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**Abstract Algebra**  
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Derek J. S. Robinson

# Abstract Algebra

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An Introduction with Applications

2nd Edition

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In Memory of My Parents



# Preface

The origins of algebra are usually traced back to Muhammad ben Musa al-Khwarizmi, who worked at the court of the Caliph al-Ma'mun in Baghdad in the early 9th Century. The word derives from the Arabic al-jabr, which refers to the process of adding the same quantity to both sides of an equation. The work of Arabic scholars was known in Italy by the 13th Century and a lively school of algebraists arose there. Much of their interest was centered on the solution of polynomial equations. This preoccupation of mathematicians lasted until the beginning of the 19th Century, when the possibility of solving the general equation of the fifth degree in terms of radicals was finally disproved by Ruffini and Abel.

This early work led to the introduction of some of the main structures of abstract algebra, groups, rings and fields. These structures have been studied intensively over the past two hundred years. For an interesting historical account of the origins of algebra the reader may consult the book by van der Waerden [17].

Until quite recently algebra was very much the domain of the pure mathematician, and applications were few and far between. But the situation has changed, in part as a result of the rise of information technology, where the precision and power inherent in the language and concepts of algebra have proved to be invaluable. Today many specialists in computer science and engineering, as well as physics and chemistry, routinely take courses in abstract algebra. The present work represents an attempt to meet the needs of both mathematicians and scientists who seek to acquire a knowledge of algebra and its applications.

As to what is expected of the reader, a basic knowledge of matrix algebra is assumed and also at least the level of mathematical maturity consistent with completion of three semesters of calculus. The objective is to introduce the reader to the principal structures of abstract algebra and to give an account of some of its more convincing applications. In particular there are sections on solution of equations by radicals, ruler and compass constructions, Polya counting theory, Steiner systems, orthogonal latin squares and error correcting codes. The book should be suitable for students in the final year of undergraduate or first year of (post)graduate studies at a university in North America or the United Kingdom.

The principal change to the book from the first edition is the addition of two new chapters. The first of these is an introduction to the theory of modules, a topic that combines the concepts of group and ring. Enough of the theory is developed to establish the structure theorem for finitely generated modules over principal ideal domains. Then applications to matrices and linear operators are presented. The second new chapter gives an introduction to tensor products, an essential tool in many advanced parts of algebra. Also Hilbert's Basis Theorem is proved and a more detailed account of Hall's theory of finite solvable groups is given. The original chapter on vector spaces has been modified by substituting an account of the theory of eigenvalues and eigenvectors of

linear operators for the section on orthogonality. Some of these changes have inevitably had the effect of raising the level of abstraction in parts of the book. However, the original aim of making abstract algebra accessible to as many readers as possible is maintained in this edition.

Naturally the opportunity has been taken to correct errors and obscurities in the first edition. I am grateful to those readers who took the time and trouble to send in lists of corrections, and here particular thanks are due to Adolfo Ballester-Bolinches and Dieter Kilsch. Of course, as usual, full credit for all errors belongs to the author.

There is more than enough material here for a two semester course in abstract algebra. If just one semester is available, Chapters One through Eight and Chapter Eleven could be covered. The first two chapters contain topics that will be familiar to many readers and can be covered more quickly. In addition, a good deal of the material in Chapter Eight will not be new to a reader who has taken a first course in linear algebra. A word about proofs is in order. Sometimes students from outside mathematics question the need for mastering the art of rigorous proof, although this is perhaps becoming less common. One response is that the only way to be sure that a statement is correct, or that a computer program will always deliver the correct answer, is to prove it. As a rule complete proofs are given and they should be read. The first two chapters, which contain much elementary material, are a good place for the reader to develop and polish theorem proving skills. Each section of the book is followed by a selection of problems of varying degrees of difficulty.

The second edition of this book, like the first, is based on courses given over many years at the University of Illinois at Urbana-Champaign, the National University of Singapore and the University of London. I am grateful to my colleagues for good advice and many stimulating conversations: these have led to numerous improvements in the text. My thanks are due to Otto Kegel and Manfred Karbe for assistance with the first edition. In preparing this second edition I have been aided by Leonardo Milla and Friederike Dittberner at Walter de Gruyter, whose advice and assistance have greatly helped. Finally, I thank my family for their patience and encouragement, which are essential in a project such as this.

Derek Robinson

Urbana, Illinois,  
November 2014



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# 1 Sets, relations and functions

The concepts introduced in this chapter are truly fundamental and underlie almost every branch of mathematics. Most of the material is quite elementary and will be familiar to many readers. Nevertheless readers are encouraged to review the material and to check notation and definitions. Because of its nature the pace of this chapter is somewhat faster than in subsequent chapters.

## 1.1 Sets and subsets

By a *set* we shall mean any well-defined collection of objects, which are called the *elements* of the set. Some care must be exercised in using the term “set” because of Bertrand Russell’s famous paradox, which shows that not every collection can be regarded as a set. Russell considered the collection  $C$  of all sets which are not elements of themselves. If  $C$  is allowed to be a set, a contradiction arises when one inquires whether or not  $C$  is an element of itself. Now plainly there is something suspicious about the idea of a set being an element of itself and we shall take this as evidence that the qualification “well-defined” needs to be taken seriously. A collection that is not a set is called a *proper class*.

Sets will be denoted by capital letters and their elements by lower case letters. The standard notation

$$a \in A$$

means that  $a$  is a element of the set  $A$ , or  $a$  *belongs* to  $A$ . The negation of  $a \in A$  is denoted by  $a \notin A$ . Sets can be defined either by writing their elements out between braces, as in  $\{a, b, c, d\}$ , or alternatively by giving a formal description of the elements, the general format being

$$A = \{a \mid a \text{ has property } P\},$$

i.e.,  $A$  is the set of all objects with the property  $P$ . If  $A$  is a finite set, the number of its elements is written

$$|A|.$$

**Subsets.** Let  $A$  and  $B$  be sets. If every element of  $A$  is an element of  $B$ , we write

$$A \subseteq B$$

and say that  $A$  is a *subset* of  $B$ , or that  $A$  is *contained* in  $B$ . If  $A \subseteq B$  and  $B \subseteq A$ , so that  $A$  and  $B$  have exactly the same elements, then  $A$  and  $B$  are said to be *equal*,

$$A = B.$$

The negation of this is  $A \neq B$ . The notation  $A \subset B$  is used if  $A \subseteq B$  and  $A \neq B$ ; then  $A$  is called a *proper subset* of  $B$ .

**Some special sets.** A set with no elements at all is called an *empty set*. An empty set  $E$  is a subset of any set  $A$ ; for if this were false, there would be an element of  $E$  that is not in  $A$ , which is certainly wrong. As a consequence, *there is exactly one empty set*: for if  $E$  and  $E'$  are two empty sets, then  $E \subseteq E'$  and  $E' \subseteq E$ , so that  $E = E'$ . The unique empty set is written

$$\emptyset.$$

Some further standard sets with reserved notations are

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C},$$

which are respectively the sets of natural numbers  $0, 1, 2, \dots$ , integers, rational numbers, real numbers and complex numbers.

**Set operations.** Next we recall the familiar set operations of union, intersection and complement. Let  $A$  and  $B$  be sets. The *union*  $A \cup B$  is the set of all objects which belong to  $A$  or  $B$ , or possibly to both; the *intersection*  $A \cap B$  consists of all objects that belong to both  $A$  and  $B$ . Thus

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\},$$

while

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

It should be clear how to define the union and intersection of an arbitrary collection of sets  $\{A_\lambda \mid \lambda \in \Lambda\}$ ; these are written

$$\bigcup_{\lambda \in \Lambda} A_\lambda \quad \text{and} \quad \bigcap_{\lambda \in \Lambda} A_\lambda,$$

respectively. The *relative complement* of  $B$  in  $A$  is

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Frequently one has to deal only with subsets of some fixed set  $U$ , called the *universal set*. If  $A \subseteq U$ , then the *complement* of  $A$  in  $U$  is

$$\bar{A} = U - A.$$

We list for future reference the fundamental properties of unions, intersections and complements: most of these should be familiar.

**(1.1.1)** Let  $A, B, C, B_\lambda$  ( $\lambda \in \Lambda$ ) be sets. Then the following statements are valid:

- (i)  $A \cup B = B \cup A$  and  $A \cap B = B \cap A$ , (*commutative laws*).
- (ii)  $(A \cup B) \cup C = A \cup (B \cup C)$  and  $(A \cap B) \cap C = A \cap (B \cap C)$ , (*associative laws*).
- (iii)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  and  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ , (*distributive laws*).
- (iv)  $A \cup A = A = A \cap A$ .

(v)  $A \cup \emptyset = A, A \cap \emptyset = \emptyset.$

(vi)  $A - (\bigcup_{\lambda \in \Lambda} B_\lambda) = \bigcap_{\lambda \in \Lambda} (A - B_\lambda)$  and  $A - (\bigcap_{\lambda \in \Lambda} B_\lambda) = \bigcup_{\lambda \in \Lambda} (A - B_\lambda),$  (*De Morgan's Laws*).<sup>1</sup>

The easy proofs of these results are left to the reader as an exercise.

**Set products.** Let  $A_1, A_2, \dots, A_n$  be sets. By an  $n$ -tuple of elements from  $A_1, A_2, \dots, A_n$  is to be understood a sequence of elements  $a_1, a_2, \dots, a_n$  with  $a_i \in A_i$ . The  $n$ -tuple is usually written  $(a_1, a_2, \dots, a_n)$  and the set of all  $n$ -tuples is denoted by

$$A_1 \times A_2 \times \cdots \times A_n.$$

This is the *set product* (or *cartesian product*) of  $A_1, A_2, \dots, A_n$ . For example  $\mathbb{R} \times \mathbb{R}$  is the set of coordinates of points in the plane.

The following result is a basic counting tool.

**(1.1.2)** *If  $A_1, A_2, \dots, A_n$  are finite sets, then*

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|.$$

*Proof.* In forming an  $n$ -tuple  $(a_1, a_2, \dots, a_n)$  we have  $|A_1|$  choices for  $a_1, |A_2|$  choices for  $a_2, \dots, |A_n|$  choices for  $a_n$ . Each choice of an  $a_i$  yields a different  $n$ -tuple. Therefore the total number of  $n$ -tuples is  $|A_1| \cdot |A_2| \cdots |A_n|.$   $\square$

**The power set.** The *power set* of a set  $A$  is the set of all subsets of  $A$ , including the empty set and  $A$  itself; it is denoted by

$$\mathcal{P}(A).$$

The power set of a finite set is always a larger set, as the next result shows.

**(1.1.3)** *If  $A$  is a finite set, then  $|\mathcal{P}(A)| = 2^{|A|}.$*

*Proof.* Let  $A = \{a_1, a_2, \dots, a_n\}$  with distinct  $a_i$ 's. Also put  $I = \{0, 1\}$ . Each subset  $B$  of  $A$  is to correspond to an  $n$ -tuple  $(i_1, i_2, \dots, i_n)$  with  $i_j \in I$ . Here the rule for forming the  $n$ -tuple corresponding to  $B$  is this:  $i_j = 1$  if  $a_j \in B$  and  $i_j = 0$  if  $a_j \notin B$ . Conversely, every  $n$ -tuple  $(i_1, i_2, \dots, i_n)$  with  $i_j \in I$  determines a subset  $B$  of  $A$ , defined by  $B = \{a_j \mid 1 \leq j \leq n, i_j = 1\}$ . It follows that the number of subsets of  $A$  equals the number of elements in  $I \times I \times \cdots \times I$ , where the number of factors is  $n$ . By (1.1.2) we obtain  $|\mathcal{P}(A)| = 2^n = 2^{|A|}.$   $\square$

<sup>1</sup> Augustus De Morgan (1806–1871)

The power set  $\mathcal{P}(A)$ , together with the operations  $\cup$  and  $\cap$ , constitutes what is known as a *Boolean<sup>2</sup> algebra*; such algebras have become very important in logic and computer science.

### Exercises (1.1)

- (1) Prove as many parts of (1.1.1) as possible.
- (2) Let  $A, B, C$  be sets such that  $A \cap B = A \cap C$  and  $A \cup B = A \cup C$ . Prove that  $B = C$ .
- (3) If  $A, B, C$  are sets, establish the following:
  - (i)  $(A - B) - C = A - (B \cup C)$ .
  - (ii)  $A - (B - C) = (A - B) \cup (A \cap B \cap C)$ .
- (4) Let  $A$  and  $B$  be finite sets. Prove that  $|\mathcal{P}(A \times B)| = |\mathcal{P}(A)|^{|B|}$ .
- (5) Let  $A$  and  $B$  be finite sets with more than one element in each. Prove that  $|\mathcal{P}(A \times B)|$  is larger than both  $|\mathcal{P}(A)|$  and  $|\mathcal{P}(B)|$ .
- (6) The *disjoint union*  $A \oplus B$  of sets  $A$  and  $B$  is defined by the rule  $A \oplus B = A \cup B - A \cap B$ , so its elements are those that belong to exactly one of  $A$  and  $B$ . Prove the following statements:
  - (i)  $A \oplus A = \emptyset, A \oplus B = B \oplus A$ .
  - (ii)  $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ .
  - (iii)  $(A \oplus B) \cap C = (A \cap C) \oplus (B \cap C)$ .
- (7) If  $A$  and  $B$  be finite sets, show that  $|\mathcal{P}(A \cup B)| = \frac{|\mathcal{P}(A)| \cdot |\mathcal{P}(B)|}{|\mathcal{P}(A \cap B)|}$ .

## 1.2 Relations, equivalence relations, partial orders

In mathematics it is often not sufficient to deal with the individual elements of a set: for it may be critical to understand how elements of the set are related to each other. This leads us to formulate the concept of a relation.

Let  $A$  and  $B$  be sets. Then a *relation*  $R$  between  $A$  and  $B$  is a subset of the set product  $A \times B$ . The definition is clarified by use of a suggestive notation: if  $(a, b) \in R$ , then  $a$  is said to be *related to*  $b$  by  $R$  and we write

$$a R b.$$

The most important case is of a relation  $R$  between  $A$  and itself; this is called a *relation on the set*  $A$ .

### Example (1.2.1)

- (i) Let  $A$  be a set and define  $R = \{(a, a) \mid a \in A\}$ . Thus  $a_1 R a_2$  means that  $a_1 = a_2$  and  $R$  is the relation of equality on  $A$ .

---

<sup>2</sup> George Boole (1815–1864)

- (ii) Let  $P$  be the set of all points and  $L$  the set of all lines in the plane. A relation  $R$  from  $P$  to  $L$  is defined by:  $p R \ell$  if the point  $p$  lies on the line  $\ell$ .
- (iii) A relation  $R$  on the set of integers  $\mathbb{Z}$  is defined by:  $a R b$  if  $a - b$  is even.

The next result confirms what one might suspect, that a finite set has many relations.

**(1.2.1)** *If  $A$  is a finite set, the number of relations on  $A$  equals  $2^{|A|^2}$ .*

For this is the number of subsets of  $A \times A$  by (1.1.2) and (1.1.3).

The concept of a relation on a set is evidently a very broad one. In practice the relations of greatest interest are those which have special properties. The most common of these are listed next. Let  $R$  be a relation on a set  $A$ .

- (i)  $R$  is *reflexive* if  $a R a$  for all  $a \in A$ .
- (ii)  $R$  is *symmetric* if  $a R b$  always implies that  $b R a$ .
- (iii)  $R$  is *antisymmetric* if  $a R b$  and  $b R a$  imply that  $a = b$ ;
- (iv)  $R$  is *transitive* if  $a R b$  and  $b R c$  imply that  $a R c$ .

Relations which are reflexive, symmetric and transitive are called *equivalence relations*; they are of fundamental importance. Relations which are reflexive, antisymmetric and transitive are also important; they are called *partial orders*. Here are some examples of relations of various types.

**Example (1.2.2)**

- (i) Equality on a set is both an equivalence relation and a partial order.
- (ii) A relation  $R$  on  $\mathbb{Z}$  is defined by:  $a R b$  if and only if  $a - b$  is even. This is an equivalence relation, but it is not a partial order.
- (iii) If  $A$  is any set, the relation of containment  $\subseteq$  is a partial order on the power set  $P(A)$ .
- (iv) A relation  $R$  on  $\mathbb{N}$  is defined by  $a R b$  if  $a$  divides  $b$ . Here  $R$  is a partial order on  $\mathbb{N}$ .

**Equivalence relations and partitions.** The structure of an equivalence relation on a set will now be analyzed. The essential conclusion will be that an equivalence relation causes the set to split up into non-overlapping non-empty subsets.

Let  $E$  be an equivalence relation on a set  $A$ . First of all define the *E-equivalence class* of an element  $a$  of  $A$  to be the subset

$$[a]_E = \{x \mid x \in A \text{ and } x E a\}.$$

By the reflexive law  $a \in [a]_E$ , so

$$A = \bigcup_{a \in A} [a]_E$$

and  $A$  is the union of all the equivalence classes.

Next suppose that the equivalence classes  $[a]_E$  and  $[b]_E$  both contain an element  $x$ . Assume that  $y \in [a]_E$ ; then  $y E a$ ,  $a E x$  and  $x E b$ , by the symmetric law. Hence  $y E b$  by two applications of the transitive law. Therefore  $y \in [b]_E$  and we have proved that

$[a]_E \subseteq [b]_E$ . By the same reasoning  $[b]_E \subseteq [a]_E$ , so that  $[a]_E = [b]_E$ . It follows that distinct equivalence classes are disjoint, i.e., they have no elements in common.

What has been shown so far is that the set  $A$  is the union of the  $E$ -equivalence classes and that distinct equivalence classes are disjoint. A decomposition of  $A$  into disjoint non-empty subsets is called a *partition* of  $A$ . Thus  $E$  determines a partition of  $A$ .

Conversely, suppose that a partition of  $A$  into non-empty disjoint subsets  $A_\lambda$ , ( $\lambda \in \Lambda$ ), is given. We would like to construct an equivalence relation on  $A$  corresponding to the partition. Now each element of  $A$  belongs to a unique subset  $A_\lambda$ ; thus we may define  $a E b$  to mean that  $a$  and  $b$  belong to the same subset  $A_\lambda$ . It follows immediately from the definition that the relation  $E$  is an equivalence relation; what is more, the equivalence classes are just the subsets  $A_\lambda$  of the original partition. We summarize these conclusions in:

### (1.2.2)

- (i) *If  $E$  is an equivalence relation on a set  $A$ , the  $E$ -equivalence classes form a partition of  $A$ .*
- (ii) *Conversely, each partition of  $A$  determines an equivalence relation on  $A$  for which the equivalence classes are the subsets in the partition.*

Thus the concepts of equivalence relation and partition are in essence the same. In the equivalence relation (ii) above there are two equivalence classes, the sets of even and odd integers; of course these form a partition of  $\mathbb{Z}$ .

**Partial orders.** Suppose that  $R$  is a partial order on a set  $A$ , i.e.,  $R$  is a reflexive, anti-symmetric, transitive relation on  $A$ . Instead of writing  $a R b$  it is customary to employ a more suggestive symbol and write

$$a \preceq b.$$

The pair  $(A, \preceq)$  then constitutes a *partially ordered set* (or *poset*).

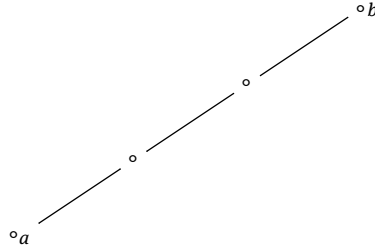
The effect of a partial order is to impose a hierarchy on the set  $A$ . When the set is finite, this can be visualized by drawing a picture of the poset called a *Hasse<sup>3</sup> diagram*. It consists of vertices and edges drawn in the plane, the vertices representing the elements of  $A$ . A sequence of upwardly sloping edges from  $a$  to  $b$ , as in the diagram below, indicates that  $a \preceq b$ . Elements  $a, b$  not connected by such a sequence of edges do not satisfy  $a \preceq b$  or  $b \preceq a$ . In order to simplify the diagram as far as possible, it is

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<sup>3</sup> Helmut Hasse (1898–1979)



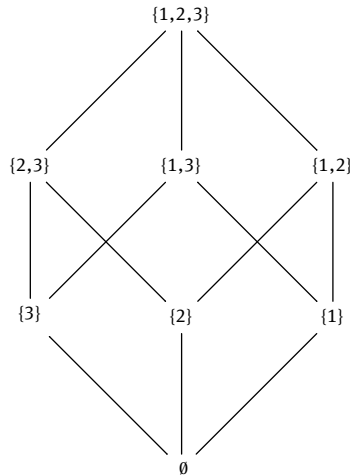
agreed that unnecessary edges are to be omitted.



A very familiar poset is the power set of a set  $A$  with the partial order  $\subseteq$ , i. e.  $(\mathcal{P}(A), \subseteq)$ .

**Example (1.2.3)** Draw the Hasse diagram of the poset  $(\mathcal{P}(A), \subseteq)$  where  $A = \{1, 2, 3\}$ .

This poset has  $2^3 = 8$  vertices, which can be visualized as the vertices of a cube standing on one corner.

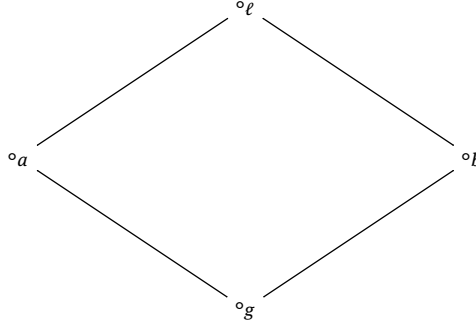


One reason why partially ordered sets are important in algebra is that they provide a useful representation of substructures of standard algebraic structures, for example subsets, subgroups, subrings etc..

A partial order  $\leq$  on a set  $A$  is called a *linear order* if, given  $a, b \in A$ , either  $a \leq b$  or  $b \leq a$  holds. Then  $(A, \leq)$  is called a *linearly ordered set* or *chain*. The Hasse diagram of a chain is a single sequence of edges sloping upwards. Obvious examples of chains are  $(\mathbb{Z}, \leq)$  and  $(\mathbb{R}, \leq)$  where  $\leq$  is the usual “less than or equal to”. Finally, a linear order on  $A$  is called a *well order* if each non-empty subset  $X$  of  $A$  contains a *least* element  $a$ , i.e., such that  $a \leq x$  for all elements  $x \in X$ . While it might seem obvious that  $\leq$  is a well order on the set of all positive integers, this actually an axiom, the Well-Ordering Law, which is discussed in Section (2.1).

**Lattices.** Consider a poset  $(A, \leq)$ . If  $a, b \in A$ , a *least upper bound* (or lub) of  $a$  and  $b$  is an element  $\ell \in A$  such that  $a \leq \ell$  and  $b \leq \ell$ , and if  $a \leq x$  and  $b \leq x$ , with  $x$  in  $A$ , then

$\ell \leq x$ . Similarly a *greatest lower bound* (or glb) of  $a$  and  $b$  is an element  $g \in A$  such that  $g \leq a$  and  $g \leq b$ , while  $x \leq a$  and  $x \leq b$  imply that  $x \leq g$ . If  $\ell$  and  $g$  exist, the Hasse diagram of  $(A, \leq)$  will contain the subdiagram



A poset in which each pair of elements has an lub and a glb is called a *lattice*. For example,  $(\mathcal{P}(S), \subseteq)$  is a lattice since the lub and glb of  $A$  and  $B$  are just  $A \cup B$  and  $A \cap B$  respectively.

**The composite of relations.** Since a relation is a subset, two relations may be combined by forming their union or intersection. However, there is a more useful way of combining relations called *composition*: let  $R$  and  $S$  be relations between  $A$  and  $B$  and between  $B$  and  $C$  respectively. Then the *composite relation*

$$S \circ R$$

is the relation between  $A$  and  $C$  defined by:  $a (S \circ R) c$  if and only if there exists  $b \in B$  such that  $a R b$  and  $b S c$ .

For example, assume that  $A = \mathbb{Z}$ ,  $B = \{a, b, c\}$ ,  $C = \{\alpha, \beta, \gamma\}$ . Define relations  $R = \{(1, a), (2, b), (4, c)\}$ ,  $S = \{(a, \alpha), (b, \gamma), (c, \beta)\}$ . Then  $S \circ R = \{(1, \alpha), (2, \gamma), (4, \beta)\}$ .

In particular one can form the composite of any two relations  $R$  and  $S$  on a set  $A$ . Notice that the condition for a relation  $R$  to be transitive can now be expressed in the form  $R \circ R \subseteq R$ .

A result of fundamental importance is the associative law for composition of relations.

**(1.2.3)** Let  $R, S, T$  be relations between  $A$  and  $B$ ,  $B$  and  $C$ , and  $C$  and  $D$  respectively. Then  $T \circ (S \circ R) = (T \circ S) \circ R$ .

*Proof.* Let  $a \in A$  and  $d \in D$ . Then  $a (T \circ (S \circ R)) d$  means that there exists  $c \in C$  such that  $a (S \circ R) c$  and  $c T d$ , i.e., there exists  $b \in B$  such that  $a R b$ ,  $b S c$  and  $c T d$ . Therefore  $b (T \circ S) d$  and  $a ((T \circ S) \circ R) d$ . Thus  $T \circ (S \circ R) \subseteq (T \circ S) \circ R$ , and in a similar way  $(T \circ S) \circ R \subseteq T \circ (S \circ R)$ .  $\square$

### Exercises (1.2)

- (1) Determine whether the binary relations  $R$  defined on the set  $A$  below are reflexive, symmetric, antisymmetric or transitive.
  - (i)  $A = \mathbb{R}$  and  $a R b$  means  $a^2 = b^2$ .
  - (ii)  $A = \mathbb{R}$  and  $a R b$  means  $a - b \leq 2$ .
  - (iii)  $A = \mathbb{Z} \times \mathbb{Z}$  and  $(a, b) R (c, d)$  means  $a + d = b + c$ .
  - (iv)  $A = \mathbb{Z}$  and  $a R b$  means that  $b = a + 3c$  for some integer  $c$ .
- (2) A relation  $\sim$  on  $\mathbb{R} - \{0\}$  is defined by  $a \sim b$  if  $ab > 0$ . Show that  $\sim$  is an equivalence relation and identify the equivalence classes.
- (3) Let  $A = \{1, 2, \dots, n\}$  where  $n$  is a positive integer. Define  $a \leq b$  to mean that  $a$  divides  $b$ . Show that  $(A, \leq)$  is a poset. Draw the Hasse diagram for the case  $n = 12$ .
- (4) Let  $(A, \leq)$  be a poset and let  $a, b \in A$ . Show that  $a$  and  $b$  have at most one lub and at most one glb.
- (5) Given linearly ordered sets  $(A_i, \leq_i)$ ,  $i = 1, 2, \dots, k$ , suggest a way to make  $A_1 \times A_2 \times \dots \times A_k$  into a linearly ordered set.
- (6) How many equivalence relations are there on sets with 1, 2, 3 or 4 elements?
- (7) Suppose that  $A$  is a set with  $n$  elements. Show that there are exactly  $2^{n^2-n}$  reflexive relations on  $A$  and  $2^{n(n+1)/2}$  symmetric ones.
- (8) Let  $R$  be a relation on a set  $A$ . Define *powers* of  $R$  recursively by  $R^1 = R$  and  $R^{n+1} = R^n \circ R$  for  $n = 1, 2, \dots$ .
  - (i) If  $R$  is transitive, show that  $\dots R^n \subseteq R^{n-1} \subseteq \dots \subseteq R^2 \subseteq R$ .
  - (ii) If in addition  $R$  is reflexive, show that  $R = R^2 = R^3 = \dots$ .
  - (iii) If  $R$  is a transitive relation on a finite set with  $n$  elements, prove that  $R^m = R^{m+1} = \dots$  where  $m = n^2 + 1$ .

## 1.3 Functions

A more familiar concept than a relation is a function. While functions are to be found throughout mathematics, they are usually first encountered in calculus as real-valued functions of a real variable. Functions can provide convenient descriptions of complex objects and processes in mathematics and the information sciences.

Let  $A$  and  $B$  be sets. A *function* or *mapping* or *map* from  $A$  to  $B$ , in symbols

$$\alpha : A \rightarrow B,$$

is a rule which assigns to each element  $a$  of  $A$  a unique element  $\alpha(a)$  of  $B$ , called the *image of  $a$*  under  $\alpha$ . The sets  $A$  and  $B$  are the *domain* and *codomain* of  $\alpha$  respectively. The *image of the function  $\alpha$*  is

$$\text{Im}(\alpha) = \{\alpha(a) \mid a \in A\},$$

which is a subset of the codomain. The set of all functions from  $A$  to  $B$  will occasionally be written  $\text{Fun}(A, B)$ .

### Examples of functions.

- (i) The functions that appear in calculus are those whose domain and codomain are subsets of  $\mathbb{R}$ . Such a function can be visualized by drawing its graph in the usual way.
- (ii) Given a function  $\alpha : A \rightarrow B$ , we define

$$R_\alpha = \{(a, \alpha(a)) \mid a \in A\} \subseteq A \times B.$$

Thus  $R_\alpha$  is a relation between  $A$  and  $B$ . Observe that  $R_\alpha$  is a special kind of relation since each  $a$  in  $A$  is related to a *unique* element of  $B$ , namely  $\alpha(a)$ .

Conversely, suppose that  $R$  is a relation between  $A$  and  $B$  such that each  $a \in A$  is related to a unique  $b \in B$ . We may define a corresponding function  $\alpha_R : A \rightarrow B$  by  $\alpha_R(a) = b$  where  $a R b$ . Thus functions from  $A$  to  $B$  may be regarded as special types of relation between  $A$  and  $B$ .

This observation permits us to form the composite of two functions  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  by forming the composite of the corresponding relations: thus  $\beta \circ \alpha : A \rightarrow C$  is defined by

$$\beta \circ \alpha(a) = \beta(\alpha(a)).$$

- (iii) *The characteristic function of a subset.* Let  $A$  be a fixed set. For each subset  $X$  of  $A$  define a function  $\alpha_X : A \rightarrow \{0, 1\}$  by the rule

$$\alpha_X(a) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{if } a \notin X. \end{cases}$$

Then  $\alpha_X$  is called the *characteristic function* of the subset  $X$ . Conversely, every function  $\alpha : A \rightarrow \{0, 1\}$  is the characteristic function of some subset of  $A$  – which subset?

- (iv) The *identity function* on a set  $A$  is the function  $\text{id}_A : A \rightarrow A$  defined by  $\text{id}_A(a) = a$  for all  $a \in A$ .

**Injectivity and surjectivity.** There are two special types of function of critical importance. A function  $\alpha : A \rightarrow B$  is called *injective* (or *one-one*) if  $\alpha(a) = \alpha(a')$  always implies that  $a = a'$ , i.e., distinct elements of  $A$  have distinct images in  $B$  under  $\alpha$ . Next  $\alpha : A \rightarrow B$  is *surjective* (or *onto*) if each element of  $B$  is the image under  $\alpha$  of at least one element of  $A$ , i.e.,  $\text{Im}(\alpha) = B$ . Finally,  $\alpha : A \rightarrow B$  is said to be *bijective* (or a *one-one correspondence*) if it is both injective and surjective.

Here are some examples of various types of functions.

- (i)  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  where  $\alpha(x) = 2^x$  is injective but not surjective.
- (ii)  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  where  $\alpha(x) = x^3 - 4x$  is surjective but not injective. Here surjectivity is best seen by drawing the graph of  $y = x^3 - 4x$ . Note that any line parallel to the  $x$ -axis meets the curve at least once. But  $\alpha$  is not injective since  $\alpha(0) = 0 = \alpha(2)$ .

(iii)  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  where  $\alpha(x) = x^3$  is bijective.

(iv)  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  where  $\alpha(x) = x^2$  is neither injective nor surjective.

**Inverse functions.** Functions  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow A$  are said to be *mutually inverse* if  $\alpha \circ \beta = \text{id}_B$  and  $\beta \circ \alpha = \text{id}_A$ . Also  $\beta$  is an *inverse* of  $\alpha$ . Suppose that  $\beta'$  is another inverse of  $\alpha$ . Then, with the aid of the associative law, we have

$$\beta = \text{id}_A \circ \beta = (\beta' \circ \alpha) \circ \beta = \beta' \circ (\alpha \circ \beta) = \beta' \circ \text{id}_B = \beta'.$$

Therefore  $\alpha$  has a unique inverse, if it has one at all. We will write

$$\alpha^{-1} : B \rightarrow A$$

for the unique inverse of  $\alpha$  when it exists.

It is important to be able to recognize functions which possess inverses.

**(1.3.1)** A function  $\alpha : A \rightarrow B$  has an inverse if and only if it is bijective.

*Proof.* Assume that  $\alpha^{-1} : A \rightarrow B$  exists. If  $\alpha(a_1) = \alpha(a_2)$ , then, applying  $\alpha^{-1}$  to each side, we arrive at  $a_1 = a_2$ , which shows that  $\alpha$  is injective. Next, to show that  $\alpha$  is surjective, let  $b \in B$ . Then  $b = \text{id}_B(b) = \alpha(\alpha^{-1}(b)) \in \text{Im}(\alpha)$ , showing that  $\text{Im}(\alpha) = B$  and  $\alpha$  is surjective. Thus  $\alpha$  is bijective.

Conversely, let  $\alpha$  be bijective. If  $b \in B$ , there is precisely one element  $a$  in  $A$  such that  $\alpha(a) = b$  since  $\alpha$  is bijective. Define  $\beta : B \rightarrow A$  by  $\beta(b) = a$ . Then  $\alpha\beta(b) = \alpha(a) = b$  and  $\alpha\beta = \text{id}_B$ . Also  $\beta\alpha(a) = \beta(b) = a$ ; since every  $a$  in  $A$  arises in this way,  $\beta\alpha = \text{id}_A$  and  $\beta = \alpha^{-1}$ .  $\square$

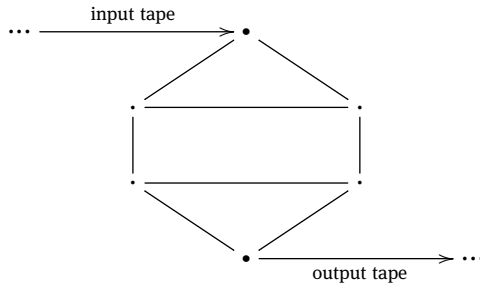
The next result records some useful facts about inverses.

**(1.3.2)**

- (i) If  $\alpha : A \rightarrow B$  is bijective, then so is  $\alpha^{-1} : B \rightarrow A$  and  $(\alpha^{-1})^{-1} = \alpha$ .
- (ii) If  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  are bijective functions, then  $\beta \circ \alpha : A \rightarrow C$  is bijective and  $(\beta \circ \alpha)^{-1} = \alpha^{-1} \circ \beta^{-1}$ .

*Proof.* The equations  $\alpha \circ \alpha^{-1} = \text{id}_B$  and  $\alpha^{-1} \circ \alpha = \text{id}_A$  tell us that  $\alpha$  is the inverse of  $\alpha^{-1}$ . Check directly that  $\alpha^{-1} \circ \beta^{-1}$  is the inverse of  $\beta \circ \alpha$  by using the associative law twice: thus  $(\beta \circ \alpha) \circ (\alpha^{-1} \circ \beta^{-1}) = ((\beta \circ \alpha) \circ \alpha^{-1}) \circ \beta^{-1} = (\beta \circ (\alpha \circ \alpha^{-1})) \circ \beta^{-1} = (\beta \circ \text{id}_B) \circ \beta^{-1} = \beta \circ \beta^{-1} = \text{id}_C$ . Similarly  $(\alpha^{-1} \circ \beta^{-1}) \circ (\beta \circ \alpha) = \text{id}_A$ .  $\square$

**Application to automata.** As an illustration of how the language of sets and functions may be used to describe information systems, we give a brief account of automata. An *automaton* is a theoretical device that is a basic model of a digital computer. It consists of an input tape and an output tape together with two “heads”, which are able to read symbols on the input tape and print symbols on the output tape. At any instant the system is in one of a number of states. When the automaton reads a symbol on the input tape, it goes to another state and writes a symbol on the output tape.



To make this idea precise we define an automaton  $A$  to be a 5-tuple

$$(I, O, S, \nu, \sigma)$$

where  $I$  and  $O$  are the respective sets of input and output symbols,  $S$  is the set of states,  $\nu : I \times S \rightarrow O$  is the *output function* and  $\sigma : I \times S \rightarrow S$  is the *next state function*. The automaton operates in the following manner. If it is in state  $s \in S$  and input symbol  $i \in I$  is read, the automaton prints the symbol  $\nu(i, s)$  on the output tape and goes to state  $\sigma(i, s)$ . Thus the mode of operation is determined by the three sets  $I, O, S$  and the two functions  $\nu, \sigma$ .

### Exercises (1.3)

- (1) Which of the following functions are injective, surjective, bijective?
  - (i)  $\alpha : \mathbb{R} \rightarrow \mathbb{Z}$  where  $\alpha(x) = [x]$ , the largest integer  $\leq x$ .
  - (ii)  $\alpha : \mathbb{R}^{>0} \rightarrow \mathbb{R}$  where  $\alpha(x) = \log_{10}(x)$ . (Here  $\mathbb{R}^{>0} = \{x \mid x \in \mathbb{R}, x > 0\}$ ).
  - (iii)  $\alpha : A \times B \rightarrow B \times A$  where  $\alpha((a, b)) = (b, a)$ .
- (2) Prove that a composite of injective functions is injective and a composite of surjective functions is surjective.
- (3) Let  $\alpha : A \rightarrow B$  be a function between finite sets. Show that if  $|A| > |B|$ , then  $\alpha$  cannot be injective, and if  $|A| < |B|$ , then  $\alpha$  cannot be surjective.
- (4) Define  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  by  $\alpha(x) = \frac{x^3}{x^2+1}$ . Prove that  $\alpha$  is bijective.
- (5) Give an example of two functions  $\alpha, \beta$  on a set  $A$  such that  $\alpha \circ \beta = \text{id}_A$  but  $\beta \circ \alpha \neq \text{id}_A$ .
- (6) Let  $\alpha : A \rightarrow B$  be an injective function. Show that there is a surjective function  $\beta : B \rightarrow A$  such that  $\beta \circ \alpha = \text{id}_A$ .
- (7) Let  $\alpha : A \rightarrow B$  be a surjective function. Show that there is an injective function  $\beta : B \rightarrow A$  such that  $\alpha \circ \beta = \text{id}_B$ .
- (8) Describe a simplified version of an automaton with no output tape in which each output is the new state. (This is called a *state output automaton*).
- (9) Let  $\alpha : A \rightarrow B$  be a function. Define a relation  $E_\alpha$  on  $A$  by the rule:  $a E_\alpha a'$  means that  $\alpha(a) = \alpha(a')$ . Prove that  $E_\alpha$  is an equivalence relation on  $A$ . Then show that, conversely, if  $E$  is any equivalence relation on a set  $A$ , then  $E = E_\alpha$  for some function  $\alpha$  with domain  $A$ .

## 1.4 Cardinality

If we want to compare two sets, a natural basis for comparison is the “size” of each set. If the sets are finite, their sizes are just the numbers of elements in the set. But how can one measure the size of an infinite set? A reasonable point of view would be to hold that two sets have the same size if their elements can be paired off. Certainly two finite sets have the same number of elements precisely when their elements can be paired. The point to observe is that this idea also applies to infinite sets, making it possible to give a rigorous definition of the size of an infinite set, its *cardinal*.

Let  $A$  and  $B$  be two sets. Then  $A$  and  $B$  are said to be *equipollent* if there is a bijection  $\alpha : A \rightarrow B$ ; thus the elements of  $A$  and  $B$  may be paired off as  $(a, \alpha(a))$ ,  $a \in A$ . It follows from (1.3.2) that equipollence is an equivalence relation on the class of all sets. Thus each set  $A$  belongs to a unique equivalence class, which will be written

$$|A|$$

and called the *cardinal* of  $A$ . Informally we can think of  $|A|$  as the collection of all sets with the same “size” as  $A$ . A *cardinal number* is the cardinal of some set.

If  $A$  is a finite set with exactly  $n$  elements, then  $A$  is equipollent to the set  $\{0, 1, \dots, n-1\}$  and  $|A| = |\{0, 1, \dots, n-1\}|$ . It is reasonable to identify the finite cardinal  $|\{0, 1, \dots, n-1\}|$  with the non-negative integer  $n$ . For then cardinal numbers appear as infinite versions of the non-negative integers.

Let us sum up our very elementary conclusions so far.

### (1.4.1)

- (i) Every set  $A$  has a unique cardinal number  $|A|$ .
- (ii) Two sets are equipollent if and only if they have the same cardinal.
- (iii) The cardinal of a finite set may be identified with the number of its elements.

Since we plan to use cardinals to compare the sizes of sets, it makes sense to define a “less than or equal to” relation  $\leq$  on cardinals. Define

$$|A| \leq |B|$$

to mean that there is an injective function  $\alpha : A \rightarrow B$ . Of course we will write  $|A| < |B|$  if  $|A| \leq |B|$  and  $|A| \neq |B|$ .

It is important to verify that this definition of  $\leq$  depends only on the cardinals  $|A|$  and  $|B|$ , not on the choice of sets  $A$  and  $B$ . Indeed, if  $A' \in |A|$  and  $B' \in |B|$ , then there are bijections  $\alpha' : A' \rightarrow A$  and  $\beta' : B \rightarrow B'$ ; by composing these with the injection  $\alpha : A \rightarrow B$  we obtain the injection  $\beta' \circ \alpha \circ \alpha' : A' \rightarrow B'$ . Thus  $|A'| \leq |B'|$ .

Next we prove a famous result about inequality of cardinals.

**(1.4.2)** (The Cantor-Bernstein<sup>4</sup> Theorem) *If  $A$  and  $B$  are sets such that  $|A| \leq |B|$  and  $|B| \leq |A|$ , then  $|A| = |B|$ .*

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<sup>4</sup> Georg Cantor (1845–1918), Felix Bernstein (1878–1956)

The proof of (1.4.2) is our most challenging proof so far and some readers may prefer to skip it. However, the basic idea behind it is not difficult to grasp.

*Proof.* By hypothesis there are injective functions  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow A$ . These will be used to construct a bijective function  $\gamma : A \rightarrow B$ , which will show that  $|A| = |B|$ .

Consider an arbitrary element  $a$  in  $A$ ; either  $a = \beta(b)$  for some unique  $b \in B$  or else  $a \notin \text{Im}(\beta)$ : here we use the injectivity of  $\beta$ . Similarly, either  $b = \alpha(a')$  for a unique  $a' \in A$  or else  $b \notin \text{Im}(\alpha)$ . Continuing this process, we trace back the “ancestry” of the element  $a$ . There are three possible outcomes:

- (i) we reach an element of  $A - \text{Im}(\beta)$ ;
- (ii) we reach an element of  $B - \text{Im}(\alpha)$ ;
- (iii) the process continues without end.

Partition the set  $A$  into three subsets corresponding to possibilities (i), (ii), (iii) and call them  $AA$ ,  $AB$ ,  $A_\infty$  respectively. In a similar fashion the set  $B$  decomposes into three disjoint subsets  $BA$ ,  $BB$ ,  $B_\infty$ ; for example, if  $b \in BA$ , we can trace  $b$  back to an element of  $A - \text{Im}(\beta)$ .

Now we are in a position to define the function  $\gamma : A \rightarrow B$ . First observe that the restriction of  $\alpha$  to  $AA$  is a bijection from  $AA$  to  $BA$ , and the restriction of  $\alpha$  to  $A_\infty$  is a bijection from  $A_\infty$  to  $B_\infty$ . Also, if  $x \in AB$ , there is a unique element  $x' \in BB$  such that  $\beta(x') = x$ . Now define

$$\gamma(x) = \begin{cases} \alpha(x) & \text{if } x \in AA \\ \alpha(x) & \text{if } x \in A_\infty \\ x' & \text{if } x \in AB. \end{cases}$$

Then  $\gamma$  is the desired bijection. □

**(1.4.3)** *The relation  $\leq$  is a partial order on cardinal numbers.*

For we have proved antisymmetry in (1.4.2), while reflexivity and transitivity are clearly true. In fact one can do better since  $\leq$  is even a *linear* order. This is because of:

**(1.4.4)** (The Law of Trichotomy) *If  $A$  and  $B$  are sets, then exactly one of the following must hold:*

$$|A| < |B|, \quad |A| = |B|, \quad |B| < |A|.$$

The proof will not be given at this point since it depends on advanced material – see (14.1.7) below for a proof.

The next result establishes the existence of arbitrarily large cardinal numbers.

**(1.4.5)** *If  $A$  is any set, then  $|A| < |\mathcal{P}(A)|$ .*

*Proof.* The easy step is to show that  $|A| \leq |\mathcal{P}(A)|$ . This is because the assignment  $a \mapsto \{a\}$  sets up an injection from  $A$  to  $\mathcal{P}(A)$ .

Next assume that  $|A| = |\mathcal{P}(A)|$ , so that there is a bijection  $\alpha : A \rightarrow \mathcal{P}(A)$ . Of course at this point we are looking for a contradiction. The trick is to consider the subset



$B = \{a \mid a \in A, a \notin \alpha(a)\}$  of  $A$ . Then  $B \in \mathcal{P}(A)$ , so  $B = \alpha(a)$  for some  $a \in A$ . Now either  $a \in B$  or  $a \notin B$ . If  $a \in B$ , then  $a \notin \alpha(a) = B$ ; if  $a \notin B = \alpha(a)$ , then  $a \in B$ . This is our contradiction.  $\square$

**Countable sets.** The cardinal of the set of natural numbers  $\mathbb{N} = \{0, 1, 2, \dots\}$  is denoted by

$$\aleph_0.$$

Here  $\aleph$  is the Hebrew letter aleph. A set  $A$  is said to be *countable* if  $|A| \leq \aleph_0$ . Essentially this means that the elements of  $A$  can be “labelled” by attaching to each element a natural number as a label. An uncountable set cannot be so labelled.

We need to explain what is meant by an infinite set for the next result to be meaningful. A set  $A$  will be called *infinite* if it has a subset that is equipollent with  $\mathbb{N}$ , i.e., if  $\aleph_0 \leq |A|$ . An *infinite cardinal* is the cardinal of an infinite set.

**(1.4.6)**  $\aleph_0$  is the smallest infinite cardinal.

*Proof.* If  $A$  is an infinite set, then  $A$  has a subset  $B$  such that  $\aleph_0 = |B|$ . Hence  $\aleph_0 \leq |A|$ .  $\square$

It follows that if  $A$  is a countable set, either  $A$  is finite or  $|A| = \aleph_0$ . As the final topic of the chapter we consider the cardinals of the sets  $\mathbb{Q}$  and  $\mathbb{R}$ .

**(1.4.7)**

(i) The set  $\mathbb{Q}$  of rational numbers is countable.

(ii) The set  $\mathbb{R}$  of real numbers is uncountable.

*Proof.* (i) Each positive rational number has the form  $\frac{m}{n}$  where  $m$  and  $n$  are positive integers. Arrange these rationals in a rectangular array, with  $\frac{m}{n}$  in the  $m$ th row and  $n$ th column. Of course each rational will occur infinitely often because of cancellation. Now follow the path indicated by the arrows in the diagram below.

$$\begin{array}{ccccccc}
 \frac{1}{1} & \rightarrow & \frac{1}{2} & \frac{1}{3} & \rightarrow & \frac{1}{4} & \dots \\
 & \swarrow & & & \swarrow & & \\
 \frac{2}{1} & & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \dots & \\
 \downarrow & \nearrow & \swarrow & & & & \\
 \frac{3}{1} & & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \dots & \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \dots & 
 \end{array}$$

This creates a sequence in which every positive rational number appears infinitely often. Delete repetitions in the sequence. Insert 0 at the beginning of the sequence and insert  $-r$  immediately after  $r$  for each positive rational  $r$ . Now every rational occurs exactly once in the sequence. Hence  $\mathbb{Q}$  is countable.

(ii) It is enough to show that the set  $I$  of all real numbers  $r$  such that  $0 \leq r \leq 1$  is uncountable: this is because  $|I| \leq |\mathbb{R}|$ . Assume that  $I$  is countable, so that it can be

written in the form  $\{r_1, r_2, r_3, \dots\}$ . Write each  $r_i$  as a decimal, say

$$r_i = 0 \cdot r_{i1} r_{i2} \dots$$

where  $0 \leq r_{ij} \leq 9$ . We reach a contradiction by producing a number in the set  $I$  which does not equal any  $r_i$ . Define

$$s_i = \begin{cases} 0 & \text{if } r_{ii} \neq 0 \\ 1 & \text{if } r_{ii} = 0 \end{cases}$$

and let  $s$  be the decimal  $0 \cdot s_1 s_2 \dots$ ; then certainly  $s \in I$ . Hence  $s = r_i$  for some  $i$ , so that  $s_i = r_{ii}$ ; but this is impossible by the definition of  $s_i$ .  $\square$

### Exercises (1.4)

- (1) A finite set cannot be equipollent to a proper subset.
- (2) A set is infinite if and only if it has the same cardinal as some proper subset.
- (3) If there is a surjection from a set  $A$  to a set  $B$ , then  $|B| \leq |A|$ .
- (4) Show that  $|\mathbb{Z}| = \aleph_0$  and  $|\mathbb{Z} \times \mathbb{Z}| = \aleph_0$ .
- (5) Let  $A_1, A_2, \dots$  be countably many, countable sets. Prove that  $\bigcup_{i=1,2,\dots} A_i$  is a countable set. [Hint: write  $A_i = \{a_{i0}, a_{i1}, \dots\}$  and follow the method of the proof of (1.4.7(i)).]
- (6) Suggest reasonable definitions of the sum and product of two cardinal numbers. [Hint: try using the union and set product]
- (7) Let  $S$  denote the set of all *restricted sequences* of integers  $a_1, a_2, a_3, \dots$ , i.e.,  $a_i = 0$  for all but a finite number of  $i$ . Prove that  $|S| = \aleph_0$ .
- (8) Let  $A$  be a countably infinite set and let  $\mathcal{P}_f(A)$  denote the set of all finite subsets of  $A$ .
  - (i) Prove that  $|\mathcal{P}_f(A)| = |A|$ , so that  $\mathcal{P}_f(A)$  is countable.
  - (ii) Prove that on the other hand  $\mathcal{P}(A)$  is uncountable.

## 2 The integers

The role of the integers is central in algebra, as it is in all parts of mathematics. One reason for this is that the set of integers  $\mathbb{Z}$ , together with the standard arithmetic operations of addition and multiplication, serves as a model for several of the fundamental structures of algebra, including groups and rings. In this chapter the most basic properties of the integers are developed.

### 2.1 Well-ordering and mathematical induction

We begin by listing the properties of the fundamental arithmetic operations on  $\mathbb{Z}$ , addition and multiplication. In the following  $a, b, c$  are arbitrary integers.

- (i)  $a + b = b + a$ ,  $ab = ba$ , (commutative laws);
- (ii)  $(a + b) + c = a + (b + c)$ ,  $(ab)c = a(bc)$ , (associative laws);
- (iii)  $(a + b)c = ac + bc$ , (distributive law);
- (iv)  $0 + a = a$  and  $1 \cdot a = a$ , (existence of identities);
- (v) each integer  $a$  has a *negative*  $-a$  with the property  $a + (-a) = 0$ ;
- (vi) if  $ab = 0$ , then  $a = 0$  or  $b = 0$ .

Next we list properties of the relation  $\leq$  on  $\mathbb{Z}$ .

- (vii)  $\leq$  is a linear order on  $\mathbb{Z}$ , i.e., the relation  $\leq$  is reflexive, antisymmetric and transitive; in addition, for any pair of integers  $a, b$  either  $a \leq b$  or  $b \leq a$ ;
- (viii) if  $a \leq b$  and  $c \geq 0$ , then  $ac \leq bc$ ;
- (ix) if  $a \leq b$ , then  $-b \leq -a$ .

These properties are assumed as axioms. But there is a further property of the linearly ordered set  $(\mathbb{Z}, \leq)$  which is independent of the above axioms and is quite vital for the development of the elementary theory of the integers.

**The Well-Ordering Law.** Let  $k$  be a fixed integer and put  $U = \{n \mid n \in \mathbb{Z}, n \geq k\}$ . Suppose that  $S$  is a non-empty subset of  $U$ . Then the *Well-Ordering Law* (WO) asserts that  $S$  has a smallest element. Thus  $\leq$  is a well order on  $U$  in the sense of (1.2).

While this may seem a harmless assumption, it cannot be deduced from axioms (i)–(ix) and must be adopted as an additional axiom. The importance of WO for us is that it provides a sound basis for the method of proof by mathematical induction. This is embodied in

**(2.1.1) (The Principle of Mathematical Induction)** Let  $k$  be an integer and let  $U = \{n \mid n \in \mathbb{Z}, n \geq k\}$ . Assume that  $S$  is a subset of  $U$  with the properties

- (i)  $k \in S$ ;
- (ii) if  $n \in S$ , then  $n + 1 \in S$ .

Then  $S$  equals  $U$ .

*Proof.* Once again the assertion sounds fairly obvious, but in order to prove it, we must use WO. To see how WO applies, assume that  $S \neq U$ , so that  $S' = U - S$  is not empty. Then WO guarantees that  $S'$  has a smallest element, say  $s$ . Notice that  $k < s$  since  $k \in S$  by hypothesis. Thus  $k \leq s - 1$  and  $s - 1 \notin S'$  because  $s$  is minimal in  $S'$ . Hence  $s - 1 \in S$ , which by (ii) above implies that  $s \in S$ , a contradiction. Thus (2.1.1) is established.  $\square$

**The method of proof by induction.** Suppose that  $k$  is a fixed integer and that for each integer  $n \geq k$  there is a proposition  $p(n)$ , which is either true or false. Assume that the following hold:

- (i)  $p(k)$  is true;
- (ii) if  $p(n)$  is true, then  $p(n + 1)$  is true.

Then we can conclude that  $p(n)$  is true for all  $n \geq k$ .

For let  $S$  be the set of all integers  $n \geq k$  for which  $p(n)$  is true. Then the hypotheses of PMI (Principle of Mathematical Induction) apply to  $S$ . The conclusion is that  $S$  equals  $\{n \mid n \in \mathbb{Z}, n \geq k\}$ , i.e.,  $p(n)$  is true for all  $n \geq k$ .

Here is a simple example of proof by mathematical induction.

**Example (2.1.1)** Use mathematical induction to show that  $8^{n+1} + 9^{2n-1}$  is a multiple of 73 for all positive integers  $n$ .

Let  $p(n)$  denote the statement:  $8^{n+1} + 9^{2n-1}$  is a multiple of 73. Then  $p(1)$  is certainly true since  $8^{n+1} + 9^{2n-1} = 73$  when  $n = 1$ . Assume that  $p(n)$  is true; we have to deduce that  $p(n + 1)$  is true. Now we may rewrite  $8^{(n+1)+1} + 9^{2(n+1)-1}$  in the form

$$\begin{aligned} 8^{n+2} + 9^{2n+1} &= 8(8^{n+1} + 9^{2n-1}) + 9^{2n+1} - 8 \cdot 9^{2n-1} \\ &= 8(8^{n+1} + 9^{2n-1}) + 73 \cdot 9^{2n-1}. \end{aligned}$$

Since both terms in the last expression are multiples of 73, so is  $8^{n+2} + 9^{2n+1}$ . Thus  $p(n + 1)$  is true and by PMI the statement  $p(n)$  is true for all  $n \geq 1$ .

**(2.1.2) (Alternate Form of PMI)** Let  $k$  be an integer and let  $U = \{n \mid n \in \mathbb{Z}, n \geq k\}$ . Assume that  $S$  is a subset of  $U$  with the properties

- (i)  $k \in S$ ;
- (ii) if  $m \in S$  for all integers  $m$  such that  $k \leq m < n$ , then  $n \in S$ .

Then  $S = U$ .

This variant of PMI follows from WO just as the original form does. There are situations where proof by induction cannot be easily used but the alternate form is effective. In such a case one has a proposition  $p(n)$  for  $n \geq k$  such that:

- (i)  $p(k)$  is true;
- (ii) if  $p(m)$  is true whenever  $k \leq m < n$ , then  $p(n)$  is true.

The conclusion is that  $p(n)$  is true for all  $n \geq k$ .

A good example of a proposition where this type of induction proof is successful is the Fundamental Theorem of Arithmetic – see (2.2.7).

Our approach to the integers in this section has been quite naive: we have simply stated as axioms all the properties that we need. For a good axiomatic treatment of the construction of the integers, including an account of the axioms of Peano, see [6].

### Exercises (2.1)

- (1) Use induction to establish the following summation formulas for  $n \geq 1$ .
  - (i)  $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ ;
  - (ii)  $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ ;
  - (iii)  $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{1}{2}n(n+1)\right)^2$ .
- (2) Deduce the alternate form of PMI from WO.
- (3) Prove that  $2^n > n^3$  for all integers  $n \geq 10$ .
- (4) Prove that  $2^n > n^4$  for all integers  $n \geq 17$ .
- (5) Prove by mathematical induction that 6 divides  $n^3 - n$  for all integers  $n \geq 0$ .
- (6) Use the alternate form of mathematical induction to show that any  $n$  cents worth of postage, where  $n \geq 12$ , can be made up by using only 4-cent and 5-cent stamps. [Hint: first verify the statement for  $n \leq 15$ ].

## 2.2 Division in the integers

In this section we establish the basic properties of the integers that relate to division, notably the Division Algorithm, the existence of greatest common divisors and the Fundamental Theorem of Arithmetic.

Recall that if  $a, b$  are integers, then  $a$  divides  $b$ , in symbols

$$a \mid b,$$

if there is an integer  $c$  such that  $b = ac$ . The following properties of division are simple consequences of the definition, as the reader should verify.

### (2.2.1)

- (i) *The relation of division is a partial order on  $\mathbb{Z}$ .*
- (ii) *If  $a \mid b$  and  $a \mid c$ , then  $a \mid bx + cy$  for all integers  $x, y$ .*
- (iii)  *$a \mid 0$  for all  $a$ , while  $0 \mid a$  if and only if  $a = 0$ .*
- (iv)  *$1 \mid a$  for all  $a$ , while  $a \mid 1$  if and only if  $a = \pm 1$ .*

**The division algorithm.** The first result about the integers of real significance is the *Division Algorithm*; it codifies the time-honored process of dividing one integer by

another to obtain a quotient and remainder. It should be noted that the proof of the result uses WO.

**(2.2.2)** Let  $a, b$  be integers with  $b \neq 0$ . Then there exist unique integers  $q$  (the quotient) and  $r$  (the remainder) such that  $a = bq + r$  and  $0 \leq r < |b|$ .

*Proof.* Let  $S$  be the set of all non-negative integers of the form  $a - bq$  where  $q \in \mathbb{Z}$ . In the first place we need to observe that  $S$  is not empty. Indeed, if  $b > 0$  and we choose an integer  $q \leq \frac{a}{b}$ , then  $a - bq \geq 0$ ; if  $b < 0$ , choose an integer  $q \geq \frac{a}{b}$ , so that again  $a - bq \geq 0$ . Applying the Well-Ordering Law to the set  $S$ , we conclude that it contains a smallest element, say  $r$ . Then  $r = a - bq$  for some integer  $q$  and  $a = bq + r$ .

Now suppose that  $r \geq |b|$ . If  $b > 0$ , then  $a - b(q + 1) = r - b < r$ , while if  $b < 0$ , then  $a - b(q - 1) = r + b < r$ . In each case a contradiction is reached since we have found an integer in  $S$  which is less than  $r$ . Hence  $r < |b|$ .

Finally, we must show that  $q$  and  $r$  are unique. Suppose that  $a = bq' + r'$  where  $q', r' \in \mathbb{Z}$  and  $0 \leq r' < |b|$ . Then  $bq + r = bq' + r'$  and  $b(q - q') = r' - r$ . Thus  $|b| \cdot |q - q'| = |r - r'|$ . If  $q \neq q'$ , then  $|r - r'| \geq |b|$ , whereas  $|r - r'| < |b|$  since  $0 \leq r, r' < |b|$ . Therefore  $q = q'$  and it follows at once that  $r = r'$ .  $\square$

When  $a < 0$  or  $b < 0$ , care must be taken to ensure that a negative remainder is not obtained. For example, if  $a = -21$  and  $b = -4$ , then  $-21 = (-4)6 + 3$ , so that  $q = 6$  and  $r = 3$ .

**Greatest common divisors.** Let  $a_1, a_2, \dots, a_n$  be integers. An integer  $c$  which divides every  $a_i$  is called a *common divisor* of  $a_1, a_2, \dots, a_n$ . Our next goal is to establish the existence of a *greatest common divisor*.

**(2.2.3)** Let  $a_1, a_2, \dots, a_n$  be integers. Then there is a unique integer  $d \geq 0$  with the properties:

- (i)  $d$  is a common divisor of  $a_1, a_2, \dots, a_n$ ;
- (ii) every common divisor of  $a_1, a_2, \dots, a_n$  divides  $d$ ;
- (iii)  $d = a_1x_1 + \dots + a_nx_n$  for some integers  $x_i$ .

*Proof.* If all of the  $a_i$  are 0, we can take  $d = 0$  since this fits the description. So assume that at least one  $a_i$  is non-zero. Then the set  $S$  of all non-negative integers  $a_1x_1 + a_2x_2 + \dots + a_nx_n$  with  $x_i \in \mathbb{Z}$  is non-empty. By WO there is a least element in  $S$ , say  $d = a_1x_1 + a_2x_2 + \dots + a_nx_n$ . If an integer  $c$  divides each  $a_i$ , then  $c \mid d$  by (2.2.1). Thus it only remains to show that  $d \mid a_i$  for all  $i$ .

By the Division Algorithm we can write  $a_i = dq_i + r_i$  where  $q_i, r_i \in \mathbb{Z}$  and  $0 \leq r_i < d$ . Then

$$r_i = a_i - dq_i = a_1(-x_1q_i) + \dots + a_i(1 - x_iq_i) + \dots + a_n(-x_nq_i).$$

If  $r_i \neq 0$ , then  $r_i \in S$ , which contradicts the minimality of  $d$  in  $S$ . Hence  $r_i = 0$  and  $d \mid a_i$  for all  $i$ .

Finally, we show that  $d$  is unique. If  $d'$  is another integer satisfying (i) and (ii), then  $d \mid d'$  and  $d' \mid d$ , so that  $d = d'$  since  $d, d' \geq 0$ .  $\square$

The integer  $d$  of (2.2.3) is called the *greatest common divisor* of  $a_1, a_2, \dots, a_n$ , in symbols

$$d = \gcd\{a_1, a_2, \dots, a_n\}.$$

If  $d = 1$ , the integers  $a_1, a_2, \dots, a_n$  are said to be *relatively prime*; of course this means that the integers have no common divisors except  $\pm 1$ .

**The Euclidean<sup>1</sup> Algorithm.** The proof of the existence of gcd's which has just been given is not constructive, i.e., it does not provide a method for calculating gcd's. However, there is a well known procedure called the *Euclidean Algorithm* which is effective in this respect.

Assume that  $a, b$  are integers with  $b \neq 0$ . Apply the Division Algorithm to divide  $a$  by  $b$  to get quotient  $q_1$  and remainder  $r_1$ . Next divide  $b$  by  $r_1$  to get quotient  $q_2$  and remainder  $r_2$ ; then divide  $r_1$  by  $r_2$  to get quotient  $q_3$  and remainder  $r_3$ . And so on. By WO there is a smallest non-zero remainder, say  $r_{n-1}$ . Thus  $r_n = 0$  and we have a system of integer equations

$$\begin{cases} a = bq_1 + r_1, \\ b = r_1q_2 + r_2, \\ r_1 = r_2q_3 + r_3, \\ \vdots \quad \quad \quad \vdots \\ r_{n-3} = r_{n-2}q_{n-1} + r_{n-1}, \\ r_{n-2} = r_{n-1}q_n + 0. \end{cases}$$

Here  $0 \leq r_1 < |b|$ ,  $0 \leq r_i < r_{i-1}$  and  $r_{n-1}$  is the smallest non-zero remainder. With this notation we can state:

**(2.2.4) (The Euclidean Algorithm)** *The greatest common divisor of  $a$  and  $b$  equals the last non-zero remainder  $r_{n-1}$ .*

*Proof.* Starting with the second last equation in the system above, we can solve back for  $r_{n-1}$ , obtaining eventually an expression of the form  $r_{n-1} = ax + by$ , where  $x, y \in \mathbb{Z}$ . This shows that any common divisor of  $a$  and  $b$  must divide  $r_{n-1}$ . We can also use the system of equations above to show successively that  $r_{n-1} \mid r_{n-2}$ ,  $r_{n-1} \mid r_{n-3}$ ,  $\dots$ , etc., and finally  $r_{n-1} \mid b$ ,  $r_{n-1} \mid a$ . It follows that  $r_{n-1} = \gcd\{a, b\}$  by uniqueness of gcd's.  $\square$

<sup>1</sup> Euclid of Alexandria (325–265 BC)

**Example (2.2.1)** Find  $\gcd(76, 60)$ . We compute successively:  $76 = 60 \cdot 1 + 16$ ,  $60 = 16 \cdot 3 + 12$ ,  $16 = 12 \cdot 1 + 4$ ,  $12 = 4 \cdot 3 + 0$ . Hence  $\gcd\{76, 60\} = 4$ , the last non-zero remainder. By reading back from the third equation we obtain the predicted expression for the gcd,  $4 = 76 \cdot 4 + 60 \cdot (-5)$ .

The Euclidean algorithm can also be applied to calculate gcd's of more than two integers by using the formula

$$\gcd\{a_1, a_2, \dots, a_{m+1}\} = \gcd\{\gcd\{a_1, a_2, \dots, a_m\}, a_{m+1}\}$$

and induction on  $m$ : see Exercise (2.2.1).

A very useful tool in working with divisibility is:

**(2.2.5) (Euclid's Lemma)** *Let  $a, b, m$  be integers. If  $m$  divides  $ab$  and  $m$  is relatively prime to  $a$ , then  $m$  divides  $b$ .*

*Proof.* By hypothesis  $\gcd\{a, m\} = 1$ , so by (2.2.3) there are integers  $x, y$  such that  $1 = mx + ay$ . Multiplying by  $b$ , we obtain  $b = mbx + aby$ . Since  $m$  divides  $ab$ , it divides the right side of the equation. Hence  $m$  divides  $b$ .  $\square$

Recall that a *prime number* is an integer  $p > 1$  such that  $\pm 1$  and  $\pm p$  are its only divisors. If  $p$  is a prime and  $a$  is any integer, then either  $\gcd\{a, p\} = 1$  or  $p \mid a$ . Thus (2.2.5) has the consequence.

**(2.2.6)** *If a prime  $p$  divides  $ab$  where  $a, b \in \mathbb{Z}$ , then  $p$  divides  $a$  or  $b$ .*

**The Fundamental Theorem of Arithmetic.** It is a basic result that every integer greater than 1 can be expressed as a product of primes. The proof of this result is a good example of proof by the alternate form of mathematical induction.

**(2.2.7)** *Every integer  $n > 1$  can be expressed as a product of primes. Moreover the expression is unique up to the order of the factors.*

*Proof.* (i) *Existence.* We show that  $n$  is a product of primes, which is certainly true if  $n = 2$ . Assume that every integer  $m$  satisfying  $2 \leq m < n$  is a product of primes. If  $n$  itself is a prime, there is nothing to prove. Otherwise  $n = n_1 n_2$  where  $1 < n_1 < n$ . Then  $n_1$  and  $n_2$  are both products of primes, whence so is  $n = n_1 n_2$ . The result now follows by the alternate form of mathematical induction (2.1.2).

(ii) *Uniqueness.* In this part we have to show that  $n$  has a unique expression as a product of primes. Again this is clearly correct for  $n = 2$ . Assume that if  $2 \leq m < n$ , then  $m$  is uniquely expressible as a product of primes. Next suppose that  $n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s$  where the  $p_i$  and  $q_j$  are primes. Then  $p_1 \mid q_1 q_2 \cdots q_s$  and by (2.2.6) the prime  $p_1$  must divide, and hence equal, one of the  $q_j$ 's; we can assume  $p_1 = q_1$  by relabelling the  $q_j$ 's if necessary. Now cancel  $p_1$  to get  $m = p_2 \cdots p_r = q_2 \cdots q_s$ . Since  $m = n/p_1 < n$ , we deduce that  $p_2 = q_2, \dots, p_r = q_r$ , and  $r = s$ , after further relabelling of the  $q_j$ 's. Hence the result is proven.  $\square$



A convenient expression for an integer  $n > 1$  is

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

where the  $p_i$  are distinct primes and  $e_i > 0$ . That the  $p_i$  and  $e_i$  are unique up to order follows from (2.2.7).

Finally in this section we prove the famous theorem of Euclid on the infinitude of primes.

**(2.2.8)** *There exist infinitely many prime numbers.*

*Proof.* Suppose this is false and let  $p_1, p_2, \dots, p_k$  be the list of all the primes. The trick is to produce a prime that is not on the list. To do this put  $n = p_1 p_2 \cdots p_k + 1$ . Now no  $p_i$  can divide  $n$ , otherwise  $p_i \mid 1$ . But  $n$  is certainly divisible by at least one prime, so we reach a contradiction.  $\square$

**Example (2.2.2)** *If  $p$  is a prime, then  $\sqrt{p}$  is not a rational number.*

For, assume that  $\sqrt{p}$  is a rational and  $\sqrt{p} = \frac{m}{n}$  where  $m, n$  are integers; evidently there is nothing to be lost in assuming that  $m$  and  $n$  are relatively prime since any common factor can be cancelled. Squaring both sides, we obtain  $p = m^2/n^2$  and  $m^2 = pn^2$ . Hence  $p \mid m^2$  and Euclid's Lemma shows that  $p \mid m$ . Write  $m = pm_1$  for some integer  $m_1$ . Then  $p^2 m_1^2 = pn^2$ , so  $pm_1^2 = n^2$ . Thus  $p \mid n^2$  and  $p \mid n$ ; but this means  $m$  and  $n$  are not relatively prime, a contradiction.

### Exercises (2.2)

(1) Let  $a_1, a_2, \dots, a_m$  be integers. Prove that

$$\gcd\{a_1, a_2, \dots, a_{m+1}\} = \gcd\{\gcd\{a_1, a_2, \dots, a_m\}, a_{m+1}\}.$$

(2) Prove that  $\gcd\{ka_1, ka_2, \dots, ka_m\} = k \cdot \gcd\{a_1, a_2, \dots, a_m\}$  where the  $a_i$  and  $k \geq 0$  are integers.

(3) Use the Euclidean Algorithm to compute the following gcd's:

$$\gcd\{840, 410\}, \quad \gcd\{24, 328, 472\}.$$

Then express each gcd as a linear combination of the relevant integers.

(4) Consider the equation  $ax + by = c$  where  $a, b, c$  are given integers.

- (i) Prove that there is a solution in integers  $x, y$  if and only if  $d = \gcd\{a, b\}$  divides  $c$ .
- (ii) Write  $d = ua + vb$  where  $u, v \in \mathbb{Z}$ . Prove that the general solution of the equation is  $x = \frac{uc}{d} + \frac{mb}{d}$ ,  $y = \frac{vc}{d} - \frac{ma}{d}$  where  $m$  is an arbitrary integer.

- (5) Find all solutions in integers of the equation  $6x + 11y = 1$ .
- (6) If  $p$  and  $q$  are distinct primes, prove that  $\sqrt{pq}$  is irrational.
- (7) Let  $a_1, a_2, \dots, a_m$  be positive integers and write  $a_i = p_1^{e_{i1}} p_2^{e_{i2}} \cdots p_n^{e_{in}}$  where the  $e_{ij}$  are integers  $\geq 0$  and the primes  $p_i$  are all different. Show that  $\gcd\{a_1, a_2, \dots, a_m\} = p_1^{f_1} p_2^{f_2} \cdots p_n^{f_n}$  where  $f_j = \min\{e_{1j}, e_{2j}, \dots, e_{mj}\}$ .
- (8) A *least common multiple* (or lcm) of integers  $a_1, a_2, \dots, a_m$  is an integer  $\ell \geq 0$  such that each  $a_i$  divides  $\ell$  and  $\ell$  divides any integer which is divisible by every  $a_i$ .
- (i) Let  $a_i = p_1^{e_{i1}} p_2^{e_{i2}} \cdots p_n^{e_{in}}$  where the  $e_{ij}$  are integers  $\geq 0$  and the primes  $p_i$  are all different. Prove that lcm's exist and are unique by establishing the formula  $\text{lcm}\{a_1, a_2, \dots, a_m\} = p_1^{g_1} p_2^{g_2} \cdots p_n^{g_n}$  with  $g_j = \max\{e_{1j}, e_{2j}, \dots, e_{mj}\}$ .
- (ii) Prove that  $\gcd\{a, b\} \cdot \text{lcm}\{a, b\} = ab$ .
- (9) Let  $r$  be a rational number and let  $a$  and  $b$  be relatively prime integers. If  $ar$  and  $br$  are integers, prove that  $r$  is also an integer.
- (10) Let  $a$  and  $b$  be integers with  $b > 0$ . Prove that there are integers  $u, v$  such that  $a = bu + v$  and  $-\frac{b}{2} \leq v < \frac{b}{2}$ . [Hint: start with the Division Algorithm].
- (11) Prove that  $\gcd\{4n + 5, 3n + 4\} = 1$  for all integers  $n$ .
- (12) Prove that  $\gcd\{2n + 6, n^2 + 3n + 2\} = 2$  or  $4$  for any integer  $n$  and show that both possibilities can occur.
- (13) Show that if  $2^n + 1$  is prime, then  $n$  must have the form  $2^l$ . (Such primes are called *Fermat<sup>2</sup> primes*).
- (14) The only integer  $n$  which is expressible as  $a^3(3a + 1)$  and  $b^2(b + 1)^3$  with  $a, b$  relatively prime and positive is 2000.

## 2.3 Congruences

The notion of congruence was introduced by Gauss<sup>3</sup> in 1801, but it had long been implicit in ancient writings concerned with the computation of dates.

Let  $m$  be a positive integer. Two integers  $a, b$  are said to be *congruent modulo  $m$* , in symbols

$$a \equiv b \pmod{m},$$

if  $m$  divides  $a - b$ . Thus congruence modulo  $m$  is a relation on  $\mathbb{Z}$  and an easy check reveals that it is an equivalence relation. Hence the set  $\mathbb{Z}$  splits up into equivalence classes, which in this context are called *congruence classes modulo  $m$* : see (1.2.2). The unique congruence class to which an integer  $a$  belongs is written

$$[a] \text{ or } [a]_m = \{a + mq \mid q \in \mathbb{Z}\}.$$

<sup>2</sup> Pierre de Fermat (1601–1665)

<sup>3</sup> Carl Friedrich Gauss (1777–1855)

By the Division Algorithm any integer  $a$  can be written in the form  $a = mq + r$  where  $q, r \in \mathbb{Z}$  and  $0 \leq r < m$ . Thus  $a \equiv r \pmod{m}$  and  $[a] = [r]$ . Therefore  $[0], [1], \dots, [m-1]$  are all the congruence classes modulo  $m$ . Furthermore, if  $[r] = [r']$  where  $0 \leq r, r' < m$ , then  $m \mid r - r'$ , which can only mean that  $r = r'$ . Thus we have proved:

**(2.3.1)** *Let  $m$  be any positive integer. Then there are exactly  $m$  congruence classes modulo  $m$ , namely  $[0], [1], \dots, [m-1]$ .*

**Congruence arithmetic.** We will write

$$\mathbb{Z}_m$$

for the set of all congruence classes modulo  $m$ . Next we define operations of addition and multiplication for congruence classes, thereby introducing the possibility of arithmetic in  $\mathbb{Z}_m$ .

The *sum* and *product* of congruence classes modulo  $m$  are defined by the rules

$$[a] + [b] = [a + b] \quad \text{and} \quad [a] \cdot [b] = [ab].$$

These definitions are surely the natural ones. However, some care must be exercised in framing definitions of this type. A congruence class can be represented by any one of its elements: we need to ensure that the sum and product specified above depend only on the congruence classes themselves, not on the chosen representatives.

To this end, let  $a' \in [a]$  and  $b' \in [b]$ . It must be shown that  $[a + b] = [a' + b']$  and  $[ab] = [a'b']$ . Now  $a' = a + mu$  and  $b' = b + mv$  for some  $u, v \in \mathbb{Z}$ . Therefore  $a' + b' = (a + b) + m(u + v)$  and  $a'b' = ab + m(av + bu + muv)$ ; from these equations it follows that  $a' + b' \equiv a + b \pmod{m}$  and  $a'b' \equiv ab \pmod{m}$ . Thus  $[a' + b'] = [a + b]$  and  $[a'b'] = [ab]$ , as required.

Now that we know the sum and product of congruence classes to be well-defined, it is a routine task to establish the basic properties of these operations.

**(2.3.2)** *Let  $m$  be a positive integer and let  $[a], [b], [c]$  be congruence classes modulo  $m$ . Then*

- (i)  $[a] + [b] = [b] + [a]$  and  $[a] \cdot [b] = [b] \cdot [a]$ ;
- (ii)  $([a] + [b]) + [c] = [a] + ([b] + [c])$  and  $([a][b])[c] = [a]([b][c])$ ;
- (iii)  $([a] + [b])[c] = [a][c] + [b][c]$ ;
- (iv)  $[0] + [a] = [a]$  and  $[1][a] = [a]$ ;
- (v)  $[a] + [-a] = [0]$ .

Since all of these properties are valid in  $\mathbb{Z}$  as well as  $\mathbb{Z}_m$  – see (2.1) – we recognize some common features of the arithmetics on  $\mathbb{Z}$  and  $\mathbb{Z}_m$ . This commonality can be expressed by saying that  $\mathbb{Z}$  and  $\mathbb{Z}_m$  are both *commutative rings with identity*, as will be explained in Chapter Six.

**Fermat's Theorem.** Before proceeding to this well-known theorem, we will establish a frequently used property of the binomial coefficients. If  $n$  and  $r$  are integers satisfying  $0 \leq r \leq n$ , the *binomial coefficient*  $\binom{n}{r}$  is the number of ways of choosing  $r$  objects from a set of  $n$  distinct objects. There is the well-known formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1) \cdots (n-r+1)}{r!}.$$

The property needed is:

**(2.3.3)** *If  $p$  is a prime and  $0 < r < p$ , then  $\binom{p}{r} \equiv 0 \pmod{p}$ .*

*Proof.* Write  $\binom{p}{r} = pm$  where  $m$  is the rational number

$$\frac{(p-1)(p-2) \cdots (p-r+1)}{r!}.$$

Notice that each prime appearing as a factor of the numerator or denominator of  $m$  is smaller than  $p$ . Write  $m = \frac{u}{v}$  where  $u$  and  $v$  are relatively prime integers. Then  $v\binom{p}{r} = pmv = pu$  and by Euclid's Lemma  $v$  divides  $p$ . Now  $v \neq p$ , so  $v = 1$  and  $m = u \in \mathbb{Z}$ . Hence  $p$  divides  $\binom{p}{r}$ .  $\square$

We are now able to prove what is often called *Fermat's Little Theorem*, to distinguish it from the well known Fermat's Last Theorem.

**(2.3.4)** *If  $p$  is a prime and  $x$  is any integer, then  $x^p \equiv x \pmod{p}$ .*

*Proof.* Since  $(-x)^p \equiv -x^p \pmod{p}$ , whether or not  $p$  is odd, there is no loss in assuming that  $x \geq 0$ . We will use induction on  $x$  to show that  $x^p \equiv x \pmod{p}$ , which certainly holds for  $x = 0$ . Assume it is true for  $x$ . Then by the Binomial Theorem

$$(x+1)^p = \sum_{r=0}^p \binom{p}{r} x^r \equiv x^p + 1 \pmod{p}$$

since  $p$  divides  $\binom{p}{r}$  if  $0 < r < p$ . Because  $x^p \equiv x \pmod{p}$ , it follows that  $(x+1)^p \equiv x+1 \pmod{p}$ . The induction is now complete.  $\square$

**Solving Congruences.** Just as we solve equations for unknown real numbers, we can try to solve congruences for unknown integers. The simplest case is that of a *linear congruence* with a single unknown  $x$ ; this has the form  $ax \equiv b \pmod{m}$ , where  $a, b, m$  are given integers.

**(2.3.5)** *Let  $a, b, m$  be integers with  $m > 0$ . Then there is a solution  $x$  of the congruence  $ax \equiv b \pmod{m}$  if and only if  $\gcd\{a, m\}$  divides  $b$ .*

*Proof.* Set  $d = \gcd\{a, m\}$ . If  $x$  is a solution of congruence  $ax \equiv b \pmod{m}$ , then  $ax = b + mq$  for some  $q \in \mathbb{Z}$ , from which it follows that  $d$  must divide  $b$ . Conversely, assume that  $d \mid b$ . By (2.2.3) there are integers  $u, v$  such that  $d = au + mv$ . Multiplying this equation by the integer  $b/d$ , we obtain  $b = a(ub/d) + m(vb/d)$ . Put  $x = ub/d$ , which is an integer; then  $ax \equiv b \pmod{m}$  and  $x$  is a solution of the congruence.  $\square$

The most important case is for  $b = 1$ .

**Corollary (2.3.6)** *Let  $a, m$  be integers with  $m > 0$ . Then the congruence  $ax \equiv 1 \pmod{m}$  has a solution  $x$  if and only if  $a$  is relatively prime to  $m$ .*

It is worthwhile translating the last result into the language of congruence arithmetic. Given an integer  $m > 0$  and a congruence class  $[a]$  modulo  $m$ , there is a congruence class  $[x]$  such that  $[a][x] = [1]$  if and only if  $a$  is relatively prime to  $m$ . Thus we can tell which congruence classes modulo  $m$  have “inverses”: they are classes  $[x]$  where  $0 < x < m$  and  $x$  is relatively prime to  $m$ . The number of invertible congruence classes modulo  $m$  is denoted by

$$\phi(m).$$

Here  $\phi$  is called *Euler's<sup>4</sup> function*. Next we consider systems of linear congruences.

**(2.3.7)** (The Chinese Remainder Theorem) *Let  $a_1, a_2, \dots, a_k$  and  $m_1, m_2, \dots, m_k$  be integers with  $m_i > 0$ . Assume that  $\gcd\{m_i, m_j\} = 1$  if  $i \neq j$ . Then there is a common solution  $x$  of the system of congruences*

$$\begin{cases} x \equiv a_1 \pmod{m_1} \\ x \equiv a_2 \pmod{m_2} \\ \vdots \\ x \equiv a_k \pmod{m_k}. \end{cases}$$

When  $k = 2$ , this striking result was discovered by the Chinese mathematician Sun Tse, who lived sometime between the Third and Fifth centuries AD.

*Proof of (2.3.7).* Put  $m = m_1 m_2 \cdots m_k$  and  $m'_i = m/m_i$ . Then  $m_i$  and  $m'_i$  are relatively prime, so by (2.3.6) there exists an integer  $\ell_i$  such that  $m'_i \ell_i \equiv 1 \pmod{m_i}$ . Now let  $x = a_1 m'_1 \ell_1 + \cdots + a_k m'_k \ell_k$ . Then

$$x \equiv a_i m'_i \ell_i \equiv a_i \pmod{m_i}$$

since  $m_i \mid m'_j$  if  $i \neq j$ .  $\square$

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<sup>4</sup> Leonhard Euler (1707–1783)

As an application of (2.3.7) a well-known formula for Euler's function will be derived.

**(2.3.8)**

- (i) If  $m$  and  $n$  are relatively prime positive integers, then  $\phi(mn) = \phi(m)\phi(n)$ .  
(ii) If  $m = p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}$  with  $l_i > 0$  and distinct primes  $p_i$ , then

$$\phi(m) = \prod_{i=1}^k (p_i^{l_i} - p_i^{l_i-1}).$$

*Proof.* (i) Let  $U_m$  denote the set of invertible congruence classes in  $\mathbb{Z}_m$ . Thus  $|U_m| = \phi(m)$ . Define a map  $\alpha : U_{mn} \rightarrow U_m \times U_n$  by the rule  $\alpha([a]_{mn}) = ([a]_m, [a]_n)$ . First of all observe that  $\alpha$  is well-defined. Next suppose that  $\alpha([a]_{mn}) = \alpha([a']_{mn})$ . Then  $[a]_m = [a']_m$  and  $[a]_n = [a']_n$ , equations which imply that  $a - a'$  is divisible by  $m$  and  $n$ , and hence by  $mn$ . Therefore  $[a]_{mn} = [a']_{mn}$  and  $\alpha$  is an injective function.

In fact  $\alpha$  is also surjective. For, if  $[a]_m \in U_m$  and  $[b]_n \in U_n$  are given, the Chinese Remainder Theorem assures us that there is an integer  $x$  such that  $x \equiv a \pmod{m}$  and  $x \equiv b \pmod{n}$ . Hence  $[x]_m = [a]_m$  and  $[x]_n = [b]_n$ , so that  $\alpha([x]_{mn}) = ([a]_m, [b]_n)$ . Therefore  $\alpha$  is a bijection and consequently  $|U_{mn}| = |U_m \times U_n| = |U_m| \cdot |U_n|$ , as required.

(ii) Suppose that  $p$  is a prime and  $n > 0$ . There are  $p^{n-1}$  multiples of  $p$  among the integers  $0, 1, \dots, p^n - 1$ ; therefore  $\phi(p^n) = p^n - p^{n-1}$ . Finally apply (2.3.8)(i) to obtain the formula indicated.  $\square$

We end the chapter with several examples which illustrate the utility of congruences.

**Example (2.3.1)** Show that an integer is divisible by 3 if and only if the sum of its digits is a multiple of 3.

Let  $n = a_0 a_1 \dots a_k$  be the decimal representation of an integer  $n$ . Thus  $n = a_k + a_{k-1}10 + a_{k-2}10^2 + \cdots + a_0 10^k$  where  $0 \leq a_i < 10$ . The key observation is that  $10 \equiv 1 \pmod{3}$ , i.e.,  $[10] = [1]$ . Hence  $[10^i] = [10]^i = [1]^i = [1]$ , i.e.,  $10^i \equiv 1 \pmod{3}$  for all  $i \geq 0$ . It therefore follows that  $n \equiv a_0 + a_1 + \cdots + a_k \pmod{3}$ . The assertion is an immediate consequence of this congruence.

**Example (2.3.2)** (Days of the week) Congruences have long been used implicitly to compute dates. As an example, let us determine what day of the week September 25 of the year 2020 will be.

To keep track of the days assign the integers  $0, 1, 2, \dots, 6$  as labels for the days of the week, say Sunday = 0, Monday = 1,  $\dots$ , Saturday = 6. Suppose that we reckon from January 5, 2014, which was a Sunday. All we have to do is count the number of days from this date to September 25, 2020. Allowing for leap years, this number is 2455. Now  $2455 \equiv 5 \pmod{7}$  and 5 is the label for Friday. Therefore September 25, 2020 will be a Friday.

**Example (2.3.3)** (The Basket of Eggs Problem) What is the smallest number of eggs a basket can contain if, when eggs are removed  $k$  at time, there is one egg left when  $k = 2, 3, 4, 5$  or  $6$  and there are no eggs left when  $k = 7$ ? (This ancient problem is mentioned in an Indian manuscript of the 7th Century).

Let  $x$  be the number of eggs in the basket. The conditions require that  $x \equiv 1 \pmod{k}$  for  $k = 2, 3, 4, 5, 6$  and  $x \equiv 0 \pmod{7}$  for  $k = 7$ . Clearly this amounts to  $x$  satisfying the four congruences  $x \equiv 1 \pmod{3}$ ,  $x \equiv 1 \pmod{4}$ ,  $x \equiv 1 \pmod{5}$  and  $x \equiv 0 \pmod{7}$ . Furthermore these are equivalent to the congruences

$$x \equiv 1 \pmod{60} \quad \text{and} \quad x \equiv 0 \pmod{7}.$$

By the Chinese Remainder Theorem there is a solution to this pair of congruences: we have to find the smallest positive solution. Applying the method of the proof of (2.3.7), we have  $m_1 = 60$ ,  $m_2 = 7$ ,  $m = 420$  and thus  $m'_1 = 7$ ,  $m'_2 = 60$ . Also  $\ell_1 = 43$ ,  $\ell_2 = 2$ . Therefore one solution is given by  $x = 1 \cdot 7 \cdot 43 + 0 \cdot 60 \cdot 2 = 301$ . If  $y$  is any other solution, observe that  $y - x$  must be divisible by  $60 \times 7 = 420$ . Hence the general solution is  $x = 301 + 420q$ ,  $q \in \mathbb{Z}$ . So the smallest positive solution is 301.

The next example is a refinement of Euclid's Theorem on the infinity of primes – see (2.2.8).

**Example (2.3.4)** Prove that there are infinitely many primes of the form  $3n + 2$  where  $n$  is an integer  $\geq 0$ .

In fact the proof is a variant of Euclid's method. Suppose the result is false and let the odd primes of the form  $3n + 2$  be  $p_1, p_2, \dots, p_k$ . Now consider the positive integer  $m = 3p_1p_2 \cdots p_k + 2$ . Notice that  $m$  is odd and it is not divisible by any  $p_i$ . Therefore  $m$  is a product of odd primes different from  $p_1, \dots, p_k$ . Hence  $m$  must be a product of primes of the form  $3n + 1$  since every integer is of the form  $3n$ ,  $3n + 1$  or  $3n + 2$ . It follows that  $m$  itself must have the form  $3n + 1$  and thus  $m \equiv 1 \pmod{3}$ . On the other hand,  $m \equiv 2 \pmod{3}$ , so we have reached a contradiction.

Actually this exercise is a special case of a famous theorem of Dirichlet<sup>5</sup>: every arithmetic progression  $an + b$ , where  $n = 0, 1, 2, \dots$ , and the integers  $a$  and  $b$  are positive and relatively prime, contains infinitely many primes.

**Example (2.3.5)** (The RSA Cryptosystem) This is a secure system for message encryption which has been widely used for transmitting sensitive data since its invention in 1977 by R. Rivest, A. Shamir and L. Adleman. It has the advantage of being a public key system in which only the decyphering function is not available to the public.

Suppose that a message is to be sent from A to B. The parameters required are two distinct large primes  $p$  and  $q$ . Put  $n = pq$  and  $m = \phi(n)$ ; therefore  $m = (p - 1)(q - 1)$

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<sup>5</sup> Johann Peter Gustav Lejeune Dirichlet (1805–1859)

by (2.3.8). Let  $a$  be an integer in the range 1 to  $m$  which is relatively prime to  $m$ . Then by (2.3.6) there is a unique integer  $b$  satisfying  $0 < b < m$  and  $ab \equiv 1 \pmod{m}$ . The sender A is assumed to know the integers  $a$  and  $n$ , while the receiver B knows  $b$  and  $n$ .

The message to be sent is first converted to an integer  $x$  which is not divisible by  $p$  or  $q$  and satisfies  $0 < x < n$ . Then A encyphers  $x$  by raising it to the power  $a$  and then reducing modulo  $n$ . In this form the message is transmitted to B. On receiving the transmitted message, B raises it to the power  $b$  and reduces modulo  $n$ . The result will be the original message  $x$ . What is being claimed here is that  $x^{ab} \equiv x \pmod{n}$ , since  $0 < x < n$ . To see why this holds, first write

$$ab = 1 + lm = 1 + l(p-1)(q-1)$$

with  $l$  an integer. Then

$$x^{ab} = x^{1+l(p-1)(q-1)} = x(x^{p-1})^{l(q-1)} \equiv x \pmod{p}$$

since  $x^{p-1} \equiv 1 \pmod{p}$  by Fermat's Theorem. Hence  $p$  divides  $x^{ab} - x$ , and in a similar way  $q$  also divides this number. Therefore  $n = pq$  divides  $x^{ab} - x$  as claimed.

Even if  $n$  and  $a$  become public knowledge, it will be difficult to break the system by finding  $b$ . For this would require computation of the inverse of  $[a]$  in  $\mathbb{Z}_m$ . To do this using the Euclidean Algorithm, the result that lies behind (2.3.6), one would need to know the primes  $p$  and  $q$ . But the problem of factorizing the integer  $n = pq$  in order to discover the primes  $p$  and  $q$  is considered to be computationally very hard. Thus the RSA-system remains secure until more efficient ways of factorizing large numbers become available.

### Exercises (2.3)

- (1) Establish the properties of congruences listed in (2.3.2).
- (2) In  $\mathbb{Z}_{24}$  find the inverses of  $[7]$  and  $[13]$ .
- (3) Show that if  $n$  is an odd integer,  $n^2 \equiv 1 \pmod{8}$ .
- (4) Find the general solution of the congruence  $6x \equiv 11 \pmod{5}$ .
- (5) What day of the week will April 1, 2030 be?
- (6) Find the smallest positive solution  $x$  of the system of congruences  $x \equiv 4 \pmod{3}$ ,  $x \equiv 5 \pmod{7}$ ,  $x \equiv 6 \pmod{11}$ .
- (7) Prove that there are infinitely many primes of the form  $4n + 3$ .
- (8) Prove that there are infinitely many primes of the form  $6n + 5$ .
- (9) In a certain culture the festivals of the snake, the monkey and the fish occur every 6, 5 and 11 years respectively. The next festivals occur in 3, 4 and 1 years respectively. How many years must pass before all three festivals occur in the same year?
- (10) Prove that no integer of the form  $4n + 3$  can be written as the sum of two squares of integers.



### 3 Introduction to groups

Groups constitute one of the most important and natural structures in algebra. They also feature in other areas of mathematics such as geometry, topology and combinatorics. In addition groups arise in many areas of science, typically in situations where symmetry is important, as in atomic physics and crystallography. More general algebraic structures which have recently come to prominence due to the rise of information science include semigroups and monoids. This chapter serves as an introduction to these types of structure.

There is a continuing debate as to whether it is better to introduce groups or rings first in an introductory course in algebra: here we take the point of view that groups are logically the simpler objects since they involve only one binary operation, whereas rings have two. Accordingly rings are left until Chapter Six.

Historically the first groups to be studied consisted of *permutations*, i.e., bijective functions on a set. Indeed for most of the 19th century “group” was synonymous with “group of permutations”. Since permutation groups have the great advantage that their elements are concrete and easy to compute with, we begin this chapter with a discussion of permutations.

#### 3.1 Permutations

If  $X$  is any non-empty set, a bijective function  $\pi : X \rightarrow X$  is called a *permutation* of  $X$ . Thus by (1.3.1)  $\pi$  has a unique inverse function  $\pi^{-1} : X \rightarrow X$ , which is also a permutation. The set of all permutations of the set  $X$  is denoted by

$$\text{Sym}(X),$$

which stands for *the symmetric group on  $X$* .

If  $\pi$  and  $\sigma$  are permutations of  $X$ , their composite  $\pi \circ \sigma$  is also a permutation; this is because it has an inverse, namely the permutation  $\sigma^{-1} \circ \pi^{-1}$  by (1.3.2). In the future for the sake of simplicity we will usually write

$$\pi\sigma$$

for  $\pi \circ \sigma$ . Of course  $\text{id}_X$ , the identity function on  $X$ , is a permutation.

At this juncture we pause to note some features of the set  $\text{Sym}(X)$ : this set is “closed” with respect to forming inverses and composites, by which we mean that if  $\pi, \sigma \in \text{Sym}(X)$ , then  $\pi^{-1}$  and  $\pi \circ \sigma$  belong to  $\text{Sym}(X)$ . In addition  $\text{Sym}(X)$  contains the identity permutation  $\text{id}_X$ , which has the property  $\text{id}_X \circ \pi = \pi = \pi \circ \text{id}_X$ . And finally, the associative law for permutations is valid,  $(\pi \circ \sigma) \circ \tau = \pi \circ (\sigma \circ \tau)$ . In fact what these properties assert is that the pair  $(\text{Sym}(X), \circ)$  is a group, as defined in (3.2). Thus the permutations of a set afford a very natural example of a group.

**Permutations of finite sets.** We now begin the study of permutations of a finite set with  $n$  elements,

$$X = \{x_1, x_2, \dots, x_n\}.$$

Let  $\pi \in \text{Sym}(X)$ . Since  $\pi$  is injective,  $\pi(x_1), \pi(x_2), \dots, \pi(x_n)$  are all different and therefore constitute all  $n$  elements of the set  $X$ , but possibly in some order different from  $x_1, x_2, \dots, x_n$ . Thus we can think of a permutation as a rearrangement of the order  $x_1, x_2, \dots, x_n$ . A convenient way to denote the permutation  $\pi$  is

$$\pi = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ \pi(x_1) & \pi(x_2) & \dots & \pi(x_n) \end{pmatrix}$$

where the second row consists of the images under  $\pi$  of the elements of the first row. It should be clear to the reader that nothing essential is lost if we take  $X$  to be the set  $\{1, 2, \dots, n\}$ . With this choice of  $X$ , it is usual to write

$$S_n$$

for  $\text{Sym}(X)$ ; this is called the *symmetric group of degree  $n$* .

Computations with elements of  $S_n$  are easily performed by working directly from the definitions. An example will illustrate this.

**Example (3.1.1)** Let

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 5 & 3 & 4 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

be elements of  $S_6$ . Hence

$$\pi\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 5 & 2 & 1 & 3 \end{pmatrix}, \quad \sigma\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 4 & 3 \end{pmatrix}$$

and

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 4 & 1 \end{pmatrix}.$$

Here  $\pi\sigma$  has been computed using the definition  $\pi\sigma(i) = \pi(\sigma(i))$ , while  $\pi^{-1}$  is readily obtained by reading up from 1, 2, ..., 6 in the second row of  $\pi$  to obtain the second row of  $\pi^{-1}$ . Notice that  $\pi\sigma \neq \sigma\pi$ , i. e., multiplication of permutations is not commutative in general.

A simple count establishes the number of permutations of a finite set.

**(3.1.1)** If  $X$  is a set with  $n$  elements, then  $|\text{Sym}(X)| = n!$ .

*Proof.* Consider the number of ways of constructing the second row of a permutation

$$\pi = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}$$

There are  $n$  choices for  $y_1$ , but only  $n - 1$  choices for  $y_2$  since  $y_1$  cannot be chosen again. Next we cannot choose  $y_1$  or  $y_2$  again, so there are  $n - 2$  choices for  $y_3$ , and so on; finally, there is just one choice for  $y_n$ . Each choice of a  $y_i$  leads to a different permutation. Therefore the number of different permutations of  $X$  is  $n(n - 1)(n - 2) \cdots 1 = n!$ .  $\square$

**Cyclic permutations.** Let  $\pi \in S_n$ , so that  $\pi$  is a permutation of the set  $\{1, 2, \dots, n\}$ . The *support* of  $\pi$  is defined to be the set of all  $i$  such that  $\pi(i) \neq i$ , in symbols

$$\text{supp}(\pi).$$

Let  $r$  be an integer satisfying  $1 \leq r \leq n$ . Then  $\pi$  is called an  $r$ -cycle if  $\text{supp}(\pi) = \{i_1, i_2, \dots, i_r\}$ , with distinct  $i_j$ , where  $\pi(i_1) = i_2$ ,  $\pi(i_2) = i_3, \dots, \pi(i_{r-1}) = i_r$  and  $\pi(i_r) = i_1$ . To visualize the permutation think of the integers  $i_1, i_2, \dots, i_r$  as being arranged in this order anticlockwise round a circle. Then  $\pi$  has the effect of rotating the circle in the anticlockwise direction. Of course  $\pi$  fixes all the other integers: often  $\pi$  is written in the form

$$\pi = (i_1 i_2 \cdots i_r)(i_{r+1}) \cdots (i_n)$$

where the presence of a 1-cycle ( $j$ ) means that  $\pi(j) = j$ . The notation may be abbreviated by omitting all 1-cycles, although if this is done, the integer  $n$  may need to be specified.

In particular a 2-cycle has the form  $(ij)$ : it interchanges  $i$  and  $j$  and fixes all other integers. 2-cycles are frequently called *transpositions*.

**Example (3.1.2)** The permutation  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix}$  is the 3-cycle  $(125)(3)(4)$ , that is,  $(125)$ . While

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 5 & 8 & 7 & 2 & 3 & 4 \end{pmatrix}$$

is not a cycle, it is the composite of three cycles of length  $> 1$ , namely  $(162) \circ (357) \circ (48)$ , as one can see by following what happens to each of the integers  $1, 2, \dots, 8$  when the permutation is applied. In fact this is an instance of an important general result, that any permutation is expressible as a composite of cycles: this will be established in (3.1.3).

It should be observed that there are  $r$  different ways to write an  $r$ -cycle since any element of the cycle can be the initial element: indeed  $(i_1 i_2 \dots i_r) = (i_2 i_3 \dots i_r i_1) = \cdots = (i_r i_1 i_2 \dots i_{r-1})$ .

Two permutations  $\pi, \sigma$  in  $S_n$  are said to be *disjoint* if their supports are disjoint, i.e., they do not both move the same element. An important fact about disjoint permutations is that they commute, in contrast to permutations in general.

**(3.1.2)** If  $\pi$  and  $\sigma$  are disjoint permutations in  $S_n$ , then  $\pi\sigma = \sigma\pi$ .

*Proof.* Let  $i \in \{1, 2, \dots, n\}$ ; we show that  $\pi\sigma(i) = \sigma\pi(i)$ . If  $i \notin \text{supp}(\pi) \cup \text{supp}(\sigma)$ , then plainly  $\pi\sigma(i) = i = \sigma\pi(i)$ . Suppose that  $i \in \text{supp}(\pi)$ ; then  $i \notin \text{supp}(\sigma)$  and  $\sigma(i) = i$ . Thus  $\pi\sigma(i) = \pi(i)$ . Also  $\sigma\pi(i) = \pi(i)$ ; for otherwise  $\pi(i) \in \text{supp}(\sigma)$  and so  $\pi(i) \notin \text{supp}(\pi)$ , which leads to  $\pi(\pi(i)) = \pi(i)$ . However,  $\pi^{-1}$  can be applied to both sides of this equation to give  $\pi(i) = i$ , a contradiction since  $i \in \text{supp}(\pi)$ .  $\square$

**Powers of a permutation.** Since we know how to form products of permutations using composition, it is natural to define powers of a permutation. Let  $\pi \in S_n$  and let  $i$  be a non-negative integer. Then the  $i$ th power  $\pi^i$  is defined recursively by the rules:

$$\pi^0 = \text{id}, \quad \pi^{i+1} = \pi^i \pi.$$

The point to note here is that the rule allows us to compute successive powers of the permutation as follows:  $\pi^1 = \pi$ ,  $\pi^2 = \pi\pi$ ,  $\pi^3 = \pi^2\pi$ , etc. Powers are used in the proof of the following fundamental theorem.

**(3.1.3)** *Let  $\pi \in S_n$ . Then  $\pi$  is expressible as a product of disjoint cycles and the cycles appearing in the product are unique.*

*Proof.* We deal with the existence of the expression first. If  $\pi$  is the identity, then obviously  $\pi = (1)(2) \cdots (n)$ . Assume that  $\pi \neq \text{id}$  and choose an integer  $i_1$  such that  $\pi(i_1) \neq i_1$ . Now the integers  $i_1, \pi(i_1), \pi^2(i_1), \dots$  belong to the finite set  $\{1, 2, \dots, n\}$  and so they cannot all be different; say  $\pi^r(i_1) = \pi^s(i_1)$  where  $r > s \geq 0$ . Applying  $(\pi^{-1})^s$  to both sides of the equation and using associativity, we find that  $\pi^{r-s}(i_1) = i_1$ . Hence by the Well-Ordering Law there is a least positive integer  $m_1$  such that  $\pi^{m_1}(i_1) = i_1$ .

Next we argue that the integers  $i_1, \pi(i_1), \pi^2(i_1), \dots, \pi^{m_1-1}(i_1)$  are all different. For if not and  $\pi^r(i_1) = \pi^s(i_1)$  where  $m_1 > r > s \geq 0$ , then, just as above, we can argue that  $\pi^{r-s}(i_1) = i_1$ ; on the other hand,  $0 < r - s < m_1$ , which contradicts the choice of  $m_1$ . It follows that  $\pi$  permutes the  $m_1$  distinct integers  $i_1, \pi(i_1), \dots, \pi^{m_1-1}(i_1)$  in a cycle, so that we have identified the  $m_1$ -cycle  $(i_1 \pi(i_1) \dots \pi^{m_1-1}(i_1))$  as a component of  $\pi$ .

If  $\pi$  fixes all other integers, then  $\pi = (i_1 \pi(i_1) \dots \pi^{m_1-1}(i_1))$  and  $\pi$  is an  $m_1$ -cycle. Otherwise there exists an integer  $i_2 \notin \{i_1, \pi(i_1), \dots, \pi^{m_1-1}(i_1)\}$  such that  $\pi(i_2) \neq i_2$ . Just as above we identify a second cycle  $(i_2 \pi(i_2) \dots \pi^{m_2-1}(i_2))$  present in  $\pi$ . This is disjoint from the first cycle. Indeed, if the cycles had a common element, they would have to coincide. It should also be clear that by a finite number of applications of this procedure we can express  $\pi$  as a product of disjoint cycles.

Next to establish uniqueness. Assume that there are two expressions for  $\pi$  as a product of disjoint cycles, say  $(i_1 i_2 \dots)(j_1 j_2 \dots) \cdots$  and  $(i'_1 i'_2 \dots)(j'_1 j'_2 \dots) \cdots$ . By (3.1.2) disjoint cycles commute. Thus without loss of generality we can assume that  $i_1$  occurs in the cycle  $(i'_1 i'_2 \dots)$ . Since any element of a cycle can be moved up to the initial position, it can also be assumed that  $i_1 = i'_1$ . Then  $i_2 = \pi(i_1) = \pi(i'_1) = i'_2$ ; similarly  $i_3 = i'_3$ , etc. The other cycles are dealt with in the same manner. Therefore the two expressions for  $\pi$  are identical.  $\square$

**Corollary (3.1.4)** If  $n > 1$ , every element of  $S_n$  is expressible as a product of transpositions.

*Proof.* Because of (3.1.3) it is sufficient to show that each cyclic permutation is a product of transpositions. That this is true follows from the easily verified identity:

$$(i_1 i_2 \cdots i_{r-1} i_r) = (i_1 i_r)(i_1 i_{r-1}) \cdots (i_1 i_3)(i_1 i_2). \quad \square$$

**Example (3.1.3)** Express  $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 1 & 4 & 2 \end{pmatrix}$  as a product of transpositions.

First of all write  $\pi$  as a product of disjoint cycles, following the method of the proof of (3.1.3) to get  $\pi = (1354)(26)$ . Also  $(1354) = (14)(15)(13)$ , so that

$$\pi = (14)(15)(13)(26).$$

On the other hand not every permutation in  $S_n$  is expressible as a product of *disjoint* transpositions. (Why not?)

**Even and odd permutations.** If  $\pi$  is a permutation in  $S_n$ , then  $\pi$  replaces the natural order of integers,  $1, 2, \dots, n$  by the new order  $\pi(1), \pi(2), \dots, \pi(n)$ . Thus  $\pi$  may cause inversions of the natural order: here an *inversion* occurs if for some  $i < j$ , we have  $\pi(i) > \pi(j)$ . To clarify the definition it is convenient to introduce a formal device.

Consider a polynomial  $f$  in indeterminates  $x_1, x_2, \dots, x_n$ , with integer coefficients. (Here we assume the reader is familiar with the concept of a polynomial). If  $\pi \in S_n$ , then  $\pi$  determines a new polynomial  $\pi f$  which is obtained by permuting the variables  $x_1, x_2, \dots, x_n$ . Thus  $\pi f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ . For example, if  $f = x_1 - x_2 - 2x_3$  and  $\pi = (12)(3)$ , then  $\pi f = x_2 - x_1 - 2x_3$ .

Now consider the polynomial

$$f(x_1, \dots, x_n) = \prod_{\substack{i,j=1 \\ i < j}}^n (x_i - x_j).$$

A typical factor in  $\pi f$  is  $x_{\pi(i)} - x_{\pi(j)}$ . Now if  $\pi(i) < \pi(j)$ , this is also a factor of  $f$ , while if  $\pi(i) > \pi(j)$ , then  $-(x_{\pi(i)} - x_{\pi(j)})$  is a factor of  $f$ . Consequently  $\pi f = +f$  if the number of inversions of the natural order in  $\pi$  is even and  $\pi f = -f$  if it is odd. This observation permits us to define the *sign* of the permutation  $\pi$  to be

$$\text{sign}(\pi) = \frac{\pi f}{f}.$$

Thus  $\text{sign}(\pi) = 1$  or  $-1$  according as the number of inversions in  $\pi$  is even or odd. Call  $\pi$  an *even permutation* if  $\text{sign}(\pi) = 1$  and an *odd permutation* if  $\text{sign}(\pi) = -1$ .

**Example (3.1.4)** The even permutations in  $S_3$  are  $(1)(2)(3)$ ,  $(123)$  and  $(132)$ , while the odd permutations are  $(1)(23)$ ,  $(2)(13)$  and  $(3)(12)$ .

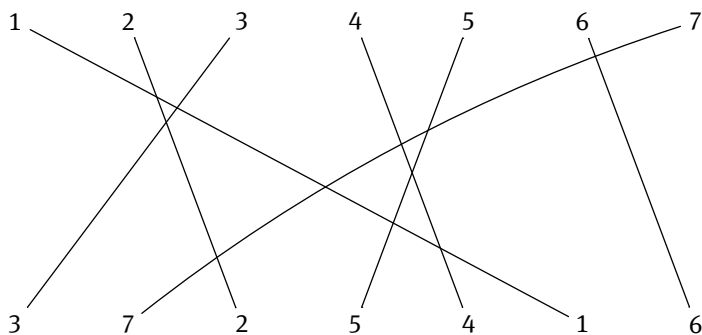
In deciding if a permutation is even or odd a *crossover diagram* is a useful tool. We illustrate this idea with an example.

**Example (3.1.5)** Is the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 2 & 5 & 4 & 1 & 6 \end{pmatrix}$$

even or odd?

To construct the crossover diagram simply join equal integers in the top and bottom rows of  $\pi$  and count the intersections or “crossovers”, taking care to avoid multiple or unnecessary intersections. A crossover indicates the presence of an inversion of the natural order.

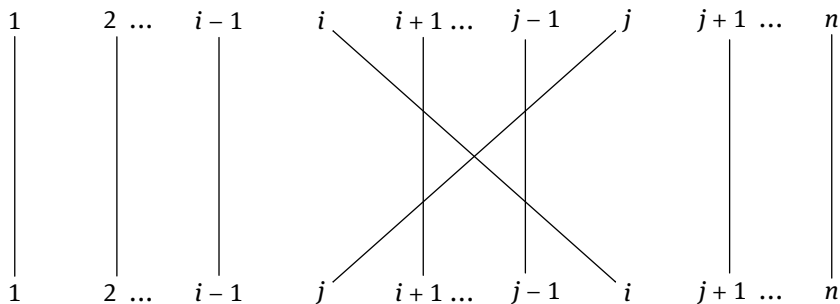


There are 11 crossovers, so  $\text{sign}(\pi) = -1$  and  $\pi$  is an odd permutation.

The next result records very significant property of transpositions.

**(3.1.5)** *Transpositions are always odd.*

*Proof.* Consider the crossover diagram for the transposition  $(ij)$  where  $i < j$ .



An easy count reveals the presence of  $1 + 2(j - i - 1)$  crossovers. Since this integer is certainly odd,  $(ij)$  is an odd permutation.  $\square$

The basic properties of the sign function are laid out next.

**(3.1.6)** Let  $\pi, \sigma \in S_n$ . Then the following hold:

- (i)  $\text{sign}(\pi\sigma) = \text{sign}(\pi) \text{sign}(\sigma)$ ;
- (ii)  $\text{sign}(\pi^{-1}) = \text{sign}(\pi)$ .

*Proof.* Let  $f = \prod_{i < j=1}^n (x_i - x_j)$ . Since  $\pi f = \text{sign}(\pi)f$ , we have

$$\begin{aligned} \pi\sigma f(x_1, \dots, x_n) &= \pi(\sigma f(x_1, \dots, x_n)) \\ &= \pi((\text{sign}(\sigma)f(x_1, \dots, x_n))) \\ &= \text{sign}(\sigma)\pi f(x_1, \dots, x_n) \\ &= \text{sign}(\sigma) \text{sign}(\pi)f(x_1, \dots, x_n). \end{aligned}$$

Since  $(\pi\sigma)f = \text{sign}(\pi\sigma)f$ , it follows that  $\text{sign}(\pi\sigma) = \text{sign}(\pi) \text{sign}(\sigma)$ . Finally, by (i) we have  $1 = \text{sign}(\text{id}) = \text{sign}(\pi\pi^{-1}) = \text{sign}(\pi) \text{sign}(\pi^{-1})$ , so that  $\text{sign}(\pi^{-1}) = 1/\text{sign}(\pi) = \text{sign}(\pi)$ .  $\square$

**Corollary (3.1.7)** A permutation  $\pi$  in  $S_n$  is even (odd) if and only if it is a product of an even (respectively odd) number of transpositions.

For, if  $\pi = \prod_{i=1}^k \pi_i$  with each  $\pi_i$  a transposition, then

$$\text{sign}(\pi) = \prod_{i=1}^k \text{sign}(\pi_i) = (-1)^k$$

by (3.1.5) and (3.1.6).

The subset of all even permutations in  $S_n$  is denoted by

$$A_n,$$

which is called the *alternating group of degree  $n$* . Obviously  $A_1 = S_1$ . For  $n > 1$  exactly half of the permutations in  $S_n$  are even, as the next result shows.

**(3.1.8)** If  $n > 1$ , there are  $\frac{1}{2}(n!)$  even permutations and  $\frac{1}{2}(n!)$  odd permutations in  $S_n$ .

*Proof.* Define a function  $\alpha : A_n \rightarrow S_n$  by the rule  $\alpha(\pi) = \pi \circ (12)$ , observing that  $\alpha(\pi)$  is odd and  $\alpha$  is injective. Every odd permutation  $\sigma$  belongs to  $\text{Im}(\alpha)$  since  $\alpha(\pi) = \sigma$  where  $\pi = \sigma \circ (12) \in A_n$ . Thus  $\text{Im}(\alpha)$  is precisely the set of all odd permutations and  $|\text{Im}(\alpha)| = |A_n|$ .  $\square$

**(3.1.9)** (Cauchy's<sup>1</sup> Formula) If  $\pi$  in  $S_n$  is the product of  $c$  disjoint cycles, including 1-cycles, then

$$\text{sign}(\pi) = (-1)^{n-c}.$$

<sup>1</sup> Augustin Louis Cauchy (1789–1857)

*Proof.* Let  $\pi = \sigma_1 \sigma_2 \cdots \sigma_c$  where the  $\sigma_i$  are disjoint cycles and  $\sigma_i$  has length  $\ell_i$ . Now  $\sigma_i$  is expressible as a product of  $\ell_i - 1$  transpositions by the proof of (3.1.4). Hence by (3.1.6) we have  $\text{sign}(\sigma_i) = (-1)^{\ell_i-1}$  and thus

$$\text{sign}(\pi) = \prod_{i=1}^c \text{sign}(\sigma_i) = \prod_{i=1}^c (-1)^{\ell_i-1} = (-1)^{n-c}$$

since  $\sum_{i=1}^c \ell_i = n$ . □

**Derangements.** We conclude the section with a discussion of a special type of permutation. A permutation of a set is called a *derangement* if it fixes no elements of the set, i.e., its support is the entire set. For example,  $(1234)(56)$  is a derangement in  $S_6$ . A natural question is: how many derangements does  $S_n$  contain? To answer the question we employ a well known combinatorial principle.

**(3.1.10)** (The Inclusion–Exclusion Principle) *If  $A_1, A_2, \dots, A_r$  are finite sets, then*

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_r| &= \sum_{i=1}^r |A_i| - \sum_{i < j=1}^r |A_i \cap A_j| + \\ &\quad \sum_{i < j < k=1}^r |A_i \cap A_j \cap A_k| + \cdots + (-1)^{r-1} |A_1 \cap A_2 \cap \cdots \cap A_r|. \end{aligned}$$

*Proof.* We have to count the number of objects that belong to at least one  $A_i$ . Our first estimate is  $\sum_{i=1}^r |A_i|$ , but this double counts elements in more than one  $A_i$ , so we subtract  $\sum_{i < j=1}^r |A_i \cap A_j|$ . But now elements belonging to three or more  $A_i$ 's have not been counted at all, so we must add  $\sum_{i < j < k=1}^r |A_i \cap A_j \cap A_k|$ . Now elements in four or more  $A_i$ 's have been double counted, and so on. After a succession of  $r$  such “inclusions” and “exclusions” we arrive at the correct formula. □

It is now relatively easy to count derangements.

**(3.1.11)** *The number of derangements in  $S_n$  is given by the formula*

$$d_n = n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right).$$

*Proof.* Let  $X_i$  denote the set of all permutations in  $S_n$  which fix the integer  $i$ , ( $1 \leq i \leq n$ ). Then the number of derangements in  $S_n$  is

$$d_n = n! - |X_1 \cup \cdots \cup X_n|.$$



Now  $|X_i| = (n-1)!$ ; also  $|X_i \cap X_j| = (n-2)!$ ,  $(i < j)$ , and  $|X_i \cap X_j \cap X_k| = (n-3)!$ ,  $(i < j < k)$ , etc. Therefore by the Inclusion–Exclusion Principle

$$d_n = n! - \left\{ \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \dots + (-1)^{n-1} \binom{n}{n}(n-n)! \right\}.$$

Here the reason is that there are  $\binom{n}{r}$  intersections  $X_{i_1} \cap X_{i_2} \cap \dots \cap X_{i_r}$  with  $i_1 < i_2 < \dots < i_r$ . The required formula appears after a minor simplification of the terms in the sum.  $\square$

Notice that  $\lim_{n \rightarrow \infty} \left( \frac{d_n}{n!} \right) = e^{-1} = 0.36787\dots$ , so roughly 36.8 % of the permutations in  $S_n$  are derangements.

**Example (3.1.6)** (The Hat Problem) There are  $n$  people attending a party each of whom wears a different hat. All the hats are checked in on arrival. Afterwards each person is given a hat at random. What is the probability that no one get the correct hat?

A distribution of hats corresponds to a permutation of the original order. The permutations that are derangements give the distributions in which everyone has the wrong hat. So the probability asked for is  $\frac{d_n}{n!}$  or roughly  $e^{-1}$ .

### Exercises (3.1)

- (1) Let  $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 5 & 3 & 6 \end{pmatrix}$  and  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 3 & 2 & 4 \end{pmatrix}$ . Compute  $\pi^{-1}$ ,  $\pi\sigma$  and  $\pi\sigma\pi^{-1}$ .
- (2) Determine which of the permutations in Exercise (3.1.1) are even and which are odd.
- (3) Prove that  $\text{sign}(\pi\sigma\pi^{-1}) = \text{sign}(\sigma)$  for all  $\pi, \sigma \in S_n$ .
- (4) Prove that if  $n > 1$ , every non-trivial element of  $S_n$  is a product of *adjacent* transpositions, i.e., transpositions of the form  $(i \ i+1)$ . [Hint: it is enough to prove the statement for a transposition  $(i \ j)$  where  $i < j$ . Now consider the composite  $(j \ j+1)(i \ j)(j \ j+1)$ .]
- (5) Prove that an element  $\pi$  in  $S_n$  satisfies  $\pi^2 = \text{id}$  if and only if  $\pi$  is a product of disjoint transpositions.
- (6) How many elements  $\pi$  in  $S_n$  satisfy  $\pi^2 = \text{id}$ ? [Hint: count the permutations which have exactly  $k$  disjoint transpositions for  $2k \leq n$  by first choosing  $2k$  integers from  $1, 2, \dots, n$  and then forming  $k$  transpositions from them.]
- (7) How many permutations in  $S_n$  contain at most one 1-cycle?
- (8) In the game of Rencontre there are two players A and B, each of whom has a regular pack of 52 cards. The players deal their cards simultaneously. If at some point they both deal the same card, this is a “rencontre” and player A wins. If no rencontre appears, player B wins. What are the probabilities of each player winning?

### 3.2 Semigroups, monoids and groups

Many of the structures that occur in algebra consist of a set together with a set of operations that can be applied to elements of the set. To make this precise, let us define a *binary operation on a set*  $S$  to be a function

$$\alpha : S \times S \rightarrow S.$$

Thus for each ordered pair  $(a, b)$  with  $a, b$  in  $S$  the function  $\alpha$  produces a unique element  $\alpha((a, b))$  of  $S$ . It is better notation if we write

$$a * b$$

instead of  $\alpha((a, b))$  and refer to the binary operation as  $*$ .

Of course binary operations abound: one need think no further than addition or multiplication in sets such as  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , or composition on the set of all functions on a given set.

The first algebraic structure of interest to us is a *semigroup*, which is a pair

$$(S, *)$$

consisting of a non-empty set  $S$  and a binary operation  $*$  on  $S$  which satisfies the *associative law*,

$$(i) \quad (a * b) * c = a * (b * c) \text{ for all } a, b, c \in S.$$

If the semigroup has an *identity element*, i.e., an element  $e$  of  $S$  such that

$$(ii) \quad a * e = a = e * a \text{ for all } a \in S,$$

then it is called a *monoid*.

Finally, the monoid is called a *group* if each element  $a$  has an inverse, i.e., an element  $a'$  of  $S$  such that

$$(iii) \quad a * a' = e = a' * a.$$

Also a semigroup  $(S, *)$  is said to be *commutative* if

$$(iv) \quad a * b = b * a \text{ for all } a, b \in S.$$

A commutative group is called an *abelian*<sup>2</sup> *group*.

Thus semigroups, monoids and groups form successively narrower classes of algebraic structures. These concepts will now be illustrated by some familiar examples.

#### Examples of semigroups, monoids and groups.

(i) The pairs  $(\mathbb{Z}, +)$ ,  $(\mathbb{Q}, +)$ ,  $(\mathbb{R}, +)$  are groups where  $+$  is ordinary addition, 0 is an identity element and an inverse of  $x$  is its negative  $-x$ .

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<sup>2</sup> After Niels Henrik Abel (1802–1829)

(ii) Next consider  $(\mathbb{Q}^*, \cdot)$ ,  $(\mathbb{R}^*, \cdot)$  where the dot denotes ordinary multiplication and  $\mathbb{Q}^*$  and  $\mathbb{R}^*$  are the sets of *non-zero* rational numbers and real numbers respectively. Here  $(\mathbb{Q}^*, \cdot)$  and  $(\mathbb{R}^*, \cdot)$  are groups, the identity element being 1 and an inverse of  $x$  being  $\frac{1}{x}$ . On the other hand,  $(\mathbb{Z}^*, \cdot)$  is only a monoid since the integer 2, for example, has no inverse in  $\mathbb{Z}^* = \mathbb{Z} - \{0\}$ .

(iii)  $(\mathbb{Z}_m, +)$  is a group where  $m$  is a positive integer. The usual addition of congruence classes is used here.

(iv)  $(\mathbb{Z}_m^*, \cdot)$  is a group where  $m$  is a positive integer: here  $\mathbb{Z}_m^*$  is the set of invertible congruence classes  $[a]$  modulo  $m$ , i.e., such that  $\gcd\{a, m\} = 1$ , and multiplication of congruence classes is used – see (2.3.6). Note that  $|\mathbb{Z}_m^*| = \phi(m)$  where  $\phi$  is Euler's function.

(v) Let  $M_n(\mathbb{R})$  be the set of all  $n \times n$  matrices with real entries. If the usual rule of addition of matrices is used,  $(M_n(\mathbb{R}), +)$  is an abelian group.

On the other hand,  $M_n(\mathbb{R})$  with matrix multiplication is only a monoid. To obtain a group we must form

$$GL_n(\mathbb{R}),$$

the subset of all invertible (or *non-singular*) matrices in  $M_n(\mathbb{R})$ : recall that these are the matrices with non-zero determinant. This group is called the *general linear group of degree  $n$  over  $\mathbb{R}$* .

(vi) For an example of a semigroup that is not a monoid we need look no further than the set of all even integers with multiplication as the group operation. Clearly there is no identity element here.

(vii) *The monoid of functions on a set.* Let  $A$  be any non-empty set, and write  $\text{Fun}(A)$  for the set of all mappings or functions  $\alpha$  on  $A$ . Then

$$(\text{Fun}(A), \circ)$$

is a monoid where  $\circ$  is functional composition. Indeed, this binary operation is associative by (1.2.3) and the identity function on  $A$  is an identity element.

If we restrict attention to the bijective functions on  $A$ , i.e., to those which have inverses, we obtain the *symmetric group* on  $A$

$$(\text{Sym}(A), \circ),$$

consisting of all the permutations of  $A$ . This example was the motivation for the definition of a group.

(viii) *Monoids of words.* For a different type of example we consider words in an alphabet  $X$ . Here  $X$  is any non-empty set and a *word* in  $X$  is just an  $n$ -tuple of elements of  $X$ , written for convenience without parentheses in the form  $x_1 x_2 \cdots x_n$ ,  $n \geq 0$ . The case  $n = 0$  is the *empty word*  $\emptyset$ . Let  $W(X)$  denote the set of all words in  $X$ .

There is a natural binary operation on  $X$ , namely juxtaposition. Thus, if  $u = x_1 \cdots x_n$  and  $v = y_1 \cdots y_m$  are words in  $X$ , define  $uv$  to be the word  $x_1 \cdots x_n y_1 \cdots y_m$ . If  $u = \emptyset$ , then

by convention  $uz = z = zu$  for all  $z$ . It is clear that this binary operation is associative and that  $\emptyset$  is an identity element. Thus  $W(X)$ , with the operation specified, is a monoid, the so-called *free monoid on  $X$* .

(ix) *Monoids and automata*. There is a somewhat unexpected connection between monoids and automata. Suppose that  $A = (I, S, \nu)$  is a state output automaton with input set  $I$ , state set  $S$  and next state function  $\nu : I \times S \rightarrow S$ : see Exercise (1.3.8). Then  $A$  determines a monoid  $M_A$  in the following way.

Let  $i \in I$  and define  $\theta_i : S \rightarrow S$  by the rule  $\theta_i(s) = \nu(i, s)$  where  $s \in S$ . Now let  $M_A$  consist of the identity function and all composites of finite sequences of  $\theta_i$ 's; thus  $M_A \subseteq \text{Fun}(S)$ . Clearly  $(M_A, \circ)$  is a monoid with respect to functional composition.

In fact one can go in the opposite direction as well. Let  $(M, *)$  be a monoid and define an automaton  $A_M = (M, M, \nu)$  where the next state function  $\nu : M \times M \rightarrow M$  is given by the rule  $\nu(x_1, x_2) = x_1 * x_2$ . Thus a connection between monoids and state output automata has been detected.

(x) *Symmetry groups*. As has been remarked, groups tend to arise wherever symmetry is of importance. The size of the group can be regarded as a measure of the amount of symmetry present. Since symmetry is at heart a geometric notion, it is not surprising that geometry provides many interesting examples of groups.

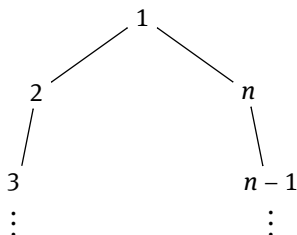
A bijective function defined on 3-dimensional space or the plane is called an *isometry* if it preserves distances between points. Natural examples of isometries are translations, rotations and reflections. Let  $X$  be a non-empty set of points in 3-space or the plane – we will refer to  $X$  as a geometric configuration. An isometry  $\alpha$  which fixes the set  $X$ , i.e., such that

$$X = \{\alpha(x) \mid x \in X\},$$

is called a *symmetry of  $X$* . Note that a symmetry can move the individual points of  $X$ .

It is easy to see that the symmetries of  $X$  form a group with respect to functional composition; this is the *symmetry group  $S(X)$*  of  $X$ . Thus  $S(X)$  is a subset of  $\text{Sym}(X)$ , usually a proper subset.

**The symmetry group of the regular  $n$ -gon.** As an illustration let us analyze the symmetries of the *regular  $n$ -gon*: this is a polygon in the plane with  $n$  edges of equal length, ( $n \geq 3$ ). It is convenient to label the vertices of the  $n$ -gon  $1, 2, \dots, n$ , so that each symmetry is represented by a permutation of the vertex set  $\{1, 2, \dots, n\}$ , i.e., by an element of  $S_n$ .



Each symmetry arises from an axis of symmetry of the figure. Of course, in order to obtain a group, we must include the identity symmetry, represented by  $(1)(2)\cdots(n)$ . There are  $n - 1$  anticlockwise rotations about the line perpendicular to the plane of the figure and through the centroid, through angles  $i(\frac{2\pi}{n})$ , for  $i = 1, 2, \dots, n - 1$ . For example, the rotation through  $\frac{2\pi}{n}$  is represented by the  $n$ -cycle  $(1\ 2\ 3\ \dots\ n)$ ; other rotations correspond to powers of this  $n$ -cycle. (Note that every clockwise rotation is achievable as an anticlockwise rotation).

Then there are  $n$  reflections in axes of symmetry in the plane. If  $n$  is odd, such axes join a vertex to the midpoint of the opposite edge. For example,  $(1)(2\ n)(3\ n-1)\cdots$  corresponds to one such reflection. However, if  $n$  is even, there are two types of reflections, in an axis joining a pair of opposite vertices and in an axis joining midpoints of opposite edges: hence there are  $\frac{1}{2}n + \frac{1}{2}n = n$  reflections in this case as well.

Since all axes of symmetry of the  $n$ -gon have now been exhausted, we conclude that the order of the symmetry group is  $1 + (n - 1) + n = 2n$ . This group is called the *dihedral group of order  $2n$* ,

$$\text{Dih}(2n).$$

Notice that  $\text{Dih}(2n)$  is a proper subset of  $S_n$  if  $2n < n!$ , i.e., if  $n \geq 4$ . Thus not every permutation of the vertices arises from a symmetry when  $n \geq 4$ .

### Simple consequences of the axioms.

We end the section by noting three elementary facts that follow quickly from the axioms.

#### (3.2.1)

- (i) (The Generalized Associative Law) *Let  $x_1, x_2, \dots, x_n$  be elements of a semigroup  $(S, *)$ . If an element  $u$  is constructed by combining these elements in the given order, using any mode of bracketing, then  $u = (\cdots((x_1 * x_2) * x_3) * \cdots) * x_n$ , so that  $u$  is independent of the positioning of the parentheses.*
- (ii) *Every monoid has a unique identity element.*
- (iii) *Every element in a group has a unique inverse.*

*Proof.* (i) We argue by induction on  $n$ , which can be assumed to be at least 3. If  $u$  is constructed from  $x_1, x_2, \dots, x_n$  in that order, then  $u = v * w$  where  $v$  is constructed from

$x_1, x_2, \dots, x_i$  and  $w$  from  $x_{i+1}, \dots, x_n$ ; here  $1 \leq i \leq n-1$ . Then  $v = (\dots(x_1 * x_2) \dots * x_i)$  by induction on  $n$ . If  $i = n-1$ , then  $w = x_n$  and the result follows at once. Otherwise  $i+1 < n$  and  $w = z * x_n$  where  $z$  is constructed from  $x_{i+1}, \dots, x_{n-1}$ . Then  $u = v * w = v * (z * x_n) = (v * z) * x_n$  by the associative law. The result is true for  $v * z$  by induction, so it is true for  $u$ .

(ii) Suppose that  $e$  and  $e'$  are two identity elements in a monoid. Then  $e = e * e'$  since  $e'$  is an identity, and  $e * e' = e'$  since  $e$  is an identity. Hence  $e = e'$ .

(iii) Let  $g$  be an element of a group and suppose  $g$  has two inverses  $x$  and  $x'$ ; we claim that  $x = x'$ . To see this observe that  $(x * g) * x' = e * x' = x'$ , while also  $(x * g) * x' = x * (g * x') = x * e = x$ . Hence  $x = x'$ .  $\square$

Because of (3.2.1)(i) above, we can without ambiguity omit all parentheses from an expression formed from elements  $x_1, x_2, \dots, x_n$  of a semigroup – an enormous gain in simplicity. Also (ii) and (iii) show that it is unambiguous to speak of *the* identity element of a monoid and *the* inverse of an element of a group.

### Exercises (3.2)

(1) Let  $S$  be the subset of  $\mathbb{R} \times \mathbb{R}$  specified below and define  $(x, y) * (x', y') = (x + x', y + y')$ . Say in each case whether  $(S, *)$  is a semigroup, a monoid, a group, or none of these, as is most appropriate.

(i)  $S = \{(x, y) \mid x + y \geq 0\}$ ;

(ii)  $S = \{(x, y) \mid x + y > 0\}$ ;

(iii)  $S = \{(x, y) \mid |x + y| \leq 1\}$ ;

(iv)  $S = \{(x, y) \mid 2x + 3y = 0\}$ .

(2) Do the sets of even or odd permutations in  $S_n$  form a semigroup when functional composition is used as the binary operation?

(3) Show that the set of all  $2 \times 2$  real matrices with non-negative entries is a monoid, but not a group, when matrix addition used.

(4) Let  $A$  be a non-empty set and define a binary operation  $*$  on the power set  $\mathcal{P}(A)$  by  $S * T = (S \cup T) - (S \cap T)$ . Prove that  $(\mathcal{P}(A), *)$  is an abelian group.

(5) Define powers in a semigroup  $(S, *)$  by the rules  $x^1 = x$  and  $x^{n+1} = x^n * x$  where  $x \in S$  and  $n$  is a non-negative integer. Prove that  $x^m * x^n = x^{m+n}$  and  $(x^m)^n = x^{mn}$  where  $m, n > 0$ .

(6) Let  $G$  be a monoid such that for each  $x$  in  $G$  there is a positive integer  $n$  such that  $x^n = e$ . Prove that  $G$  is a group.

(7) Let  $G$  be the set consisting of the permutations (12)(34), (13)(24), (14)(23) and the identity permutation (1)(2)(3)(4). Show that  $G$  is a group with exactly four elements in which each element is its own inverse. (This group is called the *Klein*<sup>3</sup> 4-group).

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3 Felix Klein (1849–1925)

- (8) Prove that the group  $S_n$  is abelian if and only if  $n \leq 2$ .  
 (9) Prove that the group  $GL_n(\mathbb{R})$  is abelian if and only if  $n = 1$ .

### 3.3 Groups and subgroups

From this point on we will concentrate on groups and we begin by improving the notation. In the first place it is customary not to distinguish between a group  $(G, *)$  and its underlying set  $G$ , provided there is no likelihood of confusion. Then there are two standard ways of writing the group operation. In the *additive notation* we write  $x + y$  for  $x * y$ ; the identity is  $0_G$  or  $0$  and the inverse of an element  $x$  is  $-x$ . The additive notation is most often used for abelian groups, i.e., groups  $(G, *)$  such that  $x * y = y * x$  for all  $x, y \in G$ .

For non-abelian groups the *multiplicative notation* is generally employed, with  $xy$  being written for  $x * y$ ; the identity element is  $1_G$  or  $1$  and the inverse of  $x$  is  $x^{-1}$ . The multiplicative notation will be used here unless the additive notation is clearly preferable, as with a group such as  $\mathbb{Z}$ .

**Isomorphism.** It is important to decide when two groups are to be regarded as essentially the same. It is possible that two groups have very different sets of elements, but their elements behave in a similar manner with respect to their respective group operations. This leads us to introduce the concept of isomorphism. Let  $G$  and  $H$  be (multiplicatively written) groups. An *isomorphism* from  $G$  to  $H$  is a bijective function  $\alpha : G \rightarrow H$  such that

$$\alpha(xy) = \alpha(x)\alpha(y)$$

for all  $x, y \in G$ . Groups  $G$  and  $H$  are said to be *isomorphic* if there exists an isomorphism from  $G$  to  $H$ , in symbols

$$G \simeq H.$$

- (3.3.1)** (i) If  $\alpha : G \rightarrow H$  is an isomorphism of groups, then so is its inverse  $\alpha^{-1} : H \rightarrow G$ .  
 (ii) Isomorphism is an equivalence relation on the class of groups.

*Proof.* To establish (i) all we need to do is prove that  $\alpha^{-1}(xy) = \alpha^{-1}(x)\alpha^{-1}(y)$ . Now  $\alpha(\alpha^{-1}(xy)) = xy$ , while

$$\alpha(\alpha^{-1}(x)\alpha^{-1}(y)) = \alpha(\alpha^{-1}(x))\alpha(\alpha^{-1}(y)) = xy.$$

Hence  $\alpha^{-1}(xy) = \alpha^{-1}(x)\alpha^{-1}(y)$  by injectivity of  $\alpha$ .

To prove (ii) note that reflexivity is obvious, while transitivity follows from the observation that a composite of isomorphisms is an isomorphism: of course (i) implies the symmetric property.  $\square$

The idea behind isomorphism is that, while the elements in two isomorphic groups may be different, they have the same properties in relation to their respective group operations. Note that isomorphic groups have the same order, where by the *order* of a group  $G$  we mean the cardinality of its set of elements  $|G|$ .

The next result records some very useful techniques for working with group elements.

**(3.3.2)** Let  $x, a, b$  be elements of a group  $G$ .

- (i) If  $xa = b$ , then  $x = ba^{-1}$ , and if  $ax = b$ , then  $x = a^{-1}b$ .
- (ii)  $(ab)^{-1} = b^{-1}a^{-1}$ .

*Proof.* From  $xa = b$  we obtain  $(xa)a^{-1} = ba^{-1}$ , i.e.,  $x(aa^{-1}) = ba^{-1}$ . Since  $aa^{-1} = 1$  and  $x1 = x$ , we get  $x = ba^{-1}$ . The second statement in (i) is dealt with similarly. By (3.2.1) to establish (ii) it is enough to show that  $b^{-1}a^{-1}$  is an inverse of  $ab$ . This can be checked directly:  $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = a1a^{-1} = aa^{-1} = 1$ ; similarly  $(b^{-1}a^{-1})(ab) = 1$ . Consequently  $(ab)^{-1} = b^{-1}a^{-1}$ .  $\square$

**The group table.** Suppose that  $(G, *)$  is a group of finite order  $n$  whose elements are ordered in some fixed manner, let us say  $g_1, g_2, \dots, g_n$ . The rule for combining elements in the group can be displayed in its *group table*. This is the  $n \times n$  rectangular array  $M$  whose  $(i, j)$  entry is  $g_i * g_j$ . Thus the  $i$ th row of  $M$  is  $g_i * g_1, g_i * g_2, \dots, g_i * g_n$ . From the group table any pair of group elements can be combined. If the group is written multiplicatively, the term *multiplication table* is used.

Notice that all the elements in a row are different: for  $g_i * g_j = g_i * g_k$  implies that  $g_j = g_k$  by (3.3.2). The same is true of the columns of  $M$ . What this means is that each group element appears exactly once in each row and exactly once in each column of  $M$ , that is, the group table is a *latin square*. Such configurations are studied in (11.4).

As an example, consider the group of order 4 whose elements are the identity permutation  $1 = (1)(2)(3)(4)$  and the permutations  $a = (12)(34)$ ,  $b = (13)(24)$ ,  $c = (14)(23)$ . This is the *Klein 4-group*, which was mentioned in Exercise (3.2.7). The multiplication table of this group is the  $4 \times 4$  array

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

**Powers of group elements.** Let  $x$  be an element of a (multiplicative) group  $G$  and let  $n$  be an integer. The  $n$ th power  $x^n$  of  $x$  is defined recursively as follows:

$$x^0 = 1, \quad x^{n+1} = x^n x, \quad x^{-n} = (x^n)^{-1}$$



where  $n \geq 0$ . (See also Exercise (3.2.5)). Of course, if  $G$  were written additively, we would write  $nx$  instead of  $x^n$ . Fundamental for the manipulation of powers is:

**(3.3.3)** (The Laws of Exponents) *Let  $x$  be an element of a group  $G$  and let  $m, n$  be integers. Then*

- (i)  $x^m x^n = x^{m+n} = x^n x^m$ ;
- (ii)  $(x^m)^n = x^{mn}$ .

*Proof.* (i) First we show that  $x^r x^s = x^{r+s}$  where  $r, s \geq 0$ , using induction on  $s$ . This is clear if  $s = 0$ . Assuming it true for  $s$ , we have

$$x^r x^{s+1} = x^r x^s x = x^{r+s} x = x^{r+s+1},$$

thus completing the induction. Next using (3.3.2) and the definition of negative powers, we deduce from  $x^r x^s = x^{r+s}$  that  $x^{-r} x^{r+s} = x^s$  and hence by inversion that  $x^{-r-s} x^r = x^{-s}$ . This shows that  $x^{-r} x^s = x^{s-r}$  for all  $r, s \geq 0$ . In a similar way  $x^r x^{-s} = x^{r-s}$  for all  $r, s \geq 0$ .

Finally, by inverting  $x^s x^r = x^{r+s}$  where  $r, s \geq 0$ , we obtain  $x^{-r} x^{-s} = x^{-r-s}$ . Thus all cases have been covered.

(ii) When  $n \geq 0$ , use induction on  $n$ : clearly it is true when  $n = 0$ . Assuming the statement true for  $n$ , we have  $(x^m)^{n+1} = (x^m)^n x^m = x^{mn} x^m = x^{m(n+1)}$  by (3.3.2)(i). Next  $(x^m)^{-n} = ((x^m)^n)^{-1} = (x^{mn})^{-1} = x^{-mn}$ , which covers the case where the second exponent is negative.  $\square$

**Subgroups.** Roughly speaking, a subgroup is a group contained within a larger group. To make this concept precise, consider a group  $(G, *)$  and a subset  $S$  of  $G$ . If the group operation  $*$  is restricted to  $S$ , we obtain a function  $'$  from  $S \times S$  to  $G$ . If  $'$  is a binary operation on  $S$ , i.e., if  $x * y \in S$  whenever  $x, y \in S$ , and if  $(S, ')$  is actually a group, then  $S$  is called a *subgroup* of  $G$ .

The first point to settle is that  $1_S$ , the identity element of  $(S, ')$ , equals  $1_G$ . Indeed  $1_S = 1_S *' 1_S = 1_S * 1_S$ , so  $1_S * 1_S = 1_S * 1_G$ . By (3.3.2) it follows that  $1_S = 1_G$ . Next let  $x \in S$  and denote the inverse of  $x$  in  $(S, ')$  by  $x_S^{-1}$ . We want to be sure that  $x_S^{-1} = x^{-1}$ . Now  $1_G = 1_S = x *' x_S^{-1} = x * x_S^{-1}$ . Hence  $x * x^{-1} = x * x_S^{-1}$  and so  $x_S^{-1} = x^{-1}$ . Thus inverses are the same in  $(S, ')$  and in  $(G, *)$ .

On the basis of these observations we are able to formulate a convenient test for a subset of a group to be a subgroup.

**(3.3.4)** *Let  $S$  be a subset of a group  $G$ . Then  $S$  is a subgroup of  $G$  if and only if the following hold:*

- (i)  $1_G \in S$ ;
- (ii)  $xy \in S$  whenever  $x \in S$  and  $y \in S$ , (closure under products);
- (iii)  $x^{-1} \in S$  whenever  $x \in S$ , (closure under inverses).

To indicate that  $S$  is a subgroup of a group  $G$  we write

$$S \leq G.$$

If in addition  $S \neq G$ , then  $S$  is a *proper* subgroup and we sometimes write  $S < G$ .

### Examples of subgroups.

(i)  $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$ . These statements follow at once via (3.3.4). For the same reason  $\mathbb{Q}^* \leq \mathbb{R}^* \leq \mathbb{C}^*$ .

(ii)  $A_n \leq S_n$ . Recall that  $A_n$  is the set of even permutations in  $S_n$ . Here the point to note is that if  $\pi$  and  $\sigma$  are even permutations, then so are  $\pi\sigma$  and  $\pi^{-1}$  by (3.1.6): of course the identity permutation is even. However, the odd permutations in  $S_n$  do not form a subgroup.

(iii) Two subgroups that are present in every group  $G$  are the *trivial* or *identity subgroup*  $\{1_G\}$ , which is written  $1$  or  $1_G$ , and the *improper subgroup*  $G$  itself. For some groups these are the only subgroups.

(iv) *Cyclic subgroups*. The interesting subgroups of a group are the proper non-trivial ones. An easy way to produce subgroups is to take all the powers of a fixed element. Let  $G$  be a group and choose  $x \in G$ . We denote the set of all powers of the element  $x$  by

$$\langle x \rangle.$$

Using (3.3.4) and the Laws of Exponents (3.3.3), we quickly verify that  $\langle x \rangle$  is a subgroup. It is called *the cyclic subgroup generated by  $x$* . Since every subgroup of  $G$  which contains  $x$  must also contain all powers of  $x$ , it follows that  $\langle x \rangle$  is *the smallest subgroup of  $G$  containing  $x$* .

A group  $G$  is said to be *cyclic* if  $G = \langle x \rangle$  for some  $x$  in  $G$ . For example,  $\mathbb{Z}$  and  $\mathbb{Z}_n$  are cyclic groups since, allowing for the additive notation,  $\mathbb{Z} = \langle 1 \rangle$  and  $\mathbb{Z}_n = \langle [1]_n \rangle$ .

Next we consider intersections of subgroups.

**(3.3.5)** If  $\{S_\lambda \mid \lambda \in \Lambda\}$  is a set of subgroups of a group  $G$ , then  $\bigcap_{\lambda \in \Lambda} S_\lambda$  is also a subgroup of  $G$ .

This follows immediately from (3.3.4). Now suppose that  $X$  is a non-empty subset of a group  $G$ . There is at least one subgroup that contains  $X$ , namely  $G$  itself. Thus we may form the intersection of all the subgroups of  $G$  that contain  $X$ . This is a subgroup by (3.3.5) which is denoted by

$$\langle X \rangle.$$

Obviously  $\langle X \rangle$  is the smallest subgroup of  $G$  containing  $X$ : it is called the *subgroup generated by  $X$* . Note that the cyclic subgroup  $\langle x \rangle$  is just the subgroup generated by the singleton set  $\{x\}$ . More generally a group  $G$  is said to be *finitely generated* if  $G = \langle X \rangle$  for some finite set  $X$ .

It is natural to enquire ask about the form of elements of  $\langle X \rangle$ .

**(3.3.6)** Let  $X$  be a non-empty subset of a group  $G$ . Then  $\langle X \rangle$  consists of all elements of  $G$  of the form

$$x_1^{e_1} x_2^{e_2} \cdots x_k^{e_k}$$

where  $x_i \in X$ ,  $e_i = \pm 1$  and  $k \geq 0$ , (the case  $k = 0$  being interpreted as  $1_G$ ).

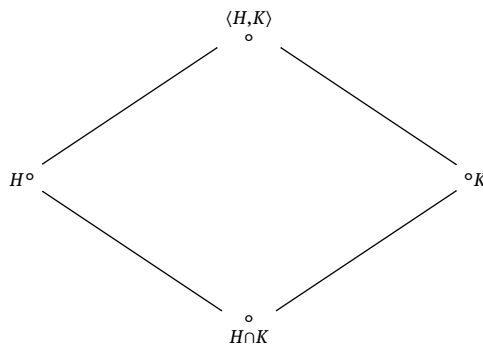
*Proof.* Let  $S$  denote the set of all elements of the specified form. It is easy to check that  $S$  contains 1 and is closed under products and inversion, by using (3.3.2). Thus  $S$  is a subgroup. Clearly  $X \subseteq S$ , so that  $\langle X \rangle \subseteq S$  since  $\langle X \rangle$  is the smallest subgroup containing  $X$ . On the other hand, any element of the form  $x_1^{e_1} \cdots x_k^{e_k}$  must belong to  $\langle X \rangle$  since  $x_i \in \langle X \rangle$ . Therefore  $S \subseteq \langle X \rangle$  and  $\langle X \rangle = S$ .  $\square$

Notice that if  $X$  is the 1-element set  $\{x\}$ , we recover the fact that the cyclic subgroup  $\langle x \rangle$  consists of all powers of  $x$ .

**The lattice of subgroups.** Let  $G$  be a group; then set inclusion is a partial order on the set of all subgroups of  $G$

$$S(G),$$

which is therefore a partially ordered set. Now if  $H$  and  $K$  are subgroups of  $G$ , they have a greatest lower bound in  $S(G)$ , namely  $H \cap K$ , and also a least upper bound  $\langle H \cup K \rangle$ , which is usually written  $\langle H, K \rangle$ . This last is true because any subgroup containing  $H$  and  $K$  must also contain  $\langle H, K \rangle$ . This means that  $S(G)$  is a lattice, in the sense of (1.2). When  $G$  is finite,  $S(G)$  can be visualized by means of its Hasse diagram; the basic component in the diagram of subgroups of a group is the subdiagram below



**The order of a group element.** Let  $x$  be an element of a group. If the subgroup  $\langle x \rangle$  has a finite number  $m$  of elements,  $x$  is said to have *finite order*  $m$ . If on the other hand  $\langle x \rangle$  is infinite, then  $x$  is called an element of *infinite order*. We shall write

$$|x|$$

for the order of  $x$ . The basic facts about orders of group elements are contained in the next result.

**(3.3.7)** *Let  $x$  be an element of a group  $G$ .*

- (i) *If all powers of  $x$  are distinct, then  $x$  has infinite order.*
- (ii) *Assume that two powers of  $x$  are equal. Then  $x$  has finite order  $m$  and  $x^\ell = 1$  if and only if  $\ell$  is divisible by  $m$ . Thus  $m$  is the smallest positive integer such that  $x^m = 1$ . Furthermore  $\langle x \rangle = \{1, x, \dots, x^{m-1}\}$ .*

*Proof.* (i) This is clearly true.

(ii) Suppose that two powers of  $x$  are equal, say  $x^i = x^j$  where  $i > j$ . Then  $x^{i-j} = 1$  by (3.3.3). Using Well-Ordering we may choose a smallest positive integer  $m$  for which  $x^m = 1$ . Now let  $\ell$  be any integer and write  $\ell = mq + r$  where  $q, r \in \mathbb{Z}$  and  $0 \leq r < m$ , using the Division Algorithm. By (3.3.3) again  $x^\ell = (x^m)^q x^r = x^r$ . By minimality of  $m$  we deduce that  $x^\ell = 1$  if and only if  $r = 0$ , i.e.,  $\ell$  is divisible by  $m$ . It follows that  $\langle x \rangle = \{1, x, x^2, \dots, x^{m-1}\}$ , so that  $x$  has finite order  $m$ . □

We will now study cyclic groups with the aim of identifying them up to isomorphism.

**(3.3.8)** *A cyclic group of order  $n$  is isomorphic with  $\mathbb{Z}_n$ . An infinite cyclic group is isomorphic with  $\mathbb{Z}$ .*

*Proof.* Let  $G = \langle x \rangle$  be a cyclic group. If  $|G| = n$ , then  $G = \{1, x, \dots, x^{n-1}\}$ . Define  $\alpha : \mathbb{Z}_n \rightarrow G$  by  $\alpha([i]) = x^i$ , which is a well-defined function because  $x^{i+nq} = x^i(x^n)^q = x^i$ . Also

$$\alpha([i] + [j]) = \alpha([i + j]) = x^{i+j} = x^i x^j = \alpha([i])\alpha([j]),$$

while  $\alpha$  is clearly bijective. Therefore, allowing for  $\mathbb{Z}_n$  being written additively and  $G$  multiplicatively, we conclude that  $\alpha$  is an isomorphism and  $\mathbb{Z}_n \simeq G$ . When  $G$  is infinite cyclic, the proof is similar, but easier, and is left to the reader. □

There is a simple way to compute the order of an element of the symmetric group  $S_n$  by using least common multiples – see Exercise (2.2.8).

**(3.3.9)** *Let  $\pi \in S_n$  and write  $\pi = \pi_1 \pi_2 \cdots \pi_k$  where the  $\pi_i$  are disjoint cycles, with  $\pi_i$  of length  $\ell_i$ . Then the order of  $\pi$  equals the least common multiple of  $\ell_1, \ell_2, \dots, \ell_k$ .*

*Proof.* By (3.1.3) there is such an expression for  $\pi$ . Also disjoint permutations commute by (3.1.2). Hence  $\pi^m = \pi_1^m \pi_2^m \cdots \pi_k^m$  for any  $m > 0$ . Now the  $\pi_i^m$  affect disjoint sets of integers, so  $\pi^m = 1$ , (i.e.,  $\pi^m = \text{id}$ ), if and only if  $\pi_1^m = \pi_2^m = \cdots = \pi_k^m = 1$ . By (3.3.7) these conditions are equivalent to  $m$  being divisible by the orders of all the  $\pi_i$ . Finally, it is easy to see by forming successive powers that the order of an  $r$ -cycle is  $r$ . Therefore  $|\pi| = \text{lcm}\{\ell_1, \ell_2, \dots, \ell_k\}$ . □

**Example (3.3.1)** What is the largest possible order of an element of  $S_8$ ?

Let  $\pi \in S_8$  and write  $\pi = \pi_1 \cdots \pi_k$  where the  $\pi_i$  are disjoint cycles. If  $\pi_i$  has length  $\ell_i$ , then  $\sum_{i=1}^k \ell_i = 8$  and  $|\pi| = \text{lcm}\{\ell_1, \dots, \ell_k\}$ . So the question is: which positive integers  $\ell_1, \dots, \ell_k$  with sum equal to 8 have the largest least common multiple? A little experimentation will convince the reader that the answer is  $k = 2$ ,  $\ell_1 = 3$ ,  $\ell_2 = 5$ . Hence 15 is the largest order of an element of  $S_8$ . For example, the permutation  $(123)(45678)$  has order 15.

We conclude with two more examples, including an application to number theory.

**Example (3.3.2)** Let  $G$  be a finite abelian group. Prove that the product of all the elements of  $G$  equals the product of all the elements of  $G$  of order 2.

The key point to notice here is that if  $x \in G$ , then  $|x| = 2$  if and only if  $x = x^{-1} \neq 1$ . Since  $G$  is abelian, in the product  $\prod_{g \in G} g$  we can group together elements of order greater than 2 with their inverses and then cancel each pair  $xx^{-1}$ . What is left is the product of the elements of order 2.

**Example (3.3.3)** (Wilson's<sup>4</sup> Theorem) If  $p$  is a prime, then  $(p-1)! \equiv -1 \pmod{p}$ .

Apply Example (3.3.2) to  $\mathbb{Z}_p^*$ , the multiplicative group of non-zero congruence classes mod  $p$ . Now the only element of order 2 in  $\mathbb{Z}_p^*$  is  $[-1]$ : for  $a^2 \equiv 1 \pmod{p}$  implies that  $a \equiv \pm 1 \pmod{p}$ , i.e.,  $[a] = [1]$  or  $[-1]$ . It follows that  $[1][2] \cdots [p-1] = [-1]$  and hence  $(p-1)! \equiv -1 \pmod{p}$ .

### Exercises (3.3)

(1) In each of the following situations say whether or not the subset  $S$  is a subgroup of the group  $G$ :

- (i)  $G = \text{GL}_n(\mathbb{R})$ ,  $S = \{A \in G \mid \det(A) = 1\}$ .
- (ii)  $G = (\mathbb{R}, +)$ ,  $S = \{x \in \mathbb{R} \mid |x| \leq 1\}$ .
- (iii)  $G = \mathbb{R} \times \mathbb{R}$ ,  $S = \{(x, y) \mid 3x - 2y = 1\}$ : here the group operation of  $G$  is addition of pairs componentwise.

(2) Let  $H$  and  $K$  be subgroups of a group  $G$ . Prove that  $H \cup K$  is a subgroup if and only if  $H \subseteq K$  or  $K \subseteq H$ .

(3) Show that no group can be the union of two proper subgroups. Then exhibit a group which is the union of three proper subgroups.

(4) Find the largest possible order of an element of  $S_{11}$ . How many elements of  $S_{11}$  have this order?

(5) The same question for  $S_{12}$ .

(6) Find the orders of the elements  $[3]$  and  $[7]$  of  $\mathbb{Z}_{11}^*$ .

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<sup>4</sup> John Wilson (1741–1793)

- (7) Prove that a group of even order must contain an element of order 2. [Hint: assume this is false and group the non-identity elements in pairs  $x, x^{-1}$ ].
- (8) Assume that for each pair of elements  $a, b$  of a group  $G$  there is an integer  $n$  such that  $(ab)^i = a^i b^i$  holds for  $i = n, n + 1$  and  $n + 2$ . Prove that  $G$  is abelian.
- (9) Let  $S$  denote the set product  $\mathbb{Z} \times \mathbb{Z}$ . Define a relation  $E$  on  $S$  by  $(a, b) E (a', b') \Leftrightarrow a - b = a' - b'$ .
- Prove that  $E$  is an equivalence relation on  $S$  and that each  $E$ -equivalence class contains a pair  $(a, b)$  with  $a, b > 0$ .
  - Define  $(a, b) + (a', b') = (a + a', b + b')$  and show that this determines a well defined binary operation on the set  $P$  of all  $E$ -equivalence classes.
  - Prove that if  $+$  denotes the binary operation in (ii), then  $(P, +)$  is an abelian group.
  - By finding a mapping from  $P$  to  $\mathbb{Z}$ , prove that  $P \simeq \mathbb{Z}$ .
- (10) Let  $\mathcal{S}$  be a non-empty set of subgroups of a group. Then  $\mathcal{S}$  is said to satisfy the *ascending chain condition* (acc) if there does not exist an infinite ascending chain of subgroups  $G_1 < G_2 < \dots$  where  $G_i \in \mathcal{S}$ . Also  $\mathcal{S}$  is said to satisfy the *maximal condition* (max) if each non-empty subset  $\mathcal{T}$  of  $\mathcal{S}$  has at least one maximal element, i.e., a subgroup in  $\mathcal{T}$  which is not properly contained in any other subgroup in  $\mathcal{T}$ . Prove that the acc and max are the same property.
- (11) A group  $G$  is said to satisfy the *maximal condition on subgroups* (max) if the set of all its subgroups  $S(G)$  satisfies max, or equivalently the acc. Prove that  $G$  satisfies max if and only if every subgroup of  $G$  is finitely generated. [Hint: use the acc form].
- (12) Prove that  $\mathbb{Z}$  satisfies max, but  $\mathbb{Q}$  does not.

## 4 Quotient groups and homomorphisms

In this chapter we probe more deeply into the nature of the subgroups of a group and we introduce functions between groups that relate their group operations.

### 4.1 Cosets and Lagrange's Theorem

Consider a group  $G$  with a fixed subgroup  $H$ . A binary relation  $\sim_H$  on  $G$  is defined by the following rule:  $x \sim_H y$  means that  $x = yh$  for some  $h \in H$ . It is an easy verification that  $\sim_H$  is an equivalence relation on  $G$ . Therefore by (1.2.2) the group  $G$  splits up into disjoint equivalence classes. The equivalence class to which an element  $x$  belongs to is the subset

$$\{xh \mid h \in H\},$$

which is called a *left coset* of  $H$  in  $G$  and is written

$$xH.$$

Thus  $G$  is the disjoint union of the distinct left cosets of  $H$ . Notice that the only coset which is a subgroup is  $1H = H$  since no other coset contains the identity element.

Next observe that the assignment  $h \mapsto xh$ , ( $h \in H$ ), determines a bijection from  $H$  to  $xH$ ; for  $xh_1 = xh_2$  implies that  $h_1 = h_2$ . From this it follows that

$$|xH| = |H|,$$

so that each left coset of  $H$  has the cardinal of  $H$ .

Suppose that we label the left cosets of  $H$  in some manner, say as  $C_\lambda$ ,  $\lambda \in \Lambda$ , and for each  $\lambda$  in  $\Lambda$  we choose an arbitrary element  $t_\lambda$  from  $C_\lambda$ . (If  $\Lambda$  is infinite, we are assuming at this point a set theoretic axiom called *the axiom of choice* – for this see (14.1)). Then  $C_\lambda = t_\lambda H$  and, since every group element belongs to some left coset of  $H$ , we have  $G = \bigcup_{\lambda \in \Lambda} t_\lambda H$ . Furthermore, cosets are equivalence classes and therefore are disjoint, so each element  $x$  of  $G$  has a unique expression  $x = t_\lambda h$ , where  $h \in H$ ,  $\lambda \in \Lambda$ . The set  $\{t_\lambda \mid \lambda \in \Lambda\}$  is called a *left transversal* to  $H$  in  $G$ . Thus we have found a unique way to express elements of  $G$  in terms of the transversal and elements of the subgroup  $H$ .

In a similar fashion one can define *right cosets* of  $H$  in  $G$ ; these are the equivalence classes of the equivalence relation  $\sim_H$ , where  $x \sim_H y$  means that  $x = hy$  for some  $h$  in  $H$ . The *right coset* containing  $x$  is

$$Hx = \{hx \mid h \in H\}$$

and right transversals are defined analogously.

The next result was the first significant theorem to be discovered in group theory.

**(4.1.1)** (Lagrange's<sup>1</sup> Theorem) *Let  $H$  be a subgroup of a finite group  $G$ . Then  $|H|$  divides  $|G|$  and  $\frac{|G|}{|H|}$  = the number of left cosets of  $H$  = the number of right cosets of  $H$ .*

*Proof.* Let  $\ell$  be the number of left cosets of  $H$  in  $G$ . Since the number of elements in any left coset of  $H$  is  $|H|$  and distinct left cosets are disjoint, a count of the elements of  $G$  yields  $|G| = \ell \cdot |H|$ ; thus  $\ell = |G|/|H|$ . For right cosets the argument is similar.  $\square$

**Corollary (4.1.2)** *The order of an element of a finite group divides the order of the group.*

For the order of an element equals the order of the cyclic subgroup it generates.

**The index of a subgroup.** Even in an infinite group  $G$  the sets of left and right cosets of a subgroup  $H$  have the same cardinal. Indeed the assignment  $xH \mapsto Hx$  clearly determines a bijection between these sets. This allows us to define the *index of  $H$  in  $G$*  to be simultaneously the cardinal of the set of left cosets and the cardinal of the set of right cosets of  $H$ ; the index is written

$$|G : H|.$$

When  $G$  is finite, we have already seen that  $|G : H| = |G|/|H|$  by Lagrange's Theorem.

**Example (4.1.1)** Let  $G$  be the symmetric group  $S_3$  and let  $H = \langle (12)(3) \rangle$ . Then  $|H| = 3$  and  $|G : H| = |G|/|H| = 6/2 = 3$ , so we expect to find three left cosets and three right ones. The left cosets are

$$H = \{(1)(2)(3)(4), (12)(3)\}, (123)H = \{(123), (13)(2)\}, (132)H = \{(132), (1)(23)\},$$

and the right cosets are

$$H = \{(1)(2)(3)(4), (12)(3)\}, H(123) = \{(123), (1)(23)\}, H(132) = \{(132), (13)(2)\}.$$

Notice that the left cosets are disjoint, as are the right ones; but the left and right cosets are not all disjoint.

The next result is useful in calculations with subgroups: it involves the concept of the product of cardinal numbers – see Exercise (1.4.6).

**(4.1.3)** *Let  $H \leq K \leq G$  where  $G$  is any group. Then*

$$|G : H| = |G : K| \cdot |K : H|.$$

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<sup>1</sup> Joseph Louis Lagrange (1736–1813)



*Proof.* Let  $\{t_\lambda \mid \lambda \in \Lambda\}$  be a left transversal to  $H$  in  $K$ , and let  $\{s_\mu \mid \mu \in M\}$  be a left transversal to  $K$  in  $G$ . Thus  $K = \bigcup_{\lambda \in \Lambda} t_\lambda H$  and  $G = \bigcup_{\mu \in M} s_\mu K$ . Hence

$$G = \bigcup_{\lambda \in \Lambda, \mu \in M} (s_\mu t_\lambda)H.$$

We claim that the elements  $s_\mu t_\lambda$  belong to different left cosets of  $H$ . Indeed suppose that  $s_\mu t_\lambda H = s_{\mu'} t_{\lambda'} H$ ; then, since  $t_\lambda H \subseteq K$ , we have  $s_\mu K = s_{\mu'} K$ , which implies that  $\mu = \mu'$ . Hence  $t_\lambda H = t_{\lambda'} H$ , which shows that  $\lambda = \lambda'$ . It follows that  $|G : H|$ , which is the cardinal of the set of left cosets of  $H$  in  $G$ , equals  $|M \times \Lambda|$ . By definition of the product this is  $|M| \cdot |\Lambda| = |G : K| \cdot |K : H|$ .  $\square$

**Groups of prime order.** Lagrange's Theorem is sufficiently strong to enable us to describe all groups of prime order. This is the first example of a classification theorem in group theory; it is also a first indication of the importance of arithmetic properties of the group order for the structure of a group.

**(4.1.4)** *A group  $G$  has prime order  $p$  if and only if  $G \simeq \mathbb{Z}_p$ .*

*Proof.* Assume that  $|G| = p$  and let  $1 \neq x \in G$ . Then  $|\langle x \rangle|$  divides  $|G| = p$  by (4.1.1). Hence  $|\langle x \rangle| = p = |G|$  and  $G = \langle x \rangle$ , a cyclic group of order  $p$ . Thus  $G \simeq \mathbb{Z}_p$  by (3.3.8). The converse is obvious.  $\square$

**Example (4.1.2)** Find all groups of order less than 6.

Let  $G$  be a group such that  $|G| < 6$ . If  $|G| = 1$ , then  $G$  is a trivial group. If  $|G| = 2, 3$  or 5, then (4.1.4) tells us that  $G \simeq \mathbb{Z}_2, \mathbb{Z}_3$  or  $\mathbb{Z}_5$  respectively. We are left with the case where  $|G| = 4$ . If  $G$  contains an element  $x$  of order 4, then  $G = \langle x \rangle$  and  $G \simeq \mathbb{Z}_4$  by (3.3.8). Assuming that  $G$  has no elements of order 4, we conclude from (4.1.2) that  $G$  must consist of 1 and three elements of order 2, say  $a, b, c$ .

Now  $ab$  cannot equal 1, otherwise  $b = a^{-1} = a$ . Also it is clear that  $ab \neq a$  and  $ab \neq b$ . Hence  $ab$  must equal  $c$ ; also  $ba = c$  by the same argument. Similarly we can prove that  $bc = a = cb$  and  $ca = b = ac$ .

At this point the reader should recognize that  $G$  is very like the Klein 4-group

$$V = \{(1)(2)(3)(4), (12)(34), (13)(24), (14)(23)\}.$$

In fact the assignments  $1_G \mapsto 1_V, a \mapsto (12)(34), b \mapsto (13)(24), c \mapsto (14)(23)$  determine an isomorphism from  $G$  to  $V$ . Our conclusion is that *up to isomorphism there are exactly six groups with order less than 6, namely  $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, V, \mathbb{Z}_5$ .*

The following application of Lagrange's Theorem furnishes another proof of Fermat's Theorem – see (2.3.4).

**Example (4.1.3)** Use a group theoretic argument to prove that if  $p$  is a prime and  $n$  is any integer, then  $n^p \equiv n \pmod{p}$ .

Apply (4.1.2) to  $\mathbb{Z}_p^*$ , the multiplicative group of non-zero congruence classes modulo  $p$ . If  $[n] \neq [0]$ , then (4.1.2) implies that the order of  $[n]$  divides  $|\mathbb{Z}_p^*| = p - 1$ . Thus  $[n]^{p-1} = [1]$ , i.e.,  $n^{p-1} \equiv 1 \pmod{p}$ . Multiply by  $n$  to get  $n^p \equiv n \pmod{p}$ , and observe that this also holds if  $[n] = [0]$ .

According to Lagrange's Theorem the order of a subgroup of a finite group divides the group order. However, the natural converse of this statement is false: there need not be a subgroup with order equal to a positive divisor of the group order. This is demonstrated by the following example.

**Example (4.1.4)** The alternating group  $A_4$  has order 12, but it has no subgroups of order 6.

Write  $G = A_4$ . First note that each non-trivial element of  $G$  is either a 3-cycle or the product of two disjoint transpositions. Also all of the latter with the identity form the Klein 4-group  $V$ .

Suppose that  $H$  is a subgroup of  $G$  with order 6. Assume first that  $H \cap V = 1$ . Then there are  $6 \times 4 = 24$  distinct elements of the form  $h\nu$ ,  $h \in H$ ,  $\nu \in V$ ; for if  $h_1\nu_1 = h_2\nu_2$  with  $h_i \in H$ ,  $\nu_i \in V$ , then  $h_2^{-1}h_1 = \nu_2\nu_1^{-1} \in H \cap V = 1$ , so that  $h_1 = h_2$  and  $\nu_1 = \nu_2$ . This is impossible, so  $H \cap V \neq 1$ .

Let us say  $H \cap V$  contains  $\pi = (12)(34)$ . Now  $H$  must also contain a 3-cycle since there are 8 of these in  $G$ , say  $\sigma = (123) \in H$ . Hence  $H$  contains  $\tau = \sigma\pi\sigma^{-1} = (14)(23)$ . Thus  $H$  contains  $\pi\tau = (13)(24)$  and it follows that  $V \subseteq H$ . Other choices of elements leads to the same conclusion. However,  $|V|$  does not divide  $|H|$ , a final contradiction.

**Subgroups of cyclic groups.** Usually a group has many subgroups and it can be a difficult task to find all of them. Thus it is of interest that the subgroups of a cyclic group are easy to describe. The first observation is that such subgroups are themselves cyclic.

**(4.1.5)** *A subgroup of a cyclic group is cyclic.*

*Proof.* Let  $H$  be a subgroup of a cyclic group  $G = \langle x \rangle$ . If  $H = 1$ , then obviously  $H = \langle 1 \rangle$ ; thus we may assume that  $H \neq 1$ , so that  $H$  contains some  $x^m \neq 1$ ; since  $H$  must also contain  $(x^m)^{-1} = x^{-m}$ , we may as well assume that  $m > 0$ . Now choose  $m$  to be the *smallest* positive integer for which  $x^m \in H$ ; of course we have used the Well-Ordering Law here.

Certainly it is true that  $\langle x^m \rangle \subseteq H$ . We will prove the reverse inclusion, which will show that  $H = \langle x^m \rangle$ . Let  $h \in H$  and write  $h = x^i$ . By the Division Algorithm  $i = mq + r$  where  $q, r \in \mathbb{Z}$  and  $0 \leq r < m$ . By the Laws of Exponents (3.3.3)  $x^i = x^{mq}x^r$ . Hence  $x^r = x^{-mq}x^i$ , which belongs to  $H$  since  $x^m \in H$  and  $x^i \in H$ . From the minimality of  $m$  it follows that  $r = 0$  and  $i = mq$ . Therefore  $h = x^i \in \langle x^m \rangle$ .  $\square$

The next result tells us how to construct the subgroups of a given cyclic group.

**(4.1.6)** Let  $G = \langle x \rangle$  be a cyclic group.

- (i) If  $G$  is infinite, each subgroup of  $G$  has the form  $G_i = \langle x^i \rangle$  where  $i \geq 0$ . Furthermore, the  $G_i$  are all distinct and  $G_i$  has infinite order if  $i > 0$ .
- (ii) If  $G$  has finite order  $n$ , then it has exactly one subgroup of order  $d$  for each positive divisor  $d$  of  $n$ , namely  $\langle x^{n/d} \rangle$ .

*Proof.* Assume first that  $G$  is infinite and let  $H$  be a subgroup of  $G$ . By (4.1.5)  $H$  is cyclic, say  $H = \langle x^i \rangle$  where  $i \geq 0$ . Thus  $H = G_i$ . If  $x^i$  had finite order  $m$ , then  $x^{im} = 1$ , which, since  $x$  has infinite order, can only mean that  $i = 0$  and  $H = 1$ . Thus  $H$  is certainly infinite cyclic if  $i > 0$ . Next  $G_i = G_j$  implies that  $x^i \in \langle x^j \rangle$  and  $x^j \in \langle x^i \rangle$ , i.e.,  $j \mid i$  and  $i \mid j$ , so that  $i = j$ . Therefore all the  $G_i$ 's are different.

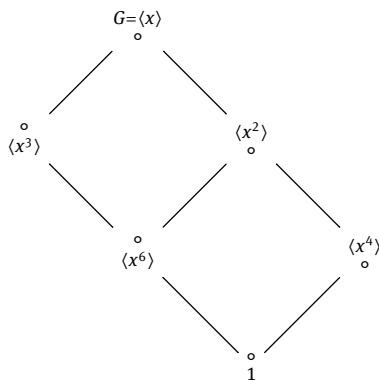
Next let  $G$  have finite order  $n$  and suppose  $d$  is a positive divisor of  $n$ . Now  $(x^{\frac{n}{d}})^d = x^n = 1$ , so  $\ell = |x^{\frac{n}{d}}|$  must divide  $d$  by (3.3.7). But also  $x^{\frac{n\ell}{d}} = 1$  and hence  $n$  divides  $\frac{n\ell}{d}$ , i.e.,  $d$  divides  $\ell$ . It follows that  $\ell = d$  and thus  $K = \langle x^{n/d} \rangle$  has order exactly  $d$ .

To complete the proof, suppose that  $H = \langle x^r \rangle$  is another subgroup with order  $d$ . Then  $x^{rd} = 1$ , so  $n$  divides  $rd$  and  $\frac{n}{d}$  divides  $r$ . This shows that  $H = \langle x^r \rangle \leq \langle x^{n/d} \rangle = K$ . But  $|H| = |K| = d$ , from which it follows that  $H = K$ . Consequently there is exactly one subgroup of order  $d$ .  $\square$

Recall from (3.3) that the set of all subgroups of a group is a lattice and may be represented by a Hasse diagram. In the case of a finite cyclic group, (4.1.6) shows that the lattice corresponds to the lattice of divisors of the group order.

**Example (4.1.5)** Display the Hasse diagram for the subgroups of a cyclic group of order 12.

Let  $G = \langle x \rangle$  have order 12. By (4.1.6) the subgroups of  $G$  correspond to the positive divisors of 12, i.e., 1, 2, 3, 4, 6, 12; indeed, if  $i \mid 12$ , the subgroup  $\langle x^{12/i} \rangle$  has order  $i$ . It is now easy to draw the Hasse diagram:



Next comes a useful formula for order of an element in a cyclic group.

**(4.1.7)** Let  $G = \langle x \rangle$  be a cyclic group with finite order  $n$ . Then the order of the element  $x^i$  is

$$\frac{n}{\gcd\{i, n\}}.$$

*Proof.* In the first place  $(x^i)^m = 1$  if and only if  $n \mid im$ , i.e.,  $\frac{n}{d} \mid (\frac{i}{d})m$  where  $d = \gcd\{i, n\}$ . Since  $\frac{n}{d}$  and  $\frac{i}{d}$  are relatively prime, by Euclid's Lemma this is equivalent to  $\frac{n}{d}$  dividing  $m$ . Therefore  $(x^i)^m = 1$  if and only if  $\frac{n}{d}$  divides  $m$ , which shows that  $x^i$  has order  $\frac{n}{d}$ , as claimed.  $\square$

**Corollary (4.1.8)** Let  $G = \langle x \rangle$  be a cyclic group of finite order  $n$ . Then  $G = \langle x^i \rangle$  if and only if  $\gcd\{i, n\} = 1$ .

For  $G = \langle x \rangle$  if and only if  $x^i$  has order  $n$ , i.e.,  $\gcd\{i, n\} = 1$ . This means that the number of possible generators of  $G$  equals the number of integers  $i$  satisfying  $1 \leq i < n$  and  $\gcd\{i, n\} = 1$ . This number is  $\phi(n)$  where  $\phi$  is the Eulerian function introduced in (2.3).

Every non-trivial group has at least two subgroups, itself and the trivial subgroup: which groups have these two subgroups and no more? The question is easily answered using (4.1.7) and Lagrange's Theorem.

**(4.1.9)** A group  $G$  has just two subgroups if and only if  $G \simeq \mathbb{Z}_p$  for some prime  $p$ .

*Proof.* Assume that  $G$  has only the two subgroups 1 and  $G$ . Let  $1 \neq x \in G$ ; then  $1 \neq \langle x \rangle \leq G$ , so  $G = \langle x \rangle$  and  $G$  is cyclic. Now  $G$  cannot be infinite; for then it would have infinitely many subgroups by (4.1.6). Thus  $G$  has finite order  $n$ , say. Now if  $n$  is not a prime, it has a divisor  $d$  where  $1 < d < n$ , and  $\langle x^{n/d} \rangle$  is a subgroup of order  $d$ , which is impossible. Therefore  $G$  has prime order  $p$  and  $G \simeq \mathbb{Z}_p$  by (4.1.4). Conversely, if  $G \simeq \mathbb{Z}_p$ , then  $|G| = p$  and Lagrange's Theorem shows that  $G$  has no non-trivial proper subgroups.  $\square$

**Products of subgroups.** If  $H$  and  $K$  are subsets of a group  $G$ , the *product* of  $H$  and  $K$  is defined to be the subset

$$HK = \{hk \mid h \in H, k \in K\}.$$

For example, if  $H = \{h\}$  and  $K$  is a subgroup, then  $HK$  is just the left coset  $hK$ . Products of more than two subsets are defined in the obvious way:

$$H_1 H_2 \cdots H_m = \{h_1 h_2 \cdots h_m \mid h_i \in H_i\}.$$

Even if  $H$  and  $K$  are subgroups, their product  $HK$  need not be a subgroup. For example, in  $S_3$  let  $H = \langle (12) \rangle$  and  $K = \langle (13) \rangle$ . Then  $HK = \{(1)(2)(3)(4), (12), (13), (132)\}$ . But  $HK$  cannot be a subgroup since 4 does not divide 6, the order of  $S_3$ .

The following result tells us when the product of two subgroups is a subgroup.

**(4.1.10)** *Let  $H$  and  $K$  be subgroups of a group  $G$ . Then  $HK$  is a subgroup if and only if  $HK = KH$ , and in this event  $\langle H, K \rangle = HK$ .*

*Proof.* Assume first that  $HK \leq G$ . Then  $H \leq HK$  and  $K \leq HK$ , so  $KH \subseteq HK$ . By taking the inverse of each side of this inclusion, we deduce that  $HK \subseteq KH$ . Hence  $HK = KH$ . Moreover  $\langle H, K \rangle \subseteq HK$  since  $H \leq HK$  and  $K \leq HK$ , while  $HK \subseteq \langle H, K \rangle$  is always true. Therefore  $\langle H, K \rangle = HK$ .

Conversely, assume that  $HK = KH$ ; we will verify that  $HK$  is a subgroup by using (3.3.4). Let  $h_1, h_2 \in H$  and  $k_1, k_2 \in K$ . Then  $(h_1 k_1)^{-1} = k_1^{-1} h_1^{-1} \in KH = HK$ . Also  $(h_1 k_1)(h_2 k_2) = h_1(k_1 h_2)k_2$ ; now  $k_1 h_2 \in KH = HK$ , so that  $k_1 h_2 = h_3 k_3$  where  $h_3 \in H$ ,  $k_3 \in K$ . Thus  $(h_1 k_1)(h_2 k_2) = (h_1 h_3)(k_3 k_2) \in HK$ . Obviously  $1 \in HK$ . Since we have proved that the subset  $HK$  is closed under products and inversion, it is a subgroup.  $\square$

It is customary to say that the subgroups  $H$  and  $K$  *permute* if  $HK = KH$ . The next result is frequently used in calculations with subgroups.

**(4.1.11)** (Dedekind's<sup>2</sup> Modular Law) *Let  $H, K, L$  be subgroups of a group and assume that  $K \subseteq L$ . Then*

$$(HK) \cap L = (H \cap L)K.$$

*Proof.* In the first place  $(H \cap L)K \subseteq L$  since  $K \subseteq L$ ; therefore  $(H \cap L)K \subseteq (HK) \cap L$ . To prove the converse, let  $x \in (HK) \cap L$  and write  $x = hk$  where  $h \in H$ ,  $k \in K$ . Hence  $h = xk^{-1} \in LK = L$ , from which it follows that  $h \in H \cap L$  and  $x = hk \in (H \cap L)K$ . Thus  $(HK) \cap L \subseteq (H \cap L)K$  and the result follows.  $\square$

Notice that (4.1.11) is a special case of the distributive law  $\langle H, K \rangle \cap L = \langle H \cap L, K \cap L \rangle$ . However, this law does not hold in general, (see Exercise (4.1.1) below).

Frequently one wants to count the elements in a product of finite subgroups, which makes the next result useful.

**(4.1.12)** *If  $H$  and  $K$  are finite subgroups of a group, then*

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.$$

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<sup>2</sup> Richard Dedekind (1831–1916)

*Proof.* Define a function  $\alpha : H \times K \rightarrow HK$  by the rule  $\alpha((h, k)) = hk$  where  $h \in H, k \in K$ ; evidently  $\alpha$  is surjective. Now  $\alpha((h_1, k_1)) = \alpha((h_2, k_2))$  holds if and only if  $h_1 k_1 = h_2 k_2$ , i.e.,  $h_2^{-1} h_1 = k_2 k_1^{-1} = d \in H \cap K$ . Thus  $h_2 = h_1 d^{-1}$  and  $k_2 = d k_1$ . It follows that the elements of  $H \times K$  which have the same image under  $\alpha$  as  $(h_1, k_1)$  are those of the form  $(h_1 d^{-1}, d k_1)$  where  $d \in H \cap K$ . Now compute the number of the elements of  $H \times K$  by counting their images under  $\alpha$  and allowing for the number of elements with the same image. This gives  $|H \times K| = |HK| \cdot |H \cap K|$ . Of course  $|H \times K| = |H| \cdot |K|$ , so the result is proved.  $\square$

The final result of this section provides important information about the index of the intersection of two subgroups.

**(4.1.13)** (Poincaré<sup>3</sup>) *Let  $H$  and  $K$  be subgroups of finite index in a group  $G$ . Then  $H \cap K$  has finite index and*

$$|G : H \cap K| \leq |G : H| \cdot |G : K|,$$

*with equality if  $|G : H|$  and  $|G : K|$  are relatively prime.*

*Proof.* To each left coset  $x(H \cap K)$  assign the pair of left cosets  $(xH, xK)$ . This is a well-defined function; for, if we were to replace  $x$  by  $xd$  with  $d \in H \cap K$ , then  $xH = xdH$  and  $xK = xdK$ . The function is also injective; for  $(xH, xK) = (yH, yK)$  implies that  $xH = yH$  and  $xK = yK$ , i.e.,  $y^{-1}x \in H \cap K$ , so that  $x(H \cap K) = y(H \cap K)$ . It follows that the number of left cosets of  $H \cap K$  in  $G$  is at most  $|G : H| \cdot |G : K|$ .

Now assume that  $|G : H|$  and  $|G : K|$  are relatively prime. Since  $|G : H \cap K| = |G : H| \cdot |H : H \cap K|$  by (4.1.3), we see that  $|G : H|$  divides  $|G : H \cap K|$ , as does  $|G : K|$  for a similar reason. But  $|G : H|$  and  $|G : K|$  are relatively prime, which means that  $|G : H \cap K|$  is divisible by  $|G : H| \cdot |G : K|$ . It follows that  $|G : H \cap K|$  must equal  $|G : H| \cdot |G : K|$ .  $\square$

### Exercises (4.1)

- (1) Show that the distributive law for subgroups  $\langle H, K \rangle \cap L = \langle H \cap L, K \cap L \rangle$  is false in general.
- (2) If  $H$  is a subgroup of a finite group, show that there are  $|H|^{[G:H]}$  left transversals to  $H$  in  $G$  and the same number of right transversals.
- (3) Let  $H$  be a subgroup of a group  $G$  such that  $G - H$  is finite. Prove that either  $H = G$  or  $G$  is finite.
- (4) Display the Hasse diagram for the subgroup lattices of the following groups:  $\mathbb{Z}_{18}$ ,  $\mathbb{Z}_{24}$ ,  $V$  (the Klein 4-group),  $S_3$ .
- (5) Let  $G$  be a group with exactly three subgroups. Show that  $G \simeq \mathbb{Z}_{p^2}$  where  $p$  is a prime. [Hint: first prove that  $G$  is cyclic].

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<sup>3</sup> Henri Poincaré (1854–1912)

- (6) Let  $H$  and  $K$  be subgroups of a finite group  $G$  with relatively prime indexes in  $G$ . Prove that  $G = HK$ . [Hint: use (4.1.12) and (4.1.13)].
- (7) If the product of subsets is used as the binary operation, show that the set of all non-empty subsets of a group is a monoid.
- (8) Let  $H$  and  $K$  be subgroups of a finite group with relatively prime orders. Show that  $H \cap K = 1$  and  $|HK|$  divides the order of  $\langle H, K \rangle$ .
- (9) Let  $G = \langle x \rangle$  be an infinite cyclic group and put  $H = \langle x^\ell \rangle$ ,  $K = \langle x^j \rangle$ . Prove that  $H \cap K = \langle x^\ell \rangle$  and  $\langle H, K \rangle = \langle x^d \rangle$  where  $\ell = \text{lcm}\{i, j\}$  and  $d = \text{gcd}\{i, j\}$ .
- (10) Let  $G$  be a finite group of order  $n$  and let  $d$  be the minimum number of generators of  $G$ . Prove that  $n \geq 2^d$ , so that  $d \leq [\log_2 n]$ .
- (11) By applying Lagrange's Theorem to the group  $\mathbb{Z}_n^*$ , prove that  $x^{\phi(n)} \equiv 1 \pmod{n}$  where  $n$  is any positive integer and  $x$  is an integer relatively prime to  $n$ . Here  $\phi$  is Euler's function. (This is a generalization of Fermat's theorem (2.3.4)).
- (12) Let  $H$  be a subgroup with finite index in a finitely generated group  $G$ . Use the argument that follows to prove that  $H$  is finitely generated. Let  $G = \langle g_1, \dots, g_n \rangle$  and let  $\{t_1, \dots, t_m\}$  be a left transversal to  $H$  in  $G$  with  $t_1 = 1$ . Without loss assume that each  $g_i^{-1}$  is also a generator. Write  $g_i = t_{\ell_i} h_i$  with  $h_i \in H$ . Also write  $g_i t_j = t_{r(i,j)} h_{ij}$  with  $h_{ij} \in H$ .
- Prove that  $g_i g_j = g_{r(i,\ell_j)} h_{i\ell_j} h_j$ .
  - Let  $h = g_{i_1} g_{i_2} \cdots g_{i_k} \in H$ . By applying (i) repeatedly to segments of  $h$ , prove that  $h \in \langle h_i, h_{j\ell} \mid i, j = 1, \dots, n, \ell = 1, \dots, m, \rangle$ . Conclude that  $H$  is finitely generated.

## 4.2 Normal subgroups and quotient groups

We focus next on a special type of subgroup called a *normal subgroup*. Such subgroups are important since they can be used to construct new groups, the so-called *quotient groups*. Normal subgroups are characterized in the following result.

**(4.2.1)** *Let  $H$  be a subgroup of a group  $G$ . Then the following statements about  $H$  are equivalent:*

- $xH = Hx$  for all  $x$  in  $G$ ;
- $xhx^{-1} \in H$  whenever  $h \in H$  and  $x \in G$ .

*Proof.* First assume that (i) holds and let  $x \in G$  and  $h \in H$ . Then  $xh \in xH = Hx$ , so  $xh = h_1x$  for some  $h_1 \in H$ ; hence  $xhx^{-1} = h_1 \in H$ , which establishes (ii).

Now assume that (ii) holds. Again let  $x \in G$  and  $h \in H$ . Then  $xhx^{-1} = h_1 \in H$ , so  $xh = h_1x \in Hx$  and therefore  $xH \subseteq Hx$ . Next  $x^{-1}hx = x^{-1}h(x^{-1})^{-1} = h_2 \in H$ , which shows that  $hx = xh_2 \in xH$  and  $Hx \subseteq xH$ . Thus  $xH = Hx$  and (i) is proved.  $\square$

A subgroup  $H$  with the equivalent properties in (4.2.1) is called a *normal subgroup* of  $G$ . The notation

$$H \triangleleft G$$

is used to denote the fact that  $H$  is a normal subgroup of a group  $G$ . Also  $xhx^{-1}$  is called the *conjugate* of  $h$  by  $x$ . Thus  $H \triangleleft G$  is valid if and only if  $H$  contains all conjugates of its elements by elements of  $G$ .

**Example (4.2.1)**

(i) Obvious examples of normal subgroups are:  $1 \triangleleft G$  and  $G \triangleleft G$ , and it is possible that these are the only normal subgroups present. If 1 and  $G$  are the only normal subgroups of a non-trivial group  $G$ , then  $G$  is said to be a *simple group*. This is one of the great mis-nomers of mathematics since simple groups can have extremely complicated structure.

(ii)  $A_n \triangleleft S_n$ .

For, if  $\pi \in A_n$  and  $\sigma \in S_n$ , then by (3.1.6) we have

$$\text{sign}(\sigma\pi\sigma^{-1}) = \text{sign}(\sigma) \text{sign}(\pi)(\text{sign}(\sigma))^{-1} = (\text{sign}(\sigma))^2 = 1,$$

so that  $\sigma\pi\sigma^{-1} \in A_n$ .

(iii) In an abelian group  $G$  every subgroup  $H$  is normal.

This is because  $xhx^{-1} = hxx^{-1} = h$  for all  $h \in H$ ,  $x \in G$ .

(iv) Recall that  $\text{GL}_n(\mathbb{R})$  is the group of all non-singular  $n \times n$  real matrices. The subset of matrices in  $\text{GL}_n(\mathbb{R})$  with determinant equal to 1 is denoted by

$$\text{SL}_n(\mathbb{R}).$$

First observe that this is a subgroup, the so-called *special linear group* of degree  $n$  over  $\mathbb{R}$ ; indeed, if  $A, B \in \text{SL}_n(\mathbb{R})$ , then  $\det(AB) = \det(A) \det(B) = 1$  and  $\det(A^{-1}) = (\det(A))^{-1} = 1$ . In addition

$$\text{SL}_n(\mathbb{R}) \triangleleft \text{GL}_n(\mathbb{R}):$$

for if  $A \in \text{SL}_n(\mathbb{R})$  and  $B \in \text{GL}_n(\mathbb{R})$ ,

$$\det(BAB^{-1}) = \det(B) \det(A) (\det(B))^{-1} = \det(B) 1 \det(B)^{-1} = 1.$$

In these computations two standard results about determinants have been used:

$$\det(XY) = \det(X) \det(Y) \quad \text{and} \quad \det(X^{-1}) = (\det(X))^{-1}.$$

(v) A subgroup of  $S_3$  that is *not* normal is  $\langle (12)(3) \rangle$ .

(vi) *The normal closure.* Let  $X$  be a non-empty subset of a group  $G$ . The *normal closure*

$$\langle X^G \rangle$$



of  $X$  in  $G$  is the subgroup generated by all the conjugates  $gxg^{-1}$  with  $g \in G$  and  $x \in X$ . Clearly this is the smallest normal subgroup of  $G$  which contains  $X$ .

(vii) Finally, we introduce two important normal subgroups that can be formed in any group  $G$ . The *center* of  $G$ ,

$$Z(G),$$

consists of all  $x$  in  $G$  such that  $xg = gx$  for every  $g$  in  $G$ . The reader should check that  $Z(G) \triangleleft G$ . Plainly a group  $G$  is abelian if and only if  $G = Z(G)$ .

Next, if  $x, y$  are elements of a group  $G$ , their *commutator* is the element

$$[x, y] = xyx^{-1}y^{-1},$$

The significance of commutators arises from the fact that  $[x, y] = 1$  if and only if  $xy = yx$ , i.e.,  $x$  and  $y$  commute. The *derived subgroup* or *commutator subgroup* of  $G$  is the subgroup  $G'$  generated by all the commutators,

$$G' = \langle [x, y] \mid x, y \in G \rangle.$$

An easy calculation reveals that  $z[x, y]z^{-1} = [zxz^{-1}, zyz^{-1}]$ , which implies that  $G' \triangleleft G$ . Clearly a group  $G$  is abelian if and only if  $G' = 1$ .

**Quotient groups.** Next we will explain how to form a new group from a normal subgroup  $N$  of a group  $G$ . This is called the *quotient group* of  $N$  in  $G$ ,

$$G/N.$$

The elements of  $G/N$  are the cosets  $xN = Nx$ , while the group operation is given by the natural rule

$$(xN)(yN) = (xy)N, \quad (x, y \in G).$$

Our first concern is to check that this binary operation on  $G/N$  is properly defined; it should depend on the two cosets  $xN$  and  $yN$ , not on the choice of coset representatives  $x$  and  $y$ . To prove this, let  $x_1 \in xN$  and  $y_1 \in yN$ , so that  $x_1 = xa$  and  $y_1 = yb$  where  $a, b \in N$ . Then

$$x_1y_1 = xayb = xy(y^{-1}ay)b \in (xy)N$$

since  $y^{-1}ay = y^{-1}a(y^{-1})^{-1} \in N$  by normality of  $N$ . Thus  $(xy)N = (x_1y_1)N$ .

It is straightforward to verify that the binary operation just defined is associative. The role of the identity in  $G/N$  is played by  $1N = N$ , while  $x^{-1}N$  is the inverse of  $xN$ , as is readily checked. It follows that  $G/N$  is a group. Note that the elements of  $G/N$  are *subsets*, not elements, of  $G$ , so that  $G/N$  is not a subgroup of  $G$ . If  $G$  is finite, so is  $G/N$  with order

$$|G/N| = |G : N| = \frac{|G|}{|N|}.$$

### Example (4.2.2)

(i)  $G/1$  is the set of all  $x1 = \{x\}$ , i.e., one-element subsets of  $G$ . Also  $\{x\}\{y\} = \{xy\}$ . In fact this quotient is not really a new group since  $G \simeq G/1$  via the isomorphism in which

$x \mapsto \{x\}$ . Another trivial example of a quotient group is  $G/G$ , which is a group of order 1, with the single element  $G$ .

(ii) Let  $n$  be a positive integer. Then  $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ . For, allowing for the additive notation, the coset of  $n\mathbb{Z}$  containing  $x$  is  $x + n\mathbb{Z} = \{x + nq \mid q \in \mathbb{Z}\}$ , which is just the congruence class of  $x$  modulo  $n$ .

(iii) If  $G$  is any group, the quotient group  $G/G'$  is an abelian group: indeed  $(xG')(yG') = xyG' = (yx)(x^{-1}y^{-1}xy)G' = yxG' = (yG')(xG')$ . Also, if  $G/N$  is any other abelian quotient group, then

$$(xy)N = (xN)(yN) = (yN)(xN) = (yx)N,$$

which implies that  $[x^{-1}, y^{-1}] = x^{-1}y^{-1}xy \in N$  for all  $x, y \in N$ . Since  $G'$  is generated by all commutators  $[x^{-1}, y^{-1}]$ , it follows that  $G' \leq N$ . Therefore  $G/G'$  is the “largest” abelian quotient group of  $G$ .

(iv) *The circle group.* Let  $r$  be a real number and suppose that the plane is rotated through angle  $2r\pi$  in an anti-clockwise direction about an axis through the origin and perpendicular to the plane. This results in a symmetry of the unit circle, which we will call  $r'$ .

Now define  $G = \{r' \mid r \in \mathbb{R}\}$ , a subset of the symmetry group of the unit circle. Note that  $r'_1 \circ r'_2 = (r_1 + r_2)'$  and  $(r')^{-1} = (-r)'$ . This shows that  $G$  is actually a subgroup of the symmetry group; indeed it is the subgroup of all rotations. Our aim is to identify  $G$  as a quotient group.

It is claimed that the assignment  $r + \mathbb{Z} \mapsto r'$  determines a function  $\alpha : \mathbb{R}/\mathbb{Z} \rightarrow G$ : first we need to make sure that the function is well-defined. To this end let  $n$  be an integer and observe that  $(r + n)' = r' \circ n' = r'$  since  $n'$  is a rotation through angle  $2n\pi$ , i.e., it is the identity rotation. Clearly  $\alpha$  is surjective; it is also injective because  $r'_1 = r'_2$  implies that  $2r_1\pi = 2r_2\pi + 2n\pi$  for some integer  $n$ , i.e.,  $r_1 = r_2 + n$ , and hence  $r_1 + \mathbb{Z} = r_2 + \mathbb{Z}$ . Thus  $\alpha$  is a bijection. Finally  $\alpha((r_1 + \mathbb{Z}) + (r_2 + \mathbb{Z})) = \alpha((r_1 + r_2) + \mathbb{Z})$ , which equals

$$(r_1 + r_2)' = r'_1 \circ r'_2 = \alpha(r_1 + \mathbb{Z}) \circ \alpha(r_2 + \mathbb{Z}).$$

Therefore, allowing for the additive and multiplicative notations for the respective groups  $\mathbb{R}/\mathbb{Z}$  and  $G$ , we conclude that  $\alpha$  is an isomorphism from the quotient group  $\mathbb{R}/\mathbb{Z}$  to the circle group  $G$ . Hence  $G \simeq \mathbb{R}/\mathbb{Z}$ .

**Subgroups of quotient groups.** Suppose that  $N$  is a normal subgroup of a group  $G$ ; it is natural to enquire about the subgroups of the quotient group  $G/N$ . It is to be expected that they are related in some simple fashion to the subgroups of  $G$ .

Assume that  $H$  is a subgroup of  $G/N$  and define a corresponding subset of  $G$ ,

$$\bar{H} = \{x \in G \mid xN \in H\}.$$

It is easy to verify that  $\bar{H}$  is a subgroup of  $G$ . Also the definition of  $\bar{H}$  shows that  $N \subseteq \bar{H}$ .

Conversely, suppose we start with a subgroup  $K$  of  $G$  which contains  $N$ . Since  $N \triangleleft G$  implies that  $N \triangleleft K$ , we can form the quotient group  $K/N$ , which is evidently a subgroup of  $G/N$ . Notice that if  $N \leq K_1 \leq G$ , then  $K/N = K_1/N$  implies that  $K = K_1$ . Thus the assignment  $K \mapsto K/N$  determines an injective function from the set of subgroups of  $G$  that contain  $N$  to the set of subgroups of  $G/N$ . The function is also surjective since  $\bar{H} \mapsto H$  in the notation of the previous paragraph; therefore it is a bijection.

These arguments establish the following fundamental theorem.

**(4.2.2)** (The Correspondence Theorem) *Let  $N$  be a normal subgroup of a group  $G$ . Then the assignment  $K \mapsto K/N$  determines a bijection from the set of subgroups of  $G$  that contain  $N$  to the set of subgroups of  $G/N$ . Furthermore,  $K/N \triangleleft G/N$  if and only if  $K \triangleleft G$ .*

All of this has been proven except the last statement, which follows from the observation that  $(xN)(kN)(xN)^{-1} = (xkx^{-1})N$  for  $k \in K$  and  $x \in G$ .

**Example (4.2.3)** Let  $n$  be a positive integer. The subgroups of  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$  are of the form  $K/n\mathbb{Z}$  where  $n\mathbb{Z} \leq K \leq \mathbb{Z}$ . Now by (4.1.5) there is an integer  $m > 0$  such that  $K = \langle m \rangle = m\mathbb{Z}$ , and clearly  $m$  divides  $n$  since  $n\mathbb{Z} \leq K$ . Thus the Correspondence Theorem tells us that the subgroups of  $\mathbb{Z}_n$  correspond to the positive divisors of  $n$ , a fact we already know from (4.1.6).

**Example (4.2.4)** Let  $N$  be a normal subgroup of a group  $G$ . Call  $N$  a *maximal normal subgroup* of  $G$  if  $N \neq G$  and if  $N < L \triangleleft G$  implies that  $L = G$ . In short “maximal normal” means “maximal proper normal”. It follows from the Correspondence Theorem that if  $N$  is a proper normal subgroup of  $G$ , then  $G/N$  is simple if and only if there are no normal subgroups of  $G$  lying strictly between  $N$  and  $G$ , i.e.,  $N$  is maximal normal in  $G$ . Thus maximal normal subgroups lead in a natural way to simple groups.

**Direct products.** Consider two normal subgroups  $H$  and  $K$  of a (multiplicatively written) group  $G$  such that  $H \cap K = 1$ . Let  $h \in H$  and  $k \in K$ . Then  $[h, k] = (hkh^{-1})k^{-1} \in K$  since  $K \triangleleft G$ ; also  $[h, k] = h(kh^{-1}k^{-1}) \in H$  since  $H \triangleleft G$ . But  $H \cap K = 1$ , so  $[h, k] = 1$ , i.e.,  $hk = kh$ . Thus elements of  $H$  commute with elements of  $K$ .

If in addition  $G = HK$ , then  $G$  is said to be the *internal direct product* of  $H$  and  $K$ , in symbols

$$G = H \times K.$$

Each element of  $G$  is *uniquely* expressible in the form  $hk$ , ( $h \in H, k \in K$ ). For if  $hk = h'k'$  with  $h' \in H, k' \in K$ , then  $(h')^{-1}h = k'k^{-1} \in H \cap K = 1$ , so that  $h = h'$  and  $k = k'$ . Notice also the form taken by the group operation in  $G$ ,

$$(h_1k_1)(h_2k_2) = (h_1h_2)(k_1k_2) \quad (h_i \in H, k_i \in K),$$

since  $k_1h_2 = h_2k_1$ .

For example, consider the Klein 4-group

$$V = \{(1)(2)(3)(4), (12)(34), (13)(24), (14)(23)\} :$$

here  $V = A \times B = B \times C = A \times C$  where  $A = \langle (12)(34) \rangle$ ,  $B = \langle (13)(24) \rangle$ ,  $C = \langle (14)(23) \rangle$ .

The direct product concept may be extended to an arbitrary set of normal subgroups  $\{G_\lambda \mid \lambda \in \Lambda\}$  of a group  $G$  where

- (i)  $G_\lambda \cap \langle G_\mu \mid \mu \in \Lambda, \mu \neq \lambda \rangle = 1$  for all  $\lambda \in \Lambda$ ;
- (ii)  $G = \langle G_\lambda \mid \lambda \in \Lambda \rangle$ .

By the argument in the case of two subgroups, elements from different  $G_\lambda$ 's commute. Also every element of  $G$  has a unique expression of the form  $g_1 g_2 \cdots g_m$  where  $g_i \in G_{\lambda_i}$  and the  $\lambda_i \in \Lambda$  are distinct. (The reader should supply a proof). The direct product is denoted by

$$G = \text{Dr}_{\lambda \in \Lambda} G_\lambda$$

or, in case  $\Lambda$  is a finite set  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ , by

$$G_{\lambda_1} \times G_{\lambda_2} \times \cdots \times G_{\lambda_n}.$$

For additively written groups the term *direct sum* is used and the notation for direct sums is

$$\bigoplus_{\lambda \in \Lambda} G_\lambda \quad \text{and} \quad G_{\lambda_1} \oplus G_{\lambda_2} \oplus \cdots \oplus G_{\lambda_n}.$$

**External direct products.** Up to now a direct product can only be formed from subgroups within a given group. We show next how to form the direct product of groups that are not necessarily subgroups of the same group. For simplicity we deal only with the case of a finite set of groups  $\{G_1, G_2, \dots, G_m\}$ , but see Exercise (4.2.13) for the infinite case.

First we form the set product

$$G = G_1 \times G_2 \times \cdots \times G_m,$$

consisting of all  $m$ -tuples  $(g_1, g_2, \dots, g_m)$  with  $g_i \in G_i$ . Next a binary operation on  $G$  is defined by the rule

$$(g_1, g_2, \dots, g_m)(g'_1, g'_2, \dots, g'_m) = (g_1 g'_1, g_2 g'_2, \dots, g_m g'_m)$$

where  $g_i, g'_i \in G_i$ . With this operation  $G$  becomes a group, with identity element  $(1_{G_1}, 1_{G_2}, \dots, 1_{G_m})$  and inverses given by

$$(g_1, g_2, \dots, g_m)^{-1} = (g_1^{-1}, g_2^{-1}, \dots, g_m^{-1}).$$

Call  $G$  the *external direct product* of the  $G_i$ ; it is also written

$$G_1 \times G_2 \times \cdots \times G_m.$$

Although  $G_i$  is not a subgroup of  $G$ , there is an obvious subgroup of  $G$  which is isomorphic with  $G_i$ . Let  $\bar{G}_i$  consist of all elements of the form  $\bar{g}_i = (1_{G_1}, 1_{G_2}, \dots, g, \dots, 1_{G_m})$  where  $g \in G_i$  appears as the  $i$ th entry of  $\bar{g}_i$ . Then  $G_i \cong \bar{G}_i \leq G$ , because of the assignment  $g_i \mapsto \bar{g}_i$ . Also, if  $x = (g_1, g_2, \dots, g_m)$  is any element of  $G$ , then  $x = \bar{g}_1 \bar{g}_2 \cdots \bar{g}_m$ , by the product rule in  $G$ . Hence  $G = \bar{G}_1 \bar{G}_2 \cdots \bar{G}_m$ . It is easy to verify that  $\bar{G}_i \triangleleft G$  and  $\bar{G}_i \cap \langle \bar{G}_j \mid j = 1, \dots, m, j \neq i \rangle = 1$ , which shows that  $G$  is also the internal direct product

$$G = \bar{G}_1 \times \bar{G}_2 \times \cdots \times \bar{G}_m$$

of subgroups isomorphic with  $G_1, G_2, \dots, G_m$ . Thus the external direct product can be regarded as an internal direct product.

**Example (4.2.5)** Let  $C_1, C_2, \dots, C_k$  be finite cyclic groups of orders  $n_1, n_2, \dots, n_k$  where the  $n_i$  are pairwise relatively prime. Form the external direct product

$$D = C_1 \times C_2 \times \cdots \times C_k.$$

Therefore  $|D| = n_1 n_2 \cdots n_k = n$ , say. Now let  $C_i = \langle x_i \rangle$  and put  $x = (x_1, x_2, \dots, x_m) \in D$ . We claim that  $x$  generates  $D$ , so that  $D$  is a cyclic group of order  $n$ .

To prove this statement it is enough to show that an arbitrary element  $x_1^{u_1} \cdots x_k^{u_k}$  of  $D$  is of the form  $x^r$  for some  $r$ . This amounts to proving that  $x_i^r = x_i^{u_i}$  for each  $i$ , so there is a solution  $r$  of the system of linear congruences  $r \equiv u_i \pmod{n_i}$ ,  $i = 1, 2, \dots, k$ . This is true by the Chinese Remainder Theorem (2.3.7) since  $n_1, n_2, \dots, n_k$  are relatively prime.

For example, let  $n$  be a positive integer and write  $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  where the  $p_i$  are distinct primes and  $e_i > 0$ . Then the preceding discussion shows that  $\mathbb{Z}_{p_1^{e_1}} \times \mathbb{Z}_{p_2^{e_2}} \times \cdots \times \mathbb{Z}_{p_k^{e_k}}$  is a cyclic group of order  $n$  and hence is isomorphic with  $\mathbb{Z}_n$ .

### Exercises (4.2)

- (1) Identify all the normal subgroups of the groups  $S_3$ ,  $\text{Dih}(8)$  and  $A_4$ .
- (2) Let  $H$  be a subgroup of a group  $G$  with index 2. Prove that  $H \triangleleft G$ . Is this true if 2 is replaced by 3?
- (3) Let  $H \triangleleft K \leq G$  and  $L \leq G$ . Show that  $H \cap L \triangleleft K \cap L$ . Also, if  $L \triangleleft G$ , prove that  $HL/L \triangleleft KL/L$ .
- (4) Let  $H \leq G$  and  $N \triangleleft G$ . Prove that  $HN$  is a subgroup of  $G$ .
- (5) Assume that  $H \leq K \leq G$  and  $N \triangleleft G$ . If  $H \cap N = K \cap N$  and  $HN = KN$ , prove that  $H = K$ .
- (6) Show that normality is not a transitive relation in general, i.e.,  $H \triangleleft K \triangleleft G$  need not imply that  $H \triangleleft G$ . [Hint: consider  $\text{Dih}(8)$ ].
- (7) If  $H, K, L$  are arbitrary groups, prove that

$$H \times (K \times L) \cong H \times K \times L \cong (H \times K) \times L.$$

- (8) Let  $G = H \times K$  where  $H, K \leq G$ . Prove that  $G/H \simeq K$  and  $G/K \simeq H$ .
- (9) Let  $G = \langle x \rangle$  be a cyclic group of order  $n$ . If  $d \geq 0$ , prove that  $G/\langle x^d \rangle$  is cyclic with order  $\gcd\{n, d\}$ .
- (10) Prove that  $Z(S_n) = 1$  if  $n \neq 2$ .
- (11) Prove that  $S'_n \neq S_n$  if  $n \neq 1$ .
- (12) Prove that the center of the group  $\text{GL}_n(\mathbb{R})$  of all  $n \times n$  non-singular real matrices is the subgroup of all scalar matrices.
- (13) (*External direct products of infinitely many groups*). Let  $\{G_\lambda, \lambda \in \Lambda\}$  be a set of groups. A *restricted choice function* for the set is a mapping  $f : \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda$  such that  $f(\lambda) \in G_\lambda$  and  $f(\mu) = 1_{G_\mu}$  for all but a finite number of  $\mu$ . Let  $G$  be the set of all restricted choice functions and define a binary operation on  $G$  by  $fg(\lambda) = f(\lambda)g(\lambda)$ . Then  $G$  is called the *restricted external direct product*  $\text{Dr}_{\lambda \in \Lambda} G_\lambda$  of the groups  $G_\lambda$ .
- Prove that  $G$  is a group.
  - For  $\lambda \in \Lambda$  define  $f_\lambda : G_\lambda \rightarrow G$  as follows: if  $x \in G_\lambda$ , then  $f_\lambda(x)$  sends  $\lambda$  to  $x$  and  $\mu$  to  $1_{G_\mu}$  for  $\mu \neq \lambda$ . Prove that  $\tilde{G}_\lambda = \{f_\lambda(x) \mid x \in G_\lambda\}$  is a normal subgroup of  $G$  and that  $\tilde{G}_\lambda \simeq G_\lambda$ .
  - Prove that  $G$  is the internal direct product of the subgroups  $\tilde{G}_\lambda$ .
- (14) It is also possible to form the *unrestricted external direct product* of the  $G_\lambda, \lambda \in \Lambda$ . Follow the construction in Exercise (4.2.13), but allow *all* choice functions  $f : \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda$ , (so the condition that  $f(\mu) = 1_{G_\mu}$  for all but a finite number of  $\mu$  is omitted). In this way obtain a group  $\overline{\text{Dr}}_{\lambda \in \Lambda} G_\lambda$  containing  $\text{Dr}_{\lambda \in \Lambda} G_\lambda$  as a subgroup. Prove that the subgroups  $\tilde{G}_\lambda$  in Exercise (4.2.13) are normal subgroups of  $\overline{\text{Dr}}_{\lambda \in \Lambda} G_\lambda$ , as is  $\text{Dr}_{\lambda \in \Lambda} G_\lambda$ .

### 4.3 Homomorphisms

A homomorphism between two groups is a function that links the operations of the groups. More precisely, if  $G$  and  $H$  are groups, a function  $\alpha : G \rightarrow H$  is called a *homomorphism* if

$$\alpha(xy) = \alpha(x)\alpha(y)$$

for all  $x, y \in G$ . The reader will recognize that a bijective homomorphism is what we have been calling an *isomorphism*.

**Example (4.3.1)** We list some standard examples of homomorphisms.

- $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}_n$  where  $\alpha(x) = [x]_n$ . Here  $n$  is a positive integer. Allowing for the additive notation, what is claimed here is that  $\alpha(x + y) = \alpha(x) + \alpha(y)$ , i.e.  $[x + y]_n = [x]_n + [y]_n$ ; this is just the definition of the sum of congruence classes.
- The determinant function  $\det : \text{GL}_n(\mathbb{R}) \rightarrow \mathbb{R}^*$  in which  $A \mapsto \det(A)$ , is a homomorphism, the reason being the well known identity  $\det(AB) = \det(A)\det(B)$ .

(iii) The sign function  $\text{sign} : S_n \rightarrow \{\pm 1\}$  in which  $\pi \mapsto \text{sign}(\pi)$ , is a homomorphism since  $\text{sign}(\pi\sigma) = \text{sign}(\pi)\text{sign}(\sigma)$  by (3.1.6).

(iv) *The canonical homomorphism.* This example provides the first evidence of a link between homomorphisms and normal subgroups. Let  $N$  be a normal subgroup of a group  $G$  and define a function

$$\alpha : G \rightarrow G/N$$

by the rule  $\alpha(x) = xN$ . Then  $\alpha(xy) = \alpha(x)\alpha(y)$ , i.e.,  $(xy)N = (xN)(yN)$ , by definition of the group operation in  $G/N$ . Thus  $\alpha$  is a homomorphism.

(v) For any pair of groups  $G, H$  there is always at least one homomorphism from  $G$  to  $H$ , namely the *trivial homomorphism* in which  $x \mapsto 1_H$  for all  $x$  in  $G$ . Another obvious example is the *identity homomorphism* from  $G$  to  $G$ , which is the identity function on  $G$ .

Next come two very basic properties that all homomorphism possess.

**(4.3.1)** Let  $\alpha : G \rightarrow H$  be a homomorphism of groups. Then:

- (i)  $\alpha(1_G) = 1_H$ ;
- (ii)  $\alpha(x^n) = (\alpha(x))^n$  for all  $n \in \mathbb{Z}$ .

*Proof.* Applying  $\alpha$  to the equation  $1_G 1_G = 1_G$ , we obtain  $\alpha(1_G)\alpha(1_G) = \alpha(1_G)$ , which on cancellation yields  $\alpha(1_G) = 1_H$ .

If  $n > 0$ , an easy induction on  $n$  shows that  $\alpha(x^n) = (\alpha(x))^n$ . Next  $xx^{-1} = 1_G$ , so that  $\alpha(x)\alpha(x^{-1}) = \alpha(1_G) = 1_H$ ; from this it follows that  $\alpha(x^{-1}) = (\alpha(x))^{-1}$ . Finally, if  $n \geq 0$ , we have  $\alpha(x^{-n}) = \alpha((x^n)^{-1}) = (\alpha(x^n))^{-1} = ((\alpha(x))^n)^{-1} = (\alpha(x))^{-n}$ , which completes the proof.  $\square$

**Image and kernel.** Let  $\alpha : G \rightarrow H$  be a group homomorphism. The *image* of  $\alpha$  is the subset  $\text{Im}(\alpha) = \{\alpha(x) \mid x \in G\}$ . Another significant subset associated with  $\alpha$  is the *kernel*, which is defined by

$$\text{Ker}(\alpha) = \{x \in G \mid \alpha(x) = 1_H\}.$$

The fundamental properties of image and kernel appear in the following result.

**(4.3.2)** If  $\alpha : G \rightarrow H$  is a homomorphism of groups, the image  $\text{Im}(\alpha)$  is a subgroup of  $H$  and the kernel  $\text{Ker}(\alpha)$  is a normal subgroup of  $G$ .

*Proof.* By (4.3.1)  $1_H \in \text{Im}(\alpha)$ . Let  $x, y \in G$ ; then  $\alpha(x)\alpha(y) = \alpha(xy)$  and  $(\alpha(x))^{-1} = \alpha(x^{-1})$ . These equations show that  $\text{Im}(\alpha)$  is a subgroup of  $H$ .

Next, if  $x, y \in \text{Ker}(\alpha)$ , then  $\alpha(xy) = \alpha(x)\alpha(y) = 1_H 1_H = 1_H$ , and  $\alpha(x^{-1}) = (\alpha(x))^{-1} = 1_H^{-1} = 1_H$ ; thus  $\text{Ker}(\alpha)$  is a subgroup of  $G$ . Finally, we establish the critical fact that  $\text{Ker}(\alpha)$  is normal in  $G$ . Let  $x \in \text{Ker}(\alpha)$  and  $g \in G$ ; then

$$\alpha(gxg^{-1}) = \alpha(g)\alpha(x)\alpha(g)^{-1} = \alpha(g)1_H\alpha(g)^{-1} = 1_H,$$

so that  $gxg^{-1} \in \text{Ker}(\alpha)$ , as required.  $\square$

**Example (4.3.2)** Let us compute the image and kernel of some of the homomorphisms in Example (4.3.1).

- (i)  $\det : \mathrm{GL}_n(\mathbb{R}) \rightarrow \mathbb{R}^*$ . The kernel is  $\mathrm{SL}_n(\mathbb{R})$ , the special linear group, and the image is  $\mathbb{R}^*$  since each non-zero real number is the determinant of a diagonal matrix in  $\mathrm{GL}_n(\mathbb{R})$ .
- (ii)  $\mathrm{sign} : S_n \rightarrow \{\pm 1\}$ . The kernel is the alternating group  $A_n$  and the image is the group  $\{\pm 1\}$ , unless  $n = 1$ .
- (iii) The kernel of the canonical homomorphism from  $G$  to  $G/N$  is, as one would expect, the normal subgroup  $N$ . The image is  $G/N$ .

Clearly one can tell from the image of a homomorphism whether it is surjective. In fact the kernel of a homomorphism shows whether it is injective.

**(4.3.3)** Let  $\alpha : G \rightarrow H$  be a group homomorphism. Then:

- (i)  $\alpha$  is surjective if and only if  $\mathrm{Im}(\alpha) = H$ ;
- (ii)  $\alpha$  is injective if and only if  $\mathrm{Ker}(\alpha) = 1_G$ ;
- (iii)  $\alpha$  is an isomorphism if and only if  $\mathrm{Im}(\alpha) = H$  and  $\mathrm{Ker}(\alpha) = 1_G$ .

*Proof.* Of course (i) is true by definition. As for (ii), if  $\alpha$  is injective and  $x \in \mathrm{Ker}(\alpha)$ , then  $\alpha(x) = 1_H = \alpha(1_G)$ , so that  $x = 1_G$  by injectivity of  $\alpha$ . Conversely, assume that  $\mathrm{Ker}(\alpha) = 1_G$ . If  $\alpha(x) = \alpha(y)$ , then  $\alpha(xy^{-1}) = \alpha(x)\alpha(y)^{-1} = 1_H$ , which means that  $xy^{-1} \in \mathrm{Ker}(\alpha) = 1_G$  and  $x = y$ . Thus (ii) is proven, while (iii) follows at once from (i) and (ii).  $\square$

**The Isomorphism Theorems.** We come now to three fundamental results about homomorphisms and quotient groups which are traditionally known as the *Isomorphism Theorems*.

**(4.3.4)** (First Isomorphism Theorem) If  $\alpha : G \rightarrow H$  is a homomorphism of groups, then  $G/\mathrm{Ker}(\alpha) \simeq \mathrm{Im}(\alpha)$  via the mapping  $x\mathrm{Ker}(\alpha) \mapsto \alpha(x)$ .

Thus the image of a homomorphism may be regarded as a quotient group: conversely, every quotient group is the image of the associated canonical homomorphism. What this means is that up to isomorphism quotient groups and homomorphic images are the same objects.

*Proof of (4.3.4)* Let  $K = \mathrm{Ker}(\alpha)$ . We wish to define a function  $\theta : G/K \rightarrow \mathrm{Im}(\alpha)$  by the natural rule  $\theta(xK) = \alpha(x)$ , but first we need to check that this makes sense. If  $k \in K$ , then  $\alpha(xk) = \alpha(x)\alpha(k) = \alpha(x)$ , showing that  $\theta(xK)$  depends only on the coset  $xK$  and not on the choice of  $x$  from  $xK$ . Thus  $\theta$  is a well-defined function.

Next  $\theta((xy)K) = \alpha(xy) = \alpha(x)\alpha(y) = \theta(xK)\theta(yK)$ , so  $\theta$  is a homomorphism. It is obvious that  $\mathrm{Im}(\theta) = \mathrm{Im}(\alpha)$ . Finally,  $\theta(xK) = 1_H$  if and only if  $\alpha(x) = 1_H$ , i.e.,  $x \in K$  or



equivalently  $xK = K = 1_{G/K}$ . Therefore  $\text{Ker}(\theta)$  is the identity subgroup of  $G/K$  and  $\theta$  is an isomorphism from  $G/K$  to  $\text{Im}(\alpha)$ .  $\square$

**(4.3.5)** (Second Isomorphism Theorem) *Let  $G$  be a group with a subgroup  $H$  and a normal subgroup  $N$ . Then  $HN \leq G$ ,  $H \cap N \triangleleft H$  and  $HN/N \simeq H/H \cap N$ .*

*Proof.* We begin by defining a function  $\theta : H \rightarrow G/N$  by the rule  $\theta(h) = hN$ , ( $h \in H$ ). It is easy to check that  $\theta$  is a homomorphism. Also  $\text{Im}(\theta) = \{hN \mid h \in H\} = HN/N$ , which is a subgroup of  $G/N$  by (4.3.2); therefore  $HN \leq G$ . Next  $h \in \text{Ker}(\theta)$  if and only if  $hN = N$ , i.e.,  $h \in H \cap N$ . Therefore  $\text{Ker}(\theta) = H \cap N$  and  $H \cap N \triangleleft H$  by (4.3.2). Apply the First Isomorphism Theorem to the homomorphism  $\theta$  to obtain  $H/H \cap N \simeq HN/N$ .  $\square$

**(4.3.6)** (Third Isomorphism Theorem) *Let  $M$  and  $N$  be normal subgroups of a group  $G$  such that  $N \subseteq M$ . Then  $M/N \triangleleft G/N$  and  $(G/N)/(M/N) \simeq G/M$ .*

*Proof.* Define  $\theta : G/N \rightarrow G/M$  by the rule  $\theta(xN) = xM$ ; the reader should verify that  $\theta$  is a well-defined homomorphism. Also  $\text{Im}(\theta) = G/M$  and  $\text{Ker}(\theta) = M/N$ ; the result now follows via (4.3.4).  $\square$

Thus a quotient group of a quotient group of  $G$  is essentially a quotient group of  $G$ , which represents a considerable simplification. Next these theorems are illustrated by some examples.

**Example (4.3.3)** Let  $m, n$  be positive integers. Then, allowing for the additive notation, we deduce from (4.3.5) that

$$(m\mathbb{Z} + n\mathbb{Z})/n\mathbb{Z} \simeq m\mathbb{Z}/(m\mathbb{Z} \cap n\mathbb{Z}).$$

What does this say about the integers  $m, n$ ? Obviously  $m\mathbb{Z} \cap n\mathbb{Z} = \ell\mathbb{Z}$  where  $\ell$  is the least common multiple of  $m$  and  $n$ . Next  $m\mathbb{Z} + n\mathbb{Z}$  consists of all  $ma + nb$  where  $a, b \in \mathbb{Z}$ . From (2.2.3) we see that this is just  $d\mathbb{Z}$  where  $d = \gcd\{m, n\}$ . So the assertion is that  $d\mathbb{Z}/n\mathbb{Z} \simeq m\mathbb{Z}/\ell\mathbb{Z}$ . Now  $d\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/(\frac{n}{d})\mathbb{Z}$  via the mapping  $dx + n\mathbb{Z} \mapsto x + \frac{n}{d}\mathbb{Z}$ . Similarly  $m\mathbb{Z}/\ell\mathbb{Z} \simeq \mathbb{Z}/(\frac{\ell}{m})\mathbb{Z}$ . Therefore  $\mathbb{Z}/(\frac{n}{d})\mathbb{Z} \simeq \mathbb{Z}/(\frac{\ell}{m})\mathbb{Z}$ . Since isomorphic groups have the same order, it follows that  $\frac{n}{d} = \frac{\ell}{m}$  or  $mn = d\ell$ . Hence (4.3.5) implies that

$$\gcd\{m, n\} \cdot \text{lcm}\{m, n\} = mn,$$

(see also Exercise (2.2.8)).

**Example (4.3.4)** Consider the determinantal homomorphism  $\det : \text{GL}_n(\mathbb{R}) \rightarrow \mathbb{R}^*$ , which has kernel  $\text{SL}_n(\mathbb{R})$  and image  $\mathbb{R}^*$ . Then by (4.3.4)

$$\text{GL}_n(\mathbb{R})/\text{SL}_n(\mathbb{R}) \simeq \mathbb{R}^*.$$

**Automorphisms.** An *automorphism* of a group  $G$  is an isomorphism from  $G$  to itself. Thus an automorphism of  $G$  is a permutation of the set of group elements which is also a homomorphism. The set of all automorphisms of  $G$ ,

$$\text{Aut}(G),$$

is therefore a subset of the symmetric group  $\text{Sym}(G)$ . The first observation to make is:

**(4.3.7)** *If  $G$  is a group, then  $\text{Aut}(G)$  is a subgroup of  $\text{Sym}(G)$ .*

*Proof.* The identity permutation is certainly an automorphism. Also, if  $\alpha \in \text{Aut}(G)$ , then  $\alpha^{-1} \in \text{Aut}(G)$  by (3.3.1). Finally, if  $\alpha, \beta \in \text{Aut}(G)$ , then  $\alpha\beta$  is certainly a permutation of  $G$ , while  $\alpha\beta(xy) = \alpha(\beta(x)\beta(y)) = \alpha\beta(x)\alpha\beta(y)$ , which leads to  $\alpha\beta \in \text{Aut}(G)$ , so  $\text{Aut}(G)$  is a subgroup.  $\square$

In fact  $\text{Aut}(G)$  is usually quite a small subgroup of  $\text{Sym}(G)$ , as will be seen in some of the ensuing examples.

**Example (4.3.5)** Let  $A$  be any additively written abelian group and define  $\alpha : A \rightarrow A$  by  $\alpha(x) = -x$ . Then  $\alpha$  is an automorphism since

$$\alpha(x + y) = -(x + y) = -x - y = \alpha(x) + \alpha(y),$$

while  $\alpha^2 = 1$ , so  $\alpha^{-1} = \alpha$ .

Now suppose we choose  $A$  to be  $\mathbb{Z}$  and let  $\beta$  be any automorphism of  $A$ . Thus  $\beta(1) = n$  for some integer  $n$ . Notice that  $\beta$  is completely determined by  $n$  since  $\beta(m) = \beta(m1) = m\beta(1) = mn$  by (4.3.1)(ii). Also  $\beta(x) = 1$  for some integer  $x$  since  $\beta$  is surjective. Furthermore  $1 = \beta(x) = \beta(x1) = x\beta(1) = xn$  and it follows that  $n = \pm 1$ . Hence there are just two possibilities for  $\beta$ , namely the identity and the automorphism  $\alpha$  of the last paragraph. Therefore  $|\text{Aut}(\mathbb{Z})| = 2$  and  $\text{Aut}(\mathbb{Z}) \simeq \mathbb{Z}_2$ . On the other hand, it is not hard to show that the group  $\text{Sym}(\mathbb{Z})$  is uncountable.

**Inner automorphisms.** An easy way to construct automorphisms is to use a fixed element of the group to form conjugates. If  $g$  is an element of a group  $G$ , define a function  $\tau(g)$  on  $G$  by the rule

$$\tau(g)(x) = gxg^{-1}, \quad (x \in G).$$

Recall that  $gxg^{-1}$  is the *conjugate* of  $x$  by  $g$ . Since

$$\tau(g)(xy) = g(xy)g^{-1} = (gxg^{-1})(gyg^{-1}) = (\tau(g)(x))(\tau(g)(y)),$$

we see that  $\tau(g)$  is a homomorphism. Now  $\tau(g^{-1})$  is clearly the inverse of  $\tau(g)$ , therefore  $\tau(g)$  is an automorphism of  $G$ : it is known as the *inner automorphism induced by  $g$* . Thus we have discovered a function

$$\tau : G \rightarrow \text{Aut}(G).$$

The next observation is that  $\tau$  is a homomorphism, called the *conjugation homomorphism*; for

$$\tau(gh)(x) = (gh)x(gh)^{-1} = g(hxh^{-1})g^{-1},$$

which is also the image of  $x$  under the composite  $\tau(g)\tau(h)$ . Thus  $\tau(gh) = \tau(g)\tau(h)$  for all  $g, h \in G$ .

The image of  $\tau$  is the set of all inner automorphisms of  $G$ , which is denoted by

$$\text{Inn}(G).$$

This is a subgroup of  $\text{Aut}(G)$  by (4.3.2). What can be said about  $\text{Ker}(\tau)$ ? An element  $g$  belongs to  $\text{Ker}(\tau)$  if and only if  $\tau(g)(x) = x$  for all  $x$  in  $G$ , i.e.,  $gxg^{-1} = x$ , or  $gx = xg$ . Therefore the kernel of  $\tau$  is exactly  $Z(G)$ , the center of  $G$ , which consists of the elements of  $G$  that commute with every element of  $G$ .

These conclusions are summed up in the following important result.

**(4.3.8)** *Let  $G$  be a group and let  $\tau : G \rightarrow \text{Aut}(G)$  be the conjugation homomorphism. Then  $\text{Ker}(\tau) = Z(G)$  and  $\text{Im}(\tau) = \text{Inn}(G)$ . Hence  $\text{Inn}(G) \simeq G/Z(G)$ .*

The final statement follows on applying the First Isomorphism Theorem to the homomorphism  $\tau$ .

Usually a group possesses non-inner automorphisms. For example, if  $A$  is an (additively written) abelian group, every inner automorphism is trivial since  $\tau(g)(x) = g + x - g = g - g + x = x$ . On the other hand, the assignment  $x \mapsto -x$  determines an automorphism of  $A$  which is not trivial unless  $2x = 0$  for all  $x$  in  $A$ .

**(4.3.9)** *The relation  $\text{Inn}(G) \triangleleft \text{Aut}(G)$  holds for any group  $G$ .*

*Proof.* Let  $\alpha \in \text{Aut}(G)$  and  $g \in G$ ; we claim that  $\alpha\tau(g)\alpha^{-1} = \tau(\alpha(g))$ , which will establish normality. For if  $x \in G$ , we have

$$\tau(\alpha(g))(x) = \alpha(g)x(\alpha(g))^{-1} = \alpha(g)x\alpha(g^{-1}),$$

which equals

$$\alpha(g\alpha^{-1}(x)g^{-1}) = \alpha(\tau(g)(\alpha^{-1}(x))) = (\alpha\tau(g)\alpha^{-1})(x),$$

as required. □

On the basis of (4.3.9) we can form the quotient group

$$\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G),$$

which is termed the *outer automorphism group* of  $G$ , (although its elements are not actually automorphisms). Thus all automorphisms of  $G$  are inner precisely when  $\text{Out}(G) = 1$ .

A group  $G$  is said to be *complete* if the conjugation homomorphism  $\tau : G \rightarrow \text{Aut}(G)$  is an isomorphism: this is equivalent to requiring that  $Z(G) = 1$  and  $\text{Out}(G) = 1$ . It will

be shown in Chapter Five that the symmetric group  $S_n$  is always complete unless  $n = 2$  or 6.

Finally, we point out that the various groups and homomorphisms introduced above fit neatly together in a sequence of groups and homomorphisms

$$1 \rightarrow Z(G) \xrightarrow{\iota} G \xrightarrow{\tau} \text{Aut}(G) \xrightarrow{\nu} \text{Out}(G) \rightarrow 1.$$

Here  $\iota$  is the inclusion map,  $\tau$  is the conjugation homomorphism and  $\nu$  is the canonical homomorphism associated with the normal subgroup  $\text{Inn}(G)$ . Of course  $1 \rightarrow Z(G)$  and  $\text{Out}(G) \rightarrow 1$  are trivial homomorphisms.

The sequence above is an example of an *exact sequence*, whose feature is that at each group in the interior of the sequence the image of the homomorphism on the left equals the kernel of the homomorphism on the right. For example at  $\text{Aut}(G)$  we have  $\text{Im}(\tau) = \text{Inn}(G) = \text{Ker}(\nu)$ . Exact sequences play a prominent role in algebra, especially in the theory of modules: for more on this see (9.1).

In general it is hard to determine the automorphism group of a given group. A useful aid in the process of deciding which permutations of the group are actually automorphisms is the following simple fact.

**(4.3.10)** *Let  $G$  be a group,  $g \in G$  and  $\alpha \in \text{Aut}(G)$ . Then  $g$  and  $\alpha(g)$  have the same order.*

*Proof.* By (4.3.1)  $\alpha(g^m) = \alpha(g)^m$ . Since  $\alpha$  is injective, it follows that  $\alpha(g)^m = 1$  if and only if  $g^m = 1$ . Hence  $|g| = |\alpha(g)|$ .  $\square$

**The automorphism group of a cyclic group.** As a first example we consider the automorphism group of a cyclic group  $G = \langle x \rangle$ . If  $G$  is infinite, then  $G \simeq \mathbb{Z}$  and we saw in Example (4.3.5) that  $\text{Aut}(G) \simeq \mathbb{Z}_2$ . Assume from now on that  $G$  has finite order  $m$ .

First of all notice that  $\alpha$  is completely determined by  $\alpha(x)$  since  $\alpha(x^i) = \alpha(x)^i$ . Also  $|\alpha(x)| = |x| = m$  by (4.3.10). Thus (4.1.7) shows that  $\alpha(x) = x^i$  where  $1 \leq i < m$  and  $i$  is relatively prime to  $m$ . Consequently  $|\text{Aut}(G)| \leq \phi(m)$  where  $\phi$  is Euler's function, since  $\phi(m)$  is the number of such integers  $i$ .

Conversely, suppose that  $i$  is an integer satisfying  $1 \leq i < m$  and  $\gcd\{i, m\} = 1$ . Then the assignment  $g \mapsto g^i$ , ( $g \in G$ ), determines a homomorphism  $\alpha_i : G \rightarrow G$  because  $(g_1 g_2)^i = g_1^i g_2^i$ , the group  $G$  being abelian. Since  $|x^i| = m$ , the element  $x^i$  generates  $G$  and so this homomorphism is surjective. But  $G$  is finite, so we may conclude that  $\alpha_i$  is also injective and thus  $\alpha_i \in \text{Aut}(G)$ . It follows that  $|\text{Aut}(G)| = \phi(m)$ , the number of such  $i$ 's.

It is not hard to identify the group  $\text{Aut}(G)$ . Recall that  $\mathbb{Z}_m^*$  is the multiplicative group of congruence classes  $[a]_m$  where  $a$  is relatively prime to  $m$ . Now there is a natural function  $\theta : \mathbb{Z}_m^* \rightarrow \text{Aut}(G)$  given by  $\theta([i]_m) = \alpha_i$  where  $\alpha_i$  is defined as above;  $\theta$  is well-defined since  $\alpha_{i+\ell m} = \alpha_i$  for all  $\ell$ . In addition  $\theta$  is a homomorphism because

$\alpha_{ij} = \alpha_i \alpha_j$ , and the preceding discussion shows that it is surjective and hence bijective. We have therefore established:

**(4.3.11)** *Let  $G = \langle x \rangle$  be a cyclic group of order  $m$ . Then  $\mathbb{Z}_m^* \simeq \text{Aut}(G)$  via the assignment  $[i]_m \mapsto (g \mapsto g^i)$ .*

In particular this establishes:

**Corollary (4.3.12)** *The automorphism group of a cyclic group is abelian.*

The next example is more challenging.

**Example (4.3.6)** Show that the order of the automorphism group of the dihedral group  $\text{Dih}(2p)$  where  $p$  is an odd prime is  $p(p-1)$ .

Recall that  $\text{Dih}(2p)$  is the symmetry group of a regular  $p$ -gon – see (3.2). First we need a good description of the elements of  $G = \text{Dih}(2p)$ . If the vertices of the  $p$ -gon are labelled  $1, 2, \dots, p$ , then  $G$  contains the  $p$ -cycle  $\sigma = (1\ 2\ \dots\ p)$ , which corresponds to an anticlockwise rotation through angle  $\frac{2\pi}{p}$ . It also contains the permutation  $\tau = (1)(2\ p)(3\ p-1)\dots(\frac{p+1}{2}\ \frac{p+3}{2})$ , which represents a reflection in the line through the vertex 1 and the midpoint of the opposite edge.

The elements  $\sigma^r, \sigma^r \tau$ , where  $r = 0, 1, \dots, p-1$ , are all different and there are  $2p$  of them. Since  $|G| = 2p$ , we conclude that

$$G = \{\sigma^r, \sigma^r \tau \mid r = 0, 1, \dots, p-1\}.$$

Notice that  $(\sigma^r \tau)^2 = 1$  and in fact  $\sigma^r \tau = 1$  is a reflection, while  $\sigma^r$  is a rotation of order 1 or  $p$ .

Next let  $\alpha \in \text{Aut}(G)$ . By (4.3.10)  $\alpha(\sigma)$  has order  $p$ , and hence  $\alpha(\sigma) = \sigma^r$  where  $1 \leq r < p$ ; also  $\alpha(\tau)$  has order 2 and so it must equal  $\sigma^s \tau$  where  $0 \leq s < p$ . Observe that  $\alpha$  is determined by its effect on  $\sigma$  and  $\tau$  since  $\alpha(\sigma^i) = \alpha(\sigma)^i$  and  $\alpha(\sigma^i \tau) = \alpha(\sigma)^i \alpha(\tau)$ . It follows that there are at most  $p(p-1)$  possibilities for  $\alpha$  and hence that  $|\text{Aut}(G)| \leq p(p-1)$ .

To show that  $p(p-1)$  is the order of the automorphism group we need to construct some automorphisms. Now it is easy to see that  $Z(G) = 1$ ; thus by (4.3.8)  $\text{Inn}(G) \simeq G/Z(G) \simeq G$ . Therefore  $|\text{Inn}(G)| = 2p$ , and since  $\text{Inn}(G) \leq \text{Aut}(G)$ , it follows from Lagrange's Theorem that  $p$  divides  $|\text{Aut}(G)|$ .

Next for  $0 < r < p$  we define an automorphism  $\alpha_r$  of  $G$  by the rules

$$\alpha_r(\sigma) = \sigma^r \quad \text{and} \quad \alpha_r(\tau) = \tau.$$

To verify that  $\alpha_r$  is a homomorphism one needs to check that  $\alpha_r(xy) = \alpha_r(x)\alpha_r(y)$ ; this is not difficult, but it does involve some case distinctions, depending on the form of  $x$  and  $y$ . Now  $\alpha_r$  is clearly surjective because  $\sigma^r$  generates  $\langle \sigma \rangle$ ; thus it is an automorphism. Notice also that  $\alpha_r \alpha_s = \alpha_{rs}$ , so that  $[r]_p \mapsto \alpha_r$  determines a homomorphism from  $\mathbb{Z}_p^*$  to  $H = \{\alpha_r \mid 1 \leq r < p\}$ . This mapping is surjective, while if  $\alpha_r = 1$ , then  $r \equiv 1 \pmod{p}$ ,

i.e.,  $[r]_p = [1]_p$ . Hence the assignment  $[r]_p \mapsto \alpha_r$  determines an isomorphism from  $\mathbb{Z}_p^*$  to  $H$ . Therefore  $H$  has order  $p - 1$  and  $p - 1$  divides  $|\text{Aut}(G)|$ . Consequently  $p(p - 1)$  divides the order of  $\text{Aut}(G)$  and hence  $|\text{Aut}(G)| = p(p - 1)$ .

Since  $|\text{Inn}(G)| = |G| = 2p$ , we see that

$$|\text{Out}(G)| = \frac{p(p-1)}{2p} = \frac{p-1}{2}.$$

Thus  $|\text{Out}(G)| = 1$  if and only if  $p = 3$ . Since also  $Z(G) = 1$ , as a consequence  $\text{Dih}(2p)$  is a complete group if and only if  $p = 3$ .

**Semidirect products.** Suppose that  $G$  is a group with a normal subgroup  $N$  and a subgroup  $H$  such that

$$G = NH \quad \text{and} \quad N \cap H = 1.$$

Then  $G$  is said to be the *internal semidirect product* of  $N$  and  $H$ . As a simple example, consider the alternating group  $G = A_4$ ; this has a normal subgroup of order 4, namely the Klein 4-group  $V$ , and also the subgroup  $H = \langle (123)(4) \rangle$  of order 3. Thus  $V \cap H = 1$  and  $|VH| = |V| \cdot |H| = 12$  by (4.1.12). Hence  $G = VH$  and  $G$  is the semidirect product of  $V$  and  $H$ .

Now let us analyze the structure of a semidirect product  $G = NH$ . In the first place each element  $g \in G$  has a unique expression  $g = nh$  with  $n \in N$  and  $h \in H$ . For if  $g = n'h'$  is another such expression,  $(n')^{-1}n = h'h^{-1} \in N \cap H = 1$ , which shows that  $n = n'$  and  $h = h'$ . Secondly, conjugation in  $N$  by an element  $h$  of  $H$  produces an automorphism of  $N$ , say  $\theta(h)$ . Thus  $\theta(h)(n) = hnh^{-1}$ , ( $n \in N$ ). Furthermore it is easily verified that  $\theta(h_1h_2) = \theta(h_1)\theta(h_2)$ , ( $h_i \in H$ ). Therefore  $\theta : H \rightarrow \text{Aut}(N)$  is a homomorphism.

Let us see whether, on the basis of the preceding analysis, we can reconstruct the semidirect product from the groups  $N$  and  $H$  and a given homomorphism  $\theta : H \rightarrow \text{Aut}(N)$ . This will be the *external semidirect product*. The underlying set of this group is to be the set product  $N \times H$ , so that

$$G = \{(n, h) \mid n \in N, h \in H\}.$$

A binary operation on  $G$  is defined by the rule

$$(n_1, h_1)(n_2, h_2) = (n_1\theta(h_1)(n_2), h_1h_2).$$

The motivation for this rule is the way that products are formed in an internal semidirect product  $NH$ , which is  $(n_1h_1)(n_2h_2) = n_1(h_1n_2h_1^{-1})h_1h_2$ . The identity element of  $G$  is  $(1_N, 1_H)$  and the inverse of  $(n, h)$  is to be  $(\theta(h^{-1})(n^{-1}), h^{-1})$ : the latter is motivated by the fact that in an internal semidirect product  $NH$  inverses are formed according to the rule  $(nh)^{-1} = h^{-1}n^{-1} = (h^{-1}n^{-1}h)h^{-1}$ . We omit the entirely routine verification of the group axioms for  $G$ .

Next we look for subgroups of  $G$  which resemble the original groups  $N$  and  $H$ . There are natural candidates,

$$\tilde{N} = \{(n, 1_H) \mid n \in N\} \quad \text{and} \quad \tilde{H} = \{(1_N, h) \mid h \in H\}.$$

It is straightforward to show that these are subgroups isomorphic with  $N$  and  $H$  respectively. The group operation of  $G$  shows that

$$(n, 1_H)(1_N, h) = (n\theta(1_H)(1_N), h) = (n, h) \in \tilde{N}\tilde{H}$$

since  $\theta(1_H)$  is the identity automorphism of  $N$ . It follows that  $G = \tilde{N}\tilde{H}$ , while it is evident that  $\tilde{N} \cap \tilde{H} = 1$ .

To show that  $G$  is the semidirect product of  $\tilde{N}$  and  $\tilde{H}$ , it is only necessary to check normality of  $\tilde{N}$  in  $G$ . Let  $n, n_1 \in N$  and  $h \in H$ . Then by definition

$$\begin{aligned} (n, h)(n_1, 1_H)(n, h)^{-1} &= (n, h)(n_1, 1_H)(\theta(h^{-1})(n^{-1}), h^{-1}) \\ &= (n\theta(h)(n_1), h)(\theta(h^{-1})(n^{-1}), h^{-1}) \\ &= (n\theta(h)(n_1)\theta(h)(\theta(h^{-1})(n^{-1})), 1_H) \\ &= (n\theta(h)(n_1)n^{-1}, 1_H) \in \tilde{N}. \end{aligned}$$

In particular conjugation in  $\tilde{N}$  by  $(1_N, h)$  sends  $(n_1, 1_H)$  to  $(\theta(h)(n_1), 1_H)$ . Therefore conjugation in  $\tilde{N}$  by  $(1_N, h)$  induces the automorphism  $\theta(h)$  in  $N$ .

In the special case where  $\theta$  is chosen to be the trivial homomorphism, elements of  $\tilde{N}$  and  $\tilde{H}$  commute, so that  $G$  becomes the direct product. Thus the semidirect product is a generalization of the direct product of two groups. Semidirect products provide an important means of constructing new groups.

**Example (4.3.7)** Let  $N = \langle n \rangle$  and  $H = \langle h \rangle$  be cyclic groups with respective orders 3 and 4. Suppose we wish to form a semidirect product  $G$  of  $N$  and  $H$ . For this purpose choose a homomorphism  $\theta : H \rightarrow \text{Aut}(N)$ ; there is little choice here since  $N$  has only one non-identity automorphism, namely  $n \mapsto n^{-1}$ . Accordingly define  $\theta(h)$  to be this automorphism. The resulting group  $G$  is known as the *dicyclic group of order 12*. Observe that this group is not isomorphic with  $A_4$  or  $\text{Dih}(12)$  since, unlike these groups,  $G$  has an element of order 4.

### Exercises (4.3)

- (1) Let  $H \triangleleft K \leq G$  and let  $\alpha : G \rightarrow G_1$  be a homomorphism. Show that  $\alpha(H) \triangleleft \alpha(K) \leq G_1$  where  $\alpha(H) = \{\alpha(h) \mid h \in H\}$ .
- (2) If  $G$  and  $H$  are groups with relatively prime orders, show that the only homomorphism from  $G$  to  $H$  is the trivial one.
- (3) Let  $G$  be a simple group. Show that if  $\alpha : G \rightarrow H$  is a homomorphism, either  $\alpha$  is trivial or  $H$  has a subgroup isomorphic with  $G$ .

- (4) Prove that  $\text{Aut}(V) \simeq S_3$  where  $V$  is the Klein 4-group.
- (5) Prove that  $\text{Aut}(\mathbb{Q}) \simeq \mathbb{Q}^*$  where  $\mathbb{Q}^*$  is the multiplicative group of non-zero rationals. [Hint: an automorphism is determined by its effect on 1].
- (6) Let  $G$  and  $A$  be groups, with  $A$  abelian written additively. Let  $\text{Hom}(G, A)$  denote the set of all homomorphisms from  $G$  to  $A$ . Define a binary operation  $+$  on  $\text{Hom}(G, A)$  by  $\alpha + \beta(x) = \alpha(x) + \beta(x)$ , ( $x \in G$ ). Prove that with this operation  $\text{Hom}(G, A)$  is an abelian group. Then prove that  $\text{Hom}(\mathbb{Z}, A) \simeq A$ .
- (7) Let  $G = \langle x \rangle$  have order 8. Write down all the automorphisms of  $G$  and verify that  $\text{Aut}(G) \simeq V$ : conclude that the automorphism group of a cyclic group need not be cyclic.
- (8) If  $G$  and  $H$  are finite groups of relatively prime orders, prove that  $\text{Aut}(G \times H) \simeq \text{Aut}(G) \times \text{Aut}(H)$ .
- (9) Use Exercise (4.3.8) to prove that  $\phi(mn) = \phi(m)\phi(n)$  where  $\phi$  is Euler's function and  $m, n$  are relatively prime integers. (A different proof of this fact was given in (2.3.8)).
- (10) An  $n \times n$  matrix is called a *permutation matrix* if each row and each column contains a single 1 and all other entries are 0. If  $\pi \in S_n$ , form an  $n \times n$  permutation matrix  $M(\pi)$  by defining  $M(\pi)_{ij}$  to be 1 if  $\pi(j) = i$  and 0 otherwise.
- (i) Prove that the assignment  $\pi \mapsto M(\pi)$  determines an injective homomorphism from  $S_n$  to  $\text{GL}_n(\mathbb{Q})$ .
  - (ii) Deduce that the  $n \times n$  permutation matrices form a group which is isomorphic with  $S_n$ .
  - (iii) How can one tell from  $M(\pi)$  whether the permutation  $\pi$  is even or odd?
- (11) Show that each of the groups  $\text{Dih}(2n)$  and  $S_4$  is a semidirect product of groups of smaller orders.
- (12) Use the groups  $\mathbb{Z}_3 \times \mathbb{Z}_3$  and  $\mathbb{Z}_2$  to form three non-isomorphic groups of order 18 each with a normal subgroup of order 9.



## 5 Groups acting on sets

Until the end of the Nineteenth Century, a group was usually synonymous with a permutation group, so that the elements acted in a natural way on a set. While group theory has since become more abstract, it remains true that groups are at their most useful when their elements act on a set. In this chapter we develop the basic theory of group actions and illustrate its utility by giving applications both within group theory and to combinatorics.

### 5.1 Group actions

Let  $G$  be a group and  $X$  a non-empty set. A *left action* of  $G$  on  $X$  is a function

$$\alpha : G \times X \rightarrow X,$$

written for convenience  $\alpha((g, x)) = g \cdot x$ , with the following properties for all  $g_i \in G$  and  $x \in X$ :

- (i)  $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$
- (ii)  $1_G \cdot x = x.$

Here one should think of the group element  $g$  as operating or acting on a set element  $x$  to produce the set element  $g \cdot x$ .

There is a corresponding definition of a *right action* of  $G$  on  $X$  as a function  $\beta : X \times G \rightarrow X$ , with  $\beta((x, g))$  written  $x \cdot g$ , such that  $x \cdot 1_G = x$  and  $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$  for all  $x \in X$  and  $g_i \in G$ .

For example, suppose that  $G$  is a subgroup of the symmetric group  $\text{Sym}(X)$ , in which event  $G$  is called a *permutation group on  $X$* . Define  $\pi \cdot x$  to be  $\pi(x)$  where  $\pi \in G$  and  $x \in X$ ; this is a left action of  $G$  on  $X$ . There may of course be other ways for  $G$  to act on  $X$ , so we are dealing here with a wide generalization of a permutation group.

**Permutation representations.** Let  $G$  be a group and  $X$  a non-empty set. A homomorphism

$$\sigma : G \rightarrow \text{Sym}(X)$$

is called a *permutation representation* of  $G$  on  $X$ . Thus the homomorphism  $\sigma$  represents elements of the abstract group  $G$  by concrete objects, namely permutations of  $X$ . A permutation representation provides a useful way of visualizing the elements of an abstract group.

What is the connection between group actions and permutation representations? In fact the two concepts are essentially identical. To see why, suppose that a permutation

representation  $\sigma : G \rightarrow \text{Sym}(X)$  is given; then there is a corresponding left action of  $G$  on  $X$  defined by

$$g \cdot x = \sigma(g)(x),$$

where  $g \in G, x \in X$ ; it is easy to check that this is an action.

Conversely, if we start with a left action of  $G$  on  $X$ , say  $(g, x) \mapsto g \cdot x$ , there is a corresponding permutation representation  $\sigma : G \rightarrow \text{Sym}(X)$  defined by

$$\sigma(g)(x) = g \cdot x,$$

where  $g \in G, x \in X$ . Again it is an easy verification that the mapping  $\sigma$  is a homomorphism and hence is a permutation representation of  $G$  on  $X$ .

The foregoing discussion makes the following result clear.

**(5.1.1)** *Let  $G$  be a group and  $X$  a non-empty set. Then there is a bijection from the set of left actions of  $G$  on  $X$  to the set of permutation representations of  $G$  on  $X$ .*

If  $\sigma$  is a permutation representation of a group  $G$  on a set  $X$ , then  $G/\text{Ker}(\sigma) \simeq \text{Im}(\sigma)$  by the First Isomorphism Theorem (4.3.4). Thus  $G/\text{Ker}(\sigma)$  is isomorphic with a permutation group on  $X$ . If  $\text{Ker}(\sigma) = 1$ , then  $G$  itself is isomorphic with a permutation group on  $X$ , in which case the representation  $\sigma$  is said to be *faithful*. The term faithful can also be applied to a group action by means of the associated permutation representation.

Next we will describe some natural ways in which a group can act on a set.

**Action on a group by multiplication.** A group  $G$  can act on its underlying set  $G$  by left multiplication, that is to say,

$$g \cdot x = gx,$$

where  $g, x \in G$ ; this is an action since  $1_G \cdot x = 1_G x = x$  and

$$g_1 \cdot (g_2 \cdot x) = g_1(g_2 x) = (g_1 g_2)x = (g_1 g_2) \cdot x.$$

This action is called the *left regular action* of  $G$  and the corresponding permutation representation

$$\lambda : G \rightarrow \text{Sym}(G),$$

which is given by  $\lambda(g)(x) = gx$ , is called the *left regular representation* of  $G$ . Observe that  $\lambda(g) = 1$  if and only if  $gx = x$  for all  $x \in G$ , i.e.,  $g = 1$ . Thus  $\text{Ker}(\lambda) = 1$  and  $\lambda$  is a faithful permutation representation.

It follows at once that  $G$  is isomorphic with  $\text{Im}(\lambda)$ , which is a subgroup of  $\text{Sym}(G)$ . We have therefore proved the following result, which demonstrates in a striking fashion the significance of permutation groups.

**(5.1.2)** (Cayley's<sup>1</sup> Theorem) *An arbitrary group  $G$  is isomorphic with a subgroup of  $\text{Sym}(G)$  via the left regular representation in which  $g \mapsto (x \mapsto gx)$  where  $x, g \in G$ .*

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<sup>1</sup> Arthur Cayley (1821–1895)

**Action on cosets.** For the next example of an action take a fixed subgroup  $H$  of a group  $G$  and let  $\mathcal{L}$  be the set of all left cosets of  $H$  in  $G$ . A left action of  $G$  on  $\mathcal{L}$  is defined by the rule

$$g \cdot (xH) = (gx)H,$$

where  $g, x \in G$ . Again it is simple to verify that this is a left action.

Now consider the corresponding permutation representation  $\lambda : G \rightarrow \text{Sym}(\mathcal{L})$ . Then  $g \in \text{Ker}(\lambda)$  if and only if  $gxH = xH$  for all  $x$  in  $G$ , i.e.,  $x^{-1}gx \in H$  or  $g \in xHx^{-1}$ . It follows that

$$\text{Ker}(\lambda) = \bigcap_{x \in G} xHx^{-1}.$$

Thus we have:

**(5.1.3)** *The kernel of the permutation representation of  $G$  on the set of left cosets of  $H$  by left multiplication is*

$$\bigcap_{x \in G} xHx^{-1},$$

*which is the largest normal subgroup of  $G$  contained in  $H$ .*

For the final statement in (5.1.3), note that the intersection is normal in  $G$ . Also, if  $N \triangleleft G$  and  $N \leq H$ , then  $N \leq xHx^{-1}$  for all  $x \in G$ . The normal subgroup  $\bigcap_{x \in G} xHx^{-1}$  is called the *normal core* of  $H$  in  $G$ .

Here is an application of the action on left cosets.

**(5.1.4)** *Suppose that  $H$  is a subgroup of a finite group  $G$  such that  $|G : H|$  equals the smallest prime dividing  $|G|$ . Then  $H \triangleleft G$ . In particular, a subgroup of index 2 is always normal.*

*Proof.* Let  $|G : H| = p$  and let  $K$  be the kernel of the permutation representation of  $G$  arising from the left action of  $G$  on the set of left cosets of  $H$ . Then  $K \leq H < G$  and  $p = |G : H|$  divides  $|G : K|$  by (4.1.3). Now  $G/K$  is isomorphic with a subgroup of the symmetric group  $S_p$ , so  $|G : K|$  divides  $|S_p| = p!$  by (4.1.1). But  $|G : K|$  divides  $|G|$  and thus cannot be divisible by a smaller prime than  $p$ . Therefore  $|G : K| = p = |G : H|$  and  $H = K \triangleleft G$ .  $\square$

**Action by conjugation.** Another natural way in which a group  $G$  can act on its underlying set is by conjugation. Define

$$g \cdot x = gxg^{-1},$$

where  $g, x \in G$ ; by a simple check this is a left action. Again we ask about the kernel of the action. An element  $g$  belongs to the kernel if and only if  $gxg^{-1} = x$ , i.e.,  $gx = xg$ , for all  $x \in G$ : this is the condition for  $g$  to belong to  $Z(G)$ , the center of  $G$ . It follows that  $Z(G)$  is the kernel of the conjugation representation.

A group  $G$  can also act on its set of subgroups by conjugation; thus if  $H \leq G$ , define

$$g \cdot H = gHg^{-1} = \{ghg^{-1} \mid h \in H\}.$$

In this case the kernel consists of all group elements  $g$  such that  $gHg^{-1} = H$  for all  $H \leq G$ . This normal subgroup is called the *norm* of  $G$ ; clearly it contains the center  $Z(G)$ .

### Exercises (5.1)

- (1) Complete the proof of (5.1.1).
- (2) Let  $(x, g) \mapsto x \cdot g$  be a *right* action of a group  $G$  on a set  $X$ . Define  $\rho : G \rightarrow \text{Sym}(X)$  by  $\rho(g)(x) = x \cdot g^{-1}$ . Prove that  $\rho$  is a permutation representation of  $G$  on  $X$ . Why is the inverse necessary here?
- (3) Establish a bijection between the set of right actions of a group  $G$  on a set  $X$  and the set of permutation representations of  $G$  on  $X$ .
- (4) A right action of a group  $G$  on its underlying set is defined by  $x \cdot g = xg$ . Verify that this is an action and describe the corresponding permutation representation of  $G$ , (it is called the *right regular representation* of  $G$ ).
- (5) Prove that a permutation representation of a simple group is either faithful or trivial.
- (6) The left regular representation of a finite group is surjective if and only if the group has order 1 or 2.
- (7) Define a “natural” right action of a group  $G$  on the set of right cosets of a subgroup  $H$  and then identify the kernel of the associated representation.
- (8) Show that up to isomorphism the number of groups of order  $n$  is at most  $(n!)^{\lfloor \log_2 n \rfloor}$ . [Hint: a group of order  $n$  can be generated by  $\lfloor \log_2 n \rfloor$  elements by Exercise (4.1.10). Now apply (5.1.2)].

## 5.2 Orbits and stabilizers

In this section we proceed to develop the theory of group actions, introducing the fundamental concepts of *orbit* and *stabilizer*.

Let  $G$  be a group and  $X$  a non-empty set, and suppose that a left action of  $G$  on  $X$  is given. A binary relation  $\sim_G$  on  $X$  is defined by the rule:

$$a \sim_G b \text{ if and only if } g \cdot a = b$$

for some  $g \in G$ . A simple verification shows that  $\sim_G$  is an equivalence relation on the set  $X$ . The  $\sim_G$ -equivalence class containing  $a$  is evidently

$$G \cdot a = \{g \cdot a \mid g \in G\},$$

which is called the  $G$ -orbit of  $a$ . Thus  $X$  is the union of the distinct  $G$ -orbits and distinct  $G$ -orbits are disjoint: these statements follow from general facts about equivalence relations – see (1.2.2).

If  $X$  is the only  $G$ -orbit, the action of  $G$  on  $X$  – and the corresponding permutation representation of  $G$  – is called *transitive*. Thus the action of  $G$  is transitive if for each pair of elements  $a, b$  of  $X$ , there exists a  $g$  in  $G$  such that  $g \cdot a = b$ . For example, the left regular representation is transitive, as is the left action of a group on the set of left cosets of a subgroup.

Another important notion is that of a stabilizer. The *stabilizer* in  $G$  of an element  $a \in X$  is defined to be

$$\text{St}_G(x) = \{g \in G \mid g \cdot x = x\}.$$

It is easy to verify that  $\text{St}_G(a)$  is a subgroup of  $G$ . If  $\text{St}_G(a) = 1$  for all  $a \in X$ , the action is called *semiregular*. An action which is both transitive and semiregular is termed *regular*.

We illustrate these concepts by examining the group actions introduced in (5.1).

**Example (5.2.1)** Let  $G$  be any group.

- (i) The left regular action of  $G$  is regular. Indeed  $(yx^{-1})x = y$  for any  $x, y \in G$ , so it is transitive, while  $gx = x$  implies that  $g = 1$  and regularity follows.
- (ii) In the conjugation action of  $G$  on its underlying set the stabilizer of  $x$  consists of all  $g$  in  $G$  such that  $gxg^{-1} = x$ , i.e.,  $gx = xg$ . This subgroup is called the *centralizer* of  $x$  in  $G$ : it is denoted by

$$C_G(x) = \{g \in G \mid gx = xg\}.$$

- (iii) In the conjugation action of  $G$  on its underlying set the  $G$ -orbit of  $x$  is  $\{gxg^{-1} \mid g \in G\}$ , i.e., the set of all conjugates of  $x$  in  $G$ . This is called the *conjugacy class* of  $x$ . The number of conjugacy classes in a finite group is called the *class number*.
- (iv) In the action of  $G$  by conjugation on its set of subgroups, the  $G$ -orbit of  $H \leq G$  is just the set of all conjugates of  $H$  in  $G$ , i.e.,  $\{gHg^{-1} \mid g \in G\}$ . The stabilizer of  $H$  in  $G$  is an important subgroup termed the *normalizer* of  $H$  in  $G$ ,

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}.$$

Centralizers and normalizers feature throughout group theory.

Next we will prove two basic theorems on group actions. The first one counts the number of elements in an orbit.

**(5.2.1)** Let  $G$  be a group acting on a set  $X$  on the left and let  $x \in X$ . Then the assignment  $g\text{St}_G(x) \mapsto g \cdot x$  determines a bijection from the set of left cosets of  $\text{St}_G(x)$  in  $G$  to the orbit  $G \cdot x$ . Hence  $|G \cdot x| = |G : \text{St}_G(x)|$ .

*Proof.* In the first place the assignment  $g \operatorname{St}_G(x) \mapsto g \cdot x$  determines a well-defined function. For if  $s \in \operatorname{St}_G(x)$ , then  $gs \cdot x = g \cdot (s \cdot x) = g \cdot x$ . Next  $g_1 \cdot x = g_2 \cdot x$  implies that  $(g_2^{-1}g_1) \cdot x = x$ , so  $g_2^{-1}g_1 \in \operatorname{St}_G(x)$ , i.e.,  $g_1 \operatorname{St}_G(x) = g_2 \operatorname{St}_G(x)$ . Hence the function is injective, while it is obviously surjective.  $\square$

**Corollary (5.2.2)** *Let  $G$  be a finite group acting on a finite set  $X$ . If the action is transitive,  $|X|$  divides  $|G|$ . If the action is regular,  $|X| = |G|$ .*

*Proof.* If the action is transitive,  $X$  is the only  $G$ -orbit, so  $|X| = |G : \operatorname{St}_G(x)|$  for any  $x \in X$  by (5.2.1); hence  $|X|$  divides  $|G|$ . If the action is regular, then in addition  $\operatorname{St}_G(x) = 1$  and thus  $|X| = |G|$ .  $\square$

The corollary tells us that if  $G$  is a transitive permutation group of degree  $n$ , i.e., acting on a set with elements, then  $n$  divides  $|G|$ , while  $|G| = n$  if  $G$  is regular.

The second main theorem on actions counts the number of orbits and has many applications. If a group  $G$  acts on a set  $X$  on the left and  $g \in G$ , the *fixed point set* of  $g$  is defined to be

$$\operatorname{Fix}(g) = \{x \in X \mid g \cdot x = x\}.$$

**(5.2.3)** (The Frobenius-Burnside Theorem)<sup>2</sup> *Let  $G$  be a finite group acting on a finite set  $X$  (on the left). Then the number of  $G$ -orbits in  $X$  equals*

$$\frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|,$$

*i.e., the average number of fixed points of elements of  $G$ .*

*Proof.* Consider how often an element  $x$  of  $X$  is counted in the sum  $\sum_{g \in G} |\operatorname{Fix}(g)|$ . This happens once for each  $g$  in  $\operatorname{St}_G(x)$ . Thus by (5.2.1) the element  $x$  contributes  $|\operatorname{St}_G(x)| = |G|/|G \cdot x|$  to the sum. The same is true of each element of the orbit  $|G \cdot x|$ , so that the total contribution of this orbit to the sum is  $(|G|/|G \cdot x|) \cdot |G \cdot x| = |G|$ . It follows that  $\sum_{g \in G} |\operatorname{Fix}(g)|$  must equal  $|G|$  times the number of orbits, so the result is proven.  $\square$

We illustrate the Frobenius-Burnside Theorem with a simple example.

