Deep Learning for Smoothing in Dynamical Systems

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Outline

- Many dynamical systems can be described by a *state space model*:

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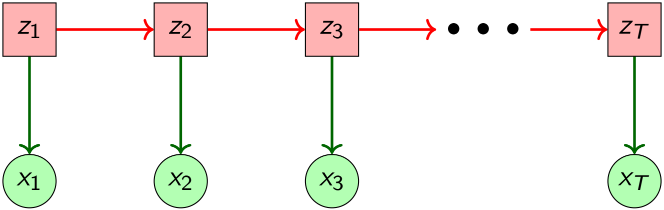
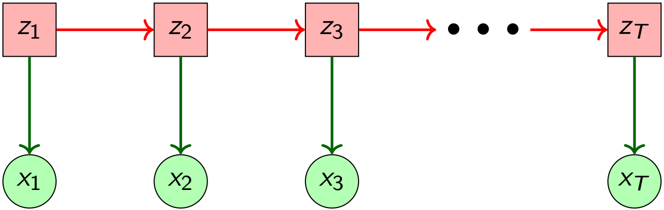
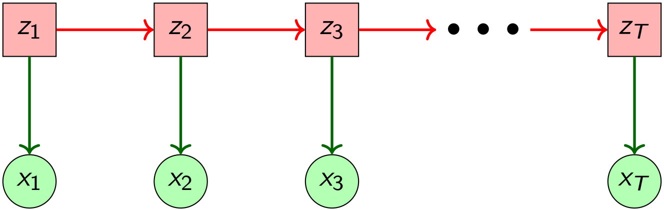


Dynamical Systems and State Space Models



$$\begin{cases} z_{t+1} = f(z_t) + v_{1_t} \\ x_t = g(z_t) + v_{2_t}, \end{cases} \iff \begin{cases} Z_{t+1}|z_t \sim p(z_{t+1}|z_t) \\ X_t|z_t \sim p(x_t|z_t) \end{cases}$$



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- $z_1, z_2, ..., z_T$ are *hidden states*.
- $x_1, x_2, ..., x_T$ are the *measured* quantities.

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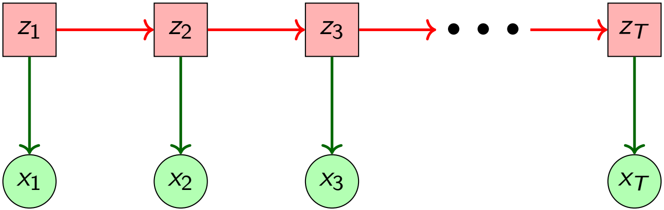

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- $z_1, z_2, ..., z_T$ are *hidden states*.
- $x_1, x_2, ..., x_T$ are the *measured* quantities.
- v_1 and v_2 are noise terms.

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Filtering and Smoothing

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- Filtering - online problem

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- if f and g linear and v_1, v_2 Gaussian white noise the optimal solution is the *Kalman filter/smoother*

Smoothing

- Offline problem - finding all z as a function of x

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- $\mathbf{x} = (x_1, x_2, ..., x_T)$
- $\mathbf{z} = (z_1, z_2, ..., z_T)$

$$p(\mathbf{x}, \mathbf{z}) = p(z_0)\Big[\prod_{t=1}^{T} p(z_t|z_{t-1})\Big]\Big[\prod_{t=1}^{T} p(x_t|z_t)\Big]$$

Variational Inference

- Variational inference

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- $p(\mathbf{x})$ is hard to calculate
- Idea: use approximate posterior $q_\theta(\mathbf{z}|\mathbf{x}) \approx p(\mathbf{z}|\mathbf{x})$

Variational Inference

- Chose $q_\theta(\mathbf{x}|\mathbf{z})$ s.t. it maximises the Evidence Lower Bound (ELBO)

$$\mathcal{L} = \mathbb{E}_{q_\theta}[\log p(\mathbf{x}, \mathbf{z})] - \mathbb{E}_{q_\theta}[\log q_\theta(\mathbf{z}|\mathbf{x})]$$

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- This is equivalent to minimising the Kullback–Leibler divergence

$$D_{KL}(q||p) = \int_\Omega p(\omega) \log \frac{p(\omega)}{q(\omega)} d\omega$$

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Black Box Variational Inference

