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P 1

Suppose f is a real funcvtion defined on $\mathbb{R}^1$ which satisfies

$$\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

(sol) 1.1

No. It could be the case the $f(x)$ does not equal the left and right limits, a discontinuity of the first type.

P 2

If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$.

(sol) 2.1

Let E be arbitrary. Let $p \in \overline{E}$. If $p \in E$, then $f(p) \in \bar{(}f(E))$. So let p be a limit point of E but $p \notin E$. Want to show that $f(p)$ is a limit point of $f(E)$. Fix $epsilon$. Since f is continuous, there exists a $\delta > 0$ such that $d(f(p), f')) < \epsilon \implies d(p, p') < \delta$. Since p is a limit point of E, $\forall \delta > 0, \exists p' \in E : d(p, p') < \delta$. This implies $f(p)$ is also a limit point of $f(E)$. $\square$

P 3

Let f be a continuous function on a metric space X. let $Z(f)$(the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

(sol) 3.1

The target set is the singletone set of $\{0\}$, which is closed. Since f is continuous, $f^{-1}(\{0\})$ is also closed. $\square$

P 4

Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$. (In order words, a continuous mapping is determined by its values on a dense subset of its domain.)

(sol) 4.1

Want to show

$$f(X) = \overline{f(E)}$$

From problem 2, we know that $f(\overline{E}) \subset \overline{f(E)}$. Since E is dense in X, we have $\overline{E} = X$. So $f(X) \subset \overline{f(E)}$. Since $f(E) \subset f(X)$, and X is the whole space, $f(X)$ is closed and contains $f(E)$ so it also contains $f(\overline{E})$.
The second part follows from uniqueness of limits.

P 5

If f is a real continuous function defined on a closed set $E \subset \mathbb{R}^1$, prove that there exist continuous real functions g on $\mathbb{R}^1$ such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called *continuous extensions* of f from E to $\mathbb{R}^1$.) Show that the result becomes false

(sol) 5.1

We can simply let g be constant equal to the value of the edges of E. Since E is closed, there exists a minimum and maximum value. Let

$$g(x) = \begin{cases} min(E), & \text{if } x < min(E) \\ f(x), & \text{if } x \in E \\ max(E), & \text{if } x > max(E) \end{cases} \tag{1}$$

By construction the function is continuous. If "closed" is emitted, then there may not exist a min or max of E.

P 6

If f is defined on E, the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is the set of real numbers, and f is real-valued, the graph of f is a subsetof the plane.
Suppose E is compact, and prove that f is continuous if and only if its graph is compact.

(sol) 6.1

Suppose E is compact and f is continuous. Want to show: $(x, f(x))$ is compact.
Since f is continuous, $f(E)$ is compact. Since both E and $f(E)$ are compact, there exists finite open covers for both. The cartesian product of these covers is still finite.
Suppose both E and $(x, f(x))$ are compact. Using the fact that the projection functions are continuous, We get that both E and $f(E)$ are compact, which implies f is continuous. (?) $\square$

P 7

Skip for now.

(sol) 7.1

P 8

Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}^1$.
Prove that f is bounded on E.
Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

(sol) 8.1

Since f is uniformly continuous. Let $\epsilon = 1$, so that

$$\exists \delta > 0, \forall p, q \in E, d(p, q) < \delta \implies d(f(p), f(q)) < \epsilon$$

We then chunk E into N segments that are at most δ width. In this case, $N = \lceil \frac{(a-b)}{\delta} \rceil$.
Let $x \in E$ be arbitrary. It will fall into one of these segments and by construction will be less than δ away from one of the boundary points, call it z. Then since $d(x, z) < \delta$, $d(f(x), f(z)) < 1$. Since there are finite number of segments, there exists a maximum boundary point $f(z)$. Since x was arbitrary, it follows that $f(E)$ is bounded by the max of boundary points $f(z) + 1$. $\square$.
For the second part, consider $f(x) = x$ on $\mathbb{R}$.

P 9

Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\epsilon > 0$, there exists a $\delta > 0$ such that diam $f(E) < \epsilon$ for all $E \subset X$ with diam $E < \delta$.

(sol) 9.1

$$\forall E \subset X, (\forall p, q \in E, d(f(p), f(q)) < \epsilon) \wedge (\forall p, qinE, d(p, q) < \delta)$$

can be rewritten as:

$$\forall E \subset X, \forall p, q \in E, (d(f(p), f(q)) < \epsilon) \wedge (d(p, q) < \delta)$$

equals

$$\forall p, q \in X, (d(f(p), f(q)) < \epsilon) \wedge (d(p, q) < \delta)$$

□

P 10

Theorem 4.19: Let f be a continuous mapping of a compact metric space X into a metric space Y. f is uniformly continuous on X.

