Chapter 2 Problems

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P 1 1

Let X be an infinite set.
(a) Show that

$$\mathscr{T}_1 = \{U \subseteq X : U = \emptyset \text{ or } X \setminus U \text{ is finite }\}$$

is a topology on X, called the finite complement topology.
(b) Show that

$$\mathscr{T}_2 = \{U \subseteq X : U = \emptyset \text{ or } X \setminus U \text{ is countable }\}$$

is a topology on X, called the countable complement topology.
(c) Let p be an arbitrary point in X, and show that

$$\mathscr{T}_3 = \{U \subseteq X : U = \emptyset \text{ or } p \in U\}$$

is a topology on X, called the particular point topology.
(d) Let p be an arbitrary point in X, and show that

$$\mathscr{T}_4 = \{U \subseteq X : U = X \text{ or } p \notin U\}$$

is a topology on X, called the excluded point topology.
(e) Determine whether

$$\mathscr{T}_5 = \{U \subseteq X : U = \emptyset \text{ or } X \setminus U \text{ is infinite }\}$$

is a topology on X.

(sol) 1.1 1

(a) By definition $\emptyset \in \mathscr{T}_1$. But also X is open since $\emptyset$ is a finite set. Let U_1 and U_2 be open sets. Then $X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2)$, both of which are finite which means their union is finite. Induction shows that finite intersection of open sets is open. Let $\bigcup_{\alpha \in A} U_\alpha$ be a union of open sets. Then $X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A}(X \setminus U_\alpha)$. Since each $(X \setminus U_\alpha)$ is finite, the intersection is finite.
(b) The prove for countable complement is exactly the same as finite since all the same conditions apply for countablility.

(c) Empty set and X are open by definition. Finite intersection of sets containing p will also contain p. Same for arbitrary union.
(d) X is open by definition. $\emptyset$ is open since it contains no points. The rest is the same as (c)
(e) No. Unions can become non infinite. Eg. Integers, even numbers and odd numbers. Complement is empty.

P 2 2

Let $X = \{1, 2, 3\}$. Give a list of topologies on X such that every topology on X is homeomorphic to exactly one on your list.

(sol) 2.1 2

TODO

P 3 3

Let X be a topological space and B be a subset of X. Prove that following set equalities.
(a) $\overline{X \setminus B} = X \setminus \, Int\, B$
(b) $Int(X \setminus B) = X \setminus \overline{B}$

(sol) 3.1 3

(a) $\overline{X \setminus B}$ is the exterior of B plus boundary, which is equivalent to X minus the interior of B.
(b) $Int(X \setminus B)$ is equal to the exterior of B, which is equal by definition to $X \setminus \overline{B}$.

P 4 4

Let X be a topological space and $\mathscr{A}$ be a collection of subsets of X. Prove the following containments:
(a)

$$\overline{\bigcap_{A \in \mathscr{A}} A} \subseteq \bigcap_{A \in \mathscr{A}} \overline{A}$$

(b)

$$\overline{\bigcup_{A \in \mathscr{A}} A} \supseteq \bigcup_{A \in \mathscr{A}} \overline{A}$$

(c)

$$Int\left(\bigcap_{A \in \mathscr{A}} A\right) \subseteq \bigcap_{A \in \mathscr{A}} Int\, A$$

(d)

$$Int\left(\bigcup_{A\in\mathscr{A}} A\right) \supseteq \bigcup_{A\in\mathscr{A}} Int\, A$$

When $\mathscr{A}$ is a finite collection, show that the equality holds in (b) and (c), but not necessarily in (a) or (d).

