Comprehensive Analysis of European Option Pricing, Sensitivities, and Implied Volatility under the Black-Scholes Framework

A K M Rokonuzzaman Sonet

Abstract

This project explores the pricing of European call and put options using the Black-Scholes model. It analyzes the behavior of option prices and their sensitivities (Greeks) with respect to time to expiration and spot prices. Additionally, the implied volatility is estimated using three numerical methods—Newton-Raphson with Vega, Newton-Raphson with Finite Difference, and Brent's root-finding algorithm—and compared through the volatility smile. The implementation is done in Python using NumPy, SciPy, and Matplotlib.

1 Introduction

The Black-Scholes model is a fundamental tool in quantitative finance, offering a closed-form solution to price European options. This study implements the model to analyze the impact of input parameters on option prices and Greeks, followed by the numerical inversion to estimate implied volatilities.

2 Mathematical Framework

2.1 Black-Scholes Formulas

The prices of European call and put options are given, respectively, by:

$$C = S_0\Phi(d_1) - Ke^{-rt}\Phi(d_2), \quad P = Ke^{-rt}\Phi(-d_2) - S_0\Phi(-d_1)$$

where,

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}, \quad d_2 = d_1 - \sigma\sqrt{t}$$

where,

- C: European call option price
- S_0: Current spot price of the underlying asset
- K: Strike price
- r: Continuously compounded risk-free interest rate
- t: Time to expiration (in years)

- σ: Annualized volatility of the underlying asset
- $\Phi(\cdot)$: Standard normal cumulative distribution function

3 The Greeks: Sensitivity Measures

The Greeks are partial derivatives of the option price with respect to underlying parameters. They quantify how sensitive an option is to changes in market conditions. Below are the primary Greeks analyzed in this study.

3.1 Delta

Delta (Δ) measures the sensitivity of the option price to changes in the underlying asset price.

$$\Delta_{\text{call}} = \Phi(d_1), \quad \Delta_{\text{put}} = \Phi(d_1) - 1$$

Interpretation: Delta for a call option lies between 0 and 1, indicating the probability the option will expire in-the-money. For a put, it lies between -1 and 0.

3.2 Gamma

Gamma (Γ) measures the rate of change of delta with respect to the underlying asset price.

$$\Gamma = \frac{\phi(d_1)}{S_0 \sigma \sqrt{t}}$$

Interpretation: Gamma is highest when the option is at-the-money and declines as the option becomes deep in- or out-of-the-money.

3.3 Theta

Theta (Θ) represents the rate of change of the option price with respect to time (time decay).

$$\Theta_{\text{call}} = -\frac{S_0 \phi(d_1) \sigma}{2\sqrt{t}} - rKe^{-rt}\Phi(d_2)$$

$$\Theta_{\text{put}} = -\frac{S_0 \phi(d_1) \sigma}{2\sqrt{t}} + rKe^{-rt}\Phi(-d_2)$$

Interpretation: Theta is usually negative, reflecting the loss of option value as expiration approaches.

3.4 Vega

Vega measures sensitivity of the option price to changes in volatility.

$$\text{Vega} = S_0 \phi(d_1)\sqrt{t}$$

Interpretation: Options are most sensitive to changes in volatility when at-the-money and with longer time to expiration.

3.5 Rho

Rho (ρ) measures the sensitivity of the option price to changes in the interest rate.

$$\rho_{\text{call}} = Kte^{-rt}\Phi(d_2), \quad \rho_{\text{put}} = -Kte^{-rt}\Phi(-d_2)$$

Interpretation: Call options increase in value as interest rates rise, while put options decrease.

4 Option Prices vs Time to Expiration

The following figures (1 and 2) display how call and put prices change as a function of time to expiration for fixed spot prices S_0, strike K, and volatility σ.

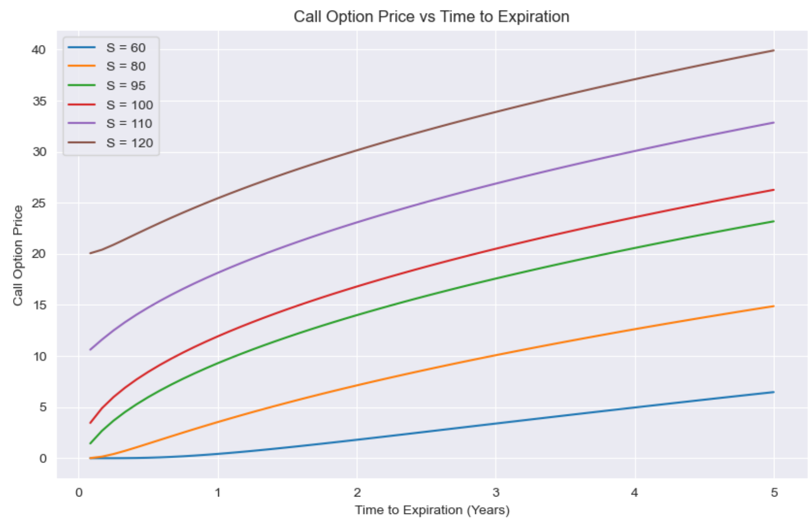

Figure 1: Call Option Price vs Time to Expiration

Observation: As time increases, both call and put prices increase due to the higher uncertainty in the future.