Complete the details of the following alternative proof of Theorem 4.19:
If f is not uniformly continuous, then for some $\epsilon > 0$ there are sequences $\{p_n\}, \{q_n\}$ in X such that $d_X(p_n, q_n) \longrightarrow 0$ but $d_Y(f(p_n), f(q_n)) > \epsilon$, Use Theorem 2.37 to obtain a contradiction.
2.37: If E is an infinite subset of a compact set K, then E has a limit point in K.

(sol) 10.1

Assume f is not uniformly continuous. Then fix $\epsilon > 0$ such that there are sequences $\{p_n\}, \{q_n\}$ in X such that $d_X(p_n, q_n) \longrightarrow 0$ but $d_Y(f(p_n), f(q_n)) > \epsilon$. It follows that $f(p_n), f(q_n)$ never converge to the same point, so $\{p_n\}, \{q_n\}$ do not converge to a limit point. But $\{p_n\}, \{q_n\}$ are infinite subsets of a compact metric space, and therefore must have a limit point. $\Rightarrow\Leftarrow$ □

P 11

Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof of the theorem stated in Exercise 13.

(sol) 11.1

Let $\{x_n\}$ in X be a Cauchy sequence. I.e.

$$\forall \delta > 0, \exists N, \forall n, m > N, d(x_n, x_m) < \delta$$

Want to show: $\{f(x_n)\}$ is a Cauchy sequence in Y. Since f is uniformly continuous, $\forall \epsilon > 0, d(x_n, x_m) < \delta \implies d(f(x_n), f(x_m)) < \epsilon$ So for each ϵ there also exists an M such that $d(f(x_n), f(x_m)) < \epsilon$ if $m, n > M$. $\square$

P 12

A uniformly continuous function of a uniformly continuous function is uniformly continuous. State this more precisely and prove it.

(sol) 12.1

Let f be a uniformly continuous function from metric space X to metric space Y, and g be a uniformly continuous from metric space Y to metric space Z. Then $g \circ f$ from X to Z is uniformly continuous.
Proof.
Want to show:

$$\forall \epsilon > 0, \exists \delta > 0 s.t. \forall p, q \in X, d(p, q) < \delta \implies d(g \circ f(p), g \circ f(q)) < \epsilon$$

Fix $\epsilon > 0$. Since g is uniformly continuous, there exist a $\gamma > 0$... such that

$$\forall a, b \in Y, d(a, b) < \gamma \implies d(g(a), g(b)) < \epsilon$$

Since f is uniformly continuous, there exists a $\delta > 0$ such that

$$\forall x, y \in X, d(x, y) < \delta \implies d(g(a), g(b)) < \gamma$$

$\square$

P 13

Let E be a dense subset of a metric space X, and let f be a uniformly continuous real function defined on E. Prove that f has a continuous extension from E to X.
Hint: For each $p \in X$ and each positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < 1/n$. Use Exercise 9 to show that the intersection of the closures of the sets $f(V_1(p)), f(V_2(p)), \ldots$ consists of a single point, say $g(p)$, of $\mathbb{R}^1$. Prove that the function g so defined on X is the desired extension of f.

(sol) 13.1

Following the hint, from Exercise 9, since diam of $V_n(p)$ goes to 0, and f is uniformly continuous, $f(V_1(p))$ also goes to 0, which implies the set only consist of a single point, call it $g(p)$. It is clear that if $p \in E$, then $f(p) = g(p)$. Then, assume $p \notin E$. Since E is dense in X, p then must be a limit point of E. Want to show that g is continuous. Since p is a limit point of E, $g(p)$ is defined, and by construction contiuous as $g(p)$. □

P 14

Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

(sol) 14.1

Consider $g(x) = x - f(x)$. Contradiction follows from the intermediate value theorem.

P 15

Call a mapping of X into Y open if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping in $\mathbb{R}^1$ into $\mathbb{R}^1$ is monotonic.

(sol) 15.1

Since f is open, there exists an f^{-1} that is continuous. If f is not monotonic, then something something f^{-1} not a function or not continous.

P 16

Let $\lceil x \rceil$ denote the largest integer contained in x, that is $\lceil x \rceil$ is the integer such that $x - 1 < \lceil x \rceil \leq x$; and let $(x) = x - \lceil x \rceil$ denote the fractional part of x. WHat discontinuities do the above functions have?

(sol) 16.1

Integers. They ”jump” to the next int, or drop to 0 from almost 1.

P 17

Let f be a real function on (a, b). Prove that the set of points at which f has a simple discontinuity is at most countable.
Hint: Let E be the set on which $f(x-) < f(x+)$. With each point x of E associate a triple (p, q, r) of rational numbers such that
(a) $f(x-) < p < f(x+)$, (b) $a < q < t < x$ implies $f(t) < p$, (c) $x < t < r < b$ implies $f(t) > p$.

