# 简介

这是我个人用于复习的「***Linear Algebra Done Right 3E/4E, by Sheldon Axler***」笔记，一本习题选答与课文补注。因为使用中文会给我编撰这份笔记带来额外的中英文输入法切换的工作成本，况且对于专业学习者，直接使用英文不会造成任何困扰。但英文词句的冗长性拖慢我复习的效率，所以我对许多常用术语作了简写。这份笔记的内容范围和标识说明，我已经在自述中写得很清楚，不再赘述。这份笔记尚处于缓慢的编撰进度中。

我目前还没有能力和资格评论原书好坏以及线性代数课程教材选用的问题。但作为原书的学习者，我可以说，这本书作为初学线性代数的第一教材，虽然不需要其他辅助教材，但要求学习者有足够的耐心和毅力：课文一次看不懂就多看几遍，一天看不懂就分三天看；习题一个小时做不出来，隔六个小时再尝试，一天做不出来，就隔天再尝试。我虽然没有学过除此以外的其他任何线性代数教材，但我认为这样钻研原书是值得的。

## Goto

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| A | A | A | | A | A | A | A | A | A |
| B | B | B | | $B^{I}$ | B | B | B | B | B |
| | | | | $B^{II}$ | | | | | |
| C | C | C | | C | C | C | C | | |
| | | D | | | D | D | D | | |
| | | E | | E* | | | | | |
| | | F | | | | F* | | | |

## Abbreviation Table

| | | | |
|---|---|---|---|
| def | definition | vec | vector |
| vecsp | vector space | subsp | subspace |
| add | addition/additive | multi | multiplication/multiplicative/multiple |
| assoc | associative/associativity | distr | distributive properties/property |
| inv | inverse | existns | existence |
| uniqnes | uniqueness | linely inde | linearly independent/independence |
| linely dep | linearly dependent/dependence | dim | dimension(al) |
| coeff | coefficient | degree | deg |
| req | require(d)/requiring | $B_V$ | basis of $V$ |
| inje | injective | surj | surjective |
| col | column | with resp | with respect |
| standard basis | std basis | iso | isomorphism/isomorphic |
| correspd | correspond(ing) | poly | polynomial |
| eigval | eigenvalue | eigvec | eigenvector |
| mini poly | minimal polynomial | char poly | characteristic polynomial |

# 1·B

**1** *Prove that* $\forall v \in V, -(-v) = v.$

**Solution:** $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v.$

Or. Because $-(-v) + (-v) = 0$ 又 $v + (-v) = 0$. Now by the uniqnes of add inv. □

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**2** *Suppose* $a \in \mathbf{F}, v \in V$, *and* $av = 0$. *Prove that* $a = 0$ *or* $v = 0$.

**Solution:** Suppose $a \neq 0, \exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0.$ □

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**3** *Suppose* $v, w \in V$. *Explain why* $\exists!\, x \in V, v + 3x = w$.

**Solution:** $v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \frac{1}{3}(w - v).$ □

Or. [*Existns*] Let $x = \dfrac{1}{3}(w - v)$.

[*Uniqnes*] If $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) − (II) : $3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. □

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**5** *Show that in the def of a vecsp, the add inv condition can be replaced by* [1.29].

***Hint***: *Suppose V satisfies all conds in the def, except we've replaced the add inv cond with* [1.29].
*Prove that the add inv is true.*

Using [1.31]. $0v = 0$ for all $v \in V \Longleftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0.$ □

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**6** *Let* $\infty$ *and* $-\infty$ *denote two distinct objects, neither of which is in* $\mathbf{R}$.
*Define an add and scalar multi on* $\mathbf{R} \cup \{\infty, -\infty\}$ *as you could guess.*
*The operations of real numbers is as usual. While for* $t \in \mathbf{R}$ *define*

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

(I) $t + \infty = \infty + t = \infty + \infty = \infty,$
(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty,$
(III) $\infty + (-\infty) = (-\infty) + \infty = 0.$

*With these operations of add and scalar multi, is* $\mathbf{R} \cup \{\infty, -\infty\}$ *a vecsp over* $\mathbf{R}$? *Explain.*

**Solution:** Not a vecpsp, since the add and scalar mult is not assoc and distr.
By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty)).$
Or. By Distr: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0.$ □

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• **Tips:** *About the Field* $\mathbf{F}$ : *Many choices.*
**Example:** $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \ldots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$. [ Using Euler's Theorem. ]

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Ended

# 1·C

7 8 9 11 12 13 15 16 17 18 21 23 24

• **Note For [1.45]:** If $\mathbf{F} = \{0, 1\}$. Prove that if $U + W$ is a direct sum, then $U \cap W = \{0\}$.
Because $\forall v \in U \cap W, \exists!\,(u, w) \in U \times W, v = u + w$.
If $U \cap W \neq \{0\}$, then $(u, w)$ can be $(v, 0)$ or $(0, v)$, contradicts the uniqnes. □

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• **TIPS:** Suppose $U, W \subseteq V$. And $U, W, V$ are vecsps $\Rightarrow U, W$ are subsps of $V$.
Then $U + W$ is also a subsp of $V$. Because $\forall u \in U, w \in U, u + w \in V$ since $u, w \in V$.

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**7** *Give a nonempty $U \subseteq \mathbf{R}^2$,*
*$U$ is closed under taking add invs and under add, but is not a subsp of $\mathbf{R}^2$.*

**SOLUTION:** $\big($ $0 \in U$; $v \in U \Rightarrow -v \in U$. And operations on $U$ are the same as $\mathbf{R}^2$. $\big)$ Let $\mathbf{Z}^2, \mathbf{Q}^2$.

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**8** *Give a nonempty $U \subseteq \mathbf{R}^2$, $U$ is closed under scalar multi, but is not a subsp of $\mathbf{R}^2$.*

**SOLUTION:** Let $U = \{(x, y) \in \mathbf{R}^2 : x = 0 \vee y = 0\}$.

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**9** *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if $\exists\, p \in \mathbf{N}^+$, $f(x) = f(x + p)$ for all $x \in \mathbf{R}$.*
*Is the set of periodic functions $\mathbf{R} \to \mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$ ? Explain.*

**SOLUTION:** Denote the set by $S$.

Suppose $h(x) = \cos x + \sin \sqrt{2}x \in S$, since $\cos x, \sin \sqrt{2}x \in S$.

Assume $\exists\, p \in \mathbf{N}^+$ such that $h(x) = h(x + p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin \sqrt{2}p = \cos p - \sin \sqrt{2}p$

$\Rightarrow \sin \sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Contradiction! $\square$

OR. Because $[\mathrm{I}] : \cos x + \sin \sqrt{2}x = \cos(x + p) + \sin\left(\sqrt{2}x + \sqrt{2}p\right)$. By differentiating twice,

$$[\mathrm{II}] : \cos x + 2\sin \sqrt{2}x = \cos(x + p) + 2\sin\left(\sqrt{2}x + \sqrt{2}p\right).$$

$$\left.\begin{array}{ll} [\mathrm{II}] - [\mathrm{I}] : & \sin \sqrt{2}x = \sin\left(\sqrt{2}x + \sqrt{2}p\right) \\ 2[\mathrm{I}] - [\mathrm{II}] : & \cos x = \cos(x + p) \end{array}\right\} \Rightarrow \text{Let } x = 0,\ p = \frac{m\pi}{\sqrt{2}} = 2k\pi.\ \text{Contradicts.}$$ $\square$

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• *Suppose $U, W, V_1, V_2, V_3$ are subsps of $V$.*

**15** $U + U \ni u + w \in U$. **16** $U + W \ni u + w = w + u \in W + U$. $\square$

**17** $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3)$. $\square$

• $(U + W)_{\mathbf{C}} \ni (u_1 + w_1) + \mathrm{i}(u_2 + w_2) = (u_1 + \mathrm{i}u_2) + (w_1 + \mathrm{i}w_2) \in U_{\mathbf{C}} + W_{\mathbf{C}}$. $\square$

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**18** *Does the add on the subsps of $V$ have an add identity? Which subsps have add invs?*

**SOLUTION:** Suppose $\Omega$ is the unique add identity.

(a) For any subsp $U$ of $V$. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

(b) Now suppose $W$ is an add inv of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. $\square$

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**11** *Prove that the intersection of every collection of subsps of $V$ is a subsp of $V$.*

**SOLUTION:** Suppose $\{U_\alpha\}_{\alpha \in \Gamma}$ is a collection of subsps of $V$; here $\Gamma$ is an index set.

We show that $\bigcap_{\alpha \in \Gamma} U_\alpha$, which equals the set of vecs that are in $U_\alpha$ for each $\alpha \in \Gamma$, is a subsp of $V$.

(一) $0 \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha \in \Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Closed under add.

(三) $u \in \bigcap_{\alpha \in \Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Closed under scalar multi.

Thus $\bigcap_{\alpha \in \Gamma} U_\alpha$ is nonempty subset of $V$ that is closed under add and scalar multi. $\square$

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**12** *Suppose $U, W$ are subsps of $V$. Prove that $U \cup W$ is a subsp of $V \iff U \subseteq W$ or $W \subseteq U$.*

**SOLUTION:** (a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subsp of $V$.

(b) Suppose $U \cup W$ is a subsp of $V$. Assume that $U \not\subseteq W$, $U \not\supseteq W$ ( $U \cup W \neq U$ and $W$ ).

Then $\forall a \in U \wedge a \notin W, \forall b \in W \wedge b \notin U$, we have $a + b \in U \cup W$.

$a + b \in U \Rightarrow b = (a + b) + (-a) \in U$, contradicts $\Rightarrow W \subseteq U$. | Contradicts the

$a + b \in W \Rightarrow a = (a + b) + (-b) \in W$, contradicts $\Rightarrow U \subseteq W$. | assumption. □

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**13** *Prove that the union of three subsps of $V$ is a subsp of $V$ if and only if one of the subsps contains the other two.*

*This exercise is not true if we replace $\mathbf{F}$ with a field containing only two elements.*

**SOLUTION:**

Suppose $U_1, U_2, U_3$ are subsps of $V$. Denote $U_1 \cup U_2 \cup U_3$ by $\mathcal{U}$.

(a) Suppose that one of the subsps contains the other two.

Then $\mathcal{U} = U_1, U_2$ or $U_3$ is a subsp of $V$.

(b) Suppose that $U_1 \cup U_2 \cup U_3$ is a subsp of $V$.

Distinctively notice that $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.

Also note that, if $U \cup W = V$ is a vecsp, then in general $U$ and $W$ are not subsps of $V$.

Hence this literal trick is invalid.

(I) If any $U_j$ is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.

By applying Problem (12) we conclude that one $U_j$ contains the other two. Thus we are done.

(II) Assume that no $U_j$ is contained in the union of the other two, and no $U_j$ contains the union of the other two. Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3$; $v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if any $v + \lambda u \in W \cap U_1$ then $v + \lambda u - \lambda u = v \in U_1$.

Now $W \subseteq U_1 \cup U_2 \cup U_3 \Rightarrow W \subseteq U_2 \cup U_3$. $\forall v + \lambda u \in W, v + \lambda u \in U_i, i = 2, 3$.

If $U_2 \subseteq U_3$ or $U_2 \supseteq U_3$, then $\mathcal{U} = U_1 \cup U_i, i = 2, 3$. By Problem (12) we are done.

Otherwise, both $U_2, U_3 \neq \{0\}$. Because $W \subseteq U_2 \cup U_3$ *has at least three elements.*

There must be some $U_i$ that contains at least two elements of $W$.

$\exists$ distinct $\lambda_1, \lambda_2 \in \mathbf{F}, v + \lambda_1 u, v + \lambda_2 u \in U_i, i \in \{2, 3\}$.

Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Contradicts. □

**EXAMPLE:** Let $\mathbf{F} = \mathbf{Z}_2$. $U_1 = \{u, 0\}, U_2 = \{v, 0\}, U_3 = \{v + u, 0\}$. While $\mathcal{U} = \{0, u, v, v + u\}$ is a subsp.

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• **EXAMPLE:** *Suppose $U = \{(x, x, y, y) \in \mathbf{F}^4\}, W = \{(x, x, x, y) \in \mathbf{F}^4\}$. Prove that $U + W = \{(x, x, y, z) \in \mathbf{F}^4\}$.*

Let T denote $\{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$. By def, $U + W \subseteq T$.

And $T \ni (x, x, y, z) \Rightarrow (0, 0, y - x, y - x) + (x, x, x, -y + x + z) \in U + W$. Hence $T \subseteq U + W$. □

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**21** *Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5\}$. Find a $W$ such that $\mathbf{F}^5 = U \oplus W$.*

**SOLUTION:** Let $W = \{(0, 0, z, w, u) \in \mathbf{F}^5\}$. Then $U \cap W = \{0\}$.

And $\mathbf{F}^5 \ni (x, y, z, w, u) \Rightarrow (x, y, x + y, x - y, 2x) + (0, 0, z - x - y, w - x - y, u - 2x) \in U + W$.

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**23** *Give an example of vecsps $V_1, V_2, U$ such that $V_1 \oplus U = V_2 \oplus U$, but $V_1 \neq V_2$.*

**Solution:** $V = \mathbf{F}^2, U = \{(x,x) \in \mathbf{F}^2\}, V_1 = \{(x,0) \in \mathbf{F}^2\}, V_2 = \{(0,x) \in \mathbf{F}^2\}$.

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• **Tips:** *Suppose $V_1 \subseteq V_2$ in Exercise (23). Prove or give a counterexample: $V_1 = V_2$.*

**Solution:**

Because the subset $V_1$ of vecsp $V_2$ is closed under add and scalar multi, $V_1$ is a subspace of $V_2$.
Suppose $W$ is such that $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$.
If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, contradicts. Hence $W = \{0\}, V_1 = V_2$. ☐

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• *Suppose $V_1, V_2, U_1, U_2$ are vecsps, $V_1 \oplus U_1 = V_2 \oplus U_2$, $V_1 \subseteq V_2$, $U_2 \subseteq U_1$.*
*Prove or give a counterexample: $V_1 = V_2$, $U_1 = U_2$.*

| $V_1$ | $U_1$ | |
|---|---|---|
| | $V_2$ | $U_2$ |

**Solution:** A counterexample: [ Using notations in Chapter 2. ]
Let $V = \mathbf{F}^3, B_V = (e_1, e_2, e_3)$, $V_1 = \operatorname{span}(e_1)$, $U_1 = \operatorname{span}(e_2, e_3)$, $V_2 = \operatorname{span}(e_1, e_2)$, $U_2 = \operatorname{span}(e_3)$.
Now $V_1 \subseteq V_2$, $U_2 \subseteq U_1$ and $V_1 \oplus U_1 = V_2 \oplus U_2$. But $V_1 \neq V_2$, $U_1 \neq U_2$. ☐

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**24** *Let $V_E = \{f \in \mathbf{R}^\mathbf{R} : f \text{ is even}\}, V_O = \{f \in \mathbf{R}^\mathbf{R} : f \text{ is odd}\}$. Show that $V_E \oplus V_O = \mathbf{R}^\mathbf{R}$.*

**Solution:** (a) $V_E \cap V_O = \{f \in \mathbf{R}^\mathbf{R} : f(x) = f(-x) = -f(-x)\} = \{0\}$.

$$\text{(b)} \left| \begin{array}{l} \text{Let } f_e(x) = \dfrac{1}{2}[g(x) + g(-x)] \Longrightarrow f_e \in V_E \\ \text{Let } f_o(x) = \dfrac{1}{2}\left[g(x) - g(-x)\right] \Longrightarrow f_o \in V_O \end{array} \right\} \Rightarrow \forall g \in \mathbf{R}^\mathbf{R},\ g(x) = f_e(x) + f_o(x).$$ ☐

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**Ended**

## 2·A   1 2 6 10 11 14 16 17 | 4E: 3,14

**2** (a) $[P]$ *A list $(v)$ of length 1 in $V$ is linely inde $\Longleftrightarrow v \neq 0$.* $[Q]$
(b) $[P]$ *A list $(v, w)$ of length 2 in $V$ is linely inde $\Longleftrightarrow \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$.* $[Q]$

**Solution:**

(a) $Q \overset{1}{\Rightarrow} P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ linely inde.
$P \overset{2}{\Rightarrow} Q : (v)$ linely inde $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.

Or. $\neg Q \overset{3}{\Rightarrow} \neg P : v = 0 \Rightarrow av = 0$ while we can let $a \neq 0 \Rightarrow (v)$ is linely dep.
$\neg P \overset{4}{\Rightarrow} \neg Q : (v)$ linely dep $\Rightarrow av = 0$ while $a \neq 0 \Rightarrow v = 0$.

