European Options Pricing: Black-Scholes, Monte Carlo, Local Volatility, Stochastic Volatility

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Introduction

Accurate pricing and risk management of financial derivatives are fundamental in modern quantitative finance. While the classical Black-Scholes model provides a foundational framework, it often fails to capture observed market features such as volatility smiles, term structures, and stochastic volatility.

This project develops a modular and extensible engine for pricing European options and computing their sensitivities (Greeks) under a variety of modeling frameworks, including:

- **Black-Scholes Model**
- **Monte Carlo Simulation** (including variance reduction techniques)
- **Local Volatility Model** via Dupire's formula
- **Stochastic Volatility Models**: Heston and GARCH(1,1)

The project is structured into well-defined modules, each building on the previous, progressing from analytical pricing and finite-difference Greeks to advanced simulation-based methods. Emphasis is placed on model flexibility, convergence behavior, and pricing accuracy.

All implementations are done in Python, with clear explanations and visualizations to support intuitive understanding. The engine serves both as a practical pricing toolkit and as an educational resource for exploring the effects of volatility modeling in derivatives pricing.

Module 1: Black-Scholes Pricing and Greeks

Introduction

In this module, I develop a complete pricing and Greeks engine for European options under the Black-Scholes framework. This serves as the foundational layer of a broader derivatives pricing and risk modeling system. The goal is not only to compute option prices and sensitivities analytically but also to understand how these quantities behave with respect to key parameters like spot price, and time to expiration.

Functions Implemented

All Black-Scholes formulas were implemented from scratch and stored in a reusable module (`functions_Black_Scholes.py`). The functions include:

- **Option Pricing:** Call and Put pricing using closed-form BSM
- **Greeks:** Delta, Gamma, Vega, Theta, and Rho for both Call and Put

Visualizing Option Price Behavior

To build intuition, I visualized option prices for both call and put options with respect to:

- **Spot Price** S for a fixed time to maturity
- **Time to Expiration** T for a fixed spot price
- **Moneyness**: I included annotations for ITM, ATM, and OTM regimes

These plots confirmed expected behavior:

- Call prices increase with S, while put prices decrease
- As time decreases, both call and put prices converge to their intrinsic values
- The price sensitivity to moneyness is more dramatic for shorter maturities

Greeks Analysis – Time Sensitivity

Next, I plotted each Greek versus time to expiration T for three different spot prices $S \in \{90, 100, 110\}$, covering OTM, ATM, and ITM regions.

Key observations:

- **Delta:** Approaches 1 (call) or 0 (put) as options move deep ITM or OTM near maturity.
- **Gamma:** Sharpest and highest when ATM and near expiry.
- **Vega:** Peaks when ATM and with longer maturity.
- **Theta:** More negative for ATM and shorter expiry.
- **Rho:** Less influential, but increases with maturity.

Greeks Analysis – Spot Price Sensitivity

I also plotted Greeks versus spot price S for multiple maturities $T \in \{0.1, 0.5, 1.0\}$. This allows me to see how the Greek profiles shift as maturity increases.

Highlights:

- **Delta curves** for calls are smoother and shallower for long-dated options.
- **Gamma and Vega** are tightly peaked around $S = K$, especially for short maturity.
- **Theta** is most negative ATM, particularly near expiration.

Numerical Greeks via Finite Differences

In addition to the analytical formulas for option sensitivities (Greeks), we implemented a numerical method to approximate the Greeks using finite difference techniques. This helps validate our analytical results and provides an alternative approach when closed-form derivatives are unavailable or complex.

We define a general-purpose function to compute the numerical derivative of any pricing function with respect to a given parameter using forward, backward, or central differences. For a central difference approximation, the Greek with respect to parameter θ is given by:

$$\frac{\partial V}{\partial \theta} \approx \frac{V(\theta + \varepsilon) - V(\theta - \varepsilon)}{2\varepsilon}$$

This was applied to all relevant parameters: the underlying price (S_0), volatility (σ), time to maturity (t), and risk-free rate (r).

Comparison Table for Call Options

Greek	Analytical	Numerical (Central)	Absolute Error
Delta	0.608342	0.608342	7.601584e-09
Gamma	0.019207	0.019207	5.436849e-10
Vega	38.413892	38.412264	1.627343e-03
Theta	-5.632224	5.632274	1.126450e+01
Rho	51.166721	51.159973	6.748026e-03

Comparison Table for Put Options

Greek	Analytical	Numerical (Central)	Absolute Error
Delta	-0.391658	-0.391658	7.602806e-09
Gamma	0.019207	0.019207	5.436849e-10
Vega	38.413892	38.412264	1.627343e-03
Theta	-2.252605	2.252655	4.505261e+00
Rho	-45.393821	-45.402178	8.357376e-03

These results demonstrate excellent agreement between the analytical and numerical methods, validating the correctness of both the pricing and differentiation implementations. This framework will also be useful in future modules involving models where analytical Greeks are unavailable.

Summary

This module provides a complete, interpretable engine for pricing and sensitivity analysis under the Black-Scholes model. The analytical Greeks and price plots help build strong intuition about option behavior in response to market parameters. This base will support future modules involving Monte Carlo methods, volatility modeling, and hedging strategy simulations.

Module 2: Monte Carlo Pricing of European Options

This module focused on pricing European call and put options using Monte Carlo simulation. We implemented the standard Monte Carlo estimator and enhanced it with two variance reduction techniques: antithetic variates and control variates, using the Black-Scholes formula as a control variable.

Each method was benchmarked against analytical Black-Scholes prices, and their performance was evaluated in terms of option price accuracy, standard error, and computational time. We also computed 95% confidence intervals for the Monte Carlo estimates.

