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P 1 1

Define

$$f(x) = \begin{cases} e^{-1/x^2}, & (x \neq 0) \\ 0, & (x = 0) \end{cases} \tag{1}$$

Prove that f has derivaties of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$.

(sol) 1.1

P 2 2

Let a_{ij} be the number in the ith row and jth column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & -1 & 0 & 0 & \ldots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \ldots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0, & (i < j) \\ -1, & (i = j) \\ 2^{j-1}, & (i > j) \end{cases} \tag{2}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \qquad \sum_j \sum_i a_{ij} = 0$$

(sol) 2.1

Summing vertically first

$$\sum_j \sum_i a_{ij}$$

we have that

$$-1 + \sum_{k=1}^{\infty} 2^{-k} = -1 + 1 = 0$$

for each i thus $\sum_j \sum_i a_{ij} = 0$.
Summing horizontally first

$$\sum_j \sum_i a_{ji}$$

we have

$$\sum_{k=0}^{\infty} -2^{-k} = -1 - \sum_{k=1}^{\infty} 2^{-k} = -1 - 1 = -2$$

□

P 3 3

Prove that

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

if $a_{ij} \geq 0$ for all i and j (the case $+\infty = +\infty$ may occur).

(sol) 3.1

Follows from theorem 8.3 for the convergent case. The divergent case is such that if one sum diverges both of them do so equality follows. □

P 4 4

Prove that following limit relations:

$$(a) \lim_{x \to 0} \frac{b^x - 1}{x} = \log b$$

$$(b) \lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

$$(c) \lim_{x \to 0} (1+x)^{1/x} = e$$

$$(d) \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

(sol) 4.1

(a) Using L'hospital's rule, the denominator becomes 1 and numerator $\log b$.
(b) Same as in a), we get $\lim_{x\to 0} \frac{1}{1+x} = 1$
(c)

$$\begin{aligned}\lim_{x\to 0}(1+x)^{1/x} &= \lim_{x\to 0} e^{\frac{\log(1+x)}{x}}\\ &= e^{\lim_{x\to 0}\frac{\log(1+x)}{x}} = e^1 = e\end{aligned}$$

(d)

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\left(\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right)^x\\ &= \left(\lim_{y\to 0}\left(1+y\right)^{\frac{1}{y}}\right)^x\\ &= e^x\end{aligned}$$

$\square$

P 5

(sol) 5.1