Chapter 7

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P 1 2

(sol) 1.1

Since both sequences converge uniformly (to f and g), fix ϵ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

for all x when $n > N$ and

$$|g_m(x) - g(x)| < \frac{\epsilon}{2}$$

for all x when $m > M$. Let $T = max(N, M)$. Then if $n > T$,

$$|f_n(x) + g_n(x) - f(x) - g(x)| = |f_n(x) - f(x) + g_n(x) - g(x)|$$

$$\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| = \epsilon$$

Next assume that both sequences are of bounded functions. From problem 1, we can conclude that both sequences are uniformly bounded, that is there exists T such that

$$|f_i(x)| < T$$

for all x and i and there exists P such that

$$|g_i(x)| < P$$

for all x and i. Then let $M = max(T, P)$. Then

$$|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)|$$

$$= |g_n(x)(f_n(x) - f(x)) + f(x)(g_n(x) - g(x))| \leq |M(f_n(x) - f(x)) + M(g_n(x) - g(x))|$$

$$\leq M|(f_n(x) - f(x))| + M|(g_n(x) - g(x))|$$

and since the sequences are uniformly continuous, for all $\epsilon > 0$

$$< M\epsilon + M\epsilon = 2M\epsilon$$

thus $\{f_n g_n\}$ is uniformly convergent. $\square$

P 2 3

Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E.

(sol) 2.1

$f_n(x) = x$ for all x converges to $f(x) = x$ uniformly with $\epsilon = 0$ for all n. $g_n(x) = \frac{1}{n}$ for all x is the same. So $f_n(x)g_n(x) = 0$. But then for any n, there exists an x equal to it, thus for all n there exists some x such that $f_n(x)g_n(x) = \frac{x}{n} = 1 > \epsilon$. $\square$

P 3 4

Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

(sol) 3.1

The series is almost convergent (comparing with $\frac{1}{n^2}$), except for a few points. When $x = 0$ we get $\sum_{n=1}^{\infty} 1$ which diverges. There are countable number of points where a term is undefined, namely $-\frac{1}{n^2}$ making the denominator 0.
For any $x > 0$ it converges uniformly since given some $\epsilon > 0$, $\frac{1}{1+n^2 x} \leq \frac{1}{n^2 \epsilon}$ and the right hand side converges uniformly. The same is true for $x < 0$ except for the aforementioned countable points.
Any interval containing 0 on the boundary does not converge uniformly. We can always pick an x so that the partial sums never converge.
f does not converge at $x = 0$ so it's not bounded.

P 4 5

Let

$$f_n(x) = \begin{cases} 0, & \left(x < \frac{1}{n+1}\right) \\ \sin^2 \frac{\pi}{x}, & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n}\right) \\ 0, & \left(\frac{1}{n} < x\right) \end{cases} \tag{1}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that the absolute convergence, even for all x, does not imply uniform convergence.

(sol) 4.1

P 5

(sol) 5.1