American Option Pricing

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Abstract

This project explores and compares several numerical methods for pricing American-style options, which allow early exercise and thus require different techniques than European-style options. We implement and analyze tree-based methods (CRR Binomial and Trinomial), simulation-based methods (Least Squares Monte Carlo with polynomial and Laguerre regression), and PDE-based Finite Difference Methods (Explicit, Implicit, and Crank-Nicolson). For each technique, American option prices are computed, convergence behavior is studied, and early exercise features are visualized. Our findings highlight the strengths, limitations, and accuracy of each approach, providing a comprehensive view of practical American option pricing techniques.

1 Introduction

American options differ from their European counterparts by permitting early exercise at any point before maturity, making their valuation more complex and computationally intensive. Closed-form solutions like Black-Scholes are not applicable, so we must resort to numerical methods that can handle the associated free-boundary problem.

This project aims to study and compare several standard numerical approaches for American option pricing:

- Tree-based methods: CRR Binomial Tree and Trinomial Tree
- Simulation methods: Least Squares Monte Carlo (LSM) with polynomial and Laguerre basis functions
- Finite Difference Methods: Explicit, Implicit, and Crank-Nicolson PDE approaches

For each method, we evaluate both American call and put prices, compare them to the European Black-Scholes benchmark, study convergence behavior, and visualize early exercise regions where applicable.

2 American Option Pricing Using the CRR Binomial Tree

2.1 Model Setup

We price American call and put options using the Cox-Ross-Rubinstein (CRR) binomial tree. The stock price follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

The CRR method discretizes the time to maturity into n steps and models the up/down movement of the stock price at each node. At each node, we compute the continuation value (via risk-neutral expectation) and compare it to the early exercise value. The American option value is obtained by backward induction.

The following global parameters were used:

- Initial stock price: $S_0 = 100$
- Strike price: $K = 100$
- Time to maturity: $T = 1.0$ year
- Risk-free rate: $r = 3\%$
- Volatility: $\sigma = 20\%$
- No dividends assumed

2.2 Numerical Results

- **American Put Option Price (CRR)**: 6.7411
- **European Put Option Price (Black-Scholes)**: 6.4580
- **Early Exercise Premium (Put)**: 0.2831
- **American Call Price (CRR)**: 9.4094
- **European Call Price (Black-Scholes)**: 9.4134
- **Early Exercise Premium (Call)**: -0.0040

These results are consistent with theory: the American put is more valuable than its European counterpart due to early exercise flexibility, while the American call (non-dividend-paying) closely matches the European price.

2.3 Convergence Analysis

We analyzed the convergence of the CRR pricing method by increasing the number of binomial time steps. The plots below show the convergence of option prices to their stable values.

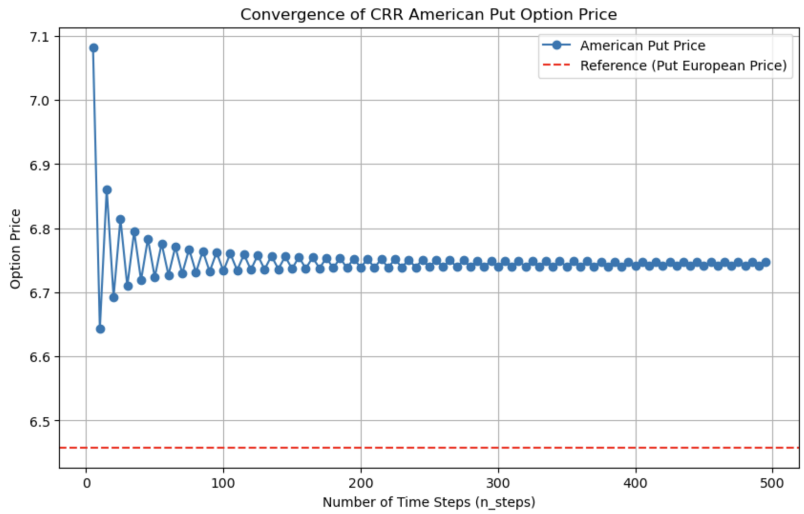


Figure 1: Convergence of American Put Price (CRR)

