Chapter 7

Campinghedgehog

June 14, 2023

P 1   1

Prove that every bounded uniformly convergent sequence of bounded functions is uniformly bounded.

(sol) 1.1

Pick N such that if $n, m >= N$

$$|f_n(x) - f_m(x)| < 1$$

for all x. Then let $M = max(M_1, ..., M_N)$ where $|f_i| < M_i$. Then let $P = M + 1$, then if $i < N$, $|f_i| < M$ and if $i > N$ then $|f_i| < |f_N| + 1 \leq M + 1$.   □

P 2   2

(sol) 2.1

Since both sequences converge uniformly (to f and g), fix ϵ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

for all x when $n > N$ and

$$|g_m(x) - g(x)| < \frac{\epsilon}{2}$$

for all x when $m > M$. Let $T = max(N, M)$. Then if $n > T$,

$$|f_n(x) + g_n(x) - f(x) - g(x)| = |f_n(x) - f(x) + g_n(x) - g(x)|$$

$$\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| = \epsilon$$

Next assume that both sequences are of bounded functions. From problem 1, we can conclude that both sequences are uniformly bounded, that is there exists T such that

$$|f_i(x)| < T$$

for all x and i and there exists P such that

$$|g_i(x)| < P$$

for all x and i. Then let $M = max(T, P)$. Then

$$|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)|$$
$$= |g_n(x)(f_n(x) - f(x)) + f(x)(g_n(x) - g(x))| \leq |M(f_n(x) - f(x)) + M(g_n(x) - g(x))|$$
$$\leq M|(f_n(x) - f(x))| + M|(g_n(x) - g(x))|$$

and since the sequences are uniformly continuous, for all $\epsilon > 0$

$$< M\epsilon + M\epsilon = 2M\epsilon$$

thus $\{f_n g_n\}$ is uniformly convergent. $\qquad\square$

P 3   3

Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E.

(sol) 3.1

$f_n(x) = x$ for all x converges to $f(x) = x$ uniformly with $\epsilon = 0$ for all n. $g_n(x) = \frac{1}{n}$ for all x is the same. So $f_n(x)g_n(x) = 0$. But then for any n, there exists an x equal to it, thus for all n there exists some x such that $f_n(x)g_n(x) = \frac{x}{n} = 1 > \epsilon$. $\qquad\square$

P 4   4

Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

(sol) 4.1

The series is almost convergent (comparing with $\frac{1}{n^2}$), except for a few points. When $x = 0$ we get $\sum_{n=1}^{\infty} 1$ which diverges. There are countable number of points where a term is undefined, namely $-\frac{1}{n^2}$ making the denominator 0.
For any $x > 0$ it converges uniformly since given some $\epsilon > 0$, $\frac{1}{1+n^2 x} \leq \frac{1}{n^2 \epsilon}$ and the right hand side converges uniformly. The same is true for $x < 0$ except for the aformentioned countable points.
Any interval containing 0 on the boundary does not converge uniformly. We can always pick an x so that the partial sums never converge.
f does not converge at $x = 0$ so it's not bounded.

P 5 5

Let

$$f_n(x) = \begin{cases} 0, & \left(x < \frac{1}{n+1} \right) \\ \sin^2 \frac{\pi}{x}, & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n} \right) \\ 0, & \left(\frac{1}{n} < x \right) \end{cases} \tag{1}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that the absolute convergence, even for all x, does not imply uniform convergence.

(sol) 5.1

The function converges (point-wise at least) to $f = 0$, since rationals are dense in reals, fixing any x there is always an n such that $f_n(x) = 0$.

The series cannot converge uniformly because for any n, there exists some m such that $x = \frac{1}{2m+\frac{1}{2}}$ such that $\frac{1}{n+1} < x < \frac{1}{n+1}$, thus $f(x) = \sin^2(2\pi m + \frac{\pi}{2}) = 1$. $\qquad \square$

P 6 6

Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

(sol) 6.1

Skip for now.

P 7 7

For $n = 1, 2, 3, ..., x$ real, put

$$f_n(x) = \frac{x}{1 + nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n \to \infty} f'_n(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

(sol) **7.1**

Using the quotient rule and differentiating f_n and setting it equal to 0 shows that f_n has its max at $x = \frac{1}{\sqrt{n}}$, at which $f_n(x) = \frac{1}{2\sqrt{n}}$. Thus the sequence f_n is bounded by $\frac{1}{2\sqrt{n}}$ which converges uniformly to $f = 0$.
Since

$$f'_n(x) = \frac{1 - nx^2}{(1 + nx^2)^2}$$

so

$$f'_n(0) = 1$$

so $\lim_{n \to \infty} f'_n(x) = 1$ but since $f = 0$, $f' = 0$.   □

P 8   8

If

$$I(x) = \begin{cases} 0, & (x \leq 0) \\ 1, & (x > 0) \end{cases} \tag{2}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \qquad (a \leq x \leq b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

