week 11 day 1

Graphical models: continuation
Algebraic & Geometric Methods in Statistics

Sonja Petrović
Created for Math/Stat 561

Mar 22, 2023.

Material

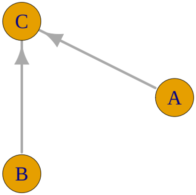
- Chapter 13: Graphical models
- We are following after Miles Bakenhus' course project lecture on sections 13.1 and 13.2.
- We will review a couple of examples from the basics of graphical models (think of the first st of these slides as your study worksheet in class).
- We will then see a few more examples
- Discuss the *discrete distributions* and connection to algebra&geometry.

Examples

- Genes:
  - three genes in this example A,B,C
- Relationships:
  - A regulates C
  - B regulates C

BIOLOGY

- genes
- relationships



GRAPH

- vertices
- edges

$$P(A, B, C) =$$

$$P(A)P(B)P(C|A, B)$$

PROBABILISTIC MODEL

- random variables
- statistical dependencies

Correlation vs causation

- Genes regulated as $X \to Y \to Z$
- Z and X are correlated, but do not interact directly

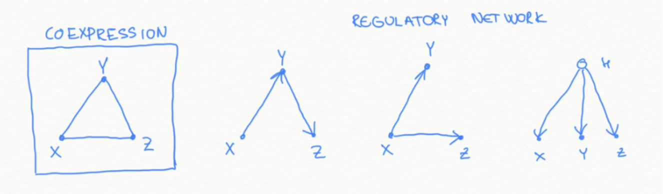


Figure 1: Source: K. Kubjas

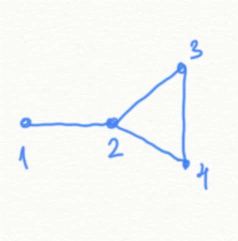
Separator

Poll:

Let G be a graph with nodes $\{1,2,3,4\}$ and edges (1,2), (2,3), (2,4), (3,4).

Which of the following sets are separators for the nodes 1 and 4?

1. $\{2\}$
2. $\{3\}$
3. $\{2,3\}$
4. $\{1,2,3,4\}$



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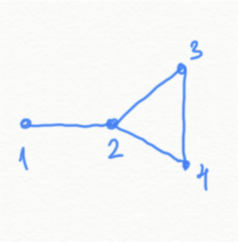

Answer

Correct answers: 1. and 3.

Reminder: conditinoal independence definition

[board]

Pairwise Markov property

Let $G = (V, E)$ be an undirected graph.

Definition

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Question (example)

The pairwise Markov property associated to G below is:

1. $\{1 \perp\!\!\!\perp 3|(2,4), 1 \perp\!\!\!\perp 4|(2,3)\}$
2. $\{1 \perp\!\!\!\perp 3|2, 1 \perp\!\!\!\perp 4|2\}$
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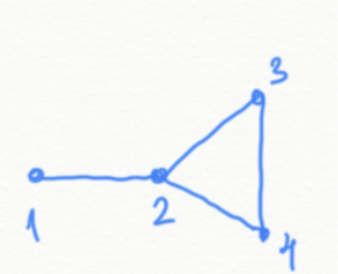

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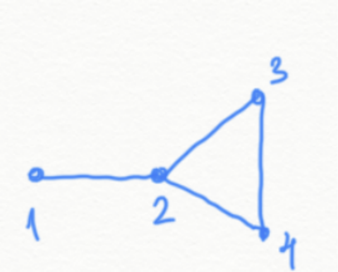
Correct answer: 1.



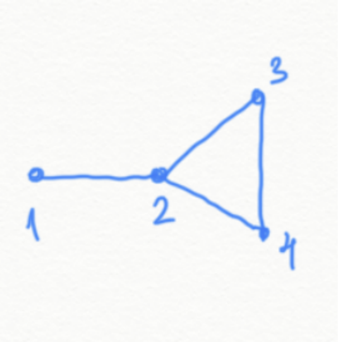
Multivariate Gaussian random variables

- The CI statement $X_u \perp\!\!\!\perp X_v | X_{V\setminus\{u,v\}}$ is equivalent to the matrix $\Sigma_{V\setminus\{u\},V\setminus\{v\}}$having rank $|V\{u,v\}|$, or equivalently $det(\Sigma_{V\setminus\{u\},V\setminus\{v\}}) = 0$.
- This is equivalent to $(\Sigma^{-1})_{u,v} = 0$.
- The pairwise Markov property holds for a *Gaussian distribution* if and only if the entries of the concentration matrix corresponding to non-edges are zero.