**Comment:** (1) with (3) and (2) with (4) will do as well. ☐

(b) $P \overset{1}{\Rightarrow} Q : (v, w)$ linely inde $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.
$Q \overset{2}{\Rightarrow} P :$ no scalar multi $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v, w)$ linely inde.

Or. $\neg P \overset{3}{\Rightarrow} \neg Q : (v, w)$ linely dep $\Rightarrow$ if $av + bw = 0$, then $a$ or $b \neq 0 \Rightarrow$ scalar multi
$\neg Q \overset{4}{\Rightarrow} \neg P :$ scalar multi $\Rightarrow$ if $av + bw = 0$, then $a$ or $b \neq 0 \Rightarrow$ linely dep.

**Comment:** (1) with (3) and (2) with (4) will do as well. ☐

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**1** *Prove that* $[P]$ $(v_1, v_2, v_3, v_4)$ *spans* $V \iff \left(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\right)$ *also spans* $V$ $[Q]$.

**SOLUTION:**

Notice that $V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists\, a_1, \dots, a_n \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n$.

Assume that $\forall v \in V, \exists\, a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbf{F}$, ( that is, if $\exists\, a_i$, then we are to find $b_i$, vice versa )

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4. \end{aligned}$$

Now we can let $b_i = \sum_{r=1}^{i} a_r$ if we are to prove $Q$ with $P$ already assumed;

or let $a_i = b_i - b_{i-1}$ with $b_0 = 0$, if we are to prove $P$ with $Q$ already assumed. $\square$

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**6** *Prove that* $[P]$ $(v_1, v_2, v_3, v_4)$ *is linely inde*

$\iff \left(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\right)$ *is linely inde.* $[Q]$

**SOLUTION:**

$$\begin{aligned} P \Rightarrow Q :\ & a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0 \\ & \Rightarrow a_1 v_1 + (a_2 - a_1) v_2 + (a_3 - a_2) v_3 + (a_4 - a_3) v_4 = 0 \Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \\ Q \Rightarrow P :\ & a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0 \\ & \Rightarrow a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \cdots + a_4) v_4 = 0 \\ & \Rightarrow a_1 = a_1 + a_2 = a_1 + a_2 + a_3 = a_1 + \cdots + a_4 = 0. \end{aligned}$$ $\square$

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• *Suppose* $(v_1, \dots, v_m)$ *is a list of vecs in* $V$. *For each* $k$, *let* $w_k = v_1 + \cdots + v_k$.

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

(b) *Show that* $[P]$ $(v_1, \dots, v_m)$ *is linely inde* $\iff$ $(w_1, \dots, w_m)$ *is linely inde* $[Q]$.

**SOLUTION:**

(a) Assume $a_1 v_1 + \cdots + a_m v_m = b_1 w_1 + \cdots + b_m w_m = b_1 v_1 + \cdots + b_k(v_1 + \cdots + v_k) + \cdots + b_m(v_1 + \cdots + v_m)$.

Then $a_k = b_k + \cdots + b_m;\ \ a_{k+1} = b_{k+1} + \cdots + b_m \Rightarrow b_k = a_k - a_{k+1};\ b_m = a_m$. Similar to Problem (1).

(b) $P \Rightarrow Q :\ \ b_1 w_1 + \cdots + b_m w_m = 0 = a_1 v_1 + \cdots + a_m v_m$, where $0 = a_k = b_k + \cdots + b_m$.

$Q \Rightarrow P :\ \ a_1 v_1 + \cdots + a_m v_m = 0 = b_1 w_1 + \cdots + b_m w_m = 0$, where $0 = b_m = a_m$, $0 = b_k = a_k - a_{k+1}$.

OR. Because $W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

By [2.21](b), a list of length $(m-1)$ spans $W$, then by [2.23],

$(w_1, \dots, w_m)$ linely dep $\Longrightarrow$ $(v_1, \dots, v_m)$ linely dep. Conversely it is true as well. $\square$

---

**10** *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* $V$ *and* $w \in V$.

*Prove that if* $(v_1 + w, \dots, v_m + w)$ *is linely depe, then* $w \in \operatorname{span}(v_1, \dots, v_m)$.

**SOLUTION:**

Suppose $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0, \exists\, a_i \neq 0 \Rightarrow a_1 v_1 + \cdots + a_m v_m = -(a_1 + \cdots + a_m) w$.

Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1 v_1 + \cdots + a_m v_m = 0$ while $a_i \neq 0$ for some $i$, contradicts. $\square$

OR. By contrapositive: Prove that $w \notin \operatorname{span}(v_1, \dots, v_m) \Longrightarrow (v_1 + w, \dots, v_m + w)$ is linely inde.

Suppose $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0 \Rightarrow a_1 v_1 + \cdots + a_m v_m = -(a_1 + \cdots + a_m) w$.

Now by assumption, $a_1 + \cdots + a_m = 0$. Then $a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$. $\square$

OR. $\exists\, j \in \{1, \dots, m\}, v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and we are done.

If $j \geqslant 2$, then $\exists\, a_i \in \mathbf{F}, v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1 v_1 + \cdots + a_{j-1} v_{j-1}$.

Where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, contradicts.

Now $w = \lambda^{-1}(a_1 v_1 + \cdots + a_{j-1} v_{j-1} - v_j) \Rightarrow w \in \operatorname{span}(v_1, \dots, v_m)$. $\square$

---

**11** *Suppose $(v_1, \dots, v_m)$ is linely inde in $V$ and $w \in V$.*
*Show that $[P]$ $(v_1, \dots, v_m, w)$ is linely inde $\iff w \notin \operatorname{span}(v_1, \dots, v_m)$ $[Q]$.*

**Solution:** $\neg Q \Rightarrow \neg P$ : Suppose $w \in \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1, \dots, v_m, w)$ is linely depe.
$\neg P \Rightarrow \neg Q$ : Suppose $(v_1, \dots, v_m, w)$ is linely dep. Then by [2.21](a), $w \in \operatorname{span}(v_1, \dots, v_m)$. □

---

**14** *Prove that $[P]$ $V$ is infinite-dim $\iff$ $[Q]$ | there is a sequence $(v_1, v_2, \dots)$ in $V$ such that $(v_1, \dots, v_m)$ is linely inde for each $m \in \mathbf{N}^+$.*

**Solution:**

$P \Rightarrow Q$ : Suppose $V$ is infinite-dim, so that no list spans $V$.
**Step 1** Pick a $v_1 \neq 0$, $(v_1)$ linely inde.
**Step m** Pick a $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$, by Problem (11), $(v_1, \dots, v_m)$ is linely inde.
This process recursively defines the desired sequence $(v_1, v_2, \dots)$.

$\neg P \Rightarrow \neg Q$ : Suppose $V$ is finite-dim and $V = \operatorname{span}(w_1, ..., w_m)$.
Let $(v_1, v_2, \dots)$ be a sequence in $V$, then $(v_1, v_2, \dots, v_{m+1})$ must be linely dep.

Or. $Q \Rightarrow P$ : Suppose there is such a sequence.
Choose an $m$. Suppose a linely inde list $(v_1, \dots, v_m)$ spans $V$.
Similar to [2.16]. $\exists\, v_{m+1} \in V \backslash \operatorname{span}(v_1, \dots, v_m)$. Hence no list spans $V$. □

---

**16** *Prove that the vecsp of all continuous functions in $\mathbf{R}^{[0,1]}$ is infinite-dim.*

**Solution:** Denote the vecsp by $U$.

Choose one $m \in \mathbf{N}^+$. Suppose $a_0, \dots, a_m \in \mathbf{R}$ are such that $p(x) = a_0 + a_1 x + \dots + a_m x^m = 0$, $\forall x \in [0, 1]$.
Then $p$ has infinitely many roots and hence each $a_k = 0$, otherwise $\deg p \geqslant 0$, contradicts [4.12].
Thus $(1, x, \dots, x^m)$ is linely inde in $\mathbf{R}^{[0,1]}$. Similar to [2.16], $U$ is infinite-dim. □

Or. Note that $\dfrac{1}{1} > \dfrac{1}{2} > \dots > \dfrac{1}{m}$, $\forall m \in \mathbf{N}^+$. Suppose $f_m = \begin{cases} x - \dfrac{1}{m}, & x \in \left( \dfrac{1}{m}, 1 \right] \\ 0, & x \in \left[ 0, \dfrac{1}{m} \right] \end{cases}$

Then $f_1\left(\dfrac{1}{m}\right) = \dots = f_m\left(\dfrac{1}{m}\right) = 0 \neq f_{m+1}\left(\dfrac{1}{m}\right)$. Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. By Problem (14). □

---

**17** *Suppose $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$.*
*Prove that $(p_0, p_1, \dots, p_m)$ is not linely inde in $\mathcal{P}_m(\mathbf{F})$.*

**Solution:**

Suppose $(p_0, p_1, \dots, p_m)$ is linely inde. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z$.
Notice that $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \dots + a_m p_m(z)$, for if not, let $z = 2$. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has length $(m+1)$.
Hence $(p_0, p_1, \dots, p_m)$ is linely depe in $\mathcal{P}_m(\mathbf{F})$.
For if not, then because $(1, z, \dots, z^m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$,
by the steps in [2.23] trivially, $(p_0, p_1, \dots, p_m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. □

Or. Note that $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(\underbrace{1, z, \dots, z^m}_{\text{of length } (m+1)})$. Then $(p_0, p_1, \dots, p_m, z)$ of length $(m+2)$ is linely dep.

As shown above, $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$. And hence by [2.21](a), $(p_0, p_1, \dots, p_m)$ is linely dep. □

---

**Ended**

## 2·B   1 7 8 | 4E: 5 9 11

**7** *Prove or give a counterexample: If $(v_1, v_2, v_3, v_4)$ is a basis of $V$ and $U$ is a subsp of $V$ such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then $(v_1, v_2)$ is a basis of $U$.*

**Solution:** A counterexample: Let $V = \mathbf{R}^4$ and $B_V = (e_1, e_2, e_3, e_4)$ be std basis.
Let $v_1 = e_1, v_2 = e_2, v_3 = e_3 + e_4, v_4 = e_4$. Then $(v_1, \dots, v_4)$ is a basis of $\mathbf{R}^4$.
Let $U = \operatorname{span}(e_1, e_2, e_3) = \operatorname{span}(v_1, v_2, v_3 - v_4)$. Then $v_3 \notin U$ and $(v_1, v_2)$ is not a basis of $U$. □

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• **Note For "$\complement_V U \cup \{0\}$":** "$\complement_V U \cup \{0\}$" is supposed to be a subsp $W$ such that $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cup \{0\} \\ u \pm w \in \complement_V U \cup \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cup \{0\}$. Contradicts.

To fix this, denote the set $\{W_1, W_2, \cdots\}$ by $\mathcal{S}_V U$, where for each $W_i$, $V = U \oplus W_i$. See also in (1.C.23).

---

• **Tips:** *Suppose $V$ is finite-dim with* $\dim V = n$ *and $U$ is a subsp of $V$ with $U \neq V$.*
*Prove that $\exists\, B_V = (v_1, \dots, v_n)$ such that each $v_k \notin U$.*

Note that $U \neq V \Rightarrow n \geqslant 1$. We will construct $B_V$ via the following process.

**Step 1.** $\exists\, v_1 \in V \backslash U \Rightarrow v_1 \neq 0$. If $\operatorname{span}(v_1) = V$ then we stop.

**Step k.** Suppose $(v_1, \dots, v_{k-1})$ is linely inde in $V$, each of which belongs to $V \backslash U$.
Note that $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$. And if $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U = V$, then by (1.C.12),
$\big[$ because $\operatorname{span}(v_1, \dots, v_{k-1}) \not\subseteq U$, $\big]$ $U \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) \Rightarrow \operatorname{span}(v_1, \dots, v_{k-1}) = V$.
Hence because $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$, it must be case that $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U \neq V$.
Thus $\exists\, v_k \in V \backslash U$ such that $v_k \notin \operatorname{span}(v_1, \dots, v_{k-1})$.
By (2.A.11), $(v_1, \dots, v_k)$ is linely inde in $V$. If $\operatorname{span}(v_1, \dots, v_k) = V$, then we stop.

Because $V$ is finite-dim, this process will stop after $n$ steps. □

Or. Suppose $U \neq \{0\}$. Let $B_U = (u_1, \dots, u_m)$. Extend to a basis $(u_1, \dots, u_n)$ of $V$.
Then let $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$. □

---

**1** *Find all vecsps on whatever* $\mathbf{F}$ *that have exactly one basis.*

**Solution:** The trivial vecsp $\{0\}$ will do. Indeed, the only basis of $\{0\}$ is the empty list ( ).
Now consider the field $\{0, 1\}$ containing only the add identity and multi identity,
with $1 + 1 = 0$. Then the list $(1)$ is the unique basis. Now the vecsp $\{0, 1\}$ will do.
**Comment:** All vecsp on such $\mathbf{F}$ of dim 1 will do.
And more generally, consider $\mathbf{F} = \mathbf{Z}_m$, $\forall m - 1 \in \mathbf{N}^+$. For each $s, t \in \{1, \dots, m\}$,
$\mathbf{F} = \operatorname{span}(K_s) = \operatorname{span}(K_t)$. More than one basis. So are $\mathbf{Q}, \mathbf{R}, \mathbf{C}$ and all vecsps on such $\mathbf{F}$.
Consider other $\mathbf{F}$. Note that this $\mathbf{F}$ contains at least and strictly more than 0 and 1. Failed. □

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• (4E 9) *Suppose $(v_1, \dots, v_m)$ is a list of vecs in $V$. For $k \in \{1, \dots, m\}$, let $w_k = v_1 + \cdots + v_k$. Show that $[P]$ $B_V = (v_1, \dots, v_m) \iff B_W = (w_1, \dots, w_m)$. $[Q]$*

**Solution:** Notice that $B_U = (u_1, \dots, u_n) \iff \forall u \in U, \exists!\, a_i \in \mathbf{F}, u = a_1 u_1 + \cdots + a_n u_n$.
$P \Rightarrow Q: \forall v \in V, \exists!\, a_i \in \mathbf{F},\ v = a_1 v_1 + \cdots + a_m v_m \Rightarrow v = b_1 w_1 + \cdots + b_m w_m, \exists!\, b_k = a_k - a_{k+1}, b_m = a_m$.
$Q \Rightarrow P: \forall v \in V, \exists!\, b_i \in \mathbf{F},\ v = b_1 w_1 + \cdots + b_m w_m \Rightarrow v = a_1 v_1 + \cdots + a_m v_m, \exists!\, a_k = \sum_{j=k}^{m} b_j$. □
**Comment:** See also ??? in (3.F).

---

• (4E 5) *Suppose $U, W$ are finite-dim, $V = U + W$, $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$. Prove that $\exists\, B_V$ consisting of vecs in $U \cup W$.*

**SOLUTION:** $V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(\overbrace{u_1, \dots, u_m, w_1, \dots, w_n}^{\text{Reduce}})$. By [2.31]. ☐

---

**8** *Suppose $V = U \oplus W$, $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$. Prove that $B_V = \big(u_1, \dots, u_m, w_1, \dots, w_n\big)$.*

**SOLUTION:** $\forall v \in V, \exists\, !\, u \in U, w \in W \Rightarrow \exists\, !\, a_i, b_i \in \mathbf{F}, v = u + w = \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i$.

OR. $V = \operatorname{span}(u_1, \dots, u_m) \oplus \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}\big(u_1, \dots, u_m, w_1, \dots, w_n\big)$.

Note that $\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0 \Rightarrow \sum_{i=1}^{m} a_i u_i = -\sum_{i=1}^{n} b_i w_i \in U \cap W = \{0\}$. ☐

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• (9.A.3,4 OR 4E 11) *Suppose $V$ is on $\mathbf{R}$, and $v_1, \dots, v_n \in V$. Let $B = (v_1, \dots, v_n)$.*

(a) *Show that $[P]$ $B$ is linely inde in $V \iff B$ is linely inde in $V_{\mathbf{C}}$. $[Q]$*

(b) *Show that $[P]$ $B$ spans $V \iff B$ spans $V_{\mathbf{C}}$. $[Q]$*

(a) $P \Rightarrow Q$ : Note that each $v_k \in V_{\mathbf{C}}$. $\quad Q \Rightarrow P$ : If $\lambda_k \in \mathbf{R}$ with $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then each $\operatorname{Re} \lambda_k = \lambda_k = 0$.

