Delta–Hedging under Non-Constant Volatility

A K M Rokonuzzaman Sonet

1 Introduction

The goal of this project is to examine the profit distribution of a short European call position that is *delta-hedged* when the underlying equity's volatility is **not constant**. Three stylised volatility processes were investigated:

1. **Discrete Regime Model** — each day, $\sigma_t \in \{0.20,\ 0.30,\ 0.45\}$ with probabilities $(0.5, 0.3, 0.2)$;
2. **Heston Model** — mean-reverting stochastic variance with parameters $(\kappa, \theta, \xi, \rho)$;
3. **GARCH(1,1)** — discrete clustering with (ω, α, β).

Simulation horizon is one year (252 steps); the call is struck at-the-money ($K = S_0 = 105$) and interest rate $r = 3.5\%$. Deltas are recomputed daily via the Black–Scholes formula; profits are in USD, scaled to `1000` contracts.

2 Methodology

2.1 Stock Path Generation

- **Custom Regime:** IID draw of σ_t each step, standard GBM for returns.
- **Heston:** Euler–Maruyama, $\mathrm{d}V_t = \kappa(\theta - V_t)\,\mathrm{d}t + \xi\sqrt{V_t}\,\mathrm{d}W_t^{(v)}$ with $\mathrm{d}W^{(s)}\mathrm{d}W^{(v)} = \rho\,\mathrm{d}t$.
- **GARCH(1,1):** $\sigma_{t+1}^2 = \omega + \alpha\epsilon_t^2 + \beta\sigma_t^2$; log-returns use σ_t.

2.2 Delta Hedging P&L

For path j and step i ($\Delta t = \frac{1}{252}$):

$$\mathrm{P\&L}_{i,j} = \left(S_{i+1,j} - e^{r\Delta t} S_{i,j}\right)\Delta_{i,j}\ e^{-r(i+1)\Delta t}, \quad \Delta_{i,j} = N\left(\frac{\ln(S_{i,j}/K) + (r + \frac{1}{2}\sigma_{i,j}^2)\tau}{\sigma_{i,j}\sqrt{\tau}}\right).$$

Total hedging cost is $\sum_i \mathrm{P\&L}_{i,j}$; profit = premium − hedgeCost − payoff.

3 Results

Observations

- **Drift sensitivity:** Higher μ shifts profits left; bearish drift is favourable to option sellers.

Table 1: Key Profit Metrics (1000 contracts)

Model	Drift μ	Mean \$	Profit %	Max Loss \$
Discrete σ	-0.4	-258	44	-8466
Discrete σ	0.4	123	54	-8071
Heston (BS premium)	0.4	172	58	-7260
Heston (BS premium)	-0.4	647	65	-6615
Heston (Heston premium)	-0.4	247	57	
Heston (Heston premium)	0.4	-193	49	
GARCH (GARCH premium)	-0.40	-4.51	51	$-16{,}250$
GARCH (GARCH premium)	0.00	-17.65	51	
GARCH (GARCH premium)	0.40	194	50	$-15{,}140$

- **Model-risk:** Using a BS premium under GARCH or Heston systematically *overstates* profit—consistent with BS underpricing clustered/fat-tailed volatility.
- **Hedging error:** Even daily delta-hedging leaves P&L variance large; max losses exceed $15k for 1000 contracts.

4 Conclusion

Delta hedging neutralises small price moves but not volatility shocks. Under realistic stochastic-volatility (Heston, GARCH), option sellers face:

1. Profits highly sensitive to drift assumptions;
2. Significant tail risk, even when premiums are model-consistent;
3. Potential mispricing if Black–Scholes premiums are used in clustered-volatility markets.

Future work could implement vega (sigma) hedging with variance swaps or VIX futures to reduce the volatility exposure.