(sol) **8.1**

Since I is just an indicator function, it follows that

$$f_N(x) = \sum_{n=1}^{N} c_n I(x - x_n) \leq \sum_{n=1}^{N} |c_n|$$

. Since $\sum |c_n|$ converges absolutely, it follows that f_n converges uniformly.
f will be continuous if each f_n is. And each f_n is continuous if $x \neq x_n$, since if it "jumps" via the indicator function and misses or includes an extra term greater than some ϵ in the summation if so.

P 9   9

Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n \to \infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

(sol) 9.1

Decouple x_n from f_n and call it x_k. Then we have

$$\lim_{k\to\infty} \lim_{n\to\infty} f_n(x_k) = \lim_{k\to\infty} f(x_k)$$

by uniform convergence. Then

$$\lim_{k\to\infty} f(x_k) = f(x)$$

. Or more rigorously, Fix ϵ. Pick N such that if $n > N$, $|f_n(x) - f(x)| < \epsilon$ for all x (by uniform convergence). Then pick M such that if $m > M$, $|x_m - x| < \delta$ such that $|f(x_m) - f(x)| < \epsilon$ (since f is continuous). Then pick $k > max(N, M)$, thus

$$|f_k(x_k) - f(x)| = |f_k(x_k) - f(x_k) + f(x_k) - f(x)|$$

$$\leq |f_k(x_k) - f(x_k)| + |f(x_k) - f(x)| < 2\epsilon$$

Probem 5 is a counter example.

P 10   10

Letting (x) denote that fractional part of the real number x, consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \qquad (x \text{ real})$$

Find all discontinuities of f, and show that they form a countable dense set. Show that f is neverthless Riemann-integrable on every bounded interval.

(sol) 10.1

Discontinuities of f would include all discontinuities of the partial sums

$$f_N(x) = \sum_{n=1}^{N} \frac{(nx)}{n^2}$$

The discontinuities are when (nx) "jumps" back to 0. This happens in all rationals since there will always be an n equal to the denominator of x, thus f is discontinuities at all rationals. And the rationals are countable and dense in the reals.
f is Riemann-integrable if each f_n in the sequence is also Rieman-integrable. There are a finite number of discontinuities given an f_n since it is only a sum of N terms. Thus f is also Riemann-integrable.   □

P 11 11

Suppose $\{f_n\}$, $\{g_n\}$ are defined on E, and
(a) $\sum f_n$ has uniformly bounded partial sums;
(b) $g_n \to 0$ uniformly on E;
(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq ...$ for every $x \in E$
Prove that $\sum f_n g_n$ converges uniformly on E. Hint: Compare with Theorem 3.42.

(sol) 11.1

Choose M such that $F_N = \sum_{n=1}^{N} f_n(x) < M$ for all x. Fix ϵ. From b), choose N such that $g_N < \frac{\epsilon}{2M}$. Then for $p, q \geq N$,

$$|\sum_{n=p}^{q} f_n g_n| = |\sum_{n=p}^{q-1} F_n(g_n - g_{n+1}) + F_q g_q - F_{p-1} g_p|$$

$$\leq M|\sum_{n=p}^{q-1}(g_n - g_{n+1}) + g_q + b_p|$$

$$= 2g_p M \leq 2g_N M \leq \epsilon$$

for all x. $\qquad\qquad\square$

P 12 12

Suppose g and $f_n(n = 1, 2, 3, ...)$ are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$ and

$$\int_0^\infty g(x)\, dx < \infty$$

Prove that

$$\lim_{n \to \infty} \int_0^\infty f_n(x)\, dx = \int_0^\infty f(x)\, dx$$

(sol) 12.1

Want to first show that $\int_0^\infty f_n(x)\, dx$ exists for all n. For $a < b$,

$$|\int_a^t f_n(x)\, dx \int_b^t f_n(x)\, dx| = |\int_b^a f_n(x)\, dx|$$

$$\leq \int_b^a |f_n(x)|\, dx \leq \int_b^a g(x)\, dx$$

And since $int_0^\infty g_n(x)\, dx$ exists, we can choose a, b such that $int_b^a g(x)\, dx < \epsilon$ for all ϵ. Similarly for the lower end point. It follows from Cauchy criterion that $\int_0^\infty f_n(x)\, dx$

exists.

