# 1·B

- (OR [9.2,9.3]. OR Problem (1) in 9.A)

*Suppose $V$ is a real vector space. The complexification of $V$, denoted by $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair $(u, v)$, where $u, v \in V$, but we write this as $u + iv$.*

- *Addition on $V_{\mathbb{C}}$ is defined by*
$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
*for all $u_1, v_1, u_2, v_2 \in V$.*
- *Complex scalar multiplication on $V_{\mathbb{C}}$ is defined by*
$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$
*for all $a, b \in \mathbf{R}$ and all $u, v \in V$.*

*Prove that with the definitions above, $V_{\mathbb{C}}$ is a complex vector space.*

*Think of $V$ as a subset of $V_{\mathbb{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbb{C}}$ from $V$ can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.*

**SOLUTION:**

- Commutativity: $(u_1 + \mathrm{i}v_1) + (u_2 + \mathrm{i}v_2) = (u_2 + \mathrm{i}v_2) + (u_1 + \mathrm{i}v_1)$.
- Associativity:

(I) $[(u_1 + \mathrm{i}v_1) + (u_2 + \mathrm{i}v_2)] + (u_3 + \mathrm{i}v_3) = (u_1 + \mathrm{i}v_1) + [(u_2 + \mathrm{i}v_2) + (u_3 + \mathrm{i}v_3)]$.

(II) $\begin{cases} [(a+b\mathrm{i})(c+d\mathrm{i})](u+\mathrm{i}v) = [(ac-bd)+(ad+bc)\mathrm{i}](u+\mathrm{i}v) = [(ac-bd)u-(ad+bc)v] + \mathrm{i}[(ac-bd)v+(ad+bc)u] \\ (a+b\mathrm{i})[(c+d\mathrm{i})(u+\mathrm{i}v)] = (a+b\mathrm{i})[(cu-dv)+\mathrm{i}(cv+du)] = [a(cu-dv)-b(cv+du)] + \mathrm{i}[a(cv+du)+b(cu-dv)] \end{cases}$

- Additive identity.
- Additive inverse: $(u_1 + \mathrm{i}v_1) + (-u_1 + \mathrm{i}(-v_1)) = 0$.
- Multiplication identity.
- Distributive properties:

(I) $\begin{cases} (a+b\mathrm{i})[(u_1+\mathrm{i}v_1)+(u_2+\mathrm{i}v_2)] = (a+b\mathrm{i})[(u_1+u_2)+\mathrm{i}(v_1+v_2)] \\ \qquad\qquad = [a(u_1+u_2)-b(v_1+v_2)] + \mathrm{i}[a(v_1+v_2)+b(u_1+u_2)] \\ (a+b\mathrm{i})(u_1+\mathrm{i}v_1)+(a+b\mathrm{i})(u_2+\mathrm{i}v_2) = [(au_1-bv_1)+\mathrm{i}(av_1+bu_1)] + [(au_2-bv_2)+\mathrm{i}(av_2+bu_2)] \end{cases}$

(II) $\begin{cases} [(a+b\mathrm{i})+(c+d\mathrm{i})](u+\mathrm{i}v) = [(a+c)+(b+d)\mathrm{i}](u+\mathrm{i}v) = [(a+c)u-(b+d)v] + \mathrm{i}[(a+c)v+(b+d)u] \\ (a+b\mathrm{i})(u+\mathrm{i}v)+(c+d\mathrm{i})(u+\mathrm{i}v) = [(au-bv)+\mathrm{i}(av+bu)] + [(cu-dv)+\mathrm{i}(cv+du)] \end{cases}$

□

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- *Suppose $S$ is a nonempty set. Let $V^S$ denote the set of functions from $S$ to $V$. Define a natural addition and scalar multiplication on $V^S$, and show that $V^S$ is a vector space with these definitions.*

**SOLUTION:**

- Addition on $V^S$ is defined by $(f + g)(x) = f(x) + g(x)$ for any $x \in S$ and $f, g \in V^S$.
- Scalar Multiplication on $V^S$ is defined by $(\lambda f)(x) = \lambda f(x)$ for any $x \in S, \lambda \in \mathbf{F}, f \in V^S$.

Commutativity. Associativity.

Additive identity: $0\,(x) = 0$.

Additive inverse: $f(x) + (-f)(x) = 0$.

Multiplication identity: $I(x) = x$.

Distributive properties: $(\lambda(f + g))(x) = \lambda(f(x) + g(x)) = (\lambda f)(x) + (\lambda g)(x)$;
$((\lambda + \mu)f)(x) = (\lambda + \mu)f(x) = \lambda f(x) + \mu f(x)$. □

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**1** *Prove that $-(-v) = v$ for every $v \in V$.*

**SOLUTION:** $\left.\begin{array}{l} (-(-v)) + (-v) = 0 \\ v + (-v) = 0 \end{array}\right\} \Rightarrow$ By the uniqueness of additive inverse. □

OR. $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$ □

**2** *Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.*

**Solution:** If $a = 0$, then we are done.

Otherwise, $\exists\ a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. $\square$

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**3** *Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.*

**Solution:**

[Existence] Let $x = \dfrac{1}{3}(w - v)$.

[Uniqueness] Suppose $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II).

Then (I) $-$ (II) : $3(x_1 - x_2) = 0 \Rightarrow$ By Problem (2), $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$. $\square$

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**5** *Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that $0v = 0$ for all $v \in V$. Here the $0$ on the left side is the number $0$, and the $0$ on the right side is the additive identity of $V$.*

*The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.*

**Solution:** Using [1.31]. $0v = 0$ for all $v \in V \Leftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. $\square$

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**6** *Let $\infty$ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define*

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

and (I) $t + \infty = \infty + t = \infty + \infty = \infty$,

(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,

(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

**Solution:** Not a vector space. By Associativity: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

Or By Distributive properties: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. $\square$

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**Ended**

# 1·C

**2** (1.35)

(b) *The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$*

Denote the set by $U$. $\forall x \in [0, 1]$ we have $\left.\begin{array}{l} (一)\ 0 \in U;\ f(x) = 0 \Leftrightarrow f = 0 \\ (二)\ \forall f, g \in U,\ (f+g)(x) = f(x) + g(x) \\ (三)\ \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R},\ (\lambda f)(x) = \lambda f(x) \end{array}\right\} \Rightarrow \square$

(c) *The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$*

Denote the set by $U$. $\left.\begin{array}{l} (一)\ 0 \in U \\ (二)\ \forall f, g \in U,\ (f' + g') = f' + g' \\ (三)\ \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R},\ (\lambda f)' = \lambda (f)' \end{array}\right\} \Rightarrow \square$

(d) *The set of differentiable real-valued functions $f$ on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.*

Denote the set by $U$. Suppose $b = 0$. Then

$$\left.\begin{array}{l}(\text{一})\ 0\in U\\ (\text{二})\ \forall f,g\in U,\ (f+g)'(2)=f'(2)+g'(2)=0\\ (\text{三})\ \forall f\in U,\forall\lambda\in\mathbf{F}=\mathbf{R},\ (\lambda f)'(2)=\lambda f'(2)\end{array}\right\}\Rightarrow U \text{ is a subspace of } \mathbf{R}^{(0,3)}.$$

Suppose $U$ is a subspace of $\mathbf{R}^{(0,3)}$. Suppose $f=0\Rightarrow f\in U$. Then $f'(2)=0=b$. $\square$

(e) *The set of all sequences with limit 0, of complex numbers, is a subspace of* $\mathbf{C}^{\infty}$.

Denote the set by $A$.

$$\left.\begin{array}{l}(\text{一})\ (0,0,\dots)\in A\\ (\text{二})\ \forall\ (a_1,a_2,\dots),\ (b_1,b_2,\dots)\in A\iff \lim\limits_{n\to\infty}a_n=0,\ \lim\limits_{n\to\infty}b_n=0.\\ \qquad\text{Thus } \lim\limits_{n\to\infty}(a_n+b_n)=\lim\limits_{n\to\infty}a_n+\lim\limits_{n\to\infty}b_n=0\Rightarrow(a_1+b_1,a_2+b_2,\dots)\in A\\ (\text{三})\ \forall\ (a_1,a_2,\dots)\in A,\forall\lambda\in\mathbf{F}=\mathbf{C},\ \lim\limits_{n\to\infty}(\lambda a_n)=\lambda\lim\limits_{n\to\infty}a_n=0\Rightarrow\lambda(a_1,a_2,\dots)\in A\end{array}\right\}\Rightarrow\square$$

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**4** *Suppose $b\in\mathbf{R}$. Show that the set of continuous real-valued functions $f$ on the interval $[0,1]$ such that $\int_0^1 f=b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b=0$*

**Solution:** Denote the set by $V_b$.

(a) Suppose $V_b$ is a subspace of $\mathbf{R}^{[0,1]}$, then $\forall f\in V_b$, we have $\int_0^1 f=b$.

Because $kf\in V_b$ for any $k\in\mathbf{R}$. Hence $\int_0^1(kf)=k\int_0^1 f=kb=b\Leftrightarrow b=0$.

Or. Because $g=0\in V_b\Rightarrow\int_0^1 g=0=b$.

(b) Suppose $b=0$. $\forall f,g\in V_b=V_0,\lambda\in\mathbf{R}$, $\int_0^1(f+\lambda g)=\int_0^1 f+\int_0^1 g=0$. $\square$

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**5** *Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?*

**Answer:** No. Because $\mathbf{R}^2$ is not closed on scalar multiplication on $\mathbf{C}$.

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**6** (a) *Is $\{(a,b,c)\in\mathbf{R}^3 : a^3=b^3\}$ a subspace of $\mathbf{R}^3$?*

**Answer:** True. As can be easily checked ( note that $a^3=b^3\Leftrightarrow a=b$ ).

(b) *Is $\{(a,b,c)\in\mathbf{C}^3 : a^3=b^3\}$ a subspace of $\mathbf{C}^3$?*

**Answer:** No. Note that $(\dfrac{-1\pm\sqrt{3}\mathrm{i}}{2})^3=1$.

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**7** *Prove or give a counterexample: If $U$ is a nonempty subset of $\mathbf{R}^2$ such that $U$ is closed under addition and under taking additive inverses (meaning $-u\in U$ whenever $u\in U$), then $U$ is a subspace of $\mathbf{R}^2$.*

**Solution:** Let $U=\mathbf{Z}^2$, $(\mathbf{Z}^*)^2$, $(\mathbf{Q}^*)^2$, $\mathbf{Q}^2\backslash\{0\}$, or $\mathbf{R}^2\backslash\{0\}$.

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**8** *Give an example of a nonempty subset $U$ of $\mathbf{R}^2$ such that $U$ is closed under scalar multiplication, but $U$ is not a subspace of $\mathbf{R}^2$.*

**Solution:** $U=\{(x,y)\in\mathbf{R}^2 : x=0\vee y=0\}$.

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**9** *A function $f:\mathbf{R}\to\mathbf{R}$ is called periodic if there exists a positive number $p$ such that $f(x)=f(x+p)$ for all $x\in\mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$ ? Explain.*

**Solution:** Denote the set by $S$.

Suppose $h(x)=\sin\sqrt{2}x+\cos x\in S$, since both $f(x)=\sin\sqrt{2}$, $g(x)=\cos x$ are periodic functions $\mathbf{R}\to\mathbf{R}$.

Assume $\exists\ p\in\mathbf{N}^+$ such that $h(x)=h(x+p),\forall x\in\mathbf{R}$. Let $x=0\Rightarrow h(0)=h(\pm p)=1$.

Thus $1=\cos p+\sin\sqrt{2}p=\cos p-\sin\sqrt{2}p$

$\Rightarrow\sin\sqrt{2}p=0,\ \cos p=1\Rightarrow p=2k\pi,k\in\mathbf{Z}$, while $p=\dfrac{m\pi}{\sqrt{2}},m\in\mathbf{Z}$.

Hence $2k=\dfrac{m}{\sqrt{2}}\Rightarrow\sqrt{2}=\dfrac{m}{2k}\in\mathbf{Q}$. Contradiction! $\square$

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**11** *Prove that the intersection of every collection of subspaces of $V$ is a subspace of $V$.*

**SOLUTION:**

Suppose $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collection of subspaces of $V$; here $\Gamma$ is an arbitrary index set.

We need to prove that $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vectors that are in $U_\alpha$ for each $\alpha \in \Gamma$, is a subspace of $V$.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under addition.

(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under scalar multiplication.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of $V$ that is closed under addition and scalar multiplication.

Hence $\bigcap_{\alpha\in\Gamma} U_\alpha$ is a subspace of $V$. $\square$

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**12** *Prove that the union of two subspaces of $V$ is a subspace of $V$ if and only if one of the subspaces is contained in the other.*

**SOLUTION:** Suppose $U$ and $W$ are subspaces of $V$.

(a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subspace of $V$.

(b) Suppose $U \cup W$ is a subspace of $V$. Suppose $U \nsubseteq W$ and $U \nsupseteq W$ ( $U \cup W \neq U$ and $W$ ).

Then $\forall a \in U$ but $a \notin W$; $b \in W$ but $b \notin U$. $a + b \in U \cup W$.

$$\left.\begin{array}{l}\text{(1) Consider } a+b \in U \Rightarrow b = (a+b) + (-a) \in U \text{, contradicts!}\\ \text{(2) Consider } a+b \in W \Rightarrow a = (a+b) + (-b) \in W \text{, contradicts!}\end{array}\right\} \Rightarrow U \cup W = U \text{ or } W\text{. Contradicts!}$$

Thus $U \subseteq W$ and $U \supseteq W$. $\square$

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**13** *Prove that the union of three subspaces of $V$ is a subspace of $V$ if and only if one of the subspaces contains the other two.*

*This exercise is surprisingly harder than Exercise 12, possibly because this exercise is not true if we replace* **F** *with a field containing only two elements.*

**SOLUTION:** Suppose $A, B, C$ are subspaces of $V$.

(a) If any two of them are subsets of the third one, then $A \cup B \cup C = A, B$ or $C$, which is a subspace of $V$.

(b)* If $A \cup B \cup C$ is a subspace of $V$, suppose $\left\{\begin{array}{c} A \nsupseteq B \text{ and } C \\ B \nsupseteq A \text{ and } C \\ C \nsupseteq A \text{ and } B \end{array}\right\} \Longleftrightarrow A \cap B \cap C \neq A, B$ and $C$.

$\forall a \in A$ but $a \notin B, C$; $\forall b \in B$ but $b \notin A, C$; $\forall c \in C$ but $c \notin A, B$; by assumption, $a + b + c \in A \cup B \cup C$.

(I) $A \cup B$ is a subspace $\Rightarrow$ By Problem (12), $A \subseteq B$ or $A \supseteq B$.

(II) $A \cup C$ is a subspace $\Rightarrow$ By Problem (12), $A \subseteq C$ or $A \supseteq C$.

(III) $B \cup C$ is a subspace $\Rightarrow$ By Problem (12), $B \subseteq C$ or $B \supseteq C$.

Any two of (I), (II) and (III) must be true.

$$\text{(一). (I) and (II) are true. Then } \left.\begin{array}{r} C \subseteq B \subseteq A \\ \text{or } C \supseteq B \supseteq A \\ \text{or } B \supseteq A, C \\ \text{or } \underbrace{B \subseteq A, C}_{(1)} \end{array}\right\} \Rightarrow \left\{\begin{array}{r} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array}\right.$$

$$\text{(二). (II) and (III) are true. Then } \left.\begin{array}{r} A \subseteq C \subseteq B \\ \text{or } A \supseteq C \supseteq B \\ \text{or } C \supseteq A, B \\ \text{or } \underbrace{C \subseteq A, B}_{(2)} \end{array}\right\} \Rightarrow \left\{\begin{array}{r} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array}\right.$$

(三). (I) and (III) are true. Then $\left.\begin{array}{r} B \subseteq A \subseteq C \\ \text{or } B \supseteq A \supseteq C \\ \text{or } A \supseteq B, C \\ \text{or } \underbrace{A \subseteq B, C}_{(3)} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \quad A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array}\right.$

Among these, any two of (1), (2) and (3) must be true. $\left.\begin{array}{l} \left.\begin{array}{c} (1) \\ (2) \end{array}\right\} \Rightarrow B \subseteq C \subseteq A \\ \left.\begin{array}{c} (2) \\ (3) \end{array}\right\} \Rightarrow C \subseteq A \subseteq B \\ \left.\begin{array}{c} (1) \\ (3) \end{array}\right\} \Rightarrow B \subseteq A \subseteq C \end{array}\right\} \Rightarrow \ \square$

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• *Suppose $U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$ and $W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$.*
*Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.*

**SOLUTION:**

(a) $U + W = \{(x + y, x - y, 2(x + y)) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} = \{(x', y', 2x')) \in \mathbf{F}^3 : x', y' \in \mathbf{F}\}$.

(b) $U + W$ is a plane of which $(1, 0, 2), (0, 1, 0)$ is a basis. $\square$

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**15** *Suppose $U$ is a subspace of $V$. What is $U + U$?*

**SOLUTION:** $\left.\begin{array}{r} \forall x, y \in U, x + y \in U \Rightarrow U + U \subseteq U \\ \forall x \in U, 0 \in U, x + 0 \in U + U \Rightarrow U \subseteq U + U \end{array}\right\} \Rightarrow U + U = U. \ \square$

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**16** *Suppose $U$ and $W$ are subspaces of $V$. Prove that $U + W = W + U$?*

**SOLUTION:** $\forall x \in U, y \in W,\ \left.\begin{array}{r} x + y = y + x \in W + U \Rightarrow U + W \subseteq W + U \\ y + x = x + y \in U + W \Rightarrow W + U \subseteq U + W \end{array}\right\} \Rightarrow U + W = W + U. \ \square$

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**17** *Suppose $V_1, V_2, V_3$ are subspaces of $V$. Prove that $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$.*

**SOLUTION:**

Let $x \in V_1, y \in V_2, z \in V_3$. Denote $(V_1 + V_2) + V_3$ by $L$, $V_1 + (V_2 + V_3)$ by $R$.

$\left.\begin{array}{l} \forall u \in L,\ \exists\, x, y, z,\ u = (x + y) + z = x + (y + z) \in R \Rightarrow L \subseteq R \\ \forall u \in R,\ \exists\, x, y, z,\ u = x + (y + z) = (x + y) + z \in L \Rightarrow R \subseteq L \end{array}\right\} \Rightarrow (V_1 + V_2) + V_3 = V_1 + (V_2 + V_3). \ \square$

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**18** *Does the operation of addition on the subspaces of $V$ have an additive identity?*
*Which subspaces have additive inverses?*

**SOLUTION:**

Suppose $\Omega$ is the additive identity.

For any subspace $U$ of $V$. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

Now suppose $W$ is an additive inverse of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. $\square$

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**19** *Prove or give a counterexample: If $V_1, V_2, U$ are subspaces of $V$ such that $V_1 + U = V_2 + U$, then $V_1 = V_2$.*

**SOLUTION:** An counterexample:

$V = \mathbf{F}^3, U = \{(x, 0, 0) \in \mathbf{F}^3 : x \in \mathbf{F}\}$,

$V_1 = \{(x, x, y)) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}, V_2 = \{(x, y, z)) \in \mathbf{F}^3 : x, y, z \in \mathbf{F}\}$.

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**EXAMPLE**: *Suppose* $U = \{(x,x,y,y) \in \mathbf{F}^4 : x,y \in \mathbf{F}\}, W = \{(x,x,x,y) \in \mathbf{F}^4 : x,y \in \mathbf{F}\}$.
*Prove that* $U + W = \{(x,x,y,z) \in \mathbf{F}^4 : x,y,z \in \mathbf{F}\}$.

**SOLUTION**: Let T denote $\{(x,x,y,z) \in \mathbf{F}^4 : x,y,z \in \mathbf{F}\}$.

(a) By definition, $U+W = \{(x_1+x_2, x_1+x_2, y_1+x_2, y_1+y_2) \in \mathbf{F}^4 : (x_1,x_1,y_1,y_1) \in U, (x_2,x_2,x_2,y_2) \in W\}$.
$\Rightarrow \forall v \in U+W,\ \exists\, t \in T,\ \ v = t \Rightarrow U+W \subseteq T$.

(b) $\forall x,y,z \in \mathbf{F}$, let $u = (0,0,y-x,y-x) \in U,\ \ w = (x,x,x,-y+x+z) \in W$
$\Rightarrow (x,x,y,z) = u + w \in U + W$. Hence $\forall t \in T,\ \exists\, u \in U, w \in W,\ \ t = u + w \Rightarrow T \subseteq U + W$. $\square$

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**21** *Suppose* $U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5 : x,y \in \mathbf{F}\}$.
*Find a subspace* $W$ *of* $\mathbf{F}^5$ *such that* $\mathbf{F}^5 = U \oplus W$.

**SOLUTION:**

(a) Let $W = \{(0,0,z,w,u) \in \mathbf{F}^5 : z,w,u \in \mathbf{F}\}$. Then $W \cap U = \{0\}$.

(b) $\forall x,y,z,w,u \in \mathbf{F}$, let $u = (x,y,x+y,x-y,2x) \in U$,
$$w = (0,0,z-x-y,w-x-y,u-2x) \in W$$
$\Rightarrow (x,y,z,w,u) = u + w \Rightarrow \mathbf{F}^5 \subseteq U + W$. $\square$

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**22** *Suppose* $U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5 : x,y \in \mathbf{F}\}$.
*Find three subspaces* $W_1, W_2, W_3$ *of* $\mathbf{F}^5$*, none of which equals* $\{0\}$,
*such that* $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

**SOLUTION:**

(1) Let $W_1 = \{(0,0,z,0,0) \in \mathbf{F}^5 : z \in \mathbf{F}\}$. Then $W_1 \cap U = \{0\}$.
Let $U_1 = U \oplus W_1$. Then $U_1 = \{(x,y,z,x-y,2x) \in \mathbf{F}^5 : x,y,z \in \mathbf{F}\}$. ( Check it! )

(2) Let $W_2 = \{(0,0,0,w,0) \in \mathbf{F}^5 : w \in \mathbf{F}\}$. Then $W_2 \cap U_1 = \{0\}$.
Let $U_2 = U_1 \oplus W_2$. Then $U_2 = \{(x,y,z,w,2x) \in \mathbf{F}^5 : x,y,z,w \in \mathbf{F}\}$.

(3) Let $W_3 = \{(0,0,0,0,u) \in \mathbf{F}^5 : u \in \mathbf{F}\}$. Then $W_3 \cap U_2 = \{0\}$.
Let $U_3 = U_2 \oplus W_3$. Then $U_3 = \{(x,y,z,w,u) \in \mathbf{F}^5 : x,y,z,w,u \in \mathbf{F}\}$.

Thus $\mathbf{F}^5 = ((U \oplus W_1) \oplus W_2) \oplus W_3$. $\square$

---

**23** *Prove or give a counterexample: If* $V_1, V_2, U$ *are subspaces of* $V$ *such that*
$V = V_1 \oplus U$ *and* $V = V_2 \oplus U$*, then* $V_1 = V_2$.

**HINT:** *When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in* $\mathbf{F}^2$.

**SOLUTION:** An counterexample:
$V = \mathbf{F}^2, U = \{(x,x) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_1 = \{(x,0) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_2 = \{(0,x) \in \mathbf{F}^2 : x \in \mathbf{F}\}$.

---

**24** *Let* $V_e$ *denote the set of real-valued even functions on* $\mathbf{R}$
*and let* $V_o$ *denote the set of real-valued odd functions on* $\mathbf{R}$. *Show that* $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

**SOLUTION:**

(a) $V_e \cap V_o = \{f : f(x) = f(-x) = -f(-x)\} = \{0\}$.

(b) $$\left.\begin{array}{r} f_e \in V_e \Leftrightarrow f_e(x) = f_e(-x) \Leftarrow \text{let } f_e(x) = \dfrac{g(x)+g(-x)}{2} \\ f_o \in V_o \Leftrightarrow f_o(x) = -f_o(-x) \Leftarrow \text{let } f_o(x) = \dfrac{g(x)-g(-x)}{2} \end{array}\right\} \Rightarrow \forall g \in \mathbb{R}^{\mathbb{R}}, g(x) = f_e(x) + f_o(x).\ \square$$

---

**ENDED**

# 2·A

**2** (a) *Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.*

(b) *Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.*

**SOLUTION:**

(a) Suppose $v \neq 0$. Then let $av = 0, a \in \mathbf{F}$. Getting $a = 0$. Thus $(v)$ is linearly independent.
Suppose $(v)$ is linearly independent. $av = 0 \Rightarrow a = 0$. Then $v \neq 0$, for if not, $a \neq 0 \Rightarrow av = 0$. Contradicts.

(b) Denote the list by $(v, w)$, where $v, w \in V$. If $(v, w)$ is linearly independent, suppose $av + bw = 0 \Rightarrow a = b = 0$.
Without loss of generality, suppose $v \neq cw \ \forall c \in \mathbf{F}$. Then let $av + bw = 0$, getting $a = b = 0 \Rightarrow (v, w)$ is linearly independent.

---

**1** *Prove that if $(v_1, v_2, v_3, v_4)$ spans $V$, then the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V.*

**SOLUTION:** Assume that $\forall v \in V, \ \exists \ a_1, \ldots, a_4 \in \mathbf{F}$,

$$\begin{aligned} v &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 \\ &= b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4, \text{ letting } b_i = \sum_{r=1}^{i} a_r. \end{aligned}$$

Thus $\forall v \in V, \ \exists \ b_i \in \mathbf{F}, v = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4$.

Hence the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V. $\square$

---

**6** *Suppose $(v_1, v_2, v_3, v_4)$ is linearly independent in V. Prove that the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ is also linearly independent.*

**SOLUTION:** $a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4v_4 = 0$

$\Rightarrow a_1v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0$

$\Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \Rightarrow a_1 = \cdots = a_4 = 0 \Rightarrow \ \square$

---

**7** *Prove that if $(v_1, v_2, \ldots, v_m)$ is a linearly independent list of vectors in $V$, then $(5v_1 - 4v_2, v_2, v_3, \ldots, v_m)$ is linearly independent.*

**SOLUTION:** $a_1(5v_1 - 4v_2) + a_2v_2 + a_3v_3 + a_4v_4 = 0$

$\Rightarrow 5a_1v_1 + (a_2 - 4a_1)v_2 + a_3v_3 + a_4v_4 = 0$

$\Rightarrow 5a_1 = a_2 - 4a_1 = a_3 = a_4 = 0 \Rightarrow a_1 = \cdots = a_4 = 0 \ \square$

---

• *Suppose $(v_1, \ldots, v_m)$ is a list of vectors in $V$. For $k \in \{1, \ldots, m\}$, let $w_k = v_1 + \cdots + v_k$.*

(a) *Show that span$(v_1, \ldots, v_m) =$ span$(w_1, \ldots, w_m)$.*

(b) *Show that $(v_1, \ldots, v_m)$ is linearly independent if and only if $(w_1, \ldots, w_m)$ is linearly independent.*

**SOLUTION:**

(a) Let span$(v_1, \ldots, v_m) = U$. Assume that $\forall v \in U, \ \exists \ a_i \in \mathbf{F}$,

$v = a_1v_1 + \cdots + a_mv_m = b_1w_1 + \cdots + b_mw_m = \sum_{j=1}^{m}(\sum_{i=j}^{m} b_i)v_j$

$\Rightarrow b_1 = a_1, \ \ b_i = a_i - \sum_{r=1}^{i-1} b_r$. Thus $\exists \ b_i \in \mathbf{F}$ such that $v = b_1w_1 + \cdots + b_mw_m$.

又 Each $w_i \in U \Rightarrow$ span$(v_1, \ldots, v_m) =$ span$(w_1, \ldots, w_m)$.

(b) $a_1w_1 + \cdots + a_mw_m = 0$

$\Rightarrow (a_1 + \cdots + a_m)v_1 + \cdots + (a_i + \cdots + a_m)v_i + \cdots + a_mv_m = 0$

$\Rightarrow a_m = \cdots = (a_m + \cdots + a_i) = \cdots = (a_m + \cdots + a_1) = 0. \ \square$

---

**10** *Suppose $(v_1, \dots, v_m)$ is linearly independent in $V$ and $w \in V$.*

(a) *Prove that if $(v_1 + w, \dots, v_m + w)$ is linearly dependent, then $w \in span(v_1, \dots, v_m)$.*

(b) *Show that $(v_1, \dots, v_m, w)$ is linearly independent $\iff w \notin span(v_1, \dots, v_m)$.*

**SOLUTION:**

(a) Suppose $a_1(v_1+w)+\cdots+a_m(v_m+w) = 0,\ \exists\ a_i \neq= 0 \Rightarrow a_1v_1+\cdots+a_mv_m = 0 = -(a_1+\cdots+a_m)w$.

Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1v_1 + \cdots + a_mv_m = 0$ while $a_i \neq 0$ for some $i$, contradicts.

Hence $w \in \operatorname{span}(v_1, \dots, v_m)$.

(b) Suppose $w \in \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1, \dots, v_m, w)$ is linearly dependent.

Thus have we proven the "$\Rightarrow$" by its contrapositive.

Suppose $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then by [2.23], $(v_1, \dots, v_m, w)$ is linearly independent. $\square$

---

**14** *Prove that $V$ is infinite-dim if and only if there is a sequence $(v_1, v_2, \dots)$ in $V$ such that $(v_1, \dots, v_m)$ is linearly independent for every $m \in \mathbf{N}^+$.*

**SOLUTION:** Similar to [2.16].

Suppose there is a sequence $(v_1, v_2, \dots)$ in $V$ such that $(v_1, \dots, v_m)$ is linearly independent for any $m \in \mathbf{N}^+$.

Choose an $m$. Suppose a linearly independent list $(v_1, \dots, v_m)$ spans $V$.

Then there exists $v_{m+1} \in V$ but $v_{m+1} \notin \operatorname{span}(v_1, \dots, v_m)$. Hence no list spans $V$. Thus $V$ is infinite-dim.

Conversely it is true as well. For if not, $V$ must be finite-dim, contradicting the assumption. $\square$

---

**15** *Prove that $\mathbf{F}^\infty$ is infinite-dim.*

**SOLUTION:** Let $e_i = (0, \dots, 0, 1, 0, \dots) \in \mathbf{F}^\infty$ for every $m \in \mathbf{N}^+$, where '1' is on the i$^{th}$ entry of $e_i$.

Suppose $\mathbf{F}^\infty$ is finite-dim. Then let $\operatorname{span}(e_1, \dots, e_m) = V$. But $e_{m+1} \notin \operatorname{span}(e_1, \dots, e_m)$. Contradicts. $\square$

---

**16** *Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.*

**SOLUTION:** Denote the vec-sp by $U$. Note that for each $m \in \mathbf{N}^+$, $(1, x, \dots, x^m)$ is linearly independent.

Because if $a_0, \dots, a_m \in \mathbf{R}$ are such that $a_0 + a_1x + \cdots + a_mx^m = 0,\ \forall x \in [0,1]$, then the polynomial has infinitely many roots and hence $a_0 = \cdots = a_m = 0$. } Similar to [2.16], $U$ is infinite-dim.

OR. Note that for $a_n = \dfrac{1}{n},\quad a_1 < a_2 < \cdots < a_m,\ \ \forall m \in \mathbf{N}^+$.

Suppose $f_n = \begin{cases} x - \dfrac{1}{n}, & x \in [\frac{1}{n}, 1) \\ 0, & x \in [0, \frac{1}{n}) \end{cases}$. Then for any $m$, $f_1(\frac{1}{m}) = \cdots = f_m(\frac{1}{m})$, while $f_{m+1}(\frac{1}{m}) \neq 0$.

Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. Thus by Problem (14), $U$ is infinite-dim.

---

**17** *Suppose $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$. Prove that $(p_0, p_1, \dots, p_m)$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.*

**SOLUTION:**

Suppose $(p_0, p_1, \dots, p_m)$ is linearly independent. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z\ \forall z \in \mathbf{F}$.

But $\forall a_i \in \mathbf{F}$, $z \neq a_0p_0(z) + \cdots + a_mp_m(z)$, for if not, let $z = 2$, contradicts. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.

Then $\operatorname{span}(p_0, p_1, \dots, p_m) \subsetneqq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has length $m + 1$.

Hence $(p_0, p_1, \dots, p_m)$ is linearly dependent in $\mathcal{P}_m(\mathbf{F})$.

For if not, notice that the list $(1, z, \dots, z^m)$ spans $\mathcal{P}_m(\mathbf{F})$,

thus by [2.23], $(p_0, p_1, \dots, p_m)$ spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. $\square$

---

**ENDED**

# 2·B

**NOTE FOR** *linearly independent sequence and [2.34].*

"$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expression.

If we allow using "infinite list", then we must guarantee that $(v_1, \dots, v_n, \dots)$ is a spanning "list" such that for all $v \in V$, there exists a certain positive integer such that $v = a_1 v_{\alpha_1} + \cdots + a_n v_{\alpha_n}$, where $\Gamma = \{\alpha_1, \dots, \alpha_n\}$ is an finite index set. The key point is, how do we find such a "list"?

**NOTE FOR "$\complement_V U \cap \{0\}$":** "$\complement_V U \cap \{0\}$" is supposed to be "$W$", where $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{matrix} w \in \complement_V U \cap \{0\} \\ u \pm w \in \complement_V U \cap \{0\} \end{matrix}\right\} \Rightarrow u \in \complement_V U \cap \{0\}$. Contradicts.

**NEW NOTATION:** Denote the set $\{W_1, W_2 \dots\}$ by $\mathcal{S}_V U$, where for each $W_i$, $V = U \oplus W_i$. See also in (1.C.23).

---

**1** *Find all vector spaces that have exactly one basis.* **SOLUTION:** $\mathbf{F} = \mathbf{C}, \mathbf{R}, \mathbf{Q}, \{0,1\}$, $\mathcal{P}_0(\mathbf{F})$.

---

**6** *Suppose $(v_1, v_2, v_3, v_4)$ is a basis of $V$. Prove that $(v_1+v_2, v_2+v_3, v_3+v_4, v_4)$ is also a basis.*

**SOLUTION:** $\forall v \in V,\ \exists!\ a_1, \dots, a_4 \in \mathbf{F},\ v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$.

Assune that $v = b_1(v_1+v_2)+b_2(v_2+v_3)+b_3(v_3+v_4)+b_4v_4$. Then $v = b_1v_1+(b_1+b_2)v_2+(b_2+b_3)v_3+(b_3+b_4)v_4$.

$\Rightarrow\ \exists!\ b_1 = a_1,\ \ b_2 = a_2 - b_1,\ \ b_3 = a_3 - b_2,\ \ b_4 = a_4 - b_3 \in \mathbf{F}$. $\square$

---

**7** *Prove or give a counterexample: If $v_1, v_2, v_3, v_4$ is a basis of $V$ and $U$ is a subspace of $V$ such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \in U$, then $v_1, v_2$ is a basis of U.*

**SOLUTION:** Let $V = \mathbf{F}^4$, $v_1 = (1,0,0,0)$, $v_2 = (0,1,0,0)$, $v_3 = (0,0,1,1)$, $v_4 = (0,0,0,1)$.

And $U = \{(x,y,z,0) \in \mathbf{R}^4 : x,y,z \in \mathbf{F}\}$. We have an counterexample.

---

• *Suppose $V$ is finite-dim and $U$, $W$ are subspaces of $V$ such that $V = U + W$. Prove that there exists a basis of $V$ consisting of vectors in $U \cup W$.*

**SOLUTION:** Let $(u_1, \dots, u_m)$ and $(w_1, \dots, w_n)$ be bases of $U$ and $W$ respectively.

Then $V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.

Hence, by [2.31], we get a basis of $V$ consisting of vectors in $U$ or $W$. $\square$

---

**8** *Suppose $U$ and $W$ are subspaces of $V$ such that $V = U \oplus W$. Suppose also that $(u_1, \dots, u_m)$ is a basis of U and $(w_1, \dots, w_n)$ is a basis of $W$. Prove that $(u_1, \dots, u_m, w_1, \dots, w_n)$ is a basis of $V$.*

**SOLUTION:**

$\forall v \in V,\ \exists!\, a_i, b_i \in \mathbf{F},\ v = (a_1u_1 + \cdots + a_mu_m) + (b_1w_1 + \cdots + b_nw_n)$

$\Rightarrow (a_1u_1 + \cdots + a_mu_m) = -(b_1w_1 + \cdots + b_nw_n) \in U \cap W = \{0\}$. Thus $a_1 = \cdots = a_m = b_1 = \cdots = b_n$. $\square$

---

• (OR 9.4) *Suppose $V$ is a real vector space. Show that if $(v_1, \dots, v_n)$ is a basis of $V$ (as a real vector space), then $(v_1, \dots, v_n)$ is also a basis of the complexification $V_{\mathbb{C}}$ (as a complex vector space).*

*See Section 1B (4e) for the definition of the complexification $V_{\mathbb{C}}$.*

**SOLUTION:**

$\forall u + \mathrm{i}v \in V_{\mathbb{C}},\ \exists!\ u, v \in V, a_i, b_i \in \mathbf{R}$,

$u + \mathrm{i}v = (a_1v_1 + \cdots + a_nv_n) + \mathrm{i}(b_1v_1 + \cdots + b_nv_n) = (a_1 + b_1\mathrm{i})v_1 + \cdots + (a_n + b_n\mathrm{i})v_n$

$\Rightarrow\ u + \mathrm{i}v = c_1v_1 + \cdots + c_nv_n,\ \exists!\, c_i = a_i + b_i\mathrm{i} \in \mathbf{C}$

$\Rightarrow$ By the uniqueness of $c_i$ and [2.29], $(v_1, \dots, v_n)$ is a basis of $V_{\mathbb{C}}$. $\square$

---

**ENDED**

# 2·C

**1** *Suppose $V$ is finite-dim and $U$ is a subspace of $V$ such that* $\dim V = \dim U$.

Let $(u_1, \ldots, u_m)$ be a basis of $U$. Then $n = \dim U = \dim V$. 又 $u_i \in V$.

Then by [2.39], $(u_1, \ldots, u_m)$ is a basis of $V$. Thus $V = U$.

---

**2** *Show that the subspaces of* $\mathbf{R}^2$ *are precisely* $\{0\}$, *all lines in* $\mathbf{R}^2$ *containing the origin, and* $\mathbf{R}^2$.

**SOLUTION:**

Suppose $U$ is a subspace of $\mathbf{R}^2$. Let $\dim U = n$.

If $n = 0$, then $U = \{0\}$.

If $n = 1$, then $U = \operatorname{span}(v)$, where $v$ is a vector in $\mathbf{R}^2$. Thus $U$ can be any line in $\mathbf{R}^2$ containing the origin.

If $n = 2$, then $U = \operatorname{span}(v, w)$, where $v, w$ are vectors in $\mathbf{R}^2$ and $(v, w)$ is linearly independent $\Rightarrow U = \mathbf{R}^2$. $\square$

---

**3** *Show that the subspaces of* $\mathbf{R}^3$ *are precisely* $\{0\}$, *all lines in* $\mathbf{R}^3$ *containing the origin, all planes in* $\mathbf{R}^3$ *containing the origin, and* $\mathbf{R}^3$.

**SOLUTION:**

Suppose $U$ is a subspace of $\mathbf{R}^3$. Let $\dim U = n$.

If $n = 0$, then $U = \{0\}$.

If $n = 1$, then $U = \operatorname{span}(v)$, where $v$ is a vector in $\mathbf{R}^3$. Thus $U$ can be any line in $\mathbf{R}^3$ containing the origin.

If $n = 2$, then $U = \operatorname{span}(v, w)$, where $v, w$ are vectors in $\mathbf{R}^3$ and $(v, w)$ is linearly independent.

Thus $U$ can be any plane in $\mathbf{R}^3$ containing the origin.

If $n = 3$, then $U = \operatorname{span}(u, v, w)$, where $u, v, w$ are vectors in $\mathbf{R}^3$ and $(u, v, w)$ is linearly independent

$\Rightarrow U = \mathbf{R}^3$. $\square$

---

**7** (a) *Let* $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. *Find a basis of* $U$.

(b) *Extend the basis in* $(a)$ *to a basis of* $\mathcal{P}_4(\mathbf{F})$.

(c) *Find a subspace* $W$ *of* $\mathcal{P}_4(\mathbf{F})$ *such that* $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

**SOLUTION:**

Suppose $p(z) = az^4 + bz^3 + cz^2 + dz + e$ and $p(2) = p(5) = p(6)$.

$$\text{Then} \begin{cases} p(2) = 16a + 8b + 4c + 2d + e \text{ (I)} \\ p(5) = 625a + 125b + 25c + 5d + e \text{ (II)} \\ p(6) = 1296a + 216b + 36c + 6d + e \text{ (III)} \end{cases}$$

You don't have to compute to know that the dimension of the set of soultions is 3.

(a) A basis: $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. $\square$

---

**9** *Suppose* $(v_1, \ldots, v_m)$ *is linearly independent in* $V$ *and* $w \in V$.

*Prove that* $\dim$ *span*$(v_1 + w, \ldots, v_m + w) \geq m - 1$.

**SOLUTION:**

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \ldots, v_n + w)$, for each $i = 1, \ldots, m$.

$(v_1, \ldots, v_m)$ is linearly independent $\Rightarrow (v_1, v_2 - v_1, \ldots, v_m - v_1)$ is linearly independent

$\Rightarrow (v_2 - v_1, \ldots, v_m - v_1)$ is linearly independent of length $m - 1$.

又 By the contrapositive of (2.A.10), $w \notin \operatorname{span}(v_1, \ldots, v_m) \Rightarrow (v_1 + w, \ldots, v_m + w)$ is linearly independent.

$\therefore\ m \geq \dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$. $\square$

---

**10** *Suppose $m$ is a positive integer and $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each $p_k$ has degree $k$. Prove that $(p_0, p_1, \ldots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

**SOLUTION:** Using mathematical induction on $m$.

(i) For $p_0, \deg p_0 = 0 \Rightarrow \text{span}\,(p_0) = \text{span}\,(1)$.

(ii) Suppose for $i \geq 1, \text{span}\,(p_0, p_1, \ldots, p_i) = \text{span}(1, x, \ldots, x^i)$.

Then $\text{span}(p_0, p_1, \ldots, p_i, p_{i+1}) \subseteq \text{span}\,(1, x, \ldots, x^i, x^{i+1})$.

又 $\deg p_{i+1} = i + 1, \quad p_{i+1}(x) = a_{i+1}x^{i+1} + r_{i+1}(x); \quad a_{i+1} \neq 0, \quad \deg r_{i+1} \leq i$.

$\Rightarrow x^{i+1} = \frac{1}{a_{i+1}}(p_{i+1}(x) - r_{i+1}(x)) \in \text{span}\,(1, x, \ldots, x^i, p_{i+1}) = \text{span}\,(p_0, p_1, \ldots, p_i, p_{i+1})$.

$\therefore \; x^{i+1} \in \text{span}\,(p_0, p_1, \ldots, p_i, p_{i+1}). \Rightarrow \text{span}\,(1, x, \ldots, x^i, x^{i+1}) \subseteq \text{span}\,(p_0, p_1, \ldots, p_i, p_{i+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \text{span}\,(1, x, \ldots, x^m) = \text{span}(p_0, p_1, \ldots, p_m)$. $\square$

---

• *Suppose $m$ is a positive integer. For $0 \leq k \leq m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show that $(p_0, \ldots, p_m)$ is a basis of $\mathcal{P}(\mathbf{F})$.*

*The basis in this exercise leads to what are called Bernstein polynomials. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on $[0, 1]$.*

**SOLUTION:** Using mathematical induction.

(i) $k = 0, 1, 2, \; p_m(x) = x^m, \;\; p_{m-1}(x) = x^{m-1} - x^m, \;\; p_{m-2}(x) = x^{m-2} + x^m - 2x^{m-1}$.

(ii) $k \geq 2$. Suppose for $p_{m-k}(x), \;\exists\,!\; a_i \in \mathbf{F}, x^{m-k} = p_{m-k}(x) + a_m x^m + \cdots + a_{m-k+1}x^{m-k+1}$.

Then for $p_{m-k-1}(x), \;\exists\,!\; c_i \in \mathbf{F}$,

$$x^{m-k-1} = p_{m-k-1}(x) + \mathcal{C}_{k+1}^{1}(-1)^2 x^{m-k} + \cdots + \mathcal{C}_{k+1}^{k}(-1)^{k+1}x^{m-1} + (-1)^{k-2}x^m$$

$$\Rightarrow c_{m-i} = \mathcal{C}_{k+1}^{k+1-i}(-1)^{k-i}.$$

Thus for each $x^i, \;\exists\,!\; b_i \in \mathbf{F}, x^i = b_m p_m(x) + \cdots + b_{m-i}p_{m-i}(x)$.

$\Rightarrow \text{span}(x^m, \ldots, x, 1) = \text{span}\underbrace{(p_m, \ldots, p_1, p_0)}_{\text{Basis}}$. $\square$

---

• *Suppose $V$ is finite-dim and $V_1, V_2, V_3$ are subspaces of $V$ with $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*

**SOLUTION:**

$$\left.\begin{aligned} &\dim V_1 + \dim V_2 > 2\dim V - \dim V_3 \geq \dim V \Rightarrow V_1 \cap V_2 \neq \{0\} \\ &\dim V_2 + \dim V_3 > 2\dim V - \dim V_1 \geq \dim V \Rightarrow V_2 \cap V_3 \neq \{0\} \\ &\dim V_1 + \dim V_3 > 2\dim V - \dim V_2 \geq \dim V \Rightarrow V_1 \cap V_3 \neq \{0\} \end{aligned}\right\} \Rightarrow V_1 \cap V_2 \cap V_3 \neq \{0\}. \;\square$$

---

• *Suppose $V$ is finite-dim and $U$ is a subspace of $V$ with $U \neq V$. Let $n = \dim V, m = \dim U$. Prove that there exist $(n-m)$ subspaces of $V$, say $U_1, \ldots, U_{n-m}$, each of dimension $(n-1)$, such that $\bigcap_{i=1}^{n-m} U_i = U$.*

**SOLUTION:** Let $(v_1, \ldots, v_m)$ be a basis of $U$, extend to a basis of $V$ as $(v_1, \ldots, v_m, \ldots, v_n)$.

Define $U_i = \text{span}\,(v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+i-1}, v_{m+i+1}, \ldots, v_n)$ for each $i$. Thus we are done.

**EXAMPLE:** Suppose $\dim V = 6, \dim U = 3$.

$$\underbrace{(\underbrace{v_1, v_2, v_3}_{\text{Basis of U}}, v_4, v_5, v_6)}_{\text{Basis of V}}, \text{ define } \left.\begin{aligned} U_1 &= \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_5, v_6) \\ U_2 &= \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_6) \\ U_3 &= \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_5) \end{aligned}\right\} \Rightarrow \dim U_i = 6 - 1, \; i = \underbrace{1, 2, 3}_{6-3=3}.$$

$\square$

---

**14** *Suppose that $V_1, \ldots, V_m$ are finite-dim subspaces of $V$.*
*Prove that $V_1 + \cdots + V_m$ is finite-dim and $\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m$.*

**SOLUTION:**

Choose a basis $\mathcal{E}_i$ of $V_i \Rightarrow V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim U_i = \operatorname{card} \mathcal{E}_i$.

Then $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$.

又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup E_m) \leq \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leq \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.

Thus $\dim(V_1 + \cdots + V_m) \leq \dim U_1 + \cdots + \dim U_m$.

•*The inequality above is an equality if and only if $V_1 + \cdots + V_m$ is a direct sum.*

For each $i$, $(V_1 + \cdots + V_i) \cap V_{i+1} = \{0\} \Longleftrightarrow V_1 + \cdots + V_m$ is a direct sum $\Longleftrightarrow \square$

---

**17** *Suppose $V_1, V_2, V_3$ are subspaces of a finite-dim vector space, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

*Explain why you might think and prove the formula above or give a counterexample.*

**SOLUTION:**

*Looks like:* given three sets $A, B$ and $C$.

*Note that:* $\operatorname{card}(X \cup Y) = \operatorname{card}(X) + \operatorname{card}(Y) - \operatorname{card}(X \cap Y)$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

*Then:* $\operatorname{card}((A \cup B) \cup C) = \operatorname{card}(A \cup B) + \operatorname{card} C - \operatorname{card}((A \cup B) \cap C)$.

*And:* $\operatorname{card}((A \cup B) \cap C) = \operatorname{card}((A \cap C) \cup (B \cap C)) = \operatorname{card}(A \cap C) + \operatorname{card}(B \cap C) - \operatorname{card}(A \cap B \cap C)$.

*Thus:* $\operatorname{card}((A \cup B) \cup C) = \operatorname{card} A + \operatorname{card} B + \operatorname{card} C + \operatorname{card}(A \cap B \cap C) - \operatorname{card}(A \cap B) - \operatorname{card}(A \cap C) - \operatorname{card}(B \cap C)$.

Because $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned} \dim(V_1 + V_2 + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \quad (1) \\ &= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) \quad (2) \\ &= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) \quad (3) \end{aligned}$$

Notice that $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$.

For example, $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

• **COROLLARY:** *If $V_1, V_2$ and $V_3$ are finite-dim vector spaces, then* $\dfrac{(1) + (2) + (3)}{3}$ :

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3}$$
$$- \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.$$

*The formula above may seem strange because the right side does not look like an integer.* $\square$

---

**ENDED**

# 3·A

**2** *Suppose $b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by*

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c \sin p(0)\right).$$

*Show that $T$ is linear if and only if $b = c = 0$.*

**SOLUTION:**

(a) Suppose $b = c = 0$, then $\forall p, q \in \mathcal{P}(\mathbf{R}), T(p+q) = (3(p+q)(4) + 5(p+q)'(6), \int_{-1}^{2} x^3 (p+q)(x)\mathrm{d}x)$.

Because $(p+q)(x) = p(x) + q(x), (p+q)'(x) = p'(x) + q'(x)$,

$\int_{-1}^{2} x^3 (p+q)(x)\mathrm{d}x = \int_{-1}^{2} x^3 p(x)\mathrm{d}x + \int_{-1}^{2} x^3 q(x)\mathrm{d}x$.

$\Rightarrow T(p+q) = Tp + Tq$. Similarly, $\forall \lambda \in \mathbf{F}$, $\lambda Tp = T(\lambda p)$. Thus $T$ is linear.

(b) Suppose $T$ is linear, denote the linear map in (a) by $S \Rightarrow (T-S)$ is linear.

$\Rightarrow (T-S)(p) = (bp(1)p(2), c\sin p(0))$ is linear.

Consider $p(x) = q(x) = \frac{\pi}{2},\ \forall x \in \mathbf{R}$.

$\Rightarrow ((T-S)(p+q) = (T-S)(\pi) = (b\pi^2, 0) = (T-S)(\frac{\pi}{2}) + (T-S)(\frac{\pi}{2}) = (b\frac{\pi^2}{2}, 2c) \Rightarrow b = c = 0.$ □

---

• **TIPS:** $T: V \to W$ *is linear* $\iff \begin{cases} \forall v, u \in V, T(v+u) = Tv + Tu \\ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv) \end{cases} \iff T(v + \lambda u) = Tv + \lambda Tu.$

---

**3** *Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists\ A_{j,k} \in \mathbf{F}$ such that*

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

*for any $(x_1, \ldots, x_n) \in \mathbf{F}^n$.*

**SOLUTION:**

Let $T(1,0,0,\ldots,0,0) = (A_{1,1},\ldots,A_{m,1})$,
$T(0,1,0,\ldots,0,0) = (A_{1,2},\ldots,A_{m,2})$,
$\vdots$
$T(0,0,0,\ldots,0,1) = (A_{1,n},\ldots,A_{m,n})$.

Note that $(1,0,\ldots,0,0), \cdots, (0,0,\ldots,0,1)$ is a basis of $\mathbf{F}^n$.
Then by [3.5], we are done. □

---

**4** *Suppose $T \in \mathcal{L}(V, W)$ and $(v_1, \ldots, v_m)$ is a list of vectors in $V$ such that $(Tv_1, \ldots, Tv_m)$ is linearly independent in $W$. Prove that $(v_1, \ldots, v_m)$ is linearly independent.*

**SOLUTION:**

Suppose $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$.□

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**5** *Prove that $\mathcal{L}(V, W)$ is a vector space,*

**SOLUTION:** Note that $\mathcal{L}(V, W)$ is a subspace of $W^V$. □

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**7** *Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if* $\dim V = 1$ *and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.*

**SOLUTION:**

Let $u$ be a nonzero vector in $V \Rightarrow V = \operatorname{span}(u)$.

Because $Tu \in V \Rightarrow Tu = \lambda u$ for some $\lambda$.

Suppose $v \in V \Rightarrow v = au,\ \exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. □

---

**8** *Give an example of a function $\varphi: \mathbf{R}^2 \to \mathbf{R}$ such that $\varphi(av) = a\varphi(v)$ for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but $\varphi$ is not linear.*

**SOLUTION:**

Define $T(x,y) = \begin{cases} x + y, \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, \quad \text{otherwise.} \end{cases}$

OR. Define $T(x,y) = \sqrt[3]{(x^3+y^3)}$. □

---

**9** *Give an example of a function $\varphi: \mathbf{C} \to \mathbf{C}$ such that $\varphi(w+z) = \varphi(w) + \varphi(z)$ for all $w, z \in \mathbf{C}$ but $\varphi$ is not linear.*

*(Here **C** is thought of as a complex vector space.)*

**SOLUTION:**

Suppose $V_{\mathbb{C}}$ is the complexification of a vector space $V$. Suppose $\varphi: V_{\mathbb{C}} \to V_{\mathbb{C}}$.

Define $\varphi(u + \mathrm{i}v) = u = \Re(u + \mathrm{i}v)$

OR. Define $\varphi(u + \mathrm{i}v) = v = \Im(u + \mathrm{i}v)$. □

---

• OR (3.D.16) *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.*
*Prove that $T$ is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.*

**SOLUTION:**

Assume that $(v, Tv)$ is linearly dependent for every $v \in V$, then by (2.A.2.(b)), $Tv = \lambda_v v$ for some $\lambda_v \in \mathbf{F}$.

To prove that $\lambda_v$ is independent of $v$

( in other words, for any two distinct nonzero vectors $v$ and $w$ in V, we have $\lambda_v \neq \lambda_w$ ), we discuss in two cases:

$$\left.\begin{array}{l} (-) \text{ If } (v,w) \text{ is linearly independent, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = a_v v + a_w w \\ \qquad\qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=) \text{ Otherwise, suppose } w = cv,\ a_w w = Tw = cTv = ca_v v = a_v w \Rightarrow (a_w - a_v)w \end{array}\right\} \Rightarrow a_w = a_v.$$

Now we prove the assumption by contradiction. Suppose $(v, Tv)$ is linearly independent for every nonzero vector $v \in V$.

Fix one $v$. Extend to $(v, Tv, u_1, \ldots, u_n)$ a basis of $V$.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$.

Hence a contradiction arises. $\square$

---

**10** *Suppose $U$ is a subspace of $V$ with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$*
*( which means that $Su \neq 0$ for some $u \in U$ ).*

*Define $T : V \to W$ by $Tv = \begin{cases} Sv, \text{ if } v \in U, \\ 0, \quad \text{if } v \in V \backslash U. \end{cases}$ Prove that $T$ is not a linear map on $V$.*

**SOLUTION:**

Suppose $T$ is a linear map. And $v \in V\backslash U,\ u \in U$ such that $Su \neq 0$.

Then $v + u \in V\backslash U$, ( for if not, $v = (v + u) - u \in U$ ) while $T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$.

Hence we get a contradiction. $\square$

---

**11** *Suppose $V$ is finite-dim. Prove that every linear map on a subspace of $V$ can be extended to a linear map on $V$. In other words, show that if $U$ is a subspace of $V$ and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.*

**SOLUTION:** Define $T \in \mathcal{L}(V, W)$ by $T(a_1u_1 + \cdots + a_nu_n + a_{n+1}u_{n+1} + \cdots + a_mu_m) = a_1Su_1 + \cdots + a_nSu_n$. Where:
Let $(u_1, \ldots, u_n)$ be a basis of $U$, extend to a basis of $V$ as $(u_1, \ldots, u_n, \ldots, u_m)$.

---

**12** *Suppose $V$ is finite-dim with $\dim V > 0$, and $W$ is infinite-dim.*
*Prove that $\mathcal{L}(V, W)$ is infinite-dim.*

**SOLUTION:**

Let $(v_1, \ldots, v_n)$ be a basis of $V$. Let $(w_1, \ldots, w_m)$ be linearly independent in $W$ for any $m \in \mathbf{N}^+$.

Define $T_{x,y} \in \mathcal{L}(V, W)$ by $T_{x,y}(v_x) = w_y, \forall x \in \{1, \ldots, n\}, y \in \{1, \ldots, m\}$.

Suppose $a_1T_{x,1} + \cdots + a_mT_{x,m} = 0$. Then $(a_1T_{x,1} + \cdots + a_mT_{x,m})(v_x) = 0 = a_1w_1 + \cdots + a_mw_m$.

$\Rightarrow a_1 = \cdots = a_m = 0$. 又 $m$ is arbitrarily chosen.

Thus $(T_{x,1}, \ldots, T_{x,m})$ is a linearly independent list in $\mathcal{L}(V, W)$ for any $x$ and length $m$. Hence by (2.A.14). $\square$

---

**13** *Suppose $(v_1, \ldots, v_m)$ is a linearly dependent list of vectors in $V$.*
*Suppose also that $W \neq \{0\}$. Prove that there exist $(w_1, \ldots, w_m) \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.*

**SOLUTION:** We show it by contradiction.

By linear independence lemma, $\exists\ j \in \{1, \ldots, m\}$ such that $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$.

Fix $j$. Let $w_j \neq 0$, while $w_1 = \cdots = w_{j-1} = w_{j+1} = w_m = 0$.

Define $T$ by $Tv_k = w_k$ for all $k$. Suppose $a_1v_1 + \cdots + a_mv_m = 0$ ( where $a_j \neq 0$ ).

Then $T(a_1v_1 + \cdots + a_mv_m) = 0 = a_1w_1 + \cdots + a_mw_m = a_jw_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. $\square$

---

• *Suppose $V$ is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

*A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.*

**SOLUTION:**

Let $(v_1, \dots, v_n)$ be a basis of $V$. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E}\backslash\{0\}$.

Suppose $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \cdots + a_n v_n$, where $a_k \neq 0$.

Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y}(v_x) = v_y, R_{x,y}(v_z) = 0\,(\, z \neq x\,)$. Then for any $x, y \in \mathbf{N}^+$,

$(R_{k,y}S)(v_i) = a_k v_y \Rightarrow ((R_{k,y}S) \circ R_{x,i})(v_x) = a_k v_y$, and $((R_{k,y}S) \circ R_{x,i})(v_z) = 0$ for $z \neq x$.

Thus $R_{k,y}SR_{x,i} = a_k R_{x,y}$. Denote by $T_{x,y}$.

Getting $(\frac{1}{a_k}T_{1,1} + \cdots + \frac{1}{a_k}T_{n,n})v_j = v_j \Rightarrow \sum\limits_{r=1}^{n} \frac{1}{a_k}T_{r,r} = I$.

又 By assumption, $T_{x,y} \in \mathcal{E} \Rightarrow I \in \mathcal{E}$.

Hence for any $T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. ☐

---

**ENDED**

# 3·B

**2** *Suppose $S, T \in \mathcal{L}(V)$ are such that range $S \subseteq$ null $T$. Prove that $(ST)^2 = 0$.*

**SOLUTION:** $TS = 0 \Rightarrow STST = (ST)^2 = 0$. ☐

---

**3** *Suppose $(v_1, \dots, v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1 v_1 + \cdots + z_m v_m$.*

(a) *What property of $T$ corresponds to $(v_1, \dots, v_m)$ spanning $V$?*

(b) *What property of $T$ corresponds to $(v_1, \dots, v_m)$ being linearly independent?*

**ANSWER:** (a) Surjectivity; (b) Injectivity. ☐

---

**4** *Show that $U = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim null\, T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.*

**SOLUTION:** Let $(v_1, v_2, v_3, v_4, v_5)$ be a basis of $\mathbf{R}^5$, $(w_1, w_2, w_3, w_4)$ be a basis of $\mathbf{R}^4$.

Define $T_1, T_2 \in U$ as $T_1 v_1 = 0,\ T_1 v_2 = 0,\ T_1 v_3 = 0,\ T_1 v_4 = w_4,\ T_1 v_5 = w_1;$

$T_2 v_1 = 0,\ T_2 v_2 = 0,\ T_2 v_3 = w_3,\ T_2 v_4 = 0,\ T_2 v_5 = w_4.$ Thus $T_1 + T_2 \notin U$.

*For $U' = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim null\, T > 0\}$,*

define $T_1, T_2 \in U'$ as $T_1 v_1 = 0,\ T_1 v_2 = w_2,\ T_1 v_3 = w_3,\ T_1 v_4 = w_4,\ T_1 v_5 = w_1;$

$T_2 v_1 = w_1,\ T_2 v_2 = w_2,\ T_2 v_3 = 0,\ T_2 v_4 = w_3,\ T_2 v_5 = w_4.$ Thus $T_1 + T_2 \notin U'$. ☐

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**7** *Suppose $V$ is finite-dim with $2 \leq \dim V \leq \dim W$, if $W$ is finite-dim.*
*Show that $U = \{\, T \in \mathcal{L}(V, W) : T$ is not injective $\}$ is not a subspace of $\mathcal{L}(V, W)$.*

**SOLUTION:**

Let $(v_1, \dots, v_n)$ be a basis of $V$, $(w_1, \dots, w_m)$ be linearly independent in $W$.

( Let $\dim W = m$, if $W$ is finite, otherwise, we choose $m \in \{n, n+1, \dots\}$ arbitrarily; $2 \leq n \leq m$ ).

Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 :\quad v_1 \mapsto 0, \qquad v_2 \mapsto w_2, \qquad v_i \mapsto w_i.$

Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 :\quad v_1 \mapsto w_1, \qquad v_2 \mapsto 0, \qquad v_i \mapsto w_i, \qquad i = 3, \dots, n.$

Thus $T_1 + T_2 \notin U$. ☐

**COMMENT:** If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V, W), T$ is injective. Hence $U = \varnothing$.

If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0 v_0 = 0$.

If $V$ is infinite-dim, the result is true as well.

