REAL ANALYSIS

LECTURE NOTES

Abstract. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:

[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.

[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

1. Measurable sets and their properties

1.1. Increasing unions and decreasing Intersections.

Proposition 1.1. *Let $\{E_k\}_{j\geq 1}$ be a sequence of measurable sets in $\mathbb{R}^n$.*

(i) *If $E_k \nearrow E$, then $m(E) = \lim_{N\to\infty} m(E_N)$.*
(ii) *If $E_k \searrow E$ and $m(E_k) < \infty$ for some k, then $m(E) = \lim_{N\to\infty} m(E_N)$.*

Remark 1.1. *Consider $E_k = (k, \infty) \subset \mathbb{R}$. Then $E_k \searrow \emptyset$. Hence $m(E) < \lim_{N\to\infty} E_N$, which shows (ii) in Proposition 1.1 may fail without the assumption $m(E_k) < \infty$.*

Proof. We show (i). Write $E = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \cup \cdots = \bigcup_{k=1}^{\infty}(E_k \setminus E_{k-1})$, where $E_0 = \emptyset$. Since $E_{k+1} \setminus E_k$ are disjoint, by the additivity,

$$m(E) = \sum_{k=1}^{\infty} m(E_k \setminus E_{k-1}) = \lim_{N\to\infty} \sum_{k=1}^{N} m(E_k \setminus E_{k-1}) = \lim_{N\to\infty} m(E_N).$$

For (ii), observe that $E_1 \setminus E_k \searrow E_1 \setminus E$. Without loss of generality, we simply assume $m(E_1) < \infty$. Applying part (i) to $\{E_1 \setminus E_k\}_{k\geq 1}$, we see that

$$m(E_1 \setminus E) = \lim_{N\to\infty} m(E_1 \setminus E_N)$$

which yields

$$m(E) = m(E_1) - \lim_{N\to\infty} m(E_1 \setminus E_N) = \lim_{N\to\infty} \Big(m(E_1) - m(E_1 \setminus E_N)\Big) = \lim_{N\to\infty} m(E_N).$$

□

Corollary 1.1. *Let $\{E_k\}_{j\geq 1}$ be a sequence of measurable sets in $\mathbb{R}^n$. Then*

$$m(\liminf_{k\to\infty} E_k) = \lim_{k\to\infty} m\big(\bigcap_{j\geq k} E_j\big) \leq \liminf_{k\to\infty} m(E_k),$$

and, if $m(\bigcup_{j\geq k} E_j) < \infty$ for some k,

$$m(\limsup_{k\to\infty} E_k) = \lim_{k\to\infty} m\big(\bigcup_{j\geq k} E_j\big) \geq \limsup_{k\to\infty} m(E_k).$$

The results hold for exterior measure of a sequence of sets (not necessarily measurable).

Proposition 1.2. *Let $\{E_k\}_{j\geq 1}$ be a sequence of sets in $\mathbb{R}^n$.*

(i) *If $E_k \nearrow E$, then $m_*(E) = \lim_{N\to\infty} m_*(E_N)$.*
(ii) *$m_*(\liminf_{k\to\infty} E_k) = \lim_{k\to\infty} m_*\big(\bigcap_{j\geq k} E_j\big) \leq \liminf_{k\to\infty} m_*(E_k)$.*

Proof. By Observation 3 of the exterior measure, for $\varepsilon > 0$, there are open sets $\mathcal{O}_k \supset E_k$ with $m(\mathcal{O}_k) \leq m_*(E_k) + \varepsilon$. Since E_k is increasing, $E_k \subset \bigcap_{j\geq k} \mathcal{O}_j$. By Corollary 1.1,

$$m_*(E) \leq m(\liminf_{k\to\infty} \mathcal{O}_k) \leq \liminf_{k\to\infty} m(\mathcal{O}_k) \leq \liminf_{k\to\infty} m(E_k) + \varepsilon,$$

which gives $m_*(E) \leq \liminf_{k\to\infty} m(E_k)$. On the other hand, by the monotonicity,

$$m_*(E) \geq \limsup_{k\to\infty} m_*(E_k).$$

This proves part (i).

