ETH Zurich HS 2019
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 5

Exercise 5.1

(a) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables converging in probability to a random variable X. Show that $(X_n)_{n\in\mathbb{N}}$ converges to X in distribution.

(b) The converse does not hold in general. Instead, show that if the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution to a *constant* random variable $X = c$, then $(X_n)_{n\in\mathbb{N}}$ converges in probability to c.

Exercise 5.2 Compute the characteristic functions of the following distributions:

(a) The triangular distribution $(1-|x|)1_{[-1,1]}(x)\mathrm{d}x$.

(b) The Cauchy distribution $\frac{\alpha}{\pi}\frac{1}{x^2+\alpha^2}\mathrm{d}x$ with parameter $\alpha > 0$.
Hint: Use a contour integral.

Exercise 5.3 Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of probability measures with

$$\int_{\mathbb{R}} x^k \, P_n(dx) \overset{n\to\infty}{\longrightarrow} \alpha_k \in \mathbb{R} \qquad \text{for all } k \in \mathbb{N}.$$

Assume that there exists *exactly one* probability measure P with the k-th moment $\alpha_k = \int_{\mathbb{R}} x^k \, P(dx)$. Show that $(P_n)_{n\in\mathbb{N}}$ converges weakly towards P.

Hint: First show that $(P_n)_{n\in\mathbb{N}}$ is tight. Then show that each subsequence of $(P_n)_{n\in\mathbb{N}}$ has a sub-subsequence that converges weakly towards P.

Remark: Note that in general, there is no uniqueness of such P. A counter-example would be $Y = e^X$ with X a standard normal random variable.

Submission: until 14:15, Oct 29., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint