Assignment 1

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Problem 1.1

$\dot{q} = T_q(q)\omega$
$I_{CG}\dot{\omega} - S(I_{CG}\omega)\omega = \tau$

$q = [\eta \quad \epsilon_1 \quad \epsilon_2 \quad \epsilon_3]^T = [1 \quad 0 \quad 0 \quad 0]^T$
$\tau = 0$

$\eta = \sqrt{1 - \epsilon^T\epsilon}$

$$I_{CG} = mr^2 I_{3x3} = mr^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 180 \cdot 4 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 720 & 0 & 0 \\ 0 & 720 & 0 \\ 0 & 0 & 720 \end{bmatrix}$$
$$= \begin{bmatrix} mr^2 & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{\eta} \\ \dot{\epsilon} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\epsilon^T \\ \eta I_{3x3} + S(\epsilon) \end{bmatrix} \omega_{b,n}^b$$

$$T_q(q) = \frac{1}{2}\begin{bmatrix} -\epsilon_1 & -\epsilon_2 & -\epsilon_3 \\ \eta & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & \eta & \epsilon_1 \\ -\epsilon_2 & \epsilon_1 & \eta \end{bmatrix}$$

$$\dot{q} = \frac{1}{2}\begin{bmatrix} -\epsilon_1 & -\epsilon_2 & -\epsilon_3 \\ \eta & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & \eta & \epsilon_1 \\ -\epsilon_2 & \epsilon_1 & \eta \end{bmatrix}\omega = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\omega$$
$$\dot{q} = 0 \Rightarrow \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\omega = 0 \Rightarrow \omega = [0 \quad 0 \quad 0]^T$$

$$\Rightarrow x_0 = \begin{bmatrix} \epsilon^T \\ \omega^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$I_{CG}\dot{\omega} - S(I_{CG}\omega)\omega = \tau$
$I_{CG}\dot{\omega} = S(I_{CG}\omega)\omega$

$I_{CG}\dot{\omega} = \tau + S(I_{CG}\omega)\omega$
$\dot{\omega} = I_{CG}^{-1}S(I_{CG}\omega)\omega + I_{CG}^{-1}\tau$

$\Rightarrow \dot{\omega} = I_{CG}^{-1}S(I_{CG}\omega)\omega$

$$I_{CG}^{-1}S(I_{CG}\omega) = \frac{1}{mr^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -mr^2\omega & mr^2\omega \\ mr^2\omega & 0 & -mr^2\omega \\ -mr^2\omega & mr^2\omega & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{\epsilon}^T \\ \dot{\omega}^T \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\omega^T \\ I_{CG}^{-1}S(I_{CG}\omega)\omega \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\omega^T \\ 0\omega \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} x$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/mr^2 & 0 & 0 \\ 0 & 1/mr^2 & 0 \\ 0 & 0 & 1/mr^2 \end{bmatrix} \tau$$

Problem 1.2

$$\tau = -K_d \omega - k_p \epsilon, \qquad k_p > 0$$

$$\tau = -\begin{bmatrix} k_d & 0 & 0 \\ 0 & k_d & 0 \\ 0 & 0 & k_d \end{bmatrix} \omega - k_p \epsilon$$

$$\dot{x} = Ax + B\tau = Ax + B\left(-K_d \omega - k_p \epsilon\right)$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{mr^2} & 0 & 0 \\ 0 & \frac{1}{mr^2} & 0 \\ 0 & 0 & \frac{1}{mr^2} \end{bmatrix} \left(-k_d \omega - k_p \epsilon\right)$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{mr^2}\left(-k_d \omega_1 - k_p \epsilon_1\right) \\ \frac{1}{mr^2}\left(-k_d \omega_2 - k_p \epsilon_2\right) \\ \frac{1}{mr^2}\left(-k_d \omega_3 - k_p \epsilon_3\right) \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{k_p}{mr^2} & 0 & 0 & -\frac{1}{mr^2} k_d & 0 & 0 \\ 0 & -\frac{k_p}{mr^2} & 0 & 0 & -\frac{1}{mr^2} k_d & 0 \\ 0 & 0 & -\frac{k_p}{mr^2} & 0 & 0 & -\frac{1}{mr^2} k_d \end{bmatrix} x$$

Inserted into MATLAB, we calculate the eigenvalues to be complex, all in the left half plane.