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- Idea: Sample z from q_θ and calculate noisy gradient

Black Box Variational Inference: Noisy Gradient

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- Let $q_\theta(z_t|x_t, ..., x_T) = \mathcal{N}(\mu_t, \sigma_t^2)$

$$z_t = \mu_t + \sigma_t \epsilon =: g_\theta(\epsilon)$$
$$\epsilon \sim \mathcal{N}(0, 1) =: p(\epsilon)$$

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- Approximate
$\nabla_\theta \mathcal{L} = \mathbb{E}_{p(\epsilon)}[\nabla f_\theta(x, g_\theta(\epsilon))] \approx \frac{1}{N} \sum_{i=1}^{N} \nabla f_\theta(x, g_\theta(\epsilon))$

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- This is called the *reparametrisation trick*

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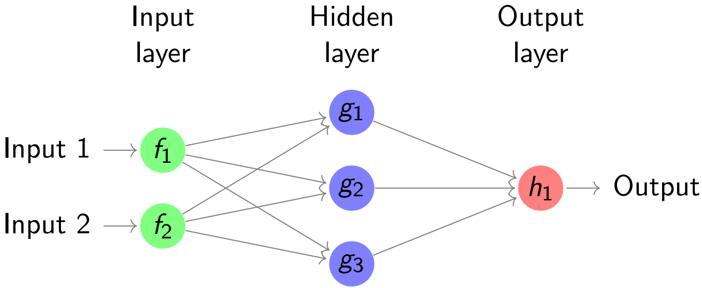

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Neural Networks

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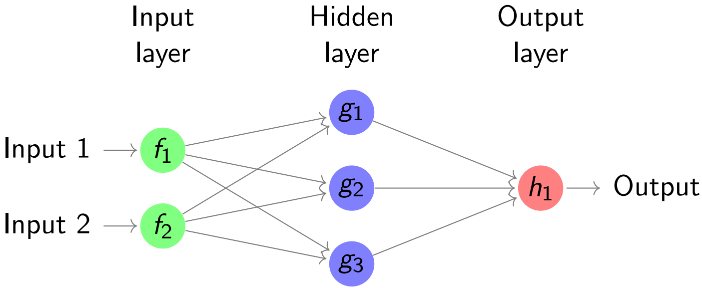

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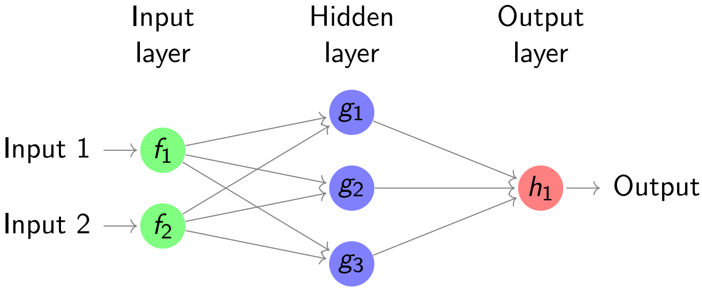

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- b is called a bias and ω is called a weight, K is some non-linear function, called an activation function.
- Typically some loss function is minimised w.r.t. the weights and biases.

Summary so far

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- State Space Models

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- Black Box Variational Inference
- Neural Networks

Implementation

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- All of the methods here were implemented with Python and TensorFlow.
- TensorFlow is an open source software library commonly used in deep learning.

BBVI for Smoothing Using RNN

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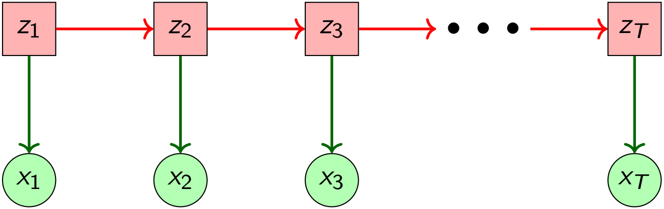

