第十八章　隐函数定理及其应用

§1　隐函数

1. 方程 $\cos x + \sin y = e^{xy}$ 能否在原点的某邻域内确定隐函数 $y = f(x)$ 或 $x = g(y)$?

解　令 $F(x,y) = \cos x + \sin y - e^{xy}$,则有

(Ⅰ)$F(x,y)$ 在原点的某邻域内连续;

(Ⅱ)$F(0,0) = 0$;

(Ⅲ)$F_x = -\sin x - ye^{xy}, F_y = \cos y - xe^{xy}$ 均在上述邻域内连续;

(Ⅳ)$F_y(0,0) = 1 \neq 0, F_x(0,0) = 0$.

故由隐函数存在唯一性定理知,方程 $\cos x + \sin y = e^{xy}$ 在原点的某邻域内可确定隐函数 $y = f(x)$.

2. 方程 $xy + z\ln y + e^{xz} = 1$ 在点$(0,1,1)$ 的某邻域内能否确定出某一个变量为另外两个变量的函数?

解　令 $F(x,y,z) = xy + z\ln y + e^{xz} - 1$,则

(Ⅰ)$F(x,y,z)$ 在点$(0,1,1)$ 的某邻域内连续;

(Ⅱ)$F(0,1,1) = 0$;

(Ⅲ)$F_x = y + ze^{xz}, F_y = x + \dfrac{z}{y}, F_z = \ln y + xe^{xz}$ 均在上述邻域内连续;

(Ⅳ)$F_x(0,1,1) = 2 \neq 0, F_y = (0,1,1) = 1 \neq 0, F_z(0,1,1) = 0$.

故由定理 18.3 知,在点$(0,1,1)$ 的某邻域内原方程能确定出函数 $x = f(y,z)$ 和 $y = g(x,z)$.

3. 求由下列方程所确定的隐函数的导数.

(1)$x^2y + 3x^4y^3 - 4 = 0$,求$\dfrac{dy}{dx}$;

(2) $\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$，求$\frac{dy}{dx}$；

(3) $e^{-xy}-2z+e^z=0$，求$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$；

(4) $a+\sqrt{a^2-y^2}=ye^u, u=\frac{x+\sqrt{a^2-y^2}}{a}$，$(a>0)$ 求$\frac{dz}{dx},\frac{d^2z}{dx^2}$

(5) $x^2+y^2+z^2-2x+2y-4z-5=0$，求$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$；

(6) $z=f(x+y+z,xyz)$，求$\frac{\partial z}{\partial x},\frac{\partial x}{\partial y},\frac{\partial y}{\partial z}$.

解　(1) 方程两边对 x 求导，则

$$2xy+x^2\frac{dy}{dx}+12x^3y^3+9x^4y^2\frac{dy}{dx}=0$$

所以$\frac{dy}{dx}=-\frac{2y+12x^2y^3}{x+9x^3y^2}$

(2) 方程两边对 x 求导数，则

$$\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{2x+2y\frac{dy}{dx}}{2\sqrt{x^2+y^2}}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{x\frac{dy}{dx}-y}{x^2}$$

所以$\frac{dy}{dx}=\frac{x+y}{x-y}\quad(x\neq y)$.

(3) 设 $F(x,y,z)=e^{-xy}-2z+e^z$，则

$$F_x=-ye^{-xy},F_y=-xe^{-xy},F_z=-2+e^z.$$

所以$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{ye^{-xy}}{e^z-2},\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{xe^{-xy}}{e^z-2}$

(4) 令 $F(x,y)=a+\sqrt{a^2-y^2}-ye^{\frac{x+\sqrt{a^2-y^2}}{a}}$

则　$F_x=-\frac{y}{a}e^u\quad F_y=-\left(e^u+ye^u\frac{-y}{a\sqrt{a^2-y^2}}\right)-\frac{y}{\sqrt{a^2-y^2}}$

将 $e^u=\frac{1}{y}(a+\sqrt{a^2-y^2})$ 代入上式，即：$F_y=\frac{y}{a}-\frac{a}{y}-\frac{\sqrt{a^2-y^2}}{y}$

$$\therefore \frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{y}{\sqrt{a^2-y^2}}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=-\frac{\sqrt{a^2-y^2}\frac{dy}{dx}-y\frac{y}{\sqrt{a^2-y^2}}\frac{dy}{dx}}{a^2-y^2}$$

$$=\frac{a^2y}{(a^2-y^2)^2}$$

(5) 令 $F(x,y,z)=x^2+y^2+z^2-2x+2y-4z-5$,则

$$F_x=2x-2,F_y=2y+2,F_z=2z-4.$$

所以$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{1-x}{z-2},\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{y+1}{2-z}$.

(6) 把 z 看成x,y 的函数,两边对 x 求偏导数,则有

$$\frac{\partial z}{\partial x}=f_1\left(1+\frac{\partial z}{\partial x}\right)+f_2\left(yz+xy\frac{\partial z}{\partial x}\right)$$

所以$\frac{\partial z}{\partial x}=\frac{f_1+yzf_2}{1-f_1-xyf_2}$;

把 x 看成y,z 的函数,两边对 y 求偏导数,则

$$0=f_1\left(1+\frac{\partial x}{\partial y}\right)+f_2\left(yz\frac{\partial x}{\partial y}+xz\right).$$

所以$\frac{\partial x}{\partial y}=-\frac{f_1+xzf_2}{f_1+yzf_2}$

把 y 看成z,x 的函数,对 z 求偏导数,则

$$1=f_1\left(\frac{\partial y}{\partial z}+1\right)+f_2\left(xy+xz\frac{\partial y}{\partial z}\right)$$

所以 $\frac{\partial y}{\partial z}=\frac{1-f_1-xyf_2}{f_1+xzf_2}$

4. 设 $z=x^2+y^2$,其中 $y=f(x)$ 为由方程 $x^2-xy+y^2=1$ 所确定的隐函数,求$\frac{dz}{dx}$ 及$\frac{d^2z}{dx^2}$.

解　由方程 $x^2-xy+y^2=1$,得$\frac{dy}{dx}=\frac{2x-y}{x-2y}$

因$\dfrac{dz}{dx}=2x+2y\dfrac{dy}{dx}=\dfrac{2(x^2-y^2)}{x-2y}$

故$\dfrac{d^2z}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dz}{dx}\right)$

$$=\frac{2\left(2x-2y\dfrac{dy}{dx}\right)(x-2y)-2(x^2-y^2)\left(1-2\dfrac{dy}{dx}\right)}{(x-2y)^2}$$

$$=\frac{4x-2y}{x-2y}+\frac{6x^2(2x-y)}{(x-2y)^3}$$

5. 设$u=x^2+y^2+z^2$,其中$z=f(x,y)$由方程$x^3+y^3+z^3=3xyz$所确定的隐函数,求u_x及u_{xx}.

解　由$x^3+y^3+z^3=3xyz$所确定的隐函数$z=f(x,y)$得$z_x=\dfrac{x^2-yz}{xy-z^2}$.故

$$u_x=2x+2zz_x=2\left(x+\frac{zx^2-yz^2}{xy-z^2}\right)$$

$$u_{xx}=\frac{\partial}{\partial x}u_x$$

$$=2\left[1+\frac{(z_xx^2+2zx-2yzz_x)(xy-z^2)}{(xy-z^2)^2}-\frac{(zx^2-yz^2)(y-2zz_x)}{(xy-z^2)^2}\right]$$

$$=\frac{2xz(y^3-3xyz+x^3+z^3)}{(xy-z^2)^3}$$

6. 求下列方程所确定的隐函数的偏导数:

(1)$x+y+z=e^{-(x+y+z)}$,求z对于x,y的一阶与二阶偏导数;

(2)$F(x,x+y,x+y+z)=0$,求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$和$\dfrac{\partial^2 z}{\partial x^2}$.

解　(1) 令$F(x,y,z)=x+y+z-e^{-(x+y+z)}$,则

$$F_x=1+e^{-(x+y+z)}=F_y=F_z.$$

故$\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial y}=-1,\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial y^2}=0$

(2) 把 z 看成 x,y 的函数,两边对 x 求偏导数,得

$F_1+F_2+F_3\left(1+\dfrac{\partial z}{\partial x}\right)=0$,故

$$\frac{\partial z}{\partial x}=-\frac{F_1+F_2+F_3}{F_3}$$

原方程两边关于 y 求偏导数,得 $\quad F_2+F_3\left(1+\dfrac{\partial z}{\partial y}\right)=0$

故 $\quad \dfrac{\partial z}{\partial y}=-\dfrac{F_2+F_3}{F_3}$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$=-\frac{F_{11}+F_{12}+F_{21}+F_{22}+F_{31}+F_{32}+(F_{13}+F_{23}+F_{33})\times\left(1+\dfrac{\partial z}{\partial x}\right)}{F_3}$$

$$+(F_1+F_2+F_3)\left[F_{31}+F_{32}+F_{33}\left(1+\frac{\partial z}{\partial x}\right)\right]F_3^{-2}$$

$$=-F_3^{-3}[F_3^2(F_{11}+2F_{12}+F_{22})]$$

$$-2(F_1+F_2)F_3(F_{13}+F_{23})+(F_1+F_2)^2F_{33}]$$

7. 证明:设方程 $F(x,y)=0$ 所确定的隐函数 $y=f(x)$ 具有二阶导数,则当 $F_y\neq 0$ 时,有

$$F_y^3y''=\begin{vmatrix} F_{xx} & F_{xy} & F_x \\ F_{xy} & F_{yy} & F_y \\ F_x & F_y & 0 \end{vmatrix}$$

