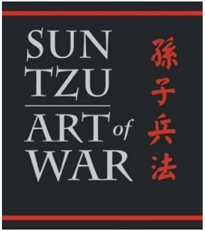
Algorithm Design and Analysis

Day 3
Divide and Conquer

2015, AUT - CJLU

Day 3: Divide and Conquer

Part I: Divide-and-Conquer – Integer Multiplication

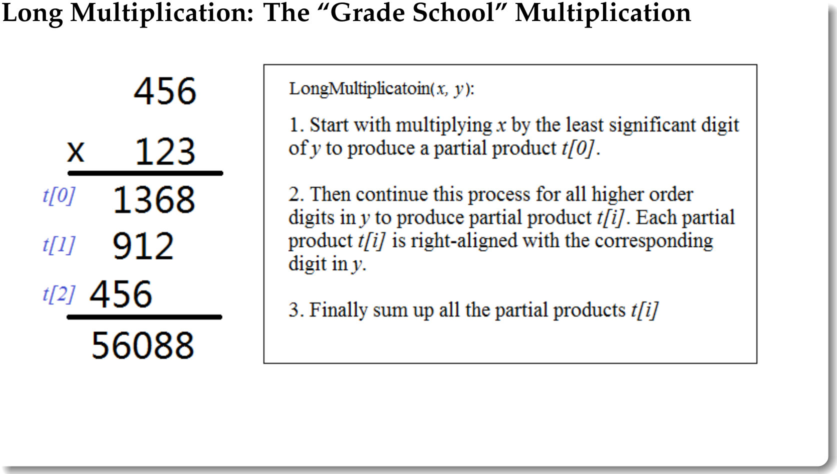


故用兵之法，十則圍之，五則攻之，倍則分之， 敵則能戰之，少則能守之，不若則能避之。

``It is the rule in war, if ten times the enemy's strength, surround them; if five times, attack them; if double, be able to divide them; if equal,engage them; if fewer, be able to evade them; if weaker, be able to avoid them.''

---``Chapter III Strategic Attack" 500BC

Multiplication



Long Multiplication: The "Grade School" Multiplication

```
          456
x         123
-------------
t[0]     1368
t[1]      912
t[2]  456
-------------
        56088
```

LongMultiplicatoin(x, y):

1. Start with multiplying x by the least significant digit of y to produce a partial product *t[0]*.

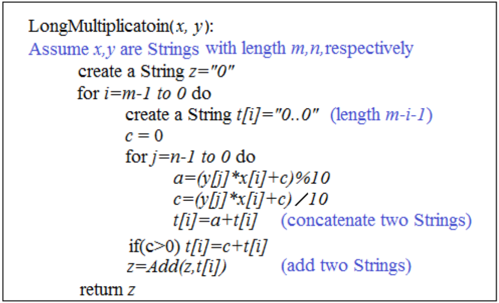
2. Then continue this process for all higher order digits in y to produce partial product *t[i]*. Each partial product *t[i]* is right-aligned with the corresponding digit in y.

3. Finally sum up all the partial products *t[i]*

Multiplication

Long Multiplication: The “Grade School” Multiplication

$$
\begin{array}{rr}
 & 456 \\
\times & 123 \\
\hline
t[0] & 1368 \\
t[1] & 9120 \\
t[2] & 45600 \\
\hline
 & 56088
\end{array}
$$

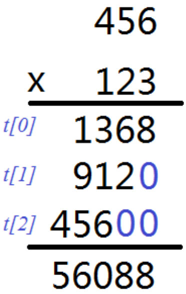


```
LongMultiplicatoin(x, y):
Assume x,y are Strings with length m,n,respectively
    create a String z="0"
    for i=m-1 to 0 do
        create a String t[i]="0..0"  (length m-i-1)
        c = 0
        for j=n-1 to 0 do
            a=(y[j]*x[i]+c)%10
            c=(y[j]*x[i]+c)/10
            t[i]=a+t[i]    (concatenate two Strings)
        if(c>0) t[i]=c+t[i]
        z=Add(z,t[i])      (add two Strings)
    return z
```

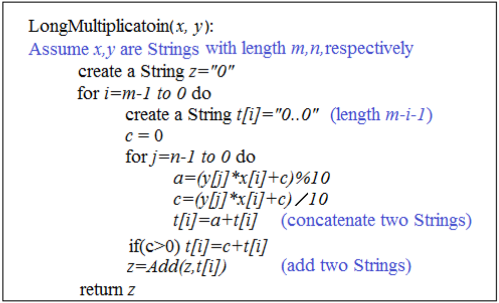
Complexity: What is the running time?

Multiplication

Long Multiplication: The "Grade School" Multiplication



```
         456
   x     123
   ---------
t[0]    1368
t[1]    9120
t[2]   45600
   ---------
       56088
```



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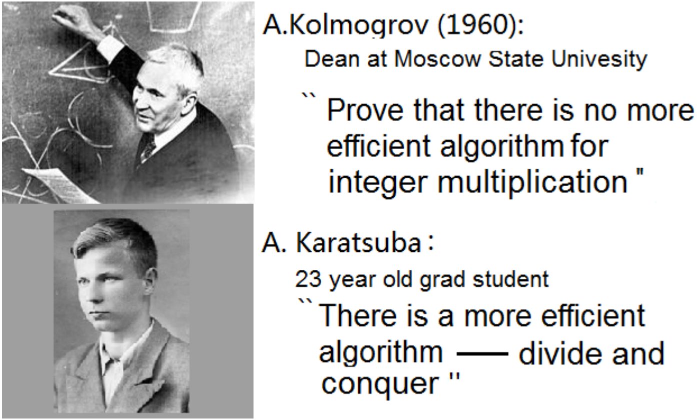
```

Complexity: What is the running time? $O(n^2)$

Multiplication

Kolmogorov's Conjecture



A.Kolmogrov (1960):

Dean at Moscow State Univesity

`` Prove that there is no more efficient algorithm for integer multiplication "

A. Karatsuba :

23 year old grad student

`` There is a more efficient algorithm —— divide and conquer "

Faster Multiplication

Observation

- $(2x + 3)(5x + 7) = 2 \times 5x^2 + (2 \times 7 + 3 \times 5)x + 3 \times 7$

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  $\Rightarrow (2x+3)(5x+7) =$
  $2\times 5x^2 + 3\times 7 + ((2+3)\times(5+7) - 2\times 5 - 3\times 7)x$

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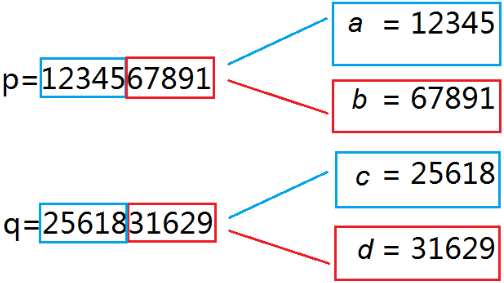
  3 multiplications, 6 additions

General Version

$$(ax+b)(cx+d) = acx^2 + bd + ((a+b)(c+d) - ac - bd)x$$

Karatsuba's Algorithm

Example:



$p = 1234567891$

$a = 12345$

$b = 67891$

$q = 2561831629$

$c = 25618$

$d = 31629$

Karatsuba's Algorithm

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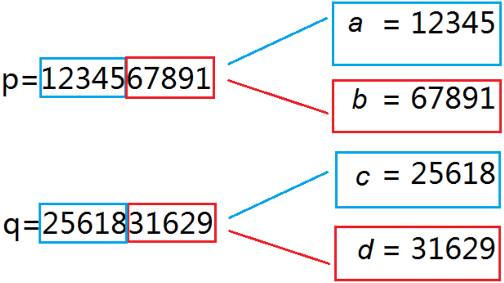


p=1234567891

a = 12345

b = 67891

q=2561831629

c = 25618

d = 31629

$$\begin{aligned} p \times q &= (a \times 10^5 + b) \times (c \times 10^5 + d) \\ &= ac \times 10^{10} + bd + ((a+b)(c+d) - ac - bd) \times 10^5 \end{aligned}$$

Karatsuba’s Algorithm

