AN ELEMENTARY PROOF OF THE EXISTENCE OF ISOTHERMAL PARAMETERS ON A SURFACE

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1. Introduction. Let

$$(1) \qquad ds^2 = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2,$$

$$EG - F^2 > 0, \, E > 0,$$

be a positive definite Riemann metric of two dimensions defined in a neighborhood of a surface with the local coordinates x, y. By isothermal parameters we mean local coordinates u, v relative to which the metric takes the form

$$(2) \qquad ds^2 = \lambda(u, v)(du^2 + dv^2), \qquad \lambda(u, v) > 0.$$

In order that isothermal parameters exist it is necessary to impose on the metric some regularity assumptions. In fact, it was shown recently by Hartman and Wintner[1] that it is not sufficient to assume the functions E, F, G to be continuous. So far the weakest conditions under which the isothermal parameters are known to exist were found by Korn and Lichtenstein.[2] To formulate their theorem we recall that a function $f(x, y)$ in a domain D of the (x, y)-plane is said to satisfy a Hölder condition of order λ, $0 < \lambda \leq 1$, if the inequality

$$(3) \qquad |f(x, y) - f(x', y')| < Cr^\lambda$$

holds for any two points (x, y), (x', y') of D, where C is a constant and r is the Euclidean distance between these two points. With this definition the theorem of Korn-Lichtenstein can be stated as follows:

Suppose, in a domain D of the (x, y)-plane, the functions E, F, G satisfy a Hölder condition of order λ, $0 < \lambda < 1$. *Then every point of D has a neighborhood whose local coordinates are isothermal parameters.*

This theorem is rather useful in the global theory of surfaces, when analyticity assumptions are not desirable. We shall present here what seems to be an elementary and rather straightforward proof. Needless to say, the essential ideas of this proof are contained in the works of Korn and Lichtenstein. We believe, however, that some simplifications are achieved by the consistent use of the complex notation.[3]

We first observe that the isothermal parameters are invariant under conformal transformations, that is, multiplications of the Riemann metric (1) by a positive factor. Under a conformal transformation the angle between two vectors remains invariant. If we further assume that the coordinates x, y form a positive system, that is, if we allow only those transformations of local coordinates for which the Jacobian is positive, the angle can be defined, together with its orientation. To express these relations analytically, it will be convenient to introduce complex-valued differential forms, that is, forms $\omega = \alpha + i\beta$, where α, β are real differential forms. We shall write $\bar{\omega} = \alpha - i\beta$.

Since the quadratic differential form in the right-hand side of (1) is positive definite, we can write

$$(4) \qquad ds^2 = \theta_1^2 + \theta_2^2,$$

where θ_1, θ_2 are real linear differential forms:

$$(5) \qquad \begin{aligned} \theta_1 &= a_1 dx + b_1 dy, \\ \theta_2 &= a_2 dx + b_2 dy. \end{aligned}$$

Assuming that $a_1 b_2 - a_2 b_1 > 0$, the forms θ_1, θ_2 are determined up to a proper orthogonal transformation, that is, one with determinant $+1$. We put

$$(6) \qquad \phi = \theta_1 + i\theta_2,$$

so that

$$(7) \qquad ds^2 = \phi\bar{\phi}.$$

The form ϕ is then determined up to a complex factor of absolute value 1. A conformal transformation of the Riemann metric is given,

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[3] After this paper was submitted for publication, it has come to my attention that a similar proof was given by L. Bers in his mimeographed notes on Riemann surfaces, New York University, 1951–1952.

in terms of the form ϕ, by the multiplication of ϕ by an arbitrary nonzero complex factor. In the recent terminology we say that the complex linear differential form ϕ, determined up to a nonzero complex factor, defines an almost complex structure. As discussed above, such a structure allows the introduction of the notion of oriented angle.

