Chapter 4 Exercises

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P 1 4.3

Suppose X is a connected topological space, and $\sim$ is an equivalence relation on X such that every equivalence class is open. Show that there is exactly one equivalence class, namely X itself.

(sol) 1.1 4.3

By definition, equivalence relations partition a set into disjoint subsets. If there were more than 1 non-empty partitions, then by proposition 4.1, X would be disconnected. $\square$

P 2 4.4

Prove that a topological space X is disconnected if and only if there exists a nonconstant continuous function from X to the discrete space $\{0, 1\}$.

(sol) 2.1 4.4

The contraposition of propostion 4.2 implies that if there exists a non-constant continuous function from X to $\{0, 1\}$, then X is disconnected. Thus we only need to show that if X is disconnected, then there exists a continuous non-constant function from $X \to \{0, 1\}$.
Let X be disconnected. Then $\exists U, V \subseteq X$ such that $X = U \cup V$ and $U \cap V = \emptyset$. Then the function f such that $f(U) = \{0\}$ and $f(V) = \{1\}$ is a continuous function because $\{0\}$ is both open and closed and so is U. Same for V and $\{1\}$. Furthermore f is not constant. $\square$

P 3 4.5

Prove that a topological space is disconnected if and only if it is homeomorphic to a disjoint union of two or more topological spaces.

(sol) 3.1 4.5

Let X be a disconnected topological space. Then $\exists U, V \subseteq X$ such that $X = U \cup V$ and $U \cap V = \emptyset$. Let $Y = U \coprod V$. Then the identity map from $X \to Y$ where if $x \in U$ $x \mapsto (0, x)$ and $y \in V, y \mapsto (1, y)$ is a homeomorphism between X and Y. It is clearly bijective, then the open sets are also bijective since both U and V are open, U and V considered as topological spaces themselves have the same open sets as the subspace topology, which has the same open sets as X intersect either U or V respectively.
Assume that X is homeomorphic to a disjoint union of two or more topological spaces, call it $Y = U \coprod V$. Then $f^{-1}(U)$ is open and $f^{-1}(V)$ is open and they are disjoint and their union is all of X. $\square$

P 4

(sol) 4.1