Chapter 6
Heapsort

Algorithm Analysis

School of CSEE

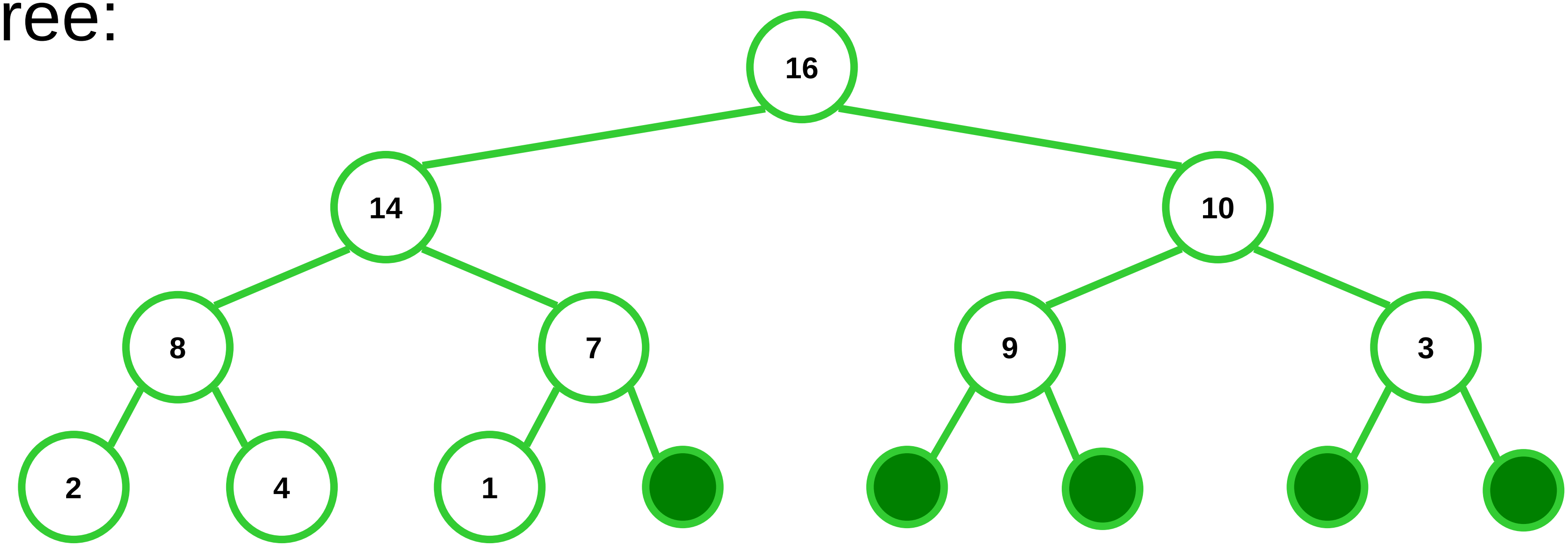
Sorting Revisited

- So far we've talked about two algorithms to sort an array of numbers
  - What is the advantage of merge sort? ➔ faster
  - What is the advantage of insertion sort? ➔ sort in place
- Next on the agenda: *Heapsort*
  - Combines advantages of both previous algorithms

Sort in place : Only a constant number of array elements are stored outside the input array at any time.

Heaps

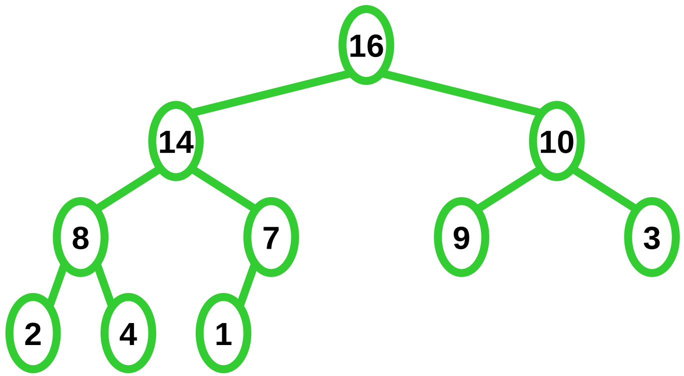
- A *heap* can be seen as a nearly complete binary tree:



  – We can think of unfilled slots as null pointers

Heaps

- In practice, heaps are usually implemented as arrays:



=

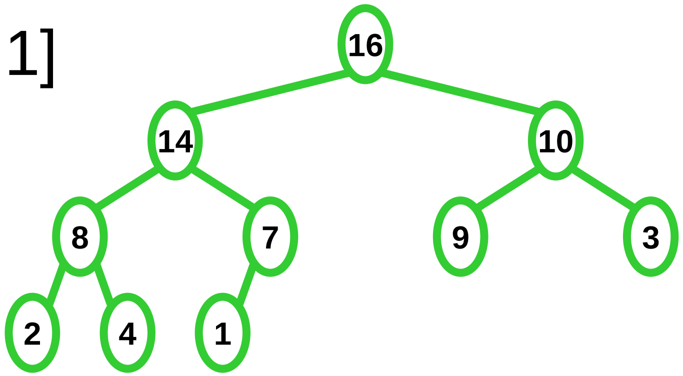
A =

16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

Heaps

- To represent a nearly complete binary tree as an array:
  - The root node is A[1]
  - Node *i* is A[*i*]
  - The parent of node *i* is A[*i*/2] (note: integer divide)
  - The left child of node *i* is A[2*i*]
  - The right child of node *i* is A[2*i* + 1]

A =	*16*	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*	=
		2		4				8	9		



Heap

- A max tree is a tree in which the key value in each node is no smaller than the key values in its children.
- A min tree is a tree in which the key value in each node is no larger than the key values in its children.
- A max-heap is a nearly complete binary tree that is also a max tree.
- A min-heap is a nearly complete binary tree that is also a min tree.

The Heap Property

- Heaps satisfy the *heap property*:

  $A[Parent(i)] \geq A[i]$ for all nodes $i > 1$

  – In other words, the value of a node is at most the value of its parent

  – *Where is the largest element in a heap stored?*

- Definitions:

  – The *height* of a node in the tree = the number of edges on the longest downward path to a leaf

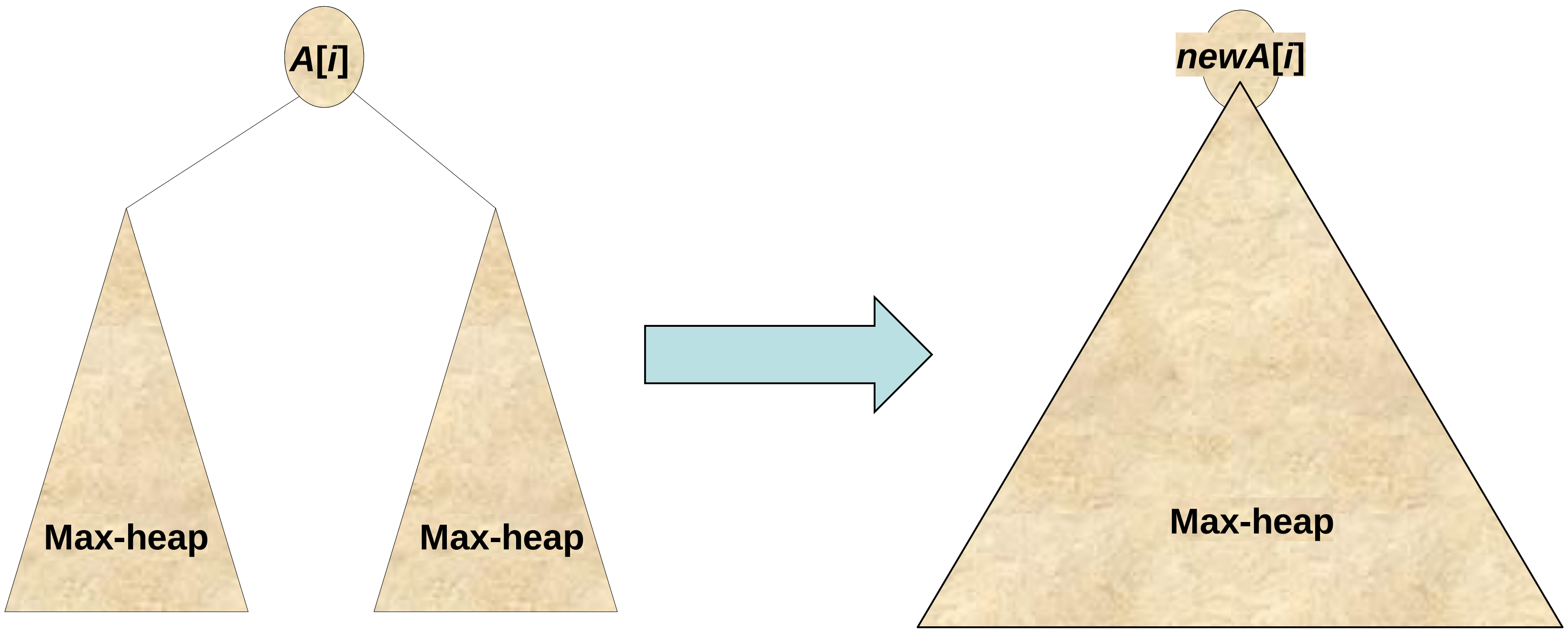
  – The *height* of a tree = the height of its root = $\Theta(\lg n)$.

