Algorithm Design and Analysis

Day 9
Graphs and Matrices

AUT

Part I: Network Centrality

Network Science

- The challenge of big data: Data sets become so large and complex that we need novel efficient methods to store, process and analyse them.

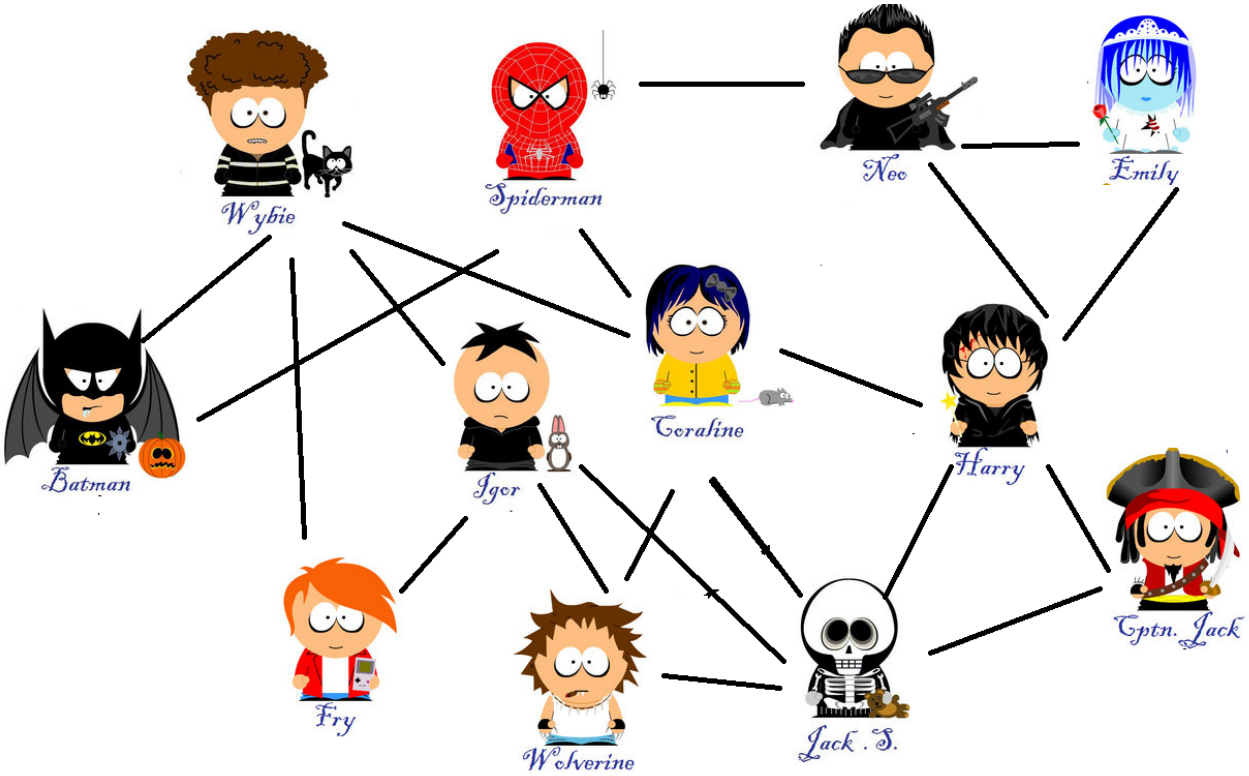
  e.g. TradeMe has over 3.69 million users.

  Facebook has more than 1.23 billion active users.

  Internet has more than 30 trillion webpages.
- Network science: The analysis of complex networks (such as social networks, telecommunication networks, biological networks) for identification of patterns and predictions.
- Social network analysis: Examine the structures and relationships of social entities such as people, organisations, web sites, etc.

Network Centrality

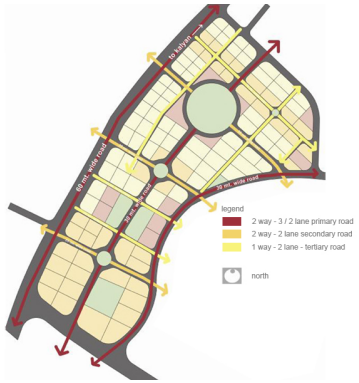
A network consists of a set of individuals, represented as nodes, and interactions between the individuals, represented as edges. Thus a network can be viewed as a graph.



Network Centrality

Some questions:

- If I am to send my cool selfie to 3 people on Facebook, so that this picture can go viral, whom should I pick?
- If I were to nominate a leader for a team of 100, whom should I ask?
- If I want to pick a place for a billboard of my company, where in the road network should I pick?
- If I want to sell hot dog at a tourist attraction, which location should I park my hot dog stand?



Network Centrality

Centrality

Let $G = (V, E)$ be an undirected graph. A centrality measure is a function $c : V \to \mathbb{R}$ that maps every node to a real value:

- For $v \in V$, $c(v)$ is the centrality of v, which represents the amount of influence (or level of importance) of v in G
- Say $V = \{v_1, v_2, \ldots, v_n\}$. The centrality vector is $\vec{c} = (c(v_1), c(v_2), \ldots, c(v_n)) \in \mathbb{R}^n$.

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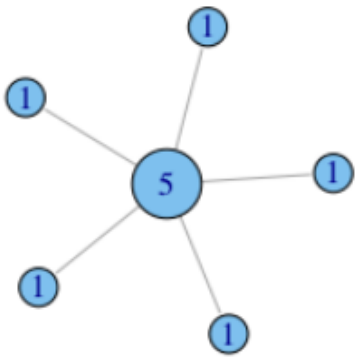
Importance of Centrality:

- Help disseminating information in the network faster (online advertisement)
- Help topping epidemics
- Help protecting the network from breaking

Centrality Measure 1: Degree Centrality

The degree centrality of a node v is the degree of v.



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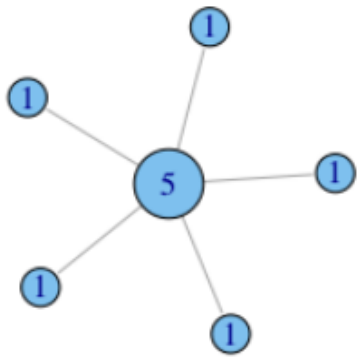

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- Advantage: Simple to compute.
- Disadvantage: Not able to compare centrality between nodes with the same degree.

  Purely local measure

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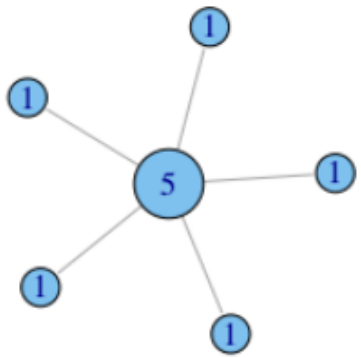

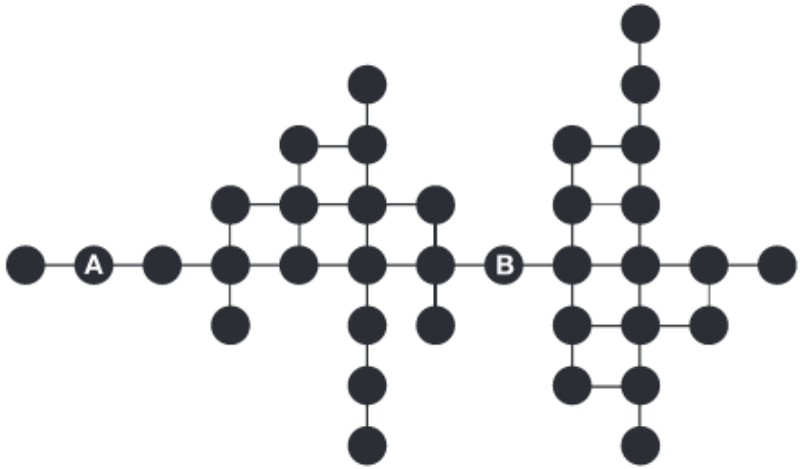

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Centrality Measure 2: Eigenvector Centrality

Intuitions: Influence diffusion

- If your friends are influential, you would also be influential.
- The influence of a node v is affected by the influence of its neighbours.
- Each neighbour contribute $\frac{1}{\lambda}$ of its own influence to v's influence.