$\neg P \Rightarrow \neg Q$ : $\exists\, v_j = a_{j-1} v_{j-1} + \cdots + a_1 v_1 \in V_{\mathbf{C}}$.

$\neg Q \Rightarrow \neg P$ : $\exists\, v_j = \lambda_{j-1} v_{j-1} + \cdots + \lambda_1 v_1 \Rightarrow v_j = (\operatorname{Re} \lambda_{j-1}) v_{j-1} + \cdots + (\operatorname{Re} \lambda_1) v_1 \in V$.

(b) $P \Rightarrow Q$ : $\forall u + \mathrm{i}v \in V_{\mathbf{C}},\ u, v \in V \Rightarrow \exists\, a_i, b_i \in \mathbf{R}, u + \mathrm{i}v = \sum_{i=1}^{n} (a_i + \mathrm{i} b_i) v_i$.

$Q \Rightarrow P$ : $\forall v \in V, \exists\, a_i + \mathrm{i}b_i \in \mathbf{C},\ v + \mathrm{i}0 = \big(\sum_{i=1}^{n} a_i v_i\big) + \mathrm{i}\,\big(\sum_{i=1}^{n} b_i v_i\big) \Rightarrow v \in \operatorname{span}(v_1, \dots, v_m)$.

$\neg Q \Rightarrow \neg P$ : $\exists\, v \in V, v \notin \operatorname{span}(B) \Rightarrow v + \mathrm{i}0 \notin \operatorname{span}(B)$ while $v + \mathrm{i}0 \in V_{\mathbf{C}}$.

$\neg Q \Rightarrow \neg P$ : $\exists\, u + \mathrm{i}v \in V_{\mathbf{C}}, u + \mathrm{i}v \notin \operatorname{span}(B) \Rightarrow u$ or $v \notin \operatorname{span}(B)$. Note that $u, v \in V$. ☐

---

• **NOTE FOR *linely inde sequence and* [2.34]:** "$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expression. If we allow using "infinite list", then we must guarantee that $(v_1, \dots, v_n, \dots)$ is a spanning "list" such that $\forall v \in V, \exists$ smallest $n \in \mathbf{N}^+$, $v = a_1 v_1 + \cdots + a_n v_n$. Moreover, given a list $(w_1, \cdots, w_n, \cdots)$ in $W$, we can prove that $\exists\, !\, T \in \mathcal{L}(V, W)$ with each $T v_k = w_k$, which has less restrictions than [3.5]. But the key point is, how can we guarantee that such a "list" exists. TODO: More details.

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**ENDED**

# 2·C

1 7 9 10 14,16 15 17 | 4E: 10 14,15 16

**15** *Suppose $V$ is finite-dim and $\dim V = n \geqslant 1$. Prove that $\exists$ one-dim subsps $V_1, \dots, V_n$ of $V$ such that $V = V_1 \oplus \cdots \oplus V_n$.*

**SOLUTION:** Suppose $B_V = (v_1, \dots, v_n)$. Define $V_i$ by $V_i = \operatorname{span}(v_i)$ for each $i \in \{1, \dots, n\}$.

Then $\forall v \in V, \exists\, !\, a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n \Rightarrow \exists\, !\, u_i \in V_i, v = u_1 + \cdots + u_n$ ☐

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• **NOTE FOR Problem (15):**

*Suppose $v \in V \backslash \{0\}$, and $\dim V = n \geqslant 1$. Prove that $\exists\, B_V = (v_1, \dots, v_n), v = v_1 + \cdots + v_n$.*

**SOLUTION:** If $n = 1$ then let $v_1 = v$ and we are done. Suppose $n > 1$.

Extend $(v)$ to a basis $(v, v_1, \dots, v_{n-1})$ of $V$. Let $v_n = v - v_1 - \cdots - v_{n-1}$.

又 $\operatorname{span}(v, v_1, \dots, v_{n-1}) = \operatorname{span}(v_1, \dots, v_n)$. Hence $(v_1, \dots, v_n)$ is also a basis of $V$. ☐

**COMMENT:** Let $B_V = (v_1, \dots, v_n)$ and suppose $v = u_1 + \cdots + u_n$, where each $u_i = a_i v_i \in V_i$.

But $(u_1, \dots, u_n)$ might not be a basis, because there might be some $u_i = 0$.

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**1** [COROLLARY for [2.38,39]] *Suppose $U$ is a subsp of $V$ such that $\dim V = \dim U$. Then $V = U$.*

Let $B_U = (u_1, \dots, u_m)$. Then $m = \dim V$. 又 $u_i \in V$. By [2.39], $B_V = (u_1, \dots, u_m)$. $\square$

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- Let $v_1, \dots, v_n \in V$ and $\dim \operatorname{span}(v_1, \dots, v_n) = n$. Then $(v_1, \dots, v_n)$ is a basis of $\operatorname{span}(v_1, \dots, v_n)$.
  *Notice that $(v_1, \dots, v_n)$ is a spanning list of* $\operatorname{span}(v_1, \dots, v_n)$ *of length* $n = \dim \operatorname{span}(v_1, \dots, v_n)$.

---

**7** (a) *Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of $U$.*
(b) *Extend the basis in (b) to a basis of $\mathcal{P}_4(\mathbf{F})$.*
(c) *Find a subsp $W$ of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.*

**SOLUTION:** Using Problem (10).

NOTICE that $\nexists\, p \in \mathcal{P}(\mathbf{F})$ of deg 1 and 2, while $p \in U$. Thus $\dim U \leqslant \dim \mathcal{P}_4(\mathbf{F}) - 2 = 3$.

(a) Consider $B = \big(1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.
Let $a_0 + a_3(z-2)(z-5)(z-6) + a_4 z(z-2)(z-5)(z-6) = 0 \Rightarrow a_0 = a_3 = a_4 = 0$.
Thus the list $B$ is linely inde in $U$. Now $\dim U \geqslant 3 \Rightarrow \dim U = 3$. Thus $B_U = B$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $\big(1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. $\square$

---

**9** *Suppose $(v_1, \dots, v_m)$ is linely inde in $V$ and $w \in V$.*
*Prove that* $\dim \operatorname{span}\big(v_1 + w, \dots, v_m + w\big) \geq m - 1$.

**SOLUTION:** Using the result of (2.A.10, 11).

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$, for each $i = 1, \dots, m$.

$(v_1, \dots, v_m)$ linely inde $\Rightarrow \big(v_1, v_2 - v_1, \dots, v_m - v_1\big)$ linely inde $\Rightarrow \underbrace{\big(v_2 - v_1, \dots, v_m - v_1\big)}_{\text{of length } (m-1)}$ linely inde.

又 If $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $\big(v_1 + w, \dots, v_m + w\big)$ is linely inde.

Hence $m \geqslant \dim \operatorname{span}\big(v_1 + w, \dots, v_m + w\big) \geq m - 1$. $\square$

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- (4E 16) *Suppose $V$ is finite-dim, $U$ is a subsp of $V$ with $U \neq V$. Let $n = \dim V, m = \dim U$.*
  *Prove that $\exists\, (n-m)$ subsps $U_1, \dots, U_{n-m}$, each of dim $(n-1)$, such that $\bigcap\limits_{i=1}^{n-m} U_i = U$.*

**SOLUTION:** Let $B_U = (v_1, \dots, v_m)$, $B_V = \big(v_1, \dots, v_m, u_1, \dots, v_{n-m}\big)$.

Define $U_i = \operatorname{span}\big(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m}\big)$ for each $i$. Then $U \subseteq U_i$ for each $i$.

And because $\forall v \in \bigcap\limits_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \dots + b_{n-m} u_{n-m} \in U_i \Rightarrow$ each $b_i = 0 \Rightarrow v \in U$.

Hence $\bigcap\limits_{i=1}^{n-m} U_i \subseteq U$. $\square$

---

- **NOTE FOR Problem 10:** For each nonconst $p \in \operatorname{span}(1, z, \dots, z^m)$, $\exists$ smallest $m \in \mathbf{N}^+$, which is $\deg p$.

(a) If $p_0, p_1, \dots, p_m$ are such that all $a_{k,k} \neq 0$, and
$p_0 = a_{0,0}$, each $p_k = a_{0,k} + a_{1,k} z + \dots + a_{k,k} z^k$.
Then the upper-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m} \\ 0 & a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix}$.

(b) If $p_0, p_1, \dots, p_m$ are such that all $a_{k,k} \neq 0$, and
$p_0 = a_{0,0} + \dots + a_{m,0} x^m$, each $p_k = a_{k,k} x^k + \dots + a_{m,k} x^m$.
Then the lower-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,0} & a_{m,1} & \cdots & a_{m,m} \end{pmatrix}$.

**COMMENT:** Define $\xi_k(p)$ by the coeff of $z^k$ in $p \in \mathcal{P}_m(\mathbf{F})$.
Then $\mathcal{M}\Big(\xi_k, (1, z, \dots, z^m), (1)\Big) = \mathcal{E}^{(1,k)} \in \mathbf{F}^{1, m+1}$.

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**10** *Suppose $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each $p_k$ has degree $k$. Prove that $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

**Solution:** Using mathematical induction on $m$.

(i) $k = 1$. $\deg p_0 = 0$; $\deg p_1 = 1 \Rightarrow \operatorname{span}(p_0, p_1) = \operatorname{span}(1, x)$.

(ii) $1 \leqslant k \leqslant m-1$. Assume that $\operatorname{span}(p_0, p_1, \dots, p_k) = \operatorname{span}(1, x, \dots, x^k)$.

Then $\operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \subseteq \operatorname{span}(1, x, \dots, x^k, x^{k+1})$.

又 $\deg p_{k+1} = k+1$, $p_{k+1}(x) = a_{k+1}x^{k+1} + r_{k+1}(x)$; $a_{k+1} \neq 0$, $\deg r_{k+1} \leqslant k$.

$\Rightarrow x^{k+1} = \dfrac{1}{a_{k+1}}\Big(p_{k+1}(x) - r_{k+1}(x)\Big) \in \operatorname{span}(1, x, \dots, x^k, p_{k+1}) = \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

$\therefore x^{k+1} \in \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \Rightarrow \operatorname{span}(1, x, \dots, x^k, x^{k+1}) \subseteq \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m)$. $\square$

Or. 用比较系数法. Denote the coeff of $x^k$ in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Suppose $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show that $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is linely inde.

**Step 1.** For $k = m$, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m$, $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.

Now $L = a_{m-1}p_{m-1}(x) + \cdots + a_0 p_0(x)$.

**Step k.** For $0 \leqslant k \leqslant m$, we have $a_m = \cdots = a_{k+1} = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = 0$, then we are done. Otherwise, we have $L = a_{k-1}p_{k-1}(x) + \cdots + a_0 p_0(x)$. $\square$

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• **Tips:** *Suppose $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ are such that the lowest term of each $p_k$ is of deg $k$. Prove that $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

**Solution:** Using mathematical induction on $m$.

Let each $p_k$ be defined by $p_k(x) = a_{k,k}x^k + \cdots + a_{m,k}x^m$, where $a_{k,k} \neq 0$.

(i) $k = 1$. $p_m(x) = a_{m,m}x^m$; $p_{m-1}(x) = a_{m-1,m-1}x^{m-1} + a_{m,m-1}x^m \Longrightarrow \operatorname{span}(x^m, x^{m-1}) = \operatorname{span}(p_m, p_{m-1})$.

(ii) $1 \leqslant k \leqslant m-1$. Assume that $\operatorname{span}(x^m, \dots, x^{m-k}) = \operatorname{span}(p_m, \dots, p_{m-k})$.

Then $\operatorname{span}(p_m, \dots, p_{m-(k+1)}) \subseteq \operatorname{span}(x^m, \dots, x^{m-(k+1)})$.

又 $p_{m-(k+1)}$ has the form $a_{m-(k+1),m-(k+1)}x^{m-(k+1)} + r_{m-(k+1)}(x)$;

where the lowest term of $r_{m-(k+1)} \in \mathcal{P}_m(\mathbf{F})$ is of deg $(m-k)$.

$$\Rightarrow x^{m-(k+1)} = \frac{1}{a_{m-(k+1),m-(k+1)}}\Big(p_{m-(k+1)}(x) - r_{m-(k+1)}(x)\Big) \in \operatorname{span}(x^m, \dots, x^{m-k}, p_{m-(k+1)}) = \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$$

$\therefore x^{m-(k+1)} \in \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$

$\Rightarrow \operatorname{span}(x^m, \dots, x^{m-k}, x^{m-(k+1)}) \subseteq \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}(p_m, \dots, p_1, p_0)$. $\square$

Or. 用比较系数法. Denote the coeff of $x^k$ in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Suppose $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show that $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is linely inde.

**Step 1.** For $k = 0$, $\xi_0(L) = a_0 \xi_0(p_0) = \xi_0(R) = 0$ 又 $\deg p_0 = 0$, $\xi_0(p_0) \neq 0 \Rightarrow a_0 = 0$.

Now $L = a_1 p_1(x) + \cdots + a_m p_m(x)$.

**Step k.** For $0 \leqslant k \leqslant m$, we have $a_{k-1} = \cdots = a_0 = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = m$, then we are done. Otherwise, we have $L = a_{k+1}p_{k+1}(x) + \cdots + a_m p_m(x)$. $\square$

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- **NOTE FOR [2.11]:** ***Good definition for a general term always aviods undefined behaviours.***
  If $\deg p = 0$, then $p(z) = a_0 \neq 0$, but not literally $a_0 z^0$, by which if $p$ is defined, then it comes to $0^0$.
  To make it clear, we specify that *in* $\mathcal{P}(\mathbf{F})$, $a_0 z^0 = a_0$, where $z^0$ appears just for notational convenience.
  Because by definition, the term $a_0 z^0$ in a poly only represents the const term of the poly, which is $a_0$.
  For convenicence, we assume that $z^0 = 1$ in formula deduction and poly def. Absolutely without $0^0$.

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- (4E 10) *Suppose $m$ is a positive integer. For $0 \leqslant k \leqslant m$, let $p_k(x) = x^k(1-x)^{m-k}$.*
  *Show that $(p_0, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

**SOLUTION:** *We may see $p_0 = 1$ and $p_m(x) = x^m$, from the expansion below, by the* NOTE FOR *[2.11] above.*

Note that each $p_k(x) = \sum_{j=0}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j = \underbrace{(-1)^0 \cdot x^k \cdot 1^0}_{\text{of deg k}} + \underbrace{\sum_{j=1}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j}_{\text{of deg m; denote it by } q_k(x)}$.
And, each $q_k \in \operatorname{span}(x^{k+1}, \dots, x^m)$. Using TIPS above. $\square$

OR. Similar to the TIPS above. We will recursively prove that each $x^{m-k} \in \operatorname{span}(p_m, \dots, p_{m-k})$.
(i) $k = 1$. $p_m(x) = x^m \in \operatorname{span}(p_m)$; $\quad p_{m-1}(x) = x^{m-1} - x^m \Rightarrow x^{m-1} \in \operatorname{span}(p_{m-1}, p_m)$.
(ii) $k \in \{1, \dots, m-1\}$. Suppose for each $k \in \{0, \dots, k\}$, we have $x^{m-k} \in \operatorname{span}(p_{m-k}, \dots, p_m), \exists!\, a_m \in \mathbf{F}$.
Note that $x^{m-(k+1)} = p_{m-(k+1)}(x) + \sum_{j=1}^{k+1} C_{k+1}^{j}(-1)^{j+1} x^{m-(k+1)+j} \in \operatorname{span}(p_{m-(k+1)}, x^{m-k}, \dots, x^m)$.
Thus $x^{m-(k+1)} \in \operatorname{span}(p_{m-(k+1)}, p_{m-k}, \dots, p_m)$. $\square$

**COMMENT:** The base step and the inductive step can be independent.

OR. For any $m, k \in \mathbf{N}^+$ such that $k \leqslant m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k(1-x)^{m-k}$.
Define the statement $S(m)$ by $S(m) : (p_{0,m}, \dots, p_{m,m})$ is linely inde ( and therefore is a basis ).
We use induction on to show that $S(m)$ holds for all $m \in \mathbf{N}^+$.
(i) $m = 0$. $p_{0,0} = 1$, and $a p_{0,0} = 0 \Rightarrow a = 0$.
$m = 1$. Let $a_0(1-x) + a_1 x = 0, \forall x \in \mathbf{F}$. Then take $x = 1, x = 0 \Rightarrow a_1 = a_0 = 0$.
(ii) $1 \leqslant m$. Assume that $S(m)$ and $S(m-1)$ holds. Now we show that $S(m+1)$ holds.
Suppose $\sum_{k=0}^{m+1} a_k p_{k,m+1}(x) = \sum_{k=0}^{m+1} a_k [x^k(1-x)^{m+1-k}] = 0, \forall x \in \mathbf{F}$.
Now $a_0(1-x)^{m+1} + \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k} + a_{m+1} x^{m+1} = 0, \forall x \in \mathbf{F}$.
While $x = 0 \Rightarrow a_0 = 0$; and $x = 1 \Rightarrow a_{m+1} = 0$.
Then $0 = \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k}$
$= x(1-x) \sum_{k=1}^{m} a_k x^{k-1}(1-x)^{m-k}$, note that $m - k = (m-1) - (k-1)$
$= x(1-x) \sum_{k=0}^{m-1} a_{k+1} x^k (1-x)^{m-1-k} = x(1-x) \sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x)$.
Hence $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0, \forall x \in \mathbf{F} \backslash \{0,1\}$. Which has infinitely many zeros.
Moreover, $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0$. By assumption, $a_1 = \cdots = a_{m-1} = a_m = 0$.
Thus $(p_{0,m+1}, \dots, p_{m+1,m+1})$ is linely inde and $S(m+1)$ holds. $\square$

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**14** *Suppose $V_1, \dots, V_m$ are finite-dim. Prove that* $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$.