证 由题设条件可得 $\quad y'=-\dfrac{F_x}{F_y}(F_y\neq 0)$

故 $\quad y''=-[(F_{xx}+F_{xy}y')F_y-F_x(F_{yx}+F_{yy}y')]F_y^{-2}$

$$=(2F_xF_yF_{xy}-F_y^2F_{xx}-F_x^2F_{yy})F_y^{-3}\quad (F_y\neq 0)$$

所以 $\quad F_y^3y''=2F_xF_yF_{xy}-F_y^2F_{xx}-F_x^2F_{yy}$

$$=\begin{vmatrix} F_{xx} & F_{xy} & F_x \\ F_{xy} & F_{yy} & F_y \\ F_x & F_y & 0 \end{vmatrix}\qquad (F_y\neq 0)$$

8. 设 f 是一元函数，试问应对 f 提出什么条件，方程 $2f(xy)=f(x)+f(y)$ 在点 $(1,1)$ 的邻域内就能确定出惟一的 y 为 x 的函数？

解　设 $F(x,y)=f(x)+f(y)-2f(xy)$，则

$$F_x=f'(x)-2yf'(xy),F_y=f'(y)-2xf'(xy)$$

且　$F(1,1)=f(1)+f(1)-2f(1)=0$

$$F_y(1,1)=f'(1)-2f'(1)=-f'(1)$$

因此只需 $f'(x)$ 在 $x=1$ 的某邻域内连续，则 F,F_x,F_y 在 $(1,1)$ 的某邻域内连续.

所以，当 $f'(x)$ 在 $x=1$ 的某邻域内连续，且 $f'(1)\neq 0$ 时，方程 $2f(xy)=f(x)+f(y)$ 就能唯一确定 y 为 x 的函数.

§2　隐函数组

1. 试讨论方程组

$$\begin{cases}x^2+y^2=\dfrac{z^2}{2}\\ x+y+z=2\end{cases}$$

在点 $(1,-1,2)$ 的附近能否确定形如 $x=f(z),y=g(z)$ 的隐函数组？

解　令 $F(x,y,z)=x^2+y^2-\dfrac{z^2}{2},G(x,y,z)=x+y+z=2$

则　(Ⅰ)F,G 在点 $(1,-1,2)$ 的某邻域内连续；

(Ⅱ)$F(1,-1,2)=0,G(1,-1,2)=0$；

(Ⅲ)$F_x=2x,F_y=2y,F_3=-z,G_x=G_y=G_z=1$，均在点 $(1,-1,2)$ 的邻域内连续；

(Ⅳ)$\left.\dfrac{\partial(F,G)}{\partial(x,y)}\right|_{(1,-1,2)}=\begin{vmatrix}F_x(1,-1,2) & F_y(1,-1,2)\\ G_x(1,-1,2) & G_y(1,-1,2)\end{vmatrix}$

$=\begin{vmatrix}2 & -2\\ 1 & 1\end{vmatrix}=4\neq 0$

故由隐函数组定理，在点 $(1,-1,2)$ 的附近所给方程组能确定形如 $x=f(z),y=g(z)$ 的隐函数组.

2. 求下列方程组所确定的隐函数组的导数.

(1) $\begin{cases} x^2+y^2+z^2=a^2 \\ x^2+y^2=ax \end{cases}$，　求$\dfrac{dy}{dx},\dfrac{dz}{dx}$

(2) $\begin{cases} x-u^2-yv=0, \\ y-v^2-xu=0 \end{cases}$　求$\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial y}$;

(3) $\begin{cases} u=f(ux,v+y), \\ v=g(u-x,v^2y) \end{cases}$　求$\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$

解　(1) 设方程组确定的隐函数组为$\begin{cases} y=g(x), \\ z=z(x) \end{cases}$　对方程组两边关于 x 求导,得

$$\begin{cases} 2x+2y\dfrac{dy}{dx}+2z\dfrac{dz}{dx}=0 \\ 2x+2y\dfrac{dy}{dx}=a \end{cases}$$

解此方程组得,$\dfrac{dy}{dx}=\dfrac{a-2x}{2y},\dfrac{dz}{dx}=-\dfrac{a}{2z}$

(2) 方程组关于 x 求偏导,得

$$\begin{cases} 1-2u\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}=0, \\ -2v\dfrac{\partial v}{\partial x}-u-x\dfrac{\partial u}{\partial x}=0 \end{cases}$$

解得:$\dfrac{\partial u}{\partial x}=\dfrac{2v+yu}{4uv-xy},\dfrac{\partial v}{\partial x}=\dfrac{2u^2+x}{xy-4uv}$

方程组关于 y 求偏导数,得

$$\begin{cases} -2u\dfrac{\partial u}{\partial y}-v-y\dfrac{\partial v}{\partial y}=0 \\ 1-2v\dfrac{\partial v}{\partial y}-x\dfrac{\partial u}{\partial y}=0 \end{cases}$$

解得　$\dfrac{\partial u}{\partial y}=\dfrac{2v^2+y}{xy-4uv},\dfrac{\partial v}{\partial y}=\dfrac{2u+xv}{4uv-xy}$

(3) 把 u,v 看成x,y 的函数,对 x 求偏导数

$$\begin{cases} \dfrac{\partial u}{\partial x}=f_1(u+x\dfrac{\partial u}{\partial x})+f_2\dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x}=g_1(\dfrac{\partial u}{\partial x}-1)+g_2(2vy\dfrac{\partial v}{\partial x}) \end{cases}$$

解之得　$$\frac{\partial u}{\partial x}=\frac{u(1-2vyg_2)f_1-f_2g_1}{(1-xf_1)(1-2vyg_2)-f_2g_1},$$
$$\frac{\partial v}{\partial x}=\frac{-(1-xf_1)g_1+uf_1g_1}{(1-xf_1)(1-2vyg_2)-f_2g_1}$$

3. 求下列函数组所确定的反函数组的偏导数：

(1) $\begin{cases}x=e^u+u\sin v,\\ y=e^u-u\cos v,\end{cases}$ 求 u_x,u_y,v_x,v_y；

(2) $\begin{cases}x=u+v,\\ y=u^2+v^2,\\ z=u^3+v^3\end{cases}$ 求 z_x.

解　(1) 因 $\dfrac{\partial(x,y)}{\partial(u,v)}=[1+e^u\sin v-e^u\cos v]u$，所以由反函数组定理，得

$$u_x=\frac{\partial y}{\partial v}\Big/\frac{\partial(x,y)}{\partial(u,v)}=\frac{\sin v}{1+e^u(\sin v-\cos v)},$$
$$v_x=-\frac{\partial y}{\partial u}\Big/\frac{\partial(x,y)}{\partial(u,v)}=\frac{\cos v-e^u}{[1+e^u(\sin v-\cos v)]u},$$
$$u_y=-\frac{\partial x}{\partial v}\Big/\frac{\partial(x,y)}{\partial(u,v)}=\frac{-\cos v}{1+e^u(\sin v-\cos v)},$$
$$v_y=\frac{\partial x}{\partial u}\Big/\frac{\partial(x,y)}{\partial(u,v)}=\frac{e^u+\sin v}{[1+e^u(\sin v-\cos v)]u},$$

(2) 关于 x 求偏导数，$\begin{cases}1=u_x+v_x,\\ 0=2uu_x+2vv_x,\\ z_x=3u^2u_x+3v^2v_x\end{cases}$，解之得 $z_x=-3uv$.

4. 设函数 $z=z(x,y)$ 由方程组

$x=e^{u+v},y=e^{u-v},z=uv$（u,v 为参量）所定义的函数，求当 $u=0,v=0$ 时的 dz.

解　因 $dz=z_xdx+z_ydy$

$$z_x=u_xv+uv_x,\quad z_y=u_yv+uv_y$$

所以　当 $u=0,v=0$ 时，$dz=0$.

5. 设以 u,v 为新的自变量变换下列方程

(1) $(x+y)\dfrac{\partial z}{\partial x}-(x-y)\dfrac{\partial z}{\partial y}=0$，设 $u=\ln\sqrt{x^2+y^2},v=\arctan\dfrac{y}{x}$；

(2)$x^2\dfrac{\partial^2 z}{\partial x^2}-y^2\dfrac{\partial^2 z}{\partial y^2}=0$,设 $u=xy, v=\dfrac{x}{y}$

解 (1) 因$\dfrac{\partial u}{\partial x}=\dfrac{x}{x^2+y^2}$, $\dfrac{\partial u}{\partial y}=\dfrac{y}{x^2+y^2}$,

$\dfrac{\partial v}{\partial x}=-\dfrac{y}{x^2+y^2}$, $\dfrac{\partial v}{\partial y}=\dfrac{x}{x^2+y^2}$

所以 $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=\dfrac{x}{x^2+y^2}\dfrac{\partial z}{\partial u}-\dfrac{y}{x^2+y^2}\dfrac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{y}{x^2+y^2}\frac{\partial z}{\partial u}+\frac{x}{x^2+y^2}\frac{\partial z}{\partial v}$$

将$\dfrac{\partial z}{\partial x}$,$\dfrac{\partial z}{\partial y}$代入原方程,并化简得,$\dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial v}$

(2) $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=y\dfrac{\partial z}{\partial u}+\dfrac{1}{y}\dfrac{\partial z}{\partial v}$,

$\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}=x\dfrac{\partial z}{\partial u}-\dfrac{x}{y^2}\dfrac{\partial z}{\partial v}$

