Title

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The process described in part (c) of Exercise 22 can of course be applied to functions that map $(0, \infty)$ to $(0, \infty)$.
Fix some $\alpha > 1$, and put

$$f(x) = \frac{1}{2}(x + \frac{\alpha}{x}), \qquad g(x) = \frac{\alpha + x}{1 + x}$$

Both f and g have $\sqrt(\alpha)$ as their only fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g, why the convergence in Exercise 16, Chap. 3, is so much more rapid than it is in Exercise 17. (Compare f' and g', draw the zig-zags suggested in Exercise 22.)
Do the same when $0 < \alpha < 1$.

(sol) 1.1

Looking at the zig-zag, $f(x)$ does not spiral and stays on the positive side, and $g(x)$ goes back and forth, so presumably $f(x)$ converges quite a bit faster. For $\alpha < 1$, $g(x)$ no longer goes back and forth, but it's not clear if it converges as fast as $f(x)$. I want to say no because when x gets close to the fixed point, f will still be halved each time, but g does not have a constant rate at which it converges.

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Suppose f is twice diffrentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f"(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$.
Complete the following outline of Newton's method for computing ξ.
(a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of f.
(b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n\to\infty} x_n = \xi$$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.
(d) If $A = \frac{M}{2\delta}$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}$$

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does $g'(x)$ behave for x near ξ?
(f) Apply to $f(x) = x^{\frac{1}{3}}$

(sol) 2.1

(a) Geometrically this is taking the tangent line at x_n, finding the intersection with the x-axis, and applying that new x again.
(b) Since both $f(x_n)$ and $f'(x_n)$ are positive, $x_{n+1} < x_n$. Since $\{x_n\}$ is monotonically decreasing, either it diverges to $-\infty$ or converges to a limit point. Now for some x_n, we have by the mean value theorem

$$\exists z \in (x_{n+1}, x_n) : f(x_n) - f(x_{n+1}) = f'(z)(x_n - x_{n+1})$$

and by definition of f, $f'(x_n)(x_n - x_{n+1}) = f(x_n)$, and since f'' is always positive, $f'(z) < f'(x_n)$, thus

$$f(x_{n+1}) = f(x_n) - f'(z)(x_n - x_{n+1}) > f(x_n) - f'(x_n)(x_n - x_{n+1}) = f(x_n) - f(x_n) = 0$$

so for all $x_n, f(x_n) > 0$, so $f(x_n)$ converges to 0, so $\{x_n\}$ converges to ξ.
(c) Let $\beta = \xi$, $\alpha = x_n$, $n = 2$. Then Taylor's theorem state that there exist a $t_n \in (\xi, x_n)$ such that

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

$$\implies -f(x_n) - f'(x_n)(\xi - x_n) = \frac{f''(t_n)}{2}(-(x_n - \xi))^2$$

$$\implies -f(x_n) - f'(x_n)\xi + f'(x_n)x_n = \frac{f"(t_n)}{2}(x_n - \xi)^2$$

divide by $f'(x_n)$

$$\implies -\frac{f(x_n)}{f'(x_n)} - \xi + x_n = \frac{f"(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

$$\implies (x_n - \frac{f(x_n)}{f'(x_n)}) - \xi = \frac{f"(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

$$\implies x_{n+1} - \xi = \frac{f"(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

(d) Will prove by induction. The base case of $n = 1$ just follows from part c. The inductive step is as follows: using part c again

$$\exists t_{n+1} : x_{n+2} - \xi = \frac{f"(t_{n+1})}{2f'(t_{n+1})}(x_{n+1} - \xi)^2$$

$$\leq A(x_{n+1} - \xi)^2 \leq A(\frac{1}{A}(x_1 - \xi)^{2^n})^2 = \frac{1}{A}(x_1 - \xi)^{2^{(n+1)}}$$

$\square$(e) Follows directly from definition. $g'(x)$ goes to 1.
(f) it keeps jumping back and forth and doesnt converge

P 3

(sol) 3.1