The determination of isothermal parameters u, v is equivalent to that of a complex-valued function $w = u + iv$, such that

$$(8) \qquad\qquad dw = (1/\rho)\phi.$$

For we have then

$$(9) \qquad ds^2 = \phi\bar{\phi} = |\rho|^2 dw d\bar{w} = |\rho|^2 (du^2 + dv^2).$$

Conversely, the isothermal parameters u, v determine a function w satisfying (8).

2. **Preliminaries.** We shall first make estimates of certain integrals, which will be needed in the proof.

Let (ξ, η) be any point in the (x, y)-plane, and let

$$(10) \qquad r = + ((x - \xi)^2 + (y - \eta)^2)^{1/2}.$$

If $g(r)$ is a function of class 1, we have, by exterior differentiation,

$$d\left\{ g(r) \frac{-(y - \eta)dx + (x - \xi)dy}{r^2} \right\} = \frac{g'(r)}{r} dx \wedge dy.$$

It follows by Stokes Theorem that, if D is a domain bounded by a curve C and if $(\xi, \eta) \notin D$, we have

$$(11) \qquad \iint_D \frac{g'(r)}{r} dxdy = \int_C g(r) \frac{-(y - \eta)dx + (x - \xi)dy}{r^2}.$$

This formula remains true, even if $(\xi, \eta) \in D$ (but not on C), provided that the integral in the left-hand side is convergent. The integral

$$(12) \qquad \nu = \frac{1}{2\pi} \int_C \frac{-(y - \eta)dx + (x - \xi)dy}{r^2}, \qquad\qquad (\xi, \eta) \notin C$$

is an integer and is usually called the order of the point (ξ, η) relative to the curve C.

In particular, if $g(r) = r^\lambda$, $\lambda \neq 0$, formula (11) becomes

$$(13) \qquad \lambda \iint_D \frac{r^\lambda dxdy}{r^2} = \int_C r^\lambda \frac{-(y - \eta)dx + (x - \xi)dy}{r^2},$$

and is true either when $(\xi, \eta) \notin D$ or when $\lambda > 0$. It follows that, if the

vector joining (ξ, η) to a point $(x, y) \in C$ turns monotonely in the same sense, we have

$$(14) \qquad \left| \iint_D \frac{r^\lambda dx dy}{r^2} \right| \leq \frac{2\pi}{\lambda} \Delta^\lambda, \qquad\qquad \lambda > 0,$$

$$(15) \qquad \left| \iint_D \frac{r^\lambda dx dy}{r^2} \right| \leq \frac{2\pi}{|\lambda|} \delta^\lambda, \qquad\qquad \lambda < 0,$$

where Δ and δ denote respectively an upper bound and a lower bound of the distance from (ξ, η) to a point $(x, y) \in C$. We emphasize that formula (15) is valid only under the assumption $(\xi, \eta) \notin D$.

3. **Main lemma.** To simplify our formulas we shall use the complex coordinate $z = x + iy$ in the (x, y)-plane. If $f(x, y)$ is a complex-valued function of class 1, we define $f_z, f_{\bar{z}}$ by the equation

$$(16) \qquad df = f_z dz + f_{\bar{z}} d\bar{z} = f_z(dx + idy) + f_{\bar{z}}(dx - idy).$$

They are therefore related to the ordinary partial derivatives f_x, f_y by the equations

$$(17) \qquad f_z = (f_x - if_y)/2, \qquad f_{\bar{z}} = (f_x + if_y)/2.$$

We shall write the function as $f(z, \bar{z})$, thus emphasizing the fact that it is in general not analytic in z.

On the other hand, we can define these operators $f_z, f_{\bar{z}}$ directly, without using the partial derivatives f_x, f_y. For instance, we can adopt the definitions:

$$(18) \qquad \begin{aligned} f_z &= \lim_{h \to 0} \{ f(z + h/2, \bar{z} + h/2) - f(z, \bar{z}) \\ &\quad + if(z - hi/2, \bar{z} + hi/2) - if(z, \bar{z}) \}/h, \\ f_{\bar{z}} &= \lim_{h \to 0} \{ f(z + h/2, \bar{z} + h/2) - f(z, \bar{z}) \\ &\quad + if(z + hi/2, \bar{z} - hi/2) - if(z, \bar{z}) \}/h, \end{aligned}$$

which clearly give (17), when f_x, f_y exist.