Problem 1.3

$\tau = -K_d\omega - k_p\epsilon$

We introduce a reference by setting the control law to:
$\tau = K_d(\omega_r - \omega) + k_p(\epsilon_r - \epsilon)$

Problem 1.4

$\tau = -K_d\omega - k_p\tilde{\epsilon}$

$\tilde{q} \coloneqq \begin{bmatrix}\tilde{\eta}\\ \tilde{\epsilon}\end{bmatrix} = \bar{q}_d \times q$

$\bar{q} = \begin{bmatrix}\eta\\ -\epsilon^T\end{bmatrix}$

$q_1 \times q_2 = \begin{bmatrix}\eta_1\eta_2 - \epsilon_1^T\epsilon_2\\ \eta_1\epsilon_2 + \eta_2\epsilon_1 + S(\epsilon_1)\epsilon_2\end{bmatrix}$

$\Rightarrow \tilde{q} \coloneqq \begin{bmatrix}\eta_d\eta + \epsilon_d^T\epsilon\\ \eta_d\epsilon - \eta\epsilon_d - S(\epsilon_d)\epsilon\end{bmatrix}$

$S(\epsilon_d)\epsilon = \begin{bmatrix}0 & -\epsilon_{3,d} & \epsilon_{2,d}\\ \epsilon_{3,d} & 0 & -\epsilon_{1,d}\\ -\epsilon_{2,d} & \epsilon_{1,d} & 0\end{bmatrix}\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\end{bmatrix} = \begin{bmatrix}-\epsilon_{3,d}\,\epsilon_2 + \epsilon_{2,d}\epsilon_3\\ \epsilon_{3,d}\epsilon_1 - \epsilon_{1,d}\epsilon_3\\ -\epsilon_{2,d}\epsilon_1 + \epsilon_{1,d}\epsilon_2\end{bmatrix}$

$\Rightarrow \tilde{q} \coloneqq \begin{bmatrix}\eta_d\eta + \epsilon_d^T\epsilon\\ \eta_d\epsilon - \eta\epsilon_d - \begin{bmatrix}-\epsilon_{3,d}\,\epsilon_2 + \epsilon_{2,d}\epsilon_3\\ \epsilon_{3,d}\epsilon_1 - \epsilon_{1,d}\epsilon_3\\ -\epsilon_{2,d}\epsilon_1 + \epsilon_{1,d}\epsilon_2\end{bmatrix}\end{bmatrix}$

When $q = q_d$:

$\tilde{q} \coloneqq \begin{bmatrix}\eta^2 + \epsilon^2\\ -\epsilon_3\,\epsilon_2 + \epsilon_2\epsilon_3\\ \epsilon_3\epsilon_1 - \epsilon_1\epsilon_3\\ -\epsilon_2\epsilon_1 + \epsilon_1\epsilon_2\end{bmatrix} = \begin{bmatrix}\eta^2 + \epsilon^2\\ 0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}1 - \epsilon^2 + \epsilon^2\\ 0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}$

Problem 1.5

Ok

Problem 1.6

$\phi(t) = 0$
$\theta(t) = 15\cos(0.1t)$
$\psi(t) = 10\sin(0.05t)$

$\dot{\phi}(t) = 0$
$\dot{\theta}(t) = -1.5\sin(0.1t)$
$\dot{\psi}(t) = 0.5\cos(0.05t)$

$$\omega_d = T_{\Theta_d}^{-1}(\Theta_d) = \begin{bmatrix} 1 & 0 & -\sin\theta_d \\ 0 & \cos\phi_d & \cos\theta_d\sin\phi_d \\ 0 & -\sin\phi_d & \cos\theta_d\cos\phi_d \end{bmatrix}\begin{bmatrix} 0 \\ -1.5\sin(0.1t) \\ 0.5\cos(0.05t) \end{bmatrix}$$

$$\Rightarrow \omega_d = \begin{bmatrix} -0.5\, sin\,\theta_d\, cos\,0.05t \\ -1.5\, sin\,0.1t\, cos\,\phi_d + cos\,\theta_d\, sin\,\phi_d \cdot 0.5\, cos\,0.05t \\ 1.5\, sin\,\phi_d\, sin\,0.1t + cos\,\theta_d \cos\phi_d\, 0.5\cos 0.05t \end{bmatrix}$$

Problem 1.7

$\omega_d = 0, \qquad \epsilon_d = constant, \qquad \eta_d = constant$

$\tau = -K_d\omega - k_p\tilde{\epsilon}$

$$V = \frac{1}{2}\tilde{\omega}^T I_{CG}\tilde{\omega} + 2k_p(1-\tilde{\eta})$$

Want to prove that V is positive:

$I_{CG} > 0 \;\Rightarrow \tilde{\omega} I_{CG}\tilde{\omega} > 0$

$k_p > 0$
$\tilde{\omega} = \omega - \omega_d = \omega$

$\tilde{\eta}$ is a quaternion and thus $\tilde{\eta} \leq 1 \Rightarrow (1-\tilde{\eta}) \geq 0$
As such, V is positive.

We then also see that

$$\lim_{\omega\to\infty} V = \infty,$$

and thus V is radially unbounded.

