ETH Zurich HS 2019
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 6

Exercise 6.1 Let $(X_i)_{i\geq 1}$ be i.i.d. with symmetric stable distribution of parameter $\alpha \in (0,2)$, see lecture notes p. 63.

(a) Find the distribution of $n^{-1/\alpha}(X_1 + \cdots + X_n)$.

(b) Does $\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$ converge in distribution?

Exercise 6.2 Let $\{X_j\}_{j=1,\dots,n}$, $n \geq 1$ be random variables and let us denote by ϕ_j the characteristic function of X_j. Prove that $\{X_j\}_{j=1,\dots,n}$ are independent if and only if for all $\xi_1, \dots, \xi_n \in \mathbb{R}$.

$$E\left[\exp\left\{i\sum_{j=1}^{n}\xi_j X_j\right\}\right] = \prod_{j=1}^{n}\phi_j(\xi_j).$$

Hint: For $d \geq 1$, and ν a probability measure on $\mathbb{R}^d$, one can define the characteristic function $\phi_\nu : \mathbb{R}^d \to \mathbb{R}$ of ν, as

$$\phi_\nu(\lambda) = \int_{\mathbb{R}^d} \exp(i\lambda \cdot x)\nu(dx),$$

where $\lambda \cdot x$ denotes the scalar product in $\mathbb{R}^d$, and then use (without proof) the following uniqueness property of characteristic functions of $\mathbb{R}^d$-valued random variables: if ν and μ are probability measures on $\mathbb{R}^d$ with the same characteristic function, then $\nu = \mu$, (cf. (2.3.13) the uniqueness property for one-dimensional random variables in the lecture notes).

Exercise 6.3 Let $X_1, X_2, \dots$ be independent random variables for which there exists a constant $M > 0$, such that $|X_n| \leq M$, P-a.s. for $n = 1, 2, \dots$. We write $S_n = X_1 + \ldots + X_n$. Show that, if $\sum \mathrm{Var}(X_n) = \infty$, then there exist constants a_n, b_n such that $(S_n - b_n)/a_n$ converges in distribution towards a standard normal random variable.

Exercise 6.4 (Optional.) Show that when Y_k, $k \geq 1$ are independent uniformly bounded random variables such that $\sum_k Y_k$ converges P-a.s., then $\sum_k \mathrm{Var}(Y_k) < \infty$.
Hint: consider independent copies $\tilde{Y}_k$, $k \geq 1$ of the Y_k, $k \geq 1$ and use Exercise 6.3 with $X_k = Y_k - \tilde{Y}_k$, $k \geq 1$.

Submission: until 14:15, Nov 5., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint

Solution 6.1 Let $S_n = \sum_{i=1}^n X_i$.

(a) Note that $\frac{1}{n^{1/\alpha}}(X_1 + \cdots + X_n) = n^{-1/\alpha} S_n$. Using that the random variables are i.i.d. and that the characteristic function is given by $\varphi_{X_1}(t) = \exp(-c|t|^\alpha)$ with $c > 0$,

$$\begin{aligned}\varphi_{\frac{S_n}{n^{1/\alpha}}}(t) &= \varphi_{S_n}(t/n^{1/\alpha}) = \prod_{i=1}^n \varphi_{X_i}(t/n^{1/\alpha}) = \varphi_{X_1}(t/n^{1/\alpha})^n \\ &= \left(e^{-c|t|^\alpha/n}\right)^n = e^{-c|t|^\alpha} = \varphi_{X_1}(t),\end{aligned}$$

showing that $\frac{1}{n^{1/\alpha}} S_n$ is distributed as X_1.

