Title

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P 1 1

Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that f is Riemann integrable and that $\int f \, d\alpha = 0$.

(sol) 1.1

Since f is bounded and only has 1 discontinuity, and since α is continuous where f is not, theorem 6.10 applies and f is Riemann integrable.
Because f is bounded, and α is continuous at x_0, $\Delta\alpha \to 0$ implies $\int f \to 0$. $\square$

P 2 2

Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x) \, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this to Exercise 1).

(sol) 2.1

Since $f \geq 0$, let $f(x_0) = M > 0$. Since f is continuous, by the intermediate value theorem, it assumes a point $f(x_1) = T < M$. Then there exists a partition such that $L(P, f) > 0$ which implies $\int_a^b f(x) \, dx > 0$ which is a contradiction. $\square$

P 3 3

Define three functions: $\beta_1, \beta_3, \beta_3$ as follows: $\beta_j = 0$ if $x < 0$, $\beta_j = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0, \beta_2(0) = 1, \beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.
(a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f \, d\beta_1 = f(0)$$

(b) State and prove a similar result for β_2.
(c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.
(d) If f is continuous at 0 prove that

$$\int f\,d\beta_1 = \int f\,d\beta_2 = \int f\,d\beta_3 = f(0)$$

(sol) 3.1

(a) Assume $f(0+) = f(0)$. We only need to consider an open neighborhood of 0 since theorem 6.10 takes care of everywhere else. Fix $\epsilon > 0$ and let $\delta > 0$ such that $x < \delta \implies |f(x) - f(0)| < \epsilon$. Let P be a partition such that $\Delta x_i < \delta$ for all i. We only need to consider $x \geq 0$. In particular only the partition such that $x_i = 0$. Then

$$U(P, f, \beta_1) - L(P, f, \beta_1) = (M_i - m_i)\Delta\beta_1(x_i) = (M_i - m_i) < \epsilon$$

And thus $f \in \mathcal{R}(\beta_1)$.
Assume $f \in \mathcal{R}(\beta_1)$. If $f(0+) \neq f(0)$, then we see that it $\Delta\beta_1$ around a neighborhood of 0+ is always 1, so the upper and lower sums never converge to each other.
(b) same as a
(c) same as a and b
(d) follows from fact that if a limit exists, the left and right hand limits are equal.

P 4 4

If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that f is not integrable on $a[b]$ for any $a < b$.

(sol) 4.1

The rationals and irrationals are dense in each other, thus the upper sum is always equal to 1 while the lower sum is always 0. $\square$

P 5 5

Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?

(sol) 5.1

No. Take the last problems function but let $f(x) = 1$ if x is rational and -1 if irrational. f^2 is the constant function 1 which is integrable, but f is not. If f^3 is integrable then so is f since the cube root is continuous on the real line. $\square$

P 6 6

Let P be the Contor set constructed in Sec. 2.44. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P. Prove that $f \in \mathcal{R}$ on $[0, 1]$. HINT: P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.

(sol) 6.1

Cantor set has measure 0. Rest follows as in 6.10.

P 7 7

Suppose f is a real function on $(0, 1]$ and $f \in \mathcal{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x)\,dx = \lim_{c \to 0} \int_c^1 f(x)\,dx$$

if this limit exists (and is finite).
(a) If $f \in \mathcal{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.
(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

(sol) 7.1

Skip for now

P 8 8

Suppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

if this limit exists (and is finite). Int that case, we say that the integral on the left converges. If it also converges after f has been replaced by $|f|$, it is said to converge absolutely.
Assume that $f(x) \geq 0$ and that f decreases monotonically on $[, \infty)$. Prove that

$$\int_1^\infty f(x)\,dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series)

(sol) 8.1

Assume $\sum_{n=1}^{\infty} f(n)$ converges. Then so does $\int_1^{\infty} f(x)\,dx$ since the partition $\{1, 2, ..., n\}$ has an Upper sum equal to the sum. Lower sums yield the other way. □

P 9 9

skip for now

(sol) 9.1

P 10 10

Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Prove that following statements.
(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.
(b) If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq$, $g \geq 0$, and

$$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha$$

then

$$\int_a^b fg \, d\alpha \leq 1$$

(c) If f and g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}$$

(d) Show that Holder's inequality is also true for the "improper" integrals described in exercise 7 and 8.