LEMMA. *Let D be a disc of radius R about the origin in the $(x, y\text{-})$plane. Let $f(z, \bar{z})$ be a complex-valued continuous function in D, which satisfies the condition*

$$(19) \qquad |f(\zeta_1, \bar{\zeta}_1) - f(\zeta_2, \bar{\zeta}_2)| \leq Br_{12}^\lambda, \qquad r_{12} = |\zeta_1 - \zeta_2|,$$

for any two points $\zeta_1, \zeta_2 \in D$, where λ and B are constants, $0 < \lambda < 1$. Let the function $F(\zeta, \bar{\zeta})$ be defined by

$$(20) \quad -2\pi i F(\zeta, \bar{\zeta}) = \iint\limits_{D} \frac{f(z, \bar{z}) d\bar{z} dz}{z - \zeta} = 2i \iint\limits_{D} \frac{f(z, \bar{z}) dx dy}{z - \zeta}, \quad \zeta \in D.$$

Then: (1) F_ζ *and* $F_{\bar{\zeta}}$ *exist, and* $F_{\bar{\zeta}} = f(\zeta, \bar{\zeta})$; (2) *If* $|f(z, \bar{z})| \leq A$, $z \in D$, *the following inequalities are valid*:

$$(21) \qquad\qquad\qquad |F(\zeta, \bar{\zeta})| \leq 4RA,$$

$$(22) \qquad\qquad\qquad |F_\zeta(\zeta, \bar{\zeta})| \leq (2^{\lambda+1}/\lambda) R^\lambda B,$$

$$(23) \qquad |F(\zeta_1, \bar{\zeta}_1) - F(\zeta_2, \bar{\zeta}_2)| \leq 2(A + (2^{\lambda+1}/\lambda) R^\lambda B) r_{12},$$

$$(24) \qquad\qquad |F_\zeta(\zeta_1, \bar{\zeta}_1) - F_\zeta(\zeta_2, \bar{\zeta}_2)| \leq \mu(\lambda) B r_{12}^\lambda,$$

where $\mu(\lambda) > 0$ *is independent of* ζ_1, ζ_2.

To prove the lemma we write, according to the second equation of (18),

$$-2\pi i F_{\bar{\zeta}} = \lim_{h \to 0} \frac{1}{2} \iint\limits_{D} (f(z, \bar{z}) - f(\zeta, \bar{\zeta})) \left\{ \frac{1}{(z - \zeta)(z - \zeta - h/2)} \right.$$
$$\left. - \frac{1}{(z - \zeta)(z - \zeta - hi/2)} \right\} d\bar{z} dz + f(\zeta, \bar{\zeta}) \frac{\partial}{\partial \bar{\zeta}} \iint\limits_{D} \frac{d\bar{z} dz}{z - \zeta}.$$

Computation gives

$$(25) \qquad\qquad \iint\limits_{D} \frac{d\bar{z} dz}{z - \zeta} = -2\pi i \bar{\zeta}.$$

From these it follows that $F_{\bar{\zeta}} = f(\zeta, \bar{\zeta})$. Similarly, we prove that F_ζ exists and is given by

$$(26) \qquad\qquad -2\pi i F_\zeta = \iint\limits_{D} \frac{f(z, \bar{z}) - f(\zeta, \bar{\zeta})}{(z - \zeta)^2} d\bar{z} dz.$$

From (20) we have, by using (14) with $\lambda = 1$,

$$2\pi |F| \leq 2A \iint\limits_{D} \frac{dx dy}{r} \leq 8\pi A R,$$

which gives (21). Similarly, from (26), we have

$$2\pi |F_\zeta| \leq 2B \iint\limits_{D} \frac{r^\lambda}{r^2} dx dy \leq \frac{2^{\lambda+2}}{\lambda} \pi B R^\lambda,$$

which gives (22).

