

Math 561 Assignment 2*

SP, using Miles' template!

Due date: 10 Feb 2023.

Submit solutions to at least 4 problems out of problems 1-5, and at least one of 6&7.

1. Let

$$P^{(X_3=0)} := \begin{pmatrix} 0.05 & 0.15 \\ 0.075 & 0.225 \end{pmatrix}, \quad P^{(X_3=1)} := \begin{pmatrix} 0.125 & 0.125 \\ 0.125 & 0.125 \end{pmatrix}$$

Consider three binary random variables, X_1, X_2, X_3 each taking values in the set $\{0, 1\}$ with joint probabilities $P(X_1 = i, X_2 = j, X_3 = 0) = P_{i,j}^{(X_3=0)}$ and $P(X_1 = i, X_2 = j, X_3 = 1) = P_{i,j}^{(X_3=1)}$.

- Find the marginal distribution P_{X_1} of X_1 (Recall that in the discrete case, integration is substituted by summation.)
 - Find the conditional distribution $P_{X_2, X_3 | X_1}$ of (X_2, X_3) given X_1 .
 - Is X_2 conditionally independent of X_3 given X_1 ?
 - Is X_1 conditionally independent of X_2 given X_3 ?
2. Consider four binary random variables X_1, X_2, X_3, X_4 and the collection $\mathcal{C} = \{X_1 \perp\!\!\!\perp (X_3, X_4) \mid X_2\}$. Write down the corresponding conditional independence ideal $I_{\mathcal{C}}$. Hint: The polynomials in the CI ideal are 2×2 minors of some matrices. Can you figure out which matrices?
3. **Exercise 4.7 from the book.** For four binary random variables, consider the conditional independence model $C = \{1 \perp\!\!\!\perp 3 \mid (2, 4), 2 \perp\!\!\!\perp 4 \mid (2, 3)\}$. Compute the primary decomposition of I_C and describe the components.

Advanced version of this problem (i.e., you may choose this instead). **Exercise 4.10.** Consider the marginal independence model for four binary random variables $C = \{1 \perp\!\!\!\perp 2, 2 \perp\!\!\!\perp 3, 3 \perp\!\!\!\perp 4, 1 \perp\!\!\!\perp 4\}$.

- Compute the primary decomposition of I_C .
 - Show that there is exactly one component of $V(I_C)$ that intersects the probability simplex.
 - Give a parametrization of that component.
4. **Exercise 4.12.** Consider the Gaussian conditional independence model $C = \{1 \perp\!\!\!\perp 2 \mid 3, 1 \perp\!\!\!\perp 3 \mid 4, 1 \perp\!\!\!\perp 4 \mid 2\}$. Compute the primary decomposition of I_C and use this to determine what, if any, further conditional independence statements are implied.
5. **Coding Problem!** Write an R (preferred) or Python (acceptable) script to generate the discrete and/or Gaussian conditional independence ideal from (1) one CI statement and (2) a list of CI statements. The output should be as the output of Macaulay2 shown in slides in Lectures 4 and 5.

6. **Exercise 6.2.** Consider the vector $h = (1, 1, 1, 2, 2, 2)$ and the matrix $A = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{bmatrix}$.

- Compute generators for the toric ideals I_A and $I_{A,h}$.
- What familiar statistical model is the discrete exponential family $\mathcal{M}_{A,h}$?

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7. **Exercise 6.3.** Consider the monomial parametrization

$$p_{ijk} = \alpha_{ij}\beta_{ik}\gamma_{jk}$$

for $i \in [r_1]$, $j \in [r_2]$, $k \in [r_3]$. Describe the matrix A associated to this monomial parametrization. How does A act as a linear transformation on 3-way arrays (3-dimensional tables)? Compute the vanishing ideal I_A for $r_1 = r_2 = r_3 = 3$.