ETH Zurich HS 2019
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 3

Exercise 3.1 Assume that $X_k = -\frac{1}{k^{1.5}} + \frac{Z_k}{k^\alpha}$, for $k \geq 1$, where Z_k are i.i.d random variables with $P[Z_k = 1] = P[Z_k = -1] = P[Z_k = 0] = \frac{1}{3}$ and $\alpha > 0$. Discuss the convergence of the random series $\sum_{k\geq 1} X_k$.

Exercise 3.2 Let $\mathcal{M}$ be the set of the real-valued random variables on the probability space $(\Omega, \mathcal{A}, P)$. We define on $\mathcal{M}$ an equivalence relation as follows:

$$X \sim Y \quad :\iff \quad P(X = Y) = 1$$

We denote by $\mathcal{M}/\sim$ the set of equivalence classes in $\mathcal{M}$ with respect to $\sim$ and we denote by $[X]$ the equivalence class of $X \in \mathcal{M}$.

(a) Show that

$$\begin{aligned} d : (\mathcal{M}/\sim) \times (\mathcal{M}/\sim) &\rightarrow \mathbb{R} \\ ([X],[Y]) &\mapsto E\left[|X-Y| \wedge 1\right] \end{aligned}$$

is a metric on $\mathcal{M}/\sim$.

(b) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ and let X be an element of $\mathcal{M}$. Show that $([X_n])_{n\in\mathbb{N}}$ converges to $[X]$ with respect to the metric d if and only if $(X_n)_{n\in\mathbb{N}}$ converges to X in probability.

Exercise 3.3 Let X_i, $i \geq 1$, be identically distributed, integrable random variables and define $S_n = \sum_{i=1}^n X_i$ for each $n \in \mathbb{N}$. Show that:

$$\lim_{M\to\infty} \sup_{n\geq 1} E\left[\frac{|S_n|}{n} 1_{\left\{\frac{|S_n|}{n} > M\right\}}\right] = 0.$$

Note: This family $\left\{\frac{|S_n|}{n}, n \in \mathbb{N}\right\}$ is thus so-called "uniformly integrable". See (3.6.14) in the lecture notes. Thanks to Theorem 3.41 and the strong law of large numbers, one has that: if $X_i, i \geq 1$, are also pairwise independent, (in addition to being identically distributed as in the question), then $\frac{S_n}{n}$ converges P-a.s. and in L^1 towards $E[X_1]$ for $n \to \infty$.

Submission: until 14:15, Oct 15., during exercise class or in the tray outside of HG G 53.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Afa-Fül	Tue 13-14	HG F 26.5	Angelo Abächerli
Gan-Math	Tue 13-14	ML H 41.1	Zhouyi Tan
Meh-Schu	Tue 14-15	HG F 26.5	Angelo Abächerli
Schü-Zur	Tue 14-15	ML H 41.1	Dániel Bálint

Solution 3.1 Note first that the event $\{\sum_{k\geq 1} X_k \text{ converges}\}$ belongs to the asymptotic σ-algebra $\mathcal{F}_\infty$ associated with independent random variables X_k, $k \geq 1$. Therefore it follows from Theorem 1.30 (Kolmogorov's 0-1 law) that we have either $P[\sum_{k\geq 1} X_k \text{ converges}] = 0$ or $P[\sum_{k\geq 1} X_k \text{ converges}] = 1$.

Using now the same notation as in the statement of Theorem 1.37 (Kolmogorov's three-series theorem), we choose $A = 2$ and define $Y_k := X_k 1_{\{|X_k| \leq A\}}$ for $k \geq 1$. From the definition of X_k and the fact that $|Z_k| \leq 1$ we actually have $|X_k| \leq 2$ and thus $Y_k = X_k$ for all $k \geq 1$. This means in particular that condition $i)$ in (1.4.17) is satisfied. Moreover, $\mathrm{E}[Y_k] = -\frac{1}{k^{1.5}}$, so $\sum_{k\geq 1} \mathrm{E}[Y_k]$ converges and hence condition $ii)$ is also satisfied.

Now for condition $iii)$, since $\mathrm{Var}(Z_k) = \frac{2}{3}$ for all $k \geq 1$, we have

$$\mathrm{Var}(X_k) = \mathrm{Var}\left(-\frac{1}{k^{1.5}} + \frac{Z_k}{k^\alpha}\right) = \frac{1}{k^{2\alpha}}\mathrm{Var}(Z_k) = \frac{2}{3k^{2\alpha}}.$$

If $\alpha \leq \frac{1}{2}$, then we have $\sum_{k\geq 1} \mathrm{Var}(Y_k) = \sum_{k\geq 1} \frac{2}{3k^{2\alpha}} = \infty$, which implies that condition $iii)$ fails. Hence by Theorem 1.37, we obtain that $\sum_{k\geq 1} X_k$ cannot converge P-a.s., or in other words, $P[\sum_{k\geq 1} X_k \text{ converges}] < 1$. So by the intoductury remark it follows that $P[\sum_{k\geq 1} X_k \text{ converges}] = 0$. Similarly, if $\alpha > \frac{1}{2}$, $\sum_{k\geq 1} \mathrm{Var}(Y_k) < \infty$ and hence condition $iii)$ is satisified. Hence by Theorem 1.37, we obtain that $\sum_{k\geq 1} X_k$ converges P-a.s., or in other words, $P[\sum_{k\geq 1} X_k \text{ converges}] = 1$.

