Title

Author

Date

P 1   1

Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is constant.

(sol) 1.1

Divide both sides by $(x - y)$:

$$\frac{|f(x) - f(y)|}{(x - y)} \leq (x - y)$$

for all x, y. Which implies $f'(x) = 0$ for all x. $\qquad \square$

P 2   2

Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b)$$

(sol) 2.1

Since $f'(x) > 0$ for all $x \in (a, b)$, $\lim_{t \to x} \frac{f(t) - f(x)}{t - x} > 0$ for all x. Pick $\delta > 0$ such that $a < x - \delta < t < x < b$. Then $f(t) - f(x) < 0$ since $t < x$, which implies $f(t) < f(x)$. Want to show that g is differentiable, i.e., for all x in (a, b)

$$\lim_{t \to x} \frac{g(f(t)) - g(f(x))}{f(t) - f(y)}$$

exists.

The above limit is equal to

$$\lim_{t \to x} \frac{t - x}{f(t) - f(x)}$$

which exists by limit laws, and is equal to $\frac{1}{f'(x)}$. $\qquad \square$

P 3 3

Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \le M$). Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissible values of ϵ can be determined which depends only on M.)

(sol) 3.1

Want to show $x \ne y \implies f(x) \ne f(y)$. Let $x \ne y$. Fix $\epsilon < \frac{1}{M}$. Suppose to the contrary that there exists x, y such that $x \ne y f(x) = f(y)$.

$$x + \frac{g(x)}{M} = y + \frac{g(y)}{M}$$

which implies

$$(x - y) = \frac{g(y) - g(x)}{M} \implies M = |\frac{g(y) - g(x)}{(x - y)}|$$

which contradicts the fact that $M > g'$. $\qquad \square$

P 4 4

If

$$C_0 + \frac{C_1}{2} + ... + \frac{C_{n-1}}{n} + \frac{C_n}{n + 1} = 0$$

where $C_0, ..., C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + ... + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

(sol) 4.1

Let $P(x)$ be defined:

$$P(x) = C_0 x + \frac{C_1}{2} x^2 + ... + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n + 1} x^{n+1}$$

then $P'(x) = C_0 + C_1 x + ... + C_{n-1} x^{n-1} + C_n x^n$. Also, $P(0) = 0$ is obvious, and $P(1) = 0$ by hypothesis. Since $P(x)$ $(0 < x < 1)$ is bounded, it acheives either a local min or max, which implies $P'(x) = 0$ for some $x \in (0, 1)$. $\square$ Also Rolle's theorem.

P 5   5

Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

(sol) 5.1

Fix $\epsilon > 0$. Then there exists a N such that if $x > N$ then $|f'(x)| < \epsilon$. Using the mean value theorem, there exists a $x' \in (x, x+1)$ such that $f(x+1) - f(x) = (x+1-x)f'(x')$. Since $x' > N$ we have that $|f'(x')| < \epsilon$. $\qquad\square$

P 6   6

Suppose
(a) f is continuous for $x \geq 0$,
(b) $f'(x)$ exists for $x > 0$,
(c) $f(0) = 0$,
(d) f' is monotonically increasing.
Put

$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$

and prove that g is monotonically increasing.

(sol) 6.1

Diffentiating $g(x)$ with the quotient rule gives

$$\frac{x f'(x) - f(x)}{x^2}$$

Using the mean value theorem, for all x, there exists some $x' \in (0, x)$

$$f(x) - f(0) = (x - 0)f'(x') \implies f(x) = x f'(x')$$

Since f' is monotonically increasing, $x > x'$ so $f(x) < x f'(x)$. Thus

$$\frac{x f'(x) - f(x)}{x^2} > 0 \implies g'(x) > 0$$

for all x. $\qquad\square$

P 7   7

Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$$

(This holds also for complex functions.)