所以 $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$

$=y\left(\dfrac{\partial^2 z}{\partial u^2}\dfrac{\partial u}{\partial x}+\dfrac{\partial^2 z}{\partial u\partial v}\dfrac{\partial v}{\partial x}\right)+\dfrac{1}{y}\left(\dfrac{\partial^2 z}{\partial u\partial v}\dfrac{\partial u}{\partial x}+\dfrac{\partial^2 z}{\partial v^2}\dfrac{\partial v}{\partial x}\right)$

$=y^2\dfrac{\partial^2 z}{\partial u^2}+2\dfrac{\partial^2 z}{\partial u\partial v}+\dfrac{1}{y^2}\dfrac{\partial^2 z}{\partial v^2}$

$\dfrac{\partial^2 z}{\partial y^2}=x\left(\dfrac{\partial^2 z}{\partial u^2}\dfrac{\partial u}{\partial y}+\dfrac{\partial^2 z}{\partial u\partial v}\dfrac{\partial v}{\partial y}\right)+\dfrac{2x}{y^3}\dfrac{\partial z}{\partial v}-\dfrac{x}{y^2}\left(\dfrac{\partial^2 z}{\partial u\partial v}\dfrac{\partial u}{\partial y}+\dfrac{\partial^2 z}{\partial v^2}\dfrac{\partial v}{\partial y}\right)$

$$=x^2\frac{\partial^2 z}{\partial u^2}+\frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}-\frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v}+\frac{2x}{y^3}\frac{\partial z}{\partial v}$$

将上述$\dfrac{\partial^2 z}{\partial x^2}$,$\dfrac{\partial^2 z}{\partial y^2}$代入原方程,并化简得

$$2xy\frac{\partial^2 z}{\partial u\partial v}=\frac{\partial z}{\partial v}\quad 即\ 2u\frac{\partial^2 z}{\partial u\partial v}=\frac{\partial z}{\partial v}$$

6. 设函数 $u=u(x,y)$ 由方程组

$$u=f(x,y,z,t), g(y,z,t)=0, h(z,t)=0$$

所确定,求$\dfrac{\partial u}{\partial x}$ 和$\dfrac{\partial u}{\partial y}$

解 方程组分别关于 x,y 求偏导数.

$$由\begin{cases}\dfrac{\partial u}{\partial x}=f_x+f_z\dfrac{\partial z}{\partial x}+f_t\dfrac{\partial t}{\partial x}\\ g_z\dfrac{\partial z}{\partial x}+g_t\dfrac{\partial t}{\partial x}=0\\ h_z\dfrac{\partial z}{\partial x}+h_t\dfrac{\partial t}{\partial x}=0\end{cases}$$

解得　$\dfrac{\partial u}{\partial x}=f_x$

$$由\begin{cases}\dfrac{\partial u}{\partial y}=f_y+f_z\dfrac{\partial z}{\partial y}+f_t\dfrac{\partial t}{\partial y},\\ g_y+g_z\dfrac{\partial z}{\partial y}+g_t\dfrac{\partial t}{\partial y}=0,\\ h_z\dfrac{\partial z}{\partial y}+h_t\dfrac{\partial t}{\partial y}=0\end{cases}$$

解得　$\dfrac{\partial u}{\partial y}=f_y+\left(\dfrac{\partial(h,f)}{\partial(z,t)}\Big/\dfrac{\partial(g,h)}{\partial(z,t)}\right)g_y$

7. 设 $u=u(x,y,z),v=v(x,y,z)$ 和 $x=x(s,t)$, $y=y(s,t),z=z(s,t)$ 都具有连续的一阶偏导数. 证明:

$$\frac{\partial(u,v)}{\partial(s,t)}=\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(s,t)}+\frac{\partial(u,v)}{\partial(y,z)}\frac{\partial(y,z)}{\partial(s,t)}+\frac{\partial(u,v)}{\partial(z,x)}\frac{\partial(z,x)}{\partial(s,t)}$$

证　右端 $=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}\begin{vmatrix}x_s & x_t\\ y_s & y_t\end{vmatrix}+\begin{vmatrix}u_y & u_z\\ v_y & v_z\end{vmatrix}\begin{vmatrix}y_s & y_t\\ z_s & z_t\end{vmatrix}$

$+\begin{vmatrix}u_z & u_x\\ v_z & v_x\end{vmatrix}\begin{vmatrix}z_s & z_t\\ x_s & x_t\end{vmatrix}$

$=\begin{vmatrix}u_xx_s+u_yy_s & u_xx_t+u_yy_t\\ v_xx_s+v_yy_s & v_xx_t+v_yy_t\end{vmatrix}+\begin{vmatrix}u_yy_s+u_zz_s & u_yy_t+u_zz_t\\ v_yy_s+v_zz_s & u_yy_t+v_zz_t\end{vmatrix}$

$+\begin{vmatrix}u_zz_s+u_xx_s & u_zz_t+u_xx_t\\ v_zz_s+v_xx_s & v_zz_t+v_xx_t\end{vmatrix}$

$=(u_xx_s+u_yy_s+u_zz_s)(v_xx_t+v_yy_t+v_zz_t)-(u_xx_t+u_yy_t+u_yy_t$ $+u_zz_t)(v_xx_s+v_yy_s+v_sz_s)$

$=u_sv_t-u_tv_s=\begin{vmatrix}u_s & u_t\\ v_s & v_t\end{vmatrix}=\dfrac{\partial(u,v)}{\partial(s,t)}=$ 左端

8. 设 $u=\dfrac{y}{\tan x},v=\dfrac{y}{\sin x}$. 证明:当 $0<x<\dfrac{\pi}{2},y>0$ 时, u,v 可

以用来作为曲线坐标；解出 x,y 作为 u,v 的函数；画出 xy 平面上 $u=1,v=2$ 所对应的坐标曲线；计算$\dfrac{\partial(u,v)}{\partial(x,y)}$和$\dfrac{\partial(x,y)}{\partial(u,v)}$并验证它们互为倒数.

解　$u_x=-\dfrac{y}{\sin^2 x}, u_y=\dfrac{1}{\tan x}, v_x=-\dfrac{y\cos x}{\sin^2 x}, v_y=\dfrac{1}{\sin x}$.

所以$\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}=-\dfrac{y}{\sin x}$

故当 $0<x<\dfrac{\pi}{2}, y>0$ 时，u_x,u_y,v_x,v_y 都连续且$\dfrac{\partial(u,v)}{\partial(x,y)}<0$. 由反函数组定理，存在反函数组 $x=x(u,v), y=y(u,v)$ 从而 u,v 可以用来作为曲线坐标.

由$\begin{cases} u=\dfrac{y}{\tan x}, \\ v=\dfrac{y}{\sin x} \end{cases}$　解得　$\begin{cases} x=\arccos\dfrac{u}{v} \\ y=\sqrt{v^2-u^2} \end{cases}$

$u=1, v=2$ 分别对应 xy 平面上坐标曲线 $y=\tan x, y=2\sin x$；如图所示：

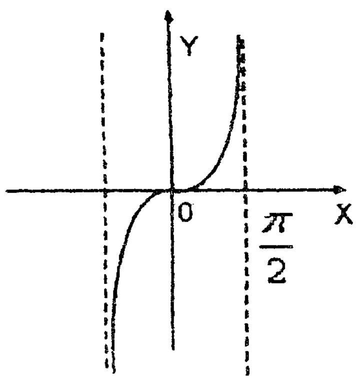

图 18—1

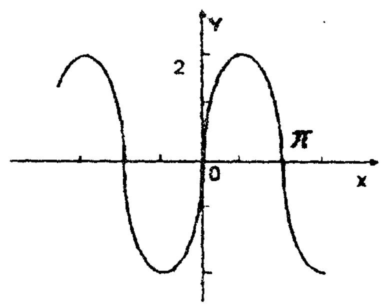

图 18—2

$$因\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{-1}{v\sqrt{1-(\frac{u}{v})^2}} & \dfrac{1}{\sqrt{1-(\frac{u}{v})^2}}\cdot\dfrac{u}{v^2}\\ -\dfrac{u}{\sqrt{v^2-u^2}} & \dfrac{v}{\sqrt{v^2-u^2}}\end{vmatrix}$$

$$=-\frac{1}{v}=-\frac{\sin x}{y}$$

而前面已算得$\dfrac{\partial(u,v)}{\partial(x,y)}=-\dfrac{y}{\sin x}$,所以$\dfrac{\partial(u,v)}{\partial(x,y)}\cdot\dfrac{\partial(x,y)}{\partial(u,v)}=1$

即$\dfrac{\partial(u,v)}{\partial(x,y)}$与$\dfrac{\partial(x,y)}{\partial(u,v)}$互为倒数.

9. 将以下式子中的(x,y,z)变换成球面坐标(r,θ,φ)的形式:

$$\triangle_1 u=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2,$$

$$\triangle_2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

解　将$\begin{cases}x=r\sin\theta\cos\varphi\\ y=r\sin\theta\sin\varphi\\ z=r\cos\theta\end{cases}$看成由$\begin{cases}x=\rho\cos\varphi\\ y=\rho\sin\varphi\\ z=z\end{cases}$ ① 和 $\begin{cases}z=r\cos\theta\\ \rho=r\sin\theta\\ \varphi=\varphi\end{cases}$ ② 复合而成.

