Delta Hedging under Constant and Stochastic Volatility Models

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Abstract

This report investigates delta hedging performance for European call options under both constant and stochastic volatility assumptions. We explore various hedging strategies including Black-Scholes delta, Monte Carlo delta, the effect of transaction costs, optimal hedge frequency, and model-consistent finite-difference deltas in static and dynamic forms under Heston and GARCH models. Performance is evaluated through hedging error metrics, illustrating how model assumptions and rebalancing choices impact replication accuracy.

1 Introduction

Delta hedging is a common technique to manage the directional risk of an option position by holding a dynamic amount of the underlying asset. The effectiveness of delta hedging depends on how accurately the delta reflects the option's sensitivity to changes in the underlying asset's price, and how often the hedge is rebalanced.

In this project, we simulate hedging strategies under idealized (Black-Scholes) and realistic (stochastic volatility) models. We use historical-style Monte Carlo paths and compare multiple delta estimation techniques and rebalance strategies to assess hedging performance in practical settings.

2 Portfolio and Stock Price Dynamics

Underlying Asset Model

We assume the underlying stock S_t follows a stochastic differential equation of the form:

$$dS_t = \mu S_t \, dt + \sigma_t S_t \, dW_t,$$

where:

- μ is the drift rate of the asset,
- σ_t is the (possibly time-dependent) volatility,
- W_t is a standard Brownian motion.

In the Black-Scholes model, volatility is constant: $\sigma_t = \sigma$. For stochastic volatility models like Heston and GARCH, σ_t evolves randomly over time.

Delta Hedging Portfolio

At each time step t, we construct a hedging portfolio with:

- Δ_t shares of the underlying asset,
- The remainder in a cash account earning the risk-free rate r.

The portfolio value is given by:

$$\Pi_t = \Delta_t S_t + B_t,$$

where B_t is the bank account balance. The portfolio is rebalanced at discrete intervals to match the target delta Δ_t based on the chosen model (Black-Scholes, Monte Carlo, Heston, etc.).

Hedging Error

At maturity T, the hedging error is computed as:

$$\text{Error} = \Pi_T - \text{Payoff}(S_T),$$

where $\text{Payoff}(S_T) = \max(S_T - K, 0)$ for a European call option.

3 Model Assumptions and Option Parameters

We consider European call options with the following base parameters:

- Spot Price $S_0 = 100$
- Strike Price $K = 100$
- Risk-Free Rate $r = 0.03$
- Time to Maturity $T = 1$ year
- Volatility $\sigma = 0.2$ (for constant-vol scenarios)
- Time Steps: 252
- Number of Paths: up to 50,000

4 Black-Scholes Delta Hedging

We begin with delta hedging under the classical Black-Scholes model using analytical delta formulas. The hedge is rebalanced daily over 1 year (252 steps).

Observation:

- On average, the hedge was nearly unbiased, indicating that delta hedging accurately replicates the option payoff under ideal assumptions.
- The strategy exhibited modest hedging error volatility, reflecting the impact of discrete-time rebalancing.
- The worst-case scenario resulted in a 3.0364 shortfall, while the best-case yielded a 2.8282 surplus, highlighting tail risk due to market fluctuations between hedge updates.

Table 1: Black-Scholes Delta Hedging Results

	Mean Error	Std Dev	Max Loss	Max Gain
Black-Scholes Delta	0.0020	0.4407	-3.0364	2.8282

To evaluate the effect of hedging frequency, we repeat the simulation for multiple rebalance intervals ranging from daily (1 day) to monthly (21 days).

Hedging Performance at Different Frequencies:

- As the rebalancing interval increases, hedging error variability grows substantially.
- The strategy remains nearly unbiased across frequencies, but risk grows due to the reduced responsiveness of the hedge to underlying price changes.
- Standard error grows roughly linearly with time between rebalances, reflecting estimation uncertainty and execution lag.

Table 2: Hedging Error vs Hedging Frequency

Hedging Freq (days)	Mean Error	Std Dev	Std Error
1	0.002007	0.440698	0.001971
2	-0.001376	0.617335	0.002761
5	-0.003550	0.958349	0.004286
10	-0.001706	1.345997	0.006019
21	-0.000998	1.949714	0.008719

5 Hedging with Transaction Costs

We now incorporate proportional transaction costs into the hedging strategy. At each rebalance, a cost of $c \cdot |\Delta_t - \Delta_{t-1}| \cdot S_t$ is deducted from the portfolio, where c is the transaction cost rate. We use a cost rate of $c = 0.01$ (1%).