(sol) **7.1**

$$lim_{t \to x} \frac{f(t)}{g(t)} = lim_{t \to x} \frac{f(t) - f(x)}{g(t) - g(x)}$$

$$= lim_{t \to x} \frac{\frac{f(t) - f(x)}{t-x}}{\frac{g(t) - g(x)}{t-x}}$$

$$= \frac{lim_{t \to x} \frac{f(t) - f(x)}{t-x}}{lim_{t \to x} \frac{g(t) - g(x)}{t-x}}$$

$$= \frac{f'(x)}{g'(x)}$$

$\square$

P 8   8

Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \le x \le b$, $a \le t \le b$. (This could be expressed by saying that f is uniformly differentiable on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

(sol) **8.1**

Since f' is continuous, fix $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $|y - x| < \delta$ then $|f'(y) - f'(x)| < \epsilon$. Let $0 < |t - x| < \delta$. By the mean value theorem there exists a z between t and x such that:

$$f(t) - f(x) = (t - x)f'(z)$$

for some z in $[a, b]$. So $\frac{f(t) - f(x)}{t-x} = f'(z)$. Since $|z - x| < \delta$ we have that $|f'(z) - f'(x)| < \epsilon$. $\square$

P 9   9

Let f be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \ne 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

(sol) **9.1**

$$f'(0) = \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0}$$

Using L'hopital's rule:

$$\lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} \frac{f'(t) - f(0)'}{t'}$$

$$= \lim_{t \to 0} \frac{f'(t)}{1} = 3$$

□

P 10   **10**

Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where A and B are complex numbers, $B \neq 0$. Prove that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{A}{B}$$

Compare with Example 5.18. Hint:

$$\frac{f(x)}{g(x)} = \left\{ \frac{f(x)}{x} - A \right\} \cdot \frac{x}{g(x)} + A \cdot \frac{x}{g(x)}$$

Apply theorem 5.13 to the real and imaginary parts of $f(x)/x$ and $g(x)/x$.

(sol) **10.1**

Not sure what it means to apply to the real and imaginary parts. But if we do that, then it follows that $\lim_{x \to 0} \frac{f(x)}{x} = A$ and $\lim_{x \to 0} \frac{g(x)}{x} = B$. If we plug it into the big equation above, we get our conclusion, along with using the inverse of a limit rule thing. □.

P 11   **11**

Suppose f is defined in a neighborhood of x, and suppose $f"(x)$ exists. Show that

$$\lim h \to 0 \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f"(x)$$

Show by example that the limit may exist even if $f"(x)$ does not. Hint: Use Theorem 5.13.

(sol) 11.1

Applying theorem 5.13 once we get that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

applying again:

$$= \lim_{h\to 0} \frac{f''(x+h) + f''(x-h)}{2} = f''(x)$$

P 12   12

If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$, for all real x, and show that $f^{(3)}(0)$ does not exist.

(sol) 12.1

$f'(x) = 3|x|^2$, $f''(x) = 6|x|$ And $f^{(3)}(0)$ does not exist since the left and right hand limits of the derivative of the second derivative are not equal, therefore the limit itself does not exist. □

P 13   13

Skip for now.

(sol) 13.1

P 14   14

Let f be a diffentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex iff and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

(sol) 14.1

($\implies$) Suppose f is convex. Want to show that f' is monotonically increasing. Skip for now.

P 15   15

Suppose $a \in \mathbb{R}^1$, f is a twice-diffentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a, ∞).
Prove that

$$M_1^2 \leq 4M_0 M_2$$

(sol) 15.1

Following the hint, let $h > 0$, by Taylor's theorem:

$$f'(x) = \frac{1}{2h}[f(x+h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence

$$|f'(x)| \le hM_2 + \frac{M_0}{h}$$

which implies

$$|f'(x)|^2 \le h^2 M_2^2 + 2M_0 M_2 + \frac{M_0^2}{h^2}$$

Since h was arbitrary, let $h = \sqrt{\frac{M_0}{M_2}}$, which is greater than 0. Then

$$|f'(x)|^2 \le \frac{M_0}{M_2}M_2^2 + 2M_0 M_2 + M_0^2 \frac{M_2}{M_0}$$

$$|f'(x)|^2 \le 4M_0 M_2$$

□

P 16   16

Suppose f is twice-diffentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$. Hint: Let $a \to \infty$ in Exercise 15.