**SOLUTION:** For each $V_i$, let $B_{V_i} = \mathcal{E}_i$. Then $V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim V_i = \operatorname{card} \mathcal{E}_i$.
Now $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.

**COROLLARY:** $V_1 + \cdots + V_m$ is direct
$\iff$ For each $k \in \{1, \dots, m-1\}$, $(V_1 \oplus \cdots \oplus V_k) \cap V_{k+1} = \{0\}$, $(\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$
$\iff \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$
$\iff \dim(V_1 \oplus \cdots \oplus V_m) = \dim V_1 + \cdots + \dim V_m$. $\square$

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**17** *Suppose $V_1, V_2, V_3$ are subsps of a finite-dim vecsp, then*

$$\dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

*Explain why you might think and prove the formula above or give a counterexample.*

**SOLUTION:**

[ *Similar to* ] Given three sets $A, B$ and $C$.

Because $|X \cup Y| = |X| + |Y| - |X \cap Y|$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

Now $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$.

And $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$.

Hence $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$.

Note that $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned}\dim(V_1+V_2+V_3) &= \dim(V_1+V_2) + \dim(V_3) - \dim\big((V_1+V_2) \cap V_3\big) \quad (1)\\ &= \dim(V_2+V_3) + \dim(V_1) - \dim\big((V_2+V_3) \cap V_1\big) \quad (2)\\ &= \dim(V_1+V_3) + \dim(V_2) - \dim\big((V_1+V_3) \cap V_2\big) \quad (3).\end{aligned}$$

Notice that in general, $(X+Y) \cap Z \neq X \cap Z + Y \cap Z$.

For example, $X = \{(x,0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0,y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z,z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

**COMMENT:** If $X \subseteq Y$, then $(X+Y) \cap Z = Y \cap Z$; $\dim(X+Y+Z) = \dim(Y) + \dim(Z) - \dim(Y \cap Z)$, and the wrong formual holds. Similar for $Y \subseteq Z, X \subseteq Z$, and $X, Y \subseteq Z$.

However, $(X+Y) \cap Z \supseteq (X \cap Z) + (Y \cap Z)$ holds. Because $\forall v \in (X \cap Z) + (Y \cap Z)$,
$\exists\, u = x_1 = z_1 \in X \cap Z, w = y_2 = z_2 \in Y \cap Z$, $v = u + w = x_1 + y_2 = z_1 + z_2 \in (X+Y) \cap Z$.

**COMMENT:** $\dim\big((X+Y) \cap Z\big) \geqslant \dim(X \cap Z) + \dim(Y \cap Z) - \dim(X \cap Y \cap Z)$.

- **COROLLARY:** *Suppose $V_1, V_2, V_3$ are finite-dim, then* $\frac{(1)+(2)+(3)}{3}$:

$$\dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} - \frac{\dim\big((V_1+V_2) \cap V_3\big) + \dim\big((V_1+V_3) \cap V_2\big) + \dim\big((V_2+V_3) \cap V_1\big)}{3}.$$

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- **TIPS:** Because $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big)$.
  And $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$. We have (1), and (2), (3) similarly.
  (1) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2+V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big)$.
  (2) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1+V_3) - \dim\big(V_2 + (V_1 \cap V_3)\big)$.
  (3) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1+V_2) - \dim\big(V_3 + (V_1 \cap V_2)\big)$.

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- *Suppose $V_1, V_2, V_3$ are subsps of $V$ with*
  (a) $\dim V = 10$, $\dim V_1 = \dim V_2 = \dim V_3 = 7$. *Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*
  By TIPS, $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 \underline{- 2\dim V} > 0$.
  (b) $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. *Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*
  By TIPS, $\dim(V_1 \cap V_2 \cap V_3) \underline{\underline{\geq 2\dim V}} - \dim(V_2 + V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big) \geq 0$. ☐

---

**ENDED**

# 3·A   3 4 5 7 8 10 11 12 13 | 4E: 10 11 17

- **TIPS 1:** $T: V \to W$ is linear $\iff$ $\left|\begin{array}{l} (一)\ \forall v,u \in V, T(v+u) = Tv + Tu; \\ (二)\ \forall v,u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv). \end{array}\right| \iff T(v+\lambda u) = Tv + \lambda Tu.$

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- (9.A.2,6 OR 4E 3.B.33) *Suppose that $V, W$ are on $\mathbf{R}$, and $T \in \mathcal{L}(V,W)$. Show that*
  (a) $T_{\mathrm{C}} \in \mathcal{L}(V_{\mathrm{C}}, W_{\mathrm{C}})$. (b) null $(T_{\mathrm{C}}) = (\text{null}\, T)_{\mathrm{C}}$, range $(T_{\mathrm{C}}) = (\text{range}\, T)_{\mathrm{C}}$. (c) $T_{\mathrm{C}}$ is inv $\iff$ $T$ is inv.

**SOLUTION:** (a) $T_{\mathrm{C}}\big((u_1 + \mathrm{i}v_1) + (x+\mathrm{i}y)(u_2 + \mathrm{i}v_2)\big) = T(u_1 + xu_2 - yv_2) + \mathrm{i}T(v_1 + xv_2 + yu_2)$
$= T_{\mathrm{C}}(u_1 + \mathrm{i}v_1) + (x+\mathrm{i}y)T_{\mathrm{C}}(u_2 + \mathrm{i}v_2)$.

(b) $u + \mathrm{i}v \in \text{null}\,(T_{\mathrm{C}}) \iff u, v \in \text{null}\, T \iff u + \mathrm{i}v \in (\text{null}\, T)_{\mathrm{C}}$.
$w + \mathrm{i}x \in \text{range}\,(T_{\mathrm{C}}) \iff w, x \in \text{range}\, T \iff w + \mathrm{i}x \in (\text{range}\, T)_{\mathrm{C}}$.

(c) $\forall w, x \in W, \exists!\, u, v \in V, T_{\mathrm{C}}(u + \mathrm{i}v) = w + \mathrm{i}x \iff Tu = w, Tv = x$. OR. By (b). ☐

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- (9.A.5) *Suppose $V$ is on $\mathbf{R}$, and $S, T \in \mathcal{L}(V,W)$. Prove that $(S + \lambda T)_{\mathrm{C}} = S_{\mathrm{C}} + \lambda T_{\mathrm{C}}$.*

**SOLUTION:** $(S + \lambda T)_{\mathrm{C}}(u + \mathrm{i}v) = (S + \lambda T)(u) + \mathrm{i}(S + \lambda T)(v)$
$= Su + \mathrm{i}Sv + \lambda(Tu + \mathrm{i}Tv) = (S_{\mathrm{C}} + \lambda T_{\mathrm{C}})(u + \mathrm{i}v)$. ☐

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- *Suppose $U, V, W$ are on $\mathbf{R}$, $S \in \mathcal{L}(V,W), T \in \mathcal{L}(U,V)$. Prove that $(ST)_{\mathrm{C}} = S_{\mathrm{C}}T_{\mathrm{C}}$.*

**SOLUTION:** $\forall u + \mathrm{i}x \in U_{\mathrm{C}}, (ST)_{\mathrm{C}}(u + \mathrm{i}x) = STu + \mathrm{i}STx = S_{\mathrm{C}}(Tu + \mathrm{i}Tx) = (S_{\mathrm{C}}T_{\mathrm{C}})(u + \mathrm{i}x)$. ☐

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- **NOTE FOR Restriction:** *$U$ is a subsp of $V$.*
  (a) $\forall S, T \in \mathcal{L}(V,W), \lambda \in \mathbf{F}$, $(T + \lambda S)|_U = T|_U + \lambda S|_U$.
  (b) $\forall S \in \mathcal{L}(W,X), T \in \mathcal{L}(V,W)$, $(ST)|_U = ST|_U$.

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- (4E 1.B.7) *Suppose $V \neq \emptyset$ and $W$ is a vecsp. Let $W^V = \{f : V \to W\}$.*
  (a) *Define a natural add and scalar multi on $W^V$.*
  (b) *Prove that $W^V$ is a vecsp with these definitions.*

**SOLUTION:**

(a) $W^V \ni f + g : x \to f(x) + g(x)$; where $f(x) + g(x)$ is the vec add on $W$.
$W^V \ni \lambda f : x \to \lambda f(x)$; where $\lambda f(x)$ is the scalar multi on $W$.

(b) Commutativity: $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$.
Associativity: $\big((f+g)+h\big)(x) = \big(f(x) + g(x)\big) + h(x)$
$= f(x) + \big(g(x) + h(x)\big) = \big(f + (g+h)\big)(x)$.
Additive Identity: $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$.
Additive Inverse: $(f+g)(x) = f(x) + g(x) = f(x) + \big(-f(x)\big) = 0 = 0(x)$.
Distributive Properties:
$\big(a(f+g)\big)(x) = a(f+g)(x) = a\big(f(x) + g(x)\big)$
$= af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x)$.
Similarly, $\big((a+b)f\big)(x) = (af + bf)(x)$.
So far, we have used the same properties in $W$.
Which means that ***if** $W^V$ **is a vecsp, then** $W$ **must be a vecsp.***
Multiplication Identity: $(1f)(x) = 1f(x) = f(x)$. ( NOTICE that the smallest $\mathbf{F}$ is $\{0,1\}$. ) ☐

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• **Tips 2:** $T \in \mathcal{L}(V,W) \iff T \in \mathcal{L}(V, \operatorname{range} T) \iff T \in \mathcal{L}(V,U)$, if range $T$ is a subsp of $U$.

**Corollary:** $\{T \in \mathcal{L}(V,W) : \operatorname{range} T \subseteq U\} = \{T \in \mathcal{L}(V,U)\} = \mathcal{L}(V,U)$.

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**5** *Because $\mathcal{L}(V,W) = \{T : V \to W \mid T \text{ is linear}\}$ is a subsp of $W^V$, $\mathcal{L}(V,W)$ is a vecsp.*

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**3** *Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists A_{j,k} \in \mathbf{F}$ such that for any $(x_1, \dots, x_n) \in \mathbf{F}^n$,*

$$T(x_1, \dots, x_n) = \begin{pmatrix} A_{1,1}x_1 + \cdots + A_{1,n}x_n, \\ \vdots \quad\quad \ddots \quad\quad \vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n \end{pmatrix}$$

**Solution:**

Let $T(1,0,0,\dots,0,0) = (A_{1,1}, \dots, A_{m,1})$, Note that $(1,0,\dots,0,0), \cdots, (0,0,\dots,0,1)$ is a basis of $\mathbf{F}^n$.
$T(0,1,0,\dots,0,0) = (A_{1,2}, \dots, A_{m,2})$, Then by [3.5], we are done. ☐
$\vdots$
$T(0,0,0,\dots,0,1) = (A_{1,n}, \dots, A_{m,n})$.

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**4** *Suppose $T \in \mathcal{L}(V,W)$, and $v_1, \dots, v_m \in V$ such that $(Tv_1, \dots, Tv_m)$ is linely inde in $W$. Prove that $(v_1, \dots, v_m)$ is linely inde.*

**Solution:** Suppose $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$. ☐

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**7** *Show that every linear map from a one-dim vecsp to itself is a multi by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then $\exists \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$.*

**Solution:** Let $u$ be a nonzero vec in $V \Rightarrow V = \operatorname{span}(u)$. Because $Tu \in V \Rightarrow Tu = \lambda u$ for some $\lambda$.
Suppose $v \in V \Rightarrow v = au$, $\exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. ☐

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**8** *Give a map $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that $\forall a \in \mathbf{R}, v \in \mathbf{R}^2, \varphi(av) = a\varphi(v)$ but $\varphi$ is not linear.*

**Solution:** Define $T(x,y) = \begin{cases} x+y, & \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, & \text{otherwise.} \end{cases}$ Or. Define $T(x,y) = \sqrt[3]{(x^3+y^3)}$. ☐

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**9** *Give a map $\varphi : \mathbf{C} \to \mathbf{C}$ such that $\forall w, z \in \mathbf{C}, \varphi(w+z) = \varphi(w) + \varphi(z)$ but $\varphi$ is not linear.*

**Solution:** Define $\varphi(u+iv) = u = \operatorname{Re}(u+iv)$ Or. Define $\varphi(u+iv) = v = \operatorname{Im}(u+iv)$. ☐

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• *Prove that if $q \in \mathcal{P}(\mathbf{R})$ and $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by $\underbrace{Tp = q \circ p}_{\text{composition}}$, then $T$ is not linear.*

**Solution:** ***Composition and product are not the same in $\mathcal{P}(\mathbf{F})$.***

Notice that $(p \circ q)(x) = p(q(x))$, while $(pq)(x) = p(x)q(x) = q(x)p(x)$.

Because in general, $[q \circ (p_1 + \lambda p_2)](x) = q(p_1(x) + \lambda p_2(x)) \neq (qp_1)(x) + \lambda(qp_2)(x)$.

**Example:** Let $q$ be defined by $q(x) = x^2$, then $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$. ☐

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**10** *Suppose $U$ is a subsp of $V$ with $U \neq V$. Suppose $S \in \mathcal{L}(U,W)$ with $S \neq 0$ (which means that $\exists\, u \in U, Su \neq 0$). Define $T : V \to W$ by $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \backslash U. \end{cases}$ Prove that $T$ is not a linear map on $V$.*

**Solution:** Suppose $T$ is a linear map. And $v \in V \backslash U$, $u \in U$ such that $Su \neq 0$.
Then $v + u \in V \backslash U$, for if not, $v = (v+u) - u \in U$;
while $T(v+u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$. Contradicts. ☐

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**11** *Suppose $U$ is a subsp of $V$ and $S \in \mathcal{L}(U, W)$.*

*Prove that $\exists\, T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U$.* ( OR. $\exists\, T \in \mathcal{L}(V, W), T|_U = S$. )

*In other words, every linear map on a subsp of $V$ can be **extended** to a linear map on the entire $V$.*

**SOLUTION:** Suppose $W$ is such that $V = U \oplus W$. Then $\forall v \in V, \exists\,!\, u_v \in U, w_v \in W, v = u_v + w_v$.

Define $T \in \mathcal{L}(V, W)$ by $T(u_v + w_v) = Su_v$. □

OR. [ *Finite-dim Req* ] Define by $T\Big( \sum_{i=1}^{m} a_i u_i \Big) = \sum_{i=1}^{n} a_i S u_i$. Let $B_V = ( \overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m )$. □

---

**12** *Suppose nonzero $V$ is finite-dim and $W$ is infinite-dim. Prove that $\mathcal{L}(V, W)$ is infinite-dim.*

**SOLUTION:** Using (2.A.14).

Let $B_V = (v_1, \dots, v_n)$ be a basis of $V$. Let $(w_1, \dots, w_m)$ be linely inde in $W$ for any $m \in \mathbf{N}^+$.

Define $T_{x,y} : V \to W$ by $T_{x,y}(v_z) = \delta_{z,x} w_y$, $\forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$

$\forall v = \sum_{i=1}^{n} a_i v_i,\ u = \sum_{i=1}^{n} b_i v_i,\ \lambda \in \mathbf{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) w_y = T_{x,y}(v) + \lambda T_{x,y}(u)$.

Linearity checked. Now suppose $a_1 T_{x,1} + \dots + a_m T_{x,m} = 0$.

Then $(a_1 T_{x,1} + \dots + a_m T_{x,m})(v_x) = 0 = a_1 w_1 + \dots + a_m w_m \Rightarrow a_1 = \dots = a_m = 0$. 又 $m$ arbitrary.