```
Algorithm multiply($x, y$)
INPUT: $x, y$ are length-$n$ (string representations of) integers
OUTPUT: The (string representation of) product of $x, y$
    if $n = 1$ return $xy$
    $a = leftmost\ \lceil n/2 \rceil$ bits of $x$, $b = rightmost\ \lfloor n/2 \rfloor$ bits of $x$
    $c = leftmost\ \lceil n/2 \rceil$ bits of $y$, $d = rightmost\ \lfloor n/2 \rfloor$ bits of $y$
    $p_1 \leftarrow multiply(a, c)$
    $p_2 \leftarrow multiply(b, d)$
    $p_3 \leftarrow multiply(a + b, c + d)$
    $p_3 \leftarrow p_3 - p_1 - p_2$
    Shift $p_1$ to the left by $n$-bits, $p_3$ to the left by $\lfloor n/2 \rfloor$-bits
    return $p_1 + p_2 + p_3$
```

Karatsuba's Algorithm

Time Complexity

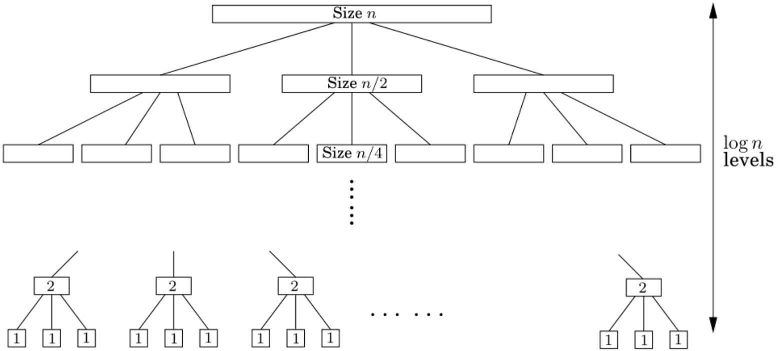
Let $T(n)$ be the time complexity of multiplying two length-n integers.

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 3T(\lceil n/2 \rceil) + cn & \text{otherwise} \end{cases}$$

where c is a constant.

Divide-and-Conquer

Tree of Recursive Calls

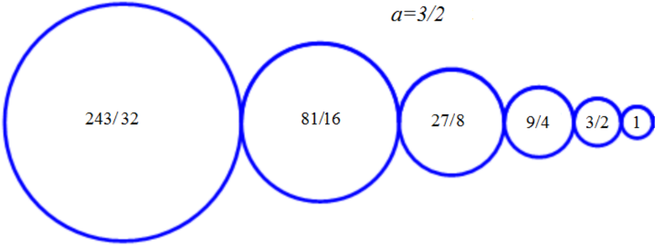


Solving Recurrence

Time Complexity

$$T(n) = nc(1 + \frac{3}{2} + \frac{3^2}{2^2} + \ldots + \frac{3^{k-1}}{2^{k-1}} + \frac{3^k}{2^k})$$

Geometric Series



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Let $m = 1 + a + a^2 + a^3 + \ldots + a^k$, where $a > 0, a \neq 1$, be a geometric series.

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$\Rightarrow am = a + a^2 + a^3 + \ldots + a^{k+1}$
$\Rightarrow am - m = a^{k+1} - 1$
$\Rightarrow m(a-1) = a^{k+1} - 1$
$\Rightarrow m = \frac{a^{k+1}-1}{a-1}$

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Therefore

$$T(n) = nc\frac{(3/2)^{k+1} - 1}{3/2 - 1} \leq 3nc\frac{3^k}{2^k}$$

Solving Recurrence

Time Complexity

Recall $n = 2^k$. Therefore $k = \log_2 n$. We have

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$$T(n) \leq 3nc\frac{3^{\log_2 n}}{2^{\log_2 n}} = 3nc\frac{3^{\log_2 n}}{n} = 3c \times 3^{\log_2 n}$$

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$$T(n) \leq 3nc\frac{3^{\log_2 n}}{2^{\log_2 n}} = 3nc\frac{3^{\log_2 n}}{n} = 3c \times 3^{\log_2 n}$$

Note that $3 = 2^{\log_2 3}$. So

$$\begin{aligned} T(n) &\leq 3c \times (2^{\log_2 3})^{\log_2 n} \\ &= 3c \times 2^{\log_2 3 \times \log_2 n} \\ &= 3c \times 2^{\log_2 n \times \log_2 3} \\ &= 3c \times (2^{\log_2 n})^{\log_2 3} \\ &= 3c \times n^{\log_2 3} \\ &\leq 3c \times n^{1.59} \end{aligned}$$

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Therefore $T(n)$ is $O(n^{1.59})$, a big improvement from $O(n^2)$!!

Divide-and-Conquer

- Karatsuba's algorithm solves integer multiplication in time $O(n^{1.59})$.
- The technique used is called divide-and-conquer

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Divide-and-Conquer as an algorithm design technique

The divide-and-conquer technique solves a computational problem by dividing it into one or more subprograms of smaller size, conquering each of them by solving them recursively, and then combining their solutions into a solution for the original problem.

Divide-and-Conquer

General Divide-and-Conquer Strategy

```
if $n \leq n_0$ then
    directly solve problem without dividing
else
    divide problem into $a$ subproblems of size $n/b$ each
    for $i \leftarrow 0$ to $a-1$ do
        recursively solve the $i$th subproblem
    combine the $a$ solutions into a solution of the original
    problem
```

Divide-and-Conquer

General Divide-and-Conquer Strategy

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```

Running Time Analysis

Come up with a recurrence: $T(n) = aT(n/b) + f(n)$

Example 1: Karatsuba's Algorithm

Karatsuba's algorithm

Given two input numbers x, y:

```
if $x, y$ both have length 1 then
    directly multiply $x, y$
else
    divide each $x$ and $y$ into two numbers and
    obtain 3 subproblems of size $n/2$ each
    for each subprogram do
        recursively solve the $i$th subproblem
    add the 3 solutions
```

Time complexity: $T(n) = 3T(n/2) + cn$.

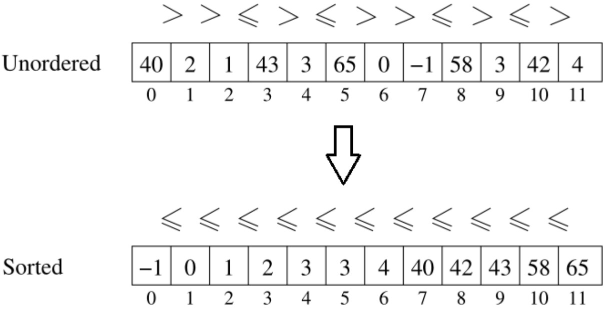
Day 3: Divide and Conquer

Part II: Sorting

The Sorting Problem

The Sorting Problem

Arrange an array of integers so that every adjacent pair of values is in the correct order.



Unordered

$> \quad > \quad \leqslant \quad > \quad \leqslant \quad > \quad > \quad \leqslant \quad > \quad \leqslant \quad >$

40	2	1	43	3	65	0	−1	58	3	42	4
0	1	2	3	4	5	6	7	8	9	10	11

⇩

Sorted

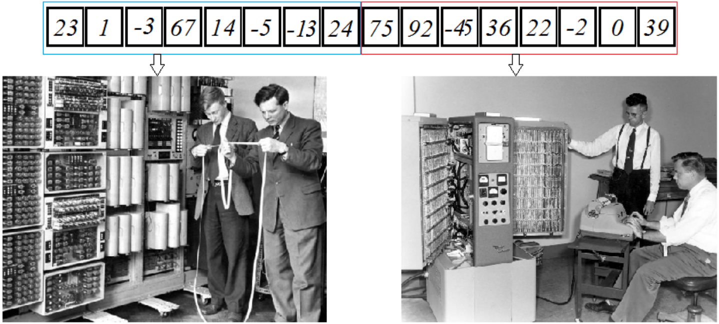
$\leqslant \quad \leqslant \quad \leqslant \quad \leqslant \quad \leqslant \quad \leqslant \quad \leqslant \quad \leqslant \quad \leqslant \quad \leqslant \quad \leqslant$

−1	0	1	2	3	3	4	40	42	43	58	65
0	1	2	3	4	5	6	7	8	9	10	11

Merge Sort

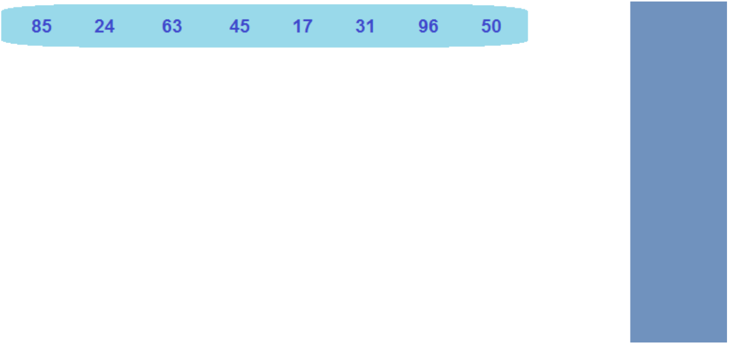
Merge Sort Algorithm

1. To sort an array, partition it into two parts; and give each part to a different machine
2. Each machine sorts its part recursively
3. Merge the two solutions



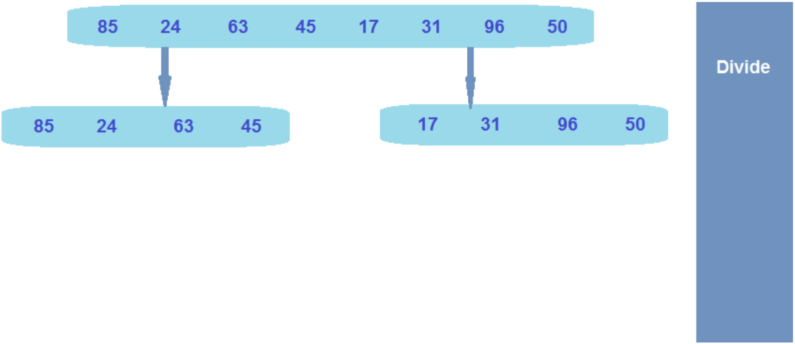
23	1	-3	67	14	-5	-13	24	75	92	-45	36	22	-2	0	39

Merge Sort



85 24 63 45 17 31 96 50

Merge Sort



85 24 63 45 17 31 96 50

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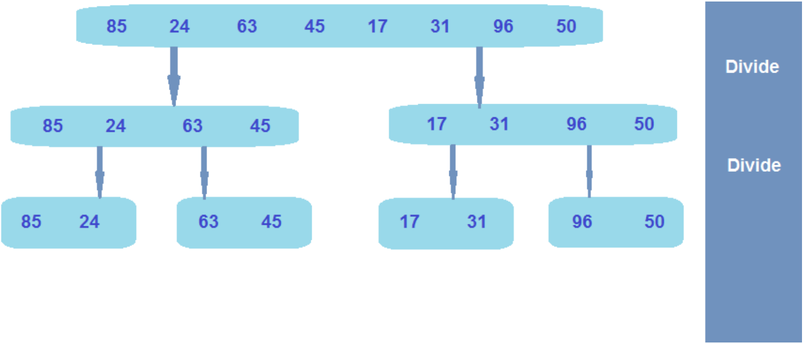
17 31 96 50

Divide

Merge Sort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Divide
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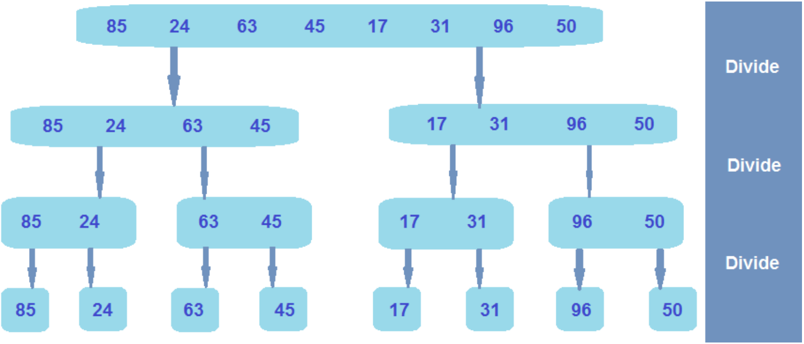
Merge Sort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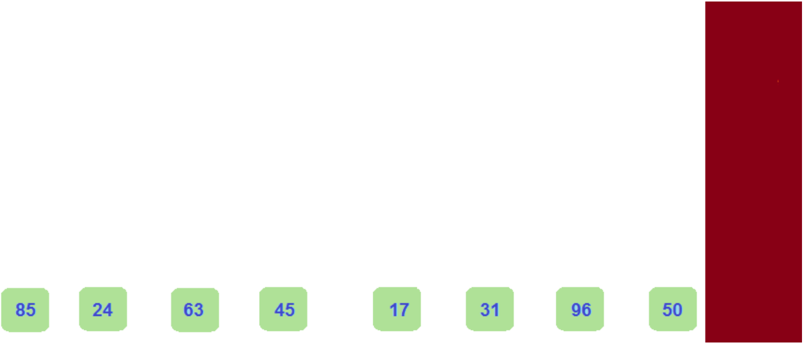

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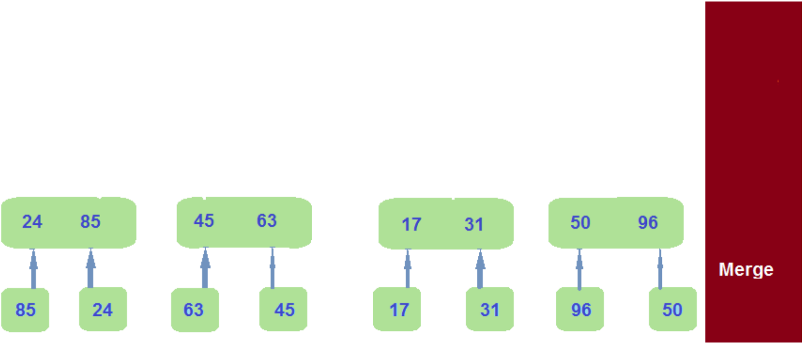

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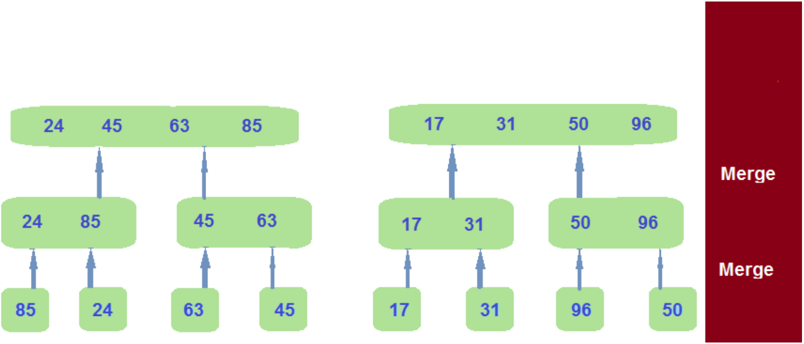
24 45 63 85
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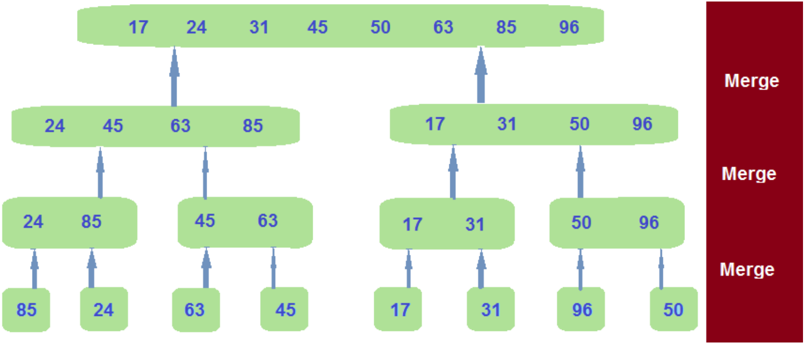
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Merge Sort

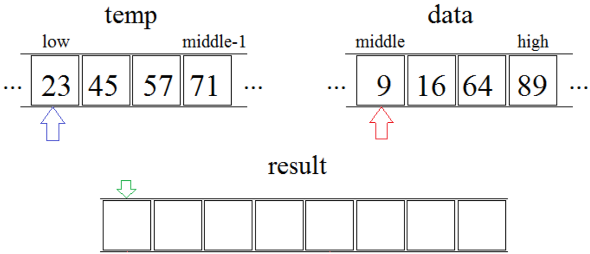
The Partition Procedures

