Algorithms Design and Analysis

Day 5 Traversing a Graph

2015, AUT-CJLU

Part I: Depth First Search

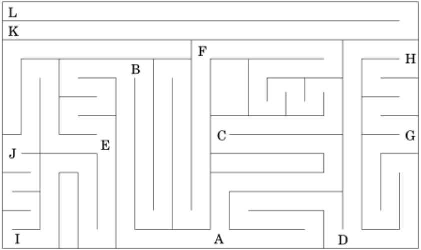
Graph Traversal

Question

If I use a digraph to store a collection of data, how can I search for information in the graph?

Answer

Traverse through each node of the graph.



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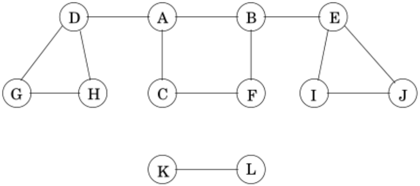

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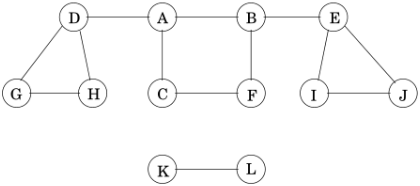

String: Keep track of the path we are currently on

Chalk: Mark a node after we have finished visiting it

Simulating String and Chalk

Question

Can we use an algorithm to simulate the "string+chalk" procedure to traverse a graph?

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Strategy: Each node is processed in two stages:

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Can we use an algorithm to simulate the "string+chalk" procedure to traverse a graph?

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Graph Traversal Problem

INPUT: A (representation of) digraph G
OUTPUT: Enumeration of all nodes in the digraph

We would like a traversal algorithm that reveals also the link topology of the graph.

Depth First Search

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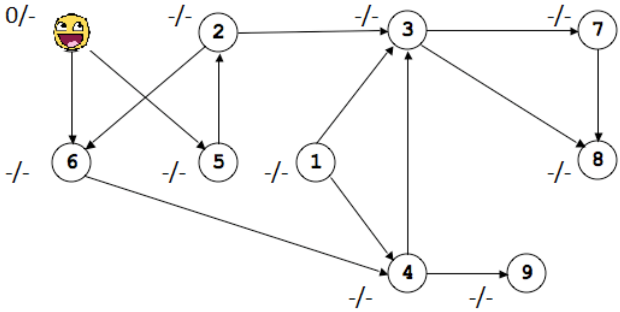

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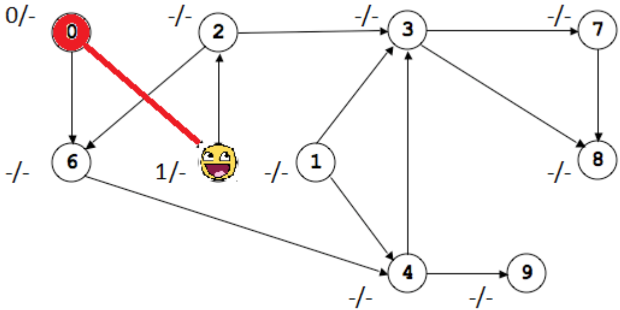

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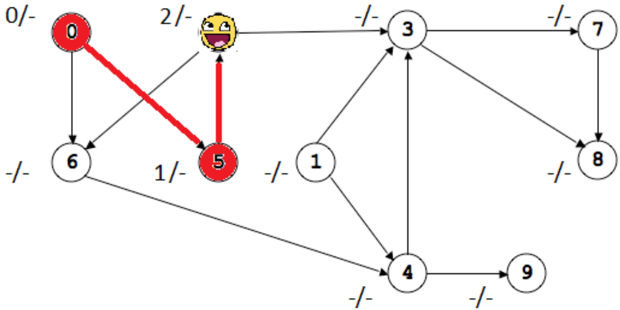

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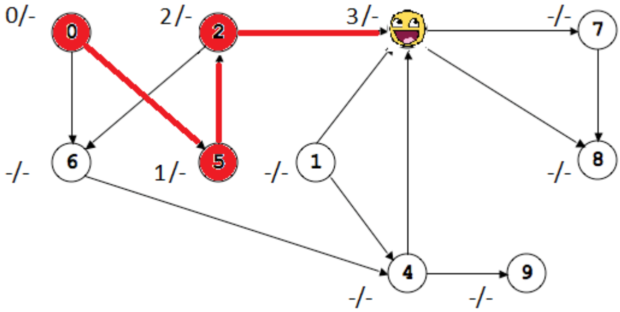

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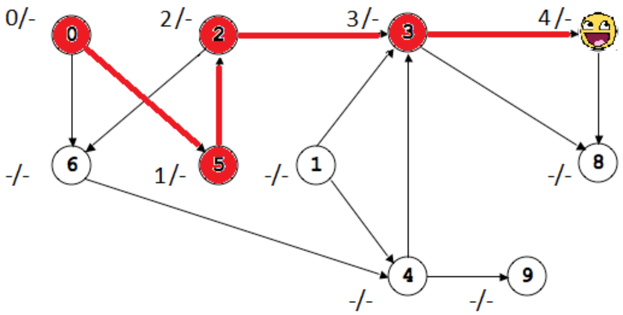