(sol) 10.1

(a)

$$\ln(uv) = \ln(u) + \ln(v) = \frac{\ln(u) * p}{p} + \frac{\ln(v) * q}{q}$$

$$= \frac{\ln(u^p)}{p} + \frac{\ln(v^q)}{q} \leq \ln(\frac{u^p}{p} + \frac{v^q}{q})$$

The last inequality due to concavity of ln.
(b) from part a let

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

then

$$\int_a^b fg \leq \int_a^b \frac{f^p}{p} + \int_a^b \frac{g^q}{q}$$
$$= \frac{\int_a^b f^p}{p} + \frac{\int_a^b g^q}{q}$$
$$= \frac{1}{p} + \frac{1}{q} = 1$$

(c) replace f and g with $\frac{|f|}{(\int_a^b |f|^p \, d\alpha)^{1/p}}$ and $\frac{|g|}{(\int_a^b |g|^q \, d\alpha)^{1/q}}$ and integrate both sides. The right hand side becomes 1, and then multipling both sides by the left hand side's denominator, and the result follows.
(d) skip for now.

P 11 11

Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$||u||_2 = \left\{ \int_a^b |u|^2 \, d\alpha \right\}^{1/2}$$

Suppose $f, g, h \in \mathcal{R}(\alpha)$ and prove the triangle inequality

$$||f - h||^2 \leq ||f - g||_2 + ||g - h||_2$$

(sol) 11.1

$$||f - h||_2 = \left(\int_a^b |f - h|^2 \right)^{1/2}$$
$$= \left(\int_a^b |(f - g) + (g - h)|^2 \right)^{1/2}$$
$$= \left(\int_a^b |(f - g)|^2 + 2|f - g||g - h| + |(g - h)|^2 \right)^{1/2}$$
$$= \left(\int_a^b |(f - g)|^2 + 2 \int_a^b |f - g||g - h| + \int_a^b |(g - h)|^2 \right)^{1/2}$$

$$\leq \left(||(f-g)||_2 + 2||f-g||_2||g-h||_2 + ||(g-h)||_2 \right)^{1/2}$$

$$= \left((||(f-g)|| + ||(g-h)||)^2 \right)^{1/2}$$

$\square$

P 12 12

With the notations of Exercise 11, suppose $f \in \mathcal{R}(\alpha)$ and $\epsilon > 0$. Prove that there exists a continuous function g on $[a,b]$ such that $||f-g||_2 < \epsilon$.
Hint: Let $P = \{x_0, ..., x_n\}$ be a suitable partition of $[a,b]$, define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$.

(sol) 12.1

g is continuous since it's a linear combination of f which is integrable so with respect to some partition, g is continuous. Then

$$f(t) - g(t) = \frac{x_i - t}{\Delta x_i}(f(t) - f(x_{i-1})) + \frac{t - x_{i-1}}{\Delta x_i}(f(t) - f(x_i))$$

Then

$$\int_a^b |f-g|^2 \, d\alpha = \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} |f-g|^2 \, d\alpha$$

since f is integrable, we can choose a partition such that $M_i - m_i < 2M$ where M is $sup(f)$, and $\Delta\alpha(x) < \frac{\epsilon^2}{4M^2}$

$$\leq \sum_{i=0}^{N} (M_i - m_i)^2 \Delta\alpha$$

$$\leq 4M^2 \sum_{i=0}^{N} \Delta\alpha$$

$$\leq 4M^2 (\frac{\epsilon^2}{4M^2}) = \epsilon^2$$

$\square$

P 13 13

Define

$$f(x) = \int_x^{x+1} sin(t^2)dt$$

(a) Prove that $|f(x)| < \frac{1}{x}$ if $x > 0$.
Hint: put $t^2 = u$ and integrate by parts to show that $f(x)$ is equal to

$$\frac{\cos(x^2)}{2x} - \frac{cos[x+1]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}$$

