REAL ANALYSIS
LECTURE NOTES

ABSTRACT. The Notes indicate what we do in the lectures, but are not a complete replacement of the book and lectures. The text is from two books of *Real Analysis*:
[1] Xingwei Zhou & Wenchang Sun: Real Variable Analysis, the third edition, Science Press, 2014.
[2] E. Stein & R. Shakarchi: Real Analysis, Princeton University Press, 2005.

1. MEASURABLE FUNCTIONS

Let us turn our attention to the objects that lie at the heart of integration theory: measurable functions. Recall that the characteristic function of a set E is given by

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

Definition 1.1. *A simple function is a function of the form $f = \sum_{k=1}^N a_k \chi_{E_k}$, where each E_k is a measurable set of finite measure, and the a_k are constants.*

These functions will be the basic functions used to define the Lebesgue integral.

Definition 1.2. *A step function is a function of the form $f = \sum_{k=1}^N a_k \chi_{R_k}$, where each R_k is a rectangle, and the a_k are constants.*

These functions are the basic ones used to define the Riemann integral.

1.1. **Definition and basic properties.**

The real-valued function f on a measurable set $E \subset \mathbb{R}^n$ in our context is allowed to take on the infinite values $\pm\infty$, so that $f(x)$ belongs to the extended real numbers

$$-\infty \le f(x) \le \infty.$$

We say a function f is finite-valued if $-\infty < f(x) < \infty$ for all x.

Unless otherwise clear from context, when we use notation $f : E \to [-\infty, \infty]$, we always refer that E is a measurable set of $\mathbb{R}^n$.

In the theory that follows, and the many applications of it, we shall almost always find ourselves in situation where a function takes on infinite values on at most a set of measure zero.

Definition 1.3. *A function f defined on a measurable set $E \subset \mathbb{R}^n$ is measurable, if for all $a \in \mathbb{R}$, the set*

$$f^{-1}([-\infty, a)) = \{x \in E : f(x) < a\}$$

is measurable.

To simplify our notation, we denote the set $\{x \in E : -\infty \leq f(x) < a\}$ simply by $\{f < a\}$ whenever no confusion is possible.

Remark 1.1. *Let $f : E \to [-\infty, +\infty]$, where E is a measurable set of $\mathbb{R}^n$.*

(i) *f is measurable if and only if $\{f \leq a\}$ is measurable for all $a \in \mathbb{R}$;*
Proof: Note that $\{f \leq a\} = \bigcap_{k\geq 1}\{f < a+\frac{1}{k}\}$ and $\{f < a\} = \bigcup_{k\geq 1}\{f \leq a-\frac{1}{k}\}$.

(ii) *f is measurable if and only if $\{f \geq a\}$ is measurable for all $a \in \mathbb{R}$;*
Proof: Note that $\{f \geq a\} = \{f < a\}^c$.

(iii) *f is measurable if and only if $\{f > a\}$ is measurable for all $a \in \mathbb{R}$;*
Proof: Note that $\{f > a\} = \{f \leq a\}^c$.

(iv) *f is measurable if and only if $-f$ is measurable.*

(v) *f is measurable if and only if $\{f = -\infty\}$, $\{f = \infty\}$ are measurable, and $\{a < f < b\}$ is measurable for all $a, b \in \mathbb{R}$;*

(vi) *Conclusions of (v) hold for whichever combination of strict or weak inequalities one chooses.*

Property 1 The finite-valued function f is measurable if and only if $f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O} \subset \mathbb{R}$, and if and only if $f^{-1}(F)$ is measurable for every closed set $F \subset \mathbb{R}$.

(This is also true for extended valued functions if we make the additional hypothesis that both $\{f = \infty\}$ and $\{f = -\infty\}$ are measurable.)

Proof. The point is that every open set $\mathcal{O} \subset \mathbb{R}$ is a countable union of open intervals, and every closed set $F \subset \mathbb{R}$ is the complement of an open set.

$\square$

Remark 1.2. *If f is finite-valued, then f is measurable if and only if $f^{-1}(\mathcal{B})$ is measurable for every Borel set $\mathcal{B}$ of $\mathbb{R}$.*

This is again true for extended valued functions if one additionally requires that $\{f = \infty\}$ and $\{f = -\infty\}$ are measurable.

Proof: Since open sets are Borel, one direction is immediate.