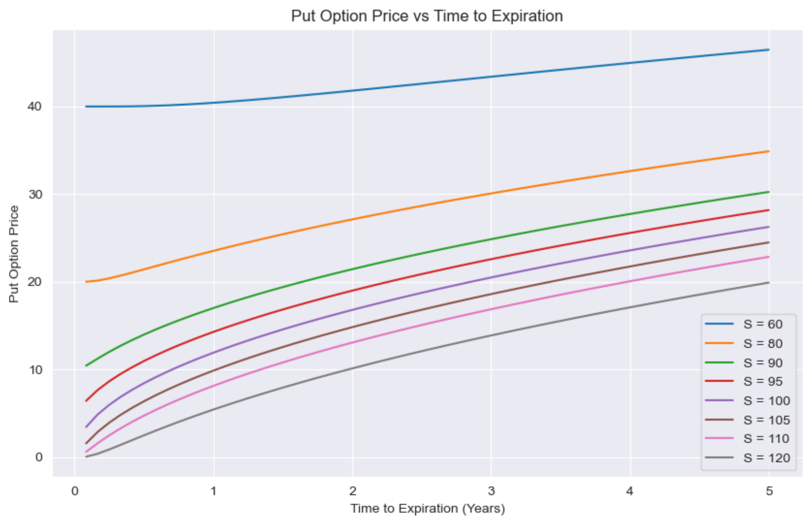

Figure 2: Put Option Price vs Time to Expiration

5 Option Prices vs Spot Price

Next, we examine how option prices vary with the spot price at a fixed time to expiration (t years) (Figure 3 and 4)

Figure 3: Call Option Price vs Spot Price

Observation: Call price increases and put price decreases with rising spot price.

Figure 4: Put Option Price vs Spot Price

6 Greeks Analysis

In this section, we analyze how the five primary Greeks behave under the Black-Scholes framework for both European call and put options. The Greeks are partial derivatives of the option price and provide insight into how the option responds to changes in various market parameters.

6.1 Delta

Delta measures the sensitivity of the option price with respect to the underlying asset price.

$$\Delta_{\text{call}} = \Phi(d_1), \quad \Delta_{\text{put}} = \Phi(d_1) - 1$$

Interpretation: For call options, delta increases with the underlying price and approaches 1 as the option becomes deep in-the-money. For puts, delta is negative and approaches -1 in the same situation.

Call Option Delta vs Spot Price:

Interpretation: As shown in Figure 5, as the spot price increases, Delta rises towards 1. The shorter the time to expiration, the steeper the curve around the strike price, indicating more abrupt changes in sensitivity for near-term options.

Call Option Delta vs Time:

Interpretation: In Figure 6, we observe that Delta stabilizes over longer maturities. For in-the-money options, Delta remains high as time increases; for out-of-the-money options, Delta is low and grows slowly.

Put Option Delta vs Spot Price:

Figure 5: Call Option Delta vs Spot Price at Different Expirations

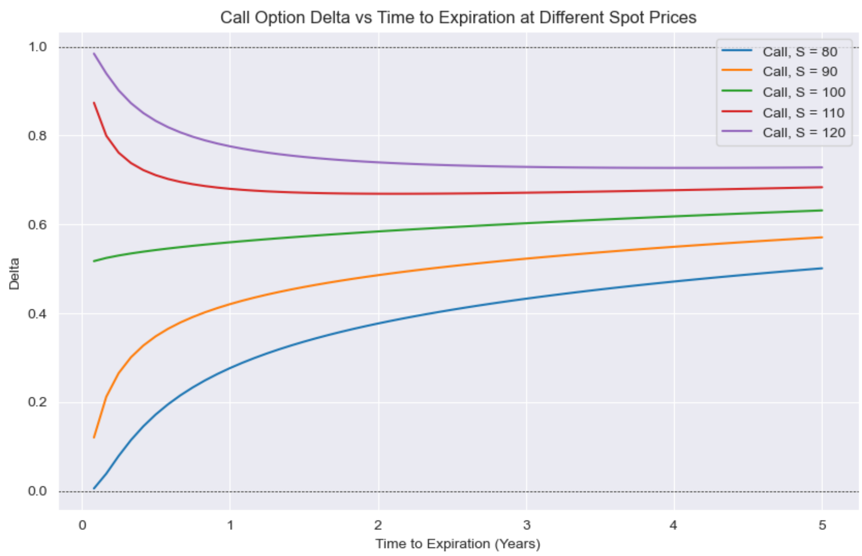

Figure 6: Call Option Delta vs Time to Expiration at Different Spot Prices

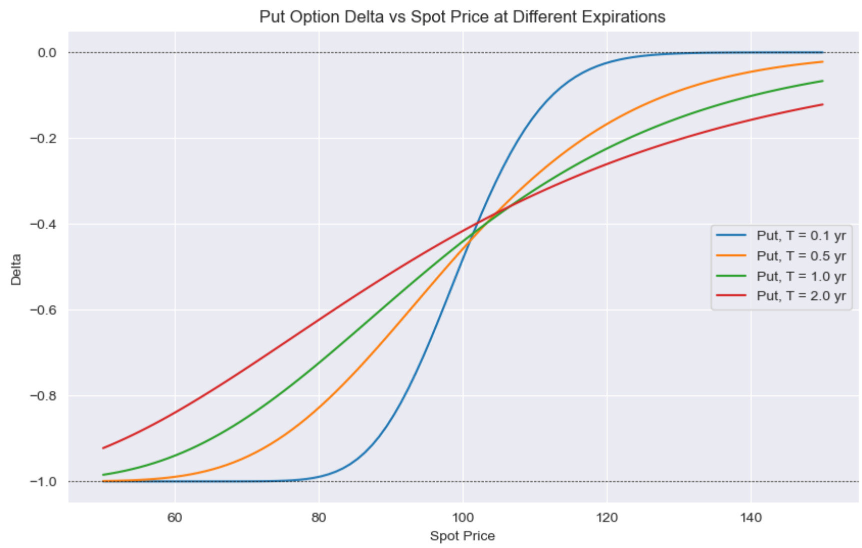

Figure 7: Put Option Delta vs Spot Price at Different Expirations

Interpretation: As illustrated in Figure 7, Put Delta is negative, steepest at-the-money, and becomes less sensitive as time increases.