Maintaining the heap property

- MAX-HEAPIFY maintains the max-heap property.
  - Before MAX-HEAPIFY, $A[i]$ may be smaller than its children.
  - Assume left and right subtrees of i are max-heaps.
  - After MAX-HEAPIFY, subtree rooted at i is a max-heap.

MAX-HEAPIFY lets the value at $A[i]$ "float down" in the max-heap so that the subtree rooted at index i becomes a max-heap.

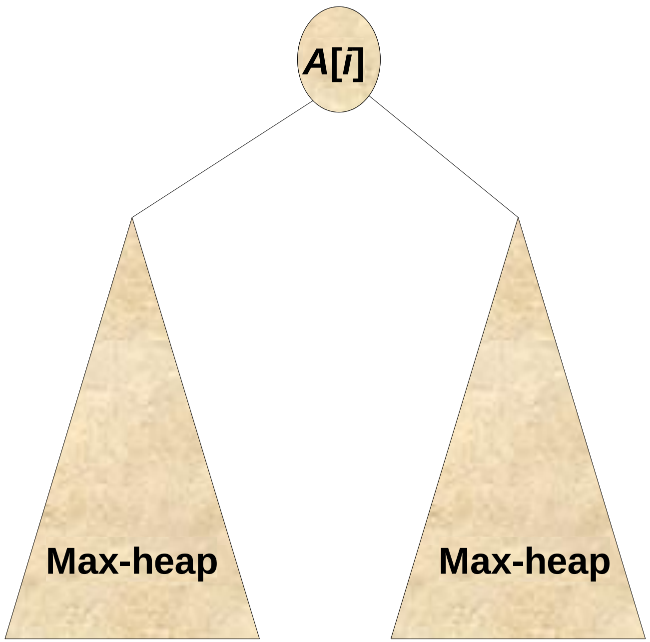
Maintaining the Max-Heap



initially

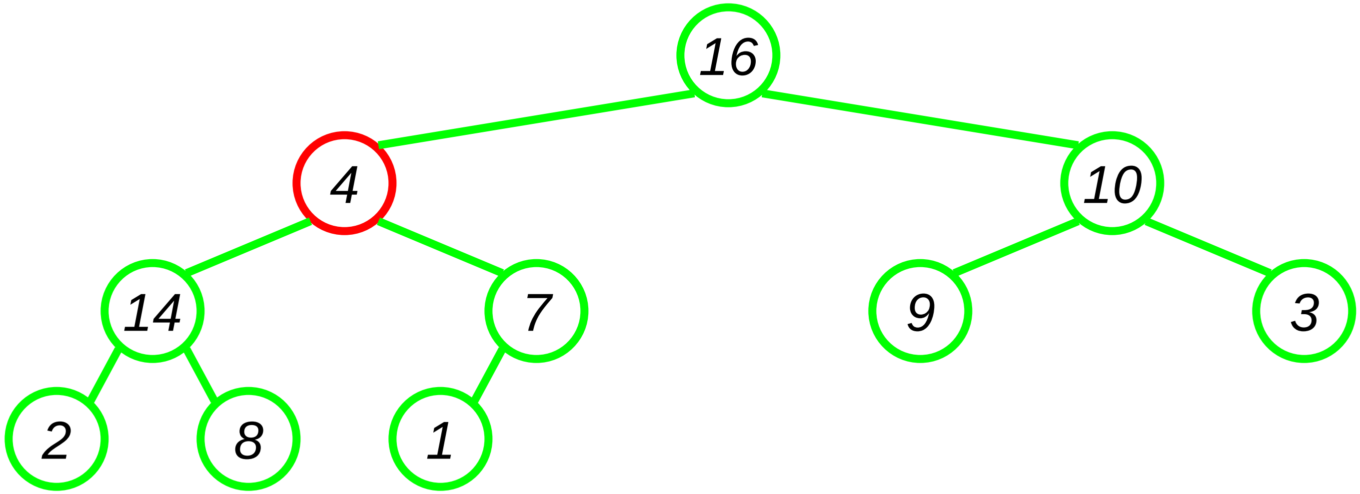
After Max-Heapify

Maintaining the Max-Heap




```
Max-Heapify(A, i)
    l = left(i)
    r = right(i)
    if l <= heap-size[A] and A[l] > A[i]
       then largest =
       else largest =
    if r <= heap-size[A] and A[r] > A[largest]
       then largest =
    if largest ≠ i
       then exchange A[i] with A[largest]
            Max-Heapify(A, largest)
```