Thus $(T_{x,1}, \dots, T_{x,m})$ is a linely inde list in $\mathcal{L}(V, W)$ for any $x$ and length $m$. Hence by (2.A.14). □

---

**13** *Suppose $(v_1, \dots, v_m)$ is linely depe in $V$ and $W \neq \{0\}$.*

*Prove that $\exists\, w_1, \dots, w_m \in W, \nexists\, T \in \mathcal{L}(V, W)$ such that $Tv_k = w_k, \forall k = 1, \dots, m$.*

**SOLUTION:**

We prove by contradiction. By linear dependence lemma, $\exists\, j \in \{1, \dots, m\}, v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$.

Fix $j$. Let $w_j \neq 0$, while $w_1 = \dots = w_{j-1} = w_{j+1} = w_m = 0$. Define $T \in \mathcal{L}(V, W)$ by $Tv_k = w_k$ for each $k$.

Suppose $a_1 v_1 + \dots + a_m v_m = 0$, where $a_j \neq 0$.

Then $T(a_1 v_1 + \dots + a_m v_m) = 0 = a_1 w_1 + \dots + a_m w_m = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. □

OR. We prove the contrapositive: Suppose $\forall w_1, \dots, w_m \in W, \exists\, T \in \mathcal{L}(V, W), Tv_k = w_k$ for each $w_k$.

Now we show that $(v_1, \dots, v_n)$ is linely inde. Suppose $\exists\, a_i \in \mathbf{F}, a_1 v_1 + \dots + a_n v_n = 0$.

Choose one $w \in W \backslash \{0\}$. By assumption, for $(\overline{a_1} w, \dots, \overline{a_m} w), \exists\, T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$ for each $v_k$.

Now we have $0 = T\Big( \sum_{k=1}^{m} a_k v_k \Big) = \sum_{k=1}^{m} a_k T v_k = \sum_{k=1}^{m} a_k \overline{a_k} w = \Big( \sum_{k=1}^{m} |a_k|^2 \Big) w$.

Then $\sum_{k=1}^{m} |a_k|^2 = 0 \Longrightarrow$ each $a_k = 0$. Hence $(v_1, \dots, v_n)$ is linely inde. □

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• (4E 3.A.17) *Suppose $V$ is finite-dim. Show that all two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

*A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.*

**SOLUTION:** Let $B_V = (v_1, \dots, v_n)$. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \backslash \{0\}$.

Suppose $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \dots + a_n v_n$, where $a_k \neq 0$.

Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y} :\ v_x \mapsto v_y,\ v_z \mapsto 0\,(\,z \neq x\,)$. OR. $R_{x,y} v_z = \delta_{z,x} v_y$.

Then $(R_{1,1} + \dots + R_{n,n}) v_j = v_j \Rightarrow \sum_{r=1}^{n} R_{r,r} = I$. Assume that each $R_{x,y} \in \mathcal{E}$.

Hence $\forall T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. Now we prove the assumption.

Notice that $\forall x, y \in \mathbf{N}^+,\ (R_{k,y} S)(v_i) = a_k v_y \Rightarrow \big( (R_{k,y} S) \circ R_{x,i} \big)(v_z) = \delta_{z,x}(a_k v_y)$.

Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Now $S \in \mathcal{E} \Rightarrow R_{k,y} S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}$. □

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• (4E 3.B.32) *Suppose $V$ is finite-dim with $n = \dim V > 1$.*
*Show that if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is linear and $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$, then $\varphi = 0$.*

**Solution:** Using notations in (4E 3.A.17). Using the result in Note For [3.60].

Suppose $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \dots, n\}, \varphi(R_{i,j}) \neq 0$. Because $R_{i,j} = R_{x,j} \circ R_{i,x},\ \forall x = 1, \dots, n$
$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.
Again, because $R_{i,x} = R_{y,x} \circ R_{i,y},\ \forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0,\ \forall x, y = 1, \dots, n$.
Let $k \neq i, j \neq l$ and then $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$
$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. □

Or. Note that by (4E 3.A.17), $\exists\, S, T \in \mathcal{L}(V), ST - TS \neq 0$.
Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \operatorname{null}\varphi \neq \{0\}$.
Note that $\forall\, E \in \operatorname{null}\varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \operatorname{null}\varphi$.
Hence $\operatorname{null}\varphi$ is a nonzero two-sided ideal of $\mathcal{L}(V)$. □

---

• *Suppose $V$ is finite-dim. $T \in \mathcal{L}(V)$ is such that $\forall S \in \mathcal{L}(V), ST = TS$.*
*Prove that $\exists\, \lambda \in \mathbf{F}, T = \lambda I$.*

**Solution:** If $V = \{0\}$, then we are done. Now suppose $V \neq \{0\}$.

Assume that $\forall v \in V, (v, Tv)$ is linely depe, then by (2.A.2.(b)), $\exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.
To prove that $\lambda_v$ is independent of $v$, we discuss in two cases:

$$\left.\begin{array}{l} (-) \text{ If } (v,w) \text{ is linely inde, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \\ \qquad\qquad\qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=) \text{ Otherwise, suppose } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w \end{array}\right\} \Rightarrow \lambda_w = \lambda_v.$$

Now we prove the assumption. Assume that $\exists\, v \in V, (v, Tv)$ is linely inde. Let $B_V = (v, Tv, u_1, \dots, u_n)$.
Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1 u_1 + \dots + c_n u_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Contradicts. □

Or. Let $B_V = (v_1, \dots, v_m)$.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \dots = \varphi(v_m) = 1$. Let $\lambda = \varphi(Tv_1) \in \mathbf{F}$.
For any $v \in V$, define $S_v \in \mathcal{L}(V)$ by $S_v u = \varphi(u)v$.
Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. □

Or. For each $k \in \{1, \dots, n\}$, define $S_k \in \mathcal{L}(V)$ by $S_k v_j = \begin{cases} v_k, & j = k, \\ 0, & j \neq k. \end{cases}$ Or. $S_k v_j = \delta_{j,k} v_k$

Note that $S_k\Big(\sum_{i=1}^{n} a_i v_i\Big) = a_k v_k$. Then $S_k v = v \iff \exists\,!\, a_k \in \mathbf{F}, v = a_k v_k$.
Hence $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$.
Define $A^{(j,k)} \in \mathcal{L}(V)$ by $A^{(j,k)} v_j = v_k,\ A^{(j,k)} v_k = v_j,\ A^{(j,k)} v_x = 0, x \neq j, k$.

Then $$\left|\begin{array}{r} A^{(j,k)} Tv_j = TA^{(j,k)} v_j = Tv_k = a_k v_k \\ A^{(j,k)} Tv_j = A^{(j,k)} a_j v_j = a_j A^{(j,k)} v_j = a_j v_k \end{array}\right\} \Rightarrow a_k = a_j.$$ Hence $a_k$ is inde of $v_k$. □

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• **Tips 3:** *Suppose $T \in \mathcal{L}(V, W)$. Prove that $Tv \neq 0 \Rightarrow v \neq 0$.*

**Solution:** Assume that $v = 0$. Then $Tv = T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$.
Or. $T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$. Contradicts. □

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• *Given the fact that $\mathcal{L}(V,W)$ is a vecsp. Prove or give a counterexample: $V,W$ are vecsps.*

*We can guarantee that $\{0\} \subseteq \mathcal{L}(V,W), \{0\} \subseteq V, \{0\} \subseteq W$.*

*And by [3.2], the additivity and homogeneity imply that $V$ is closed under add and scalar multi.*

*( We cannot even guarantee that $W^V$ is a vecsp. )*

**SOLUTION:** TODO: Too tricky to be answered by AI.

(I) If $W^V = \{0\}$. Then $\mathcal{L}(V,W) = \{0\}$.

And $W = \{0\}$, for if not, $\exists\, w \in W \backslash \{0\}$, define a map $f$ by $f(x) = w, \forall x \in V$.

And $V$ might not be a vecsp. Example: ???

(II) If $W^V$ is a nonzero vecsp. Then $W$ is a vecsp.

(a) If $\mathcal{L}(V,W) = \{0\}$, then we cannot guarantee that $V$ is a vecsp. Example: ???

(b) If not, then $\exists\, T \in \mathcal{L}(V,W)$, $T \neq 0$. Which means $\exists\, v \in V, Tv \neq 0 \Rightarrow v \neq 0$.

Then both $W$ and $V$ have a nonzero element.

(i) If $\exists$ inje $T \in \mathcal{L}(V,W)$, then $T(u+v) = T(v+u) \Rightarrow u+v = v+u$. etc. Hence $V$ is a vecsp.

(ii) If not, then we cannot guarantee that $V$ is a vecsp. Example: ???

(III) If $W^V$ is not a vecsp, then $W$ is not a vecsp. Example: ??? □

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**ENDED**

# 3·B

3 7 8 9 10 11 12 16 17 18 19 20 21 22 23 24 25 26 28 29 30 | 4E: 21 24 27 31 32 33

**3** *Suppose $(v_1,\dots,v_m)$ in $V$. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1,\dots,z_m) = z_1v_1 + \cdots + z_mv_m$.*

(a) *The surj of $T$ correspds to $(v_1,\dots,v_m)$ spanning $V$.*

(b) *The inje of $T$ correspds to $(v_1,\dots,v_m)$ being linely inde.*

**COMMENT:** Let $(e_1,\dots,e_m)$ be the std basis of $\mathbf{F}^m$. Then $Te_k = v_k$.

(a) range $T = \operatorname{span}(v_1,\dots,v_m) = V$; (b) $(v_1,\dots,v_m)$ is linely inde $\iff$ $T$ is inje.

---

**7** *Suppose $V$ is finite-dim with $2 \leqslant \dim V$. And $\dim V \leqslant \dim W = m$, if $W$ is finite-dim. Show that $U = \{T \in \mathcal{L}(V,W) : \operatorname{null} T \neq \{0\}\}$ is not a subsp of $\mathcal{L}(V,W)$.*

**SOLUTION:** *The set of all inje $T \in \mathcal{L}(V,W)$ is a not subsp either.*

Let $(v_1,\dots,v_n)$ be a basis of $V$, $(w_1,\dots,w_m)$ be linely inde in $W$. $[\,2 \leqslant n \leqslant m.\,]$

Define $T_1 \in \mathcal{L}(V,W)$ as $T_1: \quad v_1 \mapsto 0, \quad v_2 \mapsto w_2, \quad v_i \mapsto w_i.$

Define $T_2 \in \mathcal{L}(V,W)$ as $T_2: \quad v_1 \mapsto w_1, \quad v_2 \mapsto 0, \quad v_i \mapsto w_i, \quad i = 3,\dots,n.$

Thus $T_1 + T_2 \notin U$. □

**COMMENT:** If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V,W), T$ is inje. Hence $U = \emptyset$.

If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $\forall v \in V, T_0v = 0 \Rightarrow T_0 = 0$.

---

**8** *Suppose $W$ is finite-dim with $\dim W \geqslant 2$. And $n = \dim V \geqslant \dim W$, if $V$ is finite-dim. Show that $U = \{T \in \mathcal{L}(V,W) : \operatorname{range} T \neq W\}$ is not a subsp of $\mathcal{L}(V,W)$.*

**SOLUTION:** *The set of all surj $T \in \mathcal{L}(V,W)$ is not a subsp either.* ***Using the generalized version of*** **[3.5]**.

Let $(v_1,\dots,v_n)$ be linely inde in $V$, $(w_1,\dots,w_m)$ be a basis of $W$. $[\,n \in \{m, m+1,\dots\}; 2 \leqslant m \leqslant n.\,]$

Define $T_1 \in \mathcal{L}(V,W)$ as $T_1: \quad v_1 \mapsto 0, \quad v_2 \mapsto w_2, \quad v_j \mapsto w_j, \quad v_{m+i} \mapsto 0.$

Define $T_2 \in \mathcal{L}(V,W)$ as $T_2: \quad v_1 \mapsto w_1, \quad v_2 \mapsto 0, \quad v_j \mapsto w_j, \quad v_{m+i} \mapsto 0.$

( For each $j = 2,\dots,m;\ i = 1,\dots,n-m$, if $V$ is finite, otherwise let $i \in \mathbf{N}^+$. ) Thus $T_1 + T_2 \notin U$. □

**COMMENT:** If $\dim W = 0$, then $W = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V,W), T$ is surj. Hence $U = \emptyset$.

If $\dim W = 1$, then $W = \operatorname{span}(w_0)$. Thus $U = \operatorname{span}(T_0)$, where each $T_0v_i = 0 \Rightarrow T_0 = 0$.

**9** *Suppose* $(v_1,\dots,v_n)$ *is linely inde. Prove that* $\forall$ *inje* $T$, $(Tv_1,\dots,Tv_n)$ *is linely inde.*

**SOLUTION:** $a_1Tv_1+\cdots+a_nTv_n=0=T\Big(\sum_{i=1}^n a_iv_i\Big)\iff\sum_{i=1}^n a_iv_i=0\iff a_1=\cdots=a_n=0.$ $\square$

---

**10** *Suppose* $\operatorname{span}(v_1,\dots,v_n)=V$. *Show that* $\operatorname{span}(Tv_1,\dots,Tv_n)=\operatorname{range}T$.

**SOLUTION:** (a) $\operatorname{range}T=\{Tv:v\in\operatorname{span}(v_1,\dots,v_n)\}\Rightarrow Tv_1,\dots,Tv_n\in\operatorname{range}T$. By [2.7].

OR. $\operatorname{span}(Tv_1,\dots,Tv_n)\ni a_1Tv_1+\cdots+a_nTv_n=T(a_1v_1+\cdots+a_nv_n)\in\operatorname{range}T$.

(b) $\forall w\in\operatorname{range}T, w=Tv,\exists v\in V\Rightarrow\exists a_i\in\mathbf{F}, v=\sum_{i=1}^n a_iv_i, w=a_1Tv_1+\cdots+a_nTv_n$. $\square$

---

**11** *Suppose* $S_1,\dots,S_n\in\mathcal{L}(V)$ *and* $S=S_1S_2\cdots S_n$ *makes sense.* Then using induction:

(a) $\operatorname{range}S_1\supseteq\operatorname{range}(S_1S_2)\supseteq\cdots\supseteq\operatorname{range}(S)$; (b) $\operatorname{null}S_n\subseteq\operatorname{null}(S_{n-1}S_n)\subseteq\cdots\subseteq\operatorname{null}(S)$.

- Define $X_p=\{T\in\mathcal{L}(V):p(T)\text{ holds}\}$; $P_p:X_p$ is closed under vec multi; $Q_p:X_p$ is a group.

(1) $S$ surj $\iff$ each $S_k$ surj. $P_{surj}$ holds. (2) $S$ inje $\iff$ each $S_k$ inje. $P_{inje}$ holds.

(3) $P_{inv}$ and $Q_{inv}$ hold. $Q_p$ in (1) and (2) holds $\iff$ $V$ is finite-dim.

(4) $P_{inje\ or\ surj}$ holds $\iff$ $V$ is finite-dim $\iff$ $Q_{inje\ or\ surj}$ holds.

---

- *Suppose* $S,T\in\mathcal{L}(V)$. *Prove or give a counterexample*:

(a) $\operatorname{null}S\subseteq\operatorname{null}T\Rightarrow\operatorname{range}T\subseteq\operatorname{range}S$; (b) $\operatorname{range}T\subseteq\operatorname{range}S\Rightarrow\operatorname{null}S\subseteq\operatorname{null}T$.

**SOLUTION:** Let $B_V=(v_1,v_2,v_3)$. Counterexamples:

(a) Let $S:v_1\mapsto0;\ v_2\mapsto0;\ v_3\mapsto v_2.$ $T:v_1\mapsto0;\ v_2\mapsto0;\ v_3\mapsto v_3.$ | Then $\operatorname{null}S=\operatorname{null}T$, but $\operatorname{range}T=\operatorname{span}(v_3)\not\subseteq\operatorname{span}(v_2)=\operatorname{null}T$.

(b) Let $S:v_1\mapsto v_2;\ v_2\mapsto v_2;\ v_3\mapsto v_2.$ $T:v_1\mapsto0;\ v_2\mapsto0;\ v_3\mapsto v_2.$ | Then $\operatorname{range}T=\operatorname{range}S$, but $\operatorname{null}S=\operatorname{span}(v_1-v_2,v_2-v_3,v_3-v_1)\not\subseteq\operatorname{span}(v_1,v_2)=\operatorname{null}T$.