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- Each neighbour contribute $\frac{1}{\lambda}$ of its own influence to v's influence.

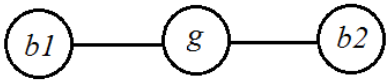
Let c_v denote the "influence index" of node v. We call c_v the centrality of v.

$$c_v = \frac{1}{\lambda}\left(c_{u_1} + c_{u_2} + \cdots + c_{u_k}\right)$$

where $u_1, u_2, \ldots, u_k$ are the neighbours of v.

Centrality Measure 2: Eigenvector Centrality

Example 1:



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Centrality Measure 2: Eigenvector Centrality

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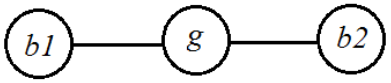

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{cases} c(b_1) & = & \frac{1}{\lambda}c(g) & \\ c(g) & = \frac{1}{\lambda}c(b_1) & & + \frac{1}{\lambda}c(b_2) \\ c(b_2) & = & \frac{1}{\lambda}c(g) & \end{cases}$$

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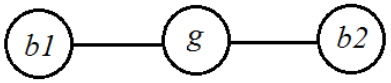

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$\Rightarrow$

$$\begin{pmatrix} c(b_1) \\ c(g) \\ c(b_2) \end{pmatrix} = \frac{1}{\lambda}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{pmatrix} c(b_1) \\ c(g) \\ c(b_2) \end{pmatrix}$$

Centrality Measure 2: Eigenvector Centrality

What are the values of λ and $c(b_1), c(g), c(b_2)$?

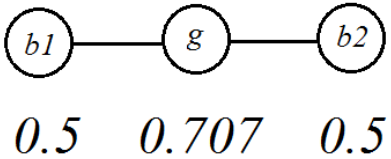
Centrality Measure 2: Eigenvector Centrality

What are the values of λ and $c(b_1), c(g), c(b_2)$?

$$\begin{pmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{pmatrix}$$

Hence $\lambda = \sqrt{2}$ and

$$\begin{pmatrix} c(b_1) \\ c(g) \\ c(b_2) \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{pmatrix} \approx \begin{pmatrix} 0.5 \\ 0.707 \\ 0.5 \end{pmatrix}$$



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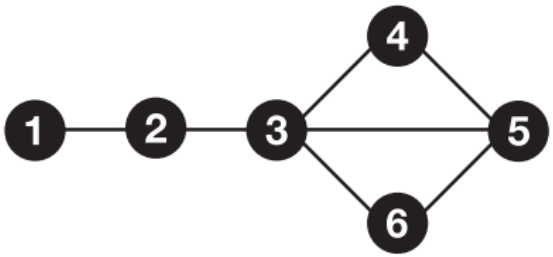
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Centrality Measure 2: Eigenvector Centrality

Example 2: Spread of a Disease

Links between Villages



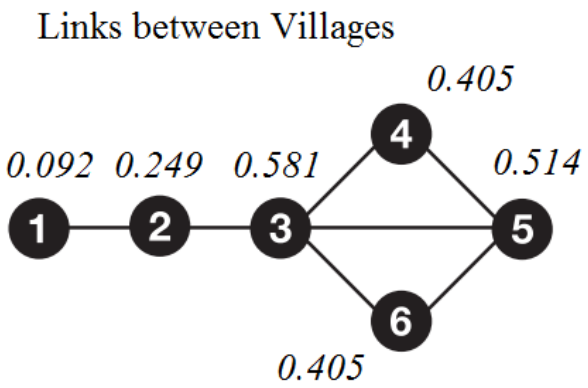
The centrality of town represents potential to spread diseases.

$$\begin{pmatrix} c(1) \\ c(2) \\ c(3) \\ c(4) \\ c(5) \\ c(6) \end{pmatrix} = \frac{1}{\lambda} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} c(1) \\ c(2) \\ c(3) \\ c(4) \\ c(5) \\ c(6) \end{pmatrix}$$

Centrality Measure 2: Eigenvector Centrality

$$\begin{pmatrix} 0.092 \\ 0.249 \\ 0.581 \\ 0.405 \\ 0.514 \\ 0.405 \end{pmatrix} \approx \frac{1}{2.7} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 0.092 \\ 0.249 \\ 0.581 \\ 0.405 \\ 0.514 \\ 0.405 \end{pmatrix}$$

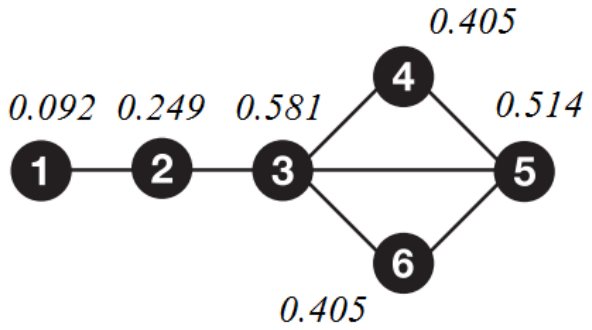
Hence $\lambda \approx 2.7$ and the centralities of villages are given below:



Part II: Eigenvalues and Eigenvectors

Network Centrality

Recall the last example:



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Let $G = (V, E)$ be a network where $V = \{v_1, v_2, \ldots, v_n\}$. The centrality vector is a vector $\vec{c} = (c(v_1), c(v_2), \ldots, c(v_n))$ that satisfies the following equation:

$$\vec{c} = \frac{1}{\lambda} A\vec{c}$$

where A is the adjacency matrix of G.

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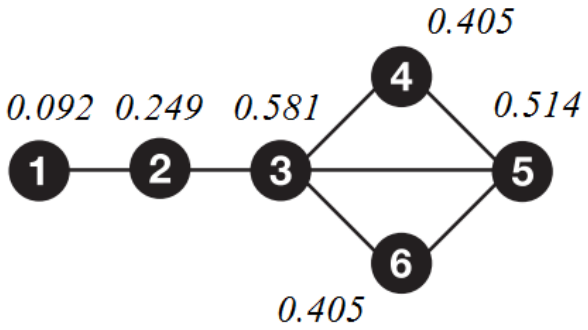

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We need to solve the matrix equation:

$$A\vec{c} = \lambda\vec{c}$$

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

For any $n \times n$ matrix A, a scalar λ is called an eigenvalue of A if there is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.
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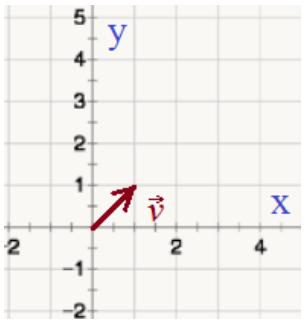
- e.g. $2, -1$ are eigenvalues of $\begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}$, with corresponding eigenvectors $\vec{v}_1 = \binom{1}{1}$ and $\vec{v}_2 = \binom{-1}{1}$ respectively.