**Example (5.2.2)** The group

$$G = \{(1)(2)(3)(4), (12)(3)(4), (1)(2)(34), (12)(34)\}$$

acts on the set  $X = \{1, 2, 3, 4\}$  in the natural way, as a permutation group. There are two  $G$ -orbits, namely  $\{1, 2\}$  and  $\{3, 4\}$ . Count the fixed points of the elements of  $G$  by

<sup>2</sup> Ferdinand Georg Frobenius (1849–1917), William Burnside (1852–1927)

looking for 1-cycles. Thus the four elements of the group have respective numbers of fixed points 4, 2, 2, 0. Therefore the number of  $G$ -orbits should be

$$\frac{1}{|G|} \left( \sum_{g \in G} |\text{Fix}(g)| \right) = \frac{1}{4} (4 + 2 + 2 + 0) = 2,$$

which is the correct answer.

**Example (5.2.3)** Show that the average number of fixed points of elements of  $S_n$  is 1.

The symmetric group  $S_n$  acts on the set  $\{1, 2, \dots, n\}$  in the natural way and the action is clearly transitive. By (5.2.3) the average number of fixed points equals the number of  $S_n$ -orbits, which is 1 by transitivity of the action.

### Exercises (5.2)

- (1) If  $g$  is an element of a finite group  $G$ , show that the number of conjugates of  $g$  divides  $|G : \langle g \rangle|$ .
- (2) If  $H$  is a subgroup of a finite group  $G$ , show that the number of conjugates of  $H$  divides  $|G : H|$ .
- (3) Let  $G = \langle (1\ 2\ \dots\ p), (1)(2\ p)(3\ p-1)\dots \rangle$  be the dihedral group  $\text{Dih}(2p)$  where  $p$  is an odd prime. Check the validity of (5.2.3) for the group  $G$  acting on the set  $\{1, 2, \dots, p\}$  as a permutation group.
- (4) Let  $G$  be a finite group acting as a finite set  $X$ . If the action is semiregular, prove that  $|G|$  divides  $|X|$ .
- (5) Prove that the class number of a finite group  $G$  is given by the formula

$$\frac{1}{|G|} \left( \sum_{x \in G} |C_G(x)| \right).$$

- (6) Prove that the class number of a direct product  $H \times K$  equals the product of the class numbers of  $H$  and  $K$ .
- (7) Let  $G$  be a finite group acting transitively on a finite set  $X$  where  $|X| > 1$ . Prove that  $G$  contains at least  $|X| - 1$  *fixed-point-free elements*, i.e., elements  $g$  such that  $\text{Fix}(g)$  is empty.
- (8) Let  $H$  be a proper subgroup of a finite group  $G$ . Prove that  $G \neq \bigcup_{x \in G} xHx^{-1}$ . [Hint: consider the action of  $G$  on the set of left cosets of  $H$  by multiplication. The action is transitive, so Exercise (5.2.7) may be applied].
- (9) Let  $X$  be a subset of a group  $G$ . Define the *centralizer*  $C_G(X)$  of  $X$  in  $G$  to be the set of elements of  $G$  that commute with every element of  $X$ . Prove that  $C_G(X)$  is a subgroup and then show that  $C_G(C_G(C_G(X))) = C_G(X)$ .
- (10) Let  $G$  be a finite group with class number  $h$ . An element is chosen at random from  $G$  and replaced. Then another group element is chosen. Prove that the probability of the two elements commuting is  $\frac{h}{|G|}$ . What would the answer be if the first group element were not replaced? [Hint: use Exercise (5.2.5)].

### 5.3 Applications to the structure of groups

The aim of this section is to demonstrate that group actions can be a highly effective tool for investigating the structure of groups. The first result provides important arithmetic information about the conjugacy classes of a finite group.

**(5.3.1)** *Let  $G$  be a finite group with distinct conjugacy classes  $C_1, C_2, \dots, C_h$ . Then*

- (i)  $|C_i| = |G : C_G(x_i)|$  for any  $x_i$  in  $C_i$ ; thus  $|C_i|$  divides  $|G|$ .
- (ii)  $|G| = |C_1| + |C_2| + \dots + |C_h|$ , (the Class Equation).

Here (i) follows on applying (5.2.1) to the conjugation action of  $G$  on its underlying set. For in this action the  $G$ -orbit of  $x$  is its conjugacy class, while the stabilizer of  $x$  is the centralizer  $C_G(x)$ ; thus  $|G \cdot x| = |G : \text{St}_G(x)| = |G : C_G(x)|$ . Finally, (ii) holds because the  $C_i$  are disjoint.

There are other ways to express the class equation. Choose any  $x_i \in C_i$  and put  $n_i = |C_G(x_i)|$ . Then  $|C_i| = |G|/n_i$ . On division by  $|G|$ , the class equation becomes

$$\frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_h} = 1,$$

an interesting diophantine equation for the orders of the centralizers.

It is an easy observation that a one-element set  $\{x\}$  is a conjugacy class of  $G$  if and only if  $x$  is its only conjugate in  $G$ , i.e.,  $x$  belongs to the center of the group  $G$ . Now suppose we order the conjugacy classes in such a way that  $|C_i| = 1$  for  $i = 1, 2, \dots, r$  and  $|C_i| > 1$  if  $r < i \leq h$ . With this notation the class equation takes the form:

$$(5.3.2) \quad |G| = |Z(G)| + |C_{r+1}| + \dots + |C_h|.$$

A natural question is: what are the conjugacy classes of the symmetric group  $S_n$ ? First note that any two  $r$ -cycles in  $S_n$  are conjugate. For

$$\pi(i_1 i_2 \dots i_r) \pi^{-1} = (j_1 j_2 \dots j_r)$$

where  $\pi$  is any permutation in  $S_n$  such that  $\pi(i_1) = j_1, \pi(i_2) = j_2, \dots, \pi(i_r) = j_r$ . From this remark and (3.1.3) it follows that *any two permutations which have the same cycle type are conjugate in  $S_n$* . Here “cycle type” refers to the numbers of 1-cycles, 2-cycles, etc. which are present in the disjoint cycle decomposition. Conversely, it is easy to see that conjugate permutations have the same cycle type. Thus we have the answer to our question.

**(5.3.3)** *The conjugacy classes of the symmetric group  $S_n$  are the sets of permutations with the same cycle type.*

It follows that the class number of  $S_n$  is the number of different cycle types, which equals

$$\lambda(n),$$



the number of partitions of  $n$ , i.e., the number of ways of writing the positive integer  $n$  as a sum of positive integers when order of summands is not significant. This is a well-known number theoretic function which has been studied intensively.

**Example (5.3.1)** The symmetric group  $S_6$  has 11 conjugacy classes. For  $\lambda(6) = 11$ , as is seen by writing out the partitions of 11.

As a deeper application of our knowledge of the conjugacy classes of  $S_n$  we will prove next:

**(5.3.4)** The symmetric group  $S_n$  has no non-inner automorphisms if  $n \neq 6$ .

*Proof.* Since  $S_2$  has only the trivial automorphism, we can assume that  $n > 2$  as well as  $n \neq 6$ . First a general remark: in any group  $G$  the automorphism group  $\text{Aut}(G)$  permutes the conjugacy classes of  $G$ . Indeed, if  $\alpha \in \text{Aut}(G)$ , then  $\alpha(xgx^{-1}) = \alpha(x)\alpha(g)\alpha(x)^{-1}$ , so  $\alpha$  maps the conjugacy class of  $g$  to that of  $\alpha(g)$ .

Now let  $C_1$  denote the conjugacy class consisting of all the 2-cycles in  $S_n$ . If  $\pi$  is a 2-cycle,  $\alpha(\pi)$  also has order 2 and so is a product of, say,  $k$  disjoint 2-cycles. Hence  $\alpha(C_1) = C_k$  where  $C_k$  is the conjugacy class of all (disjoint) products of  $k$  2-cycles. The first step in the proof is to show by a counting argument that  $k = 1$ , i.e.,  $\alpha$  maps 2-cycles to 2-cycles. Assume to the contrary that  $k \geq 2$ .

Clearly  $|C_1| = \binom{n}{2}$ , and more generally

$$|C_k| = \binom{n}{2k} \frac{(2k)!}{(2!)^k k!}.$$

For, in order to form a product of  $k$  disjoint 2-cycles, first choose the  $2k$  integers from  $1, 2, \dots, n$  in  $\binom{n}{2k}$  ways. Then divide these  $2k$  elements into  $k$  pairs, with order of pairs unimportant; this can be done in  $\frac{(2k)!}{(2!)^k k!}$  ways. Forming the product, we obtain the formula for  $|C_k|$ .

Since  $\alpha(C_1) = C_k$ , it must be the case that  $|C_1| = |C_k|$  and hence

$$\binom{n}{2} = \binom{n}{2k} \frac{(2k)!}{(2!)^k k!}.$$

After cancellation this becomes

$$(n-2)(n-3) \cdots (n-2k+1) = 2^{k-1}(k!).$$

This is impossible if  $k = 2$ , while if  $k = 3$  it can only hold if  $n = 6$ , which is forbidden. Therefore  $k > 3$ . Clearly  $n \geq 2k$ , so  $(n-2)(n-3) \cdots (n-2k+1) \geq (2k-2)!$ . This leads to  $(2k-2)! \leq 2^{k-1}(k!)$ , which implies that  $k = 3$ , a contradiction.

The argument so far has established that  $k = 1$  and  $\alpha(C_1) = C_1$ . Write

$$\alpha((ab)) = (b'b'') \quad \text{and} \quad \alpha((ac)) = (c'c'').$$

Since  $(ac)(ab) = (abc)$ , which has order 3, also  $\alpha((ac)(ab)) = (c'c'')(b'b'')$  has order 3. Therefore  $b', b'', c', c''$  cannot all be different and we can write

$$\alpha((ab)) = (a'b') \quad \text{and} \quad \alpha((ac)) = (a'c').$$

Next suppose there is a  $d$  such that  $\alpha((ad)) = (b'c')$  with  $a' \neq b', c'$ . Then

$$(ac)(ad)(ab) = (abdc),$$

an element of order 4, whereas its image  $(a'c')(b'c')(a'b') = (a')(b'c')$  has order 2, another contradiction.

This argument shows that for each  $a$  there is a unique  $a'$  such that  $\alpha((ab)) = (a'b')$  for all  $b$  and some  $b'$ . Therefore  $\alpha$  determines a permutation  $\pi \in S_n$  such that  $\pi(a) = a'$ . Thus  $\alpha((ab)) = (a'b') = (\pi(a) \pi(b))$ , which equals the conjugate  $\pi(ab)\pi^{-1}$  because the latter interchanges  $a'$  and  $b'$  and fixes all other integers. Since  $S_n$  is generated by 2-cycles by (3.1.4), it follows that  $\alpha$  is conjugation by  $\pi$ , so it is an inner automorphism.  $\square$

Recall that a group is complete if the conjugation homomorphism  $\tau : G \rightarrow \text{Aut}(G)$  is an isomorphism, i.e.,  $\text{Ker}(\tau) = Z(G) = 1$  and  $\text{Aut}(G) = \text{Inn}(G)$  by (4.3.8). Now  $Z(S_n) = 1$  if  $n \neq 2$  – see Exercise (4.2.10). Hence we obtain:

**Corollary (5.3.5)** *The symmetric group  $S_n$  is complete if  $n \neq 2$  or 6.*

Of course,  $S_2$  is not complete since it is abelian. It is known that the group  $S_6$  has a non-inner automorphism, so it too is not complete.

**Finite  $p$ -groups.** If  $p$  is a prime number, a finite group is called a  $p$ -group if its order is a power of  $p$ . Finite  $p$ -groups form an important and highly complex class of finite groups. A first indication that these groups have special features is provided by the following result.

**(5.3.6)** *If  $G$  is a non-trivial finite  $p$ -group, then  $Z(G) \neq 1$ .*

*Proof.* Consider the class equation of  $G$  in the form

$$|G| = |Z(G)| + |C_{r+1}| + \cdots + |C_h|,$$

– see (5.3.1) and (5.3.2). Here  $|C_i|$  divides  $|G|$  and hence is a power of  $p$ ; also  $|C_i| > 1$ . If  $Z(G) = 1$ , then it would follow that  $|G| \equiv 1 \pmod{p}$ , which is impossible because  $|G|$  is a power of  $p$ . Therefore  $Z(G) \neq 1$ .  $\square$

This behavior is in contrast to finite groups in general, which can easily have trivial center: for example,  $Z(S_3) = 1$ .

**Corollary (5.3.7)** *If  $p$  is a prime, every group of order  $p^2$  is abelian.*

*Proof.* Let  $G$  be a group of order  $p^2$ . Then  $|Z(G)| = p$  or  $p^2$  by (5.3.6) and (4.1.1). If  $|Z(G)| = p^2$ , then  $G = Z(G)$  is abelian. Thus we can assume that  $|Z(G)| = p$ , so that  $|G/Z(G)| = p$ . By (4.1.4) both  $G/Z(G)$  and  $Z(G)$  are cyclic, say  $G/Z(G) = \langle aZ(G) \rangle$  and  $Z(G) = \langle b \rangle$ . It follows that each element of  $G$  has the form  $a^i b^j$  where  $i, j$  are integers. However,

$$(a^i b^j)(a^{i'} b^{j'}) = a^{i+i'} b^{j+j'} = (a^{i'} b^{j'})(a^i b^j)$$

since  $b \in Z(G)$ , which shows that  $G$  is abelian and  $Z(G) = G$ , a contradiction.  $\square$

On the other hand, there are non-abelian groups of order  $2^3 = 8$ , for example  $\text{Dih}(8)$ , so (5.3.7) does not generalize to groups of order  $p^3$ .

**Sylow's<sup>3</sup> Theorem.** Group actions will now be used to give a proof of *Sylow's Theorem*, which is probably the most celebrated and frequently used result in elementary group theory.

Let  $G$  be a finite group and  $p$  a prime, and write  $|G| = p^a m$  where  $p$  does not divide the integer  $m$ . Thus  $p^a$  is the highest power of  $p$  dividing  $|G|$ . Lagrange's Theorem guarantees that the order of a  $p$ -subgroup of  $G$  is at most  $p^a$ . That  $p$ -subgroups of this order actually occur is the first part of Sylow's Theorem. A subgroup of  $G$  with the order  $p^a$  is called a *Sylow  $p$ -subgroup*.

**(5.3.8)** (Sylow's Theorem) *Let  $G$  be a finite group and let  $p^a$  denote largest power of the prime  $p$  that divides  $|G|$ . Then the following are true.*

- (i) *Every  $p$ -subgroup of  $G$  is contained in some subgroup of order  $p^a$ : in particular, Sylow  $p$ -subgroups exist.*
- (ii) *If  $n_p$  is the number of Sylow  $p$ -subgroups,  $n_p \equiv 1 \pmod{p}$ .*
- (iii) *Any two Sylow  $p$ -subgroups are conjugate in  $G$ .*

*Proof.* Write  $|G| = p^a m$  where  $p$  does not divide the integer  $m$ . Three group actions will be used during the course of the proof.

(a) Let  $\mathcal{S}$  be the set of all subsets of  $G$  with exactly  $p^a$  elements. Then  $\mathcal{S}$  has  $s$  elements where

$$s = \binom{p^a m}{p^a} = \frac{m(p^a m - 1) \cdots (p^a m - p^a + 1)}{1 \cdot 2 \cdots (p^a - 1)}.$$

First we prove that  $p$  does not divide  $s$ . To this end consider the rational number  $\frac{p^a m - i}{i}$  where  $1 \leq i < p^a$ . If  $p^j \mid i$ , then  $j < a$  and hence  $p^j \mid p^a m - i$ . On the other hand, if  $p^j \mid p^a m - i$ , then  $j < a$  since otherwise  $p^a \mid i$ . Therefore  $p^j \mid i$ . It follows that the integers  $p^a m - i$  and  $i$  involve the same highest power of  $p$ , which can of course be cancelled in the fraction  $\frac{p^a m - i}{i}$ ; thus no  $p$ 's occur in this rational number. It follows that  $p$  does not divide  $s$ , as claimed.

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<sup>3</sup> Peter Ludwig Mejdell Sylow (1832–1918)

Now we introduce the first group action. The group  $G$  acts on the set  $\mathcal{S}$  via left multiplication, i.e.,  $g \cdot X = gX$  where  $X \subseteq G$  and  $|X| = p^a$ . Thus  $\mathcal{S}$  splits up into disjoint  $G$ -orbits. Since  $|\mathcal{S}| = s$  is not divisible by  $p$ , there must be at least one  $G$ -orbit  $\mathcal{S}_1$  such that  $|\mathcal{S}_1|$  is not divisible by  $p$ . Choose  $X \in \mathcal{S}_1$  and put  $P = \text{St}_G(X)$ , which is, of course, a subgroup. Then  $|G : P| = |\mathcal{S}_1|$ , from which it follows that  $p$  does not divide  $|G : P|$ . However  $p^a$  divides  $|G| = |G : P| \cdot |P|$ , which implies that  $p^a$  divides  $|P|$ .

Now fix  $x$  in  $X$ ; then the number of elements  $gx$  with  $g \in P$  equals  $|P|$ . Also  $gx \in X$ ; hence  $|P| \leq |X| = p^a$  and consequently  $|P| = p^a$ . Therefore  $P$  is a Sylow  $p$ -subgroup of  $G$  and we have shown that Sylow  $p$ -subgroups exist.

(b) Let  $\mathcal{T}$  denote the set of all conjugates of the Sylow  $p$ -subgroup  $P$  constructed in (a). We argue next that  $|\mathcal{T}| \equiv 1 \pmod{p}$ .

The group  $P$  acts on the set  $\mathcal{T}$  by conjugation, i.e.,  $g \cdot Q = gQg^{-1}$  where  $g \in P$  and  $Q \in \mathcal{T}$ ; clearly  $|gQg^{-1}| = |Q| = |P| = p^a$ . In this action  $\{P\}$  is a  $P$ -orbit since  $gPg^{-1} = P$  if  $g \in P$ . Suppose that  $\{P_1\}$  is another one-element  $P$ -orbit. Then  $P_1 \triangleleft \langle P, P_1 \rangle$ ; for  $xP_1x^{-1} = P_1$  if  $x \in P \cup P_1$ , so  $N_{\langle P, P_1 \rangle}(P_1) = \langle P, P_1 \rangle$ . By (4.3.5)  $PP_1$  is a subgroup and its order is

$$|PP_1| = \frac{|P| \cdot |P_1|}{|P \cap P_1|},$$

which is certainly a power of  $p$ . But  $P \subseteq PP_1$  and  $P$  already has the maximum order allowed for a  $p$ -subgroup. Therefore  $P = PP_1$ , so  $P_1 \subseteq P$  and hence  $P_1 = P$  since  $|P_1| = |P|$ .

Consequently there is only one  $P$ -orbit of  $\mathcal{T}$  with a single element. Every other  $P$ -orbit has order a power of  $p$  greater than 1. Therefore  $|\mathcal{T}| \equiv 1 \pmod{p}$ .

(c) Finally, let  $P_2$  be an arbitrary  $p$ -subgroup of  $G$ . We aim to show that  $P_2$  is contained in some conjugate of the Sylow  $p$ -subgroup  $P$  in (a); this will complete the proof of Sylow's Theorem.

Let  $P_2$  act on  $\mathcal{T}$  by conjugation, where as before  $\mathcal{T}$  is the set of all conjugates of  $P$ . Assume that  $P_2$  is not contained in any member of  $\mathcal{T}$ . If  $\{P_3\}$  is a one-element  $P_2$ -orbit of  $\mathcal{T}$ , then, arguing as in (b), we see that  $P_2P_3$  is a  $p$ -subgroup containing  $P_3$ , so  $P_3 = P_2P_3$  because  $|P_3| = p^a$ . Thus  $P_2 \subseteq P_3 \in \mathcal{T}$ , contrary to assumption. It follows that there are no one-element  $P_2$ -orbits in  $\mathcal{T}$ ; this means that  $|\mathcal{T}| \equiv 0 \pmod{p}$ , which contradicts the conclusion of (b).  $\square$

An important special case of Sylow's Theorem is:

**(5.3.9) (Cauchy's Theorem)** *If the order of a finite group  $G$  is divisible by a prime  $p$ , then  $G$  has an element of order  $p$ .*

*Proof.* Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Then  $P \neq 1$  since  $p$  divides  $|G|$ . Choose  $1 \neq g \in P$ ; then  $|g|$  divides  $|P|$ , and hence  $|g| = p^m$  where  $m > 0$ . Thus  $g^{p^{m-1}}$  has order  $p$ , as required.  $\square$

While Sylow's Theorem does not tell us the exact number of Sylow  $p$ -subgroups, it provides valuable information which may be sufficient to determine how many

there are. Let us review what is known. Suppose  $P$  is a Sylow  $p$ -subgroup of a finite group  $G$ . Then, since every Sylow  $p$ -subgroup is a conjugate of  $P$ , the number of Sylow  $p$ -subgroups of  $G$  equals the number of conjugates of  $P$ , which by (5.2.1) is

$$n_p = |G : N_G(P)|,$$

where  $N_G(P)$  is the normalizer of  $P$  in  $G$  – see (5.2). Hence  $n_p$  divides  $|G : P|$  since  $P \leq N_G(P)$ . Also of course

$$n_p \equiv 1 \pmod{p}.$$

**Example (5.3.2)** Find the numbers of Sylow  $p$ -subgroups of the alternating group  $A_5$ .

Let  $G = A_5$ . We can assume that  $p$  divides  $|G|$ , so that  $p = 2, 3$  or  $5$ . Note that a non-trivial element of  $G$  has one of three cycle types,

$$(**)(**)(*), \quad (***)(*)(*), \quad (*****)$$

If  $p = 2$ , then  $n_2 \mid \frac{60}{4} = 15$  and  $n_2 \equiv 1 \pmod{2}$ , so  $n_2 = 1, 3, 5$  or  $15$ . There are  $5 \times 3 = 15$  elements of order 2 in  $G$ , with three of them in each Sylow 2-subgroup. Hence  $n_2 \geq 5$ . If  $n_2 = 15$ , then  $P = N_G(P)$  where  $P$  is a Sylow 2-subgroup, since  $P \leq N_G(P) \leq G$  and  $|G : N_G(P)| = 15 = |G : P|$ . But this is wrong since  $P$  is normalized by a 3-cycle – note that the Klein 4-group is normal in  $A_4$ . Consequently  $n_2 = 5$ .

Next  $n_3 \mid \frac{60}{3} = 20$  and  $n_3 \equiv 1 \pmod{3}$ . Thus  $n_3 = 1, 4$  or  $10$ . Now  $G$  has  $\binom{5}{3} \times 2 = 20$  elements of order 3, which shows that  $n_3 > 4$ . Hence  $n_3 = 10$ . Finally,  $n_5 \mid 12$  and  $n_5 \equiv 1 \pmod{5}$ , so  $n_5 = 6$  since  $n_5 = 1$  would give only four elements of order 5.

The next result provides some very important information about the group  $A_5$ .

**(5.3.10)** *The alternating group  $A_5$  is simple.*

*Proof.* Let  $G = A_5$  and suppose  $N$  is a proper non-trivial normal subgroup of  $G$ . The possible orders of elements of  $G$  are 1, 2, 3, or 5, (note that 4-cycles are odd). If  $N$  contains an element of order 3, it contains a Sylow 3-subgroup of  $G$ , and by normality it contains all such. Hence  $N$  contains all 3-cycles. Now the easily verified equations  $(ab)(ac) = (acb)$  and  $(ac)(bd) = (abc)(abd)$ , together with the fact that every permutation in  $G$  is a product of an even number of transpositions, shows that  $G$  is generated by 3-cycles. Therefore  $N = G$ , which is a contradiction.

Next suppose  $N$  has an element of order 5; then  $N$  contains a Sylow 5-subgroup and hence all 5-cycles. But  $(12345)(12543) = (132)$ , which gives the contradiction that  $N$  contains a 3-cycle.

The argument thus far tells us that each element of  $N$  has order a power of 2, which implies that  $|N|$  is a power of 2 by Cauchy's Theorem. Since  $|N|$  divides  $|G| = 60$ , this order must be 2 or 4. We leave it to the reader to disprove these possibilities. This final contradiction shows that  $G$  is a simple group.  $\square$

More generally,  $A_n$  is simple for all  $n \geq 5$ : this is proved in (10.1.7) below. We will see in Chapter Twelve that the simplicity of  $A_5$  is intimately connected with the insolubility of polynomial equations of degree 5 by radicals.

**Example (5.3.3)** Find all groups of order 21.

Let  $G$  be a group of order 21. Then  $G$  contains elements  $a$  and  $b$  with orders 7 and 3 respectively by (5.3.9). Now the order of  $\langle a \rangle \cap \langle b \rangle$  divides both 7 and 3, i.e.,  $\langle a \rangle \cap \langle b \rangle = 1$ , and thus  $|\langle a \rangle \langle b \rangle| = |a| \cdot |b| = 21$ , which means that  $G = \langle a \rangle \langle b \rangle$ . Next  $\langle a \rangle$  is a Sylow 7-subgroup of  $G$ , and  $n_7 \equiv 1 \pmod{7}$  and  $n_7 \mid 3$ . Hence  $n_7 = 1$ , so that  $\langle a \rangle \triangleleft G$  and  $bab^{-1} = a^i$  where  $1 \leq i < 7$ . If  $i = 1$ , then  $G$  is abelian and  $|ab| = 21$ . In this case  $G = \langle ab \rangle \cong \mathbb{Z}_{21}$ .

Next assume  $i \neq 1$ . Now  $b^3 = 1$  and  $bab^{-1} = a^i$ , with  $2 \leq i < 7$ , imply that  $a = b^3ab^{-3} = a^{i^3}$ . Hence  $7 \mid i^3 - 1$ , which shows that  $i = 2$  or  $4$ . Now  $[2]_7 = [4]_7^{-1}$  since  $8 \equiv 1 \pmod{7}$ . Since we can replace  $b$  by  $b^{-1}$  if necessary, there is nothing to be lost in assuming that  $i = 2$ .

Thus far we have discovered that  $G = \{a^u b^v \mid 0 \leq u < 7, 0 \leq v < 3\}$  and that the relations  $a^7 = 1 = b^3$ ,  $bab^{-1} = a^2$  hold. But is there really such a group? An example is easily found by using permutations. Put  $\pi = (1234567)$  and  $\sigma = (235)(476)$ ; thus  $\langle \pi, \sigma \rangle$  is a subgroup of  $S_7$ . One quickly verifies that  $\pi^7 = 1 = \sigma^3$  and  $\sigma\pi\sigma^{-1} = \pi^2$ . A brief computation reveals that the assignments  $a \mapsto \pi$ ,  $b \mapsto \sigma$  determine an isomorphism from  $G$  to the group  $\langle \pi, \sigma \rangle$ . It follows that *up to isomorphism there are exactly two groups of order 21*.

**Example (5.3.4)** Show that there are no simple groups of order 300.

Suppose that  $G$  is a simple group of order 300. Since  $300 = 2^2 \cdot 3 \cdot 5^2$ , a Sylow 5-subgroup  $P$  has order 25. Now  $n_5 \equiv 1 \pmod{5}$  and  $n_5$  divides  $300/25 = 12$ . Thus  $n_5 = 1$  or  $6$ . But  $n_5 = 1$  implies that  $P \triangleleft G$ , which is impossible. Hence  $n_5 = 6$  and  $|G : N_G(P)| = 6$ . The left action of  $G$  on the set of left cosets of  $N_G(P)$  (see (5.1)) leads to a homomorphism  $\theta$  from  $G$  to  $S_6$ . Also  $\text{Ker}(\theta) = 1$  since  $G$  is simple. Thus  $\theta$  is injective and  $G \cong \text{Im}(\theta) \leq S_6$ . However,  $|G| = 300$ , which does not divide  $|S_6| = 6!$ , so we have a contradiction.

### Exercises (5.3)

- (1) A finite  $p$ -group cannot be simple unless its order is  $p$ .
- (2) Let  $G$  be a group of order  $pq$  where  $p$  and  $q$  are primes such that  $p \not\equiv 1 \pmod{q}$  and  $q \not\equiv 1 \pmod{p}$ . Prove that  $G$  is cyclic.
- (3) Show that if  $p$  is a prime, a group of order  $p^2$  is isomorphic with  $Z_{p^2}$  or  $Z_p \times Z_p$ .
- (4) Let  $P$  be a Sylow  $p$ -subgroup of a finite group  $G$  and let  $N \triangleleft G$ . Prove that  $P \cap N$  and  $PN/N$  are Sylow  $p$ -subgroups of  $N$  and  $G/N$  respectively.
- (5) Show that there are no simple groups of orders 312.

- (6) Let  $G$  be a finite simple group which has a subgroup of index  $n$ . Prove that  $G$  is isomorphic with a subgroup of  $S_n$ .
- (7) Prove that there are no simple groups of order 1960. [Hint: assume there is one and find  $n_7$ ; then apply Exercise (5.3.6)].
- (8) Prove that there are no simple groups of order 616. [Hint: assume there is one. Show that for this group one must have  $n_{11} = 56$  and  $n_7 \geq 8$ ; then count the elements of orders 7 and 11].
- (9) Prove that every group of order 561 is cyclic. [Hint: show that there is a cyclic normal subgroup  $\langle x \rangle$  of order  $11 \times 17 = 187$ ; then use the fact that 3 does not divide  $|\text{Aut}(\langle x \rangle)|$ ].
- (10) Let  $G$  be a group of order  $2m$  where  $m$  is odd. Prove that  $G$  has a normal subgroup of order  $m$ . [Hint: let  $\lambda$  be the left regular representation of  $G$ . By (5.3.9) there is an element  $g$  of order 2 in  $G$ . Now argue that  $\lambda(g)$  must be an odd permutation].
- (11) Find all finite groups with class number at most 2.
- (12) Show that every group of order 10 is isomorphic with  $\mathbb{Z}_{10}$  or  $\text{Dih}(10)$ . [Follow the method of Example (5.3.3)].
- (13) Show that up to isomorphism there are two groups of order 55.
- (14) If  $H$  is a proper subgroup of a finite  $p$ -group  $G$ , prove that  $H < N_G(H)$ . [Hint: use induction on  $|G| > 1$ , noting that  $H \triangleleft HZ(G)$ ].
- (15) Let  $P$  be a Sylow  $p$ -subgroup of a finite group  $G$  and let  $H$  be a subgroup of  $G$  containing  $N_G(P)$ . Prove that  $H = N_G(H)$ . [Hint: if  $g \in N_G(H)$ , then  $P$  and  $gPg^{-1}$  are conjugate in  $H$ ].
- (16) Let  $G$  be a finite group and suppose it is possible to choose one element from each conjugacy class in such a way that all the selected elements commute. Prove that  $G$  is abelian. [Hint: use (5.3.2)].

## 5.4 Applications to combinatorics

Group actions can be used effectively to solve certain types of counting problem. As an example of such a problem, suppose we wish to color the six faces of a cube and five colors are available. How many different coloring schemes are there? At first sight one might answer  $5^6$  since each of the six faces can be colored in five different ways. However, this answer is not correct since by merely rotating the cube it is possible to pass from one coloring scheme to another. Clearly two such coloring schemes are not really different. Thus not all of the  $5^6$  colorings schemes are distinct.

Let us pursue further the idea of rotating the cube. The group of rotations of the cube acts on the set of all possible coloring schemes. If two colorings belong to the same orbit, they should be considered identical since one arises from the other by a

suitable rotation. What we really need to do is count the number of orbits of colorings and for this purpose the Frobenius-Burnside Theorem (5.2.3) is ideally suited.

**Labelling problems.** Our problem is really about the labelling of sets. Let  $X$  and  $L$  be two non-empty sets, with  $L$  referred to as the set of *labels*. Suppose that a label is to be assigned to each element of the set  $X$ , so that we need to specify a function

$$\alpha : X \rightarrow L :$$

call such a function  $\alpha$  a *labelling* of  $X$  by  $L$ . Thus the set of all such labellings of  $X$  by  $L$  is

$$\text{Fun}(X, L).$$

Now suppose that  $G$  is a group acting on the set  $X$  (on the left). Then  $G$  can be made to act on the set of labellings in a natural way by the rule

$$(g \cdot \alpha)(x) = \alpha(g^{-1} \cdot x),$$

where  $g \in G$ ,  $x \in X$  and  $\alpha \in \text{Fun}(X, L)$ . What this equation asserts is that the labelling  $g \cdot \alpha$  assigns to the set element  $g \cdot x$  the same label as  $\alpha$  assigns to  $x$ . The example of the cube should convince the reader that this is the correct action.

First we verify that this really is an action of  $G$  on  $\text{Fun}(X, L)$ . To do this let  $g_1, g_2 \in G$ ,  $x \in X$  and  $\alpha \in \text{Fun}(X, L)$ ; then

$$\begin{aligned} (g_1 \cdot (g_2 \cdot \alpha))(x) &= (g_2 \cdot \alpha)(g_1^{-1} \cdot x) = \alpha(g_2^{-1} \cdot (g_1^{-1} \cdot x)) \\ &= \alpha((g_1 g_2)^{-1} \cdot x) \\ &= ((g_1 g_2) \cdot \alpha)(x). \end{aligned}$$

Hence  $g_1 \cdot (g_2 \cdot \alpha) = (g_1 g_2) \cdot \alpha$ . Also  $1_G \cdot \alpha(x) = \alpha(1_G \cdot x) = \alpha(x)$ , so that  $1_G \cdot \alpha = \alpha$ . Therefore we have an action of  $G$  on  $\text{Fun}(X, L)$ .

Our goal is to count the  $G$ -orbits in  $\text{Fun}(X, L)$ , which is achieved in the following fundamental result.

**(5.4.1)** (Polya<sup>4</sup>) *Let  $G$  be a finite group acting on a finite set  $X$ , and let  $L$  be a finite set of labels. Then the number of  $G$ -orbits of labellings of  $X$  by  $L$  is*

$$\frac{1}{|G|} \left( \sum_{g \in G} \ell^{m(g)} \right)$$

where  $\ell = |L|$  and  $m(g)$  is the number of disjoint cycles in the permutation of  $X$  corresponding to  $g$ .

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<sup>4</sup> George Polya (1887–1985)



*Proof.* By (5.2.3) the number of  $G$ -orbits of labellings is

$$\frac{1}{|G|} \left( \sum_{g \in G} |\text{Fix}(g)| \right)$$

where  $\text{Fix}(g)$  is the set of labellings fixed by  $g$ . We have to count these labellings. Now  $\alpha \in \text{Fix}(g)$  if and only if  $g \cdot \alpha(x) = \alpha(x)$ , i.e.,  $\alpha(g^{-1} \cdot x) = \alpha(x)$  for all  $x \in X$ . This equation asserts that  $\alpha$  is constant on the  $\langle g \rangle$ -orbit  $\langle g \rangle \cdot x$ . Now the  $\langle g \rangle$ -orbits arise from the disjoint cycles involved in the permutation of  $X$  corresponding to  $g$ . Therefore, to construct a labelling in  $\text{Fix}(g)$  all we need to do is assign a label to each cycle of  $g$ . This can be done in  $\ell^{m(g)}$  ways where  $m(g)$  is the number of cycles; consequently  $|\text{Fix}(g)| = \ell^{m(g)}$  and we have our formula.  $\square$

Polya's Theorem will now be applied to solve some counting problems.

**Example (5.4.1)** How many ways are there to design a necklace of 11 beads if  $c$  different colors of beads are available?

Here it is assumed that the beads are identical apart from color. The necklace can be visualized as a regular 11-gon with the beads as vertices. The labels are the  $c$  colors and one color has to be assigned to each vertex. Clearly a symmetry of the 11-gon can be applied without changing the design of the necklace. Recall that the group of symmetries  $G$  is a dihedral group  $\text{Dih}(22)$  – see (3.2). It consists of the identity, rotations through  $(\frac{2\pi}{11})i$ , for  $i = 1, 2, \dots, 10$ , and reflections in a line joining a vertex to the midpoint of the opposite edge.

For each element  $g \in G$  we count the number  $m(g)$  of  $\langle g \rangle$ -orbits in the set of vertices  $X = \{1, 2, \dots, 11\}$ , so that Polya's formula can be applied. The results of the count are displayed in tabular form below.

Type of element	Cycle type	Number of elements	$m$
identity	eleven 1-cycles	1	11
rotation through $\frac{2\pi i}{11}$ , $1 \leq i \leq 10$	one 11-cycle	10	1
reflection	one 1-cycle, five 2-cycles	11	6

From the table and Polya's formula we deduce that the number of different designs is

$$\frac{1}{22} (c^{11} + 11c^6 + 10c) = \frac{1}{22} c(c^5 + 1)(c^5 + 10).$$

Next we tackle the cube-coloring problem with which the section began.

**Example (5.4.2)** How many ways are there to color the faces of a cube using  $c$  different colors?

In this problem the relevant group is the rotation group  $G$  of the cube since this group acts on the set of colorings. In fact  $G \cong S_4$ : the easiest way to see this is to observe that each rotation permutes the four diagonals of the cube. But this observation is not needed to solve the problem.

Let  $L$  be the set of  $c$  colors and let  $X$  consist of the six faces of the cube. To identify the rotations in  $G$ , we examine the various axes of symmetry of the cube. For each rotation record the cycle type and number of cycles in the corresponding permutation of  $X$ . Again the results are conveniently displayed in a table.

Type of element	Cycle type	Number of elements	$m$
identity	six 1-cycles	1	6
rotation about line through centroids of opposite faces through $\pi/2$	two 1-cycles, one 4-cycle	3	3
$\pi$	two 1-cycles, two 2-cycles	3	4
$\frac{3\pi}{2}$	two 1-cycles, one 4-cycle	3	3
rotation about diagonal through $\frac{2\pi}{3}$	two 3-cycles	4	2
$\frac{4\pi}{3}$	two 3-cycles	4	2
rotation about line joining midpoints of opposite edges through $\pi$	three 2-cycles	6	3

The count of elements confirms that  $|G| = 24$ , and Polya's formula gives the answer  $\frac{1}{24}(c^6 + 3c^3 + 3c^4 + 3c^3 + 4c^2 + 4c^2 + 6c^3)$ , which factorizes as

$$\frac{1}{24}c^2(c+1)(c^3 - c^2 + 4c + 8).$$

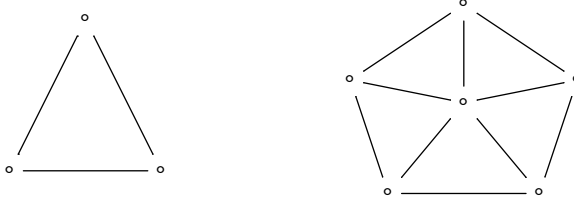
When  $c = 5$ , the formula yields 800, so there are 800 different ways to color the faces of a cube using 5 colors.

It is apparent from these examples that Polya's theorem enables us to solve some complex combinatorial problems which might otherwise be intractable.

**Counting graphs.** We conclude the chapter by describing how Polya's method can be used to count graphs. First some brief remarks about graphs.

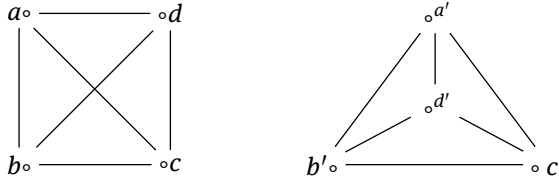
A *graph*  $\Gamma$  consists of a non-empty set  $V$  of *vertices* and a relation  $E$  on  $V$  which is symmetric and *irreflexive*, i.e.,  $v \notin v$  for all  $v \in V$ . If  $u E v$ , call the 2-element set  $\{u, v\}$  an *edge* of  $\Gamma$ . Since  $E$  is symmetric, we can identify  $E$  with the set of all edges of  $\Gamma$ .

A graph can be visualized by representing the vertices by points in the plane and the edges by lines joining appropriate vertices. Simple examples of graphs are:



Note that loops and multiple edges are not permitted. Graph theory has many applications outside mathematics, for example to transportation systems, telephone networks and electrical circuits.

Two graphs  $\Gamma_i = (V_i, E_i)$ ,  $i = 1, 2$ , are said to be *isomorphic* if there is a bijection  $\theta : V_1 \rightarrow V_2$  such that  $\{u, v\} \in E_1$  if and only if  $\{\theta(u), \theta(v)\} \in E_2$ . Two graphs may appear to be different, yet be isomorphic: for example, the graphs



are isomorphic because of the bijection  $a \mapsto a'$ ,  $b \mapsto b'$ ,  $c \mapsto c'$ ,  $d \mapsto d'$ .

The problem of interest to us is to find the number of non-isomorphic graphs on a given set of  $n$  vertices. For this purpose it is enough to count isomorphism classes of graphs with vertex set  $V = \{1, 2, \dots, n\}$ . The first step is to observe that a graph  $\Gamma = (V, E)$  is determined by its *edge function*

$$\alpha_\Gamma : V^{[2]} \rightarrow \{0, 1\}$$

where  $V^{[2]}$  is the set of all 2-element sets  $\{u, v\}$ , with  $u \neq v$  in  $V$ , and

$$\alpha_\Gamma(\{u, v\}) = \begin{cases} 0 & \text{if } (u, v) \notin E \\ 1 & \text{if } (u, v) \in E \end{cases}.$$

Thus we can think of a graph as a labelling of  $V^{[2]}$  by  $\{0, 1\}$ . The symmetric group  $S_n$  acts on the vertex set  $V$  in the natural way and this leads to an action of  $S_n$  on  $V^{[2]}$  in which

$$\pi \cdot \{u, v\} = \{\pi(u), \pi(v)\}$$

where  $\pi \in S_n$ . Thus  $S_n$  acts on the set of all edge functions for  $V$ , i.e., on

$$F = \text{Fun}(V^{[2]}, \{0, 1\}).$$

It is a consequence of the definition of isomorphism that graphs  $\Gamma_1 = (V, E_1)$  and  $\Gamma_2 = (V, E_2)$  are isomorphic if and only if there exists a  $\pi \in S_n$  such that  $\pi \cdot \alpha_{\Gamma_1} = \alpha_{\Gamma_2}$ ,

i.e.,  $\alpha_{\Gamma_1}$  and  $\alpha_{\Gamma_2}$  belong to the same  $S_n$ -orbit of  $F$ . Thus we have to count the  $S_n$ -orbits of  $F$ . Now (5.4.1) can be applied to this situation with  $G = S_n$ ,  $X = V^{[2]}$  and  $L = \{0, 1\}$ . This allows us to derive a formula for the number of isomorphism classes of graphs with vertex set  $V$ .

**(5.4.2)** *The number of non-isomorphic graphs with a given set of  $n$  vertices is given by*

$$g(n) = \frac{1}{n!} \left( \sum_{\pi \in S_n} 2^{m(\pi)} \right)$$

where  $m(\pi)$  is the number of disjoint cycles present in the permutation of  $V^{[2]}$  induced by  $\pi$ .

To use this result one must be able to compute  $m(\pi)$ , the number of  $S_n$ -orbits in  $V^{[2]}$ . While formulas for  $m(\pi)$  are available, we will be content to calculate these numbers directly for small values of  $n$ .

**Example (5.4.3)** Show that there are exactly 11 non-isomorphic graphs with 4 vertices.

What we have to do is to compute  $m(\pi)$  for  $\pi$  of each cycle type in  $S_4$ . Note that  $|V^{[2]}| = \binom{4}{2} = 6$ . Of course  $m(1) = 6$ . If  $\pi$  is a 4-cycle, say  $(1234)$ , there are two cycles in the permutation of  $V^{[2]}$  produced by  $\pi$ , namely  $(\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\})$  and  $(\{1, 3\}, \{2, 4\})$ ; thus  $m(\pi) = 2$ . Also there are six 4-cycles in  $S_4$ .

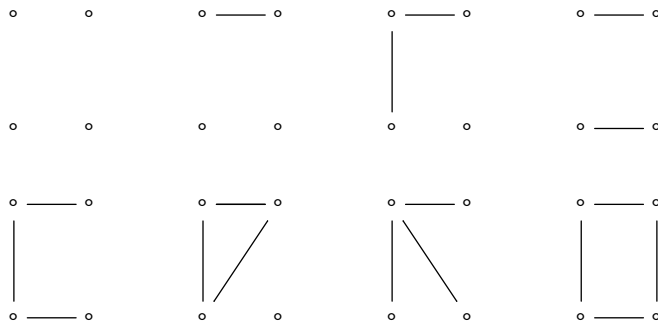
If  $\pi$  is a 3-cycle, say  $(123)(4)$ , there are two cycles,  $(\{1, 2\}, \{2, 3\}, \{1, 3\})$  and  $(\{1, 4\}, \{2, 4\}, \{3, 4\})$ , thus  $m(\pi) = 2$ : there are eight such 3-cycles.

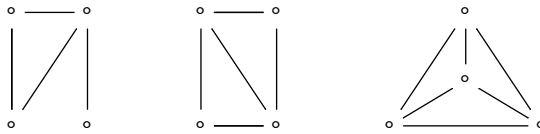
If  $\pi$  has two 2-cycles, say  $\pi = (12)(34)$ , there are four cycles  $(\{1, 2\})$ ,  $(\{3, 4\})$ ,  $(\{1, 3\}, \{2, 4\})$ ,  $(\{1, 4\}, \{2, 3\})$ ; so  $m(\pi) = 4$ . There are three such  $\pi$ 's.

Finally, there are six transpositions  $\pi$  and it is easy to see that for each one  $m(\pi) = 4$ . The formula in (5.4.2) therefore yields

$$g(4) = \frac{1}{4!} (2^6 + 6 \cdot 2^2 + 8 \cdot 2^2 + 3 \cdot 2^4 + 6 \cdot 2^4) = 11.$$

This result can be verified by actually enumerating the graphs.





Notice that all these graphs are *planar*, i.e., they can be drawn in the plane in such a way that no edges cross except at vertices.

### Exercises (5.4)

- (1) Show that there are  $\frac{1}{10}c(c^2+1)(c^2+4)$  ways to label the vertices of a regular pentagon using  $c$  labels.
- (2) The same problem for the *edges* of the pentagon.
- (3) A baton has  $n$  bands of equal width. Show that there are  $\frac{1}{2}(c^n + c^{\lceil \frac{n+1}{2} \rceil})$  ways to color it using  $c$  colors. [The baton can be rotated through  $180^\circ$ ].
- (4) The faces of a regular tetrahedron are to be painted using  $c$  colors. Prove that there are  $\frac{1}{12}c^2(c^2 + 11)$  ways to do it.
- (5) A necklace has  $p$  beads of identical shape and size where  $p$  is an odd prime number. Beads of  $c$  colors available. How many necklace designs are possible?
- (6) How many ways are there to place eight identical checkers on an  $8 \times 8$  chessboard of squares if rotation of the board is allowed?
- (7) Prove that the number of isomorphism types of graphs with  $n$  vertices is at most  $2^{n(n-1)/2}$ .
- (8) Show that there are four isomorphism types of graphs with three vertices.
- (9) Show that there are 34 isomorphism types of graphs with five vertices.
- (10) Prove that the number of ways to design a necklace with  $n$  beads of  $c$  different colors is

$$\frac{1}{2n} \left( \sum_{\substack{i=1 \\ i|n}}^n \phi(i) c^{\frac{n}{i}} \right) + \frac{1}{4} (c^{\lceil \frac{n+1}{2} \rceil} + c^{\lceil \frac{n+2}{2} \rceil}),$$

where  $\phi$  is Euler's function.

## 6 Introduction to rings

A *ring* is a set, together with two binary operations called *addition* and *multiplication* which are subject to a number of natural requirements. Thus, from the logical point of view, a ring is a more complex object than a group, which is a set with a single binary operation. Yet some of the most familiar mathematical objects are rings – for example, the sets of integers, real polynomials, continuous functions – and for this reason some readers may feel more at home with rings than with groups. One motivation for the study of rings is to see how far properties of the ring of integers extend to rings in general.

### 6.1 Elementary properties of rings

A *ring* is a triple

$$(R, +, \times)$$

where  $R$  is a set and  $+$  and  $\times$  are binary operations on  $R$ , called *addition* and *multiplication*, such that the following properties hold: here  $a \times b$  is written  $ab$ :

- (i)  $(R, +)$  is an abelian group;
- (ii)  $(R, \times)$  is a semigroup;
- (iii) the left and right distributive laws hold,

$$a(b + c) = ab + ac, \quad (a + b)c = ac + bc, \quad (a, b, c \in R).$$

If in addition the commutative law for multiplication holds,

- (iv)  $ab = ba$  for all  $a, b \in R$ ,

the ring is said to be *commutative*.

If  $R$  contains an element  $1_R \neq 0_R$  such that  $1_R a = a = a 1_R$  for all  $a \in R$ , then  $R$  is called a *ring with identity* and  $1_R$  is the (clearly unique) *identity element* of  $R$ . Care must be taken to distinguish between the additive identity (or zero element)  $0_R$ , which exists in any ring  $R$ , and the multiplicative identity  $1_R$  in a ring  $R$  with identity. These will often be written simply 0 and 1. As with groups, we usually prefer to speak of “the ring  $R$ ”, rather than the triple  $(R, +, \times)$ .

There are many familiar examples of rings at hand.

#### Examples (6.1.1)

- (i)  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  are commutative rings with identity where the ring operations are the usual addition and multiplication of arithmetic.
- (ii) Let  $m$  be a positive integer. Then  $\mathbb{Z}_m$ , the set of congruence classes modulo  $m$ , is a commutative ring with identity where the ring operations are addition and multiplication of congruence classes.

- (iii) The set of all continuous real-valued functions defined on the interval  $[0, 1]$  is a ring when addition and multiplication are given by  $f + g(x) = f(x) + g(x)$  and  $fg(x) = f(x)g(x)$ . This is a commutative ring in which the identity element is the constant function 1.
- (iv) Let  $R$  be any ring with identity and define  $M_n(R)$  to be the set of all  $n \times n$  matrices with entries in  $R$ . The usual rules for adding and multiplying matrices are to be used. By the elementary properties of matrices  $M_n(R)$  is a ring with identity. It is not hard to see that  $M_n(R)$  is commutative if and only if  $R$  is commutative and  $n = 1$ .

Of course the ring axioms must be verified in these examples, but this presents little difficulty.

**Rings of polynomials.** Next we introduce rings of polynomials, which are one of the most fruitful sources of rings.

First we must give a clear definition of a polynomial, not involving vague terms like “indeterminate”. In essence a polynomial is just the list of its coefficients, of which only finitely many can be non-zero. We proceed to refine this idea. Let  $R$  be a ring with identity. A *polynomial over  $R$*  is a sequence of elements  $a_i \in R$ , one for each natural number  $i$ ,

$$f = (a_0, a_1, a_2, \dots)$$

such that  $a_i = 0$  for all but a finite number of  $i$ ; the  $a_i$  are called the *coefficients* of  $f$ . The *zero polynomial* is  $(0_R, 0_R, 0_R, \dots)$ . If  $f = (a_0, a_1, \dots)$  is not zero, there is a largest integer  $i$  such that  $a_i \neq 0$ ; thus  $f = (a_0, a_1, \dots, a_i, 0, 0, \dots)$ . The integer  $i$  is called the *degree* of  $f$ , in symbols

$$\deg(f).$$

It is convenient to assign to the zero polynomial the degree  $-\infty$ . A polynomial whose degree is  $\leq 0$ , i.e., one of the form  $(a_0, 0, 0, \dots)$ , is called a *constant* polynomial.

The definitions of addition and multiplication of polynomials are just the familiar rules from elementary algebra, but adapted to the current notation. Let  $f = (a_0, a_1, \dots)$  and  $g = (b_0, b_1, \dots)$  be polynomials over  $R$ . Their sum and product are defined by

$$f + g = (a_0 + b_0, a_1 + b_1, \dots, a_i + b_i, \dots)$$

and

$$fg = (a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, \dots, \sum_{j=0}^n a_jb_{n-j}, \dots).$$

Notice that these really are polynomials; for all but a finite number of the coefficients are 0. Negatives are defined by  $-f = (-a_0, -a_1, -a_2, \dots)$ .

**(6.1.1)** *If  $f$  and  $g$  are polynomials over a ring with identity, then  $f + g$  and  $fg$  are polynomials. Also*

- (i)  $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$ ;

(ii)  $\deg(fg) \leq \deg(f) + \deg(g)$ .

This follows quickly from the definitions of sum and product. It is also quite routine to verify that the ring axioms hold for polynomials with these binary operations. Thus we have:

**(6.1.2)** *If  $R$  is a ring with identity, then so is the ring of all polynomials over  $R$ .*

Of course, the multiplicative identity in the polynomial ring over  $R$  is the constant polynomial  $(1_R, 0_R, 0_R, \dots)$ .

Now we would like to recover the traditional notation for polynomials, involving an “indeterminate”  $t$ . This is accomplished as follows. Let  $t$  denote the polynomial  $(0, 1, 0, 0, \dots)$ ; then the product rule shows that  $t^2 = (0, 0, 1, 0, \dots)$ ,  $t^3 = (0, 0, 0, 1, 0, \dots)$  etc. If we define the *multiple* of a polynomial by a ring element  $r$  by the rule

$$r(a_0, a_1, \dots) = (ra_0, ra_1, \dots),$$

then it follows that

$$(a_0, a_1, \dots, a_n, 0, 0, \dots) = a_0 + a_1 t + \dots + a_n t^n,$$

which is called a *polynomial in  $t$* . Thus we can return with confidence to the traditional notation for polynomials knowing that it is soundly based. The ring of polynomials in  $t$  over  $R$  will be written

$$R[t].$$

Polynomial rings in more than one indeterminate are defined recursively by the equation

$$R[t_1, \dots, t_n] = (R[t_1, \dots, t_{n-1}])[t_n],$$

where  $n > 1$ . A typical element of  $R[t_1, \dots, t_n]$  is a multinomial expression

$$\sum_{\ell_i=0,1,\dots} r_{\ell_1 \dots \ell_n} t_1^{\ell_1} \dots t_n^{\ell_n},$$

where the  $\ell_i$  are non-negative integers and  $r_{\ell_1 \dots \ell_n} \in R$  equals zero for all but a finite number of  $(\ell_1, \ell_2, \dots, \ell_n)$ .

We list next some elementary and frequently used consequences of the ring axioms.

**(6.1.3)** *Let  $R$  be any ring. Suppose that  $a, b$  are elements of  $R$  and that  $n$  is an integer. Then:*

- (i)  $a0 = 0 = 0a$ ;
- (ii)  $a(-b) = (-a)b = -(ab)$ ;
- (iii)  $(na)b = n(ab) = a(nb)$ .



*Proof.* By the distributive law  $a(0 + 0) = a0 + a0$ . Hence  $a0 = a0 + a0$  and so  $a0 = 0$  after cancellation. Similarly  $0a = 0$ . This proves (i). As for (ii) we have  $a(-b) + ab = a(-b + b) = a0 = 0$ . Thus  $a(-b) = -(ab)$ . Similarly  $(-a)b = -(ab)$ . To prove (iii) assume that  $n \geq 0$ ; then  $(na)b = n(ab)$  by an easy induction on  $n$ . Next  $(-na)b + nab = (-na + na)b = 0b = 0$ , so  $(-na)b = -(nab)$ . Similarly  $a(-nb) = -(nab)$ , which completes the proof.  $\square$

**Units in rings.** Suppose that  $R$  is a ring with identity. An element  $r \in R$  is called a *unit* if it has a *multiplicative inverse*, i.e., an element  $s \in R$  such that  $rs = 1 = sr$ . Notice that  $0$  cannot be a unit since  $0s = 0 \neq 1$  for all  $s \in S$  by (6.1.3). Also, if  $r$  is a unit, it has a *unique* inverse, written  $r^{-1}$ : this is proved in the same way as (3.2.1)(iii).

Now suppose that  $r_1$  and  $r_2$  are two units of  $R$ . Then  $r_1 r_2$  is also a unit since  $(r_1 r_2)^{-1} = r_2^{-1} r_1^{-1}$ , as is seen by forming products with  $r_1 r_2$ . Also of course  $(r^{-1})^{-1} = r$ , so that  $r^{-1}$  is a unit. Since  $1$  is its own inverse, we can state:

**(6.1.4)** *If  $R$  is a ring with identity, the set of units of  $R$  is a multiplicative group in which the group operation is ring multiplication.*

The group of units of  $R$  is written

$$U(R)$$

or sometimes  $R^*$ . Here are some simple examples of groups of units.

**Example (6.1.2)**

- (i)  $U(\mathbb{Z}) = \{\pm 1\}$ , a group of order 2.
- (ii)  $U(\mathbb{Q}) = \mathbb{Q} - 0$ , the multiplicative group of non-zero rational numbers.
- (iii) If  $m > 0$ , then  $U(\mathbb{Z}_m)$  is the multiplicative group  $\mathbb{Z}_m^*$  of all congruence classes  $[i]_m$  where  $\gcd(i, m) = 1$ . This is an abelian group of order  $\phi(m)$ .
- (iv)  $U(\mathbb{R}[t])$  is the group of non-zero constant polynomials. For if  $fg = 1$ , the polynomials  $f$  and  $g$  must be constant.

**Exercises (6.1)**

- (1) Which of the following are rings?
  - (i) The sets of even and odd integers, with the usual arithmetic operations;
  - (ii) the set of all differentiable functions on  $[0, 1]$  where  $f + g(x) = f(x) + g(x)$  and  $fg(x) = f(x)g(x)$ ;
  - (iii) the set of all singular  $2 \times 2$  real matrices, with the usual matrix operations.
- (2) Let  $S$  be a non-empty set. Define two binary operations on the power set  $\mathcal{P}(S)$  by  $X + Y = (X \cup Y) - (X \cap Y)$  and  $X \cdot Y = X \cap Y$ . Prove that  $(\mathcal{P}(S), +, \cdot)$  is a commutative ring with identity. Show also that  $X^2 = X$  and  $2X = 0_{\mathcal{P}(S)}$ .

- (3) A ring  $R$  is called *Boolean* if  $r^2 = r$  for all  $r \in R$ , (cf. Exercise (6.1.2)). If  $R$  is a Boolean ring, prove that  $2r = 0$  and that  $R$  is commutative.
- (4) Let  $A$  be an arbitrary (additively written) abelian group. Prove that  $A$  is the underlying additive group of some commutative ring.
- (5) Find the unit groups of the following rings:
- (i)  $\{\frac{m}{2^n} \mid m, n \in \mathbb{Z}\}$ , with the usual addition and multiplication;
  - (ii)  $M_n(\mathbb{R})$  with the standard matrix operations;
  - (iii) the ring of continuous functions on  $[0, 1]$ .
- (6) Prove that the Binomial Theorem is valid in any commutative ring  $R$ , i.e.,  $(a + b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}$  where  $a, b \in R$  and  $n$  is a non-negative integer.
- (7) Let  $R$  be a ring with identity. Suppose that  $a$  is an element of  $R$  with a *unique left inverse*  $b$ , i.e.,  $b$  is the unique element in  $R$  such that  $ba = 1$ . Prove that  $ab = 1$ , so that  $a$  is a unit. [Hint: consider the element  $ab - 1 + b$ .]
- (8) Let  $R$  be a ring with identity. Explain how to define a *formal power series* over  $R$  of the form  $\sum_{n=0}^{\infty} a_n t^n$  with  $a_n \in R$ . Then verify that these form a ring with identity with respect to appropriate sum and product operations. (This is called the *ring of formal power series* in  $t$  over  $R$ , in symbols  $R[[t]]$ ).
- (9) Let  $R$  be a ring with identity. Prove that  $M_n(\mathbb{R})$  is a commutative ring if and only if  $R$  is commutative and  $n = 1$ .

## 6.2 Subrings and ideals

In Chapter Three the concept of a subgroup of a group was introduced and already this has proved to be valuable in the study of groups. We aim to pursue a similar course for rings by introducing subrings.

Let  $(R, +, \times)$  be a ring and  $S$  a subset of the underlying set  $R$ . Then  $S$  is called a *subring* of  $R$  if  $(S, +_S, \times_S)$  is a ring where  $+_S$  and  $\times_S$  denote the binary operations  $+$  and  $\times$  when restricted to  $S$ . In particular  $S$  is a subgroup of the additive group  $(R, +)$ . With the aid of (3.3.4), we obtain a more useful description of a subring.

**(6.2.1)** *Let  $S$  be a subset of a ring  $R$ . Then  $S$  is a subring of  $R$  if and only if  $S$  contains  $0_R$  and is closed with respect to addition, multiplication and the formation of negatives, i.e., if  $a, b \in S$ , then  $a + b \in S$ ,  $ab \in S$  and  $-a \in S$ .*

### Example (6.2.1)

- (i)  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$  are successively larger subrings of the ring of complex numbers  $\mathbb{C}$ .
- (ii) The set of even integers  $2\mathbb{Z}$  is a subring of  $\mathbb{Z}$ . Notice that it does not contain the identity element, which is not a requirement for a subring.
- (iii) In any ring  $R$  there are at least two subrings, the *zero subring*  $0 = \{0_R\}$  and the *improper subring*  $R$  itself.

- (iv) Let  $S = \frac{1}{2}\mathbb{Z}$ , i.e.,  $S = \{\frac{m}{2} \mid m \in \mathbb{Z}\}$ . Then  $S$  is an additive subgroup of the ring  $\mathbb{Q}$ , but it is not a subring since  $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \notin S$ . Thus the concept of a subring is more special than that of an additive subgroup.

**Ideals.** It is reasonable to expect there to be an analogy between groups and rings in which subgroups correspond to subrings. The question then arises: what is to correspond to normal subgroups? This is where ideals enter the picture.

Let  $R$  be an arbitrary ring. A *left ideal* of  $R$  is an additive subgroup  $L$  such that  $ra \in L$  whenever  $r \in R$  and  $a \in L$ . Similarly a *right ideal* of  $R$  is an additive subgroup  $S$  such that  $ar \in S$  whenever  $r \in R$  and  $a \in S$ . If  $I$  is both a left and right ideal of  $R$ , it is called a *2-sided ideal*, or simply an *ideal* of  $R$ . Thus an ideal is an additive subgroup which is closed with respect to multiplication of its elements by *arbitrary ring elements* on the left and the right. Notice that *left ideals and right ideals are subrings*.

### Example (6.2.2)

- (i) Let  $R$  be a ring and let  $x \in R$ . Define subsets of  $R$

$$Rx = \{rx \mid r \in R\} \text{ and } xR = \{xr \mid r \in R\}.$$

Then  $Rx$  and  $xR$  are respectively a left ideal and a right ideal of  $R$ . For the first statement  $Rx$  is a subgroup since  $r_1x + r_2x = (r_1 + r_2)x$  and  $-(rx) = (-r)x$ ; also  $s(rx) = (sr)x$  for all  $r \in R$ , so  $Rx$  is a left ideal. Similarly  $xR$  is a right ideal. If  $R$  is a commutative ring,  $Rx = xR$  is an ideal. An ideal of this type is called a *principal ideal*.

- (ii) Every subgroup of  $\mathbb{Z}$  has the form  $n\mathbb{Z}$  where  $n \geq 0$  by (4.1.5). Hence every subgroup of  $\mathbb{Z}$  is a principal ideal.
- (iii) On the other hand,  $\mathbb{Z}$  is a subring, but not an ideal, of  $\mathbb{Q}$  since  $\frac{1}{2}(1) \notin \mathbb{Z}$ . Thus *subrings are not always ideals*.

Thus we have a hierarchy of distinct substructures of rings:

$$\text{left (right) ideal} \Rightarrow \text{ideal} \Rightarrow \text{subring} \Rightarrow \text{subgroup}.$$

**(6.2.2)** *The intersection of a non-empty set of subrings (left ideals, right ideals) of a ring  $R$  is a subring (respectively left ideal, right ideal) of  $R$ .*

The easy proofs are left to the reader. Let  $R$  be any ring and let  $X$  be a non-empty subset of  $R$ . On the basis of (6.2.2) we can assert that the intersection of all the subrings of  $R$  which contain  $X$  is a subring, clearly the smallest subring containing  $X$ . This is called the *subring generated by  $X$*  and it will be denoted by

$$\text{Rg}\langle X \rangle.$$

If  $X = \{x_1, x_2, \dots, x_n\}$ , this subring is denoted by  $\text{Rg}\langle x_1, x_2, \dots, x_n \rangle$ . When  $R$  has an identity element, the general form of elements of  $\text{Rg}\langle X \rangle$  is not hard to determine.

**(6.2.3)** *Let  $R$  be a ring with identity and let  $X$  be a non-empty subset of  $R$ . Then  $\text{Rg}\langle X \rangle$  consists of all elements of the form*

$$\sum_{\ell_1, \ell_2, \dots, \ell_n} m_{\ell_1, \ell_2, \dots, \ell_n} x_1^{\ell_1} \cdots x_n^{\ell_n}$$

where  $x_i \in X$ ,  $n \geq 0$ ,  $m_{\ell_1, \ell_2, \dots, \ell_n} \in \mathbb{Z}$  and the  $\ell_i$  are non-negative integers.

Again the easy proof is left to the reader. A ring  $R$  is said to be *finitely generated* if  $R = \text{Rg}\langle x_1, x_2, \dots, x_n \rangle$  for some finite set of elements  $\{x_1, \dots, x_n\}$ . In a similar vein we define the left, right or two-sided *ideal generated* by a non-empty subset  $X$  of a ring  $R$  to be the intersection of all the respective ideals that contain  $X$ .

**(6.2.4)** *Let  $R$  be a ring with identity and let  $X$  be a non-empty subset of  $R$ . Then the left ideal generated by  $X$  consists of all elements of the form*

$$\sum_{i=1}^n r_i x_i$$

where  $x_i \in X$ ,  $r_i \in R$ ,  $n \geq 0$ .

There are similar statements for right and two-sided ideals. The simple proofs are left as an exercise. The left ideal of  $R$  generated by  $X$  is denoted by

$$RX.$$

A left ideal  $I$  of a ring  $R$  is said to be *finitely generated as a left ideal* if it can be generated by finitely many elements  $x_1, x_2, \dots, x_n$ . If  $R$  has an identity, the ideal  $I$  has the form  $I = R\{x_1, x_2, \dots, x_n\} = Rx_1 + Rx_2 + \cdots + Rx_n$ .

If  $R$  is a commutative ring with identity, the ideal  $R\{x_1, x_2, \dots, x_n\}$  is often written  $(x_1, x_2, \dots, x_n)$ . In particular

$$(x)$$

is the *principal ideal*  $R\{x\}$ , consisting of all elements of the form  $rx$  where  $r \in R$ .

**Homomorphisms of rings.** It is still not apparent why ideals as defined above should be the analogs of normal subgroups. The decisive test of the appropriateness of the definition will come when ring homomorphisms are defined. If we are right, the kernel of a homomorphism will be an ideal.

It is fairly obvious how one should define a *homomorphism from a ring  $R$  to a ring  $S$* : this is a function  $\theta : R \rightarrow S$  which relates the ring operations in the sense that

$$\theta(a + b) = \theta(a) + \theta(b) \quad \text{and} \quad \theta(ab) = \theta(a)\theta(b)$$

for all  $a, b \in R$ . Thus in particular  $\theta$  is a homomorphism of groups.

If in addition  $\theta$  is bijective,  $\theta$  is called an *isomorphism of rings*. If there is an isomorphism from ring  $R$  to ring  $S$ , then  $R$  and  $S$  are said to be *isomorphic rings*, in symbols

$$R \simeq S.$$

**Example (6.2.3)**

- (i) Let  $m$  be a positive integer. The function  $\theta_m : \mathbb{Z} \rightarrow \mathbb{Z}_m$  defined by  $\theta_m(x) = [x]_m$  is a ring homomorphism. This is a consequence of the way in which sums and products of congruence classes were defined.
- (ii) The *zero homomorphism*  $0 : R \rightarrow S$  sends every  $r \in R$  to  $0_S$ . Also the *identity isomorphism* from  $R$  to  $R$  is just the identity function on  $R$ .

**Complex numbers.** For a more interesting example of a ring isomorphism, consider the set  $R$  of matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad (a, b \in \mathbb{R}).$$

These are quickly seen to form a subring of the matrix ring  $M_2(\mathbb{R})$ . Now define a function  $\theta : R \rightarrow \mathbb{C}$  by the rule

$$\theta\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) = a + ib$$

where  $i = \sqrt{-1}$ . Then  $\theta$  is a ring homomorphism: for

$$\begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 - b_1 b_2 & a_1 b_2 + a_2 b_1 \\ -a_1 b_2 - a_2 b_1 & a_1 a_2 - b_1 b_2 \end{bmatrix},$$

which is mapped by  $\theta$  to  $(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$ , i.e., to the product  $(a_1 + ib_1)(a_2 + ib_2)$ . An easier calculation shows that  $\theta$  sends

$$\begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix}$$

to  $(a_1 + ib_1) + (a_2 + ib_2)$ .

Certainly  $\theta$  is surjective; it is also injective since  $a + ib = 0$  implies that  $a = 0 = b$ . Therefore  $\theta$  is an isomorphism and we obtain the interesting fact that  $R \simeq \mathbb{C}$ . Thus complex numbers can be represented by real  $2 \times 2$  matrices. In fact this provides a way to define complex numbers without resorting to the square root of  $-1$ .

Next we consider the nature of the kernel and image of a ring homomorphism. The following result should be compared with (4.3.2).

**(6.2.5)** *If  $\theta : R \rightarrow S$  is a homomorphism of rings, then  $\text{Ker}(\theta)$  is an ideal of  $R$  and  $\text{Im}(\theta)$  is a subring of  $S$ .*

*Proof.* We know already from (4.3.2) that  $\text{Ker}(\theta)$  and  $\text{Im}(\theta)$  are subgroups. Let  $k \in \text{Ker}(\theta)$  and  $r \in R$ . Then  $\theta(kr) = \theta(k)\theta(r) = 0_S$  and  $\theta(rk) = \theta(r)\theta(k) = 0_S$  since  $\theta(k) = 0_S$ . Therefore  $\text{Ker}(\theta)$  is an ideal of  $R$ . Furthermore  $\theta(r_1)\theta(r_2) = \theta(r_1r_2)$ , so that  $\text{Im}(\theta)$  is a subring of  $S$ .  $\square$

**(6.2.6)** If  $\theta : R \rightarrow S$  is an isomorphism of rings, then so is  $\theta^{-1} : S \rightarrow R$ .

*Proof.* We know from (3.3.1) that  $\theta^{-1}$  is an isomorphism of groups. It must still be shown that  $\theta^{-1}(s_1s_2) = \theta^{-1}(s_1)\theta^{-1}(s_2)$ , ( $s_i \in S$ ). Observe that the image of each side under  $\theta$  is  $s_1s_2$ . Since  $\theta$  is injective, it follows that  $\theta^{-1}(s_1s_2) = \theta^{-1}(s_1)\theta^{-1}(s_2)$ .  $\square$

**Quotient rings.** Since ideals appear to be the natural ring theoretic analog of normal subgroups, we expect to be able to define the quotient of a ring by an ideal. Let  $I$  be an ideal of a ring  $R$ . Certainly  $I$  is a normal subgroup of the additive abelian group  $R$ , so we can form the quotient group  $R/I$ . This is an additive abelian group whose elements are the cosets of  $I$ . To make  $R/I$  into a ring, a rule for multiplying cosets must be specified: the natural one to try is

$$(r_1 + I)(r_2 + I) = r_1r_2 + I, \quad (r_i \in R).$$

To prove that this is well-defined, let  $i_1, i_2 \in I$  and note that

$$(r_1 + i_1)(r_2 + i_2) = r_1r_2 + (r_1i_2 + i_1r_2 + i_1i_2) \in r_1r_2 + I$$

since  $I$  is an ideal. Thus the rule is independent of the choice of coset representatives  $r_1$  and  $r_2$ .

A further easy check shows that the ring axioms hold; therefore  $R/I$  is a ring, the *quotient ring* of  $I$  in  $R$ . Note also that the assignment  $r \mapsto r + I$  is a surjective ring homomorphism from  $R$  to  $R/I$  with kernel  $I$ ; this is the *canonical homomorphism*, (cf. (4.3)).

As one might expect, there are isomorphism theorems for rings similar to those for groups.

**(6.2.7)** (First Isomorphism Theorem) If  $\theta : R \rightarrow S$  is a homomorphism of rings, then  $R/\text{Ker}(\theta) \simeq \text{Im}(\theta)$ .

**(6.2.8)** (Second Isomorphism Theorem) If  $I$  is an ideal and  $S$  is a subring of a ring  $R$ , then  $S + I$  is a subring of  $R$  and  $S \cap I$  is an ideal of  $S$ . Also  $S + I/I \simeq S/S \cap I$ .

**(6.2.9)** (Third Isomorphism Theorem) Let  $I$  and  $J$  be ideals of a ring  $R$  with  $I \subseteq J$ . Then  $J/I$  is an ideal of  $R/I$  and  $(R/I)/(J/I) \simeq R/J$ .

Fortunately we can apply the isomorphism theorems for groups – see (4.3.4), (4.3.5), (4.3.6). The isomorphisms constructed in the proofs of these theorems still

stand if we allow for the additive notation. Thus we have only to check that they are homomorphisms of rings.

For example, take the case of (6.2.7). From (4.3.4) we know that the assignment  $r + \text{Ker}(\theta) \mapsto \theta(r)$  yields a group isomorphism  $\alpha : R / \text{Ker}(\theta) \mapsto \text{Im}(\theta)$ . Also

$$\alpha((r_1 + \text{Ker}(\theta))(r_2 + \text{Ker}(\theta))) = \alpha(r_1 r_2 + \text{Ker}(\theta)) = \theta(r_1 r_2),$$

which is equal to  $\theta(r_1)\theta(r_2) = \alpha(r_1 + \text{Ker}(\theta))\alpha(r_2 + \text{Ker}(\theta))$ . Therefore we conclude that  $\alpha$  is an isomorphism of rings: this proves (6.2.7). It is left to the reader to complete the proofs of the other two isomorphism theorems.

**(6.2.10)** (The Correspondence Theorem) *Let  $I$  be an ideal of a ring  $R$ . Then the assignment  $S \mapsto S/I$  determines a bijection from the set of subrings of  $R$  that contain  $I$  to the set of subrings of  $R/I$ . Furthermore  $S/I$  is an ideal of  $R/I$  if and only if  $S$  is an ideal of  $R$ .*

*Proof.* The correspondence between subgroups described in (4.2.2) applies here. It remains only to verify that  $S$  is a subring (ideal) if and only if  $S/I$  is. It is left to the reader to fill in the details.  $\square$

## Exercises (6.2)

(1) Classify the following subsets of a ring  $R$  as an additive subgroup, subring or ideal, as is most appropriate:

- (i)  $\{f \in \mathbb{R}[t] \mid f(a) = 0\}$  where  $R = \mathbb{R}[t]$  and  $a \in \mathbb{R}$  is fixed;
- (ii) the set of twice differentiable functions on  $[0, 1]$  which satisfy the differential equation  $f'' + f' = 0$ : here  $R$  is the ring of continuous functions on  $[0, 1]$ ;
- (iii)  $n\mathbb{Z}$  where  $R = \mathbb{Z}$ ;
- (iv)  $\frac{1}{2}\mathbb{Z}$  where  $R = \mathbb{Q}$ .
- (v) the set of real  $n \times n$  matrices with zero first row where  $R = M_n(\mathbb{R})$ .

(2) Prove (6.2.2).

(3) Prove (6.2.3) and (6.2.4).

(4) Which of the following rings are finitely generated?  $\mathbb{Z}$ ;  $\mathbb{Q}$ ;  $\mathbb{Z}[t_1, t_2, \dots, t_n]$ .

(5) Let  $R$  be a ring with identity. If  $I$  is a left ideal containing a unit, show that  $I = R$ .

(6) Let  $I$  and  $J$  be ideals of a ring  $R$  such that  $I \cap J = 0$ . Prove that  $ab = 0$  for all  $a \in I$ ,  $b \in J$ .

(7) Let  $a \in \mathbb{R}$  and define  $\theta_a : \mathbb{R}[t] \rightarrow \mathbb{R}$  by  $\theta_a(f) = f(a)$ . Prove that  $\theta_a$  is a ring homomorphism. Identify  $\text{Im}(\theta_a)$  and  $\text{Ker}(\theta_a)$ .

(8) Let  $\alpha : R \rightarrow S$  be a surjective ring homomorphism and assume that  $R$  has an identity element and  $S$  is not the zero ring. Prove that  $S$  has an identity element.

(9) Give examples of a left ideal that is not a right ideal and a right ideal that is not a left ideal.

- (10) Give an example of an ideal of a commutative ring with identity that is not principal.
- (11) What is the form of elements of the left ideal generated by a subset  $X$  in a ring  $R$  that does not have an identity element?
- (12) Prove that the subring of  $\mathbb{Q}$  consisting of all  $\frac{m}{2^n}$  is a finitely generated ring.

### 6.3 Integral domains, division rings and fields

The purpose of this section is to introduce some special types of ring with desirable properties. Specifically we are interested in rings having a satisfactory theory of division. For this reason it is necessary to exclude the phenomenon in which the product of two non-zero ring elements is zero.

If  $R$  is a ring, a *left zero divisor* is a non-zero element  $a$  such that  $ab = 0$  for some  $b \neq 0$  in  $R$ . Of course  $b$  is called a *right zero divisor*. Clearly the presence of zero divisors will make it difficult to construct a reasonable theory of division.

**Example (6.3.1)** Let  $n$  be a positive integer. The zero divisors in  $\mathbb{Z}_n$  are the congruence classes  $[m]$  where  $m$  and  $n$  are not relatively prime and  $1 < m < n$ . Thus  $\mathbb{Z}_n$  has zero divisors if and only if  $n$  is not a prime.

For, if  $m$  and  $n$  are not relatively prime and  $d > 1$  is a common divisor of  $m$  and  $n$ , then  $[m][\frac{n}{d}] = [\frac{m}{d}][n] = [0]$  since  $[n] = [0]$ , while  $[m] \neq 0$  and  $[\frac{n}{d}] \neq [0]$ ; thus  $[m]$  is a zero divisor.

Conversely, suppose that  $[m]$  is a zero divisor and  $[m][\ell] = [0]$  where  $[\ell] \neq [0]$ . Then  $n \mid m\ell$ ; thus, if  $m$  and  $n$  are relatively prime,  $n \mid \ell$  and  $[\ell] = [0]$  by Euclid's Lemma. This contradiction shows that  $m$  and  $n$  cannot be relatively prime.

Next we introduce an important class of rings with no zero divisors. An *integral domain* (or more briefly a *domain*) is a commutative ring with identity which has no zero divisors. For example,  $\mathbb{Z}$  is a domain, while  $\mathbb{Z}_n$  is a domain if and only if  $n$  is a prime, by Example (6.3.1). Domains can also be characterized by a cancellation property.

**(6.3.1)** Let  $R$  be a commutative ring with identity. Then  $R$  is a domain if and only if the cancellation law is valid in  $R$ , that is,  $ab = ac$  and  $a \neq 0$  always imply that  $b = c$ .

*Proof.* If  $ab = ac$  and  $b \neq c$ ,  $a \neq 0$ , then  $a(b - c) = 0$ , so that  $a$  is a zero divisor and  $R$  is not a domain. Conversely, if  $R$  is not a domain and  $ab = 0$  with  $a, b \neq 0$ , then  $ab = a0$ , so the cancellation law fails.  $\square$

The next result shows that it is much simpler to work with polynomials if the coefficient ring is a domain.

**(6.3.2)** Let  $R$  be an integral domain and let  $f, g \in R[t]$ . Then

$$\deg(fg) = \deg(f) + \deg(g).$$



Hence  $fg \neq 0$  if  $f \neq 0$  and  $g \neq 0$ , so that  $R[t]$  is an integral domain.

*Proof.* If  $f = 0$ , then  $fg = 0$  and  $\deg(f) = -\infty = \deg(fg)$ ; hence the formula is valid in this case. Assume that  $f \neq 0$  and  $g \neq 0$ , and let  $at^m$  and  $bt^n$  be the terms of highest degree in  $f$  and  $g$  respectively; thus  $a \neq 0$  and  $b \neq 0$ . Then  $fg = abt^{m+n} + \text{terms of lower degree}$ , and  $ab \neq 0$  since  $R$  is a domain. Therefore  $\deg(fg) = m+n = \deg(f) + \deg(g)$ .  $\square$

Recall that a unit in a ring with identity is an element with a multiplicative inverse. A ring with identity in which every non-zero element is a unit is termed a *division ring*. Commutative division rings are called a *fields*. Clearly  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are examples of fields, while  $\mathbb{Z}$  is not a field. Fields are one of the most frequently used types of ring since the ordinary operations of arithmetic can be performed in a field.

Notice that a *division ring cannot have zero divisors*: for if  $ab = 0$  and  $a \neq 0$ , then  $b = a^{-1}ab = a^{-1}0 = 0$ . Thus the rings without zero divisors include domains and division rings.

**The ring of quaternions.** The examples of division rings given so far are commutative, i.e., they are fields. We will now describe a famous example of a non-commutative division ring, the *ring of Hamilton's<sup>1</sup> quaternions*. First of all consider the following  $2 \times 2$  matrices over  $\mathbb{C}$ ,

$$I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

where  $i = \sqrt{-1}$ . These are known in physics as the *Pauli<sup>2</sup> spin matrices*. Simple matrix computations show that the following relations hold:

$$I^2 = J^2 = K^2 = -1, \quad IJ = K = -JI, \quad JK = I = -KJ, \quad KI = J = -IK.$$

Here 1 is being used to denote the identity  $2 \times 2$  matrix and it should be distinguished from the matrix  $I$ .

If  $a, b, c, d$  are rational numbers, we can form the matrix

$$a1 + bI + cJ + dK = \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix},$$

which is called a *rational quaternion*. Let  $R$  be the set of all rational quaternions. Then  $R$  is a subring of the matrix ring  $M_2(\mathbb{C})$  containing the identity: for

$$\begin{aligned} (a1 + bI + cJ + dK) + (a'1 + b'I + c'J + d'K) \\ = (a + a')1 + (b + b')I + (c + c')J + (d + d')K, \end{aligned}$$

<sup>1</sup> William Rowan Hamilton (1805–1865)

<sup>2</sup> Wolfgang Ernst Pauli (1900–1958)

while  $(a1 + bI + cJ + dK)(a'1 + b'I + c'J + d'K)$  equals

$$(aa' - bb' - cc' - dd')1 + (ab' + a'b + cd' - c'd)I \\ + (ac' + a'c + b'd - bd')J + (ad' + a'd + bc' - b'c)K,$$

as is seen by multiplying out and using the properties of  $I, J, K$  above.

The significant property of the ring  $R$  is that each non-zero element is a unit. For, if  $0 \neq Q = a1 + bI + cJ + dK$ , then

$$\det(Q) = \begin{vmatrix} a + bi & c + di \\ -c + di & a - bi \end{vmatrix} = a^2 + b^2 + c^2 + d^2 \neq 0,$$

and by elementary matrix algebra

$$Q^{-1} = \frac{1}{\det(Q)} \begin{bmatrix} a - bi & -c - di \\ c - di & a + bi \end{bmatrix} \in R.$$

This allows us to state:

**(6.3.3)** *The ring of rational quaternions is a non-commutative division ring.*

Notice that the ring of quaternions is infinite. This is no accident since, by a famous theorem of Wedderburn,<sup>3</sup> *a finite division ring is a field*. This will not be proved here; however, we will prove the corresponding statement for domains, which is much easier.

**(6.3.4)** *A finite integral domain is a field.*

*Proof.* Let  $R$  be a finite domain and let  $0 \neq r \in R$ ; we need to show that  $r$  has an inverse. Consider the function  $\alpha : R \rightarrow R$  defined by  $\alpha(x) = rx$ . Now  $\alpha$  is injective since  $rx = ry$  implies that  $x = y$  by (6.3.1). However,  $R$  is a finite set, so it follows that  $\alpha$  must also be surjective. Therefore  $1 = rx$  for some  $x \in R$  and  $x = r^{-1}$ .  $\square$

Next we consider the role of ideals in commutative ring theory. A first observation is that the presence of proper non-zero ideals is counter-indicative for the existence of units.

**(6.3.5)** *Let  $R$  be a commutative ring with identity. Then the set of non-units of  $R$  is equal to the union of all the proper ideals of  $R$ .*

*Proof.* Suppose that  $r$  is not a unit of  $R$ ; then  $Rx = \{rx \mid x \in R\}$  is a proper ideal containing  $r$  since  $1 \notin Rx$ . Conversely, if a unit  $r$  belongs to an ideal  $I$ , then for any  $x$  in  $R$  we have  $x = (xr^{-1})r \in I$ , showing that  $I = R$ . Thus a unit cannot belong to a proper ideal.  $\square$

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<sup>3</sup> Joseph Henry Maclagan Wedderburn (1881–1948)

Recalling that fields are exactly the commutative rings with identity in which each non-zero element is a unit, we deduce:

**Corollary (6.3.6)** *A commutative ring with identity is a field if and only if it has no proper non-zero ideals.*

**Maximal ideals and prime ideals.** Let  $R$  be a commutative ring with identity. A *maximal ideal* of  $R$  is a proper ideal  $I$  such that the only ideals containing  $I$  are  $I$  itself and  $R$ . Thus a maximal ideal is a *maximal proper ideal*. For example, if  $p$  is a prime,  $p\mathbb{Z}$  is a maximal ideal of  $\mathbb{Z}$ : for  $|\mathbb{Z}/p\mathbb{Z}| = p$  and (6.2.10) shows that no ideal can occur strictly between  $p\mathbb{Z}$  and  $\mathbb{Z}$ .

A related concept is that of a prime ideal. If  $R$  is a commutative ring with identity, a *prime ideal* of  $R$  is a proper ideal  $I$  with the property:  $ab \in I$  implies that  $a \in I$  or  $b \in I$ , where  $a, b \in R$ .

There are enlightening characterizations of prime and maximal ideals in terms of quotient rings.

**(6.3.7)** *Let  $I$  be a proper ideal of a commutative ring  $R$  with identity.*

- (i)  *$I$  is a prime ideal of  $R$  if and only if  $R/I$  is an integral domain;*
- (ii)  *$I$  is a maximal ideal of  $R$  if and only if  $R/I$  is a field.*

*Proof.* Let  $a, b \in R$ ; then  $ab \in I$  if and only if  $(a + I)(b + I) = I = 0_{R/I}$ . Thus  $I$  is prime precisely when  $R/I$  has no zero divisors, i.e., it is a domain, so (i) is established. By (6.2.10)  $I$  is maximal in  $R$  if and only if  $R/I$  has no proper non-zero ideals and by (6.3.6) this is equivalent to  $R/I$  being a field.  $\square$

Since every field is a domain, there follows at once:

**Corollary (6.3.8)** *Every maximal ideal of a commutative ring with identity is a prime ideal.*

On the other hand, prime ideals need not be maximal. Indeed, if  $R$  is any domain, the zero ideal is certainly prime, but it is not maximal unless  $R$  is a field. More interesting examples of non-maximal prime ideals can be constructed in polynomial rings.

**Example (6.3.2)** Let  $R = \mathbb{Q}[t_1, t_2]$ , the ring of polynomials in  $t_1, t_2$  with rational coefficients. Let  $I$  be the subset of all polynomials in  $R$  which are multiples of  $t_1$ . Then  $I$  is a prime ideal of  $R$ , but it is not maximal.

For consider the function  $\alpha : R \rightarrow \mathbb{Q}[t_2]$  which carries a polynomial  $f(t_1, t_2)$  to  $f(0, t_2)$ . This is a surjective ring homomorphism. Now if  $f(0, t_2) = 0$ , then  $f$  is a multiple of  $t_1$ , which shows that the kernel of  $\alpha$  is  $I$ . From (6.2.7) we deduce that  $R/I \cong \mathbb{Q}[t_2]$ . Since  $\mathbb{Q}[t_2]$  is a domain, but not a field, it follows from (6.3.7) that  $I$  is a prime ideal of  $R$  which is not maximal.

**The characteristic of an integral domain.** Let  $R$  be a domain and let  $S = \langle 1 \rangle$ , the additive subgroup of  $R$  generated by 1. Suppose for the moment that  $S$  is finite, with order  $n$  say; we claim that  $n$  must be a prime. For suppose that  $n = n_1 n_2$  where  $n_i \in \mathbb{Z}$  and  $1 < n_i < n$ . Then  $0 = n1 = (n_1 n_2)1 = (n_1 1)(n_2 1)$  by (6.1.3). However,  $R$  is a domain, so  $n_1 1 = 0$  or  $n_2 1 = 0$ , which shows that  $n$  divides  $n_1$  or  $n_2$ , a contradiction. Therefore  $n$  is a prime.

This observation is the essence of:

**(6.3.9)** *Let  $R$  be an integral domain and put  $S = \langle 1 \rangle$ . Then either  $S$  is infinite or else it has prime order  $p$ . In the latter event  $pa = 0$  for all  $a \in R$ .*

To prove the final statement, simply note that  $pa = (p1_R)a = 0a = 0$ .

If  $R$  is an integral domain and  $\langle 1_R \rangle$  has prime order  $p$ , then  $R$  is said to have *characteristic  $p$* . The other possibility is that  $\langle 1_R \rangle$  is infinite, in which event  $R$  is said to have *characteristic 0*. Thus the characteristic of  $R$ ,

$$\text{char}(R),$$

is either 0 or a prime. For example,  $\mathbb{Z}_p$  and  $\mathbb{Z}_p[t]$  are domains with characteristic  $p$ , while  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{R}[t]$  all have characteristic 0.

**The field of fractions of an integral domain.** Suppose that  $F$  is a field and  $R$  is a subring of  $F$  containing  $1_F$ . Then  $R$  is a domain since there cannot be zero divisors in  $F$ . Conversely, one can ask if every domain arises in this way as a subring of a field. We will answer the question positively by showing how to construct the *field of fractions of a domain*. It will be helpful for the reader to keep in mind that the procedure to be described is a generalization of the way in which the rational numbers are constructed from the integers.

Let  $R$  be any integral domain. First we have to decide how to define a fraction over  $R$ . Consider the set

$$S = \{(a, b) \mid a, b \in R, b \neq 0\}.$$

Here  $a$  will correspond to the numerator and  $b$  to the denominator of the fraction. A binary relation  $\sim$  on  $S$  will now be introduced which allows for cancellation between numerator and denominator:

$$(a_1, b_1) \sim (a_2, b_2) \Leftrightarrow a_1 b_2 = a_2 b_1.$$

Of course this relation is motivated by a familiar arithmetic rule:  $\frac{m_1}{n_1} = \frac{m_2}{n_2}$  if and only if  $m_1 n_2 = m_2 n_1$ .

We verify that  $\sim$  is an equivalence relation on  $S$ . Only transitivity requires a comment: suppose that  $(a_1, b_1) \sim (a_2, b_2)$  and  $(a_2, b_2) \sim (a_3, b_3)$ ; then  $a_1 b_2 = a_2 b_1$  and  $a_2 b_3 = a_3 b_2$ . Multiply the first equation by  $b_3$  and use the second equation to derive

$a_1 b_3 b_2 = a_2 b_3 b_1 = a_3 b_2 b_1$ . Cancel  $b_2$  to obtain  $a_1 b_3 = a_3 b_1$ ; thus  $(a_1, b_1) \sim (a_3, b_3)$ . Now define a *fraction* over  $R$  to be a  $\sim$ -equivalence class

$$\frac{a}{b} = [(a, b)]$$

where  $a, b \in R$ ,  $b \neq 0$ . Note that  $\frac{ac}{bc} = \frac{a}{b}$  since  $(a, b) \sim (ac, bc)$ ; thus cancellation can be performed within a fraction.

Let  $F$  denote the set of all fractions over  $R$ : we wish to make  $F$  into a ring. To this end define addition and multiplication in  $R$  by the rules

$$\frac{a}{b} + \frac{a'}{b'} = \frac{ab' + a'b}{bb'} \quad \text{and} \quad \left(\frac{a}{b}\right)\left(\frac{a'}{b'}\right) = \frac{aa'}{bb'}.$$

Here we have been guided by the ordinary arithmetic rules for adding and multiplying fractions. However, it is necessary to show that these operations are well-defined, i.e., there is no dependence on the chosen representative  $(a, b)$  of the equivalent class  $\frac{a}{b}$ . For example, take the case of addition. Let  $(a, b) \sim (c, d)$  and  $(a', b') \sim (c', d')$ : then in fact  $(ab' + a'b, bb') \sim (cd' + c'd, dd')$  because

$$(ab' + a'b)dd' = ab'dd' + a'bdd' = bcb'd' + b'c'bd = (cd' + c'd)bb'.$$

The next step is to verify the ring axioms: as an example we will check the validity of the distributive law

$$\left(\frac{a}{b} + \frac{c}{d}\right)\left(\frac{e}{f}\right) = \left(\frac{a}{b}\right)\left(\frac{e}{f}\right) + \left(\frac{c}{d}\right)\left(\frac{e}{f}\right),$$

leaving the reader to verify the other axioms. By definition

$$\left(\frac{a}{b}\right)\left(\frac{e}{f}\right) + \left(\frac{c}{d}\right)\left(\frac{e}{f}\right) = \frac{ae}{bf} + \frac{ce}{df} = \frac{aedf + cebf}{bdf^2} = \frac{ade + bce}{bdf},$$

which equals

$$\left(\frac{ad + bc}{bd}\right)\left(\frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right)\left(\frac{e}{f}\right),$$

as claimed.

Once all the axioms have been checked, we know that  $F$  is a ring; note that the zero element of  $F$  is  $0_F = \frac{0_R}{1_R}$ . Clearly  $F$  is commutative and it has identity element  $1_F = \frac{1_R}{1_R}$ . Furthermore, if  $a, b \neq 0$ ,

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = \frac{ab}{ab} = \frac{1_R}{1_R} = 1_F,$$

so that, as expected, the inverse of  $\frac{a}{b}$  is  $\frac{b}{a}$ . Therefore  $F$  is a field, the *field of fractions of the domain  $R$* .

In order to relate  $F$  to  $R$  we introduce the natural function

$$\theta : R \rightarrow F$$

defined by  $\theta(a) = \frac{a}{1}$ . It is straightforward to check that  $\theta$  is an injective ring homomorphism. Therefore  $R \simeq \text{Im}(\theta)$  and of course  $\text{Im}(\theta)$  is a subring of  $F$  containing  $1_F$ . Thus the original domain  $R$  is isomorphic with a subring of the field  $F$ . Our conclusions are summed up in the following result.

**(6.3.10)** *Let  $R$  be an integral domain and let  $F$  denote the set of all fractions over  $R$ , with the addition and multiplication specified above. Then  $F$  is a field and the assignment  $a \mapsto \frac{a}{1}$  determines an injective ring homomorphism from  $R$  to  $F$ .*

**Example (6.3.3)**

- (i) When  $R = \mathbb{Z}$ , the field of fractions is, up to isomorphism, the field of rational numbers  $\mathbb{Q}$ . This example motivated the general construction.
- (ii) Let  $K$  be any field and put  $R = K[t]$ ; this is a domain by (6.3.2). The field of fractions  $F$  of  $R$  is the *field of rational functions* in  $t$  over  $K$ ; these are formal quotients of polynomials in  $t$  over  $K$

$$\frac{f}{g}$$

where  $f, g \in R$ ,  $g \neq 0$ . The notation  $K\{t\}$  is often used to denote the field of rational functions in  $t$  over  $K$ .

**Exercises (6.3)**

- (1) Find all zero divisors in the following rings:  $\mathbb{Z}_6$ ,  $\mathbb{Z}_{15}$ ,  $\mathbb{Z}_2[t]$ ,  $\mathbb{Z}_4[t]$ ,  $M_n(\mathbb{R})$ .
- (2) Let  $R$  be a commutative ring with identity such that the degree formula  $\deg(fg) = \deg(f) + \deg(g)$  is valid in  $R[t]$ . Prove that  $R$  is a domain.
- (3) If  $R$  is a division ring, prove that the only left ideals and right ideals are  $0$  and  $R$ .
- (4) Let  $R$  be a ring with identity. If  $R$  has no left or right ideals except  $0$  and  $R$ , prove that  $R$  is a division ring.
- (5) Let  $\theta : D \rightarrow R$  be a non-zero ring homomorphism. If  $D$  is a division ring, show that it is isomorphic with a subring of  $R$ .
- (6) Let  $I_1, I_2, \dots, I_k$  be non-zero ideals of a domain. Prove that  $I_1 \cap I_2 \cap \dots \cap I_k \neq 0$ . Is this necessarily true for an infinite set of non-zero ideals?
- (7) Let  $I$  be the principal ideal  $(\mathbb{Z}[t])t$  of  $\mathbb{Z}[t]$ . Prove that  $I$  is prime but not maximal.
- (8) The same problem for  $I = (\mathbb{Z}[t])(t^2 - 2)$ .
- (9) Let  $F$  be a field. If  $a, b \in F$  and  $a \neq 0$ , define a function  $\theta_{a,b} : F \rightarrow F$  by the rule  $\theta_{a,b}(x) = ax + b$ . Prove that the set of all  $\theta_{a,b}$ 's is a group with respect to functional composition.
- (10) Let  $F$  be the field of fractions of a domain  $R$  and let  $\alpha : R \rightarrow F$  be the canonical injective homomorphism  $r \mapsto \frac{r}{1}$ . Suppose that  $\beta : R \rightarrow K$  is an injective ring homomorphism from  $R$  to some other field  $K$ . Prove that there is an injective homomorphism  $\theta : F \rightarrow K$  such that  $\theta\alpha = \beta$ . (Thus in a sense  $F$  is the smallest field containing an isomorphic copy of  $R$ .)

## 6.4 Finiteness conditions on ideals

In this section we introduce certain finiteness properties of ideals that are possessed by some important types of ring.

**(6.4.1)** *Let  $\mathcal{J}$  be a non-empty set of left ideals of a ring  $R$ . Then the following statements about  $\mathcal{J}$  are equivalent.*

- (i) *The set  $\mathcal{J}$  satisfies the ascending chain condition, i.e., there does not exist an infinite ascending chain of left ideals  $I_1 \subset I_2 \subset \cdots$  with  $I_i \in \mathcal{J}$ .*
- (ii) *The set  $\mathcal{J}$  satisfies the maximal condition, i.e., every non-empty subset of  $\mathcal{J}$  has a maximal element, that is to say, an element which is not properly contained in any other element of  $\mathcal{J}$ .*

*Proof.* Assume that  $\mathcal{J}$  satisfies condition (i) and suppose that  $\mathcal{S}$  is a non-empty subset of  $\mathcal{J}$  that does not contain a maximal element. Let  $I_1 \in \mathcal{S}$ ; then there exists  $I_2 \in \mathcal{S}$  which is strictly larger than  $I_1$  since  $I_1$  is not maximal in  $\mathcal{J}$ . Similarly there exists  $I_3 \in \mathcal{S}$  which is strictly larger than  $I_2$ , and so on. But clearly this leads to an infinite ascending chain  $I_1 \subset I_2 \subset \cdots$  in  $\mathcal{J}$ , a contradiction.

Conversely, assume that  $\mathcal{J}$  satisfies condition (ii). If there is an infinite ascending chain  $I_1 \subset I_2 \subset \cdots$  in  $\mathcal{J}$ , the maximal condition can be applied to the set  $\{I_1, I_2, \dots\}$  to give a maximal element. This is obviously impossible.  $\square$

We remark that similar properties for subgroups of a group were introduced in Exercise (3.3.10).

There is of course a corresponding result for right ideals. The case of greatest interest to us is when  $\mathcal{J}$  is the set of all left ideals of the ring  $R$ . If this set satisfies one of the two equivalent conditions of (6.4.1), then  $R$  is called a *left noetherian*<sup>4</sup> ring. There is a corresponding definition of a *right noetherian ring*. In case of a commutative ring, the ring is simply said to be *noetherian*. The following result sheds some light on the nature of the noetherian condition.

**(6.4.2)** *Let  $R$  be a ring with identity. Then  $R$  is left noetherian if and only if every left ideal of  $R$  is finitely generated as a left ideal of  $R$ .*

*Proof.* First suppose that  $I$  is a left ideal of  $R$  which is not finitely generated. Certainly  $I \neq 0$ , so there exists  $r_1 \in I - 0$  and  $I \neq Rr_1$  since  $I$  is not finitely generated. Let  $r_2 \in R - Rr_1$ . Then  $I \neq Rr_1 + Rr_2$ . Let  $r_3 \in I - (Rr_1 + Rr_2)$  and note that  $Rr_1 + Rr_2 \neq Rr_1 + Rr_2 + Rr_3$ , and so on. But this leads to an infinite ascending chain of left ideals  $Rr_1 \subset Rr_1 + Rr_2 \subset Rr_1 + Rr_2 + Rr_3 \subset \cdots$  and  $R$  is not left noetherian.

Conversely, assume  $R$  is not left noetherian, so that there exists an infinite ascending chain of left ideals  $I_1 \subset I_2 \subset \cdots$ . Set  $I = \bigcup_{i=1,2,\dots} I_i$ , which is clearly a left ideal of  $R$ .

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<sup>4</sup> Emmy Noether (1882–1935)

Then  $I$  cannot be generated by finitely many elements  $r_1, r_2, \dots, r_k$ , since all the  $r_i$  would belong to some  $I_j$ , which leads to the contradiction  $I_j = I_{j+1}$ .  $\square$

Obvious examples of noetherian rings include the ring of integers and any field. Much more interesting examples are provided by (6.4.3) below, which is probably the most celebrated result in the theory of noetherian rings.

**(6.4.3)** (Hilbert's<sup>5</sup> Basis Theorem) *Let  $R$  be a commutative noetherian ring with identity. Then the polynomial ring  $R[t_1, t_2, \dots, t_n]$  is also noetherian.*

*Proof.* In the first place is enough to prove the theorem for  $n = 1$ . For assume that this case has been dealt with and that  $n > 1$ . Now  $R[t_1, t_2, \dots, t_n] = S[t_n]$  where  $S = R[t_1, t_2, \dots, t_{n-1}]$  and  $S$  is noetherian by induction on  $n$ . Therefore the result is true by the case  $n = 1$ . From now on we will work with the ring  $T = R[t]$ .

By (6.4.2) it suffices to prove that an arbitrary ideal  $J$  of  $T$  is finitely generated as an ideal. Suppose that  $J$  is not finitely generated; then  $J \neq 0$  and there is a polynomial  $f_1 \in J - 0$  of smallest degree  $d_1$ . Since  $J$  is not finitely generated,  $J \neq J_1 = (f_1)$  and  $J - J_1$  contains a polynomial  $f_2$  of smallest degree  $d_2$ . Furthermore  $J \neq J_2 = (f_1) + (f_2)$  and  $J - J_2$  contains a polynomial  $f_3$  of smallest degree  $d_3$ , and so on. This gives rise to infinite sequences of ideals  $J_1 \subset J_2 \subset \dots$  where  $J_i = (f_1) + (f_2) + \dots + (f_i)$ , and non-zero polynomials  $f_1, f_2, \dots$  with  $\deg(f_i) = d_i$  and  $d_1 \leq d_2 \leq \dots$ ; moreover  $f_{i+1} \notin J_i$ . Let us write  $f_i = a_i t^{d_i} +$  terms of lower degree, where  $0 \neq a_i \in R$ .

Set  $I_i = (a_1) + (a_2) + \dots + (a_i)$ , so that  $I_1 \subseteq I_2 \subseteq \dots$  is an ascending sequence of ideals of  $R$ . This sequence must have finite length since  $R$  is noetherian, so  $I_m = I_{m+1}$  for some integer  $m$ . Hence  $a_{m+1} \in I_m$  and consequently there is an expression  $a_{m+1} = r_1 a_1 + r_2 a_2 + \dots + r_m a_m$  with  $r_i \in R$ . Now define a new polynomial  $g \in R[t]$  by

$$g = f_{m+1} - \sum_{i=1}^m (r_i f_i) t^{d_{m+1}-d_i}.$$

Thus  $g \in J_{m+1}$ . Observe that  $g \notin J_m$  since  $f_{m+1} \notin J_m$ . The highest power of  $t$  that could occur in  $g$  is certainly  $t^{d_{m+1}}$ , but by inspection we see that its coefficient is

$$a_{m+1} - r_1 a_1 - r_2 a_2 - \dots - r_m a_m = 0.$$

Therefore  $\deg(g) < d_{m+1} = \deg(f_{m+1})$ , which is contrary to the choice of  $f_{m+1}$  as a polynomial of smallest degree in  $J - J_m$ . This contradiction establishes the theorem.  $\square$

**Corollary (6.4.4)** *The rings  $\mathbb{Z}[t_1, t_2, \dots, t_n]$  and  $F[t_1, t_2, \dots, t_n]$  are noetherian, where  $F$  is any field.*

Using this result we can find a large class of noetherian rings.

**(6.4.5)** *Every finitely generated commutative ring with identity is noetherian.*

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<sup>5</sup> David Hilbert (1862–1943)



*Proof.* Let  $R$  be the ring in question and suppose that it has generators  $x_1, x_2, \dots, x_n$ . By (6.2.3) every element of  $R$  has the form

$$\sum_{\ell_i \geq 0} m_{\ell_1, \ell_2, \dots, \ell_n} x_1^{\ell_1} x_2^{\ell_2} \cdots x_n^{\ell_n}$$

where  $m_{\ell_1, \ell_2, \dots, \ell_n} \in \mathbb{Z}$  and the sum is over all non-negative integers  $\ell_1, \ell_2, \dots, \ell_n$ . Let  $S = \mathbb{Z}[t_1, t_2, \dots, t_n]$  and define a map  $\theta : S \rightarrow R$  by

$$\theta\left(\sum_{\ell_i \geq 0} m_{\ell_1, \ell_2, \dots, \ell_n} t_1^{\ell_1} t_2^{\ell_2} \cdots t_n^{\ell_n}\right) = \sum_{\ell_i \geq 0} m_{\ell_1, \ell_2, \dots, \ell_n} x_1^{\ell_1} x_2^{\ell_2} \cdots x_n^{\ell_n}.$$

Then  $\theta$  is a ring homomorphism since sums and products of elements in  $R$  and in  $S$  are formed by the same rules, and clearly  $\theta$  is also surjective. Hence  $S/\text{Ker}(\theta) \simeq R$  by (6.2.7). By (6.4.4) the ring  $S$  is noetherian and thus every quotient of  $S$  is also noetherian, which establishes the result.  $\square$

### Exercises (6.4)

- (1) Prove that every non-zero commutative noetherian ring has at least one maximal (proper) ideal.
- (2) If  $R$  is a non-zero commutative noetherian ring, prove that  $R$  has a quotient ring which is a field.
- (3) Let  $R$  be a commutative noetherian ring and  $I$  an ideal of  $R$ . Prove that  $R/I$  is also noetherian.
- (4) Let  $R$  be the ring of all rational numbers of the form  $\frac{m}{2^n}$  where  $m, n \in \mathbb{Z}$ . Show that  $R$  is a noetherian ring.
- (5) Prove that the ring  $\mathbb{Z}[t_1, t_2, \dots]$  of polynomials in *infinitely* many indeterminates  $t_i$  cannot be noetherian by finding an ideal which is not finitely generated.
- (6) Prove that if  $R$  is a commutative noetherian ring with identity, the ring of formal power series  $R[[t]]$  is noetherian: see Exercise (6.1.8). [Hint: follow the proof of Hilbert's Basis Theorem].
- (7) Prove that if  $R$  is a commutative ring with identity which can be generated by  $n$  elements, then  $R \simeq \mathbb{Z}[t_1, t_2, \dots, t_n]/(f_1, f_2, \dots, f_k)$  for certain polynomials  $f_i$ . Conclude that  $R$  is determined up to isomorphism by finitely many polynomials in  $t_1, t_2, \dots, t_n$ .

## 7 Division in commutative rings

The aim of this chapter is to construct a theory of division in rings that mirrors, as closely as possible, the familiar theory of division in the ring of integers. To simplify matters let us agree to restrict attention to commutative rings – in non-commutative rings questions of left and right divisibility arise. Also, remembering from (6.3) the phenomenon of zero divisors, we will further restrict ourselves to integral domains. In fact even this class of rings is too wide, although it provides a reasonable target for our theory. For this reason we will introduce some well-behaved types of domains.

### 7.1 Euclidean domains

Let  $R$  be a commutative ring with identity and let  $a, b \in R$ . Then  $a$  is said to *divide*  $b$ , in symbols

$$a \mid b,$$

if  $ac = b$  for some  $c \in R$ . From the definition there quickly follow some elementary facts about division.

**(7.1.1)** *Let  $R$  be a commutative ring with identity and let  $a, b, c, x, y$  be elements of  $R$ . Then:*

- (i)  $a \mid a$  and  $a \mid 0$  for all  $a \in R$ ;
- (ii)  $0 \mid a$  if and only if  $a = 0$ ;
- (iii) if  $a \mid b$  and  $b \mid c$ , then  $a \mid c$ , so division is a transitive relation;
- (iv) if  $a \mid b$  and  $a \mid c$ , then  $a \mid bx + cy$  for all  $x, y \in R$ ;
- (v) if  $u$  is a unit,  $u \mid a$  for all  $a \in R$ , while  $a \mid u$  if and only if  $a$  is a unit.

For example, taking the case of (iv), we have  $b = ad$  and  $c = ae$  for some  $d, e \in R$ . Then  $bx + cy = a(dx + ey)$ , so that  $a$  divides  $bx + cy$ . The other proofs are equally simple exercises which are left to the reader.

One situation we expect to encounter in a ring is a pair of elements each of which divides the other: such elements are called *associates*.

**(7.1.2)** *Let  $R$  be an integral domain and let  $a, b \in R$ . Then  $a \mid b$  and  $b \mid a$  if and only if  $b = au$  where  $u$  is a unit of  $R$ .*

*Proof.* Let  $u$  be a unit; then  $a \mid au$ . Also  $(au)u^{-1} = a$ , so  $au \mid a$ . Conversely, assume that  $a \mid b$  and  $b \mid a$ . If  $a = 0$ , then  $b = 0$  and the statement is certainly true, so let  $a \neq 0$ . Now  $a = bc$  and  $b = ad$  for some  $c, d \in R$ . Therefore  $a = bc = adc$  and by (6.3.1) we obtain  $dc = 1$ , so that  $d$  is a unit.  $\square$

For example, two integers  $a$  and  $b$  are associates if and only if  $b = \pm a$ .

**Irreducible elements.** Let  $R$  be a commutative ring with identity. An element  $a$  of  $R$  is called *irreducible* if it is neither 0 nor a unit and if its only divisors are units and associates of  $a$ , i.e., the elements that we know must divide  $a$ . Thus irreducible elements have as few divisors as possible.

**Example (7.1.1)**

- (i) The irreducible elements of  $\mathbb{Z}$  are the prime numbers and their negatives.
- (ii) A field has no irreducible elements since every non-zero element is a unit.
- (iii) If  $F$  is a field, the irreducible elements of the polynomial ring  $F[t]$  are the so-called *irreducible polynomials*, i.e., the non-constant polynomials which are not expressible as a product of polynomials of lower degree.

Almost every significant property of division in  $\mathbb{Z}$  depends ultimately on the Division Algorithm. Thus it is natural to focus on rings in which some version of this property is valid. This motivates us to introduce a special class of domains, the so-called Euclidean domains.

A domain  $R$  is called *Euclidean* if there is a function

$$\delta : R - \{0_R\} \rightarrow \mathbb{N}$$

with the following properties:

- (i)  $\delta(a) \leq \delta(ab)$  if  $0 \neq a, b \in R$ ;
- (ii) if  $a, b \in R$  and  $b \neq 0$ , there exist  $q, r \in R$  such that  $a = bq + r$  where either  $r = 0$  or  $\delta(r) < \delta(b)$ .

The standard example of a Euclidean domain is  $\mathbb{Z}$  where  $\delta$  is the absolute value function, i.e.,  $\delta(a) = |a|$ . Note that property (i) holds since  $|ab| = |a| \cdot |b| \geq |a|$  if  $b \neq 0$ . Of course (ii) is the usual statement of the Division Algorithm for  $\mathbb{Z}$ .

New and important examples of Euclidean domains are given by the next result.

**(7.1.3)** *If  $F$  is a field, the polynomial ring  $F[t]$  is a Euclidean domain with associated function  $\delta$  given by  $\delta(f) = \deg(f)$ .*

*Proof.* We already know from (6.3.2) that  $R = F[t]$  is a domain. Also, by the same result, if  $f, g \neq 0$ , then  $\delta(fg) = \deg(fg) = \deg(f) + \deg(g) \geq \deg(f) = \delta(f)$ . Hence property (i) is valid. To establish the validity of (ii), put

$$S = \{f - gq \mid q \in R\}.$$

If  $0 \in S$ , then  $f = gq$  for some  $q \in R$  and we may take  $r$  to be 0. Assuming that  $0 \notin S$ , we note that every element of  $S$  has degree  $\geq 0$ , so by the Well-Ordering Principle there is an element  $r$  in  $S$  with smallest degree, say  $r = f - gq$  where  $q \in R$ . Thus  $f = gq + r$ .

Suppose that  $\deg(r) \geq \deg(g)$ . Write  $g = at^m + \cdots$  and  $r = bt^n + \cdots$  where  $m = \deg(g)$ ,  $n = \deg(r)$ ,  $0 \neq a, b \in F$  and the dots represent terms of lower degree in  $t$ . Since

$m \leq n$ , we can form the polynomial

$$s = r - (a^{-1}bt^{n-m})g \in R.$$

Now the term in  $t^n$  cancels in  $s$ , so either  $s = 0$  or  $\deg(s) < n$ . But  $s = f - (q + a^{-1}bt^{n-m})g \in S$  and hence  $s \neq 0$ , so  $\deg(s) < n$ , which contradicts the minimality of  $n = \deg(r)$ . Therefore  $\deg(r) < \deg(g)$ , as required.  $\square$

A less familiar example of a Euclidean domain is the ring of Gaussian integers. A *Gaussian integer* is a complex number of the form

$$u + iv$$

where  $u, v \in \mathbb{Z}$  and of course  $i = \sqrt{-1}$ . It is easily seen that the Gaussian integers form a subring of  $\mathbb{C}$  containing 1 and hence constitute a domain.

**(7.1.4)** *The ring  $R$  of Gaussian integers is a Euclidean domain.*

*Proof.* In this case an associated function  $\delta : R - \{0\} \rightarrow \mathbb{N}$  is defined by the rule

$$\delta(u + iv) = |u + iv|^2 = u^2 + v^2.$$

We must show that  $\delta$  satisfies the two requirements for a Euclidean domain. In the first place, if  $0 \neq a, b \in R$ , then  $\delta(ab) = |ab|^2 = |a|^2|b|^2 \geq |a|^2$  since  $|b| \geq 1$ .

Verification of the second requirement is harder. First write  $ab^{-1} = u' + iv'$  where  $u', v'$  are rational numbers. Now choose integers  $u$  and  $v$  that are as close as possible to  $u'$  and  $v'$  respectively; thus  $|u - u'| \leq \frac{1}{2}$  and  $|v - v'| \leq \frac{1}{2}$ . Next

$$a = b(u' + iv') = b(u + iv) + b(u'' + iv'')$$

where  $u'' = u' - u$  and  $v'' = v' - v$ . Finally, let  $q = u + iv$  and  $r = b(u'' + iv'')$ . Then  $a = bq + r$ ; also  $q \in R$  and hence  $r = a - bq \in R$ . If  $r \neq 0$ , then, since  $|u''| \leq \frac{1}{2}$  and  $|v''| \leq \frac{1}{2}$ ,

$$\delta(r) = |b|^2|u'' + iv''|^2 = |b|^2(u''^2 + v''^2) \leq |b|^2\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2}|b|^2,$$

so that  $\delta(r) < |b|^2 = \delta(b)$ . Therefore  $\delta(r) < \delta(b)$  as required.  $\square$

### Exercises (7.1)

- (1) Complete the proof of (7.1.1).
- (2) Identify the irreducible elements in the following rings:
  - (i) the ring of rational numbers with odd denominators;
  - (ii)  $\mathbb{Z}[t]$ .

- (3) Let  $R$  be a commutative ring with identity. If  $R$  has no irreducible elements, show that either  $R$  is a field or there exists an infinite strictly increasing chain of principal ideals  $I_1 \subset I_2 \subset \cdots$  in  $R$ . Deduce that if  $R$  is noetherian, it is a field.
- (4) Let  $R = F[[t]]$  be the ring of formal power series in  $t$  over a field  $F$ , (see Exercise (6.1.8)). Prove that the irreducible elements of  $R$  are those of the form  $tf$  where  $f \in R$  and  $f(0) \neq 0$ .
- (5) Let  $f = t^5 - 3t^2 + t + 1$  and  $g = t^2 + t + 1$  be polynomials in  $\mathbb{Q}[t]$ . Find  $q, r \in \mathbb{Q}[t]$  such that  $f = gq + r$  and  $\deg(r) \leq 1$ .
- (6) Let  $R$  be a Euclidean domain with associated function  $\delta : R - \{0\} \rightarrow \mathbb{N}$ .
- (i) Show that  $\delta(a) \geq \delta(1)$  for all  $a \neq 0$  in  $R$ .
  - (ii) If  $a$  is a unit of  $R$ , prove that  $\delta(a) = \delta(1)$ .
  - (iii) Conversely, show that if  $\delta(a) = \delta(1)$ , then  $a$  is a unit of  $R$ .
- (7) Prove that  $t^3 + t + 1$  is irreducible in  $\mathbb{Z}_2[t]$ , but  $t^3 + t^2 + t + 1$  is reducible.

## 7.2 Principal ideal domains

Let  $R$  be a commutative ring with identity. If  $r \in R$ , recall from (6.2) that the subset  $Rr = \{rx \mid x \in R\} = (r)$  is an ideal of  $R$  containing  $r$  called a principal ideal. If every ideal of  $R$  is principal, then  $R$  is a *principal ideal ring*. A domain in which every ideal is principal is called a *principal ideal domain* or PID: these rings form an extremely important class of domains. For example,  $\mathbb{Z}$  is a PID; for an ideal of  $\mathbb{Z}$  is a cyclic subgroup and thus has the form  $\mathbb{Z}n$  where  $n \geq 0$ .

A good source of PID's is indicated by the next result.

**(7.2.1)** *Every Euclidean domain is a principal ideal domain.*

*Proof.* Let  $R$  be a Euclidean domain with associated function  $\delta : R - 0 \rightarrow \mathbb{N}$  and let  $I$  be an ideal of  $R$ ; we need to show that  $I$  is principal. If  $I$  is the zero ideal,  $I = (0)$  and  $I$  is principal. So we assume that  $I \neq 0$  and apply the Well-Ordering Law to pick an  $x$  in  $I - 0$  such that  $\delta(x)$  is minimal. Now certainly  $(x) \subseteq I$ ; the claim is that  $I \subseteq (x)$ . To substantiate this, let  $y \in I$  and write  $y = xq + r$  with  $q, r \in R$  where either  $r = 0$  or  $\delta(r) < \delta(x)$ . This is possible since  $\delta$  is the associated function for the Euclidean domain  $R$ . If  $r = 0$ , then  $y = xq \in (x)$ . Otherwise  $\delta(r) < \delta(x)$ ; but this is impossible since  $r = y - xq \in I$ , which contradicts the choice of  $x$  in  $I - 0$ . Therefore  $I = (x)$ .  $\square$

The following important result is a consequence of (7.1.3) and (7.2.1).

**Corollary (7.2.2)** *If  $F$  is a field, then  $F[t]$  is a principal ideal domain.*

Another example of a PID is the ring of Gaussian integers by (7.2.1) and (7.1.4). Our next objective is to show that PID's have good division properties, despite the lack of a division algorithm.

**Greatest common divisors.** Let  $a, b$  be elements in a domain  $R$ . A *greatest common divisor* (or gcd) of  $a$  and  $b$  is a ring element  $d$  such that the following hold:

- (i)  $d \mid a$  and  $d \mid b$ ;
- (ii) if  $c \mid a$  and  $c \mid b$  for some  $c \in R$ , then  $c \mid d$ .

The definition here has been carried over directly from the integers – see (2.2).

Notice that if  $d$  and  $d'$  are two gcd's of  $a, b$ , then  $d \mid d'$  and  $d' \mid d$ , so that  $d$  and  $d'$  are associate. Thus by (7.1.2)  $d' = du$  with  $u$  a unit of  $R$ . It follows that gcd's are unique only up to a unit. Of course in the case of  $\mathbb{Z}$ , where the units are  $\pm 1$ , we were able to make gcd's unique by insisting that they be positive. This course of action is not possible in arbitrary domains since there is no concept of positivity.

There is no reason why gcd's should exist in a domain. However, the situation is very satisfactory for PID's.

**(7.2.3)** *Let  $a$  and  $b$  be elements of a principal ideal domain  $R$ . Then  $a$  and  $b$  have a greatest common divisor  $d$  which has the form  $d = ax + by$  with  $x, y \in R$ .*

*Proof.* Define  $I = \{ax + by \mid x, y \in R\}$  and observe that  $I$  is an ideal of  $R$ . Hence  $I = (d)$  for some  $d \in I$ , with  $d = ax + by$  say. If  $c \mid a$  and  $c \mid b$ , then  $c \mid ax + by = d$  by (7.1.1). Also  $a \in I = (d)$ , so  $d \mid a$ , and similarly  $d \mid b$ . Hence  $d$  is a gcd of  $a$  and  $b$ .  $\square$

Elements  $a$  and  $b$  of a domain  $R$  are said to be *relatively prime* if 1 is a gcd of  $a$  and  $b$ , which means that  $ax + by = 1$  for some  $x, y \in R$ .

**(7.2.4)** (Euclid's Lemma) *Let  $a, b, c$  be elements of a principal ideal domain and assume that  $a \mid bc$  where  $a$  and  $b$  are relatively prime. Then  $a \mid c$ .*

**Corollary (7.2.5)** *If  $R$  is a principal ideal domain and  $p \mid bc$  where  $p, b, c \in R$  and  $p$  is irreducible, then  $p \mid b$  or  $p \mid c$ .*

The proofs of these results are exactly the same as those given in (2.2) for  $\mathbb{Z}$ .

**Maximal ideals in principal ideal domains.** In a PID the maximal ideals and the prime ideals coincide and admit a nice description in terms of irreducible elements.

**(7.2.6)** *Let  $I$  be a non-zero ideal of a principal ideal domain  $R$ . Then the following statements about  $I$  are equivalent:*

- (i)  $I$  is maximal;
- (ii)  $I$  is prime;
- (iii)  $I = (p)$  where  $p$  is an irreducible element of  $R$ .

*Proof.* (i)  $\Rightarrow$  (ii). This was proved in (6.3.8).

(ii)  $\Rightarrow$  (iii). Assume that  $I$  is prime. Since  $R$  is a PID, we have  $I = (p)$  for some  $p \in R$ . Note that  $p$  cannot be a unit since  $I \neq R$ . Suppose that  $p = ab$  where neither  $a$  nor  $b$  is

associate to  $p$ . Then  $ab \in I$  and  $I$  is prime, so  $a \in I$  or  $b \in I$ , i.e.,  $p \mid a$  or  $p \mid b$ . Since we also have  $a \mid p$  and  $b \mid p$ , we obtain the contradiction that  $a$  or  $b$  is associate to  $p$ . This shows that  $p$  is irreducible.

(iii)  $\Rightarrow$  (i). Assume that  $I = (p)$  with  $p$  irreducible, and let  $I \subseteq J \subseteq R$  where  $J$  is an ideal of  $R$ . Then  $J = (x)$  for some  $x \in R$ , and  $p \in (x)$ , so that  $x \mid p$ . Hence either  $x$  is a unit or it is associate to  $p$ , so that  $J = R$  or  $J = I$ . Therefore  $I$  is maximal as claimed.  $\square$

**Corollary (7.2.7)** *Let  $F$  be a field. Then the maximal ideals of the polynomial ring  $F[t]$  are exactly those of the form  $(f)$  where  $f$  is an irreducible polynomial which is monic, (i.e., its leading coefficient is 1).*

This is because  $F[t]$  is a PID by (7.2.2) and the irreducible elements of  $F[t]$  are just the irreducible polynomials. The corollary provides us with an important method for constructing a field from an irreducible polynomial  $f \in F[t]$ : indeed  $F[t]/(f)$  is a field. This will be exploited in (7.4) below.

We conclude the section by noting a property of PID's which will be crucial when we address the issue of unique factorization in (7.4).

**(7.2.8)** *Every principal ideal domain is noetherian.*

*Proof.* Let  $R$  be a PID. By definition every ideal of  $R$  is principal and hence can be generated by a single element. Therefore  $R$  is noetherian by (6.4.2).  $\square$

## Exercises (7.2)

- (1) Prove (7.2.4) and (7.2.5).
- (2) Show that  $\mathbb{Z}[t]$  is not a PID.
- (3) Show that  $F[t_1, t_2]$  is not a PID for any field  $F$ .
- (4) Let  $R$  be a commutative ring with identity. If  $R[t]$  is a PID, prove that  $R$  must be a field.
- (5) Let  $f = t^3 + t + 1 \in \mathbb{Z}_2[t]$ . Show that  $\mathbb{Z}_2[t]/(f)$  is finite field and find its order.
- (6) Prove that the ring of rational numbers with odd denominators is a PID.
- (7) Prove that  $F[[t]]$ , the ring of formal power series in  $t$  over a field  $F$ , is a PID by describing its ideals.
- (8) Let  $R$  be a commutative noetherian ring with identity. Assume that  $R$  has the property that each pair of elements  $a, b$  has a greatest common divisor which is a linear combination of  $a$  and  $b$ . Prove that  $R$  is a PID. [Hint: let  $I$  be an ideal of  $R$ . Note that  $I$  is a finitely generated ideal and reduce to the case where it is generated by two elements].
- (9) Prove that the Chinese Remainder Theorem holds in a Euclidean domain, (cf. (2.3.7)).
- (10) Describe the Euclidean algorithm for a Euclidean domain.

### 7.3 Unique factorization in integral domains

The present section is concerned with domains in which there is unique factorization in terms of irreducible elements. Our model here is the Fundamental Theorem of Arithmetic (2.2.7), which asserts that such factorizations exist in  $\mathbb{Z}$ . First it is necessary to clarify what is meant by uniqueness of factorization.

Let  $R$  be a domain and let  $S$  denote the set of all irreducible elements in  $R$ , which might of course be empty. Observe that “being associate to” is an equivalence relation on  $S$ , so that  $S$  splits up into equivalence classes. Choosing one element from each equivalence class, we form a subset  $C$  of  $S$ . (Strictly speaking this procedure involves the Axiom of Choice – see (14.1)). Now observe that the set  $C$  has the following properties:

- (i) every irreducible element of  $R$  is associate to some element of  $C$ ;
- (ii) distinct elements of  $C$  are not associate.

A subset  $C$  with these properties is called a *complete set of irreducibles* for  $R$ . We have just established the following simple fact.

**(7.3.1)** *Every integral domain has a (possibly empty) complete set of irreducible elements.*

Our interest in complete sets of irreducibles stems from the observation that if there is to be *unique* factorization in terms of irreducibles, then only irreducibles from a complete set can be used: otherwise there will be different factorizations of the type  $ab = (ua)(u^{-1}b)$  where  $a, b$  are irreducible and  $u$  is a unit.

An integral domain  $R$  is called a *unique factorization domain*, or UFD, if there exists a complete set of irreducibles  $C$  for  $R$  such that each non-zero element  $a$  of  $R$  has an expression of the form

$$a = up_1p_2 \cdots p_k$$

where  $u$  is a unit and  $p_i \in C$ , and furthermore this expression is unique up to order of the factors.

At present the only example of a UFD we know is  $\mathbb{Z}$ , where  $C$  can be taken to be the set of prime numbers. The next theorem provides us with many more examples.

**(7.3.2)** *Every principal ideal domain is a unique factorization domain.*

*Proof.* Let  $R$  be a PID and let  $C$  be any complete set of irreducibles of  $R$ . It will be shown that there is unique factorization for elements of  $R$  in terms of units and elements of  $C$ . This is accomplished in three steps, the first of which establishes the existence of irreducibles when  $R$  contains a non-zero, non-unit element, i.e.,  $R$  is not a field.

(i) *If  $a$  is a non-zero, non-unit element of  $R$ , it is divisible by at least one irreducible element of  $R$ .*

Suppose this is false. Then  $a$  itself must be reducible, so  $a = a_1a'_1$  where  $a_1$  and  $a'_1$  are non-units and  $(a) \subseteq (a_1)$ . Also  $(a) \neq (a_1)$ . For otherwise  $a_1 \in (a)$ , so that  $a \mid a_1$ , as well as  $a_1 \mid a$ ; by (7.1.2) this implies that  $a'_1$  is a unit. Therefore  $(a) \subset (a_1)$ .



Next  $a_1$  cannot be irreducible since  $a_1 \mid a$ . Thus  $a_1 = a_2 a'_2$  where  $a_2, a'_2$  are non-units and it follows that  $(a_1) \subset (a_2)$  by the argument just given. Continuing in this way, we recognize that the procedure cannot terminate: for otherwise an irreducible divisor of  $a$  will appear. Hence there is an infinite strictly ascending chain of ideals  $(a) \subset (a_1) \subset (a_2) \subset \cdots$ ; but this is impossible since  $R$  is noetherian by (7.2.8).

(ii) *If  $a$  is a non-zero, non-unit element of  $R$ , then  $a$  is a product of irreducibles.*

Again suppose this is false. By (i) there is an irreducible  $p_1$  dividing  $a$ , with  $a = p_1 a_1$  say. Now  $a_1$  cannot be a unit, so there is an irreducible  $p_2$  dividing  $a_1$ , with say  $a_1 = p_2 a_2$  and  $a = p_1 p_2 a_2$ , and so on indefinitely. However,  $(a) \subset (a_1) \subset (a_2) \subset \cdots$  is a strictly ascending infinite chain of ideals, which again contradicts (7.2.8).

(iii) *If  $a$  is a non-zero element of  $R$ , then  $a$  is the product of a unit and irreducible elements in  $C$ .*

This is clear if  $a$  is a unit – no irreducibles are needed. Otherwise by (ii)  $a$  is a product of irreducibles, each of which is associate to an element of  $C$ . The result now follows on replacing each irreducible factor of  $a$  by an irreducible in  $C$  multiplied by a unit.

(iv) The final step in the proof establishes uniqueness. Suppose that

$$a = up_1 p_2 \cdots p_k = vq_1 q_2 \cdots q_\ell$$

where  $u, v$  are units of  $R$  and  $p_i, q_j \in C$ . Argue by induction on  $k$ : if  $k = 0$ , then  $a = u$ , a unit, so  $\ell = 0$  and  $u = v$ . Now assume that  $k > 0$ .