---

**16** *Suppose* $T\in\mathcal{L}(V)$ *such that* $\operatorname{null}T,\operatorname{range}T$ *are finite-dim. Prove that* $V$ *is finite-dim.*

**SOLUTION:** Let $B_{\operatorname{range}T}=(Tv_1,\dots,Tv_n), B_{\operatorname{null}T}=(u_1,\dots,u_m)$.

$\forall v\in V,\exists!\,a_i\in\mathbf{F}, T(v-a_1v_1-\cdots-a_nv_n)=0\Rightarrow\exists!\,b_i\in\mathbf{F}, v-\sum_{i=1}^n a_iv_i=\sum_{i=1}^m b_iu_i$. $\square$

---

**17** *Suppose* $V,W$ *are finite-dim. Prove that* $\exists$ *inje* $T\in\mathcal{L}(V,W)\iff\dim V\leqslant\dim W$.

**SOLUTION:** (a) Suppose $\exists$ inje $T$. Then $\dim V=\dim\operatorname{range}T\leqslant\dim W$.

(b) Suppose $\dim V\leqslant\dim W$. Let $B_V=(v_1,\dots,v_n), B_W=(w_1,\dots,w_m)$.

Define $T\in\mathcal{L}(V,W)$ by $Tv_i=w_i,\ i=1,\dots,n\ (=\dim V)$. $\square$

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**18** *Suppose* $V,W$ *are finite-dim. Prove that* $\exists$ *surj* $T\in\mathcal{L}(V,W)\iff\dim V\geqslant\dim W$.

**SOLUTION:** (a) Suppose $\exists$ surj $T$. Then $\dim V=\dim W+\dim\operatorname{null}T\Rightarrow\dim W\leqslant\dim V$.

(b) Suppose $\dim V\geqslant\dim W$. Let $B_V=(v_1,\dots,v_n), B_W=(w_1,\dots,w_m)$.

Define $T\in\mathcal{L}(V,W)$ by $T\big(a_1v_1+\cdots+a_mv_m+\cdots+a_nv_n\big)=a_1w_1+\cdots+a_mw_m$. $\square$

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**19** *Suppose* $V,W$ *are finite-dim,* $U$ *is a subsp of* $V$.

*Prove that* $\exists\,T\in\mathcal{L}(V,W)$, $\operatorname{null}T=U\iff\underbrace{\dim U}_{m}\geqslant\underbrace{\dim V}_{m+n}-\underbrace{\dim W}_{p}$.

**SOLUTION:**

(a) Suppose $\exists\,T\in\mathcal{L}(V,W),\operatorname{null}T=U$. Then $\dim U+\dim\operatorname{range}T=\dim V\leqslant\dim U+\dim W$.

(b) Let $B_U=(u_1,\dots,u_m), B_V=(u_1,\dots,u_m,v_1,\dots,v_n), B_W=(w_1,\dots,w_p)$. Suppose that $p\geqslant n$.

Define $T\in\mathcal{L}(V,W)$ by $T(a_1v_1+\cdots+a_nv_n+b_1u_1+\cdots+b_mu_m)=a_1w_1+\cdots+a_nw_n$. $\square$

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• **TIPS 1:** *Suppose $U$ is a subsp of $V$. Then* $\forall T \in \mathcal{L}(V,W), U \cap \operatorname{null} T = \operatorname{null} T|_U$.

• **TIPS 2:** *Suppose $T \in \mathcal{L}(V,W)$ and $T|_U$ is inje. Let $V = M + N,\ U = X + Y$.*
*Then* $\operatorname{range} T = \operatorname{range} T|_M + \operatorname{range} T|_N = \operatorname{range} T|_X + \operatorname{range} T|_Y$.
(a) *Show that if $U = X \oplus Y$, then* $\operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_Y$.
(b) *Give an example such that $V = M \oplus N$,* $\operatorname{range} T \neq \operatorname{range} T|_M \oplus \operatorname{range} T|_N$.

**SOLUTION:** Assume that for some $v \in V$, there exist two distinct pairs $(x_1, y_1), (x_2, y_2)$ in $X \times Y$ such that $Tv = Tx_1 + Ty_1 = Tx_2 + Ty_2$. Because $\forall v \in X \oplus Y, \exists!\,(x,y) \in X \times Y, v = x + y$.
Now $T(x_1 + y_1) = T(x_2 + y_2) \Longrightarrow x_1 + y_1 = x_2 + y_2 \Longrightarrow x_1 = x_2, y_1 = y_2$. Contradicts.
Thus $\forall Tv \in \operatorname{range} T, \exists!\, Tx \in \operatorname{range} T|_X, Ty \in \operatorname{range} T|_Y, Tv = Tx + Ty$. □

**EXAMPLE:** Let $B_V = (v_1, v_2, v_3), B_W = (w_1, w_2),\ T : v_1 \mapsto 0,\ v_2 \mapsto w_1,\ v_3 \mapsto w_2$.
Let $B_M = (v_1 - v_2, v_3), B_N = (v_2)$. Then $\operatorname{range} T|_M = \operatorname{span}(w_1, w_2), \operatorname{range} T|_N = \operatorname{span}(w_1)$

**COMMENT:** Also $\operatorname{null} T|_M = \operatorname{null} T|_N = \{0\}$. Hence $\operatorname{null} T \neq \operatorname{null} T|_M \oplus \operatorname{null} T|_N$.

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**12** *Prove that $\forall T \in \mathcal{L}(V,W), \exists$ subsp $U$ of $V$ such that*
$U \cap \operatorname{null} T = \operatorname{null} T|_U = \{0\}$, $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$.
*Which is equivalent to $T|_U : U \to \operatorname{range} T$ being an iso.*

**SOLUTION:** By [2.34] ( note that $V$ can be infinite-dim ), $\exists$ subsp $U$ of $V$ such that $V = U \oplus \operatorname{null} T$.
$\forall v \in V, \exists!\, w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w + u) = Tu \in \{Tu : u \in U\}$. □

**COROLLARY:** $[P]\quad T|_U : U \to \operatorname{range} T$ *is an iso* $\Longleftrightarrow U \oplus \operatorname{null} T = V. \quad [Q]$
We have shown $Q \Rightarrow P$. Now we show that $P \Rightarrow Q$ to complete the proof.
$\forall v \in V, Tv \in \operatorname{range} T = \operatorname{range} T|_U \Rightarrow \exists!\, u \in U, Tv = Tu \Rightarrow v - u \in \operatorname{null} T$.
Thus $v = (v - u) + u \in U + \operatorname{null} T$. 又 $u \in U \cap \operatorname{null} T \Longleftrightarrow T|_U(u) = 0 \Longleftrightarrow u = 0$. □

OR. $\neg Q \Rightarrow \neg P$ : Because $U \oplus \operatorname{null} T \subsetneq V$. We show $\operatorname{range} T \neq \operatorname{range} T|_U$ by contradiction.
Let $X \oplus (U \oplus \operatorname{null} T) = V$. Now $\operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_U$. And $X$ is nonzero.
Assume that $\operatorname{range} T = \operatorname{range} T|_U$. Then $\operatorname{range} T|_X = \{0\}$. While $T|_X$ is inje. Contradicts.
OR. $\operatorname{range} T|_X \subseteq \operatorname{range} T|_U \Rightarrow \forall x \in X, Tx \in \operatorname{range} T|_U, \exists\, u \in U, Tu = Tx \Rightarrow x = 0$.

Also, $\neg P \Rightarrow \neg Q$ : (a) $\operatorname{range} T|_U \subsetneq \operatorname{range} T$; OR (b) $U \cap \operatorname{null} T \neq \{0\}$.
For (a), $\exists\, x \in V \backslash U, Tx \neq 0 \Longleftrightarrow x \notin \operatorname{null} T$. Thus $U + \operatorname{null} T \subsetneq V$. For (b), immediately. □

**COMMENT:** If $T|_U : U \to \operatorname{range} T$ is an iso. Let $R \oplus U = V$. Then $R$ might not be $\operatorname{null} T$.
OR. Extend $B_U$ to $B_V = (u_1, \dots, u_n, r_1, \dots, r_m)$, then $(r_1, \dots, r_m)$ might not be a $B_{\operatorname{null} T}$.

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• **TIPS 3:** Suppose $T \in \mathcal{L}(V,W)$ and $U$ is a subsp such that $V = U \oplus \operatorname{null} T$. Let $\operatorname{null} T = X \oplus Y$.
Now $\forall v \in V, \exists!\, u_v \in U, (x_v, y_v) \in X \times Y, v = u_v + x_v + y_v$. Define $i \in \mathcal{L}(V,U)$ by $i(v) = u_v + x_v$.
<u>Then $T = T \circ i$.</u> Because $\forall v \in V, T(v) = T(u_v + x_v + y_v) = T(u_v) = T(u_v + x_v) = T\big(i(v)\big) = (T \circ i)(v)$.

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• **TIPS 4:** Suppose $T \in \mathcal{L}(V,W), T \neq 0$. Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
By (3.A.4), $R = (v_1, \dots, v_n)$ is linely inde in $V$. Let $\operatorname{span} R = U$. We will prove that $U \oplus \operatorname{null} T = V$.
(a) $T\big(\sum_{i=1}^n a_i v_i\big) = 0 \Longrightarrow \sum_{i=1}^n a_i T v_i = 0 \Rightarrow a_1 = \cdots = a_n = 0 \Longrightarrow U \cap \operatorname{null} T = \{0\}$.
(b) $Tv = \sum_{i=1}^n a_i T v_i \Rightarrow v - \sum_{i=1}^n a_i v_i \in \operatorname{null} T \Longrightarrow v = \big(v - \sum_{i=1}^n a_i v_i\big) + \big(\sum_{i=1}^n a_i v_i\big) \Rightarrow U + \operatorname{null} T = V$.
OR. $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$. Then by the COROLLARY in Problem (12). □

**COROLLARY:** Conversely, if $U \oplus \operatorname{null} T = V$ and $B_U = (v_1, \dots, v_n)$, then $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
Because $\operatorname{range} T = \operatorname{range} T|_U = \operatorname{span}(Tv_1, \dots, Tv_n)$, 又 $T$ is inje.

• (4E 21) *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V,W)$, $Y$ is a subsp of $W$. Let $\{v \in V : Tv \in Y\}$.*
(a) *Prove that $\{v \in V : Tv \in Y\}$ is a subsp of $V$.*
(b) *Prove that $\dim\{v \in V : Tv \in Y\} = \dim \operatorname{null} T + \dim(Y \cap \operatorname{range} T)$.*

**SOLUTION:** Let $\mathcal{K}_Y = \{v \in V : Tv \in Y\}$.

(a) $\forall u, w \in \mathcal{K}_Y, [\, Tu, Tw \in Y \,], \lambda \in \mathbf{F}, T(u + \lambda w) = Tu + \lambda Tw \in Y \Longrightarrow \mathcal{K}_Y$ is a subsp of $V$.

(b) Define the range-restricted map $R$ of $T$ by $R = T|_{\mathcal{K}_Y} \in \mathcal{L}(\mathcal{K}_Y, Y)$. Now $\operatorname{range} R = Y \cap \operatorname{range} T$.
And $v \in \operatorname{null} T \Longleftrightarrow Tv = 0 \in Y \Longleftrightarrow Rv = 0 \in \operatorname{range} T \Longleftrightarrow v \in \operatorname{null} R$. By [3.22]. $\square$

**COMMENT:** Now $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = \mathcal{K}_Y$. Where $B_{Y \cap \operatorname{range} T} = (Tv_1, \dots, Tv_m)$.
In particular, $\dim \mathcal{K}_{\operatorname{range} T} = \dim \operatorname{null} T + \dim \operatorname{range} T \Longrightarrow \mathcal{K}_{\operatorname{range} T} = V$.

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**28** *Suppose $T \in \mathcal{L}(V,W)$. Let $B_{\operatorname{range} T} = (w_1, \dots, w_m)$.*
*Prove that $\exists\, \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ such that $\forall v \in V, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.*

**SOLUTION:** Suppose $v_1, \dots, v_m \in V$ such that $Tv_i = w_i$ for each $v_i$. Then $(v_1, \dots, v_m)$ is linely inde.
And $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = V$. Now $\forall v \in V, \exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} T,\ v = \sum_{i=1}^{m} a_i v_i + u$.
Define $\varphi_i \in \mathcal{L}(V, \mathbf{F})$ by $\varphi_i(v_j) = \delta_{i,j},\ \varphi_i(u) = 0$ for all $u \in \operatorname{null} T$.
Linearity: $\forall v, w \in V\ [\exists!\, a_i, b_i \in \mathbf{F}], \lambda \in \mathbf{F}, \varphi_i(v + \lambda w) = a_i + \lambda b_i = \varphi(v) + \lambda \varphi(w)$. $\square$

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**29** *Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$. Suppose $\varphi(u) \neq 0$. Prove that $V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}$.*

**SOLUTION:** Let $B_{\operatorname{range} \varphi} = (\varphi(u))$. Then by TIPS (4), $\operatorname{span}(u) \oplus \operatorname{null} \varphi = V$. $\square$

OR. (a) $v = cu \in \operatorname{null} \varphi \cap \operatorname{span}(u) \Rightarrow c\varphi(u) = 0 \Rightarrow v = 0$. Now $\operatorname{null} \varphi \cap \operatorname{span}(u) = \{0\}$.

(b) $\forall\, v \in V, v = \underbrace{\left(v - \dfrac{\varphi(v)}{\varphi(u)} u\right)}_{\in \operatorname{null} \varphi} + \dfrac{\varphi(v)}{\varphi(u)} u \Longrightarrow V = \operatorname{null} \varphi + \operatorname{span}(u)$. $\square$

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**30** *Suppose $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$ and $\operatorname{null} \varphi_1 = \operatorname{null} \varphi_2 = \operatorname{null} \varphi$. Prove that $\exists\, c \in \mathbf{F}, \varphi_1 = c\varphi_2$*

**SOLUTION:**

If $\operatorname{null} \varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $\varphi(u) \neq 0 \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$.
By Problem (29), $V = \operatorname{null} \varphi \oplus \operatorname{span}(u)$. Hence $\forall v \in V, \exists!\, w \in \operatorname{null} \varphi, a \in \mathbf{F}, v = w + a_v u$.
Now $\varphi_1(v) = a\varphi_1(u),\ \ \varphi_2(v) = a\varphi_2(u) \Rightarrow a = \dfrac{\varphi_1(v)}{\varphi_1(u)} = \dfrac{\varphi_2(v)}{\varphi_2(u)} \Longrightarrow \dfrac{\varphi_1(u)}{\varphi_2(u)} = \dfrac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}$. $\square$

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• (4E 31) *Suppose $V$ is finite-dim, $X$ is a subsp of $V$, and $Y$ is a finite-dim subsp of $W$.*
*Prove that if $\dim X + \dim Y = \dim V$, then $\exists\, T \in \mathcal{L}(V,W), \operatorname{null} T = X, \operatorname{range} T = Y$.*

**SOLUTION:** Let $V = U \oplus X, B_U = (v_1, \dots, v_m), B_Y = (w_1, \dots, w_m)$.
Define $T \in \mathcal{L}(V,W)$ by $Tv_i = w_i, Tx = 0$ for each $v_i$ and all $x \in X$.
Because $\forall v \in V, \exists!\, a_i \in \mathbf{F}, x \in X,\ v = \sum_{i=1}^{m} a_i v_i + x$.
Now $v \in \operatorname{null} T \Longleftrightarrow Tv = a_1 w_1 + \cdots + a_m w_m = 0 \Longleftrightarrow v = x \in X$. Hence $\operatorname{null} T = X$.
And $Y \ni w = a_1 w_1 + \cdots + a_m w_m = a_1 Tv_1 + \cdots + a_m Tv_m \in \operatorname{range} T$. Hence $\operatorname{range} T = Y$.
OR. NOTICE that $V = U \oplus \operatorname{null} T$. By the COROLLARY in Problem (12), $\operatorname{range} T = \operatorname{range} T|_U$.
又 $\dim \operatorname{range} T|_U = \dim U = \dim Y$; $\operatorname{range} T \subseteq Y$.
OR. Let $B_X = (x_1, \dots, x_n)$. Now $\operatorname{range} T = \operatorname{span}(Tv_1, \dots, Tv_m, Tx_1, \dots, Tx_n) = \operatorname{span}(w_1, \dots, w_m) = Y$. $\square$

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**20, 21** (a) *Prove that if $ST = I \in \mathcal{L}(V)$, then $T$ is inje and $S$ is surj.*
(b) *Suppose $T \in \mathcal{L}(V, W)$. Prove that if $T$ is inje, then $\exists\, S \in \mathcal{L}(W, V),\ ST = I$.*
(c) *Suppose $S \in \mathcal{L}(W, V)$. Prove that if $S$ is surj, then $\exists\, T \in \mathcal{L}(V, W),\ ST = I$.*

**SOLUTION:**

(a) $Tv = 0 \Rightarrow S(Tv) = 0 = v$. OR. $\operatorname{null} T \subseteq \operatorname{null} ST = \{0\}$.
$\forall v \in V, ST(v) = v \in \operatorname{range} S$. OR. $V = \operatorname{range} ST \subseteq \operatorname{range} S$.