Part(ii) is a consequence of part (i).

□

Exercise 1.1. *Let A_t, $t \in (0,1)$, be sets in $\mathbb{R}^n$, and $A_{t_1} \subset A_{t_2}$ when $t_1 < t_2$. Show*

(i) *$m_*(\bigcup_{t\in(0,1)} A_t) = \lim_{t\to 1-} m_*(A_t)$;*
(ii) *$m_*(\bigcap_{t\in(0,1)} A_t) \leq \lim_{t\to 0+} m_*(A_t)$;*
(iii) *Suppose A_t are measurable. Then $m(\bigcup_{t\in(0,1)} A_t) = \lim_{t\to 1-} m(A_t)$; and*

$$m\big(\bigcap_{t\in(0,1)} A_t\big) = \lim_{t\to 0+} m(A_t),$$

provided $m(A_{t_0}) < \infty$ for some $t_0 \in (0,1)$.

Exercise 1.2. *Let $E \subset \mathbb{R}$, $0 < m_*(E) < \infty$. Show $f(x) = m_*((-\infty, x)\cap E)$ is Lipschitz continuous, and $I = \{m_*(F) : F \subset E\}$ is a bounded closed interval.*

1.2. **Cubes/Rectangles and sets with positive measure.**

One may image that a set with positive measure would stay compactly in a region, and thus there would be a cube so that E occupies a big portion of it. This is true.

Proposition 1.3. *Let E be a measurable set of $\mathbb{R}^n$ with $m(E) > 0$. The for each $0 < \lambda < 1$, there exists a cube Q such that*

$$m(E \cap Q) > \lambda |Q|. \tag{1.1}$$

Proof. This is an exercise. □

Remark 1.2. *Proposition 1.3 holds for all $E \subset \mathbb{R}^n$ with $m_*(E) > 0$. The measurability is not necessary.*

Proposition 1.4. *Let E be a measurable set of $\mathbb{R}^n$ with $m(E) > 0$. Then the difference set of E, which is defined by*

$$E_\# := E - E = \{z : \ z = x - y \text{ for some } x, y \in E\},$$

contains an open ball centred at the origin.

Proof. This is an exercise.

□

In general we have the following result.

Proposition 1.5. *Let E_1 and E_2 be two measurable sets, $m(E_1) > 0$ and $m(E_2) > 0$. Show that $E := \{x - y : \ x \in E_1, \ y \in E_2\}$ has non-empty interior.*

Remark 1.3. *As a corollary, we see that if $m(E_1) > 0$ and $m(E_2) > 0$, then $A + B$ has non-empty interior. For this end, one applies the above result to E_1 and $-E_2$ and notices that $m(-E_2) = m(E_2)$.*

Proof of Proposition 1.5. The strategy is as follows: show that there is a $p \in \mathbb{R}^n$ such that $m(E_1 \cap (E_2)_p) > 0$, where $(E_2)_p = \{x + p : x \in E_2\}$ is the translation of E_2. Once this is obtained, we take $\tilde{E} = E_1 \cap (E_2)_p$, then applying Proposition 1.4 to conclude that $\tilde{E}_\#$ contains an open ball centred at the origin. Since $\tilde{E}_\# \subset E_{-p}$, we deduce that E contains an interior point p.

□

1.3. **Approximating measurable sets.**

An important geometric and analytic insight into the nature of measurable sets is that, in effect, an arbitrary measurable set can be well approximated by the open sets that contain it, and alternatively, by the closed sets it contains.

Theorem 1.1. *Let E be a measurable set in $\mathbb{R}^n$. Then, for every $\varepsilon > 0$:*

(i) *There is an open set $\mathcal{O} \supset E$ with $m(\mathcal{O} - E) \leq \varepsilon$.*
(ii) *There is a closed set $F \subset E$ with $m(E - F) \leq \varepsilon$.*
(iii) *If $m(E) < \infty$, then there is a compact set $K \subset E$ with $m(E - K) \leq \varepsilon$.*
(iv) *If $m(E) < \infty$, then there is a finite union $F = \bigcup_{j=1}^N Q_j$ of closed cubes such that*

$$m(E \Delta F) \leq \varepsilon.$$

Proof. Part (i) is directly from the definition of measurability of E.