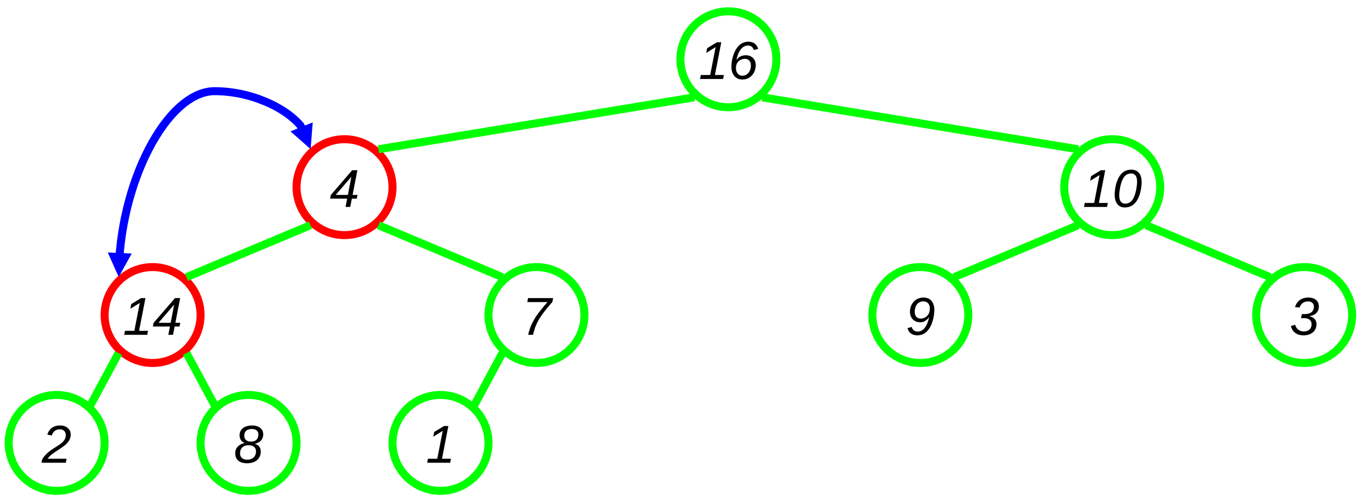
Max-Heapify() Example



A =

16	4	10	14	7	9	3	2	8	1

Max-Heapify() Example