Put Option Delta vs Time:

Interpretation: Figure 8 shows that Delta for deep in- or out-of-the-money options remains relatively flat as time increases.

6.2 Gamma

Gamma measures the rate of change of Delta with respect to the underlying asset price.

$$\Gamma = \frac{\phi(d_1)}{S\sigma\sqrt{t}}$$

Interpretation: Gamma is highest for at-the-money options and near expiration. It highlights the convexity of the option price.

Gamma vs Spot Price:

Interpretation: Figure 9 shows that Gamma is sharply peaked at-the-money and increases with shorter time to expiration.

Gamma vs Time to Expiration:

Interpretation: From Figure 10, we see Gamma decays over time across all spot prices. The at-the-money option initially has the highest Gamma, which decreases rapidly as time increases.

6.3 Vega

Vega measures the sensitivity of the option price to changes in volatility.

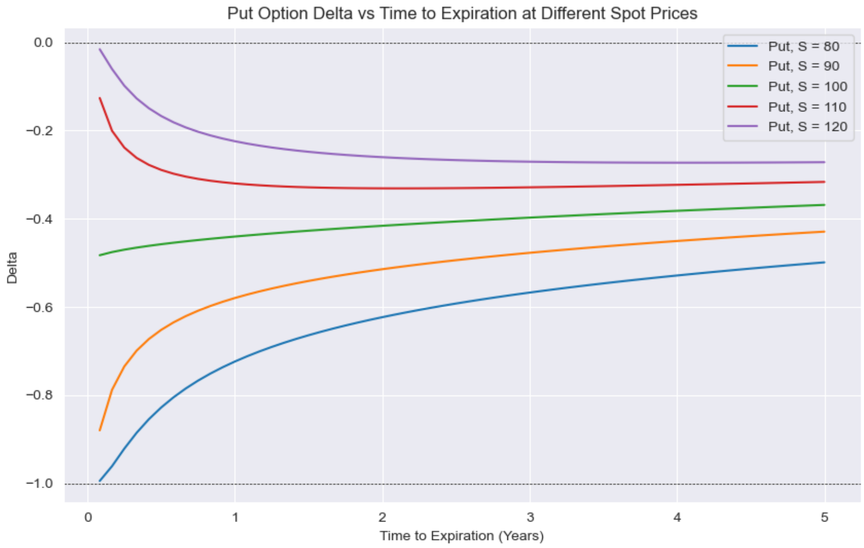

Figure 8: Put Option Delta vs Time to Expiration at Different Spot Prices

Figure 9: Option Gamma vs Spot Price at Different Expirations

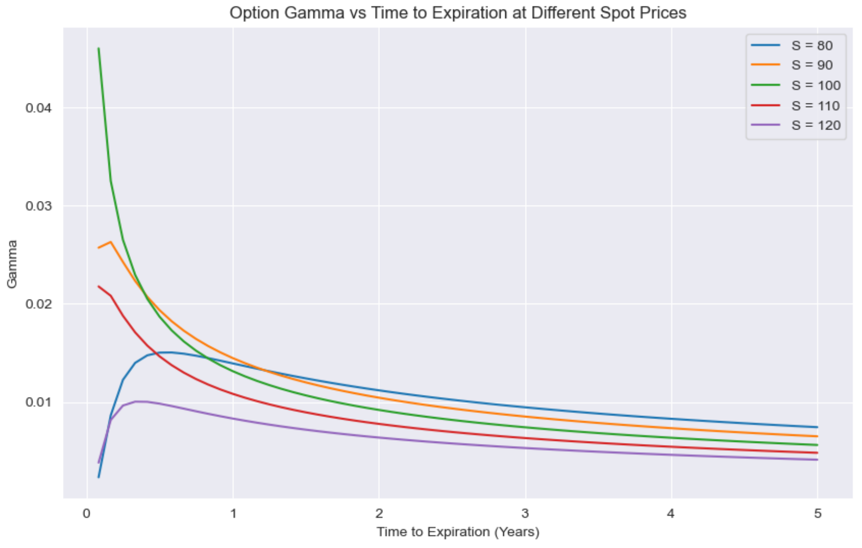

Figure 10: Option Gamma vs Time to Expiration at Different Spot Prices

$$\text{Vega} = S\phi(d_1)\sqrt{t}$$

Interpretation: Vega is highest when the option is at-the-money and decreases as the option becomes deep in- or out-of-the-money.

Vega vs Spot Price:

Interpretation: In Figure 11, we observe that Vega peaks near the strike price (typically at-the-money). As expiration increases, the peak flattens and Vega values increase. Longer-dated options are more sensitive to changes in volatility.

Vega vs Time to Expiration:

Interpretation: As shown in Figure 12, Vega increases with time to expiration for all spot prices. The growth is faster initially and then gradually levels off, especially for at-the-money options.