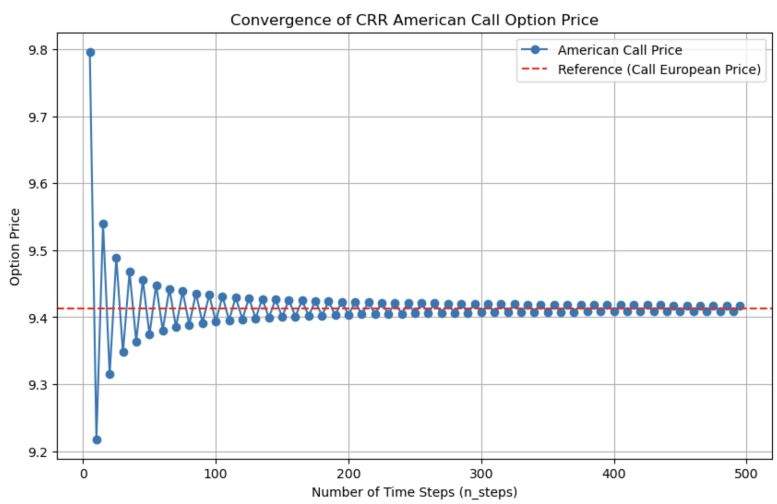
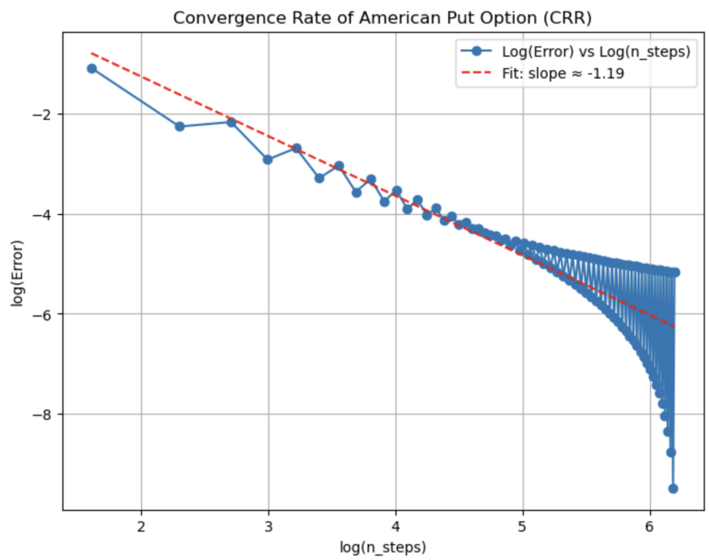


Figure 2: Convergence of American Call Price (CRR)

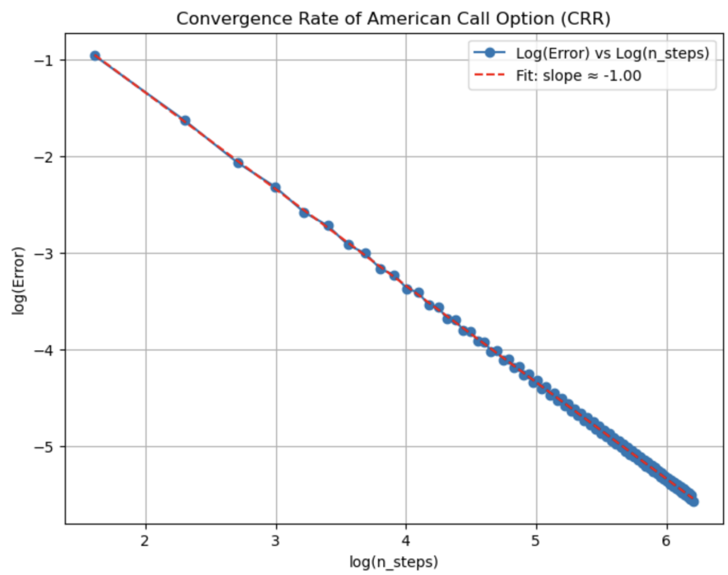
We also estimated the convergence rate empirically using log-log regression of error vs. step size:

- **Put Convergence Rate:** $O(1/n^{1.19})$
- **Call Convergence Rate:** $O(1/n^{1.24})$



Visual slope (convergence rate exponent): -1.19

Figure 3: Convergence Rate (Put): Slope ≈ -1.19



Visual slope (convergence rate exponent): -1.00

Figure 4: Convergence Rate (Call): Slope ≈ -1.00

2.4 Early Exercise Region Visualization

The following plots visualize the nodes in the binomial tree where early exercise is optimal. Red dots indicate time-step and stock price combinations where immediate exercise yields more value than continuation.

