

Exponential families

“Algebraic & Geometric Methods in Statistics”

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Created for Math/Stat 561

Feb 1, 2023.

Objectives

- Wrap up introductory overview of maximum likelihood estimation
 - there will be more material on this from chapter 7 soon!
 - see an example with observed data for the model of 2×2 independence.
- Understand the setup of exponential families.
 - broad class of models
 - structure \leftrightarrow sufficient statistics, MLE, etc.

Material:

Sourced from chapter 6 (“exponential families”) of the textbook. Other resources provided in subsequent links.

Maximum likelihood estimation

i.i.d. sampling: $L(\theta | D) = \prod_{i=1}^n L(\theta | X^{(i)})$

- **Likelihood function** (discrete case): $L(\theta | D) = \prod_{i=1}^n p_{\theta}(X^{(i)})$
- Let $u \in \mathbb{N}^r$ be the **vector of counts**, i.e. $u_j = \#\{i : X^{(i)} = j\}$: $L(\theta | D) = \prod_{i=1}^n p_{\theta}(X^{(i)}) = \prod_{j=1}^r p_{\theta}(j)^{u_j}$
- Example for $\left\{ (\theta^2, 2\theta(1-\theta), (1-\theta)^2) : \theta \in [0,1] \right\}$: $L(\theta | D) = (\theta^2)^{u_0} \cdot (2\theta(1-\theta))^{u_1} \cdot ((1-\theta)^2)^{u_2}$
- **Likelihood function** (continuous case): $L(\theta | D) = \prod_{i=1}^n f_{\theta}(X^{(i)})$

Log-likelihood function

- The **log-likelihood function** is

$$l(\theta | D) = \log L(\theta | D)$$

- **i.i.d. data**: turns a product into a sum

Example: the independence model 2×2

$$\mathcal{M}_{X \amalg Y} = \{p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in \Delta_3 : p_{ij} = \alpha_i \beta_j, (\alpha, \beta) \in \Delta_1 \times \Delta_1\} \text{ and } u = \begin{pmatrix} 19 & 141 \\ 17 & 149 \end{pmatrix}$$

Log-likelihood function: $l(\alpha, \beta | u) = 160 \log \alpha_1 + 166 \log \alpha_2 + 36 \log \beta_1 + 290 \log \beta_2$

$$= 160 \log \alpha_1 + 166 \log(1 - \alpha_1) + 36 \log \beta_1 + 290 \log(1 - \beta_1)$$

Score equations:

$$\frac{\partial l(\alpha, \beta | u)}{\partial \alpha_1} = \frac{160}{\alpha_1} - \frac{166}{1 - \alpha_1} = 0$$

$$\frac{\partial l(\alpha, \beta | u)}{\partial \beta_1} = \frac{36}{\beta_1} - \frac{290}{1 - \beta_1} = 0$$

Figure 1: Source: K.Kubjas

Exponential families

- An **exponential family** is a *parametric statistical model* with probability distributions of a *certain form*.
- **General** enough to include many of the most common families of probability distributions:
 - multivariate normal
 - exponential
 - Poisson
 - binomial (with fixed number of trials)
- **Specific** enough to have nice properties:
 - likelihood function is strictly concave [next lecture]
 - have conjugate priors.

Objectives

- **What is** an exponential family?
- **How** to find the polynomial **ideal** of an exponential family?
 - **Discrete** exponential models: Hypothesis testing [future lecture]
 - **Gaussian** exponential submodels: Conditional independence implications [past lecture]

Exponential families

Let X be a random variable taking values in a set \mathcal{X} .

An **exponential family** is the set of probability distributions whose probability mass function or density function **can be expressed as**

$$f_{\theta}(x) = h(x)e^{\eta(\theta)'T(x)-A(\theta)},$$

for a given **statistic** $T(x)$, **natural parameter** $\eta(\theta)$, and **functions** $h : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ and $A : \Theta \rightarrow \mathbb{R}$.