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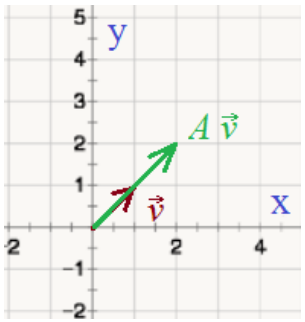

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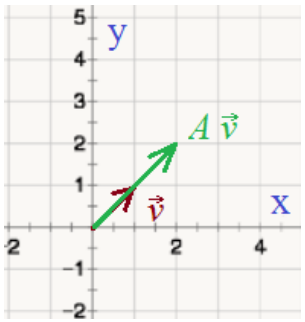

- e.g. $1.1, 0.6$ are eigenvalues of $\begin{bmatrix} 1.1 & 0 \\ 0.2 & 0.6 \end{bmatrix}$, with corresponding eigenvectors $\vec{v}_1 = \binom{5}{2}$, $\vec{v}_2 = \binom{0}{1}$ respectively.

Eigenvalues and Eigenvectors

Note:

- If λ is an eigenvalue of A with corresponding eigenvector $\vec{v}$, then λ is the scaling ratio $\vec{v}$ in the transformation $\vec{v} \mapsto A\vec{v}$
- If $\vec{v}$ is an eigenvector, then for any constant c, $c\vec{v}$ is also an eigenvector (corresponding to the same eigenvalue).

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Questions:

- Will an eigenvalue always exist? If so, is it unique? If not, how many eigenvalues a matrix would have?
- Will an eigenvalue correspond to a unique vector (up to scalar multiplication)?
- How can we compute eigenvalues and their corresponding eigenvectors (efficiently) for large matrices?

Eigenvalues and Eigenvectors

- Consider arrangements of $\{1, 2, 3, \ldots, n\}$ in a row, e.g. when $n = 5$

  13245 34152 25431

  Such arrangements are called permutations.
- For each permutation p, let p_i denotes the ith elements in p, e.g., when $p = 13245$,

$$p_1 = 1 \;\; p_2 = 3 \;\; p_3 = 2 \;\; p_4 = 4 \;\; p_5 = 5$$

- An inversion in a permutation p is a pair (p_i, p_j) where $i < j$ and $p_i > p_j$, e.g., when $p = 32415$, the inversions are

$$(3, 2), (3, 1), (2, 1)$$

- A permutation is called even if it has an even number of inversions and odd otherwise.
- Let $\varepsilon(p) = 1$ if p is even; and $\varepsilon(p) = -1$ if p is odd, e.g., $\varepsilon(32415) = -1$

Eigenvalues and Eigenvectors

Let P_n be the set of all permutation on $\{1, \ldots, n\}$.
The determinant of an $n \times n$ matrix A is

$$\sum_{p \in P} \varepsilon(p) \Pi_{i=1}^{n} A_{p_i i}$$

E.g.

- $$\det\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\right) = 1 \times 3 - 4 \times 2 = -5$$

- $$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}\right)$$
$$= 1 \times 1 \times 2 + 2 \times 0 \times 2 + 3 \times 2 \times 1 - 2 \times 1 \times 3 - 1 \times 0 \times 1 - 2 \times 2 \times 2$$
$$= -6$$

Eigenvalues and Eigenvectors

Facts about determinants

- For square matrix A, A is invertible if and only if $\det(A) \neq 0$
- The equation $A\vec{v} = 0$ has non-zero solution if and only if $\det(A) = 0$.

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- A value λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Eigenvalues and Eigenvectors

Characteristic Polynomial

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Eigenvalues and Eigenvectors

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By the fundamental theorem of algebra, the characteristic polynomial has exactly n (possibly identical, possibly imaginary) roots.

Therefore, any $n \times n$ matrix A has at most n eigenvalues in C, each eigenvalue corresponds to at least one eigenvector (up to scalar multiplication).

Eigenvalues and Eigenvectors

Example 1: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

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$$= (1-\lambda)^2 - 4 = (\lambda+1)(\lambda-3)$$

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For $\lambda_1 = -1$, solve

$$\begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix} \vec{v_1} = 0$$

Thus eigenvector corresponding to -1: $\vec{v_1} = (1, -1)^T$

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For $\lambda_2 = 3$, solve

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} \vec{v_2} = 0$$

Thus eigenvector corresponding to 3: $\vec{v_2} = (1, 1)^T$

Eigenvalues and Eigenvectors

Example 2: $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right)$$
$$= (\lambda - 1)(\lambda - 3)^2$$

Eigenvalues: $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 3$

Eigenvalues and Eigenvectors

Example 2: $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Characteristic polynomial:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right)$$
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Eigenvectors: We say that 1 has multiplicity 1. It has eigenvector $(-1, 0, 2)^T$.
We say that 3 has multiplicity 2. It has two linearly independent eigenvectors $(1, 0, 0)^T$ and $(0, 1, 0)^T$.

Eigenvector Centrality

Eigenvector Centrality

Eigenvalues and eigenvectors lay the foundation of the notion of eigenvector centrality: The centrality vector is an eigenvector of the adjacency matrix of the network.

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Theorem [Eigenvector Centrality]

Let G be a connected undirected graph and A be the adjacency matrix of G. Then

(a) All eigenvalues of A are real numbers

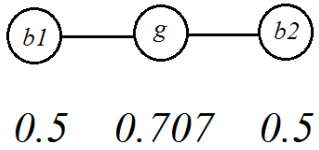
(b) The largest eigenvalue λ of A is positive and has multiplicity 1

(c) There is an eigenvector $\vec{c}$ corresponding to λ all of whose entries are positive real numbers

The eigenvector $\vec{c}$ is the centrality vector. The ith entry in the eigenvector $\vec{c}$ is the eigenvector centrality of i in the network G.

Eigenvector Centrality

Example.



$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Characteristic polynomial:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix}\right) = -\lambda^3 + 2\lambda$$

Therefore the eigenvalues are $\lambda_1 = \sqrt{2}$, $\lambda_2 = 0$, $\lambda_3 = -\sqrt{2}$

Eigenvector corresponding to $\sqrt{2}$:

Solve the equation

$$\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \begin{pmatrix} c(b_1) \\ c(g) \\ c(b_2) \end{pmatrix} = 0$$

To get $c(b_1) = 0.5$, $c(g) = 0.707$, $c(b_2) = 0.5$.

Computing Eigenvector Centrality

Question: How to compute centrality efficiently?

- Precise algorithm using characteristic polynomial: Impossible for large graph
- Numerical algorithms (to approximate solutions).

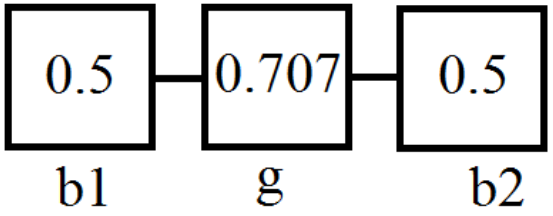
  We will look at power iteration algorithm, which is an easy numerical algorithm for computing centrality.

Computing Eigenvector Centrality

We can imagine a network as a transition system: At any time, the system configuration is described by a vector, indicating "importance" of all nodes.