Since  $p_1 \mid a = vq_1 q_2 \cdots q_\ell$ , Euclid's Lemma shows that  $p_1$  must divide one of  $q_1, \dots, q_\ell$ . Relabelling the  $q_j$ 's, we may assume that  $p_1 \mid q_1$ . Thus  $p_1$  and  $q_1$  are associate members of  $C$ , which can only mean that  $p_1 = q_1$ . Hence, on cancelling  $p_1$ , we obtain  $a' = up_2 \cdots p_k = vq_2 \cdots q_\ell$ . By the induction hypothesis  $k - 1 = \ell - 1$ , so  $k = \ell$  and, after further relabelling,  $p_i = q_i$  for  $i = 2, 3, \dots, k$ , and  $u = v$ . Therefore uniqueness has been established.  $\square$

**Corollary (7.3.3)** *If  $F$  is a field, the polynomial ring  $F[t]$  is a unique factorization domain.*

This is because  $F[t]$  is a PID by (7.2.2). The natural choice for a complete set of irreducibles in  $F[t]$  is the set of all monic irreducible polynomials. Thus we have unique factorization in  $F[t]$  in terms of constants and monic irreducible polynomials. Another example of a UFD is the ring of Gaussian integers  $\{a + b\sqrt{-1} \mid a, b \in \mathbb{Z}\}$ , which by (7.1.4) is a Euclidean domain and hence a PID. However, some domains of similar appearance are not UFD's.

**Example (7.3.1)** Let  $R$  be the subring of  $\mathbb{C}$  consisting of all  $a + b\sqrt{-3}$  where  $a, b \in \mathbb{Z}$ . Then  $R$  is not a unique factorization domain.

First observe that  $\pm 1$  are the only units of  $R$ . For, let  $0 \neq r = a + b\sqrt{-3} \in R$ . Then

$$r^{-1} = \frac{1}{a^2 + 3b^2} (a - b\sqrt{-3}),$$

which is in  $R$  if and only if  $\frac{a}{a^2+3b^2}$  and  $\frac{b}{a^2+3b^2}$  are integers. This happens only when  $b = 0$  and  $\frac{1}{a} \in \mathbb{Z}$ , i.e.,  $r = a = \pm 1$ . It follows that no two of the elements  $2, 1 + \sqrt{-3}, 1 - \sqrt{-3}$  are associate.

Next we claim that  $2, 1 + \sqrt{-3}, 1 - \sqrt{-3}$  are irreducible elements of  $R$ . Fortunately all three elements can be handled simultaneously. Suppose that

$$(a + \sqrt{-3}b)(c + \sqrt{-3}d) = 1 \pm \sqrt{-3} \text{ or } 2$$

where  $a, b, c, d \in \mathbb{Z}$ . Taking the modulus squared of both sides, we obtain  $(a^2 + 3b^2)(c^2 + 3d^2) = 4$  in every case. But this implies that  $a^2 = 1$  and  $b = 0$  or  $c^2 = 1$  and  $d = 0$ , i.e., either  $a + \sqrt{-3}b$  or  $c + \sqrt{-3}d$  is a unit.

Finally, unique factorization fails because

$$4 = 2 \cdot 2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$$

and  $2, 1 + \sqrt{-3}, 1 - \sqrt{-3}$  are non-associate irreducibles. It follows that  $R$  is not a UFD.

Two useful properties of UFD's are recorded in the next result.

**(7.3.4)** *Let  $R$  be a unique factorization domain. Then:*

- (i) *gcd's exist in  $R$ ;*
- (ii) *Euclid's Lemma holds in  $R$ .*

*Proof.* To prove (i) let  $a = up_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$  and  $b = vp_1^{f_1}p_2^{f_2}\cdots p_k^{f_k}$  where  $u, v$  are units of  $R$ , the  $p_i$  belong to a complete set of irreducibles for  $R$ , and  $e_i, f_i \geq 0$ . Define  $d = p_1^{g_1}p_2^{g_2}\cdots p_k^{g_k}$  where  $g_i$  is the minimum of  $e_i$  and  $f_i$ . Then  $d$  is a gcd of  $a$  and  $b$ . For clearly  $d \mid a$  and  $d \mid b$ , and, on the other hand, if  $c \mid a$  and  $c \mid b$ , the unique factorization property shows that  $c$  must have the form  $wp_1^{h_1}p_2^{h_2}\cdots p_k^{h_k}$  where  $w$  is a unit and  $0 \leq h_i \leq g_i$ . Hence  $c \mid d$ . The proof of (ii) is left to the reader as an exercise.  $\square$

Although polynomial rings in more than one variable over a field are not PID's – see Exercise (7.2.3) – they are in fact UFD's. It is our aim in the remainder of the section to prove this important result.

**Primitive polynomials.** Let  $R$  be a UFD and let  $0 \neq f \in R[t]$ . Since gcd's exist in  $R$  by (7.3.4), we can form the gcd of the coefficients of  $f$ ; this is called the *content* of  $f$ ,

$$c(f).$$

Keep in mind that content is unique only up to a unit of  $R$ , and equations involving content have to be interpreted in this light. If  $c(f) = 1$ , i.e.,  $c(f)$  is a unit, the polynomial  $f$  is said to be *primitive*. For example  $2 + 4t - 3t^3 \in \mathbb{Z}[t]$  is a primitive polynomial. Next two useful results about the content of polynomials will be established.

**(7.3.5)** *Let  $0 \neq f \in R[t]$  where  $R$  is a unique factorization domain. Then  $f = cf_0$  where  $c = c(f)$  and  $f_0 \in R[t]$  is primitive.*

*Proof.* Write  $f = a_0 + a_1t + \cdots + a_nt^n$ ; then  $c(f) = \gcd\{a_0, a_1, \dots, a_n\} = c$ , say. Write  $a_i = cb_i$  with  $b_i \in R$  and put  $f_0 = b_0 + b_1t + \cdots + b_nt^n \in R[t]$ . Thus  $f = cf_0$ . If  $d = \gcd\{b_0, b_1, \dots, b_n\}$ , then  $d \mid b_i$  and so  $cd \mid cb_i = a_i$ . Since  $c$  is the gcd of the  $a_i$ , it follows that  $cd$  divides  $c$ , which shows that  $d$  is a unit and  $f_0$  is primitive.  $\square$

**(7.3.6)** Let  $R$  be a unique factorization domain and let  $f, g$  be non-zero polynomials over  $R$ . Then  $c(fg) = c(f)c(g)$ . In particular, if  $f$  and  $g$  are primitive, then so is  $fg$ .

*Proof.* Consider first the special case where  $f$  and  $g$  are primitive. If  $fg$  is not primitive,  $c(fg)$  is not a unit, so it must be divisible by an irreducible element  $p$  of  $R$ . Write  $f = \sum_{i=0}^m a_it^i$  and  $g = \sum_{j=0}^n b_jt^j$ , so that

$$fg = \sum_{k=0}^{m+n} c_k t^k$$

where  $c_k = \sum_{i=0}^k a_ib_{k-i}$ . (Here  $a_i = 0$  if  $i > m$  and  $b_j = 0$  if  $j > n$ ). Since  $f$  is primitive,  $p$  cannot divide all its coefficients and there is an integer  $r \geq 0$  such that  $p \mid a_0, a_1, \dots, a_{r-1}$ , but  $p \nmid a_r$ . Similarly there is an  $s \geq 0$  such that  $p$  divides each of  $b_0, b_1, \dots, b_{s-1}$ , but not  $b_s$ . Now consider  $c_{r+s}$ , which can be written

$$(a_0b_{r+s} + a_1b_{r+s-1} + \cdots + a_{r-1}b_{s+1}) + a_rb_s + (a_{r+1}b_{s-1} + \cdots + a_{r+s}b_0).$$

We know that  $p \mid c_{r+s}$ ; also  $p$  divides both the expressions in parentheses in the expression above. It follows that  $p \mid a_rb_s$ . By Euclid's Lemma for UFD's (see (7.3.4)), it follows that  $p \mid a_r$  or  $p \mid b_s$ , both of which are impossible. By this contradiction  $fg$  is primitive.

Now we are ready for the general case. Using (7.3.5), we write  $f = cf_0$  and  $g = dg_0$  where  $c = c(f)$ ,  $d = c(g)$  and the polynomials  $f_0, g_0$  are primitive in  $R[t]$ . Then  $fg = cd(f_0g_0)$  and, as has just been shown,  $f_0g_0$  is primitive. In consequence  $c(fg) = cd = c(f)c(g)$ .  $\square$

The next result is frequently helpful in deciding whether a polynomial is irreducible.

**(7.3.7)** (Gauss's Lemma) Let  $R$  be a unique factorization domain and let  $F$  denote its field of fractions. If  $f \in R[t]$ , then  $f$  is irreducible over  $R$  if and only if it is irreducible over  $F$ .

*Proof.* We can assume that  $R \subseteq F$ . Of course irreducibility over  $F$  implies irreducibility over  $R$ . It is the converse implication that requires proof. Assume that  $f$  is irreducible over  $R$  but reducible over  $F$ . We can assume that  $f$  is primitive on the basis of (7.3.5). Then  $f = gh$  where  $g, h \in F[t]$  are non-constant. Since  $F$  is the field of fractions of  $R$ , there exist elements  $a, b \neq 0$  in  $R$  such that  $g_1 = ag \in R[t]$  and  $h_1 = bh \in R[t]$ . Write  $g_1 = c(g_1)g_2$  where  $g_2 \in R[t]$  is primitive. Then  $ag = c(g_1)g_2$ , so we can divide both

sides by  $\gcd\{a, c(g_1)\}$ . On these grounds it is permissible to assume that  $c(g_1)$  and  $a$  are relatively prime, and for similar reasons the same can be assumed of  $c(h_1)$  and  $b$ .

Next  $(ab)f = (ag)(bh) = g_1h_1$ . Taking the content of each side and using (7.3.6), we obtain  $ab = c(g_1)c(h_1)$  since  $f$  is primitive. But  $c(g_1)$  and  $a$  are relatively prime, so  $a \mid c(h_1)$ , and for a similar reason  $b \mid c(g_1)$ . Therefore we have the factorization  $f = (b^{-1}g_1)(a^{-1}h_1)$  in which both factors are polynomials over  $R$ . But this contradicts the irreducibility of  $f$  over  $R$  and the proof is complete.  $\square$

For example, to show that a polynomial in  $\mathbb{Z}[t]$  is  $\mathbb{Q}$ -irreducible, it is enough to show that it is  $\mathbb{Z}$ -irreducible, usually an easier task.

**Polynomial rings in several variables.** Let us now use the theory of content to show that unique factorization occurs in polynomial rings with more than one variable. Here one should keep in mind that such rings are not PID's and so are not covered by (7.3.2). The main result is:

**(7.3.8)** *If  $R$  is a unique factorization domain, then so is the polynomial ring  $R[t_1, \dots, t_k]$ .*

*Proof.* In the first place we need only prove the theorem for  $k = 1$ . Indeed if  $k > 1$ , we have

$$R[t_1, \dots, t_k] = (R[t_1, \dots, t_{k-1}])[t_k],$$

so that induction on  $k$  will succeed once the case  $k = 1$  is settled. From now on consider the ring  $S = R[t]$ . The first step in the proof is to establish:

(i) *Any non-constant polynomial  $f$  in  $S$  is expressible as a product of irreducible elements of  $R$  and primitive irreducible polynomials over  $R$ .*

The key idea in the proof is to introduce the field of fractions  $F$  of  $R$ , and exploit the fact that  $F[t]$  is known to be a PID and hence a UFD. First of all write  $f = c(f)f_0$  where  $f_0 \in S$  is primitive, using (7.3.5). Here  $c(f)$  is either a unit or a product of irreducibles of  $R$ . Thus we can assume that  $f$  is primitive. Regarding  $f$  as an element of the UFD  $F[t]$ , we write  $f = p_1p_2 \cdots p_m$  where  $p_i \in F[t]$  is irreducible over  $F$ . Now find  $a_i \neq 0$  in  $R$  such that  $f_i = a_ip_i \in S$ . Writing  $c(f_i) = c_i$ , we have  $f_i = c_iq_i$  where  $q_i \in R[t]$  is primitive. Hence  $p_i = a_i^{-1}f_i = a_i^{-1}c_iq_i$  and  $q_i$  is  $F$ -irreducible since  $p_i$  is  $F$ -irreducible. Thus  $q_i$  is certainly  $R$ -irreducible.

Combining these expressions for  $p_i$ , we find that

$$f = (a_1^{-1}a_2^{-1} \cdots a_m^{-1}c_1c_2 \cdots c_m)q_1q_2 \cdots q_m,$$

and hence  $(a_1a_2 \cdots a_m)f = (c_1c_2 \cdots c_m)q_1q_2 \cdots q_m$ . Now take the content of both sides of this equation to get  $a_1a_2 \cdots a_m = c_1c_2 \cdots c_m$  up to a unit, since  $f$  and the  $q_i$  are primitive. Consequently  $f = uq_1q_2 \cdots q_m$  for some unit  $u$  of  $R$ . This is what was to be proved.

(ii) The next step is to assemble a complete set of irreducibles for  $S$ . First take a complete set of irreducibles  $C_1$  for  $R$ . Then consider the set of all primitive irreducible

polynomials in  $S$ . Now being associate is an equivalence relation on this set, so we can choose an element from each equivalence class. This yields a set of non-associate primitive irreducible polynomials  $C_2$  with the property that every primitive irreducible polynomial in  $R[t]$  is associate to an element of  $C_2$ . Now put

$$C = C_1 \cup C_2.$$

Since distinct elements of  $C$  cannot be associate,  $C$  is a complete set of irreducibles for  $S$ . If  $0 \neq f \in S$ , it follows from step (i) that  $f$  is expressible as a product of elements of  $C$  and a unit of  $R$ .

(iii) There remains the question of uniqueness. Suppose that

$$f = ua_1a_2 \cdots a_k f_1 f_2 \cdots f_r = vb_1b_2 \cdots b_\ell g_1 g_2 \cdots g_s$$

where  $u, v$  are units,  $a_k, b_\ell \in C_1$  and  $f_i, g_j \in C_2$ . By Gauss's Lemma (7.3.7) the  $f_i$  and  $g_j$  are  $F$ -irreducible. Since  $F[t]$  is a UFD and  $C_2$  is a complete set of irreducibles for  $F[t]$ , we conclude that  $r = s$  and  $f_i = w_i g_i$ , (after possible relabelling), where  $w_i \in F$ . Write  $w_i = c_i d_i^{-1}$  where  $c_i, d_i \in R$ . Then  $d_i f_i = c_i g_i$ , which, on taking contents, yields  $d_i = c_i$  up to a unit. This implies that  $w_i$  is a unit of  $R$ . Therefore  $f_i$  and  $g_i$  are associate and thus  $f_i = g_i$ .

Cancelling the  $f_i$  and  $g_i$ , we are left with  $ua_1a_2 \cdots a_k = vb_1b_2 \cdots b_\ell$ . Since  $R$  is a UFD with a complete set of irreducibles  $C_1$ , it follows that  $k = \ell$ ,  $u = v$  and  $a_i = b_i$  (after further relabelling). This completes the proof.  $\square$

This theorem provides us with some important new examples of UFD's.

**Corollary (7.3.9)** *The following rings are unique factorization domains:*

$$\mathbb{Z}[t_1, \dots, t_k] \text{ and } F[t_1, \dots, t_k]$$

where  $F$  is any field.

### Exercises (7.3)

- (1) Prove that a UFD satisfies the ascending chain condition on *principal* ideals, i.e., there does not exist an infinite strictly ascending chain of principal ideals.
- (2) If  $R$  is a UFD and  $C$  is *any* complete set of irreducible elements for  $R$ , show that there is unique factorization in terms of  $C$ .
- (3) If  $C_1$  and  $C_2$  are two complete sets of irreducibles for a domain  $R$ , prove that  $|C_1| = |C_2|$ .
- (4) Show that the domain  $\{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$  is not a UFD.
- (5) Prove that  $t^3 + at + 1 \in \mathbb{Z}[t]$  is reducible over  $\mathbb{Q}$  if and only if  $a = 0$  or  $-2$ .
- (6) Explain why the ring of rational numbers with odd denominators is a UFD and find a complete set of irreducibles for it.
- (7) The same question for the power series ring  $F[[t]]$  where  $F$  is a field.
- (8) Prove that Euclid's Lemma is valid in any UFD.

## 7.4 Roots of polynomials and splitting fields

Let  $R$  be a commutative ring with identity, let  $f = b_0 + b_1t + \cdots + b_nt^n \in R[t]$  and let  $a \in R$ . Then the *value* of the polynomial  $f$  at  $a$  is defined to be

$$f(a) = b_0 + b_1a + \cdots + b_na^n \in R.$$

Thus we have a function  $\theta_a : R[t] \rightarrow R$  which evaluates polynomials at  $a$ , i.e.,  $\theta_a(f) = f(a)$ . Now  $f + g(a) = f(a) + g(a)$  and  $(fg)(a) = f(a)g(a)$ , because the ring elements  $f(a)$  and  $g(a)$  are added and multiplied by the same rules as the polynomials  $f$  and  $g$ . It follows that  $\theta_a : R[t] \rightarrow R$  is a ring homomorphism. Its kernel consists of all  $f \in R[t]$  such that  $f(a) = 0$ , that is, all polynomials that have  $a$  as a *root*.

The following criterion for an element to be a root of a polynomial should be familiar from elementary algebra.

**(7.4.1)** (The Remainder Theorem) *Let  $R$  be an integral domain, let  $f \in R[t]$  and let  $a \in R$ . Then  $a$  is a root of  $f$  if and only if  $t - a$  divides  $f$  in the ring  $R[t]$ .*

*Proof.* If  $t - a$  divides  $f$ , then  $f = (t - a)g$  where  $g \in R[t]$ . Then  $f(a) = \theta_a(f) = \theta_a((t - a)g) = \theta_a(t - a)\theta_a(g) = 0$ . Hence  $a$  is a root of  $f$ .

Conversely, assume that  $f(a) = 0$  and let  $F$  denote the field of fractions of  $R$ . Since  $F[t]$  is a Euclidean domain, we can divide  $f$  by  $t - a$  to get a quotient and remainder in  $F[t]$ , say  $f = (t - a)q + r$  where  $q, r \in F[t]$  and  $\deg(r) < 1$ , i.e.,  $r$  is constant. However, notice that by the usual long division process  $q$  and  $r$  actually belong to  $R[t]$ . Finally, apply the evaluation homomorphism  $\theta_a$  to  $f = (t - a)q + r$  to obtain  $0 = r$  since  $r$  is constant. Therefore  $t - a$  divides  $f$ .  $\square$

**Corollary (7.4.2)** *The kernel of the evaluation homomorphism  $\theta_a$  is the principal ideal  $(t - a)$ .*

This is simply a restatement of (7.4.1).

**The multiplicity of a root.** Let  $R$  be a domain and suppose that  $f \in R[t]$  is not constant and has a root  $a$  in  $R$ ; thus  $t - a \mid f$ . There is a largest positive integer  $n$  such that  $(t - a)^n \mid f$ , since the degree of a divisor of  $f$  cannot exceed  $\deg(f)$ . In this situation  $a$  is said to be a *root of  $f$  with multiplicity  $n$* . If  $n > 1$ , then  $a$  is called a *multiple root* of  $f$ .

**(7.4.3)** *Let  $R$  be a domain and let  $0 \neq f \in R[t]$  have degree  $n$ . Then the sum of the multiplicities of all the roots of  $f$  that lie in  $R$  is at most  $n$ .*

*Proof.* Let  $a$  be a root of  $f$ . By (7.4.2)  $t - a$  divides  $f$  and  $f = (t - a)g$  where  $g \in R[t]$  has degree  $n - 1$ . By induction on  $n$  the sum of the multiplicities of the roots of  $g$  is at most  $n - 1$ . Now a root of  $f$  either equals  $a$  or else is a root of  $g$ . Consequently the sum of the multiplicities of the roots of  $f$  is at most  $1 + (n - 1) = n$ .  $\square$

**Example (7.4.1)**

- (i) The polynomial  $t^2 + 1 \in \mathbb{Q}[t]$  has no roots in  $\mathbb{Q}$ , so the sum of the multiplicities of the roots of a polynomial can be less than the degree.
- (ii) Consider the polynomial  $t^4 - 1 \in R[t]$  where  $R$  is the ring of rational quaternions (see (6.3)). Then  $f$  has 8 roots in  $R$ , namely  $\pm 1, \pm I, \pm J, \pm K$ . Therefore (7.4.3) is not valid for non-commutative rings, which is another reason to keep our rings commutative.

Next comes another well-known theorem.

**(7.4.4)** (The Fundamental Theorem of Algebra) *Let  $f$  be a non-zero polynomial of degree  $n$  over the field of complex numbers  $\mathbb{C}$ . Then the sum of the multiplicities of the roots of  $f$  in  $\mathbb{C}$  equals  $n$ , i.e.,  $f$  is a product of  $n$  linear factors over  $\mathbb{C}$ .*

The proof of this theorem will be postponed until Chapter Twelve – see (12.3.6). Despite its name, all the known proofs of the theorem employ some analysis.

**Derivatives.** Derivatives are useful in detecting multiple roots of polynomials. Since we are not dealing with polynomials over  $\mathbb{R}$  here, limits cannot be used. For this reason we adopt the following formal definition of the *derivative*  $f'$  of the polynomial  $f \in R[t]$  where  $R$  is a commutative ring with identity. If  $f = a_0 + a_1t + \cdots + a_nt^n$ , then

$$f' = a_1 + 2a_2t + \cdots + na_nt^{n-1} \in R[t].$$

On the basis of this definition the usual rules of differentiation can be established.

**(7.4.5)** *Let  $f, g \in R[t]$  and  $c \in R$  where  $R$  is a commutative ring with identity. Then*

- (i)  $(f + g)' = f' + g'$ ;  
(ii)  $(cf)' = cf'$ ;  
(iii)  $(fg)' = f'g + fg'$ .

*Proof.* Only the statement (iii) will be proved. Write  $f = \sum_{i=0}^m a_i t^i$  and  $g = \sum_{j=0}^n b_j t^j$ ; then

$$fg = \sum_{i=0}^{m+n} \left( \sum_{k=0}^i a_k b_{i-k} \right) t^i.$$

The coefficient of  $t^{i-1}$  in  $(fg)'$  is therefore equal to  $i \left( \sum_{k=0}^i a_k b_{i-k} \right)$ .

On the other hand, the coefficient of  $t^{i-1}$  in  $f'g + fg'$  is

$$\sum_{k=0}^{i-1} (k+1) a_{k+1} b_{i-k-1} + \sum_{k=0}^{i-1} (i-k) a_k b_{i-k},$$

which equals

$$ia_i b_0 + \sum_{k=0}^{i-2} (k+1) a_{k+1} b_{i-k-1} + ia_0 b_i + \sum_{k=1}^{i-1} (i-k) a_k b_{i-k}.$$

On adjusting the summation in the second sum, this becomes

$$ia_ib_0 + \sum_{k=0}^{i-2} (k+1)a_{k+1}b_{i-k-1} + \sum_{k=0}^{i-2} (i-k-1)a_{k+1}b_{i-k-1} + ia_0b_i,$$

which reduces to

$$i\left(a_0b_i + \sum_{k=0}^{i-2} a_{k+1}b_{i-k-1} + a_ib_0\right) = i\left(\sum_{k=0}^i a_kb_{i-k}\right).$$

It follows that  $(fg)' = f'g + fg'$ . □

**Corollary (7.4.6)**  $(f^m)' = mf^{m-1}f'$  where  $m$  is a positive integer.

This is proved by induction on  $m$  using (7.4.5). A criterion for a polynomial to have multiple roots can now be given.

**(7.4.7)** Let  $R$  be a domain and let  $a \in R$  be a root of a polynomial  $f \in R[t]$ . Then  $a$  is a multiple root if and only if  $t - a$  divides both  $f$  and  $f'$ .

*Proof.* Let  $\ell$  be the multiplicity of the root  $a$ . Then  $\ell \geq 1$  and  $f = (t - a)^\ell g$  where  $t - a \nmid g \in R[t]$ . Hence  $f' = \ell(t - a)^{\ell-1}g + (t - a)^\ell g'$  by (7.4.5) and (7.4.6). If  $a$  is a multiple root of  $f$ , then  $\ell \geq 2$  and  $f'(a) = 0$ ; by (7.4.1)  $t - a$  divides  $f'$ , as well as  $f$ .

Conversely, suppose that  $t - a \mid f' = \ell(t - a)^{\ell-1}g + (t - a)^\ell g'$ . If  $\ell = 1$ , then  $t - a$  divides  $g$ , a contradiction. Therefore  $\ell > 1$  and  $a$  is a multiple root. □

**Example (7.4.2)** Let  $F$  be a field whose characteristic does not divide the positive integer  $n$ . Then  $t^n - 1 \in F[t]$  has no multiple roots in  $F$ .

For, with  $f = t^n - 1$ , we have  $f' = nt^{n-1} \neq 0$  since  $\text{char}(F)$  does not divide  $n$ . Hence  $t^n - 1$  and  $nt^{n-1}$  are relatively prime and thus  $f$  and  $f'$  have no common roots. Therefore  $f$  has no multiple roots by (7.4.7).

**Splitting fields.** We will now consider roots of polynomials over a field  $F$ . If  $f \in F[t]$  is not constant, we know that  $f$  has at most  $\deg(f)$  roots in  $F$ , including multiplicities, by (7.4.3). However,  $f$  need not have any roots in  $F$ , as the example  $t^2 + 1 \in \mathbb{R}[t]$  shows. On the other hand,  $t^2 + 1$  has two roots in the larger field  $\mathbb{C}$ .

The question to be addressed is this: can we construct a field  $K$ , larger than  $F$  in some sense, in which  $f$  has exactly  $\deg(f)$  roots up to multiplicity, i.e., over which  $f$  splits into a product of linear factors? A smallest such field is called a *splitting field* of  $f$ . In the case of the polynomial  $t^2 + 1 \in \mathbb{R}[t]$ , the situation is quite clear; its splitting field is  $\mathbb{C}$  since  $t^2 + 1 = (t + i)(t - i)$  where  $i = \sqrt{-1}$ . However, for a general field  $F$  we do not have a convenient larger field like  $\mathbb{C}$  at hand. Thus splitting fields will have to be constructed from scratch.



We begin by formulating precisely the definition of a splitting field. If  $F$  is a field, by a *subfield* of  $F$  is meant a subring containing the identity element which is closed under forming inverses of non-zero elements. Let  $f$  be a non-constant polynomial over  $F$ . A *splitting field* for  $f$  over  $F$  is a field  $K$  containing an isomorphic copy  $F_1$  of  $F$  as a subfield such that the polynomial in  $F_1[t]$  corresponding to  $f$  can be expressed in the form

$$a(t - c_1)(t - c_2) \cdots (t - c_n)$$

where  $a$  is in  $F_1$  and  $K$  is a smallest field containing  $F_1$  and the elements  $c_1, c_2, \dots, c_n$ . There is nothing to be lost in assuming that  $F \subseteq K$  since  $F$  can be replaced by the isomorphic field  $F_1$ . Thus  $F$  is a subfield of  $K$ .

Our first objective is to demonstrate that splitting fields actually exist.

**(7.4.8)** *If  $f$  is a non-constant polynomial over a field  $F$ , then  $f$  has a splitting field over  $F$ .*

*Proof.* We argue by induction on  $n = \deg(f)$ ; note that we may assume  $n > 1$  since otherwise  $F$  itself is a splitting field for  $f$ . Assume the result is true for all polynomials of degree less than  $n$ . Consider first the case where  $f$  is reducible, so  $f = gh$  where  $g, h$  in  $F[t]$  both have degree less than  $n$ . By induction hypothesis  $g$  has a splitting field over  $F$ , say  $K_1$ , which we may suppose contains  $F$  as a subfield. For the same reason  $h$  has a splitting field over  $K_1$ , say  $K$ , with  $K_1 \subseteq K$ . Clearly  $f$  is a product of linear factors over  $K$ . Hence  $K$  is a splitting field of  $f$ .

Therefore we can assume  $f$  is irreducible. By (7.2.6) the ideal  $(f)$  is maximal in  $F[t]$  and consequently the quotient ring

$$K_1 = F[t]/(f)$$

is a field. Next the assignment  $a \mapsto a + (f)$ , where  $a \in F$ , determines an injective ring homomorphism from  $F$  to  $K_1$ . The image is a subfield  $F_1$  of  $K_1$  and  $F \simeq F_1$ . Thus we may regard  $f$  as a polynomial over  $F_1$ .

The critical observation to make is that  $f$  has a root in  $K_1$ , namely  $a_1 = t + (f)$ ; for  $f(a_1) = f(t) + (f) = (f) = 0_{K_1}$ . By (7.4.1)  $f = (t - a_1)g$  where  $g \in K_1[t]$ , and of course  $\deg(g) = n - 1$ . By induction hypothesis  $g$  has a splitting field  $K$  containing  $K_1$ . Since  $a_1 \in K_1 \subseteq K$ , we see that  $K$  is a splitting field for  $f$ : for any subfield of  $K$  containing  $F$  and the roots of  $f$  must contain  $K_1$  since each element of  $K_1$  has the form  $h + (f) = h(a_1)$  for some  $h \in F[t]$ . This completes the proof.  $\square$

**Example (7.4.3)** Let  $f = t^3 - 2 \in \mathbb{Q}[t]$ . The roots of  $f$  are  $2^{1/3}$ ,  $c2^{1/3}$ ,  $c^2 2^{1/3}$  where  $c = e^{2\pi i/3}$ , a complex cube root of unity. Then  $f$  has as its splitting field the smallest subfield of  $\mathbb{C}$  containing  $\mathbb{Q}$ ,  $2^{1/3}$  and  $c$ .

The next example shows how finite fields can be constructed from irreducible polynomials.

**Example (7.4.4)** Show that  $f = t^3 + 2t + 1 \in \mathbb{Z}_3[t]$  is irreducible and use it to construct a field of order 27. Prove that this is a splitting field of  $f$ .

First of all notice that the only way a cubic polynomial can be reducible is if it has a linear factor, i.e., it has a root in the field. But we easily verify that  $f$  has no roots in  $\mathbb{Z}_3 = \{0, 1, 2\}$  since  $f(0) = f(1) = f(2) = 1$ . (For conciseness we have written  $i$  for the congruence class  $[i]$ ). It follows that  $f$  is irreducible and

$$K = \mathbb{Z}_3[t]/(f)$$

is a field.

If  $g \in \mathbb{Z}_3[t]$ , then by the Division Algorithm  $g = fq + r$  where  $q, r \in \mathbb{Z}_3[t]$  and  $r = 0$  or  $0 \leq \deg r < 3$ . Hence  $g + (f) = r + (f)$ . This shows that every element of  $K$  has the form  $a_0 + a_1t + a_2t^2 + (f)$  where  $a_i \in \mathbb{Z}_3$ . Thus  $|K| \leq 3^3 = 27$ . On the other hand, all such elements are distinct. Indeed, if  $r + (f) = s + (f)$  with  $r$  and  $s$  both of degree  $< 3$ , then  $f \mid r - s$ , so that  $r = s$ . Therefore  $|K| = 27$  and we have constructed a field of order 27.

As in the proof of (7.4.8), we see that  $f$  has the root  $a = t + (f)$  in  $K$ . To prove that  $K$  is actually a splitting field, note that  $f$  has two further roots in  $K$ , namely  $a + 1$  and  $a - 1$ . Thus  $f = (t - a)(t - a - 1)(t - a + 1)$ .

Further discussion of fields is postponed until Chapter Eleven. However, we have seen enough to realize that irreducible polynomials play a vital role in the theory of fields. Thus a practical criterion for irreducibility is sure to be useful. Probably the best known test for irreducibility is:

**(7.4.9) (Eisenstein's<sup>1</sup> Criterion)** Let  $R$  be a unique factorization domain and let  $f = a_0 + a_1t + \cdots + a_nt^n$  be a non-constant polynomial over  $R$ . Suppose that there is an irreducible element  $p$  of  $R$  such that  $p \mid a_0, p \mid a_1, \dots, p \mid a_{n-1}$ , but  $p \nmid a_n$  and  $p^2 \nmid a_0$ . Then  $f$  is irreducible over  $R$ .

*Proof.* Assume that  $f$  is reducible and

$$f = (b_0 + b_1t + \cdots + b_rt^r)(c_0 + c_1t + \cdots + c_st^s)$$

where  $b_i, c_j \in R$ ,  $r, s < n$ , and  $r + s = n$ . By hypothesis  $p \mid a_0 = b_0c_0$ , but  $p^2 \nmid a_0$ ; thus  $p$  must divide exactly one of  $b_0$  and  $c_0$ , say  $p \mid b_0$  and  $p \nmid c_0$ . Also  $p$  does not divide  $a_n = b_rc_s$ , so it cannot divide  $b_r$ . Therefore there is a smallest positive integer  $k \leq r$  such that  $p \nmid b_k$ . Now  $p$  divides each of  $b_0, b_1, \dots, b_{k-1}$ , and also  $p \mid a_k$  because  $k \leq r < n$ . Since  $a_k = (b_0c_k + b_1c_{k-1} + \cdots + b_{k-1}c_1) + b_kc_0$ , (where  $c_i = 0$  if  $i > s$ ), it follows that  $p \mid b_kc_0$ . By Euclid's Lemma – which by (7.3.4) is valid in a UFD – either  $p \mid b_k$  or  $p \mid c_0$ , both of which are forbidden.  $\square$

<sup>1</sup> Ferdinand Gotthold Max Eisenstein (1823–1852)

Eisenstein's Criterion is often applied in conjunction with Gauss's Lemma (7.3.7) to give a test for irreducibility over the field of fractions of a domain.

**Example (7.4.5)** Prove that  $t^5 - 9t + 3$  is irreducible over  $\mathbb{Q}$ .

First of all  $f = t^5 - 9t + 3$  is irreducible over  $\mathbb{Z}$  by Eisenstein's Criterion with  $p = 3$ . Then Gauss's Lemma shows that  $f$  is irreducible over  $\mathbb{Q}$ .

**Example (7.4.6)** Show that if  $p$  is a prime, the polynomial  $f = 1 + t + t^2 + \cdots + t^{p-1}$  is irreducible over  $\mathbb{Q}$ .

By Gauss's Lemma it suffices to prove that  $f$  is  $\mathbb{Z}$ -irreducible. Since (7.4.9) is not immediately applicable to  $f$ , we resort to a trick. Consider the polynomial  $g = f(t+1)$ ; then

$$g = 1 + (t+1) + \cdots + (t+1)^{p-1} = \frac{(t+1)^p - 1}{t},$$

by the formula for the sum of a geometric series. On expanding  $(t+1)^p$  by the Binomial Theorem – see Exercise (6.1.6) – we arrive at the formula

$$g = t^{p-1} + \binom{p}{p-1}t^{p-2} + \cdots + \binom{p}{2}t + \binom{p}{1}.$$

Now  $p \mid \binom{p}{i}$  if  $0 < i < p$  by (2.3.3). Therefore  $g$  is irreducible over  $\mathbb{Z}$  by Eisenstein's Criterion. Clearly this implies that  $f$  is irreducible over  $\mathbb{Z}$ . (The polynomial  $f$  is called the *cyclotomic polynomial of order  $p$* ).

### Exercises (7.4)

- (1) Let  $f \in F[t]$  have degree  $\leq 3$  where  $F$  is a field. Show that  $f$  is reducible over  $F$  if and only if it has a root in  $F$ .
- (2) Find the multiplicity of the root 2 of the polynomial  $t^3 + 2t^2 + t + 2 \in \mathbb{Z}_5[t]$ .
- (3) List all irreducible polynomials of degree at most 3 in  $\mathbb{Z}_2[t]$ .
- (4) Use  $t^3 + t + 1 \in \mathbb{Z}_5[t]$  to construct a field of order 125.
- (5) Let  $f = 1 + t + t^2 + t^3 + t^4 \in \mathbb{Q}[t]$ .
  - (i) Prove that  $K = \mathbb{Q}[t]/(f)$  is a field.
  - (ii) Show that every element of  $K$  can be uniquely written in the form  $a_0 + a_1x + a_2x^2 + a_3x^3$  where  $x = t + (f)$  and  $a_i \in \mathbb{Q}$ .
  - (iii) Prove that  $K$  is a splitting field of  $f$ . [Hint: note that  $x^5 = 1$  and check that  $x^2, x^3, x^4$  are roots of  $f$ ].
  - (iv) Compute  $(1 + x^2)^3$  and  $(1 + x)^{-1}$  in  $K$ .
- (6) Show that  $t^6 + 6t^5 + 4t^4 + 2t + 2$  is irreducible over  $\mathbb{Q}$ .
- (7) Show that  $t^6 + 12t^5 + 49t^4 + 96t^3 + 99t^2 + 54t + 15$  is irreducible over  $\mathbb{Q}$ . [Hint: use a suitable change of variable].
- (8) Let  $F = \mathbb{Z}_p\{t_1\}$ , the field of rational functions, and  $R = F[t]$  where  $t$  and  $t_1$  are distinct indeterminates. Prove that  $t^n - t_1^2t + t_1 \in R$  is irreducible over  $F$  for all  $n \geq 1$ .

- (9) Find a polynomial of degree 4 in  $\mathbb{Z}[t]$  which has  $\sqrt{3} - \sqrt{2}$  as a root and is irreducible over  $\mathbb{Q}$ .
- (10) Prove that if  $n$  is a positive integer that is not a prime, then  $1 + t + t^2 + \cdots + t^{n-1}$  is reducible over any field.
- (11) Show that  $\mathbb{Q}[t]$  contains an irreducible polynomial of every degree  $n \geq 1$ .
- (12) Let  $R$  be a commutative ring with identity containing a zero divisor. Find a linear polynomial in  $R[t]$  which has at least two roots in  $R$ , so that (7.4.3) fails for  $R$ .

## 8 Vector spaces

We have already encountered groups and rings, two of the most commonly used algebraic structures. A third structure of great importance is a vector space. Vector spaces appear throughout mathematics and they also turn up in many applied areas, for example, in quantum theory and coding theory.

### 8.1 Vector spaces and subspaces

Let  $F$  be a field. A *vector space over  $F$*  is an additively written abelian group  $V$  with an action of  $F$  on  $V$  called *scalar multiplication*, that is, a function from  $F \times V$  to  $V$  written  $(a, v) \mapsto av$ , ( $a \in F$ ,  $v \in V$ ), such that the following axioms hold for all  $u, v \in V$  and  $a, b \in F$ .

- (i)  $a(u + v) = au + av$ ;
- (ii)  $(a + b)v = av + bv$ ;
- (iii)  $(ab)v = a(bv)$ ;
- (iv)  $1_F v = v$ .

Notice that (iii) and (iv) assert that the multiplicative group of  $F$  acts on the set  $V$  in the sense of (5.1). Elements of  $V$  are called *vectors* and elements of  $F$  *scalars*. When there is no chance of confusion, it is usual to refer to the set  $V$  as the vector space.

First of all we record two elementary consequences of the axioms.

**(8.1.1)** Let  $v$  be a vector in a vector space  $V$  over a field  $F$  and let  $a \in F$ . Then:

- (i)  $0_F v = 0_V$  and  $a 0_V = 0_V$ ;
- (ii)  $(-1_F)v = -v$ .

*Proof.* Put  $a = 0_F = b$  in vector space axiom (ii) to get  $0_F v = 0_F v + 0_F v$ . Hence  $0_F v = 0_V$  by the cancellation law for the group  $(V, +)$ . Similarly, setting  $u = 0_V = v$  in (i) yields  $a 0_V = 0_V$ . This establishes (i).

Using axioms (ii) and (iv) and property (i), we obtain

$$v + (-1_F)v = 1_F v + (-1_F)v = (1_F + (-1_F))v = 0_F v = 0_V.$$

Therefore  $(-1_F)v$  equals  $-v$ , which completes the proof. □

#### Examples of vector spaces

Before proceeding further we review some standard sources of vector spaces.

(i) *Vector spaces of matrices.* Let  $F$  be a field and define

$$M_{m,n}(F)$$

to be the set of all  $m \times n$  matrices over  $F$ . This is already an abelian group with respect to ordinary matrix addition. There is also a natural scalar multiplication here: if  $A = [a_{ij}] \in M_{m,n}(F)$  and  $f \in F$ , then  $fA$  is the matrix which has  $fa_{ij}$  as its  $(i, j)$  entry. That the vector space axioms hold is guaranteed by elementary results from matrix algebra.

Two special cases of interest are the vector spaces

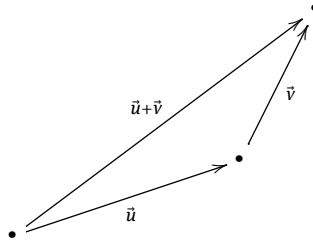
$$F^m = M_{m,1}(F) \quad \text{and} \quad F_n = M_{1,n}(F).$$

Thus  $F^m$  is the vector space of  $m$ -column vectors over  $F$ , while  $F_n$  is the vector space of  $n$ -row vectors over  $F$ .

The space  $\mathbb{R}^n$  is called *Euclidean  $n$ -space*. For  $n \leq 3$  there is a well-known geometric interpretation of  $\mathbb{R}^n$ . Consider for example  $\mathbb{R}^3$ . A vector in  $\mathbb{R}^3$

$$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is represented by a line segment  $\vec{v}$  in 3-dimensional space drawn from an arbitrary initial point  $(p, q, r)$  to the point  $(p + a, q + b, r + c)$ . With this interpretation of vectors, the rule of addition of vectors  $u$  and  $v$  in  $\mathbb{R}^3$  is equivalent to the well-known *triangle rule* for addition of line segments  $\vec{u}$  and  $\vec{v}$ ; this is illustrated in the diagram below.



A detailed account of the geometric interpretations of euclidean 2-space and 3-space may be found in any text on linear algebra – see for example [12].

(ii) *Vector spaces of polynomials*. The set  $F[t]$  of all polynomials in  $t$  over a field  $F$  is a vector space over  $F$  with the usual addition and scalar multiplication of polynomials.

(iii) *Fields as vector spaces*. Suppose that  $F$  is a *subfield* of a field  $K$ , i.e.,  $F$  is a subring of  $K$  containing 1 which is closed with respect to taking inverses of non-zero elements. We can regard  $K$  as a vector space over  $F$ , using the field operations as vector space operations. At first sight this example may seem confusing since elements of  $F$  are simultaneously vectors and scalars. However, this point of view will be particularly valuable when we come to investigate the structure of fields in Chapter Eleven.

**Subspaces.** In analogy with subgroups of groups and subrings of rings, it is natural to introduce the concept of a subspace of a vector space. Let  $V$  be a vector space over a

field  $F$  and let  $S$  be a subset of  $V$ . Then  $S$  is called a *subspace* of  $V$  if, when we restrict the vector space operations of  $V$  to  $S$ , we obtain a vector space over  $F$ . Taking note of the analysis of the subgroup concept in (3.3) – see especially (3.3.4) – we conclude that *a subspace is a subset of  $V$  containing  $0_V$  which is closed with respect to addition and multiplication by scalars.*

Obvious examples of subspaces of  $V$  are  $0 = 0_V$ , the *zero subspace* which contains just the zero vector, and  $V$  itself, the *improper subspace*. A more interesting source of examples is given in:

**Example(8.1.1)** Let  $A$  be an  $m \times n$  matrix over a field  $F$  and define  $S$  to be the subset of all  $X$  in  $F^n$  such that  $AX = 0$ . Then  $S$  is a subspace of  $F^n$ , verification of the closure properties being very easy. The subspace  $S$  is called the *null space* of the matrix  $A$ .

**Linear combinations of vectors.** Suppose that  $V$  is a vector space over a field  $F$  and  $v_1, v_2, \dots, v_k$  are vectors in  $V$ . A *linear combination* of these vectors is a vector of the form

$$a_1v_1 + a_2v_2 + \cdots + a_kv_k$$

where  $a_1, a_2, \dots, a_k \in F$ . If  $X$  is any non-empty set of vectors in  $V$ , we will write either  $F\langle X \rangle$  or, if we do not wish to emphasize the field,  $\langle X \rangle$  for the set of all linear combinations of vectors in the set  $X$ . It is a fundamental fact that this is always a subspace.

**(8.1.2)** Let  $X$  be a non-empty subset of a vector space  $V$  over a field  $F$ . Then  $F\langle X \rangle$  is the smallest subspace of  $V$  that contains  $X$ .

*Proof.* In the first place it is easy to verify that  $F\langle X \rangle$  is closed with respect to addition and scalar multiplication; of course it also contains the zero vector  $0_V$ . Therefore  $F\langle X \rangle$  is a subspace. Also it contains  $X$  since  $x = 1_Fx \in F\langle X \rangle$  for all  $x \in X$ . Finally, any subspace that contains  $X$  automatically contains every linear combination of vectors in  $X$ , i.e., it must contain  $F\langle X \rangle$  as a subset.  $\square$

The subspace  $\langle X \rangle$  is called the subspace *generated* (or *spanned*) by  $X$ . If  $V = \langle X \rangle$ , then  $X$  is said to *generate* the vector space  $V$ . If  $V$  can be generated by some finite set of vectors, we say that  $V$  is a *finitely generated vector space*. What this means is that every vector in  $V$  can be expressed as a linear combination of the vectors in some finite set.

**Example (8.1.2)**  $F^n$  is a finitely generated vector space. To see why, consider the so-called *elementary vectors* in  $F^n$ ,

$$E_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, E_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

A general vector in  $F^n$ ,

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

can be written as  $a_1E_1 + a_2E_2 + \cdots + a_nE_n$ . Hence  $F^n = \langle E_1, E_2, \dots, E_n \rangle$  and  $F^n$  is finitely generated.

On the other hand, infinitely generated, i.e., non-finitely generated, vector spaces are not hard to find.

**Example (8.1.3)** The vector space  $F[t]$  is infinitely generated.

Indeed suppose that  $F[t]$  could be generated by finitely many polynomials  $p_1, p_2, \dots, p_k$  and let  $m$  be the maximum degree of the  $p_i$ . Then clearly  $t^{m+1}$  cannot be expressed as a linear combination of  $p_1, \dots, p_k$ , so a contradiction is reached.

### Exercises (8.1)

(1) Which of the following are vector spaces? The operations of addition and scalar multiplication are the natural ones.

- (i) The set of all real  $2 \times 2$  matrices with determinant 0.
- (ii) The set of all solutions  $y(x)$  of the homogeneous linear differential equation  $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$ , where the  $a_i(x)$  are given real-valued functions of  $x$ .
- (iii) The set of all solutions  $X$  of the matrix equation  $AX = B$ .

(2) In the following cases say whether  $S$  is a subspace of the vector space  $V$ .

- (i)  $V = \mathbb{R}^2$ ,  $S = \text{all } \begin{bmatrix} a^2 \\ a \end{bmatrix}, a \in \mathbb{R}$ ;
- (ii)  $V$  is the vector space of all continuous functions on the interval  $[0, 1]$  and  $S$  consists of all infinitely differentiable functions in  $V$ ;
- (iii)  $V = F[t]$ ,  $S = \{f \in V \mid f(a) = 0\}$  where  $a$  is a fixed element of  $F$ .

(3) Verify that the rule for adding the vectors in  $\mathbb{R}^3$  corresponds to the usual triangle rule for addition of line segments.

(4) Does  $\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$  belong to the subspace of  $M_2(\mathbb{R})$  generated by the matrices  $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 2 \\ -\frac{1}{3} & 4 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$ ?

(5) Let  $V$  be a vector space over a finite field. Prove that  $V$  is finitely generated if and only if it is finite.



## 8.2 Linear independence, basis and dimension

A concept of critical importance in vector space theory is linear independence. For an understanding of this topic some knowledge of systems of linear equations, and in particular row and column operations on matrices, is essential and will be assumed.

Let  $V$  be a vector space over a field  $F$  and let  $X$  be a non-empty subset of  $V$ . Then  $X$  is called *linearly dependent* if there exist distinct vectors  $x_1, x_2, \dots, x_k$  in  $X$  and scalars  $a_1, a_2, \dots, a_k \in F$ , not all the  $a_i$  being zero, such that

$$a_1x_1 + a_2x_2 + \dots + a_kx_k = 0.$$

This amounts to saying that some  $x_i$  can be expressed as a linear combination of the others. For if, say,  $a_i \neq 0$ , we can solve for  $x_i$ , obtaining

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^k (-a_i^{-1})a_jv_j.$$

A subset which is not linearly dependent is called *linearly independent*. For example, the elementary vectors  $E_1, E_2, \dots, E_n$  form a linearly independent subset of  $F^n$  for any field  $F$ .

**Homogeneous linear systems.** To make significant progress with linear independence, some knowledge of systems of linear equations is needed. Let  $F$  be a field and consider a system of  $m$  homogeneous linear equations over  $F$

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = 0 \end{cases}$$

Here  $a_{ij} \in F$  and  $x_1, x_2, \dots, x_n$  are the unknowns

Clearly the system has the *trivial solution*  $x_1 = x_2 = \dots = x_n = 0$ . The interesting question is whether there are any non-trivial solutions. A detailed account of the theory of systems of linear equations can be found in any book on linear algebra, for example [12].

The linear system can be written in the matrix form

$$AX = 0,$$

where  $A = [a_{ij}]_{m,n}$  is the coefficient matrix and  $X$  is the  $n$ -column vector formed by the unknowns  $x_1, x_2, \dots, x_n$ . The following result is sufficient for our present purposes.

**(8.2.1)** *The homogenous linear system  $AX = 0$  has a non-trivial solution  $X$  if and only if the rank of the coefficient matrix  $A$  is less than the number of unknowns.*

*Proof.* Write  $A = [a_{ij}]$ . We adopt the method of systematic elimination known as *Gaussian elimination*. It may be assumed that  $a_{11} \neq 0$ ; for, if this is not true, replace equation 1 by the first equation in which  $x_1$  appears. Since equation 1 can be multiplied by  $a_{11}^{-1}$ , we may also assume that  $a_{11} = 1$ . Then, by subtracting multiples of equation 1 from equations 2 through  $m$ , the unknown  $x_1$  can be eliminated from these equations.

Next find the first of equations 2 through  $m$  which contains an unknown with smallest subscript  $> 1$ , say  $x_{i_2}$ . Move this equation up to second position. Now make the coefficient of  $x_{i_2}$  equal to 1 and subtract multiples of equation 2 from equations 3 through  $m$  so as to eliminate  $x_{i_2}$ . Repeat this procedure until the remaining equations involve no further unknowns, i.e., they are of the trivial form  $0 = 0$ . Let us say this happens after  $r$  steps. At this point the matrix of coefficients is in *row echelon form* with  $r$  linearly independent rows. The integer  $r$  is the *rank* of  $A$ .

Unknowns other than  $x_1 = x_{i_1}, x_{i_2}, \dots, x_{i_r}$  can be given arbitrary values. The non-trivial equations may then be used to solve back for  $x_{i_r}, x_{i_{r-1}}, \dots, x_{i_1}$  successively. Therefore there is a non-trivial solution if and only if  $r < n$ ; for then at least one unknown can be given an arbitrary value.  $\square$

**Corollary (8.2.2)** *A homogeneous linear system  $AX = 0$  of  $n$  equations in  $n$  unknowns has a non-trivial solution if and only if  $\det(A) = 0$ .*

For  $\det(A) = 0$  if and only if the rank of  $A$  is less than  $n$ . This result is used to establish the fundamental theorem on linear dependence in vector spaces.

**(8.2.3)** *Let  $v_1, v_2, \dots, v_k$  be vectors in a vector space  $V$  over a field  $F$ . Then any set of  $k + 1$  or more vectors in the subspace  $\langle v_1, v_2, \dots, v_k \rangle$  is linearly dependent.*

*Proof.* Let  $u_1, u_2, \dots, u_{k+1} \in S = \langle v_1, \dots, v_k \rangle$ . It is enough to show that  $\{u_1, u_2, \dots, u_{k+1}\}$  is a linearly dependent set. This amounts to finding field elements  $a_1, a_2, \dots, a_{k+1}$ , not all of them zero, such that  $a_1 u_1 + a_2 u_2 + \dots + a_{k+1} u_{k+1} = 0$ .

Since  $u_i \in S$ , there is an expression  $u_i = d_{1i} v_1 + d_{2i} v_2 + \dots + d_{ki} v_k$  where  $d_{ji} \in F$ . On substituting for the  $u_i$ , we obtain

$$a_1 u_1 + a_2 u_2 + \dots + a_{k+1} u_{k+1} = \sum_{i=1}^{k+1} a_i \left( \sum_{j=1}^k d_{ji} v_j \right) = \sum_{j=1}^k \left( \sum_{i=1}^{k+1} d_{ji} a_i \right) v_j.$$

Therefore  $a_1 u_1 + a_2 u_2 + \dots + a_{k+1} u_{k+1} = 0$  if the  $a_i$  satisfy the equations

$$\sum_{i=1}^{k+1} d_{ji} a_i = 0, \quad j = 1, \dots, k.$$

But this is a system of  $k$  linear homogeneous equations in the  $k + 1$  unknowns  $a_i$ , so the rank of the coefficient matrix  $[d_{ij}]$  is at most  $k$ . By (8.2.1) there is a non-trivial solution  $a_1, a_2, \dots, a_{k+1}$ . Therefore  $\{u_1, u_2, \dots, u_{k+1}\}$  is linearly dependent, as claimed.  $\square$

**Corollary (8.2.4)** *If a vector space  $V$  can be generated by  $k$  elements, then every subset of  $V$  with  $k + 1$  or more elements is linearly dependent.*

**Bases.** A basis of a vector space  $V$  is a non-empty subset  $X$  such that:

- (i)  $X$  is linearly independent;
- (ii)  $X$  generates  $V$ .

These are contrasting properties in the sense that (i) means that  $X$  is not too large and (ii) that  $X$  is not too small.

For example, the elementary vectors  $E_1, E_2, \dots, E_n$  form a basis of the vector space  $F^n$  called the *standard basis*. More generally a basis of  $M_{m,n}(F)$  is obtained by taking all the  $m \times n$  matrices over  $F$  with a single non-zero entry which is equal to  $1_F$ .

A important property of a basis is unique expressibility.

**(8.2.5)** *If  $\{v_1, v_2, \dots, v_n\}$  is a basis of a vector space  $V$  over a field  $F$ , then every vector  $v$  in  $V$  is uniquely expressible in the form  $v = a_1v_1 + \dots + a_nv_n$  with  $a_i \in F$ .*

*Proof.* In the first place such expressions for  $v$  exist by definition. If  $v$  in  $V$  had two such expressions  $v = \sum_{i=1}^n a_i v_i = \sum_{i=1}^n b_i v_i$ , we would have  $\sum_{i=1}^n (a_i - b_i) v_i = 0$ , from which it follows that  $a_i = b_i$  by linear independence of the  $v_i$ .  $\square$

This result shows that a basis may be used to introduce coordinates in a vector space. Suppose that  $V$  is a vector space over field  $F$  and that  $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$  is a basis of  $V$  with its elements written in a specific order, i.e., an *ordered basis*. Then by (8.2.5) each  $v \in V$  has a unique expression  $v = \sum_{i=1}^n c_i v_i$  with  $c_i \in F$ . Thus  $v$  is determined by the column vector in  $F^n$  whose entries are  $c_1, c_2, \dots, c_n$ ; this is called the *coordinate column vector* of  $v$  with respect to  $\mathcal{B}$  and is written

$$[v]_{\mathcal{B}}.$$

Coordinate vectors provide a concrete representation of vectors in an abstract vector space.

**The existence of bases.** There is nothing in the definition of a basis to make us certain that bases exist. Our first task will be to show that this is true for any finitely generated non-zero vector space. Notice that the zero space does not have a basis since it has no linearly independent subsets.

**(8.2.6)** *Let  $V$  be a finitely generated vector space and suppose that  $X_0$  is a linearly independent subset of  $V$ . Then  $X_0$  is contained in a basis of  $V$ .*

*Proof.* Suppose that  $V$  can be generated by  $m$  vectors. Then by (8.2.4) a linearly independent subset of  $V$  cannot contain more than  $m$  vectors. It follows that  $X_0$  is contained in a largest linearly independent subset  $X$ ; for otherwise it would be possible to form ever larger finite linearly independent subsets containing  $X_0$ .

We complete the proof by showing that  $X$  generates  $V$ . If this is false, there is a vector  $u$  in  $V - F\langle X \rangle$ . Then  $u \notin X$ , so  $X \neq X \cup \{u\}$  and  $X \cup \{u\}$  is linearly dependent by maximality of  $X$ . Writing  $X = \{v_1, \dots, v_k\}$ , we conclude that there is a relation of the type

$$a_1 v_1 + \dots + a_k v_k + bu = 0,$$

where  $a_1, \dots, a_k, b \in F$  and not all of these scalars are 0. Now  $b$  cannot equal 0: for otherwise  $a_1 v_1 + \dots + a_k v_k = 0$  and hence  $a_1 = \dots = a_k = 0$  since the  $v_i$  are known to be linearly independent. Therefore  $b \neq 0$  and  $u = -b^{-1}a_1 v_1 - \dots - b^{-1}a_k v_k \in F\langle X \rangle$ , which is a contradiction.  $\square$

**Corollary (8.2.7)** *Every finitely generated non-zero vector space  $V$  has a basis.*

The reason is that since  $V \neq 0$ , we can choose a non-zero vector  $v$  from  $V$  and apply (8.2.6) with  $X_0 = \{v\}$ . In fact every infinitely generated vector space has a basis, but advanced methods are needed to prove this – see (14.1.1) below.

**Dimension.** A vector space usually has many bases: it is an important fact that all of them have the same number of elements.

**(8.2.8)** *Let  $V$  be a finitely generated non-zero vector space. Then any two bases of  $V$  have the same number of elements.*

*Proof.* In the first place a basis of  $V$  is necessarily finite by (8.2.4). Next let  $\{u_1, \dots, u_m\}$  and  $\{v_1, \dots, v_n\}$  be two bases. Then  $V = \langle v_1, \dots, v_n \rangle$  and by (8.2.4) there cannot be a linearly independent subset of  $V$  with more than  $n$  elements. Therefore  $m \leq n$ . By the same reasoning  $n \leq m$ , so we obtain  $m = n$ , as required.  $\square$

This result enables us to define the *dimension*

$$\dim(V)$$

of a finitely generated vector space  $V$ . If  $V = 0$ , define  $\dim(V)$  to be 0, and if  $V \neq 0$ , let  $\dim(V)$  be the number of elements in a basis of  $V$ . By (8.2.8) this definition is unambiguous. In the future we shall speak of *finite dimensional vector spaces* instead of finitely generated ones.

**(8.2.9)** *Let  $X_1, X_2, \dots, X_k$  be vectors in  $F^n$  where  $F$  is a field. Let  $A = [X_1 X_2 \dots X_k]$  be the  $n \times k$  matrix which has the  $X_i$  as columns. Then  $\dim(\langle X_1, \dots, X_k \rangle) = r$  where  $r$  is the rank of the matrix  $A$ .*

*Proof.* We will use some elementary facts about matrices here. In the first place,  $S = \langle X_1, \dots, X_k \rangle$  is the *column space* of  $A$ , and it is unaffected when column operations are applied to  $A$ . By applying column operations to  $A$ , just as we did for row operations during Gaussian elimination in the proof of (8.2.1), we can replace  $A$  by a matrix with

the same column space  $S$  which has the so-called *column echelon form* with  $r$  non-zero columns. Here  $r$  is the rank of  $A$ . Since the  $r$  columns are linearly independent, they form a basis of  $S$  (if  $r > 0$ ). Hence  $\dim(S) = r$ .  $\square$

Next we consider the relation between the dimension of a vector space and that of a subspace.

**(8.2.10)** *If  $V$  is a vector space with finite dimension  $n$  and  $U$  is a subspace of  $V$ , then  $\dim(U) \leq \dim(V)$ . Furthermore  $\dim(U) = \dim(V)$  if and only if  $U = V$ .*

*Proof.* If  $U = 0$ , then  $\dim(U) = 0 \leq \dim(V)$ . Assume that  $U \neq 0$  and let  $X$  be a basis of  $U$ . By (8.2.6) the subset  $X$  is contained in a basis  $Y$  of  $V$ . Hence  $\dim(U) = |X| \leq |Y| = \dim(V)$ . Finally, suppose that  $\dim(U) = \dim(V)$ , but  $U \neq V$ . Then  $U \neq 0$ . As before, a basis  $X$  of  $U$  is contained in a basis  $Y$  of  $V$ . Since  $|X| = |Y|$ , it follows that  $X = Y$ . Therefore  $U = V$ , a contradiction.  $\square$

The next result can simplify the task of showing that a subset of a finite dimensional vector space is a basis.

**(8.2.11)** *Let  $V$  be a finite dimensional vector space with dimension  $n$  and let  $X$  be a subset of  $V$  with  $n$  elements. Then the following statements about  $X$  are equivalent:*

- (i)  $X$  is a basis of  $V$ ;
- (ii)  $X$  is linearly independent;
- (iii)  $X$  generates  $V$ .

*Proof.* Of course (i) implies (ii). Assume that (ii) holds. Then  $X$  is a basis of  $\langle X \rangle$ , the subspace it generates; hence  $\dim(\langle X \rangle) = n = \dim(V)$  and (8.2.10) shows that  $\langle X \rangle = V$ . Thus (ii) implies (iii).

Finally, assume that (iii) holds. If  $X$  is not a basis of  $V$ , it must be linearly dependent, so one of its elements can be written as a linear combination of the others. Hence  $V$  can be generated by fewer than  $n$  elements, which is a contradiction by (8.2.4).  $\square$

**Change of basis.** As has been previously remarked, vector spaces usually have many bases and a vector may be represented with respect to each basis by a coordinate column vector. A natural question is: how are these coordinate vectors related?

Let  $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$  and  $\mathcal{B}' = \{v'_1, v'_2, \dots, v'_n\}$  be two ordered bases of a finite dimensional vector space  $V$  over a field  $F$ . Then each  $v'_i$  can be expressed as a linear combination of  $v_1, v_2, \dots, v_n$ , say

$$v'_i = \sum_{j=1}^n s_{ji} v_j,$$

where  $s_{ji} \in F$ . The change of basis  $\mathcal{B}' \rightarrow \mathcal{B}$  is described by the *transition matrix*  $S = [s_{ji}]$ . Observe that  $S$  is  $n \times n$  and its  $i$ th column is the coordinate vector  $[v'_i]_{\mathcal{B}}$ .

To understand how  $S$  determines the change of basis  $\mathcal{B}' \rightarrow \mathcal{B}$ , choose an arbitrary vector  $v$  from  $V$  and write  $v = \sum_{i=1}^n c'_i v'_i$  where  $c'_1, c'_2, \dots, c'_n$  are the entries of the coordinate vector  $[v]_{\mathcal{B}'}$ . Replace  $v'_i$  by  $\sum_{j=1}^n s_{ji} v_j$  to get

$$v = \sum_{i=1}^n c'_i \left( \sum_{j=1}^n s_{ji} v_j \right) = \sum_{j=1}^n \left( \sum_{i=1}^n s_{ji} c'_i \right) v_j.$$

Therefore the entries of the coordinate vector  $[v]_{\mathcal{B}}$  are  $\sum_{i=1}^n s_{ji} c'_i$  for  $j = 1, 2, \dots, n$ . This shows that

$$[v]_{\mathcal{B}} = S[v]_{\mathcal{B}'},$$

i.e., left multiplication by the transition matrix  $S$  transforms coordinate vectors with respect to  $\mathcal{B}'$  into those with respect to  $\mathcal{B}$ .

Notice that the transition matrix  $S$  must be non-singular. For otherwise, by standard matrix theory there would exist a non-zero  $X \in F^n$  such that  $SX = 0$ ; however, if  $u \in V$  is defined by  $[u]_{\mathcal{B}'} = X$ , then  $[u]_{\mathcal{B}} = SX = 0$ , which can only mean that  $u = 0$  and  $X = 0$ . From  $[v]_{\mathcal{B}} = S[v]_{\mathcal{B}'}$  we deduce that  $S^{-1}[v]_{\mathcal{B}} = [v]_{\mathcal{B}'}$ . Thus  $S^{-1}$  is the transition matrix for the change of basis  $\mathcal{B} \rightarrow \mathcal{B}'$ . These conclusions are summed up in the next result.

**(8.2.12)** Let  $\mathcal{B}$  and  $\mathcal{B}'$  be ordered bases of an  $n$ -dimensional vector space  $V$ . Define  $S$  to be the  $n \times n$  matrix whose  $i$ th column is the coordinate vector of the  $i$ th vector of  $\mathcal{B}'$  with respect to  $\mathcal{B}$ . Then  $S$  is non-singular and for all  $v$  in  $V$

$$[v]_{\mathcal{B}} = S[v]_{\mathcal{B}'} \quad \text{and} \quad [v]_{\mathcal{B}'} = S^{-1}[v]_{\mathcal{B}}.$$

**Example (8.2.1)** Let  $V$  be the vector space of all real polynomials in  $t$  with degree at most 2. Then  $\mathcal{B} = \{1, t, t^2\}$  is clearly a basis of  $V$  and so is  $\mathcal{B}' = \{1+t, 2t, 4t^2-2\}$ , since it is quickly seen that this set is linearly independent. Write the coordinate vectors of  $1+t, 2t, 4t^2-2$  with respect to  $\mathcal{B}$  as columns of the matrix

$$S = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

This is the transition matrix for the change of basis  $\mathcal{B}' \rightarrow \mathcal{B}$ . The transition matrix for  $\mathcal{B} \rightarrow \mathcal{B}'$  is

$$S^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

For example, to express  $f = a + bt + ct^2$  in terms of the basis  $\mathcal{B}'$ , we compute

$$[f]_{\mathcal{B}'} = S^{-1}[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + \frac{c}{2} \\ -\frac{1}{2}a + \frac{1}{2}b - \frac{1}{4}c \\ \frac{1}{4}c \end{bmatrix}.$$

Thus  $f = (a + \frac{c}{2})(1+t) + (-\frac{1}{2}a + \frac{1}{2}b - \frac{1}{4}c)(2t) + \frac{1}{4}c(4t^2 - 2)$ , which is clearly correct.

**Dimension of the sum and intersection of subspaces.** Since a vector space  $V$  is an additively written abelian group, one can form the sum of two subspaces  $U$  and  $W$ ; thus

$$U + W = \{u + w \mid u \in U, w \in W\}.$$

It is easily verified that  $U + W$  is a subspace of  $V$ . Also  $U \cap W$  is a subspace. There is a useful formula connecting the dimensions of  $U + W$  and  $U \cap W$ .

**(8.2.13)** *If  $U$  and  $W$  are subspaces of a finite dimensional vector space  $V$ , then*

$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W).$$

*Proof.* If  $U = 0$ , then  $U + W = W$  and  $U \cap W = 0$ ; in this case the formula is certainly true. Thus we can assume that  $U \neq 0$  and  $W \neq 0$ .

Choose a basis for  $U \cap W$ , say  $z_1, \dots, z_r$ , if  $U \cap W \neq 0$ ; should  $U \cap W$  be 0, just ignore the  $z_i$ . By (8.2.6) we can extend  $\{z_1, \dots, z_r\}$  to bases of  $U$  and of  $W$ , say

$$\{z_1, \dots, z_r, u_{r+1}, \dots, u_m\} \quad \text{and} \quad \{z_1, \dots, z_r, w_{r+1}, \dots, w_n\}.$$

Now the vectors  $z_1, z_2, \dots, z_r, u_{r+1}, \dots, u_m, w_{r+1}, \dots, w_n$  surely generate  $U + W$ : for any vector in  $U + W$  is expressible as a linear combination of them. In fact these elements are also linearly independent, so they form a basis of  $U + W$ . To establish this claim, suppose there is a linear relation

$$\sum_{i=1}^r e_i z_i + \sum_{j=r+1}^m c_j u_j + \sum_{k=r+1}^n d_k w_k = 0$$

where  $e_i, c_j, d_k$  are scalars. Then

$$\sum_{k=r+1}^n d_k w_k = \sum_{i=1}^r (-e_i) z_i + \sum_{j=r+1}^m (-c_j) u_j,$$

which belongs to  $U$  and to  $W$  and so to  $U \cap W$ . Hence  $\sum_{k=r+1}^n d_k w_k$  is a linear combination of the  $z_i$ . But  $z_1, \dots, z_r, w_{r+1}, \dots, w_n$  are linearly independent, which implies that  $d_k = 0$  for all  $k$ . The linear relation now reduces to

$$\sum_{i=1}^r e_i z_i + \sum_{j=r+1}^m c_j u_j = 0.$$

But  $z_1, \dots, z_r, u_{r+1}, \dots, u_m$  are linearly independent. Therefore all the  $c_j$  and  $e_i$  equal zero, which establishes the claim of linear independence.

Finally,  $\dim(U + W)$  equals the number of the vectors  $z_1, \dots, z_r, u_{r+1}, \dots, u_m, w_{r+1}, \dots, w_n$ : this is,  $r + (m - r) + (n - r) = m + n - r$ , which equals  $\dim(U) + \dim(W) - \dim(U \cap W)$ , so the required formula follows.  $\square$

**Direct sums of vector spaces.** Since a vector space  $V$  is an additive abelian group, we can form the direct sum of subspaces  $U_1, U_2, \dots, U_k$  – see (4.2). This is an additive abelian group which is written

$$U = U_1 \oplus U_2 \cdots \oplus U_k.$$

Thus  $U = \{u_1 + u_2 + \cdots + u_k \mid u_i \in U_i\}$  and  $U_i \cap \sum_{j \neq i} U_j = 0$ . Clearly  $U$  is a subspace of  $V$ . Note that by (8.2.13) and induction on  $k$

$$\dim(U_1 \oplus U_2 \oplus \cdots \oplus U_k) = \dim(U_1) + \dim(U_2) + \cdots + \dim(U_k).$$

Next if  $\{v_1, v_2, \dots, v_n\}$  be a basis of  $V$ , then  $V = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \cdots \oplus \langle v_n \rangle$ , so that we have established:

**(8.2.14)** *An  $n$ -dimensional vector space is the direct sum of  $n$  1-dimensional subspaces.*

This result is also true when  $n = 0$  if the direct sum is interpreted as 0.

**Quotient spaces.** Suppose that  $V$  is a vector space over a field  $F$  and  $U$  is a subspace of  $V$ . Since  $V$  is an abelian group and  $U$  is a subgroup, the quotient

$$V/U = \{v + U \mid v \in V\}$$

is already defined as an abelian group. Now make  $V/U$  into a vector space over  $F$  by defining scalar multiplication in the natural way,

$$a(v + U) = av + U, \quad (a \in F).$$

This is evidently a well-defined operation. After an easy check of the axioms, we conclude that  $V/U$  is a vector space over  $F$ , the *quotient space* of  $U$  in  $V$ . The dimension of a quotient space is easily computed.

**(8.2.15)** *Let  $U$  be a subspace of a finite dimensional space  $V$ . Then  $\dim(V/U) = \dim(V) - \dim(U)$ .*

*Proof.* If  $U = 0$ , the statement is obviously true. Assuming  $U \neq 0$ , we choose a basis  $\{v_1, v_2, \dots, v_m\}$  of  $U$  and extend it to a basis of  $V$ , say  $\{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$ . We will argue that  $\{v_{m+1} + U, \dots, v_n + U\}$  is a basis of  $V/U$ .

Assume that  $\sum_{i=m+1}^n a_i(v_i + U) = 0_{V/U} = U$  where  $a_i$  is a scalar. Then  $\sum_{i=m+1}^n a_i v_i \in U$ , so this element is a linear combination of  $v_1, \dots, v_m$ . It follows by linear independence that each  $a_i = 0$ , which shows that  $\{v_{m+1} + U, \dots, v_n + U\}$  is linearly independent. Next, if  $v \in V$ , write  $v = \sum_{i=1}^n a_i v_i$ , with scalars  $a_i$ , and observe that  $v + U = \sum_{i=m+1}^n a_i(v_i + U)$  since  $v_1, \dots, v_m \in U$ . It follows that  $v_{m+1} + U, \dots, v_n + U$  form a basis of  $V/U$  and  $\dim(V/U) = n - m = \dim(V) - \dim(U)$ , as required.  $\square$



To conclude this section let us show that the mere existence of a basis in a finite dimensional vector space is enough to prove two important results about abelian groups and finite fields.

Let  $p$  be a prime. An additively written abelian group  $A$  is called an *elementary abelian  $p$ -group* if  $pa = 0$  for all  $a$  in  $A$ , i.e., each element of  $A$  has order 1 or  $p$ . For example, the Klein 4-group is an elementary abelian 2-group. The structure of finite elementary abelian  $p$ -groups is given by the next result.

**(8.2.16)** *Let  $A$  be a finite abelian group. Then  $A$  is an elementary abelian  $p$ -group if and only if  $A$  is a direct sum of copies of  $\mathbb{Z}_p$ .*

*Proof.* The essential idea of the proof is to view  $A$  as a vector space over the field  $\mathbb{Z}_p$ . Here the scalar multiplication is the natural one, namely  $(i + p\mathbb{Z})a = ia$  where  $i \in \mathbb{Z}$ ,  $a \in A$ . One has to verify that this operation is well-defined, which is true since  $(i + pm)a = ia + mpa = ia$  for all  $a \in A$ . Since  $A$  is finite, it is a finite dimensional vector space over  $\mathbb{Z}_p$ . By (8.2.14)  $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$  where each  $A_i$  is a 1-dimensional subspace; thus  $|A_i| = p$  and  $A_i \cong \mathbb{Z}_p$ . Conversely, any direct sum of copies of  $\mathbb{Z}_p$  certainly satisfies  $pa = 0$  for every element  $a$  and so is an elementary abelian  $p$ -group.  $\square$

The second application is to prove that the number of elements in a finite field is always a prime power. This is in marked contrast to the behavior of groups and rings, examples of which exist with any finite order.

**(8.2.17)** *Let  $F$  be a finite field. Then  $|F|$  is a power of a prime.*

*Proof.* By (6.3.9) the field  $F$  has characteristic a prime  $p$  and  $pa = 0$  for all  $a \in F$ . Thus, as an additive group,  $F$  is elementary abelian  $p$ . It now follows from (8.2.16) that  $|F|$  is a power of  $p$ .  $\square$

### Exercises (8.2)

(1) Show that  $X_1 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ ,  $X_2 = \begin{bmatrix} -5 \\ 2 \\ -3 \end{bmatrix}$ ,  $X_3 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$  form a basis of  $\mathbb{R}^3$ , and express the elementary vectors  $E_1, E_2, E_3$  in terms of  $X_1, X_2, X_3$ .

(2) Find a basis for the null space of the matrix  $\begin{bmatrix} 2 & 3 & 1 & 1 \\ -3 & 1 & 4 & -7 \\ 1 & 2 & 1 & 0 \end{bmatrix}$ .

(3) Find the dimension of the vector space  $M_{m,n}(F)$  where  $F$  is an arbitrary field.

(4) Let  $v_1, v_2, \dots, v_n$  be vectors in a vector space  $V$ . Assume that each element of  $V$  is uniquely expressible as a linear combination of  $v_1, v_2, \dots, v_n$ . Prove that the  $v_i$ 's form a basis of  $V$ .

(5) Let  $\mathcal{B} = \{E_1, E_2, E_3\}$  be the standard ordered basis of  $\mathbb{R}^3$  and let

$$\mathcal{B}' = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Show that  $\mathcal{B}'$  is a basis of  $\mathbb{R}^3$  and find the transition matrices for the changes of bases  $\mathcal{B}' \rightarrow \mathcal{B}$  and  $\mathcal{B} \rightarrow \mathcal{B}'$ .

(6) Let  $V$  be a vector space of dimension  $n$  and let  $i$  be an integer such that  $0 \leq i \leq n$ . Prove that  $V$  has at least one subspace of dimension  $i$ .

(7) The same as Exercise (8.2.6) with “subspace” replaced by “quotient space”.

(8) Let  $U$  be a subspace of a finite dimensional vector space  $V$ . Prove that there is a subspace  $W$  such that  $V = U \oplus W$ .

(9) Let  $V$  be a vector space of dimension  $2n$  and assume that  $U$  and  $W$  are subspaces of dimensions  $n$  and  $n + 1$  respectively. Prove that  $U \cap W \neq 0$ .

(10) Let the vectors  $v_1, v_2, \dots, v_m$  generate a vector space  $V$ . Prove that some subset of  $\{v_1, v_2, \dots, v_m\}$  is a basis of  $V$ .

### 8.3 Linear mappings

Just as there are homomorphisms of groups and of rings, there are homomorphisms of vector spaces. Traditionally these are called *linear mappings* or *transformations*. Let  $V$  and  $W$  be vector spaces over the same field  $F$ . Then a function

$$\alpha : V \rightarrow W$$

is called a *linear mapping from  $V$  to  $W$*  if the following rules are valid for all  $v_1, v_2 \in V$  and  $a \in F$ :

- (i)  $\alpha(v_1 + v_2) = \alpha(v_1) + \alpha(v_2)$ ;
- (ii)  $\alpha(av_1) = a\alpha(v_1)$ .

If  $\alpha$  is also bijective, it is called an *isomorphism of vector spaces*. Should there exist an isomorphism between vector spaces  $V$  and  $W$  over a field  $F$ , then  $V$  and  $W$  are said to be *isomorphic* and we write

$$V \stackrel{F}{\cong} W \text{ or } V \simeq W.$$

Notice that a linear mapping is automatically a homomorphism of additive groups by (i) above, so all results established for group homomorphisms may be carried over to linear mappings. A linear mapping  $\alpha : V \rightarrow V$  is called a *linear operator* on  $V$ .

**Example (8.3.1)** Let  $A$  be an  $m \times n$  matrix over a field  $F$  and define a function  $\alpha : F^n \rightarrow F^m$  by the rule  $\alpha(X) = AX$  where  $X \in F^n$ . Simple properties of matrices show that  $\alpha$  is a linear mapping.

**Example (8.3.2)** Let  $V$  be an  $n$ -dimensional vector space over a field  $F$  and let  $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$  be an ordered basis of  $V$ . Recall that to each vector  $v$  in  $V$  there corresponds a unique coordinate vector  $[v]_{\mathcal{B}}$  with respect to  $\mathcal{B}$ .

Use this correspondence to define a function  $\alpha : V \rightarrow F^n$  by  $\alpha(v) = [v]_{\mathcal{B}}$ . By simple calculations we see that  $[u + v]_{\mathcal{B}} = [u]_{\mathcal{B}} + [v]_{\mathcal{B}}$  and  $[av]_{\mathcal{B}} = a[v]_{\mathcal{B}}$  where  $u, v \in V$ ,  $a \in F$ . Hence  $\alpha$  is a linear mapping. Clearly  $[v]_{\mathcal{B}} = 0$  implies that  $v = 0$ ; thus  $\alpha$  is injective and it is obviously surjective. The conclusion is that  $\alpha$  is an isomorphism and  $V \stackrel{F}{\cong} F^n$ .

We state this conclusion as:

**(8.3.1)** *If  $V$  is a vector space with dimension  $n$  over a field  $F$ , then  $V \stackrel{F}{\cong} F^n$ . Thus two finite dimensional vector spaces over  $F$  are isomorphic if and only if they have the same dimension.*

Here the converse statement follows from the observation that isomorphic vector spaces have the same dimension.

An important way of defining a linear mapping is by specifying its effect on a basis.

**(8.3.2)** *Let  $\{v_1, \dots, v_n\}$  be a basis of a vector space  $V$  over a field  $F$  and let  $w_1, \dots, w_n$  be any  $n$  vectors in another  $F$ -vector space  $W$ . Then there is a unique linear mapping  $\alpha : V \rightarrow W$  such that  $\alpha(v_i) = w_i$  for  $i = 1, 2, \dots, n$ .*

*Proof.* Let  $v \in V$  and write  $v = \sum_{i=1}^n a_i v_i$ , with  $a_i \in F$ . Define a function  $\alpha : V \rightarrow W$  by the rule

$$\alpha(v) = \sum_{i=1}^n a_i w_i.$$

Then an easy check shows that  $\alpha$  is a linear mapping, and of course  $\alpha(v_i) = w_i$ . If  $\alpha' : V \rightarrow W$  is another such linear mapping, then  $\alpha' = \alpha$ ; for  $\alpha'(v) = \sum_{i=1}^n a_i \alpha'(v_i) = \sum_{i=1}^n a_i w_i = \alpha(v)$ .  $\square$

Our experience with groups and rings suggests it may be worthwhile to examine the kernel and image of a linear mapping.

**(8.3.3)** *Let  $\alpha : V \rightarrow W$  be a linear mapping. Then  $\text{Ker}(\alpha)$  and  $\text{Im}(\alpha)$  are subspaces of  $V$  and  $W$  respectively.*

*Proof.* Since  $\alpha$  is a group homomorphism, it follows from (4.3.2) that  $\text{Ker}(\alpha)$  and  $\text{Im}(\alpha)$  are additive subgroups. We leave the reader to complete the proof by showing that these subgroups are also closed under scalar multiplication.  $\square$

Just as for groups and rings, there are isomorphism theorems for vector spaces.

**(8.3.4)** (First Isomorphism Theorem) *If  $\alpha : V \rightarrow W$  is a linear mapping between vector spaces over a field  $F$ , then  $V/\text{Ker}(\alpha) \stackrel{F}{\cong} \text{Im}(\alpha)$ .*

**(8.3.5)** (Second Isomorphism Theorem) *Let  $U$  and  $W$  be subspaces of a vector space over a field  $F$ . Then  $(U + W)/W \stackrel{F}{\cong} U/(U \cap W)$ .*

**(8.3.6)** (Third Isomorphism Theorem) *Let  $U$  and  $W$  be subspaces of a vector space over a field  $F$  such that  $U \subseteq W$ . Then  $(V/U)/(W/U) \stackrel{F}{\cong} V/W$ .*

Since the isomorphism theorems for groups are applicable, all one has to prove here is that the functions introduced in the proofs of (4.3.4), (4.3.5) and (4.3.6) are linear mappings, i.e., they act appropriately on scalar multiples.

For example, in (8.3.4) the function in question is  $\theta : V/\text{Ker}(\alpha) \rightarrow \text{Im}(\alpha)$  where  $\theta(v + \text{Ker}(\alpha)) = \alpha(v)$ . Then

$$\theta(a(v + \text{Ker}(\alpha))) = \theta(av + \text{Ker}(\alpha)) = \alpha(av) = a\alpha(v) = a\theta(v + \text{Ker}(\alpha)).$$

It follows that  $\theta$  is a linear mapping.

There is an important formula connecting the dimensions of kernel and image.

**(8.3.7)** *If  $\alpha : V \rightarrow W$  is a linear mapping between finite dimensional vector spaces, then  $\dim(\text{Ker}(\alpha)) + \dim(\text{Im}(\alpha)) = \dim(V)$ .*

This follows directly from (8.3.4) and (8.2.15). There is an immediate application to the null space of a matrix.

**Corollary (8.3.8)** *Let  $A$  be an  $m \times n$  matrix with rank  $r$  over a field  $F$ . Then the dimension of the null space of  $A$  is  $n - r$ .*

*Proof.* Let  $\alpha$  be the linear mapping from  $F^n$  to  $F^m$  defined by  $\alpha(X) = AX$ . Then  $\text{Ker}(\alpha)$  is the null space of  $A$  and it is readily seen that  $\text{Im}(\alpha)$  is just the column space. By (8.2.9)  $\dim(\text{Im}(\alpha)) = r$ , the rank of  $A$ , and by (8.3.7)  $\dim(\text{Ker}(\alpha)) = n - r$ .  $\square$

As another application of (8.3.7) we give a different proof of the dimension formula for sum and intersection of subspaces – see (8.2.13).

**(8.3.9)** *If  $U$  and  $W$  are subspaces of a finite dimensional vector space, then  $\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$ .*

*Proof.* By (8.3.5)  $(U + W)/W \cong U/(U \cap W)$ . Hence, taking dimensions and applying (8.2.15), we find that  $\dim(U + W) - \dim(W) = \dim(U) - \dim(U \cap W)$ , and the result follows.  $\square$

**Vector spaces of linear mappings.** It is useful to endow sets of linear mappings with the structure of a vector space. Suppose that  $V$  and  $W$  are vector spaces over the same field  $F$ . We will write

$$L(V, W)$$

for the set of all linear mappings from  $V$  to  $W$ . Define addition and scalar multiplication in  $L(V, W)$  by the natural rules

$$(\alpha + \beta)(v) = \alpha(v) + \beta(v), \quad (a \cdot \alpha)(v) = a(\alpha(v)),$$

where  $\alpha, \beta \in L(V, W)$ ,  $v \in V$ ,  $a \in F$ . It is simple to verify that  $\alpha + \beta$  and  $a \cdot \alpha$  are linear mappings. The basic result about  $L(V, W)$  is:

**(8.3.10)** *Let  $V$  and  $W$  be vector spaces over a field  $F$ . Then:*

- (i)  $L(V, W)$  is a vector space over  $F$ ;
- (ii) if  $V$  and  $W$  are finite dimensional, then so is  $L(V, W)$  and  $\dim(L(V, W)) = \dim(V) \cdot \dim(W)$ .

*Proof.* We omit the routine proof of (i) and concentrate on (ii). Let  $\{v_1, \dots, v_m\}$  and  $\{w_1, \dots, w_n\}$  be bases of  $V$  and  $W$  respectively. By (8.3.2), for  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ , there is a unique linear mapping  $\alpha_{ij} : V \rightarrow W$  such that

$$\alpha_{ij}(v_k) = \begin{cases} w_j & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}.$$

Thus  $\alpha_{ij}$  sends basis element  $v_i$  to basis element  $w_j$  and all other  $v_k$ 's to 0. First we show that the  $\alpha_{ij}$  are linearly independent in the vector space  $L(V, W)$ .

Let  $a_{ij} \in F$ ; then by definition of  $\alpha_{ij}$  we have for each  $k$

$$\left( \sum_{i=1}^m \sum_{j=1}^n a_{ij} \alpha_{ij} \right)(v_k) = \sum_{j=1}^n \sum_{i=1}^m a_{ij} (\alpha_{ij}(v_k)) = \sum_{j=1}^n a_{kj} w_j. \quad (*)$$

Therefore  $\sum_{i=1}^m \sum_{j=1}^n a_{ij} \alpha_{ij} = 0$  if and only if  $a_{kj} = 0$  for all  $j, k$ . It follows that the  $\alpha_{ij}$  are linearly independent.

Finally, we claim that the  $\alpha_{ij}$  actually generate  $L(V, W)$ . To prove this let  $\alpha \in L(V, W)$  and write  $\alpha(v_k) = \sum_{j=1}^n a_{kj} w_j$  where  $a_{kj} \in F$ . Then from the equation (\*) above we see that  $\alpha = \sum_{i=1}^m \sum_{j=1}^n a_{ij} \alpha_{ij}$ . Therefore the  $\alpha_{ij}$ 's form a basis of  $L(V, W)$  and  $\dim(L(V, W)) = mn = \dim(V) \cdot \dim(W)$ .  $\square$

**The dual space.** If  $V$  is a vector space over a field  $F$ , the vector space

$$V^* = L(V, F)$$

is called the *dual space* of  $V$ ; here  $F$  is regarded as a 1-dimensional vector space over  $F$ . The elements of  $V^*$  are linear mappings from  $V$  to  $F$ , which are called *linear functionals* on  $V$ .

**Example (8.3.3)** Let  $Y \in F^n$  be fixed and define  $\alpha : F^n \rightarrow F$  by the rule  $\alpha(X) = Y^T X$  where  $Y^T$  is the transpose of  $Y$ . Then  $\alpha$  is a linear functional on  $F^n$ .

If  $V$  is an  $n$ -dimensional vector space over  $F$ ,

$$\dim(V^*) = \dim(L(V, F)) = \dim(V)$$

by (8.3.10). Thus  $V$ ,  $V^*$  and the *double dual*  $V^{**} = (V^*)^*$  all have the same dimension, so these vector spaces are isomorphic by (8.3.1).

In fact there is a *canonical* linear mapping  $\theta : V \rightarrow V^{**}$ . Let  $v \in V$  and define  $\theta(v) \in V^{**}$  by the rule

$$\theta(v)(\alpha) = \alpha(v)$$

where  $\alpha \in V^*$ . Thus  $\theta(v)$  evaluates each linear functional on  $V$  at  $v$ . Regarding the function  $\theta$ , we prove:

**(8.3.11)** *If  $V$  is a finite dimensional vector space, then  $\theta : V \rightarrow V^{**}$  is an isomorphism.*

*Proof.* In the first place  $\theta(v) \in V^{**}$  for all  $v \in V$ : indeed, if  $\alpha, \beta \in V^*$ ,

$$\theta(v)(\alpha + \beta) = (\alpha + \beta)(v) = \alpha(v) + \beta(v) = \theta(v)(\alpha) + \theta(v)(\beta).$$

Also  $\theta(v)(a \cdot \alpha) = (a \cdot \alpha)(v) = a(\alpha(v)) = a(\theta(v)(\alpha))$  where  $a$  is a scalar.

Next for any  $\alpha \in V^*$  and  $v_i \in V$ , we have

$$\begin{aligned} \theta(v_1 + v_2)(\alpha) &= \alpha(v_1 + v_2) \\ &= \alpha(v_1) + \alpha(v_2) \\ &= \theta(v_1)(\alpha) + \theta(v_2)(\alpha) \\ &= (\theta(v_1) + \theta(v_2))(\alpha), \end{aligned}$$

which shows that  $\theta(v_1 + v_2) = \theta(v_1) + \theta(v_2)$ . We leave the reader to verify that  $\theta(a \cdot v) = a(\theta(v))$  where  $a \in F$ ,  $v \in V$ . Hence  $\theta$  is a linear mapping from  $V$  to  $V^{**}$ .

Next suppose that  $\theta(v) = 0$ . Then  $0 = \theta(v)(\alpha) = \alpha(v)$  for *all*  $\alpha \in V^*$ . This can only mean that  $v = 0$ : for if  $v \neq 0$ , then  $v$  can be included in a basis of  $V$ . Then by (8.3.2) we can construct a linear functional  $\alpha$  such that  $\alpha(v) = 1_F$  and other basis elements are mapped by  $\alpha$  to 0. It follows that  $\theta$  is injective.

Finally,  $\dim(V) = \dim(V^*) = \dim(V^{**})$  and also  $\dim(V) = \dim(\text{Im}(\theta))$  since  $\theta$  is injective. By (8.2.10) we have  $\text{Im}(\theta) = V^{**}$ , so that  $\theta^*$  is an isomorphism.  $\square$

**Representing linear mappings by matrices.** A linear mapping between finite dimensional vector spaces can be described by matrix multiplication, which provides us with a concrete way of representing linear mappings.

Let  $V$  and  $W$  be vector spaces over a field  $F$  with respective finite dimensions  $m > 0$  and  $n > 0$ . Choose ordered bases for  $V$  and  $W$ , say  $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$  and  $\mathcal{C} = \{w_1, w_2, \dots, w_n\}$  respectively. Now let  $\alpha \in L(V, W)$ ; then

$$\alpha(v_i) = \sum_{j=1}^n a_{ji} w_j, \quad i = 1, 2, \dots, m,$$

where  $a_{ji} \in F$ . This enables us to form the  $n \times m$  matrix over  $F$

$$A = [a_{ji}],$$

which is to represent  $\alpha$ . Notice that the  $i$ th column of  $A$  is precisely the coordinate column vector of  $\alpha(v_i)$  with respect to the basis  $\mathcal{C}$ . Thus we have a function

$$\theta : L(V, W) \rightarrow M_{n,m}(F)$$

defined by the rule that column  $i$  of  $\theta(\alpha)$  is  $[\alpha(v_i)]_{\mathcal{C}}$ .

To understand how the matrix  $A = \theta(\alpha)$  reproduces the effect of  $\alpha$  on an arbitrary vector  $v = \sum_{i=1}^m b_i v_i$  of  $V$ , we compute

$$\alpha(v) = \sum_{i=1}^m b_i (\alpha(v_i)) = \sum_{i=1}^m b_i \left( \sum_{j=1}^n a_{ji} w_j \right) = \sum_{j=1}^n \left( \sum_{i=1}^m a_{ji} b_i \right) w_j.$$

Hence the coordinate column vector of  $\alpha(v)$  with respect to  $\mathcal{C}$  has entries  $\sum_{i=1}^m a_{ji} b_i$ , for  $j = 1, \dots, n$ , i.e., it is

$$A \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = A[v]_{\mathcal{B}}.$$

Thus we arrive at the basic formula

$$[\alpha(v)]_{\mathcal{C}} = A[v]_{\mathcal{B}} = \theta(\alpha)[v]_{\mathcal{B}}.$$

Concerning the function  $\theta$  we prove:

**(8.3.12)** *If  $V$  and  $W$  are finite dimensional vector spaces over a field  $F$ , the function  $\theta : L(V, W) \rightarrow M_{n,m}(F)$  is an isomorphism of vector spaces.*

*Proof.* In the first place  $\theta$  is a linear mapping. For, let  $\alpha, \beta \in L(V, W)$  and  $v \in V$ ; then the formula above shows that

$$\theta(\alpha + \beta)[v]_{\mathcal{B}} = [(\alpha + \beta)(v)]_{\mathcal{C}} = [\alpha(v) + \beta(v)]_{\mathcal{C}} = [\alpha(v)]_{\mathcal{C}} + [\beta(v)]_{\mathcal{C}},$$

which equals

$$\theta(\alpha)[v]_{\mathcal{B}} + \theta(\beta)[v]_{\mathcal{B}} = (\theta(\alpha) + \theta(\beta))[v]_{\mathcal{B}}.$$

Hence  $\theta(\alpha + \beta) = \theta(\alpha) + \theta(\beta)$ , and in a similar fashion it may be shown that  $\theta(a \cdot \alpha) = a(\theta(\alpha))$  where  $a \in F$ .

Next if  $\theta(\alpha) = 0$ , then  $[\alpha(v)]_{\mathcal{C}} = 0$ , so  $\alpha(v) = 0$  for all  $v \in V$  and  $\alpha = 0$ . Hence  $\theta$  is injective. If  $V$  and  $W$  have respective dimensions  $m$  and  $n$ , then  $L(V, W) \simeq \text{Im}(\theta) \subseteq M_{n,m}(F)$ . But the vector spaces  $L(V, W)$  and  $M_{n,m}(F)$  both have dimension  $mn$  – see (8.3.10). Therefore  $\text{Im}(\theta) = M_{n,m}(F)$  by (8.2.10) and  $\theta$  is an isomorphism.  $\square$

**Example (8.3.4)** Consider the dual space  $V^* = L(V, F)$ , where  $V$  is an  $n$ -dimensional vector space over a field  $F$ . Choose an ordered basis  $\mathcal{B}$  of  $V$  and use the basis  $\{1_F\}$  for  $V$ . Then a linear functional  $\alpha \in V^*$  is represented by an  $n$ -row vector, i.e., by  $X^T$  where  $X \in F^n$ , according to the rule  $\alpha(v) = X^T[v]_{\mathcal{B}}$ . Thus the effect of a linear functional is produced by left multiplication of coordinate vectors by a row vector, (cf. Example (8.3.3)).

**The effect of a change of basis.** We have seen that any linear mapping between finite dimensional vector spaces can be represented by multiplication by a matrix. However, the matrix depends on the choice of ordered bases of the vector spaces. The precise nature of this dependence will now be investigated.

Let  $\mathcal{B}$  and  $\mathcal{C}$  be ordered bases of respective finite dimensional vector spaces  $V$  and  $W$  over a field  $F$ , and let  $\alpha : V \rightarrow W$  be a linear mapping. Then  $\alpha$  is represented by a matrix  $A$  over  $F$  where  $[\alpha(v)]_{\mathcal{C}} = A[v]_{\mathcal{B}}$ . Now suppose now that two different ordered bases  $\mathcal{B}'$  and  $\mathcal{C}'$  are chosen for  $V$  and  $W$  respectively. Then  $\alpha$  will be represented by another matrix  $A'$ . The question is: how are  $A$  and  $A'$  related?

To answer the question we introduce the transition matrices  $S$  and  $T$  for the respective changes of bases  $\mathcal{B} \rightarrow \mathcal{B}'$  and  $\mathcal{C} \rightarrow \mathcal{C}'$  (see (8.2.12)). Thus for any  $v \in V$  and  $w \in W$  we have

$$[v]_{\mathcal{B}'} = S[v]_{\mathcal{B}} \quad \text{and} \quad [w]_{\mathcal{C}'} = T[w]_{\mathcal{C}}.$$

Therefore

$$[\alpha(v)]_{\mathcal{C}'} = T[\alpha(v)]_{\mathcal{C}} = TA[v]_{\mathcal{B}} = TAS^{-1}[v]_{\mathcal{B}'},$$

and it follows that  $A' = TAS^{-1}$ . We record this conclusion in:

**(8.3.13)** *Let  $V$  and  $W$  be non-zero finite dimensional vector spaces over the same field. Let  $\mathcal{B}, \mathcal{B}'$  be ordered bases of  $V$  and  $\mathcal{C}, \mathcal{C}'$  ordered bases of  $W$ . Suppose further that  $S$  and  $T$  are the transition matrices for the changes of bases  $\mathcal{B} \rightarrow \mathcal{B}'$  and  $\mathcal{C} \rightarrow \mathcal{C}'$  respectively. If the linear mapping  $\alpha : V \rightarrow W$  is represented by matrices  $A$  and  $A'$  with respect to the respective pairs of bases  $(\mathcal{B}, \mathcal{C})$  and  $(\mathcal{B}', \mathcal{C}')$ , then  $A' = TAS^{-1}$ .*

The case where  $\alpha$  is a linear operator on  $V$  is especially important. Here  $V = W$  and we can take  $\mathcal{B} = \mathcal{C}$  and  $\mathcal{B}' = \mathcal{C}'$ . Thus  $S = T$  and  $A' = SAS^{-1}$ , i.e.,  $A$  and  $A'$  are *similar matrices*. Consequently, *matrices that represent the same linear operator are similar*.

**The algebra of linear operators.** Let  $V$  be a vector space over a field  $F$  and suppose also that  $V$  is a ring with respect to some product operation. Then  $V$  is said to be an *F-algebra* if, in addition to the vector space and ring axioms, the following law is valid:

$$a(uv) = (au)v = u(av)$$

for all  $a \in F$ ,  $u, v \in V$ . For example, the set of all  $n \times n$  matrices  $M_n(F)$  is an  $F$ -algebra with respect to the usual matrix operations.



Now let  $V$  be any vector space over a field  $F$ ; we will write

$$L(V)$$

for the vector space  $L(V, V)$  of all linear operators on  $V$ . Our aim is to make  $L(V)$  into an  $F$ -algebra: it is already an  $F$ -vector space. There is a natural product operation on  $L(V)$ , namely functional composition. Indeed, if  $\alpha_1, \alpha_2 \in L(V)$ , then  $\alpha_1\alpha_2 \in L(V)$  by an easy check. We claim that with this product operation  $L(V)$  becomes an  $F$ -algebra.

The first step is to verify that  $L(V)$  is a ring. This is fairly routine; for example, if  $\alpha_i \in L(V)$  and  $v \in V$ ,

$$\alpha_1(\alpha_2 + \alpha_3)(v) = \alpha_1(\alpha_2(v) + \alpha_3(v)) = \alpha_1\alpha_2(v) + \alpha_1\alpha_3(v),$$

which equals  $(\alpha_1\alpha_2 + \alpha_1\alpha_3)(v)$ . Hence  $\alpha_1(\alpha_2 + \alpha_3) = \alpha_1\alpha_2 + \alpha_1\alpha_3$ .

Once the ring axioms have been verified, we have to check that  $a(\alpha_1\alpha_2) = (a\alpha_1)\alpha_2 = \alpha_1(a\alpha_2)$  for  $a \in F$ . This is not hard to see; indeed all three mappings send  $v$  to  $a(\alpha_1(\alpha_2(v)))$ . Therefore  $L(V)$  is an  $F$ -algebra.

A function  $\alpha : A_1 \rightarrow A_2$  between two  $F$ -algebras is called an *algebra isomorphism* if it is bijective and it is both a linear mapping of vector spaces and a homomorphism of rings.

**(8.3.14)** *Let  $V$  be a vector space with finite dimension  $n$  over a field  $F$ . Then  $L(V)$  and  $M_n(F)$  are isomorphic as  $F$ -algebras.*

*Proof.* Choose an ordered basis  $\mathcal{B}$  of  $V$  and let  $\Phi : L(V) \rightarrow M_n(F)$  be the function which associates to a linear operator  $\alpha$  the  $n \times n$  matrix that represents  $\alpha$  with respect to  $\mathcal{B}$ . Thus  $[\alpha(v)]_{\mathcal{B}} = \Phi(\alpha)[v]_{\mathcal{B}}$  for all  $v \in V$ . Clearly  $\Phi$  is bijective, so to prove that it is an  $F$ -algebra isomorphism we need to establish  $\Phi(\alpha + \beta) = \Phi(\alpha) + \Phi(\beta)$ ,  $\Phi(a \cdot \alpha) = a \cdot \Phi(\alpha)$  and  $\Phi(\alpha\beta) = \Phi(\alpha)\Phi(\beta)$ .

For example, take the third statement. If  $v \in V$ , then

$$\Phi(\alpha\beta)[v]_{\mathcal{B}} = [\alpha\beta(v)]_{\mathcal{B}} = \Phi(\alpha)[\beta(v)]_{\mathcal{B}} = \Phi(\alpha)(\Phi(\beta)[v]_{\mathcal{B}}) = (\Phi(\alpha)\Phi(\beta))[v]_{\mathcal{B}}.$$

Therefore  $\Phi(\alpha\beta) = \Phi(\alpha)\Phi(\beta)$ . The other statements are dealt with in a similar fashion.  $\square$

Thus (8.3.14) tells us in a precise way that linear operators on an  $n$ -dimensional vector space over  $F$  behave in very much the same manner as  $n \times n$  matrices over  $F$ .

### Exercises (8.3)

(1) Which of the following functions are linear mappings?

- (i)  $\alpha : \mathbb{R}_3 \rightarrow \mathbb{R}$  where  $\alpha([x_1 \ x_2 \ x_3]) = \sqrt{x_1^2 + x_2^2 + x_3^2}$ ;
- (ii)  $\alpha : M_{m,n}(F) \rightarrow M_{n,m}(F)$  where  $\alpha(A) = A^T$ , the transpose of  $A$ ;
- (iii)  $\alpha : M_n(F) \rightarrow F$  where  $\alpha(A) = \det(A)$ .

- (2) A linear mapping  $\alpha : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  sends  $[x_1 \ x_2 \ x_3 \ x_4]^T$  to  $[x_1 - x_2 + x_3 - x_4 \ 2x_1 + x_2 - x_3 \ x_2 - x_3 + x_4]^T$ . Find the matrix which represents  $\alpha$  when the standard bases of  $\mathbb{R}^4$  and  $\mathbb{R}^3$  are used.
- (3) Answer Exercise (8.3.2) when the ordered basis  $\{[1 \ 1 \ 1]^T, [0 \ 1 \ 1]^T, [0 \ 0 \ 1]^T\}$  of  $\mathbb{R}^3$  is used, together with the standard basis of  $\mathbb{R}^4$ .
- (4) Find bases for the kernel and image of the following linear mappings:
- (i)  $\alpha : F_4 \rightarrow F$  where  $\alpha$  maps a column vector to the sum of its entries;
  - (ii)  $\alpha : \mathbb{R}[t] \rightarrow \mathbb{R}[t]$  where  $\alpha(f) = f'$ , the derivative of  $f$ ;
  - (iii)  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $\alpha([x \ y]^T) = [2x + 3y \ 4x + 6y]^T$ .
- (5) Prove that a linear mapping  $\alpha : V \rightarrow W$  is injective if and only if  $\alpha$  maps linearly independent subsets of  $V$  to linearly independent subsets of  $W$ .
- (6) Prove that a linear mapping  $\alpha : V \rightarrow W$  is surjective if and only if  $\alpha$  maps generating sets of  $V$  to generating sets of  $W$ .
- (7) Let  $U$  and  $W$  be subspaces of a finite dimensional vector space  $V$ . Prove that there is a linear operator  $\alpha$  on  $V$  such that  $\text{Ker}(\alpha) = U$  and  $\text{Im}(\alpha) = W$  if and only if  $\dim(U) + \dim(W) = \dim(V)$ .
- (8) Suppose that  $\alpha : V \rightarrow W$  is a linear mapping. Explain how to define a corresponding "induced" linear mapping  $\alpha^* : W^* \rightarrow V^*$ . Then prove that  $(\alpha\beta)^* = \beta^*\alpha^*$ .
- (9) Let  $U \xrightarrow{\alpha} V \xrightarrow{\beta} W \rightarrow 0$  be an *exact sequence* of vector spaces and linear mappings. (This means that  $\text{Im}(\alpha) = \text{Ker}(\beta)$  and  $\text{Im}(\beta) = \text{Ker}(W \rightarrow 0) = W$ , i.e.,  $\beta$  is surjective). Prove that the corresponding sequence of dual spaces and induced linear mappings  $0 \rightarrow W^* \xrightarrow{\beta^*} V^* \xrightarrow{\alpha^*} U^*$  is exact, i.e.,  $\beta^*$  is injective and  $\text{Im}(\beta^*) = \text{Ker}(\alpha^*)$ . (For more general results of this kind see (9.1.19)).

## 8.4 Eigenvalues and eigenvectors

Let  $\alpha$  be a linear operator on a vector space  $V$  over a field  $F$ . An *eigenvector* of  $\alpha$  is a non-zero vector  $v$  of  $V$  such that  $\alpha(v) = cv$  for some  $c \in F$  called an *eigenvalue*. For example, if  $\alpha$  is a rotation in  $\mathbb{R}^3$ , the eigenvectors of  $\alpha$  are the non-zero vectors parallel to the axis of rotation and the eigenvalues are all equal to 1. A large amount of information about a linear operator is carried by its eigenvectors and eigenvalues. In addition the theory of eigenvectors and eigenvalues has many applications, for example to systems of linear recurrence relations and systems of linear differential equations.

Let  $A$  be an  $n \times n$  matrix over a field  $F$ . Define  $\alpha$  to be the linear operator on  $F^n$  which sends  $X$  to  $AX$ . Then an eigenvector of  $\alpha$  is a non-zero vector  $X \in F^n$  such that  $AX = cX$  for some  $c \in F$ . We will call  $X$  an eigenvector and  $c$  an eigenvalue of the matrix  $A$ .

Conversely, suppose we start with a linear operator  $\alpha$  on a finite dimensional vector space  $V$  over a field  $F$ . Choose an ordered basis  $\mathcal{B}$  for  $V$ , so that  $\alpha$  is represented by an

$n \times n$  matrix  $A$  with respect to  $\mathcal{B}$  and  $[\alpha(v)]_{\mathcal{B}} = A[v]_{\mathcal{B}}$ . Let  $v$  be an eigenvector for  $\alpha$  with corresponding eigenvalue  $c \in F$ . Then  $\alpha(v) = cv$ , which translates into  $A[v]_{\mathcal{B}} = c[v]_{\mathcal{B}}$ . Thus  $[v]_{\mathcal{B}}$  is an eigenvector and  $c$  an eigenvalue of  $A$ .

These considerations show that the theory of eigenvalues and eigenvectors can be developed for either matrices or linear operators on a finite dimensional vector space. We will follow both approaches here, as is convenient.

**Example (8.4.1)** Let  $D$  denote the vector space of infinitely differentiable real valued functions on the interval  $[a, b]$ . Consider the linear operator  $\alpha$  on  $D$  defined by  $\alpha(f) = f'$ , the derivative of the function  $f$ . The condition for  $f \neq 0$  to be an eigenvector of  $\alpha$  is that  $f' = cf$  for some constant  $c$ . The general solution of this simple differential equation is  $f = de^{cx}$  where  $d$  is a constant. Thus the eigenvectors of  $\alpha$  are the functions  $de^{cx}$  with  $d \neq 0$ , while the eigenvalues are all real numbers  $c$ .

**Example (8.4.2)** A linear operator  $\alpha$  on the vector space  $\mathbb{C}^2$  is defined by  $\alpha(X) = AX$  where

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}.$$

Thus  $\alpha$  is represented with respect to the standard basis by the matrix  $A$ . The condition for a vector  $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  to be an eigenvector of  $A$  (or  $\alpha$ ) is that  $AX = cX$  for some scalar  $c$ . This is equivalent to  $(cI_2 - A)X = 0$ , which asserts that  $X$  is a solution of the linear system

$$\begin{bmatrix} c-2 & 1 \\ -2 & c-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By (8.2.2) this linear system has a non-trivial solution  $[x_1, x_2]^T$  if and only if the determinant of the coefficient matrix vanishes, i.e.,

$$\begin{vmatrix} c-2 & 1 \\ -2 & c-4 \end{vmatrix} = 0.$$

On expansion this becomes  $c^2 - 6c + 10 = 0$ . The roots of this quadratic equation are  $c_1 = 3 + i$  and  $c_2 = 3 - i$  where  $i = \sqrt{-1}$ , so these are the eigenvalues of  $A$ .

The eigenvectors for each eigenvalue are found by solving the linear systems  $(c_1 I_2 - A)X = 0$  and  $(c_2 I_2 - A)X = 0$ . For example, in the case of  $c_1$  we have to solve

$$\begin{cases} (1+i)x_1 + x_2 = 0 \\ -2x_1 + (-1+i)x_2 = 0 \end{cases}$$

The general solution of this system is  $x_1 = \frac{d}{2}(-1+i)$ ,  $x_2 = d$  where  $d$  is an arbitrary scalar. Thus the eigenvectors of  $A$  associated with the eigenvalue  $c_1$  are the non-zero vectors of the form

$$d \begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix}.$$

Notice that these, together with the zero vector, form a 1-dimensional subspace of  $\mathbb{C}^2$ . In a similar manner the eigenvectors for the eigenvalue  $3 - i$  are found to be the vectors of the form

$$d \begin{bmatrix} -\frac{1+i}{2} \\ 1 \end{bmatrix}$$

where  $d \neq 0$ . Again these form with the zero vector a subspace of  $\mathbb{C}^2$ .

This example is an illustration of the general procedure for finding eigenvectors and eigenvalues.

**The characteristic equation.** Let  $A$  be an  $n \times n$  matrix over a field  $F$  and let  $X$  be a non-zero  $n$ -column vector over  $F$ . The condition for  $X$  to be an eigenvector of  $A$  is  $AX = cX$  or

$$(cI_n - A)X = 0,$$

where  $c$  is the corresponding eigenvalue. Thus the eigenvectors associated with  $c$ , together with the zero vector, form the null space of the matrix  $cI_n - A$ . This subspace is called the *eigenspace* of the eigenvalue  $c$ .

Next  $(cI_n - A)X = 0$  is a homogeneous linear system of  $n$  equations in  $n$  unknowns, namely the entries of  $X$ . By (8.2.2) the condition for there to be a non-trivial solution of the system is

$$\det(cI_n - A) = 0.$$

Conversely, if  $c \in F$  satisfies this equation, there is a non-zero solution of the system and  $c$  is an eigenvalue. These considerations show that the determinant

$$\det(tI_n - A) = \begin{vmatrix} t - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & t - a_{22} & \cdots & -a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ -a_{n1} & -a_{n2} & \cdots & t - a_{nn} \end{vmatrix}$$

plays a critical role. This is a polynomial of degree  $n$  in  $t$  with coefficients in  $F$  called the *characteristic polynomial* of  $A$ . The equation obtained by setting the characteristic polynomial equal to zero is the *characteristic equation*. Thus the eigenvalues of  $A$  are the roots of the characteristic polynomial which lie in the field  $F$ .

One should keep in mind that  $A$  may well have no eigenvalues in  $F$ . For example, the characteristic polynomial of the real matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is  $t^2 + 1$ , which has no real roots, so the matrix has no eigenvalues in  $\mathbb{R}$ .

In general the eigenvalues of a linear operator or a matrix lie in the splitting field of the characteristic polynomial – see (7.4). If  $F = \mathbb{C}$ , all roots of the characteristic

equation lie in  $\mathbb{C}$  by the Fundamental Theorem of Algebra. Because of this we can be sure that a complex matrix has all its eigenvalues in  $\mathbb{C}$ .

Let us sum up our conclusions about the eigenvalues of matrices so far.

**(8.4.1)** Let  $A$  be an  $n \times n$  matrix over a field  $F$ .

- (i) The eigenvalues of  $A$  in  $F$  are precisely the roots of the characteristic polynomial  $\det(tI_n - A)$  which lie in  $F$ .
- (ii) The eigenvectors of  $A$  associated with the eigenvalue  $c$  are the non-zero vectors in the null space of the matrix  $cI_n - A$ .

**Example (8.4.3)** Find the eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

The characteristic polynomial of  $A$  is

$$\begin{vmatrix} t - a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ 0 & t - a_{22} & -a_{23} & \cdots & -a_{2n} \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & t - a_{nn} \end{vmatrix},$$

which equals  $(t - a_{11})(t - a_{22}) \cdots (t - a_{nn})$ . The eigenvalues of the matrix are therefore just the diagonal entries  $a_{11}, a_{22}, \dots, a_{nn}$ .

**Example (8.4.4)** Consider the  $3 \times 3$  matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

The characteristic polynomial of  $A$  is

$$\begin{vmatrix} t - 2 & 1 & 1 \\ 1 & t - 2 & 1 \\ 1 & 1 & t \end{vmatrix} = t^3 - 4t^2 + t + 6.$$

By inspection one root of this cubic polynomial is  $-1$ . Dividing the polynomial by  $t + 1$  using long division, we obtain the quotient  $t^2 - 5t + 6 = (t - 2)(t - 3)$ . Hence the characteristic polynomial factorizes completely as  $(t + 1)(t - 2)(t - 3)$  and the eigenvalues of  $A$  are  $-1, 2$  and  $3$ .

To find the corresponding eigenvectors, solve the three linear systems  $(-I_3 - A)X = 0$ ,  $(2I_3 - A)X = 0$  and  $(3I_3 - A)X = 0$ . On solving these, we find that the respective eigenvectors are the non-zero scalar multiples of the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

so that eigenspaces all have dimension 1.

**Properties of the characteristic polynomial.** Let us see what can be said about the characteristic polynomial of an arbitrary  $n \times n$  matrix  $A = [a_{ij}]$  over a field  $F$ . This is

$$p(t) = \begin{vmatrix} t - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & t - a_{22} & \cdots & -a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ -a_{n1} & -a_{n2} & \cdots & t - a_{nn} \end{vmatrix}.$$

At this point recall the definition of a determinant as an alternating sum of  $n!$  terms, each term being a product of  $n$  entries, one from each row and column. The term of  $p(t)$  with highest degree in  $t$  arises from the product

$$(t - a_{11})(t - a_{22}) \cdots (t - a_{nn})$$

and is clearly  $t^n$ . The terms of degree  $n - 1$  are easily identified as they arise from the same product. Thus the coefficient of  $t^{n-1}$  is  $-(a_{11} + a_{22} + \cdots + a_{nn})$ . The sum of the diagonal entries of  $A$  is called the *trace* of  $A$ ,

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn},$$

so the term in  $p(t)$  of degree  $n - 1$  is  $-\text{tr}(A)t^{n-1}$ .

The constant term in  $p(t)$  is  $p(0) = \det(-A) = (-1)^n \det(A)$ . Our knowledge of  $p(t)$  so far is summarized by the formula

$$p(t) = t^n - \text{tr}(A)t^{n-1} + \cdots + (-1)^n \det(A).$$

The other coefficients in the characteristic polynomial are not so easy to describe, but they are in fact expressible in terms of subdeterminants of  $\det(A)$ . For example, take the case of  $t^{n-2}$ . A term in  $t^{n-2}$  arises in two ways: from the product  $(t - a_{11})(t - a_{22}) \cdots (t - a_{nn})$  or from products like  $-a_{12}a_{21}(t - a_{33}) \cdots (t - a_{nn})$ . So a typical contribution to the coefficient of  $t^{n-2}$  is

$$(a_{11}a_{22} - a_{12}a_{21}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

From this one can see that the term of degree  $n - 2$  in  $p(t)$  is  $t^{n-2}$  times the sum of all the  $2 \times 2$  sub-determinants of the form

$$\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

where  $i < j$ .

In general it can be shown by similar considerations that the following is true.

**(8.4.2)** *The characteristic polynomial of the  $n \times n$  matrix  $A$  is*

$$\sum_{i=0}^n (-1)^i d_i t^{n-i}$$

where  $d_i$  is the sum of all the  $i \times i$  subdeterminants of  $\det(A)$  whose principal diagonals are part of the principal diagonal of  $A$ .

Next let  $c_1, c_2, \dots, c_n$  be the eigenvalues of  $A$  in the splitting field of its characteristic polynomial  $p(t)$ . Since  $p(t)$  is monic, we have

$$p(t) = (t - c_1)(t - c_2) \cdots (t - c_n).$$

The constant term in this product is evidently  $(-1)^n c_1 c_2 \cdots c_n$ , while the term in  $t^{n-1}$  has coefficient  $-(c_1 + \cdots + c_n)$ . On the other hand, we found these coefficients to be  $(-1)^n \det(A)$  and  $-\operatorname{tr}(A)$  respectively. Thus we have found two important relations between the eigenvalues and the entries of  $A$ .

**Corollary (8.4.3)** *If  $A$  is a square matrix, the product of the eigenvalues equals  $\det(A)$  and the sum of the eigenvalues equals  $\operatorname{tr}(A)$ .*

Let  $A$  and  $B$  be  $n \times n$  matrices over a field  $F$ . Recall that  $A$  and  $B$  are similar over  $F$  if there is an invertible  $n \times n$  matrix  $S$  over  $F$  such that  $B = SAS^{-1}$ . The next result indicates that similar matrices have much in common.

**(8.4.4)** *Similar matrices have the same characteristic polynomial. Hence they have the same eigenvalues, trace and determinant.*

*Proof.* Let  $A$  and  $S$  be  $n \times n$  matrices over a field with  $S$  invertible. Then the characteristic polynomial of the matrix  $SAS^{-1}$  is

$$\begin{aligned} \det(tI - SAS^{-1}) &= \det(S(tI - A)S^{-1}) = \det(S) \det(tI - A) \det(S)^{-1} \\ &= \det(tI - A). \end{aligned}$$

(Here we have used the property of determinants,  $\det(PQ) = \det(P) \det(Q)$ ). The statements about trace and determinant follow from (8.4.3).  $\square$

On the other hand, similar matrices need not have the same eigenvectors. Indeed the condition for  $X$  to be an eigenvector of  $SAS^{-1}$  with eigenvalue  $c$  is  $(SAS^{-1})X = cX$ , which is equivalent to  $A(S^{-1}X) = c(S^{-1}X)$ . Thus  $X$  is an eigenvector of  $SAS^{-1}$  if and only if  $S^{-1}X$  is an eigenvector of  $A$ .

**Diagonalizable matrices.** We now consider when a square matrix is similar to a diagonal matrix. This is an important question since diagonal matrices have much simpler properties than arbitrary matrices. For example, when a diagonal matrix is raised to the  $m$ th power, the effect is merely to raise each element on the diagonal to the  $m$ th power, whereas there is no simple expression for the  $m$ th power of an arbitrary matrix. Suppose we want to compute  $A^m$  where  $A$  is similar to a diagonal matrix  $D$ , with say  $A = SDS^{-1}$ . Then  $A^m = (SDS^{-1})^m = SD^mS^{-1}$  after cancellation. Thus it is possible to calculate  $A^m$  quite simply if we have explicit knowledge of  $S$  and  $D$ .

Let  $A$  be a square matrix over a field  $F$ . Then  $A$  is said to be *diagonalizable* over  $F$  if it is similar to a diagonal matrix  $D$  over  $F$ , that is, there is an invertible matrix  $S$  over  $F$  such that  $A = SDS^{-1}$  or equivalently  $D = S^{-1}AS$ . We also say that  $S$  *diagonalizes*  $A$ .

The terminology extends naturally to linear operators on a finite dimensional vector space  $V$ . A linear operator  $\alpha$  on  $V$  is said to be *diagonalizable* if there is a basis  $\{v_1, \dots, v_n\}$  such that  $\alpha(v_i) = c_i v_i$  where  $c_i \in F$ , for  $i = 1, \dots, n$ . Thus  $\alpha$  is represented by the diagonal matrix  $\text{diag}(c_1, c_2, \dots, c_n)$  with respect to this basis.

It is an important observation that if a matrix  $A$  is diagonalizable and its eigenvalues are  $c_1, \dots, c_n$ , then  $A$  must be similar to the diagonal matrix with  $c_1, \dots, c_n$  on the principal diagonal. This is because similar matrices have the same eigenvalues and the eigenvalues of a diagonal matrix are just the entries on the principal diagonal.

We aim to find a criterion for a square matrix to be diagonalizable. A key step in the search is next.

**(8.4.5)** *Let  $A$  be an  $n \times n$  matrix over a field  $F$  and let  $c_1, \dots, c_r$  be distinct eigenvalues of  $A$  with associated eigenvectors  $X_1, \dots, X_r$ . Then  $\{X_1, \dots, X_r\}$  is a linearly independent subset of  $F^n$ .*

*Proof.* Assume the theorem is false; then there is a positive integer  $i$  such that  $\{X_1, \dots, X_i\}$  is linearly independent, but adjunction of the vector  $X_{i+1}$  produces the linearly dependent set  $\{X_1, \dots, X_i, X_{i+1}\}$ . Hence there are scalars  $d_1, \dots, d_{i+1}$ , not all of them zero, such that

$$d_1 X_1 + \dots + d_i X_i + d_{i+1} X_{i+1} = 0.$$

Premultiply both sides of this equation by  $A$  and use the equations  $AX_j = c_j X_j$  to get

$$c_1 d_1 X_1 + \dots + c_i d_i X_i + c_{i+1} d_{i+1} X_{i+1} = 0.$$

On subtracting  $c_{i+1}$  times the first equation from the second, we arrive at the equation

$$(c_1 - c_{i+1})d_1 X_1 + \dots + (c_i - c_{i+1})d_i X_i = 0.$$

Since  $X_1, \dots, X_i$  are linearly independent, the coefficients  $(c_j - c_{i+1})d_j$  must vanish. But  $c_1, \dots, c_{i+1}$  are all different, so it follows that  $d_j = 0$  for  $j = 1, \dots, i$ . Hence  $d_{i+1} X_{i+1} = 0$  and  $d_{i+1} = 0$ , contrary to assumption, so the theorem is proved.  $\square$



A criterion for diagonalizability can now be established.

**(8.4.6)** *Let  $A$  be an  $n \times n$  matrix over a field  $F$ . Then  $A$  is diagonalizable over  $F$  if and only if  $A$  has  $n$  linearly independent eigenvectors in  $F^n$ .*

*Proof.* First of all assume that  $A$  has  $n$  linearly independent eigenvectors in  $F^n$ , say  $X_1, X_2, \dots, X_n$ , and let the associated eigenvalues be  $c_1, c_2, \dots, c_n$ . Define  $S$  to be the  $n \times n$  matrix whose columns are the eigenvectors; thus

$$S = [X_1 \ X_2 \ \dots \ X_n].$$

The first thing to note is that  $S$  is invertible since its columns are linearly independent. Forming the product of  $A$  and  $S$  in partitioned form, we find that

$$AS = [AX_1 \ AX_2 \ \dots \ AX_n] = [c_1X_1 \ c_2X_2 \ \dots \ c_nX_n],$$

so that

$$AS = [X_1 \ X_2 \ \dots \ X_n] \begin{bmatrix} c_1 & 0 & 0 & \dots & 0 \\ 0 & c_2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \cdot & \dots & c_n \end{bmatrix} = SD,$$

where  $D = \text{diag}(c_1, c_2, \dots, c_n)$  is the diagonal matrix with diagonal entries  $c_1, \dots, c_n$ . Therefore  $A = SDS^{-1}$  and  $A$  is diagonalizable.

Conversely, assume that  $A$  is diagonalizable and  $S^{-1}AS = D = \text{diag}(c_1, c_2, \dots, c_n)$ . Here the  $c_i$  must be the eigenvalues of  $A$ . Then  $AS = SD$ , which implies that  $AX_i = c_iX_i$  where  $X_i$  is the  $i$ th column of  $S$ . Therefore  $X_1, X_2, \dots, X_n$  are eigenvectors of  $A$  with associated eigenvalues  $c_1, c_2, \dots, c_n$ . Since  $X_1, X_2, \dots, X_n$  are columns of the invertible matrix  $S$ , they are linearly independent. Consequently  $A$  has  $n$  linearly independent eigenvectors.  $\square$

**Corollary (8.4.7)** *An  $n \times n$  complex matrix with  $n$  distinct eigenvalues is diagonalizable.*

This follows at once from (8.4.5) and (8.4.6). On the other hand, it is easy to find matrices that are not diagonalizable: for example, the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Indeed, if  $A$  were diagonalizable, it would be similar to the identity matrix  $I_2$ , since both eigenvalues of  $A$  equal to 1. But then  $A = SI_2S^{-1} = I_2$  for some  $S$ , a contradiction.

A feature of the proof of (8.4.6) is that it provides a method for finding a matrix  $S$  which diagonalizes  $A$ . It suffices to find a largest set of linearly independent eigenvectors of  $A$ ; if there are enough of them, they can be taken to form the columns of the matrix  $S$ .

**Example (8.4.5)** Find a matrix which diagonalizes

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}.$$

From Example (8.4.2) we know that the eigenvalues of  $A$  are  $3 \pm i$ , so  $A$  is diagonalizable over  $\mathbb{C}$  by (8.4.7). Also corresponding eigenvectors for  $A$  were found which form the matrix

$$S = \begin{bmatrix} \frac{-1+i}{2} & -\frac{1+i}{2} \\ 1 & 1 \end{bmatrix}.$$

From the preceding theory we may be sure that

$$S^{-1}AS = \begin{bmatrix} 3+i & 0 \\ 0 & 3-i \end{bmatrix}.$$

**Triangularizable matrices.** It has been seen that not every complex square matrix is diagonalizable. Compensating for this failure is the fact such a matrix is always similar to an upper triangular matrix.

Let  $A$  be a square matrix over a field  $F$ . Then  $A$  is said to be *triangularizable over  $F$*  if there is an invertible matrix  $S$  over  $F$  such that  $A = STS^{-1}$  or equivalently  $S^{-1}AS = T$ , where  $T$  is upper triangular. It will also be convenient to say that  $S$  *triangularizes  $A$* . Note that the diagonal entries of the triangular matrix  $T$  will necessarily be the eigenvalues of  $A$ . This is because of Example (8.4.3) and the fact that similar matrices have the same eigenvalues. Thus a necessary condition for  $A$  to be triangularizable over  $F$  is that all its eigenvalues belong to  $F$ . In fact the converse is also true.

**(8.4.8)** A square matrix  $A$  over a field  $F$  all of whose eigenvalues lie in  $F$  is triangularizable over  $F$ .

*Proof.* We show by induction on  $n$  that  $A$  is triangularizable. If  $n = 1$ , there is nothing to prove, so let  $n > 1$ . Assume the result is true for  $(n-1) \times (n-1)$  matrices.

By hypothesis  $A$  has at least one eigenvalue  $c$  in  $F$ , with associated eigenvector  $X$  say. Since  $X \neq 0$ , it is possible to adjoin vectors to  $X$  to produce a basis of  $F^n$ , say  $\{X = X_1, X_2, \dots, X_n\}$ ; here we have used (8.2.6). Left multiplication of the vectors of  $F^n$  by  $A$  gives rise to linear operator  $\alpha$  on  $F^n$ . With respect to the basis  $\{X_1, \dots, X_n\}$ , the linear operator  $\alpha$  is represented by a matrix with the special form

$$B_1 = \begin{bmatrix} c & A_2 \\ 0 & A_1 \end{bmatrix}$$

where  $A_1$  and  $A_2$  are matrices over  $F$  and  $A_1$  has  $n-1$  rows and columns. The reason for the special form is that  $\alpha(X_1) = AX_1 = cX_1$  since  $X_1$  is an eigenvector of  $A$  with associated eigenvalue  $c$ . The matrices  $A$  and  $B_1$  are similar since they represent the

same linear operator  $\alpha$ . Suppose that in fact  $B_1 = S_1^{-1}AS_1$  where  $S_1$  is an invertible  $n \times n$  matrix.

Observe that the eigenvalues of  $A_1$  are among those of  $B_1$  and hence  $A$ , so they are all distinct. By induction on  $n$  there is an invertible matrix  $S_2$  with  $n-1$  rows and columns such that  $B_2 = S_2^{-1}A_1S_2$  is upper triangular. Now write

$$S = S_1 \begin{bmatrix} 1 & 0 \\ 0 & S_2 \end{bmatrix}.$$

This is a product of invertible matrices, so it is invertible. An easy matrix computation shows that

$$S^{-1}AS = \begin{bmatrix} 1 & 0 \\ 0 & S_2^{-1} \end{bmatrix} (S_1^{-1}AS_1) \begin{bmatrix} 1 & 0 \\ 0 & S_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & S_2^{-1} \end{bmatrix} B_1 \begin{bmatrix} 1 & 0 \\ 0 & S_2 \end{bmatrix}.$$

From this we obtain

$$S^{-1}AS = \begin{bmatrix} 1 & 0 \\ 0 & S_2^{-1} \end{bmatrix} \begin{bmatrix} c & A_2 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & S_2 \end{bmatrix} = \begin{bmatrix} c & A_2S_2 \\ 0 & S_2^{-1}A_1S_2 \end{bmatrix} = \begin{bmatrix} c & A_2S_2 \\ 0 & B_2 \end{bmatrix} = T.$$

The matrix  $T$  is upper triangular, so the theorem is proved.  $\square$

The preceding proof provides a method for triangularizing a matrix.

**Example (8.4.6)** Triangularize the matrix  $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$  over  $\mathbb{C}$ .

The characteristic polynomial of  $A$  is  $t^2 - 4t + 4$ , so both eigenvalues equal 2. Solving  $(2I_2 - A)X = 0$ , we find that all the eigenvectors of  $A$  are scalar multiples of  $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Therefore by (8.4.6) the matrix  $A$  is *not* diagonalizable.

Let  $\alpha$  be the linear operator on  $\mathbb{C}^2$  arising from left multiplication by  $A$ . Adjoin a vector to  $X_2$  to  $X_1$  to get a basis  $\mathcal{B}_2 = \{X_1, X_2\}$  of  $\mathbb{C}^2$ : for example let  $X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Denote by  $\mathcal{B}_1$  the standard basis of  $\mathbb{C}^2$ . The change of basis  $\mathcal{B}_2 \rightarrow \mathcal{B}_1$  has transition matrix  $S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ , so  $S_1 = S^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$  is the transition matrix of the change of basis  $\mathcal{B}_1 \rightarrow \mathcal{B}_2$ . Therefore by (8.3.13) the matrix that represents  $\alpha$  with respect to the basis  $\mathcal{B}_2$  is  $S_1AS_1^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = T$ . Hence  $A = S_1^{-1}TS_1 = STS^{-1}$  and  $S$  triangularizes  $A$ .

To conclude the chapter we show how to solve a system of linear recurrences by using matrix diagonalization.

**Example (8.4.7)** In a population of rabbits and weasels it is observed that each year the number of rabbits is equal to four times the number of rabbits less twice the number

of weasels in the previous year. The number of weasels in any year equals the sum of the numbers of rabbits and weasels in the previous year. If the initial numbers of rabbits and weasels were 100 and 10 respectively, find the numbers of each species after  $n$  years.

Let  $r_n$  and  $w_n$  denote the respective numbers of rabbits and weasels after  $n$  years. The information given translates into the two *linear recurrence relations*

$$\begin{cases} r_{n+1} = 4r_n - 2w_n \\ w_{n+1} = r_n + w_n \end{cases}$$

together with the initial conditions  $r_0 = 100$ ,  $w_0 = 10$ . We have to solve this system of linear recurrence relations for  $r_n$  and  $w_n$ .

To see how eigenvalues enter into the problem, write the system of recurrences in matrix form. Put  $X_n = \begin{bmatrix} r_n \\ w_n \end{bmatrix}$  and  $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ . Then the two recurrences are equivalent to the single matrix equation

$$X_{n+1} = AX_n,$$

while the initial conditions assert that  $X_0 = \begin{bmatrix} 100 \\ 10 \end{bmatrix}$ . These equations enable us to calculate successive vectors  $X_n$ ; for  $X_1 = AX_0$ ,  $X_2 = A^2X_0$  and in general  $X_n = A^nX_0$ .

