Chapter 7

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P 1 2

(sol) 1.1

Since both sequences converge uniformly (to f and g), fix ϵ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

for all x when $n > N$ and

$$|g_m(x) - g(x)| < \frac{\epsilon}{2}$$

for all x when $m > M$. Let $T = max(N, M)$. Then if $n > T$,

$$|f_n(x) + g_n(x) - f(x) - g(x)| = |f_n(x) - f(x) + g_n(x) - g(x)|$$

$$\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| = \epsilon$$

Next assume that both sequences are of bounded functions. From problem 1, we can conclude that both sequences are uniformly bounded, that is there exists T such that

$$|f_i(x)| < T$$

for all x and i and there exists P such that

$$|g_i(x)| < P$$

for all x and i. Then let $M = max(T, P)$. Then

$$|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)|$$

$$= |g_n(x)(f_n(x) - f(x)) + f(x)(g_n(x) - g(x))| \leq |M(f_n(x) - f(x)) + M(g_n(x) - g(x))|$$

$$\leq M|(f_n(x) - f(x))| + M|(g_n(x) - g(x))|$$

and since the sequences are uniformly continuous, for all $\epsilon > 0$

$$< M\epsilon + M\epsilon = 2M\epsilon$$

thus $\{f_n g_n\}$ is uniformly convergent. $\square$

P 2 3

Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E.

(sol) 2.1

$f_n(x) = x$ for all x converges to $f(x) = x$ uniformly with $\epsilon = 0$ for all n. $g_n(x) = \frac{1}{n}$ for all x is the same. So $f_n(x)g_n(x) = 0$. But then for any n, there exists an x equal to it, thus for all n there exists some x such that $f_n(x)g_n(x) = \frac{x}{n} = 1 > \epsilon$. $\square$

P 3 4

Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

(sol) 3.1

The series is almost convergent (comparing with $\frac{1}{n^2}$), except for a few points. When $x = 0$ we get $\sum_{n=1}^{\infty} 1$ which diverges. There are countable number of points where a term is undefined, namely $-\frac{1}{n^2}$ making the denominator 0.
For any $x > 0$ it converges uniformly since given some $\epsilon > 0$, $\frac{1}{1+n^2 x} \leq \frac{1}{n^2 \epsilon}$ and the right hand side converges uniformly. The same is true for $x < 0$ except for the aformentioned countable points.
Any interval containing 0 on the boundary does not converge uniformly. We can always pick an x so that the partial sums never converge.
f does not converge at $x = 0$ so it's not bounded.

P 4 5

Let

$$f_n(x) = \begin{cases} 0, & \left(x < \frac{1}{n+1}\right) \\ \sin^2 \frac{\pi}{x}, & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n}\right) \\ 0, & \left(\frac{1}{n} < x\right) \end{cases} \tag{1}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that the absolute convergence, even for all x, does not imply uniform convergence.

(sol) 4.1

The function converges (point-wise at least) to $f = 0$, since rationals are dense in reals, fixing any x there is always an n such that $f_n(x) = 0$.
The series cannot converge uniformly because for any n, there exists some m such that $x = \frac{1}{2m+\frac{1}{2}}$ such that $\frac{1}{n+1} < x < \frac{1}{n+1}$, thus $f(x) = \sin^2(2\pi m + \frac{\pi}{2}) = 1$. $\square$

P 5 6

Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

(sol) 5.1

Skip for now.

P 6 7

For $n = 1, 2, 3, ..., x$ real, put

$$f_n(x) = \frac{x}{1+nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

(sol) 6.1

Using the quotient rule and differentiating f_n and setting it equal to 0 shows that f_n has its max at $x = \frac{1}{\sqrt{n}}$, at which $f_n(x) = \frac{1}{2\sqrt{n}}$. Thus the sequence f_n is bounded by $\frac{1}{2\sqrt{n}}$ which converges uniformly to $f = 0$.
Since

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$$

so

$$f_n'(0) = 1$$

so $\lim_{n\to\infty} f_n'(x) = 1$ but since $f = 0$, $f' = 0$. □

P 7 8

If

$$I(x) = \begin{cases} 0, & \big(x \leq 0\big) \\ 1, & \big(x > 0\big) \end{cases} \tag{2}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \qquad (a \leq x \leq b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

(sol) 7.1

Since I is just an indicator function, it follows that

$$f_N(x) = \sum_{n=1}^{N} c_n I(x - x_n) \leq \sum_{n=1}^{N} |c_n|$$

. Since $\sum |c_n|$ converges absolutely, it follows that f_n converges uniformly.
f will be continuous if each f_n is. And each f_n is continuous if $x \neq x_n$, since if it "jumps" via the indicator function and misses or includes an extra term greater than some ϵ in the summation if so.

