Chapter 3 Exercises

Campinghedgehog

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P 1 Exercise 3.12

(c) If (p_i) is a sequence of points in S and $p \in S$, then $p_i \to p$ in S if and only if $p_i \to p$ in X.
(d) Every subspace of a Hausdorff space is Hausdorff.
(e) Every subspace of a first countable space is first countable.
(f) Every subspace of a second countable space is second countable.

(sol) 1.1 Exercise 3.12

(c) Assume $p_i \to p$ in S. Since for all open neighborhoods V_p in S, there exists U_p in X such that $V_p \subseteq U_p$, $p_i \to p$ in X as well.
Assume $p_i \to p$ in X. Since it is a sequence of points in S, by definition, for each open set intersect S, blah blah blah, $p_i \to p$ in S as well.
(d) Let X be Hausdorff. Let S be a subspace. Let $x, y \in S$. Then there exists U_x an U_y open sets of X such that they are disjoint. Since they are disjoint, intersecting them with S still leaves them disjoint.
(d) Every point has a neighborhood basis in X; taking the intersection of that neighborhood with S, each basis element intersected with S is open in S and by definition is a basis for that neighborhood.
(e) By part by, the basis for the subspace is the collection of basis elements intersected with S, which is countable since the original basis is countable.

P 2 Exercise 3.13

Let X be a topological space and let S be a subspace of X. Show that the inclusion map $S \hookrightarrow X$ is a topological embedding.

(sol) 2.1 Exercise 3.13

Restricting the co-domain to just S yields the bijective identity, which is a homeomorphism.

P 3 Exercise 3.14

Give an example of a topological embedding that is neither an open map nor a closed map.

(sol) 3.1 Exercise 3.14

The inclusion map is a topological embedding. So for any topological space X, choose a subset that is neither closed nor open, and thus the inclusion map from such a subset to X is neither open nor closed.

P 4 Exercise 3.15

A surjective topological embedding is a homeomorphism.

(sol) 4.1 Exercise 3.15

The restriction of a surjective topological embedding is the whole co-domain itself, thus the whole map is both injective and surjective so it's bijective and thus a homeomorphism with respect to the whole co-domain.

P 5 Exercise 3.25

$$\mathscr{B} = \{U_1 \times \cdots \times U_n : U_i \text{ is an open subset of } X_i,\ i = 1, \ldots, n\}$$

Prove that $\mathscr{B}$ is a basis for a topology.

(sol) 5.1 Exercise 3.25

Let $\mathscr{T}$ be the topology as follows: a subset of $X_1 \times \cdots \times X_n$ is open if and only if the projections into $X_1, \ldots, X_n$ are open.
By definition, each set in $\mathscr{B}$ is open. Let U, V be arbitrary open subsets of the product space. Then it's components U_is and V_is are all open in X_i. Then $U_i \cap V_i$ is open in X_i. Then

$$U_1 \cap V_1 \times \cdots \times U_n \cap V_n$$

is open in the product space and is contained in

$$(U_1 \times ... \times U_n) \cap (V_1 \times ... \times V_n)$$

Thus $\mathscr{B}$ is a basis for $\mathscr{T}$. $\square$

P 6 3.29

Prove the preceding corollary using only the characteristic property of the product topology.
If $X_1, ..., X_n$ are topological spaces, each canonical projection $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is continuous.

(sol) 6.1 3.29

Let $Y = X_1, ..., X_n$ and let f be the identity map. Since the identity is always continuous, each $f_i = \pi_i \circ f$ is also continuous. And since f is identity, $f_i = \pi_i$. So the result follows. $\square$

P 7 3.32

Prove proposition 3.31:
Let $X_1, ..., X_n$ be topological spaces.
(a) The product topology is "associative" in the sense that the three topologies on the set $X_1 \times X_3 \times X_3$, obtained by thinking of it as $X_1 \times X_3 \times X_3$ or $(X_1 \times X_3) \times X_3$ or $X_1 \times (X_3 \times X_3)$ are equal.
(b) For any $i \in \{1, ..., n\}$ and any points $x_j \in X_j$, $j \neq i$, the map $X_i \to X_1 \times \cdots \times X_n$ given by

$$f(x) = (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)$$

is a topological embedding of X_i into the product space.
(c) Each canonical projection $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is an open map.
(d) If for each i, $\mathscr{B}_i$ is a basis for the topology of X_i, then the set

$$\{B_1 \times \cdots \times B_n : B_i \in \mathscr{B}_i\}$$

is abasis for the product topology.
(e) If S_i is a subspace of X_i for $i = 1, \ldots, n$, then the product topology and the subspace topology on $S_1 \times \cdots \times S_n \subseteq X_1 \times \cdots \times X_n$ are equal.
(f) If each X_i is Hausdorff, so is $X_1 \times \cdots \times X_n$.
(g) If each X_i is first countable, so is $X_1 \times \cdots \times X_n$.
(h) If each X_i is second countable, so is $X_1 \times \cdots \times X_n$.