6.4 Theta

Theta measures the rate at which an option's price decays as time passes — often referred to as "time decay." Call and put options typically have negative Theta, meaning they lose value as expiration approaches.

$$\Theta_{\text{call}} = -\frac{S\phi(d_1)\sigma}{2\sqrt{t}} - rKe^{-rt}\Phi(d_2)$$

$$\Theta_{\text{put}} = -\frac{S\phi(d_1)\sigma}{2\sqrt{t}} + rKe^{-rt}\Phi(-d_2)$$

Figure 11: Option Vega vs Spot Price at Different Expirations

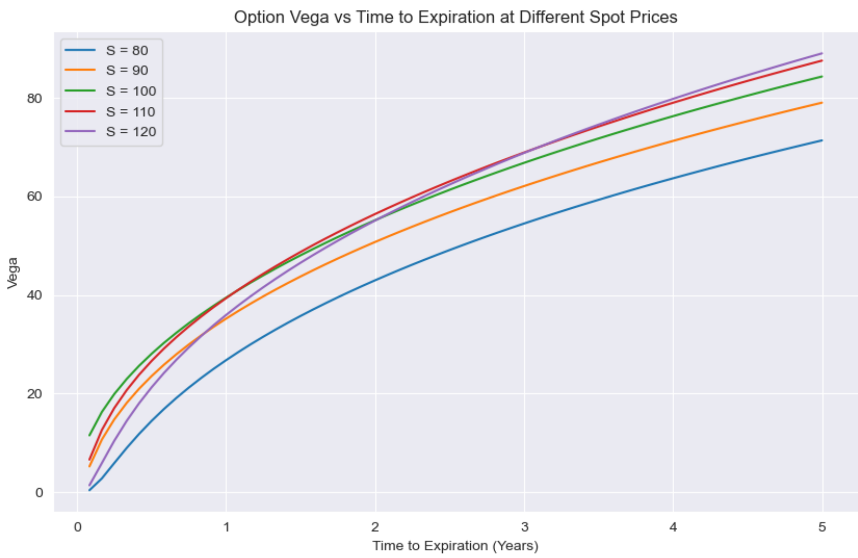

Figure 12: Option Vega vs Time to Expiration at Different Spot Prices

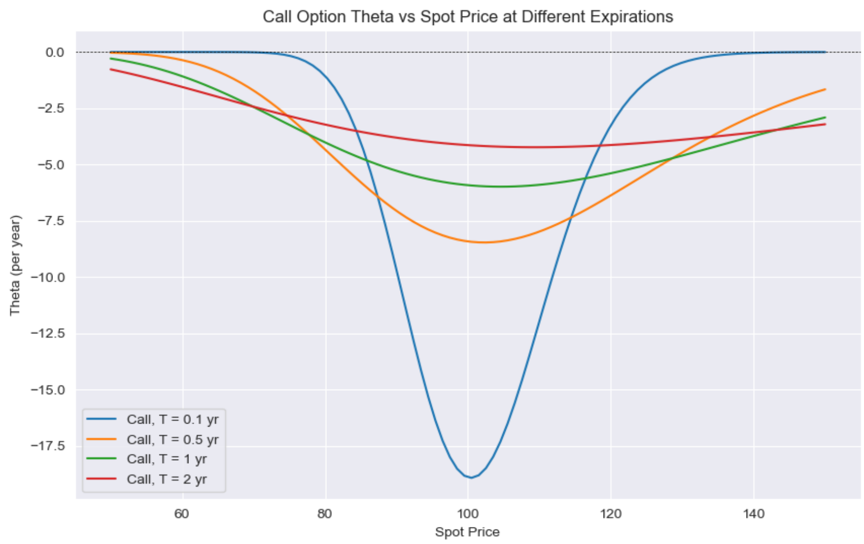

Figure 13: Call Option Theta vs Spot Price at Different Expirations

Interpretation: Theta is typically negative. The rate of time decay accelerates as expiration approaches, especially for at-the-money options.

Call Option Theta vs Spot Price:

Interpretation: In Figure 13, Theta is most negative near the strike price — where time decay is most severe. As the option becomes deep in- or out-of-the-money, Theta decreases in magnitude, especially for long-dated options.

Call Option Theta vs Time to Expiration:

Interpretation: As shown in Figure 14, call Theta becomes increasingly negative for at-the-money options as expiration nears. For deep in- or out-of-the-money options, Theta is closer to zero.

Put Option Theta vs Spot Price:

Interpretation: Figure 15 shows a symmetric pattern with call Theta. Theta is lowest when the spot price is near the strike, and decays more rapidly for near-term options.

Put Option Theta vs Time to Expiration:

Interpretation: In Figure 16, Theta becomes sharply negative for at-the-money options close to expiration, then converges gradually toward zero for long expirations.

6.5 Rho

Rho measures the sensitivity of an option's price to changes in the risk-free interest rate. Call options have positive Rho (value increases with rate), while put options have negative Rho (value decreases with rate).

$$\rho_{\text{call}} = Kte^{-rt}\Phi(d_2), \quad \rho_{\text{put}} = -Kte^{-rt}\Phi(-d_2)$$