Solution 3.2

(a) We verify the criteria for d to be a metric

1. It is clear that d is well-defined;
2. From the definition of d we know that $\forall X, Y\ d([X],[Y]) = d([Y],[X])$;
3. It also follows from the definition of d that $\forall X\ d([X],[X]) = 0$;
4. That $d([X],[Y]) = 0$ for $X, Y \in L^0$ implies $X = Y$ P-.a.s., which further implies $[X] = [Y]$ in $\mathcal{M}/\sim$;
5. To prove that $\forall X, Y, Z \in L^0\ d([X],[Z]) \leq d([X],[Y]) + d([Y],[Z])$, it is sufficient to note that for all $a, b, c \in \mathbb{R}$,

$$|a - c| \wedge 1 \leq |a - b| \wedge 1 + |b - c| \wedge 1.$$

(b) Assume $d([X_n],[X]) \to 0$. With Chebyshev's inequality it follows that

$$P[|X_n - X| > \varepsilon] = P[|X_n - X| \wedge 1 > \varepsilon] \leq \frac{E[|X_n - X| \wedge 1]}{\varepsilon} \to 0.$$

For the converse, assume $P[|X_n - X| > \varepsilon] \to 0$ for each $\varepsilon > 0$. Then, it follows that

$$\begin{aligned} E[|X_n - X| \wedge 1] &\leq E[|X_n - X| \wedge 1, |X_n - X| < \varepsilon] \\ &\quad + E[|X_n - X| \wedge 1, |X_n - X| \geq \varepsilon] \\ &\leq \varepsilon + P[|X_n - X| \geq \varepsilon] < 2\varepsilon, \end{aligned}$$

for sufficiently large n.

Solution 3.3 Let $\widetilde{S}_n = \sum_{i=1}^n |X_i|$. Since $\widetilde{S}_n \geq |S_n|$, we have,

$$1_{\left\{\frac{|S_n|}{n} > M\right\}} \leq 1_{\left\{\frac{\widetilde{S}_n}{n} > M\right\}}$$

which implies that

$$E\left[\frac{|S_n|}{n} 1_{\left\{\frac{|S_n|}{n} > M\right\}}\right] \leq E\left[\frac{\widetilde{S}_n}{n} 1_{\left\{\frac{\widetilde{S}_n}{n} > M\right\}}\right].$$

Hence we can assume, without loss of generality, that $X_i \geq 0$ for all i. Then we have that for $A > 0$:

$$\begin{aligned}
E\left[\frac{S_n}{n} 1_{\left\{\frac{S_n}{n} > M\right\}}\right] &= E\left[\frac{1}{n}\left(\sum_{i=1}^n X_i 1_{\{X_i > A\}}\right) 1_{\left\{\frac{S_n}{n} > M\right\}}\right] + E\left[\frac{1}{n}\left(\sum_{i=1}^n X_i 1_{\{X_i \leq A\}}\right) 1_{\left\{\frac{S_n}{n} > M\right\}}\right] \\
&\leq E\left[\frac{1}{n}\sum_{i=1}^n X_i 1_{\{X_i > A\}}\right] + E\left[\frac{1}{n}\sum_{i=1}^n A\, 1_{\left\{\frac{S_n}{n} > M\right\}}\right] \\
&= E\left[X_1 1_{\{X_1 > A\}}\right] + A\ P\left[\frac{S_n}{n} > M\right] \\
&\overset{(*)}{\leq} E\left[X_1 1_{\{X_1 > A\}}\right] + \frac{A}{M} E\left[\frac{S_n}{n}\right] \\
&= E\left[X_1 1_{\{X_1 > A\}}\right] + \frac{A}{M} E[X_1],
\end{aligned}$$

where we have used the fact that $X_i \geq 0$ for all i and applied Chebyshev's inequality (1.2.13) at $(*)$.

Now we take $A = \sqrt{M}$. Then:

$$\varlimsup_{M\to\infty} \sup_{n\geq 1} E\left[\frac{S_n}{n} 1_{\{\frac{S_n}{n} > M\}}\right] \leq \varlimsup_{M\to\infty} E\left[X_1 1_{\{X_1 > \sqrt{M}\}}\right] + \varlimsup_{M\to\infty} \frac{1}{\sqrt{M}} E[X_1] = 0.$$

Where the last equality follows by dominated convergence and the fact that X_1 is integrable.