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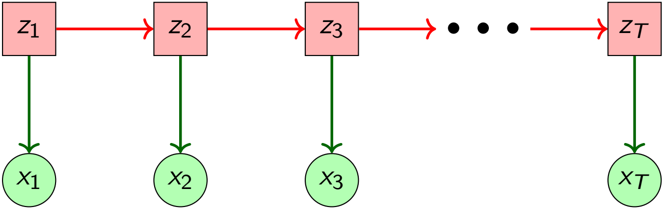
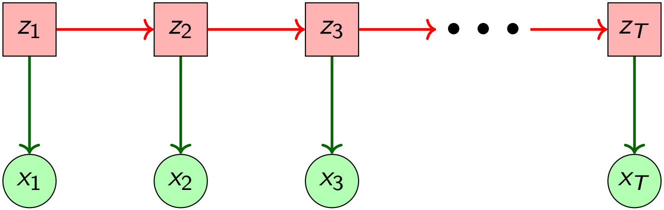


- BBVI: Black box variational inference

$$q_\theta(\mathbf{z}|\mathbf{x}) \xrightarrow{\text{Optimize } \theta} \text{Maximize ELBO} \longrightarrow q_\theta(\mathbf{z}|\mathbf{x}) \approx p(\mathbf{z}|\mathbf{x})$$

BBVI for Smoothing Using RNN

- Factorize $q_\theta(\mathbf{z}|\mathbf{x})$



$$\begin{cases} q_\theta(\mathbf{z}|\mathbf{x}) = q_\theta(z_1|x_1, \dots, x_T) \prod_{t=2}^{T} q_\theta(z_t|z_{t-1}, x_t, \dots, x_T) \\ q_\theta(z_t|z_{t-1}, x_t, \dots, x_T) \sim \mathcal{N}(\mu_t(z_{t-1}, x_t, \dots, x_T), \sigma_t^2(z_{t-1}, x_t, \dots, x_T)) \end{cases}$$

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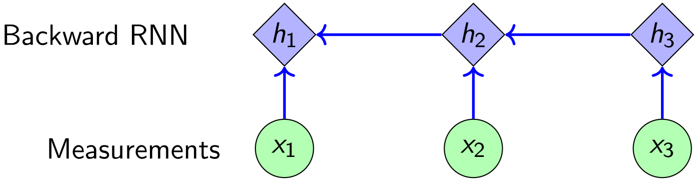

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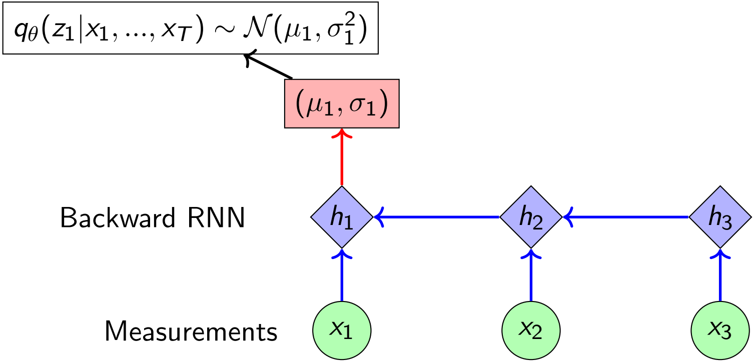

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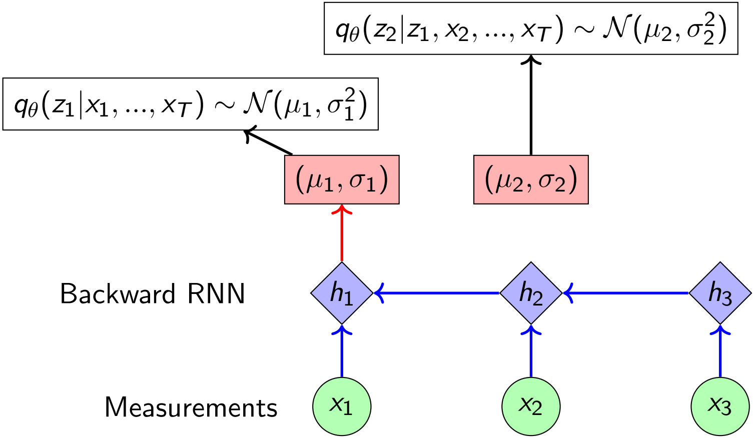