---

**8** *Suppose $W$ is finite-dim with* $\dim V \geq \dim W \geq 2$*, if $V$ is finite-dim.*
*Show that $U = \{ T \in \mathcal{L}(V,W) : T$ is not surjective $\}$ is not a subspace of $\mathcal{L}(V,W)$.*

**SOLUTION:**

Let $(v_1, \dots, v_n)$ be linearly independent in $V$, $(w_1, \dots, w_m)$ be a basis of $W$.
( Let $n = \dim V$, if $V$ is finite, otherwise we choose $n \in \{m, m+1, \dots\}$; $2 \leq m \leq n$ ).
Define $T_1 \in \mathcal{L}(V,W)$ as $T_1: \quad v_1 \mapsto 0, \qquad v_2 \mapsto w_2, \qquad v_j \mapsto w_j, \qquad v_{m+i} \mapsto 0.$
Define $T_2 \in \mathcal{L}(V,W)$ as $T_2: \quad v_1 \mapsto w_1, \qquad v_2 \mapsto 0, \qquad v_j \mapsto w_j, \qquad v_{m+i} \mapsto 0.$
For each $j = 2, \dots, m;\ i = 1, \dots, n-m$, if $V$ is finite, otherwise let $i \in \mathbf{N}^+$.
Thus $T_1 + T_2 \notin U$. $\square$

**COMMENT:** If $\dim W = 0$, then $W = \{0\} = \operatorname{span}()$. $\forall\, T \in \mathcal{L}(V,W), T$ is surjective. Hence $U = \varnothing$.
If $\dim W = 1$, then $W = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0 v_0 = 0$.
If $W$ is infinite-dim, the result is true as well.

---

**9** *Suppose $T \in \mathcal{L}(V,W)$ is injective and $(v_1, \dots, v_n)$ is linearly independent in $V$.*
*Prove that $(Tv_1, \dots, Tv_n)$ is linearly independent in $W$.*

**SOLUTION:**

$$a_1 T v_1 + \cdots + a_n T v_n = 0 = T(\sum_{i=1}^{n} a_i v_i) \iff \sum_{i=1}^{n} a_i v_i = 0 \iff a_1 = \cdots = a_n = 0. \ \square$$

---

**10** *Suppose $(v_1, \dots, v_n)$ spans $V$ and $T \in \mathcal{L}(V,W)$. Show that $(Tv_1, \dots, Tv_n)$ spans range $T$.*

**SOLUTION:**

(a) $\operatorname{range} T = \{ Tv : v \in V \} = \{ Tv : v \in \operatorname{span}(v_1, \dots, v_n)\}$
$\Rightarrow Tv_1, \dots, Tv_n \in \operatorname{range} T \Rightarrow$ By [2.7] and [3.19], $\operatorname{span}(Tv_1, \dots, Tv_n) \subseteq \operatorname{range} T$.
(b) $\forall w \in \operatorname{range} T,\ \exists v \in V, Tv = w$. 又 $\forall v \in V,\ \exists\, a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n$
$\Rightarrow w = Tv = a_1 T v_1 + \cdots + a_n T v_n \Rightarrow \operatorname{range} T \subseteq \operatorname{span}(Tv_1, \dots, Tv_n)$. $\square$

---

**11** *Suppose $S_1, \dots, S_n$ are injective linear maps and $S_1 S_2 \dots S_n$ makes sence.*
*Prove that $S_1 S_2 \dots S_n$ is injective.*

**SOLUTION:** $S_1 S_2 \dots S_n(v) = 0 \iff S_2 S_3 \dots S_n(v) = 0 \iff \cdots \iff S_n(v) = 0 \iff v = 0.$ $\square$

---

**12** *Suppose that $V$ is finite-dim and that $T \in \mathcal{L}(V,W)$. Prove that*
*there exists a subspace $U$ of $V$ such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{ Tu : u \in U \}$.*

**SOLUTION:**

By [2.34], there exists a subspace $U$ of $V$ such that $V = U \oplus \operatorname{null} T$.
$\forall v \in V,\ \exists!\ w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w+u) = Tu \in \{ Tu : u \in U \} \Rightarrow \square$

**COMMENT:** $V$ can be infinite-dim. See the above of [2.34].

---

**16** *Suppose there exists a linear map on $V$*
*whose null space and range are both finite-dim. Prove that $V$ is finite-dim.*

**SOLUTION:**

Denote the linear map by $T$. Let $(Tv_1, \dots, Tv_n)$ be a basis of range $T$, $(u_1, \dots, u_m)$ be a basis of null $T$.
Then for all $v \in V, T(\underbrace{v - a_1 v_1 - \cdots - a_n v_n}_{u \,\in\, \operatorname{null} T}) = 0$, where $Tv = a_1 T v_1 + \cdots + a_n T v_n$.
$\Rightarrow u = b_1 u_1 + \cdots + b_m u_m \Rightarrow v = a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m$.
Getting $V \subseteq \operatorname{span}(v_1, \dots, v_n, u_1, \dots, u_m)$. Thus $V$ is finite-dim. $\square$

---

**17** *Suppose $V$ and $W$ are both finite-dim. Prove that there exists an injective $T \in \mathcal{L}(V, W)$ if and only if* $\dim V \leq \dim W$.

**SOLUTION:**

(a) Suppose there exists an injective $T$. Then $\dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim \operatorname{range} T \leq \dim W$.

(b) Suppose $\dim V \leq \dim W$, letting $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ be bases of $V$ and $W$ respectively.
Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i, \;\; i = 1, \ldots, n\,(= \dim V\,)$. $\square$

---

**18** *Suppose V and W are both finite-dim. Prove that there exists a surjective $T \in \mathcal{L}(V, W)$ if and only if* $\dim V \geq \dim W$.

**SOLUTION:**

(a) Suppose there exists a surjective $T$. Then $\dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim W + \dim \operatorname{null} T$
$\Rightarrow \dim W = \dim V - \dim \operatorname{null} T \leq \dim V$.

(b) Suppose $\dim V \geq \dim W$, letting $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ be bases of $V$ and $W$ respectively.
Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_m v_m + a_{m+1} v_{m+1} + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m$. $\square$

---

**19** *Suppose $V$ and $W$ are finite-dim and that $U$ is a subspace of $V$.
Prove that $\exists\, T \in \mathcal{L}(V, W)$, null$\,T = U \iff$* $\dim U \geq \dim V - \dim W$.

**SOLUTION:**

(a) Suppose $\exists\, T \in \mathcal{L}(V, W), \;\operatorname{null} T = U$. Then $\dim \operatorname{null} T = \dim U \geq \dim V - \dim W$.

(b) Suppose $\underbrace{\dim U}_{m} \geq \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_{p}$ $(\Rightarrow \dim W = p \geq n = \dim V - \dim U\,)$.
Let $(u_1, \ldots, u_m)$ be a basis of $U$, extend to a basis of $V$ as $(u_1, \ldots, u_m, v_1, \ldots, v_n)$.
Let $(w_1, \ldots, w_p)$ be a basis of $W$.
Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m) = a_1 w_1 + \cdots + a_n w_n$. $\square$

---

• **TIPS:** *Suppose $T \in \mathcal{L}(V, W)$ and $R = (Tv_1, \ldots, Tv_n)$ is linearly independent in range$\,T$.*
*( Let* $\dim$ *range$\,T = n$, if range$\,T$ is finite, otherwise choose $n$ arbitrarily. ).*
*By (3.A.4), $L = (v_1, \ldots, v_n)$ is linearly independent in $V$.*

**NEW NOTATION:** *Denote $\mathcal{K}_R$ by span$L$, if range$\,T$ is finite-dim,
otherwise, denote it by an vector space in the set $\mathcal{S}_V$null$\,T$.*

**NEW THEOREM:**

$$\mathcal{K}_R \oplus \operatorname{null} T = V \Leftarrow \begin{cases} \text{(a) } T(\sum\limits_{i=1}^{n} a_i v_i) = 0 \Rightarrow \sum\limits_{i=1}^{n} a_i T v_i = 0 \Rightarrow a_1 = \cdots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \operatorname{null} T = \{0\}. \\ \text{(b) } \forall v \in V, Tv = \sum\limits_{i=1}^{n} a_i T v_i \Rightarrow Tv - \sum\limits_{i=1}^{n} a_i T v_i = T(v - \sum\limits_{i=1}^{n} a_i v_i) = 0 \\ \quad \Rightarrow v - \sum\limits_{i=1}^{n} a_i v_i \in \operatorname{null} T \Rightarrow v = (v - \sum\limits_{i=1}^{n} a_i v_i) + (\sum\limits_{i=1}^{n} a_i v_i) \Rightarrow \mathcal{K}_R + \operatorname{null} T = V. \end{cases}$$

**COMMENT:** $\operatorname{null} T \in \mathcal{S}_V \mathcal{K}_R$.

---

• *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V, W)$, and $U$ is a subspace of $W$.
Prove that $\mathcal{K}_U = \{\, v \in V : Tv \in U \,\}$ is a subspace of $V$
and* $\dim \mathcal{K}_U = \dim$ *null$\,T$* $+ \dim(U \cap$ *range$\,T$*$)$.

**SOLUTION:** For any $u, w \in \mathcal{K}_U$ and $\lambda \in \mathbf{F}$, $T(u + \lambda w) = Tu + \lambda Tw \in U \Rightarrow T$ is linear
Define $S \in \mathcal{L}(\mathcal{K}_U, U)$ as $Rv = Tv$ for all $v \in \mathcal{K}_U$. Hence range $R = U \cap \operatorname{range} T$.
Suppose $Tv = 0$ for some $v \in V$. 又 $0 \in U \Rightarrow Rv = 0$. Thus $\operatorname{null} T \subseteq \operatorname{null} R$. $\square$

---

**20** *Suppose* $T \in \mathcal{L}(V, W)$. *Prove that* $T$ *is injective* $\iff \exists\ S \in \mathcal{L}(W, V),\ ST = I \in \mathcal{L}(V)$.

**SOLUTION:**

(a) Suppose $\exists\ S \in \mathcal{L}(W, V),\ ST = I$. Then if $Tv = 0 \Rightarrow ST(v) = 0 = v$. Hence $T$ is injective.

(b) Suppose $T$ is injective. $\forall w \in \operatorname{range} T,\ \exists!\, v \in V, Tv = w$. ( if $w = 0$, then $v = 0$ )

Define $S : W \to V$ by $Sw = v$ and $Su = 0,\ u \in U$. Where $W = U \oplus \operatorname{range} T$.

$\Rightarrow S(Tv + \lambda Tu) = S(T(v + \lambda u)) = v + \lambda u$ and $S(x + \nu y) = 0,\ x, y \in U$.

Thus $S|_{\operatorname{range} T + U} = S|_W \in \mathcal{L}(W, V)$ and $ST = I$. $\square$

OR. Let $R = (Tv_1, \ldots, Tv_n)$ be linearly independent in $\operatorname{range} T \subseteq W$, $(\ldots)$ and then $\mathcal{K}_R \oplus \operatorname{null} T = V$.

Supose $W = U \oplus \operatorname{range} T$. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$ and $Su = 0,\ u \in U$. Thus $ST = I$. $\square$

---

**21** *Suppose* $T \in \mathcal{L}(V, W)$. *Prove that* $T$ *is surjective* $\iff \exists\ S \in \mathcal{L}(W, V),\ TS = I \in \mathcal{L}(W)$.

**SOLUTION:**

(a) Suppose $\exists\ S \in \mathcal{L}(W, V),\ TS = I$. Then for any $w \in W, TS(w) = w \in \operatorname{range} T \Rightarrow \operatorname{range} T = W$. $\square$

(b) Suppose $T$ is surjective. $\forall w \in W,\ \exists v \in V, Tv = w$. Define $S : W \to V$ by $Sw = v$.

But $T(Sv + \lambda Su) = T(Sv) + \lambda T(Su) = v + \lambda u = T(S(v + \lambda u)) \not\Rightarrow Sv + \lambda Su = S(v + \lambda u)$.

So we let $R = (Tv_1, \ldots, Tv_n)$ be linearly independent in $\operatorname{range} T = W$, $(\ldots)$ and then $\mathcal{K}_R \oplus \operatorname{null} T = V$.

Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$. Then $TS = I$. $\square$

---

**22** *Suppose U and V are finite-dim vec-sps and* $S \in \mathcal{L}(V, W)$ *and* $T \in \mathcal{L}(U, V)$.
*Prove that* $\dim \textit{null}\, ST \leq \dim \textit{null}\, S + \dim \textit{null}\, T$.

**SOLUTION:** Define $R \in \mathcal{L}(\operatorname{null} ST, V)$ by $Ru = Tu$ for all $u \in \operatorname{null} ST \subseteq U$.

$$\left.\begin{aligned} S(Tu) = 0 = S(Ru) &\Rightarrow \operatorname{range} R \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} R \leq \dim \operatorname{null} S \\ Tu = 0 = Ru &\Rightarrow \operatorname{null} R \supseteq \operatorname{null} T \Rightarrow \dim \operatorname{null} R = \dim \operatorname{null} T \end{aligned}\right\} \Rightarrow \square$$

• **COROLLARY:**

(1) If $T$ is injective, then $\dim \operatorname{null} T = 0 \Rightarrow \dim \operatorname{null} ST \leq \dim \operatorname{null} S$.

(2) If $T$ is surjective, then $\operatorname{range} R = \operatorname{null} S \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.

(3) If $S$ is injective, then $\operatorname{range} R = \{0\} \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} R = \dim \operatorname{null} T$.

---

**23** *Suppose U and V are finite-dim vec-sps and* $S \in \mathcal{L}(V, W)$ *and* $T \in \mathcal{L}(U, V)$.
*Prove that* $\dim \textit{range}\, ST \leq \min\{\dim \textit{range}\, S, \dim \textit{range}\, T\}$.

**SOLUTION:**

$\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{span}(Su_1, \ldots, Su_{\dim \operatorname{range} T})$, letting $\operatorname{span}(u_1, \ldots, u_{\dim \operatorname{range} T}) = \operatorname{range} T$.

$\dim \operatorname{range} ST \leq \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leq \dim \operatorname{range} S \Rightarrow \square$

• **COROLLARY:**

(1) If $S$ is injective, then $\dim \operatorname{range} ST = \dim \operatorname{range} T$.

(2) If $T$ is surjective, then $\operatorname{range} ST = \operatorname{range} S$.

---

• (a) *Suppose* $\dim V = 5$ *and* $S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$. *Prove that* $\dim \textit{range}\, TS \leq 2$.
(b) *Give an example of* $S, T \in \mathcal{L}(\mathbf{F}^5)$ *with* $ST = 0$ *and* $\dim \textit{range}\, TS = 2$.

**SOLUTION:**

By Problem (23), $\dim \operatorname{range} TS \leq \min\{\underbrace{\dim \operatorname{range} S}_{5 - \dim \operatorname{null} T}, \underbrace{\dim \operatorname{range} T}_{5 - \dim \operatorname{null} S}\}$.

Suppose $\dim \operatorname{range} TS \geq 3$. Then $\min\{5 - \dim \operatorname{null} T, 5 - \dim \operatorname{null} S\} \geq 3$

$$\Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leq 2.$$

又 $\dim \operatorname{null} ST = 5 \leq \dim \operatorname{null} S + \dim \operatorname{null} T \leq 4$. Contradicts. Thus $\dim \operatorname{range} TS \leq 2$. $\square$

EXAMPLE: $V = \text{span}(v_1, \dots, v_5)$

$$T: \quad v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i \;;$$
$$S: \quad v_1 \mapsto v_4, \quad v_2 \mapsto v_5, \quad v_i \mapsto 0 \;; \qquad i = 3,4,5$$

---

• *Suppose* $\dim V = n$ *and* $S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$.

*Prove that* $\dim TS \le m = \begin{cases} \frac{n}{2}, & \textit{if } 2 \mid n. \\ \frac{n-1}{2}, & \textit{otherwise.} \end{cases}$

**SOLUTION:**

By Problem (23), $\dim \text{range}\, TS \le \min\{\underbrace{\dim \text{range}\, S}_{n-\dim \text{null}\, T}, \underbrace{\dim \text{range}\, T}_{n-\dim \text{null}\, S}\}$. Suppose $\dim \text{range}\, TS \ge m+1$.

Then $\min\{n - \dim \text{null}\, T, n - \dim \text{null}\, S\} \ge m+1$

$\Rightarrow \max\{\dim \text{null}\, T, \dim \text{null}\, S\} \le n - m - 1$.

又 $\dim \text{null}\, ST = n \le \dim \text{null}\, S + \dim \text{null}\, T \le n - m - 1$. Contradicts. Thus $\dim \text{range}\, TS \le m$. $\square$

---

**24** *Suppose that* $W$ *is finite-dim and* $S, T \in \mathcal{L}(V, W)$.

*Prove that null* $S \subseteq$ *null* $T \iff \exists\, E \in \mathcal{L}(W)$ *such that* $T = ES$.

**SOLUTION:**

Suppose $\text{null}\, S \subseteq \text{null}\, T$. Let $R = (Sv_1, \dots, Sv_n)$ be a basis of range $S \Rightarrow (v_1, \dots, v_n)$ is linearly independent.

Let $\mathcal{K}_R = \text{span}(v_1, \dots, v_n) \Rightarrow V = \mathcal{K}_R \oplus \text{null}\, S$.

Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i$, $\;Eu = 0$; for each $i = 1 \dots, n$ and $u \in \text{null}\, S$.

Hence $\forall v \in V$, $(\exists!\, a_i \in \mathbf{F}, u \in \text{null}\, S)$, $Tv = a_1Tv_1 + \cdots + a_nTv_n = E(a_1Sv_1 + \cdots + a_nSv_n) \Rightarrow T = ES$.

Suppose $\exists\, E \in \mathcal{L}(W)$ such that $T = ES$. Then $\text{null}\, T = \text{null}\, ES \supseteq \text{null}\, S$. $\square$

---

**25** *Suppose that* $V$ *is finite-dim and* $S, T \in \mathcal{L}(V, W)$.

*Prove that range* $S \subseteq$ *range* $T \iff \exists\, E \in \mathcal{L}(V)$ *such that* $S = TE$.

**SOLUTION:**

Suppose $\text{range}\, S \subseteq \text{range}\, T$. Let $(v_1, \dots, v_m)$ be a basis of $V$.

Because $\text{range}\, S \subseteq \text{range}\, T \Rightarrow Sv_i \in \text{range}\, T$ for each $i$. Suppose $u_i \in V$ for each $i$ such that $Tu_i = Tv_i$.

Thus defining $E \in \mathcal{L}(V)$ by $Ev_i = u_i$ for each $i \Rightarrow S = TE$.

Suppose $\exists\, E \in \mathcal{L}(V)$ such that $S = TE$. Then $\text{range}\, S = \text{range}\, TE \subseteq \text{range}\, T$. $\square$

---

• *Suppose* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove that* $V =$ *null* $P \oplus$ *range* $P$.

**SOLUTION:**

Let $P^2v_1, \dots, P^2v_n$ be a basis of range $P^2$. Then $(Pv_1, \dots, Pv_n)$ is linearly independent in $V$.

$$\left.\begin{array}{l} \text{Let } \mathcal{K} = \text{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \text{null}\, P^2 \\ \text{又 } \mathcal{K} = \text{range}\, P = \text{range}\, P^2; \;\; \text{null}\, P = \text{null}\, P^2 \end{array}\right\} \Rightarrow \square$$

---

**26** *Prove that the differentiation map* $D \in \mathcal{P}(\mathbf{R})$ *is surjective.*

**SOLUTION:** Note that $\deg Dx^n = n - 1$.

Because $\text{span}(Dx, Dx^2, \dots) \subseteq \text{range}\, D$. 又 By (2.A.10), $\text{span}(Dx, Dx^2, \dots) = \text{span}(1, x, \dots) = \mathcal{P}(\mathbf{R})$. $\square$

---

**27** *Suppose* $p \in \mathcal{P}(\mathbf{R})$. *Prove that there exists a polynomial* $q \in \mathcal{P}(\mathbf{R})$ *such that* $5q'' + 3q' = p$.

**SOLUTION:**

Define $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$.

Note that $\deg Bx^n = n - 1$. Similar to Problem (26), we conclude that $B$ is surjective.

Hence for any $p \in \mathcal{P}(\mathbf{R})$, there exists $q \in \mathcal{P}(\mathbf{R})$ such that $Bq = p$. $\square$

---

**28** *Suppose $T \in \mathcal{L}(V,W)$ and $(w_1,\dots,w_m)$ is a basis of range $T$. Prove that $\exists\ \varphi_1,\dots,\varphi_m \in \mathcal{L}(V,\mathbf{F})$ such that for all $v \in V, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.*

**SOLUTION:**

Suppose $(v_1,\dots,v_m)$ in $V$ such that $Tv_i = w_i$ for each $i$.

Then $(v_1,\dots,v_m)$ is linearly independent, extend it to a basis of $V$ as $(v_1,\dots,v_m,u_1,\dots,u_n)$.

Note that $\forall v \in V, v = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n,\ \exists!\, a_i, b_i \in \mathbf{F} \Rightarrow Tv = a_1w_1 + \cdots + a_mw_m$.

Define $\varphi_i : V \to \mathbf{F}$ by $\varphi_i(v) = a_iv_i$ for each $i$. We now check the linearity.

$\forall v,u \in V\ (\exists!\, a_i,b_i,c_i,d_i \in \mathbf{F}), \lambda \in \mathbf{F}, \varphi_i(v+\lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda\varphi(u)$. $\square$

---

**29** *Suppose $\varphi \in \mathcal{L}(V,\mathbf{F})$ and $\varphi \neq 0$. Suppose $u \in V$ is not in null $\varphi$. Prove that $V = \text{null}\,\varphi \oplus \{au : a \in \mathbf{F}\}$.*

**SOLUTION:**

(a) Suppose $v = cu \in \text{null}\,\varphi \cap \{au : a \in \mathbf{F}\}$, where $c \in \mathbf{F}$.

Then $\varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$. Hence $\text{null}\,\varphi \cap \{au : a \in \mathbf{F}\}$.

(b) Suppose $v \in V$. Then $v = (v - \frac{\varphi(v)}{\varphi(u)}u) + \frac{\varphi(v)}{\varphi(u)}u \Rightarrow \varphi(v) = 0$.

$$\left.\begin{array}{l} v - \frac{\varphi(v)}{\varphi(u)}u \in \text{null}\,\varphi \\ \frac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array}\right\} \Rightarrow V = \text{null}\,\varphi \oplus \{au : a \in \mathbf{F}\}.\ \square$$

*This may seems strange. Here we explain why.*

$\varphi \neq 0 \Rightarrow \exists$ a linearly independent list $(v_1,\dots,v_n \in V)$ such that $\varphi(v_i) = a_i \neq 0$.

Choose a $v_k$ arbitrarily. Then $\varphi(v_k - \frac{\varphi(v_k)}{\varphi(v_j)}v_j) = 0$ for each $j = 1,\dots,k-1,k+1,\dots,n$.

Thus span $\{v_k - \frac{\varphi(v_k)}{\varphi(v_j)}v_j\}_{j\neq k} \subseteq \text{null}\,\varphi$. Hence there is only one nonzero vector in every vec-sp in $\mathcal{S}_V$null $\varphi$.

---

**30** *Suppose $\varphi_1,\varphi_2 \in \mathcal{L}(V,\mathbf{F})$ and null $\varphi_1$ = null $\varphi_2$ = null $\varphi$. Prove that $\exists\ c \in \mathbf{F}, \varphi_1 = c\varphi_2$*

**SOLUTION:**

If $\text{null}\,\varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $u \in V/\text{null}\,\varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$.

By Problem (29), $V = \text{null}\,\varphi \oplus \text{span}\,(u)$.

Hence for any $v \in V, v = w + a_vu,\ \exists!\, w \in \text{null}\,\varphi, a_v \in \mathbf{F}$.

$$\varphi_1(v) = a_v\varphi_1(u),\ \ \varphi_2(v) = a_v\varphi_2(u) \Rightarrow a_v = \frac{\varphi_1(v)}{\varphi_1(u)} = \frac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \frac{\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}.$$

Thus $\varphi_1 = c\varphi_2$. $\square$

---

**31** *Give an example of $T_1, T_2 \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^2)$ such that null $T_1$ = null $T_2$ and that $T_1$ is not a scalar multiple of $T_2$.*

**SOLUTION:**

Let $(v_1,\dots,v_5)$ be a basis of $\mathbf{R}^5$, $(w_1,w_2)$ be a basis of $\mathbf{R}^2$. Define $T,S \in \mathcal{L}(V,W)$ by

$$\left.\begin{array}{lll} Tv_1 = w_1, & Tv_2 = w_2, & Tv_3 = Tv_4 = Tv_5 = 0 \\ Sv_1 = w_1, & Sv_2 = 2w_2, & Sv_3 = Sv_4 = Sv_5 = 0 \end{array}\right\} \Rightarrow \text{null}\,T = \text{null}\,S.$$

Suppose $T = \lambda S$. Then $w_1 = Tv_1 = \lambda Sv_1 = \lambda w_1 \Rightarrow \lambda = 1$.

While $w_2 = Tv_2 = \lambda Sv_2 = 2\lambda w_2 \Rightarrow \lambda = \frac{1}{2}$. Contradicts. $\square$

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• *Suppose $V$ is finite-dim, $X$ is a subspace of $V$, and $Y$ is a finite-dim subspace of $W$. Prove that there exists $T \in \mathcal{L}(V, W)$ such that null $T = X$ and range $T = Y$ if and only if* $\dim X + \dim Y = \dim V$.

**SOLUTION:**

(a) Suppose $\dim X + \dim Y = \dim V$. Let $(u_1, \ldots, u_n)$ be a basis of $X$, $R = (w_1, \ldots, w_m)$ be a basis of $Y$.

Extend $(u_1, \ldots, u_n)$ to a basis of $V$ as $(u_1, \ldots, u_n, v_1, \ldots, v_m)$.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_m v_m + b_1 v_1 + \cdots + b_n v_n) = a_1 w_1 + \cdots + a_m w_m$.

Now we show that null $T = X$ and range $T = Y$

Suppose $v \in V$. Then $\exists!\, a_i, b_j \in \mathbf{F}, v = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_n u_n$.

$$\left.\begin{aligned} v \in \text{null}\, T &\Rightarrow Tv = 0 \\ &\Rightarrow a_1 = \cdots = a_m = 0 \\ &\Rightarrow v \in X \Rightarrow \text{null}\, T \subseteq X. \\ v \in X &\Rightarrow v \in \text{null}\, T \Rightarrow \text{null}\, T \supseteq X. \end{aligned}\right\} \Rightarrow \text{null}\, T = X.$$

$$\left.\begin{aligned} w \in \text{range}\, T &\Rightarrow \exists\; v \in V, Tv = w \Rightarrow \text{let } v = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_n u_n \\ &\qquad \Rightarrow Tv = w = a_1 w_1 + \cdots + a_m w_m \Rightarrow w \in Y \Rightarrow \text{range}\, T \subseteq Y. \\ w \in Y &\Rightarrow w = a_1 T v_1 + \cdots + a_m T v_m = T(a_1 v_1 + \cdots + a_m v_m) \\ &\qquad \Rightarrow w \in \text{range}\, T \Rightarrow \text{range}\, T \supseteq Y. \end{aligned}\right\} \Rightarrow \text{range}\, T = Y.$$

(b) Conversely it is true as well.

$\square$

---

• *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V, W)$. Let $(Tv_1, \ldots, Tv_n)$ be a basis of range $T$. Extend $(v_1, \ldots, v_n)$ to a basis of $V$ as $(v_1, \ldots, v_n, u_1, \ldots, u_m)$. Prove or give a counterexample: $(u_1, \ldots, u_m)$ is a basis of null $T$.*

**SOLUTION:** An counterexample:

Suppose $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$. Then span $(Tv_1, Tv_2, Tv_3) =$ span $(w_1)$.

Extend $(v_i)$ to $(v_1, v_2, v_3)$ for each $i$. But none of $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is a basis of null $T$.

---

• *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V, W)$. Let $(u_1, \ldots, u_m)$ be a basis of null $T$. Extend $(u_1, \ldots, u_m)$ to a basis of $V$ as $(u_1, \ldots, u_m, v_1, \ldots, v_n)$. Prove or give a counterexample: $(Tv_1, \ldots, Tv_n)$ spans range $T$.*

**SOLUTION:**

$\forall w \in \text{range}\, T,\ \exists\, v \in V,\ (\exists!\, a_i, b_i \in \mathbf{F}), Tv = T(a_1 v_1 + \cdots + a_n v_n) = w$

$\Rightarrow w \in \text{span}\,(Tv_1, \ldots, Tv_n) \Rightarrow \text{range}\, T \subseteq \text{span}\,(Tv_1, \ldots, Tv_n)$. $\square$

COMMENT: If $T$ is injective, then $(Tv_1, \ldots, Tv_n)$ is a basis of range $T$.

---

• *Suppose $V$ is finite-dim with* $\dim V > 1$. *Show that if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is a linear map such that $\varphi(ST) = \varphi(S) \cdot \varphi(T)$ for all $S, T \in \mathcal{L}(V)$, then $\varphi = 0$.*

HINT: *The description of the two-sided ideals of $\mathcal{L}(V)$ in Section 3A might be useful.*

**SOLUTION:** Using notations in (3.A.• the last).

Suppose $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \dots, n\}$, $\varphi(R_{i,j}) \neq 0$.

Because $R_{i,j} = R_{x,j} \circ R_{i,x},\ \forall x = 1, \dots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, because $R_{i,x} = R_{y,x} \circ R_{i,y},\ \forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0$ for any $x, y = 1, \dots, n$.

Let $l \neq i, k \neq j$ and then $\varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. $\square$

---

• *Suppose that $V$ and $W$ are real vector spaces and $T \in \mathcal{L}(V, W)$.*
*Define $T_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ by $T_{\mathbb{C}}(u + iv) = Tu + iTv$ for all $u, v \in V$.*

(a) *Show that $T_{\mathbb{C}}$ is a (complex) linear map from $V_{\mathbb{C}}$ to $W_{\mathbb{C}}$.*

(b) *Show that $T_{\mathbb{C}}$ is injective $\iff$ $T$ is injective.*

(c) *Show that range $T_{\mathbb{C}} = W_{\mathbb{C}} \iff$ range $T = W$.*

*See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbb{C}}$.*

*The linear map $T_{\mathbb{C}}$ is called the complexification of the linear map $T$.*

**SOLUTION:**

(a) $\forall u_1 + \mathrm{i}v_1, u_2 + \mathrm{i}v_2 \in V_{\mathbb{C}}, \lambda \in \mathbf{F}$,

$T((u_1 + \mathrm{i}v_1) + \lambda(u_2 + \mathrm{i}v_2)) = T((u_1 + \lambda u_2) + \mathrm{i}(v_1 + \lambda v_2)) = T(u_1 + \lambda u_2) + \mathrm{i}T(v_1 + \lambda v_2)$

$= Tu_1 + \mathrm{i}Tv_1 + \lambda Tu_2 + \mathrm{i}\lambda Tv_2 = T(u_1 + \mathrm{i}v_1) + \lambda T(u_2 + \mathrm{i}v_2)$. $\square$

(b) $\left.\begin{array}{l}\text{Suppose } T_{\mathbb{C}} \text{ is injective. Let } T(u) = 0 \Rightarrow T_{\mathbb{C}}(u + \mathrm{i}0) = Tu = 0 \Rightarrow u = 0. \\ \text{Suppose } T \text{ is injective. Let } T_{\mathbb{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + \mathrm{i}v = 0.\end{array}\right\} \Rightarrow \square$

(c) $\left.\begin{array}{l}\text{Suppose } T_{\mathbb{C}} \text{ is surjective. } \forall w, x \in W,\ \exists\, u, v \in V, T(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = w + \mathrm{i}x \\ \qquad \Rightarrow Tu = w, Tv = x \Rightarrow \text{T is surjective.} \\ \text{Suppose } T \text{ is surjective. } \forall w, x \in W, \exists\, u, v \in V, Tu = w, Tv = x \\ \qquad \Rightarrow \forall w + \mathrm{i}x \in W_{\mathbb{C}}, \exists\, u + \mathrm{i}v \in V, T(u + \mathrm{i}v) = w + \mathrm{i}x \Rightarrow T_{\mathbb{C}} \text{ is surjective.}\end{array}\right\} \Rightarrow \square$

---

**ENDED**

# 3·C

• **NOTE FOR [3.47]:** $LHS = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k} = RHS.$

---

• **NOTE FOR [3.48]:**

$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\ 8 & 9 & 10\end{pmatrix} = \begin{pmatrix} \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} & \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix} \\ \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} & \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}1\times5+2\times8 & 1\times6+2\times9 & 1\times7+2\times10\\ 3\times5+4\times8 & 3\times6+4\times9 & 3\times7+4\times10\end{pmatrix} = \begin{pmatrix}21 & 24 & 27\\ 47 & 54 & 61\end{pmatrix}$$

---

• **NOTE FOR [3.49]:** $\because [(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$

$\therefore (AC)_{\cdot,k} = A_{\cdot,\cdot}C_{\cdot,k} = AC_{\cdot,k}$

---

• **EXERCISE 10:** $\because [(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$

$\therefore (AC)_{j,\cdot} = A_{j,\cdot}C_{\cdot,\cdot} = A_{j,\cdot}C.$

---

• *Suppose* $C \in \mathbf{F}^{m,c}$, $R \in \mathbf{F}^{c,p}$.