```
private static void mergeSortRecursive(int data[], int temp[],
                                        int low, int high)
// pre: 0 <= low <= high < data.length
// post: values in data[low..high] are in ascending order
{
    int n = high-low+1;
    int middle = low + n/2;
    int i;
    if (n < 2) return;
    // move lower half of data into temporary storage
    for (i = low; i < middle; i++)
        temp[i] = data[i];
    // sort lower half of array
    mergeSortRecursive(temp,data,low,middle-1);
    // sort upper half of array
    mergeSortRecursive(data,temp,middle,high);
    // merge halves together
    merge(data,temp,low,middle,high);
}
```

The Merge Procedures

Suppose

- `temp[low..middle − 1]` are ascending
- `data[middle..high]` are ascending



Merge Sort

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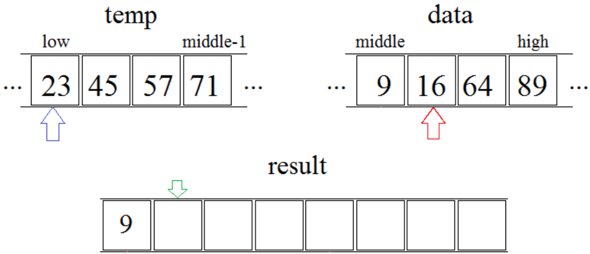

temp

low … middle-1

… 23 45 57 71 …

data

middle … high

… 9 16 64 89 …

result

9

Merge Sort

The Merge Procedures

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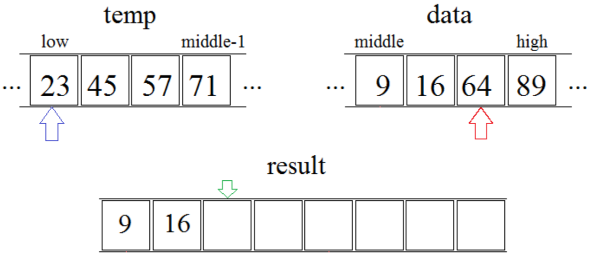

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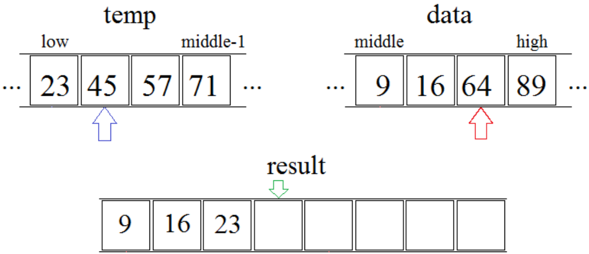

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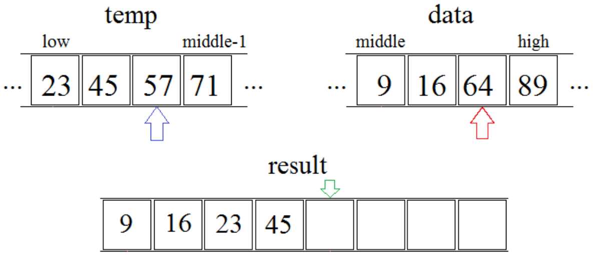

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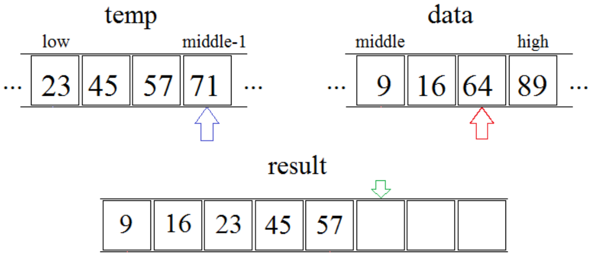
- data[middle..high] are ascending



Merge Sort

The Merge Procedures

Suppose

- `temp[low..middle − 1]` are ascending
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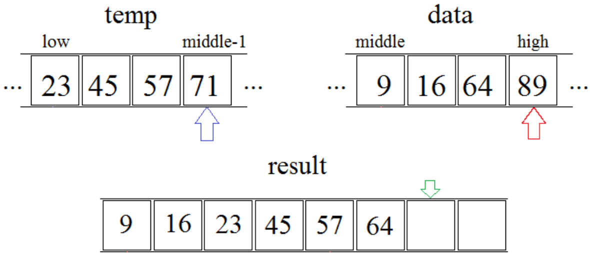

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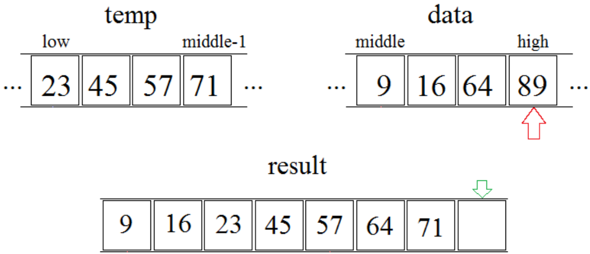

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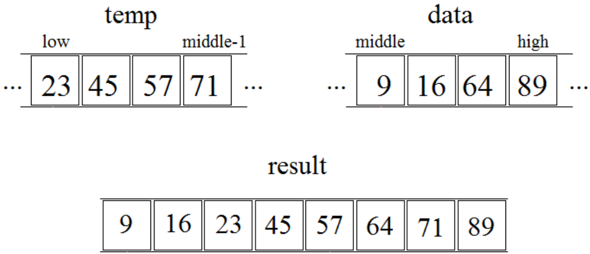


temp

low … middle-1

… 23 | 45 | 57 | 71 …

data

middle … high

… 9 | 16 | 64 | 89 …

result

9 | 16 | 23 | 45 | 57 | 64 | 71 | 89

Merge Sort

The Merge Procedures

```
private static void merge(int data[], int temp[],
                          int low, int middle, int high){
    int ri=0; //result index
    int ti=low;//temp index
    int di=middle;//data index
    int[] result = new int[high-low+1];
    while (ti<middle && di<=high){
        if (data[di]<temp[ti]){
            result[ri++] = data[di++];//smaller is in data
        } else{
            result[ri++] = temp[ti++];//smaller is in temp
        }
    }     while(ti<middle) result[ri++]=temp[ti++];
    while(di<=high) result[ri++]=data[di++];
    for(int i=0;i<high;i++) data[low+i]=result[i];
}
```

Merge Sort