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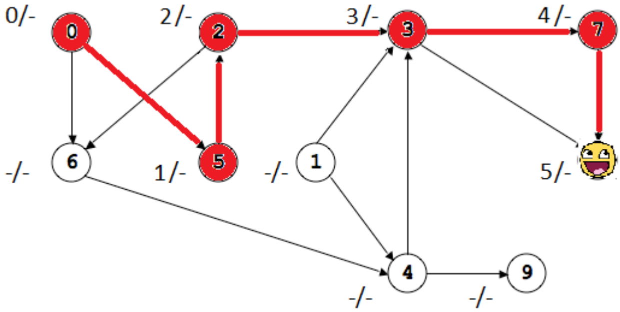

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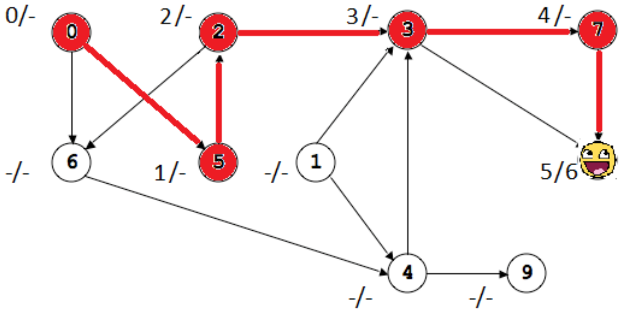

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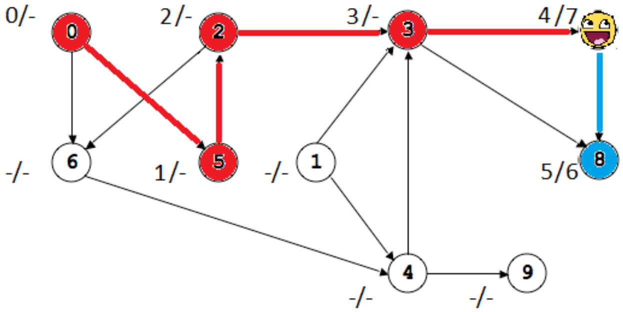

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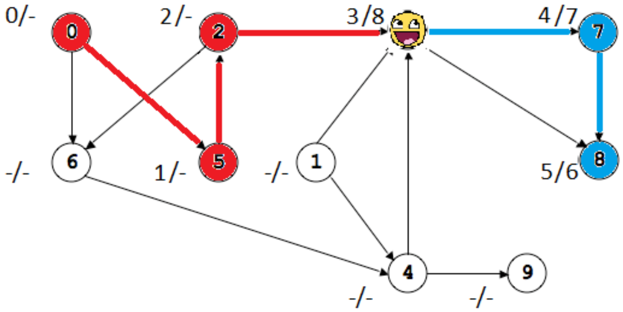

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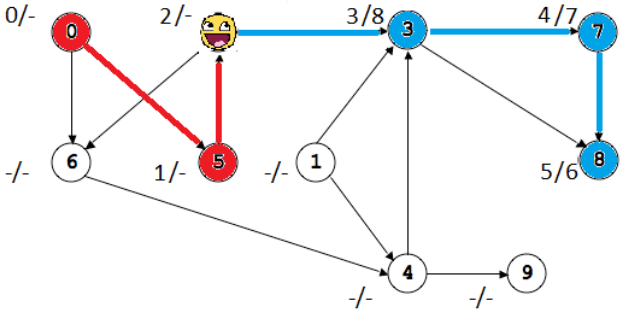

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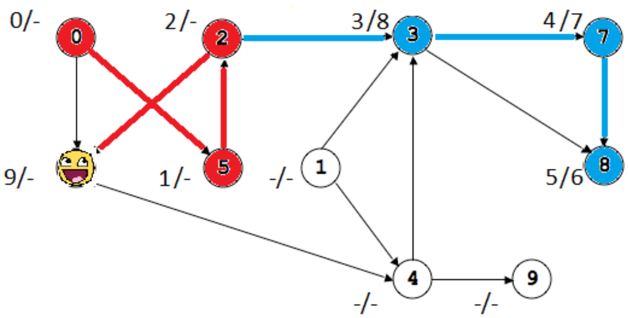

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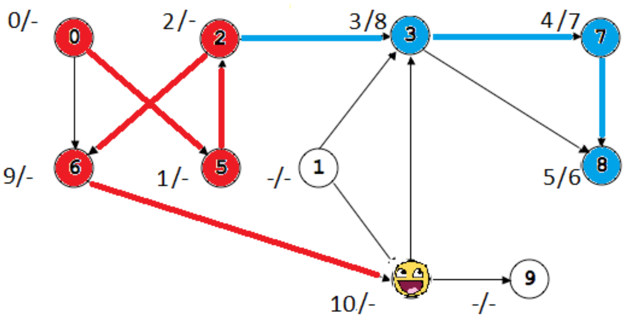