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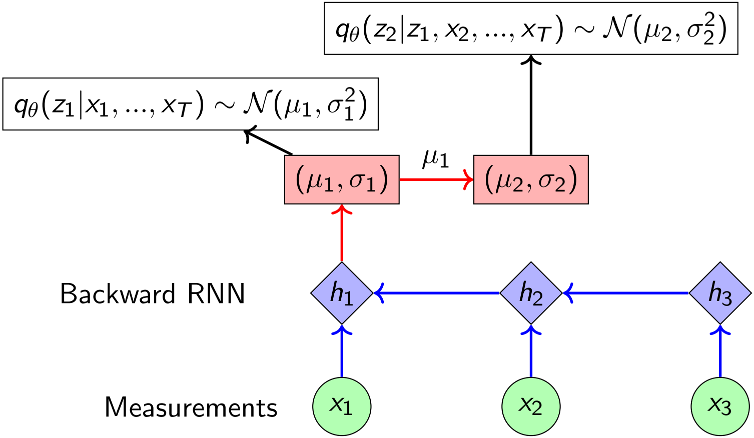

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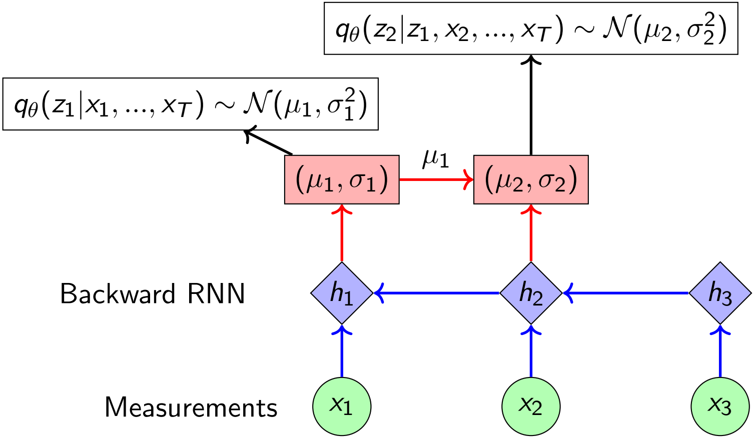

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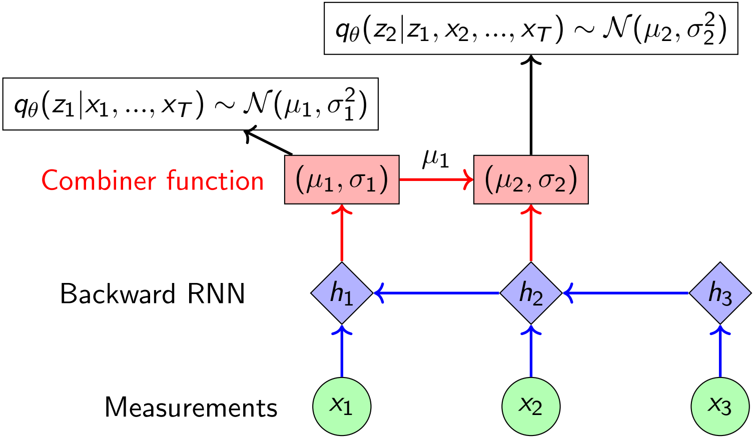

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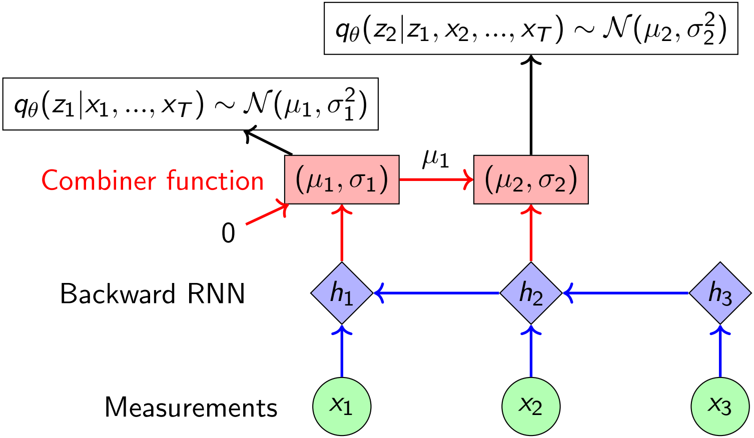