We then want to show that
$\dot{V} = -k_d\omega^T\omega$

$$V = \frac{1}{2}\tilde{\omega}^T I_{CG}\tilde{\omega} + 2k_p(1-\tilde{\eta}) = \frac{1}{2}\omega^T I_{CG}\omega + 2k_p(1-\tilde{\eta})$$

$$I_{CG}\dot{\omega} - S(I_{CG}\omega)\omega = \tau$$

$$\Rightarrow \dot{\omega} = \frac{\tau + \omega S(I_{CG}\omega)}{I_{CG}}$$
$$\dot{\tilde{\eta}} = -\frac{1}{2}\tilde{\epsilon}^T\tilde{\omega}$$

$\tau = -K_d\omega - k_p\tilde{\epsilon}$

$\dot{V} = \omega^T I_{CG}\dot{\omega} - 2k_p\dot{\tilde{\eta}}$
$= \omega^T\big(\tau + \omega S(I_{CG}\omega)\big) + k_p\tilde{\epsilon}^T\widetilde{\omega}$
$= \omega^T\left(-K_d\omega - k_p\tilde{\epsilon} + \omega S(I_{CG}\omega)\right) + k_p\tilde{\epsilon}^T\widetilde{\omega}$
$= \omega^T(-K_d\omega) - \omega^T k_p\tilde{\epsilon} + k_p\epsilon^T\widetilde{\omega} + \omega^T\omega S(I_{CG}\omega)$
$= \omega^T(-K_d\omega) + \omega^T\omega S(I_{CG}\omega)$

$\Rightarrow \dot{V} = -k_d\omega^T\omega$

Problem 2.1

$\dot{\eta} = R(\psi)v$
$\Rightarrow \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}u\\ v\\ r\end{bmatrix} = \begin{bmatrix}u\cos\psi - v\sin\psi\\ u\sin\psi + v\cos\psi\\ r\end{bmatrix}$
$\Rightarrow \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}u\cos\psi - v\sin\psi\\ u\sin\psi + v\cos\psi\end{bmatrix}$

Here we assume that θ and ϕ are small enough so that $\cos\theta \approx 1,\ \sin\theta \approx 0$

We have that:
$\cos(\psi+\beta) = \cos(\psi)\cos(\beta) - \sin(\psi)\sin(\beta)$
$\Rightarrow U cos(\psi+\beta) = U(\cos\psi\cos\beta - \sin\psi\sin\beta)$

$\dot{x} = u\cos\psi - v\sin\psi$
To match the above expression, we can write:
$u\cos\psi = U\cos\psi\cos\beta$
$v\sin\psi = U\sin\psi\sin\beta$
$\Rightarrow u = U\cos\beta\,, \qquad v = U\sin\beta$

Which gives that:
$\dot{x} = \mathrm{u}\cos\psi - v\sin\psi$
$= U\cos\psi\cos\beta - U\sin\psi\sin\beta$
$= U(\cos\psi\cos\beta - \sin\psi\sin\beta)$
$= U(cos\,\psi\, cos\,\beta - sin\,\psi\sin\beta)$
$= U\,cos(\psi+\beta) = U\cos(\chi)$

Likewise,
$\dot{y} = u\sin\psi + v\cos\psi$
$= U\sin\psi\cos\beta + U\cos\psi\sin\beta$
$= U(sin\,\psi\, cos\,\beta + cos\,\psi\sin\beta)$
$= U\,sin(\psi+\beta) = U\sin(\chi)$

Problem 2.2

If B is small, $\psi + B \approx \psi$
With ψ also being small, $\sin\psi \approx \psi$ and $\cos\psi \approx 1$

y becomes the cross-track error when the path we are following is directly north from $y = 0$.

Problem 2.3

$T\dot{r} + r = K\delta + b$
$\dot{\psi} = r$

$y(s) = h_1(s)\delta(s) + h_2(s)b(s)$

$\mathcal{L}$:
$TsR + R = K\delta(s) + b(s) \Rightarrow R(1 + Ts) = K\delta(s) + b(s)$
$\Rightarrow R = \dfrac{K\delta(s) + b(s)}{1 + Ts}$
$s\psi(s) = R \Rightarrow \psi(s) = \dfrac{R}{s}$

$\dot{y} = U\psi$
$sy(s) = U \cdot \psi(s)$

$y(s) = \dfrac{U \cdot \psi(s)}{s}$

$y(s) = \dfrac{1}{s} U \dfrac{R}{s} = \dfrac{1}{s^2} U \dfrac{K\delta(s) + b(s)}{1 + Ts} = \dfrac{U \cdot K}{s^2(1 + Ts)} \delta(s) + \dfrac{U}{s^2(1 + Ts)} b(s)$

asd