Numerical Methods for Partial Differential Equations
A.Y. 2024/2025

Written exam - August 28, 2025

Maximum score: 26. Duration: 2h 15m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded to the dedicated WeBeep folder. **All uploaded files (including source code) must be appropriately referenced when answering to questions on paper** (e.g. "The solution is depicted in `exercise-1-plot.png`").

Exercise 1.

Consider the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu = f & 0 < x < 1,\ t > 0\ , \\ u(x=0,t) = 0 & t > 0\ , \\ \varepsilon \dfrac{\partial u}{\partial x} - bu = 0 & x = 1,\ t > 0\ , \\ u(x, t=0) = u_0(x) & 0 < x < 1\ , \end{cases} \tag{1}$$

where

$$Lu = -\frac{\partial}{\partial x}\left(\varepsilon \frac{\partial u}{\partial x} - bu\right) + ku\ ,$$

$\varepsilon > 0$, $k > 0$ are two positive constants, and b is a C^1 function.

1.1. [*1 pt*] Find the weak formulation of problem (1), with the right choice of the function space V.

1.2. [*3 pt*] Prove that under suitable assumptions the bilinear form $a(v, v)$ defined at point 1.1 is strongly coercive.

1.3. [*1 pt*] Approximate problem (1) with finite elements of degree $r = 2$ in space and the Crank-Nicolson method in time.

1.4. [*2 pt*] When $b = 0$ and $f = 0$, discuss the stability properties of the fully discrete problem derived at point 1.3 and write the corresponding error estimate.

1.5. [*4 pt*] Using `deal.II`, implement a solver for problem (1). The solver should support the θ-method for time discretization, for arbitrary values of $\theta \in [0, 1]$. Use the internal `deal.II` functions for generating the mesh. Consider the following data:

$$
\begin{aligned}
&\varepsilon = 1 \,, \\
&b = x - 1 \,, \\
&k = 1 \,, \\
&f = \frac{\pi}{2} \sin\left(\frac{\pi}{2}x\right) \cos\left(\frac{\pi}{2}t\right) + \left(\frac{\pi^2}{4} + 2\right) \sin\left(\frac{\pi}{2}x\right) \sin\left(\frac{\pi}{2}t\right) + \frac{\pi}{2}(x-1) \cos\left(\frac{\pi}{2}x\right) \sin\left(\frac{\pi}{2}t\right) \,, \\
u_0(x) &= 0 \,.
\end{aligned}
$$

Solve up to a final time $T = 1$ with a time step $\Delta t = 0.1$. Use finite elements of degree $r = 2$ for space discretization, over a mesh with $N = 40$ elements, and set $\theta = 0.5$. Upload the source code of the solver.

1.6. [*2 pt*] Plot and upload the solution computed at point 1.5 as a function of space at the final time $T = 1$, and the solution as a function of time at the point $x = 0.5$ (use the filters "Plot over line" and "Plot selection over time").

1.7. [*3 pt*] Knowing that the exact solution of (1), with the data of point 1.5, is

$$u_{\text{ex}}(x, t) = \sin\left(\frac{\pi}{2}x\right) \sin\left(\frac{\pi}{2}t\right) \,,$$

analize the convergence of the numerical solution with respect to Δt. Use polynomials of degree $r = 2$, a mesh with $N = 40$ elements, and consider $\Delta t \in \{0.1, 0.05, 0.025, 0.0125\}$. Report the estimated convergence order, and discuss it in light of the theory.

1.8. [*1 pt*] Repeat point 1.7 with $N = 10$. How do the convergence results change, and why?

1.9. [*1 pt*] Repeat point 1.7 setting $\theta = 1$ (implicit Euler) and $\theta = 0$ (explicit Euler). Comment on the results you obtain.

Exercise 2.

Consider the problem

$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \boldsymbol{\nabla} p = 0 & \mathbf{x} \in \Omega \subset \mathbb{R}^2, \ t > 0 \,, \\
\operatorname{div} \mathbf{u} = 0 & \mathbf{x} \in \Omega, \ t > 0 \,, \\
\mathbf{u}(\mathbf{x}, t) = \mathbf{0} & \mathbf{x} \in \partial\Omega \,, t > 0 \,, \\
\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0(\mathbf{x}) & \mathbf{x} \in \Omega \,.
\end{cases}
\tag{2}
$$

2.1. [*2 pt*] Approximate (2) in time by using the implicit Crank-Nicolson method. Which corresponding problem do we find at any time level $t^n = n\Delta t$?

2.2. [*2 pt*] Approximate the problem obtained at any time level t^n at point 2.1 by the Taylor-Hood finite elements of degree 2 for the velocity and 1 for the pressure and discuss its existence and uniqueness. Then provide the expected error estimate.

Exercise 3.

Consider the problem

$$\begin{cases} Lu = -\varepsilon \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} + b \dfrac{\mathrm{d}u}{\mathrm{d}x} + c\,u = f \quad 0 < x < 1 \,, \\ u(0) = u(1) = 0 \,. \end{cases} \tag{3}$$

3.1. [*2 pt*] Consider the subdomains $\Omega_1 = (0, \gamma)$ and $\Omega_2 = (\gamma, 1)$, for some $\gamma \in (0, 1)$, and with the additive Dirichlet-Neumann (DN) method with relaxation for problem (3). Is this method convergent?

3.2. [*2 pt*] Can we generalize this method to the case of many subdomains (more than two)?