In principle this provides a solution to the problem. However, it involves calculating powers of the matrix  $A$ . Fortunately  $A$  is diagonalizable since it has distinct eigenvalues 2 and 3. Corresponding eigenvectors are found to be  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ; therefore the matrix

$S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$  diagonalizes  $A$ , and

$$S^{-1}AS = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = D.$$

It is now easy to compute powers since  $A^n = (SDS^{-1})^n = SD^nS^{-1}$ . Therefore  $X_n = A^nX_0 = SD^nS^{-1}X_0$  and thus

$$X_n = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 100 \\ 10 \end{bmatrix},$$

which leads to

$$X_n = \begin{bmatrix} 180 \cdot 3^n - 80 \cdot 2^n \\ 90 \cdot 3^n - 80 \cdot 2^n \end{bmatrix}.$$

The solution to the problem can now be read off:

$$r_n = 180 \cdot 3^n - 80 \cdot 2^n \quad \text{and} \quad w_n = 90 \cdot 3^n - 80 \cdot 2^n.$$

Notice that  $r_n$  and  $w_n$  both increase without limit as  $n \rightarrow \infty$  since  $3^n$  is the dominant term; however,  $\lim_{n \rightarrow \infty} \left( \frac{r_n}{w_n} \right) = 2$ . The conclusion is that, while both populations explode, in the long run there will be twice as many rabbits as weasels.

**Exercises (8.4)**

(1) Find the eigenvectors and eigenvalues of the following matrices:

$$\begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix}; \quad \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}; \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 4 \end{bmatrix}.$$

(2) Prove that  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$  and  $\text{tr}(cA) = c \text{tr}(A)$  where  $A$  and  $B$  are  $n \times n$  matrices and  $c$  is a scalar.

(3) If  $A$  and  $B$  are  $n \times n$  matrices, show that  $AB$  and  $BA$  have the same eigenvalues.

(4) Suppose that  $A$  is a square matrix with real entries and real eigenvalues. Prove that each eigenvalue of  $A$  has an associated *real* eigenvector.

(5) A real square matrix with distinct eigenvalues is diagonalizable over  $\mathbb{R}$ : true or false?

(6) Let  $p(t)$  be the polynomial  $t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_0$  over a field  $F$ . Show that  $p(t)$  is the characteristic polynomial of the matrix

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}.$$

(This is called the *companion matrix* of  $p(t)$ ):

(7) Find matrices which diagonalize the following matrices:

$$(a) \begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix}; \quad (b) \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

(8) For which values of  $a$  and  $b$  is the matrix  $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$  diagonalizable over  $\mathbb{C}$ ?

(9) Prove that a complex  $2 \times 2$  matrix is *not* diagonalizable if and only if it is similar to a matrix of the form  $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$  where  $b \neq 0$ .

(10) Let  $A$  be a diagonalizable matrix and assume that  $S$  is a matrix which diagonalizes  $A$ . Prove that a matrix  $T$  diagonalizes  $A$  if and only if it is of the form  $T = CS$  where  $C$  is a matrix such that  $AC = CA$ .

(11) If  $A$  is a non-singular matrix with eigenvalues  $c_1, \dots, c_n$ , show that the eigenvalues of  $A^{-1}$  are  $c_1^{-1}, \dots, c_n^{-1}$ .

(12) Let  $\alpha$  be a linear operator on a complex  $n$ -dimensional vector space  $V$ . Prove that there is a basis  $\{v_1, \dots, v_n\}$  of  $V$  such that  $\alpha(v_i)$  is a linear combination of  $v_1, v_2, \dots, v_i$  for  $i = 1, \dots, n$ .

- (13) Let  $T : P_n(\mathbb{R}) \rightarrow P_n(\mathbb{R})$  be the linear operator corresponding to differentiation. Show that all the eigenvalues of  $T$  are zero. What are the eigenvectors?
- (14) Let  $c_1, \dots, c_n$  be the eigenvalues of a complex matrix  $A$ . Prove that the eigenvalues of  $A^m$  are  $c_1^m, \dots, c_n^m$  where  $m$  is any positive integer.
- (15) Prove that a square matrix and its transpose have the same eigenvalues.
- (16) Use matrix diagonalization to solve the following system of linear recurrences:

$$\begin{cases} x_{n+1} = 2x_n + 10y_n \\ y_{n+1} = 2x_n + 3y_n \end{cases}$$

with the initial conditions  $x_0 = 0, y_0 = 1$ .

## 9 Introduction to modules

After groups, rings and vector spaces, the most useful algebraic structures are probably modules. A module is an abelian group on which a ring acts subject to natural rules. Aside from their intrinsic interest as algebraic objects, modules have important applications to linear operators, canonical forms of matrices and representations of groups.

### 9.1 Elements of module theory

Let  $R$  be a ring and let  $M$  be an abelian group which is written additively. Then  $M$  is said to be a *left  $R$ -module* if there is a *left action* of  $R$  on  $M$ , i.e., a map from  $R \times M$  to  $M$ , written  $(r, a) \mapsto r \cdot a$ , ( $r \in R$ ,  $a \in M$ ), such that the following axioms are valid for all  $r, s \in R$  and  $a, b \in M$ :

- (i)  $r \cdot (a + b) = r \cdot a + r \cdot b$ ;
- (ii)  $(r + s) \cdot a = r \cdot a + s \cdot a$ ;
- (iii)  $(rs) \cdot a = r \cdot (s \cdot a)$ .

If the ring  $R$  has an identity element and if in addition

- (iv)  $1_R \cdot a = a$ ,

for all  $a \in M$ , the module  $M$  is called *unitary*. It will be a tacit assumption here that whenever a ring  $R$  has identity, an  $R$ -module is unitary.

A *right  $R$ -module* is defined in the analogous fashion via a *right action* of  $R$  on  $M$ . Sometimes it is convenient to indicate whether an  $R$ -module  $M$  is left or right by writing

$${}_R M \text{ or } M_R,$$

respectively

It is usually not necessary to study left and right  $R$ -modules separately since one can always pass to modules over the *opposite ring*

$$R^{\text{opp}}$$

of  $R$ . This is the ring with the same underlying set and operation of addition as  $R$ , but with the *opposite multiplication*, i.e.,

$$r * s = sr, \quad (r, s \in R).$$

It is easy to check that  $R^{\text{opp}}$  is a ring. Of course  $R = R^{\text{opp}}$  if  $R$  is a commutative ring. The relation between left and right modules is made clear by the next result.

**(9.1.1)** *Let  $R$  be a ring and  $M$  an  $R$ -module.*

- (i) *If  $M$  is a left  $R$ -module, it is also a right  $R^{\text{opp}}$ -module with the right action  $a \cdot r = r \cdot a$ .*

(ii) If  $M$  is a right  $R$ -module, it is also a left  $R^{opp}$ -module with the left action  $r \cdot a = a \cdot r$ .

*Proof.* (i) The axioms for a right action have to be verified, the crucial one being

$$(a \cdot r) \cdot s = (r \cdot a) \cdot s = s \cdot (r \cdot a) = (sr) \cdot a = (r * s) \cdot a = a \cdot (r * s) :$$

here  $*$  denotes the ring operation in  $R^{opp}$ . The proof of (ii) is similar.  $\square$

This result allows us to concentrate on left modules.

**Elementary properties.** The simplest consequences of the module axioms are collected in the next result, which, as will usually be the case, is stated for left modules.

**(9.1.2)** *Let  $M$  be a left  $R$ -module and let  $a \in M$ ,  $r \in R$  and  $n \in \mathbb{Z}$ . Then:*

- (i)  $r \cdot 0_M = 0_M$ ;
- (ii)  $0_R \cdot a = 0_M$ ;
- (iii)  $n(r \cdot a) = (nr) \cdot a = r \cdot (na)$ .

*Proof.* For (i) put  $a = 0_M = b$  in module axiom (i): for (ii) put  $r = 0_R = s$  in axiom (ii). The proof of (iii) requires a little more effort. If  $n > 0$ , the statements are quickly proved by induction on  $n$ . For  $n = 0$  they follow at once from (i) and (ii).

Next consider the case  $n = -1$ . The elements  $(-r) \cdot a$  and  $r \cdot (-a)$  both equal  $-(r \cdot a)$  since  $(-r) \cdot a + r \cdot a = (-r + r) \cdot a = 0_R \cdot a = 0_M$  and  $r \cdot (-a) + r \cdot a = r \cdot (-a + a) = r \cdot 0_M = 0_M$  by (i) and (ii).

Finally, let  $n < 0$ . Then  $-n(r \cdot a) = (-n)r \cdot a = r \cdot (-na)$ . Take the negative of each side and use the case  $n = -1$  to get  $n(r \cdot a) = (nr) \cdot a = r \cdot (na)$ , as required.  $\square$

In future we will write 0 for both  $0_R$  and  $0_M$ .

**Examples of modules.** Next we list some standard sources of modules.

(i) Let  $R$  be an arbitrary ring. Define a left action of  $R$  on itself by using the ring product: thus  $r \cdot s = rs$ , ( $r, s \in R$ ). The ring axioms guarantee the validity of the module axioms. In a similar way  $R$  can be made into a right  $R$ -module using the ring product. To distinguish when the ring is being regarded as a left or a right module, we will often write

$${}_R R \text{ and } R_R$$

respectively.

(ii) Let  $F$  be a field. Then a left  $F$ -module is simply a vector space over  $F$  since the vector space axioms are just those for an  $F$ -module.

(iii) An abelian group  $A$  is a left  $\mathbb{Z}$ -module in which the action is  $n \cdot a = na$ ,  $n \in \mathbb{Z}$ ,  $a \in A$ .

Conversely, if  $A$  is a  $\mathbb{Z}$ -module, the module action is  $n \cdot a = na$ . To see this set  $r = 1$  in (9.1.2)(iii), keeping in mind that  $A$  is a unitary module. Consequently, there is only one way to make an abelian group into a  $\mathbb{Z}$ -module.



These examples show that the module concept is a broad one, encompassing rings, abelian groups and vector spaces.

**Bimodules.** Let  $R$  and  $S$  be a pair of rings. An  $(R, S)$ -bimodule is an abelian group  $M$  which is simultaneously a left  $R$ -module and a right  $S$ -module, and in which the left and right actions are linked by the law

$$(r \cdot a) \cdot s = r \cdot (a \cdot s),$$

where  $r \in R, s \in S, a \in M$ . The notation

$${}_R M_S$$

will be used to indicate an  $(R, S)$ -bimodule. For example, a ring  $R$  is an  $(R, R)$ -bimodule via the ring operations. Of course, if  $R$  is a commutative ring,  $R = R^{\text{opp}}$  and there is no difference between a left  $R$ -module, a right  $R$ -module and an  $(R, R)$ -bimodule.

**Submodules.** Groups have subgroups, rings have subrings and vector spaces have subspaces, so it is to be expected that submodules will play a role in module theory.

Let  $M$  be a left  $R$ -module. An  $R$ -submodule of  $M$  is a subgroup  $N$  of  $M$  which has the additional property

$$a \in N, r \in R \Rightarrow r \cdot a \in N.$$

Notice that  $N$  itself is an  $R$ -module. There is a corresponding definition for right modules. Here are some standard examples of submodules.

- (i) If  $R$  is a ring, the submodules of  ${}_R R$  are the left ideals of  $R$ , while those of  $R_R$  are the right ideals.
- (ii) Every module has the *zero submodule*, containing only the zero element, and the *improper submodule*, namely the module itself.

**Submodules generated by subsets.** Let  $R$  be a ring and  $M$  a left  $R$ -module. It follows quickly from the definition of a submodule that *the intersection of a non-empty set of submodules of  $M$  is itself a submodule*. Now let  $X$  be a non-empty subset of  $M$ . There is at least one submodule of  $M$  containing  $X$ , namely  $M$  itself. Thus we can form the intersection of all the submodules that contain  $X$ , which is a submodule called the *submodule generated by  $X$* . It is evidently the smallest submodule of  $M$  containing  $X$ .

It is natural to ask what the elements of this submodule look like; recall that similar questions arose for subgroups, subrings, ideals and subspaces. The answer in the case of a ring with identity is given next.

**(9.1.3)** *Let  $R$  be a ring with identity and  $M$  a left  $R$ -module. If  $X$  is a non-empty subset of  $M$ , the submodule of  $M$  generated by  $X$  consists of all elements of the form*

$$r_1 \cdot x_1 + r_2 \cdot x_2 + \cdots + r_n \cdot x_n$$

where  $r_i \in R, x_i \in X, n \geq 0$ .

*Proof.* Let  $N$  be the set of all elements of the form  $r_1 \cdot x_1 + r_2 \cdot x_2 + \cdots + r_n \cdot x_n$  with  $r_i \in R$ ,  $x_i \in X$ ,  $n \geq 0$ . (Note that when  $n = 0$ , the sum is to be interpreted as 0). It is an easy verification that  $N$  is a submodule. Now  $X \subseteq N$  since  $x = 1 \cdot x \in N$  for all  $x \in X$ . Hence the submodule  $L$  generated by  $X$  is contained in  $N$ . On the other hand,  $N \subseteq L$ , since it is clear from their form that every element of  $N$  belongs to  $L$ . Therefore  $L = N$ .  $\square$

If  $R$  is a ring with identity and  $X$  is a subset of a left  $R$ -module, the notation

$$R \cdot X$$

will be used to denote the submodule generated by  $X$ . (If  $R$  is a field, so that  $R$ -modules are vector spaces, the notation used in (8.1) for  $R \cdot X$  was  $\langle X \rangle$ ).

An  $R$ -module  $M$  is said to be *finitely generated* if it can be generated by a finite subset  $X$ . An important special case is when  $X = \{x\}$ . In this situation, if  $R$  has an identity, we write  $R \cdot x$  for  $R \cdot X$ ; then  $M$  is called a *cyclic  $R$ -module*. For example, the cyclic submodules of  ${}_R R$  are the principal left ideals of  $R$ , i.e., those of the form  $Rx = \{r \cdot x \mid r \in R\}$ .

**Quotient modules and homomorphisms.** Just as for groups, rings and vector spaces, it is possible to define quotients of modules. Let  $N$  be a submodule of a left  $R$ -module  $M$ . Since  $N$  is a subgroup of the abelian group  $M$ , the quotient  $M/N = \{a + N \mid a \in M\}$ , consisting of all cosets of  $N$  in  $M$ , already has the structure of an abelian group. To make  $M/N$  into a left  $R$ -module a left action must be specified. The natural candidate is the rule

$$r \cdot (a + N) = r \cdot a + N, \quad (a \in M, r \in R).$$

As usual when an operation is to be defined on a quotient structure, the question arises as to whether it is well defined. Let  $b \in a + N$ , so that  $b = a + c$  with  $c \in N$ . Then  $r \cdot b = r \cdot a + r \cdot c \in r \cdot a + N$  since  $r \cdot c \in N$ . Hence  $r \cdot a + N = r \cdot b + N$  and the left action has been well defined. The simple task of checking the validity of the module axioms is left to the reader. The module  $M/N$  is the *quotient module* (or factor module) of  $M$  by  $N$ .

It is to be expected that there will be mappings between modules called module homomorphisms. Let  $M, N$  be two left modules over a ring  $R$ . An  *$R$ -module homomorphism* from  $M$  to  $N$  is a homomorphism of abelian groups

$$\alpha : M \rightarrow N$$

which has the additional property that  $\alpha(r \cdot a) = r \cdot \alpha(a)$  for  $r \in R$ ,  $a \in M$ . Thus the mapping  $\alpha$  connects the module structures of  $M$  and  $N$ .

A standard example is the *canonical homomorphism*  $\nu$  from an  $R$ -module  $M$  to the quotient module  $M/N$  where  $N$  is a submodule of  $M$ . This is defined by  $\nu(a) = a + N$ . We already know from group theory that  $\nu$  is a group homomorphism. To show that it is a module homomorphism simply observe that  $\nu(r \cdot a) = r \cdot a + N = r \cdot (a + N) = r \cdot \nu(a)$ .

**(9.1.4)** Let  $M$  and  $N$  be left modules over a ring  $R$  and let  $\alpha : M \rightarrow N$  be a module homomorphism. Then  $\text{Im}(\alpha)$  and  $\text{Ker}(\alpha)$  are submodules of  $N$  and  $M$  respectively.

Of course group theory tells us that  $\text{Im}(\alpha)$  and  $\text{Ker}(\alpha)$  are subgroups of  $N$  and  $M$ . It is just a matter of verifying that they are submodules, another simple task that is left to the reader.

A module homomorphism which is bijective is called a *module isomorphism*. If there is a module isomorphism between two  $R$ -modules  $M$  and  $N$ , they are said to be  *$R$ -isomorphic*, in symbols

$$M \stackrel{R}{\cong} N.$$

It is an important observation that *the inverse of a module isomorphism is also a module isomorphism* – see Exercise (9.1.4). Therefore isomorphism of  $R$ -modules is an equivalence relation.

**The isomorphism theorems for modules.** Just as in group theory there are theorems connecting module homomorphisms and quotient modules.

**(9.1.5)** (First Isomorphism Theorem) Let  $\alpha : M \rightarrow N$  be an  $R$ -module homomorphism. Then the map  $\theta : M/\text{Ker}(\alpha) \rightarrow \text{Im}(\alpha)$  defined by  $\theta(a + \text{Ker}(\alpha)) = \alpha(a)$ , ( $a \in M$ ), is an isomorphism of  $R$ -modules.

**(9.1.6)** (Second Isomorphism Theorem) Let  $M$  and  $N$  be submodules of an  $R$ -module. Then  $M + N$  and  $M \cap N$  are submodules and  $(M + N)/N \stackrel{R}{\cong} M/(M \cap N)$  via the map  $a + N \mapsto a + (M \cap N)$ , ( $a \in M$ ).

**(9.1.7)** (Third Isomorphism Theorem) Let  $L, M, N$  be submodules of an  $R$ -module such that  $L \subseteq M \subseteq N$ . Then  $M/L$  is a submodule of  $N/L$  and  $(N/L)/(M/L) \stackrel{R}{\cong} N/M$  via the map  $(a + L) + (M/L) \mapsto a + M$ , ( $a \in N$ ).

*Proof.* We know from (4.3.4), (4.3.5) and (4.3.6) that each of the specified maps is an isomorphism of groups. To complete the proofs it is a question of showing that the relevant maps are module homomorphisms. For example, take the case of (9.1.5). By definition  $\theta(r \cdot (a + \text{Ker}(\alpha))) = \theta(r \cdot a + \text{Ker}(\alpha)) = \alpha(r \cdot a) = r \cdot \alpha(a) = r \cdot \theta(a + \text{Ker}(\alpha))$ . In a similar way (9.1.6) and (9.1.7) can be established.  $\square$

We mention, without writing down the details, that there is a module version of the Correspondence Theorem – see (4.2.2). This theorem describes the submodules of a quotient module  $M/N$  as  $L/N$  where  $L$  is a submodule of  $M$  containing  $N$ .

**The structure of cyclic modules.** Sufficient machinery has been developed to permit a description of cyclic  $R$ -modules when  $R$  is a ring with identity.

**(9.1.8)** Let  $R$  be a ring with identity.

- (i) If  $M$  is a cyclic left  $R$ -module, then  $M \cong {}_R R/L$  where  $L$  is a left ideal of  $R$ .
- (ii) Conversely, if  $L$  is a left ideal of  $R$ , then  ${}_R R/L$  is the cyclic left  $R$ -module generated by  $1_R + L$ .

*Proof.* Assume that  $M$  is cyclic and  $M = R \cdot a$  where  $a \in M$ . Define a function  $\alpha : {}_R R \rightarrow M$  by  $\alpha(r) = r \cdot a$ . Check that  $\alpha$  is an  $R$ -module homomorphism. For example, let  $r_1, r \in R$ ; then  $\alpha(r_1 \cdot r) = \alpha(r_1 r) = (r_1 r) \cdot a = r_1 \cdot (r \cdot a) = r_1 \cdot \alpha(r)$ . Also  $\alpha$  is surjective since each element of  $M$  has the form  $r \cdot a = \alpha(r)$  for some  $r \in R$ . Set  $L = \text{Ker}(\alpha)$  and note that  $L$  is a left ideal by (9.1.4). Hence  ${}_R R/L \cong \text{Im}(\alpha) = M$  by 9.1.5. The converse statement is obvious.  $\square$

The kernel of the function  $\alpha$  in the proof of (9.1.8)(i) is the set  $\{r \in R \mid r \cdot a = 0\}$ : this left ideal of  $R$  is called the *annihilator of  $a$  in  $R$*  and is denoted by

$$\text{Ann}_R(a).$$

Since cyclic left  $R$ -modules have been seen to correspond to left ideals of the ring  $R$ , it is to be expected that module theory will be more complicated for rings with many ideals. The simplest situation is, of course, for fields, which have no proper non-zero ideals: in this case we are dealing with vector spaces over a field and every cyclic module is a 1-dimensional space isomorphic with the field itself.

**Direct sums of submodules.** Just as for vector spaces, there is the notion of a direct sum of submodules. Let  $M$  be a module with a family of submodules  $\{M_\lambda \mid \lambda \in \Lambda\}$ . Suppose that

$$M_\lambda \cap \sum_{\mu \neq \lambda} M_\mu = 0$$

for all  $\lambda \in \Lambda$ . Then the  $M_\lambda$  generate their internal *direct sum*, which is written

$$\bigoplus_{\lambda \in \Lambda} M_\lambda.$$

This is a subgroup of  $M$ , as we know from (4.2), (where the multiplicative notation was used). It is evidently also a submodule. We will mainly be concerned with the case where  $\Lambda$  is finite. If  $\Lambda = \{1, 2, \dots, n\}$ , we write the direct sum as

$$M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

It is also possible to form the *external direct sum* of a set of modules – see (4.2) where external direct products of groups were defined. The commonest situation is where there are finitely many modules  $\{M_1, M_2, \dots, M_n\}$ . The external direct sum of these is the set product  $M_1 \times M_2 \times \cdots \times M_n$  where elements are added componentwise and the action of the ring is on components. The external direct sum is also denoted by  $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ : we will sometimes write  $a_1 \oplus a_2 \oplus \cdots \oplus a_n$  for  $(a_1, a_2, \dots, a_n)$

to distinguish the direct sum from the set product. Also any external direct sum is isomorphic with an internal direct sum – cf. the discussion for groups in (4.2).

External direct sums can be extended to the case where there are infinitely many modules – see Exercise (9.1.13) below and also Exercises (4.2.13) and (4.2.14) for the case of groups.

**Finiteness conditions on modules.** Modules are frequently studied in conjunction with finiteness restrictions on their submodules.

**(9.1.9)** *Let  $\mathcal{S}$  be a non-empty set of submodules of a module. Then the following statements about  $\mathcal{S}$  are equivalent.*

- (i) *The set  $\mathcal{S}$  satisfies the ascending chain condition, i.e., there does not exist an infinite ascending chain of submodules  $M_1 \subset M_2 \subset \cdots$  with  $M_i \in \mathcal{S}$ .*
- (ii) *The set  $\mathcal{S}$  satisfies the maximal condition, which asserts that every non-empty subset of  $\mathcal{S}$  has a maximal element, i.e., an element which is not properly contained in any other element of  $\mathcal{S}$ .*

The corresponding result for finiteness conditions on ideals in a ring was proved in (6.4.1). The proof of (9.1.9) is very similar. A module for which the set of all submodules satisfies the equivalent conditions in (9.1.9) is said to be *noetherian*. Notice that if  $R$  is a ring, then  ${}_R R$  is a noetherian  $R$ -module if and only if  $R$  is a left noetherian ring – see (6.4).

The next result provides some insight into the nature of the noetherian condition for modules.

**(9.1.10)** *Let  $M$  be a module. Then  $M$  is noetherian if and only if every submodule of  $M$  is finitely generated.*

Again there was a similar result for rings and ideals (6.4.2); the proof of (9.1.10) is nearly identical.

A noetherian module is always finitely generated, as (9.1.10) shows, but the converse is false: finitely generated modules need not be noetherian – see Exercise (9.1.9). Therefore the next result is of interest.

**(9.1.11)** *Let  $R$  be a left noetherian ring with identity and  $M$  a finitely generated  $R$ -module. Then  $M$  is noetherian.*

*Proof.* By hypothesis there exist elements  $a_1, a_2, \dots, a_k$  such that  $M = R \cdot a_1 + R \cdot a_2 + \cdots + R \cdot a_k$ . Since  $R \cdot a \cong {}_R R / \text{Ann}_R(a)$  by (9.1.8) and  ${}_R R$  is noetherian, we see that  $R \cdot a$  is a noetherian  $R$ -module. Thus the result is true when  $k = 1$ . Let  $k > 1$  and argue by induction on  $k$ ; then  $N = R \cdot a_2 + \cdots + R \cdot a_k$  is noetherian. Next  $M = R \cdot a_1 + N$  and  $M/N \cong R \cdot a_1 / (R \cdot a_1) \cap N$  by (9.1.6), which is noetherian since  $R \cdot a_1$  is noetherian. Finally, since  $M/N$  and  $N$  are both noetherian,  $M$  is noetherian by Exercise (9.1.10).  $\square$

This result provides many examples of noetherian modules. Recall from (6.4.5) that a finitely generated commutative ring with identity is noetherian. Therefore by (9.1.11) *a finitely generated module over a finitely generated commutative ring with identity is noetherian.*

**Bases and free modules.** The concept of a basis of a vector space extends in a natural way to modules. Let  $M$  be a left module over a ring  $R$  with identity. A non-empty subset  $X$  of  $M$  is called an  $R$ -basis of  $M$  if the following hold:

- (i)  $M = R \cdot X$ , i.e.,  $X$  generates  $M$  as an  $R$ -module.
- (ii)  $X$  is  $R$ -linearly independent, i.e., if  $r_1 \cdot x_1 + r_2 \cdot x_2 + \cdots + r_k \cdot x_k = 0$  with  $r_i \in R$  and distinct  $x_i \in X$ , then  $r_1 = r_2 = \cdots = r_k = 0$ .

It is easy to see that these properties taken together are equivalent to every element of the module having a unique expression as an  $R$ -linear combination of elements of  $X$ : cf. (8.2.5) and Exercise (8.2.4).

Unlike vector spaces, modules need not have bases. Indeed there are abelian groups without non-trivial elements of finite order that have no bases.

**Example (9.1.1)** The additive group  $\mathbb{Q}$  of rational numbers does not have a basis.

For suppose that  $\mathbb{Q}$  has a basis  $X$ . If  $X$  contains two different elements  $\frac{m_1}{n_1}, \frac{m_2}{n_2}$ , then

$$m_2 n_1 \frac{m_1}{n_1} - m_1 n_2 \frac{m_2}{n_2} = 0,$$

which contradicts linear independence. Hence  $X$  has just one element  $x$  and  $\mathbb{Q} = \langle x \rangle \simeq \mathbb{Z}$ . But this is certainly wrong since  $\mathbb{Q} = 2\mathbb{Q}$ , whereas  $\mathbb{Z} \neq 2\mathbb{Z}$ .

Let  $R$  be a ring with identity and  $M$  a left  $R$ -module. If  $M$  has a basis  $X$ , then it is called a *free module on  $X$* . If  $R$  is a field, all non-zero modules are free since every vector space has a basis, but, as has been seen, not every  $\mathbb{Z}$ -module has a basis. Free  $\mathbb{Z}$ -modules are called *free abelian groups*.

We will investigate the properties of free modules next. Let  $M$  be a free  $R$ -module with a basis  $X$ . Then  $M = \sum_{x \in X} R \cdot x$  and also  $(R \cdot x) \cap \sum_{y \in X - \{x\}} R \cdot y = 0$  by uniqueness of expression as a linear combination. Hence  $M = \bigoplus_{x \in X} R \cdot x$ . Next  $R \cdot x$  is clearly a cyclic module, so  $R \cdot x \simeq {}_R R / L$  where  $L = \text{Ann}_R(x)$  by (9.1.8). If  $r \in L$ , then  $0 = r \cdot x = 0 \cdot x$ , from which it follows by uniqueness of expression that  $r = 0$  and  $L = 0$ . Thus  $R \cdot x \stackrel{R}{\simeq} {}_R R$ . These conclusions are summed up in:

**(9.1.12)** *Let  $R$  be a ring with identity and  $M$  a free  $R$ -module with a basis  $X$ . Then  $M = \bigoplus_{x \in X} M_x$  where  $M_x \stackrel{R}{\simeq} {}_R R$ .*

The significance of free modules in module theory becomes clear from the next result, which shows that every module is a homomorphic image of a free module.

**(9.1.13)** *Let  $R$  be a ring with identity and let  $M$  be a left  $R$ -module which is generated by a subset  $X = \{x_\lambda \mid \lambda \in \Lambda\}$ . If  $F$  is a free left  $R$ -module with basis  $\bar{X} = \{\bar{x}_\lambda \mid \lambda \in \Lambda\}$ , there is a*

surjective  $R$ -module homomorphism  $\theta : F \rightarrow M$  such that  $\theta(\bar{x}_\lambda) = x_\lambda$  for all  $\lambda \in \Lambda$ . Thus  $M \stackrel{R}{\cong} F/\text{Ker}(\theta)$ .

*Proof.* If  $f \in F$ , there is a unique expression  $f = r_1 \cdot \bar{x}_{\lambda_1} + r_2 \cdot \bar{x}_{\lambda_2} + \cdots + r_n \cdot \bar{x}_{\lambda_n}$  with  $r_i \in R$ ,  $\bar{x}_{\lambda_i} \in \bar{X}$ . Define  $\theta(f) = r_1 \cdot x_{\lambda_1} + r_2 \cdot x_{\lambda_2} + \cdots + r_n \cdot x_{\lambda_n}$ . Then  $\theta$  is a surjective module homomorphism from  $F$  to  $M$ .  $\square$

Next comes a useful property of free modules.

**(9.1.14)** *Let  $M$  be a left  $R$ -module with a submodule  $N$  such that  $M/N$  is free. Then there is a submodule  $F$  such that  $M = N \oplus F$  and  $F \stackrel{R}{\cong} M/N$ .*

*Proof.* Let  $X = \{x_\lambda + N \mid \lambda \in \Lambda\}$  be an  $R$ -basis of  $M/N$  and let  $F$  be the submodule of  $M$  generated by all the elements  $x_\lambda$ . Certainly  $M = N + F$ . Suppose that  $f \in N \cap F$ . Then  $f = r_1 \cdot x_{\lambda_1} + \cdots + r_n \cdot x_{\lambda_n}$  where  $r_i \in R$  and the  $x_{\lambda_i}$  are distinct. Hence

$$r_1 \cdot (x_{\lambda_1} + N) + \cdots + r_n \cdot (x_{\lambda_n} + N) = f + N = 0_{M/N}.$$

Since  $X$  is a basis of  $M/N$ , it follows that  $r_i = 0$  for all  $i$  and  $f = 0$ . Therefore  $N \cap F = 0$  and  $M = N \oplus F$ .  $\square$

Finally, we address the question of the cardinality of a basis in a free module. Recall that any two bases of a finite dimensional vector space have the same cardinal, which is termed the dimension of the space. In general it is possible for a free module to contain bases with different cardinalities. For present purposes the following positive result is sufficient.

**(9.1.15)** *Let  $M$  be a finitely generated free module over a commutative noetherian ring  $R$  with identity. Then every basis of  $M$  is finite and any two bases have the same cardinality.*

*Proof.* Since  $M$  is finitely generated, it can be generated by a finite subset of the basis. Hence the basis is finite, say  $\{x_1, x_2, \dots, x_n\}$ . Since  $R$  is noetherian, there is a maximal proper ideal  $S$  of  $R$ . Then  $K = R/S$  is a field by (6.3.7). Let  $N$  be the subgroup of  $M$  generated by all elements of the form  $s \cdot a$  where  $s \in S$ ,  $a \in M$ . Then  $N$  is a submodule and  $\bar{M} = M/N$  is a  $K$ -vector space via the action  $(r + S) \cdot (a + N) = r \cdot a + N$ ; here it is necessary to check that this action is well defined.

Next we show that  $\{x_i + N \mid i = 1, \dots, n\}$  is a basis of  $\bar{M}$ . Clearly this subset generates  $\bar{M}$ , so it remains to establish  $K$ -linear independence. Suppose that  $\sum_{i=1}^n (r_i + S) \cdot (x_i + N) = 0_{\bar{M}}$  where  $r_i \in R$ . Then  $\sum_{i=1}^n r_i \cdot x_i \in N$ , which shows that

$$\sum_{i=1}^n r_i \cdot x_i = \sum_{i=1}^n s_i \cdot x_i$$

for some  $s_i \in S$ . Since the  $x_i$  are linearly independent, it follows that  $r_i = s_i \in S$  and  $r_i + S = 0_K$ . Thus the  $x_i + N$  are linearly independent and therefore form a basis of the  $K$ -space  $\bar{M}$ . Hence  $n = \dim_K(\bar{M})$ , which shows that all  $R$ -bases of  $M$  have the same number of elements.  $\square$

The cardinality of a basis in a free module  $F$ , when this is unique, is called the *rank* of  $F$ , in symbols  $\text{rank}(F)$ . A zero module is regarded as a free module of rank 0. For an extension of (9.1.15) see (14.1.5).

**Homomorphism groups.** Let  $M$  and  $N$  be left modules over a ring  $R$ . The set of all  $R$ -module homomorphisms from  $M$  to  $N$  is written

$$\text{Hom}_R(M, N).$$

This set can be endowed with the structure of an abelian group in which the group operation is defined as follows. If  $\alpha, \beta \in \text{Hom}_R(M, N)$ , then  $\alpha + \beta : M \rightarrow N$  is given by the rule  $(\alpha + \beta)(a) = \alpha(a) + \beta(a)$  where  $a \in M$ . It is a simple verification that  $\alpha + \beta = \beta + \alpha \in \text{Hom}_R(M, N)$ . The identity element is the zero mapping and the negative of  $\alpha$  is  $-\alpha$  where  $(-\alpha)(a) = -(\alpha(a))$ . The group axioms are quickly verified.

When  $R = F$  is a field,  $\text{Hom}_F(V, W) = L(V, W)$  and  $\text{Hom}_F(V, V) = L(V)$  is the set of linear operators on the  $F$ -vector space  $V$ . In particular  $\text{Hom}_F(V, W)$  is an  $F$ -vector space. In general  $\text{Hom}_R(M, N)$  is not an  $R$ -module, but it can inherit a module structure from  $M$  or  $N$ , as is seen from the next result.

**(9.1.16)** Let  ${}_R M_S$  and  ${}_R N_T$  be bimodules with respect to rings  $R, S, T$  as indicated. Then  $\text{Hom}_R(M, N)$  is an  $(S, T)$ -bimodule in which the module actions of  $S$  and  $T$  are given by

$$(s \cdot \alpha)(a) = \alpha(a \cdot s) \text{ and } (\alpha \cdot t)(a) = \alpha(a) \cdot t$$

where  $a \in M, s \in S, t \in T$ .

*Proof.* We check the module axioms for the first action, leaving the second action to the reader. Let  $\alpha \in \text{Hom}_R(M, N)$ ,  $a_i, a \in M, r \in R, s \in S$ ; then  $(s \cdot \alpha)(a_1 + a_2) = \alpha((a_1 + a_2) \cdot s) = \alpha(a_1 \cdot s + a_2 \cdot s) = \alpha(a_1 \cdot s) + \alpha(a_2 \cdot s) = (s \cdot \alpha)(a_1) + (s \cdot \alpha)(a_2)$ . Also  $(s \cdot \alpha)(r \cdot a) = \alpha((r \cdot a) \cdot s) = \alpha(r \cdot (a \cdot s)) = r \cdot (\alpha(a \cdot s)) = r \cdot ((s \cdot \alpha)(a))$ . Hence  $s \cdot \alpha \in \text{Hom}_R(M, N)$ .

Next it must be proved that  $s \cdot (\alpha_1 + \alpha_2) = s \cdot \alpha_1 + s \cdot \alpha_2$ ,  $(s_1 + s_2) \cdot \alpha = s_1 \cdot \alpha + s_2 \cdot \alpha$  and  $s_1 \cdot (s_2 \cdot \alpha) = (s_1 s_2) \cdot \alpha$ , where  $s, s_i \in S, \alpha, \alpha_i \in \text{Hom}_R(M, N)$ . Let us take the third statement, leaving the others to the reader. If  $a \in M$ , we have  $(s_1 \cdot (s_2 \cdot \alpha))(a) = (s_2 \cdot \alpha)(a \cdot s_1) = \alpha((a \cdot s_1) \cdot s_2) = \alpha(a \cdot (s_1 s_2)) = ((s_1 s_2) \cdot \alpha)(a)$ , as required.

Finally, we check the bimodule property. Let  $\alpha \in \text{Hom}_R(M, N), s \in S, t \in T$ ; then  $((s \cdot \alpha) \cdot t)(a) = ((s \cdot \alpha)(a)) \cdot t = (\alpha(a \cdot s)) \cdot t = (\alpha \cdot t)(a \cdot s) = (s \cdot (\alpha \cdot t))(a)$  for all  $a \in M$ . Therefore  $(s \cdot \alpha) \cdot t = s \cdot (\alpha \cdot t)$ .  $\square$

Of course, if we only have  ${}_R M_S$  or  ${}_R N_T$ , then  $\text{Hom}_R(M, N)$  is merely a left  $S$ -module or a right  $T$ -module respectively.

**Induced mappings.** If a homomorphism between modules is given, it can lead to an “induced homomorphism” between homomorphism groups.

**(9.1.17)** Let  $A, B, M$  be left modules over a ring  $R$  and let  $\alpha : A \rightarrow B$  be a module homomorphism. Then the following are true.



- (i) There is a group homomorphism  $\alpha_* : \text{Hom}_R(M, A) \rightarrow \text{Hom}_R(M, B)$  such that  $\alpha_*(\theta) = \alpha\theta$ .
- (ii) There is a group homomorphism  $\alpha^* : \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(A, M)$  such that  $\alpha^*(\phi) = \phi\alpha$ .

*Proof.* Only (ii) will be proved. Let  $\phi \in \text{Hom}_R(B, M)$ . Certainly  $\alpha^*(\phi) = \phi\alpha$  is a function from  $A$  to  $M$ . We check that it is an  $R$ -module homomorphism. Let  $a, a_i \in A$  and  $r \in R$ . Firstly  $\alpha^*(\phi)(a_1 + a_2) = \phi\alpha(a_1 + a_2) = \phi(\alpha(a_1) + \alpha(a_2)) = \phi\alpha(a_1) + \phi\alpha(a_2) = \alpha^*(\phi)(a_1) + \alpha^*(\phi)(a_2)$ . Then  $(\alpha^*(\phi))(r \cdot a) = \phi\alpha(r \cdot a) = \phi(r \cdot \alpha(a)) = r \cdot (\phi\alpha(a)) = r \cdot (\alpha^*(\phi)(a))$ . Hence  $\alpha^*(\phi) \in \text{Hom}_R(A, M)$ .

Finally, we prove that  $\alpha^*$  is a group homomorphism. Let  $\phi_i \in \text{Hom}_R(B, M)$ . Then  $\alpha^*(\phi_1 + \phi_2) = (\phi_1 + \phi_2)\alpha = \phi_1\alpha + \phi_2\alpha = \alpha^*(\phi_1) + \alpha^*(\phi_2)$ , as required.  $\square$

The induced mappings just defined have notable properties when applied to composites.

**(9.1.18)** Let  $A, B, C, M$  be left modules over a ring  $R$  and let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  be  $R$ -module homomorphisms. Then the induced mappings satisfy (i)  $(\beta\alpha)_* = \beta_*\alpha_*$  and (ii)  $(\beta\alpha)^* = \alpha^*\beta^*$ .

*Proof.* For example, to prove (ii) let  $\phi \in \text{Hom}_R(C, M)$ . Then  $(\beta\alpha)^*(\phi) = \phi(\beta\alpha) = (\phi\beta)\alpha = \beta^*(\phi)\alpha = \alpha^*(\beta^*(\phi)) = \alpha^*(\beta^*(\phi))$  and hence  $(\beta\alpha)^* = \alpha^*\beta^*$ .  $\square$

**Exact sequences.** An exact sequence of modules over a ring  $R$  is a chain of  $R$ -modules and  $R$ -module homomorphisms

$$\cdots \longrightarrow A_{i-1} \xrightarrow{\alpha_{i-1}} A_i \xrightarrow{\alpha_i} A_{i+1} \longrightarrow \cdots$$

such that  $\text{Im}(\alpha_{i-1}) = \text{Ker}(\alpha_i)$  for all  $i$ . Here the chain can be finite or infinite in either direction. We note some important types of exact sequences:

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \quad \text{and} \quad A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0.$$

In the first sequence exactness at  $A$  means that  $\text{Ker}(\alpha) = 0$ , i.e.,  $\alpha$  is injective: in the second exactness at  $C$  shows that  $\beta$  is surjective. The combination of the two types

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

is called a *short exact sequence*: in this case  $A \overset{R}{\cong} \text{Im}(\alpha) = \text{Ker}(\beta)$  and  $B/\text{Ker}(\beta) \overset{R}{\cong} C$ .

The Hom construction has the critical property of preserving exactness of sequences on the left.

**(9.1.19)** (Left exactness of Hom) Let  $M$  be a left  $R$ -module where  $R$  is an arbitrary ring. Then the following hold.

(i) If  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$  is an exact sequence of left  $R$ -modules, the induced sequence of abelian groups and homomorphisms

$$0 \rightarrow \operatorname{Hom}_R(M, A) \xrightarrow{\alpha_*} \operatorname{Hom}_R(M, B) \xrightarrow{\beta_*} \operatorname{Hom}_R(M, C)$$

is exact.

(ii) If  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  is an exact sequence of left  $R$ -modules, the induced sequence of abelian groups and homomorphisms

$$0 \rightarrow \operatorname{Hom}_R(C, M) \xrightarrow{\beta^*} \operatorname{Hom}_R(B, M) \xrightarrow{\alpha^*} \operatorname{Hom}_R(A, M)$$

is exact.

*Proof.* Only (i) will be proved, the proof of (ii) being similar. Firstly,  $\alpha_*$  is injective. For suppose that  $\alpha_*(\theta) = 0$ , i.e.,  $\alpha\theta = 0$ . Since  $\alpha$  is injective, it follows that  $\theta = 0$  and hence the sequence is exact at  $\operatorname{Hom}_R(M, A)$ .

Now for exactness at  $\operatorname{Hom}_R(M, B)$ , i.e.,  $\operatorname{Ker}(\beta_*) = \operatorname{Im}(\alpha_*)$ . Since  $\operatorname{Im}(\alpha) = \operatorname{Ker}(\beta)$ , we have  $\beta_*\alpha_* = (\beta\alpha)_* = 0_* = 0$  by (9.1.18). Hence  $\operatorname{Im}(\alpha_*) \subseteq \operatorname{Ker}(\beta_*)$ . Next let  $\phi \in \operatorname{Ker}(\beta_*)$ , so we have  $0 = \beta_*(\phi) = \beta\phi$ . If  $m \in M$ , then  $\beta\phi(m) = 0$ , so  $\phi(m) \in \operatorname{Ker}(\beta) = \operatorname{Im}(\alpha)$ . Hence  $\phi(m) = \alpha(a)$  for some  $a \in A$ . In fact the element  $a$  is unique: for, if also  $\phi(m) = \alpha(a')$ , then  $a = a'$  by injectivity of  $\alpha$ . This allows us to define unambiguously a function  $\theta : M \rightarrow A$  by  $\theta(m) = a$  where  $\phi(m) = \alpha(a)$ . It is easy to see that  $\theta$  is an  $R$ -module homomorphism. Next  $(\alpha_*(\theta))(m) = \alpha\theta(m) = \alpha(a) = \phi(m)$  for all  $m \in M$ . Therefore  $\alpha_*(\theta) = \phi$  and  $\phi \in \operatorname{Im}(\alpha_*)$ , so that  $\operatorname{Ker}(\beta_*) = \operatorname{Im}(\alpha_*)$ , as was to be proved.  $\square$

### Exercises (9.1)

(1) Let  $L, M, N$  be submodules of an  $R$ -module such that  $N \subseteq M$ . Prove the following statements.

(i)  $(L \cap M)/(L \cap N)$  is  $R$ -isomorphic with a submodule of  $M/N$ .

(ii)  $(L + M)/(L + N)$  is  $R$ -isomorphic with a quotient of  $M/N$ .

(2) Let  $L, M, N$  be submodules of an  $R$ -module such that  $N \subseteq M$ . If  $L + M = L + N$  and  $L \cap M = L \cap N$ , prove that  $M = N$ .

(3) Let  $X$  be a non-empty subset of an  $R$ -module  $M$ . If the ring  $R$  does *not* have an identity element, what is the general form of an element of the submodule of  $M$  generated by  $X$ ?

(4) If  $\alpha : M \rightarrow N$  is a module isomorphism, show that  $\alpha^{-1} : N \rightarrow M$  is also a module isomorphism.

(5) State and prove the Correspondence Theorem for modules.

(6) Let  $R$  be a commutative ring with identity. Prove that  $R$  is a field if and only if every non-zero cyclic  $R$ -module is isomorphic with  $R$ .

- (7) Let  $R, S$  be rings and let  ${}_R M_S$  be a bimodule as indicated. If  $R$  has an identity element, prove that  $\text{Hom}_R({}_R R, {}_R M_S) \stackrel{S}{\cong} M$ .
- (8) Prove that  $\text{Hom}_R(-, M)$  is left exact, i.e., establish (9.1.19)(ii).
- (9) Give an example of a finitely generated module which is not noetherian. [Hint: if  $R$  is a ring with identity, then  ${}_R R$  is a finitely generated  $R$ -module].
- (10) Let  $M$  be a module with a submodule  $N$ . If  $N$  and  $M/N$  are noetherian, prove that  $M$  is noetherian.
- (11) Let  $M$  be an  $R$ -module with a submodule  $N$  such that  $M \stackrel{R}{\cong} M/N$ . If  $M$  is noetherian, prove that  $N = 0$ .
- (12) Let  $u, v$  be elements of a PID  $R$  such that  $\gcd\{u, v\} = 1$ . Prove that  $R/Ru \oplus R/Rv \stackrel{R}{\cong} R/Ruv$ . Extend this result to  $n$  mutually relatively prime elements  $u_1, u_2, \dots, u_n$ .
- (13) Following Exercises (4.2.13) and (4.2.14), explain how to define the unrestricted and restricted direct sums of an infinite set of modules.
- (14) An exact sequence of  $R$ -modules and homomorphisms  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  is said to *split* if there is a module homomorphism  $\gamma: C \rightarrow B$  such that  $\beta\gamma$  is the identity map on  $C$ . Prove that in this event  $B = \text{Im}(\alpha) \oplus \text{Im}(\gamma) \stackrel{R}{\cong} A \oplus C$ .
- (15) Prove that an exact sequence  $0 \rightarrow A \rightarrow B \rightarrow F \rightarrow 0$  always splits if  $F$  is a free module.
- (16) Prove that the exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$  in which all the maps are the natural ones does not split.

## 9.2 Modules over principal ideal domains

In this section we restrict attention to modules over commutative rings. The main objective is to determine the structure of finitely generated modules over PID's. This is one of the central results of abstract algebra and it has applications to finitely generated abelian groups, linear operators on finite dimensional vector spaces and canonical forms of matrices.

**Torsion elements.** Let  $R$  be a commutative ring with identity and let  $M$  be an  $R$ -module. Recall that we need not distinguish between left and right modules. An element  $a$  of  $M$  is called an  *$R$ -torsion element* if there exists  $r \neq 0$  in  $R$  such that  $r \cdot a = 0$ . Equivalently, the annihilator  $\text{Ann}_R(a)$  is a non-zero ideal of  $R$ . If every element of  $M$  is a torsion element,  $M$  is called an  *$R$ -torsion module*. On the other hand, if  $0$  is the only torsion element of  $M$ , the module is said to be  *$R$ -torsion-free*. (The terminology comes from topology).

For example, a torsion element of a  $\mathbb{Z}$ -module, i.e., an abelian group, is an element of finite order and a torsion-free  $\mathbb{Z}$ -module is an abelian group in which every non-trivial element has infinite order.

**(9.2.1)** Let  $R$  be an integral domain and  $M$  an  $R$ -module. Then the torsion elements of  $M$  form a submodule  $T$ , the torsion submodule, such that  $M/T$  is torsion-free.

*Proof.* Let  $a$  and  $b$  be torsion elements of  $M$ ; thus there exist  $r, s \neq 0$  in  $R$  such that  $r \cdot a = 0 = s \cdot b$ . Since  $R$  is an integral domain,  $rs \neq 0$ . Now  $rs \cdot (a \pm b) = s \cdot (r \cdot a) \pm r \cdot (s \cdot b) = 0$ , which shows that  $a \pm b \in T$ . Next let  $u \in R$ ; then  $r \cdot (u \cdot a) = u \cdot (r \cdot a) = u \cdot 0 = 0$ , so  $u \cdot a \in T$ . Hence  $T$  is a submodule.

Now suppose that  $a + T$  is a torsion element of  $M/T$ . Then  $r \cdot (a + T) = 0_{M/T} = T$  for some  $r \neq 0$  in  $R$ , that is,  $r \cdot a \in T$ . Therefore  $s \cdot (r \cdot a) = 0$  for some  $s \neq 0$  in  $R$ . Hence  $(sr) \cdot a = 0$  and  $sr \neq 0$ , from which it follows that  $a \in T$  and  $a + T = T = 0_{M/T}$ .  $\square$

**$p$ -Torsion modules.** Next the concept of a torsion module will be refined. Let  $p$  denote an irreducible element of an integral domain  $R$ . An element  $a$  of an  $R$ -module  $M$  is termed a  $p$ -torsion element if  $p^i \cdot a = 0$  for some  $i > 0$ . If every element of  $M$  is  $p$ -torsion, then  $M$  is called a  $p$ -torsion module.

**(9.2.2)** Let  $M$  be a module over a principal ideal domain  $R$  and let  $p$  be an irreducible element of  $R$ . Then the following statements are true.

- (i) The  $p$ -torsion elements form a submodule  $M_p$  of  $M$ , (called the  $p$ -torsion submodule).
- (ii) If  $R$  is a principal ideal domain, a non-zero element  $a$  in  $M$  is a  $p$ -torsion element if and only if  $\text{Ann}_R(a) = (p^i)$  for some  $i > 0$ .

*Proof.* The proof of (i) is a simple exercise. As for (ii), let  $I = \text{Ann}_R(a)$ ; then  $I = (s)$  where  $s \in R$  is a non-zero, non-unit, since  $R$  is a PID. If  $a$  is a  $p$ -torsion element,  $p^j \in (s)$  for some  $j > 0$  and hence  $s$  divides  $p^j$ . Since  $R$  is a UFD by (7.3.2), it follows that  $s = p^i u$  where  $0 < i \leq j$  and  $u$  a unit of  $R$ . Therefore  $I = (s) = (p^i)$ . The converse is clear.  $\square$

The first really significant result about torsion modules is:

**(9.2.3)** (The Primary Decomposition Theorem) Let  $M$  be a torsion module over a principal ideal domain  $R$  and let  $P$  be a complete set of irreducible elements for  $R$ . Then  $M$  is the direct sum of the  $p$ -torsion components  $M_p$  for  $p \in P$ .

*Proof.* Let  $0 \neq a \in M$ . Since  $M$  is a torsion module, there exists  $r \neq 0$  in  $R$  such that  $r \cdot a = 0$ . Note that  $r$  cannot be a unit of  $R$  since otherwise  $a = 0$ . Write  $r = up_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  where the  $p_i$  are distinct elements of  $P$ ,  $e_i > 0$  and  $u$  is a unit of  $R$ . Let  $r_i$  denote the product that remains when the factor  $p_i^{e_i}$  is deleted from  $r$ . Then  $r_1, r_2, \dots, r_k$  are relatively prime since they have no common irreducible factors. By (7.2.3) applied repeatedly, there exist  $s_i \in R$  such that  $r_1 s_1 + r_2 s_2 + \cdots + r_k s_k = 1$ . Consequently  $a = 1 \cdot a = (r_1 s_1) \cdot a + (r_2 s_2) \cdot a + \cdots + (r_k s_k) \cdot a$ . Now  $p_i^{e_i} \cdot ((r_i s_i) \cdot a) = (s_i p_i^{e_i} r_i) \cdot a = s_i \cdot (r \cdot a) = 0$ . Hence  $(r_i s_i) \cdot a \in M_{p_i}$ , from which it follows that  $M$  is the sum of the submodules  $M_p$  with  $p \in P$ .

To complete the proof it must be shown that the sum is direct. Suppose that  $b \in M_p \cap \sum_{q \in P - \{p\}} M_q$ . Then  $p^m \cdot b = 0$  for some  $m > 0$  and also there is an expression

$b = b_1 + b_2 + \cdots + b_\ell$  with  $b_i \in M_{q_i}$  and  $q_i \in P - \{p\}$ . Thus  $q_i^{m_i} \cdot b_i = 0$  for some  $m_i > 0$ , and hence  $q \cdot b = 0$  where  $q = q_1^{m_1} q_2^{m_2} \cdots q_\ell^{m_\ell}$ . Since none of the  $q_i$  can equal  $p$ , the elements  $q$  and  $p^m$  are relatively prime and hence there exist  $u, v \in R$  such that  $p^m u + qv = 1$ . Therefore

$$b = 1 \cdot b = (p^m u + qv) \cdot b = u \cdot (p^m \cdot b) + v \cdot (q \cdot b) = 0,$$

and it follows that  $M_p \cap \sum_{q \in \pi - \{p\}} M_q = 0$ , so the sum is direct.  $\square$

In essence this theorem reduces the study of torsion modules over a PID to the case of  $p$ -torsion modules.

**Submodules of free modules.** Before we can proceed further with the study of finitely generated modules over PID's, we need to gain a better understanding of free modules. As a first step we consider submodules of free modules and show these are also free. For simplicity we will discuss only free modules of finite rank, although the results are true in the infinite case as well.

**(9.2.4)** *Let  $S$  be a submodule of a finitely generated free module  $F$  over a principal ideal domain  $R$ . Then  $S$  is a free module with rank less than or equal to the rank of  $F$ .*

*Proof.* By hypothesis  $F$  has finite rank, say  $r$ . If  $S = 0$ , it is free with rank 0, so we can assume that  $S \neq 0$  and thus  $r > 0$ . Suppose first that  $r = 1$ , so that  $F \stackrel{R}{\cong} R$ . Identifying  $F$  with  $R$ , we see that  $S$  is an ideal of  $R$  and thus  $S = (s)$  for some  $s$ , since  $R$  is a PID. The assignment  $x \mapsto xs$ , ( $x \in R$ ), determines a surjective  $R$ -module homomorphism from  $R$  to  $S$ . It is also injective because  $R$  is a domain, i.e., it is a module isomorphism and  $S \stackrel{R}{\cong} R$ . Thus  $S$  is a free module of rank 1.

Next assume that  $r > 1$  and let  $\{x_1, x_2, \dots, x_r\}$  be a basis of  $F$ . Define  $F_i$  to be the submodule of  $F$  generated by  $x_1, x_2, \dots, x_i$ , so we have the chain of submodules of  $F$

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r = F.$$

Clearly  $F_i$  is free with basis  $\{x_1, x_2, \dots, x_i\}$  and rank  $i$ . Define  $S_i = S \cap F_i$ , a submodule of  $F$ ; then there is a chain of submodules  $0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_r = S$ . By (9.1.6)

$$S_{i+1}/S_i = S \cap F_{i+1}/S \cap F_i \stackrel{R}{\cong} ((S \cap F_{i+1}) + F_i)/F_i \subseteq F_{i+1}/F_i.$$

Since  $F_{i+1}/F_i \stackrel{R}{\cong} R$ , either  $S_i = S_{i+1}$  or  $S_{i+1}/S_i \stackrel{R}{\cong} R$  by the rank 1 case. From (9.1.14) we obtain  $S_{i+1} = S_i \oplus T_{i+1}$ , from which it follows that  $S = T_1 \oplus T_2 \oplus \cdots \oplus T_r$ . In addition  $T_{i+1} \stackrel{R}{\cong} S_{i+1}/S_i$  and hence either  $T_{i+1} = 0$  or  $T_{i+1} \stackrel{R}{\cong} R$ . Therefore  $S$  is a free module with rank at most  $r$ .  $\square$

An important consequence of the last result is:

**Corollary (9.2.5)** *Let  $R$  be a principal ideal domain and let  $M$  be an  $R$ -module which can be generated by  $n$  elements. If  $N$  is a submodule of  $M$ , then  $N$  can be generated by  $n$  or fewer elements.*

*Proof.* By (9.1.13) we have  $M \stackrel{R}{\cong} F/L$  where  $F$  is a free module of rank  $n$  and  $L$  is a submodule. By the Correspondence Theorem for modules,  $N \stackrel{R}{\cong} S/L$  for some submodule  $S$  of  $F$  containing  $L$ . By (9.2.4)  $S$  can be generated by  $n$  or fewer elements, from which it follows that  $N$  also has this property.  $\square$

We are now equipped with sufficient knowledge of free modules over PID's to determine the structure of finitely generated, torsion-free modules.

**(9.2.6)** *Let  $M$  be a finitely generated torsion-free module over a principal ideal domain  $R$ . Then  $M$  is a free module.*

*Proof.* We can assume that  $M \neq 0$ . Suppose that  $M$  is generated by non-zero elements  $a_1, a_2, \dots, a_n$ . If  $n = 1$ , then  $M = R \cdot a_1$  and  $M \stackrel{R}{\cong} R/\text{Ann}_R(a_1)$  by (9.1.8). However,  $\text{Ann}_R(a_1) = 0$  since  $a_1 \neq 0$  and  $M$  is torsion-free. Hence  $M \stackrel{R}{\cong} R$  and  $M$  is a free module of rank 1.

Let  $n > 1$  and use induction on  $n$ . For convenience let us write  $a = a_1$ . Denote by  $N/R \cdot a$  the torsion-submodule of  $M/R \cdot a$ . By (9.2.1) the module  $M/N$  is torsion-free, and clearly it can be generated by  $n - 1$  elements. Therefore by induction hypothesis  $M/N$  is free and (9.1.14) shows that there is a submodule  $L$  such that  $M = N \oplus L$ ; moreover,  $L \stackrel{R}{\cong} M/N$ , so  $L$  is free. Thus it is enough to prove that  $N$  is a free module.

By (9.2.5)  $N$  can be finitely generated, say by  $b_1, b_2, \dots, b_k$ . Since  $b_i \in N$ , there exists  $r_i \neq 0$  in  $R$  such that  $r_i \cdot b_i \in R \cdot a$ . Writing  $r = r_1 r_2 \cdots r_k \neq 0$ , we have  $r \cdot b_i \in R \cdot a$  for  $i = 1, 2, \dots, k$ , which implies that  $r \cdot N \subseteq R \cdot a$ . But  $R \cdot a \stackrel{R}{\cong} R$  since  $\text{Ann}_R(a) = 0$ , so  $r \cdot N$  is free by the case  $n = 1$ . Finally,  $N \stackrel{R}{\cong} r \cdot N$  via the map  $b \mapsto r \cdot b$  and consequently  $N$  is a free module.  $\square$

**Corollary (9.2.7)** *Let  $M$  be a finitely generated module over a principal ideal domain  $R$  and let  $T$  be the torsion submodule of  $M$ . Then  $M = T \oplus F$  where  $F$  is a free module of finite rank.*

*Proof.* By (9.2.1)  $M/T$  is torsion-free and it is evidently finitely generated. Hence  $M/T$  is free by (9.2.6). From (9.1.14) we deduce that  $M = T \oplus F$  where  $F \stackrel{R}{\cong} M/T$ , so  $F$  is free.  $\square$

Combining (9.2.7) with the Primary Decomposition Theorem (9.2.3), we see that the remaining obstacle to determining the structure of finitely generated modules over a PID is the case of a finitely generated  $p$ -torsion module. This is overcome in the next major result.

**(9.2.8)** *Let  $M$  be a finitely generated module over a principal ideal domain  $R$ . Assume that  $M$  is a  $p$ -torsion module for some irreducible element  $p$  of  $R$ . Then  $M$  is a direct sum of finitely many cyclic  $p$ -torsion modules.*

Notice that by (9.1.8) and (9.2.2) a cyclic  $p$ -torsion  $R$ -module is isomorphic with  $R/(p^i)$  for some  $i > 0$ . Thus (9.2.8) shows that the module  $M$  is completely determined by certain powers of irreducible elements of  $R$ .

The proof of (9.2.8) is one of the harder proofs in this book. The reader is advised to look out for the main ideas behind the proof and try not to get bogged down in the details.

*Proof of (9.2.8).* We can suppose that  $M \neq 0$ ; let it be generated by non-zero elements  $b_1, b_2, \dots, b_k$ . Then  $p^{e_i} \cdot b_i = 0$  where  $e_i > 0$ . Let  $e$  be the largest of the  $e_i$ , so that  $p^e \cdot b_i = 0$  for all  $i$  and thus  $p^e \cdot M = 0$ . Choose  $e$  to be the smallest positive integer with this property. Hence there exists  $a \in M$  such that  $p^{e-1} \cdot a \neq 0$ , and thus  $\text{Ann}_R(a) = (p^e) = \text{Ann}_R(M)$ .

The main step in the proof is to establish the following statement.

*For any  $a \in M$  such that  $\text{Ann}_R(a) = (p^e) = \text{Ann}_R(M)$ , the cyclic submodule  $R \cdot a$  is a direct summand of  $M$ .* (\*)

Let us assume the statement (\*) is false: a series of contradictions will then ensue. By (9.2.5) every submodule of  $M$  is finitely generated and hence  $M$  is noetherian by (9.1.10). We claim that  $M$  contains a submodule  $M_0$  which is maximal subject to having the following properties:

- (i)  $\bar{M} = M/M_0$  has an element  $\bar{a}$  such that  $\text{Ann}_R(\bar{a}) = (p^e) = \text{Ann}(\bar{M})$ ;
- (ii)  $R \cdot \bar{a}$  is not a direct summand of  $\bar{M}$ .

Certainly there are submodules with these properties, for example the zero submodule qualifies. The maximal condition on submodules guarantees that there is a maximal one. Since we are only looking for a contradiction, we can just well work with the module  $\bar{M}$ : thus we will assume that  $M_0 = 0$  and  $M = \bar{M}$ . Consequently (\*) is true for every proper quotient of  $M$ , but false for  $M$  itself.

Suppose first that there exists  $b \in M - R \cdot a$  such that  $p \cdot b = 0$ . Notice that  $R \cdot b$  is a module over the field  $R/(p)$  since  $p \cdot b = 0$ ; thus it is a 1-dimensional vector space over  $R/(p)$ . Therefore  $(R \cdot a) \cap (R \cdot b)$ , being a subspace of  $R \cdot b$ , is either 0 or  $R \cdot b$ . In the second case  $R \cdot b \subseteq R \cdot a$  and  $b \in R \cdot a$ , contrary to the choice of  $b$ . Thus  $(R \cdot a) \cap (R \cdot b) = 0$ . Next  $p^{e-1} \cdot (a + R \cdot b) = p^{e-1} \cdot a + R \cdot b$ , which cannot equal  $0_{M/R \cdot b}$ , since otherwise  $p^{e-1} \cdot a \in (R \cdot a) \cap (R \cdot b) = 0$ , another contradiction. Therefore  $p^{e-1} \cdot (a + R \cdot b) \neq 0_{M/R \cdot b}$  and  $\text{Ann}_R(a + R \cdot b) = (p^e) = \text{Ann}_R(M/R \cdot b)$ . This means that the module  $M/R \cdot a$  and the element  $a + R \cdot a$  satisfy the hypotheses of (\*) above. Since  $M/R \cdot b$  is a proper quotient of  $M$ , there is a direct decomposition  $M/R \cdot b = R \cdot (a + R \cdot b) \oplus N/(R \cdot b)$ . Consequently  $M = (R \cdot a) + N$ , while  $(R \cdot a) \cap N \subseteq (R \cdot a) \cap (R \cdot b) = 0$  and  $M = R \cdot a \oplus N$ , contradicting the fact that (\*) is false for  $M$ .

From the discussion of the previous paragraph, we conclude that  $R \cdot a$  contains all elements  $b$  of  $M$  such that  $p \cdot b = 0$ . Let  $c \in M - (R \cdot a)$  be chosen such that  $\text{Ann}_R(c) = (p^k)$  with  $k$  minimal. Then  $1 < k \leq e$  since  $p \cdot c$  cannot equal 0. Next  $0 = p^k \cdot c = p^{k-1} \cdot (p \cdot c)$ , and by minimality of  $k$  we have  $p \cdot c \in R \cdot a$ : now write  $p \cdot c = r \cdot a$  with  $r \in R$ . Thus  $0 = p^k \cdot c = p^{k-1} \cdot (p \cdot c) = p^{k-1} \cdot (r \cdot a) = (p^{k-1}r) \cdot a$ , from which it follows that  $p^e$  divides  $p^{k-1}r$ . Since  $k-1 < e$ , we deduce that  $p$  divides  $r$ . Write  $r = pr'$  with  $r' \in R$ . Then  $p \cdot c = r \cdot a = (pr') \cdot a$  and hence  $p \cdot (c - r' \cdot a) = 0$ . Consequently  $c - r' \cdot a \in R \cdot a$

and hence  $c \in R \cdot a$ . This contradiction finally establishes the truth of the statement (\*) above.

From this point it is but a short step to finish the proof. Writing  $a_1$  for  $a$ , we have shown that  $M = R \cdot a_1 \oplus M_1$  for some finitely generated submodule  $M_1$ . Either  $M_1 = 0$ , in which event  $M = R \cdot a_1$  and we are done, or else  $M_1 \neq 0$  and the same argument may be applied to  $M_1$ , yielding  $M_1 = R \cdot a_2 \oplus M_2$  and  $M = R \cdot a_1 \oplus R \cdot a_2 \oplus M_2$  for a suitable element  $a_2$  and finitely generated submodule  $M_2$ . The argument may be repeated if  $M_2$  is non-zero, and so on. Because the ascending chain condition is valid in the noetherian module  $M$ , we will eventually reach a direct decomposition  $M = R \cdot a_1 \oplus R \cdot a_2 \oplus \cdots \oplus R \cdot a_n$ , and the theorem is proved.  $\square$

The Structure Theorem for finitely generated modules over a PID can now be stated.

**(9.2.9)** *Let  $M$  be a finitely generated module over a principal ideal domain  $R$ . Then  $M$  is the direct sum of finitely many cyclic  $R$ -modules. More precisely*

$$M = F \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_k$$

where  $F$  is a free module of finite rank  $r \geq 0$  and

$$M_i = M_i(1) \oplus M_i(2) \oplus \cdots \oplus M_i(\ell_i), \quad i = 1, 2, \dots, k,$$

where  $M_i(j)$  is the direct sum of  $n_{ij}$  isomorphic copies of  $R/(p_i^j)$ , ( $j = 1, 2, \dots, \ell_i$ ),  $n_{ij} \geq 0$ ,  $n_{i\ell_i} > 0$  and the  $p_i$  are distinct elements in a complete set of irreducible elements for  $R$ .

*Proof.* From (9.2.7) we have  $M = F \oplus T$  where  $T$  is the torsion submodule of  $M$  and  $F$  is a finitely generated free module. Next  $T$  is finitely generated since  $M$  is noetherian, so by (9.2.3)  $T = M_1 \oplus M_2 \oplus \cdots \oplus M_k$  where  $M_i \neq 0$  is the  $p_i$ -torsion submodule of  $M$  and the  $p_i$  are distinct elements in a complete set of irreducibles. Finally, by (9.2.8)  $M_i$  is a direct sum of cyclic  $p_i$ -torsion modules each of which is isomorphic with some  $R/(p_i^j)$ . By grouping together isomorphic cyclic modules in the direct sum, we obtain the desired result.  $\square$

While the last theorem gives a clear picture of the structure of the module  $M$ , it leaves a natural question open, namely, what is the significance of the data  $r, k, p_i, \ell_i, n_{ij}$ ? The module  $M$  will usually have many direct decompositions of the type in (9.2.9), so the question arises as to whether different sets of data could arise from different decompositions. In other words we are asking if  $r, k, p_i, \ell_i, n_{ij}$  are true invariants of the module  $M$ . The answer is supplied by the result that follows.

**(9.2.10)** *Let  $M$  be a finitely generated module over a principal ideal domain  $R$  and suppose that  $M$  has two direct decompositions into cyclic submodules of the type in (9.2.9),*

$$M = F \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_k = \bar{F} \oplus \bar{M}_1 \oplus \bar{M}_2 \oplus \cdots \oplus \bar{M}_{\bar{k}},$$

with corresponding data  $r, k, p_i, \ell_i, n_{ij}$  and  $\bar{r}, \bar{k}, \bar{p}_i, \bar{\ell}_i, \bar{n}_{ij}$ . Then  $r = \bar{r}, k = \bar{k}, p_i = \bar{p}_i, \ell_i = \bar{\ell}_i, n_{ij} = \bar{n}_{ij}$ , after possible reordering of the  $\bar{M}_i$ .



*Proof.* In the first place the torsion submodule of  $M$  is evidently

$$T = M_1 \oplus M_2 \oplus \cdots \oplus M_k = \bar{M}_1 \oplus \bar{M}_2 \oplus \cdots \oplus \bar{M}_{\bar{k}}.$$

Hence  $F \stackrel{R}{\cong} M/T \stackrel{R}{\cong} \bar{F}$  and by (9.1.15) we deduce that  $r = \bar{r}$ . Also the  $p_i$  and  $\bar{p}_i$  are the irreducible elements with non-trivial torsion components in  $M$ . Thus  $k = \bar{k}$  and the  $\bar{p}_i$  can be relabelled so that  $p_i = \bar{p}_i$ . Consequently we can assume that  $M$  itself is a  $p$ -torsion module for some irreducible element  $p$ , and that

$$M = M(1) \oplus M(2) \oplus \cdots \oplus M(\ell) = \bar{M}(1) \oplus \bar{M}(2) \oplus \cdots \oplus \bar{M}(\bar{\ell}),$$

where  $M(j)$  and  $\bar{M}(j)$  are direct sums of  $n_j$  and  $\bar{n}_j$  copies of  $R/(p^j)$  respectively. Note that  $n_\ell, \bar{n}_{\bar{\ell}} > 0$ . Our task is to prove that  $n_j = \bar{n}_j$  and  $\ell = \bar{\ell}$ .

We introduce the useful notation  $M[p] = \{a \in M \mid p \cdot a = 0\}$ ; notice that  $M[p]$  an  $R$ -submodule of  $M$ , indeed it is a vector space over the field  $R/(p)$ . Observe also that  $p^m \cdot (R/(p^j)) \stackrel{R}{\cong} R/(p^{j-m})$  if  $m < j$ , while  $p^m \cdot (R/(p^j)) = 0$  if  $m \geq j$ .

A consequence of these observations is that  $p^m \cdot M(j) = 0$  if  $m \geq j$  and  $p^m \cdot M(j)$  is the direct sum of  $n_j$  copies of  $R/(p^{j-m})$  if  $m < j$ . Therefore  $(p^m \cdot M)[p]$  is an  $R/(p)$ -vector space with dimension  $n_{m+1} + n_{m+2} + \cdots + n_\ell$ . Of course, the same argument may be applied to the second direct decomposition. Now clearly  $(p^m \cdot M)[p]$  depends only on the module  $M$ , not on any particular direct decomposition of it. Therefore, on equating dimensions, we obtain the system of linear equations

$$n_{m+1} + n_{m+2} + \cdots + n_\ell = \bar{n}_{m+1} + \bar{n}_{m+2} + \cdots + \bar{n}_{\bar{\ell}}$$

for  $m = 1, 2, \dots$ . Since  $n_\ell, \bar{n}_{\bar{\ell}} > 0$ , it follows that  $\ell = \bar{\ell}$ . Back solution of the linear system yields  $n_j = \bar{n}_j$ , for  $j = 1, 2, \dots, \ell$ .  $\square$

**Elementary divisors and invariant factors.** If  $M$  is a finitely generated module over a PID  $R$ , the invariants  $p_i^j$  for which  $R/(p_i^j)$  is isomorphic with one of the direct summands of  $M$  in (9.2.9) are called the *elementary divisors* of  $M$ . The torsion submodule is determined by the elementary divisors together with their multiplicities. The elementary divisors are invariants of the module and do not depend on a particular direct decomposition.

Let us suppose that the elementary divisors are arranged to form a rectangular array as shown below,

$$\begin{array}{cccc} p_1^{r_{11}} & p_1^{r_{12}} & \cdots & p_1^{r_{1\ell}} \\ p_2^{r_{21}} & p_2^{r_{22}} & \cdots & p_2^{r_{2\ell}} \\ \vdots & \vdots & \cdots & \vdots \\ p_k^{r_{k1}} & p_k^{r_{k2}} & \cdots & p_k^{r_{k\ell}} \end{array}$$

where  $0 \leq r_{i1} \leq r_{i2} \leq \cdots \leq r_{i\ell}$ , at least one element in each row and column is different from 1, and  $\ell$  is the maximum of  $\ell_1, \ell_2, \dots, \ell_k$ . Here in order to ensure that all the

rows of the array have the same length, it may be necessary to introduce several 1's at the beginning of a row.

Now define

$$s_j = p_1^{r_{1j}} p_2^{r_{2j}} \cdots p_k^{r_{kj}}, \quad j = 1, 2, \dots, \ell,$$

the product of the elements in column  $j$ . The ring elements  $s_1, s_2, \dots, s_\ell$ , which cannot be units, are called the *invariant factors* of  $M$ . These are also invariants of the module since they are expressed in terms of the elementary divisors. The invariant factors have the noteworthy divisibility properties

$$s_1 \mid s_2 \mid \cdots \mid s_\ell$$

since  $r_{ij} \leq r_{i,j+1}$ .

We remark that if  $u, v \in R$  are relatively prime, then  $R/(u) \oplus R/(v) \stackrel{R}{\cong} R/(uv)$ , which is Exercise (9.1.12). This observation allows us to combine all the cyclic modules associated with entries in the  $j$ th column of the array of elementary divisors into a single cyclic submodule  $R/(s_j)$ . In this way we obtain an alternative form of (9.2.9).

**(9.2.11)** *Let  $M$  be a finitely generated module over a principal ideal domain  $R$ . Then*

$$M \stackrel{R}{\cong} F \oplus R/(s_1) \oplus R/(s_2) \oplus \cdots \oplus R/(s_\ell)$$

where  $F$  is a free module of finite rank and the  $s_j$  are the invariant factors of  $M$ .

Here is an example with  $R = \mathbb{Z}$  to illustrate the procedure for finding the invariant factors when the elementary divisors are known.

**Example (9.2.1)** Consider the finite abelian group

$$A = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{5^2}.$$

The elementary divisors of  $A$  are quickly identified from the direct decomposition as 2, 2, 2, 3, 5,  $5^2$ . Arrange these to form an array with 1's inserted appropriately,

$$\begin{array}{ccc} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 5 & 5^2 \end{array}$$

Forming the products of the columns, we find the invariant factors to be 2, 10, 150. Therefore  $A \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{150}$ .

**Presentations of modules** Let  $R$  be a PID and  $M$  a finitely generated  $R$ -module generated by elements  $a_1, a_2, \dots, a_n$ . Suppose that  $F$  is a free  $R$ -module with basis  $\{x_1, x_2, \dots, x_n\}$ . Then by (9.1.13) there is a surjective  $R$ -module homomorphism  $\theta : F \rightarrow M$  such that  $\theta(x_i) = a_i$  for  $i = 1, \dots, n$ . Thus  $M \stackrel{R}{\cong} F/N$  where  $N = \text{Ker}(\theta)$ . By

(9.2.5)  $N$  is a finitely generated  $R$ -module, say with generators  $y_1, y_2, \dots, y_m$ , where  $m \leq n$ , and there are expressions  $y_j = \sum_{k=1}^n u_{jk} \cdot x_k$  with  $u_{jk} \in R$ .

Conversely, suppose we start with a free  $R$  module  $F$  with basis  $\{x_1, x_2, \dots, x_n\}$  and elements  $y_1, y_2, \dots, y_m$  of  $F$  where  $y_j = \sum_{k=1}^n u_{jk} \cdot x_k$ , and  $u_{jk} \in R$ . Let  $N = R \cdot \{y_1, y_2, \dots, y_m\}$  and put  $M = F/N$ . Then  $M$  is a finitely generated  $R$ -module which may be written in the form

$$M = \langle x_1, x_2, \dots, x_n \mid y_1, y_2, \dots, y_m \rangle.$$

This called a *presentation* of the  $R$ -module  $M$ : the  $x_i$  are the *generators* and the  $y_j$  are the *relators* of the presentation. We should think of the generators  $x_1, x_2, \dots, x_n$  as being subject to the *relations*  $y_1 = 0, y_2 = 0, \dots, y_m = 0$ . The presentation is determined by the *presentation matrix*

$$U = [u_{ij}]_{m,n} \in M_{m,n}(R).$$

Since every finitely generated  $R$ -module has a presentation which determines it up to isomorphism, a natural question arises: given a presentation, how can one discover the structure of the module? We will answer the question in the case of modules over a Euclidean domain by describing a procedure which, when applied to a presentation matrix, gives the invariant factors and hence the structure of the module determined by the presentation.

The key observation is that there three types of matrix operation that can be applied to a presentation matrix  $U$  without changing the isomorphism type of the associated module  $M$ . These are:

- (I) Interchange of two rows or columns.
- (II) Addition of an  $R$ -multiple of one row to another.
- (III) Addition of an  $R$ -multiple of one column to another

Clearly interchange of two rows merely changes the order of the relators and of two columns the order of generators. Adding a multiple of row  $i$  to row  $j$  produces a new relator which is a consequence of the relator associated with row  $j$  and which also implies it.

Justification of interchange of two columns requires a little more thought. Suppose we add  $c$  times column  $i$  to column  $j$  where  $c \in R$ . This amounts to replacing the generator  $x_i$  by a new generator  $x'_i = x_i - c \cdot x_j$  with corresponding changes in the relators, as can be seen from the equation

$$u_{ri} \cdot x'_i + (u_{rj} + cu_{ri}) \cdot x_j = u_{ri} \cdot x_i + u_{rj} \cdot x_j.$$

The important point to keep in mind is that, while these operations change the presentation, they do not change the isomorphism type of the corresponding module.

If a matrix  $V$  is obtained from a matrix  $U \in M_{m,n}(R)$  by means of a finite sequence of operations of types (I), (II), (III) above, then  $V$  is said to be  *$R$ -equivalent* to  $U$ , in symbols

$$U \stackrel{R}{\equiv} V.$$

This is obviously an equivalence relation on presentation matrices. The critical result needed is the following.

**(9.2.12)** *Let  $R$  be a Euclidean domain and  $U$  an  $m \times n$  matrix with entries in  $R$ . Then  $U$  is  $R$ -equivalent to an  $m \times n$  diagonal matrix*

$$V = \text{diag}(d_1, d_2, \dots, d_k, 0, \dots, 0)$$

where  $0 \neq d_i \in R$ ,  $k \geq 0$  and  $d_1 | d_2 | \dots | d_k$ .

Note that the matrix  $V$  in (9.2.12) has  $d_1, d_2, \dots, d_k, 0, \dots, 0$  on the principal diagonal and zeroes elsewhere.

*Proof of (9.1.12).* Let  $\delta : R - \{0\} \rightarrow \mathbb{N}$  be the associated function for the Euclidean domain  $R$  and recall that  $R$  is a PID by (7.2.1). We can assume that  $U \neq 0$ . To initiate the procedure move a non-zero entry  $b_1$  to the  $(1, 1)$  position by using row and column interchanges. Suppose that  $b_1$  does not divide some entry  $c$  in row 1 or column 1: let us say the latter, the case of a row being similar. Using the division algorithm for  $R$ , write  $c = b_1 q + b_2$  where  $q, b_2 \in R$  and  $\delta(b_2) < \delta(b_1)$ . Subtract  $q$  times row 1 from the row containing  $c$ , the effect of which is to replace  $c$  by  $b_2$ . Then move  $b_2$  up to the  $(1, 1)$  position.

If  $b_2$  does not divide some entry in row 1 or column 1, repeat the procedure. Continuation of this process yields a sequence of elements  $b_1, b_2, \dots$ , in  $R$  such that  $\delta(b_1) > \delta(b_2) > \dots$ . Since the  $\delta_i$  are non-negative integers, the process must terminate and when this happens, we will have a matrix  $R$ -equivalent to  $U$  with an element  $a_1$  in the  $(1, 1)$  position which divides every entry in row 1 and column 1. By further row and column subtractions we can clear out all the entries in row 1 and column 1 except the  $(1, 1)$  entry to obtain a matrix of the form

$$\begin{bmatrix} a_1 & 0 \\ 0 & U_1 \end{bmatrix}$$

which is  $R$ -equivalent to  $U$ ; here of course  $U_1$  is an  $(m-1) \times (n-1)$  matrix. By induction on  $m$  the matrix  $U_1$  is  $R$ -equivalent to a matrix  $\text{diag}(a_2, a_3, \dots, a_k, 0, \dots, 0)$  and therefore

$$U \stackrel{R}{\equiv} D = \text{diag}(a_1, a_2, a_3, \dots, a_k, 0, \dots, 0).$$

Suppose that  $a_1$  does not divide  $a_2$ . Let  $d = va_1 + wa_2$  be a gcd of  $a_1$  and  $a_2$  with  $v, w \in R$ . Then, using the operations of types (I), (II), (III), we obtain

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \stackrel{R}{\equiv} \begin{bmatrix} a & va_1 + wa_2 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 & d \\ 0 & a_2 \end{bmatrix} \stackrel{R}{\equiv} \begin{bmatrix} d & a_1 \\ a_2 & 0 \end{bmatrix} \stackrel{R}{\equiv} \begin{bmatrix} d & 0 \\ 0 & \frac{-a_1 a_2}{d} \end{bmatrix}.$$

Note that  $d$  divides  $\frac{a_1 a_2}{d}$ . Use this routine to replace  $a_1$  by  $d$  in the diagonal matrix  $D$ . Repeating the procedure for  $a_3, \dots, a_k$ , we get  $U \stackrel{R}{\equiv} \text{diag}(d_1, \bar{a}_2, \dots, \bar{a}_k, 0, \dots, 0)$

where  $d_1$  is a gcd, and hence a linear combination, of  $a_1, a_2, a_3, \dots, a_k$ , and  $d_1$  divides each of  $\bar{a}_2, \dots, \bar{a}_k$ . By induction we conclude that

$$U \stackrel{R}{\equiv} \text{diag}(d_1, d_2, \dots, d_k, 0, \dots, 0)$$

where  $d_2|d_3|\dots|d_k$  and  $d_2$  is an  $R$ -linear combination of  $\bar{a}_2, \bar{a}_3, \dots, \bar{a}_k$ . Hence  $d_1|d_2$  since  $\bar{a}_2, \bar{a}_3, \dots, \bar{a}_k$  are divisible by  $d_1$ . This completes the proof.  $\square$

The diagonal matrix  $V$  in (9.2.12) is called the *Smith normal form*<sup>1</sup> of  $U$ . Its entries are determined only up to units. Let us apply this method to the presentation matrix  $U$  for a finitely generated module  $M = F/N$  over a Euclidean domain  $R$ . Then  $U \stackrel{R}{\equiv} V$  where  $V = \text{diag}(d_1, d_2, \dots, d_k, 0, \dots, 0)$ ,  $0 \neq d_i \in R$  and  $d_1|d_2|\dots|d_k$ . The matrix  $V$  is the Smith normal form of  $U$ ; it gives a new presentation of  $M$  which is much simpler in form, having generators  $x'_1, x'_2, \dots, x'_n$  and relators  $d_1x'_1, d_2x'_2, \dots, d_kx'_k$ . From this presentation we read off that

$$M \stackrel{R}{\cong} R/(d_1) \oplus R/(d_2) \oplus \dots \oplus R/(d_k) \oplus \underbrace{R \oplus \dots \oplus R}_{n-k}.$$

Thus  $n - k$  is the number of cyclic summands isomorphic with  $R$ , while the non-unit  $d_i$ 's are the invariant factors, up to units.

**Example (9.2.2)** Let  $A$  be the abelian group with generators  $x, y, z$  and relations

$$3x + 4y + 3z = 0, \quad 6x + 4y + 6z = 0, \quad 3x + 8y + 3z = 0.$$

In this example  $R = \mathbb{Z}$  and the presentation matrix is

$$U = \begin{bmatrix} 3 & 4 & 3 \\ 6 & 4 & 6 \\ 3 & 8 & 3 \end{bmatrix}.$$

Following the steps in the algorithm in (9.2.12), we find that

$$U \stackrel{\mathbb{Z}}{\equiv} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pm 12 & 0 \\ 0 & 0 & 0 \end{bmatrix} = V,$$

which is the Smith normal form of  $U$ . Hence  $A \simeq \mathbb{Z}_1 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}$ , i.e.,

$$A \simeq \mathbb{Z}_{12} \oplus \mathbb{Z} \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}.$$

The single invariant factor is 12 and the elementary divisors are 3, 4.

<sup>1</sup> Henry John Stephen Smith (1826–1883)

The description of finite abelian groups afforded by the preceding theory is precise enough for us to make an exact count of the groups of given order.

**(9.2.13)** Let  $n > 1$  be an integer and write  $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  where  $e_i > 0$  and the  $p_i$  are distinct primes. Then the number of isomorphism types of abelian groups of order  $n$  is

$$\lambda(e_1)\lambda(e_2)\cdots\lambda(e_k)$$

where  $\lambda(i)$  is the number of partitions of  $i$ .

*Proof.* First let  $A$  be an abelian group of order  $p^e > 1$  where  $p$  is a prime. By (9.2.8)  $A$  is the direct sum of  $\ell_1$  copies of  $\mathbb{Z}_p$ ,  $\ell_2$  copies of  $\mathbb{Z}_{p^2}$ , etc, where  $\ell_i \geq 0$  and  $e = \ell_1 + 2\ell_2 + 3\ell_3 + \cdots$ . Thus we have partition of  $e$  into  $\ell_1$  1-subsets,  $\ell_2$  2-subsets, etc. Conversely, every partition of  $e$  leads to an abelian group of order  $p^e$  and different partitions yield non-isomorphic groups since the invariant factors are different. Therefore the number of possible isomorphism types for  $A$  is  $\lambda(e)$ .

Now let  $A$  be an abelian group of order  $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ ; then  $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$  where  $A_i$  is the  $p_i$ -torsion component and  $|A_i| = p_i^{e_i}$ . There are  $\lambda(e_i)$  possible isomorphism types for  $A_i$ , so the number of isomorphism types for  $A$  is  $\lambda(e_1)\lambda(e_2)\cdots\lambda(e_k)$ .  $\square$

**Example (9.2.3)** Find all abelian groups of order 600.

Since  $600 = 2^3 \cdot 3 \cdot 5^2$ , the number of abelian groups of order 600 is  $\lambda(3)\lambda(1)\lambda(2) = 3 \times 1 \times 2 = 6$ . The isomorphism types are determined by the partitions of 3 and 2, namely  $3 = 1 + 2 = 1 + 1 + 1$ , and  $2 = 1 + 1$ . Hence the six isomorphism types are:

$$\begin{aligned} &\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{5^2}, \mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5, \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{5^2}, \\ &\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{5^2}, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5. \end{aligned}$$

Notice that  $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{5^2}$  is the cyclic group of order 600.

Of course the task of counting the non-abelian groups of given finite order is a much more formidable one.

### Exercises (9.2)

- (1) Let  $R$  be a domain with field of fractions  $F$  and  $R \subseteq F$ . Regard  $F$  as an  $R$ -module via the field operations. Prove that  $F$  is torsion-free and  $F/R$  is a torsion module.
- (2) Let  $R = \mathbb{Z}_6$ , the ring of congruence classes modulo 6. Find the torsion elements in the module  $R$ . Deduce that the torsion elements in a module do not always form a submodule.
- (3) Let  $p_1, p_2, \dots$  be the sequence of primes and let  $\langle a_i \rangle$  be an additively written group of order  $p_i$ . Define  $A$  to be the set of all sequences  $(x_1, x_2, \dots)$  where  $x_i \in \langle a_i \rangle$ . Make  $A$  into an abelian group by adding components.
  - (i) Show that the torsion subgroup  $T$  consists of all sequences in which all but a finite number of components are 0.

- (ii) Prove that  $\bar{A} = A/T$  has the property  $\bar{A} = p\bar{A}$  for all primes  $p$ .
  - (iii) Prove that  $\bigcap_p pA = 0$ .
  - (iv) Deduce from (ii) and (iii) that  $T$  is *not* a direct summand of  $A$ .
- (4) Find the elementary divisors and invariant factors of the group  $\mathbb{Z}_4 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_{35}$ .
- (5) Find all abelian groups of order 1350.
- (6) Let  $A$  be a torsion-free abelian group and define  $D = \bigcap_{n=1,2,\dots} nA$ . Prove that (i)  $A/D$  is torsion-free and (ii)  $D = nD$  for all  $n > 0$ .
- (7) A finitely generated abelian group  $A$  is given by a presentation with generators  $x, y, z, u$  and relators  $x - y - z - u, 3x + y - z + u, 2x + 3y - 2z + t$ . Find the invariant factors of  $A$  and hence its structure.
- (8) Let  $A$  be a finite abelian group and denote by  $v_n(A)$  the number of elements of  $A$  which have order exactly  $n$ .
- (i) If  $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  with distinct primes  $p_i$ , show that
 
$$v_n(A) = v_{p_1^{e_1}}(A) v_{p_2^{e_2}}(A) \cdots v_{p_k^{e_k}}(A).$$
  - (ii) Let  $A$  be a finite abelian  $p$ -group. Define  $A[p^i] = \{a \in A \mid p^i a = 0\}$ . Prove that
 
$$v_{p^e}(A) = |A[p^e]| - |A[p^{e-1}]| \text{ for } e \geq 1.$$
- (9) Let  $A$  be a finite abelian  $p$ -group. Assume that  $A$  is the direct sum of  $r_i$  cyclic groups of order  $p^i$  where  $i = 1, 2, \dots, \ell$ . Prove that  $|A[p^i]| = p^{s_i}$  where  $s_i = r_1 + 2r_2 + \cdots + (i-1)r_{i-1} + i(r_i + r_{i+1} + \cdots + r_\ell)$  for  $1 \leq i \leq \ell$ .
- (10) Let  $A$  and  $B$  be finite abelian groups. If  $v_n(A) = v_n(B)$  for all positive integers  $n$ , prove that  $A \simeq B$ . [Use Exercises (9.2.8) and (9.2.9)].

## 9.3 Applications to linear operators

One of the most convincing applications of modules over PID's is to the study of linear operators on a finite dimensional vector spaces. Since the relation between modules and linear operators is not obvious, some explanation is called for.

Let  $V$  be a finite dimensional vector space over a field  $F$  with  $n = \dim(V) > 0$  and let  $\alpha$  be a fixed linear operator on  $V$ . Set  $R = F[t]$ , the ring of polynomials in  $t$  over  $F$ , and recall that  $R$  is a PID by (7.2.2). The fundamental idea is to make  $V$  into an  $R$ -module by defining

$$f \cdot v = f(\alpha)(v), \quad (f \in R, v \in V).$$

The notation here is as follows: if  $f = a_0 + a_1 t + \cdots + a_m t^m \in R$ , then  $f(\alpha)$  is the linear operator  $a_0 1 + a_1 \alpha + \cdots + a_m \alpha^m$ . (Here  $1$  is the identity linear operator on  $V$ ). It is straightforward to check the validity the module axioms for the specified action.

Next the properties of the  $R$ -module  $V$  will be investigated. Let  $v \in V$ ; since  $\dim(V) = n$ , the subset

$$\{v, \alpha(v), \alpha^2(v), \dots, \alpha^n(v)\}$$

must be linearly dependent by (8.2.3). Hence there exist elements  $a_0, a_1, \dots, a_m$  of  $F$ , not all equal to zero, such that  $a_0v + a_1\alpha(v) + \dots + a_n\alpha^n(v) = 0$ . Put  $g = a_0 + a_1t + \dots + a_nt^n \in R$ , noting that  $g \neq 0$ . Then  $g \cdot v = g(\alpha)(v) = 0$ , so  $V$  is a torsion  $R$ -module. In fact more is true. Let  $\{v_1, v_2, \dots, v_n\}$  be a basis of  $V$ . Then there exist  $g_i \neq 0$  in  $R$  such that  $g_i \cdot v_i = 0$  for  $i = 1, 2, \dots, n$ . Put  $h = g_1g_2 \cdots g_n \neq 0$ ; then  $h \cdot v_i = 0$  for all  $i$  and thus  $h \cdot v = 0$  for all  $v \in V$ , i.e.,  $h(\alpha) = 0$ . It follows that  $\text{Ann}_R(V) \neq 0$ .

Since  $R$  is a PID,  $\text{Ann}_R(V) = (f)$  for some  $f \in R$  and clearly we may choose the polynomial  $f$  to be monic. Thus a polynomial  $g$  belongs to  $\text{Ann}_R(V)$  if and only if  $f$  divides  $g$ , and consequently  $f$  is the unique monic polynomial of smallest degree such that  $f(\alpha) = 0$ . These conclusions are summed up in:

**(9.3.1)** *Let  $\alpha$  be a linear operator on a finite dimensional vector space  $V$  over a field  $F$ . Then there is a unique monic polynomial  $f$  in  $F[t]$  of smallest degree such that  $f(\alpha) = 0$ . Moreover,  $g(\alpha) = 0$  if and only if  $f$  divides  $g$  in  $F[t]$ .*

The polynomial  $f$  is called the *minimum polynomial* of  $\alpha$ . The next step forward is to apply the Primary Decomposition Theorem (9.2.3) to the torsion module  $V$ . According to this result there is a direct decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

where  $V_i \neq 0$  is the  $p_i$ -torsion submodule of  $V$  and  $p_1, p_2, \dots, p_k$  are distinct monic irreducible elements of  $R = F[t]$ . There are only finitely many such  $V_i$  since  $V$  is finite dimensional. The restriction of  $\alpha$  to  $V_i$  is a linear operator  $\alpha_i$ , which has minimum polynomial of the form  $p_i^{e_i}$ . If  $g \in R$ , then  $g(\alpha) = 0$  if and only if  $g(\alpha_i) = 0$ , i.e.,  $p_i^{e_i} | g$ , for all  $i$ . It follows that the minimum polynomial of  $\alpha$  is  $f = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ . Thus we have proved the following theorem.

**(9.3.2)** *Let  $\alpha$  be a linear operator on a finite dimensional vector space  $V$  over a field  $F$ , and suppose that the minimum polynomial of  $\alpha$  is  $f = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ , where the  $p_i$  are distinct monic irreducibles in  $F[t]$  and  $e_i > 0$ . Then  $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$  where  $V_i$  is the  $p_i$ -torsion submodule of  $V$ . Moreover,  $p_i^{e_i}$  is the minimum polynomial of  $\alpha_i = \alpha|_{V_i}$ .*

**The case of an algebraically closed field.** Up to this point the field has been arbitrary. However, important simplifications occur for an algebraically closed field  $F$ : for then an irreducible polynomial over  $F$  has degree 1. In particular these apply to the complex field  $\mathbb{C}$  by the Fundamental Theorem of Algebra – see (12.3.6).

Consider the situation of (9.3.2) when  $F$  is algebraically closed and  $p_i = t - a_i$  with  $a_i \in F$ . The minimum polynomial of  $\alpha$  is

$$f = (t - a_1)^{e_1} (t - a_2)^{e_2} \cdots (t - a_k)^{e_k}.$$

Let  $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$  be the primary decomposition of the  $F[t]$ -module  $V$ , with  $V_i$  the  $p_i$ -torsion component. Thus  $\alpha_i = \alpha|_{V_i}$  has minimum polynomial  $(t - a_i)^{e_i}$  and  $(\alpha_i - a_i 1)^{e_i} = 0$ . This means that  $\alpha_i - a_i 1$  is a *nilpotent* linear operator, i.e., some positive power of it equals to zero.



Define two new linear operators  $\delta, \nu$  on  $V$  by  $\delta|_{V_i} = a_i 1$ , for  $i = 1, 2, \dots, k$ , and  $\nu = \alpha - \delta$ . Then  $\nu_i = \nu|_{V_i} = \alpha_i - a_i 1$  and hence  $\nu_i^{e_i} = 0$ , which implies that  $\nu^e = 0$  where  $e$  is the largest of  $e_1, e_2, \dots, e_k$ . Thus  $\nu$  is a nilpotent linear operator on  $V$ . Notice that  $\delta_i$ , being multiplication by  $a_i$ , commutes with  $\nu_i$ , from which it follows that  $\delta\nu = \nu\delta$ .

The important feature of the linear operator  $\delta$  is that it is diagonalizable, since  $\nu$  acts on  $V_i$  by multiplication by  $a_i$ . This leads to the following result.

**(9.3.3)** *Let  $V$  be a finite dimensional vector space over an algebraically closed field  $F$  and let  $\alpha$  be a linear operator on  $V$ . Then there are linear operators  $\delta, \nu$  on  $V$  such that  $\alpha = \delta + \nu$  and  $\delta\nu = \nu\delta$ , where  $\delta$  is diagonalizable and  $\nu$  is nilpotent.*

Notice that (9.3.3) can be applied to an  $n \times n$  matrix  $A$  if we take  $\alpha$  to be the linear operator  $X \rightarrow AX$  on  $F^n$ . The statement then takes the form that  $A = D + N$  and  $DN = ND$  where  $D$  is diagonalizable and  $N$  is nilpotent.

**Example (9.3.1)** Let  $A = \begin{bmatrix} -7 & 27 \\ -3 & 11 \end{bmatrix}$ . The characteristic polynomial of  $A$  is  $(t - 2)^2$ . The minimum polynomial is also  $(t - 2)^2$ , either by direct matrix multiplication or by (9.3.5) below. Thus  $k = 1$  and  $V = V_1$  in the previous notation; hence  $D = 2I_2$ . Put  $N = A - D = \begin{bmatrix} -9 & 27 \\ -3 & 9 \end{bmatrix}$ , so that  $A = D + N$  and  $N^2 = 0$ ; also note that  $DN = ND$ .

**Rational canonical form.** It is time to apply the full force of the structure theorem for modules over a PID to a linear operator  $\alpha$  on an  $n$ -dimensional vector space  $V$  over an arbitrary field  $F$ . Bear in mind that  $V$  is a torsion module over  $R = F[t]$  via the ring action  $f \cdot v = f(\alpha)(v)$ . Thus by (9.2.11)

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell$$

where  $V_i = R \cdot v_i \cong R/(s_i)$  and  $\text{Ann}_R(v_i) = (s_i)$ . Here  $s_1, s_2, \dots, s_\ell$  are the invariant factors, which satisfy  $s_1 | s_2 | \dots | s_\ell$ . Recall that the  $s_i$  can be chosen to be monic. Let  $\alpha_i = \alpha|_{V_i}$ . If  $g \in R$ , then  $g(\alpha) = 0$  if and only if  $g(\alpha_i) = 0$ , that is,  $g \in (s_i)$  for  $i = 1, 2, \dots, \ell$ . The divisibility property of the  $s_i$  implies that this happens precisely when  $s_\ell$  divides  $g$ . Consequently, *the final invariant factor  $s_\ell$  is the minimum polynomial of  $\alpha$ .*

Next we will show that  $\dim_F(R/(s_i)) = \deg(s_i)$ . Write

$$s_i = t^{n_i} + a_{in_i-1}t^{n_i-1} + \cdots + a_{i1}t + a_{i0}, \quad (a_{ij} \in R).$$

If  $g \in R$ , then  $g = qs_i + r_i$  where  $q, r_i \in R$  and  $r_i = 0$  or  $\deg(r_i) < \deg(s_i) = n_i$ . Then  $g + (s_i) = r_i + (s_i)$ , so that  $\dim_F(R/(s_i)) \leq n_i$ . Suppose that  $1 + (s_i), t + (s_i), \dots, t^{n_i-1} + (s_i)$  are linearly dependent and  $a_0 1 + a_1 t + \cdots + a_{n_i-1} t^{n_i-1} + (s_i) = 0_{R/(s_i)}$  where not all the  $a_i \in F$  are zero. Let  $g = a_0 + a_1 t + \cdots + a_{n_i-1} t^{n_i-1}$ ; thus  $g \neq 0$ . Since  $g + (s_i) = 0_{R/(s_i)}$ , we have  $g \in (s_i)$  and  $s_i$  divides  $g$ . But  $\deg(g) < \deg(s_i)$ , which can only mean that  $g = 0$ . By this contradiction  $1 + (s_i), t + (s_i), \dots, t^{n_i-1} + (s_i)$  are linearly independent

and these elements form an  $F$ -basis of  $R/(s_i)$ . Hence  $\dim(V_i) = \dim(R/(s_i)) = n_i$  and  $\dim(V) = \sum_{i=1}^{\ell} n_i$ .

Since  $V_i \cong R/(s_i)$  via the assignment  $r \cdot v_i \rightarrow r + (s_i)$ , the subspace  $V_i$  has the basis  $\{v_i, \alpha(v_i), \alpha^2(v_i), \dots, \alpha^{n_i-1}(v_i)\}$ . Let us identify the matrix which represents  $\alpha_i$  with respect to this ordered basis. Now  $\alpha(\alpha^j(v_i)) = \alpha^{j+1}(v_i)$  if  $0 \leq j < n_i - 1$  and

$$\alpha(\alpha^{n_i-1}(v_i)) = \alpha^{n_i}(v_i) = -a_{i0}v_i - a_{i1}\alpha(v_i) - \dots - a_{in_i-1}\alpha^{n_i-1}(v_i)$$

since  $s_i(\alpha_i) = 0$ . Therefore  $\alpha_i$  is represented by the  $n_i \times n_i$  matrix

$$R_i = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_{i0} \\ 1 & 0 & \dots & 0 & -a_{i1} \\ 0 & 1 & \dots & 0 & -a_{i2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{in_i-1} \end{bmatrix}.$$

This is the companion matrix of the polynomial  $s_i$  – see Exercise (8.4.6). Note that  $s_i$  is the minimum polynomial of  $\alpha_i$  and hence of  $R_i$ .

Now form the union of the chosen bases of the  $V_i$  to obtain a basis of  $V$  with respect to which  $\alpha$  is represented by the block matrix

$$C = \begin{bmatrix} R_1 & 0 & \dots & 0 \\ 0 & R_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{\ell} \end{bmatrix}.$$

This is called the *rational canonical form* of  $\alpha$ .

Recall that the characteristic polynomial of  $\alpha$  is  $\det(tI_n - R)$ . Now

$$\det(tI_n - C) = \begin{vmatrix} tI_{n_1} - R_1 & 0 & \dots & 0 \\ 0 & tI_{n_2} - R_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & tI_{n_{\ell}} - R_{\ell} \end{vmatrix},$$

which is equal to the product  $\det(tI_{n_1} - R_1)\det(tI_{n_2} - R_2) \cdots \det(tI_{n_{\ell}} - R_{\ell})$ . Also

$$\det(tI_{n_i} - R_i) = \begin{vmatrix} t & 0 & 0 & \dots & 0 & a_{i0} \\ -1 & t & 0 & \dots & 0 & a_{i1} \\ 0 & -1 & t & \dots & 0 & a_{i2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & t + a_{in_i} \end{vmatrix},$$

which by direct determinantal expansion equals  $a_{i0} + a_{i1}t + \dots + a_{in_i-1}t^{n_i-1} + t^{n_i} = s_i$ . Therefore  $\det(tI_n - C) = s_1 s_2 \cdots s_{\ell}$ .

These conclusions are summed up in the following fundamental result.

**(9.3.4) (Rational canonical form)** *Let  $\alpha$  be a linear operator on a finite dimensional vector space  $V$  over an arbitrary field. Then the following statements hold.*

- (i)  $\alpha$  can be represented with respect to a suitable basis of  $V$  by a matrix in rational canonical form.
- (ii) The final invariant factor of  $\alpha$  is the minimum polynomial.
- (iii) The product of the invariant factors of  $\alpha$  equals the characteristic polynomial.

**Corollary (9.3.5)** (The Cayley-Hamilton Theorem) *The minimum polynomial of a linear operator divides its characteristic polynomial and these polynomials have the same irreducible factors.*

This follows directly from (9.3.4). The preceding very powerful results have been stated for a linear operator. Of course, they apply equally to an  $n \times n$  matrix  $A$  over a field  $F$ , since the mapping  $X \mapsto AX$  is a linear operator on  $F^n$ . Thus by (9.3.4) every square matrix is similar to a matrix in rational canonical form and also the Cayley-Hamilton Theorem is valid.

**Nilpotent linear operators.** Rational canonical form is particularly effective when applied to a nilpotent linear operator  $\alpha$  on an  $n$ -dimensional vector space  $V$  over an arbitrary field  $F$ . Since  $\alpha^k = 0$  for some  $k > 0$ , the minimum polynomial must divide  $t^k$  and thus has the form  $t^m$  where  $m \leq k$ . The invariant factors satisfy  $s_1 | s_2 | \cdots | s_\ell = t^m$  by (9.3.4). Hence  $s_i = t^{n_i}$  where  $n_1 \leq n_2 \leq \cdots \leq n_\ell = m$ . The characteristic polynomial of  $\alpha$  equals  $s_1 s_2 \cdots s_\ell = t^n$  and thus  $n = \sum_{i=1}^\ell n_i$ .

The companion matrix of  $s_i$  is the  $n_i \times n_i$  matrix

$$R_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and the rational canonical form of  $\alpha$  is the block matrix formed by  $R_1, R_2, \dots, R_\ell$ . This is a lower triangular matrix with zeros on the diagonal, a type of matrix called *lower zero triangular*. Applying this in matrix form, we deduce:

**(9.3.6)** *A nilpotent matrix is similar to a lower zero triangular matrix.*

Rational canonical form allows us to make an exact count of the similarity types of nilpotent  $n \times n$  matrix.

**(9.3.7)** *The number of similarity types of nilpotent  $n \times n$  matrices over any field equals  $\lambda(n)$  where  $\lambda$  is the partition function.*

*Proof.* Let  $A$  be an  $n \times n$  nilpotent matrix. Let  $m_i$  denote the number of rational blocks with exactly  $i$  1's on the subdiagonal. Thus  $m_i \geq 0$  and  $0 \leq i \leq n-1$ . Then  $n = \sum_{i=0}^{n-1} (i+1)m_i$ , so that we have a partition of  $n$ . Conversely, each partition of  $n$  allows us to assemble a nilpotent matrix, the rational blocks coming from the subsets in the partition. Moreover, different partitions give rise to non-similar matrices by uniqueness of the invariant factors.  $\square$

**Example (9.3.2)** Since  $\lambda(3) = 3$ , there are three similarity types of nilpotent  $3 \times 3$  matrices, corresponding to the partitions of 3, which are  $1 + 1 + 1$ ,  $1 + 2$ ,  $3$ . The respective types of matrix are

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

**Jordan form.** Let  $\alpha$  be a linear operator on an  $n$ -dimensional vector space  $V$  over a field  $F$  and let  $f$  denote the minimum polynomial of  $\alpha$ . Assume that  $f$  splits into linear factors over  $F$ , which by the Cayley-Hamilton Theorem amounts to requiring all eigenvalues of  $\alpha$  to be in  $F$ , which will certainly be true if  $F$  is algebraically closed.

In this case there is a simpler canonical form for  $\alpha$  called *Jordan normal form*. Write

$$f = (t - a_1)^{e_1} (t - a_2)^{e_2} \cdots (t - a_k)^{e_k}$$

where  $e_i > 0$  and the  $a_i$  are distinct elements of the field  $F$ . By (9.3.5) the roots of  $f$  are the roots of the characteristic polynomial, so  $a_1, a_2, \dots, a_k$  are the distinct eigenvalues of  $\alpha$ . By the Primary Decomposition Theorem  $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$  where  $V_i$  is the  $p_i = (t - a_i)$ -torsion submodule of  $V$ . Write  $n_i = \dim(V_i)$ , so that  $n = \sum_{i=1}^k n_i$ . Then  $\alpha_i = \alpha|_{V_i}$  has  $(t - a_i)^{e_i}$  as its minimum polynomial by (9.3.2); thus  $(\alpha_i - a_i 1_{n_i})^{e_i} = 0$  and  $\alpha_i - a_i 1_{n_i}$  is a nilpotent linear operator on  $V_i$ . By the discussion of nilpotent linear operators above,  $\alpha_i - a_i 1_{n_i}$  is represented with respect to a suitable basis of  $V_i$  by a matrix consisting of  $\ell_{ij}$   $n_{ij} \times n_{ij}$  blocks of the type

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

for  $j = 1, 2, \dots, e_i$ . Here  $n_{i1} \leq n_{i2} \leq \cdots \leq n_{ie_i}$  and  $\sum_{j=1}^{e_i} \ell_{ij} n_{ij} = \dim(V_i) = n_i$ . Consequently,  $\alpha_i$  is represented by a matrix consisting of  $\ell_{ij}$   $n_{ij} \times n_{ij}$  blocks  $J_{ij}$  with the form

$$J_{ij} = \begin{bmatrix} a_i & 0 & \cdots & 0 & 0 & 0 \\ 1 & a_i & 0 & 0 & \cdots & 0 \\ 0 & 1 & a_i & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 & a_i \end{bmatrix}.$$

Such matrices are called *Jordan blocks* and they are unique up to order since they are determined by the elementary divisors of  $\alpha$ . Therefore we can state:

**(9.3.8)** (Jordan normal form) *Let  $\alpha$  be a linear operator on a finite dimensional vector space over a field  $F$ . Assume that the minimum polynomial of  $\alpha$  splits into linear factors*

over  $F$ . Then  $\alpha$  can be represented with respect to a suitable basis by a matrix with Jordan blocks on the diagonal which are unique up to order.

The matrix form of (9.3.8) asserts that an  $n \times n$  matrix  $A$  whose minimum polynomial is a product of linear factors over  $F$  is similar to a matrix with Jordan blocks on the diagonal. Therefore, in particular,  $A$  is similar to a lower triangular matrix over  $F$ , i.e., with zeros above the diagonal – cf. (8.4.8).

**Example (9.3.3)** Find all similarity types of complex  $3 \times 3$  matrices  $A$  which satisfy the equation  $A(A - 2I)^2 = 0$ .

From the information furnished the minimum polynomial  $f$  of  $A$  divides  $t(t - 2)^2$ . Hence there are five possibilities for  $f$ , which are listed below with the corresponding Jordan canonical form  $J$  of  $A$ :

(i)  $f = t$ : in this case  $A = J = 0$ .

(ii)  $f = t - 2$ :  $J = 2I_3$ .

(iii)  $f = (t - 2)^2$ :  $J = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ .

(iv)  $f = t(t - 2)$ :  $J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$  or  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ .

(v)  $f = t(t - 2)^2$ :  $J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ .

Hence there are six types of matrix up to similarity.

**Smith normal form of a matrix.** We end the chapter by describing a method for calculating the invariant factors of a linear operator or matrix. It is stated for matrices.

**(9.3.9)** Let  $A$  be an  $n \times n$  matrix over a field  $F$ . Then the Smith normal form of the matrix  $tI - A$  is  $\text{diag}(1, 1, \dots, 1, s_1, s_2, \dots, s_\ell)$  up to signs, where  $s_1, s_2, \dots, s_\ell$  are the invariant factors of  $A$ .

*Proof.* Let  $S$  denote the rational canonical form of  $A$ . Then  $S = XAX^{-1}$  for some non-singular matrix  $X$  over  $F$ . It follows that  $S$  and  $A$  have the same invariant factors since they represent the same linear operator on  $F^n$ , but with respect to different bases. Also  $tI - S = X(tI - A)X^{-1}$ , so by the same reasoning  $tI - S$  and  $tI - A$  have the same Smith normal form. Therefore we may assume that  $A = S$ , i.e.,  $A$  is in rational canonical form.

Let  $R_1, R_2, \dots, R_\ell$  be the rational blocks in  $A$ , corresponding to the invariant factors  $s_1 | s_2 | \dots | s_\ell$  of  $A$ . It is enough to prove that the Smith normal form of  $tI - R_i$  is  $\text{diag}(1, 1, \dots, 1, s_i)$ ; for then  $tI - A$  will have  $\text{diag}(1, 1, \dots, 1, s_1, s_2, \dots, s_\ell)$  as its

Smith normal form. Let  $s_i = a_{i0} + a_{i1}t + \cdots + a_{in_i-1}t^{n_i-1} + t^{n_i}$ ; thus

$$R_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_{i0} \\ 1 & 0 & \cdots & 0 & -a_{i1} \\ 0 & 1 & \cdots & 0 & -a_{i2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{in_i-1} \end{bmatrix}.$$

Since  $F[t]$  is a Euclidean domain, we can transform the matrix

$$tI - R_i = \begin{bmatrix} t & 0 & 0 & \cdots & 0 & a_{i0} \\ -1 & t & 0 & \cdots & 0 & a_{i1} \\ 0 & -1 & t & \cdots & 0 & a_{i2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \vdots & \cdots & -1 & t + a_{in_i-1} \end{bmatrix},$$

into Smith normal form using the algorithm of (9.2.12). This is readily seen to be  $\text{diag}(1, 1, \dots, 1, s_i)$ , as the reader should verify at least for  $n_i \leq 3$ . (Note the absence of zeros since  $V$  is a torsion module). The required result now follows.  $\square$

**Example (9.3.4)** Consider the rational matrix

$$A = \begin{bmatrix} 0 & 4 & 1 \\ -1 & -4 & 2 \\ 0 & 0 & -2 \end{bmatrix}.$$

Apply suitable row and column operations to put the matrix

$$tI - A = \begin{bmatrix} t & -4 & -1 \\ 1 & t+4 & -2 \\ 0 & 0 & t+2 \end{bmatrix}$$

into its Smith normal form, which is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (t+2)^3 \end{bmatrix}.$$

Hence there is just one invariant factor  $s_1 = (t+2)^3$ . The rational canonical form of  $A$  can now be written down immediately as

$$\begin{bmatrix} 0 & 0 & -8 \\ 1 & 0 & -12 \\ 0 & 1 & -6 \end{bmatrix}.$$

The minimum polynomial is  $(t+2)^3$ , so the Jordan normal form is

$$\begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}.$$

**Exercises (9.3)**

(1) Find all similarity types of  $3 \times 3$  rational matrices  $A$  which satisfy the equation  $A^4 = A^5$ .

(2) Find the invariant factors and rational canonical form of the rational

matrix  $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & -4 \end{bmatrix}$ .

(3) Find the Jordan normal form and minimum polynomial of the rational matrix

$$\begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(4) Let  $A$  be an  $n \times n$  matrix over  $\mathbb{Q}$  and let  $p$  be a prime. Assume that  $A^p = I$ . Prove that the number of similarity types of  $A$  is  $1 + \lfloor \frac{n}{p-1} \rfloor$ . [Hint: recall from Example (7.4.6) that the rational polynomial  $1 + t + t^2 + \cdots + t^{p-1}$  is irreducible].

(5) Prove that a square matrix  $A$  over a field is similar to its transpose. (You may assume the field contains all roots of the minimum polynomial of  $A$ ).

(6) Prove that every square matrix is similar to an upper triangular matrix.

(7) Let  $A$  be a non-singular  $n \times n$  matrix over an algebraically closed field  $F$  and let  $J_1, J_2, \dots, J_k$  be the blocks in the Jordan normal form of  $A$ . Prove that  $A$  has finite order if and only if each  $J_i$  has finite order and in that case  $|A| = \text{lcm}\{|J_1|, |J_2|, \dots, |J_k|\}$ .

(8) Let  $J$  be an  $n \times n$  Jordan block over a field  $F$ , with diagonal elements equal to  $a \neq 0$ . If  $n > 1$ , prove that  $J$  has finite order if and only if  $a$  has finite order in  $F^*$  and  $p = \text{char}(F) \neq 0$ .

(9) Let  $A$  be a non-singular  $n \times n$  matrix over an algebraically closed field of characteristic 0. Let  $a_1, a_2, \dots, a_n$  be the eigenvalues of  $A$ . Prove that  $A$  has finite order if and only if each  $a_i$  has finite order and then  $|A| = \text{lcm}\{|a_1|, |a_2|, \dots, |a_n|\}$ .

(10) Find the Jordan normal form of the matrix  $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$  over  $GF(7)$ , the field

of seven elements. Then use it to prove that  $|A| = 7$ .

## 10 The Structure of groups

In this chapter we pursue the study of groups at a deeper level. A common method of investigation in algebra is to break up a complex structure into simpler substructures. The hope is that by repeated application of this procedure one will eventually arrive at substructures that are easy to understand. It may then be possible in some sense to synthesize these substructures to reconstruct the original structure. While it is rare for the procedure just described to be brought to such a perfect state of completion, the analytic-synthetic method can yield valuable information and suggest new concepts. We will consider some instances where this procedure can be employed in group theory.

### 10.1 The Jordan–Hölder Theorem

A basic concept in group theory is that of a *series in a group*  $G$ . By this is meant a finite chain of subgroups  $\mathcal{S} = \{G_i \mid i = 0, 1, \dots, n\}$  leading from the identity subgroup to  $G$ , with each term normal in its successor, that is, a chain of the form

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G.$$

The  $G_i$  are the *terms* of the series and the quotient groups  $G_{i+1}/G_i$  are the *factors*. The *length* of the series is defined to be the number of non-trivial factors. Keep in mind that  $G_i$  may not be normal in  $G$  since normality is not a transitive relation – see Exercise (4.2.6).

A subgroup  $H$  which appears in a series in a group  $G$  is called a *subnormal subgroup*; clearly this is equivalent to there being a chain of normality relations leading from  $H$  to  $G$ ,

$$H = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_m = G.$$

A partial order on the set of series in a group  $G$  is defined as follows. A series  $\mathcal{S}$  is called a *refinement* of a series  $\mathcal{T}$  if every term of  $\mathcal{T}$  is also a term of  $\mathcal{S}$ . If  $\mathcal{S}$  has at least one term that is not a term of  $\mathcal{T}$ , then  $\mathcal{S}$  is a *proper refinement* of  $\mathcal{T}$ . It is easy to see that the relation of being a refinement is a partial order on the set of all series in  $G$ .

**Example (10.1.1)** The symmetric group  $S_4$  has the series  $1 \triangleleft V \triangleleft A_4 \triangleleft S_4$  where  $V$  is the Klein 4-group. This is a refinement of the series  $1 \triangleleft A_4 \triangleleft S_4$ .

**Isomorphic series.** Two series  $\mathcal{S}$  and  $\mathcal{T}$  in a group  $G$  are called *isomorphic* if there is a bijection from the set of non-trivial factors of  $\mathcal{S}$  to the set of non-trivial factors of  $\mathcal{T}$  such that corresponding factors are isomorphic groups. Isomorphic series must have the same length, but the isomorphic factors may occur at different points in the series.

**Example (10.1.2)** In  $\mathbb{Z}_6$  the series  $0 \triangleleft \langle [2] \rangle \triangleleft \mathbb{Z}_6$  and  $0 \triangleleft \langle [3] \rangle \triangleleft \mathbb{Z}_6$  are isomorphic since  $\langle [2] \rangle \simeq \mathbb{Z}_6/\langle [3] \rangle$  and  $\langle [3] \rangle \simeq \mathbb{Z}_6/\langle [2] \rangle$ .

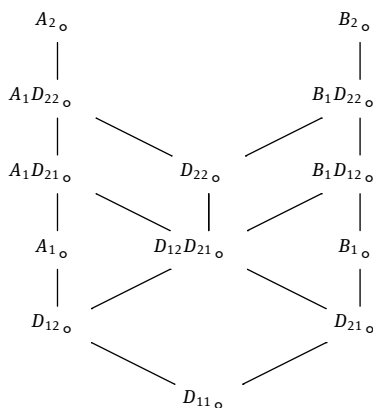


The foundation for the theory of series in groups is the following technical result. It can be viewed as a generalization of the Second Isomorphism Theorem.

**(10.1.1)** (Zassenhaus's<sup>1</sup> Lemma) *Let  $A_1, A_2, B_1, B_2$  be subgroups of a group such that  $A_1 \triangleleft A_2$  and  $B_1 \triangleleft B_2$ . Define  $D_{ij} = A_i \cap B_j$ , ( $i, j = 1, 2$ ). Then  $A_1 D_{21} \triangleleft A_1 D_{22}$  and  $B_1 D_{12} \triangleleft B_1 D_{22}$ . Furthermore*

$$A_1 D_{22} / A_1 D_{21} \simeq B_1 D_{22} / B_1 D_{12}.$$

*Proof.* The Hasse diagram below displays the relevant subgroups.



From  $B_1 \triangleleft B_2$  we obtain  $D_{21} \triangleleft D_{22}$  by intersecting with  $A_2$ . Since  $A_1 \triangleleft A_2$ , it follows that  $A_1 D_{21} \triangleleft A_1 D_{22}$  on applying the canonical homomorphism  $A_2 \rightarrow A_2/A_1$ . Similarly  $B_1 D_{12} \triangleleft B_1 D_{22}$ . Now we invoke (4.3.5) with  $H = D_{22}$  and  $N = A_1 D_{21}$  to give  $HN/N \simeq H/H \cap N$ . But  $HN = A_1 D_{22}$  and  $H \cap N = D_{22} \cap (A_1 D_{21}) = D_{12} D_{21}$  by (4.1.11). The conclusion is that  $A_1 D_{22} / A_1 D_{21} \simeq D_{22} / D_{12} D_{21}$ . By the same argument  $B_1 D_{22} / B_1 D_{12} \simeq D_{22} / D_{12} D_{21}$ , from which the result follows.  $\square$

The main use of Zassenhaus's Lemma is to prove a theorem about refinements: its statement is remarkably simple.

**(10.1.2)** (The Schreier<sup>2</sup> Refinement Theorem) *Any two series in a group have isomorphic refinements.*

*Proof.* Let  $1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_l = G$  and  $1 = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_m = G$  be two series in a group  $G$ . Define subgroups  $H_{ij} = H_i(H_{i+1} \cap K_j)$  for  $0 \leq i \leq l-1$ ,  $0 \leq j \leq m$  and

<sup>1</sup> Hans Zassenhaus (1912–1991)

<sup>2</sup> Otto Schreier (1901–1929)

$K_{ij} = K_j(H_i \cap K_{j+1})$  for  $0 \leq i \leq l$ ,  $0 \leq j \leq m-1$ . Apply (10.1.1) with  $A_1 = H_i$ ,  $A_2 = H_{i+1}$ ,  $B_1 = K_j$  and  $B_2 = K_{j+1}$ ; the conclusion is that  $H_{ij} \triangleleft H_{ij+1}$  and  $K_{ij} \triangleleft K_{i+1j}$ , and also that  $H_{ij+1}/H_{ij} \cong K_{i+1j}/K_{ij}$ . Therefore the series  $\{H_{ij} \mid i = 0, 1, \dots, l-1, j = 0, 1, \dots, m\}$  and  $\{K_{ij} \mid i = 0, 1, \dots, l, j = 0, 1, \dots, m-1\}$  are isomorphic refinements of  $\{H_i \mid i = 0, 1, \dots, l\}$  and  $\{K_j \mid j = 0, 1, \dots, m\}$  respectively.  $\square$

**Composition series.** A series which has no proper refinements is called a *composition series* and its factors are *composition factors*. If  $G$  is a finite group, we can start with any series, for example  $1 \triangleleft G$ , and keep refining it until a composition series is reached. Thus every finite group has a composition series. However, not every infinite group has a composition series, as is shown by (10.1.6) below.

A composition series can be recognized from the nature of its factors.

**(10.1.3)** *A series is a composition series if and only if all its factors are simple groups.*

*Proof.* Let  $X/Y$  be a factor of a series in a group  $G$ . If  $X/Y$  is not simple, there is a subgroup  $W$  such that  $Y < W < X$  and  $W \triangleleft X$ ; here the Correspondence Theorem (4.2.2) has been invoked. Adjoining  $W$  to the given series, we obtain a new series which is a proper refinement, with the terms  $Y \triangleleft W \triangleleft X$  replacing  $Y \triangleleft X$ .

Conversely, if a series in  $G$  has a proper refinement, there must be two consecutive terms  $Y \triangleleft X$  of the original series with additional terms of the refined series between them. Hence there is a subgroup  $W$  in the refined series such that  $Y < W < X$  and  $W \triangleleft X$ . But then  $W/Y$  is a proper non-trivial normal subgroup of  $X/Y$  and the latter cannot be simple. Hence the result is proved.  $\square$

The main result about composition series is a celebrated theorem associated with the names of two prominent 19th Century algebraists, Camille Jordan (1838–1922) and Otto Hölder (1859–1937).

**(10.1.4)** (The Jordan–Hölder Theorem) *Let  $\mathcal{S}$  be a composition series in a group  $G$  and suppose that  $\mathcal{T}$  is any series in  $G$ . Then  $\mathcal{T}$  has a refinement which is isomorphic with  $\mathcal{S}$ .*

The most important case is when  $\mathcal{T}$  itself is a composition series and the conclusion is that  $\mathcal{T}$  is isomorphic with  $\mathcal{S}$ . Thus we obtain:

**Corollary (10.1.5)** *Any two composition series in a group are isomorphic.*

*Proof of (10.1.4).* By the Refinement Theorem (10.1.2), the series  $\mathcal{S}$  and  $\mathcal{T}$  have isomorphic refinements. But  $\mathcal{S}$  is a composition series, so it is isomorphic with a refinement of  $\mathcal{T}$ .  $\square$

**Example (10.1.3)** Consider the symmetric group  $S_4$ . It has a series

$$1 \triangleleft C \triangleleft V \triangleleft A_4 \triangleleft S_4$$

where  $|C| = 2$  and  $V$  is the Klein 4-group. Now  $C$ ,  $V/C$  and  $S_4/A_4$  all have order 2, while  $A_4/V$  has order 3, so all factors of the series are simple. By (10.1.3) the series is a composition series with composition factors  $\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2$ .

The next result demonstrates that not every group has a composition series.

**(10.1.6)** *An abelian group  $A$  has a composition series if and only if it is finite.*

*Proof.* Only necessity is in doubt, so assume that  $A$  has a composition series. Each factor of the series is simple and abelian, and thus has no proper non-trivial subgroups. By (4.1.9) the factors have prime order and therefore  $A$  is finite.  $\square$

**Example (10.1.4)** *Composition series in  $\mathbb{Z}_n$ .*

Let  $n$  be an integer greater than 1. The group  $\mathbb{Z}_n$  has a composition series with factors of prime order. Since the product of the orders of the composition factors is equal to  $n$ , the group order, it follows that  $n$  is a product of primes, which is the first part of the Fundamental Theorem of Arithmetic. In fact we can also obtain the uniqueness part.

Suppose that  $n = p_1 p_2 \cdots p_k$  is an expression for  $n$  as a product of primes. Define  $H_i$  to be the subgroup of  $\mathbb{Z}_n$  generated by the congruence class  $[p_{i+1} p_{i+2} \cdots p_k]$  where  $0 \leq i < k$  and let  $H_k = \mathbb{Z}_n$ . Then

$$0 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_{k-1} \triangleleft H_k = \mathbb{Z}_n$$

is a series in  $\mathbb{Z}_n$ . Now clearly  $|H_i| = p_1 p_2 \cdots p_i$  and hence  $|H_{i+1}/H_i| = p_{i+1}$ . Thus we have constructed a composition series in  $\mathbb{Z}_n$  with factors of orders  $p_1, p_2, \dots, p_k$ .

If  $n = q_1 q_2 \cdots q_l$  is another expression for  $n$  as product of primes, there is a corresponding composition series with factors of orders  $q_1, q_2, \dots, q_l$ . By the Jordan–Hölder Theorem these composition series are isomorphic. Consequently,  $k = l$  and the  $q_j$ 's must be the  $p_i$ 's in some order. Thus we have recovered the Fundamental Theorem of Arithmetic from the Jordan–Hölder Theorem.

**Some simple groups.** The investigation so far shows that in a sense a finite group decomposes into a number of simple groups, namely its composition factors. The only simple groups we currently know are the groups of prime order and the alternating group  $A_5$  – see (5.3.10). It is definitely time to expand this list, which we do by proving:

**(10.1.7)** *The alternating group  $A_n$  is simple if and only if  $n \neq 1, 2$  or  $4$ .*

The proof uses the following property of 3-cycles.

**(10.1.8)** *If  $n \geq 3$ , the alternating group  $A_n$  is generated by 3-cycles.*

*Proof.* First of all note that 3-cycles are even and hence belong to  $A_n$ . Next each element of  $A_n$  is the product of an even number of transpositions by (3.1.7). Finally, note the equations  $(ac)(ab) = (abc)$  and  $(ab)(cd) = (adb)(adc)$ , where  $a, b, c, d$  are all different; these demonstrate that every element of  $A_n$  is a product of 3-cycles.  $\square$

*Proof of (10.1.7).* In the first place  $A_4$  has a normal subgroup of order 4, so it cannot be simple. Also  $A_1$  and  $A_2$  have order 1, so these are also excluded. However,  $A_3$  is simple because its order is 3. Thus we can assume that  $n \geq 5$  and aim to show that  $A_n$  is simple. If this is false, there is a proper, non-trivial normal subgroup  $N$ . The proof analyzes the possible forms of elements of  $N$ .

Assume first that  $N$  contains a 3-cycle  $(abc)$ . If  $(a'b'c')$  is another 3-cycle and  $\pi$  in  $S_n$  sends  $a, b, c$  to  $a', b', c'$  respectively, then  $\pi(abc)\pi^{-1} = (a'b'c')$ . If  $\pi$  is even, it follows that  $(a'b'c') \in N$ . If, on the other hand,  $\pi$  is odd, it can be replaced by the even permutation  $\pi \circ (ef)$  where  $e, f$  are different from  $a', b', c'$  – notice that this uses  $n \geq 5$ . We will still have  $\pi(abc)\pi^{-1} = (a'b'c')$ . Consequently  $N$  contains all 3-cycles and by (10.1.8)  $N = A_n$ , a contradiction. Hence  $N$  cannot contain a 3-cycle.

Assume next that  $N$  contains a permutation  $\pi$  whose disjoint cycle decomposition involves a cycle of length at least 4, say

$$\pi = (a_1 a_2 a_3 a_4 \cdots) \cdots$$

where the final dots indicate the possible presence of further disjoint cycles. Now  $N$  also contains the conjugate of  $\pi$

$$\pi' = (a_1 a_2 a_3) \pi (a_1 a_2 a_3)^{-1} = (a_2 a_3 a_1 a_4 \cdots) \cdots.$$

Therefore  $N$  contains  $\pi' \pi^{-1} = (a_1 a_2 a_4)$ : here the point to note is that the other cycles cancel. Since this conclusion is untenable, elements in  $N$  must have disjoint cycle decompositions involving cycles of length at most 3. Furthermore, such elements cannot involve just one 3-cycle, otherwise by squaring we would obtain a 3-cycle in  $N$ .

Assume next that  $N$  contains a permutation with at least two disjoint 3-cycles, say  $\pi = (abc)(a'b'c') \cdots$ . Then  $N$  contains the conjugate

$$\pi' = (a'b'c)\pi(a'b'c)^{-1} = (aba')(cc'b') \cdots,$$

and hence it contains  $\pi\pi' = (aca'bb') \cdots$ , which has been seen to be impossible. Therefore each non-trivial element of  $N$  must be the product of an even number of disjoint transpositions.