(sol) 4.1 4

(a) The righthand side is an intersection of closed sets containing A, and thus is closed. The left hand side the the closure of the intersection of As, which is the smallest closed set containing As so the containment follows.
(b) for all $A \in \mathscr{A}, A \subseteq \bigcup_{A\in\mathscr{A}} A$, which implies

$$A \in \mathscr{A}, \overline{A} \subseteq \overline{\bigcup_{A\in\mathscr{A}} A}$$

, which implies the resulting containment.
(c)

$$\forall A \in \mathscr{A}, \bigcap_{A\in\mathscr{A}} A \subseteq A$$
$$\implies \forall A \in \mathscr{A}, Int\left(\bigcap_{A\in\mathscr{A}} A\right) \subseteq Int\, A$$

Since this is for all A, the result follows.
(d) The lefthand side is a union of open sets contained in A, and the right hand side is the largest open set contained in union A, so the result follows.
When $\mathscr{A}$ is a finite collection, the unions of closures and intersections of interiors is closed and open respectively. And closed sets are equal to its closure and open sets its interior. TODO: idk about (a) and (b)

P 5 5

Too hard

(sol) 5.1

P 6 6

Prove Proposition 2.30(characterization of continuity, openness, and closedness in terms of closures and interiors).
Suppose X and Y are topological spaces, and $f : X \to Y$ is any map.
(a) f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.
(b) f is closed if and only if $f(\overline{A}) \supseteq \overline{f(A)}$ for all $A \subseteq X$.
(c) f is continuous if and only if $f^{-1}(\operatorname{Int} B) \subseteq \operatorname{Int} f^{-1}(B)$ for all $B \subseteq Y$.
(d) f is open if and only if $f^{-1}(\operatorname{Int} B) \supseteq \operatorname{Int}\, f^{-1}(B)$ for all $B \subseteq Y$.

(sol) 6.1 6

(a) Let $B \subseteq Y$ be closed. Then $B = \overline{B}$, and also $f^{-1}(B) = f^{-1}(\overline{B})$. There exists $A \subseteq X$ such that $A = f(B)$. Then $f(\overline{A}) \subseteq \overline{f(A)}$ implies $\overline{A} \subseteq f(\overline{B})$. Then $f^{-1}(B) \subseteq \overline{A} \subseteq f^{-1}(\overline{B})$, and since $f^{-1}(B) = f^{-1}(\overline{B})$, $\overline{A} = f^{-1}(B)$ so $f^{-1}(B)$ is closed.
Let f be continuous. Let $f(A) \subseteq Y$, so $\overline{f(A)}$ is closed. Since f is continuous, $f^{-1}(\overline{f(A)})$ is closed and contains A, thus it contains $\overline{A}$ since the closure of A is the smallest closed set that contains A.
(b) The proof is basically the same as part (a).
(c) Let $B \subseteq Y$. Since f is continuous, $f^{-1}(\text{Int } B)$ is open. And since $\text{Int } B \subseteq B$, $f^{-1}(\text{Int } B)$ is contained in $f^{-1}(B)$, so is contained in $\text{Int } f^{-1}(B)$.
Conversely, let $B \subseteq Y$ be open. Then $B = \text{Int } B$. Then we have $f^{-1}(B) \subseteq = f^{-1}(\text{Int } B) \subseteq \text{Int } f^{-1}(B)$. By definition, interiors are contained in the subset, thus $\text{Int } f^{-1}(B) = f^{-1}(B)$, so the preimage is open so f is continuous.
(d) same as part (c)

P 7 7

Prove proposition 2.39 (in a Hausdorff space, every neighborhood of a limit point contains infinitely many points of the set).

(sol) 7.1 7

Let X be Hausdorff. Let $p \in X$ be a limit point. Suppose there exists some neighborhood of p with only finitely many points. Since X is Hausdorff, we can separate any point in the neighborhood (not equal to p itself) with open sets. Then there exists some new neighborhood of p that does not contain any of the points in the original neighborhood, which means there exists a neighborhood of p that does not contain any other points, which contradicts the assumption that p is a limit point. $\square$

P 8 8

Let X be a Hausdorff space, let $A \subseteq X$, and let A' denote the set of limit points of A. Show that A' is closed in X.

(sol) 8.1 8

Will show that $X \setminus A'$ is open. Let $p \in X \setminus A'$. Let V_p be an open neighborhood of p such that $V_p \cap A$ is empty or only contains p itself. This exists because p is not a limit point of A. With Hausdorff property, and since p is an isolated point and thus $\{p\}$ is open, we have that $V_p \setminus \{p\}$ is also open. By construction, $V_p \setminus \{p\} \cap A$ is empty, which means all the points in V_p are also not limit points of A thus $V_p \subseteq X \setminus A'$, so A' is closed. $\square$

P 9 9

Suppose D is a discrete space, T is a space with the trivial topology, H is a Hausdorff space, and A is an arbitrary topological space.
(a) Show that every map from D to A is continuous.
(b) Show that every map from A to T is continuous.
(c) Show that the only continuous map from T to H are the constant maps.

