REAL ANALYSIS
LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

1. Lebesgue differentiation theorem

We aim to prove the following.

Theorem 1.1. *If $f \in L^1(\mathbb{R}^n)$, then*

$$\lim_{m(B)\to 0, x\in B} ⨍_B f = f(x) \to f \quad \text{as } r \to 0, \text{ for a.e.} x, \tag{1}$$

where B are open balls containing x.

1.1. Hardy-Littlewood maximal function.

Definition 1.1. *If $f \in L^1(\mathbb{R}^n)$, then the maximal function is*

$$f^*(x) = \sup_{x\in B} ⨍_B |f(y)| dy, \quad x \in \mathbb{R}^n.$$

The supremum is over all open balls B containing x.

In other words, we replace the limit in the statement of Theorem 1.1 by a supremum, and f by its absolute value.

Example 1.1.

(Eg1) *Consider $f = \chi_{[-1,1]}$. If $0 \le x < 1$, then $f^*(x) = 1$. If $x > 1$, then $f^*(x) = ⨍_{[-1,x]} f = 2/(x+1)$. By the symmetry, we see that*

$$f^*(x) = \begin{cases} 1, & |x| \le 1 \\ \dfrac{2}{1+|x|}, & |x| \ge 1. \end{cases}$$

(Eg2) *Consider* $f(x) = |x|^\alpha$ *where* $-1 < \alpha < 0$. *If* $x > 0$, *then*

$$
\begin{aligned}
f^*(x) &= \sup_{y<0} ⨍_{[y,x]} |x|^\alpha dx = \sup_{y<0} \frac{1}{x-y} \frac{x^{\alpha+1} + |y|^{\alpha+1}}{\alpha+1} \\
&= x^\alpha \sup_{t>0} \frac{1 + (t/x)^{\alpha+1}}{(\alpha+1)(1+t/x)} = cx^\alpha,
\end{aligned}
$$

where c *is independent of* x *and is given below*

$$
c = \sup_{s>0} \frac{1 + s^{\alpha+1}}{(\alpha+1)(1+s)}.
$$

By the symmetric $f^*(x) = c|x|^\alpha$.

The main properties of f^* we shall need are summarised below.

Theorem 1.2. *Suppose* $f \in L^(\mathbb{R}^n)$. *Then*

(i) f^* *is measurable.*

(ii) $f^*(x) < \infty$ *for a.e.* x.

(iii) f^* *satisfies*

$$
m(\{x : f^*(x) > t\}) \le \frac{A_n}{t} \|f\|_{L^1}, \quad \forall\, t > 0, \tag{2}
$$

where $A_n = 3^n$.

Remark 1.1. *As we shall observe that* $f^*(x) \ge |f(x)|$ *for a.e.* x*; the effect of (iii) is that, broadly speaking,* f^* *is not much larger than* $|f|$.

However, this does not mean that f^* *is integrable, and (iii) is the best substitute available. See Stein-Shakarchi's textbook pp145, Exercise 4 and 5 for couterexamples.*

An inequality of the type (1.2) is called a weak-type inequality because it is weaker than the corresponding inequality for the L^1-norms. Obviously, for arbitrary integrable function g

$$
m(\{x : |g(x)| > t\}) \le \frac{1}{t} \|g\|_{L^1}, \quad \forall\, t > 0.
$$

The proof of inequality (2) relies on an elementary version of a Vitali covering argument below.

Lemma 1.1. *Suppose $\mathcal{B} = \{B_1, \cdots, B_N\}$ is a finite collection of open balls in $\mathbb{R}^n$. Then there exists a disjoint sub-collection $B_{i_1}, \cdots, B_{i_k}$ of $\mathcal{B}$ that satisfies*

$$\bigcup_{\ell=1}^{N} B_\ell \subset \bigcup_{j=1}^{k} (3B_{i_j}),$$

where $3B$ is the ball with the same centre as B and three times the radius. It follows

$$m\Big(\bigcup_{\ell=1}^{N} B_\ell\Big) \leq 3^n \sum_{j=1}^{k} m(B_{i_j}). \tag{3}$$

Proof. Pick a ball B_{i_1} from $\mathcal{B}$ having maximal size. Discard all remaining balls in $\mathcal{B}$ that meet B_{i_1}. Pick a ball B_{i_2} from the remaining balls in $\mathcal{B}$ having maximal size. Discard all remaining balls in $\mathcal{B}$ that meet B_{i_2} . Repeat the procedure.

