Math 561 Assignment 2*

SP, using Miles' template!

Due date: 10 Feb 2023.

Submit solutions to at least 4 problems out of problems 1-5, and at least one of 6&7.

1. Let
$$P^{(X_3=0)} := \begin{pmatrix} 0.05 & 0.15 \\ 0.075 & 0.225 \end{pmatrix}, \; P^{(X_3=1)} := \begin{pmatrix} 0.125 & 0.125 \\ 0.125 & 0.125 \end{pmatrix}$$
Consider three binary random variables, X_1, X_2, X_3 each taking values in the set $\{0, 1\}$ with joint probabilities $P(X_1 = i, X_2 = j, X_3 = 0) = P^{(X_3=0)}_{i,j}$ and $P(X_1 = i, X_2 = j, X_3 = 1) = P^{(X_3=1)}_{i,j}$.
   (a) Find the marginal distribution P_{X_1} of X_1 (Recall that in the discrete case, integration is substituted by summation.)
   (b) Find the conditional distribution $P_{X_2,X_3|X_1}$ of (X_2, X_3) given X_1.
   (c) Is X_2 conditionally independent of X_3 given X_1?
   (d) Is X_1 conditionally independent of X_2 given X_3?

2. Consider four binary random variables X_1, X_2, X_3, X_4 and the collection $\mathcal{C} = \{X_1 \perp\!\!\!\perp (X_3, X_4) \mid X_2\}$. Write down the corresponding conditional independence ideal $I_{\mathcal{C}}$. Hint: The polynomials in the CI ideal are 2×2 minors of some matrices. Can you figure out which matrices?

3. **Exercise 4.7 from the book.** For four binary random variables, consider the conditional indpendence model $C = \{1 \perp\!\!\!\perp 3 \mid (2, 4),\ 2 \perp\!\!\!\perp 4 \mid (1, 3)\}$. Compute the primary decomposition of $I_{\mathcal{C}}$ and describe the components.

   Advanced version of this problem (i.e., you may choose this instead). **Exercise 4.10.** Consider the marginal indepndence model for four binary random variables $C = \{1 \perp\!\!\!\perp 2,\ 2 \perp\!\!\!\perp 3,\ 3 \perp\!\!\!\perp 4,\ 1 \perp\!\!\!\perp 4\}$.
   (a) Compute the primary decomposition of $I_{\mathcal{C}}$.
   (b) Show that there is exactly one component of $V(I_{\mathcal{C}})$ that intersects the probability simplex.
   (c) Give a parametrization of that componenet.

4. **Exercise 4.12.** Consider the Gaussian conditional independence model $C = \{1 \perp\!\!\!\perp 2|3, 1 \perp\!\!\!\perp 3|4, 1 \perp\!\!\!\perp 4|2\}$. Compute the primary decomposition of $J_{\mathcal{C}}$ and use this to determine what, if any, further conditional independence statements are implied.

5. **Coding Problem!** Write an `R` (preferred) or `Python` (acceptable) script to generate the discrete and/or Gaussian conditional independence ideal from (1) one CI statement and (2) a list of CI statemens. The output should be as the output of `Macaulay2` shown in slides in Lectures 4 and 5.

6. **Exercise 6.2.** Consider the vector $h = (1, 1, 1, 2, 2, 2)$ and the matrix $A = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{bmatrix}$.

(1) Compute generators for the toric ideals I_A and $I_{A,h}$.
(2) What familiar statistical model is the discrete exponential family $\mathcal{M}_{A,h}$?

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7. **Exercise 6.3.** Consider the monomial parametrization

$$p_{ijk} = \alpha_{ij}\beta_{ik}\gamma_{jk}$$

for $i \in [r_1]$, $j \in [r_2]$, $k \in [r_3]$. Describe the matrix A associated to this monomial parametrization. How does A act as a linear transformation on 3-way arrays (3-dimensional tables)? Compute the vanishing ideal I_A for $r_1 = r_2 = r_3 = 3$.