(sol) 16.1

Following the hint, if we let $a \to \infty$ in 15, we get that $M_0 \to 0$ and since $M_1^2 \le 4M_0 M_2$, it follows that $f'(x)$ goes to zero as $x \to \infty$.   □

P 17   17

Suppose f is a real, three times diffentiable function on $[-1, 1]$, such that

$$f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0$$

Prove that $f^{(3)}(x) \ge 3$ for some $x \in (-1, 1)$.
Note that equality holds for $\frac{1}{2}(x^3 + x^2)$.
Hint:Use Theorem 5.15, with $\alpha = 0$ and $\beta = \pm 1$, to show there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6$$

(sol) 17.1

Following the hint, we apply Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

Let $n = 3$.
With $\alpha = 0, \beta = 1$:

$$f(1) = P(1) + \frac{f^{(3)}(s)}{3!}(1)^3$$

$$1 = P(1) + \frac{f^{(3)}(s)}{3!}(1)^3$$

$$1 = \frac{f''(\alpha)}{2} + \frac{f^{(3)}(s)}{6}$$

With $\alpha = 0, \beta = -1$:

$$f(-1) = P(-1) + \frac{f^{(3)}(t)}{3!}(-1)^3$$

$$0 = P(-1) + \frac{f^{(3)}(t)}{3!}(-1)^3$$

$$0 = \frac{f''(t)}{2} - \frac{f^{(3)}(x)}{6}$$

Subtracting the latter from the former:

$$1 = \frac{f^{(3)}(s)}{6} + \frac{f^{(3)}(x)}{6}$$

$$6 = f^{(3)}(s) + f^{(3)}(x)$$

$\square$

P 18   18

Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α, β and P be as in Taylor's Theorem (5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, diffentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's Theorem:

$$f(\beta) = P(\beta)\frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n$$

(sol) **18.1**

We can show through induction that:

$$f^{(k)}(t) = (t - \beta)Q^{(k)}(t) + kQ^{(k-1)}(t)$$

Let $k = 1$. From the product rule, we get:

$$f'(t) = (t - \beta)Q'(t) + Q(t)$$

and that completes the basis case.

$$f^{(k)}(t) = (t - \beta)Q^{(k)}(t) + kQ^{(k-1)}(t)$$
$$f^{(k+1)}(t) = (t - \beta)Q^{(k+1)}(t) + Q^{(k)}(t) + kQ^{(k)}(t)$$

by the product rule again.
Multiplying both sides by $\frac{(\beta-\alpha)^k}{k!}$

$$\frac{(\beta - \alpha)^k}{k!}f^{(k)}(\alpha) = -\frac{(\beta - \alpha)^{(k+1)}}{k!}Q^{(k)}(\alpha) + \frac{(\beta - \alpha)^k}{(k-1)!}Q^{(k-1)}(\alpha)$$

Summing from 1 to $n - 1$ the right hand side telescopes to get:

$$-\frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n + f(\beta) - f(\alpha)$$

we move $-f(\alpha)$ over to the left hand side and we get:

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k = -\frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n + f(\beta)$$

The left hand side is $P(\beta)$, and we arrive at our conclusion. $\square$

P 19   **19**

Suppose f is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference equation;

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$$

Prove the following statements:
(a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.
(b) If $0 < \alpha_n < \beta_n$ and $\{\frac{\beta_n}{(\beta_n - \alpha_n)}\}$ is bounded, then $\lim D_n = f'(0)$.
(c) If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.
Give an example in which f is diffentiable in $(-1, 1)$ (but f' is not continuous at 0) and which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

(sol) 19.1

(a) Fix $\epsilon > 0$, since $f'(0)$ exists, a δ such that if $0 < |x| < \delta$ then

$$\frac{f(x) - f(0)}{x - 0} - f'(0) < \epsilon$$

D_n can be rewritten as $D_n = \frac{f(\beta_n) - f(0) + f(0) - f(\alpha_n)}{\beta_n - \alpha_n}$

$$= \frac{f(\beta_n) - f(0)}{\beta_n - \alpha_n} - \frac{f(\alpha_n) - f(0)}{\beta_n - \alpha_n} = \frac{\beta_n}{\beta_n - \alpha_n}\frac{f(\beta_n) - f(0)}{\beta_n} - \frac{\alpha}{\beta_n - \alpha_n}\frac{f(\alpha_n) - f(0)}{\alpha}$$

Then we can choose $\alpha > -\delta$ and $\beta < \delta$ so that

$$|D_n - f'(0)| \leq \frac{\beta_n}{\beta_n - \alpha_n}|\frac{f(\beta_n) - f(0)}{\beta_n} - f'(0)|$$

$$-\frac{\alpha}{\beta_n - \alpha_n}|\frac{f(\alpha_n) - f(0)}{\alpha} - f'(0)| \leq \frac{\beta_n}{\beta_n - \alpha_n}\epsilon - \frac{\alpha}{\beta_n - \alpha_n}\epsilon = \epsilon$$

$\square$

(b) How is this different from a?
Edit: Okay I see, because in a $\alpha_n < 0 < \beta_n$ the denominator is always greater than the numerator so the thing is always bounded. So here we can do something similar in a, but use the assumption of boundedness to get a delta. Choose δ such that the distance is less than $\frac{\epsilon}{2M}$. (c) Since f' is continuous, $D_n = f'(x_n)$ and $x_n \to 0$ implies $D_n = f'(0)$.