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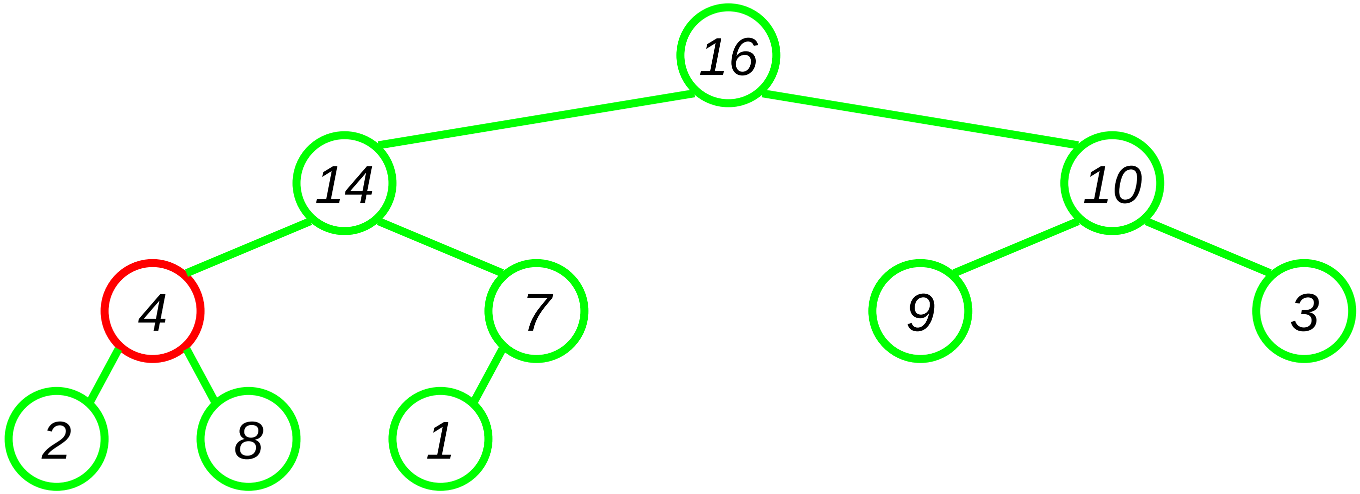
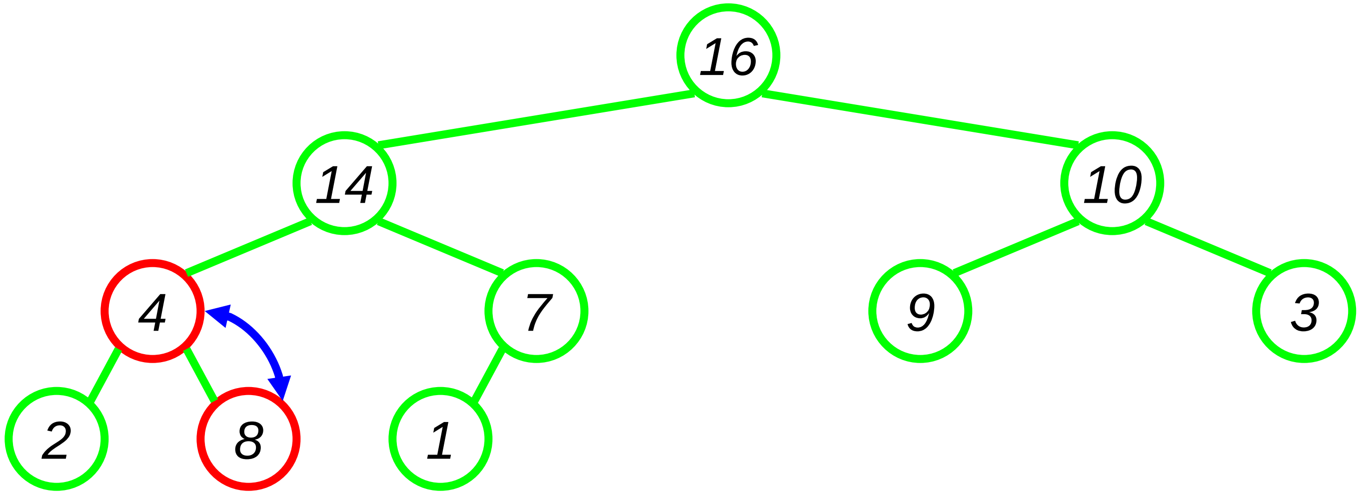
Max-Heapify() Example