对变换 ①,有

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\left(\frac{\partial u}{\partial \rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial u}{\partial \varphi}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2$$

对变换 ②,有

$$\left(\frac{\partial u}{\partial \rho}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial u}{\partial \varphi}\right)^2$$

$$=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial u}{\partial \varphi}\right)^2$$

$$=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial u}{\partial \varphi}\right)^2$$

故有　$\triangle_1 u = \left(\dfrac{\partial u}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2 + \dfrac{1}{r^2\sin^2\theta}\left(\dfrac{\partial u}{\partial \varphi}\right)^2$

对上述变换 ① 由 P_{183} 第 2 题的结果，得

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}$$

对变换(2)，有

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

因为　$r = \sqrt{\rho^2 + z^2}, \theta = \arctan\dfrac{\rho}{z}$,

所以　$\dfrac{\partial u}{\partial \rho} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial \rho} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial \rho} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\rho}{r} + \dfrac{\partial u}{\partial \theta}\cdot\dfrac{z}{r^2}$

$$= \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

故 $\triangle_2 u = \dfrac{\partial^u}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\cos\theta}{r^2\sin\theta}\dfrac{\partial u}{\partial \theta} + \dfrac{1}{r^2\sin^2\theta}\cdot\dfrac{\partial^2 u}{\partial \varphi^2}$

10. 设 $u = \dfrac{x}{r^2}, v = \dfrac{y}{r^2}, w = \dfrac{z}{r^2}$，其中 $r = \sqrt{x^2 + y^2 + z^2}$,

(1) 试求以 u, v, w 为自变量的反函数组；

(2) 计算 $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$

解　(1) 因 $u^2 + v^2 + w^2 = \dfrac{x^2 + y^2 + z^2}{r^4} = \dfrac{1}{r^2}$，所以 $r^2 = (u^2 + v^2 + w^2)^{-1}$

所以　$x = ur^2 = \dfrac{u}{u^2 + v^2 + w^2}, y = \dfrac{v}{u^2 + v^2 + w^2}, z = \dfrac{w}{u^2 + v^2 + w^2}$

$$(2)\ \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{r^2 - 2x^2}{r^4} & -\dfrac{2xy}{r^4} & -\dfrac{2xz}{r^4} \\ -\dfrac{2xy}{r^4} & \dfrac{r^2 - 2y^2}{r^4} & -\dfrac{2yz}{r^4} \\ -\dfrac{2xz}{r^4} & -\dfrac{2yz}{r^4} & \dfrac{r^2 - 2z^2}{r^4} \end{vmatrix} = -\frac{1}{r^6}$$

§3　几何应用

1. 求平面曲线 $x^{2/3}+y^{2/3}=a^{2/3}(a>0)$ 上任何一点处的切线方程,并证明这些切线被坐标轴所截取的线段等长.

解　令 $F(x,y)=x^{\frac{2}{3}}+y^{\frac{2}{3}}-a^{2/3}$　则

$$F_x(x_0,y_0)=\frac{2}{3}x_0^{-1/3},F_y(x_0,y_0)=\frac{2}{3}y_0^{-\frac{1}{3}}$$

所以,曲线上任一点(x_0,y_0)处的切线方程为:

$$x_0^{-\frac{1}{3}}(x-x_0)+y_0^{-\frac{1}{3}}(y-y_0)=0$$

代简即　$x_0^{-\frac{1}{3}}x+y_0^{-\frac{1}{3}}y=a^{\frac{2}{3}}$

此切线与 x,y 轴的交点分别为$(a^{2/3}x_0^{1/3},0),(0,a^{2/3}y_0^{1/3})$. 又因$(a^{2/3}x_0^{1/3})^2+(a^{2/3}y_0^{1/3})^2=a^{4/3}(x_0^{2/3}+y_0^{2/3})=a^{4/3}\cdot a^{2/3}=a^2$. 所以,任一点处的切线被坐标轴截取的线段等长(均为 a).

2. 求下列曲线在所示点处的切线方程与法平面:

(1)$x=a\sin^2t,y=b\sin t\cos t,z=c\cos^2t$ 在点 $t=\frac{\pi}{4}$;

(2)$2x^2+3y^2+z^2=9,z^2=3x^2+y^2$,在点$(1,-1,2)$.

解　(1)因 $x'(\frac{\pi}{4})=a,y'(\frac{\pi}{4})=0,z'(\frac{\pi}{4})=-c$,所以切线方程为:

$$\frac{x-\frac{a}{2}}{a}=\frac{y-\frac{b}{2}}{0}=\frac{z-\frac{c}{2}}{-c},$$

即
$$\begin{cases}\frac{x}{a}+\frac{z}{c}=1\\ y=\frac{b}{2}\end{cases}$$

法平面方程为　$a(x-\frac{a}{2})-c(z-\frac{c}{2})=0$,即

$$ax-cz=\frac{1}{2}(a^2-c^2)$$

(2) 令 $F(x,y,z)=2x^2+3y^2+z^2-9$

$G(x,y,z)=3x^2+y^2-z^2$

$F_x=4x, F_y=6y, F_z=2z, G_x=6x, G_y=2y, G_z=-2z$，所以

$$\left.\frac{\partial(F,G)}{\partial(x,y)}\right|_{(1,-1,2)}=28,\left.\frac{\partial(F,G)}{\partial(y,z)}\right|_{(1,-1,2)}=32,\left.\frac{\partial(F,G)}{\partial(z,x)}\right|_{(1,-1,2)}=40.$$

故切线方程为$\dfrac{x-1}{8}=\dfrac{y+1}{10}=\dfrac{z-2}{7}$.

法平面为 $8(x-1)+10(y+1)+7(z-2)=0$.

3. 求下列曲线在所示点处的切平面与法线：

(1) $y-e^{2x-z}=0$，在点$(1,1,2)$；

(2) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$，在点$\left(\dfrac{a}{\sqrt{3}},\dfrac{b}{\sqrt{3}},\dfrac{c}{\sqrt{3}}\right)$

解 (1) 令 $F(x,y,z)=y-e^{2x-z}$，则

$$F_x(1,1,2)=-2, F_y(1,1,2)=1, F_z(1,1,2)=1.$$

故切平面方程为 $-2(x-1)+(y-1)+(z-2)=0$

法线方程为$\dfrac{x-1}{-2}=y-1=z-2$

(2) 令 $F(x,y,z)=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1$，则

$$F_x\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)=\frac{2}{\sqrt{3}a},$$

$$F_y\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)=\frac{2}{\sqrt{3}b},$$

$$F_z\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)=\frac{2}{\sqrt{3}c},$$

故切面方程为 $-\dfrac{1}{a}\left(x-\dfrac{a}{\sqrt{3}}\right)+\dfrac{1}{b}\left(y-\dfrac{b}{\sqrt{3}}\right)+\dfrac{1}{c}\left(z-\dfrac{c}{\sqrt{3}}\right)=0$

即 $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=\sqrt{3}$

法线方程为 $a\left(x-\dfrac{a}{\sqrt{3}}\right)=b\left(y-\dfrac{b}{\sqrt{3}}\right)=c\left(z-\dfrac{c}{\sqrt{3}}\right)$

4. 证明对任意常数 ρ,φ，球面 $x^2+y^2+z^2=\rho^2$ 与锥面 $x^2+y^2=\tan^2\varphi\cdot z^2$ 是正交的.

解　设 (x,y,z) 是球面与锥面交线上的任一点，则

球面在该点的法向量为 $\vec{n}_1=(2x,2y,2z)$，

锥面在该点的法向量为 $\vec{n}_2=(2x,2y,-2z\tan^2\varphi)$

因为 $\vec{n}_1\cdot\vec{n}_2=4x^2+4y^2-4z^2\tan^2\varphi=0$，故对任意的常数 ρ,φ，球面与锥面正交.

5. 求曲面 $x^2+2y^2+3z^2=21$ 的切平面，使它平行于平面 $x+4y+6z=0$.

解　设曲面上过点 (x_0,y_0,z_0) 的切平面和平面 $x+4y+6z=0$ 平行，又在该点的切平面为

$$2x_0(x-x_0)+4y_0(y-y_0)+6z_0(z-z_0)=0$$

故　$\dfrac{2x_0}{1}=\dfrac{4y_0}{4}=\dfrac{6z_0}{6}$. 所以 $2x_0=y_0=z_0$ 代入曲面方程得

$$x_0^2+8x_0^2+12x_0^2=21$$

所以 $x_0=\pm1$，可见在点 $(1,2,2)$ 和点 $(-1,-2,-2)$ 处的切平面与所给平面平行.

在 $(1,2,2)$ 处切平面为　$x+4y+6z=21$，

在 $(-1,-2,-2)$ 处切平面为　$x+4y+6z=-21$.

6. 在曲线 $x=t,y=t^2,z=t^3$ 上求出一点，使曲线在此点的切线平行于平面 $x+2y+z=4$.

解　$x_t=1,y_t=2t,z_t=3t^2$.

设曲线在 $t=t_0$ 处的切线平行与平面 $x+2y+2=4$. 则有 $(1,2t_0,3t_0^2)\cdot(1,2,1)=0$，即 $1+4t_0+3t_0^2=0$，解之得 $t_0=-1$ 或 $t_0=-\dfrac{1}{3}$. 所以所求点为 $(-1,1,-1)$ 或 $\left(-\dfrac{1}{3},\dfrac{1}{9},-\dfrac{1}{27}\right)$.

7. 求函数 $u=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$ 在点 $M(1,2,-2)$ 处沿曲线 $x=t$，$y=2t^2,z=-2t^4$ 在该点切线方向导数.

解　因曲线过点$(1,2,-2)$,所以 $t_0=1$,于是 $x_t(t_0)=1,y_t(t_0)=4,z_t(t_0)=-8$.

故曲线在点 M 的切线方向的方向余弦为:$\frac{1}{9},\frac{4}{9},-\frac{8}{9}$ 而 $u_x(M)=\frac{8}{27},u_y(M)=-\frac{2}{27},u_z(M)=\frac{2}{27}$.