(b) Note that

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} = \frac{S_n}{n^{1/\alpha}} \frac{n^{1/\alpha}}{\sqrt{n}}.$$

By (a),

$$\varphi_{\frac{S_n}{n^{1/\alpha}}\frac{n^{1/\alpha}}{\sqrt{n}}}(t) = \varphi_{\frac{S_n}{n^{1/\alpha}}}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right) = \varphi_{X_1}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right).$$

Since $\alpha \in (0, 2)$,

$$\lim_{n\to\infty} \varphi_{\frac{S_n}{n^{1/\alpha}}}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right) = \lim_{n\to\infty} \varphi_{X_1}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right) = \lim_{n\to\infty} \exp(-c|n^{1/\alpha - 1/2} t|^\alpha) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{otherwise,} \end{cases}$$

which, since it is not continuous, is not the characteristic function of any distribution. Hence, by the contrapositive of (2.3.24) from the lecture notes,

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

does not converge in distribution.

Solution 6.2 Let $\mu_j := X_j \circ P$ be the distribution of X_j (cf. (1.2.15)). We consider the space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, and let $\nu = \times_{j=1}^n \mu_j$ be the product measure of μ_j on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, which is the unique probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that for $B_j \in \mathcal{B}(\mathbb{R})$,

$$\mu(B_1 \times \ldots \times B_n) = \prod_{j=1}^n \mu_j(B_j). \tag{1}$$

For more information about the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and product measure $\times_{j=1}^n \mu_j$ we refer you to Chapter 14 of the book "Probability theory, a comprehensive course" by A. Klenke (English version), in particular Theorems 14.8 and 14.14 therein.
Let $\mu = (X_1, \ldots, X_n) \circ P$ be the image measure of the random vector $(X_1, \ldots, X_n)$ (cf. (1.2.15) again), which is a distribution on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. From the equation (1) we can immediately see that $X_1, \ldots, X_n$ are independent if and only if $\mu = \nu = \times_{j=1}^n \mu_j$. Indeed, the independence condition $\prod_{j=1}^n P[X_j \in B_j] = P[X_1 \in B_1, \ldots, X_n \in B_n] = P[(X_1, \ldots, X_n) \in B_1 \times \ldots \times B_n] = \mu(B_1 \times \ldots \times B_n)$ means exactly that $\mu(B_1 \times \ldots \times B_n) = \prod_{j=1}^n \mu_j(B_j)$ for all $B_j \in \mathcal{B}(\mathbb{R})$, $j = 1, \ldots, n$. Since $\nu = \times_{j=1}^n \mu_j$ is the unique probability measure which

satisfies this property (as we have mentioned above) we must have $\mu = \nu$.
Now we can use Fubini's theorem (see e.g. Theorem 14.16 in the book "Probability theory, a comprehensive course") to obtain that ν has the characteristic function

$$\begin{aligned}\varphi_\nu(\xi_1, \xi_2, \dots, \xi_n) &= \int_{\mathbb{R}^n} e^{i\xi_1 x_1 + \dots + i\xi_n x_n} \mu_1(\mathrm{d}x_1) \dots \mu_n(\mathrm{d}x_n) \\ &= \prod_{j=1}^n \int e^{i\xi_j x_j} \mu_j(\mathrm{d}x_j) = \prod_{j=1}^n E\left[e^{i\xi_j X_j}\right].\end{aligned}$$

Hence, we have proven that:

$$X_1, \dots, X_n \text{ are independent } \iff \mu = \nu \iff \varphi_\mu = \varphi_\nu,$$

where the last equivalence follows from the fact that characteristic functions also uniquely determine distributions on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, see the hints given after Exercise 6.2. Now our claim follows by noting that the characteristic function of the vector $(X_1, \dots, X_n)$ is given by

$$\varphi_\mu(\xi_1, \dots, \xi_n) = E\left[\exp\left\{i \sum_{j=1}^n \xi_j X_j\right\}\right],$$

and (as we have shown) that

$$\varphi_\nu(\xi_1, \xi_2, \dots, \xi_n) = \prod_{j=1}^n E\left[e^{i\xi_j X_j}\right].$$

Solution 6.3 We use the Lindeberg-Feller theorem (Theorem 2.24, p. 71 in lecture notes). We define

$$Y_{n,i} = \frac{X_i - E[X_i]}{\sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}}, \quad i = 1, \dots, n$$