Inequality (23) is trivial. To prove (24) let D_0 be the intersection

of D with a circular disc of radius $2r_{12} = 2|\zeta_1 - \zeta_2|$ about ζ_2. We can write

$$- 2\pi i \{ F_{\zeta}(\zeta_1, \bar{\zeta}_1) - F_{\zeta}(\zeta_2, \bar{\zeta}_2) \}$$

$$= \iint_{D_0} \{ f(z, \bar{z}) - f(\zeta_1, \bar{\zeta}_1) \} \frac{1}{(z - \zeta_1)^2} \, d\bar{z}dz$$

$$- \iint_{D_0} \{ f(z, \bar{z}) - f(\zeta_2, \bar{\zeta}_2) \} \frac{1}{(z - \zeta_2)^2} \, d\bar{z}dz$$

(27)

$$+ \iint_{D-D_0} \{ f(z, \bar{z}) - f(\zeta_1, \bar{\zeta}_1) \} \frac{1}{(z - \zeta_1)^2} \, d\bar{z}dz$$

$$- \iint_{D-D_0} \{ f(z, \bar{z}) - f(\zeta_1, \bar{\zeta}_2) \} \frac{1}{(z - \zeta_2)^2} \, d\bar{z}dz$$

$$+ \{ f(\zeta_2, \bar{\zeta}_2) - f(\zeta_1, \bar{\zeta}_1) \} \frac{\partial}{\partial \zeta_2} \iint_{D-D_0} \frac{d\bar{z}dz}{z - \zeta_2}.$$

We observe that in this sum only the first two integrals are improper integrals, while the last three are proper integrals. To estimate their absolute values we first have

$$\left| \iint_{D_0} \{ f(z, \bar{z}) - f(\zeta_1, \bar{\zeta}_1) \} \frac{1}{(z - \zeta_1)^2} \, d\bar{z}dz \right|$$

$$\leqq 2B \iint_{D_0} \frac{dxdy}{|z - \zeta_1|^{2-\lambda}} \leqq \frac{4\pi B}{\lambda} (3r_{12})^{\lambda},$$

$$\left| \iint_{D_0} \{ f(z, \bar{z}) - f(\zeta_2, \bar{\zeta}_2) \} \frac{1}{(z - \zeta_2)^2} \, d\bar{z}dz \right|$$

$$\leqq 2B \iint_{D_0} \frac{dxdy}{|z - \zeta_2|^{2-\lambda}} \leqq \frac{4\pi B}{\lambda} (2r_{12})^{\lambda}.$$

The sum of the third and fourth integrals in (27) is equal to

$$\iint_{D-D_0} \{ f(z, \bar{z}) - f(\zeta_1, \bar{\zeta}_1) \} \int_{\zeta_2}^{\zeta_1} d \, \frac{1}{(z - \zeta)^2} \, d\bar{z}dz$$

$$= 4i \iint_{D-D_0} \int_{\zeta_2}^{\zeta_1} \{ f(z, \bar{z}) - f(\zeta, \bar{\zeta}) + f(\zeta, \bar{\zeta}) - f(\zeta_1, \bar{\zeta}_1) \} \frac{1}{(z - \zeta)^3} \, d\zeta dxdy,$$

where ζ is a point on the segment $\zeta_1\zeta_2$. It follows that its absolute value is

$$\leqq 4r_{12}(F_1 + F_2),$$

where F_1 and F_2 are respectively upper bounds of the integrals

$$\left| \iint\limits_{D-D_0} \frac{f(z, \bar{z}) - f(\zeta, \bar{\zeta})}{(z - \zeta)^3}\, dxdy \right|,$$

$$\left| \iint\limits_{D-D_0} \{f(\zeta, \bar{\zeta}) - f(\zeta_1, \bar{\zeta}_1)\} \frac{dxdy}{(z - \zeta)^3} \right|$$