Part (ii) is from the definition of measurability of E^c and the take complement.

Part (iii) follows by taking $K = F \cap \overline{B}_N$ for sufficiently large N. More precisely, we select F such that $m(E - F) < \varepsilon/2$. Let $K_n := F \cap B_n$. On the other hand, since $E - K_n \searrow E - F$ and $m(E - F) < \infty$, we have $m(E - K_N) \leq m(E - F) + \varepsilon/2$ for some large N. We are done.

For part (iv), we first take almost disjoint closed cubes Q_j such that $E \subset \bigcup_{j\geq 1} Q_j$ and $\sum_{j=1}^\infty |Q_j| \leq m(E) + \varepsilon/2$. Since $m(E) < \infty$, we next choose large N so that $\sum_{j=N+1}^\infty |Q_j| \leq \varepsilon/2$. Let $F = \bigcup_{j=1}^N Q_j$. Then

$$m(E\Delta F) = m(E - F) + m(F - E) \leq m\Big(\bigcup_{j\geq N+1} Q_j\Big) + m\Big(\bigcup_{j\geq 1} Q_j - E\Big) \leq \varepsilon.$$

□

1.4. **Invariance properties of Lebesgue measure.**

Recall that the exterior measure satisfies the some invariance properties. One can check that measurability is invariant under the translation, dilation, reflection and linear transformation; more precisely, if $E \in \mathscr{M}_{\mathbb{R}^n}$ then

(a) translation-invariance: $E_h = \{x + h : x \in E\} \in \mathscr{M}_{\mathbb{R}^n}$; and $m(E) = m(E_h)$.
(b) dilation-invariance: $\lambda E = \{\lambda x : x \in E\} \in \mathscr{M}_{\mathbb{R}^n}$; and $m(\lambda E) = |\lambda|^n m(E)$.

(c) linear transformation invariance: $\mathcal{L}(E) \in \mathscr{M}_{\mathbb{R}^n}$, where $\mathcal{L}$ is a linear transformation of $\mathbb{R}^n$; and $m(\mathcal{L}(E)) = |\det \mathcal{L}| m(E)$.

Using the definition, one can give a quick check for (a) and (b).

Exercise: Show (c). The quantitative statement $m((E)) = |\det \mathcal{L}| m(E)$ is a consequence of the Fubini's theorem, which will be discussed later on.

Furthermore, it is straightforward to see the aforementioned properties (a) and (b) hold for exterior measure and all subsets of $\mathbb{R}^n$, not necessarily measurable.

For each $E \subset \mathbb{R}^n$ (not necessarily measurable), if $\det \mathcal{L} \neq 0$, we also have

$$m_*(\mathcal{L}(E)) = |\det \mathcal{L}| m_*(E) \tag{1.2}$$

If $\det \mathcal{L} \neq 0$ and $m_*(E) < \infty$, then (1.2) still holds. Note that (1.2) implies $m(\mathcal{L}(E)) = |\det \mathcal{L}| m(E)$ for all $m \in \mathscr{M}_{\mathbb{R}^n}$ without using the Fubini's theorem.

Exercise: Prove the above statement.

Before closing this subsection, we summarise a general result that was used above when we show the measurability of $\mathcal{L}(E)$.

Proposition 1.6. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map.*

(i) *If E is an F_σ set then $T(E)$ is an F_σ set.*
(ii) *If $T(Z)$ is of measure zero for all Z with $m(Z) = 0$, then $T(E) \in \mathscr{M}_{\mathbb{R}^n}$ for all $E \in \mathscr{M}_{\mathbb{R}^n}$.*

Proof. Given an F_σ set E, we have $E = \cup_{j\geq 1} F_j = \cup_{j\geq 1} \cup_{k\geq 1} (F_j \cap \overline{B}_k) = \cup_{i\geq 1} K_i$, where F_j are closed sets and K_i are compact sets. Then $T(E) = \cup_{i\geq 1} T(K_i)$. Since $T(K_i)$ is compact, $T(E)$ is an F_σ set.

