week 11 day 1

Graphical models: continuation
Algebraic & Geometric Methods in Statistics

Sonja Petrović
Created for Math/Stat 561

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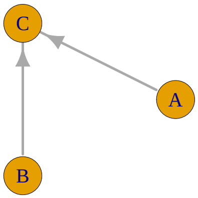
Material

- Chapter 13: Graphical models
- We are following after Miles Bakenhus' course project lecture on sections 13.1 and 13.2.
- We will review a couple of examples from the basics of graphical models (think of the first st of these slides as your study worksheet in class).
- We will then see a few more examples
- Discuss the *discrete distributions* and connection to algebra&geometry.

Examples

- Genes:
  - three genes in this example A,B,C
- Relationships:
  - A regulates C
  - B regulates C



$$P(A, B, C) =$$

$$P(A)P(B)P(C|A, B)$$

BIOLOGY

- genes
- relationships

GRAPH

- vertices
- edges

PROBABILISTIC MODEL

- random variables
- statistical dependencies

Correlation vs causation

- Genes regulated as $X \to Y \to Z$
- Z and X are correlated, but do not interact directly

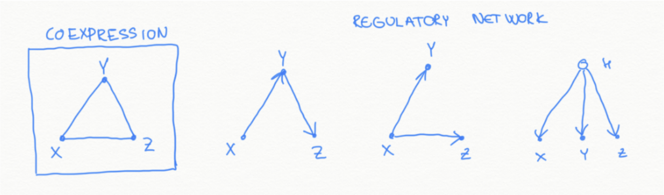


Figure 1: Source: K. Kubjas

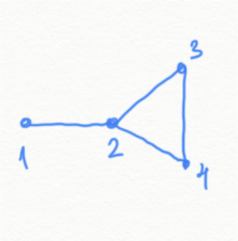
Separator

Poll:

Let G be a graph with nodes {1,2,3,4} and edges (1,2), (2,3), (2,4), (3,4).

Which of the following sets are separators for the nodes 1 and 4?

1. {2}
2. {3}
3. {2,3}
4. {1,2,3,4}



Separator

Poll:

Let G be a graph with nodes $\{1,2,3,4\}$ and edges (1,2), (2,3), (2,4), (3,4).

Which of the following sets are separators for the nodes 1 and 4?

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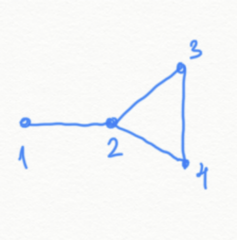

Answer

Correct answers: 1. and 3.

Reminder: conditinoal independence definition

[board]

Pairwise Markov property

Let $G = (V, E)$ be an undirected graph.

Definition

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Question (example)

The pairwise Markov property associated to G below is:

1. $\{1 \perp\!\!\!\perp 3|(2,4), 1 \perp\!\!\!\perp 4|(2,3)\}$
2. $\{1 \perp\!\!\!\perp 3|2, 1 \perp\!\!\!\perp 4|2\}$
3. $\{1 \perp\!\!\!\perp 3|(2,4)\}$
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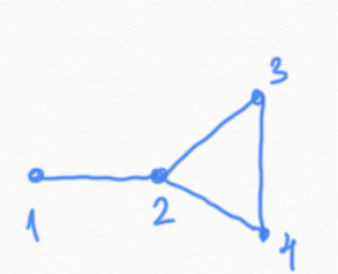

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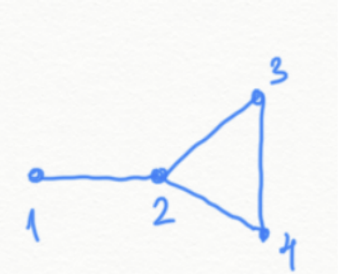
Correct answer: 1.



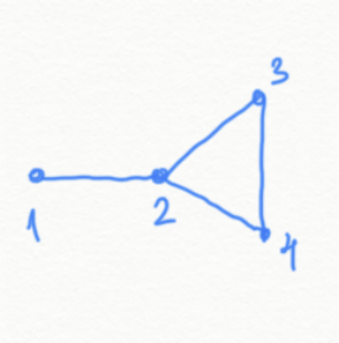
Multivariate Gaussian random variables

- The CI statement $X_u \perp\!\!\!\perp X_v | X_{V\setminus\{u,v\}}$ is equivalent to the matrix $\Sigma_{V\setminus\{u\},V\setminus\{v\}}$having rank $|V\{u,v\}|$, or equivalently $det(\Sigma_{V\setminus\{u\},V\setminus\{v\}}) = 0$.
- This is equivalent to $(\Sigma^{-1})_{u,v} = 0$.
- The pairwise Markov property holds for a *Gaussian distribution* if and only if the entries of the concentration matrix corresponding to non-edges are zero.

Question (example)

What is the form of the concentration matrices of a Gaussian distribution obeying the pairwise Markov property have?