Observation:

- Including transaction costs significantly worsens the average hedging error, introducing a consistent negative bias in final portfolio values.
- While volatility increases slightly, the primary impact is on the mean — the hedge tends to underperform due to cash leakage from frequent trading.
- Both the maximum loss worsened and the potential gain was reduced. This demonstrates the real-world trade-off between hedge accuracy and execution cost.

Table 3: Hedging Performance (Daily Rebalancing, With vs Without Cost)

	Mean Error	Std Dev	Max Loss	Max Gain
No Cost	0.002007	0.440702	-3.036373	2.828193
With Cost	-0.564169	0.489082	-4.060703	2.094601

We now analyze the effect of transaction cost under multiple hedging frequencies. Each row reflects the outcome of using a fixed delta hedging frequency under 1% cost.

Table 4: Hedging Performance with Transaction Cost by Frequency

Frequency (days)	Mean Error	Std Dev	Std Error
1	-0.5642	0.4891	0.0022
2	-0.4193	0.6419	0.0029
5	-0.2922	0.9729	0.0044
10	-0.2257	1.3578	0.0061
21	-0.1668	1.9604	0.0088

Key Observations:

- Mean hedging error becomes less negative as rebalancing becomes less frequent. This reflects reduced transaction cost drag.
- However, standard deviation and standard error increase as hedging frequency decreases, indicating higher replication error and less precision in the mean estimate.
- The strategy exhibits a clear trade-off:
  - Frequent rebalancing reduces hedging error but increases costs
  - Infrequent rebalancing saves cost but introduces more risk

6 Optimal Hedge Frequency

To balance tracking accuracy and transaction cost, we define a total cost function:

$$\text{Total Cost} = \text{Mean Hedging Error} + \lambda \cdot \text{Standard Deviation}$$

Here, $\lambda \geq 0$ is a regularization parameter that penalizes frequent rebalancing. A higher λ favors cost reduction over tracking precision.

We compute this cost for multiple rebalance intervals (1, 2, 5, 10, 21 days) and optimize over different values of λ.

Observation:

- Regardless of λ optimal frequency is daily hedging as the mean error is so less that cost become minimum with daily hedging.
- As λ increases, total minimum cost becomes less negative, reflecting higher penalties for tracking risk.
- This framework enables quantifying the trade-off between precision and execution burden, useful for practical implementation.

Table 5: Optimal Hedge Frequency Across λ Values

λ	Optimal Frequency (days)	Min Total Cost
0.00	1	-0.5642
0.25	1	-0.4419
0.50	1	-0.3196
0.75	1	-0.1974
1.00	1	-0.0751

7 Monte Carlo-Based Delta Hedging

Monte Carlo Delta Hedging

In this approach, we estimate delta using the pathwise derivative method across simulated stock paths. The delta is used to rebalance the hedge daily, while the option price is fixed using the Black-Scholes formula.

Observation:

- The strategy produced a nearly unbiased mean error but exhibited significantly higher variability.
- Hedging error showed wide tails, reflecting sensitivity to estimation noise and limited convergence from the Monte Carlo method.

- Extreme outcomes such as a $27.51 surplus or a $22.29 shortfall demonstrate the risks associated with simulation-based hedging.

Table 6: Monte Carlo Delta Hedging Results

	Mean Error	Std Dev	Max Loss	Max Gain
Monte Carlo Delta	-0.0128	3.3562	-22.2938	27.5076

Comparison with Black-Scholes Delta Hedging

To benchmark performance, we compare Monte Carlo delta hedging against analytical Black-Scholes delta hedging using the same underlying stock paths.

Observation:

- The Black-Scholes delta consistently outperforms in terms of stability and risk containment.
- Monte Carlo delta estimation introduces numerical noise that significantly increases hedging risk.
- Monte Carlo delta is a feasible alternative when the closed-form model is unavailable, but it requires a high number of paths to be reliable.
- This result underscores the trade-off between model flexibility and computational cost.