If  $\pi = (ab)(a'b') \in N$ , then  $N$  contains  $\pi' = (acb)\pi(acb)^{-1} = (ac)(a'b')$  for any  $c$  unaffected by  $\pi$ . But then  $N$  will contain  $\pi\pi' = (acb)$ , which is false. Consequently, if  $1 \neq \pi \in N$ , then  $\pi = (a_1 b_1)(a_2 b_2)(a_3 b_3)(a_4 b_4) \cdots$ , with at least four transpositions. It follows that  $N$  also contains

$$\pi' = (a_3 b_2)(a_2 b_1)\pi(a_2 b_1)(a_3 b_2) = (a_1 a_2)(a_3 b_1)(b_2 b_3)(a_4 b_4) \cdots$$

and hence  $N$  contains  $\pi\pi' = (a_1 b_2 a_3)(a_2 b_1 b_3)$ , a final contradiction.  $\square$

As a consequence of (10.1.8) there are infinitely many simple alternating groups. The simplicity of  $A_n$  will now be used to determine the composition series of  $S_n$ .

**(10.1.9)** *If  $n = 3$  or  $n \geq 5$ , then  $1 \triangleleft A_n \triangleleft S_n$  is the unique composition series of  $S_n$ .*

*Proof.* In the first place this is a composition series since  $A_n$  and  $S_n/A_n \simeq \mathbb{Z}_2$  are simple. Suppose that  $N$  is a non-trivial, proper normal subgroup of  $S_n$ . We will show that  $N = A_n$ , which will settle the matter. First note that  $N \cap A_n \triangleleft A_n$ , so that either  $N \cap A_n = 1$  or  $A_n \leq N$  since  $A_n$  is simple. Now  $|S_n : A_n| = 2$ , so if  $A_n \leq N$ , then  $N = A_n$ . Suppose that  $N \cap A_n = 1$ . Then  $NA_n = S_n$  and  $|N| = |NA_n/A_n| = |S_n/A_n| = 2$ . Thus  $N$  contains a single non-identity element  $\pi$ , (necessarily an odd permutation). Since  $N \triangleleft S_n$ , the permutation  $\pi$  belongs to the center of  $S_n$ ; however  $Z(S_n) = 1$  by Exercise (4.2.10), so a final contradiction is reached.  $\square$

**Projective linear groups.** We mention in passing another infinite family of finite simple groups. Let  $F$  be any field. It is not difficult to prove by direct matrix calculations that the center of the general linear group  $GL_n(F)$  is just the subgroup of all scalar matrices  $fI_n$  where  $f \in F$  – cf. Exercise (4.2.12). The *projective general linear group of degree  $n$  over  $F$*  is defined to be

$$PGL_n(F) = GL_n(F)/Z(GL_n(F)).$$

Recall that  $SL_n(F)$  is the special linear group consisting of all matrices in  $GL_n(F)$  with determinant equal to 1. The center of  $SL_n(F)$  can be shown to be  $Z(GL_n(F)) \cap SL_n(F)$ . Therefore by (4.3.5)

$$SL_n(F)Z(GL_n(F))/Z(GL_n(F)) \simeq SL_n(F)/Z(SL_n(F)).$$

The latter is called the *projective special linear group*

$$PSL_n(F).$$

The projective special linear groups are usually simple, as the following result shows.

**(10.1.10)** *Let  $F$  be a field and let  $n > 1$ . Then  $PSL_n(F)$  is simple if and only if  $n \geq 3$  or  $n = 2$  and  $F$  has more than three elements.*

This result can be proved by direct, if tedious, matrix calculations – see for example [12]. If  $F$  is a finite field, its order is a prime power  $q$  by (8.2.17). Moreover, by (11.3.5) below, there is up to isomorphism just one field of order  $q$ . If  $F$  is a field of order  $q$ , it is better notation to write

$$GL_n(q), PGL_n(q), PSL_n(q)$$

instead of  $GL_n(F), PGL_n(F), PSL_n(F)$ .

It is not hard to compute the orders of these groups. In the first place,  $|Z(GL_n(F))| = |F^*| = q - 1$ , where  $F^* = U(F) = F - 0$ , and also  $|Z(SL_n(F))| = \gcd\{n, q - 1\}$ . For the last

statement we need to know that  $F^*$  is cyclic: for a proof see (11.3.6) below. The orders of the projective groups can now be read off. A simple count of the non-singular  $n \times n$  matrices over  $F$  reveals that

$$|\mathrm{GL}_n(q)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}),$$

while  $|\mathrm{SL}_n(q)| = |\mathrm{GL}_n(q)|/(q - 1)$ . Thus we have formulas for the orders of the projective groups.

### (10.1.11)

- (i)  $|\mathrm{PGL}_n(q)| = |\mathrm{GL}_n(q)|/(q - 1)$ ;
- (ii)  $|\mathrm{PSL}_n(q)| = |\mathrm{SL}_n(q)|/\gcd\{n, q - 1\}$ .

For example,  $\mathrm{PSL}_2(5)$  is a simple group of order 60. In fact there is only one simple group of this order – see Exercise (10.2.18) – so  $\mathrm{PSL}_2(5)$  must be isomorphic with  $A_5$ . But  $\mathrm{PSL}_2(7)$  of order 168 and  $\mathrm{PSL}_2(8)$  of order 504 are simple groups that are not of alternating type.

**Projective groups and projective space.** We indicate briefly how the projective groups arise in geometry. Let  $V$  be an  $(n + 1)$ -dimensional vector space over a field  $F$  and let  $V^*$  denote the set of all non-zero vectors in  $V$ . An equivalence relation  $\sim$  on  $V^*$  is introduced by the following rule:  $u \sim v$  if and only if  $u = fv$  for some  $f \neq 0$  in  $F$ . Let  $[v]$  be the equivalence class of the vector  $v$ , so this is just the set of non-zero multiples of  $v$ . The set

$$\tilde{V} = \{[v] \mid v \in V^*\}$$

is called *n-dimensional projective space* over  $F$ .

Next let  $\alpha$  be a bijective linear operator on  $V$ . Then there is an induced mapping  $\tilde{\alpha} : \tilde{V} \rightarrow \tilde{V}$  defined by the rule

$$\tilde{\alpha}([v]) = [\alpha(v)].$$

Here  $\tilde{\alpha}$  is called a *collineation* on  $\tilde{V}$ . It is not hard to see that the collineations on  $\tilde{V}$  form a group  $\mathrm{PGL}(\tilde{V})$  with respect to functional composition.

It is also straightforward to verify that the assignment  $\alpha \mapsto \tilde{\alpha}$  gives rise to a surjective group homomorphism from  $\mathrm{GL}(V)$ , the group of invertible linear operators on  $V$ , to  $\mathrm{PGL}(\tilde{V})$ , with kernel equal to the subgroup of all scalar linear operators. Therefore  $\mathrm{PGL}(\tilde{V}) \simeq \mathrm{PGL}_n(F)$ , while  $\mathrm{PSL}_n(F)$  corresponds to the subgroup of collineations arising from matrices with determinant equal to 1.

**The classification of finite simple groups.** The projective special linear groups form one of a number of infinite families of finite simple groups known collectively as *the simple groups of Lie type*. They arise as groups of automorphisms of simple Lie algebras. In addition to the alternating groups and the groups of Lie type, there are 26 isolated

simple groups, the so-called *sporadic simple groups*. The smallest of these, the *Mathieu*<sup>3</sup> group  $M_{11}$ , has order 7920, while the largest one, the so-called *Monster*, has order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

or approximately  $8.08 \times 10^{53}$ .

It is now widely accepted that the alternating groups, the simple groups of Lie type and the sporadic simple groups account for all the finite non-abelian simple groups. While a complete proof of this result has yet to appear, it is the subject of a multi-volume work currently in preparation. The classification of finite simple groups is a synthesis of the work of many mathematicians and is by any standard one of the greatest scientific achievements of all time.

To conclude the section let us assess how far we have come in trying to understand the structure of finite groups. If the aim is to construct all finite groups, the Jordan–Hölder Theorem shows that two steps are necessary:

- (i) find all finite simple groups;
- (ii) construct all possible group extensions of a given finite group  $N$  by a finite simple group  $S$ .

In step (ii) we have to construct all groups  $G$  with a normal subgroup  $M$  such that  $M \simeq N$  and  $G/M \simeq S$ .

Let us accept that step (i) has been accomplished. A formal description of the extensions arising in (ii) is possible, but the general problem of deciding when two of the constructed groups are isomorphic is intractable. Thus the practicality of the scheme is questionable. However, this does not mean that the enterprise was not worthwhile since a vast amount of knowledge about finite groups has been accumulated during the course of the program.

### Exercises (10.1)

- (1) Show that isomorphic groups have the same composition factors.
- (2) Find two non-isomorphic groups with the same composition factors.
- (3) Show that  $S_3$  has a unique composition series, while  $S_4$  has exactly three composition series.
- (4) Let  $G$  be a finite group and let  $N \triangleleft G$ . How are the composition factors of  $G$  related to those of  $N$  and  $G/N$ ?
- (5) Suppose that  $G$  is a group generated by normal subgroups  $N_1, N_2, \dots, N_k$  each of which is simple. Prove that  $G$  is the direct product of certain of the  $N_i$ . [Hint: choose  $r$  maximal subject to the existence of normal subgroups  $N_{i_1}, \dots, N_{i_r}$  which generate their direct product. Then show that the direct product equals  $G$ ].

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<sup>3</sup> Émile Léonard Mathieu (1835–1890)

- (6) Let  $G$  be as in the previous exercise. If  $N \triangleleft G$ , prove that  $N$  is a direct factor of  $G$ . [Hint: write  $G = N_1 \times N_2 \times \cdots \times N_s$ . Choose  $r$  maximal subject to  $N, N_{i_1}, \dots, N_{i_r}$  generating their direct product; then prove that this direct product equals  $G$ ].
- (7) Let  $G$  be a group with a series in which each factor is either infinite cyclic or finite. Prove that any other series of this type in  $G$  has the same number of infinite factors, but not necessarily the same number of finite ones.
- (8) Suppose that  $G$  is a group with a composition series. Prove that  $G$  satisfies the *ascending and descending chain conditions for subnormal subgroups*, i.e., there cannot exist an infinite ascending chain  $H_1 < H_2 < H_3 < \cdots$  or an infinite descending chain  $H_1 > H_2 > H_3 > \cdots$  where the  $H_i$  are subnormal subgroups of  $G$ . (For more on chain conditions see Exercise (3.3.10)).
- (9) Prove that a group  $G$  which satisfies both the ascending and descending chain conditions on subnormal subgroups has a composition series. [Hint: start by choosing a minimal non-trivial subnormal subgroup of  $G$ ].
- (10) Let  $D_n$  denote the subgroup of  $S_n$  generated by all the derangements where  $n > 1$ . Prove that  $D_n = S_n$  if  $n \neq 3$ , but  $D_3 = A_3$ . Conclude that if  $n \neq 3$ , every permutation is a product of derangements. (Suggestion: first prove that  $D_n \triangleleft S_n$  and that if  $n \neq 3$ , odd derangements exist. Deal first with the case  $n = 4$ . Then note that if  $n > 4$ , then  $D_n = S_n$  by (10.1.9)).

## 10.2 Solvable and nilpotent groups

In this section we will discuss certain types of group which are wide generalizations of abelian groups, but which retain vestiges of commutativity. The basic concept is that of a *solvable group*, which is defined to be a group with a series all of whose factors are abelian. The terminology derives from the classical problem of solving algebraic equations by radicals, which is discussed in detail in Chapter Twelve. The length of a shortest series with abelian factors is called the *derived length* of the solvable group. Thus abelian groups are the solvable groups with derived length at most 1. Solvable group with derived length 2 or less are called *metabelian*.

Finite solvable groups are easily characterized in terms of their composition factors.

**(10.2.1)** *A finite group is solvable if and only if its composition factors have prime orders. In particular a simple group is solvable if and only if it has prime order.*

*Proof.* Let  $G$  be a finite solvable group, so that  $G$  has a series  $\mathcal{S}$  with abelian factors. Refine  $\mathcal{S}$  to a composition series of  $G$ . The factors of this series are simple and they are also abelian since they are isomorphic with quotients of abelian groups. By (4.1.9) a simple abelian group has prime order. Hence composition factors of  $G$  have prime orders. The converse is an immediate consequence of the definition of solvability.  $\square$



Solvability is well-behaved with respect to the formation of subgroups, quotient groups and extensions.

### (10.2.2)

- (i) If  $G$  is a solvable group, then every subgroup and every quotient group of  $G$  is solvable.
- (ii) Let  $G$  be a group with a normal subgroup  $N$  such that  $N$  and  $G/N$  are solvable. Then  $G$  is solvable.

*Proof.* (i) Let  $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$  be a series with abelian factors and let  $H$  be a subgroup of  $G$ . Then

$$1 = G_0 \cap H \triangleleft G_1 \cap H \triangleleft \cdots \triangleleft G_n \cap H = H$$

is a series in  $H$ . Let  $x, y \in G_{i+1} \cap H$ . Then the commutator  $[x, y] = xyx^{-1}y^{-1}$  belongs to  $G_i$ , because  $G_{i+1}/G_i$  is abelian, and clearly  $[x, y] \in H$ . Therefore  $[x, y] \in G_i \cap H$  and  $G_{i+1} \cap H/G_i \cap H$  is abelian, which shows that  $H$  is a solvable group.

Next let  $N \triangleleft G$ . Then  $G/N$  has the series

$$1 = G_0N/N \triangleleft G_1N/N \triangleleft \cdots \triangleleft G_nN/N = G/N.$$

Also  $(G_{i+1}N/N)/(G_iN/N) \simeq G_{i+1}N/G_iN$  by (4.3.6). The assignment  $xG_i \mapsto xG_iN$  determines a well defined, surjective homomorphism from  $G_{i+1}/G_i$  to  $G_{i+1}N/G_iN$ . Since  $G_{i+1}/G_i$  is abelian, the group  $G_{i+1}N/G_iN$  is abelian and hence  $G/N$  is solvable.

(ii) The proof is left to the reader as an exercise.  $\square$

**The derived series.** Recall from (4.2) that the derived subgroup  $G'$  of a group  $G$  is the subgroup generated by all the commutators in  $G$ ,

$$G' = \langle [x, y] \mid x, y \in G \rangle.$$

The *derived chain*  $G^{(i)}$ ,  $i = 0, 1, 2, \dots$ , is defined to be the descending sequence of subgroups formed by repeatedly taking derived subgroups: thus

$$G^{(0)} = G, \quad G^{(i+1)} = (G^{(i)})'.$$

Note that  $G^{(i)} \triangleleft G$  and  $G^{(i)}/G^{(i+1)}$  is an abelian group.

The important properties of the derived chain are that in a solvable group it reaches the identity subgroup and of all series with abelian factors it has shortest length.

**(10.2.3)** Let  $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_k = G$  be a series with abelian factors in a solvable group  $G$ . Then  $G^{(i)} \leq G_{k-i}$  for  $0 \leq i \leq k$ . In particular  $G^{(k)} = 1$ , so that the length of the derived chain equals the derived length of  $G$ .

*Proof.* The containment is certainly true when  $i = 0$ . Assume that it is true for  $i$ . Since  $G_{k-i}/G_{k-i-1}$  is abelian,  $G^{(i+1)} = (G^{(i)})' \leq (G_{k-i})' \leq G_{k-i-1}$ , as required. On setting  $i = k$ , we find that  $G^{(k)} = 1$ .  $\square$

Notice the consequence: a solvable group has a *normal series*, i.e., one in which every term is normal, with abelian factors: indeed the derived series is of this type.

It is sometimes possible to deduce solvability of a finite group from the properties of its order. Some group orders for which this can be done are given in the next result.

**(10.2.4)** *Let  $p, q, r$  be primes. Then a group whose order has the form  $p^m, p^2q^2, p^mq$  or  $pqr$  is solvable.*

*Proof.* First observe that in each case it is enough to show that there are no non-abelian simple groups with the order. For once this fact has been established, by applying it to the composition factors the general case will follow. If  $G$  is a simple group of order  $p^m \neq 1$ , then  $Z(G) \neq 1$  by (5.3.6) and  $Z(G) \triangleleft G$ , so  $G = Z(G)$  and  $G$  is abelian.

Now consider the case of a simple group  $G$  with order  $p^mq$ . We can of course assume that  $p \neq q$ . Then  $n_p \equiv 1 \pmod{p}$  and  $n_p \mid q$ , so that  $n_p = q$ , since  $n_p = 1$  would mean that there is a normal Sylow  $p$ -subgroup.

Choose two distinct Sylow  $p$ -subgroups  $P_1$  and  $P_2$  whose intersection  $I = P_1 \cap P_2$  has largest order. First of all suppose that  $I = 1$ . Then each pair of distinct Sylow  $p$ -subgroups intersects in 1, which makes it easy to count the number of non-trivial elements with order a power of  $p$ ; indeed this number is  $q(p^m - 1)$  since there are  $q$  Sylow  $p$ -subgroups. This leaves  $p^mq - q(p^m - 1) = q$  elements of order prime to  $p$ . These elements must form a single Sylow  $q$ -subgroup, which is therefore normal in  $G$ , contradicting the simplicity of the group  $G$ . It follows that  $I \neq 1$ .

By Exercise (5.3.14) or (10.2.7) below,  $I < N_i = N_{P_i}(I)$  for  $i = 1, 2$ . Thus  $I \triangleleft J = \langle N_1, N_2 \rangle$ . Suppose for the moment that  $J$  is a  $p$ -group. By Sylow's Theorem  $J$  is contained in some Sylow subgroup  $P_3$  of  $G$ . But  $P_1 \cap P_3 \geq P_1 \cap J > I$  since  $N_1 \leq P_1 \cap J$ , which contradicts the maximality of the intersection  $I$ . Therefore  $J$  is not a  $p$ -group.

By Lagrange's Theorem  $|J|$  divides  $|G| = p^mq$  and it is not a power of  $p$ , from which it follows that  $q$  must divide  $|J|$ . Let  $Q$  be a Sylow  $q$ -subgroup of  $J$ . By (4.1.12)

$$|P_1Q| = \frac{|P_1| \cdot |Q|}{|P_1 \cap Q|} = \frac{p^mq}{1} = |G|,$$

and thus  $G = P_1Q$ . Now let  $g \in G$  and write  $g = ab$  where  $a \in P_1, b \in Q$ . Then  $bIb^{-1} = I$  since  $I \triangleleft J$  and  $Q \leq J$ . Hence  $gIg^{-1} = a(bIb^{-1})a^{-1} = aIa^{-1} \leq P_1 < G$ . But this means that  $\bar{I} = \langle gIg^{-1} \mid g \in G \rangle \leq P_1 < G$  and also  $1 \neq \bar{I} \triangleleft G$ , a final contradiction.

The remaining group orders are left as exercises with appropriate hints – see Exercise (10.2.5) and (10.2.6).  $\square$

We mention two much deeper arithmetic criteria for a finite group to be solvable. The first states that *a group of order  $p^mq^n$  is solvable if  $p$  and  $q$  are primes*. This is the celebrated *Burnside  $p$ - $q$  Theorem*. It is best proved using group characters and thus lies beyond the scope of this book.

An even more difficult result is the *Odd Order Theorem*, which asserts that *every group of odd order is solvable*. This famous theorem is due W. Feit <sup>4</sup> and J.G. Thompson: the original proof, published in 1963, was over 250 pages long.

These results indicate that there are many finite solvable groups: indeed finite non-abelian simple groups should be regarded as a rarity among finite groups.

**Nilpotent groups.** Nilpotent groups form an important subclass of the class of solvable groups. A group  $G$  is said to be *nilpotent* if it has a *central series*, by which is meant a series of *normal* subgroups  $1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = G$  such that  $G_{i+1}/G_i$  is contained in the center of  $G/G_i$  for all  $i$ . The length of a shortest central series is called the *nilpotent class* of  $G$ . abelian groups are just the nilpotent groups with class  $\leq 1$ . Clearly every nilpotent group is solvable, but  $S_3$  is a solvable group that is not nilpotent since its center is trivial.

The great source of finite nilpotent groups is the groups of prime power order.

**(10.2.5)** *Let  $G$  be a group of order  $p^m$  where  $p$  is a prime. Then  $G$  is nilpotent, and if  $m > 1$ , the nilpotent class of  $G$  is at most  $m - 1$ .*

*Proof.* Define a sequence of subgroups  $\{Z_i\}$  by repeatedly forming centers. Thus  $Z_0 = 1$  and  $Z_{i+1}/Z_i = Z(G/Z_i)$ . By (5.3.6), if  $Z_i \neq G$ , then  $Z(G/Z_i) \neq 1$  and  $Z_i < Z_{i+1}$ . Since  $G$  is finite, there is a smallest integer  $n$  such that  $Z_n = G$ , and clearly  $n \leq m$ . Suppose that  $n = m$ . Then  $|Z_{m-2}| \geq p^{m-2}$  and thus  $|G/Z_{m-2}| \leq p^m/p^{m-2} = p^2$ , which means that  $G/Z_{m-2}$  is abelian by (5.3.7). This yields the contradiction  $Z_{m-1} = G$ ; therefore  $n \leq m - 1$ .  $\square$

The foregoing proof suggests a general construction, the *upper central chain* of a group  $G$ . This is the chain of subgroups defined by repeatedly forming centers,

$$Z_0(G) = 1, \quad Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)).$$

Thus  $1 = Z_0 \leq Z_1 \leq \cdots$  and  $Z_i \triangleleft G$ . If  $G$  is finite, this chain will certainly terminate, although it may it not reach  $G$ . The significance of the upper central chain for nilpotency is shown by the next result.

**(10.2.6)** *Let  $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_k = G$  be a central series in a nilpotent group  $G$ . Then  $G_i \leq Z_i(G)$  for  $0 \leq i \leq k$ . In particular,  $Z_k(G) = G$  and the length of the upper central chain equals the nilpotent class of  $G$ .*

*Proof.* We argue that  $G_i \leq Z_i(G)$  by induction on  $i$ , which is certainly true for  $i = 0$ . If it is true for  $i$ , then, since  $G_{i+1}/G_i \leq Z(G/G_i)$ , we have

$$G_{i+1}Z_i(G)/Z_i(G) \leq Z(G/Z_i(G)) = Z_{i+1}(G)/Z_i(G).$$

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<sup>4</sup> Walter Feit (1930–2004)

Thus  $G_{i+1} \leq Z_{i+1}(G)$ , which completes the induction. Consequently  $G = G_k \leq Z_k(G)$  and  $G = Z_k(G)$ .  $\square$

**Example (10.2.1)** Let  $p$  be a prime and let  $n > 1$ . Denote by  $U_n(p)$  the group of all  $n \times n$  upper unitriangular matrices over the field  $\mathbb{Z}_p$ , i.e., matrices which have 1's on the diagonal and 0's below it. Counting the matrices of this type by enumerating possible superdiagonals, we find that  $|U_n(p)| = p^{n-1} \cdot p^{n-2} \cdots p \cdot 1 = p^{n(n-1)/2}$ . Therefore  $U_n(p)$  is a nilpotent group, and in fact its class is  $n - 1$ , (see Exercise (10.2.11)).

**Characterizations of finite nilpotent groups.** There are several different descriptions of finite nilpotent groups which shed light on the nature of the property of nilpotency.

**(10.2.7)** Let  $G$  be a finite group. Then the following statements are equivalent:

- (i)  $G$  is nilpotent;
- (ii) every subgroup of  $G$  is subnormal;
- (iii) every proper subgroup of  $G$  is smaller than its normalizer;
- (iv)  $G$  is the direct product of its Sylow subgroups.

*Proof.* (i) implies (ii). Let  $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$  be a central series and let  $H$  be a subgroup of  $G$ . Then  $G_{i+1}/G_i \leq Z(G/G_i)$ , so  $HG_i/G_i \triangleleft HG_{i+1}/G_i$ . Hence there is a chain of subgroups  $H = HG_0 \triangleleft HG_1 \triangleleft \cdots \triangleleft HG_n = G$  and  $H$  is subnormal in  $G$ .

(ii) implies (iii). Let  $H < G$ ; then  $H$  is subnormal in  $G$ , so there is a chain  $H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_m = G$ . There is a least  $i > 0$  such that  $H \neq H_i$ , and then  $H = H_{i-1} \triangleleft H_i$ . Therefore  $H_i \leq N_G(H)$  and  $N_G(H) \neq H$ .

(iii) implies (iv). Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If  $P$  is not normal in  $G$ , then  $N_G(P) < G$ , and hence  $N_G(P)$  is smaller than its normalizer. But this contradicts Exercise (5.3.15). Therefore  $P \triangleleft G$  and  $P$  must be the unique Sylow  $p$ -subgroup, which will be written  $G_p$ .

Evidently  $G_p \triangleleft G$  and  $G_p \cap \langle G_q \mid q \neq p \rangle = 1$  since orders of elements from the intersecting subgroups are relatively prime. Clearly  $G$  is generated by its Sylow subgroups, so  $G$  is the direct product of the  $G_p$ .

(iv) implies (i). This follows quickly from the fact that a finite  $p$ -group is nilpotent.  $\square$

The unique Sylow  $p$ -subgroup  $G_p$  is called the  $p$ -component of the nilpotent group  $G$ .

**The Frattini<sup>5</sup> subgroup.** A very intriguing subgroup that can be formed in any group  $G$  is the *Frattini subgroup*

$$\phi(G).$$

<sup>5</sup> Giovanni Frattini (1852–1925)

This is defined to be the intersection of all the maximal subgroups of  $G$ . Here a *maximal subgroup* is a proper subgroup which is not contained in any larger proper subgroup. If  $G$  has no maximal subgroups, as is the case if  $G$  is trivial and might happen if  $G$  is infinite, then  $\phi(G)$  is defined to be  $G$ . Note that  $\phi(G)$  is normal in  $G$ . For example,  $S_3$  has one maximal subgroup of order 3 and three of order 2: these intersect in 1, so  $\phi(S_3) = 1$ .

There is another, very different, way of describing the Frattini subgroup, which involves the notion of a non-generator. An element  $g$  of a group  $G$  is called a *non-generator* if  $G = \langle g, X \rangle$  always implies that  $G = \langle X \rangle$  where  $X$  is a non-empty subset of  $G$ . Thus a non-generator can be omitted from any generating set for  $G$ .

**(10.2.8)** *If  $G$  is a finite group,  $\phi(G)$  is the set of all non-generators of  $G$ .*

*Proof.* Let  $g$  be a non-generator of  $G$  and assume that  $g$  is not in  $\phi(G)$ . Then there is at least one maximal subgroup of  $G$  which does not contain  $g$ , say  $M$ . Thus  $M$  is definitely smaller than  $\langle g, M \rangle$ , which implies that  $G = \langle g, M \rangle$  since  $M$  is maximal. Therefore  $G = M$  by the non-generator property, which is a contradiction since maximal subgroups are proper.

Conversely, let  $g \in \phi(G)$  and suppose that  $G = \langle g, X \rangle$ , but  $G \neq \langle X \rangle$ . Then  $\langle X \rangle$  must be contained in some maximal subgroup of  $G$ , say  $M$ . But  $g \in \phi(G) \leq M$ , so  $G = \langle g, M \rangle = M$ , another contradiction.  $\square$

Actually (10.2.8) is valid for infinite groups as well, but the proof requires the use of Zorn's Lemma – see Exercise (14.1.6). Next we establish an important property of the Frattini subgroup of a finite group.

**(10.2.9)** *If  $G$  is a finite group, then  $\phi(G)$  is nilpotent.*

*Proof.* The proof depends on a useful trick known as *the Frattini argument*. Write  $F = \phi(G)$  and let  $P$  be a Sylow  $p$ -subgroup of  $F$ . If  $g \in G$ , then  $gPg^{-1} \leq F$  since  $F \triangleleft G$ : also  $|gPg^{-1}| = |P|$ . Therefore  $gPg^{-1}$  is a Sylow  $p$ -subgroup of  $F$ , and as such it must be conjugate to  $P$  in  $F$  by Sylow's Theorem. Thus  $gPg^{-1} = xPx^{-1}$  for some  $x$  in  $F$ . This implies that  $x^{-1}gP(x^{-1}g)^{-1} = P$ , i.e.,  $x^{-1}g \in N_G(P)$  and  $g \in FN_G(P)$ . Thus the conclusion of the Frattini argument is that  $G = FN_G(P)$ . Now the non-generator property comes into play, allowing us to omit the elements of  $F$  one at a time, until eventually we get  $G = N_G(P)$ , i.e.,  $P \triangleleft G$ . In particular  $P \triangleleft F$ , so that all the Sylow subgroups of  $F$  are normal and  $F$  is nilpotent by (10.2.7).  $\square$

**The Frattini subgroup of a finite  $p$ -group.** The Frattini subgroup plays an especially significant role in the theory of finite  $p$ -groups. Suppose that  $G$  is a finite  $p$ -group. If  $M$  is a maximal subgroup of  $G$ , then, since  $G$  is nilpotent,  $M$  is subnormal and hence normal

in  $G$ . Furthermore  $G/M$  cannot have proper non-trivial subgroups by maximality of  $M$ . Thus  $|G/M| = p$ . Define the  $p$ th power of the group  $G$  to be

$$G^p = \langle g^p \mid g \in G \rangle.$$

Then  $G^p G' \leq M$  for all  $M$  and  $G^p G' \leq \phi(G)$ .

On the other hand,  $G/G^p G'$  is a finite abelian group in which every  $p$ th power is the identity, i.e., it is an elementary abelian  $p$ -group. By (8.2.16) such a group is a direct product of groups of order  $p$ . This fact enables us to construct maximal subgroups of  $G/G^p G'$  by omitting all but one factor from the direct product. The resulting maximal subgroups of  $G/G^p G'$  clearly intersect in the identity subgroup, which shows that  $\phi(G) \leq G^p G'$ . We have therefore proved:

**(10.2.10)** *If  $G$  is a finite  $p$ -group, then  $\phi(G) = G^p G'$ .*

Next suppose that  $V = G/G^p G'$  has order  $p^d$ ; thus  $d$  is the dimension of  $V$  as a vector space over the field  $\mathbb{Z}_p$ . Consider an arbitrary set  $X$  of generators for  $G$ . Now the subset  $\{xG^p G' \mid x \in X\}$  clearly generates  $V$  as a vector space. By Exercise (8.2.10) there is a subset  $Y$  of  $X$  such that  $\{yG^p G' \mid y \in Y\}$  is a basis of  $V$ . Of course  $|Y| = d$ . We claim that  $Y$  generates  $G$ . Certainly we have that  $G = \langle Y, G^p G' \rangle = \langle Y, \phi(G) \rangle$ . The non-generator property of  $\phi(G)$  shows that  $G = \langle Y \rangle$ .

Summing up these conclusions, we have the following basic result on finite  $p$ -groups.

**(10.2.11)** *Let  $G$  be a finite  $p$ -group and assume that  $G/\phi(G)$  has order  $p^d$ . Then every set of generators of  $G$  has a  $d$ -element subset that generates  $G$ . In particular  $G$  can be generated by  $d$  and no fewer elements.*

**Example (10.2.2)** A group  $G$  is constructed as the semidirect product of a cyclic group  $\langle a \rangle$  of order  $2^n$  with a Klein 4-group  $V = \langle x, y \rangle$  where  $n \geq 3$ ,  $xax^{-1} = a^{-1}$  and  $yay^{-1} = a^{1+2^{n-1}}$ . Thus  $|G| = 2^{n+2}$ . Observe that  $G' = \langle a^2 \rangle$  and thus  $G/G'$  is elementary abelian of order 8. Hence  $\phi(G) = G^2 G' = \langle a^2 \rangle$ . By (10.2.11) the group  $G$  can be generated by 3 and no fewer elements, and in fact  $G = \langle a, x, y \rangle$ .

### Exercises (10.2)

- (1) Let  $M \triangleleft G$  and  $N \triangleleft G$  where  $G$  is any group. If  $M$  and  $N$  are solvable, prove that  $MN$  is solvable.
- (2) Let  $M \triangleleft G$  and  $N \triangleleft G$  for any group  $G$ . If  $G/M$  and  $G/N$  are solvable, prove  $G/M \cap N$  is solvable.
- (3) Explain why a solvable group with a composition series is necessarily finite.
- (4) Let  $G$  be a finite group with two non-trivial elements  $a$  and  $b$  such that  $|a|$ ,  $|b|$ ,  $|ab|$  are relatively prime in pairs. Prove that  $G$  cannot be solvable. [Hint: put  $H = \langle a, b \rangle$  and consider  $H/H'$ ].

- (5) Prove that if  $p, q, r$  are primes, then every group of order  $pqr$  is solvable. [Hint: assume that  $G$  is a simple group of order  $pqr$  where  $p < q < r$  and show that  $n_r = pq$ ,  $n_q \geq r$  and  $n_p \geq q$ . Now count elements to obtain a contradiction].
- (6) Prove that if  $p$  and  $q$  are primes, then every group of order  $p^2q^2$  is solvable. [Hint: follow the method of proof for groups of order  $p^mq$  in (10.2.4). Deal first with the case where each pair of Sylow  $p$ -subgroups intersects in 1. Then choose two Sylow subgroups  $P_1$  and  $P_2$  such that  $I = P_1 \cap P_2$  has order  $p$  and note that  $I \triangleleft J = \langle P_1, P_2 \rangle$ ].
- (7) Establish the commutator identities

$$[x, y^{-1}] = y^{-1}([x, y]^{-1})y \quad \text{and} \quad [x, yz] = [x, y](y[x, z]y^{-1}).$$

- (8) Let  $G$  be a group and let  $z \in Z_2(G)$ . Prove that the assignment  $x \mapsto [z, x]$  determines a homomorphism from  $G$  to  $Z(G)$  whose kernel contains  $G'$ .
- (9) Let  $G$  be a group such that  $Z_1(G) < Z_2(G)$ . Use Exercise (10.2.8) to show that  $G > G'$ .
- (10) Find the upper central series of the group  $G = \text{Dih}(2^m)$  where  $m \geq 2$ . Hence compute the nilpotent class of  $G$ .
- (11) Let  $n > 1$  and let  $G = U_n(p)$ , the group of  $n \times n$  upper unitriangular matrices over  $\mathbb{Z}_p$ . Define  $G_i$  to be the subgroup of all elements of  $G$  in which the first  $i$  superdiagonals consist of 0's, where  $0 \leq i < n$ . Show that the  $G_i$  are terms of a central series of  $G$ . Then find the nilpotent class of  $G$ .
- (12) Let  $G$  be a nilpotent group with a non-trivial normal subgroup  $N$ . Prove that  $N \cap Z(G) \neq 1$ .
- (13) Let  $A$  be a maximal abelian normal subgroup of a nilpotent group  $G$ . Prove that  $C_G(A) = A$ . [Hint: assume this is false and apply Exercise (10.2.12) to  $C_G(A)/A \triangleleft G/A$ ].
- (14) If every abelian normal subgroup of a nilpotent group is finite, prove that the group is finite.
- (15) The *lower central sequence*  $\{\gamma_i(G)\}$  of group  $G$  is defined by  $\gamma_1(G) = G$  and  $\gamma_{i+1}(G) = [\gamma_i(G), G]$ . If  $G$  is a nilpotent group, prove that the lower central sequence reaches 1 and its length equals the nilpotent class of  $G$ . (If  $H, K$  are subgroups of a group, then  $[H, K]$  is the subgroup generated by all commutators  $[h, k]$ ,  $h \in H, k \in K$ ).
- (16) Find the Frattini subgroup of the groups  $A_n, S_n$  and  $\text{Dih}(2n)$  where  $n$  is odd.
- (17) Use (10.2.4) to show that a non-solvable group of order at most 100 must have order 60. (Note that the only orders requiring attention are 60, 72, 84 and 90).
- (18) Prove that  $A_5$  is the only non-solvable group with order  $\leq 100$ . [Hint: it is enough to show that a simple group of order 60 must have a subgroup of index 5. Consider the number of Sylow 2-subgroups].

### 10.3 Theorems on finite solvable groups

The final section of the chapter will take us deeper into the theory of finite solvable groups and several famous theorems will be proved.

**Schur's splitting and conjugacy theorem.** Suppose that  $N$  is a normal subgroup of a group  $G$ . A subgroup  $X$  such that  $G = NX$  and  $N \cap X = 1$  is called a *complement* of  $N$  in  $G$ . In this case  $G$  is said to *split over*  $N$  and  $G$  is the semidirect product of  $N$  and  $X$ . A *splitting theorem* is theorem asserting that a group splits over a normal subgroup. One can think of such a theorem as resolving a group into a product of potentially simpler groups. The most celebrated splitting theorem in group theory is undoubtedly Schur's theorem.

**(10.3.1)** (Schur<sup>6</sup>) *Let  $A$  be an abelian normal subgroup of a finite group  $G$  such that  $|A|$  and  $|G : A|$  are relatively prime. Then  $G$  splits over  $A$  and all complements of  $A$  are conjugate in  $G$ .*

*Proof.* (i) *Existence of a complement.* To start the proof choose an arbitrary transversal to  $A$  in  $G$ , say  $\{t_x \mid x \in Q = G/N\}$  where  $x = At_x$ . Most likely this transversal will not be a subgroup. The idea behind the proof is to transform the transversal into one which is a subgroup. Let  $x, y \in Q$ : then  $x = At_x$  and  $y = At_y$ , and in addition  $At_{xy} = xy = At_xAt_y = At_xt_y$ . Thus it is possible to write

$$t_xt_y = a(x, y)t_{xy}$$

for some  $a(x, y) \in A$ .

The associative law  $(t_xt_y)t_z = t_x(t_yt_z)$  imposes a condition on the elements  $a(x, y)$ . For, applying the above equation twice, we obtain

$$(t_xt_y)t_z = a(x, y)a(xy, z)t_{xyz},$$

and similarly

$$t_x(t_yt_z) = t_xa(y, z)t_{yz} = (t_xa(y, z)t_x^{-1})t_xt_{yz} = (t_xa(y, z)t_x^{-1})a(x, yz)t_{xyz}.$$

Now conjugation of elements of  $A$  by  $t_x$  induces an automorphism of  $A$  which depends only on  $x$ : for, if  $a, b \in A$ , then  $(bt_x)a(bt_x)^{-1} = t_xat_x^{-1}$  since  $A$  is abelian. Let us write  ${}^xa$  for  $t_xat_x^{-1}$ . Then on equating  $(t_xt_y)t_z$  and  $t_x(t_yt_z)$  and cancelling  $t_{xyz}$ , we arrive at the equation

$$a(x, y)a(xy, z) = {}^xa(y, z)a(x, yz), \quad (*)$$

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<sup>6</sup> Issai Schur (1875–1941)



which is valid for all  $x, y, z, \in Q$ . A function  $a : Q \times Q \rightarrow A$  that satisfies the condition  $(*)$  is called a *factor set or 2-cocycle*.

Next define

$$b_x = \prod_{y \in Q} a(x, y),$$

noting that the order of the factors in the product is immaterial since  $A$  is abelian. On forming the product of the equations  $(*)$  above for all  $z$  in  $Q$  with  $x$  and  $y$  fixed, we obtain the equation

$$a(x, y)^m b_{xy} = {}^x b_y b_x, \quad (**)$$

where  $m = |Q| = |G : A|$ . Note here that the product of all the  ${}^x a(y, z)$  is  ${}^x b_y$  and the product of all the  $a(x, yz)$  is  $b_x$ .

Since  $m$  is relatively prime to  $|A|$ , the mapping  $a \mapsto a^m$  is an automorphism of  $A$ . Thus we can write  $b_x$  as an  $m$ th power, say  $b_x = c_x^{-m}$  where  $c_x \in A$ . Substituting for  $b_x$  in equation  $(**)$ , we get  $(a(x, y)c_{xy}^{-1})^m = (({}^x c_y c_x)^{-1})^m$ , from which it follows that

$$c_{xy} = c_x ({}^x c_y) a(x, y).$$

We are now ready to form the new transversal. Write  $s_x = c_x t_x$  and observe that the  $s_x$ , ( $x \in Q$ ), form a transversal to  $A$ . Moreover

$$s_x s_y = c_x t_x c_y t_y = c_x ({}^x c_y) t_x t_y = c_x ({}^x c_y) a(x, y) t_{xy} = c_{xy} t_{xy} = s_{xy}.$$

This demonstrates that the transversal  $H = \{s_x \mid x \in Q\}$  is a subgroup. Since  $G = AH$  and  $A \cap H = 1$ , it follows that  $H$  is a complement of  $A$  in  $G$  and  $G$  splits over  $A$ .

(ii) *Conjugacy of complements*. Let  $H = \{s_x \mid x \in Q\}$  and  $H^* = \{s_x^* \mid x \in Q\}$  be two complements of  $A$  in  $G$ . If  $x \in Q$ , we can write  $x = As_x = As_x^*$  where  $s_x$  and  $s_x^*$  belong to  $H$  and  $H^*$  respectively. Thus  $s_x$  and  $s_x^*$  are related by an equation of the form

$$s_x^* = d(x)s_x$$

where  $d(x) \in A$ . Since  $As_{xy} = xy = As_x As_y = As_x s_y$ , we have  $s_x s_y = s_{xy}$ , and similarly  $s_x^* s_y^* = s_{xy}^*$ . In the last equation make the substitutions  $s_x^* = d(x)s_x$ ,  $s_y^* = d(y)s_y$ ,  $s_{xy}^* = d(xy)s_{xy}$  to get  $d(x)s_x d(y)s_y = d(xy)s_{xy}$  and hence

$$d(xy) = d(x)({}^x d(y))$$

for all  $x, y \in Q$ . Such a function  $d : Q \rightarrow A$  is called a *derivation or 1-cocycle*.

Put  $d = \prod_{x \in Q} d(x)$  and take the product of all the equations  $d_{xy} = d(x)({}^x d(y))$  for  $y \in Q$  with  $x$  fixed. This leads to  $d = (d(x)^m)({}^x d)$ . Writing  $d = e^m$  with  $e \in A$ , we obtain  $e^m = (d(x) {}^x e)^m$  and hence  $e = d(x)({}^x e)$ . Thus  $d(x) = e({}^x e)^{-1}$ . Since  ${}^x e = s_x e s_x^{-1}$ , we have

$$s_x^* = d(x)s_x = e({}^x e)^{-1}s_x = e(s_x e^{-1} s_x^{-1})s_x = e s_x e^{-1}.$$

Therefore  $H^* = e H e^{-1}$ , so  $H$  and  $H^*$  are conjugate. □

In fact (10.3.1) is true even when  $A$  is non-abelian, a result which is known as the *Schur-Zassenhaus Theorem*. The proof of conjugacy of complements requires the Odd Order Theorem: see for example [11] and also Exercise (10.3.7).

**Hall's theorems on finite solvable groups.** To illustrate the usefulness of Schur's splitting theorem we will make a foray into the theory of finite solvable groups by proving the following celebrated result.

**(10.3.2)** (P. Hall<sup>7</sup>) *Let  $G$  be a finite solvable group and write  $|G| = mn$  where the positive integers  $m, n$  are relatively prime. Then  $G$  has a subgroup of order  $m$  and all subgroups of this order are conjugate.*

*Proof.* (i) *Existence.* We argue by induction on  $|G| > 1$ . The group  $G$  has a non-trivial abelian normal subgroup  $A$ , for example the smallest non-trivial term of the derived series. Since  $A$  is the direct product of its primary components, we can assume that  $A$  is a  $p$ -group, with  $|A| = p^k$ , say. There are two cases to consider.

Suppose first that  $p$  does not divide  $m$ . Then  $p^k \nmid n$  because  $m$  and  $n$  are relatively prime. Since  $|G/A| = m \cdot (n/p^k)$ , the induction hypothesis may be applied to the group  $G/A$  to show that it has a subgroup of order  $m$ , say  $K/A$ . Now  $|A|$  is relatively prime to  $m = |K : A|$ , so (10.3.1) may be applied to  $K$ . Hence there is a complement of  $A$  in  $K$ : this has order  $m$ , as required.

Now assume that  $p$  divides  $m$ ; then  $p^k \mid m$  since  $p$  cannot divide  $n$ . Since  $|G/A| = (m/p^k) \cdot n$ , induction shows that  $G/A$  has a subgroup of order  $m/p^k$ , say  $H/A$ . Then  $|H| = |A| \cdot |H/A| = p^k(m/p^k) = m$ , as required.

(ii) *Conjugacy.* Let  $H$  and  $H^*$  be two subgroups of order  $m$ , and choose  $A$  as in (i). If  $p$  does not divide  $m$ , then  $A \cap H = 1 = A \cap H^*$ , and  $AH/A$  and  $AH^*/A$  are subgroups of  $G/A$  with order  $m$ . By induction on  $|G|$  these subgroups are conjugate and thus  $AH = g(AH^*)g^{-1} = A(gH^*g^{-1})$  for some  $g \in G$ . By replacing  $H^*$  by  $gH^*g^{-1}$ , we can assume that  $AH = AH^*$ . But now  $H$  and  $H^*$  are two complements of  $A$  in  $HA$ , so (10.3.1) guarantees that they are conjugate.

Finally, assume that  $p$  divides  $m$ . Then  $p$  does not divide  $n = |G : H| = |G : H^*|$ . Since  $|AH : H|$  is a power of  $p$  and it also divides  $n$ , we conclude that  $|AH : H| = 1$  and  $A \leq H$ . Similarly  $A \leq H^*$ . By induction  $H/A$  and  $H^*/A$  are conjugate in  $G/A$ , as must  $H$  and  $H^*$  be in  $G$ .  $\square$

**Hall  $\pi$ -subgroups.** Let us now assess the significance of Hall's theorem. Let  $\pi$  denote a non-empty set of primes and let  $\pi'$  be the complementary set of primes. A positive integer is called a  $\pi$ -number if it is a product of powers of primes from the set  $\pi$ . A finite group is said to be a  $\pi$ -group if its order is a  $\pi$ -number.

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Now let  $G$  be a finite solvable group and write  $|G| = mn$  where  $m$  is a  $\pi$ -number and  $n$  is a  $\pi'$ -number. Then (10.3.2) tells us that  $G$  has a subgroup  $H$  of order  $m$  and index  $n$ . Thus  $H$  is a  $\pi$ -group and  $|G : H|$  is a  $\pi'$ -number: such a subgroup is called a *Hall  $\pi$ -subgroup* of  $G$ . Thus (10.3.2) actually asserts that Hall  $\pi$ -subgroups exist in a finite solvable group for any set of primes  $\pi$ , and that any two Hall  $\pi$ -subgroups are conjugate.

Hall's theorem can be regarded as an extension of Sylow's Theorem since if  $\pi = \{p\}$ , a Hall  $\pi$ -subgroup is simply a Sylow  $p$ -subgroup. However, Sylow's Theorem is valid for any finite group, whereas Hall subgroups need not exist in an insolvable group. For example  $A_5$  has order  $60 = 3 \cdot 20$ , but it has no subgroups of order 20, as the reader should verify.

This example is no coincidence since there is in fact a strong converse of Hall's theorem: the mere existence Hall  $p'$ -subgroups for all primes  $p$  dividing the group order is enough to imply solvability of the group. Here  $p'$  is the set of all primes different from  $p$ . The proof of this result uses the Burnside  $pq$ -Theorem: a group of order  $p^m q^n$  is solvable if  $p$  and  $q$  are primes.

**(10.3.3)** (P. Hall) *Let  $G$  be a finite group and suppose that for every prime  $p$  dividing  $|G|$  there is a Hall  $p'$ -subgroup in  $G$ . Then  $G$  is solvable.*

*Proof.* Assume the theorem is false and let  $G$  be a counterexample of smallest order. We look for a contradiction. Suppose that  $N$  is proper non-trivial normal subgroup of  $G$ . If  $H$  is a Hall  $p'$ -subgroup of  $G$ , then by consideration of order and index we see that  $H \cap N$  and  $HN/N$  are Hall  $p'$ -subgroups of  $N$  and  $G/N$  respectively. Therefore  $N$  and  $G/N$  are solvable by minimality of  $|G|$ , and thus  $G$  is solvable. By this contradiction  $G$  is a simple group.

Write  $|G| = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  where  $e_i > 0$  and the  $p_i$  are distinct primes. The Burnside  $pq$ -Theorem shows that  $k > 2$ . Let  $G_i$  be a Hall  $p_i'$ -subgroup of  $G$ ; thus  $|G : G_i| = p_i^{e_i}$ . Put  $H = G_3 \cap \cdots \cap G_k$  and observe that

$$|G : H| = \prod_{i=3}^k |G : G_i| = \prod_{i=3}^k p_i^{e_i}$$

by (4.1.13). Therefore  $|H| = |G|/|G : H| = p_1^{e_1} p_2^{e_2}$  and  $H$  is solvable by Burnside's Theorem.

Since  $H \neq 1$ , it contains a minimal normal subgroup  $M$ . By Exercise (10.3.2) below  $M$  is an elementary abelian  $p$ -group where  $p = p_1$  or  $p_2$ : without loss of generality let  $p = p_1$ . Now

$$|G : H \cap G_2| = |G : H| \cdot |G : G_2| = \prod_{i=2}^k p_i^{e_i}$$

by (4.1.13) once again. Thus  $|H \cap G_2| = p_1^{e_1}$ , i.e.,  $H \cap G_2$  is a Sylow  $p_1$ -subgroup of  $H$ . Hence  $M(H \cap G_2)$  is a  $p_1$ -group, from which it follows that  $M \leq H \cap G_2$ . Also  $|H \cap G_1| = p_2^{e_2}$

by the same argument and therefore

$$|(H \cap G_1)G_2| = |H \cap G_1| \cdot |G_2| = p_2^{e_2} \frac{|G|}{p_2^{e_2}} = |G|.$$

Consequently  $G = (H \cap G_1)G_2$ . Next consider the normal closure of  $M$  in  $G$  – see Example (4.2.1)(vi). This is

$$\langle M^G \rangle = \langle M^{(H \cap G_1)G_2} \rangle = \langle M^{G_2} \rangle \leq G_2 < G,$$

since  $M \triangleleft H$ . It follows that  $\langle M^G \rangle$  is a proper non-trivial normal subgroup of  $G$ , so  $G$  is not simple, a contradiction.  $\square$

Hall's theorems are the starting point for a rich theory of finite solvable groups which has been developed over the last eight decades; the standard reference for this is [3].

### Exercises (10.3)

- (1) Give an example of a finite group  $G$  with an abelian normal subgroup  $A$  such that  $G$  does not split over  $A$ .
- (2) If  $G$  is a finite solvable group with a minimal (non-trivial) normal subgroup  $N$ , prove that  $N$  is an elementary abelian  $p$ -group for some  $p$  dividing  $|G|$ . [Hint: note that  $N' \triangleleft G$ ].
- (3) If  $M$  is a maximal subgroup of a finite solvable group  $G$ , prove that  $|G : M|$  is equal to a prime power. [Hint: use induction on  $|G|$  to reduce to the case where  $M$  contains no non-trivial normal subgroups of  $G$ . Let  $A$  be a minimal normal subgroup of  $G$ . Show that  $G = MA$  and  $M \cap A = 1$ ].
- (4) For which sets of primes  $\pi$  does the group  $A_5$  contain a Hall  $\pi$ -subgroup?
- (5) Let  $G$  be a finite solvable group and  $p$  a prime dividing the order of  $G$ . Prove that  $G$  has a maximal subgroup with index a power of  $p$ . [Hint: apply (10.3.2)].
- (6) Let  $G$  be a finite group and  $\pi$  a set of primes. Let  $L$  be a solvable normal subgroup of  $G$  and assume that  $H$  is a Hall  $\pi$ -subgroup of  $L$ . Prove that  $G = LN_G(H)$ .
- (7) Let  $G$  be a finite group with a normal subgroup  $N$ . Assume that  $|N|$  and  $|G : N|$  are relatively prime and that  $N$  is solvable. Prove that  $G$  splits over  $N$  and all complements of  $N$  are conjugate. [Hint: assume that  $N \neq 1$  and find a non-trivial abelian subgroup  $A$  of  $G$  which is contained in  $N$ . By induction on the group order the result is true for  $G/A$ ].
- (8) Let  $G$  be a finite group and let  $p$  be a prime dividing the order of  $G$ . Prove that  $p$  divides  $|G/\phi(G)|$ . [Hint: assume this is false, so  $G/\phi(G)$  is a  $p'$ -group. Since  $\phi(G)$  is nilpotent, there exists  $P \triangleleft G$  such that  $P \leq \phi(G)$ ,  $P$  is a  $p$ -group and  $G/P$  a  $p'$ -group. Now apply Exercise (10.3.7)].

# 11 The Theory of fields

Field theory is one of the most attractive parts of algebra. It contains many powerful results on the structure of fields, for example, the Fundamental Theorem of Galois Theory, which establishes a correspondence between subfields of a field and subgroups of the Galois group. In addition field theory can be applied to a wide variety of problems, some of which date from classical antiquity. Among the applications to be described here and in the following chapter are: ruler and compass constructions, solution of equations by radicals, orthogonal latin squares and Steiner systems. In short field theory is algebra at its best – deep theorems with convincing applications to problems which might otherwise be intractable.

## 11.1 Field extensions

Recall from (7.4) that a *subfield* of a field  $F$  is a subring containing 1 which is closed with respect to inversion of its non-zero elements. The following is an immediate consequence of the definition.

**(11.1.1)** *The intersection of any set of subfields of a field is a subfield.*

Suppose that  $X$  is a (non-empty) subset of a field  $F$ . By (11.1.1) the intersection of all the subfields of  $F$  that contain  $X$  is a subfield, which is evidently the smallest subfield containing  $X$ . We call this the *subfield of  $F$  generated by  $X$* . It is easy to describe the elements of the subfield generated by a given subset.

**(11.1.2)** *If  $X$  is a subset of a field  $F$ , the subfield generated by  $X$  consists of all elements of the form*

$$f(x_1, \dots, x_m)g(y_1, \dots, y_n)^{-1}$$

*where  $f \in \mathbb{Z}[t_1, \dots, t_m]$ ,  $g \in \mathbb{Z}[t_1, \dots, t_n]$ ,  $x_i, y_j \in X$  and  $g(y_1, \dots, y_n) \neq 0$ .*

To prove this, first observe that the set  $S$  of elements with the specified form is a subfield of  $F$  containing  $X$ . Then note that any subfield of  $F$  which contains  $X$  must also contain all the elements of  $S$ , so that  $S$  is the smallest subfield that contains  $X$ .

**Prime subfields.** In a field  $F$  one can form the intersection of *all* its subfields. This is the unique smallest subfield of  $F$  and it is called the *prime subfield* of  $F$ . A field which equals its prime subfield is called a *prime field*. It is easy to identify the prime fields.

**(11.1.3)** *A prime field of characteristic 0 is isomorphic with  $\mathbb{Q}$ : a prime field of prime characteristic  $p$  is isomorphic with  $\mathbb{Z}_p$ . Conversely,  $\mathbb{Q}$  and  $\mathbb{Z}_p$  are prime fields.*

*Proof.* Assume that  $F$  is a prime field and put  $I = \langle 1_F \rangle = \{n1_F \mid n \in \mathbb{Z}\}$ . Suppose first that  $F$  has characteristic 0, so  $I$  is infinite cyclic. Define a surjective mapping  $\alpha : \mathbb{Q} \rightarrow F$  by the rule  $\alpha(\frac{m}{n}) = (m1_F)(n1_F)^{-1}$ , where  $n \neq 0$ . It is easily checked that  $\alpha$  is a well defined ring homomorphism and its kernel is therefore an ideal of  $\mathbb{Q}$ . Now 0 and  $\mathbb{Q}$  are the only ideals of  $\mathbb{Q}$  and  $\alpha(1) = 1_F \neq 0_F$ , so  $\text{Ker}(\alpha) \neq \mathbb{Q}$ . It follows that  $\text{Ker}(\alpha) = 0$  and  $\mathbb{Q} \simeq \text{Im}(\alpha)$ . Since  $F$  is a prime field and  $\text{Im}(\alpha)$  is a subfield,  $\text{Im}(\alpha) = F$  and  $\alpha$  is an isomorphism. Thus  $F \simeq \mathbb{Q}$ .

Now suppose that  $F$  has prime characteristic  $p$ , so that  $|I| = p$ . In this situation we define  $\alpha : \mathbb{Z} \rightarrow F$  by  $\alpha(n) = n1_F$ . Thus  $\alpha(n) = 0_F$  if and only if  $n1_F = 0$ , i.e.,  $p$  divides  $n$ . Hence  $\text{Ker}(\alpha) = p\mathbb{Z}$  and  $\text{Im}(\alpha) \simeq \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$ . It follows that  $\mathbb{Z}_p$  is isomorphic with a subfield of  $F$  and, since  $F$  is prime,  $\mathbb{Z}_p \simeq F$ . It is left to the reader to check that  $\mathbb{Q}$  and  $\mathbb{Z}_p$  are prime fields.  $\square$

**Field extensions.** Consider two fields  $F$  and  $E$  and suppose there is an injective ring homomorphism  $\alpha : F \rightarrow E$ . Then  $F$  is isomorphic with  $\text{Im}(\alpha)$ , which is a subfield of  $E$ : under these circumstances we say that  $E$  is an *extension of  $F$* . Often we prefer to assume that  $F$  is actually a subfield of  $E$ . This is usually a harmless assumption since  $F$  can be replaced by the isomorphic field  $\text{Im}(\alpha)$ . Notice that by (11.1.3) every field is an extension of either  $\mathbb{Z}_p$  or  $\mathbb{Q}$ , according as the characteristic is a prime  $p$  or 0.

If  $E$  is an extension of  $F$ , then  $E$  can be regarded as a vector space over  $F$  by using the field operations. The vector space axioms are consequences of the field axioms. This simple idea is critical since it allows us to define the *degree of  $E$  over  $F$*  as

$$(E : F) = \dim_F(E),$$

assuming that this dimension is finite. Then  $E$  is called a *finite extension of  $F$* .

**Simple extensions.** Let  $F$  be a subfield and  $X$  a non-empty subset of a field  $E$ . The subfield of  $E$  generated by  $F \cup X$  is denoted by

$$F(X).$$

It follows readily from (11.1.2) that  $F(X)$  consists of all elements of the form

$$f(x_1, \dots, x_m)g(y_1, \dots, y_n)^{-1}$$

where  $f \in F[t_1, \dots, t_m]$ ,  $g \in F[t_1, \dots, t_n]$ ,  $x_i, y_j \in X$  and  $g(y_1, \dots, y_n) \neq 0$ . If  $X = \{x_1, x_2, \dots, x_l\}$ , we write

$$F(x_1, x_2, \dots, x_l)$$

instead of  $F(\{x_1, x_2, \dots, x_l\})$ . The most interesting case for us is when  $X = \{x\}$  and a typical element of  $F(x)$  has the form  $f(x)g(x)^{-1}$  where  $f, g \in F[t]$  and  $g(x) \neq 0$ . If  $E = F(x)$  for some  $x \in E$ , then  $E$  is said to be a *simple extension of  $F$* .

We proceed at once to determine the structure of simple extensions.

**(11.1.4)** Let  $E = F(x)$  be a simple extension of a field  $F$ . Then one of the following must hold:

- (i)  $f(x) \neq 0$  for all  $0 \neq f \in F[t]$  and  $E \simeq F\{t\}$ , the field of rational functions in  $t$  over  $F$ ;
- (ii)  $f(x) = 0$  for some monic irreducible polynomial  $f \in F[t]$  and  $E \simeq F[t]/(f)$ .

*Proof.* We assume that  $F \subseteq E$ . Define a mapping  $\theta : F[t] \rightarrow E$  by evaluation at  $x$ , i.e.,  $\theta(f) = f(x)$ . This is a ring homomorphism whose kernel is an ideal of  $F[t]$ , say  $I$ .

Assume first that  $I = 0$ , i.e.,  $f(x) = 0$  implies that  $f = 0$ . Then  $\theta$  can be extended to a function  $\alpha : F\{t\} \rightarrow E$  by the rule  $\alpha(\frac{f}{g}) = f(x)g(x)^{-1}$ ; this function is also a ring homomorphism. Notice that  $\alpha(\frac{f}{g}) = 0$  implies that  $f(x) = 0$  and hence  $f = 0$ . Therefore  $\text{Ker}(\alpha) = 0$  and  $F\{t\}$  is isomorphic with  $\text{Im}(\alpha)$ , which is a subfield of  $E$ . Now  $\text{Im}(\alpha)$  contains  $F$  and  $x$  since  $\alpha(a) = a$  if  $a \in F$  and  $\alpha(t) = x$ . Because  $E$  is a smallest field containing  $F$  and  $x$ , it follows that  $E = \text{Im}(\alpha) \simeq F\{t\}$ .

Now suppose that  $I \neq 0$ . Then  $F[t]/I$  is isomorphic with a subring of the field  $E$ , so it is a domain and hence  $I$  is a prime ideal. Since  $F[t]$  is a PID, we can apply (7.2.6) to get  $I = (f)$  where  $f$  is a monic irreducible polynomial in  $F[t]$ . Thus  $F[t]/I$  is a field which is isomorphic with  $\text{Im}(\theta)$ , a subfield of  $E$  containing  $F$  and  $x$  for reasons given above. Therefore  $F[t]/I \simeq \text{Im}(\theta) = E$ .  $\square$

**Algebraic elements.** Consider a field extension  $E$  of  $F$  and let  $x \in E$ . There are two possible forms for the subfield  $F(x)$ , as indicated in (11.1.4). If  $f(x) \neq 0$  whenever  $0 \neq f \in F[t]$ , then  $F(x) \simeq F\{t\}$  and  $x$  is said to be *transcendent over  $F$* .

The other possibility is that  $x$  is a root of a monic irreducible polynomial  $f$  in  $F[t]$ . In this case  $F(x) \simeq F[t]/(f)$  and  $x$  is said to be *algebraic over  $F$* . The polynomial  $f$  is the unique monic irreducible polynomial over  $F$  which has  $x$  as a root: for if  $g$  is another such polynomial, then  $g \in (f)$  and  $f \mid g$ , so  $f = g$  by irreducibility and monicity. We call  $f$  the *irreducible polynomial* of  $x$  over  $F$ , in symbols

$$\text{Irr}_F(x):$$

thus  $F(x) \simeq F[t]/(\text{Irr}_F(x))$ .

Now let  $f = \text{Irr}_F(x)$  have degree  $n$ . For any  $g$  in  $F[t]$  write  $g = fq + r$  where  $q, r \in F[t]$  and  $\deg(r) < n$ , by the Division Algorithm for  $F[t]$ , (see (7.1.3)). Then  $g + (f) = r + (f)$ , which shows that  $F(x)$  is generated as an  $F$ -vector space by  $1, x, x^2, \dots, x^{n-1}$ . In fact these elements are linearly independent over  $F$ . For, if  $a_0 + a_1x + \dots + a_{n-1}x^{n-1} = 0$  with  $a_i \in F$ , then  $g(x) = 0$  where  $g = a_0 + a_1t + \dots + a_{n-1}t^{n-1}$ , and hence  $f \mid g$ . But  $\deg(g) \leq n-1$ , which can only mean that  $g = 0$  and all the  $a_i$  are zero. It follows that the elements  $1, x, x^2, \dots, x^{n-1}$  form an  $F$ -basis of the vector space  $F(x)$  and hence  $(F(x) : F) = n = \deg(f)$ .

These conclusions are summarized in:

**(11.1.5)** Let  $E = F(x)$  be a simple field extension of  $F$ .

- (i) If  $x$  is transcendental over  $F$ , then  $E \simeq F\{t\}$ .  
(ii) If  $x$  is algebraic over  $F$ , then  $E \simeq F[t]/(\text{Irr}_F(x))$  and  $(E : F) = \deg(\text{Irr}_F(x))$ .

**Example (11.1.1)** Show that  $\sqrt{3} - \sqrt{2}$  is algebraic over  $\mathbb{Q}$ , by finding its irreducible polynomial and hence the degree  $(\mathbb{Q}(\sqrt{3} - \sqrt{2}) : \mathbb{Q})$ .

Put  $x = \sqrt{3} - \sqrt{2}$ . The first move is to find a rational polynomial with  $x$  as a root. Now  $x^2 = 5 - 2\sqrt{6}$ , so  $(x^2 - 5)^2 = 24$  and  $x^4 - 10x^2 + 1 = 0$ . Hence  $x$  is a root of  $f = t^4 - 10t^2 + 1$  and thus is algebraic over  $\mathbb{Q}$ . If we can show that  $f$  is irreducible over  $\mathbb{Q}$ , it will follow that  $\text{Irr}_{\mathbb{Q}}(x) = f$  and  $(\mathbb{Q}(x) : \mathbb{Q}) = 4$ .

By Gauss's Lemma (7.3.7) it is enough to show that  $f$  is irreducible over  $\mathbb{Z}$ . Now clearly  $f$  has no integer roots, for  $\pm 1$  are the only candidates and neither one is a root. Thus, if  $f$  is reducible, there must be a decomposition of the form

$$f = (t^2 + at + b)(t^2 + a_1t + b_1)$$

where  $a, b, a_1, b_1$  are integers. On equating coefficients of  $1, t^3, t^2$  on both sides, we arrive at the equations

$$bb_1 = 1, \quad a + a_1 = 0, \quad aa_1 + b + b_1 = -10.$$

Hence  $b = b_1 = \pm 1$  and  $a_1 = -a$ , so that  $-a^2 \pm 2 = -10$ . Since this equation has no integer solutions,  $f$  is irreducible.

**Algebraic extensions.** Let  $E$  be an extension of a field  $F$ . If every element of  $E$  is algebraic over  $F$ , then  $E$  is called an *algebraic extension* of  $F$ . Extensions of finite degree are an important source of algebraic extensions.

**(11.1.6)** An extension  $E$  of a field  $F$  with finite degree is algebraic.

*Proof.* Let  $x \in E$ . By hypothesis  $E$  has finite dimension as a vector space over  $F$ , say  $n$ ; consequently the set  $\{1, x, x^2, \dots, x^n\}$  is linearly dependent and there are elements  $a_0, a_1, \dots, a_n$  of  $F$ , not all zero, such that  $a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0$ . Thus  $x$  is a root of the non-zero polynomial  $a_0 + a_1t + \dots + a_nt^n$  and it is therefore algebraic over  $F$ .  $\square$

The next result is useful in calculations with degrees.

**(11.1.7)** Let  $F \subseteq K \subseteq E$  be successive field extensions. If  $K$  is finite over  $F$  and  $E$  is finite over  $K$ , then  $E$  is finite over  $F$  and  $(E : F) = (E : K) \cdot (K : F)$ .

*Proof.* Let  $\{x_1, \dots, x_m\}$  be an  $F$ -basis of  $K$  and  $\{y_1, \dots, y_n\}$  a  $K$ -basis of  $E$ . Then each  $e \in E$  can be written as  $e = \sum_{i=1}^n k_i y_i$  where  $k_i \in K$ . Also each  $k_i$  can be written  $k_i = \sum_{j=1}^m f_{ij} x_j$  with  $f_{ij} \in F$ . Therefore  $e = \sum_{i=1}^n \sum_{j=1}^m f_{ij} x_j y_i$  and it follows that the elements  $x_j y_i$  generate the  $F$ -vector space  $E$ .



Next assume there is an  $F$ -linear relation among the  $x_j y_i$ ,

$$\sum_{i=1}^n \sum_{j=1}^m f_{ij} x_j y_i = 0$$

where  $f_{ij} \in F$ . Then  $\sum_{i=1}^n \left( \sum_{j=1}^m f_{ij} x_j \right) y_i = 0$ , so that  $\sum_{j=1}^m f_{ij} x_j = 0$  for all  $i$ , since the  $y_i$  are  $K$ -linearly independent. Finally,  $f_{ij} = 0$  for all  $i$  and  $j$  by linear independence of the  $x_j$  over  $F$ . Consequently the elements  $x_j y_i$  form an  $F$ -basis of  $E$  and  $(E : F) = nm = (E : K) \cdot (K : F)$ .  $\square$

**Corollary (11.1.8)** *Let  $F \subseteq K \subseteq E$  be successive field extensions with  $E$  algebraic over  $K$  and  $K$  algebraic over  $F$ . Then  $E$  is algebraic over  $F$ .*

*Proof.* Let  $x \in E$ , so that  $x$  is algebraic over  $K$ ; let its irreducible polynomial be  $f = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n$  where  $a_i \in K$ . Put  $K_i = F(a_0, a_1, \dots, a_i)$ . Then  $a_i$  is algebraic over  $F$  and hence over  $K_{i-1}$ . Since  $K_i = K_{i-1}(a_i)$ , it follows via (11.1.5) that  $(K_i : K_{i-1})$  is finite for  $i = 0, 1, \dots, n-1$ , where  $K_{-1} = F$ . Hence  $(K_{n-1} : F)$  is finite by (11.1.7). Also  $x$  is algebraic over  $K_{n-1}$ , so that  $(K_{n-1}(x) : K_{n-1})$  is finite and therefore  $(K_{n-1}(x) : F)$  is finite. It follows via (11.1.6) that  $x$  is algebraic over  $F$ .  $\square$

**Algebraic and transcendental numbers.** Next let us consider the complex field  $\mathbb{C}$  as an extension of the rational field  $\mathbb{Q}$ . If  $x \in \mathbb{C}$  is algebraic over  $\mathbb{Q}$ , then  $x$  is called an *algebraic number*; otherwise  $x$  is a *transcendental number*. Thus the algebraic numbers are the real and complex numbers which are roots of non-zero rational polynomials.

**(11.1.9)** *The algebraic numbers form a subfield of  $\mathbb{C}$ .*

*Proof.* Let  $a$  and  $b$  be algebraic numbers. It is sufficient to show that  $a \pm b$ ,  $ab$  and  $ab^{-1}$  (if  $b \neq 0$ ) are algebraic numbers. To see this note that  $(\mathbb{Q}(a) : \mathbb{Q})$  is finite by (11.1.5). Also  $\mathbb{Q}(a, b) = (\mathbb{Q}(a))(b)$  is finite over  $\mathbb{Q}(a)$  for the same reason. Therefore  $(\mathbb{Q}(a, b) : \mathbb{Q})$  is finite by (11.1.7) and hence  $\mathbb{Q}(a, b)$  is algebraic over  $\mathbb{Q}$  by (11.1.6). The required result now follows.  $\square$

The next result shows that not every complex number is an algebraic number.

**(11.1.10)** *There are countably many algebraic numbers, but uncountably many complex numbers.*

*Proof.* Of course  $\mathbb{C}$  is uncountable by (1.4.7). To see that there are countably many algebraic numbers, observe that  $\mathbb{Q}[t]$  is countable since it is a countable union of countable sets – see Exercise (1.4.5). Also each non-zero polynomial in  $\mathbb{Q}[t]$  has finitely many roots. It follows that there are only countably many roots of non-zero polynomials in  $\mathbb{Q}[t]$ ; these are precisely the algebraic numbers.  $\square$

The existence of transcendental numbers is demonstrated by (11.1.10), but without giving a single example. Indeed it is a good deal harder to find specific examples. The best known transcendental numbers are the numbers  $\pi$  and  $e$ . The fact that  $\pi$  is transcendental underlies the impossibility of “squaring the circle” – for this see (11.2). A good reference for the transcendence of  $\pi$ ,  $e$  and many other interesting numbers is [10].

A subfield of  $\mathbb{C}$  which is a finite extension of  $\mathbb{Q}$  is called an *algebraic number field*: the elements of algebraic number fields constitute all the algebraic numbers. The theory of algebraic number fields is very well developed and is one of the most active areas of algebra.

### Exercises (11.1)

- (1) Give examples of infinite field extensions of  $\mathbb{Q}$  and of  $\mathbb{Z}_p$ .
- (2) Let  $a = 2^{\frac{1}{p}}$  where  $p$  is a prime. Prove that  $(\mathbb{Q}(a) : \mathbb{Q}) = p$  and that  $\mathbb{Q}(a)$  has only two subfields.
- (3) Let  $n$  be an arbitrary positive integer. Construct an algebraic number field of degree  $n$  over  $\mathbb{Q}$ .
- (4) Let  $a$  be a root of  $t^6 - 4t + 2 \in \mathbb{Q}[t]$ . Prove that  $(\mathbb{Q}(a) : \mathbb{Q}) = 6$ .
- (5) Let  $p$  and  $q$  be distinct primes and set  $F = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ . Prove the following statements.
  - (i)  $(F : \mathbb{Q}) = 4$ ;
  - (ii)  $F = \mathbb{Q}(\sqrt{p} + \sqrt{q})$ ;
  - (iii) the irreducible polynomial of  $\sqrt{p} + \sqrt{q}$  over  $\mathbb{Q}$  is  $t^4 - 2(p+q)t^2 + (p-q)^2$ .
- (6) Let  $K$  be a finite extension of a field  $F$  and let  $F_1$  be a subfield such that  $F \subseteq F_1 \subseteq K$ . Prove that  $F_1$  is finite over  $F$  and  $K$  is finite over  $F_1$ .
- (7) Prove that every non-constant element of  $\mathbb{Q}\{t\}$  is transcendental over  $\mathbb{Q}$ .
- (8) Let  $a = 3^{\frac{1}{2}} - 2^{\frac{1}{3}}$ . Show that  $(\mathbb{Q}(a) : \mathbb{Q}) = 6$  and find  $\text{Irr}_{\mathbb{Q}}(a)$ .
- (9) Let  $p$  be a prime and put  $a = e^{2\pi i/p}$ , a complex primitive  $p$ th root of unity. Prove that  $\text{Irr}_{\mathbb{Q}}(a) = 1 + t + t^2 + \cdots + t^{p-1}$  and  $(\mathbb{Q}(a) : \mathbb{Q}) = p - 1$ .

## 11.2 Constructions with ruler and compass

One of the most striking applications of field theory is to solve certain famous geometric problems dating back to classical Greece. Each problem asks whether it is possible to construct a geometric object *using ruler and compass only*. Here one has to keep in mind that to the ancient Greeks only mathematical objects constructed by such means had any reality, since Greek mathematics was based on geometry. We will describe four constructional problems and then translate them to field theory.

- (i) *Duplication of the cube.* A cube of side one unit is given. The problem is to construct a cube with double the volume using ruler and compass. This problem is said to have arisen when the oracle at Delphi commanded the citizens of Delos to double the size of the altar to the god Apollo, which had the shape of a cube.
- (ii) *Squaring the circle.* Here the question is whether it is possible to construct, using ruler and compass, a square whose area equals that of a circle with radius one unit? This is perhaps the most notorious of the ruler and compass problems. It is really a question about the nature of the number  $\pi$ .
- (iii) *Trisection of an angle.* Another notorious problem asks whether it is always possible to trisect a given angle using ruler and compass.
- (iv) *Construction of a regular  $n$ -gon.* Here the problem is to construct by ruler and compass a regular  $n$ -sided plane polygon with side equal to one unit where  $n \geq 3$ .

These problems defied the efforts of mathematicians for more than 2000 years despite many ingenious attempts to solve them. It was only with the rise of abstract algebra in the 18th and 19th centuries that it was realized that all four problems had negative solutions.

**Constructibility.** Our first move must be to formulate precisely what is meant by a ruler and compass construction. Let  $S$  be a set of points in the plane containing the points  $O(0, 0)$  and  $I(1, 0)$ ; note that  $O$  and  $I$  are one unit apart. A point  $P$  in the plane is said to be *constructible from  $S$  by ruler and compass* if there is a finite sequence of points  $P_0, P_1, \dots, P_n = P$  with  $P_0$  in  $S$  where  $P_{i+1}$  is obtained from  $P_0, P_1, \dots, P_i$  by a procedure of the following type:

- (i) draw a straight line joining two of  $P_0, P_1, \dots, P_i$ ;
- (ii) draw a circle with center one of  $P_0, P_1, \dots, P_i$  and radius equal to the distance between two of these points.

Then  $P_{i+1}$  is to be a point of intersection of two lines, of a line and a circle or of two circles, where the lines and circles are as described as in (i) and (ii).

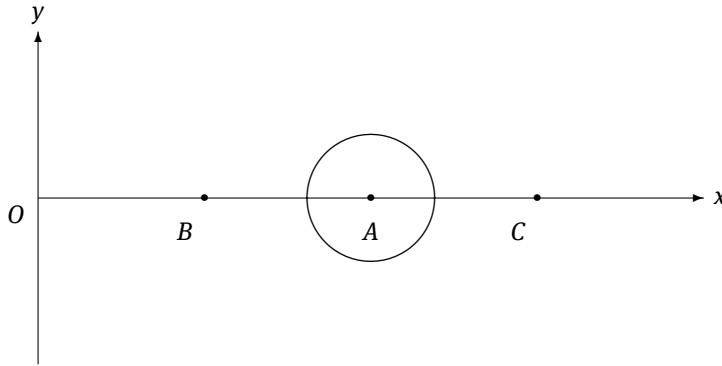
Finally, a real number  $r$  is said to be *constructible from  $S$*  if the point  $(r, 0)$  is constructible from  $S$ . The reader will realize that these definitions are designed to express precisely the intuitive idea of a construction by ruler and compass. Each of the four problems asks whether a certain real number is constructible from some given set of points. For example, in the problem of duplicating a cube of side 1, take  $S$  to be the set  $\{O, I\}$ ; the question is whether  $\sqrt[3]{2}$  is constructible from  $S$ .

We begin by showing that the real numbers which are constructible from a given set of points form a field: this explains why field theory is relevant to constructional problems.

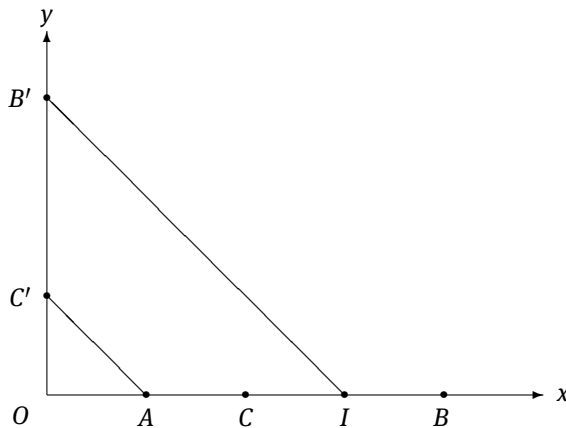
**(11.2.1)** *Let  $S$  be a set of points in the plane containing  $O(0, 0)$  and  $I(1, 0)$  and let  $S^*$  be the set of all real numbers constructible from  $S$ . Then  $S^*$  is a subfield of  $\mathbb{R}$ . Also  $\sqrt{a} \in S^*$  whenever  $a \in S^*$  and  $a > 0$ .*

*Proof.* This is entirely elementary plane geometry. Let  $a, b \in S^*$ ; we have first to prove that  $a \pm b$ ,  $ab$  and  $a^{-1}$  (if  $a \neq 0$ ) belong to  $S^*$ . Keep in mind here that by hypothesis  $a$  and  $b$  are constructible.

To construct  $a \pm b$ , where say  $a \geq b$ , draw the circle with center  $A(a, 0)$  and radius  $b$ . This intersects the  $x$ -axis at the points  $B(a - b, 0)$  and  $C(a + b, 0)$ . Hence  $a + b$  and  $a - b$  are constructible from  $S$  and belong to  $S^*$ . (If  $a < b$ , the argument is similar.)

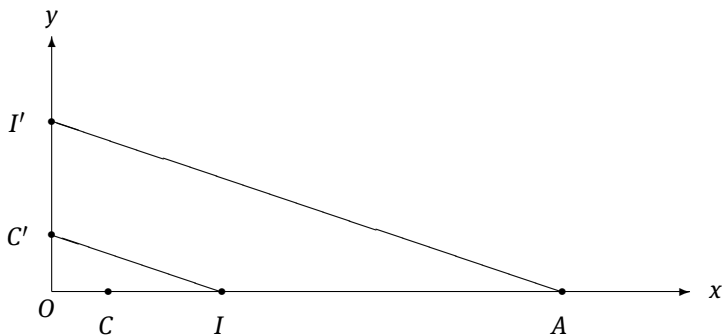


It is a little harder to construct  $ab$ . Assume that  $a \leq 1 \leq b$ : in other cases the procedure is similar. Let  $A$  and  $B$  be the points  $(a, 0)$  and  $(b, 0)$ . Mark the point  $B'(0, b)$  on the  $y$ -axis; thus  $|OB'| = |OB|$ . Draw the line  $IB'$  and then draw  $AC'$  parallel to  $IB'$  with  $C'$  on the  $y$ -axis: elementary geometry tells us how to do this. Mark  $C$  on the  $x$ -axis so that  $|OC| = |OC'|$ .

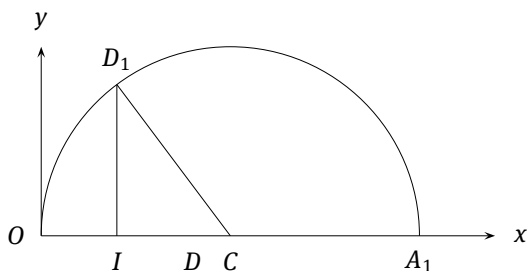


By similar triangles  $|OC'|/|OB'| = |OA|/|OI|$ ; therefore  $|OC| = |OC'| = |OA| \cdot |OB'| = ab$ . Hence  $(ab, 0)$  is constructible and  $ab \in S^*$ .

Next we show how to construct  $a^{-1}$  where, say,  $a > 1$ . Let  $A$  be the point  $(a, 0)$  and mark the point  $I'(0, 1)$  on the  $y$ -axis. Draw the line  $IC'$  parallel to  $AI'$  with  $C'$  on the  $y$ -axis. Mark  $C$  on the  $x$ -axis so that  $|OC| = |OC'|$ . Then  $|OC'|/|OI'| = |OI|/|OA|$ , so  $|OC| = |OC'| = a^{-1}$ . Thus  $(a^{-1}, 0)$  is constructible and  $a^{-1} \in S^*$ .



Finally, let  $a \in S^*$  where  $a > 0$ . We show how to construct the point  $(\sqrt{a}, 0)$ : it will then follow that  $\sqrt{a} \in S^*$ . We can assume that  $a > 1$  – otherwise replace  $a$  by  $a^{-1}$ . First mark the point  $A_1(a + 1, 0)$ . Let  $C$  be the mid-point of the line segment  $OA_1$ ; thus  $C$  is the point  $(\frac{a+1}{2}, 0)$  and it is clear how to construct this. Now draw the semicircle with center  $C$  and radius  $|OC| = \frac{a+1}{2}$ .



Then draw the perpendicular to the  $x$ -axis through the point  $I(1, 0)$  and let it meet the semicircle at  $D_1$ . Mark  $D$  on the  $x$ -axis so that  $|OD| = |ID_1|$ . Then

$$|OD|^2 = |ID_1|^2 = |D_1C|^2 - |IC|^2 = \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2 = a.$$

Hence  $|OD| = \sqrt{a}$  and  $(\sqrt{a}, 0)$  is constructible.  $\square$

It is now time to explain the field theoretic aspect of constructibility.

**(11.2.2)** Let  $S$  be a set of points in the plane containing  $O(0, 0)$  and  $I(1, 0)$ , and denote by  $F$  the subfield of  $\mathbb{R}$  generated by the coordinates of the points of  $S$ . Let  $a$  be any real number. If  $a$  is constructible from  $S$ , then  $(F(a) : F)$  is equal to a power of 2.

*Proof.* Let  $P$  be the point  $(a, 0)$ . Since  $P$  is constructible from  $S$ , there is by definition a sequence of points  $P_0, P_1, \dots, P_n = P$  with  $P_0 \in S$ , where  $P_{i+1}$  is obtained from  $P_0, P_1, \dots, P_i$  by intersecting lines and circles as explained above. Let  $P_i$  be the point  $(a_i, b_i)$  and put  $E_i = F(a_1, \dots, a_i, b_1, \dots, b_i)$  and  $E_0 = F$ . Then  $F(a) \subseteq E_n = E$ . If  $P_{i+1}$  is the point of intersection of two lines whose equations have coefficients in  $E_i$ , then  $a_{i+1}$  and  $b_{i+1}$  are in  $E_i$ , as can be seen by solving two linear equations, i.e.,  $E_i = E_{i+1}$ . If  $P_{i+1}$  is a point of intersection of a line and a circle whose equations have coefficients in  $E_i$ , then  $a_{i+1}$  is a root of a quadratic equation over  $E_i$ . Hence  $(E_i(a_{i+1}) : E_i) \leq 2$ . Clearly we can solve for  $b_{i+1}$  in terms of  $a_{i+1}$ , so  $b_{i+1} \in E_i(a_{i+1})$  and  $E_{i+1} = E_i(a_{i+1})$ . Therefore  $(E_{i+1} : E_i) \leq 2$ . If  $P_{i+1}$  is a point of intersection of two circles over  $E_i$ , subtract the equations of the circles (in standard form) to realize  $P_{i+1}$  as a point of intersection of a line and a circle. Thus  $(E_{i+1} : E_i) \leq 2$  in all cases and it follows that

$$(E : F) = \prod_{i=0}^{n-1} (E_{i+1} : E_i)$$

is a power of 2, as is  $(F(a) : F)$  since  $(E : F) = (E : F(a))(F(a) : F)$  by (11.1.7).  $\square$

The first two ruler and compass problems can now be resolved.

**(11.2.3)** *It is impossible to duplicate a cube of side 1 or to square a circle of radius 1 by ruler and compass.*

*Proof.* Let  $S$  consist of the points  $O(0, 0)$  and  $I(1, 0)$ . In the case of the cube, constructibility would imply that  $(\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q})$  is a power of 2 by (11.2.2). But  $(\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}) = 3$  since  $\text{Irr}_{\mathbb{Q}}(\sqrt[3]{2}) = t^3 - 2$ , a contradiction.

If it were possible to square the circle,  $\sqrt{\pi}$  would be constructible from  $S$ . By (11.2.2) this implies that  $(\mathbb{Q}(\sqrt{\pi}) : \mathbb{Q})$  is a power of 2, as is  $(\mathbb{Q}(\pi) : \mathbb{Q})$ , since  $(\mathbb{Q}(\pi) : \mathbb{Q}(\sqrt{\pi})) \leq 2$ . But in fact  $\pi$  is transcendental over  $\mathbb{Q}$  by a famous result of Lindemann<sup>1</sup>, so  $(\mathbb{Q}(\pi) : \mathbb{Q})$  is actually infinite. Therefore it is impossible to square the circle.  $\square$

With a little more effort we can determine which angles can be trisected.

**(11.2.4)** *An angle  $\alpha$  can be trisected by ruler and compass if and only if the polynomial  $4t^3 - 3t - \cos \alpha$  is reducible over the field  $\mathbb{Q}(\cos \alpha)$ .*

*Proof.* In this problem the angle  $\alpha$  is given, so we can construct its cosine by drawing a right angled triangle with angle  $\alpha$  and hypotenuse 1. Now let  $S$  consist of the points  $O, I$  and  $(\cos \alpha, 0)$ . Let  $F = \mathbb{Q}(\cos \alpha)$  and put  $\theta = \frac{1}{3}\alpha$ . The problem is to decide if  $\theta$ , or equivalently  $\cos \theta$ , is constructible from  $S$ . If this is the case,  $(F(\cos \theta) : F)$  must be a power of 2.

<sup>1</sup> Carl Ferdinand von Lindemann (1852–1939).

Recall the well-known trigonometric identity

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta.$$

Hence  $4 \cos^3 \theta - 3 \cos \theta - \cos \alpha = 0$ , so that  $\cos \theta$  is a root of the polynomial  $f = 4t^3 - 3t - \cos \alpha \in F[t]$ . If  $\theta$  is constructible,  $\text{Irr}_F(\cos \alpha)$  has degree a power of 2 and therefore  $f$  is reducible.

Conversely, suppose that  $f$  is reducible, so that  $\cos \theta$  is a root of a linear or quadratic polynomial over  $F$ ; thus  $\cos \theta$  has the form  $u + v\sqrt{w}$  where  $u, v, w \in F$  and  $w \geq 0$ . Since  $F \subseteq S^*$ , it follows from (11.2.1) that  $\sqrt{w} \in S^*$ . Hence  $\cos \theta \in S^*$  and  $\cos \theta$  is constructible from  $S$ , as required.  $\square$

**Example (11.2.1)** The angle  $\frac{\pi}{4}$  is trisectible by ruler and compass.

Since  $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ , the polynomial  $f$  in (11.2.4) equals  $4t^3 - 3t - \frac{1}{\sqrt{2}}$ , which has the root  $-\frac{1}{\sqrt{2}}$  in  $\mathbb{Q}(\cos(\pi/4)) = \mathbb{Q}(\sqrt{2})$ . Hence  $f$  is reducible. Now apply (11.2.4) to get the result.

**Example (11.2.2)** The angle  $\frac{\pi}{3}$  is not trisectible by ruler and compass.

In this case  $\cos \frac{\pi}{3} = \frac{1}{2}$  and  $f = 4t^3 - 3t - \frac{1}{2}$ . This polynomial is irreducible over  $\mathbb{Q}(\frac{1}{2}) = \mathbb{Q}$  since it has no rational roots. Hence  $\frac{\pi}{3}$  is not trisectible.

A complete discussion of the problem of constructing a regular  $n$ -gon calls for some Galois theory and is deferred until (12.3).

### Exercises (11.2)

- (1) Complete the proof that  $ab \in S^*$  in (11.2.1) by dealing with the cases  $1 \leq a \leq b$ , and  $a \leq b \leq 1$ .
- (2) A cube of side  $a$  can be duplicated if and only if  $2a$  is the cube of a rational number.
- (3) Consider the problem of doubling the surface area of a cube of side 1. Can a cube with double the surface area be constructed by ruler and compass?
- (4) Determine which of the following angles are trisectible: (i)  $\frac{\pi}{2}$ ; (ii)  $\frac{\pi}{6}$ ; (iii)  $\frac{\pi}{12}$ .
- (5) Let  $p$  be a prime and suppose that  $a = e^{2\pi i/p}$  is constructible from  $O(0, 0)$  and  $I(1, 0)$ . Show that  $p$  must have the form  $2^{2^c} + 1$  for some integer  $c \geq 0$ , i.e.,  $p$  is a *Fermat prime*. (The known Fermat primes occur for  $0 \leq c \leq 4$ ).

## 11.3 Finite fields

It was shown in (8.2.17) that the order of a finite field is always a power of a prime. More precisely, if  $F$  is a finite field of prime characteristic  $p$  and  $(F : \mathbb{Z}_p) = n$ , then  $|F| = p^n$ .

Our main purpose in this section is to show that there are fields with arbitrary prime power order and that fields with the same order are isomorphic.

We begin by identifying finite fields with the splitting fields of certain polynomials. Let  $F$  be a field of order  $q = p^n$  where  $p$  is a prime, namely the characteristic of  $F$ . The multiplicative group  $U(F)$  has order  $q - 1$  and Lagrange's Theorem shows that the order of every element of  $U(F)$  divides  $q - 1$ . This means that  $a^{q-1} = 1$  for every  $a \neq 0$  in  $F$ , so  $a^q - a = 0$ . Since the zero element also satisfies the last equation, every element of  $F$  is a root of the polynomial  $t^q - t \in \mathbb{Z}_p[t]$ . But  $t^q - t$  cannot have more than  $q$  roots, so we conclude that the elements of  $F$  constitute all the roots of  $t^q - t$ , so that  $F$  is a splitting field of  $t^q - t$ .

The foregoing discussion suggests that the existence of finite fields can be established by using splitting fields, a hope that is borne out by the next result.

**(11.3.1)** *Let  $q = p^n$  where  $p$  is a prime and  $n > 0$ . Then:*

- (i) *a splitting field of the polynomial  $t^q - t \in \mathbb{Z}_p[t]$  has order  $q$ ;*
- (ii) *if  $F$  is any field of order  $q$ , then  $F$  is a splitting field of  $t^q - t$  over  $\mathbb{Z}_p$ .*

*Proof.* We have already proved (ii), so let us consider the assertion (i) and write  $F$  for a splitting field of  $t^q - t$ . Define  $S = \{a \in F \mid a^q = a\}$ , i.e., the set of roots of  $t^q - t$  in  $F$ . First we show that  $S$  is a subfield of  $F$ . For this purpose let  $a, b \in S$ . Recall that  $p$  divides  $\binom{p}{i}$  if  $1 \leq i < p$  by (2.3.3); therefore the Binomial Theorem for the field  $F$  takes the form  $(a \pm b)^p = a^p \pm b^p$ , (see Exercise (6.1.6)). On taking further powers of  $p$ , we conclude that

$$(a \pm b)^q = a^q \pm b^q = a \pm b,$$

which shows that  $a \pm b \in S$ . Also  $(ab)^q = a^q b^q = ab$  and  $(a^{-1})^q = (a^q)^{-1} = a^{-1}$  if  $a \neq 0$ , so it follows that  $ab \in S$  and  $a^{-1} \in S$ . Therefore  $S$  is a subfield of  $F$ .

Next the roots of the polynomial  $t^q - t$  are all different. For  $(t^q - t)' = qt^{q-1} - 1 = -1$ , so that  $t^q - t$  and its derivative  $(t^q - t)'$  are relatively prime; therefore by (7.4.7) the polynomial  $t^q - t$  has no repeated roots and it follows that  $|S| = q$ . Finally, since  $F$  is a splitting field of  $t^q - t$ , it is generated by  $\mathbb{Z}_p$  and the roots of  $t^q - t$ . Therefore  $F = S$  and  $|F| = q$ .  $\square$

Our next objective is to show that fields with the same finite order are isomorphic. Since every finite field has been identified as a splitting field, our strategy is to prove the general result that any two splitting fields of a given polynomial are isomorphic, plainly a result of independent interest. In proving this we employ a useful lemma which shows how to extend an isomorphism between two given fields to extensions of these fields.

**(11.3.2)** *Let  $E = F(x)$  and  $E^* = F^*(x^*)$  be simple algebraic extensions of fields  $F$  and  $F^*$ . Further assume there is an isomorphism  $\alpha : F \rightarrow F^*$  such that  $\alpha(\text{Irr}_F(x)) = \text{Irr}_{F^*}(x^*)$ . Then there is an isomorphism  $\theta : E \rightarrow E^*$  such that  $\theta|_F = \alpha$  and  $\theta(x) = x^*$ .*



In the statement of this result  $\alpha$  has been extended in the obvious way to a ring isomorphism  $\alpha : F[t] \rightarrow F^*[t]$ , by the rule  $\alpha\left(\sum_{i=1}^m a_i t^i\right) = \sum_{i=1}^m \alpha(a_i) t^i$  where  $a_i \in F$ .

*Proof of (11.3.2).* Put  $f = \text{Irr}_F(x)$  and  $f^* = \text{Irr}_{F^*}(x^*)$ ; then by hypothesis  $\alpha(f) = f^*$ . This fact permits us to define a mapping

$$\theta_0 : F[t]/(f) \rightarrow F^*[t]/(f^*)$$

by the rule  $\theta_0(g + (f)) = \alpha(g) + (f^*)$ ; a simple check shows this to be a well defined isomorphism. Next by (11.1.4) we have  $F(x) \simeq F[t]/(f)$  and  $F^*(x^*) \simeq F^*[t]/(f^*)$  via the respective assignments  $g(x) \mapsto g + (f)$  and  $g^*(x^*) \mapsto g^* + (f^*)$ , ( $g \in F[t]$ ,  $g^* \in F^*[t]$ ). Composition with  $\theta_0$  yields an isomorphism  $\theta : F(x) \rightarrow F^*(x^*)$  where  $\theta(g(x)) = \alpha(g(x^*))$ , as indicated in the sequence of maps  $F(x) \rightarrow F[t]/(f) \xrightarrow{\theta_0} F^*[t]/(f^*) \rightarrow F^*(x^*)$ .  $\square$

The uniqueness of splitting fields is a special case of the next result.

**(11.3.3)** *Let  $\alpha : F \rightarrow F^*$  be an isomorphism of fields, and let  $f \in F[t]$  and  $f^* = \alpha(f) \in F^*[t]$ . If  $E$  and  $E^*$  are splitting fields of  $f$  and  $f^*$  respectively, there is an isomorphism  $\theta : E \rightarrow E^*$  such that  $\theta|_F = \alpha$ .*

*Proof.* Argue by induction on  $n = \deg(f)$ . If  $n = 1$ , then  $E = F$ ,  $E^* = F^*$  and  $\theta = \alpha$ . Assume that  $n > 1$ . Let  $a$  be a root of  $f$  in  $E$  and put  $g = \text{Irr}_F(a)$ . Choose any root  $a^*$  of  $g^* = \alpha(g) \in F^*[t]$ . Then  $g^* = \text{Irr}_{F^*}(a^*)$ . By (11.3.2) we can extend  $\alpha$  to an isomorphism  $\theta_1 : F(a) \rightarrow F^*(a^*)$  such that  $\theta_1|_F = \alpha$  and  $\theta_1(a) = a^*$ .

Now regard  $E$  and  $E^*$  as splitting fields of the polynomials  $f/(t - a)$  and  $f^*/(t - a^*)$  over  $F(a)$  and  $F^*(a^*)$  respectively. By induction on  $n$  we can extend  $\theta_1$  to an isomorphism  $\theta : E \rightarrow E^*$ ; furthermore  $\theta|_F = \theta_1|_F = \alpha$ , as required.  $\square$

**Corollary (11.3.4)** *Let  $f$  be a non-constant polynomial over a field  $F$ . Then up to isomorphism  $f$  has a unique splitting field.*

This follows from (11.3.3) by taking  $F = F^*$  and  $\alpha$  to be the identity map. Since a finite field of order  $q$  is a splitting field of  $t^q - t$ , we deduce from (11.3.4) the fundamental theorem:

**(11.3.5)** (E.H. Moore<sup>2</sup>) *Finite fields of the same order are isomorphic.*

It is customary to write

$$\text{GF}(q)$$

for the unique field of order  $q$ ; here “GF” stands for *Galois field*.

<sup>2</sup> Eliakim Hastings Moore (1862–1932)

It is a very important fact about finite fields that their multiplicative groups are cyclic. Somewhat more generally we prove:

**(11.3.6)** *If  $F$  is any field, every finite subgroup of its multiplicative group  $U(F)$  is cyclic. If  $F$  has finite order  $q$ , then  $U(F)$  is a cyclic group of order  $q - 1$ .*

*Proof.* Let  $X$  be a finite subgroup of  $U(F)$ . Then  $X$  is a finite abelian group, so by the Primary Decomposition Theorem (9.2.3),  $X = P_1 \times P_2 \times \cdots \times P_k$  where  $P_i$  is a finite  $p_i$ -group and  $p_1, p_2, \dots, p_k$  are different primes. Choose an element  $x_i$  of  $P_i$  with maximum order, say  $p_i^{\ell_i}$ , and put  $x = x_1 x_2 \cdots x_k$ . Now  $x^m = x_1^m x_2^m \cdots x_k^m$  and, since  $X$  is the direct product of the  $P_i$ , it follows that  $x^m = 1$  if and only if  $x_i^m = 1$ , i.e.,  $p_i^{\ell_i}$  divides  $m$  for all  $i$ . Consequently  $x$  has order  $d = p_1^{\ell_1} p_2^{\ell_2} \cdots p_k^{\ell_k}$  and  $|X| \geq |x| = d$ .

Next let  $y$  be any element of  $X$  and write  $y = y_1 y_2 \cdots y_k$  with  $y_i \in P_i$ . Then  $y_i^{p_i^{\ell_i}} = 1$  since  $p_i^{\ell_i}$  is the largest order of an element of  $P_i$ . Therefore  $y_i^d = 1$  for all  $i$  and  $y^d = 1$ . It follows that every element of  $X$  is a root of the polynomial  $t^d - 1$  and hence  $|X| \leq d$ . Therefore  $|X| = d = |\langle x \rangle|$  and  $X = \langle x \rangle$ .  $\square$

This result provides another way to represent the elements of a field  $F$  of order  $q$ . If  $U(F) = \langle a \rangle$ , then  $F = \{0, 1, a, a^2, \dots, a^{q-2}\}$  where  $a^{q-1} = 1$ . This representation is useful for computational purposes.

**Corollary (11.3.7)** *Every finite field  $F$  is a simple extension of its prime subfield.*

For if  $U(F) = \langle a \rangle$ , then clearly  $F = \mathbb{Z}_p(a)$  where  $p$  is the characteristic of  $F$ .

**Example (11.3.1)** Let  $F = \text{GF}(27)$  be the Galois field of order 27. Exhibit  $F$  as a simple extension of  $\text{GF}(3)$  and find a generator of  $U(F)$ .

The field  $F$  may be realized as the splitting field of the polynomial  $t^{27} - t$ , but it is simpler to choose an irreducible polynomial of degree 3 over  $\text{GF}(3)$ , for example  $f = t^3 - t + 1$ . Then  $F = (\text{GF}(3)[t])/(f)$  is a field of order  $3^3$ , which by (11.3.5) must be  $\text{GF}(27)$ . Put  $x = t + (f)$ . Then, because  $f$  has degree 3, each element  $b$  of  $F$  has the unique form  $b = a_0 + a_1 t + a_2 t^2 + (f)$ , i.e.,  $b = a_0 + a_1 x + a_2 x^2$ . Thus  $F = \text{GF}(3)(x)$  and  $\text{Irr}_{\text{GF}(3)}(x) = f = t^3 - t + 1$ .

Next we argue that  $U(F) = \langle x \rangle$ . Since  $|U(F)| = 26$ , it is enough to prove that  $|x| = 26$ . Certainly  $|x|$  divides 26, so it suffices to show that  $x^2 \neq 1$  and  $x^{13} \neq 1$ . The first statement is true because  $f \nmid t^2 - 1$ . To show that  $x^{13} \neq 1$ , use the relation  $x^3 = x - 1$  to compute  $x^{12} = (x - 1)^4 = x^2 + 2$ ; thus  $x^{13} = -1 \neq 1$ .

### Exercises (11.3)

(1) Let  $F$  be a field of order  $p^m$  where  $p$  is a prime, and let  $K$  a subfield of  $F$ . Prove that  $|K| = p^d$  where  $d$  divides  $m$ .

- (2) If  $F$  is a field of order  $p^m$  and  $d$  is a positive divisor of  $m$ , show that  $F$  has exactly one subfield of order  $p^d$ .
- (3) Find an element of order 7 in the multiplicative group of  $\mathbb{Z}_2[t]/(t^3 + t + 1) \simeq \text{GF}(8)$ .
- (4) Find elements of order 3, 5 and 15 in the multiplicative group of  $\mathbb{Z}_2[t]/(t^4 + t + 1) \simeq \text{GF}(16)$ .
- (5) Prove that  $t^{p^n} - t \in \text{GF}(p)[t]$  is the product of the distinct monic irreducible polynomials with degrees dividing  $n$ .
- (6) Let  $\psi(n)$  denote the number of monic irreducible polynomials of degree  $n$  in  $\text{GF}(p)[t]$  where  $p$  is a fixed prime.
- Prove that  $p^n = \sum_{d|n} d\psi(d)$  where the sum is over all positive divisors  $d$  of  $n$ .
  - Deduce that  $\psi(n) = \frac{1}{n} \sum_{d|n} \mu(d)p^{n/d}$  where  $\mu$  is the Möbius<sup>3</sup> function, which is defined as follows:  $\mu(1) = 1$ ,  $\mu(n)$  equals  $(-1)^r$  where  $r$  is the number of distinct prime divisors of  $n$  if  $n$  is square-free, and  $\mu(n) = 0$  otherwise. [You will need the *Möbius Inversion Formula*: if  $f(n) = \sum_{d|n} g(d)$ , then  $g(n) = \sum_{d|n} \mu(d)f(n/d)$ . For an account of the Möbius function see (12.2) below].
- (7) Find all monic irreducible polynomials over  $\text{GF}(2)$  with degrees 2, 3, 4 and 5, using Exercise (11.3.6) to check your answer.

## 11.4 Latin squares and Steiner triple systems

In this section we will describe two applications of finite fields to combinatorics, which demonstrate the efficacy of algebraic methods in solving difficult combinatorial problems.

**Latin squares.** A *latin square of order  $n$*  is an  $n \times n$  matrix with entries from a set of  $n$  symbols such that each symbol occurs exactly once in each row and exactly once in each column. Examples of latin squares are easily found.

### Example (11.4.1)

- (i) The matrices  $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$  and  $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$  are latin squares of orders 2 and 3 respectively.
- (ii) Let  $G = \{g_1, g_2, \dots, g_n\}$  be a (multiplicatively written) group of order  $n$ . Then the multiplication table of  $G$  is a latin square of order  $n$ . For, if the first row is  $g_1, g_2, \dots, g_n$ , the entries of the  $i$ th row are  $g_i g_1, g_i g_2, \dots, g_i g_n$ , which are clearly all different. A similar argument applies to the columns. On the other hand, not every latin square

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<sup>3</sup> August Ferdinand Möbius (1790–1868)

determines a group table since the associative law may not hold. In fact a latin square determines a more general algebraic structure called a *quasigroup* – for this concept see Exercises (11.4.4) and (11.4.5) below. Latin squares frequently occur in puzzles, but they also have a serious use in the design of statistical experiments. Here is an example to illustrate this use.

**Example (11.4.2)** Five types of washing powder  $P_1, P_2, P_3, P_4, P_5$  are to be tested in five machines  $A, B, C, D, E$  over five days  $D_1, D_2, D_3, D_4, D_5$ . Each washing powder is to be used once each day and tested once on each machine. How can this be done?

Here the intention is to allow for differences in the machines and in the water supply on different days, while keeping the number of tests to a minimum. A schedule of tests can be given in the form of a latin square of order 5 whose rows correspond to the washing powders and whose columns correspond to the days; the symbols are the machines. For example, we could use the latin square

$$\begin{bmatrix} A & B & C & D & E \\ B & C & D & E & A \\ C & D & E & A & B \\ D & E & A & B & C \\ E & A & B & C & D \end{bmatrix}.$$

This would mean, for example, that washing powder  $P_3$  will be used on day  $D_4$  in machine A. There are of course many other possible schedules.

**The number of latin squares.** Let  $L(n)$  denote the number of latin squares of order  $n$  which can be formed from a given set of  $n$  symbols. It is clear that  $L(n)$  must increase rapidly with  $n$ . A rough upper bound for  $L(n)$  can be found by counting derangements.

**(11.4.1)** *The number  $L(n)$  of latin squares of order  $n$  that can be formed from  $n$  given symbols satisfies the inequality*

$$L(n) \leq (n!)^n \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!} \right)^{n-1},$$

and hence  $L(n) = O((n!)^n / e^{n-1})$ .

*Proof.* Taking the symbols to be  $1, 2, \dots, n$ , we note that each row of a latin square of order  $n$  corresponds to a permutation of  $\{1, 2, \dots, n\}$ , i.e., to an element of the symmetric group  $S_n$ . Thus there are  $n!$  choices for the first row. Now rows 2 through  $n$  must be derangements of row 1 since no column can have a repeated element. Recall from (3.1.11) that the number of derangements of  $n$  symbols is

$$d_n = n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!} \right).$$

Hence rows 2 through  $n$  of the latin square can be chosen in at most  $(d_n)^{n-1}$  ways. Therefore  $L(n) \leq (n!)(d_n)^{n-1}$  and the result follows.  $\square$

It can be shown that  $L(n) \geq \frac{(n!)^{2n}}{n^{n^2}}$  – for details see [2].

**Orthogonal latin squares.** Suppose that  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are two latin squares of order  $n$ . Then  $A$  and  $B$  are called *mutually orthogonal latin squares* (or MOLS) if the  $n^2$  ordered pairs  $(a_{ij}, b_{ij})$  are all different.

**Example (11.4.3)** The latin squares

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{bmatrix}$$

are mutually orthogonal, as can be seen by listing the nine pairs of entries. On the other hand, there are no pairs of MOLS of order 2 since these would have to be of the form

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad \begin{bmatrix} a' & b' \\ b' & a' \end{bmatrix}$$

and the pair  $(a, a')$  is repeated.

One reason for the interest in mutually orthogonal latin squares is that they have statistical applications, as can be seen from an elaboration of the washing powder example.

**Example (11.4.4)** Suppose that in Example (11.4.2) there are also five washing machine operators  $\alpha, \beta, \gamma, \delta, \epsilon$ . Each operator is to test each powder once and to carry out one test per day. In addition, for reasons of economy, we do not want to repeat the same combination of machine and operator for any powder and day.

What is required here is a pair of MOLS of order 5. A latin square with the schedule of machines was given in Example (11.4.2). By a little experimentation another latin square for the machines can be found such that the pair are mutually orthogonal. The pair of MOLS is

$$\begin{bmatrix} A & B & C & D & E \\ B & C & D & E & A \\ C & D & E & A & B \\ D & E & A & B & C \\ E & A & B & C & D \end{bmatrix}, \quad \begin{bmatrix} \alpha & \beta & \gamma & \delta & \epsilon \\ \gamma & \delta & \epsilon & \alpha & \beta \\ \epsilon & \alpha & \beta & \gamma & \delta \\ \beta & \gamma & \delta & \epsilon & \alpha \\ \delta & \epsilon & \alpha & \beta & \gamma \end{bmatrix}.$$

Direct enumeration of the 25 pairs of entries from the two latin squares reveals that all are different. The two latin squares tell us the schedule of operations: thus, for example, powder  $P_3$  is to be tested on day  $D_4$  by operator  $\gamma$  in machine  $A$ .

We are interested in determining the maximum number of MOLS of order  $n$ , say

$$f(n).$$

In the first place there is an easy upper bound for  $f(n)$ .

**(11.4.2)** *If  $n \geq 1$ , then  $f(n) \leq n - 1$ .*

*Proof.* Assume that there exist  $r$  MOLS of order  $n$ , namely  $A_1, A_2, \dots, A_r$ , and let the  $(1, 1)$  entry of  $A_i$  be  $a_i$ . Consider row 2 of  $A_1$ . It has an  $a_1$  in the  $(2, i_1)$  position for some  $i_1 \neq 1$  since there is already an  $a_1$  in the first column. Hence there are  $n - 1$  possibilities for  $i_1$ . Next in  $A_2$  there is an  $a_2$  in row 2, say as the  $(2, i_2)$  entry where  $i_2 \neq 1$ ; also  $i_2 \neq i_1$  since the pair  $(a_1, a_2)$  has already occurred and cannot be repeated. Therefore there are  $n - 2$  possibilities for  $i_2$ . Continuing this line of argument until  $A_r$  is reached, we conclude that  $a_r$  is the  $(2, i_r)$  entry of  $A_r$  where there are  $n - r$  possibilities for  $i_r$ . Therefore  $n - r > 0$  and  $r \leq n - 1$ , as required.  $\square$

The question to be addressed is whether  $f(n) > 1$  for  $n > 2$ ; note that  $f(2) = 1$  since, as already observed, there cannot exist two MOLS of order 2.

**The intervention of field theory.** The mere existence of finite fields of every prime power order is enough to make a decisive advance in the construction of MOLS of prime power order.

**(11.4.3)** *Let  $p$  be a prime and  $m$  a positive integer. Then  $f(p^m) = p^m - 1$ .*

*Proof.* Let  $F$  be a field of order  $p^m$ , which exists by (11.3.1). For each  $a \neq 0$  in  $F$  define a  $p^m \times p^m$  matrix  $A(a)$  over  $F$  with rows and columns labelled by the elements of  $F$ , written in some fixed order: the  $(u, v)$  entry of  $A(a)$  is to be computed from the formula

$$[A(a)]_{u,v} = ua + v$$

where  $u, v \in F$ . In the first place  $A(a)$  is a latin square of order  $p^m$ . For  $ua + v = u'a + v$  implies that  $ua = u'a$  and  $u = u'$  since  $0 \neq a \in F$ . Also  $ua + v = ua + v'$  implies that  $v = v'$ .

Next we show that  $A(a)$ 's are mutually orthogonal. Suppose that  $A(a_1)$  and  $A(a_2)$  are not orthogonal where  $a_1 \neq a_2$ : then

$$(ua_1 + v, ua_2 + v) = (u'a_1 + v', u'a_2 + v')$$

for some  $u, v, u', v' \in F$ . Then  $ua_1 + v = u'a_1 + v'$  and  $ua_2 + v = u'a_2 + v'$ . Subtraction of the second equation from the first leads to  $u(a_1 - a_2) = u'(a_1 - a_2)$ . Since  $a_1 - a_2 \neq 0$  and  $F$  is a field, it follows that  $u = u'$  and hence  $v = v'$ . Thus we have constructed  $p^m - 1$  MOLS of order  $p^m$ , which is the maximum number permitted by (11.4.2). Therefore  $f(p^m) = p^m - 1$ .  $\square$

**Example (11.4.5)** Construct three MOLS of order 4.

In the first place  $f(4) = 3$ . To construct three MOLS, start with a field  $F$  of order 4, obtained from  $t^2 + t + 1$ , the unique irreducible polynomial of degree 2 in  $\mathbb{Z}_2[t]$ . If  $a$  is a root of this polynomial,  $F = \{0, 1, a, 1 + a\}$  where  $a^2 = a + 1$ . Now form the three MOLS  $A(1), A(a), A(1 + a)$ , using the formula indicated in the proof of (11.4.3): thus

$$A(1) = \begin{bmatrix} 0 & 1 & a & 1+a \\ 1 & 0 & 1+a & a \\ a & 1+a & 0 & 1 \\ 1+a & a & 1 & 0 \end{bmatrix},$$

$$A(a) = \begin{bmatrix} 0 & 1 & a & 1+a \\ a & 1+a & 0 & 1 \\ 1+a & a & 1 & 0 \\ 1 & 0 & 1+a & a \end{bmatrix},$$

$$A(1+a) = \begin{bmatrix} 0 & 1 & a & 1+a \\ 1+a & a & 1 & 0 \\ 1 & 0 & 1+a & a \\ a & 1+a & 0 & 1 \end{bmatrix}.$$

To construct MOLS whose order is not a prime power, a direct product construction can be used. Let  $A$  and  $B$  be latin squares of orders  $m$  and  $n$  respectively. The *direct product*  $A \times B$  is defined to be the  $mn \times mn$  matrix whose entries are pairs of elements  $(a_{ij}, b_{i'j'})$ . The matrix can be visualized in the block form

$$\begin{bmatrix} (a_{11}, B) & (a_{12}, B) & \dots & (a_{1m}, B) \\ (a_{21}, B) & (a_{22}, B) & \dots & (a_{2m}, B) \\ \vdots & \vdots & \vdots & \vdots \\ (a_{m1}, B) & (a_{m2}, B) & \dots & (a_{mm}, B) \end{bmatrix}$$

where  $(a_{ij}, B)$  means that  $a_{ij}$  is paired with each entry of  $B$  in the natural matrix order. It is easy to see that  $A \times B$  is a latin square of order  $mn$ .

**Example (11.4.6)** Given latin squares  $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$  and  $B = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{bmatrix}$ , we can form

$$A \times B = \begin{bmatrix} (a, \alpha) & (a, \beta) & (a, \gamma) & (b, \alpha) & (b, \beta) & (b, \gamma) \\ (a, \beta) & (a, \gamma) & (a, \alpha) & (b, \beta) & (b, \gamma) & (b, \alpha) \\ (a, \gamma) & (a, \alpha) & (a, \beta) & (b, \gamma) & (b, \alpha) & (b, \beta) \\ (b, \alpha) & (b, \beta) & (b, \gamma) & (a, \alpha) & (a, \beta) & (a, \gamma) \\ (b, \beta) & (b, \gamma) & (b, \alpha) & (a, \beta) & (a, \gamma) & (a, \alpha) \\ (b, \gamma) & (b, \alpha) & (b, \beta) & (a, \gamma) & (a, \alpha) & (a, \beta) \end{bmatrix},$$

which is a latin square of order 6.

Suppose that we have MOLS  $A_1, A_2, \dots, A_r$  of order  $m$  and  $B_1, B_2, \dots, B_s$  of order  $n$  where  $r \leq s$ ; then the latin squares  $A_1 \times B_1, A_2 \times B_2, \dots, A_r \times B_r$  have order  $mn$  and they are mutually orthogonal, as a check of the entry pairs shows. On the basis of this observation we can state:

**(11.4.4)** *If  $n = n_1 n_2$ , then  $f(n) \geq \min\{f(n_1), f(n_2)\}$ .*

This result can be used to give further information about the integer  $f(n)$ . Let  $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  be the primary decomposition of  $n$ . Then

$$f(n) \geq \min\{p_i^{e_i} - 1 \mid i = 1, 2, \dots, k\}$$

by (11.4.3) and (11.4.4). Therefore  $f(n) > 1$  provided that  $p_i^{e_i} \neq 2$  for all  $i$ . This will be the case if either  $n$  is odd or it is divisible by 4, i.e.,  $n \not\equiv 2 \pmod{4}$ . Hence we have:

**(11.4.5)** *If  $n \not\equiv 2 \pmod{4}$ , then  $f(n) > 1$ , so there exist at least two mutually orthogonal latin squares of order  $n$ .*

In 1782 Euler conjectured that the converse is true, i.e. if  $n \equiv 2 \pmod{4}$ , there cannot be a pair of  $n \times n$  MOLS. As evidence for this, in 1900 Tarry<sup>4</sup> was able to confirm that there does not exist a pair of  $6 \times 6$  MOLS; thus  $f(6) = 1$ . However, in the end it turned out that Euler was wrong; for in a remarkable work Bose, Shrikhande and Parker were able to prove that there is a pair of  $n \times n$  MOLS for all even integers  $n \neq 2, 6$ .

The case  $n = 6$  is Euler's celebrated *Problem of the Thirty Six Officers*. Suppose there are thirty six officers of six ranks and six regiments, with six of each regiment and six of each rank. Euler asked if it is possible for the officers to march in six rows of six, so that in each row and in each column there is exactly one officer of each rank and one of each regiment, with *no combination of rank and regiment being repeated*. Euler was really asking if there are two mutually orthogonal latin squares of order 6, the symbols of the first latin square being the ranks and those of the second the regiments of the officers. By Tarry's result the answer is negative.

**Steiner triple systems.** Another striking use of finite fields is to construct combinatorial objects known as Steiner<sup>5</sup> triple systems. We begin with a brief explanation of these. A *Steiner triple system of order  $n$*  is a pair  $(X, \mathcal{T})$  where  $X$  is a set with  $n$  elements, called *the points*, and  $\mathcal{T}$  is a set of 3-element subsets of  $X$ , called *the triples*, such that every pair of points occurs in exactly one triple. Steiner triple systems belong to a wide class of combinatorial objects called designs which are frequently used in statistics.

**Example (11.4.7)** A Steiner system of order 7.

<sup>4</sup> Gaston Tarry (1843–1913)

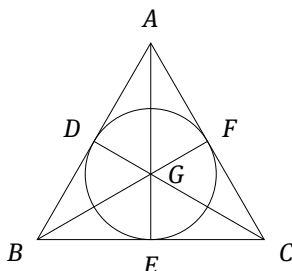
<sup>5</sup> Jakob Steiner (1796–1863).



Consider the diagram consisting of a triangle with the three medians drawn. Let  $X$  be the set of seven points consisting of the vertices, the midpoints of the sides and the centroid, labelled  $A, B, C, D, E, F, G$ . Let the triples be the sets of three points lying on each line and on the circle  $DEF$ . Thus

$$\mathcal{T} = \{ADB, BEC, CFA, AGE, BGF, CGD, DEF\},$$

where, for example, we have written  $ADF$  for the triple  $\{A, D, F\}$ . It is clear from the diagram that each pair of points belongs to a unique triple.



In fact this configuration is well known as the projective plane over  $\mathbb{Z}_2$  with seven points and seven lines.

We will consider the question: for which positive integers  $n$  do there exist Steiner triple systems of order  $n$ ? It is quite easy to derive necessary conditions on  $n$ ; these will follow from the next result.

**(11.4.6)** Suppose that  $(X, \mathcal{T})$  is a Steiner triple system of order  $n$ . Then:

- (i) each point belongs to exactly  $\frac{n-1}{2}$  triples;
- (ii) the number of triples is  $\frac{n(n-1)}{6}$ .

*Proof.* (i) Let  $x, y \in X$  with  $x$  fixed. The idea behind the proof is to count in two different ways the pairs  $(y, T)$  such that  $y \in T, y \neq x, T \in \mathcal{T}$ . There are  $n - 1$  choices for  $y$ ; then, once  $y$  has been chosen, there is a unique  $T \in \mathcal{T}$  containing  $x$  and  $y$ , so the number of such pairs is  $n - 1$ . On the other hand, let  $r$  denote the number of triples in  $\mathcal{T}$  to which  $x$  belongs. Once a  $T \in \mathcal{T}$  containing  $x$  has been chosen, there are two choices for  $y$  in  $T$ . Thus the number of pairs is  $2r$ . Therefore  $2r = n - 1$  and  $r = \frac{n-1}{2}$ .

(ii) In a similar vein we count in two different ways the pairs  $(x, T)$  such that  $x \in T$  and  $T \in \mathcal{T}$ . If  $t$  is the total number of triples, the number of pairs is  $3t$  since there are three choices for  $x$  in  $T$ . On the other hand, we may also choose  $x$  in  $n$  ways and a triple  $T$  containing  $x$  in  $\frac{n-1}{2}$  ways by (i). Therefore  $3t = \frac{n(n-1)}{2}$  and  $t = \frac{1}{6}n(n-1)$ .  $\square$

From this result we deduce a necessary condition on  $n$  for a Steiner triple system of order  $n$  to exist.

**Corollary (11.4.7)** If a Steiner triple system of order  $n$  exists, then  $n \equiv 1$  or  $3 \pmod{6}$ .

*Proof.* In the first place  $\frac{n-1}{2}$  must be an integer, so  $n$  is odd. Thus we can write  $n = 6k + \ell$  where  $\ell = 1, 3$  or  $5$ . If  $\ell = 5$ , then  $\frac{1}{6}n(n-1) = \frac{1}{3}(6k+5)(3k+2)$ , which is not an integer. Hence  $\ell = 1$  or  $3$  and  $n \equiv 1$  or  $3 \pmod{6}$ .  $\square$

The fundamental theorem on Steiner triple systems asserts that the converse of (11.4.7) is true. *If  $n \equiv 1$  or  $3 \pmod{6}$ , there is a Steiner triple system of order  $n$ .* We will prove a special case of this theorem to illustrate how field theory can be applied.

**(11.4.8)** *If  $q$  is a prime power such that  $q \equiv 1 \pmod{6}$ , there is a Steiner triple system of order  $q$ .*

*Proof.* Let  $F$  be a finite field of order  $q$ . Recall from (11.3.6) that  $U(F)$  is a cyclic group of order  $q-1$ . Since  $6 \mid q-1$  by hypothesis, it follows from (4.1.6) that  $U(F)$  contains an element  $z$  of order 6. Thus  $|U(F) : \langle z \rangle| = \frac{q-1}{6}$ . Choose a transversal to  $\langle z \rangle$  in  $U(F)$ , say  $\{t_1, t_2, \dots, t_{\frac{q-1}{6}}\}$ . Now define subsets

$$T_i = \{0, t_i, t_i z\}$$

for  $i = 1, 2, \dots, \frac{q-1}{6}$ .

The points of the Steiner triple system are to be the elements of the field  $F$ , while the set of triples is designated as

$$\mathcal{T} = \left\{ a + T_i \mid a \in F, i = 1, 2, \dots, \frac{q-1}{6} \right\}.$$

Here  $a + T_i$  denotes the set  $\{a + x \mid x \in T_i\}$ . We claim that  $(X, \mathcal{T})$  is a Steiner triple system. First we make an observation. Let  $D_i$  denote the set of differences of pairs of elements in  $T_i$ ; thus  $D_i = \{0, \pm t_i, \pm t_i z, \pm t_i(1-z)\}$ . Now  $z$  has order 6 and  $0 = z^6 - 1 = (z^3 - 1)(z + 1)(z^2 - z + 1)$ , so that  $z^2 - z + 1 = 0$  and  $z^2 = z - 1$ . Hence  $z^3 = -1$ ,  $z^4 = -z$ ,  $z^5 = 1 - z$ . From these equations it follows that  $D_i$  is simply the coset  $t_i \langle z \rangle = \{t_i z^k \mid 0 \leq k \leq 5\}$  with 0 adjoined.

To show that  $(X, \mathcal{T})$  is a Steiner triple system, we need to prove that any two distinct elements  $x$  and  $y$  of  $F$  belong to a unique triple  $a + T_i$ . Let  $f = x - y \in U(F)$ . Now  $f$  belongs to a unique coset  $t_i \langle z \rangle$ , and by the observation above  $f \in D_i$ , so that  $f$  is expressible as the difference between two elements in the set  $T_i$ , say  $f = u_i - v_i$ . Writing  $a = y - v_i$ , we have  $x = f + y = (y - v_i) + u_i \in a + T_i$  and  $y = (y - v_i) + v_i \in a + T_i$ .

Now suppose that  $x$  and  $y$  belong to another triple  $b + T_j$ , with  $x = b + d_j$  and  $y = b + e_j$  where  $d_j, e_j \in T_j$ . Then  $0 \neq f = x - y = d_j - e_j$  and hence  $f \in D_j$ . Thus  $f \in t_j \langle z \rangle$ , which means that  $j = i$ . Also there is clearly only one way to write  $f$  as the difference between two elements of  $T_i$ . Therefore  $d_i = u_i$  and  $e_i = v_i$ , from which it follows that  $a = y - v_i = y - e_i = b$ . The proof is now complete.  $\square$

The construction just described produces Steiner triple systems of order 7, 13, 19, 25. Trivially there are Steiner triple systems of orders 1 and 3. In addition there are no Steiner systems of orders 2, 4, 5, 6, 8, 10, 11, 12 by (11.4.7). In Exercise (11.4.6) below it is indicated how to construct a Steiner triple system of order 9.

**Exercises (11.4)**

- (1) Show  $L(1) = 1$ ,  $L(2) = 2$ ,  $L(3) = 12$ .
- (2) Explain how to construct the following objects: (i) four  $5 \times 5$  MOLS; (ii) eight  $9 \times 9$  MOLS.
- (3) Show that there are at least 48 MOLS of order 6125.
- (4) A *quasigroup* is a set  $Q$  together with a binary operation  $(x, y) \mapsto xy$  such that, given  $x, y \in Q$ , there is a unique  $u \in Q$  such that  $ux = y$  and a unique  $v \in Q$  such that  $xv = y$ . Prove that the multiplication table of a finite quasigroup is a latin square.
- (5) Conversely, prove that every latin square determines a finite quasigroup.
- (6) Construct a Steiner triple system of order 9 by using the following geometric procedure. Start with a  $3 \times 3$  array of 9 points. Draw all horizontals, verticals and diagonals in the figure. Then draw four curves connecting exterior points.
- (7) (*Kirkman's<sup>6</sup> schoolgirl problem*) Show that it is possible for nine schoolgirls to walk in three groups of three for four successive days in such a way that each pair of girls walks together on exactly one day.
- (8) Let  $n$  be a positive integer such that  $n \equiv 3 \pmod{6}$ . Assuming the existence of Steiner triple systems of order  $n$ , generalize the preceding problem by showing that it is possible for  $n$  schoolgirls to walk in  $\frac{n}{3}$  groups of three on  $\frac{n-1}{2}$  days without two girls walking together on more than one day.
- (9) Use the method of (11.4.8) to construct a Steiner triple system of order 13.
- (10) Construct a Steiner triple system of order 25 by starting with the field  $\mathbb{Z}_5[t]/(t^2 - t + 1)$ . [Note that a root of  $t^2 - t + 1$  has order 6].

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**6** Thomas Penyngton Kirkman (1806–1895)

## 12 Galois Theory

In this chapter the Galois group of a field extension is introduced. This establishes the critical link between field theory and group theory in which subfields correspond to subgroups of the Galois group. A major application is to the classical problem of solving polynomial equations by radicals, which is an excellent illustration of the rich rewards that can be reaped when connections are made between different mathematical theories.

### 12.1 Normal and separable extensions

We begin by introducing two special types of field extension, leading up to the concept of a Galois extension. Let  $E$  be an extension of a field  $F$  with  $F \subseteq E$ . Then  $E$  is said to be *normal* over  $F$  if it is algebraic over  $F$  and if every irreducible polynomial in  $F[t]$  having a root in  $E$  has all its roots in  $E$ ; thus the polynomial is a product of linear factors over  $E$ .

**Example (12.1.1)** Consider the field  $E = \mathbb{Q}(a)$  where  $a = 2^{1/3}$ . Then  $E$  is algebraic over  $\mathbb{Q}$  since  $(E : \mathbb{Q})$  is finite, but it is not normal over  $\mathbb{Q}$ . This is because  $t^3 - 2$  has one root  $a$  in  $E$  but not the complex roots  $a\omega, a\omega^2$  where  $\omega = e^{2\pi i/3}$ .

**Example (12.1.2)** Let  $E$  be an extension of a field  $F$  with  $(E : F) = 2$ . Then  $E$  is normal over  $F$ .

In the first place  $E$  is algebraic over  $F$ . Suppose that  $x \in E$  is a root of some monic irreducible polynomial  $f \in F[t]$ . Then  $f = \text{Irr}_F(x)$  and  $\deg(f) = (F(x) : F) \leq (E : F) = 2$ , which means that  $\deg(f) = 1$  or  $2$ . In the first case  $x$  is the only root of  $f$ . Suppose that  $\deg(f) = 2$  with say  $f = t^2 + at + b$  and  $a, b \in F$ ; if  $x'$  is another root of  $f$ , then  $xx' = b \in F$ , so that  $x' \in E$ . Therefore  $E$  is normal over  $F$ .

That there is a close connection between normal extensions and splitting fields of polynomials is demonstrated by the following fundamental result.

**(12.1.1)** *Let  $E$  be a finite extension of a field  $F$ . Then  $E$  is normal over  $F$  if and only if  $E$  is the splitting field of some polynomial in  $F[t]$ .*

*Proof.* First of all assume that  $E$  is normal over  $F$ . Since  $(E : F)$  is finite, we can write  $E = F(x_1, x_2, \dots, x_k)$ . Let  $f_i = \text{Irr}_F(x_i)$ . Now  $f_i$  has the root  $x_i$  in  $E$ , so by normality of the extension all roots of  $f_i$  are in  $E$ . Put  $f = f_1 f_2 \cdots f_k \in F[t]$ . Then  $f$  has all its roots in  $E$  and these roots with  $F$  generate the field  $E$ . Hence  $E$  is the splitting field of  $f$ .

The converse is harder to prove. Suppose that  $E$  is the splitting field of some  $f \in F[t]$ , and denote the roots of  $f$  by  $a_1, a_2, \dots, a_r$ , so that  $E = F(a_1, a_2, \dots, a_r)$ . Let  $g$  be an irreducible polynomial over  $F$  with a root  $b$  in  $E$ . Furthermore let  $K$  be the splitting field

of  $g$  over  $E$ . Then  $F \subseteq E \subseteq K$ . Let  $b^* \in K$  be another root of  $g$ . Our task is to show that  $b^* \in E$ .

Since  $g = \text{Irr}_F(b) = \text{Irr}_F(b^*)$ , there is an isomorphism  $\theta_0 : F(b) \rightarrow F(b^*)$  such that  $\theta_0(b) = b^*$  and  $\theta_0|_F$  is the identity map: here we have applied (11.3.2). Put  $g_1 = \text{Irr}_{F(b)}(a_1)$  and note that  $g_1$  divides  $f$  over  $F(b)$  since  $f(a_1) = 0$ . Now consider  $g_1^* = \theta_0(g_1) \in F(b^*)[t]$ . Then  $g_1^*$  divides  $\theta_0(f) = f$  over  $F(b^*)$ . Hence the roots of  $g_1^*$  are among  $a_1, a_2, \dots, a_r$ .