故所求方向导数为:$\frac{8}{27}\cdot\frac{1}{9}+\left(-\frac{2}{27}\right)\cdot\frac{4}{9}+\frac{2}{27}\cdot\left(-\frac{8}{9}\right)=-\frac{16}{243}$

8. 试证明:函数 $F(x,y)$ 在点 $P_0(x_0,y_0)$ 的梯度恰好是 F 的等值线在点 P_0 的法向量(设 F 有连续一阶偏导数).

证　F 的等值线为　$F(x,y)=c$.它在点 P_0 的切线方程为

$$F_x(x_0,y_0)(x-x_0)+F_y(x_0,y_0)(y-y_0)=0$$

故等值线在点 P_0 的法向量为$(F_x(x_0,y_0),F_y(x_0,y_0))$,而函数 F 在点 P_0 的梯度恰好是$(F_x(x_0,y_0),F_y(x_0,y_0))$.即结论成立.

9. 确定正数 λ,使曲面 $xyz=\lambda$ 与椭球面$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$在某一点相切.(即在该点有公共切平面)

解　设两曲面在点 $P_0(x_0,y_0,z_0)$ 相切,则曲面 $xyz=\lambda$ 在点P_0 的切平面 $y_0z_0(x-x_0)+z_0x_0(y-y_0)+x_0y_0(z-z_0)=0$ 与椭球面在点 P_0 的切平面

$$\frac{x_0}{a^2}(x-x_0)+\frac{y_0}{b^2}(y-y_0)+\frac{z_0}{c^2}(z-z_0)=0$$

应为一个平面,所以

$\frac{x_0}{a^2y_0z_0}=\frac{y_0}{b^2z_0x_0}=\frac{z_0}{c^2x_0y_0}$　即$\frac{x_0^2}{a^2}=\frac{y_0^2}{b^2}=\frac{z_0^2}{c^2}$.又$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z^2}{c^2}=1$,所以$\frac{x_0^2}{a^2}=\frac{y_0^2}{b^2}=\frac{z_0^2}{c^2}=\frac{1}{3}$

从而 $x_0^2y_0^2z_0^2=\frac{1}{27}a^2b^2c^2$

故所求的正数 λ 为:$\lambda=x_0y_0z_0=\frac{|abc|}{3\sqrt{3}}$.

10. 求曲面 $x^2+y^2+z^2=x$ 的切平面，使其垂直于平面 $x-y-\frac{1}{2}z=2$ 和 $x-y-z=2$.

解　设曲面在点 $P_0(x_0,y_0,z_0)$ 处的切平面垂直于所给两平面.

由曲面在点 P_0 处切平面方程

$(2x_0-1)(x-x_0)+2y_0(y-y_0)+2z_0(z-z_0)=0$ 知 P_0 应满足：

$$\begin{cases}(2x_0-1,2y_0,2z_0)\cdot\left(1,-1,-\dfrac{1}{2}\right)=0,\\(2x_0-1,2y_0,2z_0)\cdot(1,-1,-1)=0,\\x_0^2+y_0^2+z_0^2=x_0\end{cases}$$

解得　$x_0=\dfrac{1}{2}\pm\dfrac{1}{2\sqrt{2}},y_0=\pm\dfrac{1}{2\sqrt{2}},z_0=0.$

故所求切平面为：$x+y=\dfrac{1}{2}(1\pm\sqrt{2})$.

11. 求两曲面 $F(x,y,z)=0,G(x,y,z)=0$ 的交线在 xy 平面上的投影曲线的切线方程.

解　对方程组 $\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}$ 关于 z 求导得

$$\begin{cases}F_x\dfrac{dx}{dz}+F_y\dfrac{dy}{dz}+F_z=0\\[2ex]G_x\dfrac{dx}{dz}+G_y\dfrac{dy}{dz}+G_z=0\end{cases}$$

解得 $\dfrac{dx}{dz}=\dfrac{\partial(F,G)}{\partial(y,z)}\Big/\dfrac{\partial(F,G)}{\partial(x,y)},\dfrac{dy}{dz}=\dfrac{\partial(F,G)}{\partial(z,x)}\Big/\dfrac{\partial(F,G)}{\partial(x,y)}$.

因此交线在 xy 平面的投影曲线的切线方程为

$$(x-x_0)\Big/\frac{dx}{dz}\bigg|_{P_0}-(y-y_0)\Big/\frac{dy}{dz}\bigg|_{P_0}.$$

§4　条件极值

1. 应用拉格朗日乘数法，求下列函数的条件极值：

(1)$f(x,y)=x^2+y^2$,若 $x+y-1=0$

(2)$f(x,y,z,t)=x+y+z+t$,若 $xyzt=c^4$(其中 $x,y,z,t>0,c>0$);

(3)$f(x,y,z)=xyz$,若 $x^2+y^2+z^2=1,x+y+z=0$.

解 (1)设 $L(x,y,\lambda)=x^2+y^2+\lambda(x+y-1)$.对 L 求偏导数,并令它们都等于0,则有

$$\begin{cases} L_x=2x+\lambda=0 \\ L_y=2y+\lambda=0 \\ L_\lambda=x+y-1=0 \end{cases}$$

解之得 $x=y=\dfrac{1}{2},\lambda=-1$.

由于当 $x\to\infty,y\to\infty$ 时,$f\to\infty$.故函数必在唯一稳定点处取得极小值,极小值 $f\left(\dfrac{1}{2},\dfrac{1}{2}\right)=\dfrac{1}{2}$

(2) 设 $L(x,y,z,t,\lambda)=x+y+z+t+\lambda(xyzt-c^4)$,

令$\begin{cases} L_x=1+\lambda yzt=0 \\ L_y=1+\lambda xzt=0 \\ L_z=1+\lambda xyt=0 \\ L_t=1+\lambda xyz=0 \\ L_\lambda=xyzt-c^4=0 \end{cases}$ 解方程组得 $x=y=z=t=c$.

由于当 n 个正数的积一定时,其和必有最小值,故 f 一定在唯一稳定点 (c,c,c,c) 取得最小值也是极小值,所以极小值 $f(c,c,c,c)=4c$.

(3)设 $L(x,y,z,\lambda,u)=xyz+\lambda(x^2+y^2+z^2-1)+\mu(x+y+z)$.

$$\begin{cases} L_x=yz+2\lambda x+u=0 \\ L_y=xz+2\lambda y+u=0 \\ L_{z'}=xy+2\lambda z+u=0 \\ L_\lambda=x^2+y^2+z^2-1=0 \\ L_\mu=x+y+z=0 \end{cases}$$

解方程组得 x,y,z 的六组值为：

$$\begin{cases}x=\frac{1}{\sqrt{6}}\\ y=\frac{1}{\sqrt{6}}\\ z=-\frac{2}{\sqrt{6}}\end{cases}\quad \begin{cases}x=\frac{2}{\sqrt{6}}\\ y=-\frac{1}{\sqrt{6}}\\ z=-\frac{1}{\sqrt{6}}\end{cases}\quad \begin{cases}x=\frac{1}{\sqrt{6}}\\ y=-\frac{2}{\sqrt{6}}\\ z=\frac{1}{\sqrt{6}}\end{cases}$$

$$\begin{cases}x=-\frac{1}{\sqrt{6}}\\ y=-\frac{1}{\sqrt{6}}\\ z=\frac{2}{\sqrt{6}}\end{cases}\quad \begin{cases}x=-\frac{2}{\sqrt{6}}\\ y=-\frac{1}{\sqrt{6}}\\ z=\frac{1}{\sqrt{6}}\end{cases}\quad \begin{cases}x=-\frac{1}{\sqrt{6}}\\ y=\frac{2}{\sqrt{6}}\\ z=-\frac{1}{\sqrt{6}}\end{cases}$$

又 $f(x,y,z)=xyz$ 在有界闭集

$\{(x,y,z)\mid x^2+y^2+z^2=1, x+y+z=0\}$ 上连续，故有最值. 因此

极小值 $f\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right)$

$=f\left(-\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)=f\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$

$=-\frac{1}{3\sqrt{6}}$

极大值 $f\left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$

$=f\left(\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$

$=f\left(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)=\frac{1}{3\sqrt{6}}$

2.(1) 求表面积一定而体积最大的长方体.

(2) 求体积一定而表面积最小的长方体.

解　(1) 设长方体的长、宽、高分别为 x,y,z，表面积为 $a^2(a>0)$，则体积为 $f(x,y,z)=xyz$，限制条件为：$2(xy+yz+xz)=a^2$.

设 $L(x,y,z,\lambda)=xyz+\lambda[2(xy+yz+xz)-a^2]$

令 $\begin{cases}L_x = yz+2\lambda(y+z)=0\\ L_y = xz+2\lambda(x+z)=0\\ L_z = xy+2\lambda(x+y)=0\\ L_\lambda = 2(xy+yz+xz)-a^2=0\end{cases}$

解得 $x=y=z=\dfrac{a}{\sqrt{6}}$

因所求长方体体积有最大值，且稳定点只有一个，所以最大值 $f\left(\dfrac{a}{\sqrt{6}},\dfrac{a}{\sqrt{6}},\dfrac{a}{\sqrt{6}}\right)=\dfrac{a^3}{6\sqrt{6}}$. 故表面积一定而体积最大的长方体是正立方体.

(2) 设长方体的长、宽、高分别为 x,y,z，体积为 v，则表面积 $f(x,y,z)=2(xy+yz+xz)$，限制条件：$xyz=v$.