$\square$

P 20   20

Problem too vague. Skip for now.

(sol) 20.1

P 21   21

Let E be a closed subset of $\mathbb{R}^1$. We saw in Exercise 22, Chap. 4, that there is a real continuous function f on $\mathbb{R}^1$ whose zero set is E. Is it possible, for each closed set E, to find such an f which is diffentiable on $\mathbb{R}^1$, or one which is n times diffentiable, or even one which ahs derivatives of all orders on R^1?

(sol) 21.1

No clue. Skip for now.

P 22   22

Suppose f is a real function on $(-\infty, \infty)$. Call x a fixed point of f if $f(x) = x$.
(a) If f is diffentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point. (b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point although $0 < f'(t) < 1$ for all real t. (c)However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitarty real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3....$

(sol) 22.1

(a) Suppose for the sake of contradiction that, WLOG, $x < y$ and $f(x) = xf(y) = y$.
By the mean value theorem, there exists a $x < c < y$ such that $f'(c) = \frac{f(y)-f(x)}{}$.

$$f(x) = xf(y) = y \implies f(y) - f(x) = y - x$$

so $f'(x) = 1$.   □
(b) $(1 + e^t)^{-1} \neq 0$ for all t so there can be no fixed points.

P 23   23

The function defined by
$$f(x) = \frac{x^3 + 1}{3}$$
has three fixed pints, say α, β, γ, where

$$-2 < \alpha < -1, \qquad 0 < \beta < 1, \qquad 1 < \gamma < 2$$

For arbitrarily chosen x_1, define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.
(a) If $x_1 < \alpha$, prove that $x_n \to -\infty$ as $n \to \infty$.

(sol) 23.1

P 24   24

The process described in part (c) of Exercise 22 can of course be applied to functions that map $(0, \infty)$ to $(0, \infty)$.
Fix some $\alpha > 1$, and put

$$f(x) = \frac{1}{2}(x + \frac{\alpha}{x}), \qquad g(x) = \frac{\alpha + x}{1 + x}$$

Both f and g have $\sqrt{(\alpha)}$ as their only fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g, why the convergence in Exercise 16, Chap. 3, is so much more rapid than it is in Exercise 17. (Compare f' and g', draw the zig-zags suggested in Exercise 22.)
Do the same when $0 < \alpha < 1$.

(sol) 24.1

Looking at the zig-zag, $f(x)$ does not spiral and stays on the positive side, and $g(x)$ goes back and forth, so presumably $f(x)$ converges quite a bit faster. For $\alpha < 1$, $g(x)$ no longer goes back and forth, but it's not clear if it converges as fast as $f(x)$. I want to say no because when x gets close to the fixed point, f will still be halved each time, but g does not have a constant rate at which it converges.

P 25   25

Suppose f is twice diffrentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$.
Complete the following outline of Newton's method for computing ξ.
(a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of f.
(b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n \to \infty} x_n = \xi$$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.
(d) If $A = \frac{M}{2\delta}$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}$$

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does $g'(x)$ behave for x near ξ?

(f) Apply to $f(x) = x^{\frac{1}{3}}$

(sol) 25.1

(a) Geometrically this is taking the tangent line at x_n, finding the intersection with the x-axis, and applying that new x again.

(b) Since both $f(x_n)$ and $f'(x_n)$ are positive, $x_{n+1} < x_n$. Since $\{x_n\}$ is monotonically decreasing, either it diverges to $-\infty$ or converges to a limit point. Now for some x_n, we have by the mean value theorem

$$\exists z \in (x_{n+1}, x_n) : f(x_n) - f(x_{n+1}) = f'(z)(x_n - x_{n+1})$$

and by definition of f, $f'(x_n)(x_n - x_{n+1}) = f(x_n)$, and since $f"$ is always positive, $f'(z) < f'(x_n)$, thus

$$f(x_{n+1}) = f(x_n) - f'(z)(x_n - x_{n+1}) > f(x_n) - f'(x_n)(x_n - x_{n+1}) = f(x_n) - f(x_n) = 0$$

so for all x_n, $f(x_n) > 0$, so $f(x_n)$ converges to 0, so $\{x_n\}$ converges to ξ.