1. $$\begin{bmatrix} k_{11} & 0 & k_{13} & k_{14} \\ 0 & k_{22} & 0 & 0 \\ k_{13} & 0 & k_{33} & 0 \\ k_{14} & 0 & 0 & k_{44} \end{bmatrix}$$
2. $$\begin{bmatrix} k_{12} & k_{12} & 0 & 0 \\ k_{12} & k_{22} & k_{23} & k_{24} \\ 0 & k_{23} & k_{33} & k_{34} \\ 0 & k_{24} & k_{34} & k_{44} \end{bmatrix}$$



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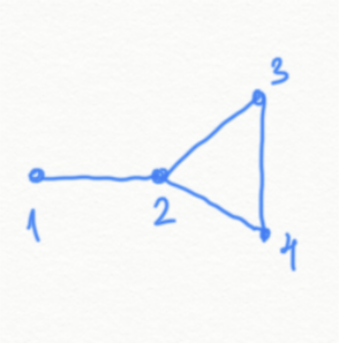
Correct answer: 2.



Global Markov property

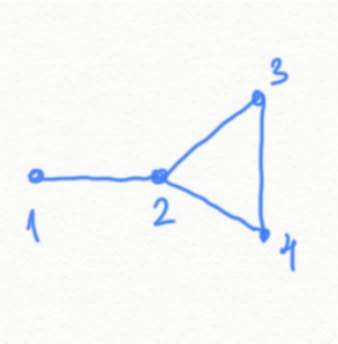
Definition (remider!)

[board]

Question (example)

The global Markov property associated to G is:

1. $\{1 \perp\!\!\!\perp (3,4) | 2\}$
2. $\{1 \perp\!\!\!\perp 3 | (2,4), \quad 1 \perp\!\!\!\perp 4 | (2,3)\}$
3. $\{1 \perp\!\!\!\perp 3 | (2,4), \quad 1 \perp\!\!\!\perp 4 | (2,3),$
   $1 \perp\!\!\!\perp (3,4) | 2\}$



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3. $\{1 \perp\!\!\!\perp 3 | (2,4), \quad 1 \perp\!\!\!\perp 4 | (2,3),$
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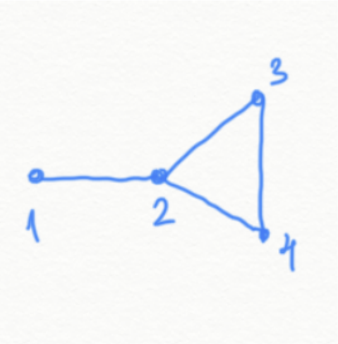
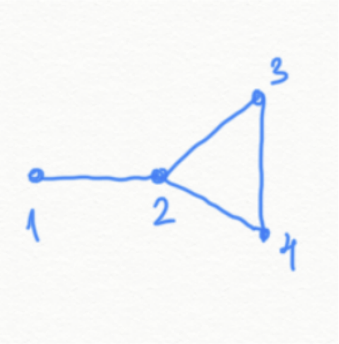
Correct answer: 3.



Markov properties

In the last lecture, Miles showed that pairwise Markov statements C_{pairs} are a subset of Global statements C_{global}.

- In our example:
- $C_{pairs} =$
  $\{1 \perp\!\!\!\perp 3|(2,4), \quad 1 \perp\!\!\!\perp 4|(2,3)\}$.
- $C_{global} = C_{pairs} \cup \{1 \perp\!\!\!\perp (3,4)|2\}$.



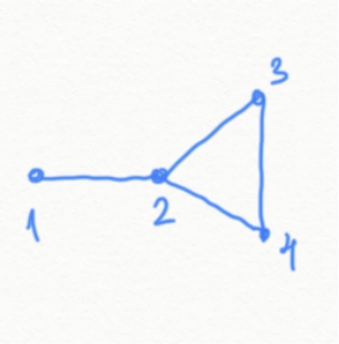
Factorization property

- We want to characterize **all** the distributions that satisfy the Markov properties for a *given graph*.
  - Hammersley-Clifford theorem relates the implicit description of a graphical model through Markov properties to a parametric description.
- Recall: definition of factorizing according to a graph via cliques. [board]
- Review Theorem 13.2.10 (recursive factorization in DAGs) with *proof*.

Question (example)

What are the maximal cliques of G?

1. $\{1\}$
2. $\{1, 2\}$
3. $\{1, 2, 3, \}$
4. $\{2, 3, 4\}$

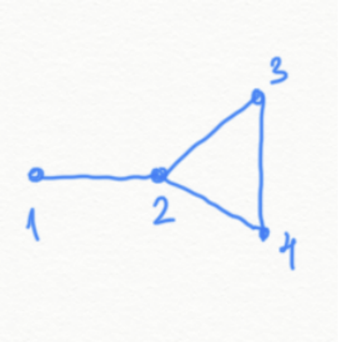


Question (example)

What are the maximal cliques of G?

1. $\{1\}$
2. $\{1, 2\}$
3. $\{1, 2, 3, \}$
4. $\{2, 3, 4\}$

Correct answers: 2 and 4.



Examples from 13.4

- (Homogeneous) Markov chain - example 13.4.1 and connection to chapter 1
- Hidden Markov model

[board notes]

Discrete distributions - and connection to algebra and geometry

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This document is created for Math/Stat 561, Spring 2023.

The slides that are not directly from the book are sourced from **Kaie Kubjas**' Algebraic Statistics course at Aaalto University.