Table 7: Monte Carlo vs. Black-Scholes Delta Hedging

Method	Mean Error	Std Dev	Max Loss	Max Gain
Black-Scholes Delta	-0.002467	0.442033	-3.600504	2.355254
Monte Carlo Delta	-0.012763	3.356195	-22.293795	27.507608

8 Delta Hedging Under Stochastic Volatility Models

We now analyze delta hedging performance under models with stochastic volatility. Specifically, we study the Heston and GARCH(1,1) models using different delta estimation methods:

- Black-Scholes Delta (benchmark)
- Pathwise Monte Carlo Volatility
- Finite-Difference (FD) Model-Based Constant Delta
- Finite-Difference (FD) Model-Based Dynamic Delta

All simulations use 50,000 paths and 252 time steps unless otherwise noted.

Black-Scholes Delta with Heston and GARCH Models

In this setup, the stock price is simulated using Heston or GARCH dynamics, but delta is computed from the Black-Scholes formula using a constant volatility.

Observation:

- While BS delta works reasonably under Heston dynamics, it performs poorly under GARCH due to the discrete volatility shocks and clustering effects.
- The GARCH strategy yielded large errors and directional bias, suggesting significant model mismatch.

Table 8: BS Delta Hedging with Stochastic Volatility Paths

Model	Mean Error	Std Dev	Max Loss	Max Gain
Heston	0.1641	1.5457	-10.5208	5.2533
GARCH	-30.1104	23.4430	-161.1550	-0.0065

Hedging with Pathwise Volatility

Next, we update the Black-Scholes delta by using the simulated instantaneous variance at each time step (pathwise volatility) from the Heston or GARCH models.

Observation:

- Substituting pathwise volatility moderately improved hedging accuracy for GARCH model and for Heston model quite similar.
- The GARCH model still showed wider tails, but the mean error flipped positive, showing partial correction of the model mismatch.

Table 9: Pathwise Volatility-Based Hedging

Model	Mean Error	Std Dev	Max Loss	Max Gain
Heston	0.1734	2.5483	-12.2074	5.2216
GARCH	1.0971	4.0241	-17.7890	15.1455

Model-Based Constant Delta (Finite Difference)

Here, we precompute delta at time $t = 0$ using model-consistent finite differences based on Monte Carlo pricing (with Heston or GARCH), and use this fixed delta throughout the hedge.

Observation:

- Compared to Black-Scholes delta, with constant FD delta performed poorly for both models
- Heston FD delta achieved a reasonably centered mean and lower error than GARCH.
- GARCH FD delta performed poorly, showing significant negative bias and explosive error spread.
- Results confirm that using a single initial delta is insufficient in stochastic models with volatility evolution.

Table 10: Constant Finite-Difference Delta Hedging

Model	Mean Error	Std Dev	Max Loss	Max Gain
Heston	-0.5682	16.7280	-99.4290	22.7699
GARCH	7.0134	172.2260	-1899.9103	142.2736

Model-Based Dynamic Delta (Finite Difference)

Finally, we recompute the model-consistent delta at each time step using finite differences, creating a fully dynamic delta hedging strategy consistent with the underlying model (Heston and GARCH).

Observation:

- For the Heston model, hedging errors exhibited moderate mean bias and large variability. The finite-difference deltas provided reasonable replication but suffered from sensitivity to simulation noise.
- In contrast, the GARCH model produced extreme hedging errors, with large mean deviation and high standard deviation. This indicates numerical instability when estimating deltas under path-dependent volatility.
- Both models highlight the limitations of raw FD delta estimation in stochastic environments, emphasizing the need for smoother or more robust methods (e.g., regression, neural networks).

Table 11: Dynamic Finite-Difference Delta Hedging

Model	Mean Error	Std Dev	Max Loss	Max Gain
Heston	-1.2546	97.4206	-414.4151	304.7238
GARCH	384.2422	6400.4843	-42344.6121	83590.6626

9 Conclusion

This project evaluated delta hedging strategies across both constant and stochastic volatility models. Black-Scholes delta hedging provided stable performance under ideal assumptions, while transaction costs highlighted the importance of optimizing hedge frequency. Monte Carlo and model-consistent deltas offered more flexibility but introduced greater variability. Among stochastic models, Heston-based hedging was moderately effective, whereas GARCH showed instability without careful calibration. Overall, the results underscore the balance between model complexity, computational cost, and hedging accuracy.