设 $L(x,y,z,\lambda)=2(xy+yz+xz)+\lambda(xyz-v)$

令 $\begin{cases}L_x = 2(y+z)+\lambda yz=0\\ L_y = 2(x+z)+\lambda zx=0\\ L_z = 2(x+y)+\lambda xy=0\\ L_\lambda = xyz-v=0\end{cases}$

解得 $x=y=z=\sqrt[3]{v}$

故体积一定而表面积最小的长方体是正立方体.

3. 求空间一点 (x_0,y_0,z_0) 到平面 $Ax+By+Cz+D=0$ 的最短距离.

解 由题意，相当于求 $f(x,y,z)=d^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$ 在条件 $Ax+By+Cz+D=0$ 下的最小值问题.

由几何学知，空间定点到平面的最短距离存在. 设 $L(x,y,z,\lambda)=f(x,y,z)+\lambda(Ax+By+Cz+D)$.

令 $$\begin{cases} L_x = 2(x - x_0) + \lambda A = 0 & (1) \\ L_y = 2(y - y_0) + \lambda B = 0 & (2) \\ L_z = 2(z - z_0) + \lambda C = 0 & (3) \\ L_\lambda = Ax + By + Cz + D = 0 & (4) \end{cases}$$

由(1),(2),(3)得

$$x = x_0 - \frac{\lambda}{2}A, y = y_0 - \frac{\lambda}{2}B, z = z_0 - \frac{\lambda}{2}C$$

代入(4)解得　$\lambda = \dfrac{2(Ax_0 + By_0 + Cz_0 + D)}{A^2 + B^2 + C^2}$

所以　$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \dfrac{1}{4}\lambda^2(A^2 + B^2 + C^2)$

$$= \frac{(Ax_0 + By_0 + Cz_0 + D)^2}{A^2 + B^2 + C^2}$$

故 $d = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ 为所求最短距离.

4. 证明:在 n 个正数的和为定值条件

$$x_1 + x_2 + x_3 + \cdots + x_n = a$$

下,这 n 个正数的乘积 $x_1x_2x_3\cdots x_n$ 的最大值为 $\dfrac{a^n}{n^n}$,并由此结果推出 n 个正数的几何中值不大于算术中值.

$$\sqrt[n]{x_1 \cdot x_2 \cdots x_n} \leqslant \frac{x_1 + x_2 + \cdots + x_n}{n}$$

证　设

$f(x_1, x_2 \cdots x_n) = x_1x_2\cdots x_n$

$L(x_1, x_2, \cdots, x_n, \lambda) = f(x_1, x_2, \cdots, x_n) + \lambda(x_1 + x_2 + \cdots + x_n - a), (x_1, x_2, \cdots, x_n > 0)$

$$\begin{cases} L_{x_1} = x_1x_2\cdots x_n/x_1 + \lambda = 0 \\ L_{x_2} = x_1x_2\cdots x_n/x_2 + \lambda = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ L_{x_n} = x_1x_2\cdots x_n/x_n + \lambda = 0 \\ L_\lambda = x_1 + x_2 + \cdots + x_n - a = 0 \end{cases}$$

解得 $x_1 = x_2 = \cdots = x_n = \dfrac{a}{n}$

由题意知,最大值在唯一稳定点取得.

所以 $f_{最大} = f\left(\dfrac{a}{n}, \dfrac{a}{n}, \cdots, \dfrac{a}{n}\right) = \dfrac{a^n}{n^n}$,故

$$\sqrt[n]{x_1x_2\cdots x_n} \leqslant \sqrt[n]{\frac{a^n}{n^n}} = \frac{a}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

因此 $\sqrt[n]{x_1x_2\cdots x_n} \leqslant \dfrac{x_1 + x_2 + \cdots + x_n}{n}$

5. 设 $a_1, a_2, \cdots, a_n$ 为已知的 n 个正数,求

$$f(x_1, x_2, \cdots, x_n) = \sum_{k=1}^{n} a_k x_k$$

在限制条件

$x_1^2 + x_2^2 + \cdots + x_n^2 \leqslant 1$ 下的最大值.

解 先求 f 在条件 $\sum\limits_{i=1}^{n} x_i^2 = a^2 (0 < a \leqslant 1)$ 下的最大值,为此,设

$$L(x_1, \lambda_2, \cdots, x_n, \lambda) = \sum_{k=1}^{n} a_k x_k + \lambda(x_1^2 + x_2^2 + \cdots + x_n^2 - a^2)(0 < a \leqslant 1)$$

令 $\begin{cases} L_{x_k} = a_k + 2\lambda x_k = 0 \quad (k = 1, 2, \cdots, n) \\ L_\lambda = \sum\limits_{k=1}^{n} x_k^2 - a^2 = 0 \end{cases}$

解得 $x_k = \mp a_k a / (\sum\limits_{k=1}^{n} a_k)^{\frac{1}{2}} \quad k = 1, 2, \cdots, n$

$\lambda = \pm \dfrac{1}{2a} (\sum\limits_{k=1}^{n} a_k^2)^{\frac{1}{2}}$

$$\sum_{k=1}^{n} a_k x_k = \mp a(\sum_{k=1}^{n} a_k^2)^{\frac{1}{2}}$$

于是 f 在条件 $\sum_{k=1}^{n} x_k^2 = a^2$ 下的最大值为 $a(\sum_{k=1}^{n} a_k^2)^{\frac{1}{2}}$. 故 f 在条件 $\sum_{i=1}^{n} x_k^2 \leqslant 1$ 下的最大值为 $\sup\limits_{0<a\leqslant 1}(\sum_{k=1}^{n} a_k^2)^{\frac{1}{2}} = (\sum_{k=1}^{n} a_k^2)^{\frac{1}{2}}$.（注此题也可用柯西不等式，方法更简.）

6. 求函数　$f(x_1,x_2,\cdots,x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$

在条件 $\sum_{k=1}^{n} a_k x_k = 1$, $(a_k > 0, k = 1,2,\cdots,n)$ 下的最小值.

解　设 $L(x_1,x_2,\cdots,x_n,\lambda)$

$$= f(x_1,x_2,\cdots,x_n) + \lambda(\sum_{k=1}^{n} a_k x_k - 1)$$

令 $$\begin{cases} L_{x_k} = 2x_k + \lambda a_k = 0 \quad (k = 1,2,\cdots,n) \\ L_\lambda = \sum_{k=1}^{n} a_k x_k - 1 = 0 \end{cases}$$

解得　$x_k = (\sum_{k=1}^{n} a_k^2)^{-1} a_k, \lambda = -2(\sum_{k=1}^{n} a_k^2)^{-1} (k = 1,2,\cdots,n)$

依题意，相当于求 n 维空间中原点到超平面 $\sum_{k=1}^{n} a_k x_k = 1$ 的最短距离. 由几何知，最短距离存在，而稳定点只有一个，故一定在唯一稳定点处取得最小值，故

$$f_{最小} = f[(\sum_{k=1}^{n} a_k^2)^{-1} a_1, (\sum_{k=1}^{n} a_k^2)^{-1} a_2, \cdots, (\sum_{k=1}^{n} a_k^2)^{-1} a_n]$$

$$= (\sum_{k=1}^{n} a_k^2)^{-1}$$

总 练 习 题

1. 方程 $y^2 - x^2(1 - x^2) = 0$ 在哪些点的邻域内可惟一地确定连续可导的隐函数 $y = f(x)$?

解　由 $y^2 = x^2(1-x^2)$ 知　$(1-x^2)\geqslant 0$　$\therefore |x|\leqslant 1$

且 $y^2 = x^2(1-x^2)\leqslant\left(\frac{x^2+1-x^2}{2}\right)^2 = \frac{1}{4}$　$\therefore |y|\leqslant\frac{1}{2}$

由 $F_x = -2x+4x^3$　$F_y = 2y$　由 $F_y\neq 0$ 知 $y\neq 0$ 即 $x\neq 0$, $x\neq\pm 1$

令 $D = \{(x,y)\mid |x|<1, |y|\leqslant\frac{1}{2}$ 且 $y\neq 0\}$,则 $F(x,y)$ 在 D 内每一邻域内有定义且连续;$F_x\cdot F_y$ 在 D 每一邻域内都连续. $F(x,y) = 0$, $Fy\neq 0$, 故方程 $y^2 - x^2(1-x^2) = 0$ 可在 D 上唯一确定隐函数 $y = f(x)$.

2. 设函数 $f(x)$ 在区间 (a,b) 内连续,函数 $\varphi(y)$ 在区间 (c,d) 内连续,而 $\varphi'(y)>0$. 问在怎样的条件下,方程 $\varphi(y) = f(x)$ 能确定函数 $y = \varphi^{-1}(f(x))$. 并研究例子:(ⅰ)$\sin y+\operatorname{sh}y = x$;
(ⅱ)$e^{-y} = -\sin^2 x$.

解　$F(x,y) = \varphi(y)-f(x)$ 在 R^2 上连续.

$$F_y = \varphi'(y)>0$$

故由课本 P_{145} 注意 2 知,若 $f[(a,b)]\cap[(c,d)]\neq\varnothing$ 即存在点 (x_0,y_0),满足 $F(x_0,y_0) = 0$,就可在 (x_0,y_0) 附近确定隐函数　$y = \varphi^{-1}[f(x)]$

(Ⅰ)设 $f(x) = x, \varphi(y) = \sin y+\operatorname{sh}y$

由于 $f(x),\varphi(y)$ 都在 R 上连续,且

$\varphi'(y) = \cos y+\operatorname{ch}y>0$. 又 $f(R)\cap\varphi(R) = R\neq\varnothing$,故由上面的结论知方程 $\sin y+\operatorname{sh}y = x$ 可确定函数 $y = y(x)$.