Let  $a_{i_1}$  be any root of  $g_1^*$ . By (11.3.2) once again, there is an isomorphism  $\theta_1 : F(b, a_1) \rightarrow F(b^*, a_{i_1})$  such that  $\theta_1(a_1) = a_{i_1}$  and  $(\theta_1)|_{F(b)} = \theta_0$ . Next write  $g_2 = \text{Irr}_{F(b, a_1)}(a_2)$  and  $g_2^* = \theta_1(g_2)$ . The roots of  $g_2^*$  are among  $a_1, a_2, \dots, a_r$ , by the argument used above. Let  $a_{i_2}$  be any root of  $g_2^*$ . Now extend  $\theta_1$  to an isomorphism  $\theta_2 : F(b, a_1, a_2) \rightarrow F(b^*, a_{i_1}, a_{i_2})$  such that  $\theta_2(a_2) = a_{i_2}$  and  $(\theta_2)|_{F(b, a_1)} = \theta_1$ .

After  $r$  applications of this argument we will have an isomorphism

$$\theta : F(b, a_1, a_2, \dots, a_r) \rightarrow F(b^*, a_{i_1}, a_{i_2}, \dots, a_{i_r})$$

such that  $\theta(a_j) = a_{i_j}$ ,  $\theta(b) = b^*$  and  $\theta|_F$  is the identity map. But  $b \in E = F(a_1, a_2, \dots, a_r)$  by hypothesis, so  $b^* = \theta(b) \in F(a_{i_1}, a_{i_2}, \dots, a_{i_r}) \subseteq E$ , as required.  $\square$

**Separable polynomials.** Contrary to what one might first think, it is possible for an irreducible polynomial to have repeated roots. This phenomenon is called *inseparability*.

**Example (12.1.3)** Let  $p$  be a prime and let  $f$  denote the polynomial  $t^p - x$  in  $\mathbb{Z}_p\{x\}[t]$ : here  $x$  and  $t$  are distinct indeterminates and  $\mathbb{Z}_p\{x\}$  is the field of rational functions in  $x$  over  $\mathbb{Z}_p$ . Then  $f$  is irreducible over  $\mathbb{Z}_p[x]$  by (7.4.9) since  $x$  is clearly an irreducible element of  $\mathbb{Z}_p[x]$ . Gauss's Lemma (7.3.7) shows that  $f$  is irreducible over  $\mathbb{Z}_p\{x\}$ . Let  $a$  be a root of  $f$  in its splitting field. Then  $f = t^p - a^p = (t - a)^p$  since  $\binom{p}{i} \equiv 0 \pmod{p}$  if  $0 < i < p$ . It follows that  $f$  has all its roots equal to  $a$ .

An irreducible polynomial  $f$  over a field  $F$  is said to be *separable* if all its roots are different, i.e.,  $f$  is a product of distinct linear factors over its splitting field. The example above shows that  $t^p - x$  is inseparable over  $\mathbb{Z}_p\{x\}$ , a field with prime characteristic. The criterion which follows shows that the phenomenon of inseparability can only occur for fields of prime characteristic.

**(12.1.2)** Let  $f$  be an irreducible polynomial over a field  $F$ .

- (i) If  $\text{char}(F) = 0$ , then  $f$  is separable.
- (ii) If  $\text{char}(F) = p > 0$ , then  $f$  is inseparable if and only if  $f = g(t^p)$  for some irreducible polynomial  $g$  over  $F$ .

*Proof.* There is no loss in supposing  $f$  to be monic. Assume first that  $\text{char}(F) = 0$  and let  $a$  be a root of  $f$  in its splitting field. If  $a$  has multiplicity greater than 1, then (7.4.7) shows that  $t - a \mid f'$  where  $f'$  is the derivative of  $f$ . Thus  $f'(a) = 0$ . Writing

$f = a_0 + a_1t + \cdots + a_nt^n$ , we have  $f' = a_1 + 2a_2t + \cdots + na_nt^{n-1}$ . But  $f = \text{Irr}_F(a)$ , so  $f$  divides  $f'$ . Since  $\deg(f') < \deg(f)$ , this can only mean that  $f' = 0$ , i.e.,  $ia_i = 0$  for all  $i > 0$  and so  $a_i = 0$ . Thus  $f$  is constant, which is impossible. Therefore  $a$  is not a repeated root and  $f$  is separable.

Now assume that  $\text{char}(F) = p > 0$  and again let  $a$  be a multiple root of  $f$ . Arguing as before, we conclude that  $ia_i = 0$  for  $i > 0$ . In this case all we can deduce is that  $a_i = 0$  if  $p$  does not divide  $i$ . Hence

$$f = a_0 + a_pt^p + a_{2p}t^{2p} + \cdots + a_{rp}t^{rp}$$

where  $rp$  is the largest positive multiple of  $p$  not exceeding  $n$ . It follows that  $f = g(t^p)$  where  $g = a_0 + a_pt + \cdots + a_{rp}t^r$ . Notice that  $g$  is irreducible since if it were reducible, so would  $f$  be.

Conversely, assume that  $f = g(t^p)$  where  $g = \sum_{i=0}^r a_it^i \in F[t]$ . We claim that  $f$  is inseparable. Let  $b_i$  be a root of  $t^p - a_i$  in the splitting field  $E$  of the polynomial  $(t^p - a_1)(t^p - a_2)\cdots(t^p - a_r)$ . Then  $a_i = b_i^p$  and hence

$$f = \sum_{i=0}^r a_it^{ip} = \sum_{i=0}^r b_i^p t^{ip} = \left( \sum_{i=0}^r b_it^i \right)^p,$$

from which it follows that every root of  $f$  has multiplicity at least  $p$ . Hence  $f$  is inseparable.  $\square$

**Separable extensions.** Let  $E$  be an extension of a field  $F$ . An element  $x$  of  $E$  is said to be *separable over  $F$*  if  $x$  is algebraic and its multiplicity as a root of  $\text{Irr}_F(x)$  is 1. If  $x$  is algebraic but inseparable, the final argument of the proof of (12.1.2) shows that its irreducible polynomial is a prime power of a polynomial, so that *all* its roots have multiplicity greater than 1. Therefore  $x \in E$  is *separable over  $F$*  if and only if  $\text{Irr}_F(x)$  is a *separable polynomial*.

If every element of  $E$  is separable over  $F$ , then  $E$  is called a *separable extension* of  $F$ . Finally, a field  $F$  is said to be *perfect* if every algebraic extension of  $F$  is separable. Since any irreducible polynomial over a field of characteristic 0 is separable, *all fields of characteristic 0 are perfect*. There is a simple criterion for a field of prime characteristic to be perfect.

**(12.1.3)** *Let  $F$  be a field of prime characteristic  $p$ . Then  $F$  is perfect if and only if  $F = F^p$  where  $F^p$  is the subfield  $\{a^p \mid a \in F\}$ .*

*Proof.* In the first place  $F^p$  is a subfield of  $F$  since  $(a \pm b)^p = a^p \pm b^p$ ,  $(a^{-1})^p = (a^p)^{-1}$  and  $(ab)^p = a^p b^p$  for  $a, b \in F$ . Now assume that  $F = F^p$ . If  $f \in F[t]$  is irreducible but inseparable, then  $f = g(t^p)$  for some  $g \in F[t]$  by (12.1.2). Let  $g = \sum_{i=0}^r a_it^i$ ; then  $a_i = b_i^p$  for some  $b_i \in F$  since  $F = F^p$ . Therefore  $f = \sum_{i=0}^r a_it^{pi} = \sum_{i=0}^r b_i^p t^{pi} = \left( \sum_{i=0}^r b_it^i \right)^p$ , which is impossible since  $f$  is irreducible. Thus  $f$  is separable. This shows that if  $E$  is an algebraic extension of  $F$ , then it is separable. Hence  $F$  is a perfect field.

Conversely, assume that  $F \neq F^p$  and choose  $a \in F - F^p$ . Consider the polynomial  $f = t^p - a$ . First we claim that  $f$  is irreducible over  $F$ . Suppose this is false, so that  $f = gh$  where  $g$  and  $h$  in  $F[t]$  are monic with smaller degrees than  $f$ . Now  $f = t^p - a = (t - b)^p$  where  $b$  is a root of  $f$  in its splitting field, so it follows that  $g = (t - b)^i$  and  $h = (t - b)^j$  where  $i + j = p$  and  $0 < i, j < p$ . Since  $\gcd\{i, p\} = 1$ , we can write  $1 = iu + pv$  for suitable integers  $u, v$ . Therefore  $b = (b^i)^u (b^p)^v = (b^i)^u a^v \in F$  since  $b^i \in F$ , and hence  $a = b^p \in F^p$ , a contradiction. Thus  $f$  is irreducible and by (12.1.2) it is inseparable. It follows that  $F$  cannot be a perfect field.  $\square$

**(12.1.4)** Every finite field is perfect.

*Proof.* Let  $F$  be a field of order  $p^m$  with  $p$  a prime. Every element  $f$  of  $F$  satisfies the equation  $t^{p^m} - t = 0$  by (11.3.1). Hence  $F = F^p$  and  $F$  is perfect.  $\square$

On the other hand, the field  $F = \mathbb{Z}_p\{t\}$  is not perfect because  $F^p = \mathbb{Z}_p\{t^p\}$  is a proper subfield of  $F$ .

It is desirable to have a practical criterion for a finite extension of prime characteristic to be separable.

**(12.1.5)** Let  $E$  be a finite extension of a field  $F$  with prime characteristic  $p$ . Then  $E$  is separable over  $F$  if and only if  $E = F(E^p)$ .

*Proof.* Assume that  $E$  is separable over  $F$  and let  $a \in E$ . Writing  $f = \text{Irr}_{F(a^p)}(a)$ , we observe that  $f$  divides  $t^p - a^p = (t - a)^p$ . Since  $f$  is a separable polynomial, it follows that  $f = t - a$  and thus  $a \in F(a^p) \subseteq F(E^p)$ .

Conversely, assume that  $E = F(E^p)$  and let  $x \in E$ ; we need to prove that  $f = \text{Irr}_F(x)$  is separable over  $F$ . If this is false, then  $f = g(t^p)$  for some  $g = \sum_{i=0}^k a_i t^i \in F[t]$ . Since  $0 = g(x^p) = a_0 + a_1 x^p + \cdots + a_k x^{kp}$ , the field elements  $1, x^p, \dots, x^{kp}$  are linearly dependent over  $F$ . On the other hand,  $k < kp = \deg(f) = (F(x) : F)$ , so that  $1, x, \dots, x^k$  must be linearly independent over  $F$ . Extend  $\{1, x, \dots, x^k\}$  to an  $F$ -basis of  $E$ , say  $\{y_1, y_2, \dots, y_n\}$ , using (8.2.6).

We have  $E = Fy_1 + Fy_2 + \cdots + Fy_n$  and thus  $E^p \subseteq Fy_1^p + Fy_2^p + \cdots + Fy_n^p$ . Therefore  $E = F(E^p) = Fy_1^p + Fy_2^p + \cdots + Fy_n^p$ . It follows that  $y_1^p, y_2^p, \dots, y_n^p$  are  $F$ -linearly independent since  $n = (E : F)$ . This shows that  $1, x^p, \dots, x^{kp}$  are  $F$ -linearly independent, a contradiction.  $\square$

**Corollary (12.1.6)** Let  $E = F(a_1, a_2, \dots, a_k)$  be an extension of a field  $F$  such that each  $a_i$  is separable over  $F$ . Then  $E$  is separable over  $F$ .

*Proof.* We may assume that  $\text{char}(F) = p > 0$ . Since  $a_i$  is separable over  $F$ , we have  $a_i \in F(a_i^p)$ , as in the first paragraph of the preceding proof. Hence  $a_i \in F(E^p)$  and  $E = F(E^p)$ . Therefore  $E$  is separable over  $F$  by (12.1.5).  $\square$

Notice the consequence of the last result: *the splitting field of a separable polynomial is a separable extension.*

We conclude this section by addressing a question which may already have occurred to the reader: when is a finite extension  $E$  of  $F$  a simple extension, i.e., when is  $E = F(x)$  for some  $x$ ? An important result on this problem is:

**(12.1.7)** (The Theorem of the Primitive Element) *Let  $E$  be a finite separable extension of a field  $F$ . Then there is an element  $a$  such that  $E = F(a)$ .*

*Proof.* The proof is easy when  $E$  is finite. For then  $E - \{0\}$  is a cyclic group by (11.3.6), generated by  $a$ , say. Then  $E = \{0, 1, a, \dots, a^{q-1}\}$  where  $q = |E|$ , and hence  $E = F(a)$ .

From now on assume  $E$  is infinite. Since  $(E : F)$  is finite,  $E = F(u_1, u_2, \dots, u_n)$  for some  $u_i$  in  $E$ . The proof proceeds by induction on  $n$ . If  $n > 2$ , then  $F(u_1, u_2, \dots, u_{n-1}) = F(v)$  for some  $v$ , by induction hypothesis, and hence  $E = F(v, u_n) = F(a)$  for some  $a$  by the case  $n = 2$ . Therefore it is enough to deal with the case  $n = 2$ . From now on write

$$E = F(u, v).$$

We introduce the polynomials  $f = \text{Irr}_F(u)$  and  $g = \text{Irr}_F(v)$ ; these are separable polynomials since  $E$  is separable over  $F$ . Let the roots of  $f$  and  $g$  be  $u = x_1, x_2, \dots, x_m$  and  $v = y_1, y_2, \dots, y_n$  respectively, in the splitting field of  $fg$  over  $F$ . Here all the  $x_i$  are different, as are all the  $y_j$ . From this we conclude that for  $j \neq 1$  there is at most one element  $z_{ij}$  in  $F$  such that

$$u + z_{ij}v = x_i + z_{ij}y_j,$$

namely  $z_{ij} = (x_i - u)(v - y_j)^{-1}$ . Since  $F$  is infinite, it is possible to choose an element  $z$  in  $F$  which is different from each of the finitely many  $z_{ij}$ . Then  $u + zv \neq x_i + zy_j$  if  $(i, j) \neq (1, 1)$ .

With this choice of  $z$ , put  $a = u + zv \in E$ . We will show that  $E = F(a)$ . Since  $g(v) = 0 = f(u) = f(a - zv)$ , the element  $v$  is a common root of the polynomials  $g$  and  $f(a - zv) \in F(a)[t]$ . Now these polynomials have no other common roots. For if  $y_j$  were one, then  $a - zy_j = x_i$  for some  $i$ , which implies that  $u + zv = a = x_i + zy_j$ ; this is contrary to the choice of  $z$ . It follows that  $t - v$  is the unique (monic) gcd of  $g$  and  $f(a - zv)$  in  $E[t]$ . Now the gcd of these polynomials actually lies in the subring  $F(a)[t]$ : for the gcd can be computed by using the Euclidean Algorithm, which is valid for  $F(a)[t]$ . Therefore  $v \in F(a)$  and  $u = a - zv \in F(a)$ . Finally  $E = F(u, v) = F(a)$ .  $\square$

Since an algebraic number field is by definition a finite extension of  $\mathbb{Q}$ , we deduce:

**Corollary (12.1.8)** *If  $E$  is an algebraic number field, then  $E = \mathbb{Q}(a)$  for some  $a$  in  $E$ .*

### Exercises (12.1)

(1) Which of the following field extensions are normal?

- (i)  $\mathbb{Q}(3^{1/3})$  of  $\mathbb{Q}$ ; (ii)  $\mathbb{Q}(3^{1/3}, e^{2\pi i/3})$  of  $\mathbb{Q}$ ; (iii)  $\mathbb{R}$  of  $\mathbb{Q}$ ; (v)  $\mathbb{C}$  of  $\mathbb{R}$ .



- (2) Let  $F \subseteq K \subseteq E$  be field extensions with all degrees finite. If  $E$  is normal over  $F$ , show that it is normal over  $K$ , but  $K$  need not be normal over  $F$ .
- (3) Let  $f \in F[t]$  where  $\text{char}(F) = p > 0$ , and assume that  $f$  is monic with degree  $p^n$ . If all roots of  $f$  are equal in its splitting field, prove that  $f = t^{p^n} - a$  for some  $a \in F$ .
- (4) Let  $E$  be a finite extension of a field  $F$  of characteristic  $p > 0$  and assume that  $(E : F)$  is not divisible by  $p$ . Prove that  $E$  is separable over  $F$ .
- (5) Let  $F \subseteq K \subseteq E$  be field extensions with all degrees finite and  $E$  separable over  $F$ . Prove that  $E$  is separable over  $K$ .
- (6) Let  $F \subseteq K \subseteq E$  be field extensions with all degrees finite. If  $E$  is separable over  $K$  and  $K$  is separable over  $F$ , show that  $E$  is separable over  $F$ .
- (7) Let  $E$  be a finite separable extension of a field  $F$ . Prove that there is a finite extension  $K$  of  $E$  such that  $K$  is separable and normal over  $F$ .

## 12.2 Automorphisms of field extensions

Fields, like groups, possess automorphisms and these play a crucial role in field theory. An *automorphism of a field*  $F$  is defined to be a bijective ring homomorphism  $\alpha : F \rightarrow F$ ; thus  $\alpha(x+y) = \alpha(x) + \alpha(y)$  and  $\alpha(xy) = \alpha(x)\alpha(y)$ . The automorphisms of a field are easily seen to form a group with respect to functional composition. If  $E$  is a field extension of  $F$ , we are interested in *automorphisms of  $E$  over  $F$* , i.e., automorphisms of  $E$  whose restriction to  $F$  is the identity function. For example, complex conjugation is an automorphism of  $\mathbb{C}$  over  $\mathbb{R}$ . The set of automorphisms of  $E$  over  $F$  is a subgroup of the group of all automorphisms of  $F$  and is denoted by

$$\text{Gal}(E/F) :$$

this is the *Galois<sup>1</sup> group* of  $E$  over  $F$ .

Suppose that  $E = F(a)$  is a simple algebraic extension of  $F$  with degree  $n$ . Then every element of  $E$  has the form  $x = \sum_{i=0}^{n-1} c_i a^i$  with  $c_i \in F$  and thus  $\alpha(x) = \sum_{i=0}^{n-1} c_i \alpha(a)^i$  where  $\alpha \in \text{Gal}(E/F)$ . If  $b$  is any root of the polynomial  $f = \text{Irr}_F(a)$ , then  $0 = \alpha(f(b)) = f(\alpha(b))$ , so that  $\alpha(b)$  is also a root of  $f$  in  $E$ . Thus each  $\alpha$  in  $\text{Gal}(E/F)$  gives rise to a permutation  $\pi(\alpha)$  of  $X$ , the set of distinct roots of  $f$  in  $E$ . What is more, the mapping

$$\pi : \text{Gal}(E/F) \rightarrow \text{Sym}(X)$$

is evidently a group homomorphism, i.e.,  $\alpha$  is a permutation representation of the Galois group on  $X$ .

In fact  $\pi$  a faithful permutation representation of  $\text{Gal}(E/F)$  on  $X$ . For, if  $\pi(\alpha)$  is the identity permutation,  $\alpha(a) = a$  and hence  $\alpha$  is the identity automorphism of  $E$ . For this

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<sup>1</sup> Évariste Galois (1811–1831)

reason it is often useful to think of the elements of  $\text{Gal}(E/F)$  as permutations of the set of distinct roots  $X$ .

Next let  $b$  be any element of  $X$ . Then  $F \subseteq F(b) \subseteq E = F(a)$ , and also  $(F(b) : F) = \deg(f) = (F(a) : F)$  by (11.1.4) since  $f = \text{Irr}_F(b)$ . It follows that  $F(b) = F(a) = E$  by (11.1.7). Since  $\text{Irr}_F(a) = f = \text{Irr}_F(b)$ , we may apply (11.3.2) to produce an automorphism  $\alpha$  of  $E$  over  $F$  such that  $\alpha(a) = b$ . Therefore the group  $\text{Gal}(E/F)$  acts *transitively* on the set  $X$ . Finally, if  $\alpha$  in  $\text{Gal}(E/F)$  fixes some  $b$  in  $X$ , then  $\alpha$  must equal the identity since  $E = F(b)$ . This shows that  $\text{Gal}(E/F)$  acts *regularly* on  $X$  and it follows from (5.2.2) that  $|X| = |\text{Gal}(E/F)|$ .

These conclusions are summed up in the following fundamental result.

**(12.2.1)** *Let  $E = F(a)$  be a simple algebraic extension of a field  $F$ . Then  $\text{Gal}(E/F)$  acts regularly on the set  $X$  of distinct roots of  $\text{Irr}_F(a)$  in  $E$ . Therefore*

$$|\text{Gal}(E/F)| = |X| \leq (E : F).$$

An extension of a field  $F$  which is finite, separable and normal is said to be *Galois* over  $F$ . For such extensions we have:

**Corollary (12.2.2)** *If  $E$  is a Galois extension of a field  $F$  with degree  $n$ , then  $\text{Gal}(E/F)$  is isomorphic with a regular subgroup of  $S_n$  and*

$$|\text{Gal}(E/F)| = n = (E : F).$$

For (12.1.7) shows that  $E = F(a)$  for some  $a \in E$ . Also  $\text{Irr}_F(a)$  has  $n$  distinct roots in  $E$  by normality and separability.

**The Galois group of a polynomial.** Suppose that  $f$  is a non-constant polynomial over a field  $F$  and let  $E$  be the splitting field of  $f$ : recall from (11.3.4) that this field is unique up to isomorphism. Then the *Galois group of the polynomial  $f$*  is

$$\text{Gal}(f) = \text{Gal}(E/F).$$

This is always a finite group by (12.2.1). The basic properties of the Galois group are given in the next result.

**(12.2.3)** *Let  $f$  be a non-constant polynomial of degree  $n$  over a field  $F$ . Then:*

- (i)  $\text{Gal}(f)$  is isomorphic with a permutation group on the set of distinct roots of  $f$ ; thus  $|\text{Gal}(f)|$  divides  $n!$ ;
- (ii) if all the roots of  $f$  are distinct, then  $f$  is irreducible if and only if  $\text{Gal}(f)$  acts transitively on the set of roots of  $f$ .

*Proof.* Let  $E$  denote the splitting field of  $f$ , so that  $\text{Gal}(f) = \text{Gal}(E/F)$ . Let  $\alpha \in \text{Gal}(f)$ . If  $a$  is a root of  $f$  in  $E$ , then  $f(\alpha(a)) = \alpha(f(a)) = 0$ , so that  $\alpha(a)$  is also a root of  $f$ . If  $\alpha$  fixes every root of  $f$ , then  $\alpha$  is the identity automorphism since  $E$  is generated by  $F$  and the roots of  $f$ . Hence  $\text{Gal}(f)$  is isomorphic with a permutation group on the set of distinct roots of  $f$ . If there are  $r$  such roots, then  $r \leq n$  and  $|\text{Gal}(f)| \mid r! \mid n!$ , so that  $|\text{Gal}(f)| \mid n!$ .

Next assume that all the roots of  $f$  are different. Let  $f$  be irreducible. If  $a$  and  $b$  are roots of  $f$ , then  $\text{Irr}_F(a) = f = \text{Irr}_F(b)$ , and by (11.3.2) there exists  $\alpha \in \text{Gal}(f)$  such that  $\alpha(a) = b$ . It follows that  $\text{Gal}(f)$  acts transitively on the roots of  $f$ .

Conversely, suppose that  $\text{Gal}(f)$  acts transitively on the roots of  $f$ , but  $f$  is reducible; write  $f = g_1 g_2 \cdots g_k$  where  $g_i \in F[t]$  is irreducible and  $k \geq 2$ . Let  $a_1$  and  $a_2$  be roots of  $g_1$  and  $g_2$  respectively. By transitivity there exists  $\alpha \in \text{Gal}(f)$  such that  $\alpha(a_1) = a_2$ . But  $0 = \alpha(g_1(a_1)) = g_1(\alpha(a_1)) = g_1(a_2)$ . Hence  $g_2 = \text{Irr}_F(a_2)$  divides  $g_1$ . Therefore  $g_2^2$  divides  $f$  and the roots of  $f$  cannot all be different, a contradiction which shows that  $f$  is irreducible.  $\square$

**Corollary (12.2.4)** *Let  $f$  be a separable polynomial of degree  $n$  over a field  $F$  and let  $E$  be its splitting field. Then  $|\text{Gal}(f)| = (E : F)$  and  $|\text{Gal}(f)|$  is divisible by  $n$ .*

*Proof.* Note that  $E$  is separable and hence Galois over  $F$  by (12.1.6). Hence  $|\text{Gal}(f)| = |\text{Gal}(E/F)| = (E : F)$  by (12.2.2). Further  $f$  is irreducible by definition, so  $\text{Gal}(f)$  acts transitively on the  $n$  roots of  $f$ ; therefore  $n$  divides  $|\text{Gal}(f)|$  by (5.2.2).  $\square$

Let us consider some polynomials whose Galois groups can be readily computed.

**Example (12.2.1)** Let  $f = t^3 - 2 \in \mathbb{Q}[t]$ . Then  $\text{Gal}(f) \simeq S_3$ .

To see this let  $E$  denote the splitting field of  $f$ ; thus  $E$  is Galois over  $\mathbb{Q}$ . Then  $E = \mathbb{Q}(2^{1/3}, e^{2\pi i/3})$  and one can easily check that  $(E : \mathbb{Q}) = 6$ , so that  $|\text{Gal}(f)| = 6$ . Since  $\text{Gal}(f)$  is isomorphic with a subgroup of  $S_3$ , it follows that  $\text{Gal}(f) \simeq S_3$ .

In fact it is not difficult to write down the six elements of the group  $\text{Gal}(f)$ . Put  $a = 2^{1/3}$  and  $\omega = e^{2\pi i/3}$ ; then  $E = \mathbb{Q}(a, \omega)$ . Since  $E = \mathbb{Q}(a)(\omega)$  and  $t^3 - 2$  is the irreducible polynomial of both  $a$  and  $a\omega$  over  $\mathbb{Q}(\omega)$ , there is an automorphism  $\alpha$  of  $E$  over  $\mathbb{Q}$  such that  $\alpha(a) = a\omega$ ,  $\alpha(\omega) = \omega$ . Clearly  $\alpha$  has order 3. Also  $\alpha^2(a) = a\omega^2$  and  $\alpha^2(\omega) = \omega$ . It is easy to identify an automorphism  $\beta$  such that  $\beta(a) = a$  and  $\beta(\omega) = \omega^2$ ; indeed  $\beta$  is just complex conjugation. Two more automorphisms of order 2 are formed by composition:  $\gamma = \alpha\beta$  and  $\delta = \alpha^2\beta$ . It is quickly seen that  $\gamma$  maps  $\omega$  to  $\omega^2$  and  $a$  to  $a\omega$ , while  $\delta$  maps  $\omega$  to  $\omega^2$  and  $a$  to  $a\omega^2$ . Thus the elements of the Galois group  $\text{Gal}(f)$  are 1,  $\alpha$ ,  $\alpha^2$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ .

**Example (12.2.2).** Let  $p$  be a prime and put  $f = t^p - 1 \in \mathbb{Q}[t]$ . Then  $\text{Gal}(f) \simeq U(\mathbb{Z}_p)$ , a cyclic group of order  $p - 1$ .

To see this put  $a = e^{2\pi i/p}$ , a primitive  $p$ th root of unity; the roots of  $f$  are 1,  $a$ ,  $a^2, \dots, a^{p-1}$  and its splitting field is  $E = \mathbb{Q}(a)$ . Now  $f = (t - 1)(1 + t + t^2 + \cdots + t^{p-1})$  and

the second factor is  $\mathbb{Q}$ -irreducible by Example (7.4.6). Hence the irreducible polynomial of  $a$  is  $1 + t + t^2 + \cdots + t^{p-1}$  and  $|\text{Gal}(f)| = (E : \mathbb{Q}) = p - 1$ .

To show that  $\text{Gal}(f)$  is cyclic, we construct a group isomorphism

$$\theta : U(\mathbb{Z}_p) \rightarrow \text{Gal}(f).$$

If  $1 \leq j < p$ , define  $\theta(j + p\mathbb{Z})$  to be  $\theta_j$  where  $\theta_j(a) = a^j$  and  $\theta_j$  is trivial on  $\mathbb{Q}$ ; this is an automorphism by (11.3.2). Obviously  $\theta_j$  is the identity only if  $j = 1$ , so  $\theta$  is injective. Since  $U(\mathbb{Z}_p)$  and  $\text{Gal}(f)$  both have order  $p - 1$ , they are isomorphic.

**Conjugacy in field extensions.** Let  $E$  be an extension of a field  $F$ . Two elements  $a$  and  $b$  of  $E$  are said to be *conjugate over  $F$*  if  $\alpha(a) = b$  for some  $\alpha \in \text{Gal}(E/F)$ . In normal extensions conjugacy amounts to the elements having the same irreducible polynomial, as the next result shows.

**(12.2.5)** *Let  $E$  be a finite normal extension of a field  $F$ . Then two elements  $a$  and  $b$  of  $E$  are conjugate over  $F$  if and only if they have the same irreducible polynomial.*

*Proof.* If  $a$  and  $b$  have the same irreducible polynomial, (11.3.2) shows that there is a field isomorphism  $\theta : F(a) \rightarrow F(b)$  such that  $\theta(a) = b$  and  $\theta$  is the identity map on  $F$ . By (12.1.1)  $E$  is the splitting field of some polynomial over  $F$  and hence over  $F(a)$ . Consequently, (11.3.3) can be applied to extend  $\theta$  to an isomorphism  $\alpha : E \rightarrow E$  such that  $\theta$  is the restriction of  $\alpha$  to  $F(a)$ . Hence  $\alpha \in \text{Gal}(E/F)$  and  $\alpha(a) = b$ , which shows that  $a$  and  $b$  are conjugate over  $F$ .

To prove the converse, suppose that  $b = \alpha(a)$  where  $a, b \in E$  and  $\alpha \in \text{Gal}(E/F)$ . Put  $f = \text{Irr}_F(a)$  and  $g = \text{Irr}_F(b)$ . Then  $0 = \alpha(f(a)) = f(\alpha(a)) = f(b)$ . Therefore  $g$  divides  $f$  and it follows that  $f = g$  since  $f$  and  $g$  are monic and irreducible.  $\square$

The next result is of critical importance in Galois theory: it asserts that the only elements of an extension that are fixed by every automorphism are the elements of the base field.

**(12.2.6)** *Let  $E$  be a Galois extension of a field  $F$  and let  $a \in E$ . Then  $\alpha(a) = a$  for all automorphisms  $\alpha$  of  $E$  over  $F$  if and only if  $a \in F$ .*

*Proof.* Assume that  $\alpha(a) = a$  for all  $\alpha \in \text{Gal}(E/F)$  and put  $f = \text{Irr}_F(a)$ . Since  $E$  is normal over  $F$ , all the roots of  $f$  are in  $E$ . If  $b$  is any such root, it is conjugate to  $a$  by (12.2.5), so there exists  $\alpha$  in  $\text{Gal}(E/F)$  such that  $\alpha(a) = b$ . Hence  $b = a$  and the roots of  $f$  are all equal. But  $f$  is separable since  $E$  is separable over  $F$ . Therefore  $f = t - a$  and  $a$  belongs to  $F$ . The converse is obvious.  $\square$

**Roots of unity.** We will postpone further development of the theory of Galois extensions until the next section and concentrate on roots of unity. Let  $F$  be a field and  $n$  a positive integer. A root  $a$  of the polynomial  $t^n - 1 \in F[t]$  is called an  *$n$ th root of unity over  $F$* ;

thus  $a^n = 1$ . If  $a^m \neq 1$  for all proper divisors  $m$  of  $n$ , then  $|a| = n$  and  $a$  is said to be a *primitive*  $n$ th root of unity. If  $\text{char}(F) = p$  divides  $n$ , there are no primitive  $n$ th roots of unity over  $F$ : for then  $t^n - 1 = (t^{n/p} - 1)^p$  and every  $n$ th root of unity has order at most  $n/p$ . However, if  $\text{char}(F)$  does not divide  $n$ , primitive  $n$ th roots of unity over  $F$  always exist, as will now be shown.

**(12.2.7)** *Let  $F$  be a field whose characteristic does not divide the positive integer  $n$  and let  $E$  be the splitting field of  $t^n - 1$  over  $F$ . Then:*

- (i) *primitive  $n$ th roots of unity exist in  $E$ ; furthermore these generate a cyclic subgroup of order  $n$ .*
- (ii)  *$\text{Gal}(E/F)$  is isomorphic with a subgroup of  $U(\mathbb{Z}_n)$  and is therefore abelian with order dividing  $\phi(n)$ .*

*Proof.* (i) Set  $f = t^n - 1$ , so that  $f' = nt^{n-1}$ . Since  $\text{char}(F)$  does not divide  $n$ , the polynomials  $f$  and  $f'$  are relatively prime. It follows via (7.4.7) that  $f$  has  $n$  distinct roots in its splitting field  $E$ , namely the  $n$ th roots of unity. Clearly these roots form a subgroup  $H$  of  $U(E)$  with order  $n$ , and by (11.3.6) it is cyclic, say  $H = \langle x \rangle$ . Here  $x$  has order  $n$  and thus it is a primitive  $n$ th root of unity.

(ii) Let  $a$  be a primitive  $n$ th root of unity in  $E$ . Then the roots of  $t^n - 1$  are  $a^i$ ,  $i = 0, 1, \dots, n-1$ , and  $E = F(a)$ . If  $\alpha \in \text{Gal}(E/F)$ , then  $\alpha$  is completely determined by  $\alpha(a) = a^i$  where  $1 \leq i < n$  and  $i$  is relatively prime to  $n$ . Furthermore, the assignment  $\alpha \mapsto i + n\mathbb{Z}$  yields an injective homomorphism from the Galois group into  $U(\mathbb{Z}_n)$ . By Lagrange's Theorem  $|\text{Gal}(E/F)|$  divides  $|U(\mathbb{Z}_n)| = \phi(n)$ .  $\square$

**Corollary (12.2.8)** *The number of primitive  $n$ th roots of unity over a field whose characteristic does not divide  $n$  is  $\phi(n)$ , where  $\phi$  is Euler's function.*

For, if  $a$  is a fixed primitive  $n$ th root of unity, the primitive  $n$ th roots of unity are just the powers  $a^i$  where  $1 \leq i < n$  and  $i$  is relatively prime to  $n$ .

**Cyclotomic polynomials.** Assume that  $F$  is a field whose characteristic does not divide the positive integer  $n$  and denote the primitive  $n$ th roots of unity over  $F$  by  $a_1, a_2, \dots, a_{\phi(n)}$ . The *cyclotomic polynomial of order  $n$  over  $F$*  is defined to be

$$\Phi_n = \prod_{i=1}^{\phi(n)} (t - a_i),$$

which is a monic polynomial of degree  $\phi(n)$ . Since every  $n$ th root of unity is a primitive  $d$ th root of unity for some divisor  $d$  of  $n$ , we have immediately that

$$t^n - 1 = \prod_{d|n} \Phi_d.$$

This leads to the formula

$$\Phi_n = \frac{t^n - 1}{\prod_{d|n, d < n} \Phi_d},$$

where  $d \parallel n$  means that  $d$  is a proper divisor of  $n$ . Using this formula, we can compute  $\Phi_n$  recursively, i.e., if we know  $\Phi_d$  for all proper divisors  $d$  of  $n$ , then we can calculate  $\Phi_n$ . The formula also shows that  $\Phi_n \in F[t]$ . For  $\Phi_1 = t - 1 \in F[t]$  and if  $\Phi_d \in F[t]$  for all proper divisors  $d$  of  $n$ , then  $\Phi_n \in F[t]$ .

**Example (12.2.3)** Since  $\Phi_1 = t - 1$ ,

$$\Phi_2 = \frac{t^2 - 1}{t - 1} = t + 1, \quad \Phi_3 = \frac{t^3 - 1}{t - 1} = t^2 + t + 1,$$

and

$$\Phi_4 = \frac{t^4 - 1}{(t - 1)(t + 1)} = t^2 + 1.$$

There is in fact an explicit formula for  $\Phi_n$ . This involves the *Möbius function*  $\mu$ , which is well-known from number theory. It is defined by the rules:

$$\mu(1) = 1, \quad \mu(p_1 p_2 \cdots p_k) = (-1)^k,$$

if  $p_1, p_2, \dots, p_k$  are distinct primes, and

$$\mu(n) = 0$$

if  $n$  is divisible by the square of a prime.

**(12.2.9)** *The cyclotomic polynomial of order  $n$  over any field whose characteristic does not divide  $n$  is given by*

$$\Phi_n = \prod_{d \mid n} (t^d - 1)^{\mu(n/d)}.$$

*Proof.* First we note an auxiliary property of the Möbius function,

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

This is obvious if  $n = 1$ , so assume that  $n > 1$  and write  $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  where the  $p_i$  are distinct primes. If  $d$  is a square-free divisor of  $n$ , then  $d$  has the form  $p_{i_1} p_{i_2} \cdots p_{i_r}$  where  $1 \leq i_1 < i_2 < \cdots < i_r \leq k$ , which corresponds to the term  $(-1)^r t_{i_1} t_{i_2} \cdots t_{i_r}$  in the product  $(1 - t_1)(1 - t_2) \cdots (1 - t_n)$ ; note also that  $\mu(d) = (-1)^r$ . Therefore we obtain the identity

$$(1 - t_1)(1 - t_2) \cdots (1 - t_n) = \sum \mu(p_{i_1} p_{i_2} \cdots p_{i_r}) t_{i_1} t_{i_2} \cdots t_{i_r},$$

where the sum is over all  $i_j$  satisfying  $1 \leq i_1 < i_2 < \cdots < i_r \leq k$ . Set all  $t_i = 1$  to get  $\sum \mu(p_{i_1} p_{i_2} \cdots p_{i_r}) = 0$ . Since  $\mu(d) = 0$  if  $d$  is not square-free, we can rewrite the last equation as  $\sum_{d \mid n} \mu(d) = 0$ .

We are now in a position to establish the formula for  $\Phi_n$ . Let

$$\Psi_n = \prod_{e|n} (t^e - 1)^{\mu(n/e)},$$

so that  $\Psi_1 = t - 1 = \Phi_1$ . Assume that  $\Psi_d = \Phi_d$  for all  $d < n$ . Then by definition of  $\Psi_d$ , we have

$$\prod_{d|n} \Psi_d = \prod_{d|n} \prod_{e|d} (t^e - 1)^{\mu(d/e)} = \prod_{f|n} (t^f - 1)^{\sum_{f|d|n} \mu(d/f)}.$$

Next for a fixed  $f$  dividing  $d$  we have

$$\sum_{f|d|n} \mu(d/f) = \sum_{\substack{d|n \\ f|d}} \mu(d/f),$$

which equals 1 or 0 according as  $f = n$  or  $f < n$ . It therefore follows that

$$\prod_{d|n} \Psi_d = t^n - 1 = \prod_{d|n} \Phi_d.$$

Since  $\Psi_d = \Phi_d$  if  $d < n$ , cancellation yields  $\Psi_n = \Phi_n$  and the proof is complete.  $\square$

**Example (12.2.4)** Use the formula of (12.2.9) to compute the cyclotomic polynomial of order 12 over  $\mathbb{Q}$ .

The formula yields

$$\Phi_{12} = (t - 1)^{\mu(12)} (t^2 - 1)^{\mu(6)} (t^3 - 1)^{\mu(4)} (t^4 - 1)^{\mu(3)} (t^6 - 1)^{\mu(2)} (t^{12} - 1)^{\mu(1)},$$

which reduces to

$$(t^2 - 1)(t^4 - 1)^{-1}(t^6 - 1)^{-1}(t^{12} - 1) = t^4 - t^2 + 1,$$

since  $\mu(12) = \mu(4) = 0$ ,  $\mu(2) = \mu(3) = -1$  and  $\mu(6) = \mu(1) = 1$ .

**Example (12.2.5)** If  $p$  is a prime,  $\Phi_p = 1 + t + t^2 + \cdots + t^{p-1}$ .

For  $\Phi_p = (t - 1)^{\mu(p)} (t^p - 1)^{\mu(1)} = \frac{t^p - 1}{t - 1} = 1 + t + t^2 + \cdots + t^{p-1}$ , since  $\mu(p) = -1$ .

Since we are interested in computing the Galois group of a cyclotomic polynomial over  $\mathbb{Q}$ , it is important to know if  $\Phi_n$  is irreducible. This is certainly true when  $n$  is prime by Example (7.4.6). The general result is:

**(12.2.10)** The cyclotomic polynomial  $\Phi_n$  is irreducible over  $\mathbb{Q}$  for all integers  $n$ .

*Proof.* Assume that  $\Phi_n$  is reducible over  $\mathbb{Q}$ ; then Gauss's Lemma (7.3.7) tells us that it must be reducible over  $\mathbb{Z}$ . Since  $\Phi_n$  is monic, it follows that it is a product of monic irreducible polynomials in  $\mathbb{Z}[t]$ . Let  $f$  be one such polynomial and choose a root  $a$  of  $f$ ; then  $f = \text{Irr}_{\mathbb{Q}}(a)$ . Now  $a$  is a primitive  $n$ th root of unity, so, if  $p$  is any prime not dividing  $n$ , then  $a^p$  is also a primitive  $n$ th root of unity and is thus a root of  $\Phi_n$ . Hence  $a^p$  is a root of some monic  $\mathbb{Q}$ -irreducible divisor  $g$  of  $\Phi_n$  in  $\mathbb{Z}[t]$ . Of course  $g = \text{Irr}_{\mathbb{Q}}(a^p)$ .

Suppose first that  $f \neq g$ . Thus  $t^n - 1 = fgh$  for some  $h \in \mathbb{Z}[t]$  since  $f$  and  $g$  are distinct  $\mathbb{Q}$ -irreducible divisors of  $t^n - 1$ . Also  $g(a^p) = 0$  implies that  $f$  divides  $g(t^p)$  and thus  $g(t^p) = fk$  where  $k \in \mathbb{Z}[t]$ . The canonical homomorphism from  $\mathbb{Z}$  to  $\mathbb{Z}_p$  induces a homomorphism from  $\mathbb{Z}[t]$  to  $\mathbb{Z}_p[t]$ ; let  $\bar{f}, \bar{g}, \bar{h}, \bar{k}$ , denote images of  $f, g, h, k$  under this homomorphism. Then  $\bar{f}\bar{k} = \bar{g}(t^p) = (\bar{g}(t))^p$  since  $x^p \equiv x \pmod{p}$  for any integer  $x$ . Now  $\mathbb{Z}_p[t]$  is a PID and hence a UFD. Since  $\bar{f}\bar{k} = \bar{g}^p$ , the polynomials  $\bar{f}$  and  $\bar{g}$  have a common irreducible divisor in  $\mathbb{Z}_p[t]$ . This means that  $\bar{f}\bar{g}\bar{h} \in \mathbb{Z}_p[t]$  is divisible by the square of this irreducible factor and hence  $t^n - 1 \in \mathbb{Z}_p[t]$  has a multiple root in its splitting field. However,  $(t^n - 1)' = nt^{n-1}$  is relatively prime to  $t^n - 1$  in  $\mathbb{Z}_p[t]$  since  $p$  does not divide  $n$ . This is a contradiction by (7.4.7). It follows that  $f = g$ .

We have proved that  $a^p$  is a root of  $f$  for all primes  $p$  not dividing  $n$ . It follows that  $a^m$  is a root of  $f$  whenever  $1 \leq m < n$  and  $\gcd\{m, n\} = 1$ . Therefore  $\deg(f) \geq \phi(n) = \deg(\Phi_n)$ , which shows that  $f = \Phi_n$  and  $\Phi_n$  is irreducible.  $\square$

We can now compute the Galois group of a cyclotomic polynomial.

**(12.2.11)** *If  $n$  is a positive integer, the Galois group of  $\Phi_n$  over  $\mathbb{Q}$  is isomorphic with  $U(\mathbb{Z}_n)$ , an abelian group of order  $\phi(n)$ .*

*Proof.* Let  $E$  denote the splitting field of  $\Phi_n$  over  $\mathbb{Q}$  and let  $a$  be a primitive  $n$ th root of unity in  $E$ . The roots of  $\Phi_n$  are  $a^i$  where  $i = 1, 2, \dots, n-1$  and  $\gcd\{i, n\} = 1$ . Hence  $E = \mathbb{Q}(a)$  and  $\Phi_n$  is the irreducible polynomial of  $a$  by (12.2.10). Thus  $|\text{Gal}(E/F)| = \deg(\Phi_n) = \phi_n$ . If  $1 \leq i < n$  and  $i$  is relatively prime to  $n$ , there is an automorphism  $\alpha_i$  of  $E$  over  $\mathbb{Q}$  such that  $\alpha_i(a) = a^i$  since  $a$  and  $a^i$  have the same irreducible polynomial. Moreover the map  $i + n\mathbb{Z} \mapsto \alpha_i$  is easily seen to be an injective group homomorphism from  $U(\mathbb{Z})$  to  $\text{Gal}(E/F)$ . Since both these groups have order  $\phi(n)$ , they are isomorphic.  $\square$

The splitting field of  $\Phi_n \in \mathbb{Q}[t]$  is called a *cyclotomic number field*. Thus the Galois group of a cyclotomic number field is abelian.

### Exercises (12.2)

- (1) Give an example of a finite simple extension  $E$  of a field  $F$  such that  $|\text{Gal}(E/F)| = 1$ , but  $E \neq F$ .
- (2) If  $E = \mathbb{Q}(\sqrt{5})$ , find  $\text{Gal}(E/F)$ .
- (3) If  $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ , find  $\text{Gal}(E/F)$ .



- (4) Find the Galois groups of the following polynomials in  $\mathbb{Q}[t]$ : (i)  $t^2 + 1$ ; (ii)  $t^3 - 4$ ; (iii)  $t^3 - 2t + 4$ .
- (5) Let  $f \in F[t]$  and suppose that  $f = f_1 f_2 \cdots f_k$  where the  $f_i$  are polynomials over the field  $F$ . Prove that  $\text{Gal}(f)$  is isomorphic with a subgroup of the direct product  $\text{Gal}(f_1) \times \text{Gal}(f_2) \times \cdots \times \text{Gal}(f_k)$ .
- (6) Prove that the Galois group of  $GF(p^m)$  over  $GF(p)$  is a cyclic group of order  $m$  and that it is generated by the automorphism in which  $a \mapsto a^p$ .
- (7) Give an example to show that  $\text{Gal}(\Phi_n)$  need not be cyclic.
- (8) Let  $p$  be a prime not dividing the positive integer  $n$ . Prove that if  $\Phi_n$  is irreducible over  $GF(p)$ , then  $\phi(n)$  is the smallest positive integer  $m$  such that  $p^m \equiv 1 \pmod{n}$ .
- (9) Show that  $\Phi_5$  is reducible over  $GF(11)$  and find an explicit factorization of it in terms of irreducibles.

## 12.3 The Fundamental Theorem of Galois theory

Armed with the techniques of the last two sections, we can now approach the celebrated theorem of the title. First some terminology: let  $E$  be an extension of a field  $F$ . By an *intermediate field* is meant a subfield  $S$  such that  $F \subseteq S \subseteq E$ . If  $H$  is a subgroup of  $\text{Gal}(E/F)$ , the *fixed field* of  $H$

$$\text{Fix}(H)$$

is the set of elements of  $E$  which are fixed by every element of  $H$ . It is quickly verified that  $\text{Fix}(H)$  is a subfield and  $F \subseteq \text{Fix}(H) \subseteq E$ , i.e.,  $\text{Fix}(H)$  is an intermediate field.

**(12.3.1)** *Let  $E$  be a Galois extension of a field  $F$ . Let  $S$  be an intermediate field and let  $H$  be a subgroup of the Galois group  $G = \text{Gal}(E/F)$ . Then:*

- (i) *the mappings  $H \mapsto \text{Fix}(H)$  and  $S \mapsto \text{Gal}(E/S)$  are mutually inverse, inclusion reversing bijections;*
- (ii)  *$(E : \text{Fix}(H)) = |H|$  and  $(\text{Fix}(H) : F) = |G : H|$ ;*
- (iii)  *$(E : S) = |\text{Gal}(E/S)|$  and  $(S : F) = |G : \text{Gal}(E/S)|$ .*

Thus the theorem asserts the existence of a bijection from the set of subfields between  $E$  and  $F$  to the set of subgroups of the Galois group  $G$ ; furthermore the bijection reverses set inclusions. Such a bijection is called a *Galois correspondence*. The Fundamental Theorem allows us to translate a problem about subfields into one about subgroups, which sometimes makes the problem easier to solve.

*Proof of (12.3.1).* (i) In the first place  $\text{Fix}(\text{Gal}(E/S)) = S$  by (12.2.6). To show that we have mutually inverse bijections we must still prove that  $\text{Gal}(E/\text{Fix}(H)) = H$ . By the Theorem of the Primitive Element (12.1.7),  $E = F(a)$  for some  $a$  in  $E$ . Define a polynomial  $f$  in  $E[t]$

by

$$f = \prod_{\alpha \in H} (t - \alpha(a)).$$

Note that all the roots of  $f$  are distinct: for  $\alpha_1(a) = \alpha_2(a)$  implies that  $\alpha_1 = \alpha_2$  since  $E = F(a)$ . Hence  $\deg(f) = |H|$ . Also elements of  $H$  permute the roots of  $f$ , so that  $\alpha(f) = f$  for all  $\alpha \in H$ . Therefore the coefficients of  $f$  lie in  $K = \text{Fix}(H)$ . In addition  $f(a) = 0$ , so  $\text{Irr}_K(a)$  divides  $f$ , and since  $E = K(a)$ , it follows that

$$(E : K) = \deg(\text{Irr}_K(a)) \leq \deg(f) = |H|.$$

Hence  $|\text{Gal}(E/K)| \leq |H|$ . But clearly  $H \leq \text{Gal}(E/K)$ , so that  $H = \text{Gal}(E/K)$ , as required.

(ii) Since  $E$  is Galois over  $\text{Fix}(H)$ , we have

$$(E : \text{Fix}(H)) = |\text{Gal}(E/\text{Fix}(H))| = |H|$$

by (12.3.1)(i). The second statement follows from

$$(E : \text{Fix}(H)) \cdot (\text{Fix}(H) : F) = (E : F) = |G| = |H| \cdot |G : H|.$$

(iii) The first statement is obvious. For the second statement we have  $(E : S)(S : F) = (E : F)$  and  $(E : S) = \text{Gal}(E/S)$ , while  $(E : F) = |G|$ . The result now follows.  $\square$

**Normal extensions and normal subgroups.** If  $E$  is a Galois extension of a field  $F$ , intermediate subfields which are normal over  $F$  surely correspond to subgroups of  $\text{Gal}(E/F)$  which are in some way special. In fact these are exactly the normal subgroups of  $\text{Gal}(E/F)$ . To prove this a simple lemma about Galois groups of conjugate subfields is called for. If  $\alpha \in \text{Gal}(E/F)$  and  $F \subseteq S \subseteq E$ , write  $\alpha(S) = \{\alpha(a) \mid a \in S\}$ . Clearly  $\alpha(S)$  is a subfield and  $F \subseteq \alpha(S) \subseteq E$ : the subfield  $\alpha(S)$  is called a *conjugate* of  $S$ .

**(12.3.2)** *Let  $E$  be an extension of a field  $F$  and let  $S$  be an intermediate field. If  $\alpha \in \text{Gal}(E/F)$ , then  $\text{Gal}(E/\alpha(S)) = \alpha \text{Gal}(E/S) \alpha^{-1}$ .*

*Proof.* Let  $\beta \in \text{Gal}(E/F)$ . Then  $\beta \in \text{Gal}(E/\alpha(S))$  if and only if  $\beta(\alpha(a)) = \alpha(a)$ , i.e.,  $\alpha^{-1}\beta\alpha(a) = a$ , for all  $a \in S$ , or equivalently  $\alpha^{-1}\beta\alpha \in \text{Gal}(E/S)$ . Hence  $\beta \in \text{Gal}(E/\alpha(S))$  if and only if  $\beta \in \alpha \text{Gal}(E/S) \alpha^{-1}$ .  $\square$

The connection between normal extensions and normal subgroups is now within reach.

**(12.3.3)** *Let  $E$  be a Galois extension of a field  $F$  and let  $S$  be an intermediate field. Then the following statements about  $S$  are equivalent:*

- (i)  $S$  is normal over  $F$ ;
- (ii)  $\alpha(S) = S$  for all  $\alpha \in \text{Gal}(E/F)$ ;
- (iii)  $\text{Gal}(E/S) \triangleleft \text{Gal}(E/F)$ .

*Proof.* (i) implies (ii). Let  $a \in S$  and write  $f = \text{Irr}_F(a)$ . Since  $S$  is normal over  $F$  and  $f$  has a root in  $S$ , all the roots of  $f$  are in  $S$ . If  $\alpha \in \text{Gal}(E/F)$ , then  $\alpha(a)$  is also a root of  $f$  since  $f(\alpha(a)) = \alpha(f(a)) = 0$ . Therefore  $\alpha(a) \in S$  and  $\alpha(S) \subseteq S$ . By the same argument  $\alpha^{-1}(S) \subseteq S$ , so that  $S \subseteq \alpha(S)$  and  $\alpha(S) = S$ .

(ii) implies (iii). Suppose that  $\alpha \in \text{Gal}(E/F)$ . By (12.3.2)

$$\alpha \text{Gal}(E/S) \alpha^{-1} = \text{Gal}(E/\alpha(S)) = \text{Gal}(E/S),$$

which shows that  $\text{Gal}(E/S) \triangleleft \text{Gal}(E/F)$ .

(iii) implies (i). Starting with  $\text{Gal}(E/S) \triangleleft \text{Gal}(E/F)$ , we have for any  $\alpha \in \text{Gal}(E/F)$  that  $\text{Gal}(E/S) = \alpha \text{Gal}(E/S) \alpha^{-1} = \text{Gal}(E/\alpha(S))$  by (12.3.2). Apply the function  $\text{Fix}$  to  $\text{Gal}(E/S) = \text{Gal}(E/\alpha(S))$  to obtain  $S = \alpha(S)$  by the Fundamental Theorem of Galois Theory. Next let  $f$  in  $F[t]$  be irreducible with a root  $a$  in  $S$  and suppose  $b$  is another root of  $f$ . Then  $b \in E$  since  $E$  is normal over  $F$ . Because  $\text{Irr}_F(a) = f = \text{Irr}_F(b)$ , there exists  $\alpha \in \text{Gal}(E/F)$  such that  $\alpha(a) = b$ . Therefore  $b \in \alpha(S) = S$ , from which it follows that  $S$  is normal over  $F$ .  $\square$

**(12.3.4)** If  $E$  is a Galois extension of a field  $F$  and  $S$  is an intermediate field which is normal over  $F$ , then

$$\text{Gal}(S/F) \simeq \text{Gal}(E/F)/\text{Gal}(E/S).$$

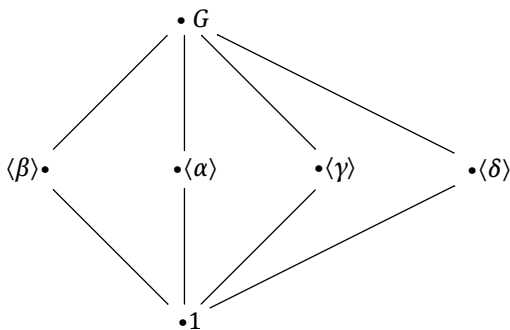
*Proof.* Let  $\alpha \in \text{Gal}(E/F)$ ; then  $\alpha(S) = S$  by (12.3.3) and thus  $\alpha|_S \in \text{Gal}(S/F)$ . What is more, the restriction map  $\alpha \mapsto \alpha|_S$  is a homomorphism from  $\text{Gal}(E/F)$  to  $\text{Gal}(S/F)$  with kernel equal to  $\text{Gal}(E/S)$ . The First Isomorphism Theorem then tells us that  $\text{Gal}(E/F)/\text{Gal}(E/S)$  is isomorphic with a subgroup of  $\text{Gal}(S/F)$ . In addition

$$|\text{Gal}(E/F)/\text{Gal}(E/S)| = (E : F)/(E : S) = (S : F) = |\text{Gal}(S/F)|$$

since  $S$  is Galois over  $F$ . Therefore  $\text{Gal}(E/F)/\text{Gal}(E/S) \simeq \text{Gal}(S/F)$ .  $\square$

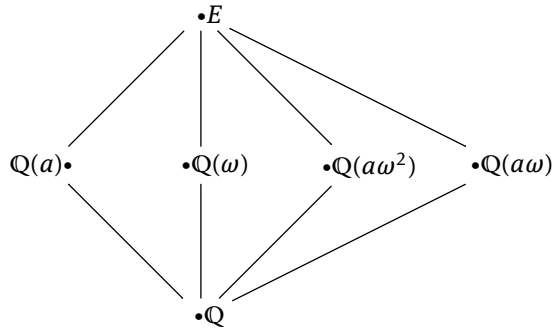
**Example (12.3.1)** Let  $E$  denote the splitting field of  $t^3 - 2 \in \mathbb{Q}[t]$ . Thus  $E = \mathbb{Q}(a, \omega)$  where  $a = 2^{1/3}$  and  $\omega = e^{2\pi i/3}$ . By Example (12.2.1)  $(E : \mathbb{Q}) = 6$  and  $G = \text{Gal}(E/F) \simeq S_3$ .

Now  $G$  has exactly six subgroups, which are displayed in the Hasse diagram below.



Here  $\alpha(a) = a\omega$  and  $\alpha(\omega) = \omega$ ;  $\beta(a) = a$  and  $\beta(\omega) = \omega^2$ ;  $\gamma(a) = a\omega$  and  $\gamma(\omega) = \omega^2$ ;  $\delta(a) = a\omega^2$  and  $\delta(\omega) = \omega$ . Each subgroup  $H$  corresponds to its fixed field  $\text{Fix}(H)$  under the Galois correspondence. For example,  $\text{Fix}(\langle\alpha\rangle) = \mathbb{Q}(\omega)$  and  $\text{Fix}(\langle\beta\rangle) = \mathbb{Q}(a)$ . The normal subgroups of  $G$  are  $1$ ,  $\langle\alpha\rangle$  and  $G$ ; the three corresponding normal extensions are  $E$ ,  $\mathbb{Q}(\omega)$  and  $\mathbb{Q}$ .

The six subfields of  $E$  are displayed in the Hasse diagram below.



Since every subgroup of an abelian group is normal, we deduce at once from (12.3.3):

**Corollary (12.3.5)** *If  $E$  is a Galois extension of a field  $F$  and  $\text{Gal}(E/F)$  is abelian, then every intermediate field is normal over  $F$ .*

For example, by (12.2.11) the Galois group of the cyclotomic polynomial  $\Phi_n \in \mathbb{Q}[t]$  is abelian. Therefore every subfield of a cyclotomic number field is normal over  $\mathbb{Q}$ .

As a demonstration of the power of Galois theory, let us prove the *Fundamental Theorem of Algebra*, which was mentioned in (7.4). All known proofs of this theorem employ some analysis. Here only the *Intermediate Value Theorem* is used: if  $f$  is a continuous function of a real variable which assumes the values  $a$  and  $b$ , then  $f$  assumes all values between  $a$  and  $b$ . In fact this result is only required for polynomial functions.

**(12.3.6)** *Let  $f$  be a non-constant polynomial over  $\mathbb{C}$ . Then  $f$  is the product of linear factors over  $\mathbb{C}$ .*

*Proof.* First note that the polynomial  $f\bar{f}$  has real coefficients. Since we can replace  $f$  by this polynomial, there is no loss in assuming that  $f$  has real coefficients. It can also be assumed that  $\deg(f) > 1$ . Let  $E$  be the splitting field of  $f$  over  $\mathbb{C}$ . Then  $E$  is the splitting field of  $(t^2 + 1)f$  over  $\mathbb{R}$ . Hence  $E$  is Galois over  $\mathbb{R}$ , the characteristic being 0. Put  $G = \text{Gal}(E/\mathbb{R})$ . Then  $|G| = (E : \mathbb{R}) = (E : \mathbb{C}) \cdot (\mathbb{C} : \mathbb{R}) = 2(E : \mathbb{C})$ , and we conclude that  $|G|$  is even.

Let  $H$  be a Sylow 2-subgroup of  $G$  and put  $F = \text{Fix}(H)$ . Then  $\mathbb{R} \subseteq F \subseteq E$  and  $(F : \mathbb{R}) = |G : H|$  is odd. Let  $a \in F$  and set  $g = \text{Irr}_{\mathbb{R}}(a)$ . Since  $\deg(g) = (\mathbb{R}(a) : \mathbb{R})$ ,

which divides  $(F : \mathbb{R})$ , we conclude that  $\deg(g)$  is odd. Also  $g$  is monic, so  $g(x) > 0$  for large positive  $x$  and  $g(x) < 0$  for large negative  $x$ . This is our opportunity to apply the Intermediate Value Theorem, the conclusion being that  $g(x) = 0$  for some real number  $x$ . But  $g$  is irreducible over  $\mathbb{R}$ , so  $\deg(g) = 1$ ; hence  $a \in \mathbb{R}$  and  $F = \mathbb{R}$ . This implies that  $H = G$  and  $G$  is a 2-group.

Let  $G_0 = \text{Gal}(E/\mathbb{C}) \leq G$ ; thus  $G_0$  is a 2-group. Now  $G_0 = 1$  implies that  $E = \mathbb{C}$  and  $f$  is a product on linear factors over  $\mathbb{C}$ . So assume that  $G_0 \neq 1$ . Hence there is a maximal (proper) subgroup  $M$  of  $G_0$ . Now  $G_0$  is nilpotent, so  $M \triangleleft G_0$  and  $|G_0 : M| = 2$  by (10.2.7)(ii). Now put  $S = \text{Fix}(M)$ . By (12.3.1) we have

$$(S : \mathbb{C}) = |\text{Gal}(E/\mathbb{C}) : \text{Gal}(E/S)| = \frac{|G_0|}{|M|} = 2.$$

Hence any  $s$  in  $S - \mathbb{C}$  has irreducible polynomial over  $\mathbb{C}$  of degree 2, say  $t^2 + at + b$ . By the quadratic formula  $s = -\frac{1}{2}(a \pm \sqrt{a^2 - 4b}) \in \mathbb{C}$  and it follows that  $S = \mathbb{C}$ , a contradiction.  $\square$

**Constructing regular  $n$ -gons.** We return to the last of the ruler and compass problems discussed in (11.2), which was left unresolved. The problem is to construct a regular  $n$ -gon of side 1 unit using ruler and compass only.

Consider a regular  $n$ -gon with vertices  $A_1, A_2, \dots, A_n$  and centroid  $C$ . Let  $\theta_n$  be the angle between lines joining the centroid  $C$  to neighboring vertices; thus  $\theta_n = \frac{2\pi}{n}$ . By elementary geometry, if  $d$  is the distance from the centroid  $C$  to a vertex, then  $d \sin \frac{1}{2}\theta_n = \frac{1}{2}$  and hence

$$d = \frac{1}{2 \sin(\frac{1}{2}\theta_n)} = \frac{1}{\sqrt{2(1 - \cos \theta_n)}}.$$

It follows from the discussion of constructibility in (11.2) that the regular  $n$ -gon is constructible by ruler and compass if and only if  $\cos \theta_n$  is constructible from the set  $\{(0, 0), (1, 0)\}$ .

The definitive result can now be proved.

**(12.3.7)** *A regular  $n$ -gon of side 1 can be constructed by ruler and compass if and only if  $n$  has the form  $2^k p_1 p_2 \cdots p_k$  where  $k \geq 0$  and the  $p_j$  are distinct Fermat primes, i.e., of the form  $2^{2^{f_j}} + 1$ .*

*Proof.* Assume that the regular  $n$ -gon is constructible, so that  $\cos \theta_n$  is constructible. Then  $(\mathbb{Q}(\cos \theta_n) : \mathbb{Q})$  must be a power of 2 by (11.2.2). Put  $c = e^{2\pi i/n}$ , a primitive  $n$ th root of unity. Then  $\cos \theta_n = \frac{1}{2}(c + c^{-1})$ , so that  $\mathbb{Q}(\cos \theta_n) \subseteq \mathbb{Q}(c)$ . Since  $c + c^{-1} = 2 \cos \theta_n$ , we have  $c^2 - 2c \cos \theta_n + 1 = 0$ . Hence  $(\mathbb{Q}(c) : \mathbb{Q}(\cos \theta_n)) = 2$  and  $(\mathbb{Q}(c) : \mathbb{Q}) = 2^d$  for some  $d$ . Recall from (12.2.10) that  $\text{Irr}_{\mathbb{Q}}(c) = \Phi_n$ , which has degree  $\phi(n)$ . Writing  $n = 2^k p_1^{e_1} \cdots p_r^{e_r}$  with distinct odd primes  $p_j$  and  $e_j > 0$ , we have  $\phi(n) = 2^{k-1}(p_1^{e_1} - p_1^{e_1-1}) \cdots (p_r^{e_r} - p_r^{e_r-1})$  by (2.3.8). This must equal  $2^d$ . Hence  $e_j = 1$  and  $p_j - 1$  is a power of 2 for all  $j$ . Since

$2^s + 1$  cannot be a prime if  $s$  is not a power of 2 (see Exercise (2.2.13)), it follows that  $p_j$  is a Fermat prime.

Conversely, assume that  $n$  has the form indicated. Since  $\mathbb{Q}(c)$  is Galois over  $\mathbb{Q}$ , we have  $(\mathbb{Q}(c) : \mathbb{Q}) = \phi(n)$ , which is a power of 2 by the formula for  $\phi(n)$ . Hence  $\text{Gal}(\mathbb{Q}(c)/\mathbb{Q})$  is a finite abelian 2-group and therefore  $G = \text{Gal}(\mathbb{Q}(\cos \theta)/\mathbb{Q})$ , being isomorphic with a quotient of it, is also a finite abelian 2-group. Therefore, all the factors in a composition series of  $G$  have order 2 and by the Fundamental Theorem of Galois Theory there is a chain of subfields

$$\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_\ell = \mathbb{Q}(\cos \theta)$$

such that  $F_{j+1}$  is Galois over  $F_j$  and  $(F_{j+1} : F_j) = 2$ .

We argue by induction on  $j$  that every element of  $F_j$  is constructible. Let  $x \in F_{j+1} - F_j$ . Then  $\text{Irr}_{F_j}(x) = t^2 + at + b$  where  $a, b \in F_j$  and thus  $x^2 + ax + b = 0$ . Hence  $(x + \frac{1}{2}a)^2 = \frac{1}{4}a^2 - b > 0$  since  $x$  is real. Writing  $x' = x + \frac{1}{2}a$ , we have  $x'^2 \in F_j$ . By induction hypothesis  $x'^2$  is constructible and (11.2.1) shows that  $x'$  is constructible, whence so is  $x$ . Finally we deduce that  $\cos \theta$  is constructible.  $\square$

**Example (12.3.2).** A regular  $n$ -gon is constructible for  $n = 3, 4, 5, 6$ , but not for  $n = 7$ .

The only known Fermat primes are 3, 5, 17,  $257 = 2^{2^3} + 1$  and  $65,537 = 2^{2^4} + 1$ . Since 7 is not a Fermat prime, it is impossible to construct a regular 7-gon using ruler and compass.

### Exercises (12.3).

- (1) For each of the following polynomials over  $\mathbb{Q}$  display the lattice of subgroups of the Galois group and the corresponding lattice of subfields of the splitting field: (i)  $t^2 - 5$ ; (ii)  $t^4 - 5$ ; (iii)  $(t^2 + 1)(t^2 + 3)$ .
- (2) Determine the normal subfields of the splitting fields in Exercise (12.3.1).
- (3) Use the Fundamental Theorem of Galois Theory and Exercise (12.2.6) to prove that  $\text{GF}(p^m)$  has exactly one subfield of order  $p^d$  for each positive divisor  $d$  of  $m$  and no subfields of other orders – see also Exercise (11.3.2).
- (4) Let  $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . Find all the subgroups of  $\text{Gal}(E/\mathbb{Q})$  and hence all subfields of  $E$ .
- (5) Find all finite fields with exactly two subfields and also those with exactly three subfields.
- (6) Let  $E$  be a Galois extension of a field  $F$  and let  $p^k$  be the largest power of a prime  $p$  dividing  $(E : F)$ . Prove that there is an intermediate field  $S$  such that  $(E : S) = p^k$ . If  $\text{Gal}(E/F)$  is solvable, prove that there is an intermediate field  $T$  such that  $(T : F) = p^k$ .
- (7) If  $E$  is a Galois extension of a field  $F$  and there is exactly one proper intermediate field, what can be said about  $\text{Gal}(E/F)$ ?

- (8) If  $E$  is a Galois extension of  $F$  and  $(E : F)$  is the square of a prime, show that each intermediate field is normal over  $F$ .
- (9) Prove that a regular  $2^k$ -gon of side 1 is constructible if  $k \geq 2$ .
- (10) For which values of  $n$  in the range 10 to 20 can a regular  $n$ -gon of side 1 be constructed?
- (11) Show that if  $a$  is a real number such that  $(\mathbb{Q}(a) : \mathbb{Q})$  is a power of 2 and  $\mathbb{Q}(a)$  is normal over  $\mathbb{Q}$ , then  $a$  is constructible from the points  $(0, 0)$  and  $(1, 0)$ .
- (12) Let  $p$  be a prime and let  $f = t^p - t - a \in F[t]$  where  $F = \text{GF}(p)$ . Denote by  $E$  the splitting field of  $f$  over  $F$ .
- If  $x$  is a root of  $f$  in  $E$ , show that the set of all roots of  $f$  is  $\{x + b \mid b \in F\}$ , and that  $E = F(x)$ .
  - Prove that  $f$  is irreducible over  $F$  if and only if  $a \neq 0$ .
  - Prove that  $|\text{Gal}(f)| = p$  unless  $a = 0$ , when  $\text{Gal}(f) = 1$ .

## 12.4 Solvability of equations by radicals

One of the oldest parts of algebra is concerned with the problem of solving equations of the form  $f(t) = 0$  where  $f$  is a non-constant polynomial over  $\mathbb{Q}$  or  $\mathbb{R}$ . The object is to find a formula for the solutions of the equation which involves the coefficients of  $f$ , square roots, cube roots, etc. The easiest cases are when  $\deg(f) \leq 2$ ; if the degree is 1, we are solving a single linear equation. If the degree is 2, there is the familiar formula for the solutions of a quadratic equation. For equations of degree 3 and 4 the problem is harder, but methods of solution had been found by the 16th Century. Thus for  $\deg(f) \leq 4$  there are explicit formulas for the roots of  $f(t) = 0$ , which in fact involve radicals of the form  $\sqrt[k]{\phantom{x}}$  for  $k \leq 4$ .

The problem of finding formulas for the solutions of equations of degree 5 and higher is one that fascinated mathematicians for hundreds of years. An enormous amount of ingenuity was expended in attempts to solve the general equation of the fifth degree. It was only with the work of Abel, Galois and Ruffini<sup>2</sup> in the early 19th Century that it became clear that all these efforts had been in vain. It is a fact that solvability of a polynomial equation is inextricably linked to the solvability of the Galois group of the polynomial. The symmetric group  $S_n$  is solvable for  $n < 5$ , but is insolvable for  $n \geq 5$ . This explains why early researchers were able to solve the general equation of degree  $n$  only for  $n \leq 4$ . Without the aid of group theory it is impossible to comprehend the reason for this failure. Our aim here is explain why the solvability of the Galois group governs the solvability of a polynomial equation.

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<sup>2</sup> Paolo Ruffini (1765–1822)

**Radical extensions.** Let  $E$  be an extension of a field  $F$ . Then  $E$  is called a *radical extension* of  $F$  if there is a chain of subfields

$$F = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_m = E$$

such that  $E_{i+1} = E_i(a_{i+1})$  where  $a_{i+1}$  has irreducible polynomial over  $E_i$  of the form  $t^{n_{i+1}} - b_i$ . It is natural to refer to  $a_{i+1}$  as a *radical* and write  $a_{i+1} = \sqrt[n_{i+1}]{b_i}$ , but here one has to keep in mind that  $a_{i+1}$  may not be uniquely determined by  $b_i$ . Since

$$E = F(\sqrt[n_1]{b_1}, \sqrt[n_2]{b_2}, \dots, \sqrt[n_m]{b_m}),$$

elements of  $E$  are expressible as polynomial functions of the radicals  $\sqrt[n_i]{b_i}$ .

Let  $f$  be a non-constant polynomial over  $F$  with splitting field  $K$ . Then  $f$ , or the equation  $f = 0$ , is said to be *solvable by radicals* if  $K$  is contained in some radical extension of  $F$ . This means that the roots of  $f$  are obtained by forming a finite sequence of successive radicals, starting with elements of  $F$ . The definition gives a precise expression of our intuitive idea of what it means for a polynomial equation to be solvable by radicals.

To make progress with the problem of describing the radical extensions it is necessary to have a better understanding of polynomials of the form  $t^n - a$ .

**(12.4.1)** Let  $F$  be a field and  $n$  a positive integer. Assume that  $F$  contains a primitive  $n$ th root of unity. Then for any  $a$  in  $F$  the group  $\text{Gal}(t^n - a)$  is cyclic with order dividing  $n$ .

*Proof.* Let  $z$  be a primitive  $n$ th root of unity in  $F$  and denote by  $b$  a root of  $f = t^n - a$  in its splitting field  $E$ . Then the roots of  $f$  are  $bz^j$ ,  $j = 0, 1, \dots, n-1$ . If  $\alpha \in \text{Gal}(f) = \text{Gal}(E/F)$ , then  $\alpha(b) = bz^{j(\alpha)}$  for some  $j(\alpha)$  and  $\alpha$  is completely determined by  $j(\alpha)$ : this is because  $\alpha|_F$  is the identity map and  $E = F(b)$  since  $z \in F$ . The assignment  $\alpha \mapsto j(\alpha) + n\mathbb{Z}$  is an injective homomorphism from  $\text{Gal}(f)$  to  $\mathbb{Z}_n$ : for  $\alpha\beta(b) = \alpha(bz^{j(\beta)}) = \alpha(b)z^{j(\beta)} = bz^{j(\alpha)+j(\beta)}$  and thus  $j(\alpha\beta) \equiv j(\alpha) + j(\beta) \pmod{n}$ . It follows that  $\text{Gal}(f)$  is isomorphic with a subgroup of  $\mathbb{Z}_n$ , whence it is a cyclic group with order dividing  $n$ .  $\square$

We will need the following simple result.

**(12.4.2)** Let  $E$  be a Galois extension of a field  $F$  and let  $K_1$  and  $K_2$  be subfields intermediate between  $F$  and  $E$ . If  $H_i = \text{Gal}(E/K_i)$ , then  $\text{Gal}(E/K_1 \cap K_2) = \langle H_1, H_2 \rangle$ .

*Proof.* Clearly  $H_1$  and  $H_2$  are contained in  $\text{Gal}(E/K_1 \cap K_2)$  and hence  $J = \langle H_1, H_2 \rangle \leq \text{Gal}(E/K_1 \cap K_2)$ . Next suppose that  $x \in E - K_i$ . Then there exists  $\alpha \in H_i$  such that  $\alpha(x) \neq x$ . Hence  $x \notin \text{Fix}(J)$  and consequently  $\text{Fix}(J) \subseteq K_1 \cap K_2$ . Taking the Galois group of  $E$  over each side and applying (12.3.1), we obtain  $J \geq \text{Gal}(E/K_1 \cap K_2)$ .  $\square$

The principal theorem is now within reach.

**(12.4.3)** Let  $f$  be a non-constant polynomial over a field  $F$  of characteristic 0. If  $f$  is solvable by radicals, then  $\text{Gal}(f)$  is a solvable group.



*Proof.* Let  $E$  denote the splitting field of  $f$  over  $F$ . By hypothesis  $E \subseteq R$  where  $R$  is a radical extension of  $F$ . Hence there are subfields  $R_i$  such that

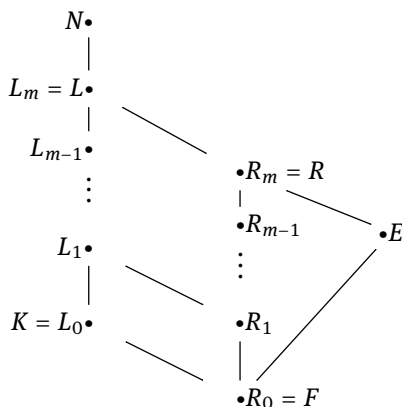
$$F = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_m = R$$

where  $R_{i+1} = R_i(a_{i+1})$  and  $\text{Irr}_{R_i}(a_{i+1}) = t^{n_{i+1}} - b_i$  with  $b_i \in R_i$ . It follows that  $(R_{i+1} : R_i) = n_{i+1}$  and hence  $(R : F) = n_1 n_2 \cdots n_m = n$ , say.

Let  $K$  and  $L$  be the splitting fields of the polynomial  $t^n - 1$  over  $F$  and  $R$  respectively. Note that  $L$  may not be normal over  $F$ . Let  $N$  be the the splitting field over  $F$  of the product of  $t^n - 1$  and all the polynomials  $\text{Irr}_F(a_i)$ ,  $i = 1, 2, \dots, m$ . Then  $L \subseteq N$  and  $N$  is normal over  $F$ . Clearly  $(N : F)$  is finite and  $N$  is separable since the characteristic is zero. Thus  $N$  is Galois over  $F$ . Put  $L_i = K(R_i)$ , so there is the chain of subfields

$$K = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m = L \subseteq N.$$

The relevant subfields are displayed in the Hasse diagram below.



Note that  $L_{i+1}$  is the splitting field of  $t^{n_{i+1}} - b_i$  over  $L_i$  since  $K$  contains all  $n_{i+1}$ th roots of unity. Thus  $L_{i+1}$  is normal and hence Galois over  $L_i$ . Now set  $G = \text{Gal}(N/F)$  and  $G_i = \text{Gal}(N/L_i)$ ; hence  $G_{i+1} \triangleleft G_i$  by (12.3.3). Also write  $V = \text{Gal}(N/K)$  and  $U = \text{Gal}(N/E)$ , noting that  $U \triangleleft G$  and  $V \triangleleft G$  since  $E$  and  $K$  are normal over  $F$ . Thus we have the truncated series of subgroups

$$\text{Gal}(N/L) = G_m \triangleleft G_{m-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = V.$$

Notice that  $G_i/G_{i+1} \simeq \text{Gal}(L_{i+1}/L_i) = \text{Gal}(t^{n_{i+1}} - b_i)$ , and the latter is cyclic by (12.4.1).

Since  $G_m \leq U$ , there is a series

$$1 = G_m U/U \triangleleft G_{m-1} U/U \triangleleft \cdots \triangleleft G_1 U/U \triangleleft G_0 U/U = UV/U,$$

and the factors of this series are cyclic since the  $G_i/G_{i+1}$  is cyclic. Therefore  $UV/U$  is a solvable group. Now from (12.4.2) we have  $\text{Gal}(N/K \cap E) = UV \triangleleft G$  and  $K \cap E$  is normal

over  $F$ . Moreover,

$$G/UV = \text{Gal}(N/F)/\text{Gal}(N/K \cap E) \simeq \text{Gal}(K \cap E/F)$$

and therefore  $G/UV \simeq \text{Gal}(K/F)/\text{Gal}(K/K \cap E)$ . Since  $\text{Gal}(K/F)$  is abelian by (12.2.7), it follows that  $G/UV$  is abelian. Therefore  $G/U$  is solvable. Finally,

$$\text{Gal}(E/F) \simeq \text{Gal}(N/F)/\text{Gal}(N/E) = G/U,$$

so that  $\text{Gal}(f) = \text{Gal}(E/F)$  is solvable, as required.  $\square$

It can be shown – although we will not do so here – that the converse of (12.4.3) is valid: see [1] or [15] for a proof. As a consequence there is the following definitive result.

**(12.4.4)** *Let  $f$  be a non-constant polynomial over a field of characteristic 0. Then  $f$  is solvable by radicals if and only if  $\text{Gal}(f)$  is a solvable group.*

Let  $n = \deg(f)$ . Then  $\text{Gal}(f)$  is isomorphic with a subgroup of the symmetric group  $S_n$  by (12.2.3). If  $n \leq 4$ , then  $S_n$ , and hence  $\text{Gal}(f)$ , is solvable. Therefore by (12.4.4) every polynomial with degree 4 or less is solvable by radicals.

On the other hand, when  $n \geq 5$ , the symmetric group  $S_n$  is not solvable since  $A_n$  is simple by (10.1.7). Thus we are led to suspect that not every polynomial equation of degree 5 is solvable by radicals. Actual examples of polynomials that are not solvable by radicals are furnished by the next result.

**(12.4.5)** *Let  $f \in \mathbb{Q}[t]$  be an irreducible polynomial of prime degree  $p$  and assume that  $f$  has exactly two complex roots. Then  $\text{Gal}(f) \simeq S_p$  and hence  $f$  is not solvable by radicals if  $p \geq 5$ .*

*Proof.* Label the roots of  $f$  in its splitting field  $a_1, a_2, \dots, a_p$ ; these are all different since  $f$  is separable. Two of these roots are complex conjugates, say  $\bar{a}_1 = a_2$ , while the other roots are all real. We can think of  $\text{Gal}(f)$  as a group of permutations of the set of roots  $\{a_1, a_2, \dots, a_p\}$  and indeed  $\text{Gal}(f)$  acts transitively since  $f$  is irreducible. Therefore  $p$  divides  $|\text{Gal}(f)|$  by (5.2.2), and Cauchy's Theorem (5.3.9) shows that there is an element of order  $p$  in  $\text{Gal}(f)$ . Hence  $\text{Gal}(f)$  contains a  $p$ -cycle, say  $\pi = (a_1 a_{i_2} \dots a_{i_p})$ . Replacing  $\pi$  by a suitable power, we may assume that  $i_2 = 2$ . Now relabel the remaining roots  $a_3, a_4, \dots, a_p$  so that  $\pi = (a_1 a_2 a_3 \dots a_p)$ .

Complex conjugation, i.e.,  $\sigma = (a_1 a_2)$ , is an element of  $\text{Gal}(f)$  with order 2. Conjugation of  $\sigma$  by powers of  $\pi$  shows that  $\text{Gal}(f)$  contains all the adjacent transpositions  $(a_i a_{i+1})$ , for  $i = 1, 2, \dots, p-1$ . But any permutation is expressible as a product of adjacent transpositions – see Exercise (3.1.4) – and therefore  $\text{Gal}(f) = S_n$ .  $\square$

**Example (12.4.1)** The polynomial  $f = t^5 - 6t + 3 \in \mathbb{Q}[t]$  is not solvable by radicals.

In the first place  $f$  is irreducible over  $\mathbb{Q}$  by Eisenstein's Criterion and Gauss's Lemma. In addition calculus tells us that the curve  $f(t) = 0$  crosses the  $t$ -axis exactly three times, so there are three real roots and two complex ones. Thus  $\text{Gal}(t^5 - 6t + 3) \simeq S_5$  and the result follows via (12.4.3).

**Example (12.4.2)** The polynomial  $f = t^5 + 8t^3 - t^2 + 12t - 2$  is solvable by radicals.

Here the situation is different since  $f$  factorizes as  $(t^2 + 2)(t^3 + 6t - 1)$ . Therefore  $\text{Gal}(f)$  is isomorphic with a subgroup of  $\text{Gal}(t^2 + 2) \times \text{Gal}(t^3 + 6t - 1)$  by Exercise (12.2.5). The latter is a solvable group. Hence by (10.2.2) the group  $\text{Gal}(f)$  is solvable and  $f$  is solvable by radicals.

**Symmetric functions.** As the final topic of the chapter, we present an account of the elementary theory of symmetric functions and explore its relationship with Galois theory. Let  $F$  be an arbitrary field and put  $E = F\{x_1, x_2, \dots, x_n\}$ , the field of rational functions over  $F$  in distinct indeterminates  $x_1, x_2, \dots, x_n$ . A *symmetric function* in  $x_1, x_2, \dots, x_n$  over  $F$  is an element  $g \in E$  such that

$$g(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}) = g(x_1, x_2, \dots, x_n)$$

for all  $\pi \in S_n$ . Thus  $g$  is unaffected by permutations of the indeterminates  $x_1, x_2, \dots, x_n$ . It is easy to verify that the symmetric functions form a subfield of  $E$ . Next consider the polynomial

$$f = (t - x_1)(t - x_2) \cdots (t - x_n) \in E[t]$$

where  $t$  is another indeterminate. Then expansion shows that

$$f = t^n - s_1 t^{n-1} + s_2 t^{n-2} - \cdots + (-1)^n s_n$$

where  $s_1 = \sum_{i=1}^n x_i$ ,  $s_2 = \sum_{i < j=1}^n x_i x_j$ , and in general

$$s_j = \sum_{i_1 < i_2 < \cdots < i_j=1}^n x_{i_1} x_{i_2} \cdots x_{i_j},$$

the last sum being over all tuples  $(i_1, i_2, \dots, i_j)$  such that  $1 \leq i_1 < i_2 < \cdots < i_j \leq n$ . It is evident that the  $s_j$  are symmetric functions: they are known as the *elementary symmetric functions* in  $x_1, x_2, \dots, x_n$ . For example, when  $n = 3$ , there are three elementary symmetric functions,

$$s_1 = x_1 + x_2 + x_3, \quad s_2 = x_1 x_2 + x_2 x_3 + x_1 x_3, \quad s_3 = x_1 x_2 x_3.$$

Put  $S = F(s_1, s_2, \dots, s_n)$ , which is a subfield of  $E$ . Then  $f \in S[t]$  and  $E$  is generated by  $S$  and the roots of  $f$ , i.e.,  $x_1, x_2, \dots, x_n$ . Hence  $E$  is the splitting field of  $f$  over  $S$ . Since all the roots of  $f$  are distinct, (12.1.6) shows that  $E$  is separable and hence Galois over  $S$ . Therefore  $\text{Gal}(f) = \text{Gal}(E/S)$  has order  $(E : S)$ . We now proceed to determine the Galois group of  $f$  over  $S$ . With the same notation the definitive result is:

**(12.4.6)**  $\text{Gal}(f) \simeq S_n$ .

*Proof.* Since  $\text{Gal}(f)$  permutes the roots  $x_1, x_2, \dots, x_n$  faithfully, we may identify it with a subgroup of  $S_n$ . Let  $\pi \in S_n$  and define  $\alpha_\pi : E \rightarrow E$  by the rule

$$\alpha_\pi(g(x_1, x_2, \dots, x_n)) = g(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)});$$

then  $\alpha_\pi$  is evidently an automorphism of  $E$ . Since  $\alpha_\pi$  fixes all the elementary symmetric functions, it fixes every element of  $S = F(s_1, s_2, \dots, s_n)$  and therefore  $\alpha_\pi \in \text{Gal}(E/S) = \text{Gal}(f)$ . Finally, all the  $\alpha_\pi$  are different, so  $\text{Gal}(f) = S_n$ .  $\square$

From this we quickly deduce a famous theorem.

**Corollary (12.4.7)** (The Symmetric Function Theorem) *If  $F$  is an arbitrary field and  $s_1, s_2, \dots, s_n$  are the elementary symmetric functions in indeterminates  $x_1, x_2, \dots, x_n$ , then  $F(s_1, s_2, \dots, s_n)$  is the field of all symmetric functions in  $x_1, x_2, \dots, x_n$ . Also the symmetric polynomials form a subring which is generated by  $F$  and the  $s_1, s_2, \dots, s_n$ .*

*Proof.* Let  $S = F(s_1, s_2, \dots, s_n) \subseteq E = F\{x_1, x_2, \dots, x_n\}$ . By (12.4.6)  $\text{Gal}(E/S)$  effectively consists of all permutations of  $\{x_1, x_2, \dots, x_n\}$ . Hence  $\text{Fix}(\text{Gal}(E/S))$  is the subfield of all symmetric functions. But by (12.3.1) this is also equal to  $S$ . The statement about polynomials is left as an exercise.  $\square$

**Generic polynomials.** Let  $F$  be an arbitrary field and write  $K$  for the rational function field in indeterminates  $x_1, x_2, \dots, x_n$  over  $F$ . The polynomial

$$f = t^n - x_1 t^{n-1} + x_2 t^{n-2} - \dots + (-1)^n x_n$$

is called a *generic polynomial*. The point to note here is that we can obtain from  $f$  any monic polynomial of degree  $n$  in  $F[t]$  by replacing  $x_1, x_2, \dots, x_n$  by suitable elements of  $F$ . It is therefore not surprising that the Galois group of  $f$  over  $K$  is as large as it could be.

**(12.4.8)** *With the above notation,  $\text{Gal}(f) \simeq S_n$ .*

*Proof.* Let  $u_1, u_2, \dots, u_n$  be the roots of  $f$  in its splitting field  $E$  over  $K$ . Then  $f = (t - u_1)(t - u_2) \cdots (t - u_n)$  and thus  $x_i = s_i(u_1, u_2, \dots, u_n)$  where  $s_i$  is the  $i$ th elementary symmetric function in  $n$  indeterminates  $y_1, y_2, \dots, y_n$ , all of which are different from  $x_1, x_2, \dots, x_n, t$ .

The assignment  $x_i \mapsto s_i$  determines a ring homomorphism

$$\phi_0 : F\{x_1, x_2, \dots, x_n\} \rightarrow F\{y_1, y_2, \dots, y_n\};$$

observe here that  $g(s_1, \dots, s_n) = 0$  implies that  $g(x_1, \dots, x_n) = 0$  because  $x_i = s_i(u_1, \dots, u_n)$ . So  $\phi_0$  is actually an isomorphism from  $K = F\{x_1, x_2, \dots, x_n\}$  to  $L = F(s_1, s_2, \dots, s_n) \subseteq F\{y_1, y_2, \dots, y_n\}$ . Set  $f^* = \phi_0(f)$  with  $f$  as above; thus

$$f^* = t^n - s_1 t^{n-1} + s_2 t^{n-2} - \dots + (-1)^n s_n = (t - y_1)(t - y_2) \cdots (t - y_n),$$

by definition of the elementary symmetric functions  $s_i$ .

By (11.3.3) we can extend  $\phi_0$  to an isomorphism  $\phi$  from  $E$ , the splitting field of  $f$  over  $K$ , to the splitting field of  $f^*$  over  $L$ . Therefore  $\phi$  induces a group isomorphism from  $\text{Gal}(f)$  to  $\text{Gal}(f^*)$ . But we know that  $\text{Gal}(f^*) \simeq S_n$  by (12.4.6). Hence  $\text{Gal}(f) \simeq S_n$ .  $\square$

**Corollary (12.4.9)** (Abel, Ruffini) *If  $F$  is a field of characteristic 0, the generic polynomial  $t^n - x_1 t^{n-1} + x_2 t^{n-2} - \cdots + (-1)^n x_n$  is insolvable by radicals over  $F(x_1, x_2, \dots, x_n)$  if  $n \geq 5$ .*

Thus, as one would expect, there is no *general formula* for the roots of a polynomial of degree  $n \geq 5$  in terms of its coefficients.

### Exercises (12.4)

- (1) Let  $F \subseteq K \subseteq E$  be field extensions with  $K$  radical over  $F$  and  $E$  radical over  $K$ . Prove that  $E$  is radical over  $F$ .
- (2) Let  $F \subseteq K \subseteq E$  be field extensions with  $E$  radical and Galois over  $F$ . Prove that  $E$  is radical over  $K$ .
- (3) Show that the polynomial  $t^5 - 3t + 2$  in  $\mathbb{Q}[t]$  is solvable by radicals.
- (4) If  $p$  is a prime larger than 11, show that  $t^5 - pt + p$  in  $\mathbb{Q}[t]$  is not solvable by radicals.
- (5) If  $f \in F[t]$  is solvable by radicals and  $g \mid f$  in  $F[t]$ , prove that  $g$  is solvable by radicals.
- (6) Let  $f = f_1 f_2$  where  $f_1, f_2 \in F[t]$  and  $F$  has characteristic 0. If  $f_1$  and  $f_2$  are solvable by radicals, show that  $f$  is too. Deduce that every non-constant reducible polynomial of degree less than 6 over  $\mathbb{Q}$  is solvable by radicals.
- (7) Let  $F$  be a field of characteristic 0 and let  $f \in F[t]$  be non-constant with splitting field  $E$ . Prove that there is a unique smallest intermediate field  $S$  such that  $S$  is normal over  $F$  and  $f$  is solvable by radicals over  $S$ . [Hint: show first that there is a unique maximum solvable normal subgroup in any finite group].
- (8) For each integer  $n \geq 5$  exhibit a polynomial of degree  $n$  over  $\mathbb{Q}$  which is insolvable by radicals.
- (9) Let  $G$  be any finite group. Prove that there is a Galois extension  $E$  of some algebraic number field  $F$  such that  $\text{Gal}(E/F) \simeq G$ . [You may assume there is an algebraic number field whose Galois group over  $\mathbb{Q}$  is isomorphic with  $S_n$ ]. (Remark: the general problem of whether every finite group is the Galois group of some algebraic number field over  $\mathbb{Q}$  is still open; it is known to be true for solvable groups.)
- (10) Write each of the following symmetric polynomials as a polynomial in the elementary symmetric functions  $s_1, s_2, s_3$  in  $x_1, x_2, x_3$ .
  - (i)  $x_1^2 + x_2^2 + x_3^2$ ;
  - (ii)  $x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_3 + x_2 x_3^2 + x_1^2 x_3 + x_1 x_3^2$ ;
  - (iii)  $x_1^3 + x_2^3 + x_3^3$ .

## 13 Tensor products

### 13.1 Definition of the tensor product

The tensor product is a very widely used construction in algebra which can be applied to modules, linear operators and matrices. We will begin by describing the tensor product of modules: here the distinction between left and right modules is essential.

Let  $R$  be an arbitrary ring and let  $M_R$  and  ${}_R N$  be right and left  $R$ -modules as indicated. Denote by  $F$  the free abelian group whose basis is the set product

$$M \times N = \{(a, b) \mid a \in M, b \in N\}.$$

Thus each element  $f$  of  $F$  can be uniquely written in the form  $f = \sum_{i=1}^k \ell_i(a_i, b_i)$  where  $\ell_i \in \mathbb{Z}$ ,  $a_i \in M$ ,  $b_i \in N$ . Define  $S$  to be the (additive) subgroup of  $F$  generated by all elements of the forms

- (i)  $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$ ,
- (ii)  $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$ ,
- (iii)  $(a \cdot r, b) - (a, r \cdot b)$ ,

where  $a, a_i \in M$ ,  $b, b_i \in N$  and  $r \in R$ . Then the *tensor product* of  $M$  and  $N$  is defined to be the quotient group

$$M \otimes_R N = F/S.$$

Thus  $M \otimes_R N$  is an abelian group generated by all elements of the form

$$a \otimes b = (a, b) + S, \quad (a \in M, b \in N);$$

the elements  $a \otimes b$  are called *tensors*. When  $R = \mathbb{Z}$ , which is a very common case, it is usual to write  $M \otimes N$  instead of  $M \otimes_{\mathbb{Z}} N$ .

The next result is an immediate consequence of the definition of the tensor product. It demonstrates the essentially bilinear nature of tensor products.

**(13.1.1)** *Let  $M_R$  and  ${}_R N$  be modules over a ring  $R$ . In the tensor product  $M \otimes_R N$  the following rules are valid:*

- (i)  $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$ ;
- (ii)  $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$ ;
- (iii)  $(a \cdot r) \otimes b = a \otimes (r \cdot b)$

where  $a, a_i \in M$ ,  $b, b_i \in N$ ,  $r \in R$ .

We record two simple consequences of (13.1.1),

$$0_M \otimes b = 0_{M \otimes_R N} = a \otimes 0_N, \quad (a \in M, b \in N).$$

These follow from (i) and (ii) on setting  $a_1 = 0_M = a_2$  and  $b_1 = 0_N = b_2$ , respectively. It should be stressed that the tensor product  $M \otimes_R N$  is only an abelian group at this point: later we will see when it can be given a module structure.

**The mapping property of tensor products.** We continue the previous notation with modules  $M_R$  and  ${}_R N$  over a ring  $R$ . A critical property of the tensor product  $M \otimes_R N$  is a certain mapping property; this involves the concept of a middle linear map, which will now be explained.

Let  $A$  be an (additively written) abelian group: a mapping  $\alpha : M \times N \rightarrow A$  is said to be *R-middle linear* if it has the three properties listed below for all  $a, a_i \in M$ ,  $b, b_i \in N$ ,  $r \in R$ :

- (i)  $\alpha((a_1 + a_2, b)) = \alpha((a_1, b)) + \alpha((a_2, b))$ ;
- (ii)  $\alpha((a, b_1 + b_2)) = \alpha((a, b_1)) + \alpha((a, b_2))$ ;
- (iii)  $\alpha((a \cdot r, b)) = \alpha((a, r \cdot b))$ .

For example, the *canonical mapping*  $v : M \times N \rightarrow M \otimes_R N$  in which  $v((a, b)) = a \otimes b$  is middle linear because of the properties listed in (13.1.1). The crucial mapping property of the tensor product is as follows.

**(13.1.2)** Let  $M_R$  and  ${}_R N$  be modules over a ring  $R$ .

- (i) Given a middle linear map  $\alpha : M \times N \rightarrow A$  with  $A$  an abelian group, there is a unique group homomorphism  $\beta : M \otimes_R N \rightarrow A$  such that  $\alpha = \beta v$  where  $v : M \times N \rightarrow M \otimes_R N$  is the canonical middle linear map in which  $(a, b) \mapsto a \otimes b$ .
- (ii) Conversely, if  $T$  is an abelian group and  $\phi : M \times N \rightarrow T$  is a middle linear map such that the pair  $(T, \phi)$  has the mapping property in (i), then  $T \simeq M \otimes_R N$ .

The assertion of (13.1.2)(i) is most easily remembered from the triangle diagram below.

$$\begin{array}{ccc} & A & \\ \alpha \nearrow & & \nwarrow \beta \\ M \times N & \xrightarrow{v} & M \otimes_R N \end{array}$$

Indeed the relation  $\alpha = \beta v$  expresses the *commutativity of the diagram*, in the sense that if we start with an element  $x \in M \times N$  and follow it in both directions around the triangle, applying the maps indicated by the arrows, we end up with the same element of  $A$ , namely  $\alpha(x) = \beta v(x)$ .

When (i) and (ii) of (13.1.2) are combined, they demonstrate that the tensor product  $M \otimes_R N$ , together with the canonical middle linear mapping  $v$ , is characterized by the mapping property. Another way of looking at the mapping property is that if a function with codomain an abelian group  $A$  is defined on tensors and arises from a middle linear mapping, then it can be extended to a homomorphism from  $M \otimes_R N$  to  $A$ . It is this form of the mapping property that makes it an indispensable tool in working with tensor products.

*Proof of (13.1.2).* (i). Let  $F$  be the free abelian group with basis  $M \times N$ . By (9.1.13) there is a homomorphism  $\beta' : F \rightarrow A$  such that  $\beta'((a, b)) = \alpha((a, b))$  for all  $a \in M$ ,  $b \in N$ . By definition  $M \otimes_R N = F/S$  where  $S$  is generated by all elements of  $F$  of the three types in the definition of the tensor product. Now  $\beta'$  maps each of the listed generators of  $S$  to 0

since  $\alpha$  is middle linear, and hence  $\beta'(s) = 0$  for all  $s \in S$ . This observation allows us to define in a unique manner a function

$$\beta : M \otimes_R N \rightarrow A$$

by the rule  $\beta(f+S) = \beta'(f)$ . Notice that  $\beta$  is a homomorphism since  $\beta'$  is one. Furthermore

$$\beta v((a, b)) = \beta((a, b) + S) = \beta'((a, b)) = \alpha((a, b))$$

for all  $a \in M, b \in N$ . Therefore  $\beta v = \alpha$ .

The uniqueness of  $\beta$  remain to be established. Suppose that  $\bar{\beta} : M \otimes_R N \rightarrow A$  is another homomorphism with the property  $\bar{\beta} v = \alpha$ . Then  $\beta v = \bar{\beta} v$ , so that  $\beta$  and  $\bar{\beta}$  agree on  $\text{Im}(v)$ , i.e., on the set of all tensors. But the tensors generate  $M \otimes_R N$ , so it follows that  $\beta = \bar{\beta}$ , which completes the proof of (i).

(ii) By the mapping property for the pair  $(M \otimes_R N, \beta)$  there is a homomorphism  $\beta : M \otimes_R N \rightarrow T$  such that  $\phi = \beta v$ , and by the mapping property for  $(T, \phi)$  there is a homomorphism  $\bar{\beta} : T \rightarrow M \otimes_R N$  such that  $v = \bar{\beta} \phi$ . Thus we have the two commutative triangles that follow:

$$\begin{array}{ccc} & T & \\ \phi \nearrow & & \nwarrow \beta \\ M \times N & \xrightarrow{v} & M \otimes_R N \\ & M \otimes_R N & \\ v \nearrow & & \nwarrow \bar{\beta} \\ M \times N & \xrightarrow{\phi} & T \end{array}$$

Therefore  $\beta \bar{\beta} \phi = \beta v = \phi$  and  $\bar{\beta} \beta v = \bar{\beta} \phi = v$ , equations that express the commutativity of the two triangles below

$$\begin{array}{ccc} & T & \\ \phi \nearrow & & \nwarrow \beta \bar{\beta} \\ M \times N & \xrightarrow{\phi} & T \\ & M \otimes_R N & \\ v \nearrow & & \nwarrow \bar{\beta} \beta \\ M \times N & \xrightarrow{v} & M \otimes_R N \end{array}$$

But clearly these triangles will also commute if  $\bar{\beta} \beta$  and  $\beta \bar{\beta}$  are replaced by the appropriate identity maps. At this point the uniqueness clause in the mapping property is invoked to show that  $\bar{\beta} \beta$  and  $\beta \bar{\beta}$  are identity maps. Hence  $\bar{\beta}$  is an isomorphism and  $T \stackrel{R}{\cong} M \otimes_R N$ .  $\square$



**Tensor products and homomorphisms.** When homomorphisms between pairs of modules are given, there are induced homomorphisms between the tensor products of these modules.

**(13.1.3)** *Let there be given modules  $M_R, M'_R$  and  ${}_R N, {}_R N'$  over a ring  $R$ , together with  $R$ -module homomorphisms  $\alpha : M \rightarrow M'$  and  $\beta : N \rightarrow N'$ . Then there is a homomorphism of groups  $\alpha \otimes \beta : M \otimes_R N \rightarrow M' \otimes_R N'$  such that*

$$\alpha \otimes \beta \left( \sum_{i=1}^k \ell_i(a_i \otimes b_i) \right) = \sum_{i=1}^k \ell_i(\alpha(a_i) \otimes \beta(b_i)),$$

where  $a_i \in M, b_i \in N, \ell_i \in \mathbb{Z}$ .

*Proof.* The first point to realise here is that one cannot simply use the formula in the statement as the definition of  $\alpha \otimes \beta$ , the reason being that there is no unique expressibility for an element of  $M \otimes_R N$  as a linear combination of tensors. However, an indirect approach using the mapping property succeeds.

To exploit this property we first introduce a function  $\theta : M \times N \rightarrow M' \otimes_R N'$  by defining  $\theta((a, b)) = \alpha(a) \otimes \beta(b)$ . Then we check the middle linearity of  $\theta$ , which is easy. By the mapping property there is a group homomorphism  $\phi : M \otimes_R N \rightarrow M' \otimes_R N'$  such that  $\phi v = \theta$  where  $v : M \times N \rightarrow M \otimes_R N$  is the canonical middle linear map  $(a, b) \mapsto a \otimes b$ . Thus the triangle below commutes

$$\begin{array}{ccc} & M' \otimes_R N' & \\ \theta \nearrow & & \nwarrow \phi \\ M \times N & \xrightarrow{v} & M \otimes_R N \end{array}$$

Now define  $\alpha \otimes \beta$  to be the map  $\phi$  and check that it has the required property:

$$\phi \left( \sum_{i=1}^k \ell_i(a_i \otimes b_i) \right) = \sum_{i=1}^k \ell_i \phi(a_i \otimes b_i) = \sum_{i=1}^k \ell_i \phi v((a_i, b_i)) = \sum_{i=1}^k \ell_i \theta((a_i, b_i)),$$

which equals  $\sum_{i=1}^k \ell_i(\alpha(a_i) \otimes \beta(b_i))$  by definition of  $\theta$ .  $\square$

This use of the mapping property is typical in situations where a mapping from a tensor product is to be defined and the problem of non-uniqueness of expression in terms of tensors must be faced.

Important special cases of (13.1.3) arise when  $\alpha$  or  $\beta$  is an identity map. Specifically, given module homomorphisms  $\alpha : M_R \rightarrow M'_R$  and  $\beta : {}_R N \rightarrow {}_R N'$ , we can form the *induced homomorphisms*

$$\alpha_* = \alpha \otimes \text{id}_N \text{ and } \beta_* = \text{id}_M \otimes \beta.$$

Thus  $\alpha_*$  and  $\beta_*$  are homomorphisms from  $M \otimes_R N$  to  $M' \otimes_R N$  and  $M \otimes_R N$  to  $M \otimes_R N'$  respectively. Moreover,  $\alpha_*(a \otimes b) = \alpha(a) \otimes b$  and  $\beta_*(a \otimes b) = a \otimes \beta(b)$  where  $a \in M, b \in N$ .

**Tensor products as modules.** As has been observed, in general a tensor product is an abelian group with no module structure other than over  $\mathbb{Z}$ . However, when the modules in a tensor product have additional module structures, this is inherited by the tensor product.

**(13.1.4)** *Let  ${}_S M_R$  and  ${}_R N_T$  be modules over rings  $R, S, T$  as indicated. Then  $M \otimes_R N$  is an  $(S, T)$ -bimodule with respect to the ring actions  $s \cdot (a \otimes b) = (s \cdot a) \otimes b$  and  $(a \otimes b) \cdot t = a \otimes (b \cdot t)$  where  $a \in M, b \in N, s \in S, t \in T$ .*

*Proof.* Fix  $s$  in  $S$  and consider the mapping  $\alpha^{(s)} : M_R \rightarrow M_R$  in which  $\alpha^{(s)}(a) = s \cdot a$ . This is a homomorphism of right  $R$ -modules, as an easy check reveals. By (13.1.3) we can form the induced homomorphism  $(\alpha^{(s)})_* : M \otimes_R N \rightarrow M \otimes_R N$ . This enables us to define a left action of  $S$  on  $M \otimes_R N$  by  $s \cdot x = (\alpha^{(s)})_*(x)$  for  $x \in M \otimes_R N$ . This is certainly well defined, but we still need to verify the module axioms. First note that  $s \cdot (a \otimes b) = (\alpha^{(s)})_*(a \otimes b) = (\alpha^{(s)}(a)) \otimes b = (s \cdot a) \otimes b$  where  $a \in M, b \in N$ .

Turning to the module axioms, we have  $s \cdot (x_1 + x_2) = (\alpha^{(s)})_*(x_1 + x_2) = (\alpha^{(s)})_*(x_1) + (\alpha^{(s)})_*(x_2) = s \cdot x_1 + s \cdot x_2$ , where  $s \in S$  and  $x_i \in M \otimes_R N$ , since  $\alpha^{(s)}$  is a homomorphism. Next let  $s_i \in S$ ; then  $(s_1 + s_2) \cdot (a \otimes b) = ((s_1 + s_2) \cdot a) \otimes b = (s_1 \cdot a + s_2 \cdot a) \otimes b = s_1 \cdot (a \otimes b) + s_2 \cdot (a \otimes b)$ . This implies that  $(s_1 + s_2) \cdot x = s_1 \cdot x + s_2 \cdot x$  for all  $x$  in  $M \otimes_R N$ , since the latter is generated by the tensors  $a \otimes b$  and  $(\alpha^{(s)})_*$  is a homomorphism. As for the last module axiom,  $s_1 \cdot (s_2 \cdot (a \otimes b)) = s_1 \cdot ((s_2 \cdot a) \otimes b) = (s_1 \cdot (s_2 \cdot a)) \otimes b = ((s_1 s_2) \cdot a) \otimes b = (s_1 s_2) \cdot (a \otimes b)$ , which is sufficient for the proof since  $M \otimes_R N$  is generated by the tensors  $a \otimes b$ .

The right action of  $T$  arises in a similar fashion from the map

$\beta^{(t)} : {}_R N \rightarrow {}_R N$ ,  $(t \in T)$ , in which  $\beta^{(t)}(b) = b \cdot t$  for  $b \in N$ . Thus  $x \cdot t$  is defined to be  $(\beta^{(t)})_*(x)$ . To complete the proof the reader should verify the bimodule condition,  $s \cdot (x \cdot t) = (s \cdot x) \cdot t$  for  $s \in S, t \in T, x \in M \otimes_R N$ , noting that it is enough to do this when  $x$  is a tensor.  $\square$

We remark that there are versions of (13.1.4) applicable to the module situations  ${}_S M_R, {}_R N$  and  $M_R, {}_R N_T$ , when  $M \otimes_R N$  is only either a left  $S$ -module or a right  $T$ -module respectively.

In the case of a commutative ring there is no difference between left and right modules, as we saw (9.1), so the tensor product is always a bimodule.

**(13.1.5)** *Let  $M$  and  $N$  be modules over a commutative ring  $R$ . Then  $M \otimes_R N$  is an  $(R, R)$ -bimodule. Furthermore,*

$$r \cdot (a \otimes b) = (r \cdot a) \otimes b = a \otimes (b \cdot r) = (a \otimes b) \cdot r$$

where  $a \in M, b \in N, r \in R$ .

**Exercises (13.1)**

- (1) Let  $R, S, T$  be rings and  ${}_S M_R, {}_R N_T$  modules. State what module structure the following tensor products possess and give the module action in each case:  $R \otimes_R N$  and  $M \otimes_R R$ .
- (2) Let  $M$  and  $N$  be  $R$ -modules where  $R$  is a commutative ring. Prove that  $M \otimes_R N \stackrel{R}{\cong} N \otimes_R M$ .
- (3) Let  $A$  be an abelian torsion group, i.e., each element of  $A$  has finite order. Prove that  $A \otimes \mathbb{Q} = 0$ .
- (4) Let  $A$  and  $B$  be abelian torsion groups such that elements from  $A$  and  $B$  have relatively prime orders. Prove that  $A \otimes B = 0$ .
- (5) Let  $R$  be a ring and let  $M_R$  and  ${}_R N$  be modules. Prove that  $M \otimes_R N \simeq N \otimes_{R^{\text{opp}}} M$ . (Here  $R^{\text{opp}}$  is the opposite ring of  $R$  – see (9.1)).
- (6) Let  $\alpha : A \rightarrow A_1, \beta : B \rightarrow B_1, \gamma : A_1 \rightarrow A_2, \delta : B_1 \rightarrow B_2$  be module homomorphisms. Prove that  $(\gamma \otimes \delta)(\alpha \otimes \beta) = (\gamma\alpha) \otimes (\delta\beta)$ .
- (7) Let  $A$  be the multiplicative group of all complex  $p$ -power roots of unity where  $p$  is a prime. Prove that  $A \otimes A = 0$ .
- (8) Let  $R$  be a ring and  $M$  a right  $R$ -module. Also let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  be homomorphisms of left  $R$ -modules. Form the induced homomorphism  $\alpha_* = \text{id}_M \otimes \alpha$  and similarly form  $\beta_*$  and  $(\beta\alpha)_*$ . Prove that  $(\beta\alpha)_* = \beta_* \alpha_*$ . (Compare this with (9.1.18)).

**13.2 Properties of tensor products.**

In this section we present a sequence of results about tensor products which aid in their calculation.

**(13.2.1)** *Let  $R$  be a ring with identity and let  $M_R$  and  ${}_R N$  be modules. Then*

- (i)  $M \otimes_R R \stackrel{R}{\cong} M$ ,  
(ii)  $R \otimes_R N \stackrel{R}{\cong} N$ ,

*via the respective isomorphisms in which  $a \otimes r \mapsto a \cdot r$  and  $r \otimes b \mapsto r \cdot b$ , ( $a \in M, b \in N, r \in R$ ).*

*Proof.* First observe that  $M \otimes_R R$  and  $R \otimes_R N$  are respectively a right  $R$ -module and a left  $R$ -module by (13.1.4). Only the first isomorphism will be proved. Consider the map from  $M \times R$  to  $M$  defined by  $(a, r) \mapsto a \cdot r$ . This is clearly middle linear, so by the mapping property there is a group homomorphism  $\alpha : M \otimes_R R \rightarrow M$  such that  $\alpha(a \otimes r) = a \cdot r$ . In fact  $\alpha$  is a homomorphism of right  $R$ -modules because

$$\alpha\left(\left(\sum_{i=1}^k \ell_i(a_i \otimes r_i)\right) \cdot r\right) = \alpha\left(\sum_{i=1}^k \ell_i(a_i \otimes (r_i r))\right) = \sum_{i=1}^k \ell_i \alpha(a_i \otimes (r_i r))$$

$$= \sum_{i=1}^k \ell_i(a_i \cdot (r_i r)) = \left( \sum_{i=1}^k \ell_i(a_i \cdot r_i) \right) \cdot r = \alpha \left( \sum_{i=1}^k \ell_i(a_i \otimes r_i) \right) \cdot r,$$

where  $a_i \in M$ ,  $r, r_i \in R$ ,  $\ell_i \in \mathbb{Z}$ .

To show that  $\alpha$  is an isomorphism we produce an inverse function. Define  $\beta : M \rightarrow M \otimes_R R$  by  $\beta(a) = a \otimes 1_R$ . This is certainly well defined and a simple check reveals that  $\alpha\beta$  and  $\beta\alpha$  are identity functions. Thus  $\beta = \alpha^{-1}$ .  $\square$

**(13.2.2)** (Associativity of tensor products) *Let  $R$  and  $S$  be rings and  $L_R, {}_R M_S, {}_S N$  modules as indicated. Then there is an isomorphism of groups*

$$\alpha : (L \otimes_R M) \otimes_S N \rightarrow L \otimes_R (M \otimes_S N)$$

such that  $\alpha((a \otimes b) \otimes c) = a \otimes (b \otimes c)$  where  $a \in L, b \in M, c \in N$ .

*Proof.* First note that these tensor products exist. Choose and fix  $c \in N$ ; then observe that the assignment  $(a, b) \mapsto a \otimes (b \otimes c)$ , where  $a \in L, b \in M$ , is an  $R$ -middle linear map from  $L \times M$  to  $L \otimes_R (M \otimes_S N)$ . By the mapping property there is a group homomorphism  $\beta_c : L \otimes_R M \rightarrow L \otimes_R (M \otimes_S N)$  such that  $\beta_c(a \otimes b) = a \otimes (b \otimes c)$ .

Next the assignment  $(x, c) \mapsto \beta_c(x)$  determines an  $S$ -middle linear map from  $(L \otimes_R M) \times N$  to  $L \otimes_R (M \otimes_S N)$  – notice that  $\beta_{c_1+c_2} = \beta_{c_1} + \beta_{c_2}$ . Hence there is a homomorphism  $\alpha : (L \otimes_R M) \otimes_S N \rightarrow L \otimes_R (M \otimes_S N)$  such that  $\alpha((a \otimes b) \otimes c) = \beta_c(a \otimes b) = a \otimes (b \otimes c)$ . By a similar argument – which the reader should supply – there is a homomorphism  $\gamma : L \otimes_R (M \otimes_S N) \rightarrow (L \otimes_R M) \otimes_S N$  such that  $\gamma(a \otimes (b \otimes c)) = (a \otimes b) \otimes c$ . Since  $\alpha$  and  $\gamma$  are inverse functions,  $\alpha$  is an isomorphism.  $\square$

Here it should be noted that if there is additional module structure in (13.2.2), the map  $\alpha$  may be a module isomorphism. Specifically, if we have  ${}_Q L_R$  or  ${}_S N_T$  with rings  $Q$  and  $T$ , then  $\alpha$  is a homomorphism of left  $Q$ -modules or of right  $T$ -modules respectively. For example, in the first case, if  $a \in L, b \in M, c \in N, q \in Q$ , then we have  $\alpha(q \cdot ((a \otimes b) \otimes c)) = \alpha(((q \cdot a) \otimes b) \otimes c) = (q \cdot a) \otimes (b \otimes c) = q \cdot (a \otimes (b \otimes c)) = q \cdot \alpha((a \otimes b) \otimes c)$ , which implies that  $\alpha$  is a  $Q$ -module homomorphism.

**(13.2.3)** (Distributivity of tensor products) *Let  $R$  be a ring and let  $L_R, {}_R M, {}_R N$  be modules. Then there is a group homomorphism*

$$\alpha : L \otimes_R (M \oplus N) \rightarrow (L \otimes_R M) \oplus (L \otimes_R N)$$

such that  $\alpha(a \otimes (b \oplus c)) = (a \otimes b) \oplus (a \otimes c)$ , where  $a \in L, b \in M, c \in N$ .

(Here, in order to improve the notation, we are writing  $b \oplus c$  for  $(b, c) \in M \oplus N$ , etc).

*Proof.* Let  $a \in L, b \in M, c \in N$ . Then the assignment  $(a, b \oplus c) \mapsto (a \otimes b) \oplus (a \otimes c)$  determines a middle linear map from  $L \times (M \oplus N)$  to  $(L \otimes_R M) \oplus (L \otimes_R N)$ , so there is a group homomorphism  $\alpha : L \otimes_R (M \oplus N) \rightarrow (L \otimes_R M) \oplus (L \otimes_R N)$  such that  $\alpha(a \otimes (b \oplus c)) = (a \otimes b) \oplus (a \otimes c)$ .

Next the canonical injections  $\iota_M : M \rightarrow M \oplus N$  and  $\iota_N : N \rightarrow M \oplus N$  lead to induced homomorphisms  $(\iota_M)_* : L \otimes_R M \rightarrow L \otimes_R (M \oplus N)$  and  $(\iota_N)_* : L \otimes_R N \rightarrow L \otimes_R (M \oplus N)$ . Combine  $(\iota_M)_*$  and  $(\iota_N)_*$  to produce a homomorphism  $\beta : (L \otimes_R M) \oplus (L \otimes_R N) \rightarrow L \otimes_R (M \oplus N)$  which sends  $(a \otimes b) \oplus 0$  to  $a \otimes (b \oplus 0)$  and  $0 \oplus (a \otimes c)$  to  $a \otimes (0 \oplus c)$ . Hence  $\beta((a \otimes b) \oplus (a \otimes c)) = a \otimes (b \oplus c)$ . Since  $\alpha$  and  $\beta$  are inverse maps,  $\alpha$  is an isomorphism.  $\square$

Once again, given the extra module structure  ${}_S L_R$  or  ${}_R M_T$  and  ${}_R N_T$ , it is easy to verify that  $\alpha$  is a left  $S$ - or a right  $T$ -module isomorphism respectively.

**Tensor products of quotients.** There is a useful technique for computing the tensor product of two quotient modules. Let  $R$  be a ring and let  $M_R, {}_R N$  be modules with respective submodules  $M_0$  and  $N_0$ . Define

$$S = \langle a \otimes b \mid a \in M_0 \text{ or } b \in N_0 \rangle,$$

which is a subgroup of  $M \otimes_R N$ . With this notation we have the fundamental result that follows.

**(13.2.4)** *There is an isomorphism*

$$\alpha : (M/M_0) \otimes_R (N/N_0) \rightarrow (M \otimes_R N)/S$$

such that  $\alpha((a + M_0) \otimes (b + N_0)) = a \otimes b + S$ .

*Proof.* In the first place the assignment  $(a + M_0, b + N_0) \mapsto a \otimes b + S$  gives rise to a well defined middle linear mapping from  $M/M_0 \times N/N_0$  to  $(M \otimes_R N)/S$ , by definition of  $S$ . Hence there is a homomorphism  $\alpha : (M/M_0) \otimes_R (N/N_0) \rightarrow (M \otimes_R N)/S$  such that  $\alpha((a + M_0) \otimes (b + N_0)) = a \otimes b + S$ . Next let  $\pi : M \rightarrow M/M_0$  and  $\sigma : N \rightarrow N/N_0$  denote the canonical homomorphisms. Now form the homomorphism  $\bar{\beta} = \pi \otimes \sigma$ ; thus  $\bar{\beta}$  sends  $a \otimes b$  to  $(a + M_0) \otimes (b + N_0)$ . Observe that  $\bar{\beta}$  maps each generator of  $S$  to 0, so that  $\bar{\beta}|_S = 0$ . Therefore we can define unambiguously a mapping

$$\beta : (M \otimes_R N)/S \rightarrow (M/M_0) \otimes_R (N/N_0)$$

by  $\beta(x + S) = \bar{\beta}(x)$ . Note that  $\beta(a \otimes b + S) = \bar{\beta}(a \otimes b) = (a + M_0) \otimes (b + N_0)$ . Finally,  $\alpha$  and  $\beta$  are inverse maps, so  $\alpha$  is an isomorphism.  $\square$

As usual when additional module structure in  $M$  or  $N$  is present,  $\alpha$  is a module isomorphism. A first application of (13.2.4) is to compute tensor products in which one factor is a cyclic module. But first recall from (9.1.8) that if  $R$  is a ring with identity, a cyclic left  $R$ -module is isomorphic with a module  ${}_R R/I$  where  $I$  is a left ideal of  $R$ , and there is a corresponding statement for cyclic right modules.

**(13.2.5)** *Let  $R$  be a ring with identity and let  $I, J$  be left and right ideals of  $R$  respectively. Let  $M_R$  and  ${}_R N$  be modules. Then*

- (i)  $M \otimes_R ({}_R R/I) \simeq M/(M \cdot I)$ ;  
(ii)  $(R_R/J) \otimes_R N \simeq N/(J \cdot N)$ .

In the statement of this result  $M \cdot I$  denotes the subgroup generated by all elements of the form  $a \cdot i$  where  $a \in M$  and  $i \in I$ , with a similar explanation for  $J \cdot N$ .

*Proof.* Only (i) will be proved. Apply (13.2.4) with  $M_0 = 0$  and  $N_0 = I$ . Then  $M \otimes_R ({}_R R/I) \simeq (M \otimes_R R)/S$  and it is just a question of identifying the subgroup  $S = \langle a \otimes i \mid a \in M, i \in I \rangle$ . By (13.2.1) the assignment  $a \otimes r \mapsto a \cdot r$  determines an isomorphism  $\alpha : M \otimes_R {}_R R \rightarrow M$ . The image of  $S$  under  $\alpha$  is generated by the elements  $a \cdot i$ , where  $a \in M, i \in I$ ; therefore  $\alpha(S) = M \cdot I$  and  $M \otimes_R ({}_R R/I) \simeq M/(M \cdot I)$ .  $\square$

**Corollary (13.2.6)** *If  $I$  and  $J$  are respectively left and right ideals of a ring  $R$  with identity, the mapping  $(r_1 + J) \otimes (r_2 + I) \mapsto r_1 r_2 + (I + J)$  yields an isomorphism*

$$({}_R R/J) \otimes_R ({}_R R/I) \simeq R/(I + J).$$

*Moreover, if  $I$  and  $J$  are two sided ideals, the isomorphism is of  $(R, R)$ -bimodules.*

*Proof.* From (13.2.5)(i) we have

$$({}_R R/J) \otimes_R ({}_R R/I) \simeq (R/J)/((R/J) \cdot I) = (R/J)/(I + J/J),$$

which by (9.1.7) is isomorphic with  $R/(I + J)$ . Composition of the isomorphisms yields the map stated. If  $I$  and  $J$  are two sided ideals, each module is an  $(R, R)$ -bimodule and clearly the isomorphism is of  $R$ -modules.  $\square$

For example, if  $m, n$  are positive integers with  $d = \gcd\{m, n\}$ , then  $d\mathbb{Z} = m\mathbb{Z} + n\mathbb{Z} = (m) + (n)$  and it follows from (13.2.6) that

$$\mathbb{Z}_m \otimes \mathbb{Z}_n = \mathbb{Z}/(m) \otimes \mathbb{Z}/(n) \simeq \mathbb{Z}/((m) + (n)) = \mathbb{Z}/(d) = \mathbb{Z}_d. \quad (*)$$

**Example (13.2.1)** Let  $A = \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{5^2}$  and  $B = \mathbb{Z} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{5^2} \oplus \mathbb{Z}_7$ . Applying the distributive property together with (13.2.1) and the isomorphism  $(*)$ , we obtain

$$A \otimes B \simeq \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{5^2} \oplus \mathbb{Z}_{5^2} \oplus \mathbb{Z}_{5^2} \oplus \mathbb{Z}_7.$$

**Tensor products of free modules.** A tensor product of free modules over a commutative ring with identity is in fact always a free module. For simplicity of presentation we will discuss only the case where the free modules are finitely generated.

**(13.2.7)** *Let  $R$  be a commutative ring with identity and let  $M$  and  $N$  be finitely generated free  $R$ -modules with respective bases  $\{x_1, x_2, \dots, x_m\}$  and  $\{y_1, y_2, \dots, y_n\}$ . Then  $M \otimes_R N$  is a free  $R$ -module with basis  $\{x_i \otimes y_j \mid i = 1, 2, \dots, m, j = 1, 2, \dots, n\}$ .*

*Proof.* We have  $N = R \cdot y_1 \oplus R \cdot y_2 \oplus \cdots \oplus R \cdot y_n$  and hence by the distributive law

$$M \otimes_R N \stackrel{R}{\cong} (M \otimes_R (R \cdot y_1)) \oplus (M \otimes_R (R \cdot y_2)) \oplus \cdots \oplus (M \otimes_R (R \cdot y_n)).$$

Now  $R \cdot y_j \stackrel{R}{\cong} R$ , since  $r \cdot y_j = 0$  implies that  $r = 0$ . Thus  $M \otimes_R (R \cdot y_j) \stackrel{R}{\cong} M \otimes_R R \stackrel{R}{\cong} M$ , and the image of  $x_i \otimes y_j$  under the composite of these isomorphisms is  $x_i$ . Therefore the  $x_i \otimes y_j$ ,  $i = 1, 2, \dots, m$ , are  $R$ -linearly independent, so they form a basis of  $M \otimes_R (R \cdot y_j)$ , which implies the result.  $\square$

**Corollary (13.2.8)** *Let  $M$  and  $N$  be free modules with finite rank over  $R$ , a commutative noetherian ring with identity. Then  $\text{rank}(M \otimes_R N) = \text{rank}(M) \cdot \text{rank}(N)$ .*

For the concept of rank see (9.1.15) and its sequel. Note that for a vector space rank equals dimension; thus if  $V$  and  $W$  are finite dimensional vector spaces over a field  $F$ , then  $V \otimes_F W$  is finite dimensional and  $\dim(V \otimes_F W) = \dim(V) \cdot \dim(W)$ .

**Tensor products of matrices.** We have seen how to form the tensor product of module homomorphisms in (13.1.3). The close connection between matrices and linear mappings suggests that there should be a corresponding way to form tensor products of matrices.

Let  $A$  and  $B$  be  $m \times n$  and  $p \times q$  matrices respectively over a field  $F$ . Then there are corresponding linear transformations  $\alpha : F^n \rightarrow F^m$  and  $\beta : F^q \rightarrow F^p$  defined by equations  $\alpha(X) = AX$  and  $\beta(Y) = BY$ . Let  $E_i^{(n)}$  denote the  $i$ th column of the  $n \times n$  identity matrix  $I_n$ . Thus  $\{E_i^{(n)} \mid i = 1, \dots, n\}$  is the standard basis of  $F^n$ . The linear transformation  $\alpha$  is represented with respect to the bases  $\{E_i^{(n)}\}$  and  $\{E_j^{(m)}\}$  by the matrix  $A$ . There is a similar statement for  $\beta$ .

By definition of the linear mapping  $\alpha \otimes \beta : F^n \otimes_F F^q \rightarrow F^m \otimes_F F^p$ ,

$$\alpha \otimes \beta(E_i^{(n)} \otimes E_j^{(q)}) = \alpha(E_i^{(n)}) \otimes \beta(E_j^{(q)}),$$

which equals

$$\sum_{k=1}^m a_{ki} E_k^{(m)} \otimes \sum_{\ell=1}^p b_{\ell j} E_\ell^{(p)} = \sum_{k=1}^m \sum_{\ell=1}^p a_{ki} b_{\ell j} (E_k^{(m)} \otimes E_\ell^{(p)}).$$

Now (13.2.7) shows that the  $E_i^{(n)} \otimes E_j^{(q)}$  form a basis for  $F^n \otimes_F F^q$ , as do the  $E_k^{(m)} \otimes E_\ell^{(p)}$  for  $F^m \otimes_F F^p$ . Let these bases be ordered lexicographically, i.e., by first subscript, then second subscript. With this choice of ordered bases we can read off the  $mp \times nq$  matrix  $M$  which represents the linear mapping  $\alpha \otimes \beta$ . The rows of  $M$  are labelled by the pairs  $[k, \ell]$ ,  $1 \leq k \leq m$ ,  $1 \leq \ell \leq p$ , and the columns by the pairs  $[i, j]$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq q$ . Therefore the  $([k, \ell], [i, j])$  entry of  $M$  is

$$a_{ki} b_{\ell j}.$$

The foregoing discussion suggests that we define the tensor product  $A \otimes B$  of  $A$  and  $B$  to be the  $mp \times nq$  matrix  $M$ . In essence the entries of  $A \otimes B$  are formed by taking

all possible products of an entry of  $A$  and an entry of  $B$ . Writing the matrix in block form, we obtain the more easily remembered formula

$$M = A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

The tensor product of matrices is sometimes called the *Kronecker product*.<sup>1</sup>

**Example (13.2.2).** Consider the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

The tensor product is

$$A \otimes B = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}.$$

**Right exactness of tensor products.** The section concludes with a discussion of the right exactness property of tensor products, a fundamental result that is used constantly in advanced work.

**(13.2.9)** Let  $M_R$  and  ${}_R N$  be modules over a ring  $R$ .

(i) Let  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  be an exact sequence of left  $R$ -modules. Then there is an exact sequence of abelian groups and induced homomorphisms

$$M \otimes_R A \xrightarrow{\alpha_*} M \otimes_R B \xrightarrow{\beta_*} M \otimes_R C \rightarrow 0.$$

(ii) Let  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  be an exact sequence of right  $R$ -modules. Then there is an exact sequence of abelian groups and induced homomorphisms

$$A \otimes_R N \xrightarrow{\alpha_*} B \otimes_R N \xrightarrow{\beta_*} C \otimes_R N \rightarrow 0.$$

*Proof.* Only (i) will be proved. The first step is to show that  $\beta_*$  is surjective. Let  $m \in M$  and  $c \in C$ . Since  $\beta$  is surjective,  $c = \beta(b)$  for some  $b \in B$ . Hence  $\beta_*(m \otimes b) = (\text{id}_M \otimes \beta)(m \otimes b) = m \otimes c$ .

<sup>1</sup> Leopold Kronecker (1823–1891)



$\beta)(m \otimes b) = m \otimes \beta(b) = m \otimes c$ . Since  $M \otimes_R C$  is generated by the tensors  $m \otimes c$ , it follows that  $\beta_*$  is surjective.

It remains to prove that  $\text{Im}(\alpha_*) = \text{Ker}(\beta_*)$ , which is harder. In the first place,  $\beta_* \alpha_* = (\beta \alpha)_* = 0_* = 0$  by Exercise (13.1.8), so that  $\text{Im}(\alpha_*) \subseteq \text{Ker}(\beta_*)$ . To establish the reverse inclusion we form the commutative triangle

$$\begin{array}{ccc} & C & \\ \beta \nearrow & & \nwarrow \gamma \\ B & \xrightarrow{\quad v \quad} & B/\text{Ker}(\beta) \end{array}$$

where  $v$  is the canonical homomorphism and  $\gamma$  is the isomorphism in the First Isomorphism Theorem which sends  $b + \text{Ker}(\beta)$  to  $\beta(b)$ . Commutativity of the diagram is easily checked, so  $\gamma v = \beta$ . This implies that  $\gamma_* v_* = (\gamma v)_* = \beta_*$ . Since  $\gamma$  is an isomorphism, so is the induced map  $\gamma_*$  and hence  $\text{Ker}(\beta_*) = \text{Ker}(v_*)$ .