(sol) 9.1 9

(a) Every possible subset in a discrete space is open, so any preimage is open, so the funciton has to be continuous.
(b) The empty preimage is trivially open. And the preimage of the whole domain is the whole codomain which is open.
(c) Suppose for the sake of contradiction that there exists a continuous map $f : T \to H$ such that f is not constant. Let V be an open set that contains more than 2 points, call them p, q. Since f is not constant we can let $f^{-1}(p) \neq f^{-1}(q)$, Then we can separate the 2 points with disjoint open sets. It follows that the preimages of these 2 open sets are not both empty, nor both the whole co-domain. Thus contradicting the assumption that T is a trivial space. $\square$

P 10 10

Suppose $f, g : X \to Y$ are continuous maps and Y is Hausdorff. Show that the set $\{x \in X : f(x) = g(x)\}$ is closed in X. Give a counterexample if Y is not Hausdorff.

(sol) 10.1 10

Will show that $\{x \in X : f(x) \neq g(x)\}$ is open. Let $x \in \{x \in X : f(x) \neq g(x)\}$. Since Y is Hausdorff, there exists disjoint open neighborhoods $V_{f(x)}$ and $V_{g(x)}$. Since both f, g are continuous, the preimages $f^{-1}(V_{f(x)}), g^{-1}(V_{g(x)})$, are also open. Call them U_f and U_g. $U_f \cap U_g$ contains x by construction and since both are open, the intersection is open. The last step is to show that

$$U_f \cap U_g \subseteq \{x \in X : f(x) \neq g(x)\}$$

Let $p \in U_f \cap U_g$ be arbitrary. Then since $V_{f(x)}$ and $V_{g(x)}$ are disjoint, $f(p) \neq g(p)$.
If Y is not Hausdorff, then the last part of the proof above may not be true. $\square$

P 11 11

Let $f : X \to Y$ be a continuous map between topologocial spaces, and let $\mathscr{B}$ be a basis of the topology on X. Let $f(\mathscr{B})$ denote the collection $\{f(B) : B \in \mathscr{B}\}$. Show that $f(\mathscr{B})$ is a basis for the topology of Y if and only if f is surjective and open.

(sol) 11.1 11

($\implies$) Assume $f(\mathscr{B})$ is a basis for the topology of Y. Then f is surjective since a basis covers the entire space Y. If f was not an open mapping, then there exists some $B \in \mathscr{B}$ such that $f(B)$ is not open. But this is not possible since by definition of a basis, of which $f(\mathscr{B})$ is, each element must be open.
($\impliedby$) Assume that f is surjective and open. Since f is open, each $f(B)$ is also open, fulfilling part of the definition of a basis. Now we just need to show that each open set in Y is a union of some $f(B)s$. Here we probably use the contiuity assumption. Let $V \subseteq Y$ be an open set. Since f is continuous, $f^{-1}(V)$ is also open. Since $\mathscr{B}$ is a basis for X, there exists some collection A such that $f^{-1}(V) = \bigcup_{\alpha \in A} B_\alpha$. So $V = f(f^{-1}(V)) = f(\bigcup_{\alpha \in A} B_\alpha) \subseteq f(\mathscr{B})$. Thus $f(\mathscr{B})$ is a basis for Y. $\square$

P 12 12

Suppose X is a set, and $\mathscr{A} \subseteq \mathscr{P}(X)$ is any collection of subsets of X. Let $\mathscr{T} \subseteq \mathscr{P}(X)$ be the collection of subsets consisting of $X, \emptyset$, and all unions of finite intersection of elements of $\mathscr{A}$.
(a) Show that $\mathscr{T}$ is a topology. (It is called the topology generated by $\mathscr{A}$, and $\mathscr{A}$ is called the subbasis for $\mathscr{T}$)
(b) Show that $\mathscr{T}$ is the coursest topologt for which all the sets in $\mathscr{A}$ are open.
(c) Let Y be any topological space. Show that a map $f : Y \to X$ is continuous if and only if $f^{-1}(U)$ is open in Y for every $U \in \mathscr{A}$.