(b) Define $S \in \mathcal{L}(\operatorname{range} T, V)$ by $Sw = T^{-1}w$, where $T^{-1}$ is the inv of $T \in \mathcal{L}(V, \operatorname{range} T)$.
Then extend to $S \in \mathcal{L}(W, V)$ by (3.A.11). Now $\forall v \in V, STv = T^{-1}Tv = v$.

OR. [ *Req V Finite-dim* ] Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n) \Rightarrow B_V = (v_1, \dots, v_n)$. Let $U \oplus \operatorname{range} T = W$.
Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i, Su = 0$ for each $v_i$ and all $u \in U$. Thus $ST = I$.

(c) By Problem (12), $\exists$ subsp $U$ of $W, W = U \oplus \operatorname{null} S,\ \operatorname{range} S = \operatorname{range} S|_U = V$.
Note that $S|_U : U \to V$ is an iso. Define $T = (S|_U)^{-1}$, where $(S|_U)^{-1} : V \to U$.
Then $ST = S \circ (S|_U)^{-1} = S|_U \circ (S|_U)^{-1} = I_V$.

OR. [ *Req V Finite-dim* ] Let $B_{\operatorname{range} S} = B_V = (Sw_1, \dots, Sw_n) \Rightarrow \operatorname{span}(w_1, \dots, w_n) \oplus \operatorname{null} S = W$.
Define $T \in \mathcal{L}(V, W)$ by $T(Sw_i) = w_i$. Now $ST(a_1Sw_1 + \cdots + a_nSw_n) = (a_1Sw_1 + \cdots + a_nSw_n)$. ☐

**COROLLARY:** For (b), if $T$ is inje and $\exists\, S,\ ST = I$, then by (a), this $S$ is surj. Similar for (c).

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**22** *Suppose $U, V$ are finite-dim, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.*
*Prove that* $\dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$.

**SOLUTION:** We show that $\dim \operatorname{null} ST = \dim \operatorname{null} S|_{\operatorname{range} T} + \dim \operatorname{null} T$.
Because (a) $\underline{\operatorname{range} T|_{\operatorname{null} ST} = \operatorname{range} T \cap \operatorname{null} S = \operatorname{null} S|_{\operatorname{range} T}}$,
(b) $\underline{\operatorname{null} T|_{\operatorname{null} ST} = \operatorname{null} T \cap \operatorname{null} ST = \operatorname{null} T}$. By [3.22] ☐

OR. NOTICE that $\underline{u \in \operatorname{null} ST \iff S(Tu) = 0 \iff Tu \in \operatorname{null} S}$.
Thus $\underline{\operatorname{null} ST = \{u \in U : Tu \in \operatorname{null} S\} = \mathcal{K}_{\operatorname{null} S \cap \operatorname{range} T} = \operatorname{null} ST}$.
By Problem (4E 21), $\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T)$. ☐

**COROLLARY:** (1) $T$ surj $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.
(2) $T$ inv $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S,\ \operatorname{null} ST = \operatorname{null} T$.
(3) $S$ inje $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} T$.

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**23** *Suppose $U, V$ are finite-dim, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.*
*Prove that* $\dim \operatorname{range} ST \leqslant \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.

**SOLUTION:** NOTICE that $\underline{\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{range} S|_{\operatorname{range} T}}$.
Let $\operatorname{range} ST = \operatorname{span}(Su_1, \dots, Su_{\dim \operatorname{range} T})$, where $B_{\operatorname{range} T} = (u_1, \dots, u_{\dim \operatorname{range} T})$.
$\dim \operatorname{range} ST \leqslant \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leqslant \dim \operatorname{range} S$. ☐

OR. $\underline{\dim \operatorname{range} ST = \dim \operatorname{range} S|_{\operatorname{range} T} = \dim \operatorname{range} T - \dim \operatorname{null} S|_{\operatorname{range} T}} \leqslant \operatorname{range} T$. ☐
**COMMENT:** $\dim \operatorname{range} ST = \dim U - \dim \operatorname{null} ST = \dim \operatorname{range} T|_U - \dim \operatorname{range} T|_{\operatorname{null} ST}$.

**COROLLARY:** (1) $S|_{\operatorname{range} T}$ inje $\iff \dim \operatorname{range} ST = \dim \operatorname{range} T$.
(2) Let $X \oplus \operatorname{null} S = V$. Then $X \subseteq \operatorname{range} T \iff \operatorname{range} ST = \operatorname{range} S$.
And $T$ is surj $\Rightarrow \operatorname{range} ST = \operatorname{range} S$.

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• **TIPS 5:** Suppose $S \in \mathcal{L}(U, V)$ is surj. Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(U, W))$ by $\mathcal{B}(T) = TS$.
Then $\mathcal{B}$ is inje. Because $\mathcal{B}(T) = TS = 0 \iff T|_{\operatorname{range} S} = 0$. OR. $\operatorname{range} TS = \operatorname{range} T = \{0\}$.

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**24** *Suppose* $S, T \in \mathcal{L}(V, W)$, *and* $\operatorname{null} S \subseteq \operatorname{null} T$. *Prove that* $\exists\, E \in \mathcal{L}(W), T = ES$.

**Solution:**

Let $\underline{V = U \oplus \operatorname{null} S}$
$\underline{\Rightarrow S|_U : U \to \operatorname{range} S \text{ is an iso.}}$
Extend $T\big(S|_U\big)^{-1}$ to $E \in \mathcal{L}(W)$.

$$\begin{tikzcd} \operatorname{range} T & U \arrow[l, "surj\ T"'] \arrow[d, "inv"', "S"] \\ & \operatorname{range} S \arrow[ul, "surj\ E"] \end{tikzcd}$$

Or. Define $E : \operatorname{range} S \to W$ by $\underline{E : Sv \mapsto Tv.}$
Extend $E \in \mathcal{L}\big(\operatorname{range} S, W\big)$ to $E \in \mathcal{L}(W)$. □

**Comment:** Let $\Delta \oplus \operatorname{null} S = \operatorname{null} T$, $U_\Delta \oplus (\Delta \oplus \operatorname{null} S) = V = U_\Delta \oplus \operatorname{null} T$. Redefine $U = U_\Delta \oplus \Delta$.

$$\begin{array}{|c|c|c|}\hline \multicolumn{2}{|c|}{U} & \operatorname{null} S \\ \hline U_\Delta & \multicolumn{2}{c|}{\operatorname{null} T} \\ \hline & \Delta & \operatorname{null} S \\ \hline \end{array}$$

$$\operatorname{range} S \xleftarrow{S} \begin{matrix} U_\Delta \xrightarrow{T} \operatorname{range} T \\ \oplus \\ \Delta \xrightarrow{T} \{0\} \end{matrix}$$

Because $\Delta = \operatorname{null} T|_U = \operatorname{null} T \cap \operatorname{range} \big(S|_U\big)^{-1}$.
Thus $E = T\big(S|_U\big)^{-1}$ is not inje $\Longleftrightarrow \Delta \neq \{0\}$.
In other words, $\operatorname{range} S|_\Delta = \operatorname{null} E$,
while $E|_{\operatorname{range} S|_{U_\Delta}} : \operatorname{range} S|_{U_\Delta} \to \operatorname{range} T$ is an iso.

**Corollary:** If $\operatorname{null} S = \operatorname{null} T$. Then $\Delta = \{0\}, U_\Delta = U$.
By (3.D.3), we can extend inje $T\big(S|_U\big)^{-1} \in \mathcal{L}\big(\operatorname{range} S, W\big)$ to inv $E \in \mathcal{L}(W)$.

Or. $\big[$ *Req* $\operatorname{range} S$ *Finite-dim* $\big]$ Let $B_{\operatorname{range} S} = (Sv_1, \dots, Sv_n)$. Then $\underline{V = \operatorname{span}(v_1, \dots, v_n) \oplus \operatorname{null} S.}$
Let $U \oplus \operatorname{range} S = W$. Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i$, $Eu = 0$ for all $u \in U$ and each $v_i$.
Hence $\forall v \in V$, $\underline{\big(\exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} S \subseteq \operatorname{null} T\,\big), Tv = a_1Tv_1 + \cdots + a_nTv_n} = E(a_1Sv_1 + \cdots + a_nSv_n)$ □

**Corollary:** $\big[$ *Req W Finite-dim* $\big]$ Suppose $\operatorname{null} S = \operatorname{null} T$. We show that $\exists$ inv $E \in \mathcal{L}(W), T = ES$.
Redefine $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i$, $E(w_j) = x_j$, for each $Tv_i$ and $w_j$. Where:
Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m), B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n), B_U = (v_1, \dots, v_m)$.
Now $V = U \oplus \operatorname{null} T = U \oplus \operatorname{null} S \Rightarrow B_{\operatorname{range} S} = \big(Sv_1, \dots, Sv_m\big)$. Let $B'_W = \big(Sv_1, \dots, Sv_m, x_1, \dots, x_n\big)$. □

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**25** *Suppose* $S, T \in \mathcal{L}(V, W)$, *and* $\operatorname{range} T \subseteq \operatorname{range} S$. *Prove that* $\exists\, E \in \mathcal{L}(V), T = SE$.

**Solution:**

Let $V = U \oplus \operatorname{null} S \Rightarrow S|_U : U \to \operatorname{range} S$ is an iso. Because $(S|_U)^{-1} : \underbrace{\operatorname{range} S}_{\supseteq \operatorname{range} T} \to U$.
$\underline{\text{Define } E = \big(S|_U\big)^{-1}T = \big(S|_U\big)^{-1}\big|_{\operatorname{range} T} T \in \mathcal{L}(V, U) \subseteq \mathcal{L}(V).}$ □

**Comment:** Let $U_1 = U$. Let $U_2 \oplus \operatorname{null} T = V = U_1 \oplus \operatorname{null} S$.
Let $U_{1\Delta} = \operatorname{range} \big(S|_{U_1}\big)\big|_{\operatorname{range} T} \subseteq U_1 = \Delta \oplus U_{1\Delta}$.
Or. Let $U_{1\Delta} = \operatorname{range} E|_{U_2}$. Let $\Delta \oplus \operatorname{range} E|_{U_2} = U_1$.
Thus $U_1 \oplus \operatorname{null} S = U_{1\Delta} \oplus \underbrace{\underline{(\Delta \oplus \operatorname{null} S)} = U_2 \oplus \underline{\operatorname{null} T}}_{\text{iso, by (3.D.Tips).}}$.

$$\begin{tikzcd} U_1 \arrow[r, "inv", "S"'] \arrow[d, equal] & \operatorname{range} S \arrow[d, equal] & \\ \Delta \arrow[r, "inv", "S"'] & \operatorname{range} S|_\Delta & \\ \oplus & \oplus & \\ U_{1\Delta} \arrow[r, "inv", "S"'] & \operatorname{range} T & U_2 \arrow[l, "inv"', "T"] \arrow[lll, bend left, "inv\ E|_{U_2}"] \end{tikzcd}$$

If $\Delta \neq \{0\}$, assume $\exists$ inv $E \in \mathcal{L}(V)$ re-extended from $E|_{U_2}$ still satisfying $T = SE$,
then let $\Delta \xrightarrow{E^{-1}} \Theta$; $\operatorname{null} S \xrightarrow{E^{-1}} \operatorname{null} T_\Theta$. Now $\Theta \oplus \operatorname{null} T_\Theta = \operatorname{null} T$.
Then $\Theta \xrightarrow{E} \Delta \neq \{0\}$, while $\operatorname{null} S \cap \Delta = \{0\}$. Thus $T|_\Theta = SE|_\Theta \neq 0$, contradicts.

**Corollary:** If $\Delta = \{0\}$, then $U_1 = U_{1\Delta} \Rightarrow \operatorname{range} S = \operatorname{range} T$. 又 $\operatorname{null} S, \operatorname{null} T$ are iso.
By (3.D.3), we can re-extend inje $E|_{U_2} \in \mathcal{L}\big(U_2, U_1 \oplus \operatorname{null} S\big)$ to inv $E \in \mathcal{L}\big(U_2 \oplus \operatorname{null} T, U_1 \oplus \operatorname{null} S\big)$.

Thus we have $\Delta \neq \{0\} \Longleftrightarrow E|_{U_2} \in \mathcal{L}(U_2, V)$ cannot be re-extended to inv $E \in \mathcal{L}(V)$ freely.

Or. $\big[$ *Req* $\operatorname{range} T$ *Finite-dim* $\big]$ Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$. Then $\underline{V = \operatorname{span}(v_1, \dots, v_n) \oplus \operatorname{null} T.}$
$\underline{\text{Let } S(u_i) = Tv_i \text{ for each } Tv_i.}$ Define $E$ by $Ev_i = u_i, Ex = 0$ for all $x \in \operatorname{null} T$ and each $v_i$. □

**Comment:** $\big[$ *Req V Finite-dim* $\big]$ Note that $\dim U_2 \leqslant \dim U_1 \Longrightarrow \dim \operatorname{null} T = p \geqslant q = \dim \operatorname{null} S$.
Let $B_{\operatorname{null} T} = (x_1, \dots, x_p), B_{\operatorname{null} S} = (y_1, \dots, y_q)$. Redefine $E : v_i \mapsto u_i$, $x_k \mapsto y_k$, $x_j \mapsto 0$,
for each $i \in \{1, \dots, \dim U_2\}, k \in \{1, \dots, \dim \operatorname{null} S\}, j \in \{\dim \operatorname{null} S + 1, \dots, \dim \operatorname{null} T\}$.
Note that $(u_1, \dots, u_n)$ is linely inde. Let $X = \operatorname{span}(x_1, \dots, x_q) \oplus \operatorname{span}(v_1, \dots, v_n)$.

Now $E|_X$ is inje, but cannot be re-extend to inv $E \in \mathcal{L}(V)$ without loss of functionality.

**Corollary:** $\big[$ *Req V Finite-dim* $\big]$ If range $T$ = range $S$, then $\dim \operatorname{null} T = \dim \operatorname{null} S = p$.
Redefine $E$ by $Ev_i = u_i,\ Ex_j = y_j$ for each $v_i$ and $x_j$. Then $E \in \mathcal{L}(V)$ is inv. □

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• Or (5.B.4) *Suppose* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove that* $V = \operatorname{null} P \oplus \operatorname{range} P$.

**Solution:** (a) If $v \in \operatorname{null} P \cap \operatorname{range} P \Rightarrow Pv = 0$ and $\exists\, u \in V, v = Pu$. Then $v = Pu = P^2u = Pv = 0$.
(b) Note that $\forall v \in V, v = Pv + (v - Pv)$ and $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$. □

Or. $\big[$ *Only in Finite-dim* $\big]$ Let $B_{\operatorname{range} P^2} = (P^2v_1, \dots, P^2v_n)$. Then $(Pv_1, \dots, Pv_n)$ is linely inde.
Let $U = \operatorname{span}(Pv_1, \dots, Pv_n) \Rightarrow V = U \oplus \operatorname{null} P^2$. While $U = \operatorname{range} P = \operatorname{range} P^2$; $\operatorname{null} P = \operatorname{null} P^2$. □

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• (a) *Suppose* $\dim V = n,\ ST = 0$ *where* $S, T \in \mathcal{L}(V)$. *Prove that* $\dim \operatorname{range} TS \leqslant \left\lfloor \dfrac{n}{2} \right\rfloor$.
(b) *Give an example of such* $S, T$ *with* $n = 5$ *and* $\dim \operatorname{range} TS = 2$.