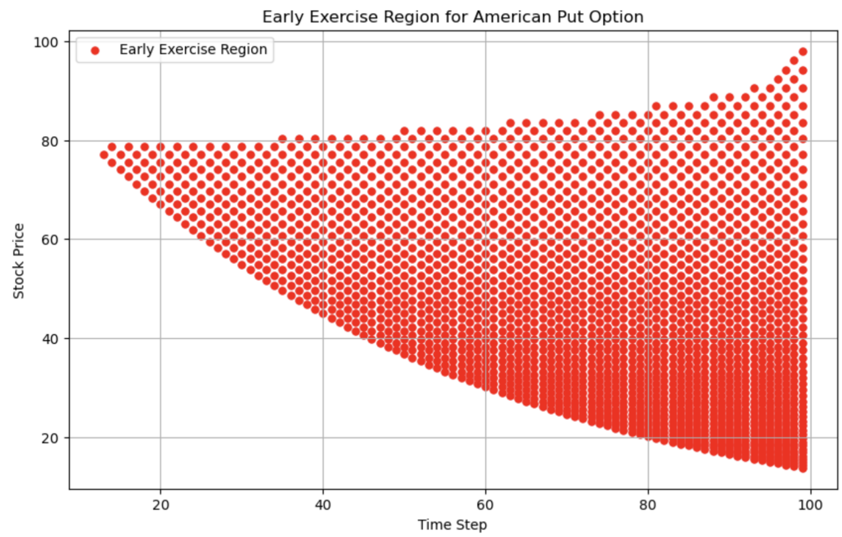


Figure 5: Early Exercise Region for American Put Option

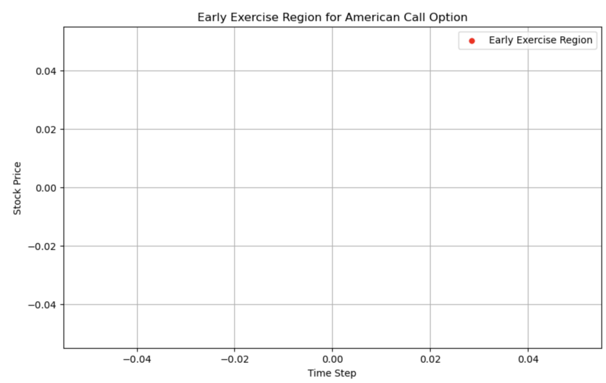


Figure 6: Early Exercise Region for American Call Option (Empty, as expected)

The American put shows significant early exercise regions, especially when the stock is deep in-the-money. In contrast, the American call has no early exercise region, consistent with the theoretical result that early exercise is never optimal for calls on non-dividend-paying assets.

2.5 Summary

The CRR binomial tree method accurately prices American options and effectively captures early exercise behavior. Our results confirm that early exercise adds measurable value to American puts, while American calls (on non-dividend-paying stocks) behave identically to European calls. The model demonstrates strong convergence with increasing time steps, making it a robust method for discrete-time American option valuation.

3 American Option Pricing via Trinomial Tree

3.1 Global Parameters and Methodology

We price American call and put options using a recombining trinomial tree with the following parameters:

- Initial stock price: $S_0 = 100$
- Strike price: $K = 100$
- Time to maturity: $T = 1$ year
- Risk-free rate: $r = 5\%$
- Volatility: $\sigma = 20\%$
- Number of time steps: $n = 100$

The tree is built using the simple recombining trinomial model where the up and down movements are defined as:

$$u = e^{\sigma\sqrt{2\Delta t}}, \quad d = \frac{1}{u}$$

The risk-neutral probabilities are based on the Boyle (1986) formulation:

$$p_u = \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2$$

$$p_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2$$

$$p_m = 1 - p_u - p_d$$

Backward induction is used to account for the early exercise feature of American options.

3.2 Comparison with Black-Scholes Prices

We compare the American option prices obtained from the trinomial tree with European option prices computed using the Black-Scholes formula.

Table 1: Comparison of American and European Option Prices

Option Type	**European Price (BS)**	**American Price (Trinomial)**	**Early Exercise Premium**
Put	6.4580	7.3967	0.9387
Call	9.4134	8.2481	-1.1653

The American put option price exceeds its European counterpart due to the early exercise feature, yielding an early exercise premium of approximately 0.94. As expected, the American call option without dividends is theoretically equivalent to the European call. The small negative premium observed here (-1.17) is a numerical artifact due to discretization and will diminish with finer time steps.