- There is an initial vector, indicating the starting importance values of all nodes
- According transition rules, the vector is changed through time

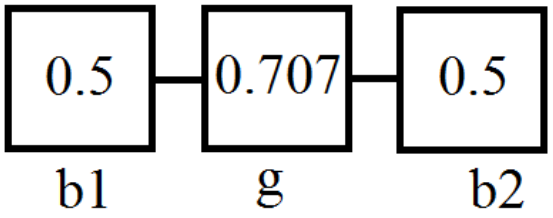
Example:



Rules: $c(b_1) = c(g)/\lambda$, $c(g) = c(b_1)/\lambda + c(b_2)/\lambda$, $c(b_2) = c(g)/\lambda$ where $\lambda = \sqrt{2}$.
Then $(0.5, 0.707, 0.5)$ is a stationary vector, which is stable through time.

Computing Eigenvector Centrality

Consider the example:



$$\text{Take } A + I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Initial vector (time 0): $(1, 1, 1)$
Time 1: $(A + I)(1, 1, 1)^T = (2, 3, 2)^T = 2 \times (1, 1.5, 1)^T$
Time 2: $(A + I)(1, 1.5, 1)^T = (2.5, 3.5, 2.5)^T = 2.5 \times (1, 1.4, 1)^T$
Time 3: $(A + I)(1, 1.4, 1)^T = (2.4, 3.4, 2.4)^T = 2.4 \times (1, 1.417, 1)^T$
Time 4:
$(A + I)(1, 1.417, 1)^T = (2.417, 3.417, 2.417)^T = 2.417 \times (1, 1.414, 1)^T$
$\Rightarrow$ Normalise $(1, 1.414, 1)^T$ to get $(0.5, 0.707, 0.5)^T$
Furthermore, the eigenvalue is approximately $2.417 - 1 = 1.417 \approx \sqrt{2}$

Computing Eigenvector Centrality

Facts.

Let G be a connected undirected graph. Let A be the adjacency matrix of G.

1. For any real number r, r is an eigenvalues of $A + I$ if and only if $r - 1$ is an eigenvalue of A.
2. $A + I$ has a positive eigenvalue λ that is bigger than the absolute value of any other eigenvalues.
3. The limit

$$\lim_{m \to \infty} \left(\frac{(A+I)^m}{\lambda^m} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right)$$

exists and is an eigenvector corresponding to the largest eigenvalue of G.

Computing Eigenvector Centrality

Algorithm PowerIteration(G)
INPUT: A connected undirected graph G
OUTPUT: Approximation of the largest eigenvalue λ of G, and an approximated eigenvector corresponding to λ

1. Let A be the adjacency matrix of G
2. Compute $B = A + I$
3. Set initial vector $\vec{v} = (1, 1, \ldots, 1)^T$
4. While $\vec{v}$ is not changing significantly

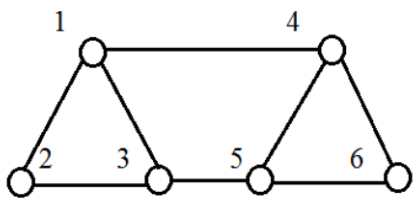
   Set $\vec{u} \leftarrow B\vec{v}$

   Let u_0 be the first component over $\vec{u}$. Let $\vec{v} \leftarrow \vec{u}/u_0$
5. Normalise $\vec{v}$ by setting $\vec{v} = \vec{v}/||\vec{v}||$
6. Return $\vec{v}$ as the eigenvector centrality

Note: Typically no more than 20 iterations of the while-loop is needed.

Computing Eigenvector Centrality

Example: Consider the following graph



$$B = A + I = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Initial vector $\vec{v} = (1, 1, 1, 1, 1, 1)^T$

Time 1: $\vec{u} = B\vec{v} = (4, 3, 4, 4, 4, 3)^T$ and $\vec{v} = \vec{u}/4 = (1, 3/4, 1, 1, 1, 3/4)^T$

Time 2: $\vec{u} = B\vec{v} = (3.75, 2.75, 3.75, 3.75, 3.75, 2.75)^T$ and $\vec{v} = (1, 0.73, 1, 1, 1, 1, 0.73)$

Time 3: $\vec{u} = B\vec{v} = (3.732, 2.732, 3.732, 3.732, 3.732, 2.732)^T$ and $\vec{v} = (1, 0.732, 1, 1, 1, 0.732)$

Time 4: $\vec{u} = B\vec{v} = (3.732, 2.732, 3.732, 3.732, 3.732, 2.732)^T$ and $\vec{v} = (1, 0.732, 1, 1, 1, 0.732)$

Computing Eigenvector Centrality

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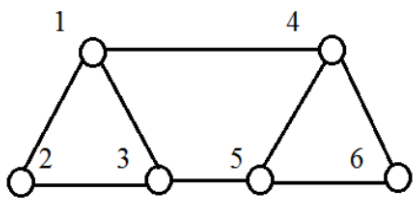

$$B = A + I = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

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Time 3: $\vec{u} = B\vec{v} = (3.732, 2.732, 3.732, 3.732, 3.732, 2.732)^T$ and $\vec{v} = (1, 0.732, 1, 1, 1, 0.732)$

Time 4: $\vec{u} = B\vec{v} = (3.732, 2.732, 3.732, 3.732, 3.732, 2.732)^T$ and $\vec{v} = (1, 0.732, 1, 1, 1, 0.732)$

Therefore the largest eigenvalue of A is $\approx 3.732 - 1 = 2.732$ with eigenvector $(1, 0.732, 1, 1, 1, 0.732)^T$

Normalise to get centrality vector $(0.444, 0.325, 0.444, 0.444, 0.444, 0.325)^T$

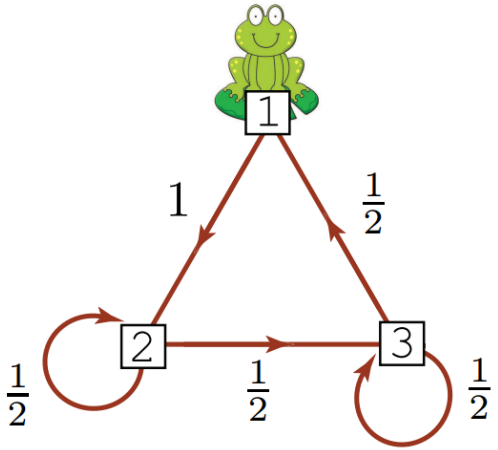
Part III: Markov Chains

Markov Chains

Markov chains is another form of transition system which consists of a stationary vector.

Example. Leaping Frog

A frog hops about on 3 lily pads. The numbers next to arrows show the probabilities with which, at the next jump, he jumps to a neighbouring lily pad. Where does it go, how long does it take to get there, and what happens in the long run?



Transition matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Transpose of the transition matrix:

$$P^T = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Markov Chains

A stochastic vector is a vector $(p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ where $\sum_{i=1}^{n} p_i = 1$ and each $p_i \in [0, 1]$.
A stochastic matrix is a square matrix A over real numbers where each entry is in the range $[0, 1]$ and each row adds up to 1.