(a) For $k = 1, \ldots, p$, $\quad (CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^{c} C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$

(b) For $j = 1, \ldots, m$, $\quad (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

EXAMPLE:

$$P = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\ 8 & 9 & 10\end{pmatrix} \in \mathbf{F}^{2,3}.$$

$$P_{\cdot,1} = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} = 5\begin{pmatrix}1\\3\end{pmatrix} + 8\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}21\\47\end{pmatrix};$$

$$P_{\cdot,2} = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} = 6\begin{pmatrix}1\\3\end{pmatrix} + 9\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}24\\54\end{pmatrix};$$

$$P_{\cdot,3} = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix} = 7\begin{pmatrix}1\\3\end{pmatrix} + 10\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}27\\61\end{pmatrix};$$

$$P_{1,\cdot} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\ 8 & 9 & 10\end{pmatrix} = 1\begin{pmatrix}5 & 6 & 7\end{pmatrix} + 2\begin{pmatrix}8 & 9 & 10\end{pmatrix} = \begin{pmatrix}21 & 24 & 27\end{pmatrix};$$

$$P_{2,\cdot} = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\ 8 & 9 & 10\end{pmatrix} = 3\begin{pmatrix}5 & 6 & 7\end{pmatrix} + 4\begin{pmatrix}8 & 9 & 10\end{pmatrix} = \begin{pmatrix}47 & 54 & 61\end{pmatrix};$$

---

• **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$

$\because (Ac)_{j,1} = \sum_{r=1}^{n} A_{j,r}c_{r,1} = [\sum_{r=1}^{n}(A_{\cdot,r}c_{r,1})]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$

$\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^{n} A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ OR. By $(Ac)_{\cdot,1} = Ac_{\cdot,1}$ Using (a) above.

---

• **EXERCISE 10:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$

$\because (aC)_{1,k} = \sum_{r=1}^{n} a_{1,r}C_{r,k} = [\sum_{r=1}^{n} a_{1,r}(C_{r,\cdot})]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$

$\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^{n} a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ OR. By $(aC)_{1,\cdot} = a_{1,\cdot}C.$ Using (b) above.

---

• COLUMN-ROW FACTORIZATION (CR Factorization)

*Suppose* $A \in \mathbf{F}^{m,n}$, $A \neq 0$. *Let* $S_c = \textit{span}\,(A_{\cdot,1}, \dots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}$, $\dim S_c = c$.

*And* $S_r = \textit{span}\,(A_{1,\cdot}, \dots, A_{n,\cdot}) \subseteq \mathbf{F}^{1,n}$, $\dim S_r = r$.

*Prove that* $A = CR$. $\exists\ C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,n}$.

**SOLUTION:** Notice that $A \neq 0 \Rightarrow c, r \geq 1$.

Let $(C_{\cdot,1}, \dots, C_{\cdot,c})$ be a basis of $S_c$, forming $C \in \mathbf{F}^{m,c}$.

Then for any $A_{\cdot,k}$, $\quad A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}, \quad \exists!\, R_{1,k}, \dots, R_{c,k} \in \mathbf{F}$.

Hence, by letting $R = \begin{pmatrix} R_{1,1} & \cdots & R_{1,n} \\ \vdots & \ddots & \vdots \\ R_{c,1} & \cdots & R_{c,n} \end{pmatrix}$, we have $A = CR$.

OR. Let $(R_{1,\cdot}, \dots, R_{c,\cdot})$ be a basis of $S_r$, forming $R \in \mathbf{F}^{c,n}$.

For any $A_{j,\cdot}$, $\quad A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot} = (CR)_{j,\cdot}, \quad \exists!\, C_{j,1}, \dots, C_{j,c} \in \mathbf{F}$. Similarly. $\square$

EXAMPLE:

$$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

(1) Because $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2\begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix}$.

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix}$ can be uniquely written as a linear combination of $A_{1,\cdot}, A_{2,\cdot}$.

Hence $\dim S_r = 2$. We choose $(A_{1,\cdot}, A_{2,\cdot})$ as the basis.

(2) Because $\begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix} - \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix}$.

Hence $\dim S_c = 2$. We choose $(A_{\cdot,2}, A_{\cdot,3})$ as the basis.

---

• COLUMN RANK EQUALS ROW RANK (Using the notation above)

For any $A_{j,\cdot} \in S_r$, $\quad A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

$\Rightarrow \operatorname{span}(A_{1,\cdot}, \dots, A_{m,\cdot}) = S_r = \operatorname{span}(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leq c = \dim S_c$.

Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leq r = \dim S_r = \dim S_c^t$. $\square$

---

• *Suppose* $T \in \mathcal{L}(V)$*, and* $u_1, \dots, u_n$ *and* $v_1, \dots, v_n$ *are bases of* $V$.

*Prove that the following are equivalent.*

(a) $T$ *is injective.*

(b) *The columns of* $\mathcal{M}(T)$ *are linearly independent in* $\mathbf{F}^{n,1}$.

(c) *The columns of* $\mathcal{M}(T)$ *span* $\mathbf{F}^{n,1}$.

(d) *The rows of* $\mathcal{M}(T)$ *span* $\mathbf{F}^{1,n}$.

(e) *The rows of* $\mathcal{M}(T)$ *are linearly independent in* $\mathbf{F}^{1,n}$.

*Here* $A = \mathcal{M}(T)$ *means* $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.

**SOLUTION:**

$T$ is injective $\iff \dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim \operatorname{range} T$

$\iff (Tu_1, \dots, Tu_n)$ is linearly independent in $V$, and therefore is a basis of $V$

$\iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n))$ is linearly independent, as well as $(A_{\cdot,1}, \dots, A_{\cdot,n})$

$\iff (A_{\cdot,1}, \dots, A_{\cdot,n})$ is a basis of $\mathbf{F}^{n,1}$.

$\Big($ 又 $\dim \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) = \dim \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = n$ $\Big)$

$\iff (A_{1,\cdot}, \dots, A_{n,\cdot})$ is a basis of $\mathbf{F}^{1,n}$. $\square$

---

• *Suppose $A$ is an m-by-n matrix with $A \neq 0$.*
*Prove that the rank of $A$ is $1$ if and only if there exist $(c_1, \ldots, c_m) \in \mathbf{F}^m$ and $(d_1, \ldots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j \cdot d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$.*

**SOLUTION:** Using the notation in CR Factorization.

(a) Suppose $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix}$. $(\ \exists\, c_j, d_k \in \mathbf{F}, \forall j, k\ )$

Then $S_c = \operatorname{span}\left( \begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \ldots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right) = \operatorname{span}\left( \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right)$.

OR. $S_r = \operatorname{span} \left( \begin{matrix} \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \end{pmatrix}, \\ \begin{pmatrix} c_2 d_1 & \cdots & c_2 d_n \end{pmatrix}, \\ \vdots \\ \begin{pmatrix} c_m d_1 & \cdots & c_m d_n \end{pmatrix} \end{matrix} \right) = \operatorname{span}\left( \begin{pmatrix} d_1 & \ldots & d_n \end{pmatrix} \right)$. Hence the rank of $A$ is 1.

(b) Suppose the rank of $A$ is $\dim S_c = \dim S_r = 1$

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $\quad d'_k = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$. $\quad (\forall j, k\ )$

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$. Letting $d_k = d'_k A_{1,1}$. $\square$

---

**1** *Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of $V$ and $W$, the matrix of $T$ has at least $\dim$ range $T$ nonzero entries.*

**SOLUTION:** Let $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ be bases of $V$ and $W$ respectively. We prove by contradiction.

Suppose $A = \mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$ has at most $(\dim \operatorname{range} T - 1)$ nonzero entries.

Then by Pigeon Hole Principle, at least one of $A_{\cdot,k} = 0$.

Thus there are at most $(\dim \operatorname{range} T - 1)$ nonzero vectors in $Tv_1, \ldots, Tv_n$.

While $\operatorname{range} T = \operatorname{span}(Tv_1, \ldots, Tv_n) \Rightarrow \dim \operatorname{range} T \leq \dim \operatorname{range} T - 1$. Hence we get a contradiction. $\square$

---

**3** *Suppose $V$ and $W$ are finite-dim and $T \in \mathcal{L}(V, W)$.*
*Prove that there exist a basis of $V$ and a basis of $W$ such that*
[ letting $A = \mathcal{M}(T)$ with respect to these bases ],
$A_{k,k} = 1, A_{i,j} = 0$, *where* $1 \leq k \leq \dim$ *range* $T, i \neq j$.

**SOLUTION:**

Let $R = (Tv_1, \ldots, Tv_n)$ be a basis of range $T$, extend it to the basis of $W$ as $(Tv_1, \ldots, Tv_n, w_1, \ldots, w_p)$.

Let $\mathcal{K}_R = \operatorname{span}(v_1, \ldots, v_n)$. Let $(u_1, \ldots, u_m)$ be a basis of null $T$.

Then $(v_1, \ldots, v_n, u_1, \ldots, u_m)$ is the basis of $V$.

Thus $T(v_k) = Tv_k, T(u_j) = 0 \Rightarrow A_{k,k} = 1, A_{i,j}$ for each $k \in \{1, \ldots, \dim \operatorname{range} T\}$ and $j \in \{1, \ldots, m\}$. $\square$

---

**4** *Suppose $(v_1, \ldots, v_m)$ is a basis of $V$ and $W$ is finite-dim. Suppose $T \in \mathcal{L}(V, W)$.*
*Prove that there exists a basis $(w_1, \ldots, w_n)$ of $W$ such that all entries in the first column of $A = \mathcal{M}(T, (v_1, \ldots, v_m), (w_1, \ldots, w_n))$ are $0$ except for possibly a $1$ in the first row, first column.*

**SOLUTION:** If $Tv_1 = 0$, then we are done. Otherwise, extend $(Tv_1)$ to a basis of $W$, as desired. $\square$

---

**5** *Suppose $(w_1,\dots,w_n)$ is a basis of $W$ and $V$ is finite-dim. Suppose $T \in \mathcal{L}(V,W)$. Prove that there exists a basis $(v_1,\dots,v_m)$ of $V$ such that all entries in the first row of $\mathcal{M}(T,(v_1,\dots,v_m),(w_1,\dots,w_n))$ are $0$ except for possibly a $1$ in the first row, first column.*

**SOLUTION:**

Let $(u_1,\dots,u_m)$ be a basis of $V$. If $A_{1,\cdot} = 0$, then let $v_i = u_i$ for each $i = 1,\dots,n$, we are done.

Otherwise, $\begin{pmatrix} A_{1,1} & \cdots & A_{1,m}\end{pmatrix} \neq 0$, choose one $A_{1,k} \neq 0$.

Let $v_1 = \dfrac{u_k}{A_{1,k}}$; $\quad v_j = u_{j-1} - A_{1,j-1}v_1 \quad$ for $j = 2,\dots,k$;

$\qquad v_i = u_i - A_{1,i}v_1 \qquad$ for $i = k+1,\dots,n$. $\square$

---

**6** *Suppose $V$ and $W$ are finite-dim and $T \in \mathcal{L}(V,W)$. Prove that* $\dim$ *range* $T = 1$ *if and only if there exist a basis of $V$ and a basis of $W$ such that with respect to these bases, all entries of $A = \mathcal{M}(T)$ equal $1$.*

**SOLUTION:**

Denote the bases of $V$ and $W$ by $B_V = (v_1,\dots,v_n)$ and $B_W = (w_1,\dots,w_m)$ respectively.

(a) Suppose $B_V, B_W$ are the bases such that all entries of $A$ equal $1$.

Then $Tv_i = w_1 + \cdots + w_m$ for all $i = 1,\dots,n$. Hence $\dim \operatorname{range} T = 1$.

(b) Suppose $\dim \operatorname{range} T = 1$. Then $\dim \operatorname{null} T = \dim V - 1$.

Let $(u_2,\dots,u_n)$ be a basis of null $T$. Extend it to a basis of $V$ as $(u_1,u_2,\dots,u_n)$.

Let $w_1 = Tv_1 - w_2 - \cdots - w_m$. Extend it to $B_W$ the basis of $W$.

Let $v_1 = u_1, v_i = u_1 + u_i$. Extend it to $B_V$ the basis of $V$. $\square$

---

**12** *Give an example of 2-by-2 matrices $A$ and $B$ such that $AB \neq BA$.*

**SOLUTION:** $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$; $\quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$.

---

**13** *Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose $A, B, C, D, E$ and $F$ are matrices whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense. Explain why $AB + AC$ and $DF + EF$ both make sense and prove that.*

**SOLUTION:** Using [3.36], [3.43].

(a) Left distributive: Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Because $[A(B+C)]_{j,k} = \sum\limits_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum\limits_{r=1}^{n}(A_{j,r}B_{r,k} + A_{j,r}C_{r,k})$.

Hence we conclude that $A(B+C) = AB + AC$.

OR. Let $(e_1,\dots,e_M)$ be the standard basis of $\mathbf{F}^M$, where $M = \max\{m,n,p\}$.

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum\limits_{j=1}^{m} A_{j,k}e_j$ for each $k = 1,\dots,n$. Then $\mathcal{M}(T) = A$.

Similarly, define $S, R$ such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

Thus $T(S+R) = TS + TR$

$\Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR)$

$\Rightarrow \mathcal{M}(T)[\mathcal{M}(S) + \mathcal{M}(R)] = \mathcal{M}(T)\mathcal{M}(S) + \mathcal{M}(T)\mathcal{M}(R)$

$\Rightarrow A(B+C) = AB + AC$.

(b) Right distributive: Similarly.

Suppose $\mathcal{M}(T) = D, \mathcal{M}(S) = E, \mathcal{M}(R) = F$.

Then $(T+S)R = TR + SR$

$\Rightarrow \mathcal{M}((T+S)R) = \mathcal{M}(TR) + \mathcal{M}(SR)$

$\Rightarrow [\mathcal{M}(T) + \mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)\mathcal{M}(R) + \mathcal{M}(S)\mathcal{M}(R)$

$\Rightarrow (D+E)F = DF + EF$. $\square$

---

**14** *Prove that matrix multiplication is associative. In other words, suppose $A, B$ and $C$ are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that $(AB)C = A(BC)$.*

*Try to find a clean proof that illustrates the following quote from Emil Artin:*

"*It is my experience that proofs involving matrices can be shortened by* 50% *if one throws the matrices out.*"

**SOLUTION:**

Because $[(AB)C]_{j,k} = (AB)_{j,\cdot}C_{\cdot,k} = \sum\limits_{s=1}^{n}(A_{j,s}B_{s,\cdot})C_{\cdot,k} = \sum\limits_{s=1}^{n} A_{j,s}(B_{s,\cdot}C_{\cdot,k}) = \sum\limits_{s=1}^{n} A_{j,s}(BC)_{s,k} = A(BC)_{j,k}$

Hence we conclude that $(AB)C = A(BC)$.

OR. Suppose $A \in \mathbf{F}^{m,n}, B \in \mathbf{F}^{n,p}, C \in \mathbf{F}^{p,s}$.

Let $(e_1, \ldots, e_M)$ be the standard basis of $\mathbf{F}^M$, where $M = \max\{m, n, p, s\}$.

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum\limits_{j=1}^{m} A_{j,k}e_j$ for each $k = 1, \ldots, n$. Then $\mathcal{M}(T) = A$.

Similarly, define $S, R$ such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

Hence $(TS)R = T(SR) \Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR))$

$$\Rightarrow [\mathcal{M}(T)\mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)[\mathcal{M}(S)\mathcal{M}(R)]$$

$$\Rightarrow (AB)C = A(BC). \ \square$$

---

**15** *Suppose $A$ is an n-by-n matrix and $1 \leq j, k \leq n$. Show that the entry in row $j$, column $k$, of $A^3$ (which is defined to mean $AAA$) is $\sum\limits_{p=1}^{n}\sum\limits_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.*

**SOLUTION:** $(AAA)_{j,k} = (AA)_{j,\cdot}A_{\cdot,k} = \sum\limits_{p=1}^{n}(A_{j,p}A_{p,\cdot})A_{\cdot,k} = \sum\limits_{p=1}^{n}\sum\limits_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.

$$\begin{aligned} OR.\quad (AAA)_{j,k} &= \sum_{r=1}^{n}(AA)_{j,r}A_{r,k} = \sum_{r=1}^{n}(\sum_{p=1}^{n} A_{j,p}A_{p,r})A_{r,k} \\ &= \sum_{r=1}^{n}(A_{j,1}A_{1,r}A_{r,k} + \cdots + A_{j,n}A_{n,r}A_{r,k}) \\ &= A_{j,1}\sum_{r=1}^{n} A_{1,r}A_{r,k} + \cdots + A_{j,n}\sum_{r=1}^{n} A_{n,r}A_{r,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}. \ \square \end{aligned}$$

---

**ENDED**

# 3·D

• *Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that $T^{-1}$ is invertible and $(T^{-1})^{-1} = T$.*

**SOLUTION:**

$$\left.\begin{array}{l} TT^{-1} = (T^{-1})^{-1}T^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = T^{-1}(T^{-1})^{-1} = I \in \mathcal{L}(W) \end{array}\right\} \Rightarrow T = (T^{-1})^{-1}, \text{ by the uniqueness of inverse.} \quad \square$$

---

**1** *Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps.*
*Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.*

**SOLUTION:**

$$\left.\begin{array}{l} (ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(V) \end{array}\right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}, \text{ by the uniqueness of inverse.} \quad \square$$

---

**9** *Suppose $V$ is finite-dim and $S, T \in \mathcal{L}(V)$.*
*Prove that $ST$ is invertible $\Longleftrightarrow$ $S$ and $T$ are invertible.*

**SOLUTION:**

Suppose $S, T$ are invertible. Then $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$. Hence $ST$ is invertible.

Suppose $ST$ is invertible. Let $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T \text{ is injective, } S \text{ is surjective.}$$

Notice that $V$ is finite-dim. Hence $S, T$ are invertible. $\square$

---

**10** *Suppose $V$ is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that $ST = I \Longleftrightarrow TS = I$.*

**SOLUTION:**

Suppose $ST = I$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T \text{ is injective, } S \text{ is surjective.}$$

Notice that $V$ is finite-dim. Thus $T, S$ are invertible.

OR. By Problem (9), $V$ is finite-dim and $ST = I$ is invertible $\Rightarrow S, T$ are invertible.

$S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v$ ( $S$ is invertible ).

OR. $ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T$. 又 $S = S \Rightarrow TS = S^{-1}S = I$.

Reversing the roles of $S$ and $T$, we conclude that $TS = I \Rightarrow ST = I$. $\square$

---

**11** *Suppose $V$ is finite-dim and $S, T, U \in \mathcal{L}(V)$ and $STU = I$.*
*Show that $T$ is invertible and that $T^{-1} = US$.*

**SOLUTION:** Using Problem (9) and (10).

$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I$.

$\Rightarrow U^{-1} = ST, \quad T^{-1} = US, \quad S^{-1} = TU.$ $\square$

---

**12** *Show that the result in Exercise 11 can fail without the hypothesis that $V$ is finite-dim.*

**SOLUTION:**

Let $V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots), T(a_1, \dots) = (0, a_1, \dots), U = I$.

Then $STU = I$ but $T^{-1}$ is not invertible.

---

**13** *Suppose $V$ is finite-dim and $R, S, T \in \mathcal{L}(V)$ are such that $RST$ is surjective. Prove that $S$ is injective.*

**SOLUTION:**

By Problem (1) and (9), Notice that $V$ is finite-dim. Then $RST$ is invertible.
$(RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. $\square$

OR. Let $X = (RST)^{-1}$, $\left| \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is injective, and therefore is invertible.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ is surjective, and therefore is invertible.} \end{array} \right.$

Thus $S = R^{-1}(RST)T^{-1}$ is invertible.

---

**15** *Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists\, A \in \mathbf{F}^{m,n}, Tx = Ax, \forall x \in \mathbf{F}^{n,1}$.*

**SOLUTION:**

Let $E_i \in \mathbf{F}^{n,1}$ for each $i = 1, \ldots, n$ ( where $M = \max\{m, n\}$ ) be such that $(E_i)_{j,1} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Then $(E_1, \ldots, E_n)$ is linearly independent and thus is a basis of $\mathbf{F}^{n,1}$.
Similarly, let $(R_1, \ldots, R_m)$ be a basis of $\mathbf{F}^{m,1}$.

Suppose $T(E_i) = A_{1,i}R_1 + \cdots + A_{m,i}R_m$ for each $i = 1, \ldots, n$. Hence by letting $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$. $\square$

COMMENT: $\mathcal{M}(T) = A$. Conversely it is true as well.

---

- OR (10.A.2) *Suppose $A, B \in \mathbf{F}^{n,n}$. Prove that $AB = I \iff BA = I$.*

**SOLUTION:** Using Problem (10) and (15).

Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Then $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.
Thus $AB = I \Leftrightarrow A(Bx) = x \iff T(Sx) = x \Leftrightarrow TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I$.$\square$

---

- **NOTE FOR [3.60]:** Suppose $(v_1, \ldots, v_n)$ is a basis of $V$ and $(w_1, \ldots, w_m)$ is a basis of $W$.

Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{ix} w_j$; $\quad \delta_{ix} = \begin{cases} 0, & i \neq x \\ 1, & i = x \end{cases}$ $\qquad$ COROLLARY: $E_{l,k}E_{i,j} = \delta_{jl}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. $(\mathcal{E}^{(j,i)})_{l,k} = \begin{cases} 0, & i \neq k \text{ or } j \neq l \\ 1, & i = k \text{ and } j = l \end{cases}$

Because $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$ are isomorphic. And $T = \mathcal{M}^{-1}\mathcal{M}(T), E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

Hence $\forall T \in \mathcal{L}(V, W)$, $\exists!\, A_{i,j} \in \mathbf{F}$ ( $\forall i = \{1, \ldots, m\}, j = \{1, \ldots, n\}$ ), $\mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)}+ & \cdots & +A_{1,n}\mathcal{E}^{(1,n)} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}\mathcal{E}^{(m,1)}+ & \cdots & +A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff T = \begin{pmatrix} A_{1,1}E_{1,1}+ & \cdots & +A_{1,n}E_{n,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}E_{1,m}+ & \cdots & +A_{m,n}E_{n,m} \end{pmatrix}.$$

$$\therefore \mathcal{L}(V, W) = \operatorname{span} \underbrace{\begin{pmatrix} E_{1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \cdots & , E_{n,m} \end{pmatrix}}_{B}; \quad \mathbf{F}^{m,n} = \operatorname{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & , \mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots & , \mathcal{E}^{(m,n)} \end{pmatrix}}_{B_M}.$$

Hence by [2.42] and [3.61], we conclude that $B$ is a basis of $\mathcal{L}(V, W)$ and that $B_M$ is a basis of $\mathbf{F}^{m,n}$.

---

◦ *Suppose $V$ is finite-dim and $S \in \mathcal{L}(V)$. Define $A \in \mathcal{L}(\mathcal{L}(V))$ by $A(T) = ST$ for $T \in \mathcal{L}(V)$.*

(a) *Show that* $\dim \textit{null}\, A = (\dim V)(\dim \textit{null}\, S)$.

(b) *Show that* $\dim \textit{range}\, A = (\dim V)(\dim \textit{range}\, S)$.

**SOLUTION:** Using NOTE FOR [3.60].

Let $(w_1, \dots, w_m)$ be a basis of range $S$, extend it to a basis of $V$ as $(w_1, \dots, w_m, \dots, w_n)$.

Let $v_i \in V$ such that $Sv_i = w_i$ for $m = 1, \dots, m$. Extend $(v_1, \dots, v_m)$ to a basis of $V$ as $(v_1, \dots, v_m, \dots, v_n)$.

Define $E_{i,j} \in \mathcal{L}(V)$ by $E_{i,j}(v_x) = \delta_{ix} w_i$.

Thus $S = E_{1,1} + \cdots + E_{m,m}$; $\quad \mathcal{M}(S, (v_1, \dots, v_n), (w_1, \dots, w_n)) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$.

Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j}(w_x) = \delta_{ix} v_i$.

Let $E_{j,k} R_{i,j} = Q_{i,k}, \quad R_{j,k} E_{i,j} = G_{i,k}$

Because $\forall T \in \mathcal{L}(V), \quad \exists!\, A_{i,j} \in \mathbf{F}\ (\forall i, j = 1, \dots, n), \quad T = \begin{pmatrix} A_{1,1}R_{1,1}+ & \cdots & +A_{1,m}R_{m,1}+ & \cdots & +A_{1,n}R_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}R_{1,m}+ & \cdots & +A_{m,m}R_{m,m}+ & \cdots & +A_{m,n}R_{n,m} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{n,1}R_{1,n}+ & \cdots & +A_{n,m}R_{m,n}+ & \cdots & +A_{n,n}R_{n,n} \end{pmatrix}$.

$$\Rightarrow A(T) = ST = (\sum_{r=1}^{m} E_{r,r})(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i}) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{i,j}Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1}+ & \cdots & +A_{1,m}Q_{m,1}+ & \cdots & +A_{1,n}Q_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}Q_{1,m}+ & \cdots & +A_{m,m}Q_{m,m}+ & \cdots & +A_{m,n}Q_{n,m} \end{pmatrix}.$$

Thus null $A$ = span $\begin{pmatrix} R_{1,m+1}, & \cdots & , R_{n,m+1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ R_{1,n}, & \cdots & , R_{n,n} \end{pmatrix}$, $\quad$ range $A$ = span $\begin{pmatrix} Q_{1,1}, & \cdots & , Q_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ Q_{1,m}, & \cdots & , Q_{n,m} \end{pmatrix}$.

Hence (a) $\dim \text{null}\, A = n \times (n - m)$; $\quad$ (b) $\dim \text{range}\, A = n \times m$. $\quad \square$

• COMMENT: Define $B \in \mathcal{L}(\mathcal{L}(V))$ by $B(T) = TS$ for $T \in \mathcal{L}(V)$.

Similarly, $B(T) = TS = (\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i})(\sum_{r=1}^{m} E_{r,r}) = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{i,j}G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1}+ & \cdots & +A_{1,m}G_{m,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}G_{1,m}+ & \cdots & +A_{m,m}G_{m,m} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{n,1}G_{1,n}+ & \cdots & +A_{n,m}G_{m,n} \end{pmatrix}$.

---

• OR (10.A.1) *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \dots, v_n)$ is a basis of $V$.*
*Prove that $\mathcal{M}(T, (v_1, \dots, v_n))$ is invertible $\Longleftrightarrow$ $T$ is invertible.*

**SOLUTION:** Notice that $\mathcal{M}$ is an isomorphism of $\mathcal{L}(V)$ onto $\mathbf{F}^{n,n}$.

(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I$. $\quad \exists!\, S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$

$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$

$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. $\quad \square$

---

• OR (10.A.4) *Suppose that* $(u_1, \dots, u_n)$ *and* $(v_1, \dots, v_n)$ *are bases of* $V$.
*Let* $T \in \mathcal{L}(V)$ *be such that* $Tv_k = u_k$ *for each* $k = 1, \dots, n$.
*Prove that* $A = \mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)) = B$.

**SOLUTION:**

$\forall\, k \in \{1, \dots, n\}, Iu_k = u_k = B_{1,k}v_1 + \cdots + B_{n,k}v_n = Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n \Rightarrow A = B.$ $\square$

OR. Note that $\mathcal{M}(T, (v_1, \dots, v_n), (u_1, \dots, u_n))$ is the identity matrix.

$$A = \mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n))\, \underbrace{\mathcal{M}(T, (v_1, \dots, v_n), (u_1, \dots, u_n))}_{=I} = B. \quad \square$$

• COMMENT: Denote $\mathcal{M}(T, (u_1, \dots, u_n))$ by $A'$.

$u_k = Iu_k = B_{1,k}v_1 + \cdots + B_{n,k}v_n, \ \forall\, k \in \{1, \dots, n\}.$

又 $Tu_k = T(B_{1,k}v_1 + \cdots + B_{n,k}v_n) = B_{1,k}u_1 + \cdots + B_{n,k}u_n = A'_{1,k}u_1 + \cdots + A'_{n,k}u_n \Rightarrow A' = B.$

OR. $A' = \mathcal{M}(T, (u_1, \dots, u_n)) = \mathcal{M}(T, (v_1, \dots, v_n), (u_1, \dots, u_n))\mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)) = B.$

---

**16** *Suppose* $V$ *is finite-dim and* $S \in \mathcal{L}(V)$.
*Prove that* $\exists\, \lambda \in \mathbf{F}, S = \lambda I \iff ST = TS$ *for every* $T \in \mathcal{L}(V)$.

**SOLUTION:** Using the notation and result in $(\circ)$.

Suppose $S = \lambda I$. Then $ST = TS = \lambda T$ for every $T \in \mathcal{L}(V)$. Conversely, if $S = 0$, then we are done.

Suppose $S \neq 0, ST = TS, \forall T \in \mathcal{L}(V)$. Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, (v_1, \dots, v_1)) = \mathcal{M}(I, (w_1, \dots, w_n), (v_1,$

Then $\forall k \in \{m+1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $n = \dim V = \dim \operatorname{range} S = m$.

Note that $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$. Thus $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \cdots + a_{n,i}v_n)$. Where:

$a_{i,j} = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n;$

For each $j$, for all $i$. Thus $a_{i,i} = a_{k,k} = \lambda, \forall\, k \neq i$.

Hence $w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} = \mathcal{M}(\lambda I, (v_1, \dots, v_n)) \Rightarrow S = \mathcal{M}^{-1}(\mathcal{M}(\lambda I)) = \lambda I.$ $\square$

---

• OR (10.A.3) *Suppose* $V$ *is finite-dim and* $T \in \mathcal{L}(V)$.
*Prove that* $T$ *has the same matrix with respect to every basis of* $V$
*if and only if* $T$ *is a scalar multiple of the identity operator.*

**SOLUTION:** [ Compare with the first solution of Problem (16) in (3.A) ]

Suppose $T = \lambda I$ for some $\lambda \in \mathbf{F}$. Then $T$ has the same matrix with respect to every basis of $V$.