```
public static void mergeSort(int data[], int n)
//pre:  0<=n <=data.length
//post:  values in data[0..n-1] are in ascending order
{
    mergeSortRecursive(data, new int[n], 0, n-1);
}
```

Merge Sort: Complexity

- The Partitioning Stage
  - The number of partitioning is $n - 1$.
- The Merging Stage
  - The number of operations is $O(n \log n)$.

The overall complexity of the algorithm: $O(n \log n)$

Merge Sort: Complexity

Optimality

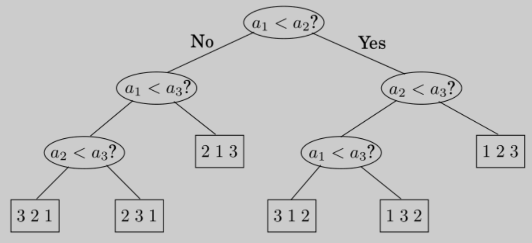
Instead of asking "Is merge sort the most efficient algorithm for sorting?",
we ask "What is the shortest time must any sorting algorithms run in?"

Comparison-Based Sorting

Merge sort is a comparison-based sorting algorithm, i.e., it sorts numbers based on a sequence of comparisons between elements from the input arrays.

Sort: Time Lower Bound

Comparison Tree



- The depth or height of the tree is the number of comparisons on the longest path from the root to a leaf.
- For $n > 0$, let $h(n)$ be the height of the comparison tree for n numbers.

Sorting: Time Lower Bound

Fact

Any comparison-based sorting algorithm uses at least $h(n)$ comparisons in the worst case.

Why? Otherwise there must be some sequence on which the algorithm fails.

A Lower Bound on Times of Comparison

What is $h(n)$ for any $n > 0$?

Sorting: Time Lower Bound

Observation 1

- The comparison tree for n numbers is a binary tree
- A binary tree with k leaves has at least height $\log k$

$\Rightarrow h(n) \geq \log k$ where k is the number of leaves.

Sorting: Time Lower Bound

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Observation 2

- Every leaf in the comparison tree is a permutation of n numbers
- There are $n!$ number of permutations with n numbers

$\Rightarrow k = n!$

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- Every leaf in the comparison tree is a permutation of n numbers
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$\Rightarrow h(n) \geq \log n!$

Sorting: Time Lower Bound

Observation 3

$$\begin{aligned} n! &= 1 \times 2 \times 3 \times \ldots \times n-1 \times n \\ &> \lfloor n/2 \rfloor \times \ldots \times n-1 \times n \\ &> (n/2)^{n/2} \end{aligned}$$

$\Rightarrow \log n! > \log(n/2)^{n/2} = n/2(\log n - 1)$
$\Rightarrow h(n)$ is $\Omega(n \log n)$

Sorting: Time Lower Bound

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Conclusion

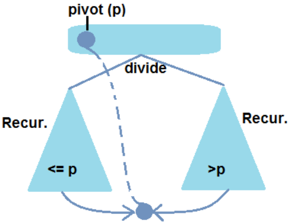
Any comparison-based sorting algorithm must use $\Omega(n \log n)$ number of comparisons in the worst case.
$\Rightarrow$ The best time complexity for any comparison-based sorting algorithm is $\Theta(n \log n)$.
$\Rightarrow$ Merge sort is optimal.

Quick Sort

Just like merge sort, quick sort is also a divide-and-conquer algorithm.

Idea

1. Divide: If S has 0 or 1 element, do nothing. Otherwise, pick a pivot/partitionElement from the array (first element). Rearrange the other elements into two parts, **those $\leq$ the pivot** and **those $>$ the pivot**.
2. Recur: Recursively sort these two parts
3. Conquer: Put the two resulting sorted arrays and the pivot in order.



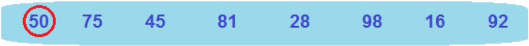
Quick Sort

Example:

50 75 45 81 28 98 16 92

Quick Sort

Example:



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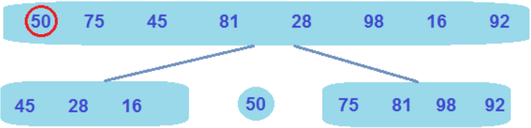


50 75 45 81 28 98 16 92

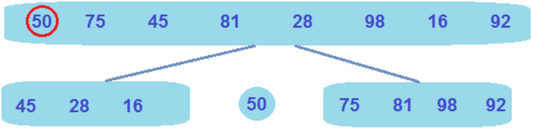
45 28 16

50

75 81 98 92

Quick Sort

Example:



50 75 45 81 28 98 16 92

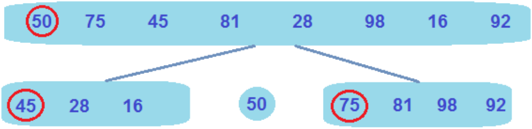
45 28 16

50

75 81 98 92

Quick Sort

Example:



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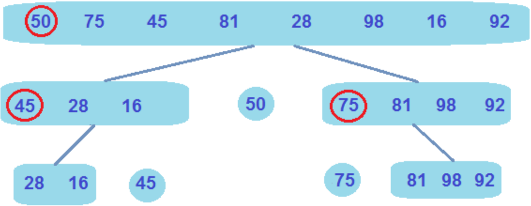

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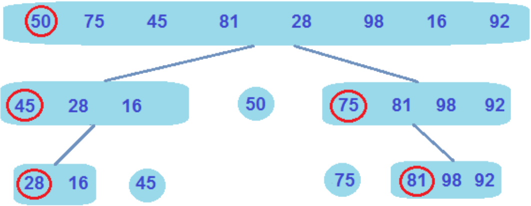

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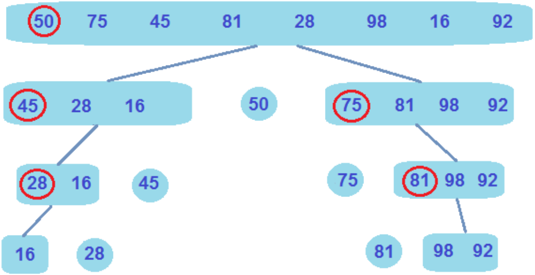

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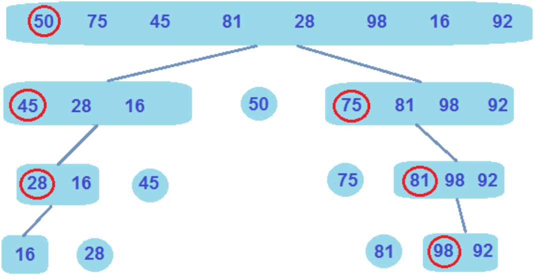

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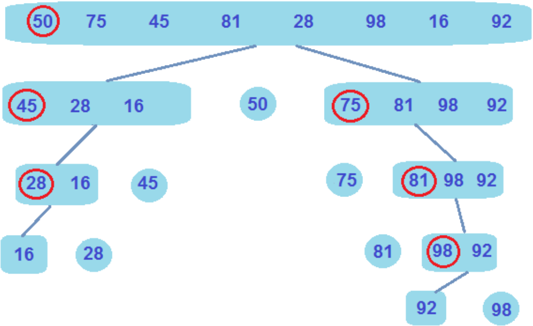

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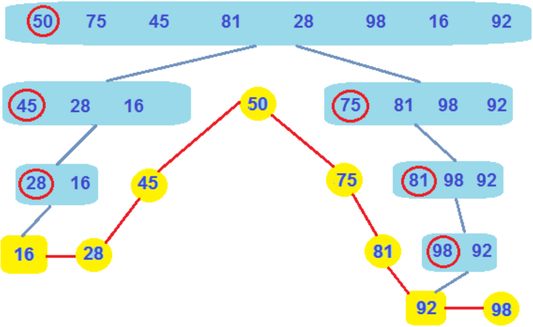


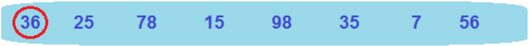
Quick Sort

Example 2:

36	25	78	15	98	35	7	56

Quick Sort

Example 2:



36 25 78 15 98 35 7 56

Quick Sort

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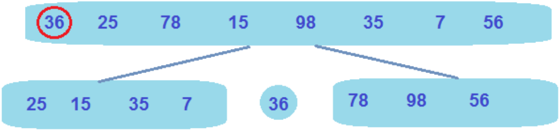

Quick Sort

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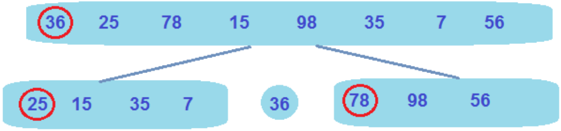

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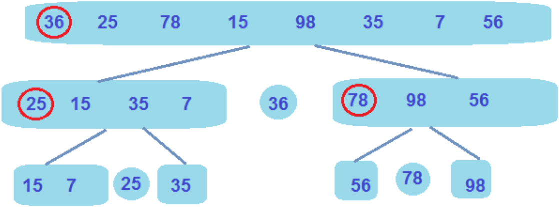

Quick Sort

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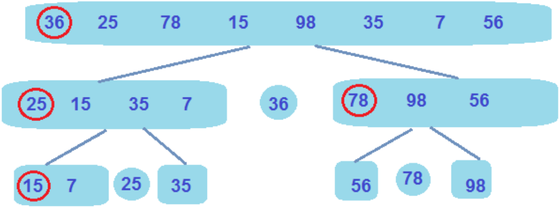

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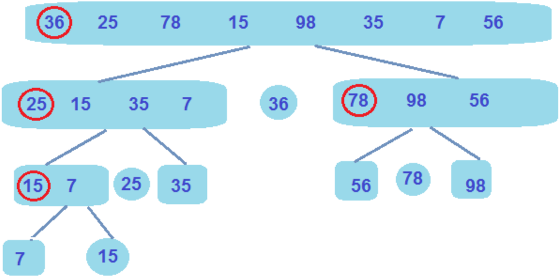

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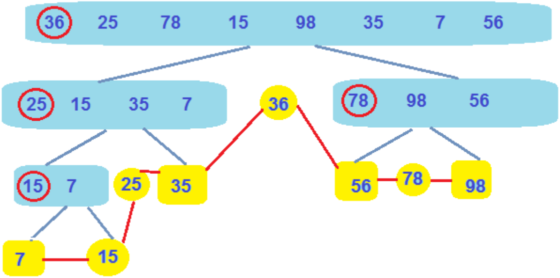

In-Place Quick Sort

- The most straight-forward implementation of quick sort requires creating two new arrays at each recursion step (otherwise we may need a lot of shiftings)

  ⇒ Takes too much memory.

- We would like to work only on the input array (without creating new arrays), just like selection, insertion and bubble sort.

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  ⇒ Takes too much memory.

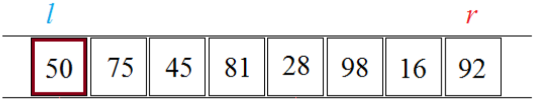
- We would like to work only on the input array (without creating new arrays), just like selection, insertion and bubble sort.
- In-place quick sort is a way of implementing quick sort so that it only works on the input array.

In-Place Quick Sort

The Partition Procedures

- Given the `data[]` array, and `left`, `right` pointers.
- Set `data[left]` as the pivot
- We want to arrange all elements ≤ pivot to its left; all elements > pivot to its right

Pivot on `left`



l							r
50	75	45	81	28	98	16	92

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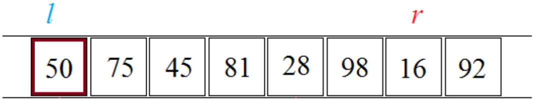

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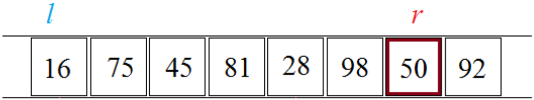
Pivot on `right`



l						r	
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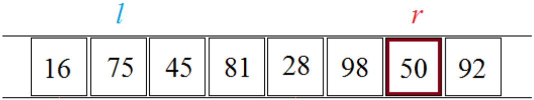

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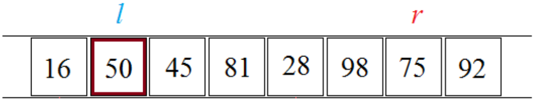

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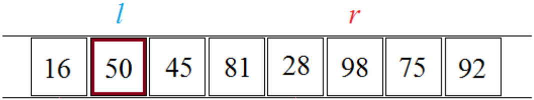

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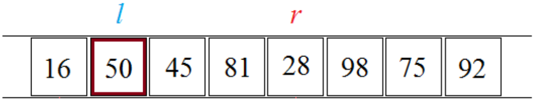

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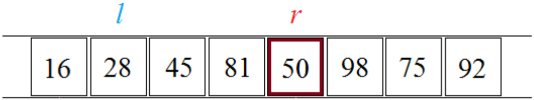
Pivot on `right`



	l			r			
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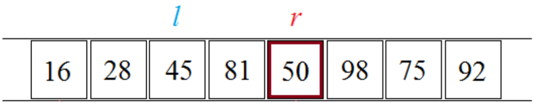

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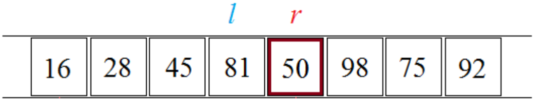
Pivot on `right`



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The Partition Procedures

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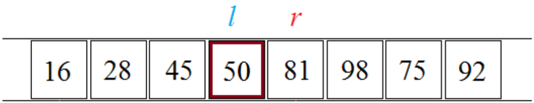
Pivot on `left`



			l	*r*			
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In-Place Quick Sort

The Partition Procedures

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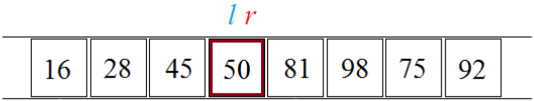
Pivot on `left` and `right`



In-Place Quick Sort

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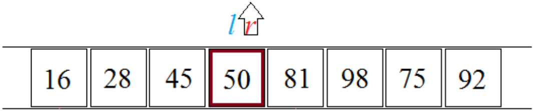
Return `right`



16	28	45	50	81	98	75	92

In-Place Quick Sort

The Partition Procedures