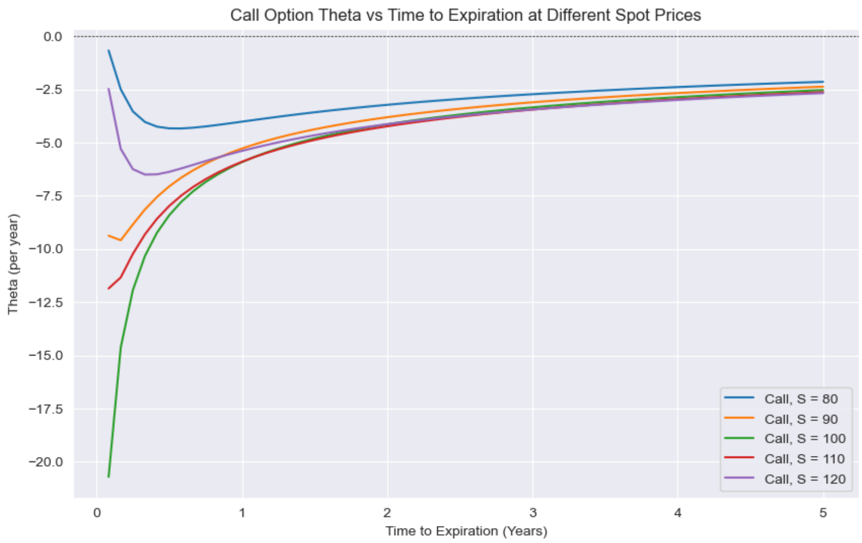

Figure 14: Call Option Theta vs Time to Expiration at Different Spot Prices

Figure 15: Put Option Theta vs Spot Price at Different Expirations

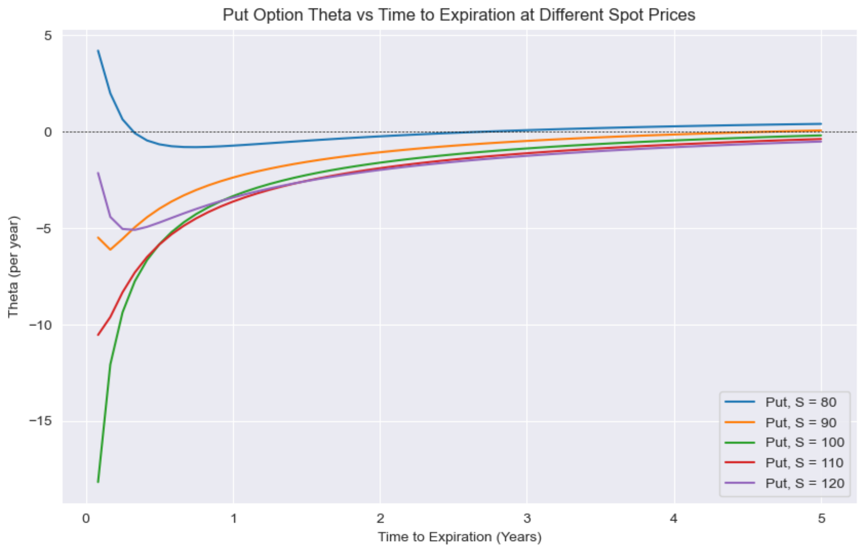

Figure 16: Put Option Theta vs Time to Expiration at Different Spot Prices

Interpretation: Rho is more significant for options with longer time to expiration. Call options increase in value with higher interest rates, while put options decrease.

Call Option Rho vs Spot Price:

Interpretation: In Figure 17, Rho increases with the spot price, particularly for long-dated options. Deep in-the-money calls benefit more from rising interest rates.

Call Option Rho vs Time to Expiration:

Interpretation: As shown in Figure 18, Rho increases with time. The longer the expiration, the more sensitive the call option is to interest rate changes.

Put Option Rho vs Spot Price:

Interpretation: In Figure 19, put Rho is negative, decreasing further for higher spot prices. Long-dated puts are more adversely impacted by rising interest rates.

Put Option Rho vs Time to Expiration:

Interpretation: Figure 20 shows that Rho for put options becomes more negative as time to expiration increases, especially for in-the-money puts.

7 Implied Volatility Analysis

In this section, we examine how implied volatilities are recovered from option prices using different numerical methods. We consider two distinct approaches to generate market prices:

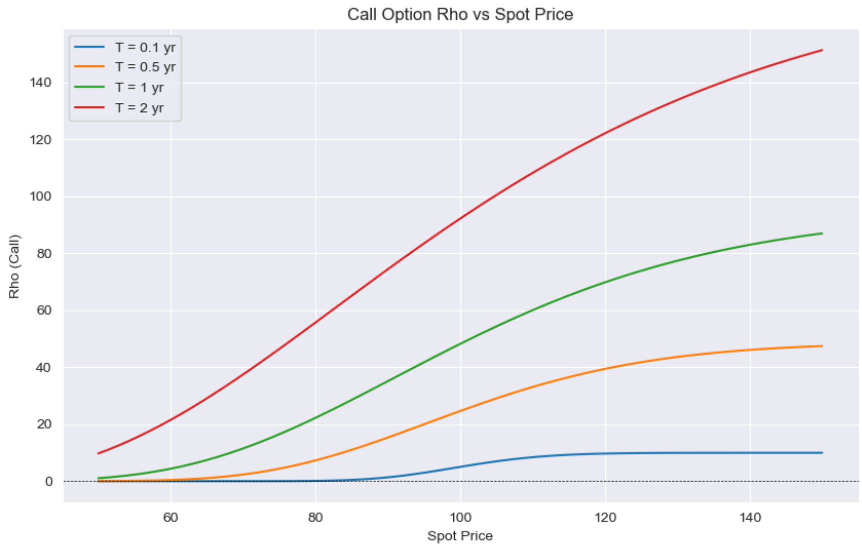

Figure 17: Call Option Rho vs Spot Price at Different Expirations

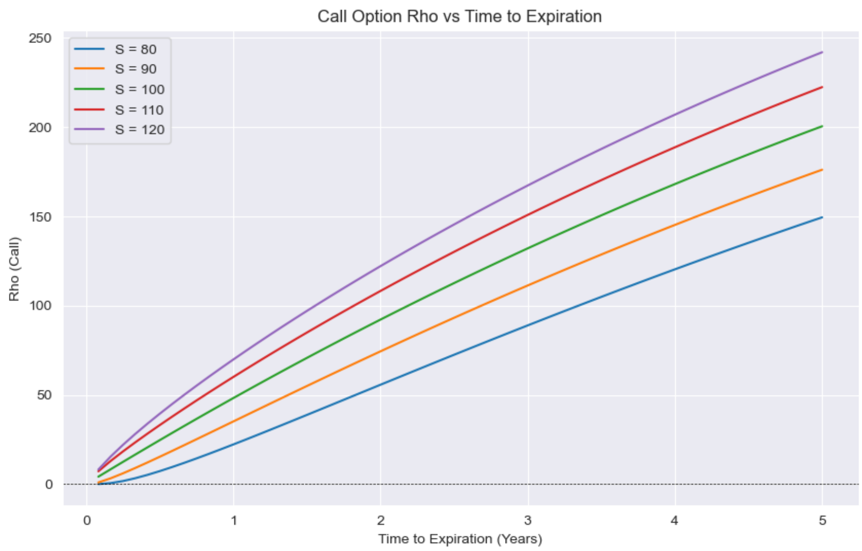

Figure 18: Call Option Rho vs Time to Expiration at Different Spot Prices

Figure 19: Put Option Rho vs Spot Price at Different Expirations

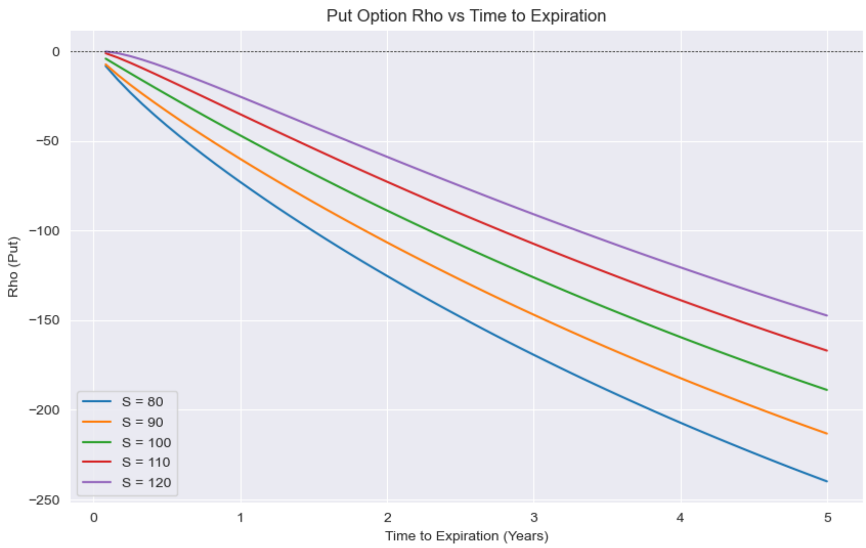

Figure 20: Put Option Rho vs Time to Expiration at Different Spot Prices

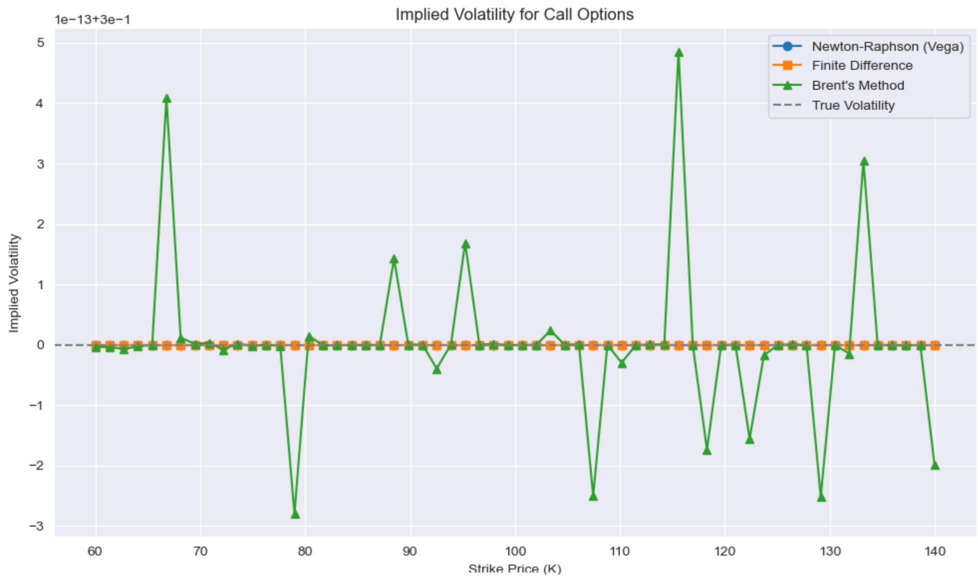

Figure 21: Implied Volatility for Call Options using constant volatility prices

7.1 Case 1: Implied Volatility from Constant Volatility Prices

We begin by generating synthetic option prices using the Black-Scholes model with a fixed volatility value ($\sigma = 0.3$). These prices are then inverted to estimate the implied volatility using three root-finding techniques:

- **Newton-Raphson Method with Vega**: Utilizes the analytical derivative of the option price with respect to volatility (vega).
- **Newton-Raphson Method with Finite Differences**: Estimates the derivative numerically.
- **Brent's Method**: A robust bracketing method that does not require derivative information.

Figures 21 and 22 display the implied volatilities computed across a range of strike prices for both call and put options.

We observe that both Newton-Raphson approaches yield accurate and stable results, while Brent's Method exhibits numerical instability in this particular setup. This highlights the sensitivity of bracketing methods to noise and tolerance thresholds, especially when the price function is nearly flat.

