Hull–White (1F) Extended Vasicek Model Bond Option Pricing

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Abstract

This short report implements and validates European option pricing on zero-coupon bonds under the one-factor Hull–White (HW1F) Extended Vasicek interest-rate model. Three ways of supplying the input discount curve are used: (i) a flat continuous-compounded rate, (ii) pillar zeros with interpolation in $z(T)$, and (iii) a toy bootstrap from a deposit and par swap quotes. We show the HW1f bond-option price *equals* a Black–76 price when the lognormal volatility is set to the HW-implied bond-price volatility. We verify put–call parity, quantify sensitivities to mean reversion a and volatility σ, and illustrate volatility model risk by comparing against an ad-hoc 10% lognormal volatility. Empirically, at-the-money (ATM-forward) pricing is vega-dominated: changing σ matters much more than curve construction or a in this setup.

1 Model overview

The HW1f short rate r_t follows

$$dr_t = \big(\theta(t) - a\, r_t\big)\, dt + \sigma\, dW_t, \tag{1}$$

with mean-reversion speed $a > 0$, volatility $\sigma > 0$, and a time-dependent drift $\theta(t)$ chosen so the model fits the input discount curve $P(0,T)$ exactly.

Zero-coupon bond prices are affine in r_t:

$$P(t,T) = A(t,T)\, \exp\{-B(t,T) r_t\}, \qquad B(t,T) = \frac{1 - e^{-a(T-t)}}{a}. \tag{2}$$

HW-implied bond-price volatility. For an option expiring at S on a bond maturing at $T > S$, the bond price under the S-forward measure is lognormal with variance

$$\sigma_P^2(t,S,T) = \sigma^2 \left(\frac{1 - e^{-a(T-S)}}{a}\right)^2 \left(\frac{1 - e^{-2a(S-t)}}{2a}\right). \tag{3}$$

In our experiments we take $t = 0$, so we write $\sigma_P \equiv \sigma_P(0,S,T)$. This σ_P is a *total* volatility over $[0,S]$ (not "per year").

2 Black–76 bond option (equivalence with HW)

Let $P(0,S)$ and $P(0,T)$ be discount factors from today to S and T. For strike K, define

$$d_1 = \frac{\ln\!\big(\frac{P(0,T)}{K \times P(0,S)}\big)}{\sigma_p} + \frac{\sigma_p}{2}, \qquad d_2 = d_1 - \sigma_p, \tag{4}$$

where σ_p is the lognormal volatility used by Black–76. Then

$$\text{Call} = P(0,T)\,\Phi(d_1) - K\,P(0,S)\,\Phi(d_2), \tag{5}$$

$$\text{Put} = K\,P(0,S)\,\Phi(-d_2) - P(0,T)\,\Phi(-d_1), \tag{6}$$

and put–call parity holds:

$$\text{Call} - \text{Put} = P(0,T) - K\,P(0,S). \tag{7}$$

3 Input curves (three constructions)

We supply $P(0,T)$ in three ways:

1. **Flat curve:** $z(T) \equiv z^\star$ (continuous compounding); $P(0,T) = e^{-z^\star T}$.
2. **Pillar zeros + interpolation:** given $\{(T_i, z_i)\}$, define $z(T)$ by linear interpolation in T and set $P(0,T) = e^{-z(T)T}$.
3. **Toy bootstrap from deposit and par swaps:** with a 1Y simple deposit r_{dep},

$$P(0,1) = \frac{1}{1 + r_{\text{dep}}}, \tag{8}$$

   and annual fixed-leg par swap quotes $\{(T_n, s_n)\}$, accruals $\alpha_i = 1$,

$$1 - P(0,T_n) = s_n \sum_{i=1}^{n} \alpha_i P(0,T_i) \quad \Longrightarrow \quad P(0,T_n) = \frac{1 - s_n \sum_{i=1}^{n-1} \alpha_i P(0,T_i)}{1 + s_n \alpha_n}. \tag{9}$$

4 Experiment design

We price a European call and put on the $T = 5$y zero-coupon bond with option expiry $S = 2$y. We use HW parameters $a = 0.05$ (per year) and $\sigma = 0.01$ (per $\sqrt{\text{year}}$). We analyze two strike regimes:

- **Deep ITM call:** $K = 0.8\,P(0,T)$ (put near zero as expected).
- **ATM-forward:** $K_{\text{ATM}} = P(0,T)/P(0,S)$ (then $P(0,T) - KP(0,S) = 0$ and Call = Put).