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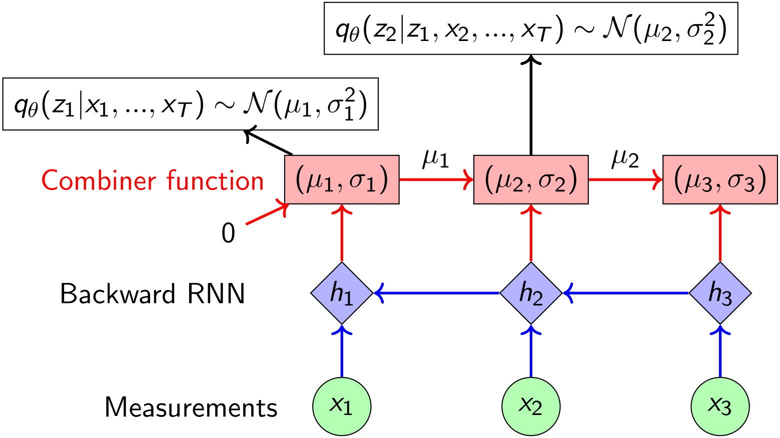

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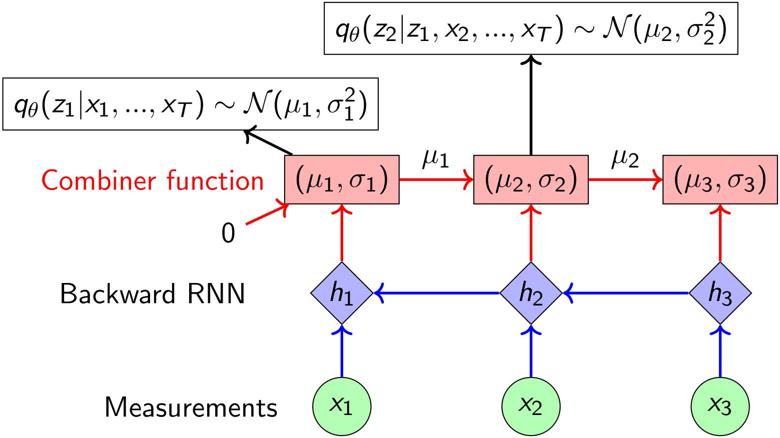

- Output: $\boldsymbol{\mu} = \mu_1, \mu_2, \ldots, \mu_T \quad \boldsymbol{\sigma} = \sigma_1, \sigma_2, \ldots, \sigma_T$

BBVI for Smoothing Using RNN

Train the RNN

- Maximising the objective function: ELBO

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\sigma}) = \mathcal{L}(\mathbf{x}, \theta)$$

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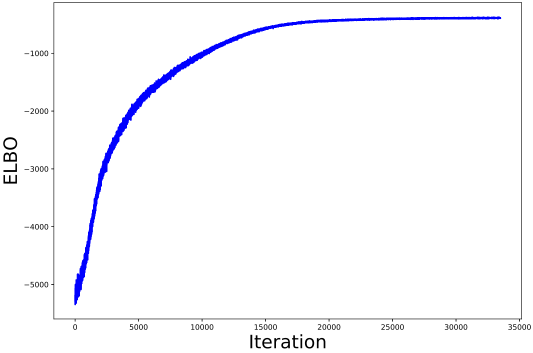

BBVI for Smoothing Using RNN

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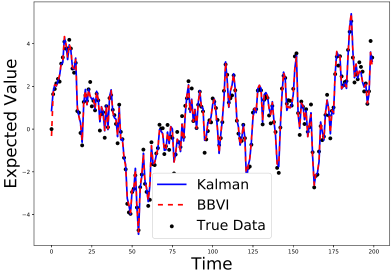
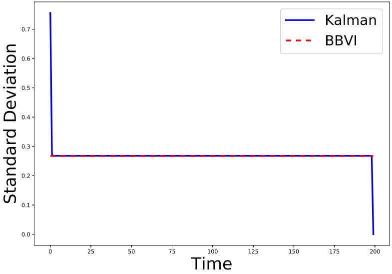
Results



(a) Expected Value



(b) Standard Deviation

Conclusions

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- Our BBVI using a RNN recovers the full posterior probability density function obtained by the Kalman smoother.

Conclusions

- Our BBVI using a RNN recovers the full posterior probability density function obtained by the Kalman smoother.
- BBVI using RNN is more flexible than the Kalman smoother, because it can also be used for non-linear SSM with noise that is not Gaussian. All we need to do is to rewrite the ELBO according to the new model.

Thank you for listening!

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Any questions?