Conversely, if $T = 0$, then we are done; Suppose $T \neq 0$. And $v$ is a nonzero vector in $V$.

Assume that $(v, Tv)$ is linearly independent.

Extend $(v, Tv)$ to a basis of $V$ as $(v, Tv, u_3, \dots, u_n)$. Let $B = \mathcal{M}(T, (v, Tv, u_3, \dots, u_n))$.

$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \forall i \neq 2.$

By assumption, $A = \mathcal{M}(T, (v, w_2, \dots, w_n) = B$ for any basis $(v, w_2, \dots, w_n)$.

Then $A_{2,1} = 1, A_{i,1} = 0\,(\,i \neq 2\,) \Rightarrow Tv = w_2$,

which is not true if we let $w_2 = u_3, w_3 = Tv, w_j = u_j\,(\,j = 4, \dots, n\,)$. Contradicts.

Hence $(v, Tv)$ is linearly dependent $\Rightarrow \forall v \in V, \exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show that $\lambda_v$ is independent of $v$, that is,

to show that for any two nonzero distinct vectors $v, w \in V, \lambda_v = \lambda_w$. Thus $T = \lambda I$ for some $\lambda \in \mathbf{F}$.

$$\left.\begin{aligned} &(v, w) \text{ is linearly independent} \Rightarrow T(v + w) = \lambda_{v+w}(v + w) \\ &\qquad = \lambda_{v+w}v + \lambda_{v+w}w \\ &\qquad = \lambda_v v + \lambda_w w = Tv + Tw \Rightarrow \lambda_{v+w} = \lambda_v = \lambda_w \\ &(v, w) \text{ is linearly dependent, } w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \Rightarrow \lambda_v = \lambda_w \end{aligned}\right\} \Rightarrow \square$$

---

**17** *Suppose $V$ is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

*A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.*

**SOLUTION:** Using NOTE FOR [3.60]. Let $(v_1, \dots, v_n)$ be a basis of $V$. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.

Then for any $E_{i,j} \in \mathcal{E}$, $(\forall x, y = 1, \dots, n)$, by assumption, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.

Again, $E_{y,x'}, E_{y',x} \in \mathcal{E}$ for all $x', y', x, y = 1, \dots, n$. Thus $\mathcal{E} = \mathcal{L}(V)$. $\square$

---

**18** *Show that $V$ and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces.*

**SOLUTION:**

Define $\varphi \in \mathcal{L}(V, \mathcal{L}(\mathbf{F}, V))$ by $\varphi(v) = \varphi_v$; where $\varphi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\varphi_v(\lambda) = \lambda v$.

$$\left.\begin{array}{l} \text{(a) } \varphi(v) = \varphi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \varphi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0. \text{ Hence } \varphi \text{ is injective.} \\ \text{(b) } \forall \psi \in \mathcal{L}(\mathbf{F}, V), \text{ let } v = \psi(1) \Rightarrow \psi(\lambda) = \lambda v = \varphi_v(\lambda), \forall \lambda \in \mathbf{F} \\ \qquad \Rightarrow \psi = \varphi_{\psi(1)} = \varphi(\psi(1)). \text{ Hence } \varphi \text{ is surjective.} \end{array}\right\} \Rightarrow \varphi \text{ is an isomorphism.} \quad \square$$

---

• *Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that $q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$.*

**SOLUTION:**

Note that $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$.

Define $T_n : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.

As can be easily checked, $T_n$ is an operator.

Now how can we prove that $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3) = 0 \iff p = 0$?

Hence $T_n$ is injective and therefore is surjective.

Thus $\forall q \in \mathcal{P}(\mathbf{R}), \deg q = m,\ \exists\, p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$.

---

**19** *Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is injective.* $\deg Tp \leq \deg p$ *for every nonzero $p \in \mathcal{P}(\mathbf{R})$.*

(a) *Prove that $T$ is surjective.*

(b) *Prove that for every nonzero $p$,* $\deg Tp = \deg p$.

**SOLUTION:**

(a) $T$ is injective $\iff T|_{\mathcal{P}_n(\mathbb{R})} : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ is injective for any $n \in \mathbf{N}^+$

$\iff T|_{\mathcal{P}_n(\mathbb{R})}$ is surjective for any $n \in \mathbf{N}^+ \iff T$ is surjective.

(b) Using mathematical induction.

(i) $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$.

$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$.

(ii) Suppose $\deg f = \deg Tf$ for all $f \in \mathcal{P}_n(\mathbf{R})$. Then suppose $\deg g = n + 1, g \in \mathcal{P}_{n+1}(\mathbf{R})$.

Assume that $\deg Tg < \deg g$ ( $\Rightarrow \deg Tg \leq n, Tg \in \mathcal{P}_n(\mathbf{R})$ ).

Then by (a), $\exists\, f \in \mathcal{P}_n(\mathbf{R}), T(f) = (Tg)$. 又 $T$ is injective $\Rightarrow f = g$.

While $\deg f = \deg Tf = \deg Tg < \deg g$. Contradicts the assumption.

Hence $\deg Tp = \deg p$ for all $p \in \mathcal{P}_{n+1}(\mathbf{R})$.

Thus $\deg Tp = \deg p$ for all $p \in \mathcal{P}(\mathbf{R})$. $\square$

---

• *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_m)$ is a list in $V$ such that $(Tv_1, \ldots, Tv_m)$ spans $V$. Prove that $(v_1, \ldots, v_m)$ spans $V$.*

**SOLUTION:**

$V = \operatorname{span}(Tv_1, \ldots, Tv_m) \Rightarrow T$ is surjective, 又 $V$ is finite-dim $\Rightarrow T$ is invertible $\Rightarrow T^{-1}$ is invertible.

$\forall v \in V,\ \exists\, a_i \in \mathbf{F}, v = a_1 Tv_1 + \cdots + a_n Tv_n$

$$\Rightarrow T^{-1}v = a_1 v_1 + \cdots + a_n v_n \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \ldots, v_n) \text{ 又 } \operatorname{range} T^{-1} = V. \quad \square$$

OR. Reduce $(Tv_1, \ldots, Tv_n)$ to a basis of $V$ as $(Tv_{\alpha_1}, \ldots, Tv_{\alpha_m})$, where $m = \dim V$ and $\alpha_i \in \{1, \ldots, m\}$. Then $(v_{\alpha_1}, \ldots, v_{\alpha_m})$ is linearly independet of length $m$, therefore is a basis of $V$, contained in the list $(v_1, \ldots, v_m)$. $\square$

---

• *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.*

$$\left.\begin{array}{l} (Tv_1, \ldots, Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1, \ldots, v_n) \text{ of } V \iff T \text{ is surjective} \\ (Tv_1, \ldots, Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1, \ldots, v_n) \text{ of } V \iff T \text{ is injective} \end{array}\right\} \iff T \text{ is invertible.}$$

---

**2** *Suppose $V$ is finite-dim and $\dim V > 1$. Prove that the set of noninvertible operators on $V$ is not a subspace of $\mathcal{L}(V)$.*

**SOLUTION:**

Suppose $\dim V = n > 1$. Let $(v_1, \ldots, v_n)$ be a basis of $V$.

Define $S, T \in \mathcal{L}(V)$ by $S(a_1v_1 + \cdots + a_nv_n) = a_1v_1$ and $T(a_1v_1 + \cdots + a_nv_n) = a_2v_1 + \cdots + a_nv_n$.

Hence $S + T = I$ is invertible.

Thus the set of noninvertible linear maps in $\mathcal{L}(V)$ is not closed under addition and therefore is not a subspace. $\square$

COMMENT: If $\dim V = 1$, then the set of noninvertible operators on $V$ equals $\{0\}$, which is a subspace of $\mathcal{L}(V)$.

---

**3** *Suppose $V$ is finite-dim, $U$ is a subspace of $V$, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible $T \in \mathcal{L}(V, V)$ such that $Tu = Su$ for every $u \in U$ if and only if $S$ is injective.*

**SOLUTION:** [ *Compare this with (3.A.11).* ]

(a) $Tu = Su$ for every $u \in U \Rightarrow u = T^{-1}Su \Rightarrow S$ is injective.

(b) Suppose $(u_1, \ldots, u_m)$ be a basis of $U$ and $S$ is injective $\Rightarrow (Su_1, \ldots, Su_m)$ is linearly independent in $V$.
Extend these to bases of $V$ as $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ and $(Su_1, \ldots, Su_m, w_1, \ldots, w_n)$.
Define $T \in \mathcal{L}(V)$ by $T(u_i) = Su_i;\ \ Tv_j = w_j$, for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$.

---

**4** *Suppose that $W$ is finite-dim and $S, T \in \mathcal{L}(V, W)$. Prove that null $S =$ null $T$ $(= U) \iff S = ET,\ \exists$ invertible $E \in \mathcal{L}(W)$.*

**SOLUTION:**

Define $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i,\ \ E(w_j) = x_j$, for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. Where:

Let $(Tv_1, \ldots, Tv_m)$ be a basis of range $T$, extend it to a basis of $W$ as $(Tv_1, \ldots, Tv_m, w_1, \ldots, w_n)$.
Let $(u_1, \ldots, u_n)$ be a basis of $U$. Then by (3.B.TIPS), $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ is a basis of $V$.
又 null $S =$ null $T \Rightarrow V = \operatorname{span}(v_1, \ldots, v_m) \oplus \operatorname{null} S \Rightarrow \operatorname{span}(Sv_1, \ldots, Sv_m) = \operatorname{range} S$.
And $\dim \operatorname{range} T = \dim \operatorname{range} S = \dim V - \operatorname{null} U = m$. Hence $(Sv_1, \ldots, Sv_m)$ is a basis of range $S$.
Thus we let $(Sv_1, \ldots, Sv_m, x_1, \ldots, x_n)$ be a basis of $W$.

Hence $E$ is invertible and $S = ET$.

Conversely, $S = ET \Rightarrow \operatorname{null} S = \operatorname{null} ET$.

Then $v \in \operatorname{null} ET \iff ET(v) = 0 \iff Tv = 0 \iff v \in \operatorname{null} T$. Hence null $ET =$ null $T =$ null $S$. $\square$

---

**5** *Suppose that $W$ is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
*Prove that range $S$ = range $T$ $(= R)$ $\Longleftrightarrow S = TE,\ \exists$ invertible $E \in \mathcal{L}(V)$.*

**SOLUTION:**

Define $E \in \mathcal{L}(V)$ as $E: \ v_i \mapsto r_i\ ; \quad u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

| | |
|---|---|
| Let $(Tv_1, \dots, Tv_m)$ and $(Sr_1, \dots, Sr_m)$ be bases of $R$ such that $\forall\, i, Tv_i = Sr_i$.<br>Let $(u_1, \dots, u_n)$ and $(s_1, \dots, s_n)$ be bases of null $T$ and null $S$ respectively.<br>Thus $(v_1, \dots, v_m, u_1, \dots, u_n)$ and $(r_1, \dots, r_m, s_1, \dots, s_n)$ are bases of $V$. | Hence $E$ is invertible and $S = TE$. |

Conversely, $S = TE \Rightarrow$ range $S$ = range $TE$.
Then $w \in$ range $S \Longleftrightarrow \exists\, v \in V, Sv = TE(v) = T(E(v)) = w \in$ range $T$. Hence range $S$ = range $T$. $\square$

---

**6** *Suppose $V$ and $W$ are finite-dim and $S, T \in \mathcal{L}(V, W)$.* [dim null $S$ = dim null $T = n$]
*Prove that $S = E_2 T E_1,\ \exists$ invertible $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W) \Longleftrightarrow$ dim null $S$ = dim null $T$.*

**SOLUTION:**

Define $E_1: \quad v_i \mapsto r_i\ ; \quad u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.
Define $E_2: Tv_i \mapsto Sr_i\ ;\ x_j \mapsto y_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

| | |
|---|---|
| Let $(Tv_1, \dots, Tv_m)$ and $(Sr_1, \dots, Sr_m)$ be bases of range $T$ and range $S$.<br>Let $(u_1, \dots, u_n)$ and $(s_1, \dots, s_n)$ be bases of null $T$ and null $S$ respectively.<br>Thus $(v_1, \dots, v_m, u_1, \dots, u_n)$ and $(r_1, \dots, r_m, s_1, \dots, s_n)$ are bases of $V$.<br>Extend $(Tv_1, \dots, Tv_m)$ and $(Sr_1, \dots, Sr_m)$ to bases of $W$ as<br>$(Tv_1, \dots, Tv_m, x_1, \dots, x_p)$ and $(Sr_1, \dots, Sr_m, y_1, \dots, y_p)$. | Thus $E_1, E_2$ are invertible and $S = E_2 T E_1$. |

Conversely, $S = E_2 T E_1 \Rightarrow$ dim null $S$ = dim null $E_2 T E_1$.
$v \in$ null $E_2 T E_1 \Longleftrightarrow E_2 T E_1(v) = 0 \Longleftrightarrow TE_1(v) = 0$. Hence null $E_2 T E_1$ = null $TE_1$ = null $S$.
又 By (3.B.22.COROLLARY), $E$ is invertible $\Rightarrow$ dim null $TE_1$ = dim null $T$ = dim null $S$. $\square$

---

**8** *Suppose $V$ is finite-dim and $T: V \to W$ is a surjective linear map of $V$ onto $W$.*
*Prove that there is a subspace $U$ of $V$ such that $T|_U$ is an isomorphism of $U$ onto $W$.*
*$T|_U$ is the function whose domain is $U$, with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.*

**SOLUTION:**

$T$ is surjective $\Rightarrow$ range $T = W \Rightarrow$ dim range $T$ = dim $W$ = dim $V$ − dim null $T$.
Let $(w_1, \dots, w_m)$ be a basis of range $T = W \Rightarrow \forall w_i, \exists!\, v_i \in V, Tv_i = w_i$.
$\Rightarrow (v_1, \dots, v_m)$ is a basis of $\mathcal{K}$. Thus dim $\mathcal{K}$ = dim $W$.
Thus $T|_{\mathcal{K}}$ maps a basis of $\mathcal{K}$ to a basis of range $T = W$. Denote $\mathcal{K}$ by $U$. $\square$

---

• *Suppose $V$ and $W$ are finite-dim and $U$ is a subspace of $V$.*
*Let $\mathcal{E} = \{\, T \in \mathcal{L}(V, W) : U \subseteq \text{null}\, T \,\}$.*
(a) *Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.*
(b) *Find a formula for* dim $\mathcal{E}$ *in terms of* dim $V$, dim $W$ *and* dim $U$.
*Hint: Define $\Phi: \mathcal{L}(V, W) \to L(U, W)$ by $\Phi(T) = T|_U$. What is null $\Phi$? What is range $\Phi$?*

**SOLUTION:**

(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, Su = Tu = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.
(b) Define $\Phi$ as in the hint.

$T \in$ null $\Phi \Longleftrightarrow \Phi(T) = 0 \Longleftrightarrow \forall u \in U, Tu = 0 \Longleftrightarrow T \in \mathcal{E}$. Hence null $\Phi = \mathcal{E}$.
$S \in \mathcal{L}(U, W) \Rightarrow \exists T \in \mathcal{L}(V, W), \Phi(T) = S$, by (3.B.11) $\Rightarrow S \in$ range $T$. Hence range $\Phi = \mathcal{L}(U, W)$.
Thus dim null $\Phi$ = dim $\mathcal{E}$ = dim $\mathcal{L}(V, W)$ − dim range $\Phi$ = (dim $V$ − dim $U$) dim $W$. $\square$

OR. Extend $(u_1, \ldots, u_m)$ a basis of $U$ to $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ a basis of $V$. Let $p = \dim W$.
( See NOTE FOR [3.60])

$$\forall T \in \mathcal{E}, k \in \{1, \ldots, m\}, TE_{k,k} = 0 \Rightarrow \underbrace{\operatorname{span}\left\{\begin{matrix} E_{1,1}, & \cdots & , E_{m,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,p}, & \cdots & , E_{m,p} \end{matrix}\right\}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

$$又\ W = \operatorname{span}\left\{\begin{matrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{matrix}\right\} \subseteq \mathcal{E}. \text{ Where } \mathcal{L}(V,W) = R \oplus W \Rightarrow \mathcal{L}(V,W) = R + \mathcal{E}.$$

Then $\dim \mathcal{E} = \dim \mathcal{L}(V,W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W.$ $\square$

---

**ENDED**

# 3·E

**2** *Suppose $V_1, \ldots, V_m$ are vec-sps such that $V_1 \times \cdots \times V_m$ is finite-dim.*
*Prove that every $V_j$ is finite-dim.*

**SOLUTION:** Denote $V_1 \times \cdots \times V_m$ by $U$. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by $U_i$.

Let $(v_1, \ldots, v_M)$ be a basis of $U$. Note that $\forall u_i \in V_i, \in U_i \subseteq U$, for each $i$.

$$\left.\begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U) \text{ by } R_i(u_i) = (0, \ldots, 0, u_i, 0, \ldots, 0). \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \ldots, u_i, \ldots, u_m) = u_i \end{array}\right\} \Rightarrow S_i|_{U_i} = R_i^{-1}|_{U_i}.$$

Thus $U_i$ and $V_i$ are isomorphic. 又 $U_i$ is a subspace of a finite-dim vec-sp $U$. $\square$

---

**3** *Give an example of a vec-sp $V$ and its two subspaces $U_1, U_2$ such that $U_1 \times U_2$ and $U_1 + U_2$ are isomorphic but $U_1 + U_2$ is not a direct sum.*

**SOLUTION:**

NOTE *that at least one of $U_1, U_2$ must be infinite-dim.*

*For if not, $U_1 \times U_2$ is finite-dim and* $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$.

*And $V$ must be infinite-dim. For if not, both $U_1$ and $U_2$ are finite-dim subspaces.*

Let $V = \mathbf{F}^\infty = U_1, U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$.

$$\left.\begin{array}{l} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T((x_1, x_2, \cdots), (x, 0, \cdots)) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = ((x_1, x_2, \cdots), (x, 0, \cdots)) \end{array}\right\} \Rightarrow S = T^{-1}. \ \square$$

---

**4** *Suppose $V_1, \ldots, V_m$ are vec-sps.*
*Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic.*

**SOLUTION:** Using the notations in Problem (2). Note that $T(u_1, \ldots, u_m) = T(u_1, 0, \ldots, 0) + \cdots + T(0, \ldots, u_m)$.

$$\left.\begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \ldots, T_m) \text{ by } \varphi(T) = (TR_1, \ldots, TR_m). \\ \text{Define } \psi : (T_1, \ldots, T_m) \mapsto T \text{ by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array}\right\} \Rightarrow \psi = \varphi^{-1}. \square$$

---

**5** *Suppose $W_1, \ldots, W_m$ are vec-sps.*
*Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic.*

**SOLUTION:** Using the notations in Problem (2).

Note that $Tv = (w_1, \ldots, w_m)$. Define $T_i \in \mathcal{L}(V, W_i)$ by $T_i(v) = w_i$.

$$\left.\begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \ldots, T_m) \text{ by } \varphi(T) = (S_1 T, \ldots, S_m T). \\ \text{Define } \psi : (T_1, \ldots, T_m) \mapsto T \text{ by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array}\right\} \Rightarrow \psi = \varphi^{-1}. \square$$

---

**6** *For $m \in \mathbf{N}^+$, define $V^m$ by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove that $V^m$ and $\mathcal{L}(\mathbf{F}^m, V)$ are isomorphic.*

**SOLUTION:**

Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m$.

Suppose $T(v_1, \dots, v_m) = 0$. Then $\forall\, (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$

$\Rightarrow (v_1, \dots, v_m) = 0 \Rightarrow T$ is injective.

Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $\forall\, (b_1, \dots, b_n) \in \mathbf{F}^m$,

$(\,T(\psi(e_1), \dots, \psi(e_m))\,)(b_1, \dots, b_m) = b_1\psi(e_1) + \cdots + b_m\psi(e_m) = \psi(b_1 e_1 + \cdots + b_m e_m) = \psi(b_1, \dots, b_m)$.

Thus $T(\psi(e_1), \dots, \psi(e_m)) = \psi$. Hence $T$ is surjective. $\square$

---

**7** *Suppose $v, x \in V$ (chosen arbitrarily) of which $U$ and $W$ are subspaces.*
*Suppose $v + U = x + W$. Prove that $U = W$.*

**SOLUTION:**

(a) $\forall u \in U,\ \exists\, w \in W, v + u = x + w$, let $u = 0$, getting $v = x + w \Rightarrow v - x \in W$.

(b) $\forall w \in W,\ \exists\, u \in U, v + u = x + w$, let $w = 0$, getting $x = v + u \Rightarrow x - v \in U$.

Thus $\pm(v - x) \in U \cap W \Rightarrow \left\{ \begin{array}{l} u = (x - v) + w \in W \Rightarrow U \subseteq W \\ w = (v - x) + u \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W.$ $\square$

---

• *Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$.*
*Prove that $A$ is a translate of $U \iff \exists\, c \in \mathbf{R}$, $A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$.*
[*Do it in your mind.*]

---

• *Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $U = \{x \in V : Tx = c\}$ is either $\varnothing$ or is a translate of null $T$.*

**SOLUTION:**

If $c \in W$ but $c \notin$ range $T$, then $U = \varnothing$ and we are done.

Suppose $c \in$ range $T$, then $\exists\, u \in V, Tu = c \Rightarrow u \in U$.

Suppose $y \in$ null $T \Rightarrow y + u \in U \iff T(y + u) = Ty + c = c$. Thus $u +$ null $T \subseteq U$. Hence $u +$ null $T = U$,

for if not, suppose $z \notin u +$ null$T$ but $Tz = c (\Leftrightarrow z \in U)$, then $\forall w \in$ null $T, z \neq u + w \Leftrightarrow z - u \notin$ null$T$.

又 $\tilde{T}(z +$ null $T) = \tilde{T}(u +$ null $T) \Rightarrow z +$ null $T = u +$ null $T \Rightarrow z - u \in$ null $T$, contradicts. $\square$

• **COROLLARY:** *The set of solutions to a system of linear equations such as [3.28] is either $\varnothing$ or a translate of the null subspace.*

---

**8** *Prove that a nonempty subset $A$ of $V$ is a translate of some subspace of $V$ if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$.*

**SOLUTION:**

Suppose $A = a + U$, where $U$ is a subspace of $V$. $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$,

$\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + [\lambda(u_1 - u_2) + u_2] \in A$.

Suppose $\lambda v + (1 - \lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $a \in A$ and let $A' = \{x - a : x \in A\}$.

Then $\forall x - a, y - a \in A', \lambda \in \mathbf{F}$,

(I) $\lambda(x - a) = [\lambda x + (1 - \lambda)a] - a \in A'$. Then let $\lambda = 2$.

(II) $\lambda(x - a) + (1 - \lambda)(y - a) = \frac{1}{2}(x - a) + \frac{1}{2}(y - a) = \frac{1}{2}x + (1 - \frac{1}{2})(y) - a \in A'$.

By (I), $2 \times [\frac{1}{2}(x - a) + \frac{1}{2}(y - a)] = (x - a) + (y - a) \in A'$.

Thus $A'$ is a subspace of $V$. Hence $a + A' = \{(x - a) + a : x \in A\} = A$ is a translate. $\square$

---

**9** *Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces $U_1, U_2$ of $V$.*
*Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of $V$ or is $\varnothing$.*

**SOLUTION:** Suppose $v+u_1, w+u_2 \in A_1 \cap A_2 \neq \varnothing$. By Problem (8),
$\forall \lambda \in \mathbf{F}$, $\lambda(v+u_1)+(1-\lambda)(w+u_2) \in A_1$ and $A_2$. Thus $A_1 \cap A_2$ is a translate of some subspace of $V$. $\square$

---

**10** *Prove that the intersection of any collection of translates of subspaces of $V$ is either a translate of some subspace or $\varnothing$.*

**SOLUTION:** Suppose $\{A_\alpha\}_{\alpha\in\Gamma}$ is a collection of translates of subspaces of $V$, where $\Gamma$ is an arbitrary index set.
Suppose $x, y \in \bigcap_{\alpha\in\Gamma} A_\alpha \neq \varnothing$, then by Problem (18), $\forall \lambda \in \mathbf{F}$, $\lambda x + (1-\lambda)y \in A_\alpha$ for every $\alpha \in \Gamma$.
Thus $\bigcap_{\alpha\in\Gamma} A_\alpha$ is a translate of some subspace of $V$. $\square$

---

**11** *Suppose $A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^{m} \lambda_i = 1\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.*

(a) *Prove that $A$ is a translate of some subspace of $V$: By Problem (8),*
$\forall \sum_{i=1}^{m} a_i v_i, \sum_{i=1}^{m} b_i v_i \in A, \lambda \in \mathbf{F}, \quad \lambda \sum_{i=1}^{m} a_i v_i + (1-\lambda)\sum_{i=1}^{m} b_i v_i = (\lambda \sum_{i=1}^{m} a_i + (1-\lambda)\sum_{i=1}^{m} b_i) v_i \in A.$ $\square$

(b) *Prove that if $B$ is a translate of some subspace of $V$ and $\{v_1,\dots,v_m\} \subseteq B$, then $A \subseteq B$.*

(c) *Prove that $A$ is a translate of some subspace of $V$ and* $\dim V < m$.

**SOLUTION:**

(b) Let $v = \lambda_1 v_+ \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on $m$ by $k$.

(i) $k=1, v=\lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.
$k=2, v=\lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1-\lambda_1$. 又 $v_1, v_2 \in B$. By problem (8), $v \in B$.

(ii) $2 \leq k \leq m$, we assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. ($\forall \lambda_i$ such that $\sum_{i=1}^{k} \lambda_i = 1$)
For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. $\forall i = 1,\dots,k$, $\exists\, \mu_i \neq 1$, fix one such $i$ by $\iota$.
Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow (\sum_{i=1}^{k+1} \frac{\mu_i}{1-\mu_\iota}) - \frac{\mu_\iota}{1-\mu_\iota} = 1.$
Let $w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k\, terms}.$
Let $\lambda_i = \frac{\mu_i}{1-\mu_\iota}$ for $i = 1,\dots,\iota-1$; $\lambda_j = \frac{\mu_{j+1}}{1-\mu_\iota}$ for $j = \iota,\dots,k$. Then,

$$\left.\begin{array}{l} \sum_{i=1}^{k} \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1-\mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \ \square$$

(c) $\forall k = 1,\dots,m$, $\forall \lambda_1,\dots,\lambda_{k-1},\lambda_{k+1},\dots,\lambda_m$, let $\lambda_k = 1-\lambda_1-\cdots-\lambda_{k-1}-\lambda_{k+1}-\cdots-\lambda_m$
$\Rightarrow \lambda_1 v_1 + \cdots + \lambda_m v_m$
$= \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} + (1-\lambda_1-\cdots-\lambda_{k-1}-\lambda_{k+1}-\cdots-\lambda_m) v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_m v_m$
$= v_k + \lambda_1(v_1-v_k) + \cdots + \lambda_{k-1}(v_{k-1}-v_k) + \lambda_{k+1}(v_{k+1}-v_k) + \cdots + \lambda_m(v_m-v_k).$
Thus $A = v_k +$ span$(v_1-v_k,\dots,v_{k-1}-v_k,v_{k+1}-v_k,\dots,v_m-v_k)$. $\square$

---

**12** *Suppose $U$ is a subspace of $V$ such that $V/U$ is finite-dim. Prove that is $V$ is isomorphic to $U \times (V/U)$.*

**SOLUTION:** Let $(v_1+U,\dots,v_n+U)$ be a basis of $V/U$. Note that
$\forall v \in V,\ \exists\,!\ a_1,\dots,a_n \in \mathbf{F}, v+U = \sum_{i=1}^{n} a_i(v_i+U) = (\sum_{i=1}^{n} a_i v_i) + U$
$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) = u \in U$ for some $u$; $v = \sum_{i=1}^{n} a_i v_i + u$.
Thus define $\varphi \in \mathcal{L}(V, U\times(V/U))$ by $\varphi(v) = (u, \sum_{i=1}^{n} a_i v_i + U)$
and $\psi \in \mathcal{L}(U\times(V/U), V)$ by $\psi(u, w+U) = u+w$; $w = \sum_{i=1}^{n} b_i v_i + U$.
So that $\psi = \varphi^{-1}$. $\square$

---

• *Suppose $V = U \oplus W$, $(w_1, \ldots, w_m)$ is a basis of $W$.*
*Prove that $(w_1 + U, \ldots, w_m + U)$ is a basis of $V/U$.*

**SOLUTION:** Note that for any $v \in V$,

$\exists\,!\; u \in U, w \in W, v = u + w$ 又 $\exists\,!\; c_i \in \mathbf{F}$ such that $w = \sum_{i=1}^{m} c_i w_i \Rightarrow v = u + \sum_{i=1}^{m} c_i w_i$.

Thus $v + U = \sum_{i=1}^{m} c_i w_i + U \Rightarrow v + U \in \operatorname{span}(w_1 + U, \ldots, w_m + U) \Rightarrow V/U \subseteq \operatorname{span}(w_1 + U, \ldots, w_m + U)$.

Now suppose $a_1(w_1 + U) + \cdots + a_m(w_m + U) = 0 + U \Rightarrow \sum_{i=1}^{m} a_i w_i \in U$ while $U \cap W = \{0\}$.

Then $\sum_{i=1}^{m} a_i w_i = 0 \Rightarrow a_1 = \cdots = a_m = 0$. $\square$

---

**13** *Suppose $(v_1 + U, \ldots, v_m + U)$ is a basis of $V/U$ and $(u_1, \ldots, u_n)$ is a basis of $U$.*
*Prove that $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ is a basis of $V$.*

**SOLUTION:** By Problem (12), $U$ and $V/U$ are finite-dim$\Rightarrow U \times (V/U)$ is finite-dim, so is $V$.

$\dim V = \dim(U \times (V/U)) = \dim U + \dim V/U = m + n$.

OR. Note that for any $v \in V, v + U = \sum_{i=1}^{m} a_i v_i + U,\ \exists\,!\; a_i \in \mathbf{F} \Rightarrow v = \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} b_i v_i,\ \exists\,!\; b_i \in \mathbf{F}$.

$\Rightarrow v \in \operatorname{span}(v_1, \ldots, v_m, u_1, \ldots, u_n) \supseteq V$.

又 Notice that $(\sum_{i=1}^{m} a_i v_i) + U = 0 + U (\Rightarrow \sum_{i=1}^{m} a_i v_i \in U) \iff a_1 = \cdots = a_m = 0$.

Hence $\operatorname{span}(v_1, \ldots, v_m) \cap U = \{0\} \Rightarrow \operatorname{span}(v_1, \ldots, v_m) \oplus U = V$

Thus $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ is linearly independent, so is a basis of $V$. $\square$

---

**14** *Suppose $U = \{(x_1, x_2, \ldots) \in \mathbf{F}^\infty : x_k \neq 0$ for only finitely many $k\}$.*

(a) *Show that $U$ is a subspace of $\mathbf{F}^\infty$. [Do it in your mind]*

(b) *Prove that $\mathbf{F}^\infty/U$ is infinite-dim.*

**SOLUTION:**

For $u = (x_1, \ldots, x_p, \ldots) \in \mathbf{F}^\infty$, denote $x_p$ by $u[p]$. For each $r \in \mathbf{N}^+$.

Define $e_r[p] = \begin{cases} 1, (p-1) \equiv 0 \,(\operatorname{mod} r) \\ 0, \text{otherwise} \end{cases}$, simply $e_r = (1, \underbrace{0, \ldots, 0}_{(p-1)\ times}, 1, \underbrace{0, \ldots, 0}_{(p-1)\ times}, 1, \ldots) \in \mathbf{F}^\infty$.