The main point is to show if f is measurable and $\mathcal{B} \in \mathscr{B}_{\mathbb{R}}$, then $f^{-1}(\mathcal{B})$ is measurable. For this let $\mathcal{S} = \{E : \ f^{-1}(E) \text{ is measurable}\}$. Then $\mathcal{S}$ contains all open set, and is a σ-algebra. It follows that $\mathscr{B}_{\mathbb{R}} \subset \mathcal{S}$.

Property 2 If f is continuous (and hence finite-valued), then f is measurable. If f is finite-valued and measurable, and ϕ is continuous, then $\phi \circ f$ is measurable.

Proof. Only need to note that $(\phi \circ f)^{-1}(-\infty, a) = f^{-1}(\phi^{-1}(-\infty, a))$. □

Remark 1.3. *As an immediate consequence, if f is measurable, then $|f|$ is measurable.*

Remark 1.4. *In general, it is not true that $f \circ \phi$ is measurable whenever f is measurable and ϕ is continuous.*

Exercise. Find an example for this.

Property 3 If $\{f_j\}_{j\geq 1}$ is a sequence of measurable functions with the same measurable domain, then

$$\sup_j f_j(x), \ \inf_j f_j(x), \ \limsup_{j\to\infty} f_j(x), \ \liminf_{j\to\infty} f_j(x)$$

are measurable functions.

Proof. Since $\sup_j f_j(x) > a$ if and only if $f_j(x) > a$ for some j, we see that

$$\{\sup_j f_j > a\} = \bigcup_{j\geq 1} \{f_j > a\}.$$

Hence $\sup_j f_j(x)$ is measurable. Since $\inf_j f_j(x) = -\inf_j(-f_j(x))$, we know that $\inf_j f_j(x)$ is measurable.

The result for the $\limsup$ and $\liminf$ follows from the two observations

$$\limsup_{j\to\infty} f_j(x) = \inf_k \{\sup_{j\geq k} f_j(x)\} \text{ and } \liminf_{j\to\infty} f_j(x) = \sup_k \{\inf_{j\geq k} f_j(x)\}.$$

□

As a consequence of Property 3, we prove the following result.

Property 4 If $\{f_j\}_{j\geq 1}$ is a sequence of measurable functions with the same measurable domain, and

$$\lim_{j\to\infty} f_j(x) = f(x),$$

then f is measurable.

Property 5 If f and g are measurable with common measurable domain, then so are f^k ($k \in \mathbb{N}$), λf for any $\lambda \in \mathbb{R}$, $f+g$ and fg.

In the last three cases we require f and g are finite-valued so that λf, $f+g$ and fg are well defined [1].

Proof. Note that $\{f^k > a\} = \{f > a^{1/k}\}$ if k is odd, and if k is even and $a \geq 0$, then $\{f^k > a\} = \{f > a^{1/k}\} \cup \{f < -a^{1/k}\}$. Hence f^k is measurable.

It is readily verified that λf is measurable.

The third is because $\{f+g > a\} = \bigcup_{r\in\mathbb{Q}} \{f > a-r\} \cap \{g > r\}$.

Finally, fg is measurable because of the previous results and the fact that

$$fg = \frac{1}{4}\big[(f+g)^2 - (f-g)^2\big].$$

□

1.2. **Almost everywhere.**

In measure theory, we can generally neglect sets of measure zero.

Definition 1.4. *A property or statement is said to hold almost everywhere (written a.e.) if it is true except on a set of measure zero.*

We say two functions f and g defined on a set E are equal almost everywhere, write

$$f(x) = g(x) \quad a.e.\ x \in E,$$

if the set $\{x \in E : f(x) \neq g(x)\}$ has measure zero. We also abbreviate this by saying that $f = g$ a.e.

One sees easily that if f is measurable and $f = g$ a.e., then g is measurable.

[1] For example, $\infty + a = \infty$, $\infty + \infty = \infty$, $\infty \times a$ (for $a \neq 0$) are well defined. But $\infty - \infty$ and $\infty \times 0$ are not well-defined

Moreover, all the properties above can be relaxed to conditions holding almost everywhere. For example, if f_j is a sequence of measurable functions and

$$\lim_{j\to\infty} f_j(x) = f(x) \textit{ a.e.},$$

then f is measure. In this light, Property 5 holds when f and g are finite-valued almost everywhere.