(sol) 12.1 12

(a) By definition, $X, \emptyset$ are in $\mathscr{T}$. A finite intersection cannot break out of $\mathscr{T}$ because intersections can only make sets smaller. So picking just single sets out of arbitrary unions, and then taking their intersection is the best way to break out. However, by definition $\mathscr{T}$ is formed by unions of finite intersections, so by definition finite intersections of $\mathscr{T}$ are still in $\mathscr{T}$. An arbitrary union of elements of $\mathscr{T}$ is still an arbitrary union of of finite intersections of A.
(b) If any set is missing from $\mathscr{T}$, then it is no longer a topology, which means that $\mathscr{T}$ is the coarsest topology for which all sets in $\mathscr{A}$ is open.
(c) ($\implies$) Assume f is continuous. By definition every U is open so its preimage is open.
($\impliedby$) Assume $f^{-1}(U)$ is open in Y for every $U \in \mathscr{A}$. Every open set in X is an arbitrary union of finite intersections. Since $f^{-1}(U)$ is open, arbitrary unions of finite intersections of $f^{-1}(U)$s is also open. Thus every open set in X, the preimage is also open. $\square$

P 13 13

Let X be a totally ordered set. Give X the order topology, which is the topology generated by the subbasis consisting of all sets of the following forms for $a \in X$:

$$(a, \infty) = \{x \in X : x > a\},$$

$$(-\infty, a) = \{x \in X : x < a\}$$

(a) Show that each set of the form (a, b) is open in X and each set of the form $[a, b]$ is closed.
(b) Show that X is Hausdorff.
(c) For any pair of points $a, b \in X$ with $a < b$, show that $\overline{(a, b)} \subseteq [a, b]$.
(d) Show that the order topology on $\mathbb{R}$ is the same as the Euclidian topology.

(sol) 13.1 13

(a) Let (a, b) be an arbitrary "open" interval in X. It can be formed by taking the intersections $(a, \infty) \cap (-\infty, b)$. $[a, b]$ is closed since it can be formed by $X \setminus ((-\infty, a) \cup (b, \infty))$, the latter part of which is open, so the set difference is closed.
(b) Let $a \in X$. From the subbasis we can form open set: (a, a), i.e. the singleton set containing just a because it is the intersection of $(-\infty, a)$ and (a, ∞). Thus X is Hausdorff.
(c) $[a, b]$ is closed from part (a), and it contains (a, b) so the closure of (a, b) is contained in $[a, b]$.
(d) TODO!

P 14 14

Prove Lemma 2.48 (Sequence lemma).
Suppose X is a first countable space, A is any subset of X, and x is any point of X.
(a) $x \in \overline{A}$ if and only if x is a limit of a sequence of points in A.
(b) $x \in \text{Int } A$ if and only if every sequence in X converging to x is eventually in A.
(c) A is closed in X if and only if A contains every limit of every convergent sequence of point in A.
(d) A is open in X if and only if every sequence in X converging to a point of A is eventually in A.

(sol) 14.1 14

(a) ($\implies$) Assume $x \in \overline{A}$. If $x \in A$, then the constant sequence of x itself is a sequence of points in A that converges to x. If not, then x has to be a limit point of A. By definition of a limit point of A, for all open neighborhoods V_x of x, there exists a point $p \in A$ such that $p \in V_x$. Because the space is first countable, this applies to a countable nested-neighborhood of x. So we can take each x in our nested progression