(b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where $|r(x)| < c/x$ and c is a constant.
(c) Find the upper and lower limits of $xf(x)$ as $x \to \infty$.
(d)

(sol) 13.1

(a) Using u substitution and integration by parts we can get the result as in the hint. Then since *cos* attains its max of 1 we let $\cos u = 1$ and integrate the remaining integral and plug in the ends points which grants

$$\frac{1 + cos(x^2)}{2x} - \frac{1 + cos[(x+1)^2]}{2(x+1)}$$

the second term can never be negative since so the maximum of the whole expression is when it is zero so

$$\leq \frac{1 + cos(x^2)}{2x}$$

similarly cosine's max is 1 so

$$\leq \frac{1+1}{2x} = \frac{1}{x}$$

The fact that cosine's min is -1 yields that

$$f(x) \geq -\frac{1}{x}$$

(b) multiply the right hand side of $f(x)$ by $2x$ and do the same thing as in (a), and the last integral can again be bounded.
(c)

P 14 14

basically same problem as 13, but converges faster.

(sol) 14.1

P 15 15

Suppose f is a real, continuously diffrentiable function on $[a,b]$, $f(a)=f(b)=0$, and

$$\int_a^b f^2(x)\,dx = 1$$

Prove that

$$\int_a^b xf(x)f'(x)\,dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2\,dx \cdot \int_a^b x^2 f^2(x)\,dx > \frac{1}{4}$$

(sol) 15.1

$$\int_a^b F(x)g(x) = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)$$

Integrate by parts and let $F = xf(x), g = f'(x)$, then

$$\int_a^b xf(x)f'(x)\,dx = -\int_a^b f(x)(xf'(x)+f(x))\,dx$$

so we have

$$\int_a^b xf(x)f'(x)\,dx = -\int_a^b xf'(x)f(x)\,dx - \int_a^b f^2(x)\,dx$$

$$2\int_a^b xf(x)f'(x)\,dx = -1$$

$$\int_a^b xf(x)f'(x)\,dx = -\frac{1}{2}$$

The inequality follows from Holder's inequality as proved in problem 10. □

P 16 16

For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(This is Riemann's zeta function, of great importance in the study of the distribution of prime numbers.) Prove that
(a) $\zeta(s) = s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx$
and that (b) $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}\,dx$
where $[x]$ denotes the greatest integer $\geq x$.
Prove that the integral in (b) converges for all $s > 0$.

(sol) 16.1

(a) the floor function makes x constant between integer intervals, thus

$$s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx = s\sum_{n=1}^\infty n\cdot\int_n^{n+1}\frac{1}{x^{s+1}}\,dx$$

$$= s\sum_{n=1}^\infty n\cdot\Big[-\frac{1}{s}\cdot\frac{1}{x^s}\Big]_n^{n+1}$$

$$= s\frac{1}{s}\sum_{n=1}^\infty n\cdot(\frac{1}{n^s}-\frac{1}{(n+1)^s})$$

the sum partially telescopes into the wanted result.
(b) By linearity of integrals, we only need to show that

$$\frac{s}{s-1} = s\cdot\int_1^\infty\frac{1}{x^s}$$

which is clear. Convergence is also clear from b.

P 17 17

Suppose α monotonically on $[a,b]$, g is continuous, and $g(x)=G'(x)$ for $a\leq x\leq b$. Prove that

$$\int_a^b \alpha(x)g(x)\,dx = G(b)\alpha(b)-G(a)\alpha(a)-\int_a^b G\,d\alpha$$

Hint: Take g real, without loss of generality.

(sol) 17.1

P 18

(sol) 18.1