Markov Chains

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Definition: Markov Chains

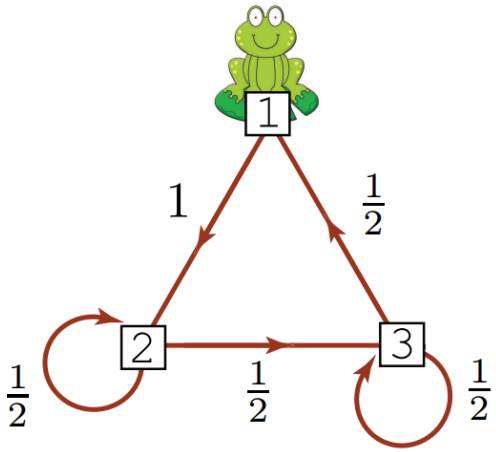
A Markov chain is a transition system that consists of the following components:

- a finite set V of nodes, called states,
- a set E of directed edges, called transitions,
- a function $P : E \to (0, 1]$ mapping edges to a transition probability, such that for any node $u \in V$,

$$\sum_{\{v|(u,v)\in E\}} P(u, v) = 1$$

  i.e., P can be represented by a stochastic matrix.
- $\vec{v}_0$ is an initial stochastic vector.

Markov Chains



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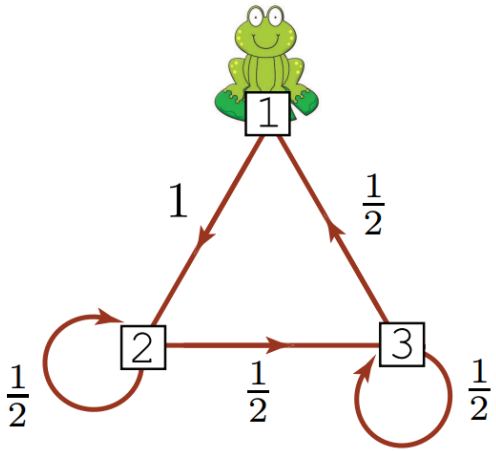
- Starting from the initial stochastic vector $(1, 0, 0)$, what is the stochastic vector after 1 step?

- What is the stochastic vector after 2 steps?

Markov Chains



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- Starting from the initial stochastic vector $(1, 0, 0)$, what is the stochastic vector after 1 step?

  Answer. $P^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

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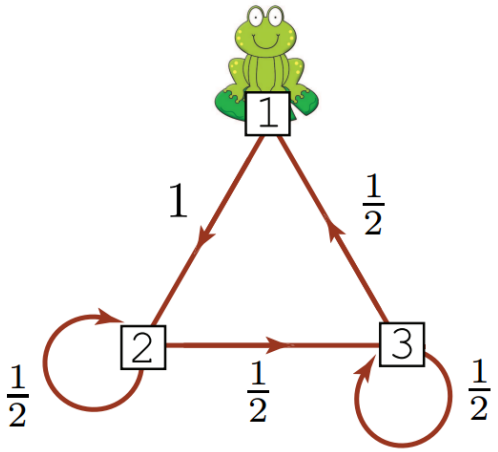
Markov Chains



Transition matrix:

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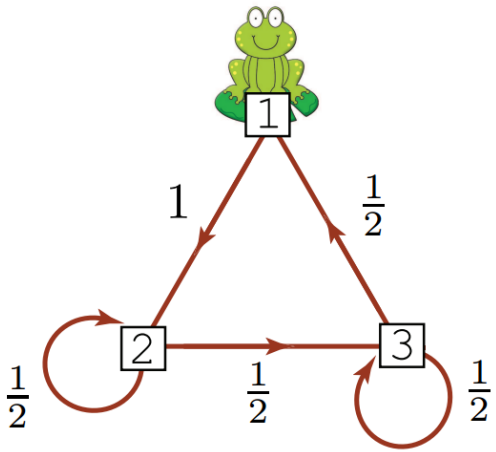
- Starting from the initial stochastic vector $(1, 0, 0)$, what is the stochastic vector after 1 step?

  Answer. $P^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

- What is the stochastic vector after 2 steps?

  Answer. $\left(P^T\right)^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = P^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$

Markov Chains



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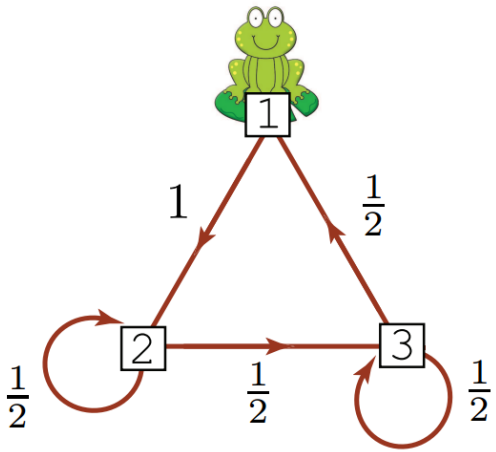
Fact

A stochastic process over a Markov chain (Q, P) is a sequence of stochastic vectors

$$\vec{v_0}\vec{v_1}\vec{v_2}\ldots$$

such that $\vec{v_{i+1}} = P^T\vec{v_i}$.

Markov Chains



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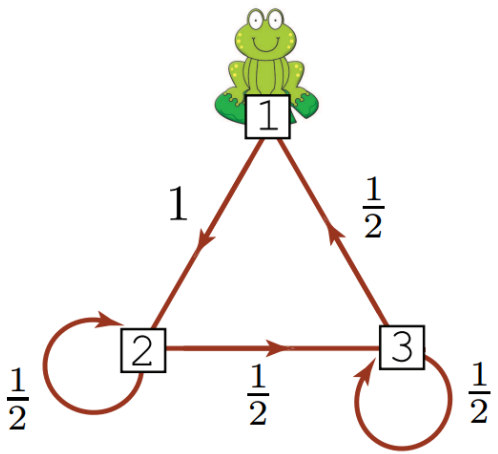
Question. What is the stochastic vector in the long run?

Markov Chains



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$$P^T = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

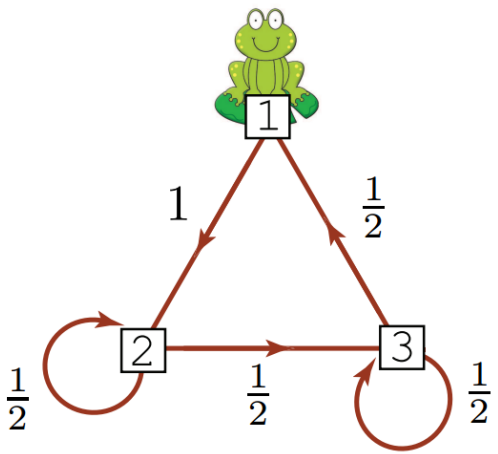
$\vec{v_1} = (0, 1, 0)^T$, $\vec{v_2} = (0, 1/2, 1/2)^T$ $\vec{v_5} = (3/16, 13/32, 23/32)$,

$\vec{v_7} \approx (0.203, 0.390, 0.406)$

$\vec{v_{10}} = (0.19921875, 0.400390625, 0.400390625)^T$

$\vec{v_{20}} \approx (0.200000007, 0.3999996, 0.3999996)^T$

Markov Chains



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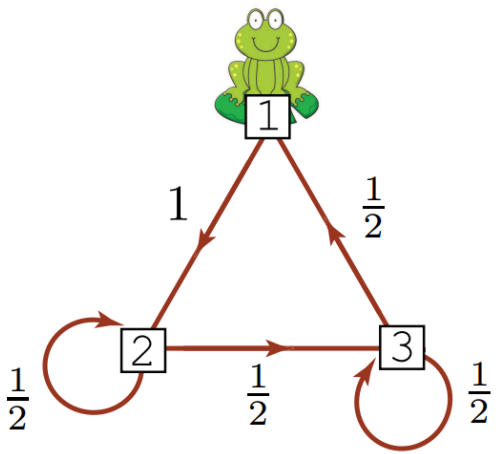
Note: $P^T \begin{pmatrix} 0.2 \\ 0.4 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.4 \\ 0.4 \end{pmatrix}$

Markov Chains



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Note: $P^T \begin{pmatrix} 0.2 \\ 0.4 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.4 \\ 0.4 \end{pmatrix}$

So the long term stochastic vector is $\begin{pmatrix} 0.2 \\ 0.4 \\ 0.4 \end{pmatrix}$

Markov Chains

Stationary Distribution

A stationary distribution $\vec{s}$ of a Markov chain $(V, E, P, \vec{v_0})$ is a stochastic vector such that

$$P^T \vec{s} = \vec{s}$$

Markov Chains

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Question: Would any Markov chain have a stationary distribution?