```
private static int partition(int data[], int left, int
right)
//pre:  left<= right
//post:  data[left] placed in the correct location
{
    while(true){
        //move right "pointer" towards left
        while(left<right && data[left] <data[right])
            right--;
        if (left<right) swap(data,left++,right);
        //move left pointer towards right
        while(left<right && data[left]<data[right])
            left++;
        if(left<right) swap(data,left,right--);
        else return right;
    }
}
```

In-Place Quick Sort

The Combine Procedures

```
private static void quickSortRecursive(int data[], int
left, int right)
//pre:  left<=right
//post:  data[left..right] in ascending order
{
    int pivot;
    if (left>=right) return;
    pivot=partition(data,left,right); //Partition
    quickSortRecursive(data,left,pivot-1); //Sort small
    quickSortRecursive(data,pivot+1,right); //Sort large
}
```

```
public static void quickSort(int data[], int n){
    quickSortRecursive(data,0,n-1);
}
```

Quick Sort: Complexity

Worst Case

- If the input array is already sorted, one side of the pivot is always empty, and the other side is $n - 1$
- There are n levels of recursion.
- Therefore $O(n^2)$.

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In the average case, quick sort runs very fast

Quick Sort: Complexity

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- Fix an input *list* of size n.
- Let $T_{list}(n)$ denote the running time for sorting *list*.
- The *pivot* is chosen as the first element in *list*.

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- Let $T_{list}(n)$ denote the running time for sorting *list*.
- The *pivot* is chosen as the first element in *list*.

A Recurrence for $T_{list}(n)$

- Let n_{small} be the number of elements smaller than *pivot*.
- Let n_{big} be the number of elements bigger than *pivot*.
- Then we have

$$T_{list}(n) = T_{list}(n_{small}) + T_{list}(n_{big}) + cn$$

Quick Sort: Complexity

Average-case analysis

Quick Sort: Complexity

Average-case analysis

- $T(n)$: the average-case running time of quick sort.

Quick Sort: Complexity

Average-case analysis

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- Then $T(n) = T(n_{small}) + T(n_{big}) + cn$.
- n_{small} may take any value in $0, 1, 2, \ldots, n-1$.
- $n_{big} = n - n_{small} - 1$.

Quick Sort: Complexity

Average-case analysis

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Equi-probability assumption

Assume that all initial orderings appear with equal probability.
$\Rightarrow$ For any $i, j \in \{0, 1, 2, \ldots, n-1\}$, $Pr[n_{small} = i] = Pr[n_{big} = j]$

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$\Rightarrow$ For any $i, j \in \{0, 1, 2, \ldots, n-1\}$, $Pr[n_{small} = i] = Pr[n_{big} = j]$

Therefore on average,

$$T(n_{small}) = [T(0) + T(1) + \ldots + T(n-1)] \div n$$
$$T(n_{big}) = [T(n-1) + T(n-2) + \ldots + T(0)] \div n$$

Quick Sort: Complexity

Quick Sort: Complexity

Average-case Analysis (continued.)

By the above arguments,

$$T(n) = 2([T(0) + \cdots + T(n-1)]) \div n + cn$$

Quick Sort: Complexity

Average-case Analysis (continued.)

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$$T(n) = 2([T(0) + \cdots + T(n-1)]) \div n + cn$$
$$nT(n) = 2([T(0) + \cdots + T(n-1)]) + cn^2$$

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Average-case Analysis (continued.)

By the above arguments,

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\begin{aligned}
T(n) &= 2([T(0) + \cdots + T(n-1)]) \div n + cn \\
nT(n) &= 2([T(0) + \cdots + T(n-1)]) + cn^2 \\
(n-1)T(n-1) &= 2([T(0) + \cdots + T(n-2)]) + c(n-1)^2
\end{aligned}
$$

Quick Sort: Complexity

Average-case Analysis (continued.)

By the above arguments,

$$\begin{aligned} T(n) &= 2([T(0) + \cdots + T(n-1)]) \div n + cn \\ nT(n) &= 2([T(0) + \cdots + T(n-1)]) + cn^2 \\ (n-1)T(n-1) &= 2([T(0) + \cdots + T(n-2)]) + c(n-1)^2 \\ nT(n) - (n-1)T(n-1) &= 2T(n-1) - 2cn + c \end{aligned}$$

Quick Sort: Complexity

Average-case Analysis (continued.)

By the above arguments,

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Quick Sort: Complexity

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T(n) &= 2([T(0) + \cdots + T(n-1)]) \div n + cn \\
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nT(n) &= (n+1)T(n-1) - c(2n-1) \\
\frac{T(n)}{(n+1)} &= \frac{T(n-1)}{n} - \frac{c(2n-1)}{n(n+1)}
\end{aligned}$$

Quick Sort: Complexity

Average-case Analysis (continued.)

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T(n) &= 2([T(0) + \cdots + T(n-1)]) \div n + cn \\
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nT(n) - (n-1)T(n-1) &= 2T(n-1) - 2cn + c \\
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\frac{T(n)}{(n+1)} &= \frac{T(n-1)}{n} - \frac{c(2n-1)}{n(n+1)} \\
\frac{T(n)}{n+1} &= \frac{T(n-1)}{n} + \frac{3c}{n+1} - \frac{c}{n}
\end{aligned}$$

Quick Sort: Complexity

Average-case Analysis (continued.)

Quick Sort: Complexity

Average-case Analysis (continued.)

Telescoping on $\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{3c}{n+1} - \frac{c}{n}$

Quick Sort: Complexity

Average-case Analysis (continued.)

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$$\begin{array}{llll}
\frac{T(n)}{n+1} & = \frac{T(n-1)}{n} & + \frac{3c}{n+1} & - \frac{c}{n} \\
\frac{T(n-1)}{n} & = \frac{T(n-2)}{n-1} & + \frac{3c}{n} & - \frac{c}{n-1} \\
\frac{T(n-2)}{n-1} & = \frac{T(n-3)}{n-2} & + \frac{3c}{n-1} & - \frac{c}{n-2} \\
\cdots & = \cdots & \cdots & \cdots \\
\frac{T(1)}{2} & = \frac{T(0)}{1} & + \frac{3c}{2} & - \frac{c}{1}
\end{array}$$

Quick Sort: Complexity

Average-case Analysis (continued.)

Telescoping on $\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{3c}{n+1} - \frac{c}{n}$

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\dots & = \dots & \dots & \dots \\
\frac{T(1)}{2} & = \frac{T(0)}{1} & + \frac{3c}{2} & - \frac{c}{1}
\end{array}$$

Cancel out the common terms, we have:

$$\frac{T(n)}{n+1} = 3c(\frac{1}{n+1} + \frac{1}{n} + \ldots + \frac{1}{2}) - c(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + \frac{1}{1})$$

Quick Sort: Complexity

Harmonic number

The n-th harmonic number is $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$.
Fact: H_n is $O(\log n)$.

Quick Sort: Complexity

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$$\frac{T(n)}{n+1} = 3c(\frac{1}{n+1} + \frac{1}{n} + \ldots + \frac{1}{2}) - c(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + \frac{1}{1})$$

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Average-case Analysis (continued.)

$$\begin{aligned}\frac{T(n)}{n+1} &= 3c(\frac{1}{n+1} + \frac{1}{n} + \ldots + \frac{1}{2}) - c(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + \frac{1}{1}) \\ &= 3c(H_{n+1} - 1) + c(H_n)\end{aligned}$$

Quick Sort: Complexity

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The n-th harmonic number is $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$.
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Average-case Analysis (continued.)

$$\begin{aligned}
\frac{T(n)}{n+1} &= 3c(\frac{1}{n+1} + \frac{1}{n} + \ldots + \frac{1}{2}) - c(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + \frac{1}{1}) \\
&= 3c(H_{n+1} - 1) + c(H_n) \\
&= 4cH_{n+1} - 3c - \frac{c}{n+1}