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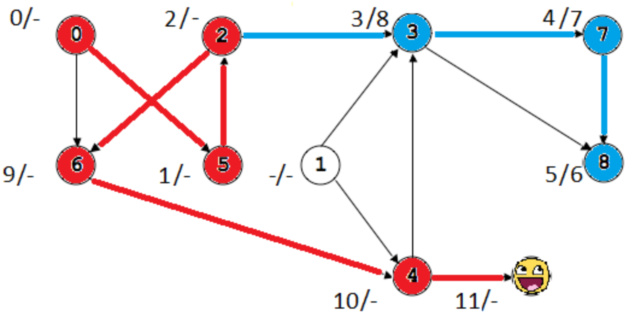

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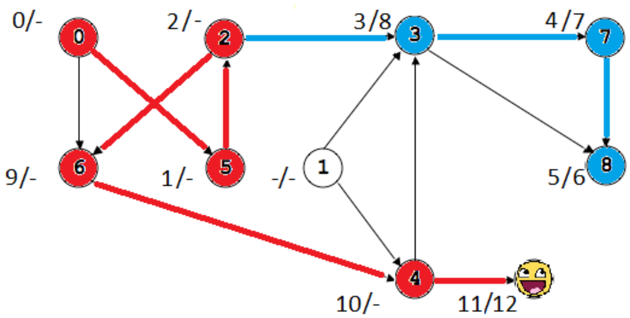

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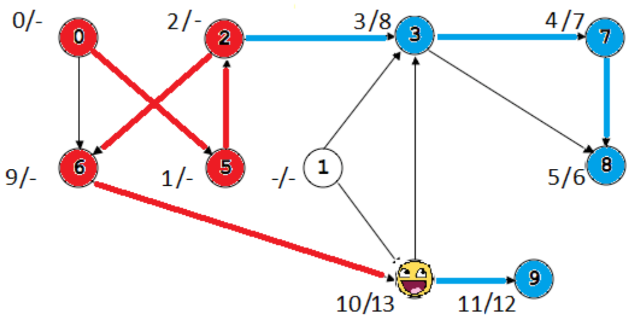

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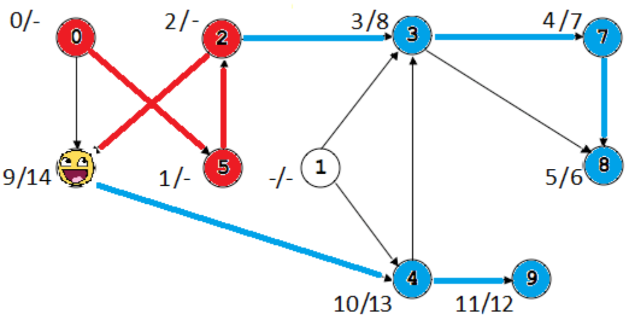

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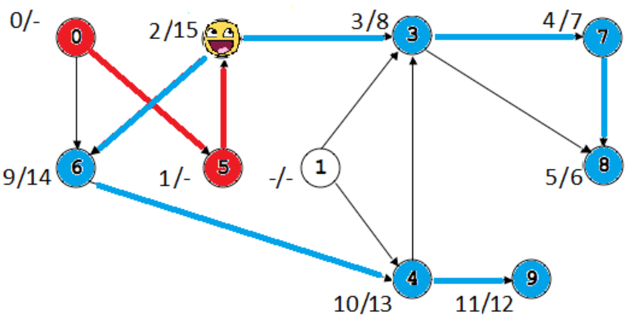

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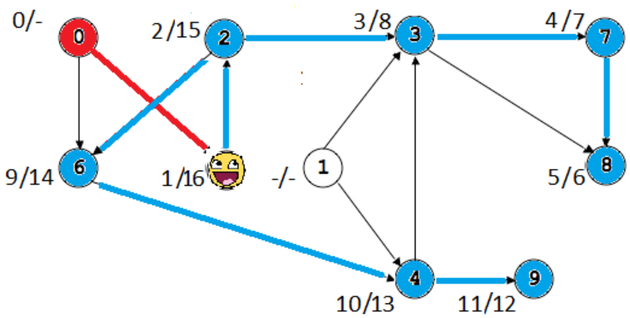

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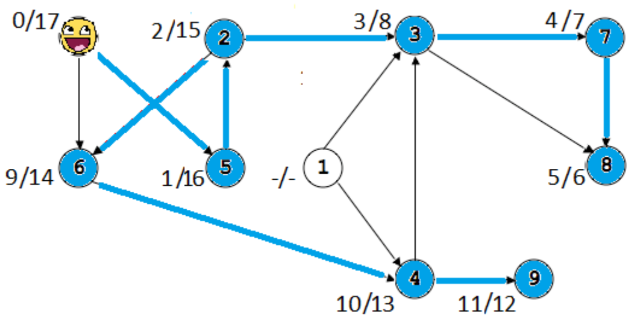

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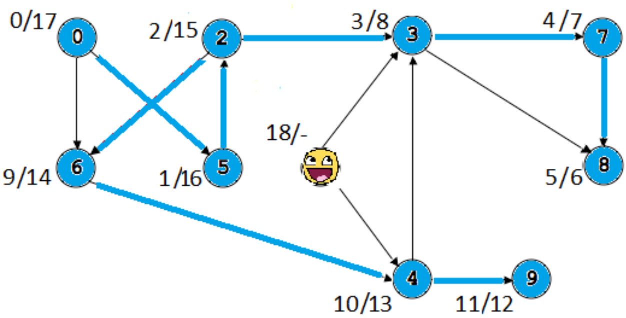