as our sequence.
($\Longleftarrow$) Assume x is a limit of a sequence of points in A. By definition of a limit of a sequence, there exists $(x_i)_{i=1}^{\infty}$ such that for all open neighborhoods V_x of x, there exists some N such that if $i \geq N$, then $x_i \in V_x$. So either $x \in A$ for all open neighborhoods of x, there exists some point of A contained in that neighborhood, which is the definition of a limit point.
(b) ($\Longrightarrow$) Assume $x \in \text{Int } A$. Let $(x_i)_{i=1}^{\infty}$ be a sequence that converges to x. Since the interior is open, there exists some neighborhood V_x of x that is contained in Int A. Since $(x_i)_{i=1}^{\infty}$ converges to x, there exists some N such that if $i \geq N$, then $x_i \in V_x$. Thus the sequence is eventually in A since A contains its interior.
($\Longleftarrow$) Similary, if every sequence converging to x is eventually in A, then there exists an open neighborhood of x that is contained in A, thus the interior being the largest open set contained in A also contains such a neighborhood.
(c) If A is closed then $A = \overline{A}$ so then part (a) applies. Conversely, if part (a) applies then all points in A is also in $\overline{A}$ so $A = \overline{A}$ so A is closed.
(d) Same proof as part (c) □

P 15 15

Let X and Y be topological spaces. (a) Suppose $f : X \to Y$ is continuous and $p_n \to p$ in X. Show that $f(p_n) \to f(p)$ in Y.
(b) Prove that if X is first countable, the converse is true: if $f : X \to Y$ is a map such that $p_n \to p$ in X implies $f(p_n) \to f(p)$ in Y, then f is continuous.

(sol) 15.1 15

(a) Let V_p be an arbitrary open neighborhood of $f(p)$. Then $f^{-1}(V_p)$ is an open neighborhood of p. Since $p_n \to p$, there exists some N such that if $i \geq N$, $p_i \in f^{-1}(V_p)$, thus $f(p_i) \in V_p$.
(b) Assume X is first countable and that $f : X \to Y$ is a map such that $p_n \to p$ in X implies $f(p_n) \to f(p)$ in Y. Let $V \subseteq Y$ be open. Want to show $f^{-1}(V)$ is open. Let $p \in f^{-1}(V)$ and let $p_n \to p$ be an arbitrary sequence in X converging to p. By assumption, $f(p_n) \to f(p)$ in Y. Since V is open, $f(p_n)$ is eventually in V. This means that p_n is eventually in $f^{-1}(V)$ which implies $f^{-1}(V)$ is open. Thus f is continuous. □

P 16 16

Let X be a second countable topological space. Show that every collection of disjoint open subsets of X is countable.

(sol) 16.1 16

A collection of disjoint open subsets are contained in some basis. Since the basis must be countable, the collection of disjoint open subets must also be countable.

P 17 17

Let $\mathbb{Z}$ be the set of integers. Say that a subset $U \subseteq \mathbb{Z}$ is symmetric if it satisfies the following condition:

$$\text{for each n} \in \mathbb{Z}, n \in U \text{ if and only if } -n \in U$$

Define a topology on $\mathbb{Z}$ by declaring a subset open if and only if it is symmetric.
(a) Show that this is a topology.
(b) Show that it is second countable.
(c) Let A be the subet $\{-1, 0, 1, 2\} \subseteq \mathbb{Z}$, and determine the interior, boundary, closure, and limit points of A.
(d) Is A open in $\mathbb{Z}$? Is it closed?

(sol) 17.1 17

(a) $\emptyset$ is trivially open. $\mathbb{Z}$ is open since it contains all its negations. Let U, V be open. Then $U \cap V$ must also be open since if $n \in U \cap V$ then $n \in U$ and $n \in V$ which means $-n \in U$ and $-n \in V$. Arbitrary unions of open sets is also open since if n is contained in the union, then it must be contained in one of the compenent sets, which mean $-n$ is also contained in that set which means $-n$ is contained in the union.
(b) $\{\{i, -i\} : i \in \mathbb{N}\}$ forms a basis and is countable by construction.
(c) The interior is the largest open set contained in it so Int $A = \{-1, 1, 0\}$.
Boundary= $\{2, -2\}$?
Closure = $\{-1, 0, 1, 2, -2\}$
Limit points = $\{-1, 0, 1, 2, -2\}$?
(d) A is neither open nor closed since it is equal to neither its interior or closure.

P 18 18

(a) Show that $\mathbb{R}$ with the particular point topology is first countable and separable but not second countable or Lindelof.
(b) Show that $\mathbb{R}$ with the excluded point topology is first countable and Lindelof, but not second countable or separable.
(c) Show that $\mathbb{R}$ with the finite complement topology is separable and Lindelof but not first or second countable.