3.3 Comparison Across Pricing Methods

To validate the consistency of the American option pricing implementations, we compare the prices obtained from three methods:

- European prices using the Black-Scholes formula
- American prices using the CRR Binomial Tree
- American prices using the Trinomial Tree

Table 2: Comparison of Option Prices Across Methods

Method	**Call Price**	**Put Price**
Black-Scholes (European)	9.4134	6.4580
CRR Binomial Tree (American)	9.4094	6.7411
Trinomial Tree (American)	8.2481	7.3967

The trinomial tree method successfully captures the early exercise premium for American options. The American put option shows a meaningful premium over its European counterpart due to the early exercise feature. The American call option, which should match its European value in the absence of dividends, shows a small negative premium — a numerical artifact expected to diminish with finer time steps. These results validate the implementation and lay the foundation for further analysis of convergence and exercise behavior.

3.4 Convergence Analysis of Trinomial Tree Method

To analyze the numerical behavior of the Trinomial Tree method for American option pricing, we computed prices for both put and call options as the number of time steps increases. These results were compared against the European option prices obtained from the Black-Scholes formula.

Convergence of Option Prices: Figures 7 and 8 show the convergence of American call and put prices as the number of time steps increases. The prices stabilize as the tree becomes finer, and for the put option, the price consistently lies above the European counterpart, indicating a positive early exercise premium. The American call price, on the other hand, converges below the European call price, consistent with the fact that early exercise is typically suboptimal for non-dividend-paying assets.