(Ⅱ)由于 $f(x) = -\sin^2 x\leqslant 0, \varphi(y) = e^{-y}>0$

所以　$f(R)\cap\varphi(R) = \varnothing$ 故方程 $e^{-y} = -\sin^2 x$ 不能确定函数 $y = y(x)$.

3. 设 $f(x,y,z) = 0, z = g(x,y)$,试求 $\frac{dy}{dx},\frac{dz}{dx}$.

解　对方程组 $\begin{cases} f(x,y,z) = 0 \\ z = g(x,y)\end{cases}$ 关于 x 求导得

$$\begin{cases} f_x + f_y \dfrac{dy}{dx} + f_z \dfrac{dz}{dx} = 0 \\ \dfrac{dz}{dx} = g_x + g_y \dfrac{dy}{dx} \end{cases}$$

解之得　$\dfrac{dy}{dx} = -\dfrac{f_x + f_z g_x}{f_y + f_z g_y}$，　$\dfrac{dz}{dx} = \dfrac{g_x f_y - g_y f_x}{f_y + f_z g_y}$

4. 已知 $G_1(x,y,z), G_2(x,y,z), f(x,y)$ 都是可微的，$g_i(x,y) = G_i(x,y,f(x,y)), i = 1,2$　证明：

$$\frac{\partial(g_1,g_2)}{\partial(x,y)} = \begin{vmatrix} -f_x & -f_y & 1 \\ G_{1x} & G_{1y} & G_{1z} \\ G_{2x} & G_{2y} & G_{2z} \end{vmatrix}$$

证　因为

$$\frac{\partial(g_1,g_2)}{\partial(x,y)} = \begin{vmatrix} G_{1x} + G_{1z}f_x & G_{1y} + G_{1z}f_y \\ G_{2x} + G_{2z}f_x & G_{2y} + G_{2z}f_y \end{vmatrix}$$

$$= f_x(G_{1z}G_{2y} - G_{1y}G_{2z}) + f_y(G_{1x}G_{2z} - G_{2x}G_{1z}) + (G_{1x}G_{2y} - G_{1y}G_{2x})$$

$$= \begin{vmatrix} -f_x & -f_y & 1 \\ G_{1x} & G_{1y} & G_{1z} \\ G_{2x} & G_{2y} & G_{2z} \end{vmatrix}$$

故原式成立.

5. 设 $x = f(u,v,w), y = g(u,v,w), z = h(u,v,w)$. 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z}$.

解　三方程分别对 x 求偏导数，得

$$\begin{cases} 1 = f_u \dfrac{\partial u}{\partial x} + f_v \dfrac{\partial v}{\partial x} + f_w \dfrac{\partial w}{\partial x} \\ 0 = g_u \dfrac{\partial u}{\partial x} + g_v \dfrac{\partial v}{\partial x} + g_w \dfrac{\partial w}{\partial x} \\ 0 = h_u \dfrac{\partial u}{\partial x} + h_v \dfrac{\partial v}{\partial x} + h_w \dfrac{\partial w}{\partial x} \end{cases}$$

解之得　$\dfrac{\partial u}{\partial x} = \dfrac{\partial(g,h)}{\partial(v,w)} \Big/ \dfrac{\partial(f,g,h)}{\partial(u,v,w)}$

同理三方程分别关于 y,z 求偏导数，则可解得

$$\frac{\partial u}{\partial y}=\frac{\partial(h,f)}{\partial(u,v)}\Big/\frac{\partial(f,g,h)}{\partial(u,v,w)},\frac{\partial u}{\partial z}=\frac{\partial(f,g)}{\partial(u,v)}\Big/\frac{\partial(f,g,h)}{\partial(u,v,w)}$$

6. 试求下列方程所确定的函数的偏导数$\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$：

(1)$x^2+u^2=f(x,u)+g(x,y,u)$

(2)$u=f(x+u,yu)$

解 (1) 把 u 看成x,y 的函数，两边对 x 求偏导数，得

$$2x+2u\frac{\partial u}{\partial x}=f_x+f_u\frac{\partial u}{\partial x}+g_x+g_u\frac{\partial u}{\partial x}$$

所以$\frac{\partial u}{\partial x}=\frac{f_x+g_x-2x}{2u-f_u-g_u}$

同理两边对 y 求偏导数得

$$\frac{\partial u}{\partial y}=\frac{g_y}{2u-f_u-g_u}$$

(2) 两边对 x 求偏导数有

$$\frac{\partial u}{\partial x}=f_1\left(1+\frac{\partial u}{\partial x}\right)+f_2\left(y\frac{\partial u}{\partial x}\right)$$

所以$\frac{\partial u}{\partial x}=\frac{f_1}{1-f_1-yf_2}$

两边对 y 求偏导数，得

$$\frac{\partial u}{\partial y}=f_1\frac{\partial u}{\partial y}+f_2\left(u+y\frac{\partial u}{\partial y}\right)$$

故 $\frac{\partial u}{\partial y}=\frac{uf_2}{1-f_1-yf_2}$

7. 据理说明：在点(0,1) 近旁是否存在连续可微的 $f(x,y)$ 和 $g(x,y)$，满足 $f(0,1)=1,g(0,1)=-1$，且

$$[f(x,y)]^3+xg(x,y)-y=0,[g(x,y)]^3+yf(x,y)-x=0$$

解 设$\begin{cases}F(x,y,u,v)=u^3+xv-y=0,\\ G(x,y,u,v)=v^3+yu-x=0\end{cases}$ 则

(Ⅰ)F,G 在以$P_0(0,1,1,-1)$ 为内点的 $\boldsymbol{R}^4$ 内连续；

(Ⅱ) F,G 在 R^4 内具有连续一阶偏导数；

(Ⅲ) $F(P_0)=0, G(P_0)=0$；

(Ⅳ) $\left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{P_0}=\left.\begin{vmatrix}3u^2 & x\\ y & 3v^2\end{vmatrix}\right|_{P_0}=9\neq 0$

由隐函数组定理知，方程组在 P_0 附近唯一地确定了在点(0,1)近旁连续可微的两个二元函数　$u=f(x,y), v=g(x,y)$. 满足 $f(0,1)=1, g(0,1)=-1$ 且

$$[f(x,y)]^3+xg(x,y)-y=0$$
$$[g(x,y)]^3+yf(x,y)-x=0$$

8. 设 (x_0,y_0,z_0,u_0) 满足方程组

$$f(x)+f(y)+f(z)=F(u)$$
$$g(x)+g(y)+g(z)=G(u)$$
$$h(x)+h(y)+h(z)=H(u)$$

这里所有的函数假定有连续的导数.

(1) 说出一个能在该点邻域内确定 x,y,z 为 u 的函数的充分条件；

(2) 在 $f(x)=x, g(x)=x^2, h(x)=x^3$ 的情形下，上述条件相当于什么？

解　(1)　设

$$\begin{cases}\overline{F}(x,y,z,u)=f(x)+f(y)+f(z)-F(u)=0\\ \overline{G}(x,y,z,u)=g(x)+g(y)+g(z)-G(u)=0\\ \overline{H}(x,y,z,u)=h(x)+h(y)+h(z)-H(u)=0\end{cases}$$

由已知条件

(Ⅰ) $\overline{F},\overline{G},\overline{H}$ 在 $\boldsymbol{R}^4$ 内连续；

(Ⅱ) $\overline{F},\overline{G},\overline{H}$ 在 $\boldsymbol{R}^4$ 内具有一阶连续偏导数；

(Ⅲ) $\overline{F}(x_0,y_0,z_0,u_0)=0, \overline{G}(x_0,y_0,z_0,u_0)=0, \overline{H}(x_0,y_0,z_0,u_0)=0$

故当

$$\left.\frac{\partial(\overline{F},\overline{G},\overline{H})}{\partial(x,y,z)}\right|_{P_0}=\begin{vmatrix} f'(x_0) & f'(y_0) & f'(z_0) \\ g'(x_0) & g'(y_0) & g'(z_0) \\ h'(x_0) & h'(y_0) & h'(z_0) \end{vmatrix}\neq 0$$

时原方程组能在 $P_0(x_0,y_0,z_0,u_0)$ 邻域内确定 x,y,z 作为 u 的函数.

(2) 在 $f(x)=x,g(x)=x^2,h(x)=x^3$ 的情况下,上述条件相当于

$$\begin{vmatrix} 1 & 1 & 1 \\ x_0 & y_0 & z_0 \\ x_0^2 & y_0^2 & z_0^2 \end{vmatrix}\neq 0$$

即 x_0,y_0,z_0 两两互异.

9. 求下列由方程所确定的隐函数的极值:

(1) $x^2+2xy+2y^2=1$

(2) $(x^2+y^2)^2=a^2(x^2-y^2),(a>0)$

解 (1) 令 $F(x,y)=x^2+2xy+2y^2-1$,则

$$F_x=2x+2y,F_y=2x+4y$$

令 $\dfrac{dy}{dx}=-\dfrac{2x+2y}{2x+4y}=0$,则有 $x=-y$,将 $x=-y$ 代入原方程得 $x^2-2x^2+x^2=1$,解此方程得 $x=\pm 1$. 于是该隐函数的稳定点为 ± 1,且 $y(1)=-1,y(-1)=1$.

又 $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=-\dfrac{1}{(2y+x)^2}\left(y+\dfrac{x^2+xy}{2y+x}\right)$.

从而 $\left.\dfrac{d^2y}{dx^2}\right|_{(1,-1)}=1>0,\left.\dfrac{d^2y}{dx^2}\right|_{(-1,1)}=-1<0$,

故当 $x=1$ 时有极小值 -1,$x=-1$ 时有极大值 1.