By Proposition 1.7, for each $E \in \mathscr{M}_{\mathbb{R}^n}$, $E = F \cup Z$, where F is an F_σ set and Z is of measure zero. Since $T(E) = T(F) \cup T(Z)$, we conclude that $T(E)$ is a union of an F_σ set and a zero measure set. Hence $T(E) \in \mathscr{M}_{\mathbb{R}^n}$.

□

1.5. σ-algebras and Borel sets.

Let X be a set.

Definition 1.1. *A σ-algebra in X is a collection of subsets of X that is closed under countable unions, countable intersections, and complements.*

Remark 1.4. *We collect some simple facts.*

(i) 2^X *is of course a σ-algebra in X.*
(ii) $\mathscr{M}_{\mathbb{R}^n}$*, the collection of measurable sets of* $\mathbb{R}^n$*, is a σ-algebra.*
(iii) *For any collection $\mathscr{C}$ of subsets of X, there is a smallest σ-algebra $\mathscr{P}(\mathscr{C})$ containing $\mathscr{C}$.*

Proof: 2^X *is certainly a σ-algebra containing $\mathscr{C}$. Note that the intersection of any family of σ-algebras is itself a σ-algebra. Hence $\mathscr{P}(\mathscr{C})$ is the intersection of the family of all σ-algebras containing $\mathscr{C}$. Clearly such intersection exists, is a σ-algebra containing $\mathscr{C}$; and moreover if $\mathscr{T}$ is any σ-algebra containing $\mathscr{C}$ then $\mathscr{P}(\mathscr{C}) \subset \mathscr{T}$.*

Definition 1.2. *The Borel σ-algebra in $\mathbb{R}^n$, denoted by $\mathscr{B}_{\mathbb{R}^n}$, is the smallest σ-algebra that contains all open sets. The elements of this σ-algebra are called Borel sets.*

Remark 1.5. *Obviously $\mathscr{B}_{\mathbb{R}^n} \subset \mathscr{M}_{\mathbb{R}^n}$. Is this inclusion is strict? The answer is "yes".*
Claim: If $\phi \in C(\mathbb{R})$ is strictly increasing/deceasing, then $\phi(E) \in \mathscr{B}_{\mathbb{R}}$, $\forall\ E \in \mathscr{B}_{\mathbb{R}}$. [1]
Proof: Let $\mathcal{B}^* = \{E :\ E \text{ and } \phi(E) \text{ are both Borel sets}\}$. We have

- *For $E \in \mathcal{B}^*$, we have $\phi(E^c) = \phi(\mathbb{R}) - \phi(E) \in \mathscr{B}_{\mathbb{R}^n}$, and so $E^c \in \mathcal{B}^*$.*
- *For a sequence $E_j \in \mathcal{B}^*$, we have $\bigcup_{j\geq 1} E_j$ and $\phi\big(\bigcup_{j\geq 1} E_j\big) = \bigcup_{j\geq 1} \phi(E_j)$ are both Borel sets.*

Hence $\mathcal{B}^*$ is a σ-algebra, and by the continuity of ϕ, containing all open sets. Therefore $\mathcal{B}^* = \mathscr{B}_{\mathbb{R}}$. This shows the Claim.

Recall the Cantor-Lebesgue function $g : [0,1] \to [0,1]$ at the end of the preliminary part. Take $\phi(x) = g(x) + x : [0,1] \to [0,2]$. We have

- *ϕ is continuous and strictly increasing.*
- *$\phi([0,1]) = [0,2]$ and, $m(\phi(G)) = 1$ where $G = [0,1] \setminus \mathcal{C}$ (note $g(G)$ is discrete).*

[1] It is not true that the image of a Borel set under a continuous function is necessarily Borel. Such images are called Souslin sets.