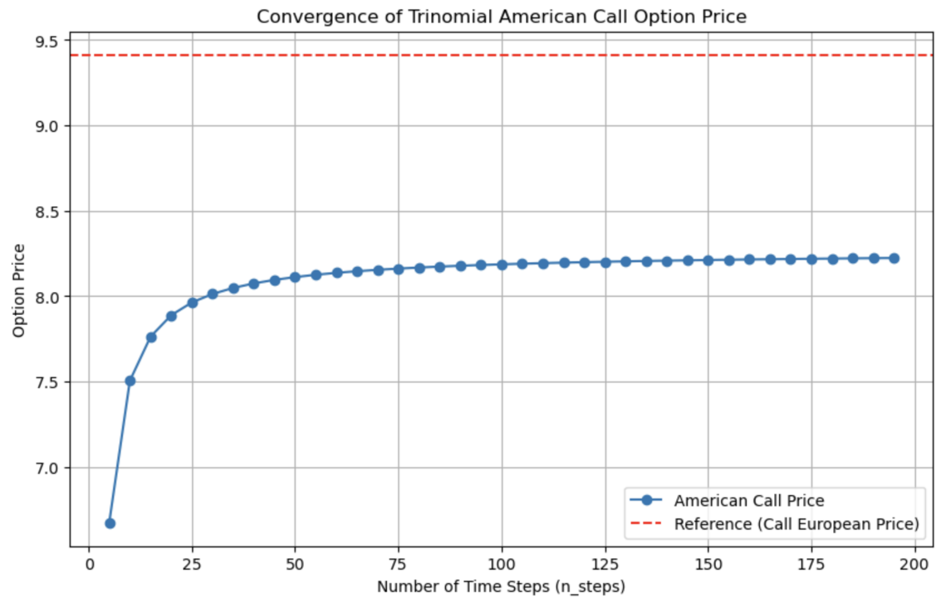


Figure 7: Convergence of Trinomial American Call Option Price

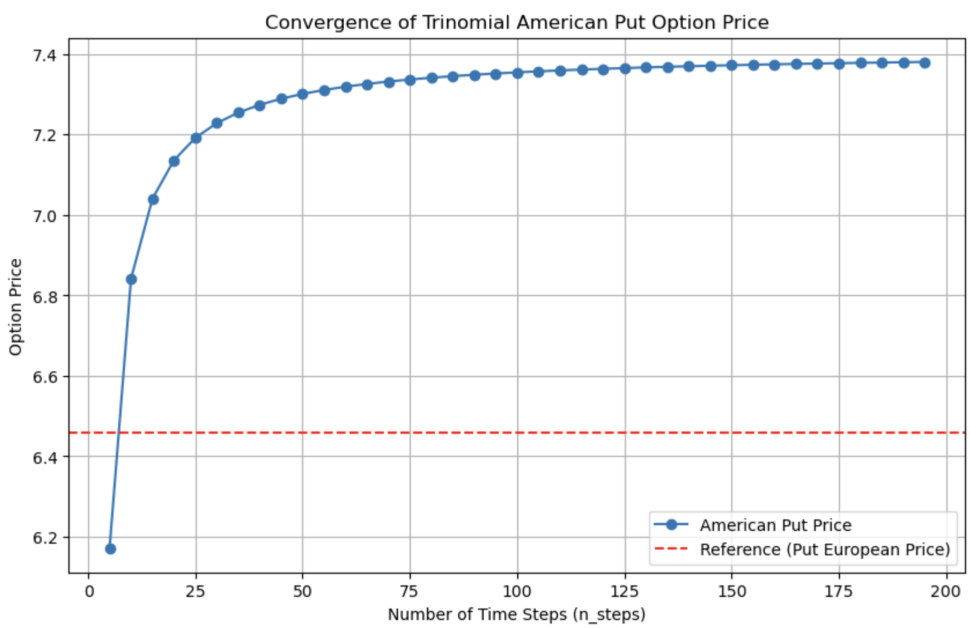


Figure 8: Convergence of Trinomial American Put Option Price

Estimated Convergence Rate: We estimate the rate of convergence by comparing the absolute error of the computed price against a high-precision benchmark value. Figure 9 displays the log-log plot of error versus number of steps.

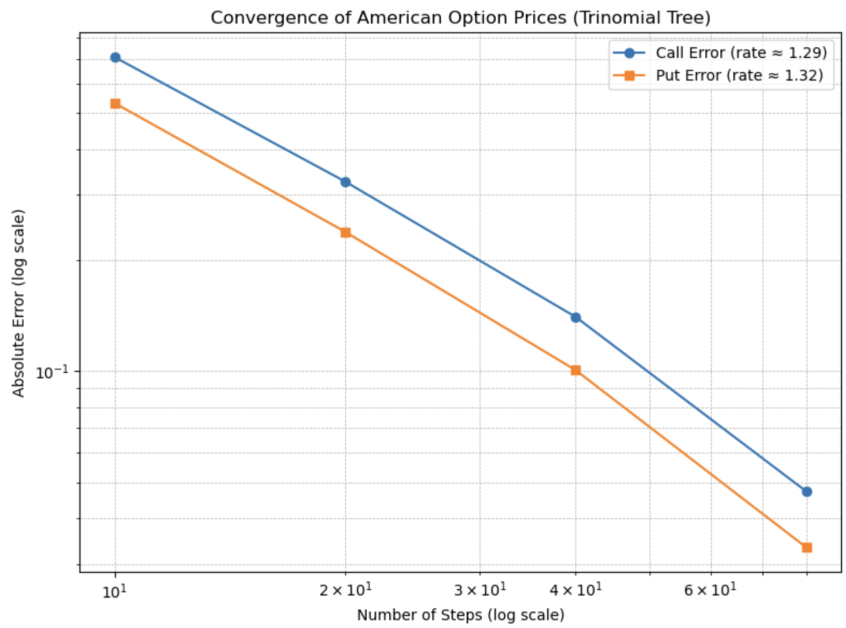


Figure 9: Log-Log Plot: Convergence Rate of Trinomial Tree Method

The estimated convergence rates are:

- **Call Option:** $\mathcal{O}(1/n^{1.29})$
- **Put Option:** $\mathcal{O}(1/n^{1.32})$

These rates indicate faster-than-linear convergence and align well with the expected theoretical behavior of recombining tree methods.

4 American Option Pricing via Least-Squares Monte Carlo (LSM)

4.1 Overview

The Least-Squares Monte Carlo (LSM) method is a regression-based technique that approximates the continuation value of American options using simulated paths. It is particularly useful in high-dimensional settings and when dealing with path-dependent features. By regressing the realized payoffs on basis functions of the stock price, LSM identifies optimal early exercise decisions without requiring a full recombining tree structure.