Markov Chains

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Question: Would any Markov chain have a stationary distribution?
Answer:

Theorem [Stationary Distribution of Markov Chains]

For any Markov chain, the matrix P^T always has largest eigenvalue 1.

Therefore any Markov chain has a stationary distribution.

Markov Chains

Question: Would any stochastic process eventually converges to the stationary distribution?

Markov Chains

Question: Would any stochastic process eventually converges to the stationary distribution?
Answer: No. There are Markov chains whose long term vector would not converge. However, the long term vector would converge to the stationary distribution for certain nicely behaving Markov chains.

Theorem [Converging behaviour of MC]

Suppose a Markov chain $(V, E, P, \vec{v_0})$ is

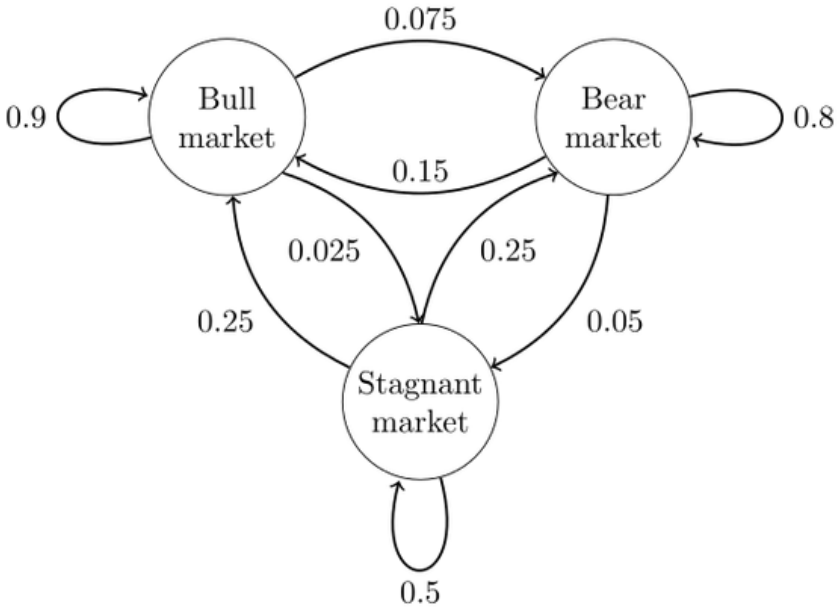
1. strongly connected (i.e., the network (V, E) contains only one strongly connected component)
2. $P(v, v) > 0$ is an edge for any node $v \in V$

Then the stochastic process of this Markov chain converges to a unique stationary distribution.

Therefore the stationary distribution is the long term probability of a random walk in Markov chains that satisfy the two conditions above.

Markov Chains

Example.



$$P = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}.$$

$$P^T = \begin{bmatrix} 0.9 & 0.15 & 0.25 \\ 0.075 & 0.8 & 0.25 \\ 0.025 & 0.05 & 0.5 \end{bmatrix}.$$

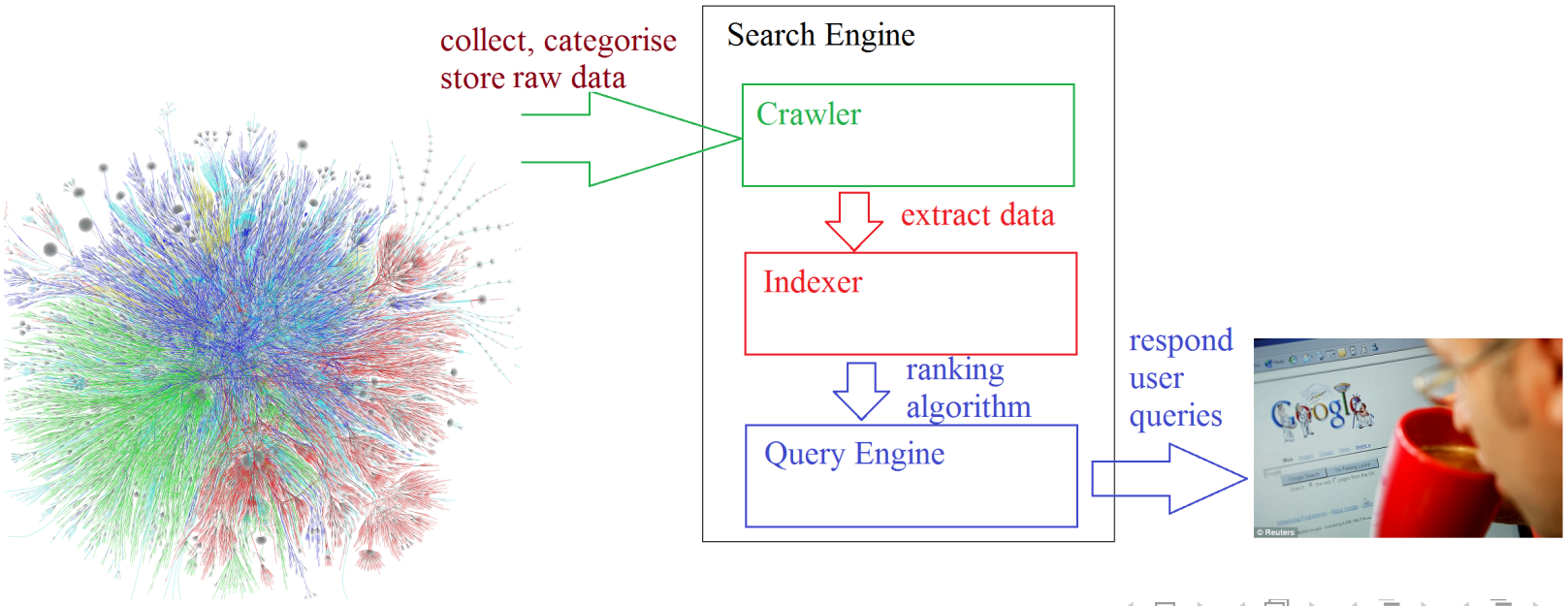
$$\vec{v_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0.9 \\ 0.075 \\ 0.025 \end{pmatrix} \to \begin{pmatrix} 0.8275 \\ 0.13375 \\ 0.03875 \end{pmatrix} \to \begin{pmatrix} 0.7745 \\ 0.17875 \\ 0.04675 \end{pmatrix} \to$$

$$\begin{pmatrix} 0.736 \\ 0.212775 \\ 0.051225 \end{pmatrix} \to \begin{pmatrix} 0.7071225 \\ 0.23822625 \\ 0.05465125 \end{pmatrix} \rightsquigarrow_\infty \begin{pmatrix} 0.625 \\ 0.3125 \\ 0.0625 \end{pmatrix}$$

Markov Chains Application: Webpage Ranking

The Web Graph

- The web graph is a graph model of the Internet, where nodes are webpages and edges represent hyperlinks between webpages.
- A search engine is broadly defined as an information retrieval system for the web, which when queried on some topic returns a list of documents which share some content or relationship to the topic.