along the segment $\zeta_1\zeta_2$. Using (15), we have

$$\left| \iint\limits_{D-D_0} \frac{f(z, \bar{z}) - f(\zeta, \bar{\zeta})}{(z - \zeta)^3}\, dxdy \right| \leqq B \iint\limits_{D-D_0} \frac{r^{\lambda-1}dxdy}{r^2}$$

$$\leqq \frac{4\pi B}{1 - \lambda} \frac{1}{r_{12}^{1-\lambda}},$$

$$\left| \iint\limits_{D-D_0} \frac{f(\zeta, \bar{\zeta}) - f(\zeta_1, \bar{\zeta}_1)}{(z - \zeta)^3}\, dxdy \right| \leqq Br_{12}^\lambda \frac{4\pi}{r_{12}} \cdot$$

The right-hand members of these inequalities can therefore be taken as F_1 and F_2 respectively.

Finally, we have, by (25),

$$\iint\limits_{D-D_0} \frac{d\bar{z}dz}{z - \zeta_2} = \iint\limits_{D} \frac{d\bar{z}dz}{z - \zeta_2} - \iint\limits_{D_0} \frac{d\bar{z}dz}{z - \zeta_2} = -2\pi i \bar{\zeta}_2$$

the integral over D_0 being equal to zero. It follows that the last term of (27) is zero.

Summing up these estimates, we get

$$2\pi \left| F_{\bar{\zeta}}(\zeta_1, \bar{\zeta}_1) - F_{\bar{\zeta}}(\zeta_2, \bar{\zeta}_2) \right|$$

$$\leqq \frac{4\pi B}{\lambda} (3r_{12})^\lambda + \frac{4\pi B}{\lambda} (2r_{12})^\lambda + 4\left(\frac{4\pi B}{1 - \lambda} + 4\pi B\right) r_{12}^\lambda.$$

This gives (24), if we set

(28) $$\mu(\lambda) = \frac{2}{\lambda} (3^\lambda + 2^\lambda) + 8\frac{2-\lambda}{1 - \lambda} \cdot$$

Thus the proof of the lemma is complete.

4. An existence and uniqueness theorem on an integro-differential equation; application to isothermal parameters.

THEOREM. *In the domain D of the complex z-plane defined by $|z| \leq R$ ($=$positive constant) let*

$$(29) \qquad\qquad Zw = a(z, \bar{z})w_{\bar{z}} + b(z, \bar{z})w_z$$

be a differential operator, whose coefficients $a(z, \bar{z})$, $b(z, \bar{z})$ satisfy a Hölder condition of order λ, $0 < \lambda < 1$, and vanish at $z = 0$. Let $\alpha(z, \bar{z})$ be a function in D, satisfying a Hölder condition of the same order λ. Then the equation

$$(30) \qquad 2\pi i w(\zeta, \bar{\zeta}) + \iint\limits_{D} \frac{(Zw + \alpha w)(z, \bar{z})}{z - \zeta}\, d\bar{z}\,dz = \sigma(\zeta) \qquad \zeta \in D,$$

where $\sigma(\zeta)$ is a complex analytic function, with $\sigma(0) = 0$, has exactly one solution $w(z, \bar{z})$, provided that R is sufficiently small.

Before proceeding to the proof of this theorem, we shall make some further estimates of integrals, based on the lemma of the last section. The hypothesis of the theorem implies the existence of a number M large enough to fulfill the following inequalities:

$$(31) \qquad\qquad 1 < M, \qquad |\alpha(\zeta, \bar{\zeta})| \leq M, \qquad |\sigma(\zeta)| \leq M,$$

$$(32) \qquad\qquad |h(\zeta_1, \bar{\zeta}_1) - h(\zeta_2, \bar{\zeta}_2)| \leq M r_{12}^{\lambda},$$

$$(33) \quad |(Z + \alpha)\sigma(\zeta_1) - (Z + \alpha)\sigma(\zeta_2)| \leq \frac{M^2}{2^{\lambda}} r_{12}^{\lambda}, \qquad r_{12} = |\zeta_1 - \zeta_2|,$$