7.2 Case 2: Implied Volatility from Smile-Based Prices

In the second approach, we introduce a strike-dependent volatility function to simulate a more realistic volatility smile:

$$\sigma(K) = 0.2 + 0.002 \cdot (K - S_0)^2$$

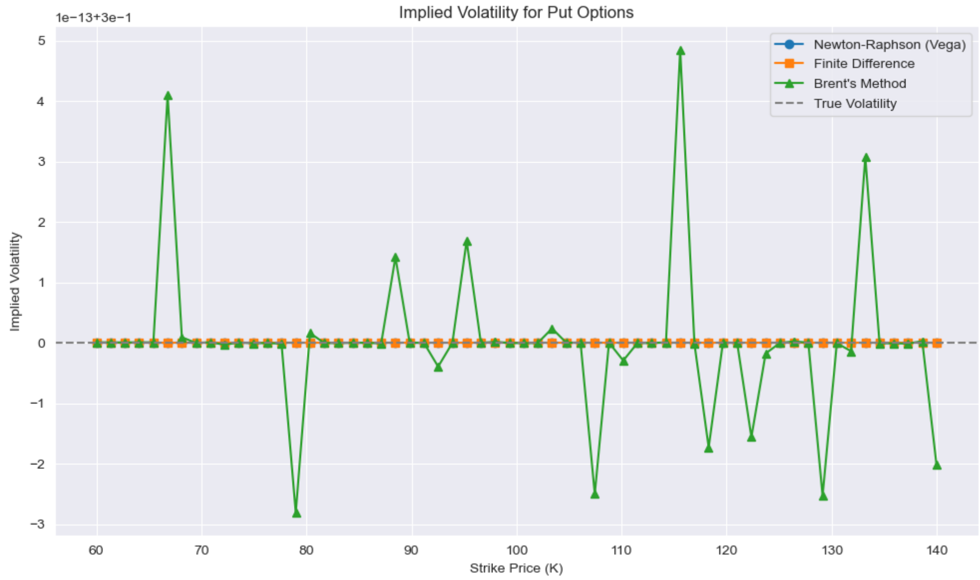

Figure 22: Implied Volatility for Put Options using constant volatility prices

This function increases volatility for both deep in-the-money and out-of-the-money options, consistent with market observations. We use this smile to generate synthetic market prices and then estimate the implied volatilities using the same three numerical methods.

Figures 23 and 24 show the resulting implied volatility smile curves for call and put options, respectively.

As expected, all three methods recover a volatility smile that closely matches the true synthetic curve. Among them, the Newton-Raphson with Vega performs consistently and is computationally efficient, while Brent's method proves reliable under noisy conditions due to its bracketing nature.

This comparative study demonstrates that implied volatility estimation is highly sensitive to both the numerical method and the shape of the price surface. While Newton-Raphson methods work well when vega is significant, Brent's method can be advantageous in flat regions where derivative-based methods struggle. When simulating realistic market behavior, incorporating a volatility smile helps illustrate the strengths and limitations of each estimation technique.

8 Conclusion

The Black-Scholes model provides a powerful framework to analyze European option pricing and sensitivities. Greek analysis reveals how sensitive option prices are to market parameters, aiding traders in hedging strategies. Implied volatility estimation methods performed consistently, showcasing their reliability in real-world applications.