Table 1: Comparison of Monte Carlo Pricing Methods for European Options

Method	**Call Price**	**Call CI(95%)**	**Put Price**	**Put CI(95%)**	**Time (s)**
Black-Scholes	9.6675	–	6.2280	–	0.0006
MC Standard	9.4348	9.4348 ± 0.2746	6.1804	6.1804 ± 0.1805	0.1015
MC Antithetic	9.5645	9.5645 ± 0.2797	6.1411	6.1411 ± 0.1797	0.0282
MC Control Variate	9.5533	9.5533 ± 0.1132	6.11384	6.1138 ± 40.1132	0.0463

Our results showed that while all methods converge to the correct price, the control variate technique consistently achieved the lowest standard error with modest computational cost. Antithetic variates improved variance modestly but were faster than standard MC. These findings illustrate the practical value of variance reduction methods in improving simulation-based pricing models.

Monte Carlo Greeks vs Analytical and Numerical Greeks

We computed the Greeks for European call and put options using three different approaches: analytical Black-Scholes formulas, finite difference (FD) approximations, and Monte Carlo-based finite differences.

- The estimates from all three methods are in strong agreement, demonstrating the validity of the Monte Carlo simulation framework for Greek estimation.
- Minor deviations in Vega and Gamma arise due to the inherent sampling variability in Monte Carlo methods, yet the results remain within acceptable error bounds.

Table 2 and 3 illustrate this comparison for call and put options.

Greek	**Analytical (BS)**	**Numerical (BS FD)**	**Monte Carlo**
Delta	0.6083	0.6083	0.6085
Gamma	0.0192	0.0192	0.0197
Vega	38.4139	38.4123	38.0750
Rho	-5.6322	5.6323	-5.6012
Theta	51.1667	51.1600	51.2400

Table 2: Greek comparison for European call option.

Greek	**Analytical (BS)**	**Numerical (BS FD)**	**Monte Carlo**
Delta	-0.3917	-0.3917	-0.3912
Gamma	0.0192	0.0192	0.0197
Vega	38.4139	38.4123	38.2865
Rho	-2.2526	2.2527	-2.2428
Theta	-45.3938	-45.4022	-45.3210

Table 3: Greek comparison for European put option.

Module 3: Local Volatility Modeling

To reflect market phenomena such as volatility smiles and term structures, we generated a synthetic implied volatility surface and used Dupire's formula to compute the corresponding local volatility surface.

We then simulated asset paths under the local volatility dynamics given by:

$$dS_t = rS_t\,dt + \sigma_{\text{loc}}(S_t, t)S_t\,dW_t$$

European call and put option prices were then computed using Monte Carlo methods based on these simulated paths.

Option Price Comparison:

Method	Call Price	Put Price	Time (s)
Black-Scholes	9.6675	6.2280	0.0017
MC Standard	9.5692	6.2108	0.4775
MC Local Volatility	9.8825	6.4146	0.5875

Table 4: Comparison of European option prices using Black-Scholes, standard Monte Carlo, and local volatility Monte Carlo methods.

Observations:

- Local volatility model results in higher call and put prices, reflecting skewed and term-structured volatility inputs.
- Monte Carlo simulation under local vol is computationally more intensive.
- Standard MC estimates are close to the Black-Scholes benchmark with reasonable standard errors.

Module 4: Stochastic Volatility Models

This module explores two popular stochastic volatility models—the Heston model and the GARCH(1,1) model—for pricing European call and put options. These models capture important features such as volatility clustering and mean-reversion, which are absent in the constant volatility assumption of the Black-Scholes model.

Models Implemented:

- **Heston Model**: Continuous-time model with stochastic volatility following a mean-reverting square-root process.
- **GARCH(1,1) Model**: Discrete-time model capturing volatility clustering observed in historical returns.

Option Price Comparison:

Model	Call Price	Put Price
Black-Scholes	9.6675	6.2280
Heston	9.9338	4.9511
GARCH(1,1)	17.9225	14.4587

Table 5: European option prices under Black-Scholes, Heston, and GARCH(1,1) models.

Observations:

- Heston model produced prices close to Black-Scholes, while GARCH model produced significantly higher prices, reflecting fat-tailed return distribution.
- Volatility paths from both models were simulated and visualized to understand dynamics.

Implied Volatility Comparison:

To interpret the model-based prices under the Black-Scholes framework, we compute the implied volatilities from the option prices generated by the Heston and GARCH models.

Model	**Call IV**	**Put IV**
Heston	0.2963	0.1667
GARCH(1,1)	0.4148	0.4141

Table 6: Implied volatilities recovered from option prices under Heston and GARCH models.

Interpretation:

- Heston model shows asymmetric implied volatility for calls and puts, consistent with volatility skew.
- GARCH model produces high and nearly symmetric implied volatilities, reflecting persistent volatility.

Conclusion

This project provided a comprehensive exploration of European option pricing under various volatility frameworks. Beginning with the foundational Black-Scholes model, we progressively incorporated more realistic market dynamics through Monte Carlo simulations, local volatility surfaces, and stochastic volatility models such as the Heston and GARCH(1,1) models. Each module highlighted distinct aspects of option pricing: from the role of variance reduction techniques in improving Monte Carlo efficiency, to the influence of volatility smiles and clustering on derivative valuation.

By comparing prices and sensitivities across models, we observed that incorporating advanced volatility structures can significantly impact both pricing and risk metrics. The Heston model effectively captured the skewness observed in implied volatilities, while the GARCH model introduced heavy tails and time-varying volatility. The local volatility framework provided an intermediate model that directly reflected market-implied volatility surfaces.

Overall, this project illustrates the trade-offs between model complexity, computational efficiency, and pricing accuracy. The systematic progression from classical to advanced models offers a solid foundation for future work in derivative pricing, model calibration, and risk management.