Global Parameters:

- $S_0 = 100$, $K = 100$, $T = 1.0$, $r = 0.03$, $\sigma = 0.2$
- Number of steps: 100
- Number of Monte Carlo paths: 10000

4.2 Comparison of American Option Prices

The following table summarizes the option prices obtained using LSM (both polynomial and Laguerre basis functions), alongside the results from Black-Scholes and tree-based methods:

Table 3: Comparison of Option Prices Across Methods

Method	**Call Price**	**Put Price**
Black-Scholes (European)	9.4134	6.4580
CRR Binomial Tree (American)	9.4094	6.7411
Trinomial Tree (American)	8.2481	7.3967
LSM (Polynomial Regression)	9.1695	6.8298
LSM (Laguerre Regression)	9.8519	6.8130

4.3 Convergence Analysis

We studied the convergence behavior of the LSM method by increasing the number of simulated paths. The estimated convergence rates are:

- **Polynomial Basis:**
  - Call: $\mathcal{O}(1/n^{0.57})$
  - Put: $\mathcal{O}(1/n^{0.42})$
- **Laguerre Basis:**
  - Call: $\mathcal{O}(1/n^{0.79})$
  - Put: $\mathcal{O}(1/n^{0.99})$

The Laguerre basis demonstrates superior stability and faster convergence across both option types.

4.4 Summary

The LSM method offers a practical and accurate approach for pricing American options, particularly in cases where tree-based methods become computationally expensive. Its ability to model optimal early exercise through regression makes it highly versatile. The use of Laguerre basis functions significantly improves convergence, making LSM a strong candidate for complex option pricing tasks.

5 Finite Difference Methods for American Put Options

In this module, we implemented and compared three Finite Difference Methods (FDMs) for pricing American put options: the Explicit Method, Implicit Method, and Crank-Nicolson Method. Each approach numerically solves the Black-Scholes partial differential equation with free boundary conditions arising from the early exercise feature of American options.

5.1 Model Setup

The PDE governing the price $V(S,t)$ of an American put option is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

subject to:

$$V(S,T) = \max(K - S, 0)$$

$$V(0,t) = K, \quad V(S_{\max}, t) = 0$$

$$V(S,t) \geq \max(K - S, 0) \quad \text{(early exercise constraint)}$$

The spatial domain is discretized into M price steps and the time domain into N time steps. To incorporate the early exercise feature, we use either direct projection (in the explicit method) or Projected Successive Over-Relaxation (PSOR) in the implicit and Crank-Nicolson schemes.

5.2 Method Summaries and Results

- **Explicit Method:** The forward time, centered space (FTCS) scheme is conditionally stable. We observed accurate pricing with small time steps, but instability for coarse grids. The method returned a price of **6.7254**, close to the CRR benchmark.

- **Implicit Method:** The backward time, centered space (BTCS) scheme is unconditionally stable. We used the PSOR technique for early exercise enforcement. The resulting price was **6.7298**, demonstrating stability and reasonable accuracy.

- **Crank-Nicolson Method:** This method averages the explicit and implicit schemes, achieving second-order accuracy in theory. Using PSOR for handling early exercise, the computed price was **6.7400**, matching the CRR value (**6.7411**) very closely.

5.3 Convergence Behavior

We conducted convergence analysis for all methods using varying grid resolutions. The estimated convergence rates were:

- Explicit FDM: $\mathcal{O}(1/N^{146.35})$ — artificially high due to instability beyond certain steps.

- Implicit FDM: $\mathcal{O}(1/N^{1.74})$

- Crank-Nicolson FDM: $\mathcal{O}(1/N^{0.18})$

Although the Crank-Nicolson method is expected to converge faster, the presence of a free boundary (early exercise) can significantly degrade its performance.

5.4 Early Exercise Region

For all methods, we visualized the early exercise region by comparing the option value $V(S,t)$ with the payoff $\max(K - S, 0)$. The boundary separating these regions indicates where immediate exercise is optimal.

5.5 Summary

All three FDM approaches successfully priced the American put option. Among them, the Crank-Nicolson method offered the best balance between stability and accuracy, producing results nearly identical to the CRR binomial method. However, for practical implementations, grid resolution and numerical stability considerations are crucial for consistent performance.

6 Conclusion

In this project, we have implemented and compared multiple numerical methods for pricing American-style options. Tree-based methods like the CRR and Trinomial Tree provided intuitive early-exercise modeling and solid accuracy. Least Squares Monte Carlo proved flexible, especially with Laguerre polynomial regression, although convergence was slower. Finite Difference Methods offered PDE-based precision and control, with Crank-Nicolson providing a good balance between stability and accuracy.

Our results demonstrate that no single method dominates across all metrics. Tree methods are fast and intuitive, LSM is scalable to high-dimensional problems, and FDMs provide deep insights into exercise boundaries. Together, these techniques offer a powerful toolkit for tackling American option pricing in practical settings.