Choose $m \in \mathbf{N}^+$ arbitrarily.

Suppose $a_1(e_1 + U) + \cdots + a_m(e_m + U) = (a_1 e_1 + \cdots + a_m e_m) + U = 0 + U = 0$.

$\Rightarrow a_1 e_1 + \cdots + a_m e_m = u$ for some $u \in U$.

Then suppose $u = (x_1, \ldots, x_t, 0, \ldots) \Rightarrow u[t + i] = 0, \forall i \in \mathbf{N}^+$,

then let $j = s \cdot m! + 1 \geq t$ ($\exists\; s \in \mathbf{N}^+$) so that $e_1[j] = \cdots = e_m[j] = 1,\ u[j + i] = 0$.

Now we have: $u[j + i] = (\Sigma_{r=1}^{m} a_r e_r)[j + i] = \Sigma_{r=1}^{m} a_r e_r[s \cdot m! + 1 + i] = a_{i_1} + \cdots + a_{i_{\tau(i)}} = 0$,

$\Rightarrow (\Sigma_{r=1}^{m} a_r e_r)[j + i] = a_{i_1} + \cdots + a_{i_{\tau(i)}} = 0$. $(\Delta)$

where $i_1, \ldots, i_{\tau(i)}$ are distinct ordered factors of $i$ ( $1 = i_1 \leq \cdots \leq i_{\tau(i)} = i$ ).

$\Big($ Note that by definition, $e_r[s \cdot m! + 1 + i] = 1 \iff s \cdot m! + i \equiv i \equiv 0 \,(\operatorname{mod} r) \iff r \mid i$. $\Big)$

Let $i' = i_{\tau(i)-1}$. Notice that $i'_l = i_l, \forall l \in \{1, \ldots, \tau(i')\}$; and $\tau(i') = \tau(i) - 1$.

Again by $(\Delta)$, $(\Sigma_{r=1}^{m} a_r e_r)[j + i'] = a_{i'_1} + \cdots + a_{i'_{\tau(i')}} = a_{i_1} + \cdots + a_{i_{\tau(i)-1}} = 0$.

Thus $a_{i_{\tau(i)}} = a_i = 0$ for any $i \in \{1, \ldots, m\}$.

Hence $(e_1, \ldots, e_m)$ is linearly independent in $\mathbf{F}^\infty$, so is $(e_1, \ldots, e_m, \ldots)$, since $m \in \mathbf{N}^+$.

又 $e_i \notin U \Rightarrow (e_1 + U, e_2 + U, \ldots)$ is linearly independent in $\mathbf{F}^\infty/U$. By [2.B.14]. $\square$

---

**15** *Suppose* $\varphi \in \mathcal{L}(V, \mathbf{F}) \setminus \{0\}$. *Prove that* $\dim V/(\text{null}\ \varphi) = 1$.

**Solution:** By [3.91] (d), $\dim \text{range}\ \varphi = 1 = \dim V/(\text{null}\ \varphi)$. □

---

## Note For [3.88, 3.90, 3.91]

For any $W \in \mathcal{S}_V U$, because $V = U \oplus W$. $\forall v \in V,\ \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.

Define $T \in \mathcal{L}(V, W)$ by $T(v) = w_v$. Hence $\text{null}\, T = U$, $\text{range}\, T = W$.

Then $\tilde{T} \in \mathcal{L}(V/\text{null}\, T, W)$ is defined as $\tilde{T}(v + U) = Tv = w_v$.

Thus $\tilde{T}$ is injective ( by [3.91(b)] ) and surjective ( $\text{range}\, \tilde{T} = \text{range}\, T = W$ ),

and therefore is an isomorphism. We conclude that $V/U$ and $W$, namely any vec-sp in $\mathcal{S}_V$, are isomorphic.

---

**16** *Suppose* $\dim V/U = 1$. *Prove that* $\exists\ \varphi \in \mathcal{L}(V, \mathbf{F})$ *such that null* $\varphi = U$ .

**Solution:**

Suppose $V_0$ is a subspace of $V$ such that $V = U \oplus V_0$. Then $V_0$ and $V/U$ are isomorphic. $\dim V_0 = 1$.

Define a linear map $\varphi : v \mapsto \lambda$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. □

---

**17** *Suppose* $V/U$ *is finite-dim.* $W$ *is a subspace of* $V$.

(a) *Show that if* $V = U + W$, *then* $\dim W \geq \dim V/U$.

(b) *Suppose* $\dim W = \dim V/U$ *and* $V = U \oplus W$. *Find such* $W$.

**Solution:**

Let $(w_1, \dots, w_n)$ be a basis of $W$

(a) $\forall v \in V,\ \exists\ u \in U, w \in W$ such that $v = u + w \Rightarrow v + U = w + U$

Then $V/U \subseteq \text{span}(w_1 + U, \dots, w_n + U) \Rightarrow V/U = \text{span}(w_1 + U, \dots, w_n + U)$.

Hence $\dim V/U = \dim \text{span}(w_1 + U, \dots, w_n + U) \leq \dim W$.

(b) Let $W \in \mathcal{S}_V U$. In other words,

reduce $(w_1{+}U, \dots, w_n{+}U)$ to a basis of $V/U$ as $(w_{\alpha_1}{+}U, \dots, w_{\alpha_m}{+}U)$ and let $W = \text{span}\,(w_{\alpha_1}, \dots, w_{\alpha_m})$. □

---

**18** *Suppose* $T \in \mathcal{L}(V, W)$ *and* $U$ *is a subspace of* $V$. *Let* $\pi$ *denote the quotient map.*

*Prove that* $\exists\ S \in \mathcal{L}(V/U, W)$ *such that* $T = S \circ \pi$ *if and only if* $U \subseteq$ *null* $T$.

**Solution:**

(a) Define $S \in \mathcal{L}(V/U, W)$ by $S(v + U) = Tv$. We have to check it is *well-defined*.

Suppose $v_1 + U = v_2 + U$, while $v_1 \neq v_2$.

Then $(v_1 - v_2) \in U \Rightarrow S((v_1 - v_2) + U) = T(v_1 - v_2) = 0 \Rightarrow Tv_1 = Tv_2$. Checked. □

(b) Suppose $\exists\ S \in \mathcal{L}(V/U, W),\ T = S \circ \pi$. Then $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \text{null}\, T$.□

---

**20** *Define* $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ *by* $\Gamma(S) = S \circ \pi (= \pi'(S))$.

(a) *Prove that* $\Gamma$ *is linear:* By [3.9] distributive properties and [3.6]. □

(b) *Prove that* $\Gamma$ *is injective:*

$$\begin{aligned} &\Gamma(S) = 0 \\ \iff &\forall v \in V, S(\pi(v)) = 0 \\ \iff &\forall v + U \in V/U, S(v + U) = 0 \\ \iff &S = 0. \ \square \end{aligned}$$

(c) *Prove that range* $\Gamma$ (= *range* $\pi'$ ) $= \{T \in \mathcal{L}(V, W) : U \subseteq$ *null* $T\}$*:*

By Problem (18). □

---

**Ended**

# 3·F

• By (18) in (3.D) *we know that* $\varphi : V \to \mathcal{L}(\mathbf{F}, V)$ *is an isomorphism. Now we prove that* $(v_1, \ldots, v_m)$ *is linearly independent* $\iff$ $(\varphi(v_1), \ldots, \varphi(v_m))$ *is linearly independent.*

**SOLUTION:**

(a) Suppose $(v_1, \ldots, v_m)$ is linearly independent and $\vartheta \in \text{span}(\varphi(v_1), \ldots, \varphi(v_m))$.

Let $\vartheta = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m)$. Then $\vartheta(1) = 0 = a_1v_1 + \cdots + a_mv_m \Rightarrow a_1 = \cdots = a_m = 0$.

OR Because $\varphi$ is injective. Suppose $a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) = 0 = \varphi(a_1v_1 + \cdots + a_mv_m)$.

Then $a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

Thus $(\varphi(v_1), \ldots, \varphi(v_m))$ is linearly independent.

(b) Suppose $(\varphi(v_1), \ldots, \varphi(v_m))$ is linearly independent and $v \in \text{span}(v_1, \ldots, v_m)$.

Let $v = 0 = a_1v_1 + \cdots + a_mv_m$. Then $\varphi(v) = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

Thus $v_1, \ldots, v_m$ is linearly independent. $\square$

---

**1** *Explain why each linear functional is surjective or is the zero map.*

**SOLUTION:** $\left.\begin{array}{r}\text{For any } \varphi \in V' \text{ and } \varphi \neq 0,\ \exists\, v \in V, \text{ such that } \varphi(v) \neq 0. \text{ (a)} \\ \text{dim range } \varphi = \dim \mathbf{F} = 1. \text{ (b)}\end{array}\right\} \Rightarrow \square$

---

**4** *Suppose* $V$ *is finite-dimensional and* $U$ *is a subspace of* $V$ *such that* $U \neq V$. *Prove that* $\exists\, \varphi \in V'$ *and* $\varphi \neq 0$ *such that* $\varphi(u) = 0$ *for every* $u \in U$.

**SOLUTION:**

Let $(u_1, \ldots, u_m)$ be a basis of $U$, extend to $(u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+n})$ a basis of $V$.

Choose $k \in \{1, \ldots, n\}$ arbitrarily. Define $\varphi \in V'$ by $\varphi(u_i) = \begin{cases} 1, & \text{if } i = m + k. \\ 0, & \text{otherwise.} \end{cases}$

OR: Equivalent to proving that $U^0 \neq \{0\}$. By [3.106], $\dim U^0 = \dim V - \dim U > 0$. $\square$

---

• *Suppose* $T \in \mathcal{L}(V, W)$ *and* $(w_1, \ldots, w_m)$ *is a basis of range*$T$. *Hence* $\forall v \in V,\ Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,\ \ \exists!\, \varphi_1(v), \ldots, \varphi_m(v)$, *thus defining functions* $\varphi_1, \ldots, \varphi_m$ *from* $V$ *to* $\mathbf{F}$. *Show that each* $\varphi_i \in V'$.

**SOLUTION:**

For each $w_i, \exists\, v_i \in V,\ Tv_i = w_i$, getting a linearly independent list $(v_1, \ldots, v_m)$.

Now we have $Tv = a_1Tv_1 + \cdots + a_mTv_m,\ \forall v \in V, \exists!\, a_i \in \mathbf{F}$.

Let $(\psi_1, \ldots, \psi_m)$ be the dual basis of range$T$. Then $(T'(\psi_i))(v) = \psi_i \circ T(v) = a_i$.

Thus letting $\varphi_i = \psi_i \circ T$. $\square$

---

• *Suppose* $\varphi,\ \beta \in V'$. *Prove that null*$\varphi \subseteq$ *null*$\beta$ *if and only if* $\beta = c\varphi.\ \exists\, c \in \mathbf{F}$.

**SOLUTION:** Using (3.B.29, 30)

(a) Suppose null$\varphi \subseteq$ null$\beta$. Choose a $u \notin$ null$\beta$. $V =$ null$\beta \oplus \{au : a \in \mathbf{F}\}$.

If null$\varphi =$ null$\beta$, then let $c = \frac{\beta(u)}{\varphi(u)}$, we are done.

Otherwise, suppose $u' \in$ null$\beta$, but $u' \notin$ null$\varphi$, then $V =$ null$\varphi \oplus \{bu' : b \in \mathbf{F}\}$.

$\forall v \in V, v = w + au = w' + bu',\ \exists!\, w, w' \in$ null$\varphi,\ a, b \in \mathbf{F}$.

Thus $\beta(v) = a\beta(u),\ \varphi(v) = b\varphi(u')$. Let $c = \frac{a\beta(u)}{b\varphi(u')}$. We are done

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$.

If $c = 0$, then null$\beta = V \supseteq$ null$\varphi$, we are done.

Otherwise, $\left.\begin{array}{l}\forall v \in \text{null}\varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \text{null}\varphi \subseteq \text{null}\beta. \\ \forall v \in \text{null}\beta, \beta(v) = 0 = \varphi(v) \Rightarrow \text{null}\beta \subseteq \text{null}\varphi.\end{array}\right\} \Rightarrow \text{null}\varphi = \text{null}\beta$

$\Rightarrow$ null$\varphi \subseteq$ null$\beta$. $\square$

---

**5** *Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic.*

**SOLUTION:** Using notations in (3.E.2).

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to V_1' \times \cdots \times V_m' \\ \quad \text{by } \varphi(T) = (T \circ R_1, \ldots, T \circ R_m) = (R_1'(T), \ldots, R_m'(T)). \\ \text{Define } \psi : V_1' \times \cdots \times V_m' \to (V_1 \times \cdots \times V_m)' \\ \quad \text{by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m = S_1'(T_1) + \cdots + S_m'(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}. \quad \square$$

---

• *Suppose $v_1, \ldots, v_n$ is a basis of $V$ and $\varphi_1, \ldots, \varphi_n$ is the dual basis of $V'$.*

$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = (\varphi_1(v), \ldots, \varphi_n(v)). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

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**35** *Prove that $(\mathcal{P}(\mathbf{R}))'$ and $\mathbf{R}^\infty$ are isomorphic.*

**SOLUTION:**

Define $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{R}))', \mathbf{R}^\infty)$ by $\theta(\varphi) = (\varphi(1), \varphi(x), \ldots, \varphi(x^n), \ldots)$.

Injectivity: $\theta(\varphi) = 0 \Rightarrow \forall x^k$ in the basis $(1, x, \ldots, x^n, \ldots)$ of $\mathcal{P}_n(\mathbf{R})$ for any $n$, $\varphi(x^k) = 0 \Rightarrow \varphi = 0$.

Surjectivity: $\forall\, (a_0, a_1, \ldots, a_n, \ldots) \in \mathbf{F}^\infty$, let $\psi$ be such that $\psi(x^k) = a_k$ and thus $\theta(\psi) = (a_0, a_1, \ldots, a_n, \ldots)$.

Hence $\theta$ is an isomorphism from $(\mathcal{P}(\mathbf{R}))'$ onto $\mathbf{R}^\infty$. $\square$

---

**7** *Suppose $m$ is a positive integer. Show that the dual basis of the basis $(1, x, \ldots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where $\varphi_k = \dfrac{p^{(k)}(0)}{k!}$. Here $p^{(k)}$ denotes the $k^{th}$ derivative of $p$, with the understanding that the $0^{th}$ derivative of $p$ is $p$.*

**SOLUTION:**

For each $j$ and $k$, $(x^j)^{(k)} = \begin{cases} j(j-1)\ldots(j-k+1) \cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\ldots(j-j+1) = j!, & j = k. \\ 0, & j \leq k. \end{cases}$ Then $(x^j)^{(k)}(0) = \begin{cases} 0, & j \neq k. \\ k!, & j = k. \end{cases}$

Thus $\varphi_k = \psi_k$, where $\psi_1, \ldots, \psi_m$ is the dual basis of $(1, x, \ldots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$. $\square$

---

**8** *Suppose $m$ is a positive integer.*

(a) By [2.C.10], $B = (1, x-5, \ldots, (x-5)^m)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) Let $\varphi_k = \frac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \ldots, m$. Then $(\varphi_0, \varphi_1, \ldots, \varphi_m)$ is the dual basis of $B$.

---

**9** *Suppose $v_1, \ldots, v_n$ is a basis of $V$ and $\varphi_1, \cdots, \varphi_n$ is the corresponding dual basis of $V'$. Suppose $\psi \in V'$. Prove that $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.*

**SOLUTION:** $\psi(v) = \psi(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i \psi(v_i) = \sum_{i=1}^{n} \psi(v_i)\varphi_i(v) = [\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n](v) \Rightarrow \square$

COMMENT: For any other basis $(u_1, \ldots, u_n)$ of $V$ and the corresponding dual basis of $(\rho_1, \ldots, \rho_n)$,
$\psi = \rho(u_1)\rho_1 + \cdots + \rho(u_n)\rho_n$.

---

• *Show that the dual map of the identity operator on $V$ is the identity operator on $V'$.*

**SOLUTION:** $I'(\varphi) = \varphi \circ I, \ \forall \varphi \in V'$. $\square$

• *Suppose $W$ is finite-dim and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \Longleftrightarrow T = 0$.*

**SOLUTION:** $T = 0 \Leftrightarrow T'(\varphi) = \varphi \circ T = 0$ for all $\varphi \in V' \Leftrightarrow T' = 0$. $\square$

---

**13** *Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$.
Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ denote the dual basis of the standard basis of $\mathbf{R}^2$ and $\mathbf{R}^3$.*

(a) *Describe the linear functionals $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$*

For any $(x, y, z) \in \mathbf{R}^3$, $(T'(\varphi_1))(x, y, z) = 4x + 5y + 6z$, $(T'(\varphi_2))(x, y, z) = 7x + 8y + 9z$.

(b) *Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of $\psi_1, \psi_2, \psi_3$.*

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$. $\square$

**14** *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for each $x \in \mathbf{R}$.*

(a) *Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe $T'(\varphi) \in \mathcal{P}(\mathbf{R})'$.*

$(T'(\varphi))(p) = [x^2 p(x) + p''(x)]'(4) = [2xp(x) + x^2 p'(x) + p'''(x)](4) = 8p(4) + 16p'(4) + p'''(4)$.

(b) *Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\varphi))(x^3)$.*

$(T'(\varphi))(x^3) = \int_0^1 (x^5 + 6x)\mathrm{d}x = \int_0^1 (\frac{1}{6}x^6 + 3x^2)'\mathrm{d}x = \frac{6}{19}$. $\square$

---

• *Suppose $V$ and $W$ are finite-dim and $T \in \mathcal{L}(V, W)$.*
*Prove that $T$ is invertible if and only if $T' \in \mathcal{L}(W', V')$ is invertible.*

**SOLUTION:** By [3.108] and [3.110]. $\square$

---

**16** *Suppose $V$ and $W$ are finite-dim. Define $\Gamma$ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(L, W)$.*
*Prove that $\Gamma$ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.*

**SOLUTION:**

$V, W$ are finite-dim $\Rightarrow$ dim $\mathcal{L}(V, W) =$ dim $\mathcal{L}(W', V')$. And by [3.101], $\Gamma$ is linear.

又 Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus $T$ is injective $\Rightarrow$ $T$ is invertible. $\square$

---

**17** *Suppose $U \subseteq V$. Explain why $U^0 = \{\varphi \in V' : U \subseteq null\varphi\}$.*

**SOLUTION:** Because for $\varphi \in V'$, $U \subseteq$ null$\varphi \iff \forall u \in U, \varphi(u) = 0$. By definition in [3.102]. $\square$

---

**18** *$U \subseteq V$. We have $U = \{0\} \iff \forall\varphi \in V', U \subseteq null\varphi \iff U^0 = V'$.*

**19** *$U$ is a subspace of $V$. Prove that $U = V \iff U_V^0 = \{0\} = V_V^0$.*

**SOLUTION:**

Suppose $U_V^0 = \{0\}$. Then $U = V$.

Conversely, suppose $U = V$, then $U_V^0 = \{\varphi \in V' : V \subseteq \text{null}\,\varphi\}$, therefore $U_V^0 = \{0\}$.

---

**20, 21** *Suppose $U$ and $W$ are subsets of $V$. Prove that $U \subseteq W \iff W^0 \subseteq U^0$.*

**SOLUTION:**

(a) $U \subseteq W \Rightarrow \forall w \in W, u \in U \cap W = U,\ \ \forall\varphi \in W^0, \varphi(w) = 0 = \varphi(u) \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.

(b) $W^0 \subseteq U^0 \Rightarrow \forall w \in W, u \in U, \varphi(w) = 0 \Rightarrow \varphi(u) = 0$. Then null$\varphi \supseteq W \Rightarrow$ null$\varphi \supseteq U$. Thus $W \supseteq U$. $\square$.

• **COROLLARY:** $W^0 = U^0 \iff U = W$.

---

**22** *Prove that $(U + W)^0 = U^0 \cap W^0$.*

**SOLUTION:**

(a) $\left.\begin{array}{c} U \subseteq U + W \\ W \subseteq U + W \end{array}\right\} \Rightarrow \left.\begin{array}{c} (U + W)^0 \subseteq U^0 \\ (U + W)^0 \subseteq W^0 \end{array}\right\} \Rightarrow (U + W)^0 \subseteq U^0 \cap W^0.$

(b) $\forall\varphi \in U^0 \cap W^0, \varphi(u + w) = 0$, where $u \in U, w \in W \Rightarrow \varphi \in (U + W)^0$. Thus $(U + W)^0 \supseteq U^0 \cap W^0$. $\square$

---

**23** *Prove that $(U \cap W)^0 = U^0 + W^0$.*

**SOLUTION:**

(a) $\left.\begin{array}{c} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array}\right\} \Rightarrow \left.\begin{array}{c} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array}\right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \supseteq U^0 \cap W^0.$

(b) $\forall\varphi \in U^0, \psi \in W^0$ and $\forall v \in U \cap W$, $(\varphi + \psi)(v) = \varphi(v) + \psi(v) = 0$. Thus $U^0 + W^0 \subseteq (U \cap W)^0$. $\square$

---

• **COROLLARY:**

*Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subspaces of $V$.*

*Then* $(\sum_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \bigcap_{\alpha_i \in \Gamma}(V_{\alpha_i}^0)$;

*And* $(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \sum_{\alpha_i \in \Gamma}(V_{\alpha_i}^0)$ .

---

**24** *Suppose $V$ is finite-dim and $U$ is a subspace of $V$.*
*Prove, using the pattern of [3.104], that dim$U$+ dim$U^0$ = dim$V$.*

**SOLUTION:**

Let $(u_1, \ldots, u_m)$ be a basis of $U$, extend to a basis of $V$ as $(u_1, \ldots, u_m, \ldots, u_n)$, and let $(\varphi_1, \ldots, \varphi_m, \ldots, \varphi_n)$ be the dual basis.

(a) Suppose $\varphi \in \text{span}(\varphi_{m+1}, \ldots, \varphi_n)$, then $\exists\, a_i \in \mathbf{F}$, $\varphi = a_{m+1}\varphi_{m+1} + \cdots + a_n\varphi_n$.
For all $u \in U$, $\varphi(u) = 0$. Thus $\varphi \in U^0$, getting $\text{span}(\varphi_{m+1}, \ldots, \varphi_n) \subseteq U^0$.

(b) Suppose $\varphi \in U^0$, then $\exists\, a_i \in \mathbf{F}$, $\varphi = a_1\varphi_1 + \cdots + a_m\varphi_m + \cdots + a_n\varphi_n$.
For all $u_i \in U$, $0 = \varphi(u_i) = \sum\limits_{i=1}^{n} \varphi(u_i) = a_i$. Then $\varphi = a_{m+1}\varphi_{m+1} + \cdots + a_n\varphi_n$.
Thus $\varphi \in \text{span}(\varphi_{m+1}, \ldots, \varphi_n)$, getting $\text{span}(\varphi_{m+1}, \ldots, \varphi_n) \supseteq U^0$.

Hence $\text{span}(\varphi_{m+1}, \ldots, \varphi_n) = U^0$, $\dim U^0 = n - m = \dim V - \dim U$. $\square$

---

**25** *Suppose $U$ is a subspace of $V$. Explain why $U = \{v \in V : \varphi(v) = 0$ for every $\varphi \in U^0\}$.*

**SOLUTION:** Note that $U = \{v \in V : v \in U\}$ is a subspace of $V$ and $\varphi(v) = 0$ for every $\varphi \in U^0 \iff v \in U$. $\square$

---

**26** *Suppose $V$ is finite-dim and $\Omega$ is a subspace of $V'$.*
*Prove that $\Omega = \{v \in V : \forall\varphi \in \Omega, \varphi(v) = 0\}^0$.*

**SOLUTION:** Using the corollary in Problem (20, 21).

Suppose $U = \{v \in V : \forall\varphi \in \Omega, \varphi(v) = 0\}$.

Getting $U^0 = \{v \in V : \forall\varphi \in \Omega, \varphi(v) = 0\}^0$. We need to show that $\Omega = U^0$.

$$\left.\begin{array}{l} \text{(a) } \forall\varphi \in \Omega, v \in U, \varphi(v) = 0 \Rightarrow \varphi \in U^0 \Rightarrow \Omega \subseteq U^0. \\ \text{(b) } v \in U \Leftrightarrow \left\{\begin{array}{l} \forall\varphi \in \Omega, \varphi(v) = 0 \\ \forall\psi \in U^0, \psi(v) = 0 \end{array}\right. \quad \text{Thus } \Omega \supseteq U^0. \end{array}\right\} \Rightarrow \square$$

---

**27** *Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ and null$T'$ = span$(\varphi)$, where $\varphi$ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that range$T$ = $\{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$.*

**SOLUTION:** By Problem (26), $\text{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \psi(p) = 0, \forall\psi \in \text{span}(\varphi)\}^0$,

Hence $\text{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \varphi(p) = p(8) = 0\}^0$, 又 $\text{span}(\varphi) = \text{null}T' = (\text{range}T)^0$.

By the corollary in Problem (20, 21), range$T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$. $\square$

---

**28, 29** *Suppose $V, W$ are finite-dim, $T \in \mathcal{L}(V, W)$.*
(a) *Suppose $\exists\, \varphi \in W'$ such that null$T'$ = span$(\varphi)$. Prove that range$T$ = null$\varphi$.*
(b) *Suppose $\exists\, \varphi \in V'$ such that range$T'$ = span$(\varphi)$. Prove that null$T$ = null$\varphi$.*

**SOLUTION:** Using Problem (26), [3.107] and [3.109].

Because $\text{span}(\varphi) = \{v \in V : \forall\psi \in \text{span}(\varphi), \psi(v) = 0\}^0 = \{v \in V : \varphi(v) = 0\}^0 = (\text{null}\varphi)^0$.

$$\left.\begin{array}{l} \text{(a) } (\text{range}T)^0 = \text{null}T' = \text{span}(\varphi) = (\text{null}\varphi)^0 \iff \text{range}T = \text{null}\varphi. \\ \text{(b) } (\text{null}T)^0 = \text{range}T' = \text{span}(\varphi) = (\text{null}\varphi)^0 \iff \text{null}T = \text{null}\varphi. \end{array}\right\} \Rightarrow \square$$

---

**31** *Suppose $V$ is finite-dim and $(\varphi_1, \ldots, \varphi_n)$ is a basis of $V'$.*
*Show that there exists a basis of $V$ whose dual basis is $(\varphi_1, \ldots, \varphi_n)$.*

**SOLUTION:** Using (3.B.29,30).

For each $\varphi_i$, $\text{null}\varphi_i \oplus \{au_i : a \in \mathbf{F}\} = V$.

Because $\varphi_1, \ldots, \varphi_m$ is linearly independent. $\text{null}\varphi_i \neq \text{null}\varphi_j$ for all $i, j \in \mathbf{N}^+$ such that $i \neq j$.

Thus $(u_1, \ldots, u_m)$ is linearly independent, for if not, then $\exists\, i, j$ such that $\text{null}\varphi_i = \text{null}\varphi_j$, contradicts.

又 $\dim V' = m = \dim V$. Then $(u_1, \ldots, u_m)$ is a basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$. $\square$.

---

• *Suppose dim and $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following three sets are equal to each other.*

(a) *span*$(\varphi_1, \dots, \varphi_m)$

(b) $((null\varphi_1) \cap \cdots \cap (null\varphi_m))^0$

(c) $\{\varphi \in V' : (\ null\varphi_1) \cap \cdots \cap (null\varphi_m) \subseteq null\varphi\}$

**SOLUTION:** By Problem (17), (b) and (c) are equivalent. By Problem (26) and the corollary in Problem (23),

$$\left.\begin{array}{r} ((\text{null}\varphi_1) \cap \cdots \cap (\text{null}\varphi_m))^0 = (\text{null}\varphi_1)^0 + \cdots + (\text{null}\varphi_m)^0. \\ \text{又 span}(\varphi_i) = \{v \in V : \forall \psi \in \text{span}(\varphi_i), \psi(v) = 0\}^0 = (\text{null}\varphi_i)^0. \end{array}\right\} \Rightarrow (a) = (b). \quad \square$$

**30 OR COROLLARY:**

*Suppose V is finite-dim and $\varphi_1, \dots, \varphi_m$ is a linearly independent list in $V'$.*
*Then dim$((null\varphi_1) \cap \cdots \cap (null\varphi_m)) = (dimV) - m$.*

---

**6** *Define $\Gamma : V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$, where $v_1, \dots, v_m \in V$.*

(a) *Show that span$(v_1, \dots, v_m) = V \iff \Gamma$ is injective.*

(b) *Show that $v_1, \dots, v_m$ is linearly independent $\iff \Gamma$ is surjective.*

**SOLUTION:**

(a) Suppose $\Gamma$ is injective. Then let $\Gamma(\varphi) = 0$, getting $\varphi = 0 \Leftrightarrow \text{null}\varphi = V = \text{span}(v_1, \dots, v_m)$.
Suppose $\text{span}(v_1, \dots, v_m) = V$. Then let $\Gamma(\varphi) = 0$, getting $\varphi(v_i) = 0$ for each $i$,
$\text{null}\varphi = \text{span}(v_1, \dots, v_m) = V$, thus $\varphi = 0$, $\Gamma$ is injective.

(b) Suppose $\Gamma$ is surjective. Then let $\Gamma(\varphi_i) = e_i$ for each $i$, where $e_1, \dots, e_m$ is the standard basis of $\mathbf{F}^m$.
Then $\varphi_1, \dots, \varphi_m$ is linearly independent, suppose $a_1v_1 + \cdots + a_mv_m = 0$,
then for each $i$, we have $\varphi_i(a_1v_1 + \cdots + a_mv_m) = a_i = 0$. Thus $v_1, \dots, v_n$ is linearly independent.
Suppose $v_1, \dots, v_m$ is linearly independent. Let $(\varphi_1, \dots, \varphi_m)$ be the dual basis of $\text{span}(v_1, \dots, v_m)$.
Thus for each $(a_1, \dots, a_m) \in \mathbf{F}^m$, we have $\varphi = a_1\varphi_1 + \cdots + a_m\varphi_m$ so that $\Gamma(\varphi) = (a_1, \dots, a_m)$. $\square$

---

• *Define $\Gamma : V \to \mathbf{F}^m$ by $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$, where $\varphi_1, \dots, \varphi_m \in V'$.*

(c) *Show that span$(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ is injective.*

(d) *Show that $\varphi_1, \dots, \varphi_m$ is linearly independent $\iff \Gamma$ is surjective.*

**SOLUTION:**

(c) Suppose $\Gamma$ is injective. Then $\Gamma(v) = 0 \Leftrightarrow \forall i, \varphi_i(v) = 0 \Leftrightarrow v \in (\text{null}\varphi_1) \cap \cdots \cap (\text{null}\varphi_m) \Leftrightarrow v = 0$.
Getting $(\text{null}\varphi_1) \cap \cdots \cap (\text{null}\varphi_m) = \{0\}$. By Problem (•) above, $\text{span}(\varphi_1, \dots, \varphi_m) = V'$
Suppose $\text{span}(\varphi_1, \dots, \varphi_m) = V'$. Again by Problem (•), $(\text{null}\varphi_1) \cap \cdots \cap (\text{null}\varphi_m) = \{0\}$.
Thus $\Gamma(v) = 0 \Rightarrow \forall i, \varphi_i(v) = 0 \Rightarrow v = 0$.