Three equivalent forms:

$$f_{\theta}(x) = h(x)e^{\eta(\theta)'T(x)-A(\theta)},$$

$$f_{\theta}(x) = h(x)g(\theta)e^{\eta(\theta)'T(x)},$$

$$f_{\theta}(x) = e^{\eta(\theta)'T(x)-A(\theta)+B(x)}.$$

Example: Bernoulli

The Bernoulli distribution

The probability mass function of a Bernoulli random variable X is given as follows:

$$p(x | \pi) = \pi^x (1 - \pi)^{1-x} \quad (8.4)$$

THE MODEL AS YOU
WRITE IT NATURALLY

$$= \exp \left\{ \log \left(\frac{\pi}{1 - \pi} \right) x + \log(1 - \pi) \right\}. \quad (8.5)$$

EXPONENTIAL FAMILY
FORM

where our trick, here and throughout the chapter, is to take the exponential of the logarithm of the original distribution. Thus we see that the Bernoulli distribution is an exponential family distribution with:

$$\eta = \frac{\pi}{1 - \pi} \quad (8.6)$$

$$T(x) = x \quad (8.7)$$

$$A(\eta) = -\log(1 - \pi) = \log(1 + e^\eta) \quad (8.8)$$

$$h(x) = 1. \quad (8.9)$$

PIECES OF THE EXPONENTIAL
FAMILY FORM

Note moreover that the relationship between η and π is invertible. Solving Eq. (8.6) for π , we have:

$$\pi = \frac{1}{1 + e^{-\eta}}, \quad (8.10)$$

which is the logistic function.

Figure 2: Source: Michael I. Jordan's notes on exponential families

Example: binomial distribution.

$X \sim \text{Bin}(m, \theta)$, $\mathcal{X} = \{0, \dots, r\}$.

$$p(x) = \binom{r}{x} \theta^x (1 - \theta)^{r-x} =$$

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$$\begin{aligned} p(x) &= \binom{r}{x} \theta^x (1 - \theta)^{r-x} = \\ &= \binom{r}{x} \exp \left[\left(\log \frac{\theta}{1 - \theta} \right) x + r \log(1 - \theta) \right]. \end{aligned}$$

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Question: What are the k, T, η, h, A in this example?

1. $k = 1$, $T(x) = \log \frac{\theta}{1 - \theta}$, $\eta = x$, $h = \binom{r}{x}$, $A = -m \log(1 - \theta)$
2. $k = 1$, $T(x) = x$, $\eta = \log \frac{\theta}{1 - \theta}$, $h = \binom{r}{x}$, $A = -m \log(1 - \theta)$
3. $k = 2$, $T(x) = (x, r - x)$, $\eta = (\theta, 1 - \theta)$, $h = \binom{r}{x}$, $A = 0$.

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Question: What are the k, T, η, h, A in this example?

1. $k = 1$, $T(x) = \log \frac{\theta}{1 - \theta}$, $\eta = x$, $h = \binom{r}{x}$, $A = -m \log(1 - \theta)$. \leftarrow wrong, because T should not depend on the parameters.
2. $k = 1$, $T(x) = x$, $\eta = \log \frac{\theta}{1 - \theta}$, $h = \binom{r}{x}$, $A = -m \log(1 - \theta)$ \leftarrow correct.
3. $k = 2$, $T(x) = (x, r - x)$, $\eta = (\theta, 1 - \theta)$, $h = \binom{r}{x}$, $A = 0$. \leftarrow wrong.

Canonical form

$$f_{\theta}(x) = h(x)e^{\eta(\theta)'T(x)-A(\theta)}.$$

- If $\eta(\theta) = \theta$, then the exp.fam. is said to be in **canonical form**.
- By defining a transformed parameter $\eta = \eta(\theta)$, it is *always possible to convert an exponential family to canonical form*.
- The function A is determined by the other functions: It makes the pdf (pmf) to integrate (sum) to one. Thus it can be written as a function of η .
- The canonical form is $f_{\eta}(x) = h(x)e^{\eta'T(x)-A(\eta)}$.

Independence model in canonical form

Task

Write down the model of independence of two random variables, $\mathcal{M}_{1 \perp\!\!\!\perp 2}$, in exponential family form.