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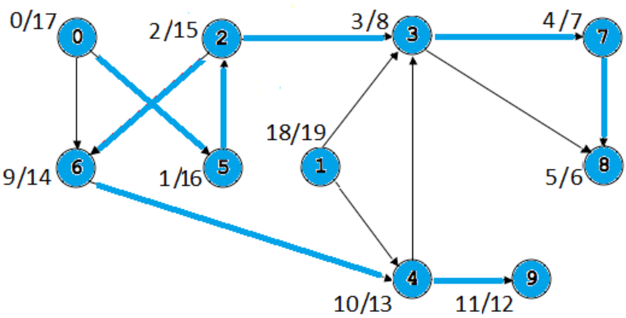

Depth First Search: Recursive Implementation

Maintain visited(v) for every $v \in V$.

Algorithm explore(G, v)

INPUT: A digraph G and a node v

```
visited(v) ← true
call discover(v) (perform operations to discover v)
for (v,u) ∈ E do
    if ¬ visited(u) do
        call explore(G,u)
call finish(v) (perform operations to finish v)
```

Algorithm dfs(G)

INPUT: A digraph G

```
for v ∈ V do
    visited(v) ← false
for v ∈ V do
    if ¬ visited(v) do
        call explore(G,v)
```

Depth First Search: Stack Implementation

Note: The current path changes in a FILO order.

Algorithm explore_stack(G, v)

INPUT: A digraph G, and a starting node v
create an empty stack S
push v to S
visited$(v) \leftarrow$ *true*

```
while  S ≠ ∅ do
    u ← top element of S
    call discover(u)
    w ← first node such that (u,w) ∈ E and visited(w) is
false
    if w does not exist then do
        call finish(u)
        pop u from S
    else do
        push w to S
        visited(w) ← true
```

Depth First Search: Complexity

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Fact

The DFS algorithm takes $O(n+m)$ time with adjacency list and $O(n^2)$ with adjacency matrix.

DFS and Reachability

Definition: Reachability

We say a node u is reachable from a node v in a graph G if there is a path that starts at v and ends at u.

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Suppose we run explore(G, v) on input graph G and node v in G, any node u is visited by the algorithm if and only if it is reachable from v.

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   True, as we only followed edges in G.

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Why?

1. If u is visited, then u is reachable.
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2. If u is reachable, then u is visited.

Proof. Suppose w is reachable but not visited.
   Then there is a path $v \rightsquigarrow w$.
   Take the last visited u on the path ($v \rightsquigarrow u \to u' \rightsquigarrow w$).
   Then we must visit u' from u. Contradiction.

Depth First Search and Search Forest

Definition [Search Forest]

- A forest is a collection of trees.
- DFS defines a forest in the digraph. We call this forest the DFS forest.
- The DFS forest contains all paths DFS used to visit nodes in G.

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Solution

Maintain a timer in the algorithm, and two times $pre(u)$ and $post(u)$ for each node u

Depth First Search and Search Forest

pre(u) **and post**(u)

procedure discover(v)	procedure finish(v)
$\text{pre}(v) \leftarrow \textit{clock}$	$\text{post}(v) \leftarrow \textit{clock}$
$\textit{clock} \leftarrow \textit{clock} + 1$	$\text{clock} \leftarrow \textit{clock} + 1$

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Observation

If u is an ancestor of v, then

$$pre(u) < pre(v) < post(v) < post(u)$$

If v is an ancestor of v, then

$$pre(v) < pre(u) < post(u) < post(v)$$

If neither case, then

$$pre(u) < post(u) < pre(v) < post(v) \text{ or}$$
$$pre(v) < post(v) < pre(u) < post(u)$$

Depth First Search and Search Forest

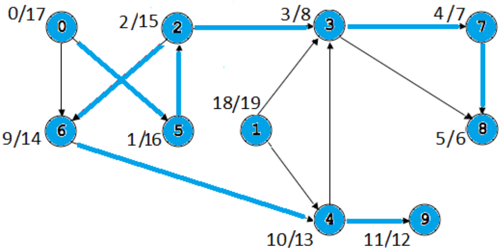
Therefore we can represent the search forest in parenthesis form:

DFS-Forest(G)

Write down a sequence of symbols $(n$ and $n)$, where $n \in \{1, \ldots, n\}$ such that:

If $pre(u) = k$, then the kth symbol is $(u$
If $post(u) = k$, then the kth symbol is $u)$

e.g. (0 (5 (2 (3 (7 (8 8) 7) 3) (6 (4 (9 9) 4) 6) 2) 5) 0) (1 1)

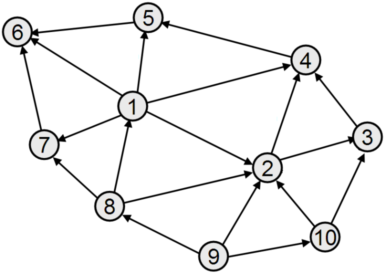


Part IV: Cyclicity and Linearisations

DFS and DAGs

Definition [DAG]

A directed acyclic graph (dag) is a digraph that does not contain a cycle.