Question (example)

What is the form of the concentration matrices of a Gaussian distribution obeying the pairwise Markov property have?

1. $$\begin{bmatrix} k_{11} & 0 & k_{13} & k_{14} \\ 0 & k_{22} & 0 & 0 \\ k_{13} & 0 & k_{33} & 0 \\ k_{14} & 0 & 0 & k_{44} \end{bmatrix}$$
2. $$\begin{bmatrix} k_{12} & k_{12} & 0 & 0 \\ k_{12} & k_{22} & k_{23} & k_{24} \\ 0 & k_{23} & k_{33} & k_{34} \\ 0 & k_{24} & k_{34} & k_{44} \end{bmatrix}$$



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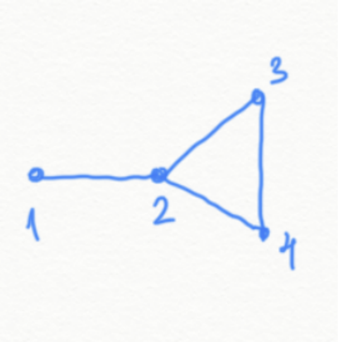


Correct answer: 2.

Global Markov property

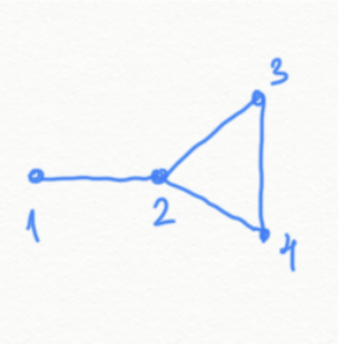
Definition (remider!)

[board]

Question (example)

The global Markov property associated to G is:

1. $\{1 \perp\!\!\!\perp (3,4)|2\}$
2. $\{1 \perp\!\!\!\perp 3|(2,4), \quad 1 \perp\!\!\!\perp 4|(2,3)\}$
3. $\{1 \perp\!\!\!\perp 3|(2,4), \quad 1 \perp\!\!\!\perp 4|(2,3),$
   $1 \perp\!\!\!\perp (3,4)|2\}$



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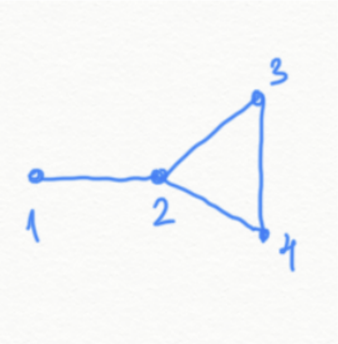
   $1 \perp\!\!\!\perp (3,4)|2\}$

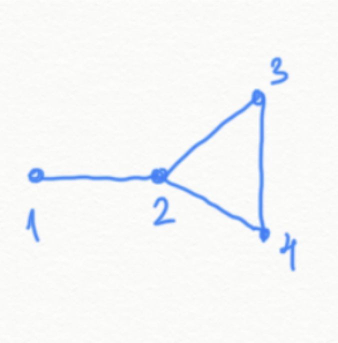
Correct answer: 3.



Markov properties

In the last lecture, Miles showed that pairwise Markov statements C_{pairs} are a subset of Global statements C_{global}.

- In our example:
- $C_{pairs} =$
  $\{1 \perp\!\!\!\perp 3|(2,4), \quad 1 \perp\!\!\!\perp 4|(2,3)\}$.
- $C_{global} = C_{pairs} \cup \{1 \perp\!\!\!\perp (3,4)|2\}$.



