Chapter 3 Exercises

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August 26, 2023

P 1 Exercise 3.12

(c) If (p_i) is a sequence of points in S and $p \in S$, then $p_i \to p$ in S if and only if $p_i \to p$ in X.
(d) Every subspace of a Hausdorff space is Hausdorff.
(e) Every subspace of a first countable space is first countable.
(f) Every subspace of a second countable space is second countable.

(sol) 1.1 Exercise 3.12

(c) Assume $p_i \to p$ in S. Since for all open neighborhoods V_p in S, there exists U_p in X such that $V_p \subseteq U_p$, $p_i \to p$ in X as well.
Assume $p_i \to p$ in X. Since it is a sequence of points in S, by definition, for each open set intersect S, blah blah blah, $p_i \to p$ in S as well.
(d) Let X be Hausdorff. Let S be a subspace. Let $x, y \in S$. Then there exists U_x an U_y open sets of X such that they are disjoint. Since they are disjoint, intersecting them with S still leaves them disjoint.
(d) Every point has a neighborhood basis in X; taking the intersection of that neighborhood with S, each basis element intersected with S is open in S and by definition is a basis for that neighborhood.
(e) By part by, the basis for the subspace is the collection of basis elements intersected with S, which is countable since the original basis is countable.

P 2 Exercise 3.13

Let X be a topological space and let S be a subspace of X. Show that the inclusion map $S \hookrightarrow X$ is a topological embedding.

(sol) 2.1 Exercise 3.13

Restricting the co-domain to just S yields the bijective identity, which is a homeomorphism.

P 3 Exercise 3.14

Give an example of a topological embedding that is neither an open map nor a closed map.

(sol) 3.1 Exercise 3.14

The inclusion map is a topological embedding. So for any topological space X, choose a subset that is neither closed nor open, and thus the inclusion map from such a subset to X is neither open nor closed.

P 4 Exercise 3.15

A surjective topological embedding is a homeomorphism.

(sol) 4.1 Exercise 3.15

The restriction of a surjective topological embedding is the whole co-domain itself, thus the whole map is both injective and surjective so it's bijective and thus a homeomorphism with respect to the whole co-domain.

P 5 Exercise 3.25

$$\mathscr{B} = \{U_1 \times \cdots \times U_n : U_i \text{ is an open subset of } X_i,\ i = 1, \ldots, n\}$$

Prove that $\mathscr{B}$ is a basis for a topology.

(sol) 5.1 Exercise 3.25

Let $\mathscr{T}$ be the topology as follows: a subset of $X_1 \times \cdots \times X_n$ is open if and only if the projections into $X_1, \ldots, X_n$ are open.
By definition, each set in $\mathscr{B}$ is open. Let U, V be arbitrary open subsets of the product space. Then it's components U_is and V_is are all open in X_i. Then $U_i \cap V_i$ is open in X_i. Then

$$U_1 \cap V_1 \times \cdots \times U_n \cap V_n$$

is open in the product space and is contained in

$$(U_1 \times ... \times U_n) \cap (V_1 \times ... \times V_n)$$

Thus $\mathscr{B}$ is a basis for $\mathscr{T}$. $\square$

P 6 3.29

Prove the preceding corollary using only the characteristic property of the product topology.
If $X_1, ..., X_n$ are topological spaces, each canonical projection $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is continuous.

(sol) 6.1 3.29

Let $Y = X_1, ..., X_n$ and let f be the identity map. Since the identity is always continuous, each $f_i = \pi_i \circ f$ is also continuous. And since f is identity, $f_i = \pi_i$. So the result follows. $\square$