\end{aligned}$$

Quick Sort: Complexity

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The *n*-th harmonic number is $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$.
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Average-case Analysis (continued.)

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\frac{T(n)}{n+1} &= 3c(\frac{1}{n+1} + \frac{1}{n} + \ldots + \frac{1}{2}) - c(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + \frac{1}{1}) \\
&= 3c(H_{n+1} - 1) + c(H_n) \\
&= 4cH_{n+1} - 3c - \frac{c}{n+1} \\
T(n) &= 4c(n+1)H_{n+1} - 3c(n+1) - c
\end{aligned}$$

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The n-th harmonic number is $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$.
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Average-case Analysis (continued.)

$$\begin{aligned}
\frac{T(n)}{n+1} &= 3c(\frac{1}{n+1} + \frac{1}{n} + \ldots + \frac{1}{2}) - c(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + \frac{1}{1}) \\
&= 3c(H_{n+1} - 1) + c(H_n) \\
&= 4cH_{n+1} - 3c - \frac{c}{n+1} \\
T(n) &= 4c(n+1)H_{n+1} - 3c(n+1) - c
\end{aligned}$$

Therefore $T(n)$ is $\Theta(n \log n)$.

Sorting Algorithms

Algorithms	Worst Case Time	Average Case Time
Merge Sort	$O(n \log n)$	$O(n \log n)$
Quick Sort	$O(n^2)$	$O(n \log n)$

Lower bound for comparison-base sorting: $O(n \log n) \Rightarrow$

- MergeSort has optimal worst case complexity
- QuickSort has optimal average case complexity

Day 3: Divide and Conquer

Part III: Analysis of Divide and Conquer Algorithms

Runtime Analysis

Divide and Conquer

The *running time* $T(n)$ of a Divide-and-Conquer algorithm can normally be specified by

$$T(n) = aT(n/b) + f(n).$$

The problem is entirely mathematical: Solve the above recursion.

Master Theorem

What is the master theorem?

The master theorem provides a direct way to solve recurrence of the form

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$, $b > 1$ are constants and $f(n)$ is a positive function.

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where $a \geq 1$, $b > 1$ are constants and $f(n)$ is a positive function.

Examples

- $T(n) = 3T(n/2) + n^2$
- $T(n) = 16T(n/2) + 3n \log n$
- $T(n) = T(n/2) + 3$
- $T(n) = \sqrt{2}T(n/4) + n^{0.51}$

Master Theorem

Intuitive Version

The master theorem allows us to solve the recurrence

$$T(n) = aT(n/b) + f(n)$$

by comparing the function $f(n)$ with $n^{\log_b a}$.
There are three cases:

Master Theorem

Intuitive Version

The master theorem allows us to solve the recurrence

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There are three cases:

- Case 1: $f(n)$ is much smaller than $n^{\log_b a}$.
  Then $T(n)$ has complexity $n^{\log_b a}$.

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There are three cases:

- Case 1: $f(n)$ is much smaller than $n^{\log_b a}$.
  Then $T(n)$ has complexity $n^{\log_b a}$.
- Case 2: $f(n)$ is the same with $n^{\log_b a}$.
  Then $T(n)$ has complexity $n^{\log_b a} \log n$.

Master Theorem

Intuitive Version

The master theorem allows us to solve the recurrence

$$T(n) = aT(n/b) + f(n)$$

by comparing the function $f(n)$ with $n^{\log_b a}$.
There are three cases:

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  Then $T(n)$ has complexity $n^{\log_b a}$.
- Case 2: $f(n)$ is the same with $n^{\log_b a}$.
  Then $T(n)$ has complexity $n^{\log_b a} \log n$.
- Case 3: $f(n)$ is much bigger than $n^{\log_b a}$.
  Then $T(n)$ has complexity $f(n)$.

Master Theorem

Master theorem

Let $\alpha \geq 1$ and $b > 1$, let $f(n)$ be a positive function, and let $T(n)$ be defined as:

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then there are three cases:

Master Theorem

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where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then there are three cases:

1. If $f(n)$ is $O(n^{\log_b a - e})$ for some constant $e > 0$, then

$$T(n) = \Theta(n^{\log_b a}).$$

Master Theorem

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2. If $f(n)$ is $\Theta(n^{\log_b a})$, then

$$T(n) = \Theta(n^{\log_b a} \log n).$$

Master Theorem

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where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then there are three cases:

1. If $f(n)$ is $O(n^{\log_b a - e})$ for some constant $e > 0$, then
$$T(n) = \Theta(n^{\log_b a}).$$
2. If $f(n)$ is $\Theta(n^{\log_b a})$, then
$$T(n) = \Theta(n^{\log_b a} \log n).$$
3. If $f(n)$ is $\Omega(n^{\log_b a + e})$ for some constant $e > 0$, and the regularity condition $af(n/b) \leq rf(n)$ for some $r < 1$ holds, then
$$T(n) = \Theta(f(n)).$$

Note: Most of the functions we see satisfy the regularity condition.

Master Theorem

Examples

- $T(n) = 9T(n/3) + n.$

- $T(n) = 4T(n/2) + n^2.$

- $T(n) = 3T(n/3) + n^2.$

Master Theorem

Examples

- $T(n) = 9T(n/3) + n.$
  $a = 9, b = 3, f(n) = n, n^{\log_b a} = n^2.$
  $f(n)$ is $O(n^{2-e})$ for some e (say $e = 0.5$).
  Hence we apply case 1.
  $T(n)$ is $\Theta(n^2)$.
- $T(n) = 4T(n/2) + n^2.$
- $T(n) = 3T(n/3) + n^2.$

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- $T(n) = 4T(n/2) + n^2$.
  $a = 4, b = 2, f(n) = n^2, n^{\log_b a} = n^2$.
  $f(n)$ is $\Theta(n^2)$. Hence we apply case 2.
  $T(n)$ is $\Theta(n^2 \log n)$.
- $T(n) = 3T(n/3) + n^2$.

Master Theorem

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  $a = 9, b = 3, f(n) = n, n^{\log_b a} = n^2$.
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  $f(n)$ is $\Theta(n^2)$. Hence we apply case 2.
  $T(n)$ is $\Theta(n^2 \log n)$.
- $T(n) = 3T(n/3) + n^2$.
  $a = 3, b = 3, f(n) = n^2, n^{\log_b a} = n^{\log_3 3} = n$.
  $f(n)$ is $\Omega(n^e)$ for some e (say $e = 0.5$).
  Hence we apply case 3.
  $T(n)$ is $\Theta(n^2)$.

Master Theorem

Examples

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  $a = 9, b = 3, f(n) = n, n^{\log_b a} = n^2$.
  $f(n)$ is $O(n^{2-e})$ for some e (say $e = 0.5$).
  Hence we apply case 1.
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  $a = 4, b = 2, f(n) = n^2, n^{\log_b a} = n^2$.
  $f(n)$ is $\Theta(n^2)$. Hence we apply case 2.
  $T(n)$ is $\Theta(n^2 \log n)$.
- $T(n) = 3T(n/3) + n^2$.
  $a = 3, b = 3, f(n) = n^2, n^{\log_b a} = n^{\log_3 3} = n$.
  $f(n)$ is $\Omega(n^e)$ for some e (say $e = 0.5$).
  Hence we apply case 3.
  $T(n)$ is $\Theta(n^2)$.

Note: Master theorem holds without assuming n is a power of b.

Master Theorem

Cases where master theorem doesn't work

- When $a < 1$.
  e.g. $T(n) = 0.5T(n/2) + n$.

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- When $f(n)$ is negative.
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- When $f(n)$ is smaller than $n^{\log_b a}$ but is not small enough:
  ($f(n)$ is $O(n^{\log_b a})$ but $f(n)$ is not $O(n^{\log_b a - e})$ for any $e > 0$)
  e.g. $T(n) = 2T(n/2) + n/\log n$.

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- When $f(n)$ is smaller than $n^{\log_b a}$ but is not small enough:
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  e.g. $T(n) = 2T(n/2) + n/\log n$.
- When $f(n)$ is bigger than $n^{\log_b a}$ but is not big enough:
  ($f(n)$ is $\Omega(n^{\log_b a})$ but $f(n)$ is not $\Omega(n^{\log_b a+e})$ for any $e > 0$)
  e.g. $T(n) = 2T(n/2) + n \log n$.