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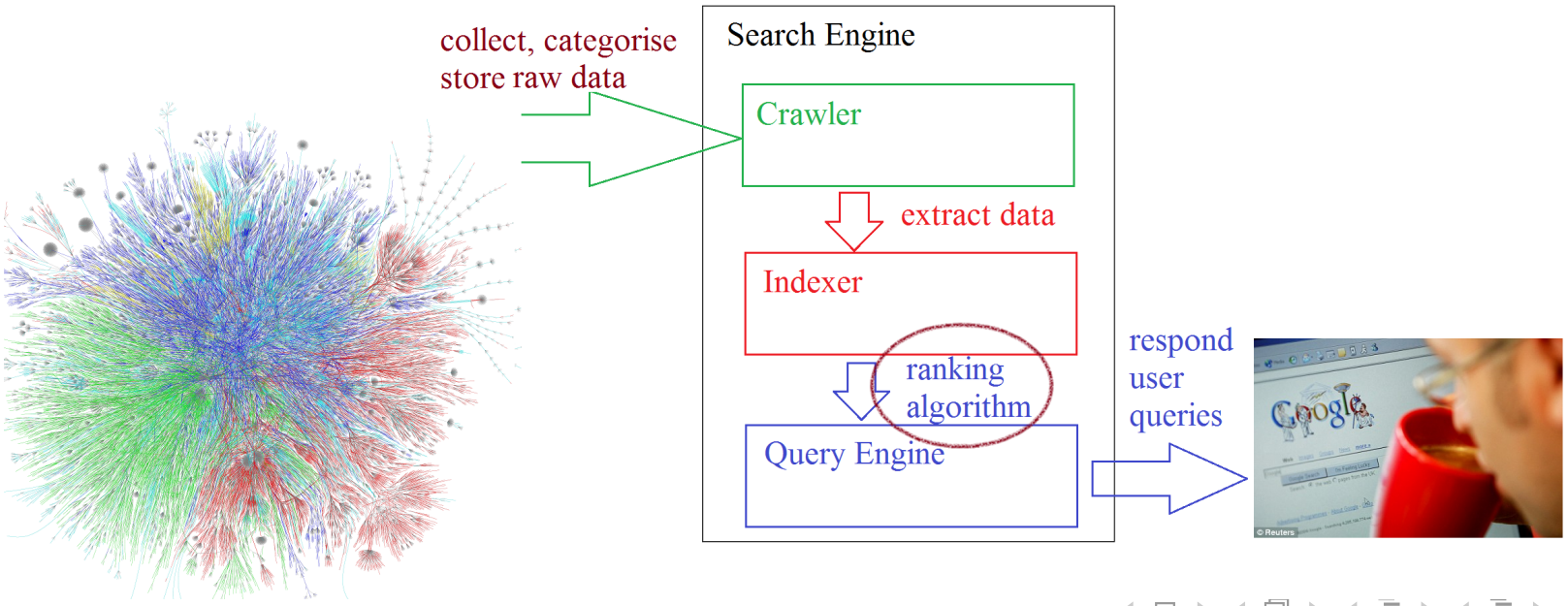

Markov Chains Application: Webpage Ranking

Ranking Algorithm

- The ranking algorithm ranks webpages in order of their relevance to the query.
- The ranking is often achieved by assigning a numerical weight to a given webpage.

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PageRank (Larry Page, Sergey Brin 1998)

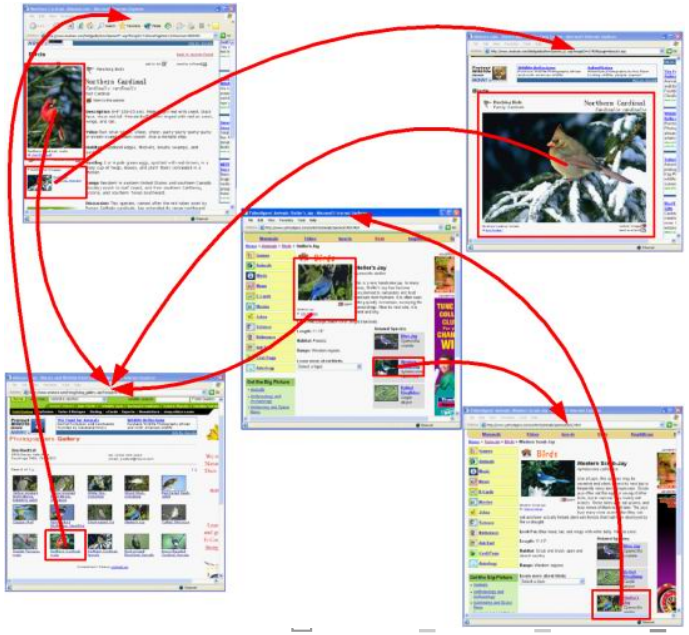
S. Brin ``*Knowledge is always good, and certainly always better than ignorance* ''

- The first ranking algorithm Google used
- One of the first link-based ranking algorithms

Markov Chains Application: Webpage Ranking

Intuition: The relevance / importance of a webpage is given by the link structure of the Web.

- Which pages does a random surfer, who is not searching for something in particular and is following links uniformly at random, visit most frequently? Such frequency will measure the importance of webpages
- Suppose the surfer is at a webpage v, we should compute probability of the surfer visiting webpage u in the next step.



Markov Chains Application: Webpage Ranking

Surfing Markov Chain and PageRank Matrix

A node i is a sink if there is not outgoing edge of i.

Markov Chains Application: Webpage Ranking

Surfing Markov Chain and PageRank Matrix

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- Case 1: If i is a sink. Let $P(i, j) = \frac{1}{n}$

Markov Chains Application: Webpage Ranking

Surfing Markov Chain and PageRank Matrix

A node i is a sink if there is not outgoing edge of i.

- Case 1: If i is a sink. Let $P(i, j) = \frac{1}{n}$
- Case 2: If i is not a sink. Fix a damping factor $c \in (0, 1]$. Let

$$P(i, j) = \begin{cases} (1 - c)/n + c/deg^{+}(i) & \text{if } (i, j) \in E \\ (1 - c)/n & \text{otherwise} \end{cases}$$

where $deg^{+}(i)$ denotes the out-degree of i.
Note: Page and Brin used $c = 0.15$

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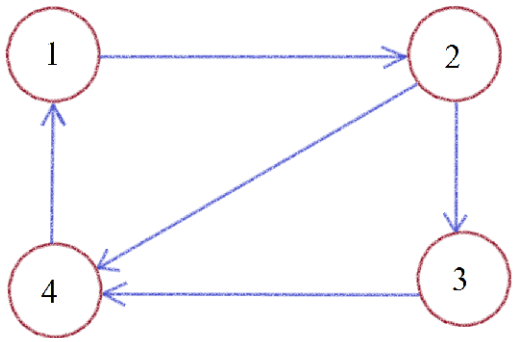
Example:



$$\begin{bmatrix} 17/80 & 29/80 & 17/80 & 17/80 \\ 17/80 & 17/80 & 23/80 & 23/80 \\ 17/80 & 17/80 & 17/80 & 29/80 \\ 29/80 & 17/80 & 17/80 & 17/80 \end{bmatrix}$$

Markov Chains Application: Webpage Ranking

Fact

The PageRank matrix is a stochastic matrix.