1.3. **Approximation by simple functions or step functions.**

Theorem 1.1. *If $f : E \to [0,\infty]$ is measurable, then there is an increasing sequence of non-negative simple functions $\{\phi_k\}_{k=1}^{\infty}$ that converges pointwise to f, namely*

$$0 \le \phi_k(x) \le \phi_{k+1}(x) \quad \textit{and} \quad \lim_{k\to\infty} \phi(x) = f(x), \ \textit{for all } x \in E.$$

Proof. We first truncate f as follows: for $x \in E$,

$$f_k(x) = \begin{cases} f(x), & |x| \le k,\ 0 \le f(x) \le k, \\ k, & |x| \le k,\ f(x) > k, \\ 0, & \text{otherwise.} \end{cases}$$

We then approximate f_k by a simple function within error 2^{-k} as follows: for $x \in E$,

$$\phi_k(x) = \frac{l}{2^k}, \ \text{ if } \frac{l}{2^k} \le f_k(x) < \frac{l+1}{2^k} \text{ for } l \ge 0 \text{ an integer.}$$

One can check that ϕ_k satisfies the required properties.

□

Theorem 1.2. *If $f : E \to [-\infty,\infty]$ is measurable, then there is a sequence of simple functions $\{\phi_k\}_{k=1}^{\infty}$ that converges that satisfies*

$$|\phi_k(x)| \le |\phi_{k+1}(x)| \quad \textit{and} \quad \lim_{k\to\infty} \phi(x) = f(x), \ \textit{for all } x \in E.$$

In particular, $|\phi_k(x)| \le |f(x)|$ for all k and $x \in E$.

Proof. Write $f(x) = f^+(x) - f^-(x)$. [2] Construct $\phi_k^{(1)}$ and $\phi_k^{(2)}$ for f^+ and f^- as in Theorem. Let $\phi_k = \phi_k^{(1)} - \phi_k^{(2)}$. One verifies that ϕ_k satisfies the needed properties.

□

[2] We use the notation: $f^+ = f\chi_{\{f>0\}}$ and $f^- = f\chi_{\{f<0\}}$. Then

$$f = f^+ - f^- \text{ and } |f| = f^+ + f^- \implies f^+ = \frac{1}{2}(|f| + f) \text{ and } f^- = \frac{1}{2}(|f| - f).$$

Theorem 1.3. *If $f : E \to [-\infty, \infty]$ is measurable, then there is a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$ such that $\psi_k \to f$ almost everywhere.*

Remark 1.5. *We only get a.e. convergence and we do not have the monotonicity properties of the previous two theorems.*

Proof. We divide the proof into several steps for clarification.

Step 1. By Theorem 1.2, there is a sequence of simple functions $\phi_k \to f$ everywhere.

Step 2. Any simple function ϕ is of the form $\sum_j a_j \chi_{A_j}$ for some finite collection of measurable sets A_j. We can require these A_j be disjoint, by considering any intersections.

(i) For every measurable set A, there is a finite union F of closed cubes such that $m(A \Delta F) < \varepsilon$.

(ii) F can be written as a sum of almost disjoint rectangles (consider the grid obtained by extending the sides of the cubes).

By taking the union $\tilde{F} \subset F$ of slightly smaller disjoint rectangles inside these rectangles we can ensure $m(F \setminus \tilde{F}) < \varepsilon$ and so $m(A \Delta \tilde{F}) < 2\varepsilon$.

(iii) By replacing each A_j by the above $\tilde{G}$ associated to A_j, we obtain a step function ψ such that $\psi = \phi$ except on a set of measure 2ε.

Step 3. Applying Step 2 to each ϕ_k, one concludes there exist step functions ψ_k such that $m(E_k) \leq 2^{-k}$, where $E_k := \{\psi_k \neq \phi_k\}$.

Step 4. Let $H_m = \bigcup_{j \geq m+1} E_j$. Then $m(H_m) \leq 2^{-m}$ and $\psi_k \to f$ except possibly on H_m.

Step 5. It follows from the Step 4 and by the arbitrariness of m, $\psi_k \to f$ except possibly on $H = \cap_{m \geq 1} H_m$. But $H_m \searrow H$ and so $m(H) = \lim_{m \to \infty} m(H_m) = 0$.

Therefore $\psi_k \to f$ almost everywhere.

$\square$