P 7 3.32

Prove proposition 3.31:
Let $X_1, ..., X_n$ be topological spaces.
(a) The product topology is "associative" in the sense that the three topologies on the set $X_1 \times X_3 \times X_3$, obtained by thinking of it as $X_1 \times X_3 \times X_3$ or $(X_1 \times X_3) \times X_3$ or $X_1 \times (X_3 \times X_3)$ are equal.
(b) For any $i \in \{1, ..., n\}$ and any points $x_j \in X_j$, $j \neq i$, the map $X_i \to X_1 \times \cdots \times X_n$ given by

$$f(x) = (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)$$

is a topological embedding of X_i into the product space.
(c) Each canonical projection $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is an open map.
(d) If for each i, $\mathscr{B}_i$ is a basis for the topology of X_i, then the set

$$\{B_1 \times \cdots \times B_n : B_i \in \mathscr{B}_i\}$$

is abasis for the product topology.
(e) If S_i is a subspace of X_i for $i = 1, \ldots, n$, then the product topology and the subspace topology on $S_1 \times \cdots \times S_n \subseteq X_1 \times \cdots \times X_n$ are equal.
(f) If each X_i is Hausdorff, so is $X_1 \times \cdots \times X_n$.
(g) If each X_i is first countable, so is $X_1 \times \cdots \times X_n$.
(h) If each X_i is second countable, so is $X_1 \times \cdots \times X_n$.

(sol) 7.1 3.32

(a) Let $f((x_1, x_2), x_3) = (x_1, x_2, x_3)$. Bijection is clear. Now just need to prove continuity of f and f^{-1}. Let $Y = (x_1, x_2), x_3)$ in the characteristic property. Then $f_1 = \pi_1 \circ \pi_{1(X_1 \times X_2)}$, $f_2 = \pi_1 \circ \pi_{2(X_1 \times X_2)}$, $f_3 = id \circ \pi_{2(X_1 \times X_2)}$. Bascially a composition of projections. All f_is are continuous because all canonical projects are are continuous and compositions of continuous functions are continuous. Since all f_i are continuous, f is also continuous. The continuity of f^{-1} can be proved similarly.

(b) f is continuous because for every open set in the product space, in particular $f^{-1}(U_i)$ must be open by definition. f^{-1} is continuous because it is the canonical projection.
(c) Same reason as why f is continuous as in (b)
(d) (d) By definition the topology on the product space is the topology for which

$$\{U_1 \times \cdots \times U_n : U_i \text{ is open in } X_i\}$$

is a basis. Since each $\mathscr{B}_i$ is a basis, each U_i is a union of some family of B_is from $\mathscr{B}_i$. The result follows.
(e) Open-ness is defined on the product space by conjuction (ands) and subspace topology is the intersection of open sets with a subset – and since intersection boils down to conjuction of set elements – and since conjuction is associative and commutative, the result follows. In order words, conjuctions of intersections is equal to intersections of conjunctions (of open sets).
(f) Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two distinct points of the product space. By assumption, for all i, there exists U_i and V_i open neighborhoods of x_i and y_i respectively, such that $U_i \cap V_i$ is empty. Since all U_i and V_i are open, $U_i \times \cdots \times U_n$ and $V_i \times \cdots \times V_n$ are open subsets of the product space. Moreover they are disjoint since each component in the cartesian product is disjoint.
(g) If each point in X_i has a countable neighborhood basis, the cartesian product is also a neighborhood basis by definition of product topology, but it must also be countable because a k-cell of countable sets must also be countable from elementary set theory.
(h) Countability is preserved for same reason as in (g)

P 8 3.34

Suppose $f_1, f_2 : X \to \mathbb{R}$ are continuous functions. Their pointwise sum $f_1 + f_2 : X \to \mathbb{R}$ and pointwise product $f_1 f_2 : X \to \mathbb{R}$ are real-valued functions defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \qquad (f_1 f_2)(x) = f_1(x) f_2(x)$$

Use the characteristic property of the product topology to show that pointwise sum and products of continuous functions are continuous.

(sol) 8.1 3.34

Using the characteristic property, since f_1, f_2 are both continuous, then the mapping $f : X \to \mathbb{R} \times \mathbb{R}$ is also continuous. Since multiplication and addition are both continuous, i.e. $m, a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the composition of f and mult or add is also continuous. $\square$

P 9 3.38

(Characteristic property of Infinite product spaces)
Let $(X_\alpha)_{\alpha\in A}$ be an indexed family of topological spaces. For any topological space Y, a map $f : Y \to \prod_{\alpha\in A} X_\alpha$ is continuous if and only if each of its component functions $f_\alpha = \pi_\alpha \circ f$ is continuous. The product topology is the unique topology on $\prod_{\alpha\in A} X_\alpha$ that satisfies this property.

