \text{ and } \liminf X_n = -\infty\}. \end{aligned}$$

Show that $P[C \cup D] = 1$.
Hint: Show that $P[C^c \cap (\{\sup_{n\in\mathbb{N}} X_n < a\} \cup \{\inf_{n\in\mathbb{N}} X_n > -a\})] = 0$, for all $a > 0$, by considering the processes $\{X_{T_A \wedge n}\}_{n\geq 0}$, for $A = [a, \infty)$ and $A = (-\infty, -a]$, where $T_A = \inf\{n \geq 0 : X_n \in A\}$.

Submission: until 14:15, Nov 26., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint

Solution 9.1 Because S, T are bounded, there exists some $k \geq 0$, such that $S \leq T \leq k$ P-almost surely. We then observe that X_S, X_T are integrable because both of them are dominated by the integrable random variable $|X_0| + \ldots + |X_k|$.

Now let $F \in \mathcal{F}_S$. We define a sequence $(C_n)_{n\geq 1}$ of non-negative, bounded random variables through

$$C_n(\omega) := 1_F(\omega) 1_{(S(\omega),T(\omega)]}(n), \quad \omega \in \Omega,\, n \geq 1.$$

Because $\{T \leq n-1\} \in \mathcal{F}_{n-1}$ and $F \cap \{S \leq n-1\} \in \mathcal{F}_{n-1}$, one has that

$$C_n = 1_F 1_{\{S<n\}} 1_{\{T\geq n\}} = 1_{F\cap\{S\leq n-1\}} 1_{\{T\leq n-1\}^c}$$

is $\mathcal{F}_{n-1}$-measurable. This implies that $(C_n)_{n\geq 1}$ is predictable.

By Theorem 3.22, p.93 of the lecture notes, it follows that $C \cdot X$ is a submartingale (with $(C\cdot X)_0 = 0$). Hence it follows that

$$0 \leq E[(C\cdot X)_k] = E\left[\sum_{n=1}^{k} C_n(X_n - X_{n-1})\right] = E\left[(X_T - X_S)1_F\right].$$

Because $F \in \mathcal{F}_S$ is arbitrary, one has that $E[X_T \mid \mathcal{F}_S] \geq X_S$, P-a.s.

Solution 9.2

(a) Since $(X_n)_{n\in\mathbb{N}}$ is a supermartingale,

$$E[X_{n+1}|\mathcal{F}_n] \leq X_n \ P\text{-a.s.}, \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$ we define the random variable $U_n := X_n - E[X_{n+1}|\mathcal{F}_n]$. Then $U_n \geq 0$ P-a.s., but by assumption,

$$E[U_n] = E\Big[X_n - E[X_{n+1}|\mathcal{F}_n]\Big] = E[X_n] - E[X_{n+1}] = 0.$$

This implies that $U_n = 0$ P-a.s. for all $n \in \mathbb{N}$, as well as $E[X_{n+1}|\mathcal{F}_n] = X_n$ P-a.s. for all $n \in \mathbb{N}$.

(b) Let X_n be an $\mathcal{F}_n$-martingale and τ a stopping time with $\tau \leq N$ for some $N \in \mathbb{N}$. Then $X_{\tau\wedge n}$ is an $\mathcal{F}_n$-martingale, by (3.4.15), so

$$E[X_\tau] = E[X_{\tau\wedge N}] = E[X_{\tau\wedge 0}] = E[X_0].$$

We now show the converse. It is sufficient to show that

$$E[X_{n+1}1_A] = E[X_n 1_A] \quad \text{for all } A \in \mathcal{F}_n.$$

Fix an arbitrary $A \in \mathcal{F}_n$. Define $\tau_1 := n+1$ and, for all $\omega \in \Omega$,

$$\tau_2(\omega) := \begin{cases} n, & \omega \in A, \\ n+1, & \omega \in A^c. \end{cases}$$

Clearly, both τ_1 and τ_2 are bounded stopping times, and

$$\begin{aligned} E[X_{n+1}1_A] + E[X_{n+1}1_{A^c}] = E[X_{n+1}] = E[X_{\tau_1}] = E[X_0] &= E[X_{\tau_2}] \\ &= E[X_n 1_A] + E[X_{n+1}1_{A^c}], \end{aligned}$$

which yields the above stated equality.

Solution 9.3 Without loss of generality, we assume that $X_0 = 0$ or we just replace X_n by $X_n - X_0$.

Note that the hitting time T_A is an $\{\mathcal{F}_n\}_{n\geq 0}$-stopping time, for any $A \in \mathcal{B}(\mathbb{R})$, as (3.3.3) in Example 3.17, p. 89 of the lecture notes. Thus, from the optional stopping theorem ((3.4.15), p. 93 of the lecture notes), $X_{T_A \wedge n}$ is an $\{\mathcal{F}_n\}_{n\geq 0}$-martingale. If we let $A = [a, \infty)$ for $a > 0$, we furthermore have that

$$X_{T_{[a,\infty)} \wedge n} \leq a + M,$$

because of the bounded increments of X_n and $X_0 = 0$. This implies that we have

$$\sup_{n\geq 0} E\left[\left(X_{T_{[a,\infty)} \wedge n}\right)^+\right] \leq a + M < \infty.$$

Thus, by the martingale convergence theorem, (3.4.23), p. 96 of the lecture notes, the martingale $X_{T_{[a,\infty)} \wedge n}$ converges to some integrable random variable. But on the event $\{\sup_{n\geq 0} X_n < a\}$, we have $T_{[a,\infty)} = \infty$, so that $X_{T_{[a,\infty)} \wedge n} = X_n$ for all n. Thus on this event, X_n converges to a finite limit. From the definition of C, we obtain

$$P\left[C^c \cap \left\{\sup_{n\geq 0} X_n < a\right\}\right] = 0, \tag{1}$$

for all $a > 0$. Similarly by considering $-X_{T_{(-\infty,-a]}}$, or by symmetry, we can obtain that for all $a > 0$

$$P\left[C^c \cap \left\{\inf_{n\geq 0} X_n > -a\right\}\right] = 0. \tag{2}$$

Now by equations (1) and (2), we have

$$P\left[C^c \cap \left(\left\{\sup_{n\geq 0} X_n < a\right\} \cup \left\{\inf_{n\geq 0} X_n > -a\right\}\right)\right] = 0. \tag{3}$$

Taking the limit $a \to \infty$, and using the continuity property of measures, we get by definition of the event D

$$P[C^c \cap D^c] = 0. \tag{4}$$

Now the claim follows by taking the complement event in (4).

Remark: This exercise is the essential ingredient of the proof of the generalised version of the second Borel-Cantelli Lemma, see Theorems 5.31 and 5.32 in Durrett's book (pp. 204-205 in 4th online edition, pp. 239-240 in 3rd edition).