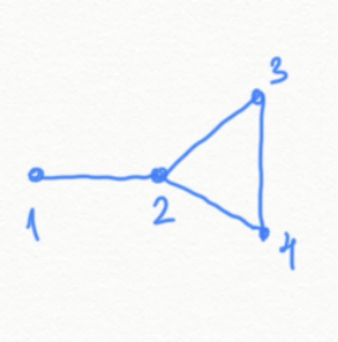
Factorization property

- We want to characterize **all** the distributions that satisfy the Markov properties for a *given graph*.
  - Hammersley-Clifford theorem relates the implicit description of a graphical model through Markov properties to a parametric description.
- Recall: definition of factorizing according to a graph via cliques. [board]
- Review Theorem 13.2.10 (recursive factorization in DAGs) with *proof*.

Question (example)

What are the maximal cliques of G?

1. $\{1\}$
2. $\{1, 2\}$
3. $\{1, 2, 3, \}$
4. $\{2, 3, 4\}$



Question (example)

What are the maximal cliques of G?

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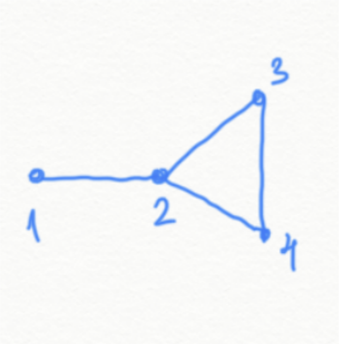
3. $\{1, 2, 3, \}$
4. $\{2, 3, 4\}$

Correct answers: 2 and 4.



Examples from 13.4

- (Homogeneous) Markov chain - example 13.4.1 and connection to chapter 1
- Hidden Markov model

[board notes]

Discrete distributions - and algebra&geometry

- $X = \{X_1, \ldots, X_m\}$ discrete random vector
- The distribution p on X factors according to G if

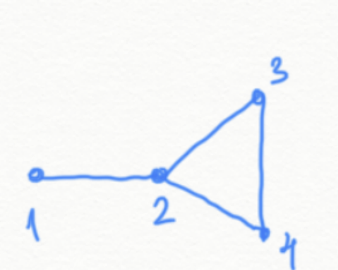
$$p_{i_1 i_2 \ldots i_m} \propto \prod_{C \in \mathcal{C}(G)} \theta^{(C)}_{i_C}.$$

- This is **a monomial parametrization**. Hence the set of distributions that factorize according to a graph G form a hierarchical **log-linear model**.

$C_{pairs} =$
$\{1 \perp\!\!\!\perp 3|(2,4), 1 \perp\!\!\!\perp 4|(2,3)\}$.
$C_{global} = C_{pairs} \cup \{1 \perp\!\!\!\perp (3,4)|2\}$.

- Spell this out: [board]

$$p(x) = \frac{1}{Z}\theta^{(12)}_{i_1 i_2}\theta^{(234)}_{i_2 i_3 i_4}.$$



Pairwise Markov proprety - algebra

$C_{pairs} = \{1 \perp\!\!\!\perp 3 | (2,4), 1 \perp\!\!\!\perp 4 | (2,3)\}$

Question (example)

How many polynomials generate the corresponding CI ideal?

Pairwise Markov proprety - algebra

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How many polynomials generate the corresponding CI ideal?

- $M_1 = \begin{bmatrix} p_{0000} & p_{0001} & p_{0010} & p_{0011} \\ p_{1000} & p_{1001} & p_{1010} & p_{1011} \end{bmatrix}$
- $M_2 = \begin{bmatrix} p_{0100} & p_{0101} & p_{0110} & p_{0111} \\ p_{1100} & p_{1101} & p_{1110} & p_{1111} \end{bmatrix}$