where ζ, ζ_1, ζ_2 are any three points of D and where $h(z, \bar{z})$ stands for each of the functions $a(z, \bar{z})$, $b(z, \bar{z})$, $\alpha(z, \bar{z})$, $\sigma(z)$. Since a, b, $(Z+\alpha)\sigma$ all vanish at 0, it follows from the corresponding Hölder inequalities that

$$(34) \qquad \begin{aligned} |a(\zeta, \bar{\zeta})| &\leq M|\zeta|^{\lambda} \leq MR^{\lambda}, \qquad |b(\zeta, \bar{\zeta})| \leq MR^{\lambda}, \\ |(Z + \alpha)\sigma(\zeta)| &\leq M^2 R^{\lambda}. \end{aligned}$$

Consider now the function $F(\zeta, \bar{\zeta})$ defined in (20). Using the notation of the lemma of §3, we have

$$\begin{aligned} |(Z + \alpha)F(\zeta, \bar{\zeta})| \\ = |a(\zeta, \bar{\zeta})F_{\bar{\zeta}} + b(\zeta, \bar{\zeta})F_{\zeta} + \alpha(\zeta, \bar{\zeta})F| \\ (35) \qquad \leq MR^{\lambda}(A + (2^{\lambda+1}/\lambda)R^{\lambda}B) + 4MRA \\ = MR^{\lambda}\{(1 + 4R^{1-\lambda})A + (2^{\lambda+1}/\lambda)R^{\lambda}B\}, \end{aligned}$$

$$\left| (Z + \alpha)F(\zeta_1, \bar\zeta_1) - (Z + \alpha)F(\zeta_2, \bar\zeta_2) \right|$$

(36)
$$= \left| a(\zeta_1, \bar\zeta_1)f(\zeta_1, \bar\zeta_1) + b(\zeta_1, \bar\zeta_1)F_\zeta(\zeta_1, \bar\zeta_1) + \alpha(\zeta_1, \bar\zeta_1)F(\zeta_1, \bar\zeta_1) \right.$$
$$\left. - a(\zeta_2, \bar\zeta_2)f(\zeta_2, \bar\zeta_2) - b(\zeta_2, \bar\zeta_2)F_\zeta(\zeta_2, \bar\zeta_2) - \alpha(\zeta_2, \bar\zeta_2)F(\zeta_2, \bar\zeta_2) \right|$$
$$\leqq Mr_{12}^\lambda \{ A + (2^{\lambda+1}/\lambda)R^\lambda B + 4RA \} + MR^\lambda(B + \mu(\lambda)B)r_{12}^\lambda$$
$$+ M(A + (2^{\lambda+1}/\lambda)R^\lambda B) \cdot 2r_{12}$$
$$\leqq Mr_{12}^\lambda \{ A + g_1(R)A + g_2(R)B \},$$

where

(37)
$$g_1(R) = 4R + 2^{2-\lambda}R^{1-\lambda},$$
$$g_2(R) = \{ (2^{\lambda+1}/\lambda) + \mu(\lambda) + 1 \}R^\lambda + (8/\lambda)R$$

are functions of R, which tend to zero with R.

Having the above inequalities, we shall prove the existence of a solution of (30) by successive approximations. For reasons which will be clear later we put the following restrictions on R:

(38)
$$4R^\lambda \leqq 1, \qquad 2^{2-\lambda}\frac{\lambda + 2}{\lambda} R^{1-\lambda} \leqq 1,$$

and we choose a constant c, depending only on λ, such that

(39)
$$\frac{\lambda + 2}{\lambda} + \frac{\lambda}{\lambda + 2} 2^\lambda \leqq c,$$
$$1 + \mu(\lambda) + \frac{3\lambda + 2}{\lambda} 2^\lambda \leqq c.$$