(sol) 9.1 3.38

The product topology is the only topology to satisfy this property because of the finite intersection of open sets are open property of topologies. The preimages of infinite spaces in the product space

P 10 3.40

Show that the disjoint union topology is indeed a topology.

(sol) 10.1 3.40

The whole disjoint space is open since the intersection with each X_α is equal to X_α which is open since X_α is a topological space. As is the empty set for the same reason. Let U, V be open sets in the disjoint union space. Then $U \cap V$ is open since $(X_\alpha \cap U) \cap (X_\alpha \cap V) = X_\alpha \cap (U \cap V)$. Openness of finite intersections follows from induction. An arbitrary union of open sets in the product space is also open since The X_α are all disjoint so each $X_\alpha \cap U$ will be open and an arbitrary union of those is open. $\square$

P 11 3.43

Let $(X_\alpha)_{\alpha\in A}$ be an indexed family of topological spaces.
(a) A subset of $\coprod_{\alpha\in A} X_\alpha$ is closed if and only if its intersection with each X_α is closed.
(b) Each canonical injection $\iota_\alpha : X_\alpha \to \coprod_{\alpha\in A} X_\alpha$ is a topological embedding and an open and closed map.
(c) If each X_α is Hausdorff, then so is $\coprod_{\alpha\in A} X_\alpha$.
(d) If each X_α is first countable, then so is $\coprod_{\alpha\in A} X_\alpha$.
(e) If each X_α is second countable and the index set A is countable, then $\coprod_{\alpha\in A} X_\alpha$ is second countable.

(sol) 11.1 3.43

(a) Assume that a subset C in the disjoint union space is closed. Then $\coprod_{\alpha\in A} X_\alpha \setminus C$ is open which mean that $X_\alpha \cap (\coprod_{\alpha\in A} X_\alpha \setminus C)$ is open in X_α. Which is equal to

$X_\alpha \cap (\coprod_{\alpha\in A} X_\alpha \cap C^c) = (X_\alpha \setminus C) \cap \coprod_{\alpha\in A} X_\alpha$ being open in X_α. Which implies C is closed in X_α.
Let C be such that each intersection with each X_α is closed. Then $X_\alpha \setminus C$ is open. And the result follows similarly as above but in reverse.
(b) Since the second element of the pair is all the same if we restrict the co-domain to the image, we bascially get the identity map which is a homeomorphism and is both open and closed.
(c) Let each X_α be Hausdorff. Let x, y be distinct points in the disjoint union space. All X_αs are disjoint by definition so there are just two cases:
1) x, y are in the same X_α. Then since X_α is Hausdorff, there exists disjoint open neighborhoods of x and y respectively.
2) If x, y are not in the same X_α, then their respective X_α is the open neighborhoods that are by definition disjoint.
(d) Let x be any point of the disjoint union space. Then by definition it must be a point in exactly one X_α. Since X_α is first countable, there exists a countable neighborhood basis of x in X_α. By definition of disjoint union topology, that neighborhood basis in X_α is also a neighborhood basis in the disjoint union topology/space, so the disjoint union space is also first countable.
(e) This follows since if A is countable then we get a countable union of countable sets which is itself countable. □

P 12 3.44

Suppose $(X_\alpha)_{\alpha\in A} X_\alpha$ is an indexed family of nonempty n-manifolds. Show that the disjoint union $\coprod_{\alpha\in A} X_\alpha$ is an n-manifold if and only if A is countable.

(sol) 12.1 3.44

Hausdorffness and second-Countability follow from Proposition 3.42. Only need to show local euclidean. For each x, it will fall into exactly one X_α which is locally euclidian and thus so is the disjoint union space since the inclusion map is a topological embedding and is open and closed mapping. □

P 13 3.45

Let X be any space and Y be a discrete space. Show that the Cartesian product $X \times Y$ is equal to the disjoint union $\coprod_{y\in Y} X_y$, and the product topology is the sameas the disjoint union topology.