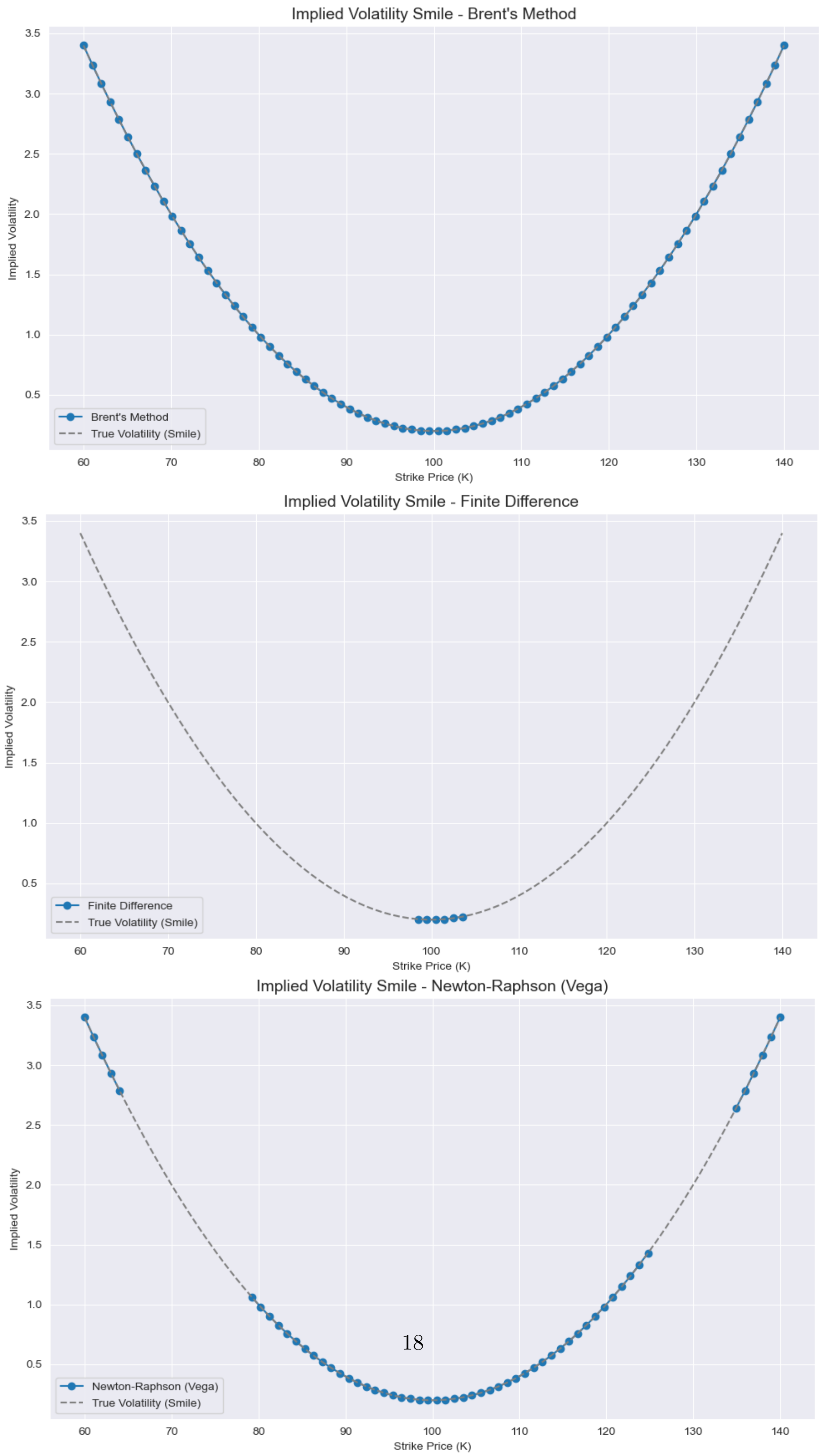

Figure 23: Implied Volatility Smile for Call Options using Brent's Method, Finite Difference, and Vega-based Newton-Raphson

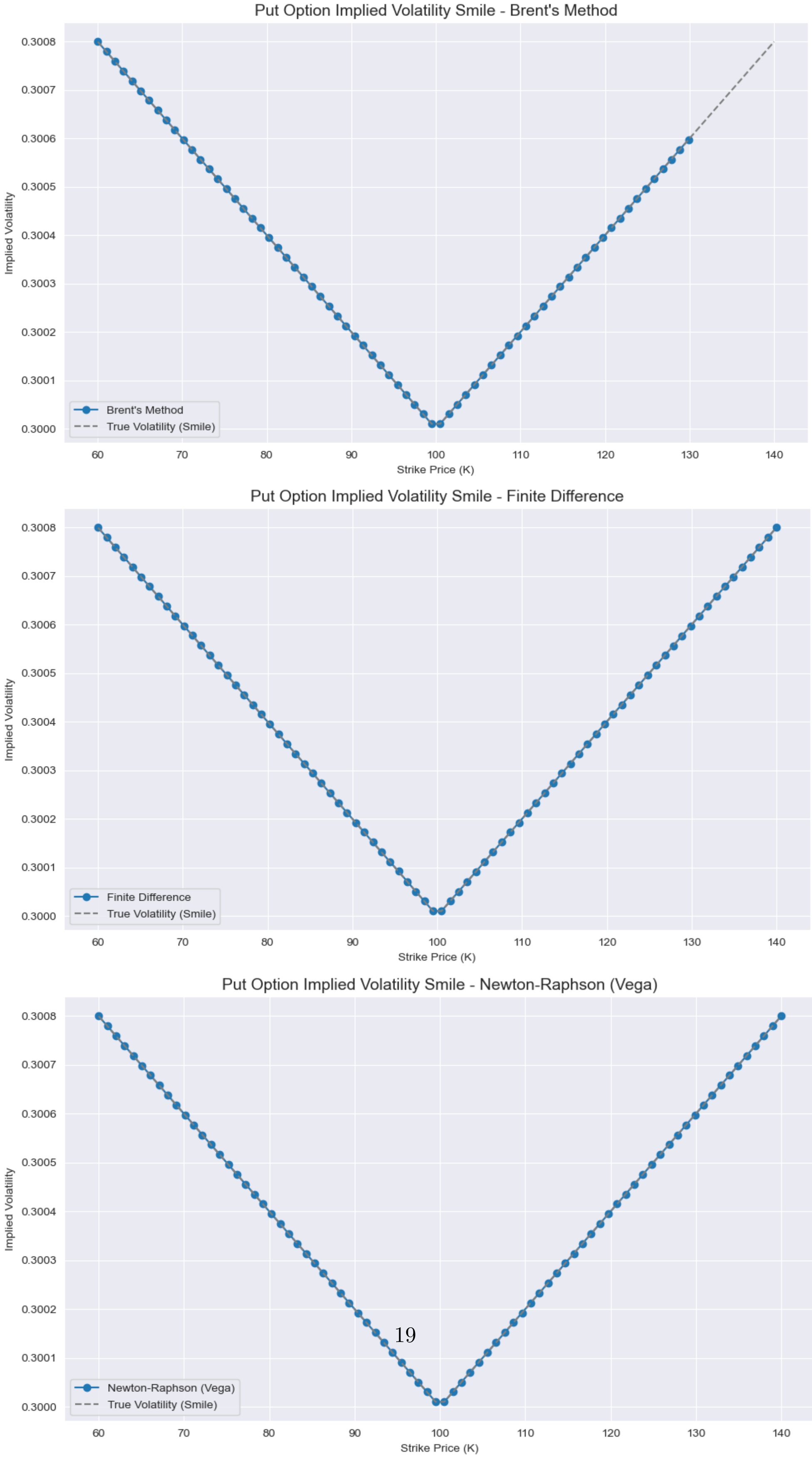

Figure 24: Implied Volatility Smile for Put Options using Brent's Method, Finite Difference, and Vega-based Newton-Raphson