2013-2014 春学期常微分试卷

1. $x\dfrac{dy}{dx}+(x+1)y=3x^2e^{-x}(x>0)$.

2. $2xy^3dx=(1-x^2y^2)dy$.

3. $(y+xy^2)dx+(x-x^2y)dy=0$.

4. $2y\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2=0, y(1)=1, \left.\dfrac{dy}{dx}\right|_{x=1}=2$.

5. $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=e^{2x}$.

6. $(1-x^2)\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-2y=0$.

7. $x^2\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+2y=\dfrac{1}{x+1}$.

8. 函数$f(x)$具有二阶导数且二阶导数连续，试求满足$\int_0^x(x-t)\cdot f(t)dt=cosx-f(x)$的函数$f(x)$.

9. $\begin{cases}\frac{dx}{dt}=y+z\\ \frac{dy}{dt}=x+z.\\ \frac{dz}{dt}=x+y\end{cases}$

2013-2014 春学期常微分试卷**参考**答案

1. $x\dfrac{dy}{dx}+(x+1)y=3x^2e^{-x}(x>0)$.

解：$\frac{dy}{dx}=-\left(1+\frac{1}{x}\right)y+3xe^{-x}$

$$y=e^{\int-\left(1+\frac{1}{x}\right)dx}\left(\int 3xe^{-x}e^{\int\left(1+\frac{1}{x}\right)dx}dx+c\right)$$

$$=\frac{1}{xe^x}\left(\int 3xe^{-x}xe^xdx+c\right)$$

$$=\frac{1}{xe^x}\left(\int 3x^2dx+c\right)$$

$$=\frac{1}{xe^x}\left(x^3+c\right)$$

$$=\frac{x^2}{e^x}+\frac{c}{xe^x}.$$

2. $2xy^3dx=(1-x^2y^2)dy$.

解：$2xy^3dx+(x^2y^2-1)dy=0$

$M=2xy^3, N=x^2y^2$

$$\varphi(y)=\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=-\frac{2}{y}$$

$$\mu=e^{\int\varphi(y)dy}=\frac{1}{y^2}$$

$$2xydx+\left(x^2-\frac{1}{y^2}\right)dy=0$$

$$x^2y+\frac{1}{y}=c$$

3. $(y+xy^2)dx+(x-x^2y)dy=0$.

解：

法一：$\frac{dy}{dx}=\frac{xy^2+y}{x^2y-x}=\frac{y+\frac{1}{x}}{x-\frac{1}{y}}=\frac{\frac{xy+1}{x}}{\frac{xy-1}{y}}=\frac{xy+1}{xy-1}\cdot\frac{y}{x}$

令$u=xy$, 则$\frac{dy}{dx}=\frac{d\left(\frac{u}{x}\right)}{dx}=\frac{1}{x}\cdot\frac{du}{dx}-\frac{1}{x^2}u=\frac{u+1}{u-1}\cdot\frac{u}{x^2}$

$\therefore\frac{du}{dx}=\frac{1}{x}\cdot\frac{2u^2}{u-1}\qquad\therefore\left(\frac{1}{u}-\frac{1}{u^2}\right)du=2\frac{dx}{x}$

$\therefore lnu+\frac{1}{u}=2lnx+c$

$\therefore ln\frac{y}{x}+\frac{1}{xy}=c$

法二：令$u=\frac{y}{x}$, 则$x\frac{du}{dx}+u=\frac{ux^2+1}{ux^2-1}\cdot u$

则$\frac{du}{dx}=\frac{2u}{ux^3-x}$ 则$\frac{dx}{du}=\frac{1}{2}(x^3-\frac{x}{u})$

$\therefore\frac{2}{x^3}\frac{dx}{du}=1-\frac{1}{u}\frac{1}{x^2}$

令$z=\frac{1}{x^2}$,则$\frac{dz}{du}=\frac{1}{u}z-1$

$\therefore z=e^{\int\frac{1}{u}du}\left(\int-e^{\int-\frac{1}{u}du}du+c\right)=u(-lnu+c)$

$\therefore\frac{1}{x^2}=\frac{y}{x}(-\ln\frac{y}{x}+c)$

$\therefore\frac{1}{xy}+ln\frac{y}{x}=c$

4. $2y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=0,y(1)=1,\left.\frac{dy}{dx}\right|_{x=1}=2.$

解：令$p=\frac{dy}{dx}$,则$\frac{dp}{dx}=\frac{dp}{dy}\cdot\frac{dy}{dx}=p\frac{dp}{dy}$

$\therefore 2yp\frac{dp}{dy}-p^2=0$ $\because y\neq0$

$\therefore\frac{dp}{dy}=\frac{1}{2y}p$ $\therefore p=ce^{\int\frac{1}{2y}dy}=c\sqrt{|y|}$