We also compare *Black*@σ_p to *Black*@10% to show volatility model risk.

5 Results

Unless noted, numbers below are from the flat 3% curve with $S = 2$, $T = 5$.

5.1 ATM-forward prices and sanity checks

With $P(0,S) = 0.941765$, $P(0,T) = 0.860708$, we have

$$K_{\text{ATM}} = \frac{P(0,T)}{P(0,S)} = 0.913931, \qquad \sigma_P = 0.037508.$$

Prices:

- **Black@σ_P (HW-consistent):** Call = 0.012878.
- **Black@10% (ad-hoc vol):** Call = 0.034323.
- **Put–call parity at ATM:** Call − Put = $P(0,T) - KP(0,S) = 0$ (holds to machine precision).

For small vol at ATM, the Black price obeys the useful approximation

$$\mathrm{Call}_{\mathrm{ATM}} \approx P(0,T)\,\phi(0)\,\sigma_P \quad \text{with} \quad \phi(0) = \frac{1}{\sqrt{2\pi}},$$

which matches the observed 0.012878.

5.2 ATM comparison across curves

For the same a, σ, S, T (so the same σ_P), changing the curve shifts $P(0,S)$ and $P(0,T)$, hence K_{ATM} and the price.

Curve	K_{ATM}	σ_P	Call@HWvol	Call@10%vol
Flat 3%	0.913931	0.037508	0.012878	0.034323
Pillar Zeros	0.904837	0.037508	0.012750	0.033981
Bootstrap (dep+swaps)	0.903354	0.037508	0.012715	0.033889

Prices move by about ∼1% across curve constructions (mainly through $P(0,T)$), whereas using a 10% vol roughly triples the ATM price—a clear volatility/model-risk effect.

5.3 Parameter sensitivity at ATM (flat curve)

We bump a and σ by ±20% and reprice at ATM:

Scenario	Price	Δ vs Base
Base (ATM, σ_P)	0.012878	—
a +20%	0.012570	−0.000308
a −20%	0.013196	+0.000318
σ +20%	0.015453	+0.002575
σ −20%	0.010303	−0.002575

Interpretation: the ATM price is *vega-dominated*. A 20% change in σ moves the price by about ±20%, whereas a 20% change in a moves the price by only ∼2.5% (higher a reduces long-dated bond-price volatility).

Deep ITM demonstration. Using $K = 0.8\,P(0,T)$, the put price is numerically ≈ 0 because the strike is far below the forward $P(0,T)/P(0,S)$. This matches intuition and parity.

6 Discussion & limitations

Volatility assumption dominates. In this configuration, the choice of volatility has a much larger impact on value than either curve-construction differences or reasonable changes in mean reversion a.

Limitations. HW1f is a one-factor Gaussian short-rate model (can produce negative rates and limited volatility smiles). The toy bootstrap used annual accruals; production OIS bootstraps handle exact day counts and tenors. No calibration to a cap/floor or swaption surface was performed here; parameters were illustrative.

7 Conclusions

- The HW1f bond option equals Black–76 when using the HW-implied bond-price volatility σ_P; this identity is confirmed numerically.
- Put–call parity holds to machine precision (checked at ATM-forward).
- Across three curve constructions, ATM prices vary only modestly ($\sim 1\%$), while replacing $\sigma_P \approx 3.75\%$ with an ad-hoc 10% roughly triples the price: volatility/model-risk dominates.
- ATM sensitivities show price is far more sensitive to σ than to a.

Appendix: quick reference

$$\begin{aligned}
\text{ATM-forward strike:} \quad & K_{\text{ATM}} = \frac{P(0,T)}{P(0,S)}. \\
\text{Put–call parity:} \quad & \text{Call} - \text{Put} = P(0,T) - K\,P(0,S). \\
\text{ATM Black approximation:} \quad & \text{Call}_{\text{ATM}} \approx P(0,T)\,\phi(0)\,\sigma_P, \quad \phi(0) = \frac{1}{\sqrt{2\pi}}. \\
\text{Par swap bootstrap (annual):} \quad & P(0,T_n) = \frac{1 - s_n \sum_{i=1}^{n-1} \alpha_i P(0,T_i)}{1 + s_n \alpha_n}, \;\; \alpha_i = 1.
\end{aligned}$$