(sol) 7.1 3.32

(a) Let $f((x_1, x_2), x_3) = (x_1, x_2, x_3)$. Bijection is clear. Now just need to prove continuity of f and f^{-1}. Let $Y = (x_1, x_2), x_3)$ in the characteristic property. Then $f_1 = \pi_1 \circ \pi_{1(X_1 \times X_2)}$, $f_2 = \pi_1 \circ \pi_{2(X_1 \times X_2)}$, $f_3 = id \circ \pi_{2(X_1 \times X_2)}$. Bascially a composition of projections. All f_is are continuous because all canonical projects are are continuous and compositions of continuous functions are continuous. Since all f_i are continuous, f is also continuous. The continuity of f^{-1} can be proved similarly.

(b) f is continuous because for every open set in the product space, in particular $f^{-1}(U_i)$ must be open by definition. f^{-1} is continuous because it is the canonical projection.
(c) Same reason as why f is continuous as in (b)
(d) (d) By definition the topology on the product space is the topology for which

$$\{U_1 \times \cdots \times U_n : U_i \text{ is open in } X_i\}$$

is a basis. Since each $\mathscr{B}_i$ is a basis, each U_i is a union of some family of B_is from $\mathscr{B}_i$. The result follows.
(e) Open-ness is defined on the product space by conjuction (ands) and subspace topology is the intersection of open sets with a subset – and since intersection boils down to conjuction of set elements – and since conjuction is associative and commutative, the result follows. In order words, conjuctions of intersections is equal to intersections of conjunctions (of open sets).
(f) Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two distinct points of the product space. By assumption, for all i, there exists U_i and V_i open neighborhoods of x_i and y_i respectively, such that $U_i \cap V_i$ is empty. Since all U_i and V_i are open, $U_i \times \cdots \times U_n$ and $V_i \times \cdots \times V_n$ are open subsets of the product space. Moreover they are disjoint since each component in the cartesian product is disjoint.
(g) If each point in X_i has a countable neighborhood basis, the cartesian product is also a neighborhood basis by definition of product topology, but it must also be countable because a k-cell of countable sets must also be countable from elementary set theory.
(h) Countability is preserved for same reason as in (g)

P 8 3.34

Suppose $f_1, f_2 : X \to \mathbb{R}$ are continuous functions. Their pointwise sum $f_1 + f_2 : X \to \mathbb{R}$ and pointwise product $f_1 f_2 : X \to \mathbb{R}$ are real-valued functions defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \qquad (f_1 f_2)(x) = f_1(x) f_2(x)$$

Use the characteristic property of the product topology to show that pointwise sum and products of continuous functions are continuous.

(sol) 8.1 3.34

Using the characteristic property, since f_1, f_2 are both continuous, then the mapping $f : X \to \mathbb{R} \times \mathbb{R}$ is also continuous. Since multiplication and addition are both continuous, i.e. $m, a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the composition of f and mult or add is also continuous. $\square$

P 9 3.38

(Characteristic property of Infinite product spaces)
Let $(X_\alpha)_{\alpha\in A}$ be an indexed family of topological spaces. For any topological space Y, a map $f : Y \to \prod_{\alpha\in A} X_\alpha$ is continuous if and only if each of its component functions $f_\alpha = \pi_\alpha \circ f$ is continuous. The product topology is the unique topology on $\prod_{\alpha\in A} X_\alpha$ that satisfies this property.

(sol) 9.1 3.38

The product topology is the only topology to satisfy this property because of the finite intersection of open sets are open property of topologies. The preimages of infinite spaces in the product space

P 10 3.40

Show that the disjoint union topology is indeed a topology.