$\because y(1)=1,\left.\frac{dy}{dx}\right|_{x=1}=2$ $\therefore c=2$

$\therefore\frac{dy}{dx}=2\sqrt{|y|}$ $\therefore\sqrt{|y|}=x+c$

$\because y(1)=1$ $\therefore c=0$

$\therefore y=x^2$

5. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^{2x}.$

解：特征方程$\lambda^2-4\lambda+4=0$，得$\lambda=2$（二重根）

$\therefore Y=c_1e^{2x}+c_2xe^{2x}$

令$y^*=x^2Ae^{2x}$,代入得$A=\frac{1}{2}$

$\therefore y=Y+y^*=c_1e^{2x}+c_2xe^{2x}+\frac{1}{2}x^2e^{2x}$

6. $(1-x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y=0.$

解：可知该方程特解$y_1=x$

则有 $Liouville$公式，得通解$y=y_1\left(c_1+c_2\int\frac{1}{y_1^2}e^{\int-\frac{2x}{1-x^2}dx}dx\right)$

$$=x(c_1+c_2\int\frac{1}{x^2}(1-x^2)dx)$$

$$= x\left(c_1 + c_2\left(-\frac{1}{x} - x\right)\right)$$

$$= c_1 x - c_2(1 + x^2)$$

7. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = \dfrac{1}{x+1}$.

解：令$x = e^t$,则$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

$\therefore \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{1}{e^t+1}$

易知特征根$\lambda_1 = -1$，$\lambda_2 = -2$，$Y = c_1e^{-t} + c_2e^{-2t} = c_1y_1 + c_2y_2$

命$y = u_1(x)y_1 + u_2(x)y_2$ 则$y' = u_1'y_1 + u_2'y_2 + y_1'u_1 + y_2'u_2$

命$u_1'y_1 + u_2'y_2 = 0$ 则$y'' = u_1''y_1 + u_2''y_2 + u_1'y_1' + u_2'y_2'$

带入原方程得$u_1'y_1' + u_2'y_2' = \frac{1}{e^t+1}$ 联立$u_1'y_1 + u_2'y_2 = 0$

得$\begin{cases} u_1' = \frac{e^t}{e^t+1} \\ u_2' = -\frac{e^{2t}}{e^t+1} \end{cases}$ $\therefore \begin{cases} u_1 = ln(e^t+1) \\ u_2 = ln(e^t+1) - e^t \end{cases}$

$\therefore y = (u_1 + c_1)y_1 + (u_2 + c_2)y_2$

$= ln(e^t+1)(e^{-t} + e^{-2t}) + (c_1+1)e^{-t} + c_2e^{-2t}$

$= ln(x+1)\left(\frac{1}{x} + \frac{1}{x^2}\right) + \frac{a_1}{x} + \frac{a_2}{x^2}$

8. 函数$f(x)$具有二阶导数且二阶导数连续，试求满足$\int_0^x (x-t)\cdot f(t)dt = cosx - f(x)$的函数$f(x)$.

解：当$x = 0$ 时，$\int_0^0 (0-t)\cdot f(t)dt = cos0 - f(0)$ $\therefore f(0) = 1$

对原方程两边求导得$\int_0^x f(t)dt = -sinx - f'(x)$

再对两边求导得$f(x) = -cosx - f''(x)$

$\therefore f''(x) + f(x) = -cosx$

其次通解$F = c_1cosx + c_2sinx$

令$f^* = x(Acosx + Bsinx)$ 带入得$\begin{cases} A = 0 \\ B = -\frac{1}{2} \end{cases}$

$\therefore f(x) = F + f^* = c_1cosx + c_2sinx - \frac{1}{2}xsinx$

9. $\begin{cases} \frac{dx}{dt} = y + z \\ \frac{dy}{dt} = x + z. \\ \frac{dz}{dt} = x + y \end{cases}$

解：$D(\lambda)=\begin{vmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{vmatrix}=-\lambda^3+3\lambda+2=-(\lambda-2)(\lambda+1)^2$

$\therefore\lambda=2$(单根)，$\lambda=-1$(二重根)

当$\lambda=2$时,$(A-E\lambda)\cdot\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\begin{vmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{vmatrix}\cdot\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$

$\therefore\ \alpha:\beta:\gamma=\begin{vmatrix}1 & 1\\ -2 & 1\end{vmatrix}:\begin{vmatrix}1 & -2\\ 1 & 1\end{vmatrix}:\begin{vmatrix}1 & 1\\ -2 & 1\end{vmatrix}=3:3:3$

$\therefore\begin{pmatrix}x\\ y\\ z\end{pmatrix}_1=\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$

当$\lambda=-1$时，$(A-E\lambda)^2\cdot\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\begin{vmatrix}3 & 3 & 3\\ 3 & 3 & 3\\ 3 & 3 & 3\end{vmatrix}\cdot\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$

$\therefore v_0^{(1)}=\begin{pmatrix}1\\ 0\\ -1\end{pmatrix},v_0^{(2)}=\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$

$\therefore v_1^{(1)}=(A-E\lambda)\cdot v_0^{(1)}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},v_1^{(2)}=(A-E\lambda)\cdot v_0^{(2)}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$

$\therefore\begin{pmatrix}x\\ y\\ z\end{pmatrix}_2=\begin{pmatrix}1\\ 0\\ -1\end{pmatrix},\begin{pmatrix}x\\ y\\ z\end{pmatrix}_3=\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$

$\therefore$综上得：$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=c_1\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}e^{2t}+c_2\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}e^{-t}+c_3\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}e^{-t}$