Because of uniform convergence, $f \leq g$ and so $\int_0^\infty f(x)\,dx$ also exists. Then

$$\left| \int_0^\infty f_n(x)\,dx - \int_0^\infty f(x)\,dx \right|$$

$$= \left| \int_0^\infty f_n(x)\,dx - \int_a^b f_n(x)\,dx + \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx + \int_a^b f(x)\,dx - \int_0^\infty f(x)\,dx \right|$$

$$\leq \left| \int_0^\infty f_n(x)\,dx - \int_a^b f_n(x)\,dx \right|$$

$$+ \left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| + \left| \int_a^b f(x)\,dx - \int_0^\infty f(x)\,dx \right|$$

$$= \left| \int_0^\infty f_n(x)\,dx - \int_a^b f_n(x)\,dx \right|$$

$$+ \left| \int_a^b f_n(x) - f(x)\,dx \right| + \left| \int_a^b f(x)\,dx - \int_0^\infty f(x)\,dx \right|$$

With uniform convergence the middle term can be made arbitrarily small. The fact that the improper integrals exist means that first and last terms can also be made arbitrarily small. $\qquad\square$

P 13   13

Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbb{R}^1$ with $0 \leq f_n(x) \leq 1$ for all x and all n.

(a) Prove that there is a function f and a sequence $\{n_k\}$ such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

for every $x \in \mathbb{R}^1$. (The existence of such a pointwise convergent subsequence is usually called Helly's selection theorem).

(b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets. Hint: (i) Some subsequence $\{f_{n_i}\}$ converges at all rational points r, say, to $f(r)$. (ii) Define $f(x)$, for any $x \in \mathbb{R}^1$, to be $\sup f(r)$, the sup taken over all $r \leq x$. (iii) Show that $f_{n_i}(x) \to f(x)$ at every x at which f is continuous. (This is where monotonicity is strongly used.) (iv) A subsequence of $\{f_{n_i}\}$ converges at every point of discontinuity of f since there are at most countably many such points. This proves (a). To prove (b), modify your proof of (iii) appropriately.

(sol) **13.1**

From Theorem 7.23 there exists a sequence of functions that converges on any countable set. Let the countable set be the rationals. Then define $f(x) = \sup f(r)$ for $r \leq x$. Points of continuity converge since the rationals are dense in the reals. For points of discontinuity, since the functions are monotonically increasing, there may only be a countable number of them; applying theorem 7.23 again to those points proves a).
Skip b for now.

P 14   14

Skip for now

(sol) **14.1**

P 15   15

Suppose f is a continuous function on $\mathbb{R}^1$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$ and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

(sol) **15.1**

It has to be a constant function since if it is not, then using equicontinuity we can find x, y with $|f_n(x) - f_n(y)| > \epsilon$. Then $|f_n(\frac{x}{n}) - f_n(\frac{y}{n})| > \epsilon$ for all n, which contradicts equicontinuity. □

P 16   16

Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

(sol) **16.1**

Fix ϵ. Choose N such that if $n > N$, then $|f_n(x) - f_n(y)| < \epsilon$ for all x, y if $|x - y| < \delta$. Then also choose a δ-cover of K. That is there exists a finite number of open neighborhoods with center points $x_1, \ldots, x_k$, with radius δ. Fixing each of the points, using point-wise convergence, choose M such that if $n, m > M$, $|f_n(x_i) - f_m(x_i)| < \epsilon$ for all i, which is finite. Then, for $n, m > max(N, M)$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)|$$

Equicontinuity implies the first and third terms being less than ϵ. The middle term is also less than ϵ since $n, m > M$. Thus

$$|f_n(x) - f_m(x)| < 3\epsilon$$

for all x. □

P 17 17

Skip for now

(sol) 17.1

P 18 18

Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)\, dt$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

(sol) 18.1

Looks like it's probably going to use theorem 7.25. The domain is compact. We need to show that it is pointwise bounded and equicontinuous to use theorem 7.25. Pointwise boundedness follows since for all n,

$$F_n(x) = \int_a^x f_n(t)\, dt \le M(b - a)$$

M comes from uniform boundedness. This applies for all x.
Next, fix ϵ. Then for some δ, if $|x - y| < \delta$,

$$|F_n(x) - F_n(y)| = |\int_a^x f_n(t)\, dt - \int_a^y f_n(t)\, dt|$$

$$= |\int_x^y f_n(t)\, dt| \le \int_x^y |f_n(t)|\, dt$$

$$\le M|x - y| < M\delta$$

So if $\delta = \frac{\epsilon}{M}$, equicontinuity follows. □

P 19   19

Let K be a compact metric space, let S be a subset of $\mathcal{C}(K)$. Prove that S is compact (with respect to the metric defined in Section 7.14) if and only if S is uniformly closed, bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on K).

(sol) **19.1**

P 20

(sol) **20.1**