(d) Suppose $\varphi_1, \dots, \varphi_m$ is linearly independent. Then by Problem (31), $(v_1, \dots, v_m)$ is linearly independent.
Thus for any $(a_1, \dots, a_m) \in \mathbf{F}$, by letting $v = \sum_{i=1}^m a_iv_i$, then $\varphi_i(v) = a_i \Rightarrow \Gamma(v) = (a_1, \dots, a_m)$.
Suppose $\Gamma$ is surjective. Let $e_1, \dots, e_m$ be a basis of $\mathbf{F}^m$.
For every $e_i$, $\exists\, v_i \in V$ such that $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v)) = e_i$,
fix $v_i$ ($\Rightarrow v_1, \dots, v_m$ is linearly independent). Thus $\varphi_i(v_i) = 1, \varphi_i(v_j) = 0$.
Hence $(\varphi_1, \dots, \varphi_m)$ is the dual basis of the basis $v_1, \dots, \varphi_m$ of $\text{span}(v_1, \dots, v_m)$. $\square$

---

**33** *Suppose $A \in \mathbf{F}^{m,n}$. Define $T : A \to A^t$. Prove that $T$ is an isomorphism of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$*

**SOLUTION:** By [3.111], T is linear. Note that $(A^t)^t = A$.

$$\left.\begin{array}{l} \text{(a) For any } B \in \mathbf{F}^{n,m}, \text{ let } A = B^t \text{ so that } T(A) = B. \text{ Thus } T \text{ is surjective.} \\ \text{(b) If } T(A) = 0 \text{ for some } A \in \mathbf{F}^{n,m}, \text{ then } A = 0. \text{ Thus } T \text{ is injective.} \\ \quad \text{for if not, } \exists\ j, k \in \mathbf{N}^+ \text{ such that } A_{j,k} \neq 0, \text{ then } T(A)_{k,j} \neq 0, \text{ contradicts.} \end{array}\right\} \Rightarrow \square$$

---

**32** *Suppose $T \in \mathcal{L}(V)$, and $(u_1, \ldots, u_m)$ and $(v_1, \ldots, v_m)$ are bases of V. Prove that*
*T is invertible $\iff$ The rows of $\mathcal{M}(T, (u_1, \ldots, u_m), (v_1, \ldots, v_m))$ form a basis of $\mathbf{F}^{1,n}$.*

**Solution:** Note that $T$ is inveritlbe $\Rightarrow T'$ is invertible. And $\mathcal{M}(T') = \mathcal{M}(T)^t = A^t$, denote it by $B$.

Let $(\varphi_1, \ldots, \varphi_m)$ be the dual basis of $(v_1, \ldots, v_m)$, $(\psi_1, \ldots, \psi_m)$ be the dual basis of $(u_1, \ldots, u_m)$.

(a) Suppose $T$ is invertible, so is $T'$. Because $T'(\varphi_1), \ldots, T'(\varphi_m)$ is linearly independent.

Noticing that $T'(\varphi_i) = B_{1,i}\psi_1 + \cdots + B_{m,i}\psi_m$.

Thus the columns of $B$, namely the rows of $A$, are linearly independent (check it by contradiction).

(b) Suppose the rows of $A$ are linearly independent, so are the columns of $B$.

Then $(T'(\varphi_1), \ldots, T'(\varphi_m))$ is a basis of range $T'$, namely $V'$. Thus $T'$ is surjective.

Hence $T'$ is invertible, so is $T$. $\square$

---

**34** *The double dual space of $V$, denoted by $V''$, is defined to be the dual space of $V'$.*
*In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \to V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.*

(a) *Show that $\Lambda$ is a linear map from $V$ to $V''$.*

(b) *Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*

(c) *Show that if $V$ is finite-dim, then $\Lambda$ is an isomorphism from $V$ onto $V''$.*

*Suppose $V$ is finite-dim. Then $V$ and $V'$ are isomorphic, but finding an isomorphism from $V$ onto $V'$ generally requires choosing a basis of V. In contrast, the isomorphism $\Lambda$ from $V$ onto $V''$ does not require a choice of basis and thus is considered more natural.*

**Solution:**

(a) $\forall \varphi \in V'$, $\forall v, w \in V, a \in \mathbf{F}$, $(\Lambda(v + aw))(\varphi) = \varphi(v + aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.

Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence $\Lambda$ is linear.

(b) $(T''(\Lambda v))(\varphi) = ((\Lambda v) \circ (T'))(\varphi) = (\Lambda v)(T'(\varphi)) = (T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = (\Lambda(Tv))(\varphi)$.

Hence $T''(\Lambda v) = (\Lambda(Tv))$, getting $T'' \circ \Lambda = \Lambda \circ T$.

(c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus $\Lambda$ is injective.

又 Because $V$ is finite-dim. dim$V$=dim$V'$=dim$V''$. Hence $\Lambda$ is an isomorphism. $\square$

---

**36** *Suppose $U$ is a subspace of $V$. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*

(a) *Show that null $i' = U^0$: null $i' = (\text{range } i)^0 = U^0 \Leftarrow \text{range } i = U$.* $\square$

(b) *Prove that if $V$ is finite-dim, then range $i' = U'$: range $i' = (\text{null } i)^0_U = (\{0\})^0_U = U'$.* $\square$

(c) *Prove that if $V$ is finite-dim, then $\tilde{i'}$ is an isomorphism from $V'/U^0$ onto $U'$:*

Note that $\tilde{i'} : V'/\text{null } i' \to \text{range } i' \;\Rightarrow\; \tilde{i'} : V'/U^0 \to U'$. By (a), (b) and [3.91(d)]. $\square$

*The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.*

---

**37** *Suppose $U$ is a subspace of $V$ and $\pi$ is the quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.*

(a) *Show that $\pi'$ is injective: Because $\pi$ is surjective. Use [3.108].* $\square$

(b) *Show that range $\pi' = U^0$.*

(c) *Conclude that $\pi'$ is an isomorphism from $(V/U)'$ onto $U^0$.*

*The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.*

*In fact, there is no assumption here that any of these vector spaces are finite-dimensional.*

**Solution:** [3.109] is not available. Using (3.E.18), also see (3.E.20).

(b) $\psi \in \text{range } \pi' \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \text{null } \psi \supseteq U \iff \psi \in U^0$. Hence range $\pi' = U^0$.

(c) $\psi \in U^0 \iff \text{null } \psi \supseteq U \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi = \pi(\varphi)$. Thus $\pi'$ is surjective. And by (a). $\square$

---

**Ended**

- **NOTE FOR [4.8]:** *division algorithm for polynomials*

  *Suppose $p, s \in \mathcal{P}(\mathbf{F})$, with $s \neq 0$. Then $\exists!\, q, r \in \mathcal{P}(\mathbf{F})$ such that $p = sq + r$ and $\deg r < \deg s$.* *Another Proof:*

  Suppose $\deg p \geq \deg s$. Then $(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of length } \deg s}, \underbrace{s, zs, \cdots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)})$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.

  Because $q \in \mathcal{P}(\mathbf{F})$, $\exists!\, a_i, b_j \in \mathbf{F}$,

$$\begin{aligned} q &= a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s \\ &= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1}}_{r} + s \underbrace{\left(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s}\right)}_{q}. \end{aligned}$$

  With $r, q$ as defined uniquely above, we are done. $\square$

---

- **NOTE FOR [4.11]:** *each zero of a polynomial corresponds to a degree-one factor;* *Another Proof:*

  First suppose $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, $\exists!\, a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.

  Then $p(z) = p(z) - p(\lambda) = a_1(z - \lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.

  Hence for each $k \in \{1, \dots, m\}$, $z^k - \lambda^k = (z - \lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.

  Thus $p(z) = \sum\limits_{j=1}^{m} a_j (z - \lambda) \sum\limits_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) \sum\limits_{j=1}^{m} a_j \sum\limits_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) q(z)$.

---

- **NOTE FOR [4.13]:** *fundamental theorem of algebra, first version*

  *Every nonconstant polynomial with complex coefficients has a zero in* $\mathbf{C}$. *Another Proof:*

  De Moivre's theorem ( which you can prove using induction on $k$ and the addition formulas for cosine and sine), states that if $k \in \mathbf{N}^+, \theta \in \mathbf{R}$, then $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$.

  Suppose $w \in \mathbf{C}, k \in \mathbf{N}^+$ and using polar coordinates. $\exists\, r \geq 0, \theta \in \mathbf{R}$ such that $r(\cos\theta + \mathrm{i}\sin\theta) = w$.

  Hence $(r^{1/k}(\cos\frac{\theta}{k} + \mathrm{i}\sin\frac{\theta}{k}))^k = w$. Thus every complex number has a $k^{th}$ *root*, a fact that we will soon use.

  Suppose a nonconstant $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.

  Then $|p(z)| \to \infty$ as $|z| \to \infty$ ( because $\dfrac{|p(z)|}{|z_m|} \to |c_m|$ as $|z| \to \infty$ ).

  Thus the continuous function $z \to |p(z)|$ has a global minimum at some point $\zeta \in \mathbf{C}$.

  To show that $p(\zeta) = 0$, suppose that $p(\zeta) \neq 0$.

  Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \dfrac{p(z + \zeta)}{p(\zeta)}$.

  The function $z \to |q(z)|$ has a global minimum value of $1$ at $z = 0$.

  Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where $k$ is the smallest positive integer such that $a_k \neq 0$.

  Let $\beta \in \mathbf{C}$ be such that $\beta^k = -\dfrac{1}{a_k}$.

  There is a constant $c > 1$ such that if $t \in (0, 1)$,

  then $|q(t\beta)| \leq |1 + a_k t^k \beta^k| + t^{k+1} c = 1 - t^k(1 - tc)$.

  Thus taking $t$ to be $1/(2c)$ in the inequality above, we have $|q(t\beta)| < 1$,

  which contradicts the assumption that the global minimum of $z \to |q(z)|$ is 1.

  Hence $p(\zeta) = 0$, as desired. $\square$

---

• *Prove that if $w, z \in \mathbf{C}$, then $||w| - |z|| \leq |w - z|$.* *The inequality here is called the* **reverse triangle inequality**.

**SOLUTION:**

$$\begin{aligned} |w-z|^2 &= (w-z)(\overline{w}-\overline{z}) \\ &= |w|^2 + |z|^2 - (w\overline{z} + \overline{w}z) \\ &= |w|^2 + |z|^2 - (\overline{\overline{w}z} + \overline{w}z) \\ &= |w|^2 + |z|^2 - 2Re(\overline{w}z) \\ &\geq |w|^2 + |z|^2 - 2|\overline{w}z| \\ &= |w|^2 + |z|^2 - 2|w||z| = ||w| - |z||^2. \qquad \square \end{aligned}$$

*Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides.*

---

• *Suppose $V$ is a complex vector space and $\varphi \in V'$.*
*Define $: V \to \mathbf{R}$ by $\sigma(v) = \Re\varphi(v)$ for each $v \in V$.*
*Show that $\varphi(v) = \sigma(v) - i\sigma(iv)$ for all $v \in V$.*

**SOLUTION:**

Notice that $\varphi(v) = \Re\varphi(v) + \mathrm{i}\Im\varphi(v) = \sigma(v) + \mathrm{i}\Im\varphi(v)$. 又 $\Re\varphi(\mathrm{i}v) = \Re[\mathrm{i}\varphi(v)] = -\Im\varphi(v) = \sigma(\mathrm{i}v)$.

Hence $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$. $\square$

---

**2** *Suppose $m$ is a positive integer. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subspace of $\mathcal{P}(\mathbf{F})$?*

**SOLUTION:**

$x^m, x^m + x^{m-1} \in U$ but $\deg[(x^m + x^{m-1}) - (x^m)] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$.

Hence $U$ is not closed under addition, and therefore is not a subspace. $\square$

---

**3** *Suppose $m$ is a positive integer. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p$ is even $\}$ a subspace of $\mathcal{P}(\mathbf{F})$?*

**SOLUTION:**

$x^2, x^2 + x \in U$ but $\deg[(x^2 + x) - (x^2)]$ is odd and hence $(x^2 + x) - (x^2) \notin U$.

Thus $U$ is not closed under addition, and therefore is not a subspace. $\square$

---

**4** *Suppose that $m$ and $n$ are positive integers with $m \leq n$, and suppose $\lambda_1, \dots, \lambda_m \in \mathbf{F}$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{F})$ such that $\deg p = n$, the zeros of $p$ are $\lambda_1, \dots, \lambda_m$.*

**SOLUTION:** Let $p(z) = (z - \lambda_1)^{n-(m-1)}(z - \lambda_2)\cdots(z - \lambda_m)$. $\square$

---

**5** *Suppose that $m \in \mathbf{N}$, $z_1, \dots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$. Prove that $\exists!\, p \in \mathcal{P}_m(\mathbf{F})$ such that $p(z_k) = w_k$ for each $k = 1, \dots, m+1$.*

*This result can be proved without using linear algebra. However, try to find the clearer, shorter proof that uses some linear algebra.*

**SOLUTION:**

Define $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by $Tq = (q(z_1), \dots, q(z_m), q(z_{m+1}))$. As can be easily checked, $T$ is linear.

We need to show that $T$ is surjective, so that such $p$ exists; and that $T$ is injective, so that such $p$ is unique.

$Tq = 0 \iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0$

$\iff q \in \mathcal{P}_m(\mathbf{F})$ is the zero polynomial, for if not,

$q$ has at least $m+1$ distinct roots, while $\deg q = m$. Contradicts (by [4.12]). Hence $T$ is injective.

$\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m + 1 = \dim \mathbf{F}^{m+1}$. 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1}$. Hence $T$ is surjective. $\square$

---

**6** *Suppose $p \in \mathcal{P}_m(\mathbf{C})$ has degree $m$. Prove that*
*$p$ has $m$ distinct zeros $\Longleftrightarrow$ $p$ and its derivative $p'$ have no zeros in common.*

**SOLUTION:**

(a) Suppose $p$ has $m$ distinct zeros. By [4.14] and $\deg p = m$, let $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$, $\exists!\, c, \lambda_i \in \mathbf{C}$.

For each $j \in \{1, \dots, m\}$, let $\dfrac{p(z)}{(z-\lambda_j)} = q_j \in \mathcal{P}_{m-1}(\mathbf{C})$, then $p(z) = (z-\lambda_j)q_j(z)$ and $q_j(\lambda_j) \neq 0$.

$p'(z) = (z-\lambda_j)q_j'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

(b) To prove the implication on the other direction, we prove the contrapositive:
Suppose $p$ has less than $m$ distinct roots.
We must show that $p$ and its derivative $p'$ have at least one zero in common.
Let $\lambda$ be a zero of $p$, then write $p(z) = (z-\lambda)^n q(z)$, $\exists!\, n \in \mathbf{N}^+, q \in \mathcal{P}_{m-n}(\mathbf{C})$.
$p'(z) = (z-\lambda)^n q'(z) + n(z-\lambda)^{n-1}q(z) \Rightarrow p'(\lambda) = 0$, $\lambda$ is a common root of $p'$ and $p$. $\square$

---

**7** *Prove that every polynomial of odd degree with real coefficients has a real zero.*

**SOLUTION:**

Using the notation proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence $\lambda_1$ exists. $\square$

OR. Using calculus but not using [4.17].
Suppose $p \in \mathcal{P}_m(\mathbf{F})$, $\deg p = m$, $m$ is odd.

Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1}a_m$ by $\delta$

Write $p(x) = x^m(\dfrac{a_0}{x^m} + \dfrac{a_1}{x^{m-1}} + \cdots + \dfrac{a_{m-1}}{x} + a_m)$.

Thus $p(x)$ is continuous, and $\lim\limits_{x\to-\infty} p(x) = -\delta\infty\,$; $\lim\limits_{x\to\infty} p(x) = \delta\infty$.

Hence we conclude that $p$ has at least one real zero. $\square$

---

**8** *For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \to \mathbf{R}$ by* $(Tp)(x) = \begin{cases} \dfrac{p(x)-p(3)}{x-3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3 \end{cases}$ *for all $x \in \mathbf{R}$.*

*Show that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$ and that $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is a linear map.*

**SOLUTION:**

For $x \neq 3$, $T(x^n) = \dfrac{x^n - 3^n}{x-3} = \sum\limits_{i=1}^{n} 3^{i-1}x^{n-i}$.

For $x = 3$, $T(x^n) = 3^{n-1}\cdot n$. Note that if $x = 3$, then $\sum\limits_{i=1}^{n} 3^{i-1}x^{n-i} = \sum\limits_{i=1}^{n} 3^{n-1} = 3^{n-1}\cdot n$.

Hence for all $x \in \mathbf{R}$ and for all $n \in \mathbf{N}$, $T(x^n) = \sum\limits_{i=1}^{n} 3^{i-1}x^{n-i} \in \mathcal{P}(\mathbf{R})$.

Because $T$ is linear, we conclude that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$.

Now we show that $T$ is linear:

$\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, T(p+\lambda q)(x) = \begin{cases} \dfrac{(p+\lambda q)(x) - (p+\lambda q)(3)}{x-3} & \text{if } x \neq 3, \\ (p+\lambda q)'(3) & \text{if } x = 3 \end{cases}$ for all $x \in \mathbf{R}$.

Notice that $(p+\lambda q)(x) - (p+\lambda q)(3) = (p(x)-p(3)) + (\lambda q(x) - \lambda q(3))$;
$(p+\lambda q)'(3) = p'(3) + \lambda q'(3)$.

Thus $T(p+\lambda q)(x) = (T(p) + \lambda T(q))(x)$ for all $x \in \mathbf{R}$. $\square$

---

**9** *Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\overline{z})}$.*
*Prove that $q$ is a polynomial with real coefficients.*

**SOLUTION:**

$p(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow p(\overline{z}) = a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0 \Rightarrow \overline{p(\overline{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\overline{z})} = \overline{p(\overline{z})}p(z) = \overline{p(\overline{z})\overline{p(\overline{\overline{z}})}} = \overline{q(\overline{z})}$.

Hence letting $q(z) = c_m x^m + \cdots + c_1 x + c_0 \;\Rightarrow\; \overline{c_k} = c_k$, $c_k \in \mathbf{R}$ for each $k$. $\square$

---

**10** *Suppose $m \in \mathbf{N}$ and $p \in \mathcal{P}_m(\mathbf{C})$ is such that*
*there are $(m+1)$ distinct real numbers $x_0, x_1, \ldots, x_m$ with $p(x_k) \in \mathbf{R}$ for each $x_k$.*
*Prove that all coefficients of $p$ are real.*

**SOLUTION:** Let $p(x_k) = y_k$ for each $k$. By Problem (5), $\exists!\, q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_k) = y_k$. Hence $p = q$. $\square$

OR. Using the Lagrange Interpolating Polynomial.

Define $q(x) = \displaystyle\sum_{j=0}^{m} \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j)$.

又 For each $j$, $x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R}) \subseteq \mathcal{P}_m(\mathbf{C})$.

Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$ for each $k \in \{0, 1, \ldots, m\}$.

Then $(q-p)$ has $(m+1)$ distinct zeros, while $(q-p) \in \mathcal{P}_m(\mathbf{C})$. Hence by [4.12], $q - p = 0 \Rightarrow p = q$. $\square$

---

**11** *Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*
(a) *Show that* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.
(b) *Find a basis of $\mathcal{P}(\mathbf{F})/U$.*

**SOLUTION:**

$U$ is a subspace of $\mathcal{P}(\mathbf{F})$ because $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, pf + \lambda pg = p(f + \lambda g) \in U$.

NOTE: Define $P :\to \mathcal{P}(\mathbf{F})$ by $(Pq)(x) = p(q(x)) = (p \circ q)(x)\ (\neq p(x)q(x)\,)$. $P$ is not linear.

(a) By [4.8], $\forall f \in \mathcal{P}(\mathbf{F})$, $\exists!\, q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r)$; $\deg r < \deg p$.

Hence $\forall f \in \mathcal{P}(\mathbf{F})$, $\exists!\, pq \in U, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), f = (pq) + (r); r \notin U$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F})$. Therefore $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p-1}(\mathbf{F})$ are isomorphic.

OR. $\forall f \in \mathcal{P}(\mathbf{F})$, $\exists!\, q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r)$; $\deg r < \deg p$.

Define $R : \mathcal{P}(\mathbf{F}) \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $(Rf)(z) = r(z)$ for each $z \in \mathbf{F}$.

$\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, R(f + \lambda g)(z) = R(f) + \lambda R(g)$.

BECAUSE: $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}$,

$\exists!\, q_1, r_1 \in \mathcal{P}(\mathbf{F}), f = (p)q_1 + (r_1)$, $\deg r_1 < \deg p$;

$\exists!\, q_2, r_2 \in \mathcal{P}(\mathbf{F}), g = (p)q_2 + (r_2)$, $\deg r_2 < \deg p$;

$\exists!\, q_3, r_3 \in \mathcal{P}(\mathbf{F}), \lambda g = (p)q_3 + (r_3) = (p)(\lambda q_2) + (\lambda r_2)$, $\deg r_3 < \deg p$ and $\deg \lambda r_2 < \deg p$.

$\Rightarrow q_3 = \lambda q_2, r_3 = \lambda r_2$.

$\exists!\, q_0, r_0 \in \mathcal{P}(\mathbf{F}), (f + \lambda g) = (p)q_0 + (r_0)$

$= (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2)$, $\deg r_0 < \deg p$ and $\deg(r_1 + \lambda r_2) < \deg p$.

$\Rightarrow q_1 + \lambda q_2 = q_0;\ \ r_1 + \lambda r_2 = r_0$.

Hence $R$ is linear.

$R(f) = 0 \iff f = pq$, $\exists!\, q \in \mathcal{P}(\mathbf{F})$. Thus null $R = U$.

$\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F})$, let $f = p + r$, then $R(f) = r$. Thus range $R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Finally, by [3.91(d)], $\mathcal{P}(\mathbf{F})/\operatorname{null} R$, namely $\mathcal{P}(\mathbf{F})/U$, and range $R$, namely $\mathcal{P}_{\deg p-1}(\mathbf{F})$, are isomorphic.

(b) $(1 + U, x + U, \ldots, x^{\deg p - 1}) + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. $\square$

---

- *Suppose nonconstant $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p$, $n = \deg q$.*
  *Use* (a)—(c) *below to prove that $\exists!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*
  (a) *Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r, s) = rp + sq$. Show that the linear map $T$ is injective.*
  (b) *Show that the linear map $T$ in* (a) *is surjective.*
  (c) *Use* (b) *to conclude that $\exists!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*

**SOLUTION:**

(a) $T$ is linear because $\forall\, r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,
$T((r_1, s_1) + \lambda(r_2, s_2)) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1, s_1) + \lambda T(r_2, s_2)$.

Suppose $T(r, s) = rp + sq = 0$. Notice that $p, q$ have no zeros in common.
Then $r = s = 0$, for if not, write $\dfrac{q(z)}{r(z)} = \dfrac{p(z)}{s(z)}$, while for any zero $\lambda$ of $q$, $\dfrac{q(\lambda)}{r(z)} = 0 \neq \dfrac{p(\lambda)}{s(z)}$. Hence $\square$

(b) $\dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) = \dim \mathcal{P}_{n-1}(\mathbf{C}) + \dim \mathcal{P}_{m-1}(\mathbf{C}) = n + m = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.
又 $T$ is injective. Hence $\dim \operatorname{range} T = \dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) - \dim \operatorname{null} T = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.
Thus $\operatorname{range} T = \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$ is surjective, and therefore is an isomorphism. $\square$

(c) Immediately. $\square$

---

**ENDED**

# 5.A

**1** *Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$.*
(a) *Prove that if $U \subseteq \operatorname{null} T$, then $U$ is invariant under $T$.*
(b) *Prove that if $\operatorname{range} T \subseteq U$, then $U$ is invariant under $T$.*

**SOLUTION:**

(a) $\forall u \in U \subseteq \operatorname{range} T, Tu = 0 \in U$. $\square$
(b) $\forall u \in U \subseteq V, Tu \in \operatorname{range} T \subseteq U$. $\square$

---

- *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*
  (a) *Prove that null $(T - \lambda I)$ is invariant under $S$, where $\lambda$ is chosen arbitrarily.*
  (b) *Prove that range $(T - \lambda I)$ is invariant under $S$, where $\lambda$ is chosen arbitrarily.*

**SOLUTION:**

Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.

(a) Suppose $v \in \operatorname{null}(T - \lambda I)$, then $(T - \lambda I)(Sv) = S(T - \lambda I)v = S(0) = 0$.
Hence $Sv \in \operatorname{null}(T - \lambda I)$ and therefore $\operatorname{null}(T - \lambda I)$ is invariant under $S$.

(b) Suppose $v \in \operatorname{range}(T - \lambda I)$, therefore $\exists\, u \in V, (T - \lambda I)u = v$.
Then $Sv = S(T - \lambda I)u = (T - \lambda I)(Su) \in \operatorname{range}(T - \lambda I)$.
Hence $Sv \in \operatorname{range}(T - \lambda I)$ and therefore $\operatorname{range}(T - \lambda I)$ is invariant under $S$. $\square$

COMMENT: Reversing the roles of $S$ and $T$, letting $\lambda = 0$, we can conclude that
null $S$ and range $S$ is invariant under $T$, which is what we will prove in Problem (2) and (3) below.

---

**2** *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that null $S$ is invariant under $T$.*

**SOLUTION:** $\forall u \in \operatorname{null} S, Su = 0 \Rightarrow TSu = 0 = STu \Rightarrow Tu \in \operatorname{null} S$. $\square$

---

**3** *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that range $S$ is invariant under $T$.*

**SOLUTION:** $\forall w \in \operatorname{range} S,\ \exists\, v \in V, Sv = w, STv = TSv = Tw \in \operatorname{range} S$. $\square$

---

**4** *Suppose $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are subspaces of $V$ invariant under $T$. Prove that $V_1 + \cdots + V_m$ is invariant under $T$.*

**SOLUTION:**

For each $i = 1, \ldots, m, \forall v_i \in V_i, Tv_i \in V_i$

Hence $\forall v = v_1 + \cdots + v_m \in V_1 + \cdots + V_m, Tv = Tv_1 + \cdots + Tv_m \in V_1 + \cdots + V_m$. $\square$

---

**5** *Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of $V$ invariant under $T$ is invariant under $T$.*

**SOLUTION:**

Suppose $\{V_\alpha\}_{\alpha \in \Gamma}$ is a collection of subspaces of $V$ invariant under $T$; here $\Gamma$ is an arbitrary index set.

We need to prove that $\bigcap_{\alpha \in \Gamma} V_\alpha$, which equals the set of vectors that are in $V_\alpha$ for each $\alpha \in \Gamma$, is invariant under $T$.

For each $\alpha \in \Gamma, \forall v_\alpha \in V_\alpha, Tv_\alpha \in V_i$.

Hence $\forall v \in \bigcap_{\alpha \in \Gamma} V_\alpha, Tv \in V_\alpha, \forall \alpha \in \Gamma \Rightarrow Tv \in \bigcap_{\alpha \in \Gamma} V_\alpha$. Thus $\bigcap_{\alpha \in \Gamma} V_\alpha$ is invariant under $T$. $\square$

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**6** *Prove or give a counterexample: If $V$ is finite-dim and $U$ is a subspace of $V$ that is invariant under every operator on $V$, then $U = \{0\}$ or $U = V$.*

**SOLUTION:**

Notice that $V$ might be $\{0\}$. In this case we are done.

Suppose $\dim V \geq 1$. We prove by contrapositive:

Suppose $U \neq \{0\}$ and $U \neq V$, then $\exists T \in \mathcal{L}(V)$ such that $U$ is not invariant under $T$.

Let $W$ be such that $V = U \oplus W$.

Let $(u_1, \ldots, u_m)$ be a basis of $U$ and $(w_1, \ldots, w_n)$ be a basis of $W$.

Hence $(u_1, \ldots, u_m, w_1, \ldots, w_n)$ is a basis of $V$.

Define $T \in \mathcal{L}(V)$ by $T(a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n) = b_1 w_1 + \cdots + b_n w_n$. $\square$

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**7** *Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of $T$.*

**SOLUTION:**

Suppose $\lambda \in \mathbf{R}$ and $(x, y) \in \mathbf{R}^2 \backslash \{0\}$ such that $T(x, y) = (-3y, x) = \lambda(x, y)$. Then $-3y = \lambda x$ and $x = \lambda y$.

Thus $-3y = \lambda^2 y \Rightarrow \lambda^2 = -3$, ignoring the possibility of $y = 0$ ( because if $y = 0$, then $x = 0$ ).

Hence the set of solution for this equation is $\varnothing$, and therefore $T$ has no eigenvalues in $\mathbf{R}$. $\square$

---

**8** *Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of $T$.*

**SOLUTION:**

Suppose $\lambda \in \mathbf{F}$ and $(w, z) \in \mathbf{F}^2$ such that $T(w, z) = (z, w) = \lambda(w, z)$. Then $z = \lambda w$ and $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, ignoring the possibility of $z = 0$ ( $z = 0 \Rightarrow w = 0$ ).

Hence $\lambda_1 = -1$ and $\lambda_2 = 1$ are all eigenvalues of $T$.

For $\lambda_1 = -1, z = -w, w = -z$; For $\lambda_2 = 1, z = w$.

Thus the set of all eigenvectors is $\{(z, -z), (z, z) : z \in \mathbf{F} \wedge z \neq 0\}$. $\square$

---

**9** *Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$.*
*Find all eigenvalues and eigenvectors of $T$.*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $(z_1, z_2, z_3) \in \mathbf{F}^3$.
Then $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$.
Thus $2z_2 = \lambda z_1, \quad 0 = \lambda z_2, \quad 5z_3 = \lambda z_3$.
We discuss in two cases:
For $\lambda = 0,\ z_2 = z_3 = 0$ and $z_1$ can be arbitrary ( $z_1 \neq 0$ ).
For $\lambda \neq 0,\ z_2 = 0 = z_1$, and $z_3$ can be arbitrary ( $z_3 \neq 0$ ), then $\lambda = 5$.
The set of all eigenvectors is $\{(0,0,z), (z,0,0) : z \in \mathbf{F} \wedge z \neq 0\}$. $\square$

---

• *Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$.*
*Prove that if $\lambda$ is an eigenvalue of $P$, then $\lambda = 0$ or $\lambda = 1$.*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue, $v \in V\backslash\{0\}$ such that $Pv = \lambda v$, then $P(Pv) = \lambda^2 v = \lambda v = Pv$. Thus $\lambda^2 = 1$. $\square$

---

**10** *Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$*
(a) *Find all eigenvalues and eigenvectors of $T$.*
(b) *Find all invariant subspaces of $V$ under $T$.*

**SOLUTION:**

(a) Suppose $v = (x_1, x_2, x_3, \ldots, x_n)$ is an eigenvector of $T$ with an eigenvalue $\lambda$.
Then $Tv = \lambda v = (x_1, 2x_2, 3x_3, \ldots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots, \lambda x_n)$.
Hence $1, \ldots, n$ are eigenvalues of $T$.
And $\{(0, \ldots, 0, x_\lambda, 0, \ldots, 0) \in \mathbf{F}^n : \lambda = 1, \ldots, n,\ x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$ is the set of all eigenvectors of $T$.
(b) Let $V_\lambda = \{(0, \ldots, 0, x_\lambda, 0, \ldots, 0) \in \mathbf{F}^n : x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$. Then $V_1, \ldots, V_n$ are invariant under $T$.
Hence by Problem (4), every sum of $V_1, \ldots, V_n$ is a invariant subspace of $V$ under $T$. $\square$

---

**11** *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of $T$.*

**SOLUTION:**

Note that in general, $\deg p' < \deg p$ ( $\deg 0 = -\infty$ ).
Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $p$.
Suppose $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p \neq p'$. Contradicts. Thus $\lambda = 0$.
Therefore $\deg \lambda p = -\infty = \deg p \Rightarrow p$ is a nonzero constant polynomial.
Hence the set of all eigenvectors is $\{C : C \in \mathbf{R} \wedge C \neq 0\} = \mathcal{P}_0(\mathbf{R})\backslash\{0\}$. $\square$

---

**12** *Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$.*
*Find all eigenvalues and eigenvectors of $T$.*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $p$, then $(Tp)(x) = xp'(x) = \lambda p(x)$.
Let $p = a_0 + a_1 x + \cdots + a_n x^n$.
Then $xp'(x) = a_1 x + 2a_2 x^2 + \cdots + na_n x^n = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n$.
Similar to Problem (10), $0, 1, \ldots, n$ are eigenvalues of $T$.
The set of all eigenvectors of $T$ is $\{cx^\lambda : \lambda = 0, 1, \ldots, n,\ c \in \mathbf{F} \wedge c \neq 0\}$. $\square$

---

**13** *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*
*Prove that $\exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}$ and $(T - \alpha I)$ is invertible.*

**SOLUTION:**

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \dfrac{1}{1000 + k}$ for each $k = 1, \ldots, \dim V + 1$.

Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigenvalues.

Hence $\exists\, k = 1, \ldots, \dim V + 1$ such that $\alpha_k$ is not an eigenvalue of $T$ and therefore $(T - \alpha_k I)$ is invertible. $\square$

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• *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$, and $\alpha \in \mathbf{F}$.*
*Prove that $\exists\, \delta > 0$ such that $(T - \lambda I)$ is invertible for all $\lambda \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.*

**SOLUTION:**

Choose $\delta > 0$ arbitrarily.

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \dfrac{\delta}{k}$ for each $k = 1, \ldots, \dim V + 1$.

Similar to Problem (13), $\exists\, k$ such that $\alpha_k$ is not an eigenvalue. $\square$

---

**14** *Suppose $V = U \oplus W$, where $U$ and $W$ are nonzero subspaces of $V$.*
*Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.*
*Find all eigenvalues and eigenvectors of $P$.*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue of $P$ with an eigenvector $(u + w)$.

Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$. By [1.44] and $V = U \oplus W$, $(\lambda - 1)u = \lambda w = 0$.

Thus if $\lambda = 1$, then $w = 0$; if $\lambda = 0$, then $u = 0$.

Hence the eigenvalues of $P$ are $0$ and $1$, the set of all eigenvectors in $P$ is $U \cup W$. $\square$

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**15** *Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.*
(a) *Prove that $T$ and $S^{-1}TS$ have the same eigenvalues.*
(b) *What is the relationship between the eigenvectors of $T$ and the eigenvectors of $S^{-1}TS$?*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $v$.

Then $S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v$.

Thus $\lambda$ is also an eigenvalue of $S^{-1}TS$ with an eigenvector $S^{-1}v$.

Suppose $\lambda$ is an eigenvalue of $S^{-1}TS$ with an eigenvector $v$.

Then $S(S^{-1}TS)v = TSv = \lambda Sv$.

Thus $\lambda$ is also an eigenvalue of $T$ with an eigenvector $Sv$. $\square$

OR. Note that $S(S^{-1}TS)S^{-1} = T$.

Hence every eigenvalue of $S^{-1}TS$ is an eigenvalue of $S(S^{-1}TS)S^{-1} = T$.

And every eigenvector $v$ of $S^{-1}TS$ is $S^{-1}v$, every eigenvector $u$ of $T$ is $Su$. $\square$

---

**17** *Give an example of an operator on $\mathbf{R}^4$ that has no (real) eigenvalues.*

**SOLUTION:**

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$. Where $(e_1, e_2, e_3, e_4)$ is the standard basis of $\mathbf{R}^4$.

Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $(x, y, z, w)$.

Then $T(x,y,z,w) = \lambda(x,y,z,w) \Rightarrow \begin{cases} (1-\lambda)x + y + z + w = 0 \\ -x + (1-\lambda)y - z - w = 0 \\ 3x + 8y + (11-\lambda)z + 5w = 0 \\ 3x - 8y - 11z + (5-\lambda)w = 0 \end{cases}$

This linear equation has no solutions.

( You can type it on `https://zh.numberempire.com/equationsolver.php` to check.)

OR. Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $(x, y, z, w)$.

Then $T(x,y,z,w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \Rightarrow \begin{cases} -y = \lambda x \\ x = \lambda y \\ -w = \lambda z \\ z = \lambda w \end{cases} \Rightarrow \begin{cases} -xy = \lambda^2 xy \\ -zw = \lambda^2 zw \end{cases}$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Otherwise, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, contradicts.

Similarly, $y = z = w = 0$. Then we fail.

Thus $T$ has no eigenvalues. $\square$

---

• *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*
*Show that $\lambda$ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.*

**SOLUTION:**

(a) Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $v$.

Let $v_1 = v$ and let $(Tv_1, \dots, Tv_n)$ be a basis of $V$, so is $(v_1, \dots, v_n)$.

Define $\varphi\mathcal{L}(V, \mathbf{F})$by $\varphi(v) = 1, \varphi(Tv_j) = 0$, for each $j = 2, \dots, n$.

Then $T'(\varphi) = \varphi \circ T$.

Thus $\forall u \in V$, $(\varphi \circ T)(a_1 v_1 + \cdots + a_n v_n) = \varphi(\lambda a_1 v + Tv_2 + \cdots + Tv_n) = \lambda a_1 = \lambda\varphi(\lambda a_1 v + Tv_2 + \cdots + Tv_n)$.

Hence $\lambda$ is an eigenvalue of $T'$.

(b) Suppose $\lambda$ is an eigenvalue $T'$ with an eigenvector $\psi$. Then $T'(\psi) = \psi \circ T = \lambda\psi$.

又 $\psi \neq 0 \Rightarrow \exists v \in V \backslash \{0\}$ such that $\psi(v) \neq 0$. Note that $\psi(Tv) = \lambda\psi(v)$.

Thus $\lambda = \dfrac{\psi(Tv)}{\psi(v)} \Rightarrow Tv = \dfrac{\psi(Tv)}{\psi(v)} v = \lambda v$. Hence $\lambda$ is an eigenvalue of $T$. $\square$

---

• TODO *Suppose $(v_1, \dots, v_n)$ is a basis of $V$ and $T \in \mathcal{L}(V)$.*
*Prove that if $\lambda$ is an eigenvalue of $T$, then*
$|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\}$*, where $\mathcal{M}(T, (v_1, \dots, v_n))$.*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue of $T$, and therefore is an eigenvalue of $\mathcal{M}(T)$, with an eigenvector $v$.

We discuss in two cases:

If $\mathcal{M}(T)$ is invertible ($\iff T$ is invertible ), then $\mathcal{M}(Tv) = \mathcal{M}(\lambda v) \Rightarrow \dfrac{1}{\lambda}\mathcal{M}(v) = \mathcal{M}(T^{-1}v)$.

Otherwise, $(T - 0I)$ is not invertible and therefore $\lambda = 0$ is an eigenvalue. And other $\lambda$s ?

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• *Suppose* $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.

(a) (OR (9.11)) $\lambda \in \mathbf{R}$. *Prove that $\lambda$ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of $T_{\mathbb{C}}$.*

(b) (OR Problem (16)) $\lambda \in \mathbf{C}$. *Prove that $\lambda$ is an eigenvalue of $T_{\mathbb{C}} \iff \overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$.*

**SOLUTION:**

(a) Suppose $v \in V$ is an eigenvector corresponding to the eigenvalue $\lambda$.

Then $Tv = \lambda v \Rightarrow T_{\mathbb{C}}(v + \mathrm{i}0) = Tv + \mathrm{i}T0 = \lambda v$.

Thus $\lambda$ is an eigenvalue of $T$.

Suppose $v + \mathrm{i}u \in V_{\mathbb{C}}$ is an eigenvector corresponding to the eigenvalue $\lambda$.

Then $T_{\mathbb{C}}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \Rightarrow Tv = \lambda v, Tu = \lambda u$. ( Note that $v$ or $u$ might be zero ).

Thus $\lambda$ is an eigenvalue of $T_{\mathbb{C}}$.

(b) Suppose $\lambda$ is an eigenvalue of $T_{\mathbb{C}}$ with an eigenvector $v + \mathrm{i}u$.

Let $(v_1, \ldots, v_n)$ be a basis of $V$. Write $v = \sum\limits_{i=1}^{n} a_i v_i, u = \sum\limits_{i=1}^{n} b_i v_i$, where $a_i, b_i \in \mathbf{R}$.

Then $T_{\mathbb{C}}(v + \mathrm{i}u) = Tv + \mathrm{i}Tu = \lambda v + \mathrm{i}\lambda u = \lambda \sum\limits_{i=1}^{n} (a_i + \mathrm{i}b_i) v_i$. Conjugating two sides, we have:

$\overline{T_{\mathbb{C}}(v + \mathrm{i}u)} = \overline{Tv + \mathrm{i}Tu} = \overline{Tv} - \mathrm{i}\overline{Tu} = Tv - \mathrm{i}Tu = T_{\mathbb{C}}(\overline{v + \mathrm{i}u}) = \overline{\lambda \sum\limits_{i=1}^{n} (a_i + \mathrm{i}b_i) v_i} = \overline{\lambda} \sum\limits_{i=1}^{n} (a_i - \mathrm{i}b_i) v_i$.

Hence $\overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$. To prove the other direction, notice that $\overline{\overline{\lambda}} = \lambda$. $\square$

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**18** *Show that the forward shift operator $T \in \mathcal{L}(\mathbf{F}^{\infty})$ defined by $T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$ has no eigenvalues.*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $(z_1, z_2, \ldots)$.

Then $T(z_1, z_2, \ldots) = (\lambda z_1, \lambda z_2, \ldots) = (0, z_1, z_2, \ldots)$.

Thus $\lambda z_1 = 0, \lambda z_2 = z_1, \ldots, \lambda z_k = z_{k-1}, \ldots$.

Let $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \lambda z_k = z_{k-1}$, therefore $(z_1, z_2, \ldots) = 0$ is not an eigenvector.

Suppose $\lambda \neq 0$. Then $\lambda z_1 = 0 \Rightarrow z_1 = 0 \Rightarrow z_2 = 0 = z_k$ for all $k \in \mathbf{N}^+$.

And then $(z_1, z_2, \ldots) = 0$ is not an eigenvector. Hence $T$ has no eigenvalues. $\square$

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**19** *Suppose $n$ is a positive integer.*
*Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n)$.*
*In other words, the entries of $\mathcal{M}(T)$ with respect to the standard basis are all $1$s.*
*Find all eigenvalues and eigenvectors of $T$.*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $(x_1, \ldots, x_n)$.

Then $T(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n)$.

Thus $\lambda x_1 = \cdots = \lambda x_n = x_1 + \cdots + x_n$.

For $\lambda = 0$, $x_1 + \cdots + x_n = 0$.

For $\lambda \neq 0$, $x_1 = \cdots = x_n$ and then $\lambda x_k = n x_k$ for each $k$.

Hence $0, n$ are eigenvectors of $T$.

And the set of all eigenvectors of $T$ is $\{(x_1, \ldots, x_n) \in \mathbf{F}^n : x_1 + \cdots + x_n = 0 \vee x_1 = \cdots = x_n\}$. $\square$

---

**20** *Define the backward shift operator $S \in \mathcal{L}(\mathbf{F}^{\infty})$ by $S(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots)$.*
(a) *Show that every element of $\mathbf{F}$ is an eigenvalue of $S$.*
(b) *Find all eigenvectors of $S$.*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue of $S$ with an eigenvector $(z_1, z_2, \ldots)$.

Then $S(z_1, z_2, z_3 \ldots) = (\lambda z_1, \lambda z_2, \ldots) = (z_2, z_3, \ldots)$.

Thus $\lambda z_1 = z_2, \lambda z_2 = z_3, \ldots, \lambda z_k = z_{k+1}, \ldots$.

For $\lambda = 0$, $\lambda z_1 = z_2 = 0 = \lambda z_2 = z_3 = \cdots = z_k$ for all $k$.

While $z_1$ can be arbitrary, so that $(z_1, 0, \ldots)$ is an eigenvector with $z_1 \neq 0$.

For $\lambda \neq 0$, $\lambda^k z_1 = \lambda^{k-1} z_2 = \cdots = \lambda z_k = z_{k+1}$ for all $k$.

Then $(z_1, \lambda z_1, \lambda^2 z_1, \ldots, \lambda^k z_1, \ldots)$ is an eigenvector with $z_1 \neq 0$.

Hence (a) each element of $\lambda \in \mathbf{F}$ is an eigenvalue of $T$.

And (b) the set of all eigenvectors of $T$ is $\{(z_1, \lambda z_1, \lambda^2 z_1, \ldots, \lambda^k z_1, \ldots) \in \mathbf{F}^\infty : \lambda \in \mathbf{F}, z_1 \neq 0\}$

---

**21** *Suppose $T \in \mathcal{L}(V)$ is invertible.*

(a) *Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$.*

*Prove that $\lambda$ is an eigenvalue of $T \iff \frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$.*

(b) *Prove that $T$ and $T^{-1}$ have the same eigenvectors.*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $v$.

Then $T^{-1}Tv = \lambda T^{-1}v = v \Rightarrow T^{-1}v = \frac{1}{\lambda}v$.

Hence $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$.

Suppose $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$ with an eigenvector $v$.

Then $TT^{-1}v = v = \frac{1}{\lambda}Tv \Rightarrow Tv = \lambda v$.

Hence $\lambda$ is an eigenvalue of $T$.

OR. Note that $(T^{-1})^{-1} = T$ and $\frac{1}{\frac{1}{\lambda}} = \lambda$. $\square$

---

**22** *Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors $u, w$ in $V$ such that $Tu = 3w$ and $Tw = 3u$. Prove that $3$ or $-3$ is an eigenvalue of $T$.*

**SOLUTION:**

COMMENT: $Tu = 3w, Tw = 3u \Rightarrow T(Tu) = 9u \Rightarrow T^2$ has an eigenvalue $9$.

$Tu = 3w, Tw = 3u \Rightarrow T(u+w) = 3(u+w), T(u-w) = 3(w-u) = -3(u-w)$.

Hence $3$ or $-3$ is an eigenvalue of $T$. $\square$

---

**23** *Suppose $V$ is finite-dim and $S, T \in \mathcal{L}(V)$.*

*Prove that $ST$ and $TS$ have the same eigenvalues.*

**SOLUTION:**

Suppose $\lambda$ is an eigenvalue of $ST$ with an eigenvector $v$.

Then $T(STv) = \lambda Tv = TS(Tv)$.

If $Tv \neq 0$, then $\lambda$ is an eigenvalue of $TS$.

Otherwise, $\lambda = 0$, $(\, v \neq 0, \lambda v = 0 = STv\,)$, then $T$ is not invertible

$\Rightarrow TS$ is not invertible $\Rightarrow (TS - 0I)$ is not invertible $\Rightarrow \lambda = 0$ is an eigenvalue of $TS$.

Reversing the roles of $T$ and $S$, we conclude that $ST$ and $TS$ have the same eigenvalues. $\square$

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**24** *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = Ax$, where elements of $\mathbf{F}^n$ are thought of as $n$-by-$1$ column vectors.*

(a) *Suppose the sum of the entries in each row of $A$ equals $1$.*

*Prove that $1$ is an eigenvalue of $T$.*

(b) *Suppose the sum of the entries in each column of $A$ equals $1$.*

*Prove that $1$ is an eigenvalue of $T$.*

**SOLUTION:**

(a) Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $Tx = Ax = \begin{pmatrix} \sum\limits_{c=1}^{n} A_{1,c}x_c \\ \vdots \\ \sum\limits_{c=1}^{n} A_{n,c}x_c \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. While $\sum\limits_{r=1}^{n} A_{R,c} = 1$ for each $R = 1, \ldots, n$.

Thus if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, and hence is an eigenvalue of $T$.

(b) Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $Tx = Ax = \begin{pmatrix} \sum\limits_{r=1}^{n} A_{1,r}x_r \\ \vdots \\ \sum\limits_{r=1}^{n} A_{n,r}x_r \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. While $\sum\limits_{r=1}^{n} A_{r,C} = 1$ for each $C = 1, \ldots, n$.

Thus $\sum\limits_{r=1}^{n} (Ax)_{r,\cdot} = \sum\limits_{r=1}^{n} (Ax)_{r,1}$

$$= \sum_{c=1}^{n} (A_{1,c} + \cdots + A_{n,c})x_c = \sum_{c=1}^{n} x_c = \lambda \begin{pmatrix} x_1 \\ + \\ \vdots \\ + \\ x_n \end{pmatrix}.$$

Hence $\lambda = 1$, for all $x$ such that $\sum\limits_{c=1}^{n} x_{c,1} \neq 0$. $\square$

OR. Prove that $(T - I)$ is not invertible, so that we can conclude $\lambda = 1$ is an eigenvalue.

Because $(T - I)x = (A - \mathcal{M}(I))x = \begin{pmatrix} \sum\limits_{r=1}^{n} A_{1,r}x_r - x_1 \\ \vdots \\ \sum\limits_{r=1}^{n} A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

Then $y_1 + \cdots + y_n = \sum\limits_{r=1}^{n} \sum\limits_{c=1}^{n} (A_{r,c}x_c - x_r) = \sum\limits_{c=1}^{n} x_c \sum\limits_{r=1}^{n} A_{r,c} - \sum\limits_{r=1}^{n} x_r = 0$.

Thus range $(T - I) \subseteq \{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbf{F}^n : y_1 + \cdots + y_n = 0 \}$. Hence $(T - I)$ is not surjective. $\square$

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• *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = xA$, where elements of $\mathbf{F}^n$ are thought of as 1-by-$n$ row vectors.*
(a) *Suppose the sum of the entries in each column of $A$ equals 1. Prove that 1 is an eigenvalue of $T$.*
(b) *Suppose the sum of the entries in each row of $A$ equals 1. Prove that 1 is an eigenvalue of $T$.*

**SOLUTION:**

(a) Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.

Then $Tx = xA = \begin{pmatrix} \sum\limits_{r=1}^{n} x_r A_{r,1} & \cdots & \sum\limits_{r=1}^{n} x_r A_{r,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. While $\sum\limits_{r=1}^{n} A_{r,C} = 1$ for each $C = 1, \ldots, n$.

Thus if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, hence is an eigenvalue of $T$.

(b) Suppose $\lambda$ is an eigenvalue of $T$ with an eigenvector $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.

Then $Tx = xA = \begin{pmatrix} \sum\limits_{c=1}^{n} x_c A_{c,1} & \cdots & \sum\limits_{c=1}^{n} x_c A_{c,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. While $\sum\limits_{c=1}^{n} A_{R,c} = 1$ for each $R = 1, \ldots, n$.

Thus $\sum\limits_{c=1}^{n} (xA)_{\cdot,c} = \sum\limits_{c=1}^{n} (xA)_{1,c} = \sum\limits_{c=1}^{n} (A_{c,1} + \cdots + A_{c,n})x_c = \sum\limits_{c=1}^{n} x_c = \lambda \begin{pmatrix} x_1 & + & \cdots & + & x_n \end{pmatrix}$.

Hence $\lambda = 1$, for all $x$ such that $\sum\limits_{r=1}^{n} x_{1,r} \neq 0$. $\square$

OR. Prove that $(T - I)$ is not invertible, so that we can conclude $\lambda = 1$ is an eigenvalue.

Because $(T-I)x = x(A-\mathcal{M}(I)) == \begin{pmatrix} \sum_{c=1}^{n} x_c A_{c,1} - x_1 & \cdots & \sum_{c=1}^{n} x_c A_{c,n} - x_n \end{pmatrix} = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}.$

Then $y_1 + \cdots + y_n = \sum_{c=1}^{n} \sum_{r=1}^{n} (x_r A_{r,c} - x_c) = \sum_{r=1}^{n} x_r \sum_{c=1}^{n} A_{r,c} - \sum_{c=1}^{n} x_c = 0.$

Thus range $(T-I) \subseteq \{ \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}. \in \mathbf{F}^n : y_1 + \cdots + y_n = 0 \}$. Hence $(T-I)$ is not surjective. □

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**25** *Suppose $T \in \mathcal{L}(V)$ and $u, w$ are eigenvectors of $T$*
*such that $u + w$ is also an eigenvector of $T$.*
*Prove that $u$ and $w$ are eigenvectors of $T$ corresponding to the same eigenvalue.*

**SOLUTION:**

Suppose $\lambda_1, \lambda_2, \lambda_0$ are eigenvalues of $T$ corresponding to $u, w, u+w$ respectively.

Then $T(u+w) = \lambda_0(u+w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w.$

Notice that $u, w, u+w$ are nonzero.

If $(u, w)$ is linearly dependent, then let $w = cu$, therefore

$$\begin{aligned} &\lambda_2 cu = Tw = cTu = \lambda_1 cu && \Rightarrow \lambda_2 = \lambda_1. \\ &\lambda_0(u+w) = T(u+w) = \lambda_1 u + \lambda_1 cu = \lambda_1(u+w) && \Rightarrow \lambda_0 = \lambda_1. \end{aligned}$$

Otherwise, $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0.$ □

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**26** *Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in $V$ is an eigenvector of $T$.*
*Prove that $T$ is a scalar multiple of the identity operator.*

**SOLUTION:**

Because $\forall v \in V, \exists! \, \lambda_v \in \mathbf{F}, Tv = \lambda_v v.$

Then for any two distinct nonzero vectors $v, w \in V$,

$T(v+w) = \lambda_{v+w}(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w.$

If $(v, w)$ is linearly independent, then let $w = cv$, therefore

$$\begin{aligned} &\lambda_v cv = cTv = Tw = \lambda_w w && \Rightarrow \lambda_w = \lambda_v. \\ &\lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v(v+cv) && \Rightarrow \lambda_{v+w} = \lambda_v. \end{aligned}$$

Otherwise, $\lambda_v = \lambda_{v+w} = \lambda_w.$ □

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**27, 28** *Suppose $V$ is finite-dim and $k \in \{1, ..., \dim V - 1\}$.*
*Suppose $T \in \mathcal{L}(V)$ is such that every subspace of $V$ of dim $k$ is invariant under $T$.*
*Prove that $T$ is a scalar multiple of the identity operator.*

**SOLUTION:**

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• *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$. Prove that*
*$T$ has an eigenvalue $\Longleftrightarrow \exists$ a subspace $U$ of $V$*
*such that $\dim U = \dim V - 1$, $U$ is invariant under $T$.*

**SOLUTION:**

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**29** *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.*
*Prove that $T$ has at most $1 + \dim$ range $T$ distinct eigenvalues.*

**SOLUTION:**

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $T$ and let $v_1, \ldots, v_m$ be the corresponding eigenvectors.

For every $\lambda_k \neq 0, T(\frac{1}{\lambda_k} v_k) = v_k$. And if $T = T - 0I$ is not invertible, then $\exists!\, \lambda_A = 0$ and $Tv_A = \lambda_A v_A = 0$.
Thus for $\lambda_k \neq 0, \forall\, k$, $(Tv_1, \ldots, Tv_m)$ is a linearly independent list of length $m$ in range $T$.
And for $\lambda_A = 0$, there is a linearly independent list of length at most $(m-1)$ in range $T$.
Hence, by [2.23], $m \leq \dim \operatorname{range} T + 1$. $\square$

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**30** *Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigenvalues of $T$.*
*Prove that $\exists\, x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.*

**Solution:** Because $9$ is not an eigenvalue. Hence $(T - 9I)$ is surjective. $\square$

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**31** *Suppose $V$ is finite-dim and $v_1, \ldots, v_m \in V$.*
*Prove that $(v_1, \ldots, v_m)$ is linearly independent*
*$\iff \exists\, T \in \mathcal{L}(V)$ such that $v_1, \ldots, v_m$ are eigenvectors of $T$*
*corresponding to distinct eigenvalues.*

**Solution:**

Suppose $(v_1, \ldots, v_m)$ is linearly independent, extend it to a basis of $V$ as $(v_1, \ldots, v_m, \ldots, v_n)$.
Then define $T \in \mathcal{L}(V)$ by $Tv_k = kv_k$ for each $k \in \{1, \ldots, m, \ldots, n\}$.
Conversely by [5.10] it is true as well. $\square$

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**32** *Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct real numbers.*
*Prove that $(e^{\lambda_1} x, \ldots, e^{\lambda_n} x)$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.*

Hint: *Let $V = \operatorname{span}(e^{\lambda_1} x, \ldots, e^{\lambda_n} x)$, and define an operator $D \in \mathcal{L}(V)$ by $Df = f'$.*
*Find eigenvalues and eigenvectors of $D$.*

**Solution:**

Define $V$ and $D \in \mathcal{L}(V)$ as in Hint. Then because for each $k$, $D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.
Thus $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of $D$. By [5.10], $(e^{\lambda_1} x, \ldots, e^{\lambda_n} x)$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$. $\square$

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• *Suppose $\lambda_1, \ldots, \lambda_n$ are distinct positive numbers.*
*Prove that $(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.*

**Solution:**

Let $V = \operatorname{span}(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.
Then because $D(\cos(\lambda_k x)) = -\lambda_k \sin(\lambda_k x)$. 又 $D(\sin(\lambda_k x)) = \lambda_k \cos(\lambda_k x)$.
Thus $D^2(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$.
Notice that $\lambda_1, \ldots, \lambda_n$ are distinct $\Rightarrow -\lambda_1^2, \ldots, -\lambda_n^2$ are distinct.
Hence $-\lambda_1^2, \ldots, -\lambda_n^2$ are distinct eigenvalues of $D^2$
with the corresponding eigenvectors $\cos(\lambda_1 x), \ldots, \cos(\lambda_n x)$ respectively.
And then $(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$. $\square$

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• *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.*
*Define $A \in \mathcal{L}(\mathcal{L}(V))$ by $A(S) = TS$ for each $S \in \mathcal{L}(V)$.*
*Prove that the set of eigenvalues of $T$ equals the set of eigenvalues of $A$.*

**Solution:**

(a) Suppose $\lambda_1, \ldots, \lambda_m$ are all eigenvalues of $T$ with eigenvectors $v_1, \ldots, v_m$ respectively.
Extend to a basis of $V$ as $(v_1, \ldots, v_m, \ldots, v_n)$.
Then for each $k \in \{1, \ldots, m\}$, $\operatorname{span}(v_k) \subseteq \operatorname{null}(T - \lambda_k I)$.
Define $S_k \in \mathcal{L}(V)$ by $S_k(v_j) = v_k$ for each $j \in \{1, \ldots, n\}$,

so that range $S_k = \text{span}(v_k)$ for each $k \in \{1, \dots, m\}$, then $A(S_k) = TS_k = \lambda_k S_k$.

Thus the eigenvalues of $T$ are eigenvalues of $A$.

(b) Suppose $\lambda_1, \dots, \lambda_m$ are all eigenvalues of $A$ with eigenvectors $S_1, \dots, S_m$ respectively.

Then for each $k \in \{1, \dots, m\}$, because $\forall v \in V, u = S_k(v) \in V \Rightarrow Tu = \lambda_k u$.

Thus the eigenvalues of $A$ are eigenvalues of $T$. $\square$

- COMMENT: Define $B \in \mathcal{L}(\mathcal{L}(V))$ by $B(S) = ST$ for all $S \in \mathcal{L}(V)$.

And the eigenvalues of $B$ are not the eigenvalues of $T$.

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- *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$, and $U$ is a subspace of $V$ invariant under $T$. The quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by*

$$(T/U)(v+U) = Tv + U \text{ for each } v \in V.$$

(a) *Show that the definition of $T/U$ makes sense (which requires using the condition that $U$ is invariant under $T$) and show that $T/U$ is an operator on $V/U$.*

(b) (OR Problem 35) *Show that each eigenvalue of $T/U$ is an eigenvalue of $T$.*

**SOLUTION:**

(a) Suppose $v + U = w + U$ ( $\Longleftrightarrow v - w \in U$ ).

Then because $U$ is invariant under $T$, $T(v-w) \in U \Longleftrightarrow Tv + U = Tw + U$.

Hence the definition of $T/U$ makes sense.

(b) Suppose $\lambda$ is an eigenvalue of $T/U$ with an eigenvector $v + U$.

Then $(T/U)(v+U) = \lambda(v+U) = Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v \in U$.

If $(T - \lambda I)v = 0 \Rightarrow Tv = \lambda v$, then we are done.

Otherwise, then $(T|_U - \lambda I) : U \to U$ is invertible,

hence $\exists !\, w \in U, (T|_U - \lambda I)(w) = (T - \lambda I)v \Rightarrow T(v-w) = \lambda(v-w)$.

Note that $v - w \neq 0$ ( for if not, $v \in U \Rightarrow v + U = 0 + U$ is not an eigenvector ).

Thus $\lambda$ is an eigenvalue of $T$. $\square$

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**36** *Prove or give an counterexample:*

*The result of* (b) *in Exercise 35 is still true if $V$ is infinite-dim.*

**SOLUTION:**

Consider $V = \text{span}(1, e^x, e^{2x}, \dots)$ in $\mathbb{R}^{\mathbb{R}}$, and a subspace $U = \text{span}(e^x, e^{2x}, \dots)$ of $V$.

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then range $T = U$ is invariant under $T$.

Consider $(T/U)(1+U) = e^x + U = 0$

$\Rightarrow$ 0 is an eigenvalue of $T/U$ but is not an eigenvalue of $T$

( null $T = \{0\}$, for if not, $\exists f \in V \backslash \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, contradicts ). $\square$

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**33** *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{range}\, T) = 0$.*

**SOLUTION:**

$\forall v + \text{range}\, T \in V/\text{range}\, T, v + \text{range}\, T \in \text{null}\,(T/(\text{range}\, T))$

$\Rightarrow \text{null}\,(T/(\text{range}\, T)) = V/\text{range}\, T \Rightarrow T/(\text{range}\, T)$ is a zero map.

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**34** *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null}\, T)$ is injective $\Longleftrightarrow (\text{null}\, T) \cap (\text{range}\, T) = \{0\}$.*

**SOLUTION:**

(a) Suppose $T/(\text{null}\, T)$ is injective.

Then $(T/(\text{null}\, T))(u + \text{null}\, T) = Tu + \text{null}\, T = 0$

$\iff Tu \in \operatorname{null} T$ 又 $Tu \in \operatorname{range} T \iff u + \operatorname{null} T = 0 \iff u \in \operatorname{null} T \iff Tu = 0.$

Thus $(\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$.

(b) Suppose $(\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$.

Then $(T/(\operatorname{null} T))(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0$

$\iff Tu \in \operatorname{null} T$ 又 $Tu \in \operatorname{range} T \iff Tu = 0 \iff u \in \operatorname{null} T \iff u + \operatorname{null} T = 0.$

Thus $T/(\operatorname{null} T)$ is injective. $\square$

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**ENDED**

# 5.B

• *Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$.*

**SOLUTION:**

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• *Suppose $T \in \mathcal{L}(V)$ has no eigenvalues and $T^4 = I$. Prove that $T^2 = -I$.*

**SOLUTION:**

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• *Suppose $T \in \mathcal{L}(V)$ and $m$ is a positive integer.*

(a) *Prove that $T$ is injective $\iff T^m$ is injective.*

(b) *Prove that $T$ is surjective $\iff T^m$ is surjective.*

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