(c) Let $\beta = \xi$, $\alpha = x_n$, $n = 2$. Then Taylor's theorem state that there exist a $t_n \in (\xi, x_n)$ such that

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f"(t_n)}{2}(\xi - x_n)^2$$

$$\implies -f(x_n) - f'(x_n)(\xi - x_n) = \frac{f"(t_n)}{2}(-(x_n - \xi))^2$$

$$\implies -f(x_n) - f'(x_n)\xi + f'(x_n)x_n = \frac{f"(t_n)}{2}(x_n - \xi)^2$$

divide by $f'(x_n)$

$$\implies -\frac{f(x_n)}{f'(x_n)} - \xi + x_n = \frac{f"(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

$$\implies (x_n - \frac{f(x_n)}{f'(x_n)}) - \xi = \frac{f"(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

$$\implies x_{n+1} - \xi = \frac{f"(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

(d) Will prove by induction. The base case of $n = 1$ just follows from part c. The inductive step is as follows: using part c again

$$\exists t_{n+1} : x_{n+2} - \xi = \frac{f"(t_{n+1})}{2f'(t_{n+1})}(x_{n+1} - \xi)^2$$

$$\leq A(x_{n+1} - \xi)^2 \leq A(\frac{1}{A}(x_1 - \xi)^{2^n})^2 = \frac{1}{A}(x_1 - \xi)^{2^{(n+1)}}$$

$\square$(e) Follows directly from definition. $g'(x)$ goes to 1.

(f) it keeps jumping back and forth and doesnt converge

P 26   26

Suppose f is diffentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. Hint: Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, \qquad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.

(sol) 26.1

Scratch work: taking the hint, we can solve for some x_0 such that $M_0 = 0$.

$$A(x_0 - a) = \frac{1}{2} \implies (x_0 - a) = \frac{1}{2A} \implies x_0 = a + \frac{1}{2A}$$

. Thus if $x_0 = a + \frac{1}{2A}$, then there exists some x in the interval such that it is the case that $M_0 \leq \frac{1}{2}M_0$ which implies $f = 0$ on $[a, x_0]$. Then let $x_n = x_{n-1} + \frac{1}{2A}$. Then at some n, $x_n > b$, so we have our conclusion. $\square$

P 27   27

Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A solution to the initial-value problem

$$y' = \phi(x, y), \qquad y(a) = c \qquad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$ and

$$f'(x) = \phi(x, f(x)) \qquad (a \leq x \leq b)$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.

Hint: APply exercise 26 to the difference of two solutions. Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{\frac{1}{2}}, \qquad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = x^2/4$. Find all other solutions.

(sol) **27.1**

Assume for sake of contradiction that there exists two different solutions f_1, f_2. Then define $g(x) = f_2(x) - f_1(x)$. Thus

$$|g'(x)| = |f_2'(x) - f_1'(x)| = |\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

$$= A|f_2(x) - f_1(x)| = A|g(x)|$$

Thus, $g = 0$ which implies $f_1 = f_2$, ie the solution is unique. $\square$

P 28   28

Formulate the prove an analogous uniqueness theorem for systems of differential equations of the form

$$y_j' = \phi_j(x, y_1, ..., y_k), \qquad y_j(a) = c_j \qquad (j = 1, ..., k)$$

(sol) **28.1**

Vector valued functions have same inequalities needed for the proof so the result follows. $\square$

P 29   29

Specialize Exercise 28 by considering the system

$$y_j' = y_{j+1} \qquad (j = 1, ..., k-1)$$

$$y_k' = f(x) - \sum_{j=1}^{k} g_j(x)y_j$$

where $f, g_1, ..., g_k$ are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x)y^{(k-1)} + ... + g_2(x)y' + g_1(x)y = f(x)$$

subject to initial conditions

$$y(a) = c_1, \qquad , y'(a) = c_2, \qquad ..., \qquad y^{(k-1)}(a) = c_k$$

(sol) **29.1**

Since all gs are continuous, the image is also bounded. So the norms are also bounded. Which gives us the assumptions of 28. $\square$