A =

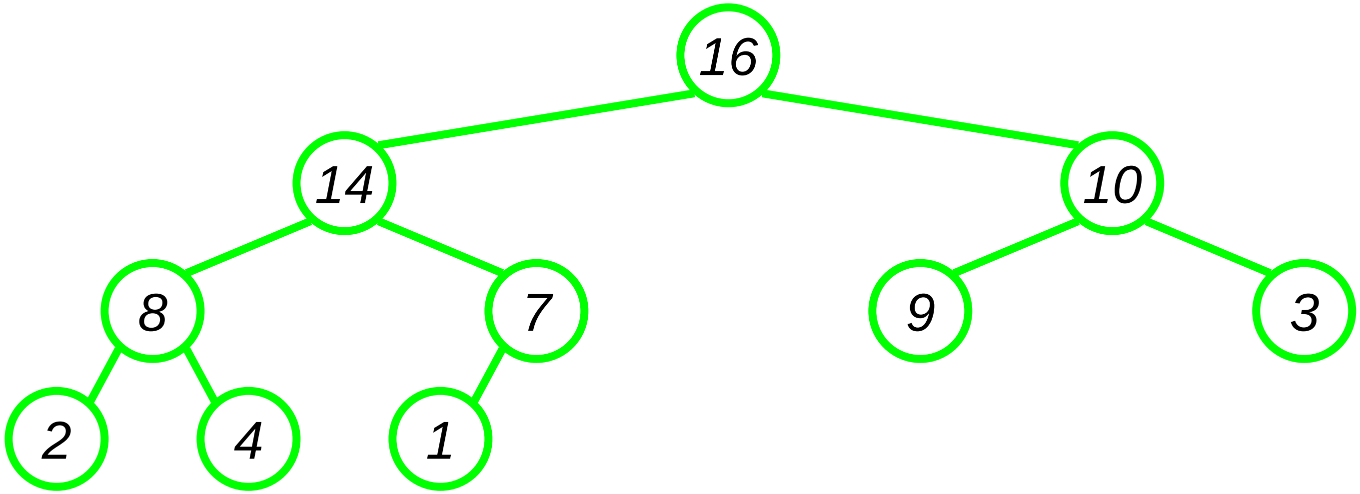
16	14	10	4	7	9	3	2	8	1

Max-Heapify() Example



A =	*16*	*14*	*10*	*4*	*7*	*9*	*3*	*2*	*8*	*1*

Max-Heapify() Example



A =

16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

Analyzing Heapify(): Formal

- Fixing up relationships between *i*, *l*, and *r* takes $\Theta(1)$ time.
- *If the heap at i has n elements, how many elements can the subtrees at l or r have?*
- Worst case = Most unbalanced case

  $2n/3$ on left subtree and $n/3$ on right subtree

  (bottom row 1/2 full)
- So time taken by **`Max-Heapify()`** is given by

  $T(n) \le T(2n/3) + \Theta(1)$

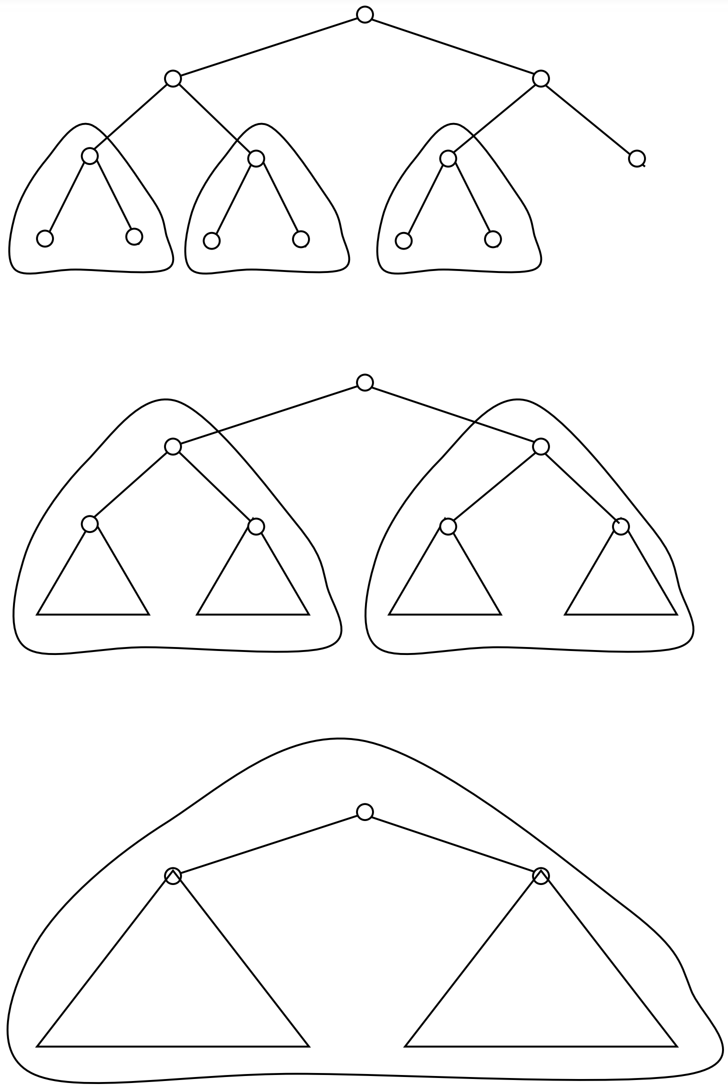
  By case 2 of the Master Theorem,

  $T(n) = O(\lg n)$

Heap Operations: Build-MAX-Heap()

- We can build a heap in a bottom-up manner by running **`Max-Heapify()`** on successive subarrays
  - Fact: for array of length *n*, all elements in range A[$\lfloor n/2 \rfloor$ + 1 .. n] are heaps (*Why?*)
  - So:
    - Walk backwards through the array from n/2 to 1, calling **`Max-Heapify()`** on each node.
    - Order of processing guarantees that the children of node *i* are heaps when *i* is processed

How to create a heap?