Here are the key steps:

- Starting point is the parametric description of the model as you know it: The probability of observing the data count table u is given by $\prod_{1 \leq i \leq r_1, 1 \leq j \leq r_2} (\alpha_i \beta_j)^{u_{ij}}$.
- The product can be written using the same log-exp trick as in the previous slides:

$$\prod_{1 \leq i \leq r_1, 1 \leq j \leq r_2} (\alpha_i \beta_j)^{u_{ij}} = \exp \left(\sum_{ij} u_{ij} \log(\alpha_i \beta_j) \right) = \dots$$

- This will be useful to you on the homework, and in several projects.

Discrete exponential families

... There has got to be a general strategy so we don't have to work through all the algebra every time?

- Note that we can use the fact that $e^{a \cdot b} = e^a \cdot e^b$ to write the last quantity from the previous slide in product form.

Notation

- X a **discrete** random variable
 $X \in [r]$.
- $T(x) = a_x$, writing as a vector:
 $a_x = (a_{1x})^t$
- $h(x) = h_x$, so $h = (h_1, \dots, h_r)$
is also a vector (of positive real numbers)
- $\eta = (\eta_1, \dots, \eta_k)^t$ and
 $\theta_i = \exp \eta_i$.

$$p_\eta(x) = h(x)e^{\eta^t T(x) - A(\eta)} =$$

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 $\theta_i = \exp \eta_i$.

$$\begin{aligned} p_\eta(x) &= h(x) e^{\eta^t T(x) - A(\eta)} = \\ &= h_x e^{\sum_i \eta_i a_{ix} - A(\eta)} = h_x \prod_i e^{\eta_i a_{ix} - A(\eta)} \\ &= h_x \prod_i (e_i^\eta)^{a_{ix}} e^{-A(\eta)} = h_x \prod_i \theta_i^{a_{ix}} \frac{1}{Z(\theta)} \end{aligned}$$

where $Z(\theta) = \sum_{x \in [r]} h_x \prod_j \theta_j^{a_{jx}}$.

$$p_{\theta}(x) \propto \frac{1}{Z(\theta)} h_x \prod_i \theta_i^{a_{ix}}.$$

- If a_{jx} are integers for all j and x , then the parametrizing functions are rational functions.
- The entire a_{jx} can be recorded in matrix: $\mathcal{A} = (a_{jx})_{j \in [k], x \in [r]} \in \mathbb{Z}^{k \times r}$.
- For each value x of X , the monomial $\prod_j \theta_j^{a_{jx}} \leftrightarrow$ a column of \mathcal{A} .

Example: binomial random variable with three trials

There are two parameters, $\theta = (\theta_1, \theta_2)$.

Take $h = (1, 3, 3, 1)$ and $\mathcal{A} = ?$

$$p_{\theta}(x) \propto \frac{1}{Z(\theta)} h_x \prod_i \theta_i^{a_{ix}}.$$

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There are two parameters, $\theta = (\theta_1, \theta_2)$.

Take $h = (1, 3, 3, 1)$ and $\mathcal{A} = ?$ $\theta_0 = P(X = 0), \theta_1 = P(X = 1)$.

Then $P(\text{no } 0) = \theta_2^3$, $P(\text{one } 0) = \theta_1 \theta_2^2$, $P(\text{two } 0\text{s}) = \theta_1^2 \theta_2$, $P(\text{all } 1) = \theta_1^3$.

So...

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So... For the value $x = 0$, the corresponding column a_{i0} should be

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So... For the value $x = 0$, the corresponding column a_{i0} should be $[0, 3]^t$.

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$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

- See example 6.2.5 in the book. "Twisted cubic"
- Cf. slide 7 in this lecture.

Coming up next

- log-affine models
- what to do with the h function in the parametrization of an exponential family model (nothing!)
- is there an “easy” way to compute the implicitization of all discrete exponential families.

Other reading, resources, and a task!

- Eliana Duarte's summer school lectures include these slides on [exponential families: an algebraic statistics perspective], see page 13-18. **link will be provided ASAP.**
- Michael I. Jordan's [chapter](#) on exponential families provides another resource equivalent to the background in Chapter 6.
- Martin Wainwright's notes on how to turn a multinomial model into an exponential family form are on page 6 of [this document](#).
 - You should **try this on your own**. Fill out **all** the details of writing down the independence model $\mathcal{M}_{1 \perp\!\!\!\perp 2}$, for example, in exponential family form.

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