Define

$$S = \langle m \otimes k \mid m \in M, k \in \text{Ker}(\beta) \rangle.$$

Then  $S = \text{Im}(\alpha_*)$  since  $\text{Ker}(\beta) = \text{Im}(\alpha)$ . Next  $S \subseteq \text{Ker}(v_*)$ ; for, if  $m \in M$  and  $k \in \text{Ker}(\beta)$ , we have  $v_*(m \otimes k) = m \otimes (k + \text{Ker}(\beta)) = m \otimes 0 = 0$ . Hence  $v_*$  induces a homomorphism  $\lambda : (M \otimes_R B)/S \rightarrow M \otimes_R (B/\text{Ker}(\beta))$  such that  $\lambda(u + S) = v_*(u)$  for  $u \in M \otimes_R B$ . Thus  $\lambda(m \otimes b + S) = v_*(m \otimes b) = m \otimes (b + \text{Ker}(\beta))$ . By (13.2.4) there is an isomorphism  $\theta : M \otimes (B/\text{Ker}(\beta)) \rightarrow (M \otimes_R B)/S$  such that  $\theta(m \otimes (b + \text{Ker}(\beta))) = m \otimes b + S$ . Notice that  $\theta$  and  $\lambda$  are mutually inverse maps, so  $\lambda = \theta^{-1}$  is an isomorphism. If  $u \in \text{Ker}(v_*)$ , then, since  $v_*$  induces  $\lambda$ , we have  $u + S \in \text{Ker}(\lambda) = 0_{M \otimes_R B/S}$  and  $u \in S$ . Hence  $\text{Ker}(v_*) \subseteq S$  and finally  $\text{Ker}(\beta_*) = \text{Ker}(v_*) \subseteq S = \text{Im}(\alpha_*)$ , so that  $\text{Ker}(\beta_*) = \text{Im}(\alpha_*)$ , as required.  $\square$

The right exactness property of tensor products should be compared with the left exactness of Hom in Chapter Nine – see (9.1.19). The “duality” between the tensor product and Hom indicated by (13.2.9) and (9.1.19) is just the beginning of a fundamental duality in homological algebra between homology and cohomology.

### Exercises (13.2)

- (1) Given a module  ${}_R N_S$  where  $R$  and  $S$  are rings and  $R$  has identity, prove that  $R \otimes_R N \stackrel{S}{\cong} N$ .
- (2) Simplify  $(\mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{Z}_{18}) \otimes (\mathbb{Q} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{24})$  as far as possible.
- (3) Show by an example that the tensor product does not have the left exactness property, i.e., if  $M$  is a right  $R$ -module and  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$  is an exact sequence of left  $R$ -modules, the induced sequence  $0 \rightarrow M \otimes_R A \xrightarrow{\alpha_*} M \otimes_R B \xrightarrow{\beta_*} M \otimes_R C$  is not exact in general. [Hint: apply  $\mathbb{Z}_2 \otimes -$  to the sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ .]
- (4) Let  $A$  and  $B$  be  $m \times m$  and  $n \times n$  matrices over a field. Prove that  $\det(A \otimes B) = (\det(A))^n (\det(B))^m$ . Deduce that the tensor product of non-singular matrices is non-singular. [Hint: define  $\bar{A}$  to be the  $mn \times mn$  block matrix whose  $(i, j)$  block is  $a_{ij} I_n$  and

let  $B^\sharp$  be the  $mn \times mn$  block matrix with  $B$  on the diagonal and 0 elsewhere. Show that  $A \otimes B = \bar{A}B^\sharp$ . Then take the determinant of both sides].

(5) Let  $Q, R, S$  be rings and  ${}_Q L_R, {}_R M_S, {}_S N$  modules as indicated. Prove that there is an isomorphism of left  $Q$ -modules  $\alpha : (L \otimes_R M) \otimes_S N \rightarrow L \otimes_R (M \otimes_S N)$ .

(6) Let  ${}_R N$  be a module over an arbitrary ring  $R$ . Suppose that  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  is an exact sequence of right  $R$ -modules. Prove that the sequence of abelian groups and induced homomorphisms

$$A \otimes_R N \xrightarrow{\alpha_*} B \otimes_R N \xrightarrow{\beta_*} C \otimes_R N \rightarrow 0$$

is exact.

(7) (Adjoint associativity). Let  $R$  and  $S$  be rings and  $A_R, {}_R B_S, C_S$  modules. (i) Explain why  $\text{Hom}_S(B, C)$  is a right  $R$ -module. (ii) Establish the isomorphism  $\text{Hom}_S(A \otimes_R B, C) \simeq \text{Hom}_R(A, \text{Hom}_S(B, C))$ .

### 13.3 Extending the ring of operators.

Suppose we have a module over a ring  $R$ : is there a way to make it into a module over a different ring  $S$ ? Of course the question is vague, but one situation in which this is possible is if a ring homomorphism  $\phi : S \rightarrow R$  is given. For, if  $M$  is a left  $R$ -module, a left action of  $S$  on  $M$  can be defined by the rule  $s \cdot a = \phi(s) \cdot a$  for  $s \in S$  and  $a \in M$ . The simple task of verifying the module axioms is left to the reader.

It is a less trivial exercise to go in the opposite direction: let  $M$  be a left  $R$ -module and let  $\phi : R \rightarrow S$  be a ring homomorphism. The question is: how can one associate a left  $S$ -module with  $M$ ? At this point tensor products come to our aid. First observe that  $S$  is a  $(S, R)$ -bimodule where the left action comes from the ring product in  $S$  and the right action of  $R$  on  $S$  is given by  $s \cdot r = s\phi(r)$ , ( $s \in S, r \in R$ ). Again verification of the module axioms is easy. Therefore we can form the tensor product  $S \otimes_R M$ , which is a left  $S$ -module by (13.1.4), and also a left  $R$ -module via  $\phi$ .

One can ask how the new  $R$ -module  $S \otimes_R M$  is related to the original module  $M$ . If  $S$  has an identity element, there is an obvious mapping

$$\theta : M \rightarrow S \otimes_R M$$

given by  $\theta(a) = 1_S \otimes a$ . Observe that  $\theta(r \cdot a) = 1_S \otimes (r \cdot a) = (1_S \cdot r) \otimes a = (1_S \phi(r)) \otimes a = (\phi(r) 1_S) \otimes a = (r \cdot 1_S) \otimes a = r \cdot (1_S \otimes a) = r \cdot (\theta(a))$ , where  $r \in R, a \in M$ . Therefore  $\theta$  is a homomorphism of left  $R$ -modules.

A case of particular interest is where  $\phi$  is injective, so that  $R$  is essentially a subring of  $S$ . In this circumstance we are extending the ring of operators on a module from the subring  $R$  to  $S$ . The interesting question is whether  $\theta$  is also injective. A detailed investigation of the problem would take us too far afield, so we will restrict ourselves to

the special, but important, case where  $R$  is a domain and  $\phi$  is the canonical injection from  $R$  into its field of fractions  $F$ . Thus we are trying to embed an  $R$ -module in an  $F$ -vector space.

**Tensor products and localizations.** Let  $R$  be a domain with field of fractions  $F$ . Recall from (6.3) that each element of  $F$  is a fraction over  $R$  with the form  $\frac{r_1}{r_2}$  where  $r_i \in R$  and  $r_2 \neq 0$ . Also there is an injective ring homomorphism  $\phi : R \rightarrow F$  in which  $r \mapsto \frac{r}{1}$ : this is by (6.3.10).

Assume now that  $M$  is a torsion-free  $R$ -module. We are interested in the mapping  $\psi : M \rightarrow F \otimes_R M$  where  $\psi(a) = 1_F \otimes a$ , the aim being to prove that it is injective. Before this can be done, a better understanding of  $F \otimes_R M$  is needed and for this purpose a "model" of this  $F$ -module will be constructed.

We start by forming the set

$$S = \{(a, r) \mid a \in M, 0 \neq r \in R\}$$

and then introduce a binary relation  $\sim$  on  $S$  by

$$(a, r) \sim (a', r') \Leftrightarrow r \cdot a' = r' \cdot a.$$

The motivation here is the rule for equality of two rational numbers. By a simple check  $\sim$  is an equivalence relation on  $S$ , but notice that for the transitive law to hold it is essential that  $M$  be torsion-free. The  $\sim$ -equivalence class of  $(a, r)$  will be written

$$\frac{a}{r}$$

and referred to as a *fraction* over  $R$ . Denote the set of all such fractions by  $R^{-1} \cdot M$ .

The plan is to turn  $R^{-1} \cdot M$  into an  $F$ -module by defining

$$\frac{a_1}{r_1} + \frac{a_2}{r_2} = \frac{r_2 \cdot a_1 + r_1 \cdot a_2}{r_1 r_2} \quad \text{and} \quad \left(\frac{r_1}{r_2}\right) \cdot \frac{a}{r} = \frac{r_1 \cdot a}{r_1 r_2}.$$

Since these are operations on equivalence classes, it is essential to verify that they are well defined, i.e., there is no dependence on the choice of elements  $(a_i, r_i)$  from  $\frac{a_i}{r_i}$  or  $(a, r)$  from  $\frac{a}{r}$ . All of this is routine calculation, which, as usual, we urge the reader to perform.

Then the module axioms must be checked. For example,

$$\left(\frac{r}{r'}\right) \cdot \left(\frac{a_1}{r_1} + \frac{a_2}{r_2}\right) = \left(\frac{r}{r'}\right) \cdot \left(\frac{r_2 \cdot a_1 + r_1 \cdot a_2}{r_1 r_2}\right) = \frac{rr_2 \cdot a_1 + rr_1 \cdot a_2}{r'r_1 r_2}.$$

Also

$$\left(\frac{r}{r'}\right) \cdot \left(\frac{a_1}{r_1}\right) + \left(\frac{r}{r'}\right) \cdot \left(\frac{a_2}{r_2}\right) = \frac{r \cdot a_1}{r'r_1} + \frac{r \cdot a_2}{r'r_2} = \frac{rr'r_2 \cdot a_1 + rr'r_1 \cdot a_2}{r'^2 r_1 r_2},$$

which is seen to equal the previous expression after allowing for cancellation of the common factor  $r'$  in the numerator and denominator.

The module  $R^{-1} \cdot M$  is called the *localization of  $M$* . The statement we are aiming for is next.

**(13.3.1)** *Let  $R$  be an integral domain and  $F$  its field of fractions. If  $M$  is a torsion-free  $R$ -module, then*

$$R^{-1} \cdot M \stackrel{F}{\cong} F \otimes_R M.$$

*Proof.* The assignment  $(\frac{r_1}{r_2}, a) \mapsto \frac{r_1 \cdot a}{r_2}$  yields a well defined  $R$ -middle linear mapping from  $F \times M$  to  $R^{-1} \cdot M$ . For example, the map sends  $(\frac{r_1}{r_2}, a_1 + a_2) - (\frac{r_1}{r_2}, a_1) - (\frac{r_1}{r_2}, a_2)$  to

$$\frac{r_1 \cdot (a_1 + a_2)}{r_2} - \frac{r_1 \cdot a_1}{r_2} - \frac{r_1 \cdot a_2}{r_2} = 0.$$

The other verifications are at a similar level of difficulty.

It follows that there is a homomorphism  $\pi : F \otimes_R M \rightarrow R^{-1} \cdot M$  such that  $\pi(\frac{r_1}{r_2} \otimes a) = \frac{r_1 \cdot a}{r_2}$ . Now check that  $\pi$  is an  $F$ -module homomorphism. Let  $r, r', r_i \in R, a \in M$ ; then

$$\pi\left(\frac{r}{r'} \cdot \left(\frac{r_1}{r_2} \otimes a\right)\right) = \pi\left(\frac{rr_1}{r'r_2} \otimes a\right) = \frac{rr_1 \cdot a}{r'r_2} = \frac{r}{r'} \cdot \left(\frac{r_1 \cdot a}{r_2}\right) = \frac{r}{r'} \cdot \left(\pi\left(\frac{r_1}{r_2} \otimes a\right)\right),$$

which is sufficient because  $F \otimes_R M$  is generated by the tensors  $\frac{r_1}{r_2} \otimes a$ .

Next define a mapping  $\psi : R^{-1} \cdot M \rightarrow F \otimes_R M$  by  $\psi(\frac{a}{r}) = \frac{1}{r} \otimes a$ . To show that  $\psi$  is well defined, suppose that  $(a, r) \sim (a', r')$ . Thus  $r \cdot a' = r' \cdot a$  and

$$\frac{1}{r} \otimes a = \frac{r'}{rr'} \otimes a = \frac{1}{rr'} \otimes (r' \cdot a) = \frac{1}{rr'} \otimes (r \cdot a') = \frac{r}{rr'} \otimes a' = \frac{1}{r'} \otimes a',$$

as required.

Finally,  $\pi$  and  $\psi$  are mutually inverse maps: for  $\pi\psi(\frac{a}{r}) = \pi(\frac{1}{r} \otimes a) = \frac{1 \cdot a}{r} = \frac{a}{r}$ , while

$$\psi\pi\left(\frac{r_1}{r_2} \otimes a\right) = \psi\left(\frac{r_1 \cdot a}{r_2}\right) = \frac{1}{r_2} \otimes (r_1 \cdot a) = \frac{r_1}{r_2} \otimes a.$$

Again this is sufficient since  $F \otimes_R M$  is generated by the tensors  $\frac{r_1}{r_2} \otimes a$ . Therefore  $\psi$  is an  $F$ -isomorphism.  $\square$

**Corollary (13.3.2)** *Let  $R$  be an integral domain and  $F$  its field of fractions. If  $M$  is a torsion-free  $R$ -module, the assignment  $a \mapsto 1 \otimes a$  determines an injective  $R$ -module homomorphism  $\theta : M \rightarrow F \otimes_R M$ .*

*Proof.* Assume that  $\theta(a) = 0$  for some  $a \in M$ , so that  $1 \otimes a = 0$ . Apply the isomorphism  $\pi$  in the proof of (13.3.1) to both sides of this equation to get  $\frac{a}{1} = \pi(1 \otimes a) = \frac{0}{1}$ , which implies that  $a = 0$ .  $\square$

This corollary provides some insight into the nature of torsion-free abelian groups, but first some terminology. A torsion-free abelian group  $A$  is said to have *finite rank* if it

has no infinite linearly independent subsets. In this event  $A$  must possess a maximal linearly independent subset  $\{a_1, a_2, \dots, a_r\}$  since otherwise there would exist infinite linearly independent subsets.

**(13.3.3)** *If  $A$  is a torsion-free abelian group, then  $A$  is isomorphic with a subgroup of the rational vector space  $V = \mathbb{Q} \otimes A$ . If  $A$  has finite rank, then  $V$  has finite dimension.*

*Proof.* Let  $\theta : A \rightarrow \mathbb{Q} \otimes A$  be the mapping  $a \rightarrow 1 \otimes a$ . By (13.3.2)  $\theta$  is an injective homomorphism, which proves the first statement. Now assume that  $A$  has finite rank and  $S = \{a_1, \dots, a_r\}$  is a maximal linearly independent subset of  $A$ . Then  $T = \theta(S)$  is linearly independent since  $\theta$  is injective. If  $\frac{r_1}{r_2} \otimes a$  is a typical tensor in  $V$  with  $r_i \in \mathbb{Z}$ ,  $a \in A$ , then  $r_2(\frac{r_1}{r_2} \otimes a) = r_1(1 \otimes a) \in \langle T \rangle$ , which shows that every element of  $\mathbb{Q} \otimes A$  is a  $\mathbb{Q}$ -linear combination of elements of  $T$  and consequently that  $T$  is a  $\mathbb{Q}$ -basis for  $V$ . Hence  $\dim_{\mathbb{Q}}(V) = r$  is finite.  $\square$

Notice that the proof shows that all maximal linearly independent subsets of  $A$  have the same number of elements, namely  $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes A)$ .

While (13.3.3) provides a familiar setting for torsion-free abelian groups of finite rank, in the sense that they “live” inside finite dimensional rational vector spaces, this placement does not materially advance the classification of these groups. In fact torsion-free abelian groups of finite rank can have extremely complex structure, far beyond that of finitely generated abelian groups: the standard reference for infinite abelian groups is [5].

### Exercises (13.3)

(1) Let  $F$  be a subfield of a field  $K$  and let  $V$  be an  $n$ -dimensional vector space over  $F$ . Prove that  $V \otimes_F K$  is an  $n$ -dimensional vector space over  $K$ .

[In the exercises that follow  $R$  is a domain with field of fractions  $F$  with  $R \subseteq F$ , and  $M$  is an  $R$ -module].

(2) Prove that the module operations specified for  $R^{-1} \cdot M$  are well defined.

(3) Prove that every element of  $F \otimes_R M$  has the form  $\frac{1}{r} \otimes a$  where  $r \in R$ ,  $a \in M$ .

(4) Let  $T$  denote the torsion submodule of  $M$ .

(i) Prove that  $F \otimes_R T = 0$ .

(ii) Prove that  $F \otimes_R M \stackrel{R}{\cong} F \otimes_R (M/T)$ . [Hint: start with the exact sequence  $0 \rightarrow T \rightarrow M \rightarrow M/T \rightarrow 0$  and apply the right exactness property of tensor products].

(5) (The flatness property of  $F$ ). Let  $\alpha : A \rightarrow B$  be an injective  $R$ -module homomorphism. Prove that the induced map  $\alpha_* : F \otimes_R A \rightarrow F \otimes_R B$  is also injective. [Hint: by Exercise (13.3.4)  $A$  and  $B$  can be assumed to be torsion-free. Form the commutative square with horizontal sides  $A \xrightarrow{\alpha} B$  and  $F \otimes_R A \xrightarrow{\alpha_*} F \otimes_R B$ , and vertical sides the canonical maps

$A \rightarrow F \otimes_R A$  and  $B \rightarrow F \otimes_R B$ . The vertical maps are injective by (13.3.2). Argue that the lower horizontal map is also injective by using Exercise (13.3.3)].

(6) If  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  is an exact sequence of  $R$ -modules, show that the induced sequence  $0 \rightarrow F \otimes_R A \xrightarrow{\alpha_*} F \otimes_R B \xrightarrow{\beta_*} F \otimes_R C \rightarrow 0$  is also exact.

(7) Prove that  $F \otimes_R F \stackrel{R}{\cong} F$ . [Hint: apply  $F \otimes_R -$  to the exact sequence  $0 \rightarrow R \rightarrow F \rightarrow F/R \rightarrow 0$ ].

## 14 Further topics

The final chapter begins with an account of Zorn's<sup>1</sup> Lemma and includes such critical applications as the existence of a basis in a vector space and the algebraic closure of a field. It also gives an introduction to free groups and presentations of groups, as well as to coding theory, an important recent application of algebra. The other sections tie up some loose ends left over from earlier chapters.

### 14.1 Zorn's Lemma with applications

The background to Zorn's Lemma lies in the kind of set theory that is being used. Up to this point we have been functioning – quite naively – in what is called the *Gödel–Bernays Theory*. In this the primitive, or undefined, notions are *class*, *membership* and *equality*. On the basis of these concepts and the accompanying axioms, the usual elementary properties of sets can be derived. In addition we have made extensive use of the Well Ordering Axiom for  $\mathbb{Z}$ , and its corollary the Principle of Mathematical Induction – see (2.1).

However, the set theory just described does not provide an adequate basis for dealing with infinite sets. For example, suppose that  $H$  is a subgroup of infinite index in a group  $G$ . We might wish to form a left transversal to  $H$  in  $G$ . This would involve making a simultaneous choice of one element from each of the infinitely many left cosets of  $H$ . That such a choice is possible is asserted by the well-known *Axiom of Choice*. However, this axiom is known to be independent of the Gödel–Bernays axioms. Thus, in order to be able to form left transversals in infinite groups, we must assume the Axiom of Choice or else something equivalent to it. For many purposes in algebra the most useful additional axiom is what has become known as *Zorn's Lemma*. Despite the name, this is an axiom that must be assumed and not a lemma.

**Zorn's Lemma.** *Let  $(S, \leq)$  be a non-empty partially ordered set with the property that every chain in  $S$  has an upper bound in  $S$ . Then  $S$  contains at least one maximal element.*

The terminology here calls for some explanation. Recall from (1.2) that a *chain* in the partially ordered set  $S$  is a subset  $C$  which is linearly ordered by the partial order  $\leq$ . An *upper bound* for  $C$  is an element  $s$  of  $S$  such that  $c \leq s$  is valid for all  $c$  in  $C$ . Finally, a *maximal element* of  $S$  is an element  $m$  such that  $m \leq s \in S$  implies that  $m = s$ . Note that in general a partially ordered set may contain several maximal elements or none at all.

We will now demonstrate how Zorn's Lemma can be used to prove a number of critical theorems in algebra.

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<sup>1</sup> Max August Zorn (1906-1993)

**Existence of a basis in a vector space.** It was shown in Chapter Eight that every finitely generated non-zero vector space has a basis – see (8.2.7). Zorn's Lemma can be used to extend this fundamental result to infinitely generated vector spaces.

**(14.1.1)** *Every non-zero vector space has a basis.*

*Proof.* Let  $V$  be a non-zero vector space over a field  $F$  and define  $\mathcal{S}$  to be the set of all linearly independent subsets of  $V$ . The set  $\mathcal{S}$  is non-empty since it contains the singleton set  $\{v\}$  where  $v$  is any non-zero vector in  $V$ . Furthermore, inclusion is a partial order on  $\mathcal{S}$ , so  $(\mathcal{S}, \subseteq)$  is a partially ordered set. To apply Zorn's Lemma, we need to verify that every chain in  $\mathcal{S}$  has an upper bound.

Let  $\mathcal{C}$  be a chain in  $\mathcal{S}$ . There is an obvious candidate for an upper bound, namely the union  $U = \bigcup_{X \in \mathcal{C}} X$ . Certainly  $U$  is linearly independent: for any relation of linear dependence in  $U$  will involve a *finite* number of elements of  $\mathcal{S}$  and so the relation will hold in some  $X \in \mathcal{C}$ . Here it is vital that  $\mathcal{C}$  be linearly ordered. Thus  $U \in \mathcal{S}$  and obviously  $U$  is an upper bound for  $\mathcal{C}$ .

It is now possible to apply Zorn's Lemma to obtain a maximal element in  $\mathcal{S}$ , say  $B$ . By definition  $B$  is linearly independent: to show that  $B$  is a basis we must prove that  $B$  generates  $V$ . Assume this is false and let  $v$  be a vector in  $V$  that is not expressible as a linear combination of vectors in  $B$ ; then certainly  $v \notin B$  and hence  $B$  is a proper subset of  $\{v\} \cup B = B'$ . By maximality of  $B$ , the set  $B'$  does not belong to  $\mathcal{S}$  and hence it is linearly dependent. Therefore there is a linear relation  $a_1u_1 + a_2u_2 + \cdots + a_mu_m + cv = 0$  where  $u_i \in B$  and  $c, a_i \in F$ , with not all the coefficients being zero. If  $c = 0$ , then  $a_1u_1 + a_2u_2 + \cdots + a_mu_m = 0$ , so that  $a_1 = a_2 = \cdots = a_m = 0$  since  $u_1, u_2, \dots, u_m$  are linearly independent. Therefore  $c \neq 0$  and we can solve the equation for  $v$ , obtaining

$$v = (-c^{-1}a_1)u_1 + (-c^{-1}a_2)u_2 + \cdots + (-c^{-1}a_m)u_m,$$

which contradicts the choice of  $v$ . Hence  $B$  generates  $V$ . □

In fact any two bases of a vector space have the same cardinal, so that it is possible to define the dimension of an infinitely generated vector space to be this cardinal. The proof below requires some facts from cardinal arithmetic.

**(14.1.2)** *Let  $X$  and  $Y$  be bases of a vector space  $V$  over a field  $F$ . Then  $|X| = |Y|$ .*

*Proof.* In the first place, if one basis is finite, the result follows from (8.2.8), so we may assume that both  $X$  and  $Y$  are infinite. Let  $\mathcal{P}_f(Y)$  denote the set of all *finite* subsets of  $Y$ . Define a function  $\alpha : X \rightarrow \mathcal{P}_f(Y)$  as follows. If  $x \in X$  and  $x = k_1y_1 + \cdots + k_ny_n$  with distinct  $y_i \in Y$  and  $0 \neq k_i \in F$ , define  $\alpha(x) = \{k_1, \dots, k_n\}$ . The function  $\alpha$  is not injective, so we will modify it.

Let  $T = \{y_1, y_2, \dots, y_n\} \subseteq Y$ , so that  $T \in \mathcal{P}_f(Y)$ . We claim that  $\bar{T} = \{x \in X \mid \alpha(x) \in T\}$  is finite. For if not, there are infinitely many elements of  $X$  that are linear combinations of  $y_1, y_2, \dots, y_n$ , contradicting (8.2.3). Let the elements of each  $\bar{T}$  be linearly ordered



in some way. Next define a function  $\beta : X \rightarrow \text{Im}(\alpha) \times \mathbb{N}$  by the rule  $\beta(x) = (T, m)$  where  $\alpha(x) = T$  – so that  $x \in \tilde{T}$  – and  $x$  is the  $m$ th element of  $\tilde{T}$ . It is clear that  $\beta$  is injective. Therefore

$$|X| \leq |\text{Im}(\alpha) \times \mathbb{N}| \leq |P_f(Y)| \cdot \aleph_0 = |Y| \cdot \aleph_0 = |Y|.$$

In a similar way  $|Y| \leq |X|$ , so by the Cantor-Bernstein Theorem (1.4.2) we arrive at  $|X| = |Y|$ .  $\square$

In the foregoing proof we used two facts about cardinals: (i)  $|\mathcal{P}_f(Y)| = |Y|$  – cf. Exercise (1.4.8); (ii)  $|Y| \cdot \aleph_0 = |Y|$  if  $Y$  is infinite. For the latter statement see for example [8].

**Maximal ideals in rings.** Recall from (6.3) that a maximal ideal  $I$  of a ring  $R$  is a largest proper ideal. If  $R$  is commutative and has an identity, then by (6.3.7) this is equivalent to  $I$  being an ideal such that  $R/I$  is a field. Maximal left ideals and maximal right ideals are defined in a similar manner. Maximal ideals were used in (7.4) to construct fields, but only in circumstances where it was clear that they existed, for example when the ascending chain on ideals held. Zorn's Lemma can be used to produce maximal ideals under more general circumstances.

**(14.1.3)** *An arbitrary ring  $R$  with identity has at least one maximal ideal.*

*Proof.* Let  $\mathcal{S}$  denote the set of all proper ideals of  $R$ . Now the zero ideal is proper since it does not contain  $1_R$ , so  $\mathcal{S}$  is not empty. Of course,  $\mathcal{S}$  is partially ordered by inclusion. Let  $\mathcal{C}$  be a chain in  $\mathcal{S}$  and define  $U$  to be  $\bigcup_{I \in \mathcal{C}} I$ . It is easily seen that  $U$  is an ideal. If  $U = R$ , then  $1_R$  belongs to some  $I$  in  $\mathcal{C}$ , from which it follows that  $R = RI \subseteq I$  and  $I = R$ . From this contradiction we infer that  $U \neq R$ , so that  $U \in \mathcal{S}$ . Now Zorn's Lemma can be applied to produce a maximal element of  $\mathcal{S}$ , i.e., a maximal ideal of  $R$ .  $\square$

In a similar manner one can show that a ring with identity has a maximal left ideal and a maximal right ideal. An immediate consequence of (14.1.3) is:

**(14.1.4)** *If  $R$  is a commutative ring with identity, it has a quotient ring which is a field.*

On the other hand, not every commutative ring has a maximal ideal.

**Example (14.1.1)** There exist non-zero commutative rings without maximal ideals.

An easy way to get an example is to take the additive abelian group  $\mathbb{Q}$  and turn it into a ring by declaring all products to be 0. Then  $\mathbb{Q}$  becomes a commutative ring in which subgroups and ideals are the same. But  $\mathbb{Q}$  cannot have a maximal subgroup: for if  $S$  were one,  $\mathbb{Q}/S$  would be a group without proper non-trivial subgroups and so  $|\mathbb{Q}/S| = p$ , a prime. But this is impossible since  $\mathbb{Q} = p\mathbb{Q}$ . It follows that the ring has no maximal ideals.

A noteworthy application of (14.1.4) is to generalize (9.1.15) and (14.1.2) to free modules over arbitrary commutative rings with identity.

**(14.1.5)** Let  $R$  be a commutative ring with identity and let  $M$  be a free  $R$ -module. If  $X$  and  $Y$  are bases of  $M$ , then  $|X| = |Y|$ .

*Proof.* By (14.1.4) there is a quotient  $K$  of  $R$  which is a field. The argument of (9.1.15) can now be applied with an appeal to (14.1.2) at the appropriate point.  $\square$

**The existence of algebraic closures.** Another important application of Zorn's Lemma is to show that for every field  $F$  there is a largest algebraic extension, its *algebraic closure*. The construction of such a largest extension is the kind of task to which Zorn's Lemma is well-suited.

Let  $E$  be a field extension of  $F$  with  $F \subseteq E$ . Then  $E$  is called an *algebraic closure* of  $F$  if the following conditions hold:

- (i)  $E$  is algebraic over  $F$ ;
- (ii) every irreducible polynomial in  $E[t]$  has degree 1.

Notice that by the second condition if  $K$  is an algebraic extension of  $E$ , then  $K = E$ , so that  $E$  is a maximal algebraic extension of  $F$ . A field that coincides with its algebraic closure is called an *algebraically closed field*. For example, the complex field  $\mathbb{C}$  is algebraically closed by the Fundamental Theorem of Algebra (12.3.6).

Our objective is to prove the following theorem:

**(14.1.6)** Every field has an algebraic closure.

*Proof.* Let  $F$  be an arbitrary field. The first step is to choose a set that is large enough to accommodate the algebraic closure. In fact what is needed is a set  $S$  with cardinal greater than  $\aleph_0 \cdot |F|$ ; for example the set  $\mathcal{P}(\mathbb{N} \times F)$  will do – see (1.4.5). In particular  $|F| < |S|$ , so there is an injection  $\alpha : F \rightarrow S$ . Now use the map  $\alpha$  to turn  $\text{Im}(\alpha)$  into a field, by defining

$$\alpha(x) + \alpha(y) = \alpha(x + y) \quad \text{and} \quad \alpha(x)\alpha(y) = \alpha(xy)$$

where  $x, y \in F$ , and  $\alpha(0_F)$  and  $\alpha(1_F)$  are the zero element and identity element respectively. Clearly  $\text{Im}(\alpha)$  is a field isomorphic with  $F$ . Thus, replacing  $F$  by  $\text{Im}(\alpha)$ , we may assume that  $F \subseteq S$ .

To apply Zorn's Lemma we need to introduce a suitable partially ordered set. Let  $\mathcal{K}$  denote the set of all subsets  $E$  such that  $F \subseteq E \subseteq S$  and the field operations of  $F$  may be extended to  $E$  in such a way that  $E$  becomes a field which is algebraic over  $F$ . Quite obviously  $F \in \mathcal{K}$ , so that  $\mathcal{K}$  is not empty. A partial order  $\leq$  on  $\mathcal{K}$  is defined as follows: if  $E_1, E_2 \in \mathcal{K}$ , then  $E_1 \leq E_2$  means that  $E_1 \subseteq E_2$  and the field operations of  $E_2$  are consistent with those of  $E_1$ . So  $E_1$  is actually a subfield of  $E_2$ . It is quite easy to see that  $\leq$  is a partial order on  $\mathcal{K}$ . Thus we have our partially ordered set  $(\mathcal{K}, \leq)$ .

Next the union  $U$  of a chain  $\mathcal{C}$  in  $\mathcal{K}$  is itself in  $\mathcal{K}$ . For, by the definition of the partial order  $\leq$ , the field operations of all members of  $\mathcal{C}$  are consistent, so they may be combined to give the field operations of  $U$ . It follows that  $U \in \mathcal{K}$  and clearly  $U$  is an

upper bound for  $\mathcal{C}$  in  $\mathcal{K}$ . Zorn's Lemma may now be applied to yield a maximal element of  $\mathcal{K}$ , say  $E$ .

By definition  $E$  is algebraic over  $F$ . What needs to be established is that any irreducible polynomial  $f$  in  $E[t]$  has degree 1. Suppose that in fact  $\deg(f) > 1$ . Put  $E' = E[t]/(f)$ , which is an algebraic extension of  $E$  and hence of  $F$  by (11.1.8). If we write  $E_0 = \{a + (f) \mid a \in E\}$ , then  $E_0 \subseteq E'$  and there is an isomorphism  $\beta : E_0 \rightarrow E$  given by  $\beta(a + (f)) = a$ .

It is at this point that the cardinality of the set  $S$  is important. One can show without too much trouble that  $|E' - E_0| < |S - E|$ , by using the inequalities  $|E| \leq \aleph_0 \cdot |F|$  and  $|E[t]| < |S|$ . Accepting this fact, we can choose an injective map  $\beta_1 : E' - E_0 \rightarrow S - E$ . Combine  $\beta_1$  with  $\beta : E_0 \rightarrow E$  to produce an injection  $\gamma : E' \rightarrow S$ . Thus  $\gamma(a + (f)) = a$  for  $a$  in  $E$ .

Next we use the map  $\gamma$  to make  $J = \text{Im}(\gamma)$  into a field, by defining  $\gamma(x_1) + \gamma(x_2) = \gamma(x_1 + x_2)$  and  $\gamma(x_1)\gamma(x_2) = \gamma(x_1x_2)$ . Then  $\gamma : E' \rightarrow J$  is an isomorphism of fields and  $\gamma(E_0) = E$ . Since  $E'$  is algebraic over  $E_0$ , it follows that  $J$  is algebraic over  $E$  and therefore  $J \in \mathcal{K}$ . However,  $E \neq J$  since  $E_0 \neq E'$ , which contradicts the maximality of  $E$  and completes the proof.  $\square$

While some details in the above proof may seem tricky, the essential idea is clear: build a largest algebraic extension of  $F$  by using Zorn's Lemma. It can be shown, although we shall not do so here, that *every field has a unique algebraic closure up to isomorphism* – see [8] for a proof.

For example, the algebraic closure of  $\mathbb{R}$  is  $\mathbb{C}$ , while the algebraic closure of  $\mathbb{Q}$  is the field of all algebraic numbers. Another example of interest is the algebraic closure of the Galois field  $\text{GF}(p)$ , which is an algebraically closed field of prime characteristic  $p$ .

As a final illustration of the power of Zorn's Lemma, we will prove a result on cardinal numbers which was stated without proof in (1.4).

**(14.1.7)** (The Law of Trichotomy) *If  $A$  and  $B$  are sets, then exactly one of the following must hold,*

$$|A| < |B|, \quad |A| = |B|, \quad |B| < |A|.$$

*Proof.* Because of the Cantor–Bernstein Theorem (1.4.2), it is enough to prove that either  $|A| \leq |B|$  or  $|B| \leq |A|$  holds. Clearly  $A$  and  $B$  can be assumed non-empty.

Consider the set  $\mathcal{F}$  of all pairs  $(X, \alpha)$  where  $X \subseteq A$  and  $\alpha : X \rightarrow B$  is an injective function. A partial order  $\leq$  on  $\mathcal{F}$  is defined by  $(X, \alpha) \leq (X', \alpha')$  if  $X \subseteq X'$  and  $\alpha'|_X = \alpha$ . It is obvious that  $\mathcal{F}$  is not empty. Let  $\mathcal{C} = \{(X_i, \alpha_i) \mid i \in I\}$  be a chain in  $\mathcal{F}$ . Put  $U = \bigcup_{i \in I} X_i$  and define  $\alpha : U \rightarrow B$  by extending the  $\alpha_i$ , which are consistent functions, to  $U$ . Then  $(U, \alpha)$  is an upper bound for  $\mathcal{C}$  in  $\mathcal{F}$ .

We can now apply Zorn's Lemma to obtain a maximal element  $(X, \alpha)$  of  $\mathcal{F}$ . We claim that either  $X = A$  or  $\text{Im}(\alpha) = B$ . For suppose that both statements are false, and let

$a \in A - X$  and  $b \in B - \text{Im}(\alpha)$ . Put  $Y = X \cup \{a\}$  and define  $\beta : Y \rightarrow B$  by  $\beta(a) = b$  and  $\beta|_X = \alpha$ . Then  $\beta$  is injective since  $b \notin \text{Im}(\alpha)$ , and clearly  $(\alpha, X) \preceq (\beta, Y)$ , which is a contradiction. Therefore, either  $X = A$  and hence  $|A| \leq |B|$  by definition of the linear ordering of cardinals, or else  $\text{Im}(\alpha) = B$ . In the latter case for each  $b$  in  $B$  choose an  $a_b$  in  $A$  such that  $\alpha(a_b) = b$ : then the map  $b \mapsto a_b$  is an injection from  $B$  to  $A$  and therefore  $|B| \leq |A|$ .  $\square$

**Axioms equivalent to Zorn's Lemma.** We mention in passing three axioms that are logically equivalent to Zorn's Lemma.

(i) *The Axiom of Well-Ordering.* Every non-empty set can be well-ordered.

Recall from (1.2) that a *well order* on a set is a linear order such that each non-empty subset has a first element. Compare the Axiom of Well-Ordering with the Well-Ordering Law in (2.1), which implies that  $\leq$  is a well-order on  $\mathbb{N}$ .

(ii) *The Principle of Transfinite Induction.* Let  $S$  be a well-ordered set and  $T$  a non-empty subset of  $S$ . Let  $t \in S$  and assume that  $t \in T$  whenever it is true that  $x \in T$  for all  $x$  in  $S$  such that  $x < t$ . Then  $T = S$ .

This result, which is the basis for the method of *proof by transfinite induction*, should be compared with the Principal of Mathematical Induction (2.1.1).

(iii) *The Axiom of Choice.* Let  $\{S_i \mid i \in I\}$  be a non-empty set whose elements  $S_i$  are non-empty sets. Then there is at least one choice function  $\alpha : I \rightarrow \bigcup_{i \in I} S_i$ , i.e., a function  $\alpha$  such that  $\alpha(i) \in S_i$ .

Informally we may express this by saying that it is possible to choose an element simultaneously from every set  $S_i$ . For a clear account of the equivalence of these axioms see [7].

### Exercises (14.1)

- (1) Let  $R$  be a ring with identity. Prove that  $R$  has a maximal left ideal and a maximal right ideal. Use these results to construct simple left and right  $R$ -modules.
- (2) Let  $R$  be a commutative ring and let  $0 \neq r \in R$ . Prove that there is an ideal  $I$  which is maximal subject to not containing  $r$ . Then prove that  $I$  is an *irreducible ideal*, i.e., it is not the intersection of two larger ideals.
- (3) Deduce from Exercise (14.1.2) that every proper ideal of a commutative ring  $R$  is an intersection of irreducible ideals. Interpret this result when  $R$  is a PID.
- (4) Let  $G$  be a non-trivial finitely generated group. Prove that  $G$  has at least one maximal subgroup. Deduce that  $\phi(G) \neq G$  where  $\phi(G)$  is the Frattini subgroup of  $G$ .
- (5) Let  $G$  be a group, let  $X$  be a subset and let  $g$  be an element of  $G$  such that  $g \notin X$ . Prove that there is a subgroup  $H$  which is maximal subject to  $X \subseteq H$  and  $g \notin H$ .

- (6) Generalize (10.2.8) by showing that in an arbitrary group  $G$  the Frattini subgroup  $\phi(G)$  consists of all non-generators. [Hint: let  $g \in \phi(G)$  and assume that  $G = \langle g, X \rangle$ , but  $G \neq \langle X \rangle$ . Apply Exercise (14.1.5)].
- (7) Let  $G$  be an arbitrary group and  $p$  a prime. Show that  $G$  has a *maximal  $p$ -subgroup*, i.e., a subgroup which is maximal subject to every element having order a power of a prime  $p$ . Then prove that the maximal  $p$ -subgroups of a finite group are just the Sylow  $p$ -subgroups.
- (8) Let  $P$  be a prime ideal of  $R$ , a commutative ring with identity. Prove that there is a largest prime ideal  $Q$  containing  $P$ . Then show that  $Q$  is a maximal ideal.
- (9) Let  $v$  be a non-zero element in a vector space  $V$ . Prove that there is a linear operator  $\alpha$  on  $V$  such that  $\alpha(v) \neq 0$ .

## 14.2 Roots of polynomials and discriminants

In this section we will complete certain topics that were begun in Chapter Twelve. In particular, the concept of the *discriminant* of a polynomial is introduced and this is applied to the Galois groups of polynomials of degree  $\leq 4$ .

**The discriminant of a polynomial.** Let  $f$  be a non-constant monic polynomial in  $t$  over a field  $F$  and let  $n = \deg(f)$ . Let the roots of  $f$  in its splitting field  $E$  be  $a_1, a_2, \dots, a_n$  and define

$$\Delta = \prod_{i < j=1}^n (a_i - a_j),$$

which is an element of  $E$ . Note that  $\Delta$  depends on the order in which the roots are written, so it is only determined up to sign. Also  $f$  has all its roots distinct if and only if  $\Delta \neq 0$ ; let us assume this to be the case. Thus  $E$  is Galois over  $F$ .

If  $\alpha \in \text{Gal}(f) = \text{Gal}(E/F)$ , then  $\alpha$  permutes the roots  $a_1, a_2, \dots, a_n$ , and  $\alpha(\Delta) = \pm\Delta$ . Indeed  $\alpha(\Delta) = \Delta$  precisely when  $\alpha$  produces an *even* permutation of the  $a_i$ 's. Thus in any event  $\alpha$  fixes

$$D = \Delta^2.$$

The element  $D$  is called the *discriminant* of  $f$ : it is independent of the order of the roots of  $f$ . Since  $D$  is fixed by every automorphism  $\alpha$  and  $E$  is Galois over  $F$ , it follows from (12.2.6) that  $D$  belongs to  $F$ . The question arises as to how  $D$  is related to the coefficients of the original polynomial  $f$ .

**(14.2.1)** Let  $f$  be a non-constant polynomial over a field  $F$ . Then the discriminant  $D$  of  $f$  is expressible as a polynomial in the coefficients of  $f$ .

*Proof.* It can be assumed that  $f$  is monic and that it has distinct roots  $a_1, a_2, \dots, a_n$  since otherwise  $D = 0$ . Then  $f = (t - a_1)(t - a_2) \cdots (t - a_n)$ , so that

$$f = t^n - s_1 t^{n-1} + s_2 t^{n-2} - \cdots + (-1)^n s_n$$

where  $s_1, s_2, \dots, s_n$  are the elementary symmetric functions of degree 1, 2,  $\dots$ ,  $n$  in  $a_1, a_2, \dots, a_n$ . Now  $D = \prod_{i < j=1}^n (a_i - a_j)^2$  is obviously a symmetric function of  $a_1, a_2, \dots, a_n$ . By the Symmetric Function Theorem (12.4.7),  $D$  is expressible as a polynomial in  $s_1, s_2, \dots, s_n$ , i.e., in the coefficients of  $f$ .  $\square$

Next we examine the discriminants of polynomials of degrees 2 and 3 over a field.

**Example (14.2.1)** Let  $f = t^2 + ut + v$ . If the roots of  $f$  are  $a_1$  and  $a_2$ , then  $\Delta = (a_1 - a_2)^2$ . This can be rewritten in the form  $D = (a_1 + a_2)^2 - 4a_1a_2$ . Now clearly  $u = -(a_1 + a_2)$  and  $v = a_1a_2$ , so we arrive at the familiar formula for the discriminant of the quadratic  $t^2 + ut + v$ ,

$$D = u^2 - 4v.$$

**Example (14.2.2)** Consider a cubic polynomial  $f = t^3 + ut^2 + vt + w$  and let  $a_1, a_2, a_3$  be its roots. Then  $D = (a_1 - a_2)^2(a_2 - a_3)^2(a_1 - a_3)^2$ . Also  $u = -(a_1 + a_2 + a_3)$ ,  $v = a_1a_2 + a_2a_3 + a_1a_3$  and  $w = -a_1a_2a_3$ . By a rather laborious calculation we can expand  $D$  and write it in terms of the elements  $u, v, w$ . What emerges is the formula

$$D = u^2v^2 - 4v^3 - 4u^3w - 27w^2 + 18uvw.$$

This expression can be simplified by a judicious change of variable. Put  $t' = t + \frac{1}{3}u$ , so that  $t = t' - \frac{1}{3}u$ . On substituting for  $t$  in  $f = t^3 + ut^2 + vt + w$ , we find that  $f = t'^3 + pt' + q$  where  $p = v - \frac{1}{3}u^2$  and  $q = w + \frac{2}{27}u^3 - \frac{1}{3}uv$ . Hence no generality is lost in assuming that  $f$  does not have a term in  $t^2$  and

$$f = t^3 + pt + q.$$

Now the formula for the discriminant reduces to the more manageable expression

$$D = -4p^3 - 27q^2.$$

Next we relate the discriminant to the Galois group of a polynomial.

**(14.2.2)** Let  $F$  be a field whose characteristic is not 2 and let  $f$  be a monic polynomial in  $F[t]$  with distinct roots  $a_1, a_2, \dots, a_n$ . Write  $\Delta = \prod_{i < j=1}^n (a_i - a_j)^2$ . If  $G = \text{Gal}(f)$  is identified with a subgroup of  $S_n$ , then  $\text{Fix}(G \cap A_n) = F(\Delta)$ .

*Proof.* Let  $H = G \cap A_n$  and note that  $H \triangleleft G$  and  $|G : H| \leq 2$ . If  $E$  is the splitting field of  $f$ , then  $F \subseteq F(\Delta) \subseteq \text{Fix}(H) \subseteq E$  since elements of  $H$ , being even permutations, fix  $\Delta$ . Now  $E$  is Galois over  $F$ , so we have

$$(F(\Delta) : F) \leq (\text{Fix}(H) : F) = |G : H| \leq 2.$$

If  $H = G$ , it follows that  $F = F(\Delta) = \text{Fix}(H)$  and  $\Delta \in F$ . The statement is therefore true in this case.

Now suppose that  $|G : H| = 2$  and let  $\alpha \in G - H$ . Then  $\alpha(\Delta) = -\Delta$  as  $\alpha$  is odd. Since  $\text{char}(F) \neq 2$ , we have  $\Delta \neq -\Delta$  and hence  $\Delta \notin F$ . Therefore  $(F(\Delta) : F) = 2$  and  $\text{Fix}(H) = F(\Delta)$ .  $\square$

**Corollary (14.2.3)** *With the above notation,  $\text{Gal}(f) \leq A_n$  if and only if  $\Delta \in F$ .*

These ideas will now be applied to investigate the Galois groups of polynomials of low degree.

**Polynomials of degree at most 4.** Let  $F$  be a field such that  $\text{char}(F) \neq 2$ .

(i) Consider a quadratic  $f = t^2 + ut + v \in F[t]$ . Then  $|\text{Gal}(f)| = 1$  or  $2$ . By (14.2.3)  $|\text{Gal}(f)| = 1$  precisely when  $\Delta \in F$ , i.e.,  $\sqrt{u^2 - 4v} \in F$ . This is the familiar condition for  $f$  to have both its roots in  $F$ . Of course  $|\text{Gal}(f)| = 2$  if and only if  $\Delta \notin F$ , which is the irreducible case.

(ii) Next let  $f$  be the cubic  $t^3 + pt + q \in F[t]$ . We saw that

$$\Delta = \sqrt{D} = \sqrt{-4p^3 - 27q^2}.$$

If  $f$  is reducible over  $F$ , it must have a quadratic factor  $f_1$  and clearly  $\text{Gal}(f) = \text{Gal}(f_1)$ , which has order 1 or 2. Thus we can assume  $f$  is irreducible. We know from (12.2.3) that  $\text{Gal}(f) \leq S_3$ , and that  $|\text{Gal}(f)|$  is divisible by 3 since it acts transitively on the roots of  $f$ . Hence  $\text{Gal}(f) = A_3$  or  $S_3$ . By (14.2.3)  $\text{Gal}(f) = A_3$  if and only if  $\Delta \in F$ ; otherwise  $\text{Gal}(f) = S_3$ .

(iii) Finally, let  $f$  be a monic polynomial of degree 4 in  $F[t]$ . If  $f$  is reducible and  $f = f_1 f_2$  with  $\deg(f_i) \leq 3$ , then  $\text{Gal}(f)$  is isomorphic with a subgroup of  $\text{Gal}(f_1) \times \text{Gal}(f_2)$ , (see Exercise (12.2.5)). The structure of  $\text{Gal}(f_i)$  is known from (i) and (ii), so assume that  $f$  is irreducible. Then  $\text{Gal}(f) \leq S_4$  and 4 divides  $|\text{Gal}(f)|$ . The subgroups of  $S_4$  whose orders are divisible by 4 are  $\mathbb{Z}_4$ ,  $V$  (the Klein 4-group),  $\text{Dih}(8)$ ,  $A_4$  and  $S_4$ ; thus  $\text{Gal}(f)$  must be one of these. In fact all five cases can occur, although we will not prove this here.

Explicit formulas for the roots of cubic and quartic equations over  $\mathbb{R}$  were found in the early 16th century by Scipione del Ferro (1465–1526), Gerolamo Cardano (1501–1576), Niccolo Tartaglia (1499–1526) and Lodovico Ferrari (1522–1565). An interesting account of their discoveries and of the mathematical life of the times can be found in [17].

### Exercises (14.2)

- (1) Find the Galois groups of the following quadratic polynomials over  $\mathbb{Q}$ : (i)  $t^2 - 5t + 6$ , (ii)  $t^2 + 5t + 1$ , (iii)  $(t + 1)^2$ .
- (2) Find the Galois group of the following cubic polynomials over  $\mathbb{Q}$ : (i)  $t^3 + 4t^2 + 2t - 7$ ; (ii)  $t^3 - t - 1$ ; (iii)  $t^3 - 3t + 1$ ; (iv)  $t^3 + 6t^2 + 11t + 5$ .

(3) Let  $f$  be a cubic polynomial over  $\mathbb{Q}$  with discriminant  $D$ . Show that  $f$  has three real roots if and only if  $D \geq 0$ . Apply this to the polynomial  $t^3 + pt + q$ .

(4) Let  $f$  be an irreducible quartic polynomial over  $\mathbb{Q}$  with exactly two real roots. Show that  $\text{Gal}(f) \cong \text{Dih}(8)$  or  $S_4$ .

(5) (*How to solve cubic equations*). Let  $f = t^3 + pt + q \in \mathbb{R}[t]$ . The following procedure, due essentially to Scipione del Ferro, will give a root of  $f$ .

(i) If  $t = u - v$  is a root of  $f$ , show that  $(u^3 - v^3) + (p - 3uv)(u - v) = -q$ .

(ii) Find a root of the form  $u - v$  by solving the equations  $u^3 - v^3 = -q$  and  $uv = \frac{p}{3}$  for  $u$  and  $v$ .

(6) The procedure of Exercise (14.2.5) yields one root  $u - v$  of  $f = t^3 + pt + q$ . Prove that the other two roots of  $f$  are  $\omega u - \omega^2 v$  and  $\omega^2 u - \omega v$  where  $\omega = e^{2\pi i/3}$ . (These are known as *Cardano's formulas*.)

(7) Use the methods of the last two exercises to find the roots of the polynomial  $t^3 + 3t + 1$ .

(8) Solve the cubic equation  $t^3 + 3t^2 + 6t + 3 = 0$  by first transforming it to one of the form  $t'^3 + pt' + q = 0$ .

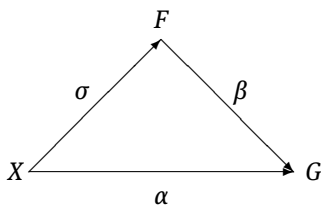
## 14.3 Presentations of groups

When groups entered the mathematical arena towards the close of the 18th century, they were exclusively permutation groups and were studied in connection with the theory of equations. A hundred years later groups arose from a different source, geometry, and usually these groups were most naturally described by listing a set of *generators* and a set of *defining relations* which the generators had to satisfy. A very simple example is where there is just one generator  $x$  and a single defining relation  $x^n = 1$  where  $n$  is a positive integer. Intuitively one would expect these to determine a cyclic group of order  $n$ .

As another example, suppose that a group has two generators  $x$  and  $y$  subject to the three relations  $x^2 = 1 = y^2$  and  $xy = yx$ . Now the Klein 4-group fits this description, with  $x = (12)(34)$  and  $y = (13)(24)$ . Thus it seems reasonable that a group with these generators and relations should be a Klein 4-group. Of course this cannot be substantiated until we have explained exactly what is meant by a group with given sets of generators and defining relations. Even when the generators are subject to no defining relations, a precise definition is still lacking: this is the important case of a *free group*. Thus our first task is to define free groups.

**Free groups.** A free group is best defined in terms of a certain *mapping property*. Let  $F$  be a group,  $X$  a non-empty set and  $\sigma : X \rightarrow F$  a function. Then  $F$ , or more precisely the pair  $(F, \sigma)$ , is said to be *free on  $X$*  if, for each function  $\alpha$  from  $X$  to a group  $G$  there is a *unique* homomorphism  $\beta : F \rightarrow G$  such that  $\beta\sigma = \alpha$ , i.e., the triangle below commutes:





First a comment on the definition. The function  $\sigma : X \rightarrow F$  is necessarily injective. For suppose that  $\sigma(x_1) = \sigma(x_2)$  and  $x_1 \neq x_2$ . Let  $G$  be any group with two or more elements and choose a function  $\alpha : X \rightarrow G$  such that  $\alpha(x_1) \neq \alpha(x_2)$ . We have  $\beta\sigma(x_1) = \beta\sigma(x_2)$  and hence  $\alpha(x_1) = \alpha(x_2)$ , a contradiction.

This indicates that we can replace  $X$  by the set  $\text{Im}(\alpha)$ , which has the same cardinality, and take  $X$  to be a subset of  $F$  with  $\sigma$  the inclusion map. What the mapping property then asserts is that every mapping from the subset  $X$  to a group  $G$  can be extended to a unique homomorphism from  $F$  to  $G$ . This property of free groups should be compared with properties of free modules. Free groups and free modules are special cases of free objects in a category.

At first sight the definition of a free group may seem abstract, but soon concrete descriptions of free groups will emerge. In the meantime the first order of business must be to show that free groups actually exist, something that is not obvious from the definition.

**(14.3.1)** *Let  $X$  be any non-empty set. Then there exist a group  $F$  and a function  $\sigma : X \rightarrow F$  such that  $(F, \sigma)$  is free on  $X$  and  $F$  is generated by  $\text{Im}(\sigma)$ .*

*Proof.* Roughly speaking, the idea of the proof is to construct  $F$  by forming “words” in  $X$  which are combined in a formal manner by juxtaposition, while at the same time allowing for cancellation of word segments like  $xx^{-1}$  or  $x^{-1}x$  where  $x \in X$ .

The first step is to choose a set disjoint from  $X$  with the same cardinality. Since the purpose of this move is to accommodate inverses of elements of  $X$ , it is appropriate to denote the set of inverses by  $X^{-1} = \{x^{-1} \mid x \in X\}$ . But keep in mind that  $x^{-1}$  is merely a symbol at this point and does not denote an inverse. By a word in  $X$  is meant any finite sequence  $w$  of elements of the set  $X \cup X^{-1}$ , written for convenience in the form

$$w = x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r},$$

where  $q_i = \pm 1$ ,  $x_i^1 = x_i \in X$  and  $r \geq 0$ . The case  $r = 0$ , when the sequence is empty, is the empty word, which is written 1. Two words are equal if they have the same entries in each position, i.e., they look exactly alike.

The product of words  $w = x_1^{q_1} \cdots x_r^{q_r}$  and  $v = y_1^{p_1} \cdots y_s^{p_s}$  is formed in the obvious way by juxtaposition, i.e.,

$$wv = x_1^{q_1} \cdots x_r^{q_r} y_1^{p_1} \cdots y_s^{p_s},$$

with the convention that  $w1 = w = 1w$ . This is clearly an associative binary operation on the set  $S$  of all words in  $X$ . The *inverse* of the word  $w$  is defined to be

$$w^{-1} = x_r^{-q_r} \cdots x_1^{-q_1},$$

with the convention that  $1^{-1} = 1$ . Thus far  $S$ , together with the product operation, is a semigroup with an identity element, i.e., a monoid. Next a device is introduced that permits the cancellation of segments of a word with the form  $xx^{-1}$  or  $x^{-1}x$ . Once this is done, instead of a monoid, we will have a group.

A relation  $\sim$  on the set  $S$  is defined in the following way:  $w \sim v$  means that it is possible to pass from  $w$  to  $v$  by means of a finite sequence of operations of the following types:

- (i) insertion of  $xx^{-1}$  or  $x^{-1}x$  as consecutive symbols in a word where  $x \in X$ ;
- (ii) deletion of any such sequences from a word.

For example,  $xyy^{-1}z \sim t^{-1}txz$  where  $x, y, z, t \in X$ . It is easy to check that  $\sim$  is an equivalence relation on  $S$ . Let  $F$  denote the set of all equivalence classes of words  $[w]$ ,  $w \in S$ . Our aim is to make  $F$  into a group: this will turn out to be a free group on the set  $X$ .

If  $w \sim w'$  and  $v \sim v'$ , then it is readily seen that  $wv \sim w'v'$ . It is therefore meaningful to define the *product* of the equivalence classes  $[w]$  and  $[v]$  by the rule

$$[w][v] = [wv].$$

It follows from this that  $[w][1] = [w] = [1][w]$  for all words  $w$ . Also  $[w][w^{-1}] = [1] = [w^{-1}][w]$ , since  $ww^{-1}$  and  $w^{-1}w$  are plainly equivalent to 1. Finally, we verify the associative law:

$$([u][v])[w] = [uv][w] = [(uv)w] = [u(vw)] = [u][vw] = [u]([v][w]).$$

Consequently,  $F$  is a group in which  $[1]$  is the identity element and  $[w^{-1}]$  is the inverse of  $[w]$ . Furthermore,  $F$  is generated by the subset  $\bar{X} = \{[x] \mid x \in X\}$ ; for, if  $w = x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}$  with  $x_i \in X$ ,  $q_i = \pm 1$ , then

$$[w] = [x_1]^{q_1} [x_2]^{q_2} \cdots [x_r]^{q_r} \in \langle \bar{X} \rangle.$$

It remains to prove that  $F$  is a free group on  $X$ . To this end define a function  $\sigma : X \rightarrow F$  by the rule  $\sigma(x) = [x]$ ; thus  $\text{Im}(\sigma) = \bar{X}$  and this subset generates  $F$ . Next let  $\alpha : X \rightarrow G$  be a map from  $X$  into some group  $G$ . To show that  $(F, \sigma)$  is free on  $X$  we need to produce a unique homomorphism  $\beta : F \rightarrow G$  such that  $\beta\sigma = \alpha$ . There is only one reasonable candidate here: define  $\beta$  by the rule

$$\beta([x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}]) = \alpha(x_1)^{q_1} \alpha(x_2)^{q_2} \cdots \alpha(x_r)^{q_r}, \quad (x_i \in X, q_i = \pm 1).$$

The first thing to observe is that  $\beta$  is well-defined: for any other element in the equivalence class  $[x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}]$  differs from  $x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}$  only by segments of the form  $xx^{-1}$

or  $x^{-1}x$ , ( $x \in X$ ), and these will contribute to the image under  $\beta$  merely  $\alpha(x)\alpha(x)^{-1}$  or  $\alpha(x)^{-1}\alpha(x)$ , i.e., the identity. It is a simple direct check that  $\beta$  is a homomorphism. Notice also that  $\beta\sigma(x) = \beta([x]) = \alpha(x)$ , so that  $\beta\sigma = \alpha$ .

Finally, we have to establish the uniqueness of  $\beta$ . If  $\beta' : F \rightarrow G$  is another homomorphism for which  $\beta'\sigma = \alpha$ , then  $\beta\sigma = \beta'\sigma$  and thus  $\beta$  and  $\beta'$  agree on  $\text{Im}(\sigma)$ . But  $\text{Im}(\sigma)$  generates the group  $F$ , so  $\beta = \beta'$ . Therefore  $(F, \sigma)$  is free on  $X$ .  $\square$

**Reduced words.** Now that free groups are known to exist, we would like to find a convenient form for their elements. Let  $F$  be the free group on the set  $X$  just constructed. A word in  $X$  is called *reduced* if it contains no pairs of consecutive symbols  $xx^{-1}$  or  $x^{-1}x$  with  $x \in X$ . The empty word is considered to be reduced. Now if  $w$  is any word, we can delete subsequences  $xx^{-1}$  and  $x^{-1}x$  from  $w$  until a reduced word is obtained. Thus each equivalence class  $[w]$  contains at least one reduced word. The important point to establish is that there is a *unique* reduced word in each equivalence class.

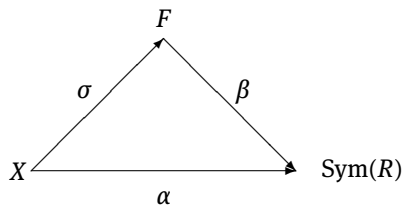
**(14.3.2)** *Each equivalence class of words on  $X$  contains a unique reduced word.*

*Proof.* There are likely to be many different ways to cancel segments  $xx^{-1}$  or  $x^{-1}x$  from a word. For this reason a direct approach to proving uniqueness would be complicated. An indirect argument will be used which avoids this difficulty.

Let  $R$  denote the set of all reduced words in  $X$ . The idea behind the proof is to introduce a permutation representation of the free group  $F$  on the set  $R$ . Let  $u \in X \cup X^{-1}$ : then a function  $u' : R \rightarrow R$  is determined by the following rule

$$u'(x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}) = \begin{cases} ux_1^{q_1} x_2^{q_2} \cdots x_r^{q_r} & \text{if } u \neq x_1^{-q_1} \\ x_2^{q_2} \cdots x_r^{q_r} & \text{if } u = x_1^{-q_1} \end{cases}.$$

Here  $x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}$  is a reduced word; observe that after applying the function  $u'$  the word is still reduced. Next  $u'$  is a permutation of  $R$  since its inverse is the function  $(u^{-1})'$ . Now let  $\alpha : X \rightarrow \text{Sym}(R)$  be defined by the assignment  $u \mapsto u'$ . By the mapping property of the free group  $F$  there is a homomorphism  $\beta : F \rightarrow \text{Sym}(R)$  such that  $\beta\sigma = \alpha$ : hence  $\alpha(x) = \beta\sigma(x) = \beta([x])$  for  $x \in X$ .



Now suppose that  $v$  and  $w$  are two equivalent reduced words; we will show that  $v = w$ . Certainly  $[v] = [w]$ , so  $\beta([v]) = \beta([w])$ . If  $v = x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}$ , then  $[v] = [x_1^{q_1}][x_2^{q_2}] \cdots [x_r^{q_r}]$  and we have

$$\beta([v]) = \beta([x_1^{q_1}])\beta([x_2^{q_2}]) \cdots \beta([x_r^{q_r}]) = \beta([x_1])^{q_1}\beta([x_2])^{q_2} \cdots \beta([x_r])^{q_r},$$

which equals

$$\alpha(x_1)^{q_1} \alpha(x_2)^{q_2} \cdots \alpha(x_r)^{q_r} = (x'_1)^{q_1} (x'_2)^{q_2} \cdots (x'_r)^{q_r}.$$

Applying the function  $\beta([v])$  to the empty word 1, which is reduced, we obtain  $x_1^{q_1} \cdots x_r^{q_r} = v$  since this word is reduced. Similarly  $\beta([w])$  sends the empty word to  $w$ . Therefore  $v = w$ .  $\square$

**Normal form.** The argument of the proof of (14.3.2) is a subtle one and it is well worth rereading. But the main point to appreciate is that (14.3.2) provides a unique way of representing the elements of the constructed free group  $F$  on the set  $X$ . Each element of  $F$  has the form  $[w]$  where  $w$  is a uniquely determined reduced word, say  $w = x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}$  where  $q_i = \pm 1$ ,  $r \geq 0$ . No consecutive terms  $xx^{-1}$  or  $x^{-1}x$  occur in  $w$ . Now  $[w] = [x_1]^{q_1} [x_2]^{q_2} \cdots [x_r]^{q_r}$ ; on combining consecutive terms of this product which involve the same  $x_i$ , we conclude that the element  $[w]$  can be uniquely written in the form

$$[w] = [x_1]^{\ell_1} [x_2]^{\ell_2} \cdots [x_s]^{\ell_s},$$

where  $s \geq 0$ ,  $\ell_i$  is a non-zero integer and  $x_i \neq x_{i+1}$ . (Strictly speaking the  $x_i$  may have been relabelled here).

To simplify the notation let us drop the distinction between  $x$  and  $[x]$ , so that now  $X \subseteq F$ . Then every element  $w$  of  $F$  has the unique form

$$w = x_1^{\ell_1} x_2^{\ell_2} \cdots x_s^{\ell_s}$$

where  $s \geq 0$ ,  $\ell_i \neq 0$  and  $x_i \neq x_{i+1}$ . This is called the *normal form* of  $w$ . For example, if  $X = \{x\}$ , each element of  $F$  has the unique normal form  $x^\ell$ , where  $\ell \in \mathbb{Z}$ . Thus  $F = \langle x \rangle$  is an infinite cyclic group.

The existence of a normal form is characteristic of free groups in the sense of the next result.

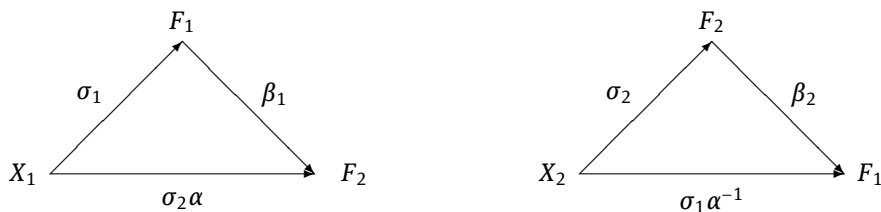
**(14.3.3)** *Let  $X$  be a subset of a group  $G$  and suppose that each element  $g$  of  $G$  can be uniquely written in the form  $g = x_1^{\ell_1} x_2^{\ell_2} \cdots x_s^{\ell_s}$  where  $x_i \in X$ ,  $s \geq 0$ ,  $\ell_i \neq 0$ , and  $x_i \neq x_{i+1}$ . Then  $G$  is free on  $X$ .*

*Proof.* Let  $F$  be the free group of equivalence classes of words in the set  $X$  constructed in (14.3.1), and let  $\sigma : X \rightarrow F$  be the associated injection; thus  $\sigma(x) = [x]$ . Apply the mapping property with the inclusion map  $\alpha : X \rightarrow G$ , i.e.,  $\alpha(x) = x$  for all  $x \in X$ . Hence there is a homomorphism  $\beta : F \rightarrow G$  such that  $\beta\sigma = \alpha$ , so  $\text{Im}(\alpha) \subseteq \text{Im}(\beta)$ . Since  $X = \text{Im}(\alpha)$  generates  $G$ , it follows that  $\text{Im}(\beta) = G$  and  $\beta$  is surjective. Finally, the uniqueness of the normal form guarantees that  $\beta$  is injective. For, if  $\beta([x_1]^{\ell_1} \cdots [x_r]^{\ell_r}) = 1$  with  $r > 0$ ,  $x_i \neq x_{i+1}$ ,  $\ell_i \neq 0$ , then  $(\beta\sigma(x_1))^{\ell_1} \cdots (\beta\sigma(x_r))^{\ell_r} = 1$ , and hence  $x_1^{\ell_1} \cdots x_r^{\ell_r} = 1$ , a contradiction. Therefore  $\beta$  is an isomorphism and  $F \simeq G$ , so that  $G$  is free on  $X$ .  $\square$

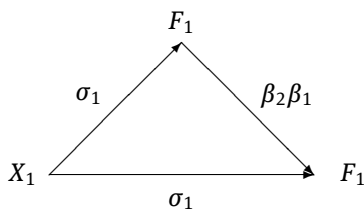
Thus far we have worked with a particular free group on a set  $X$ , the group constructed from equivalence classes of words in  $X$ . However, all free groups on the same set are isomorphic, a fact which allows us to deal only with free groups of words. This follows from the next result.

**(14.3.4)** Let  $F_i$  be a free group on  $X_i$ ,  $i = 1, 2$ , where  $|X_1| = |X_2|$ . Then  $F_1 \simeq F_2$ .

*Proof.* Let  $\sigma_1 : X_1 \rightarrow F_1$  and  $\sigma_2 : X_2 \rightarrow F_2$  be the respective injections associated with the free groups  $F_1$  and  $F_2$ , and let  $\alpha : X_1 \rightarrow X_2$  be a bijection, which exists since  $|X_1| = |X_2|$ . By the mapping property there are commutative diagrams



in which  $\beta_1$  and  $\beta_2$  are homomorphisms. Thus  $\beta_1 \sigma_1 = \sigma_2 \alpha$  and  $\beta_2 \sigma_2 = \sigma_1 \alpha^{-1}$ . Hence  $\beta_2 \beta_1 \sigma_1 = \beta_2 \sigma_2 \alpha = \sigma_1 \alpha^{-1} \alpha = \sigma_1$  and consequently the diagram below commutes,



But the identity map on  $F_1$  will also make this diagram commute, so  $\beta_2 \beta_1$  must equal this identity map by the uniqueness clause of the mapping property. In a similar fashion it can be shown that  $\beta_1 \beta_2$  equals the identity map on  $F_2$ , so that  $\beta_1 : F_1 \rightarrow F_2$  is an isomorphism.  $\square$

**Examples of free groups.** At this point free groups may appear to the reader as mysterious abstract objects, despite our success in constructing them. It is time to remedy this by exhibiting some real life examples.

**Example (14.3.1)** Consider the functions  $\alpha$  and  $\beta$  on the set  $\mathbb{C} \cup \{\infty\}$  which are defined by the rules

$$\alpha(x) = x + 2 \quad \text{and} \quad \beta(x) = \frac{1}{2 + \frac{1}{x}}.$$

Here the symbol  $\infty$  is required to satisfy the formal rules  $\frac{1}{\infty} = 0$ ,  $\frac{1}{0} = \infty$ ,  $2 + \infty = \infty$ . Thus  $\alpha(\infty) = \infty$ ,  $\beta(0) = 0$  and  $\beta(\infty) = \frac{1}{2}$ . The first thing to notice is that  $\alpha$  and  $\beta$  are

bijections since they have inverses given by  $\alpha^{-1}(x) = x - 2$  and  $\beta^{-1}(x) = \frac{1}{x-2}$ . This can be checked by computing the composites  $\alpha\alpha^{-1}$ ,  $\alpha^{-1}\alpha$ ,  $\beta\beta^{-1}$ ,  $\beta^{-1}\beta$ .

Define  $F$  to be the subgroup  $\langle \alpha, \beta \rangle$  of the symmetric group on the set  $\mathbb{C} \cup \{\infty\}$ . We are going to prove that  $F$  is a free group on  $\{\alpha, \beta\}$ . To accomplish this it is enough to show that no non-trivial reduced word in  $\alpha$  and  $\beta$  can equal 1: for then each element of  $F$  has a unique normal form and (14.3.3) can be applied.

Since direct calculations with the functions  $\alpha$  and  $\beta$  would be tedious, a geometric approach is adopted. Observe that each non-trivial power of  $\alpha$  maps the interior of the unit circle in the complex plane to its exterior. Also a non-trivial power of  $\beta$  maps the exterior of the unit circle to its interior with  $(0, 0)$  removed: the truth of the last statement is seen from the equation  $\beta(\frac{1}{x}) = \frac{1}{x+2}$ . It follows from this observation that no mapping of the form  $\alpha^{\ell_1}\beta^{m_1}\dots\alpha^{\ell_r}\beta^{m_r}$  can be trivial unless all the  $\ell_i$  and  $m_i$  are 0.

**Example (14.3.2)** An even more concrete example of a free group is provided by the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix};$$

for these generate a subgroup  $F_1$  of  $\text{GL}_2(\mathbb{Z})$  which is free on  $\{A, B\}$ .

To see why this is true, first consider a matrix

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbb{C}).$$

Thus  $ad - bc \neq 0$ . There is a corresponding permutation  $\theta(U)$  of  $\mathbb{C} \cup \{\infty\}$  defined by

$$\theta(U) : x \mapsto \frac{ax + b}{cx + d} = \frac{a + \frac{b}{x}}{c + \frac{d}{x}}.$$

Note that  $\theta(U)(\infty) = \frac{a}{c}$  if  $c \neq 0$  and  $\infty$  otherwise. This is called a *linear fractional transformation*. It is easy to verify that  $\theta(UV) = \theta(U)\theta(V)$ , so that  $\theta : \text{GL}_2(\mathbb{C}) \rightarrow \text{Sym}(\mathbb{C} \cup \{\infty\})$  is a homomorphism. Thus the linear fractional transformations form a subgroup  $\text{Im}(\theta)$  of  $\text{Sym}(\mathbb{C} \cup \{\infty\})$ . Now  $\theta(A) = \alpha$  and  $\theta(B) = \beta$ . Hence, if some non-trivial reduced word in  $A$  and  $B$  were to equal the identity matrix, the corresponding word in  $\alpha$  and  $\beta$  would equal the identity permutation, which is impossible by Example (14.3.1). Therefore  $F_1$  is free on  $\{A, B\}$  by (14.3.3).

Next we will use normal form to obtain some structural information about free groups.

**(14.3.5)** Let  $F$  be a free group on a set  $X$ . Then

- (i) each non-trivial element of  $F$  has infinite order;
- (ii) if  $F$  is not infinite cyclic, i.e.  $|X| > 1$ , then  $Z(F) = 1$ .

*Proof.* (i) Let  $1 \neq f \in F$  and suppose that  $f = x_1^{\ell_1} x_2^{\ell_2} \cdots x_s^{\ell_s}$  is the normal form. If  $x_1 = x_s$ , we can replace  $f$  by the conjugate  $x_s^{\ell_s} f x_s^{-\ell_s} = x_1^{\ell_1 + \ell_s} x_2^{\ell_2} \cdots x_{s-1}^{\ell_{s-1}}$ , which has the same order as  $f$ . For this reason there is nothing to be lost in assuming that  $x_1 \neq x_s$ . Let  $m$  be a positive integer; then

$$f^m = (x_1^{\ell_1} \cdots x_s^{\ell_s})(x_1^{\ell_1} \cdots x_s^{\ell_s}) \cdots (x_1^{\ell_1} \cdots x_s^{\ell_s}),$$

with  $m$  factors, which is in normal form since  $x_1 \neq x_s$ . It follows that  $f^m \neq 1$  and  $f$  has infinite order.

(ii) Assume that  $1 \neq f \in Z(F)$  and let  $f = x_1^{\ell_1} x_2^{\ell_2} \cdots x_s^{\ell_s}$  be the normal form of  $f$ . Then  $s > 1$ : for otherwise,  $f = x_1^{\ell_1}$  and if  $x_1 \neq x$  in  $X$ , then  $xf \neq fx$ . By conjugating  $f$  as in (i), we may assume that  $x_1 \neq x_s$ . Then  $fx_1 = x_1^{\ell_1} x_2^{\ell_2} \cdots x_s^{\ell_s} x_1$  and  $x_1 f = x_1^{\ell_1+1} x_2^{\ell_2} \cdots x_s^{\ell_s}$  are both in normal form, except that  $x_1^{\ell_1+1}$  is trivial if  $\ell_1 = -1$ ; but in any event  $fx_1 \neq x_1 f$  and so  $f \notin Z(G)$ .  $\square$

**Generators and relations.** The next result shows why free groups are worth studying: in fact they occupy a key position in group theory since their quotients account for all groups. The next result should be compared with (9.1.13) on free modules.

**(14.3.6)** *Let  $G$  be a group and  $X$  a set of generators for  $G$ . If  $F$  is a free group on the set  $X$ , there is a surjective homomorphism  $\theta : F \rightarrow G$  and hence  $G \simeq F/\text{Ker}(\theta)$ .*

*Proof.* Let  $(F, \sigma)$  be free on  $X$ . The existence of the homomorphism  $\theta$  follows on applying the mapping property of the free group  $F$  to obtain the commutative diagram

$$\begin{array}{ccc} & F & \\ \sigma \nearrow & & \searrow \theta \\ X & \xrightarrow{\iota} & G \end{array}$$

where  $\iota$  is the inclusion map. Thus  $x = \iota(x) = \theta\sigma(x) \in \text{Im}(\theta)$  for all  $x$  in  $X$ . Hence  $G = \text{Im}(\theta) \simeq F/\text{Ker}(\theta)$ .  $\square$

We are now ready to define a group given by a set of generators and defining relations. Let  $X$  be a non-empty set and  $F$  the free group on  $X$  with  $X \subseteq F$ . Let  $R$  be a subset of  $F$  and define

$$N = \langle R^F \rangle,$$

the normal closure of  $R$  in  $F$ : thus  $N$  is the subgroup generated by all conjugates in  $F$  of elements of  $R$  – see (4.2). Let

$$G = F/N.$$

Certainly the group  $G$  is generated by the elements  $xN$  where  $x \in X$ ; also  $r(xN) = r(x)N = N = 1_G$  for all  $r \in R$ . Hence the relations  $r = 1$  hold in  $G$ . Here  $r(xN)$  is the

element of  $G$  obtained from  $r$  by replacing each  $x$  by  $xN$ . Then  $G$  is called *the group with generators  $X$  and defining relations  $r = 1$  where  $r \in R$* : in symbols

$$G = \langle X \mid r = 1, \forall r \in R \rangle.$$

Elements of  $R$  are called *defining relators* and the group may also be written

$$G = \langle X \mid R \rangle.$$

The pair  $\langle X, R \rangle$  is called a *presentation* of  $G$ . An element  $w$  in the normal subgroup  $N$  is a *relator*; it is expressible as a product of conjugates of defining relators and their inverses. Also the relator  $w$  is said to be a *consequence* of the defining relators in  $R$ . Finally, a presentation  $\langle X, R \rangle$  is called *finite* if  $X$  and  $R$  are both finite.

Our first concern is to prove that every group can be defined by a presentation, which is the next result.

**(14.3.7)** *Every group has a presentation.*

*Proof.* Let  $G$  be an arbitrary group and choose a set  $X$  of generators for it, for example  $X = G$  will do. Let  $F$  be a free group on  $X$ . Then by (14.3.6) there is a surjective homomorphism  $\theta : F \rightarrow G$  and  $G \simeq F / \text{Ker}(\theta)$ . Next choose a subset  $R$  of  $\text{Ker}(\theta)$  whose normal closure in  $F$  is  $\text{Ker}(\theta)$  – for example we could take  $R$  to be  $\text{Ker}(\theta)$ . Then  $G \simeq F / \text{Ker}(\theta) = \langle X \mid R \rangle$ , which is a presentation of  $G$ .  $\square$

In the proof just given there are many possible choices for  $X$  and  $R$ , so a group has many presentations. This is one reason why it can be difficult to extract information about the structure of a group from a given presentation. Another, deeper reason for this difficulty arises from the *insolvability of the word problem*. Roughly speaking, this means that it is impossible to write a computer program which can decide if a word in the generators of a group given by a finite presentation equals the identity element. (For a very readable account of the word problem see [14]). As a consequence of this failure, special features of a group presentation will have to be exploited if we hope to derive structural information about the group from it.

Despite the difficulties inherent in working with presentations of groups, there is one very useful tool available.

**(14.3.8)** (Von Dyck's<sup>2</sup> Theorem) *Let  $G$  and  $H$  be groups with presentations  $\langle X \mid R \rangle$  and  $\langle Y \mid S \rangle$  respectively. Assume that there is given a surjective map  $\alpha : X \rightarrow Y$  such that  $\alpha(x_1)^{\ell_1} \alpha(x_2)^{\ell_2} \cdots \alpha(x_k)^{\ell_k}$  is a relator of  $H$ , i.e., a consequence of the words in  $S$ , whenever  $x_1^{\ell_1} x_2^{\ell_2} \cdots x_k^{\ell_k}$  is a defining relator of  $G$ . Then there is a surjective homomorphism  $\theta : G \rightarrow H$  such that  $\theta|_X = \alpha$ .*

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<sup>2</sup> Walter von Dyck (1856–1934)



*Proof.* Let  $F$  be the free group on  $X$ ; then  $G = F/N$  where  $N$  is the normal closure of  $R$  in  $F$ . By the mapping property of free groups there is a homomorphism  $\theta_0 : F \rightarrow H$  such that  $\theta_0|_X = \alpha$ . By hypothesis  $\theta_0(r) = 1$  for all  $r \in R$  and thus  $\theta_0(a) = 1$  for all  $a$  in  $N = \langle R^F \rangle$ . Hence  $\theta_0$  induces a homomorphism  $\theta : G \rightarrow H$  such that  $\theta(fN) = \theta_0(f)$ . Finally,  $Y \subseteq \text{Im}(\theta_0)$  since  $\alpha$  is surjective, so  $\theta_0$ , and hence  $\theta$ , is surjective.  $\square$

We will shortly show how Von Dyck's Theorem can be used to obtain information about a group from a presentation, but first it will be used to establish:

**(14.3.9)** *Every finite group has a finite presentation, i.e., a presentation with finitely many generators and finitely many relators.*

*Proof.* Let  $G = \{g_1, g_2, \dots, g_n\}$ , where  $g_1 = 1$ , be a finite group of order  $n$ . Then  $g_i g_j = g_{v(i,j)}$  and  $g_i^{-1} = g_{u(i)}$  where  $u(i), v(i, j) \in \{1, 2, \dots, n\}$ . Now let  $\tilde{G}$  be the group with generators  $\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_n$  and defining relations  $\tilde{g}_i \tilde{g}_j = \tilde{g}_{v(i,j)}$ ,  $\tilde{g}_i^{-1} = \tilde{g}_{u(i)}$ , where  $i, j = 1, 2, \dots, n$ . Clearly  $\tilde{G}$  has a finite presentation. Apply Von Dyck's Theorem to  $\tilde{G}$  and  $G$  where  $\alpha$  is the assignment  $\tilde{g}_i \mapsto g_i$ , noting that each defining relator of  $\tilde{G}$  is mapped to a relator of  $G$ . It follows that there is a surjective homomorphism  $\theta : \tilde{G} \rightarrow G$  such that  $\theta(\tilde{g}_i) = g_i$ .

Now every element  $\tilde{g}$  of  $\tilde{G}$  is expressible as a product of  $\tilde{g}_i$ 's and their inverses. Moreover, repeated use of the defining relations for  $\tilde{G}$  shows that  $\tilde{g}$  equals some  $\tilde{g}_k$  and it follows that  $|\tilde{G}| \leq n$ . But  $G \simeq \tilde{G}/\text{Ker}(\theta)$ , so  $|\text{Ker}(\theta)| = |\tilde{G}|/|G| \leq 1$ . Hence  $\text{Ker}(\theta) = 1$  and  $G \simeq \tilde{G}$ .  $\square$

Next we consider some explicit examples of groups given by a finite presentation.

**Example (14.3.3)** Let  $G = \langle x \mid x^n \rangle$  where  $n$  is a positive integer.

The free group  $F$  on  $\{x\}$  is generated by  $x$ : thus  $F \simeq \mathbb{Z}$  and  $G = F/F^n \simeq \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ , a cyclic group of order  $n$ , as expected.

**Example (14.3.4)** Let  $G = \langle x, y \mid xy = yx, x^2 = 1 = y^2 \rangle$ .

Since  $xy = yx$ , the group  $G$  is abelian; also every element of  $G$  has the form  $x^i y^j$  where  $i, j \in \{0, 1\}$ , because  $x^2 = 1 = y^2$ ; hence  $|G| \leq 4$ . On the other hand, the Klein 4-group  $V$  is generated by the permutations  $a = (12)(34)$  and  $b = (13)(24)$ , and the relations  $ab = ba$  and  $a^2 = 1 = b^2$  hold in  $V$ . Hence Von Dyck's Theorem can be applied to yield a surjective homomorphism  $\theta : G \rightarrow V$  such that  $\theta(x) = a$  and  $\theta(y) = b$ . Thus  $G/\text{Ker}(\theta) \simeq V$ . Since  $|G| \leq 4 = |V|$ , it follows that  $\text{Ker}(\theta) = 1$  and  $\theta$  is an isomorphism. Therefore  $G$  is a Klein 4-group.

For a greater challenge consider the following presentation.

**Example (14.3.5)** Let  $G = \langle x, y \mid x^2 = y^3 = (xy)^2 = 1 \rangle$ .

Our first move is to find an upper bound for  $|G|$ . Let  $H = \langle y \rangle$ ; this is a subgroup of order 1 or 3. Write  $\mathcal{S} = \{H, xH\}$ ; we will argue that  $\mathcal{S}$  is the set of all left cosets of  $H$  in  $G$ .

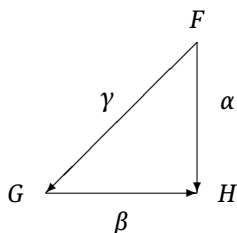
To establish this it is sufficient to show that  $x\mathcal{S} = \mathcal{S} = y\mathcal{S}$ , since it will then follow that  $\mathcal{S}$  contains every left coset of  $H$ . Certainly  $x\mathcal{S} = \mathcal{S}$  since  $x^2 = 1$ . Next  $xyxy = (xy)^2 = 1$  and hence  $yx = xy^2$ , since  $y^{-1} = y^2$ . It follows that  $yxH = xy^2H = xH$  and thus  $y\mathcal{S} = \mathcal{S}$ . Since  $|H| \leq 3$  and  $|G : H| = |\mathcal{S}| \leq 2$ , we deduce that  $|G| \leq 6$ .

Next observe that the symmetric group  $S_3$  is generated by the permutations  $a = (12)(3)$  and  $b = (123)$ , and that  $a^2 = b^3 = (ab)^2 = 1$  since  $ab = (1)(23)$ . By Von Dyck's theorem there is a surjective homomorphism  $\theta : G \rightarrow S_3$ . Since  $|G| \leq 6$ , it follows that  $\theta$  is an isomorphism and  $G \simeq S_3$ .

The method of the last two examples can be useful when a finite group is given by a presentation. The procedure is to choose a subgroup for whose order one has an upper bound, and then by coset enumeration to find an upper bound for the index. This gives an upper bound for the order of the group. The challenge is then to identify the group by comparing it with a known group for which the defining relations hold.

### Exercises (14.3)

- (1) Let  $F$  be the free group on a set  $X$ . Prove that an element  $f$  of  $F$  belongs to the derived subgroup  $F'$  if and only if the sum of the exponents of  $x$  in  $f$  is 0 for every  $x$  in  $X$ .
- (2) If  $F$  is a free group, prove that  $F/F'$  is a direct product of infinite cyclic groups.
- (3) Let  $G$  be the subgroup of  $\text{GL}_2(\mathbb{C})$  generated by  $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$  where  $a$  is real and  $a \geq 2$ . Prove that  $G$  is a free group.
- (4) (*The projective property of free groups*). Let there be given groups and homomorphisms  $\alpha : F \rightarrow H$  and  $\beta : G \rightarrow H$  where  $F$  is a free group and  $\beta$  is surjective. Show that there is a homomorphism  $\gamma : F \rightarrow G$  such that  $\beta\gamma = \alpha$ , i.e., the triangle below commutes,



- (5) Let  $G$  be a group with a normal subgroup  $N$  such that  $G/N$  is a free group. Prove that there is free subgroup  $H$  such that  $G = HN$  and  $H \cap N = 1$ .
- (6) Let  $H$  be a subgroup with finite index in a free group  $F$ . If  $1 \neq K \leq F$ , prove that  $H \cap K \neq 1$ .

[In the next three exercises identify the groups with the given presentations].

- (7)  $\langle x, y \mid x^2 = 1 = y^4, xy = yx \rangle$ .
- (8)  $\langle x, y \mid x^3 = (xy)^2 = y^3 = 1 \rangle$ .
- (9)  $\langle x, y \mid x^2 = (xy)^2 = y^5 = 1 \rangle$ .

(10) Let  $G$  be a group which has a presentation with  $n$  generators and  $r$  defining relators. If  $r < n$ , prove that  $G$  is infinite. [Hint: consider the abelian group  $G/G'$  and use Exercise (14.3.2)].

(11) Let  $F_1$  and  $F_2$  be free groups on sets  $X_1$  and  $X_2$  respectively. If  $F_1 \simeq F_2$ , prove that  $|X_1| = |X_2|$ . Thus a free group is determined up to isomorphism by the cardinality of the set on which it is free. [Hint: consider  $F_i/F_i^2$  as a vector space over  $\text{GF}(2)$ ].

## 14.4 Introduction to error correcting codes

In this, the age of information technology, enormous amounts of data are transmitted electronically over vast distances every second of every day. The data are generally in the form of bit strings, i.e., sequences of 0's and 1's. Inevitably errors occur from time to time during the process of transmission, so that the message received may differ from the one transmitted. An *error correcting code* allows the detection and correction of erroneous messages. The essential idea here is that the possible transmitted codewords should not be too close to one another, i.e., they should not agree in too many entries. This makes it more likely that an error can be detected and the original message recovered. Over the last fifty years an entire mathematical theory of error-correcting codes has evolved.

**Fundamental concepts.** Let  $Q$  be a finite set with  $q$  elements; this is called the *alphabet*. A word  $w$  of length  $n$  over  $Q$  is an  $n$ -tuple of elements of  $Q$ , written for convenience in the form

$$w = (w_1 w_2 \cdots w_n), \quad w_i \in Q.$$

The set of all words of length  $n$  over  $Q$  is called  *$n$ -dimensional Hamming<sup>3</sup> space* and is denoted by

$$H_n(q).$$

This is the set of possible messages of length  $n$ : notice that  $|H_n(q)| = q^n$ . If  $Q$  is a finite field,  $H_n(Q)$  is an  $n$ -dimensional vector space over  $Q$ . In practice  $Q$  is usually the field with two elements, when Hamming  $n$ -space is the set of all bit strings of length  $n$ .

It is important to have a measure of how far apart two words are: the natural measure to use is the number of entries in which the words differ. If  $v$  and  $w$  belong to  $H_n(q)$ , the *distance* between  $v$  and  $w$  is defined to be

$$d(v, w) = |\{i \mid v_i \neq w_i\}|,$$

i.e., the number of positions where  $v$  and  $w$  have different entries. The *weight* of a word  $v$  is its distance from the zero word,

$$wt(v) = d(v, 0),$$

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<sup>3</sup> Richard Wesley Hamming (1915–1998)

so  $wt(v)$  is just the number of non-zero entries of  $v$ . Clearly,  $d(u, v)$  is the number of errors that have been made if the word  $u$  is transmitted and it is received wrongly as  $v$ .

The basic properties of the distance function are given in the following result.

**(14.4.1)** Let  $u, v, w \in H_n(q)$ . Then:

- (i)  $d(v, w) \geq 0$  and  $d(v, w) = 0$  if and only if  $v = w$ ;
- (ii)  $d(v, w) = d(w, v)$ ;
- (iii)  $d(u, w) \leq d(u, v) + d(v, w)$ .

These properties assert that the function  $d : H_n(q) \times H_n(q) \rightarrow \mathbb{N}$  is a metric on the Hamming space  $H_n(q)$ .

*Proof of (14.4.1)* Statements (i) and (ii) are obviously true. To prove (iii) note that  $u$  can be changed to  $v$  by  $d(u, v)$  entry changes and  $v$  can then be changed to  $w$  by  $d(v, w)$  changes. Thus  $u$  can be changed to  $w$  by  $d(u, v) + d(v, w)$  entry changes. Therefore  $d(u, w) \leq d(u, v) + d(v, w)$ .  $\square$

**Codes.** A code of length  $n$  over an alphabet  $Q$  with  $q$  elements, or briefly a  $q$ -ary code of length  $n$ , is a subset  $C$  of  $H_n(Q)$  with at least two elements. The elements of  $C$ , which are transmitted in an actual message, are called *codewords*.

A code  $C$  is said to be *e-error detecting* if  $c_1, c_2 \in C$  and  $d(c_1, c_2) \leq e$  always imply that  $c_1 = c_2$ . Thus the distance between distinct codewords is always greater than  $e$ . Equivalently, a codeword cannot be transmitted and received as a different codeword if  $e$  or fewer errors have occurred. In this sense the code  $C$  is able to detect up to  $e$  errors.

Next a  $q$ -ary code of length  $n$  is called *e-error correcting* if, for every  $w$  in  $H_n(q)$ , there is at most one codeword  $c$  such that  $d(w, c) \leq e$ . This means that if a codeword  $c$  is received as a different word  $w$  and at most  $e$  errors have occurred, it is possible to recover the original codeword by examining all words  $v$  in  $H_n(q)$  such that  $d(w, v) \leq e$ : exactly one of these is a codeword and it must have been the transmitted codeword  $c$ . Clearly a code which is *e-error correcting* is *e-error detecting*.

An important parameter of a code is the shortest distance between distinct codewords; this is called the *minimum distance* of the code. The following result is basic.

**(14.4.2)** Let  $C$  be a code with minimum distance  $d$ . Then:

- (i)  $C$  is *e-error detecting* if and only if  $d \geq e + 1$ ;
- (ii)  $C$  is *e-error correcting* if and only if  $d \geq 2e + 1$ .

*Proof.* (i) Suppose that  $d \geq e + 1$ . If  $c_1, c_2$  are distinct codewords, then  $d(c_1, c_2) \geq d \geq e + 1$ . Hence  $C$  is *e-error detecting*. For the converse, assume that  $d \leq e$ . By definition of  $d$  there exist  $c_1 \neq c_2$  in  $C$  such that  $d(c_1, c_2) = d \leq e$ , so that  $C$  is not *e-error detecting*.

(ii) Assume that  $C$  is not *e-error correcting*, so there is a word  $w$  and codewords  $c_1 \neq c_2$  such that  $d(c_1, w) \leq e$  and  $d(w, c_2) \leq e$ . Then

$$d \leq d(c_1, c_2) \leq d(c_1, w) + d(w, c_2) \leq 2e$$

by (14.4.1). Hence  $d < 2e + 1$ .

Conversely, assume that  $d < 2e + 1$  and let  $c_1$  and  $c_2$  be codewords at distance  $d$  apart. Put  $f = \lfloor \frac{1}{2}d \rfloor$ , i.e., the greatest integer  $\leq \frac{1}{2}d$ ; thus  $f \leq \frac{1}{2}d \leq e$ . We claim that  $d - f \leq e$ . This is true when  $d$  is even since  $d - f = d - \frac{1}{2}d = \frac{1}{2}d \leq e$ . If  $d$  is odd,  $f = \frac{d-1}{2}$  and  $d - f = \frac{d+1}{2} < e + 1$ ; therefore  $d - f \leq e$ . Next we can pass from  $c_1$  to  $c_2$  by changing exactly  $d$  entries. Let  $w$  be the word obtained from  $c_1$  after the first  $f$  entry changes. Then  $d(c_1, w) = f \leq e$ , while  $d(c_2, w) = d - f \leq e$ . Therefore  $C$  is not  $e$ -error correcting.  $\square$

**Corollary (14.4.3)** *If a code has minimum distance  $d$ , then its maximum error detection capacity is  $d - 1$  and its maximum error correction capacity is  $\lfloor \frac{d-1}{2} \rfloor$ .*

**Example (14.4.1)** Consider the binary code  $C$  of length 5 with the three codewords

$$c_1 = (10010), \quad c_2 = (01100), \quad c_3 = (10101).$$

Clearly the minimum distance of  $C$  is 3. Hence  $C$  is 2-error detecting and 1-error correcting. For example, suppose that  $c_2$  is transmitted and is received as  $w = (11000)$ , so that two entry errors have occurred. The error can be detected since  $w \notin C$ . But  $C$  is not 2-error detecting since if  $v = (11100)$ , then  $d(c_2, v) = 1$  and  $d(c_3, v) = 2$ . Thus if  $v$  is received and up to two errors occurred, we cannot tell whether  $c_2$  or  $c_3$  was the transmitted codeword.

**Bounds for the size of a code.** It is evident from (14.4.2) that for a code to have good error correcting capability it must have large minimum distance. But the price to be paid for this is that fewer codewords are available. An interesting question is: what is the maximum size of a  $q$ -ary code with length  $n$  and minimum distance  $d$ . We begin with a lower bound, which guarantees the existence of a code of a certain size.

**(14.4.4)** (The Varshamov–Gilbert lower bound) *Let  $n, q, d$  be positive integers with  $d \leq n$ . Then there is a  $q$ -ary code of length  $n$  and minimum distance  $d$  in which the number of codewords is at least*

$$\frac{q^n}{\sum_{i=0}^{d-1} \binom{n}{i} (q-1)^i}.$$

Before embarking on the proof we introduce the important concept of the  $r$ -ball with center  $w$ ,

$$B_r(w).$$

This is the set of all words in  $H_n(q)$  at distance  $r$  or less from  $w$ . Thus a code  $C$  is  $e$ -error correcting if and only if the  $e$ -balls  $B_e(c)$  with  $c$  in  $C$  are pairwise disjoint.

*Proof of (14.4.4).* The first step is to establish a formula for the size of an  $r$ -ball,

$$|B_r(w)| = \sum_{i=0}^r \binom{n}{i} (q-1)^i.$$

To see this observe that in order to construct a word in  $B_r(w)$ , we must alter at most  $r$  entries of  $w$ . Choose the  $i$  entries to be altered in  $\binom{n}{i}$  ways and then replace each one by an element of  $Q$  in  $(q-1)^i$  ways. This gives a count of  $\binom{n}{i}(q-1)^i$  words at distance  $i$  from  $w$ ; the formula now follows at once.

To start the construction choose any  $q$ -ary code  $C_0$  of length  $n$  with minimum distance  $d$ ; for example,  $C_0$  might consist of the zero word and a single word of weight  $d$ . If the union of the  $B_{d-1}(c)$  with  $c \in C_0$  is not  $H_n(q)$ , there is a word  $w$  whose distance from every word in  $C_0$  is at least  $d$ . Let  $C_1 = C_0 \cup \{w\}$ ; this is a larger code than  $C_0$  which has the same minimum distance  $d$ . Repeat the procedure for  $C_1$  and then as often as possible. Eventually a code  $C$  with minimum distance  $d$  will be obtained which cannot be enlarged; when this occurs, we have  $H_n(q) = \bigcup_{c \in C} B_{d-1}(c)$ . Therefore

$$q^n = |H_n(q)| = \left| \bigcup_{c \in C} B_{d-1}(c) \right| \leq |C| \cdot |B_{d-1}(c)|$$

for any fixed  $c \in C$ . Hence  $|C| \geq q^n / |B_{d-1}(c)|$  and the bound has been established.  $\square$

Next we give an upper bound for the size of an  $e$ -error correcting code.

**(14.4.5)** (The Hamming upper bound) *Let  $C$  be a  $q$ -ary code of length  $n$  which is  $e$ -error correcting. Then*

$$|C| \leq \frac{q^n}{\sum_{i=0}^e \binom{n}{i} (q-1)^i}.$$

*Proof.* Since  $C$  is  $e$ -error correcting, the  $e$ -balls  $B_e(c)$  for  $c \in C$  are pairwise disjoint. Hence

$$\left| \bigcup_{c \in C} B_e(c) \right| = |C| \cdot |B_e(c)| \leq |H_n(q)| = q^n,$$

for any fixed  $c \in C$ . Therefore  $|C| \leq q^n / |B_e(c)|$ , as required.  $\square$

A  $q$ -ary code  $C$  of length  $n$  for which the Hamming upper bound is attained is called a *perfect code*. In this case

$$|C| = \frac{q^n}{\sum_{i=0}^e \binom{n}{i} (q-1)^i},$$

and clearly this happens precisely when  $H_n(q)$  is the union of the disjoint balls  $B_e(c)$ ,  $c \in C$ , i.e., every word lies at distance  $\leq e$  from exactly one codeword. Perfect codes are desirable since they have the largest number of codewords for the given error correcting capacity; however they are also quite rare.

**Example (14.4.2)** (The binary repetition code) A very simple example of a perfect code is the binary code  $C$  of length  $2e + 1$  with just two codewords,

$$c_0 = (0, 0, \dots, 0) \quad \text{and} \quad c_1 = (1, 1, \dots, 1).$$

Clearly  $C$  has minimum distance  $d = 2e + 1$  and its maximum error correction capacity is  $e$  by (14.4.3). A word  $w$  belongs  $B_e(c_0)$  if more of its entries equal 0 than 1; otherwise  $w \in B_e(c_1)$ . Thus  $B_e(c_0) \cap B_e(c_1) = \emptyset$  and  $B_e(c_0) \cup B_e(c_1) = H_{2e+1}(2)$ .

**Linear codes.** Let  $Q$  denote  $\text{GF}(q)$ , the field of  $q$  elements, where  $q$  is now a prime power. The Hamming space  $H_n(q)$  is the  $n$ -dimensional vector space  $Q_n$  of all  $n$ -row vectors over  $Q$ . A  $q$ -ary code  $C$  of length  $n$  is called *linear* if it is a subspace of  $H_n(Q)$ . Linear codes form an important class of codes; they have the advantage that they can be specified by a basis instead of listing all the codewords. Linear codes can also be described by matrices, as will be seen in the sequel.

A computational advantage of linear codes is indicated by the next result.

**(14.4.6)** *The minimum distance of a linear code equals the minimal weight of a non-zero codeword.*

*Proof.* Let  $C$  be a linear code. If  $c_1, c_2 \in C$ , then  $d(c_1, c_2) = \text{wt}(c_1 - c_2)$  and  $c_1 - c_2 \in C$ . Hence the minimum distance equals the minimum weight.  $\square$

A point to keep in mind here is that to find the minimum distance of a code  $C$  one must compute  $\binom{|C|}{2}$  distances, whereas to find the minimum weight of  $C$  only the distances from the zero word need be found, so that at most  $|C| - 1$  computations are necessary.

As with codes in general, it is desirable to have linear codes with large minimum distance and as many codewords as possible. There is a version of the Varshamov–Gilbert lower bound for linear codes.

**(14.4.7)** *Let  $d$  and  $n$  be positive integers with  $d \leq n$  and let  $q$  be a prime power. Then there is a linear  $q$ -ary code of length  $n$  and minimum distance  $d$  for which the number of codewords is at least*

$$\frac{q^n}{\sum_{i=0}^{d-1} \binom{n}{i} (q-1)^i}.$$

*Proof.* We refer to the proof of (14.4.4). To start the construction choose a linear  $q$ -ary code  $C_0$  of length  $n$  and minimum distance  $d$ ; for example, the subspace generated by a single word of weight  $d$  will suffice. If  $\bigcup_{c \in C_0} B_{d-1}(c) \neq H_n(q)$ , choose a word  $w$  in  $H_n(q)$  which belongs to no  $B_{d-1}(c)$  with  $c$  in  $C_0$ . Thus  $w \notin C_0$ . Define  $C_1$  to be the subspace generated by  $C_0$  and  $w$ . We claim that  $C_1$  still has minimum distance  $d$ . To

prove this it is sufficient to show that  $wt(c') \geq d$  for any  $c'$  in  $C_1 - C_0$ ; this is because of (14.4.6). Write  $c' = c_0 + aw$  where  $c_0 \in C_0$  and  $0 \neq a \in Q$ . Then

$$wt(c') = wt(c_0 + aw) = wt(-a^{-1}c_0 - w) = d(-a^{-1}c_0, w) \geq d$$

by choice of  $w$ , since  $-a^{-1}c_0 \in C_0$ . Note also that  $\dim(C_0) < \dim(C_1)$ .

Repeat the argument above for  $C_1$ , and then as often as possible. After at most  $n$  steps we arrive at a subspace  $C$  with minimum distance  $d$  such that  $\bigcup_{c \in C} B_{d-1}(c) = H_n(q)$ . It now follows that  $|C| \cdot |B_{d-1}(c)| \geq q^n$  for any  $c$  in  $C$ , which gives the bound.  $\square$

**Example (14.4.3)** Let  $q = 2$ ,  $d = 3$  and  $n = 31$ . According to (14.4.7) there is a linear binary code  $C$  of length 31 with minimum distance 3 such that

$$|C| \geq \frac{2^{31}}{1 + 31 + \binom{31}{2}} = 4,320,892,652.$$

In addition  $C$  is a subspace of  $H_{31}(2)$ , so its order is a power of 2. Hence  $|C| \geq 2^{23} = 8,388,608$ . In fact there is a larger code of this type with  $2^{26}$  codewords, a so-called Hamming code – see Example (14.4.7) below.

**The generator matrix and check matrix.** Let  $C$  be a linear  $q$ -ary code of length  $n$  and let  $k$  be the dimension of  $C$  as a subspace of  $H_n(q)$ . Thus  $k \leq n$  and  $|C| = q^k$ . Choose an ordered basis  $\{c_1, c_2, \dots, c_k\}$  for  $C$  and write

$$G = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}.$$

This  $k \times n$  matrix over  $Q = \text{GF}(q)$  is called a *generator matrix* for  $C$ . If  $c$  is any codeword,  $c = a_1c_1 + \dots + a_kc_k$  for suitable  $a_i \in Q$ . Thus  $c = aG$  where  $a = (a_1, \dots, a_k) \in H_k(q)$ . Hence each codeword is uniquely expressible in the form  $aG$  with  $a \in H_k(q)$ . It follows that the code  $C$  is the row space of the matrix  $G$ , i.e., the subspace of  $H_n(q)$  generated by all the rows of  $G$ . Notice that the rank of  $G$  is  $k$  since its rows are linearly independent.

Recall from (8.1) that the null space  $N$  of  $G$  consists of all  $n$ -column vectors  $x^T$  such that  $Gx^T = 0$ : here of course  $x \in H_n(q)$ . Choose an ordered basis for  $N$  and use the transposes of its elements to form the rows of a matrix  $H$ . This is called a *check matrix* for  $C$ . Since  $G$  has rank  $k$ , we can apply (8.3.8) to obtain  $\dim(N) = n - k$ , so that  $H$  is an  $(n - k) \times n$  matrix over  $Q$ . Since the columns of  $H^T$  belong to  $N$ , the null space of  $G$ , we obtain the important equation

$$GH^T = 0.$$

Keep in mind that the matrices  $G$  and  $H$  depend on choices of bases for  $C$  and  $N$ . At this point the following result about matrices is relevant.



**(14.4.8)** Let  $G$  and  $H$  be  $k \times n$  and  $(n - k) \times n$  matrices respectively over  $Q = \text{GF}(q)$ , each having linearly independent rows. Then the following statements are equivalent:

- (i)  $GH^T = 0$ ;
- (ii)  $\text{row space}(G) = \{x \in H_n(q) \mid xH^T = 0\}$ ;
- (iii)  $\text{row space}(H) = \{x \in H_n(q) \mid xG^T = 0\}$ .

*Proof.* Let  $S = \{x \in H_n(q) \mid xH^T = 0\}$ ; then  $x \in S$  if and only if  $0 = (xH^T)^T = Hx^T$ , i.e.,  $x^T$  belongs to null space( $H$ ). This implies that  $S$  is a subspace and  $\dim(S) = n - (n - k) = k$ . Now assume that  $GH^T = 0$ . If  $x \in \text{row space}(G)$ , then  $x = yG$  for some  $k$ -row vector  $y$ . Hence  $xH^T = yGH^T = 0$  and  $x \in S$ . Thus  $\text{row space}(G) \subseteq S$ . But  $\dim(\text{row space}(G)) = k = \dim(S)$ , so that  $S = \text{row space}(G)$ . Thus (i) implies (ii). It is clear that (ii) implies (i), and thus (i) and (ii) are equivalent.

Next observe that  $GH^T = 0$  if and only if  $HG^T = 0$ , by applying the transpose. Thus the roles of  $G$  and  $H$  are interchangeable, which means that (i) and (iii) are equivalent.  $\square$

Let us now return to the discussion of a linear  $q$ -ary code  $C$  of length  $n$  with generator matrix  $G$  and check matrix  $H$ . From (14.4.8) we conclude that

$$C = \text{row space}(G) = \{w \in H_n(q) \mid wH^T = 0\}.$$

So the check matrix  $H$  provides a convenient way to determine if a given word  $w$  is a codeword. At this point we have proved half of the next result.

**(14.4.9)**

- (i) If  $C$  is a linear  $q$ -ary code with generator matrix  $G$  and check matrix  $H$ , then  $GH^T = 0$  and  $C = \{w \in H_n(q) \mid wH^T = 0\}$ .
- (ii) If  $G$  and  $H$  are  $k \times n$  and  $(n - k) \times n$  matrices respectively over  $\text{GF}(q)$  with linearly independent rows and if  $GH^T = 0$ , then  $C = \{w \in H_n(q) \mid wH^T = 0\}$  is a linear  $q$ -ary code of length  $n$  and dimension  $k$  with generator matrix  $G$  and check matrix  $H$ .

*Proof.* To prove (i) note that  $C = \text{row space}(G)$  and we showed that  $GH^T = 0$ , so the result follows at once from (14.4.8). Now for (ii): clearly  $C$  is a subspace of  $H_n(q)$  and hence is linear  $q$ -ary code of length  $n$ . By (14.4.8)  $C$  is the row space of  $G$ . Hence  $\dim(C) = k$  and  $G$  is a generator matrix for  $C$ . Finally, the null space of  $G$  consists of all  $w$  in  $H_n(q)$  such that  $Gw^T = 0$ , i.e.,  $wG^T = 0$ ; this is the row space of  $H$  by (14.4.8). Hence  $G$  and  $H$  are corresponding generator and check matrices for  $C$ .  $\square$

On the basis of (14.4.9) we show how to construct a linear  $q$ -ary code of length  $n$  and dimension  $n - \ell$  with check matrix equal to a given  $\ell \times n$  matrix  $H$  over  $\text{GF}(q)$  of rank  $\ell$ . Define  $C = \{x \in H_n(q) \mid xH^T = 0\}$ ; this is a linear  $q$ -ary code. Pass from  $H$  to its reduced row echelon form  $H' = [I_\ell \mid A]$  where  $A$  is an  $\ell \times (n - \ell)$  matrix: note that interchanges of columns, i.e., of word entries, may be necessary to achieve this and

$H' = EHF$  for some non-singular  $E$  and  $F$ . Writing  $G'$  for  $[-A^T \mid I_{n-\ell}]$ , we have

$$G'H'^T = [-A^T \mid I_{n-\ell}] \begin{bmatrix} I_\ell \\ A^T \end{bmatrix} = 0.$$

Hence  $0 = G'H'^T = (G'F^T)H^TE^T$ , so  $(G'F^T)H^T = 0$  because  $E^T$  is non-singular. Put  $G = G'F^T$ ; thus  $GH^T = 0$  and by (14.4.9)  $G$  is a generator matrix and  $H$  a check matrix for  $C$ . Also  $\dim(C) = \text{rank}(G) = n - \ell$ . Note that if no column interchanges are needed to go from  $H$  to  $H'$ , then  $F = I$  and  $G = G'$ .

**Example (14.4.4)** Consider the matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

over  $\text{GF}(2)$ . Here  $q = 2$ ,  $n = 7$  and  $\ell = 3$ . The rank of  $H$  is 3, so it determines a linear binary code  $C$  of dimension  $7 - 3 = 4$ . Put  $H$  in reduced row echelon form,

$$H' = \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array} \right] = [I_3 \mid A].$$

No column interchanges were necessary here, so

$$G = G' = [-A^T \mid I_4] = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is a generator matrix for  $C$ . The rows of  $G$  form a basis for the linear code  $C$ .

A useful feature of the check matrix is that from it one can read off the minimum distance of the code.

**(14.4.10)** Let  $H$  be a check matrix for a linear code  $C$ . Then the minimum distance of  $C$  equals the largest integer  $m$  such that every set of  $m - 1$  columns of  $H$  is linearly independent.

*Proof.* Let  $d$  be the minimum distance of  $C$  and note that  $d$  is the minimum weight of a non-zero codeword, say  $d = \text{wt}(c)$ . Then  $cH^T = 0$ , which implies that there exist  $d$  linearly dependent columns of  $H$ . Hence  $m - 1 < d$  and  $m \leq d$ . Also by maximality of  $m$  there exist  $m$  linearly dependent columns of  $H$ , so  $wH^T = 0$  where  $w$  is a non-zero word with  $\text{wt}(w) \leq m$ . But  $w \in C$ ; thus  $d \leq m$  and hence  $d = m$ .  $\square$

**Example (14.4.5)** Consider the code  $C$  in Example (14.4.4). Every pair of columns of the check matrix  $H$  is linearly independent, i.e., the columns are all different. On the other

hand, columns 1, 4 and 5 are linearly dependent since their sum is zero. Therefore  $m = 3$  for this code and the minimum distance is 3 by (14.4.10). Consequently  $C$  is a 1-error correcting code.

**Using the check matrix to correct errors.** Let  $C$  be a linear  $q$ -ary code with length  $n$  and minimum distance  $d$ . Let  $H$  be a check matrix for  $C$ . Note that by (14.4.3)  $C$  is  $e$ -error correcting where  $e = \lfloor \frac{d-1}{2} \rfloor$ . Suppose that a codeword  $c$  is transmitted and received as a word  $w$  and that *at most*  $e$  errors in the entries have been made. Here is a procedure that will correct the errors and recover the original codeword  $c$ .

Write  $w = u + c$  where  $u$  is the error; thus  $wt(u) \leq e$ . Now  $|H_n(q) : C| = q^{n-k}$  where  $k = \dim(C)$ . Choose a transversal to  $C$  in  $H_n(q)$ , say  $\{v_1, v_2, \dots, v_{q^{n-k}}\}$ , by requiring that  $v_i$  be a word of *smallest length* in its coset  $v_i + C$ . (There may be more than one choice for  $v_i$ ). For any  $c_0 \in C$  we have  $(v_i + c_0)H^T = v_i H^T$ , which depends only on  $i$ . Now suppose that  $w$  belongs to the coset  $v_i + C$ . Then  $wH^T = v_i H^T$ , which is called the *syndrome* of  $w$ . Writing  $w = v_i + c_1$  with  $c_1 \in C$ , we have  $u = w - c \in v_i + C$ , so that  $wt(v_i) \leq wt(u) \leq e$  by choice of  $v_i$ . Hence  $w = u + c = v_i + c_1$  belongs to  $B_e(c) \cap B_e(c_1)$ . But this implies that  $c = c_1$  since  $C$  is  $e$ -error correcting. Therefore  $c = w - v_i$  and the transmitted codeword has been identified.

In summary here is the procedure to identify the transmitted codeword  $c$ . It is assumed that the transversal  $\{v_1, v_2, \dots, v_{q^{n-k}}\}$  has been chosen as described above, with each  $v_i$  of smallest length in its coset.

- (i) Suppose that  $w$  is the word received with at most  $e$  errors; first compute the syndrome  $wH^T$ .
- (ii) By comparing  $wH^T$  with the syndromes  $v_i H^T$ , find the unique  $i$  such that  $wH^T = v_i H^T$ .
- (iii) Then the transmitted codeword was  $c = w - v_i$ .

**Example (14.4.6)** The matrix

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

determines a linear binary code  $C$  with length 5 and dimension  $5 - 3 = 2$ ; thus  $H$  is a check matrix for  $C$ . Clearly  $C$  has minimum distance 3, so it is 1-error correcting. Also  $|C| = 2^2 = 4$  and  $|H_5(2) : C| = 2^5/4 = 8$ . By reducing  $H$  to reduced row echelon form as in Example (14.4.4), we find a generator matrix for  $C$  to be

$$G = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Thus  $C$  is generated by (01110) and (10101), so in fact  $C$  consists of (00000), (01110), (10101) and (11011).

Next enumerate the eight cosets of  $C$  in  $H_5(2)$  with  $C_1 = C$  and choose a word of minimum weight from each coset; these are shown in bold face.

$$\begin{aligned}
 C_1 &= \{(\mathbf{00000}), (01110), (10101), (11011)\} \\
 C_2 &= \{(\mathbf{10000}), (11110), (00101), (01011)\} \\
 C_3 &= \{(\mathbf{01000}), (00110), (11101), (10011)\} \\
 C_4 &= \{(\mathbf{00100}), (01010), (10001), (11111)\} \\
 C_5 &= \{(\mathbf{11000}), (10110), (01101), (00011)\} \\
 C_6 &= \{(01100), (\mathbf{00010}), (11001), (10111)\} \\
 C_7 &= \{(10100), (11010), (\mathbf{00001}), (01111)\} \\
 C_8 &= \{(11100), (\mathbf{10010}), (01001), (00111)\}
 \end{aligned}$$

The coset syndromes are computed as

$$(000), (101), (001), (110), (100), (111), (011), (010).$$

Now suppose that the word  $w = (11111)$  is received with at most one error in its entries: note that  $w \notin C$ , so  $w$  is not a codeword. The syndrome of  $w$  is  $wH^T = (110)$ , which is the syndrome of elements in the coset  $C_4$ , with coset representative  $v_4 = (00100)$ . Hence the transmitted codeword was  $c = w - v_4 = (11011)$ .

**Hamming codes.** Let  $C$  be a linear  $q$ -ary code of length  $n$  and dimension  $k$ . Assume that the minimum distance of  $C$  is at least 3, so that  $C$  is 1-error correcting. A check matrix  $H$  for  $C$  has size  $\ell \times n$  where  $\ell = n - k$ , and by (14.4.10) no column of  $H$  can be a multiple of another column.

Now consider the problem of constructing such a linear code which is as large as possible for given  $q$  and  $\ell > 1$ . Then  $H$  should have as many columns as possible, subject to no column being a multiple of another one. Now there are  $q^\ell - 1$  non-zero  $\ell$ -column vectors over  $\text{GF}(q)$ , but each of these is a multiple of  $q - 1$  other columns. So the maximum possible number of columns for  $H$  is  $n = \frac{q^\ell - 1}{q - 1}$ . Note that the columns of the identity  $\ell \times \ell$  matrix can be included among those of  $H$ , so that  $H$  has rank  $\ell$ . It follows that the matrix  $H$  determines a linear  $q$ -ary code  $C$  of length

$$n = \frac{q^\ell - 1}{q - 1}.$$

The minimum distance of  $H$  is at least 3 by construction, and in fact it is exactly 3 since we can include among the columns of  $H$  three linearly dependent ones,  $(10 \dots 0)^T$ ,  $(110 \dots 0)^T$ ,  $(010 \dots 0)^T$ . Thus  $C$  is 1-error correcting: its dimension is  $k = n - \ell$  and its order is  $q^n$ . Such a code is known as a *Hamming code*. It is not surprising that Hamming codes have optimal properties.

**(14.4.11)** *Hamming codes are perfect.*

*Proof.* Let  $C$  be a  $q$ -ary Hamming code of length  $n$  constructed from a check matrix with  $\ell$  rows. Then

$$|C| = q^{n-\ell} = q^n/q^\ell = q^n/(1+n(q-1))$$

since  $n = \frac{q^\ell-1}{q-1}$ . Thus  $C$  attains the Hamming upper bound of (14.4.5), so it is a perfect code.  $\square$

**Example (14.4.7)** Let  $q = 2$  and  $\ell = 4$ . A Hamming code  $C$  of length  $n = \frac{2^4-1}{2-1} = 15$  can be constructed from the  $4 \times 15$  check matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Here  $|C| = 2^{n-\ell} = 2^{11} = 2048$ . Similarly, by taking  $q = 2$  and  $\ell = 5$  we can construct a perfect linear binary code of length 31 and dimension 26.

**Perfect codes.** We conclude with an analysis of perfect codes which will establish the unique position of the Hamming codes.

**(14.4.12)** Let  $C$  be a perfect  $q$ -ary code where  $q = p^a$  and  $p$  is a prime. Assume that  $C$  is 1-error correcting. Then:

- (i)  $C$  has length  $\frac{q^s-1}{q-1}$  for some  $s \geq 1$ ;
- (ii) if  $C$  is linear, it is a Hamming code.

*Proof.* (i) Let  $C$  have length  $n$ . Then  $|C| = \frac{q^n}{1+n(q-1)}$  since  $C$  is perfect and 1-error correcting. Hence  $1+n(q-1)$  divides  $q^n$ , so it must be a power of  $p$ , say  $1+n(q-1) = p^r$ . By the Division Algorithm we can write  $r = sa + t$  where  $s, t \in \mathbb{Z}$  and  $0 \leq t < a$ . Then

$$1+n(q-1) = p^r = (p^a)^s p^t = q^s p^t = (q^s-1)p^t + p^t.$$

Therefore  $q-1$  divides  $p^t-1$ . However  $p^t-1 < p^a-1 = q-1$ , which shows that  $p^t = 1$  and  $1+n(q-1) = p^{as} = q^s$ . It follows that  $n = \frac{q^s-1}{q-1}$ .

(ii) Now assume that  $C$  is linear. Since  $|C| = \frac{q^n}{1+n(q-1)}$  and we have shown in (i) that  $1+n(q-1) = q^s$ , it follows that  $|C| = q^n/q^s = q^{n-s}$ . Hence  $\dim(C) = n-s$  and a check matrix  $H$  for  $C$  has size  $s \times n$ . The number of columns of  $H$  is  $n = \frac{q^s-1}{q-1}$ , which is the maximum number possible, and no column is a multiple of another one since  $C$  is 1-error correcting and thus has minimum distance  $\geq 3$ . Therefore  $C$  is a Hamming code.  $\square$

Almost nothing is known about perfect  $q$ -ary codes when  $q$  is not a prime power. Also there are very few perfect linear  $q$ -ary codes which are  $e$ -error correcting with  $e > 1$ . Apart from binary repetition codes of odd length – see Exercise (14.4.3) below –

there are just two examples, a binary code of length 23 and a ternary code of length 11. These remarkable examples, known as the *Golay codes*, are of great importance in algebra: see [18] for details.

### Exercises (14.4)

- (1) Give an example of a code for which the minimum distance is different from the minimum weight of a non-zero codeword.
- (2) Find the number of  $q$ -ary words with weights in the range  $i$  to  $i + k$ .
- (3) Let  $C$  be the set of all words  $(aa \cdots a)$  of length  $n$  where  $a \in \text{GF}(q)$ .
  - (i) Show that  $C$  is a linear  $q$ -ary code of dimension 1.
  - (ii) Find the minimum distance and error correcting capacity of  $C$ .
  - (iii) Write down a generator matrix and a check matrix for  $C$ .
  - (iv) Show that when  $q = 2$ , the code  $C$  is perfect if and only if  $n$  is odd.
- (4) Let  $C$  be a  $q$ -ary code of length  $n$  and minimum distance  $d$ . Establish the *Singleton upper bound*  $|C| \leq q^{n-d+1}$ . [Hint: two codewords with the same first  $n - d + 1$  entries are equal].
- (5) If  $C$  is a linear  $q$ -ary code of length  $n$  and dimension  $k$ , prove that the minimum distance of  $C$  is at most  $n - k + 1$ .
- (6) Let  $C$  be a linear  $q$ -ary code of length  $n$  and dimension  $k$ . Suppose that  $G$  is a generator matrix for  $C$  and that  $G' = [I_k \mid A]$  is the reduced row echelon form of  $G$ . Prove that there is a check matrix for  $C$  of the form  $[-A^T \mid I_{n-k}]$  up to a permutation of columns.
- (7) A linear binary code  $C$  has basis  $\{(101110), (011010), (001101)\}$ . Find a check matrix for  $C$  and use it to determine the error-correcting capacity of  $C$ .
- (8) A check matrix for a linear binary code  $C$  is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

- (i) Find a basis for  $C$ .
  - (ii) Find the minimum distance and error correcting capacity of  $C$ .
  - (iii) If a word  $(01111)$  is received and at most one entry is erroneous, use the syndrome method to find the transmitted codeword.
- (9) (*An alternative decoding procedure*). Let  $C$  be a linear  $q$ -ary code of length  $n$  with error correcting capacity  $e$ . Let  $H$  be a check matrix for  $C$ . Suppose that a word  $w$  is received with at most  $e$  errors. Show that the following procedure will find the transmitted codeword.
  - (i) Enumerate all words  $u$  in  $H_n(q)$  of weight  $\leq e$ ; these are the possible errors.
  - (ii) Find the syndrome  $uH^T$  of each word  $u$  from (i).

- (iii) Compute the syndrome  $wH^T$  and compare it with each  $uH^T$ : prove that there is a unique word  $u$  in  $H_n(q)$  of with weight at most  $e$  such that  $uH^T = wH^T$ .
- (iv) Show that the transmitted codeword was  $w - u$ .
- (10) Prove that the number of possible words  $u$  in Exercise (14.4.9) is a polynomial in  $n$ .
- (11) Use the method of Exercise (14.4.9) to find the transmitted codeword in Exercise (14.4.8).
- (12) (*Dual codes*). Let  $C$  be a linear  $q$ -ary code of length  $n$  and dimension  $k$ . Define the *dot product* of two words  $v, w$  in  $H_n(q)$  by  $v \cdot w = \sum_{i=1}^n v_i w_i$ . Then define  $C^\perp = \{w \in H_n(q) \mid w \cdot c = 0, \forall c \in C\}$ .
- Show that  $C^\perp$  is a linear  $q$ -ary code of length  $n$ : this is called the *dual code* of  $C$ .
  - Let  $G$  and  $H$  be a generator matrix and a check matrix for  $C$ . Prove that  $G$  is a check matrix and  $H$  a generator matrix for  $C^\perp$ .
  - Prove that  $\dim(C^\perp) = n - k$  and  $|C^\perp| = q^{n-k}$ .
- (13) Let  $C$  be a binary Hamming code of length 7. Find a check matrix for the dual code  $C^\perp$  and show that its minimum distance is 4.

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# List of symbols

$A, B, \dots$ : sets	$GL_n(R), GL_n(q)$ : general linear groups
$a, b, \dots$ : elements of sets	$SL_n(R), SL_n(q)$ : special linear groups
$a \in A$ : $a$ is an element of the set $A$	$ G : H $ : the index of $H$ in $G$
$ A $ : the cardinal of a set	$N \triangleleft G$ : $N$ is a normal subgroup of the group $G$
$A \subseteq B, A \subset B$ : $A$ is a subset, proper subset of $B$	$G/N$ : the quotient group of $N$ in $G$
$\emptyset$ : the empty set	$\simeq$ : an isomorphism
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ : the sets of natural numbers, integers, rational numbers, real numbers, complex numbers	$\text{Ker}(\alpha)$ : the kernel of a homomorphism
$\cup, \cap$ : union and intersection	$Z(G)$ : the center of the group $G$
$A_1 \times \dots \times A_n$ : a set product	$[x, y]$ : the commutator $xyx^{-1}y^{-1}$
$A - B, \bar{A}$ : complementary sets	$G'$ : the derived subgroup of a group $G$
$\mathcal{P}(A)$ : the power set	$G^{(i)}$ : the $i$ th term of the derived chain of the group $G$
$S \circ R$ : the composite of relations or functions	$Z_i(G)$ : the $i$ th term of the upper central chain of the group $G$
$[x]_E$ : the $E$ -equivalence class of $x$	$\phi(G)$ : the Frattini subgroup of a group $G$
$\alpha : A \rightarrow B$ : a function from $A$ to $B$	$N_G(H), C_G(H)$ : normalizer and centralizer of $H$ in $G$
$\text{Im}(\alpha)$ : the image of the function $\alpha$	$\text{Aut}(G), \text{Inn}(G)$ : the automorphism and inner automorphism groups of a group $G$
$\text{id}_A, \text{id}$ : the identity function on the set $A$	$\text{Out}(G)$ : the outer automorphism group of a group $G$
$\alpha^{-1}$ : the inverse of a bijective function $\alpha$	$\langle X   R \rangle$ : a presentation of a group or module
$\text{Fun}(A, B)$ : the set of all functions from $A$ to $B$	$U(R), R^*$ : the group of units of a ring $R$
$\text{Fun}(A)$ : the set of all functions on $A$	$R^{\text{opp}}$ : the opposite ring of $R$
$\text{gcd}, \text{lcm}$ : greatest common divisor, least common multiple	$RX, (x)$ : ideals generated by a set of elements
$a \equiv b \pmod{m}$ : a congruence	$R[t_1, \dots, t_n]$ : the ring of polynomials in $t_1, \dots, t_n$ over a ring $R$
$[x]_m$ or $[x]$ : the congruence class of $x$ modulo $m$	$F\{t_1, \dots, t_n\}$ : The field of rational functions in $t_1, \dots, t_n$ over a field $F$
$\mathbb{Z}_n$ : the integers modulo $n$	$M_{m,n}(R)$ : the set of $m \times n$ matrices over a ring $R$
$a   b$ : $a$ divides $b$	$\text{diag}(d_1, d_2, \dots, d_n)$ : the diagonal matrix with $d_1, d_2, \dots, d_n$ on the principal diagonal
$\phi$ : Euler's function	$\det(A), \text{tr}(A)$ : the determinant and trace of a matrix $A$
$\mu$ : the Möbius function	$\dim(V)$ : the dimension of a vector space $V$
$\lambda(n)$ : the number of partitions of $n$	$F\langle X \rangle, \langle X \rangle$ : subspace generated by a subset of an $F$ -vector space
$\text{sign}(\pi)$ : the sign of a permutation $\pi$	$[v]_{\mathcal{B}}$ : the coordinate vector of $v$ with respect to a basis $\mathcal{B}$
$(i_1 i_2 \dots i_r)$ : a cyclic permutation	$C[a, b]$ : the vector space of continuous functions on the interval $[a, b]$
$\text{St}_G(x)$ : the stabilizer of $x$ in $G$	$L(V, W), L(V)$ : vector spaces of linear mappings
$G \cdot a$ : the $G$ -orbit of $a$	$\text{GF}(q)$ : the field with $q$ elements
$\text{Fix}(G)$ : the set of points fixed by a group $G$	$(E : F)$ : the degree of $E$ over $F$
$\langle X \rangle$ : the subgroup or subspace generated by $X$	$\text{Gal}(E/F), \text{Gal}(f)$ : Galois groups
$ x $ : the order of a group element $x$	$\deg(f)$ : the degree of a polynomial $f$
$XY, X + Y$ : product, sum of subsets of a group	
$H \leq G, H < G$ : $H$ is a subgroup, proper subgroup of the group $G$	
$\text{Dr}_{\lambda \in \Lambda} G_\lambda, G_1 \times \dots \times G_n$ : direct products of groups	
$\text{Sym}(X)$ : the symmetric group on a set $X$	
$S_n, A_n$ : symmetric and alternating groups of degree $n$	
$\text{Dih}(2n)$ : the dihedral group of order $2n$	

$f'$ : the derivative of a polynomial  $f$

$\text{Irr}_F(x)$ : the irreducible polynomial of  $x$  over  $F$

$\Phi_n$ : the cyclotomic polynomial of order  $n$

${}_R M$  and  $N_R$ : left and right  $R$ -modules.

${}_R M_S$ : an  $(R, S)$ -bimodule.

$R \cdot X, R \cdot \alpha$ : submodules generated by a set of elements

$\bigoplus_{\lambda \in \Lambda} M_\lambda, M_1 \oplus \cdots \oplus M_n$ : direct sums of modules

$\text{rank}(F)$ : the rank of a free module.

$M_p$ : the  $p$ -torsion component of a module

$\text{Ann}_R(X), \text{Ann}_R(x)$ : annihilators in a ring  $R$

$\text{Hom}_R(M, N)$ : a group of homomorphisms

$\alpha_*, \alpha^*$ : induced mappings

$a \otimes b$ : a tensor

$M \otimes_R N, M \otimes N$ : tensor products of modules

$\alpha \otimes \beta, A \otimes B$ : tensor products of homomorphisms, matrices

$H_n(q)$ : Hamming  $n$ -space over a set with  $q$  elements

$B_n(v)$ : the  $n$ -ball with center  $v$

$d(a, b)$ : the distance between points  $a$  and  $b$

$\text{wt}(v)$ : the weight of the word  $v$

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