P 8 9

Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

(sol) 8.1

Decouple x_n from f_n and call it x_k. Then we have

$$\lim_{k\to\infty}\lim_{n\to\infty} f_n(x_k) = \lim_{k\to\infty} f(x_k)$$

by uniform convergence. Then

$$\lim_{k\to\infty} f(x_k) = f(x)$$

. Or more rigorously, Fix ϵ. Pick N such that if $n > N$, $|f_n(x) - f(x)| < \epsilon$ for all x (by uniform convergence). Then pick M such that if $m > M$, $|x_m - x| < \delta$ such that $|f(x_m) - f(x)| < \epsilon$ (since f is continuous). Then pick $k > max(N, M)$, thus

$$|f_k(x_k) - f(x)| = |f_k(x_k) - f(x_k) + f(x_k) - f(x)|$$

$$\leq |f_k(x_k) - f(x_k)| + |f(x_k) - f(x)| < 2\epsilon$$

Probem 5 is a counter example.

P 9 10

Letting (x) denote that fractional part of the real number x, consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \qquad (x \text{ real})$$

Find all discontinuities of f, and show that they form a countable dense set. Show that f is neverthless Riemann-integrable on every bounded interval.

(sol) 9.1

Discontinuities of f would include all discontinuities of the partial sums

$$f_N(x) = \sum_{n=1}^{N} \frac{(nx)}{n^2}$$

The discontinuities are when (nx) "jumps" back to 0. This happens in all rationals since there will always be an n equal to the denominator of x, thus f is discontinuities at all rationals. And the rationals are countable and dense in the reals.
f is Riemann-integrable if each f_n in the sequence is also Rieman-integrable. There are a finite number of discontinuities given an f_n since it is only a sum of N terms. Thus f is also Riemann-integrable. □

P 10 11

Suppose $\{f_n\}$, $\{g_n\}$ are defined on E, and
(a) $\sum f_n$ has uniformly bounded partial sums;
(b) $g_n \to 0$ uniformly on E;
(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq ...$ for every $x \in E$
Prove that $\sum f_n g_n$ converges uniformly on E. Hint: Compare with Theorem 3.42.

(sol) 10.1

Choose M such that $F_N = \sum_{n=1}^{N} f_n(x) < M$ for all x. Fix ϵ. From b), choose N such that $g_N < \frac{\epsilon}{2M}$. Then for $p, q \geq N$,

$$|\sum_{n=p}^{q} f_n g_n| = |\sum_{n=p}^{q-1} F_n(g_n - g_{n+1}) + F_q g_q - F_{p-1} g_p|$$

$$\leq M|\sum_{n=p}^{q-1}(g_n - g_{n+1}) + g_q + b_p|$$

$$= 2g_p M \leq 2g_N M \leq \epsilon$$

for all x. □

P 11 12

Suppose g and $f_n (n = 1, 2, 3, ...)$ are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$ and

$$\int_0^\infty g(x)\, dx < \infty$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx = \int_0^\infty f(x)\, dx$$

(sol) 11.1

Want to first show that $\int_0^\infty f_n(x)\, dx$ exists for all n. For $a < b$,

$$|\int_a^t f_n(x)\, dx \int_b^t f_n(x)\, dx| = |\int_b^a f_n(x)\, dx|$$

$$\leq \int_b^a |f_n(x)|\, dx \leq \int_b^a g(x)\, dx$$

And since $int_0^\infty g_n(x)\, dx$ exists, we can choose a, b such that $int_b^a g(x)\, dx < \epsilon$ for all ϵ. Similarly for the lower end point. It follows from Cauchy criterion that $\int_0^\infty f_n(x)\, dx$

exists.
Because of uniform convergence, $f \leq g$ and so $\int_0^\infty f(x)\,dx$ also exists. Then

$$\left| \int_0^\infty f_n(x)\,dx - \int_0^\infty f(x)\,dx \right|$$

$$= \left| \int_0^\infty f_n(x)\,dx - \int_a^b f_n(x)\,dx + \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx + \int_a^b f(x)\,dx - \int_0^\infty f(x)\,dx \right|$$

$$\leq \left| \int_0^\infty f_n(x)\,dx - \int_a^b f_n(x)\,dx \right|$$

$$+ \left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| + \left| \int_a^b f(x)\,dx - \int_0^\infty f(x)\,dx \right|$$

$$= \left| \int_0^\infty f_n(x)\,dx - \int_a^b f_n(x)\,dx \right|$$

$$+ \left| \int_a^b f_n(x) - f(x)\,dx \right| + \left| \int_a^b f(x)\,dx - \int_0^\infty f(x)\,dx \right|$$

With uniform convergence the middle term can be made arbitrarily small. The fact that the improper integrals exist means that first and last terms can also be made arbitrarily small. $\square$

P 12 13

(sol) 12.1

P 13

(sol) 13.1