Chapter 3 Exercises

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P 1 Exercise 3.12

(c) If (p_i) is a sequence of points in S and $p \in S$, then $p_i \to p$ in S if and only if $p_i \to p$ in X.
(d) Every subspace of a Hausdorff space is Hausdorff.
(e) Every subspace of a first countable space is first countable.
(f) Every subspace of a second countable space is second countable.

(sol) 1.1 Exercise 3.12

(c) Assume $p_i \to p$ in S. Since for all open neighborhoods V_p in S, there exists U_p in X such that $V_p \subseteq U_p$, $p_i \to p$ in X as well.
Assume $p_i \to p$ in X. Since it is a sequence of points in S, by definition, for each open set intersect S, blah blah blah, $p_i \to p$ in S as well.
(d) Let X be Hausdorff. Let S be a subspace. Let $x, y \in S$. Then there exists U_x an U_y open sets of X such that they are disjoint. Since they are disjoint, intersecting them with S still leaves them disjoint.
(d) Every point has a neighborhood basis in X; taking the intersection of that neighborhood with S, each basis element intersected with S is open in S and by definition is a basis for that neighborhood.
(e) By part by, the basis for the subspace is the collection of basis elements intersected with S, which is countable since the original basis is countable.

P 2 Exercise 3.13

Let X be a topological space and let S be a subspace of X. Show that the inclusion map $S \hookrightarrow X$ is a topological embedding.

(sol) 2.1 Exercise 3.13

Restricting the co-domain to just S yields the bijective identity, which is a homeomorphism.

P 3 Exercise 3.14

Give an example of a topological embedding that is neither an open map nor a closed map.

(sol) 3.1 Exercise 3.14

The inclusion map is a topological embedding. So for any topological space X, choose a subset that is neither closed nor open, and thus the inclusion map from such a subset to X is neither open nor closed.

P 4 Exercise 3.15

A surjective topological embedding is a homeomorphism.

(sol) 4.1 Exercise 3.15

The restriction of a surjective topological embedding is the whole co-domain itself, thus the whole map is both injective and surjective so it's bijective and thus a homeomorphism with respect to the whole co-domain.

P 5 Exercise 3.25

$$\mathscr{B} = \{U_1 \times \cdots \times U_n : U_i \text{ is an open subset of } X_i,\ i = 1, \ldots, n\}$$

Prove that $\mathscr{B}$ is a basis for a topology.

(sol) 5.1 Exercise 3.25

Let $\mathscr{T}$ be the topology as follows: a subset of $X_1 \times \cdots \times X_n$ is open if and only if the projections into $X_1, \ldots, X_n$ are open.
By definition, each set in $\mathscr{B}$ is open. Let U, V be arbitrary open subsets of the product space. Then it's components U_is and V_is are all open in X_i. Then $U_i \cap V_i$ is open in X_i. Then

$$U_1 \cap V_1 \times \cdots \times U_n \cap V_n$$

is open in the product space and is contained in

$$(U_1 \times ... \times U_n) \cap (V_1 \times ... \times V_n)$$

Thus $\mathscr{B}$ is a basis for $\mathscr{T}$. $\square$

P 6 3.29

Prove the preceding corollary using only the characteristic property of the product topology.
If $X_1, ..., X_n$ are topological spaces, each canonical projection $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is continuous.

(sol) 6.1 3.29

Let $Y = X_1, ..., X_n$ and let f be the identity map. Since the identity is always continuous, each $f_i = \pi_i \circ f$ is also continuous. And since f is identity, $f_i = \pi_i$. So the result follows. $\square$

P 7 3.32

Prove proposition 3.31:
Let $X_1, ..., X_n$ be topological spaces.
(a) The product topology is "associative" in the sense that the three topologies on the set $X_1 \times X_3 \times X_3$, obtained by thinking of it as $X_1 \times X_3 \times X_3$ or $(X_1 \times X_3) \times X_3$ or $X_1 \times (X_3 \times X_3)$ are equal.

(sol) 7.1 3.32

(a) Let $f((x_1, x_2), x_3) = (x_1, x_2, x_3)$. Bijection is clear. Now just need to prove continuity of f and f^{-1}.

P 8

(sol) 8.1