Since $\phi(\mathcal{C}) = [0,2] \setminus \phi(G)$, one sees that $\phi(\mathcal{C})$ is measurable and $m\big(\phi(\mathcal{C})\big) = 1$. Since it has positive measure, we can construct a non-measurable subset of $\phi(\mathcal{C})$, say $\mathcal{N}_{\phi(\mathcal{C})}$ [2]*. Let $E = \phi^{-1}(\mathcal{N}_{\phi(\mathcal{C})})$. Obviously $E \subset \mathcal{C}$, and so it is measurable. On the other hand, since $\phi(E) = \mathcal{N}_{\phi(\mathcal{C})}$ is not measurable, we infer by the Claim $E \notin \mathscr{B}_{\mathbb{R}}$. Hence we obtain a set $E \in \mathscr{M}_{\mathbb{R}} \setminus \mathscr{B}_{\mathbb{R}}$ as desired.*

Recall that E is called G_δ if it is a countable intersection of open sets; and is called F_σ if it is a countable union of closed sets.

One sees that G_δ sets and F_σ sets [3] are the simplest Borel sets after the open and closed sets. The Lebesgue measurable sets arise as the completion of the Borel σ-algebra.

Proposition 1.7. *A subset E of $\mathbb{R}^n$ is measurable*

(i) *if and only if E differs from a G_δ by a set of measure zero,*

(ii) *if and only if E differs from an F_σ by a set of measure zero.*

Proof. Only show if $E \in \mathscr{M}_{\mathbb{R}^n}$ then E differs from a G_δ or F_σ by a set of measure zero.

Let $\mathcal{O}_j$ be a sequence of open sets such that $E \subset \mathcal{O}_j$ and $m(\mathcal{O}_j - E) \le 1/j$, and F_j be a sequence of closed sets such that $F_j \subset E$ and $m(E - F_j) \le 1/j$.

Take $G = \cap_{j\ge 1}\mathcal{O}_j$ and $F = \cup_{j \ge 1} F_j$. Clearly G is G_δ and F is F_σ. We also have

$$m(G - E) \le m(\mathcal{O}_j - E) \le 1/j \implies m(G - E) = 0,$$

and

$$m(E - F) \le m(E - F_j) \le 1/j \implies m(E - F) = 0.$$

□

The proof of Proposition 1.7 yields indeed for any $E \subset \mathbb{R}$ (not necessarily measurale), there is a G_δ set (hence is measurable) $U \supset E$ such that $m(U) = m_*(E)$.

Proposition 1.8. *Let $E \subset \mathbb{R}^n$. There is a G_δ set G such that $E \subset G$ and $m_*(E) = m(G)$.*

---

[2]Claim: if $G \subset \mathbb{R}$ and $m_*(G) > 0$, then a subset of G is non-measurable. *Proof. Obviously there is a unit interval I such that $m_*(G \cap I) > 0$. Denote $G' = G \cap I$. Let $\mathcal{N}_k = \mathcal{N} + r_k$, $r_k \in \mathbb{Q}$. Then $G' \subset \cup_{k\ge 1}\mathcal{N}_k$. It can be verified that each measurable subset of $\mathcal{N}$ is of zero measure. If each $G' \cap \mathcal{N}_k$ is measurable, then $m_*(G') \le \sum_k m_*(G' \cap \mathcal{N}_k) = 0$. A contradiction!*

[3]The terminology G_δ comes from German "Gebiete" and "Durschnitt"; F_σ comes from French "fermé" and "somme".

Proof. This is an exercise. □

Remark 1.6. *For set G in Proposition 1.8, it does not follows that $m_*(G \setminus E) = 0$ in general. However, if H is a measurable subset of $G \setminus E$, then $m(H) = 0$. This is because $E \subset G \setminus H \subset G$, and so by the monotonicity,*

$$m_*(E) \leq m(G \setminus H) \leq m(G) \implies m(G \setminus H) = m(G) \implies m(H) = 0.$$

Exercise 1.3. *Suppose $m_*(E) < \infty$. Show that E is measurable if and only if there is a sequence of measurable sets $E_j \subset E$ such that $m(E_j) \to m_*(E)$.*

Exercise 1.4. *If $A \cup B$ is a measurable set in $\mathbb{R}^n$ and $m(A \cup B) = m_*(A) + m_*(B) < \infty$, then A and B are both measurable.*