Any ball B in $\mathcal{B}$ must meet some B_{i_j}, have equal or smaller radius, and hence be covered by $3B_{i_j}$.

□

We now prove the theorem.

Proof of Theorem 1.1.

Part (i). Suppose $x \in \{f^* > t\}$. Then $⨍_B |f| > t$ for some B containing x. Let ε be sufficiently small such that $B_\varepsilon(x) \subset B$. Then $B_\varepsilon(x) \subset \{f^* > t\}$. Hence $\{f^* > t\}$ is open and so f is measurable.

Part (ii). This is a consequence of part (iii). Because

$$m(\{f^* = \infty\}) \leq m(\{f^* > t\}) \leq \frac{3^n}{t}\|f\|_{L^1} \to 0 \ \text{ as } t \to \infty.$$

Part (iii). Let $E_t = \{x : f^*(x) > t\}$. For each $x \in E_t$, there exists a ball B_x that contains x, and such that

$$⨍_{B_x} |f| > t.$$

Therefore for each ball B_x we have

$$m(B_x) < \frac{1}{t}\int_{B_x} |f|.$$

Fix a compact subset K of E_t. Since $K \subset \cup_{x\in E_t} B_x$, we select a finite subcover of K, say $K \subset \bigcup_{\ell=1}^N B_\ell$. By Lemma 1.1, we further obtain a sub-collection $B_{i_1}, \cdots, B_{i_k}$ of disjoint balls with property (3). Hence

$$m(K) \leq 3^n \sum_{j=1}^{k} m(B_{i_j}) < \frac{3^n}{t} \int_{\bigcup_{j=1}^{k} B_{i_j}} |f| \leq \frac{3^n}{t} \|f\|_{L^1}.$$

Since this inequality is true for all compact subsets K of E_t, the proof of the weak type inequality for the maximal operator is complete.

$\square$

1.2. **Lebesgue differentiation theorem.**

The estimate obtained for the maximal function now leads to the Theorem Lebesgue differentiation theorem.

Proof of Theorem 1.1. We divide the proof into several steps.

Step 1. It suffices to show, for each $t > 0$

$$m\Big(\Big\{x : \limsup_{x \in B, m(B) \to 0} \Big| ⨍_B (f(y) - f(x)) dy \Big| > t\Big\}\Big) = 0.$$

This is because this assertion then guarantees that the set $E = \cup_{n=1}^{\infty} E_{1/n}$ has measure zero, and the limit (1) holds at all points of E^c.

Step 2. Fix t. We select a continuous function g with compact support such that

$$\|f - g\|_{L^1} < \varepsilon. \tag{4}$$

Obviously g satisfies (1). Writing

$$⨍_B f(y)dy - f(x) = ⨍_B (f(y) - g(y))dy + ⨍_B g(y)dy - g(x) + g(x) - f(x),$$

we find that

$$\limsup_{x \in B, m(B) \to 0} \Big| ⨍_B f(y)dy - f(x)\Big| \leq (f - g)^*(x) + |g(x) - f(x)|.$$

Consequently, if

$$F_t = \{x : (f - g)^*(x) > t/2\} \text{ and } G_t = \{x : |f(x) - g(x)| > t/2\},$$

then $E_t \subset F_t \cup G_t$.

Step 3. By Tchebychev's inequality,

$$m(G_t) \leq \frac{2}{t}\|f - g\|_{L^1},$$

and on the other hand, by the weak type estimate for the maximal function

$$m(F_t) \leq \frac{2A_n}{t}\|f-g\|_{L^1}.$$

It follows by (4) that

$$m(E_t) \leq \frac{2}{t}\varepsilon + \frac{2A_n}{t}\varepsilon.$$

Sending $\varepsilon \to 0$, we get $m(E_t) = 0$ as desired.

□

Remark 1.2. *As an immediate consequence of the theorem applied to $|f|$, we see that*

$$f^*(x) \geq |f(x)| \quad \textit{for a.e. } x.$$

Remark 1.3. *We say a measurable function f on $\mathbb{R}^n$ is locally integrable, if for every ball B, the function $f(x)\chi_B(x) \in L^1(\mathbb{R}^n)$.*

Theorem 1.1 holds for all $f \in L^1_{loc}(\mathbb{R}^n)$, without change of the proof.

Definition 1.2. *If $E \in \mathscr{M}_{\mathbb{R}^n}$ and $x \in \mathbb{R}^n$, then the density of E at x is*

$$\theta_E(x) = \lim_{x\in B, m(B)\to 0} \frac{m(B\cap E)}{m(B)},$$

provided the limit exists.

We say x is a point of Lebesgue density of E if $\theta_E(x) = 1$.

Corollary 1.1. *Suppose E is a measurable subset of $\mathbb{R}^n$. Then almost every $x \in E$ is a point of density of E; almost every $x \notin E$ is not a point of density of E*

Definition 1.3. *If $f \in L^1_{loc}(\mathbb{R}^n)$, the Lebesgue set of f consists of all points $\bar{x} \in \mathbb{R}^n$ for which $f(\bar{x}) < \infty$, and*

$$\lim_{\bar{x}\in B, m(B)\to 0} ⨍_B |f(y) - f(\bar{x})| = 0.$$

Such $\bar{x}$ is called the Lebesgue point of f.

Observe the following

(i) If f is continuous at $\bar{x}$, then $\bar{x}$ is a Lebesgue point of f.
(ii) If $\bar{x}$ is a Lebesgue point of f, then (1) holds at $\bar{x}$.

Theorem 1.3. *Suppose $f \in L^1_{loc}(\mathbb{R}^n)$. Then almost every point belongs to the Lebesgue set of f.*

Proof. Let $r \in \mathbb{Q}$. An application of Theorem 1.1 to $|f(x) - r| \in L^1_{\text{loc}}(\mathbb{R}^n)$ shows that there is a set E_r of measure zero such that

$$\lim_{x \in B, m(B) \to 0} ⨍_B |f(y) - r| dy = |f(x) - r| \quad \forall\, x \in E_r^c.$$

Let $E = \cup_{r \in \mathbb{Q}} E_r$. Then $m(E) = 0$.

Suppose $\bar{x} \in E^c$ and $f(\bar{x}) < \infty$. For each $\varepsilon > 0$, there is an $r \in \mathbb{Q}$ such that

$$|f(\bar{x}) - r| < \varepsilon.$$

Since

$$⨍_B |f(y) - f(\bar{x})| \leq ⨍_B |f(y) - r| dy + |f(\bar{x}) - r|,$$

we must have

$$\limsup_{\bar{x} \in B, m(B) \to 0} ⨍_B |f(y) - f(\bar{x})| dy \leq 2\varepsilon,$$

and thus $\bar{x}$ is in the Lebesgue set of f, by sending $\varepsilon \to 0$. □

A collection of sets $\{U_\alpha\}$ is said to shrink regularly to x (or has bounded eccentricity at x) if there is a constant $c > 0$ such that for each U_α there is a ball B with $x \in B$, $U_\alpha \subset B$, and $m(U_\alpha) \geq cm(B)$. Thus U_α is contained in B, but its measure is comparable to the measure of B.

Observe that if $\bar{x} \in B$ with $U_\alpha \subset B$ and $m(U_\alpha) \geq cm(B)$, then

$$⨍_{U_\alpha} |f(y) - f(\bar{x})| dy \leq \frac{1}{c} ⨍_B |f(y) - f(\bar{x})| dy.$$

We conclude the following.

Corollary 1.2. *Suppose $f \in L^1_{loc}(\mathbb{R}^n)$. If $\{U_\alpha\}$ shrinks regularly to $\bar{x}$, then*

$$\lim_{x \in U_\alpha, m(U_\alpha) \to 0} ⨍_{U_\alpha} f(y) dy = f(\bar{x}).$$

for every point $\bar{x}$ in the Lebesgue set of f.