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PageRank

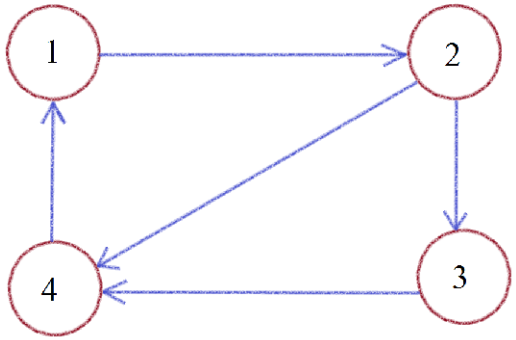
The PageRank vector is a stochastic vector $\vec{v}$ such that

$$P^T \vec{v} = \vec{v}$$

i.e. the PageRank vector is the stationary distribution of the Markov chain represented by the surfing probability matrix P.
The PageRank value of a webpage i is the probability that, at any given moment, a random surfer is visiting this page, and it is the ith entry in the PageRank vector.

Markov Chains Application: Webpage Ranking

Example:



$$\begin{bmatrix} 17/80 & 29/80 & 17/80 & 17/80 \\ 17/80 & 17/80 & 23/80 & 23/80 \\ 17/80 & 17/80 & 17/80 & 29/80 \\ 29/80 & 17/80 & 17/80 & 17/80 \end{bmatrix}$$

$$\begin{bmatrix} 17/80 & 17/80 & 17/80 & 29/80 \\ 29/80 & 17/80 & 17/80 & 17/80 \\ 17/80 & 23/80 & 17/80 & 17/80 \\ 17/80 & 23/80 & 28/80 & 17/80 \end{bmatrix} \begin{pmatrix} 0.252 \\ 0.25 \\ 0.231 \\ 0.266 \end{pmatrix} \approx \begin{pmatrix} 0.252 \\ 0.25 \\ 0.231 \\ 0.266 \end{pmatrix}$$

Stationary distribution is $(0.252, 0.25, 0.231, 0.266)^T$

Note: Essentially PageRank is also a centrality measure, called the PageRank centrality.

Computing PageRank Centrality

We apply power iteration to compute PageRank centrality.

Idea: Simulate the stochastic process by iteratively multiplying the stochastic matrix to the current vector.

Algorithm PageRank(V, E)

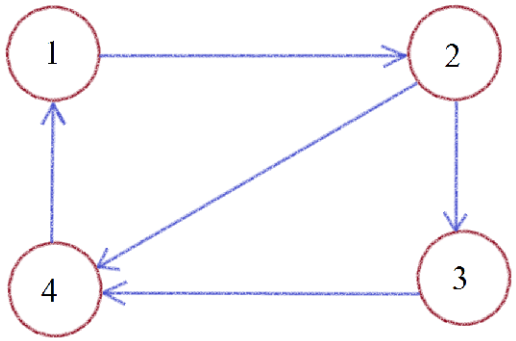
INPUT: A directed graph (V, E)
OUTPUT: PageRank vector $\vec{v}$

1. Compute PageRank Matrix P as defined above:
   - If i is a sink, let $P(i, j) = 1/n$
   - If i is not a sink, let $P(i, j) = (1 - c)/n + c/deg^+(i)$ if $(i, j) \in E$, let $P(i, j) = (1 - c)/n$ otherwise.
2. Initialize $\vec{v} = (1/n, 1/n, \ldots, 1/n)^T$
3. While the current $\vec{v}$ is changing notably do

   Set $\vec{v}$ to $P^T \times \vec{v}$
4. Return $\vec{v}$

Computing PageRank Centrality

Example:



$$\begin{bmatrix} 17/80 & 29/80 & 17/80 & 17/80 \\ 17/80 & 17/80 & 23/80 & 23/80 \\ 17/80 & 17/80 & 17/80 & 29/80 \\ 29/80 & 17/80 & 17/80 & 17/80 \end{bmatrix}$$

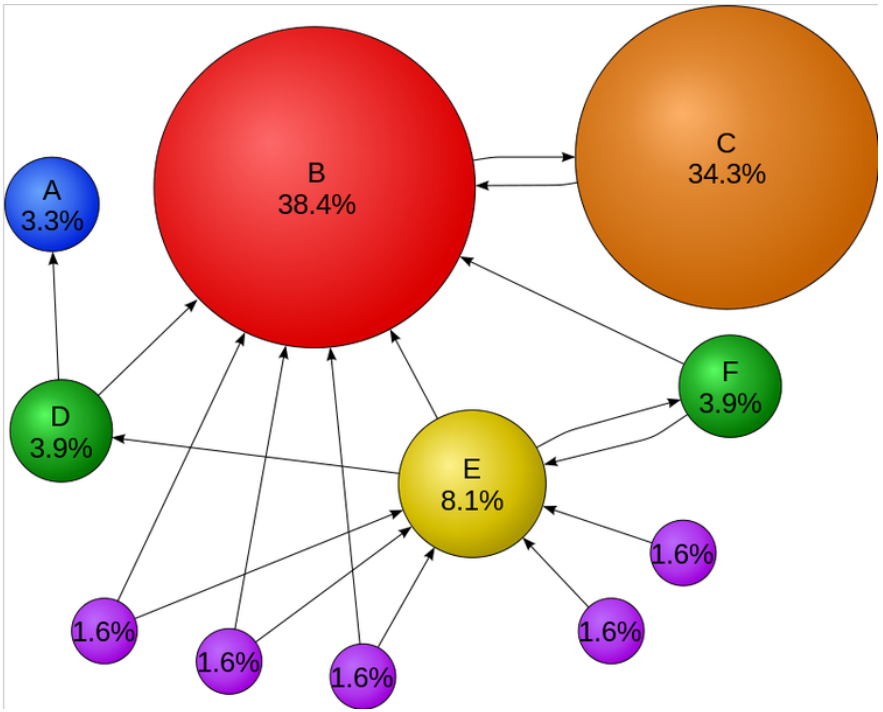
Apply power iteration:

$$\begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix} \rightarrow \begin{pmatrix} 0.25 \\ 0.25 \\ 0.23125 \\ 0.26875 \end{pmatrix} \rightarrow \begin{pmatrix} 0.2528 \\ 0.24999 \\ 0.2312 \\ 0.2659 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} 0.252 \\ 0.25 \\ 0.231 \\ 0.266 \end{pmatrix}$$

Stationary distribution is $(0.252, 0.25, 0.231, 0.266)^T$

Computing PageRank Centrality

Another Example[1]:



[1] From Wikipedia

Summary: Linked-Based Centrality Measures

A linked-based centrality notion defines importance of a node in the network by analysing the network link topology.
So far we looked at two linked-based centrality measures:

- Eigenvector centrality:

  The eigenvector corresponding to the largest eigenvalue of the adjacency matrix.

- PageRank centrality:

  The eigenvector corresponding to the eigenvalue 1 (which happens to be the largest eigenvalue) of the transpose matrix of the random surfing matrix.