Building a Heap

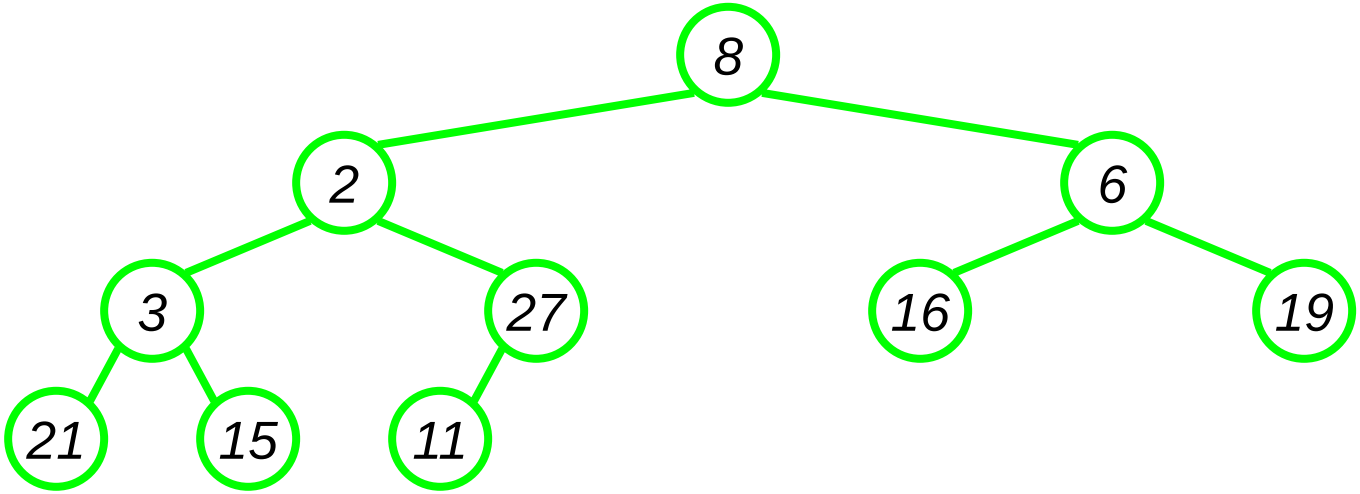
```
Build-Max-Heap(A)
1   heap-size[A]=length[A]
2   for i = ⌊length[A]/2⌋ downto 1
3       do Max-Heapify(A,i)
```

$$i = \left\lfloor \frac{length[A]}{2} \right\rfloor$$

At the start of each iteration of the "**for**" loop of lines 2-3, each node i+1, i+2, …, n is the root of a max-heap.

Build-Max-Heap() Example

- Work through example
  A = {8, 2, 6, 3, 27, 16, 19, 21, 15, 11}



Analyzing Build-Max-Heap()

- Each call to **`Max-Heapify()`** takes O(lg *n*) time
- There are O(*n*) such calls (specifically, $\lfloor n/2 \rfloor$)
- Thus the running time is O(*n* lg *n*)
  - *Is this a correct asymptotic upper bound?*
  - *Is this an asymptotically tight bound?*
- A tighter bound is O(*n*)
  - *How can this be? Is there a flaw in the above reasoning?*

Analyzing Build-Max-Heap(): Tight

- To **`Max-Heapify()`** a subtree takes O(h) time where h is the height of the subtree
  - h = O(lg n), n = # nodes in subtree
  - The height of most subtrees is small
- Fact: an n-element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h.

$$T(n) = \sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil \mathrm{O}(h)$$

$$= \mathrm{O}(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h})$$

$$= \mathrm{O}(n \sum_{h=0}^{\infty} \frac{h}{2^h})$$

$$= \mathrm{O}(n)$$

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Heapsort

- Given **`Build-Max-Heap()`**, an in-place sorting algorithm is easily constructed:
  - Maximum element is at *A*[1]
  - Discard by swapping with element at *A*[*n*]
    - Decrement heap_size[*A*]
    - *A*[*n*] now contains correct value
  - Restore heap property at *A*[1] by calling **`Max-Heapify()`**
  - Repeat, always swapping *A*[1] for *A*[heap_size(*A*)]

Heapsort

Pass 1

77	61	59	48	19	11	26	15	1	5

5	61	59	48	19	11	26	15	1	77

Record 5 will be pushed down to make heap.