(sol) 17.1

Skip for now.

P 18

Every rational x can be written in the form $x = m/n$, where $n > 0$ and m and n are intergers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function defined on $\mathbb{R}^1$ by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ irrational} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \end{cases} \tag{2}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

(sol) 18.1

Let x be irrational. Fix $\epsilon > 0$. We can always find a $\delta > 0$ since if say p is irrational then $d(f(x), f(p)) = 0 < \epsilon$. If p is rational, then the closer we get to x, the larger n will be which means $f(p) \to 0$.
Let x be rational. Since the irrationals are dense in the rationals, $f(x)$ never converges to $1/n$ since $d(x, irrational)$ is always equal to at least $1/n$. The limits exist, but $f(x) \ = lim f(x)$, i.e. simple discontinuity. □

P 19

Suppose f is a real function with domain $\mathbb{R}^1$ which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b.
Suppose also, for every rational r, that the set of all x with $f(x) = 4$ is closed. Prove that f is continuous.
Hint: If $x_n \to X_0$ but $f(x_n) > r > f(x_0)$ for some r and all n, then $f(T_n) = r$ for some t_n between x_0 and x_n; thus $t_n \to x_0$. Find a contradiction.

(sol) 19.1

From the hint, it follows that x_0 is a limit point for the sequence t_n, yet x_0 is strictly greater than r which means it is not contained in the the preimage of r and thus the preimage is not closed. □

P 20

if E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z)$$

a) Prove that $\rho_E(x) = 0$ if an only if $x \in \overline{E}$.
b) Prove that ρ_E is a uniformly continuous function on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

for all $x \in X$,$y \in Y$.
Hint: $\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)$ so that

$$\rho_E(x) \leq d(x, y) + \rho_E(y)$$

(sol) 20.1

a) Assume that $\rho_E(x) = 0$. Want to show that $x \in \overline{E}$
Since $\rho_E(x) = 0$, for all open neighborhoods of x, there exists a point in e within that neighborhood.
Assume $x \in \overline{E}$. If $x \in E$, then $d(x, x) = 0$. Assume $x \notin E$. Then x must be a limit point of E. Thus, for all $\epsilon > 0$, there exists $e \in E$ such that $d(x, e) < \epsilon$, which implies that the $\inf d(x, e)$ for e in E is equal to 0. b) Let $x, y \in X$ be arbitrary. Want to show

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

Following the hint, since $\rho_E(x) \leq d(x, z) \leq d(x, y)+d(y, z)$, by the triangle inequality, we have that

$$\rho_E(x) \leq d(x, y) + \rho_E(y)$$

which implies

$$|\rho_E(x) - \rho_E(y)| \leq |d(x, y) + \rho_E(y) - \rho_E(y)| \leq |d(x, y)| \leq d(x, y)$$

□

P 21

Suppose k and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists a $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$. Hint: ρ_F is a continuous positive function on K.
Show that the conclusion may fail for two disjoint closed sets if neither is compact.

(sol) 21.1

Since K is compact, any continuous function on K must be bounded, i.e. there exists min and max values of the image. Let the minmum value of $\rho_F(K)$ be some $k \in K$. Since K and F are disjoint, $k \notin F$. By problem 20, this implies $\rho_F(k) > 0$. Thus, for all p, q, there exists some $\delta > 0$ such that

$$d(p, q) \geq \rho_F(q) \geq \rho_F(k) \geq \delta > 0$$

$\square$If K is not compact, then it's image may not be bounded below by 0.

P 22

Skip for now.

(sol) 22.1

P 23

A real-valued function defined in (a, b) is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenver $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex.
If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

(sol) 23.1

a) Let f be a convex function. Want to show

$$\forall f(p), \forall \epsilon > 0, \exists \delta > 0, \forall q, d(p, q) < \delta \implies d(f(p), f(q)) < \epsilon$$

Fix $f(p)$. Fix $\epsilon > 0$. ff go next.

P 24

Asume that f is a continuous real function defined in (a, b) such that

$$f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$$

for all $(x, y) \in (a, b)$. Prove that f is convex. Skip for now.

(sol) 24.1

P 25

Skip for now.

(sol) 25.1

P 26

Suppose X, Y, Z are metric spaces and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$.
Prove that f is uniformly continuous if h is uniformly continuous. Hint: g^{-1} has a compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$.
Prove also that f is continuous if h is continuous.
Show by modifying Example 4.21 that the compactness of Y cannot be omitted from the hypothesis, even when X and Z are compact.

(sol) 26.1

Following the hint, since g^{-1} has a compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$, and since uniformly continous function of uniformly continuous function is uniformly continuous. Continuity follows from the same reasoning.