week 5 day 1

"Algebraic & Geometric Methods in Statistics"

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Recap: Exponential families

- An {exponential family} is a *parametric statistical model* with probability distributions of a *certain form*.
- {General} enough to include many of the most common families of probability distributions:
  - multivariate normal
  - exponential
  - Poisson
  - binomial (with fixed number of trials)
- {Specific} enough to have nice properties:
  - likelihood function is strictly concave [next lecture]
  - have conjugate priors.

Objectives

- {What is} an exponential family?
- **How to find the polynomial ideal of an exponential family?**
  - Discrete exponential models: Hypothesis testing [future lecture]
    - Gaussian exponential submodels: Conditional independence implications [past lecture]

Recap: Discrete exponential families

Notation

- X a discrete random variable $X \in [r]$.
- $T(x) = a_x$, writing as a vector: $a_x = (a_{1x}, \ldots, a_{kx})^t$. Assume $a_{jx} \in \mathbb{Z}$.
- $h(x) = h_x$, so $h = (h_1, \ldots, h_r)$ is also a vector (of positive real numbers)
- $\eta = (\eta_1, \ldots, \eta_k)^t$ and $\theta_i = \exp \eta_i$.

$$p_\theta(x) = \frac{1}{Z(\theta)} h_x \prod_i \theta_i^{a_{ix}}.$$

- The design matrix: $\mathcal{A} = (a_{jx})_{j \in [k], x \in [r]} \in \mathbb{Z}^{k \times r}$.
- For each value x of X, the monomial $\prod_j \theta_j^{a_{jx}} \leftrightarrow$ a column of $\mathcal{A}$.

Design matrix recipe

Columns of $\mathcal{A}$ are exponents of the parametrization of each given state.

Question from the previous lecture

- Consider the model $p_{ij} = \alpha_i \beta_j$ for $i \in [2]$ and $j \in [2]$.

Question from the previous lecture

- Consider the model $p_{ij} = \alpha_i \beta_j$ for $i \in [2]$ and $j \in [2]$. Binary independent random variables.
- The design matrix is

$$\mathcal{A} = \begin{array}{c c} & \begin{array}{cccc} p_{11} & p_{12} & p_{21} & p_{22} \end{array} \\ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Finally, the vector $h = [1, 1, 1, 1]^t$.

$\rightarrow$ I *assume* that **each of you** has completed this example for not 2×2 but $r_1 \times r_2$ by hand, by now.

In this lecture

- log-affine models
- what to do with the h function in the parametrization of an exponential family model (nothing!)
- is there an “easy” way to compute the implicitization of all discrete exponential families?

Log-affine, log-linear discrete exponential families

- Let $\mathcal{A} = [a_{jx}]_{j\in[k],x\in[r]} \in \mathbb{Z}^{k\times r}$ be a design matrix.

$$p_\theta(x) = \frac{1}{Z(\theta)} h_x \prod_i \theta_i^{a_{ix}}.$$

- The logarithm of the exponential family model $p_\theta(x) = \frac{1}{Z(\theta)} h_x \prod_i \theta_i^{a_{ix}}$ is \$log $p_\theta(x) = \log h_x + \sum_j a_{jx} \log \theta_j - \log Z(\theta)$.
- Assume $\mathcal{A}$ contains the vector $1 = (1, \ldots, 1)$ in the rowspan, then this is equivalent to requiring that $\log p$ belongs to the affine space $\log(h) + rowspan(A)$.

. . . "equivalent to requiring that log p belongs to the affine space $\log(h) + \mathit{rowspan}(A)$."

Definition

Definition 6.2.1. Let $A \in \mathbb{Z}^{k\times r}$ be a matrix of integers such that $\mathbf{1} \in$ rowspan(A) and let $h \in \mathbb{R}^r_{>0}$. The *log-affine model* associated to these data is the set of probability distributions

$$\mathcal{M}_{A,h} := \{p \in \operatorname{int}(\Delta_{r-1}) : \log p \in \log h + \operatorname{rowspan}(A)\}.$$

If $h = \mathbf{1}$, then $\mathcal{M}_A = \mathcal{M}_{A,\mathbf{1}}$ is called a *log-linear model*.

Figure 1: Source: textbook

Ideal of a log-linear model

Consider the *log-affine* model $\mathcal{M}_{\mathcal{A},h}$ given by the design matrix $\mathcal{A}$ and vector h.

$$p_\theta(x) \propto h_x \prod_i \theta_i^{a_{ix}}.$$

- This is a model for the joint distribution for discrete random variables, whose states we may denote by $\{1, \ldots, r\}$. So the model is a parametric form of the joint probabilities $p_1, \ldots, p_r$. In other words, the indeterminates p_i index the columns of the $\mathcal{A}$.
- $\mathcal{M}_{\mathcal{A},h}$ is the set of all joint probability vectors $(p_1, \ldots, p_r)$ of the above form.

Definition [Cf. 6.2.2. & 6.2.3. in the book]

The toric ideal of the model $\mathcal{M}_{\mathcal{A},h}$ is the ideal of the variety parametrized by $(p_1, \ldots, p_r)$. If $h = [1, \ldots, 1]$, we denote this as $I_{\mathcal{A}}$.

Proposition 6.2.4. *Let $A \in \mathbb{Z}^{k\times r}$ be a $k \times r$ matrix of integers. Then the toric ideal I_A is a binomial ideal and*

$$I_A = \langle p^u - p^v : u, v \in \mathbb{N}^r \text{ and } Au = Av \rangle.$$

If $\mathbf{1} \in \mathrm{rowspan}(A)$, then I_A is homogeneous.

Figure 2: Proposition 6.2.4. from textbook

Proof. **TBD**

EXAMPLES: 6.2.5, 6.2.6.

I think I will leave 6.2.7 for self-study/reading. (Good for hw2.)

CODE for generating ideals. :)

Other resources

- TBD.

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