61	48	59	15	19	11	26	5	1	77

Pass 2

61	48	59	15	19	11	26	5	1	77

1	48	59	15	19	11	26	5	61	77

Record 1 will be pushed down to make heap.

59	48	26	15	19	11	1	5	61	77

Heapsort

```
Heapsort(A)
1   Build-Max-Heap(A)
2   for i = length[A] downto 2
3       do exchange A[1] with A[i]
4           heap-size[A]=heap-size[A]-1
5           Max-Heapify(A,1)
```

Steps 1 : $O(n)$

Step 3-5: $O(\lg n)$

$$T(n) = O(n) + (n-1)O(\lg n) = O(n \lg n)$$

Heapsort

```
Heapsort(A)
    Build-Max-Heap(A)
    for i = length[A] downto 2
        do exchange A[1] with A[i]
            heap-size[A]=heap-size[A]-1
            Max-Heapify(A,1)
```

```
Build-Max-Heap(A)
    heap-size[A]=length[A]
    for i = ⌊length[A]/2⌋ downto 1
        do Max-Heapify(A,i)
```

```
Max-Heapify(A,i)
    l=left(i)
    r=right(i)
    if l <= heap-size[A] and A[l]>A[i]
        then largest = l
        else largest = i
    if r <= heap-size[A] and
            A[r]>A[largest]
        then largest = r
    if largest ≠ i
        then  exchange A[i] / A[largest]
                Max-Heapify(A,largest)
```

Priority Queues

- A well-implemented quicksort usually beats heapsort in practice.
- But the heap data structure is incredibly useful for implementing *priority queues.* The element to be deleted is the one with highest (or lowest) priority.

Priority Queues

- Priority queue
  - Maintains a dynamic set *S* of elements.
  - Each set element has a key.
  - Example of max-priority queue : schedule jobs on shared computers.
  - Example of min-priority queue : Utilizing the service machine time. Job with smallest service time is selected.

Dynamic Set : The set that can grow, shrink, or otherwise change over time.

Priority Queue - Insertion

(A,1)					

Insert (B,3)

(B,3)	**(A,1)**				

Insert (C,2)

(B,3)	**(A,1)**	**(C,2)**			

Insert (D,1)

(B,3)	**(A,1)**	**(C,2)**	**(D,1)**		

Insert (E,3)

(B,3)	**(E,3)**	**(C,2)**	**(D,1)**	**(A,1)**	

Priority Queue - Deletion

(B,3)	**(E,3)**	**(C,2)**	**(D,1)**	**(A,1)**	

Delete (B,3)

(E,3)	**(A,1)**	**(C,2)**	**(D,1)**		

Delete (E,3)

(C,2)	**(A,1)**	**(D,1)**			

Delete (C,2)

(A,1)	**(D,1)**				

Delete(A,1)

(D,1)					

Priority queues

- Max-priority queue supports dynamic-set operations :
    - INSERT(S, x) : inserts element x into set S.
    - MAXIMUM(S) : returns element of S with largest key.
    - EXTRACT-MAX(S) : removes and returns element of S with largest key.
    - INCREASE-KEY(S, x, k) : increases value of element x's key to k. (Assume $k \geq x$'s current key value).
  - Min-priority queue supports similar operations.

Maximum operation

Heap-Maximum(A)

1 Return $A(1)$

O(1) running time

Extracting the maximum

```
Heap-Extract-Max(A)
if heap-size[A]<1
    then error "heap underflow"
max=A[1]
A[1]=A[heapsize[A]]
Heapsize[A]= Heapsize[A]-1
Max-Heapify(A,1)
return max
```

$O(\lg n)$ running time

Increase-Key operation

```
Heap-Increase-Key(A, i, key)
if key < A[i]
    then error "new key is smaller than current key"
A[i] = key
while i > 1 and A[parent(i)] < A[i]
    do exchange A[i] with A[parent(i)]
        i = parent(i)
```

$O(\lg n)$ running time

Insert operation

Max−Heap−Insert(A, key)

1 $heapsize[A] = heapsize[A] + 1$

2 $A[heapsize[A]] = -\infty$

3 Heap−Increase−Key(A, $heapsize[A]$, key)

$O(\lg n)$ running time