(2) 设 $F(x,y)=(x^2+y^2)^2-a^2(x^2-y^2)$,令

$$\frac{dy}{dx}=-\frac{4x(x^2+y^2)-2a^2x}{4y(x^2+y^2)+2a^2y}=0$$

解得 $x=0$ 或 $y^2=\dfrac{a^2}{2}-x^2$

以 $x=0$ 代入原方程,得 $y=0$,这时 $F_y=0$,故 $x=0$ 舍去.

再以 $y^2=\frac{a^2}{2}-x^2$ 代入原方程解得 $x=\pm\sqrt{\frac{3}{8}}a$，再将 $x=\pm\sqrt{\frac{3}{8}}a$ 代入 $y^2=\frac{a^2}{2}-x^2$，解得 $y=\pm\sqrt{\frac{1}{8}}a$. 故稳定点为

$$P_1\left(\sqrt{\frac{3}{8}}a,\sqrt{\frac{1}{8}}\right),P_2\left(\sqrt{\frac{3}{8}}a,-\sqrt{\frac{1}{8}}\right)$$

$$P_3\left(-\sqrt{\frac{3}{8}}a,\sqrt{\frac{1}{8}}\right),P_4\left(-\sqrt{\frac{3}{8}}a,-\sqrt{\frac{1}{8}}\right)$$

而 $\frac{d^2y}{dx^2}=-\frac{1}{[2y(x^2+y^2)+a^2y]^2}\{[2y(x^2+y^2)+a^2y](6x^2+2y^2+4xyy'-a^2)-[2x(x^2+y^2)-a^2x](4xy+2x^2y'+6y^2y'+a^2y')\}$

在稳定点 P_1,P_2,P_3,P_4 均有 $x^2+y^2=\frac{a^2}{2}$ 及 $y'=0$ 代入 $\frac{d^2y}{dx^2}$ 的表达式中，得

$\frac{d^2y}{dx^2}=-\frac{2x^2}{a^2y}$. 可见 $\frac{d^2y}{dx^2}$ 与 y 异号.

故 $\left.\frac{d^2y}{dx^2}\right|_{\left(\pm\sqrt{\frac{3}{8}}a,\sqrt{\frac{1}{8}}a\right)}<0,\quad \left.\frac{d^2y}{dx^2}\right|_{\left(\pm\sqrt{\frac{3}{8}}a,-\sqrt{\frac{1}{8}}a\right)}>0$

所以在点 P_1,P_3 取极大值 $\sqrt{\frac{1}{8}}a$，在点 P_2,P_4 取极小值 $-\sqrt{\frac{1}{8}}a$.

10. 设 $y=F(x)$ 和一组函数 $x=\varphi(u,v),y=\psi(u,v)$，那么由方程 $\psi(u,v)=F(\varphi(u,v))$ 可以确定函数 $v=v(u)$. 试用 $u,v,\frac{dv}{du},\frac{d^2v}{du^2}$ 表示 $\frac{dy}{dx},\frac{d^2y}{dx^2}$.

解　由 $x=\varphi(u,v(u))$ 和 $y=\psi(u,v(u))$ 得

$$\frac{dy}{dx}=\frac{\psi_u+\psi_v\frac{dv}{du}}{\varphi_u+\varphi_v\frac{dv}{du}}.\text{于是}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=$$

$$\frac{1}{\left(\varphi_u+\varphi_v\frac{dv}{du}\right)^3}\left\{\left[\psi_{uu}+\psi_{uv}\frac{dv}{du}+\left(\psi_{vu}+\psi_{vv}\frac{dv}{du}\right)\frac{dv}{du}+\psi_v\frac{d^2v}{d^2u}\right]\right.$$

$$\left(\varphi_u+\varphi_v\frac{dv}{du}\right)-\left(\psi_u+\psi_v\frac{dv}{du}\right)$$

$$\left.\left[\varphi_{uu}+\varphi_{uv}\frac{dv}{du}+\left(\varphi_{vu}+\varphi_{vv}\frac{dv}{du}\right)\frac{dv}{du}+\varphi_v\frac{d^2v}{d^2u}\right]\right\}$$

11. 试证明:二次型

$f(x,y,z)=Ax^2+By^2+Cz^2+2Dyz+2Ezx+2Fxy$ 在单位球面 $x^2+y^2+z^2=1$ 上的最大值和最小值恰好是矩阵

$$\Phi=\begin{pmatrix}A & F & E\\ F & B & D\\ E & D & C\end{pmatrix}$$

的最大特征值和最小特征值.

证　设 $L(x,y,z,\lambda)=f(x,y,z)-\lambda(x^2+y^2+z^2-1)$

$$令\begin{cases}L_x=2Ax+2Fy+2Ez-2\lambda x=0 & ①\\ L_y=2Fx+2By+2Dz-2\lambda y=0 & ②\\ L_z=2Ex+2Dy+2Cz-2\lambda z=0 & ③\\ L_l=x^2+y^2+z^2-1=0 & ④\end{cases}$$

①x + ②y + ③z 结合 ④ 式,得

$$f(x,y,z)=\lambda$$

由 ①,②,③ 知 λ 是对称矩阵

$$\Phi=\begin{pmatrix}A & F & E\\ F & B & D\\ E & D & C\end{pmatrix}的特征值.$$

又 f 在有界闭集 $\{f(x,y,z)\mid x^2+y^2+z^2=1\}$ 上连续,故最大值、最小值存在.所以最大值和最小值恰好是矩阵

$$\Phi = \begin{pmatrix} A & F & E \\ F & B & D \\ E & D & C \end{pmatrix}$$

的最大特征值和最小特征值.

12. 设 n 为正整数,$x, y > 0$,用条件极值方法证明:

$$\frac{x^n + y^n}{2} \geqslant \left(\frac{x+y}{2}\right)^n$$

证　先求 $F(x,y) = \dfrac{x^n + y^n}{2}$ 在条件 $x + y = a$ 下的最小值.

设 $L(x,y,\lambda) = \dfrac{x^n + y^n}{2} + \lambda(x + y - a)$

$$令\begin{cases} L_x = \dfrac{n}{2}x^{n-1} + \lambda = 0 \\ L_y = \dfrac{n}{2}y^{n-1} + \lambda = 0 \\ L_\lambda = x + y - a = 0 \end{cases} \quad 解得\ x = y = \frac{a}{2}$$

由于当 $x \to \infty$ 或 $y \to \infty$ 时,F 都趋于 ∞,故 F 必在唯一稳定点 $\left(\dfrac{a}{2}, \dfrac{a}{2}\right)$处有最小值,即 $F_{最小} = F\left(\dfrac{a}{2}, \dfrac{a}{2}\right) = \left(\dfrac{a}{2}\right)^n$,所以

$$\frac{x^n + y^n}{2} \geqslant \left(\frac{a}{2}\right)^n = \left(\frac{x+y}{2}\right)^n$$

故　$\dfrac{x^n + y^n}{2} \geqslant \left(\dfrac{x+y}{2}\right)^n$ 成立.

13. 求出椭球 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ 在第一卦限中的切平面与三个坐标面所成四面体的最小体积.

解　由几何学知,最小体积存在,椭球面上任一点(x, y, z)处的切平面方程为

$\dfrac{2x}{a^2}(X - x) + \dfrac{2y}{b^2}(Y - y) + \dfrac{2z}{c^2}(Z - z) = 0$,切平面在坐标轴上的截距分别为:$\dfrac{a^2}{x}, \dfrac{b^2}{y}, \dfrac{c^2}{z}$. 则椭球面在第一卦限部分上任一点处的切平

面与三个坐标面围成的四面体体积为 $v = \frac{a^2b^2c^2}{6xyz}$. 故本题是求函数 $v = \frac{a^2b^2c^2}{6xyz}$ 在条件

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (x > 0, y > 0, z > 0)$$

下的最小值.

设 $L(x,y,z,\lambda) = \frac{a^2b^2c^2}{6xyz} + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

$$\text{令}\begin{cases} L_x = -\frac{a^2b^2c^2}{6x^2yz} + \frac{2\lambda x}{a^2} = 0 \\ L_y = -\frac{a^2b^2c^2}{6xy^2z} + \frac{2\lambda y}{b^2} = 0 \\ L_z = -\frac{a^2b^2c^2}{6xyz^2} + \frac{2\lambda z}{c^2} = 0 \\ L_\lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \end{cases}$$

解得 $\quad x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$.

故 $v_{\text{极小}} = v\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2}abc$

14. 设 $P_0(x_0, y_0, z_0)$ 是曲面 $F(x,y,z) = 1$ 的非奇异点, F 在 $U(P_0)$ 可微, 且为 n 次齐次函数. 证明: 此曲面在 P_0 处的切平面方程为

$$xF_x(P_0) + yF_y(P_0) + zF_z(P_0) = n$$

证 由于 F 为 n 次齐次函数, 且 $F(x,y,z) = 1$. 故有

$$xF_x + yF_y + zF_z = nF = n \tag{1}$$

曲面在 P_0 处的切平面方程为

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$$

即

$$xF_x(P_0) + yF_y(P_0) + zF_z(P_0)$$

$$= x_0F_x(P_0) + y_0F_y(P_0) + z_0F_z(P_0) \tag{2}$$

而由(1)式知　$x_0F_x(P_0) + y_0F_y(P_0) + z_0F_z(P_0) = n$

故由(2)知曲面在 P_0 处的切平面方程为

$$xF_x(P_0) + yF_y(P_0) + zf_x(P_0) = n$$