Question

Given a digraph, decide if the digraph is a dag.

DFS and DAGs

Definition [Edge Classification]

Let T be the DFS forest in G. There are four types of edges in G:

- If (u, v) belongs to the search forest, (u, v) is a tree edge;
- Otherwise if u is an ancestor of v in T, (u, v) is a forward edge;
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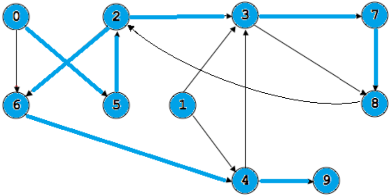

Tree edges: (0,5)(5,2),(2,6),(2,3),(3,7),(7,8),(6,4),(4,9)

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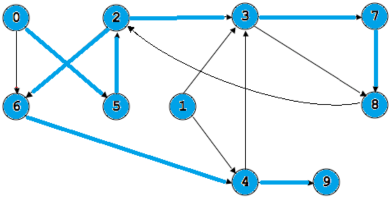

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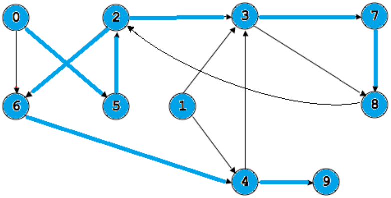

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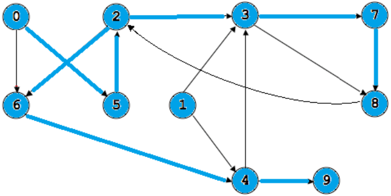

Tree edges: (0,5)(5,2),(2,6),(2,3),(3,7),(7,8),(6,4),(4,9)
Forward edges: (0,6),(3,8) Back edges: (8,2)
Cross edges: (4,3),(1,3),(1,4)

DFS and DAGs

Fact.

Let G be a digraph. Then the following are equivalent:

(1). G is a DAG

(2). the DFS forest has no back edge.

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Proof.

$\Rightarrow$ Suppose G is a dag, then the search forest doesn't have a back edge as otherwise, there will be a cycle.

$\Leftarrow$ Suppose G is not a dag, then there is a cycle C in G.
Let v be the first node discovered by the DFS in C.
Let (u, v) be the edge in C that goes into v.
Then in the search tree v is an ancestor of u.
Then (u, v) is a back edge. $\square$

DFS and DAGs

Fact

The following algorithm runs in time $O(n + m)$ and decides whether any given digraph G is a dag.

DFS and DAGs

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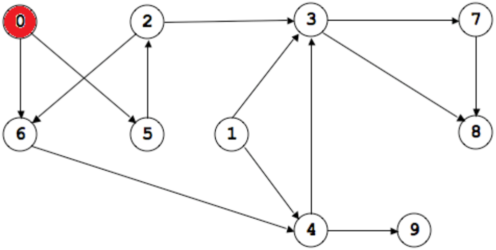
Algorithm: acyclic(G)

INPUT: A digraph G
OUTPUT: Return if G is a dag
Run DFS(G) with the following modification:
    Whenever discover a node u, do
        for every edge (u, v) out of u
            if $pre(v) < pre(u)$ and $post(v)$ is undefined
                Declare G has a cycle and return
Declare that G is a dag.

DFS and Linearisations

Definition [Linearisations]

A linearization or (topological sort) of a digraph G is a list of all nodes in G such that if G contains an edge (u, v) then u appears before v in the list.



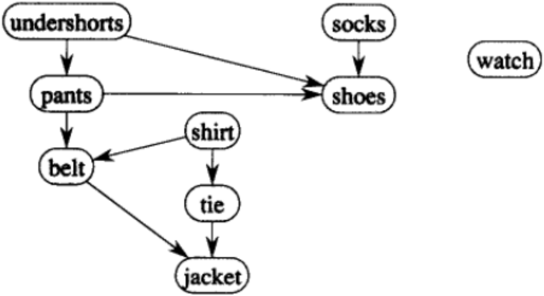
Topological Sorts:

0,5,2,6,1,4,3,7,9,8

1,0,5,2,6,4,9,3,7,8

DFS and Linearisations

In what order should I put on my cloths?

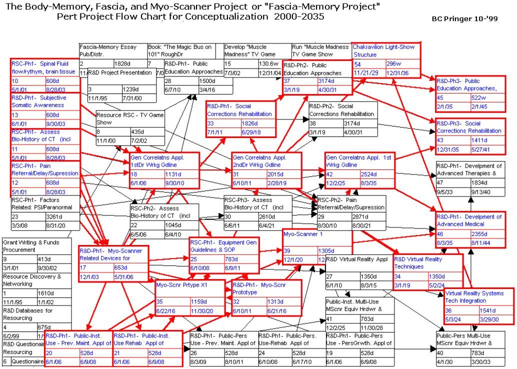


Possible orderings are linearisations of the dependency graph:
Possible order 1: Shirt, Socks, Undershorts, Watch, Pants, Tie, Belts, Jacket, Shoes
Possible order 2: Watch, Undershorts, Socks, Pants, Shoes, Shirt, Belt, Tie, Jacket

DFS and Linearisations

Application of Linearisation

- Job/Task/Instruction scheduling
- Project Evaluation and Review Technique (PERT)
- `makefiles` in Unix / `APT` in Ubuntu Linux
- Class/Package dependency in a software project



DFS and Linearisations

Question

Is there a digraph that can not be linearized?