(For the finitely many n where possibly $\sum_{j=1}^n \operatorname{Var}(X_j) = 0$, we set $Y_{n,i} \equiv 0$). Then it follows that

$$\sum_{i=1}^n E[Y_{n,i}^2] \xrightarrow{n\to\infty} 1.$$

More precisely, except for the finitely many n mentioned above,

$$\sum_{i=1}^n E[Y_{n,i}^2] = \sum_{i=1}^n \frac{E[(X_i - E[X_i])^2]}{\sum_{j=1}^n \operatorname{Var}(X_j)} = \frac{\sum_{i=1}^n \operatorname{Var}(X_i)}{\sum_{j=1}^n \operatorname{Var}(X_j)} = 1,$$

which justifies the first condition.

We now verify the second condition. For $\epsilon > 0$ we take $n_0 \in \mathbb{N}$ such that

$$\sum_{j=1}^n \operatorname{Var}(X_j) \geq \frac{(2M)^2}{\epsilon^2}, \quad \forall n \geq n_0,$$

which exists since $\sum \operatorname{Var}(X_j) = \infty$. Then by the fact that $|X_i|$ are uniformly bounded by M, one has

$$|Y_{n,i}| = \left| \frac{X_i - E[X_i]}{\sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}} \right| \le \frac{2M}{2M/\epsilon} \le \epsilon$$

for $n \ge n_0$. Hence,

$$1_{\{|Y_{n,i}|>\epsilon\}} \equiv 0, \ \forall n \ge n_0, \forall i \le n.$$

In fact,

$$\lim_{n\to\infty} \sum_{i=1}^n E\left[Y_{n,i}^2 1_{\{|Y_{n,i}|>\epsilon\}}\right] = 0.$$

Therefore all the conditions are fulfilled, whence

$$\sum_{i=1}^n Y_{n,i} \xrightarrow{n\to\infty} \mathcal{N}(0,1) \text{ in distribution.}$$

On the other hand we can rewrite $\sum_{i=1}^n Y_{n,i}$ as

$$\sum_{i=1}^n Y_{n,i} = \frac{\sum_{i=1}^n (X_i - E[X_i])}{\sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}},$$

and the claim follows with

$$a_n := \sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}, \qquad b_n := E\left[\sum_{j=1}^n X_j\right].$$

Solution 6.4 Let $\tilde{Y}_k$, $k \ge 1$ be independent copies of Y_k, $k \ge 1$ and set $X_k := Y_k - \tilde{Y}_k$ as in the hint. Then X_k, $k \ge 1$ are independent uniformly bounded variables. Suppose to the contrary that $\sum_k Y_k$ converges P-a.s. but $\sum_k \operatorname{Var}(Y_k) = \infty$. Then also $S_n := X_1 + \cdots + X_n$ converges P-a.s. as a sum of two P-a.s. convergent sequences $\sum_k Y_k$ and $\sum_k \tilde{Y}_k$, and $\sum_k \operatorname{Var}(X_k) = 2\sum_k \operatorname{Var}(Y_k) = \infty$. Define

$$a_n := \sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}, \qquad b_n := E\left[\sum_{j=1}^n X_j\right] = 0.$$

Then it follows as in the solution of Exercise 6.3 that $(S_n - b_n)/a_n = S_n/a_n$ converges in distribution towards a standard normal random variable. Since S_n is P-a.s. convergent, for each $\epsilon > 0$ there is a $N \in \mathbb{N}$ and a $M > 0$ such that for all $n \ge N$, $P[|S_n| \ge M] < \epsilon$. Since the sequence $(a_n)_n$ is monotone increasing towards infinity, we can find a $\tilde{N}$ such that for all $n \ge \tilde{N}$, $a_n \ge M$. Then for all $n \ge N \vee \tilde{N}$, $P[|S_n/a_n| \ge 1] \le P[|S_n/a_n| \ge M/a_n] < \epsilon$. Since this can be done for any $\epsilon > 0$, S_n/a_n can not converge in distribution to a standard normal random variable, which is a contradiction.