(sol) 13.1 3.45

It is clear that they have the same elements.
Since Y is a discrete space, every subset of Y is open. In particular the set of

singletons in Y form a basis for Y. Then $\{U, y\}$ where U is open in X and $y \in Y$ forms a basis for the product space. But then in the disjoint union space a set is open if the restriction is open, which boils down to $|Y|$ copies of Xs indexed by y, which is exactly the basis above. Thus they have the same topologies. □

P 14 3.46

Show that the quotient topology is indeed a topology.

(sol) 14.1 3.46

Empty set is open trivialy. The whole Y is open by surjection. And the fact that intersection and union pass through preimages imply that the quotient topology is a topology. □

P 15 3.55

Show that every wedge sum of Hausdorff spaces is Hausdorff.

(sol) 15.1 3.55

There are two cases. The case where neither distinct point is the collapsed base point, and the case where one is.
Case 1) If neither distinct points is the collapsed point, then Hausdorffness follows from Hausdorffness of disjoint union space.
Case 2) Let x be the collapsed base point and y be any other point. The preimage of x will then be $\{p_\alpha\}_{\alpha \in A}$. Then for each of these p_α, since they come from Hausdorff spaces, there exists an open neighborhood that is disjoint from the open neighborhood of y. Then the union of all the open neighborhoods of all the p_α is itself open, which means that image of the quotient map of such a set is also open in the quotient space and is disjoint from the open neighborhood of y. □

P 16 3.59

Let $q : X \to Y$ be any map. For a buset $U \subseteq X$, show that the following are equivalent.
(a) U is saturated.
(b) $U = q^{-1}(q(U))$.
(c) U is an union of fibers.
(d) If $x \in U$, then every point $x' \in X$ such that $q(x) = q(x')$ is also in U.

(sol) 16.1 3.59

($a \implies b$) Assume U is saturated. Then there exists some $V \subseteq Y$ such that $q^{-1}(Y) = U$. This assumption rules out the case where the image of U contains a point where more than one element of X maps to, where one of those points is not in U (where q is not injective). The result follows.
($b \implies c$) Assume $U = q^{-1}(q(U))$. This can be rewritten as

$$U = \bigcup\{q^{-1}(y) : y \in q(U)\}$$

($c \implies d$) Assume U is a union of fibers. Let $x \in U$. Let $x' \in X$ be such that $q(x) = q(x')$. Since U is a union of fibers, and since $q(x) = q(x')$, they map to the same element in the image, which means they are in U. ($d \implies a$) Since by definition U contains each fiber, there exists some subset in the co-domain that has the pre-image as U. □

P 17 3.61

A continuous surjective map $q : X \to Y$ is a quotient map if and only if it takes saturated open subsets to open subsets, or saturated closed subsets to closed subsets.

(sol) 17.1 3.61

If q is a quotient map, then the target's open sets are the ones for which its preimage is open in the domain. Let U be a saturated open set in X. Then we have that $U = q^{-1}(q(U))$. Then $q(U)$ is open since q is a quotient map. Therefore, q takes saturated open sets to open sets.
If q takes saturated open sets to open sets. Let $V \subseteq Y$. Need to show that V is open if and only if $q^{-1}(V)$ is open. If V is open, then $q^{-1}(V)$ is open by continuity of q. Assume that $q^{-1}(V)$ is open. By definition $q^{-1}(V)$ is a union of fibers, which means it is a saturated open set, so V is open. □

P 18 3.63

(a) Any composition of quotient maps is a quotient map.
(b) An injective quotient map is a homeopmorphism.
(c) If $q : X \to Y$ is a quotient map, a subset $K \subseteq Y$ is closed if and only if $q^{-1}(K)$ is closed in X.
(d) If $q : X \to Y$ is a quotient map and $U \subseteq X$ is a saturated open or closed subset, then the restriction $q|_U : U \to q(U)$ is a quotient map.
(e) If $\{q_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in A}$ is an indexed family of quotient maps, then the map $q : \coprod_\alpha X_\alpha \to \coprod_\alpha Y_\alpha$ whose restriction to each X_α is equal to q_α is a quotient map.