DFS and Linearisations

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What dags can be linearized?

DFS and Linearisations

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Question

What dags can be linearized?

Answer: All of them!

The Zero In-degree algorithm finds a linearization for a dag:

Algorithm: ZeroInDegree(G)

INPUT: a DAG G
OUTPUT: a linearisation of G
list $\leftarrow$ an empty list

```
while G is not empty do
    for each u in V
        if inDegree(u) = 0 then
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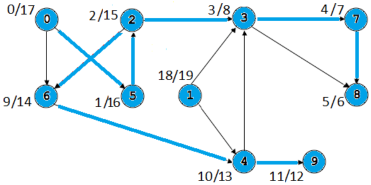
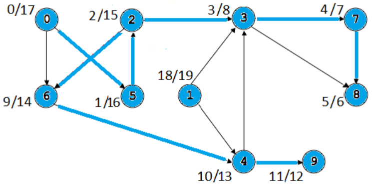
Shortcoming: The algorithm runs in time $O((n + m)n)$.

DFS and Linearisations



DFS and Linearisations



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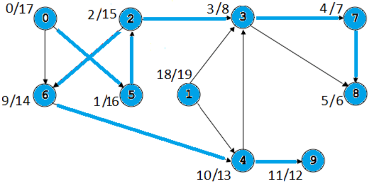
DFS and Linearisations



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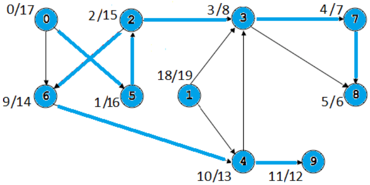

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Case 1. u is discovered earlier than v is.

Then v must be finished before u is finished.

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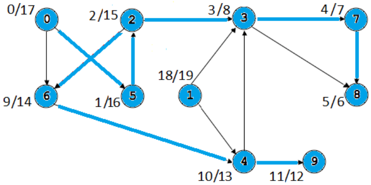

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Proof

There are two cases:

Case 1. u is discovered earlier than v is.

Then v must be finished before u is finished.

Case 2. v is discovered earlier than u is.

Since G is acyclic, there is no path that goes from v to u.

Hence v is again finished earlier than u is finished. □

DFS and Linearisations

We obtain an easy algorithm for graph linearisation in time $O(m + n)$:
Output the list of nodes in decreasing finishing order

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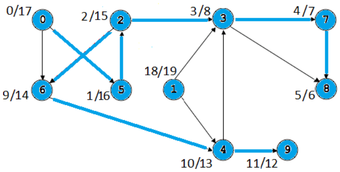


DFS-Linearize(G) :

1, 0, 5, 2, 6, 4, 9, 3, 7, 8

Further Comments

Acyclicity and Linearizability

- We established two characterizations of linearizability of a digraph:

  A digraph is linearizable if and only if

  - it is acyclic
  - the DFS forest has no back edge

- In other words

$$\text{Linearizable} \equiv \text{Acyclicity} \equiv \text{No-Back-edgeness}$$

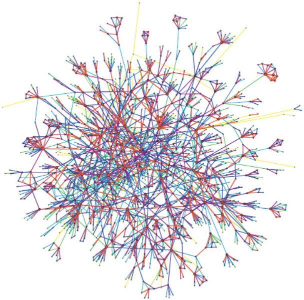
- With this understanding we are able to design algorithms for deciding these properties.

Part V: Connectivity and Components

Decomposing Graphs

Why decompose graphs?

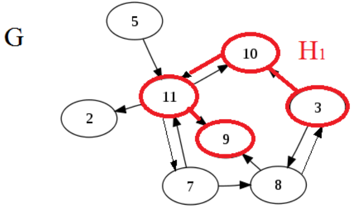
[Divide-and-Conquer] Often, when we solve a problem on graph, it is much more efficient to decompose the graph into components, solve the problem on individual components, then combine the solutions.



Subgraphs

Definition [Subgraphs]

Let $E \subseteq V^2$, and $V' \subseteq V$.
- we use $E \upharpoonright V'$ to denote the set $\{(u, v) \in E \mid u, v \in V'\}$.
- A subgraph of a digraph $G = (V, E)$ is a digraph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E \upharpoonright V'$.
- If $E' = E \upharpoonright V'$, then G' is an induced subgraph of G.



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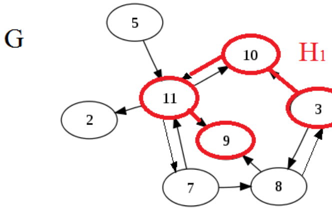

Let $V' = \{3, 9, 10, 11\}$. $E \restriction V' = \{(10, 11), (11, 10), (3, 10), (11, 9)\}$

A subgraph is $(\{3, 9, 10, 11\}, \{(3, 10), (10, 11), (11, 9)\})$

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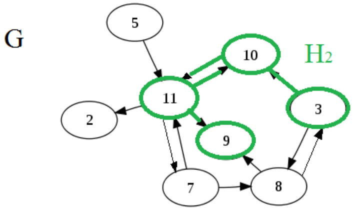

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Decomposing Undirected Graphs

Connectivity in Undirected Graphs

- Recall a node is reachable from another if there is a path linking these two nodes

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- Here reachability is an equivalence relation:
  - (reflexivity) Any node u is reachable from itself.
  - (symmetry) If v is reachable from u then u is reachable from v.