(sol) 10.1 3.40

The whole disjoint space is open since the intersection with each X_α is equal to X_α which is open since X_α is a topological space. As is the empty set for the same reason. Let U, V be open sets in the disjoint union space. Then $U \cap V$ is open since $(X_\alpha \cap U) \cap (X_\alpha \cap V) = X_\alpha \cap (U \cap V)$. Openness of finite intersections follows from induction. An arbitrary union of open sets in the product space is also open since The X_α are all disjoint so each $X_\alpha \cap U$ will be open and an arbitrary union of those is open. $\square$

P 11 3.43

Let $(X_\alpha)_{\alpha\in A}$ be an indexed family of topological spaces.
(a) A subset of $\coprod_{\alpha\in A} X_\alpha$ is closed if and only if its intersection with each X_α is closed.
(b) Each canonical injection $\iota_\alpha : X_\alpha \to \coprod_{\alpha\in A} X_\alpha$ is a topological embedding and an open and closed map.
(c) If each X_α is Hausdorff, then so is $\coprod_{\alpha\in A} X_\alpha$.
(d) If each X_α is first countable, then so is $\coprod_{\alpha\in A} X_\alpha$.
(e) If each X_α is second countable and the index set A is countable, then $\coprod_{\alpha\in A} X_\alpha$ is second countable.

(sol) 11.1 3.43

(a) Assume that a subset C in the disjoint union space is closed. Then $\coprod_{\alpha\in A} X_\alpha \setminus C$ is open which mean that $X_\alpha \cap (\coprod_{\alpha\in A} X_\alpha \setminus C)$ is open in X_α. Which is equal to

$X_\alpha \cap (\coprod_{\alpha\in A} X_\alpha \cap C^{\mathsf{c}}) = (X_\alpha \setminus C) \cap \coprod_{\alpha\in A} X_\alpha$ being open in X_α. Which implies C is closed in X_α.
Let C be such that each intersection with each X_α is closed. Then $X_\alpha \setminus C$ is open. And the result follows similarly as above but in reverse.
(b) Since the second element of the pair is all the same if we restrict the co-domain to the image, we bascially get the identity map which is a homeomorphism and is both open and closed.
(c) Let each X_α be Hausdorff. Let x, y be distinct points in the disjoint union space. All X_αs are disjoint by definition so there are just two cases:
1) x, y are in the same X_α. Then since X_α is Hausdorff, there exists disjoint open neighborhoods of x and y respectively.
2) If x, y are not in the same X_α, then their respective X_α is the open neighborhoods that are by definition disjoint.
(d) Let x be any point of the disjoint union space. Then by definition it must be a point in exactly one X_α. Since X_α is first countable, there exists a countable neighborhood basis of x in X_α. By definition of disjoint union topology, that neighborhood basis in X_α is also a neighborhood basis in the disjoint union topology/space, so the disjoint union space is also first countable.
(e) This follows since if A is countable then we get a countable union of countable sets which is itself countable. □

P 12 3.44

Suppose $(X_\alpha)_{\alpha\in A} X_\alpha$ is an indexed family of nonempty n-manifolds. Show that the disjoint union $\coprod_{\alpha\in A} X_\alpha$ is an n-manifold if and only if A is countable.

(sol) 12.1 3.44

Hausdorffness and second-Countability follow from Proposition 3.42. Only need to show local euclidean. For each x, it will fall into exactly one X_α which is locally euclidian and thus so is the disjoint union space since the inclusion map is a topological embedding and is open and closed mapping. □

P 13 3.45

Let X be any space and Y be a discrete space. Show that the Cartesian product $X \times Y$ is equal to the disjoint union $\coprod_{y\in Y} X_y$, and the product topology is the sameas the disjoint union topology.

(sol) 13.1 3.45

It is clear that they have the same elements.
Since Y is a discrete space, every subset of Y is open. In particular the set of

singletons in Y form a basis for Y. Then $\{U, y\}$ where U is open in X and $y \in Y$ forms a basis for the product space. But then in the disjoint union space a set is open if the restriction is open, which boils down to $|Y|$ copies of Xs indexed by y, which is exactly the basis above. Thus they have the same topologies. □

P 14 3.46

Show that the quotient topology is indeed a topology.

(sol) 14.1 3.46

Empty set is open trivialy. The whole Y is open by surjection. And the fact that intersection and union pass through preimages imply that the quotient topology is a topology. □

P 15 3.55

Show that every wedge sum of Hausdorff spaces is Hausdorff.

(sol) 15.1 3.55

There are two cases. The case where neither distinct point is the collapsed base point, and the case where one is.
Case 1) If neither distinct points is the collapsed point, then Hausdorffness follows from Hausdorffness of disjoint union space.
Case 2) Let x be the collapsed base point and y be any other point. The preimage of x will then be $\{p_\alpha\}_{\alpha \in A}$. Then for each of these p_α, since they come from Hausdorff spaces, there exists an open neighborhood that is disjoint from the open neighborhood of y. Then the union of all the open neighborhoods of all the p_α is itself open, which means that image of the quotient map of such a set is also open in the quotient space and is disjoint from the open neighborhood of y. □

P 16

(sol) 16.1

P 17

(sol) 17.1