(sol) 18.1 3.63

(a) Let $f : X \to Y$, $g : Y \to Z$ be quotient maps. Then the composition is surjective as well. Composition of continuity shows that the V open in Z implies $(f \circ g)^{-1}(V)$ open in X. Let U be a saturated open set in X. Then $U = f^{-1}(g^{-1}(V))$ for some $V \subseteq Z$. Need to show that V is open. Then $f(U) = g^{-1}(V)$ and $f(U)$ is open and also a union of fibers, Therefore $g(f(U)) = V$ is open in Z.
(b) A quotient map is already surjective. So a map that is both injective and surjective is bijective by the Cantor-Bernstein theorem. By definition it is continuous. By injective, all subsets are saturated, thus it is an open map, so it is a homeopmorphism.
(c) Let $K \subseteq Y$ be closed. Then $q^{-1}(K)$ is closed by continuity. Let $q^{-1}(K)$ be closed in X. Then by definition it is a saturated subset, and so it's image, K is closed.
(d) The restriction is surjective by definition. It is also continuous by the local criterion for continuity. Because U is a saturated set, any subset of it is also saturated. Thus open subset of U must also map to an open set of $q(U)$, so q is a quotient map.
(e) Let $V \subseteq \coprod_\alpha Y_\alpha$ be open. Then the intersection with each Y_α is open. Then $q_\alpha^{-1}(V_\alpha)$ is open for all α, thus $q^{-1}(V_\alpha)$ is open. The other way is similar following from the fact that a saturated open subset's intersection with each subspace is also saturated. □

P 19 3.72

(Uniqueness of the Quotient Topology)
Given a topological space X, a set Y, and a surjective map $q : X \to Y$, the quotient topology is the only topology on Y for which the characteristic property holds.

(sol) 19.1 3.72

Let τ_q, τ_p be two topologies for which the property holds, where τ_q is the quotient topology. Let $Z = Y_p$ and $f_q : Y_q \to Y_p$ be the identity map. Simlarly let $Z = Y_q$ and $f_p : Y_p \to Y_q$ be the identity. Then if both are continuous then the result follows.
$f_p \circ q_p : X \to Y_q$ is continuous (quotient map), and thus f_p is continuous.
Let $q' : X \to Y_p$, and $id : Y_d \to Y_d$. id is clearly continuous. Thus using the characteristic property, $q' = id \circ q'$ is continuous. But also $q' = f_p \circ q$, using the characteristic property again, f_p is continuous. □

P 20 3.83

Verfiy each of the following to be a topological group:
(a) the real line $\mathbb{R}$ with its additive group structure and Euclidean topology
(b) the set $\mathbb{R} \setminus \{0\}$ of nonzero real numbers under multiplication, with the Euclidean topology
(c) the general linear group $GL(n, \mathbb{R})$, which is the set of $n \times n$ invertible real matrices under matrix multiplication, with the subspace topology obtained from $\mathbb{R}^{n^2}$ (where

we identify an $n \times n$ matrix with a point in $\mathbb{R}^{n^2}$ by using the matrix entries as coordinates)
(d) any group whatsoever with the discrete topology (any such group is called a discrete group)

(sol) 20.1 3.83

(a) Addition is continuous by limit laws, negation is continuous as well.
(b) Same as part a
(c) idk
(d) by assumption it's a group. Continuity follows since any function from a discrete topology will be continuous.

P 21 3.85

Show that any subgroup of a topological group is a topological group with the subspace topology. Any finite product of topological groups is a topological group with the direct product group structure and the product topology.

(sol) 21.1 3.85

Let G be a topological group. Let $S \leq G$ be a subgroup. Then by definition S is a group. Just need to show that negation and group operation are continuous. Since S is a subset it inherits the subset topology. For any open subset in the image $V \subseteq G$, $V \cap S$ is open in S. Then $m^{-1}(V) \cap S$ is open, same as negation, so S is a topological group.
Let $G_1, \ldots, G_n$ be topological groups. Then $G_1 \times \cdots \times G_n$ is a group with the direct product group operation. Also imbue the space in the product topology. Since M is the group operation, it can be written as $(m_1, \ldots, m_n)(x) = (m_1(x), \ldots, m_n(x))$, where each m_i is continuous. By proposition 3.33, M is continuous; same goes for negation. Thus the product space is a topological group. $\square$

P 22

(sol) 22.1

P 23

(sol) 23.1