1.3. **An application: approximations to the identity.**

Definition 1.4. *Integrable functions $K_\delta \in L^1(\mathbb{R}^n)$ are said to be approximations to the identity if*

(i) $\int_{\mathbb{R}^n} K_\delta(x)dy = 1$;
(ii) $|K_\delta(x)| \le A/\delta^n$ *for all* $\delta > 0$;
(iii) $|K_\delta(x)| \le A\delta/|x|^{n+1}$ *for all* $\eta > 0$ *and* $x \in \mathbb{R}^n$.

It can be verified that approximations to the identity must be good kernels.

In previous lecture we have shown that $K_{\delta_k} * f \to f$ a.e. for a sequence of $\delta_k \to 0$, if K_δ are good kernels. The following result assets that this convergence holds for a full sequence when K_δ are moreover approximations to the identity.

Theorem 1.4. *If $\{K_\delta\}_{\delta>0}$ is an approximation to the identity and $f \in L^1(\mathbb{R}^n)$, then*

$$(K_\delta * f)(x) \to f(x) \quad \text{as } \delta \to 0$$

for every x in the Lebesgue set of f. In particular, the limit holds for a.e. x.

Lemma 1.2. *Suppose $f \in L^1(\mathbb{R}^n)$, and x is a Lebesgue point of f. Let*

$$\mathcal{A}_x(r) = ⨍_{|y|\le r} |f(x-y) - f(x)|dy, \quad r > 0.$$

Then $\mathcal{A}_x(r)$ is a continuous function of $r > 0$, and

$$\mathcal{A}_x(r) \to 0 \quad \text{as } r \to 0.$$

Moreover $\mathcal{A}_x(r)$ is bounded, that is, $\mathcal{A}_x(r) \le M$ for some $M > 0$ and all $r > 0$.

The continuity of $\mathcal{A}_x(r)$ follows by the absolute continuity of integral, and $\lim_{r\to 0} \mathcal{A}_x(r) = 0$ is because x is a Lebesgue point. To see $\mathcal{A}_x(r)$ is bounded, we estimate

$$\begin{aligned} \mathcal{A}_x(r) &\le \frac{1}{m(B_r)} \int_{|y|\le r} |f(x-y)|dy + \frac{1}{m(B_r)} \int_{|y|\le r} |f(x)|dy \\ &\le c_n r^{-n}(\|f\|_{L^1} + |f(x)|). \end{aligned}$$

Proof of Theorem 1.4. The key is to write $K_\delta * f$ as a sum of integrals over annuli as follows

$$\begin{aligned}
|(K_\delta * f)(x) - f(x)| &\le \int_{\mathbb{R}^n} |f(x-y) - f(x)||K_\delta(y)|dy \\
&= \Big[\int_{|y|\le\delta} + \sum_{k=1}^{\infty} \int_{2^k\delta<|y|\le 2^{k+1}\delta}\Big] |f(x-y) - f(x)||K_\delta(y)|dy \\
&=: I + \sum_{k=1}^{\infty} II_k.
\end{aligned}$$

The first term is estimated by, using (ii) in Definition 1.4

$$I \le C\mathcal{A}_x(\delta),$$

while the second term is estimated by, using (iii) in Definition 1.4

$$II_k \le C\delta(2^k\delta)^{-n-1} \int_{|y|\le 2^{k+1}\delta} |f(x-y) - f(y)|dy = \frac{C}{2^k}\mathcal{A}_x(2^{k+1}\delta).$$

Putting these estimates together, we find that

$$\begin{aligned}
|(K_\delta * f)(x) - f(x)| &\le C\mathcal{A}_x(\delta) + C\sum_{k=1}^{\infty} 2^{-k}\mathcal{A}_x(2^{k+1}\delta) \\
&\le C\mathcal{A}_x(\delta) + C\mathcal{A}_x(2^{N+1}\delta)\sum_{k=1}^{N} 2^{-k} + C\sup_{r>0}\mathcal{A}_x(r)\sum_{k=N+1}^{\infty} 2^{-k} \\
&\le C\mathcal{A}_x(2^{N+1}\delta) + C\sup_{r>0}\mathcal{A}_x(r)\sum_{k=N+1}^{\infty} 2^{-k} \\
&\le \varepsilon \quad \text{(firstly take } N \text{ large, then fix } N \text{ and take } \delta \text{ small).}
\end{aligned}$$

$\square$