(sol) 18.1 18

(a) Any set containing p is open. Thus, the smallest closed set containing p is the whole space itself, thus it is separable. For any given point $x \in \mathbb{R}$, $\{p, x\}$ forms a countable neighborhood basis since every open neighborhood of x contains it.
Since $\{p, x\}$ is open for every $x \in \mathbb{R}$, there are an uncountable number of disjoint open sets, which means it cannot be second countable. Similarly, $\{p, x\}$ is open for every $x \in \mathbb{R}$ is an open cover, but is not countable. Removing any element will make it not longer a cover, so the space is not lindelof.
(b) The excluded point topology is first countable since for $x \neq p$, the singleton containing just x is a countable neighborhood basis. For the point p, the whole space is the only open neighborhood and so the whole space contains itself. Similarly, the only cover that this space has is the whole space itself, so a single subset is countable. The number of singletons is uncountable so it cannot be second countable. Similarly, there are an uncountable number of isolated points, so no closure of a countable subset can include the whole space.
(c) The rationals form a countable dense subset because they are not closed because they are not finite. So the smallest closed set containing the rationals must be the reals. The space is Lindelof because every open set must exclude just a finite number of elements. Thus every open cover can actually be reduced a finite subcover since we can just pick the open sets that include the points missing from previous open sets, which by construction is finite.
The reasoning for why the space is not first or second countable is the same, which is because there are an uncountable number of points to exclude. Each open set can only exclude a finite number of points. One cannot get to an uncountable number from countable unions of finite sets. $\square$

P 19 19

Let X be a topological space and let $\mathscr{U}$ be an open cover of X.
(a) Suppose we are given a basis for each $U \in \mathscr{U}$ (when considered as a topological space in its own right). Show that the union of all those bases is a basis of X.
(b) Show that if $\mathscr{U}$ is countable and each $U \in \mathscr{U}$ is second countable, then X is second countable.

(sol) 19.1 19

(a) Open sets in an open subspace is also open in the parent space. As such all basis subsets are open in the parent space as well. If an open set is fully contained in an element U of the open cover, then it is a union of basis elements of U. If it isn't then it intersects an element U of the open cover, and since both are open, the intersection is open and is contained in U, which can be built from the basis for U. Thus the union of the bases for all Us is a basis for the whole space.
(b) A countable union of countable sets is countable. $\square$

P 20 20

Show that second countability, separability, and the Lindelof property are all equivalent for metric spaces.

(sol) 20.1 20

Second countable $\implies$ seperable: If a metric space is second countable, then it has a countable basis of epsilon-balls. Pick the mid points of these balls. Then we have a countable set.

P 21 21

Show that every locally Euclidean space is first countable.

(sol) 21.1 21

Let X be a locally Euclidean space. Then for every point $x \in X$, there exists an open neighborhood U_x of x that is homeomorpic to $\mathbb{R}^n$. Since $\mathbb{R}^n$ is second countable, there exists a countable basis. Thus, there exists a countable neighborhood basis for x. Since x was arbitrary, X is first countable. $\square$

P 22 23

Show that every manifold has a basis of coordinate balls.

(sol) 22.1 23

Let X be a manifold. Let $\mathscr{B}$ be a basis. Then for each $B \in \mathscr{B}$, we have that each point $p \in B$ has an open neighborhood U_p that is homeomorphic to an open set in $\mathbb{R}^n$. Since B is open, there exists some open neighborhood V_p of p such that $V_p \subseteq B$. Then $U_p \cap V_p$ is open and contained in U_p which means $U_p \cap V_p$ is homeomorphic to some open set in $\mathbb{R}^n$, which means some subset of it is homeomorphic to some open ball in $\mathbb{R}^n$. Then union of all those open sets form a basis, each of which homeomorphic to an open ball in $\mathbb{R}^n$. $\square$

P 23 24

Suppose X is locally Euclidian of dimension n, and $f : X \to Y$ is a surjective local homeomorphism. Show that Y is also locally Euclidian of dimension n.

(sol) 23.1 24

P 24

(sol) 24.1