We now define

(40)
$$2\pi i w_0(\zeta, \bar\zeta) = \sigma(\zeta),$$
$$2\pi i w_{n+1}(\zeta, \bar\zeta) = -\int\int_D \frac{(Zw_n + \alpha w_n)(z, \bar z)}{z - \zeta} d\bar z dz, \qquad n = 0, 1, \cdots.$$

For these functions we shall prove the inequalities:

(41)
$$\left| w_n(\zeta, \bar\zeta) \right| \leqq M(cMR^\lambda)^n,$$

(42)
$$\left| (Z + \alpha)w_n(\zeta, \bar\zeta) \right| \leqq M(cMR^\lambda)^{n+1},$$

(43)
$$\left| w_n(\zeta_1, \bar\zeta_1) - w_n(\zeta_2, \bar\zeta_2) \right| \leqq M(cMR^\lambda)^n r_{12}^\lambda,$$

(44)
$$\left| (Z + \alpha)w_n(\zeta_1, \bar\zeta_1) - (Z + \alpha)w_n(\zeta_2, \bar\zeta_2) \right| \leqq (cM^2/2)(cMR^\lambda)^n r_{12}^\lambda.$$

In particular, the last one implies that the function under the integral

sign in (40) satisfies a Hölder condition, thus allowing the definition of the next integral.

The inequalities (41)–(44) will be proved by induction on n. For $n=0$, (41) is a consequence of the third inequality of (31), (42) follows from the third inequality of (34), since $c>1$, (43) follows from (32), and (44) from (33).

We now suppose that (41)–(44) are true and proceed to prove the corresponding inequalities for $n+1$. By (21), (23), and the induction hypothesis, we have

$$\left| w_{n+1}(\zeta, \bar{\zeta}) \right| \leqq 4R \cdot M(cMR^\lambda)^{n+1} \leqq M(cMR^\lambda)^{n+1},$$

$$\left| w_{n+1}(\zeta_1, \bar{\zeta}_1) - w_{n+1}(\zeta_2, \bar{\zeta}_2) \right|$$
$$\leqq 2r_{12}\left\{ M(cMR^\lambda)^{n+1} + (2^{\lambda+1}/\lambda)R^\lambda(cM^2/2^\lambda)(cMR^\lambda)^n \right\}$$
$$= M(cMR^\lambda)^{n+1} r_{12}^\lambda (1 + 2/\lambda) \cdot 2r_{12}^{1-\lambda}.$$

The last relation gives (43) (for the index $n+1$), on account of the second inequality of (38).

Similarly, we get from (35) and the induction hypothesis,

$$\left| (Z + \alpha)w_{n+1}(\zeta, \bar{\zeta}) \right| \leqq MR^\lambda \left\{ (1 + 4R^{1-\lambda})cM^2R^\lambda \right.$$
$$\left. + (2^{\lambda+1}/\lambda)R^\lambda(cM^2/2^\lambda) \right\}(cMR^\lambda)^n$$
$$= M^2 R^\lambda (cMR^\lambda)^{n+1}(1 + 4R^{1-\lambda} + 2/\lambda)$$
$$\leqq M(cMR^\lambda)^{n+2},$$

on using the first inequality of (39). By (36), we have

$$\left| (Z + \alpha)w_{n+1}(\zeta_1, \bar{\zeta}_1) - (Z + \alpha)w_{n+1}(\zeta_2, \bar{\zeta}_2) \right|$$
$$\leqq Mr_{12}^\lambda \cdot M(cMR^\lambda)^n \left\{ (1 + g_1(R))cMR^\lambda + g_2(R)cM/2^\lambda \right\}$$
$$= M^2 r_{12}^\lambda (cMR^\lambda)^{n+1} \left\{ 1 + g_1(R) + (1/(2R)^\lambda)g_2(R) \right\}.$$

This gives (44) for $n+1$, if

$$1 + g_1(R) + (1/(2R)^\lambda)g_2(R) \leqq c/2^\lambda.$$

But the latter follows from the second inequality of (39). Thus our induction is complete.