  - (transitivity) If v is reachable from u, u is reachable from w, then v is reachable from w.

Decomposing Undirected Graphs

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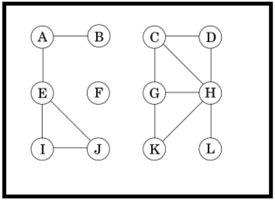
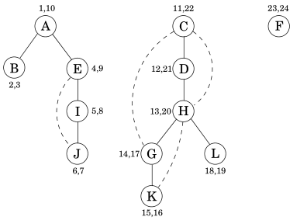
  - (transitivity) If v is reachable from u, u is reachable from w, then v is reachable from w.
- We may decompose the graph into equivalence classes: Two nodes are in the same class if they are reachable from each other
- Each equivalence class is a connected component

Decomposing Undirected Graphs

Definition [Undirected Connectivity]

A connected components is the induced subgraph of a maximal set of nodes that are pairwise reachable.
An undirected graph is connected if it contains only one connected component.





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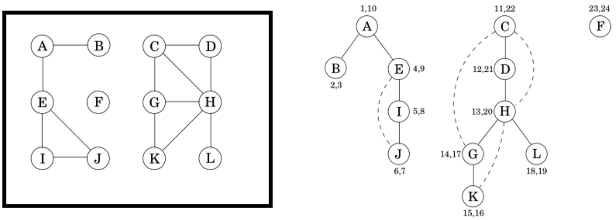

DFS and CC

- We may use DFS to decide if two nodes are in the same CC.
- Running time $O(m + n)$.

Decomposing Directed Graph

Definition [Directed Connectivity]

In a digraph G, we say that two nodes u, v are in the same strongly connected component (SCC) if there is a path from u to v and a path from v to u. A digraph is strongly connected if it contains only one SCC.

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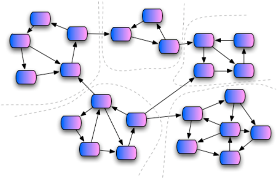

Strongly Connected Components

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- If G is undirected, then u, v are in the same SCC whenever u can reach v.
  Therefore checking SCC is same as reachability.

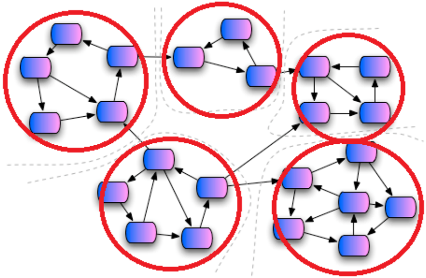
Question

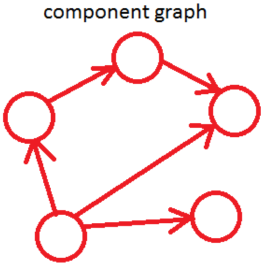
Given a digraph, find all the strongly connected components.

Meta-Graph

Definition [Meta-Graph]

Given a graph G. If we collapse all nodes in the same SCCs together, only keeping the edges between different components, then we get the meta-graph, G^{SCC}.





Meta-Graph

Let G be a digraph, and G^{SCC} be its meta-graph after collapsing every SCC into one node.

- The meta-graph G^{SCC} must be acyclic.

  Why?

Source and Sink

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Let G be a digraph, and G^{SCC} be its meta-graph after collapsing every SCC into one node.

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- We can linearize G^{SCC}.

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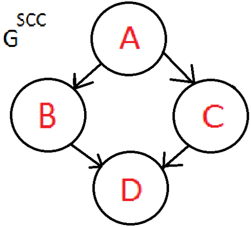
- The meta-graph G^{SCC} must be acyclic.

  Why?
- We can linearize G^{SCC}.
- Source: A meta-node in G^{SCC} with no incoming edge.
- Sink: A meta-node in G^{SCC} with no outgoing edge.

Source and Sink

Consider the following meta-graph:



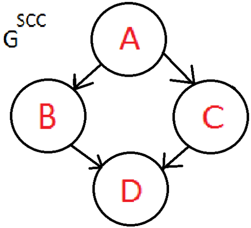
A is a source, *D* is a sink.

Observation

If we run DFS on a node in a sink, then we will find all nodes in this sink.

Finding SCC

Consider the following meta-graph:



A Plan for Finding SCC

Given G. Repeat the following:

1. Find a node u in a sink
2. Run dfs_explore(G, u)
3. Declare all visited nodes an SCC. Take those nodes out.

Finding SCC

Problem

How do we find a node in a sink?

Finding SCC

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How do we find a node in a sink?

Observe

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Problem

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- We are given the graph G, but no information about G^{SCC}.

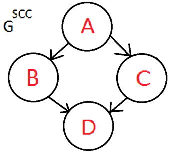
Finding SCC

Problem

How do we find a node in a sink?

Observe

- We are given the graph G, but no information about G^{SCC}.
- We can find a node in a source:
  Run DFS. Take the node that is finished last.



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How do we find a node in a sink?

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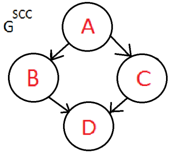