Pairwise Markov proprety - algebra

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- $M_1 = \begin{bmatrix} p_{0000} & p_{0001} & p_{0010} & p_{0011} \\ p_{1000} & p_{1001} & p_{1010} & p_{1011} \end{bmatrix}$
- $M_2 = \begin{bmatrix} p_{0100} & p_{0101} & p_{0110} & p_{0111} \\ p_{1100} & p_{1101} & p_{1110} & p_{1111} \end{bmatrix}$
- The conditional independence ideal for each statement is generated by two minors of M_1 and two minors of M_2

```
i1 : R1 = QQ[p_(0,0,0,0)..p_(1,1,1,1)]

o1 = R1

o1 : PolynomialRing

i2 : M1 = matrix{{p_(0,0,0,0),p_(0,0,0,1),p_(0,0,1,0),p_(0,0,1,1)},{p_(1,0,0,0),p_(1,0,0,1),p_(1,0,1,0),p_(1,0,1,1)}}

o2 = | p_(0,0,0,0) p_(0,0,0,1) p_(0,0,1,0) p_(0,0,1,1) |
     | p_(1,0,0,0) p_(1,0,0,1) p_(1,0,1,0) p_(1,0,1,1) |

               2       4
o2 : Matrix R1  <--- R1

i3 : M2 = matrix{{p_(0,1,0,0),p_(0,1,0,1),p_(0,1,1,0),p_(0,1,1,1)},{p_(1,1,0,0),p_(1,1,0,1),p_(1,1,1,0),p_(1,1,1,1)}}

o3 = | p_(0,1,0,0) p_(0,1,0,1) p_(0,1,1,0) p_(0,1,1,1) |
     | p_(1,1,0,0) p_(1,1,0,1) p_(1,1,1,0) p_(1,1,1,1) |

               2       4
o3 : Matrix R1  <--- R1

i4 : IP = ideal(det(M1_{0,2}),det(M1_{1,3}),det(M2_{0,2}),det(M2_{1,3}),det(M1_{0,1}),det(M1_{2,3}),det(M2_{0,1}),det(M2_{2,3}))

o4 = ideal (- p        p        + p        p        , - p        p        +
               0,0,1,0 1,0,0,0     0,0,0,0 1,0,1,0       0,0,1,1 1,0,0,1
     ------------------------------------------------------------------------
     p        p        , - p        p        + p        p        , -
      0,0,0,1 1,0,1,1       0,1,1,0 1,1,0,0   0,1,0,0 1,1,1,0
     ------------------------------------------------------------------------
     p        p        + p        p        , - p        p        +
      0,1,1,1 1,1,0,1   0,1,0,1 1,1,1,1       0,0,0,1 1,0,0,0
     ------------------------------------------------------------------------
     p        p        , - p        p        + p        p        , -
      0,0,0,0 1,0,0,1       0,0,1,1 1,0,1,0   0,0,1,0 1,0,1,1
     ------------------------------------------------------------------------
     p        p        + p        p        , - p        p        +
      0,1,0,1 1,1,0,0   0,1,0,0 1,1,0,1       0,1,1,1 1,1,1,0
     ------------------------------------------------------------------------
     p        p        )
      0,1,1,0 1,1,1,1

o4 : Ideal of R1
```

Figure 2: code will be shared after class

Global Markov property - algebra

$C_{global} = C_{pairs} \cup \{1 \perp\!\!\!\perp (3,4) | 2\}$.

- $M_1 = \begin{bmatrix} p_{0000} & p_{0001} & p_{0010} & p_{0011} \\ p_{1000} & p_{1001} & p_{1010} & p_{1011} \end{bmatrix}$
- $M_2 = \begin{bmatrix} p_{0100} & p_{0101} & p_{0110} & p_{0111} \\ p_{1100} & p_{1101} & p_{1110} & p_{1111} \end{bmatrix}$
- The conditional independence ideal for each statement is generated by **all** 2×2 minors M_1 and of M_2

Recall slide 8-9 of Lecture 4!

Factorization according to G - algebra

$$p_{i_1 i_2 \dots i_m} = \frac{1}{Z} \prod_{C \in \mathcal{C}(G)} \theta_{i_C}^{(C)}.$$

Question (example)

How many parameters does this parametrization map have?

Factorization according to G - algebra

$$p_{i_1 i_2 \dots i_m} = \frac{1}{Z} \prod_{C \in \mathcal{C}(G)} \theta^{(C)}_{i_C}.$$

Question (example)

How many parameters does this parametrization map have?

$$p_{ijkl} = a_{ij} b_{jkl}$$

- We can compute the ideal of this model I_G as follows.

It feels like this will be week 11, day 2. :)

The usual... license

This document is created for Math/Stat 561, Spring 2023.

The slides that are not directly from the book are sourced from **Kaie Kubjas**' Algebraic Statistics course at Aaalto University.