It follows that the series

$$(45) \qquad\qquad \sum_{n=0}^\infty w_n(z, \bar{z})$$

converges absolutely and uniformly in D if $CMR^\lambda < 1$, and defines a function $w(z, \bar{z})$, which satisfies (30).

To prove the uniqueness of the solution when R is sufficiently small, let $w'(z, \bar{z})$ be another solution of (30) such that $(Z+\alpha)w'(z, \bar{z})$ satisfies a Hölder condition of order λ. Then the function

$$\tilde{w}(z, \bar{z}) = w(z, \bar{z}) - w'(z, \bar{z})$$

satisfies the equation

$$-2\pi i \tilde{w}(\zeta, \bar{\zeta}) = \iint_D \frac{(Z + \alpha)\tilde{w}(z, \bar{z})}{z - \zeta} \, d\bar{z}dz.$$

Let A and B be respectively the least upper bounds of

$$\left| (Z + \alpha)\tilde{w}(\zeta, \bar{\zeta}) \right|, \zeta \in D$$

and

$$\left| (Z + \alpha)\tilde{w}(\zeta_1, \bar{\zeta}_1) - (Z + \alpha)\tilde{w}(\zeta_2, \bar{\zeta}_2) \right| / r_{12}^\lambda, \qquad \zeta_1, \zeta_2 \in D, \zeta_1 \neq \zeta_2.$$

From (35) and (36) we get

$$A \leqq MR^\lambda \{ (1 + 4R^{1-\lambda})A + (2^{\lambda+1}/\lambda)R^\lambda B \},$$
$$B \leqq M \{ A + g_1(R)A + g_2(R)B \}.$$

From these we easily conclude that, if R is sufficiently small, $A = 0$. The latter implies that $\tilde{w}(z, \bar{z}) = 0$. Thus the proof of our theorem is complete.

In order to derive from the above existence theorem the theorem of Korn-Lichtenstein we follow the notation of §1. In a neighborhood of the point $z = x + iy = 0$ we suppose the almost complex structure to be given by the complex-valued linear differential form

(46) $$\phi = (1 - a(z, \bar{z}))dz + b(z, \bar{z})d\bar{z},$$

which is determined up to a nonzero factor. By a linear transformation on x, y with constant coefficients and by multiplication of ϕ by a constant factor when necessary, we can suppose that $a(0, 0) = b(0, 0) = 0$. Equation (8) is equivalent to the equations

$$w_{\bar{z}} = b/\rho, \qquad w_z = (1 - a)/\rho.$$

Elimination of ρ gives

$$(1 - a)w_{\bar{z}} - bw_z = 0,$$

or, if we make use of the operator Z in (29),

(47) $$w_{\bar{z}} = Zw.$$

By the above existence theorem, the equation

$$(48) \qquad 2\pi i w(\zeta, \bar{\zeta}) + \iint_D \frac{Zw(z, \bar{z})}{z - \zeta}\, d\bar{z}dz = \sigma(\zeta) \qquad \zeta \in D,\ \sigma(0) = 0,$$

has a solution $w(z, \bar{z})$. If $\sigma(\zeta)$ is not a constant, say $\sigma(\zeta) = \zeta$, this solution has the property that $w_{\bar{\zeta}}(0, 0) \neq 0$. Since $Zw(z, \bar{z})$ satisfies a Hölder condition, it follows from our lemma in §3 that $w(z, \bar{z})$ satisfies (47). Thus we have proved the theorem of Korn-Lichtenstein.

REMARKS. 1. As L. Bers observed to me, the same method can be used to establish the existence of a local solution of the equation

$$(49) \qquad w_{\bar{z}} = a\bar{w}_{\bar{z}} + bw_z + \alpha w + \beta\bar{w} + \gamma,$$

where $a, b, \alpha, \beta, \gamma$ satisfy a Hölder condition and where $|a| + |b| < 1$.

2. It follows from our proof that the first partial derivatives of the isothermal parameters also satisfy a Hölder condition of order λ.

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