- But running DFS on a source does not work.

Finding SCC

Source and Sink

Does finding a source node help in finding a sink node?

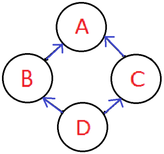
Finding SCC

Source and Sink

Does finding a source node help in finding a sink node?

Fact

Let G be a digraph. Let G^T be the transpose of G: the digraph obtained from G by reversing the direction of every edge.



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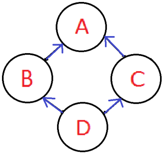

- G and G^T have the same SCCs.

Finding SCC

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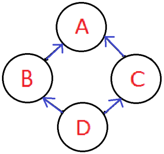
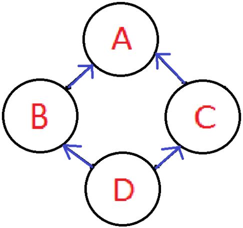
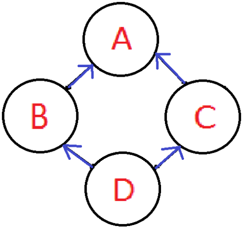


- G and G^T have the same SCCs.
- A source in G becomes a sink in G^T.

DFS and Strongly Connected Components

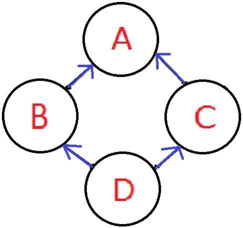


DFS and Strongly Connected Components



Example. When the edges are reversed, A, B, C, D are still SCCs.

DFS and Strongly Connected Components



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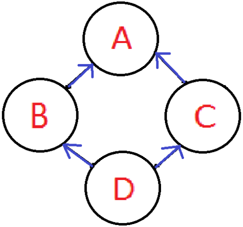
Let x be the last finished node in DFS.

Running dfs_explore(G^T, x) will compute A.

DFS and Strongly Connected Components



Example. When the edges are reversed, A, B, C, D are still SCCs.
Let x be the last finished node in DFS.

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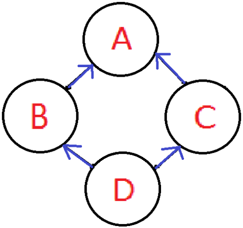
Let y be the last node that is finished in the remaining graph.

Running dfs_explore(G^T, y) will compute C.

DFS and Strongly Connected Components



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Running dfs_explore(G^T, y) will compute C.

Continue for B, D (decreasing order of *post*)

DFS and Strongly Connected Components

The following algorithm takes as input any digraph G, outputs all the SCCs of G. The algorithm runs in time $O(m+n)$.

DFS and Strongly Connected Components

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Algorithm SCC(G)

INPUT: a digraph G
OUTPUT: SCCs of G
stack $\leftarrow$ empty stack
Run $dfs(G)$, at the same time do:
    When a node is finished, push it onto a *stack*
$G^T \leftarrow G$ with all edges reversed
`for each` u `in` *stack* `(in popped order)`
    `Run dfs_explore(`G^T`,`u`)`
    `The nodes visited by` **`explore`** `is the SCC of` u`.`

DFS and Strongly Connected Components

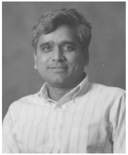
Discussion

- Essentially, the algorithm runs DFS twice: first time on G, then on G^T.
  In the second time, when no where to go, select the next node in decreasing order of finishing time of the first DFS.

DFS and Strongly Connected Components

Discussion

- Essentially, the algorithm runs DFS twice: first time on G, then on G^T.
  In the second time, when no where to go, select the next node in decreasing order of finishing time of the first DFS.
- The algorithm is called the Kosaraju-Sharir algorithm



"At some point, the learning stops and the pain begins."

------- S. Rao Kosaraju

Summary

- DFS is a linear time ($O(m + n)$) graph traversal algorithms.
- DFS implementation: Recursive or Stack-based
- Nodea are processed in two stages: discovered, finished.
- The DFS algorithm computes a DFS forest in the graph; edges in the graph are classified into tree edges, forward edges, back edges and cross edges

Using DFS we can also answer the following questions about a graph:

- Reachability: Given to nodes u, v, is u reachable from v?
- Cyclicity: Is the graph acyclic? If it contains a cycle, find a cycle.
- Linearisation: If the graph is acyclic, find a linearisation.
- Connectivity: Is the graph connected?
- Connected component: If the graph is undirected, what are the connected components of it?
  If the graph is directed, what are the strongly connected components of it?