Example 1: Karatsuba's Algorithm

Running time: $T(n) = 3T(n/2) + cn$

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Running time: $T(n) = 3T(n/2) + cn$

- $n^{\log_b a} = n^{\log_2 3} \approx n^{1.59}$
- $f(n) = cn$
- cn is $O(n^{1.59-0.1})$

Case 1: cn is much smaller than $n^{\log_2 3}$.
$\Rightarrow T(n)$ is $\Theta(n^{\log_2 3})$

Example 2: Merge Sort

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Running time: $T(n) = 2T(n/2) + cn$

- $n^{\log_2 2} = n$
- $f(n) = cn$
- cn is $\Theta(n)$

Case 2: cn has the same asymptotic growth as n.
$\Rightarrow T(n)$ is $\Theta(n \log n)$

Day 3: Divide and Conquer

Part III: Matrix Multiplication and Strassen's Algorithm

Matrix Multiplications

Problem

INPUT: Two $n \times n$ matrices A, B
OUTPUT: Their product matrix $A \times B$.

This is a crucial process in

- computer graphics
- Linear programming
- Linear dynamical systems
- etc.

Matrix Multiplications

Standard Multiplication Algorithm

The (i, j)-entry of $A \times B$ is $\sum_{k=1}^{n} A[i,k]B[k,j]$, i.e.,

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix} =$$

$$\begin{pmatrix} a_1a_2 + b_1d_2 + c_1g_2 & a_1b_2 + b_1e_2 + c_1h_2 & a_1c_2 + b_1f_2 + c_1i_2 \\ d_1a_2 + e_1d_2 + f_1g_2 & d_1b_2 + e_1e_2 + f_1h_2 & d_1c_2 + e_1f_2 + f_1i_2 \\ g_1a_2 + h_1d_2 + i_1g_2 & g_1b_2 + h_1e_2 + i_1h_2 & g_1c_2 + h_1f_2 + i_1i_2 \end{pmatrix}$$

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Standard algorithm: a three-nested loop.

- Inner-most loop: Compute value for an entry
- Middle loop: Compute values in a row
- Outer-most loop: Compute values in all rows

Matrix Multiplications

Standard Multiplication Algorithm

The (i, j)-entry of $A \times B$ is $\sum_{k=1}^{n} A[i,k]B[k,j]$, i.e.,

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix} =$$

$$\begin{pmatrix} a_1a_2 + b_1d_2 + c_1g_2 & a_1b_2 + b_1e_2 + c_1h_2 & a_1c_2 + b_1f_2 + c_1i_2 \\ d_1a_2 + e_1d_2 + f_1g_2 & d_1b_2 + e_1e_2 + f_1h_2 & d_1c_2 + e_1f_2 + f_1i_2 \\ g_1a_2 + h_1d_2 + i_1g_2 & g_1b_2 + h_1e_2 + i_1h_2 & g_1c_2 + h_1f_2 + i_1i_2 \end{pmatrix}$$

Standard algorithm: a three-nested loop.

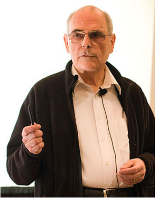
- Inner-most loop: Compute value for an entry
- Middle loop: Compute values in a row
- Outer-most loop: Compute values in all rows

Time complexity $\Theta(n^3)$

Matrix Multiplications

Let's try
Divide-and-Conquer
on this problem

Volker Strassen (1969)
Professor of Math and Stats
University of Konstanz, Germany
Knuth Prize Winner 2008



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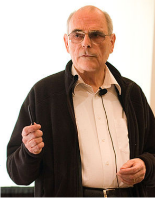

Observations

We may divide a $2n \times 2n$ matrix into four $n \times n$ sub-matrices.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{pmatrix} =$$

$$\begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{pmatrix}$$

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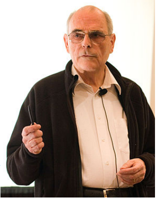

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Matrix Multiplications

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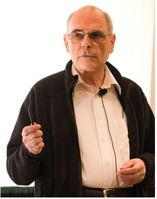

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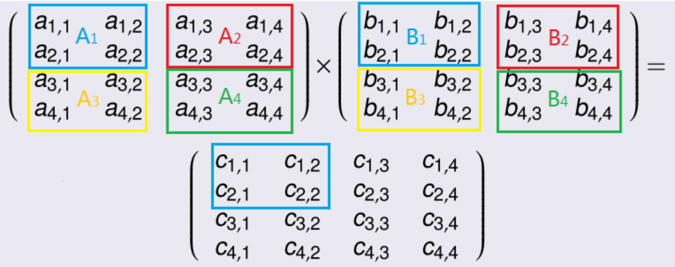

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{pmatrix} =$$

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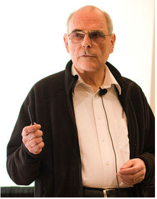

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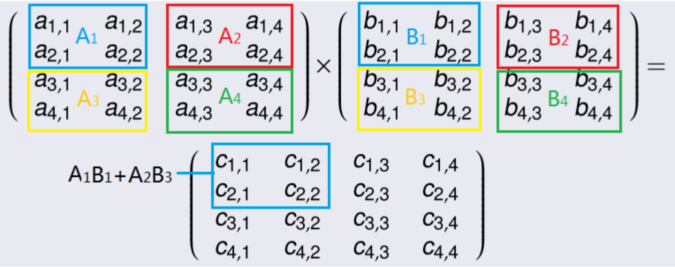

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{pmatrix} =$$

$$A_1B_1+A_2B_3 \quad \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{pmatrix}$$

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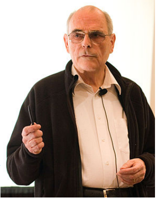

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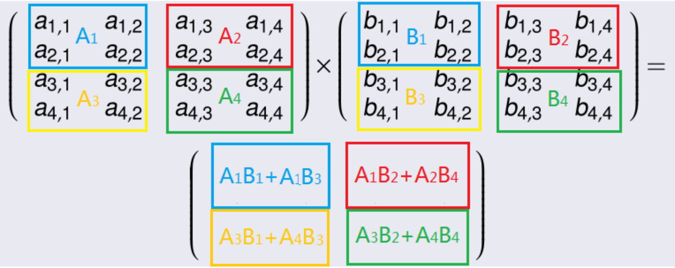

$$\begin{pmatrix} \boxed{\begin{matrix} a_{1,1} & A_1 & a_{1,2} \\ a_{2,1} & & a_{2,2} \end{matrix}} & \boxed{\begin{matrix} a_{1,3} & A_2 & a_{1,4} \\ a_{2,3} & & a_{2,4} \end{matrix}} \\ \boxed{\begin{matrix} a_{3,1} & A_3 & a_{3,2} \\ a_{4,1} & & a_{4,2} \end{matrix}} & \boxed{\begin{matrix} a_{3,3} & A_4 & a_{3,4} \\ a_{4,3} & & a_{4,4} \end{matrix}} \end{pmatrix} \times \begin{pmatrix} \boxed{\begin{matrix} b_{1,1} & B_1 & b_{1,2} \\ b_{2,1} & & b_{2,2} \end{matrix}} & \boxed{\begin{matrix} b_{1,3} & B_2 & b_{1,4} \\ b_{2,3} & & b_{2,4} \end{matrix}} \\ \boxed{\begin{matrix} b_{3,1} & B_3 & b_{3,2} \\ b_{4,1} & & b_{4,2} \end{matrix}} & \boxed{\begin{matrix} b_{3,3} & B_4 & b_{3,4} \\ b_{4,3} & & b_{4,4} \end{matrix}} \end{pmatrix} =$$

$$\begin{pmatrix} \boxed{A_1B_1 + A_1B_3} & \boxed{A_1B_2 + A_2B_4} \\ \boxed{A_3B_1 + A_4B_3} & \boxed{A_3B_2 + A_4B_4} \end{pmatrix}$$

Matrix Multiplications

Example:

$$\begin{pmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & -2 & 4 \\ -1 & 0 & 1 & 0 \\ 5 & 2 & 1 & -2 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & -1 & 1 \\ 1 & 0 & -2 & 3 \\ 2 & 3 & 1 & -3 \\ -1 & -2 & 0 & 1 \end{pmatrix}$$

Result:

$$\begin{pmatrix} 14 & 12 & -6 & 3 \\ -6 & -14 & -6 & 16 \\ -1 & 2 & 2 & -4 \\ 21 & 12 & -8 & 8 \end{pmatrix}$$

Matrix Multiplications

First Attempt

Recursively solve the 8 sub-matrices multiplications:

$$A_1B_1, A_2B_3, A_1B_2, A_2B_4, A_3B_1, A_4B_3, A_3B_2, A_4B_4$$

Then some additions ($\Theta(n^2)$-time).
Thus $T(n) = 8T(n/2) + cn^2$.

Matrix Multiplications

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By Master theorem, $T(n)$ is $\Theta(n^3)$.
No improvement from the standard way.
First attempt fails.

Matrix Multiplications

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Goal: "Group" some of the multiplications together so we need < 8 sub-matrix multiplication.

Strassen’s Algorithm

- $P_1 = A_1(B_2 - B_4)$
- $P_2 = (A_1 + A_2)B_4$
- $P_3 = (A_3 + A_4)B_1$
- $P_4 = A_4(B_3 - B_1)$
- $P_5 = (A_1 + A_4)(B_1 + B_4)$
- $P_6 = (A_2 - A_4)(B_3 + B_4)$
- $P_7 = (A_1 - A_3)(B_1 + B_2)$

$$\left(\begin{array}{c|c} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ \hline A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{array}\right) =$$

$$\left(\begin{array}{c|c} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ \hline P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{array}\right)$$

Thus we only need 7 multiplications of sub-matrices.

Strassen's Algorithm

Strassen's Algorithm

Given two input $n \times n$ matrices A, B, do the following:

- If A, B have very small dimensions, directly multiply them
- Otherwise divide A,B into $A_1, \ldots, A_4$, $B_1, \ldots, B_4$.
- Compute $P_1, \ldots, P_7$, each use one recursive call.
- Then add and subtract P_i's to get the output matrix $A \times B$

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Complexity

Let $T(n)$ be the time it takes to multiples two $n \times n$ matrices.
We have $T(n) = 7T(n/2) + cn^2$
By Master theorem, $T(n)$ is $\Theta(n^{\log 7}) \approx \Theta(n^{2.808})$
This is asymptotically better than $O(n^3)$!

Divide and Conquer Summary

- Algorithm design technique: Divide and Conquer
  - Partition the problems into subproblems
  - Combine sub-solutions to overall solution
- Analysis Technique for Divide and Conquer: Master Theorem
- Karatsuba's algorithm (Integer multiplication): $O(n^{1.59})$
- Strassen's algorithm (Matrix multiplication): $O(n^{2.808})$