**Solution:** Using Problem (23).
$\dim \operatorname{range} TS \leqslant \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$. We prove by contradiction.
Assume that $\dim \operatorname{range} TS \geqslant \left\lfloor \dfrac{n}{2} \right\rfloor + 1$. Then $\min\Big\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\Big\} \geqslant \left\lfloor \dfrac{n}{2} \right\rfloor + 1$
又 $\dim \operatorname{null} ST = n \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$ $\Bigg|$ $\Rightarrow \max\Big\{\dim \operatorname{null} T, \dim \operatorname{null} S\Big\} \leqslant n - \left\lfloor \dfrac{n}{2} \right\rfloor - 1$.
Thus $n \leqslant 2\left(n - \left\lfloor \dfrac{n}{2} \right\rfloor - 1\right) \Rightarrow \left\lfloor \dfrac{n}{2} \right\rfloor + 1 \leqslant \dfrac{n}{2}$. Contradicts. □

Or. $\dim \operatorname{null} S = n - \dim \operatorname{range} S \leqslant n - \dim \operatorname{range} TS$. 又 $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S$.
$\dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S \leqslant n - \dim \operatorname{range} TS$. Thus $2 \dim \operatorname{range} TS \leqslant n$. □

**Example:** Let $B_V = (v_1, \dots, v_5)$. Define $T:\ v_1 \mapsto 0,\ \ v_2 \mapsto 0,\ \ v_i \mapsto v_i\,;$
$S:\ v_1 \mapsto v_4,\ v_2 \mapsto v_5,\ \ v_i \mapsto 0\,;\ \ i = 3, 4, 5.$

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**26** *Suppose* $D \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$ *and* $\forall p, \deg(Dp) = (\deg p) - 1$. *Prove that* $D \in \mathcal{P}(\mathbf{R})$ *is surj.*

**Solution:** $\big[$ $D$ might not be $D : p \mapsto p'$. $\big]$ Notice that the following proof is wrong:
Because $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) \subseteq \operatorname{range} D$, and $\deg Dx^n = n - 1$.
又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) = \operatorname{span}(1, x, x^2, \cdots) = \mathcal{P}(\mathbf{R})$.

<u>Let $D(C) = 0, Dx^k = p_k$</u> of deg $(k-1)$, for all $C \in \mathbf{R} = \mathcal{P}_0(\mathbf{R})$ and for each $k \in \mathbf{N}^+$.
Because $B_{\mathcal{P}_m(\mathbf{R})} = (p_1, \dots, p_m, p_{m+1})$. And for all $p \in \mathcal{P}(\mathbf{R}), \exists\,!\, m = \deg p \in \mathbf{N}^+$.
So that $\exists\,!\, a_i \in \mathbf{R}, p = \sum_{i=1}^{m+1} a_i p_i \Rightarrow \exists\, q = \sum_{i=1}^{m+1} a_i x^i, Dq = p$. □

Or. We will <u>recursively define a sequence of polys $(p_k)_{k=0}^{\infty}$</u> where $Dp_0 = 1, Dp_k = x^k$ for each $k \in \mathbf{N}^+$.
So that $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), Dq = p, \exists\, q = \sum_{k=0}^{\deg p} a_k p_k$.
(i) Because $\deg Dx = (\deg x) - 1 = 0, Dx = C \in \mathbf{F} \backslash \{0\}$. Let $p_0 = C^{-1}x \Rightarrow Dp_0 = C^{-1}Dx = 1$.
(ii) Suppose we have defined $Dp_0 = 1, Dp_k = x^k$ for each $k \in \{1, \dots, n\}$. Because $\deg D(x^{n+2}) = n + 1$.
Let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$, with $a_{n+1} \neq 0$.
Then $a_{n+1}^{-1} D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_n Dp_n + \cdots + a_1 Dp_1 + a_0 Dp_0)$
$\Rightarrow x^{n+1} = D\Big[\underline{a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)}\Big]$. Thus defining $p_{n+1}$, so that $Dp_{n+1} = x^{n+1}$. □

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**Ended**

# 3·C

1 3 4 5 6 9 10 11 13 | 4E: 16 17

- **Note For Transpose:** [3.F.33] Define $\mathcal{T}: A \to A^t$. By [3.111], $\mathcal{T}$ is linear. Because $(A^t)^t = A$.
  $\mathcal{T}^2 = I$, $\mathcal{T} = \mathcal{T}^{-1} \Rightarrow \mathcal{T}$ is an iso of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$. Define $\mathcal{C}_k : A \to A_{\cdot,k}$, $\mathcal{R}_j : A \to A_{j,\cdot}$, $\mathcal{E}_{j,k} : A \to A_{j,k}$.
  Now we show that (a) $\underline{\mathcal{T}\mathcal{R}_j = \mathcal{C}_j\mathcal{T}}$, (b) $\underline{\mathcal{T}\mathcal{C}_k = \mathcal{R}_k\mathcal{T}}$, and (c) $\underline{\mathcal{T}\mathcal{E}_{j,k} = \mathcal{E}_{k,j}\mathcal{T}}$.
  So that furthermore, $\mathcal{T}\mathcal{C}_k\mathcal{T} = \mathcal{R}_k$, $\mathcal{T}\mathcal{R}_j\mathcal{T} = \mathcal{C}_j$, and $\mathcal{T}\mathcal{E}_{j,k}\mathcal{T} = \mathcal{E}_{k,j}$.

  Let $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{pmatrix}$.
  Note that $(A_{j,k})^t = A_{j,k} = (A^t)_{k,j}$. Thus (c) holds.
  And $(A_{\cdot,k})^t = (A_{1,k} \cdots A_{m,k}) = (A^t_{k,1} \cdots A^t_{k,m}) = (A^t)_{k,\cdot}$
  $\Longrightarrow$ (b) holds. Similar for (a).

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- **Note For [3.48]:**

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}}_{B} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

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- **Note For [3.47]:** $(AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k}$ ☐
- **Note For [3.49]:** $\left[(AC)_{\cdot,k}\right]_{j,1} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$ ☐
- **Exercise 10:** $\left[(AC)_{j,\cdot}\right]_{1,k} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$ ☐
- **Comment:** For [3.49], let $B_U = (u_1, \dots, u_p), B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.
  And $C = \mathcal{M}(T, B_U, B_V) \in \mathbf{F}^{n,p}, A = \mathcal{M}(S, B_V, B_W) \in \mathbf{F}^{m,n}$.
  Then $\mathcal{M}(Tu_k, B_V) = C_{\cdot,k} \Rightarrow \mathcal{M}(S(Tu_k), B_W) = AC_{\cdot,k}$, 又 $\mathcal{M}((ST)(u_k), B_W) = (AC)_{\cdot,k}$ ☐
  By Note For Transpose, $(AC)_{j,\cdot} = \left[\left((AC)^t\right)_{\cdot,j}\right]^t = \left(C^t(A^t)_{\cdot,j}\right)^t = \left((A^t)_{\cdot,j}\right)^t C = A_{j,\cdot}C$ ☐

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- **Note For [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$. By [4E 3.51(a)], $(Ac)_{\cdot,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ ☐
  Or. $\because (Ac)_{j,1} = \sum_{r=1}^n A_{j,r}c_{r,1} = \left[\sum_{r=1}^n (A_{\cdot,r}c_{r,1})\right]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$
  $\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^n A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ Or. $(Ac)_{j,1} = (Ac)_{j,\cdot} = A_{j,\cdot}c \in \mathbf{F}$. ☐
  Or. Let $B_V = (v_1, \dots, v_n)$. Now $Ac = \mathcal{M}(Tv, B_W) = \mathcal{M}(T(c_1v_1 + \cdots + c_nv_n)) = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$. ☐

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- **Exercise 11:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$. By [4E 3.51(b)], $(aC)_{1,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ ☐
  Or. $\because (aC)_{1,k} = \sum_{r=1}^n a_{1,r}C_{r,k} = \left[\sum_{r=1}^n a_{1,r}(C_{r,\cdot})\right]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$
  $\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^n a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ Or. $(aC)_{1,k} = (aC)_{\cdot,k} = aC_{\cdot,k} \in \mathbf{F}$. ☐
  Or. $aC = \left((aC)^t\right)^t = \left(C^ta^t\right)^t = \left[a^t_1(C^t)_{\cdot,1} + \cdots + a^t_n(C^t)_{\cdot,n}\right]^t = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$. ☐

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- [4E 3.51] Suppose $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}$. [ See also Note For [3.49] and Problem (10). ]
  (a) For $k = 1, \dots, p$, $(CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^c C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$
  (b) For $j = 1, \dots, m$, $(CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^c C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
- **Example:** $m = 2, c = 2, p = 3$.

$$(AB)_{\cdot,2} = AB_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} = A_{\cdot,1}B_{1,2} + A_{\cdot,2}B_{2,2} = 6\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix};$$

$$(AB)_{1,\cdot} = A_{1,\cdot}B = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = A_{1,1}B_{1,\cdot} + A_{1,2}B_{2,\cdot} = 1\begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 2\begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \end{pmatrix};$$

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• **Column-Row Factorization** (CR Factorization) *Suppose* $A \in \mathbf{F}^{m,n}, A \neq 0$.
*Prove, with $p$ specified below, that* $\exists\, C \in \mathbf{F}^{m,p}, R \in \mathbf{F}^{p,n}, A = CR$.
(a) *Suppose* $S_c = \operatorname{span}\big(A_{\cdot,1}, \cdots, A_{\cdot,n}\big) \subseteq \mathbf{F}^{m,1}, \dim S_c = c$, *the col rank. Let* $p = c$.
(b) *Suppose* $S_r = \operatorname{span}\big(A_{1,\cdot}, \cdots, A_{m,\cdot}\big) \subseteq \mathbf{F}^{1,n}, \dim S_r = r$, *the row rank. Let* $p = r$.

**Solution:** Using [4E 3.51]. Notice that $A \neq 0 \Rightarrow c, r \geqslant 1$.

(a) Reduce to basis $B_C = \big(C_{\cdot,1}, \cdots, C_{\cdot,c}\big)$, forming $C \in \mathbf{F}^{m,c}$. Then $\forall k \in \{1, \ldots, n\}$,
$A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$, $\exists!\, R_{1,k}, \cdots, R_{c,k} \in \mathbf{F}$, forming $R \in \mathbf{F}^{c,n}$. Thus $A = CR$.

(b) Reduce to basis $B_R = \big(R_{1,\cdot}, \cdots, R_{r,\cdot}\big)$, forming $R \in \mathbf{F}^{r,n}$. Then $\forall j \in \{1, \ldots, m\}$,
$A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,r}R_{r,\cdot} = (CR)_{j,\cdot}$, $\exists!\, C_{j,1}, \ldots, C_{j,r} \in \mathbf{F}$, forming $C \in \mathbf{F}^{m,r}$. Thus $A = CR$. $\square$

**Example:**
$$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \overset{\text{(I)}}{=\!=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \overset{\text{(II)}}{=\!=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

(I) $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2\begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix}$, using [4E 3.51(b)].

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} \in \operatorname{span}(A_{1,\cdot}, A_{2,\cdot})$, and $(A_{1,\cdot}, A_{2,\cdot})$ is linely inde. Thus $B_R = \big(A_{1,\cdot}, A_{2,\cdot}\big)$.

(II) $\begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = -\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$. Thus $B_C = \big(A_{\cdot,2}, A_{\cdot,3}\big)$.

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• **Column Rank Equals Row Rank** Using notation and result above.
For each $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
For each $A_{\cdot,k} \in S_c$, $A_{\cdot,k} = (CR)_{\cdot,k} = CR_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$
$\Rightarrow \operatorname{span}\big(A_{1,\cdot}, \cdots, A_{n,\cdot}\big) = S_r = \operatorname{span}\big(R_{1,\cdot}, \cdots, R_{c,\cdot}\big) \Rightarrow \dim S_r = r \leqslant c = \dim S_c$.
$\Rightarrow \operatorname{span}\big(A_{\cdot,1}, \cdots, A_{\cdot,m}\big) = S_c = \operatorname{span}\big(C_{\cdot,1}, \cdots, C_{\cdot,r}\big) \Rightarrow \dim S_c = c \leqslant r = \dim S_r$.
Or. Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leqslant r = \dim S_r = \dim S_c^t$. $\square$

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• *Suppose* $A \in \mathbf{F}^{m,n} \backslash \{0\}$. *Prove that* $[P]$ $\operatorname{rank} A = 1 \iff \exists\, c_j, d_k \in \mathbf{F}$, *each* $A_{j,k} = c_j \cdot d_k$. $[Q]$

**Solution:**

[ *Using CR Factorization* ]

$P \Rightarrow Q$ : Immediately.
$Q \Rightarrow P$ : Because $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}\begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix} \Longrightarrow S_r = \operatorname{span}\left\{ \begin{matrix} \big(\underline{c_1} d_1 \ \cdots \ \underline{c_1} d_n\big), \\ \vdots \\ \big(\underline{c_m} d_1 \ \cdots \ \underline{c_m} d_n\big) \end{matrix} \right\}$.

Or. $S_c = \operatorname{span}\left\{ \begin{pmatrix} c_1 \underline{d_1} \\ \vdots \\ c_m \underline{d_1} \end{pmatrix}, \ldots, \begin{pmatrix} c_1 \underline{d_n} \\ \vdots \\ c_m \underline{d_n} \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$. $\square$

[ *Not Using CR Factorization* ]

$Q \Rightarrow P$ : Using [4E 3.51(a)]. Each $A_{\cdot,k} \in \operatorname{span}\left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$. Then $\operatorname{rank} A = \dim S_c \leqslant 1$
又 $A \neq 0 \Rightarrow \dim S_c \geqslant 1$.
$P \Rightarrow Q$ : Because $\dim S_c = \dim S_r = 1$.

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $\quad d_k' = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$.

$\Rightarrow A_{j,k} = d_k' A_{j,1} = c_j A_{1,k} = c_j d_k' A_{1,1} = c_j d_k$, where $d_k = d_k' A_{1,1}$. $\square$

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• **TIPS 1:** *Suppose* $T \in \mathcal{L}(V,W)$, $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.
*Let* $L = \left(Tv_{\alpha_1}, \dots, Tv_{\alpha_k}\right)$, $M = \left(A_{\cdot,\alpha_1}, \cdots, A_{\cdot,\alpha_k}\right)$, *where each* $\alpha_i \in \{1, \dots, n\}$.
(a) *Show that* $[P]$ *$L$ is linely inde* $\iff$ *$M$ is linely inde.* $[Q]$
(b) *Show that* $[P]$ $\operatorname{span} L = W \iff \operatorname{span} M = \mathbf{F}^{m,1}$. $[Q]$ $\quad$ [ *Let* $A = \mathcal{M}(T, B_V, B_W)$. ]

**SOLUTION:**

(a) Note that $\mathcal{M}\colon Tv_k \to A_{\cdot,k}$ is an iso of $W$ onto $\mathbf{F}^{m,1}$. (b) Reduce $L$ to $B'_W$, $M$ to $B_{\mathbf{F}^{m,1}}$. Similarly. □

OR. 
$$\begin{aligned} c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} &= c_1\left(A_{1,\alpha_1} w_1 + \cdots + A_{m,\alpha_1} w_m\right) + \cdots + c_k\left(A_{1,\alpha_k} w_1 + \cdots + A_{m,\alpha_k} w_m\right) \\ &= \left(c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k}\right) w_1 + \cdots + \left(c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k}\right) w_m. \end{aligned}$$

And 
$$c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} = c_1 \begin{pmatrix} A_{1,\alpha_1} \\ \vdots \\ A_{m,\alpha_1} \end{pmatrix} + \cdots + c_k \begin{pmatrix} A_{1,\alpha_k} \\ \vdots \\ A_{m,\alpha_k} \end{pmatrix} = \begin{pmatrix} c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k} \\ \vdots \\ c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k} \end{pmatrix}.$$

(a) $P \Rightarrow Q$: Suppose $c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} = 0$. Let $v = c_1 v_{\alpha_1} + \cdots + c_k v_{\alpha_k}$.
Then $Tv = \left(c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k}\right) w_1 + \cdots + \left(c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k}\right) w_m = 0 w_1 + \cdots + 0 w_m$.
Now $c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} = 0$. Then each $c_i = 0 \Rightarrow M$ linely inde.

$Q \Rightarrow P$: Because $c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} = 0$. For each $i \in \{1, \dots, m\}$, $c_1 A_{i,\alpha_1} + \cdots + c_k A_{i,\alpha_k} = 0$.
Which is equi to $c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} = 0$. Thus each $c_i = 0 \Rightarrow L$ linely inde.

OR. $\exists A_{\cdot,\alpha_j} = c_1 A_{\cdot,\alpha_1} + \cdots + c_{j-1} A_{\cdot,\alpha_{j-1}}$
$\iff$ For each $i \in \{1, \dots, m\}$, $A_{i,\alpha_j} = c_1 A_{i,\alpha_1} + \cdots + c_{j-1} A_{i,\alpha_{j-1}}$
$$\begin{aligned} \iff Tv_{\alpha_j} &= A_{1,\alpha_j} w_1 + \cdots + A_{m,\alpha_j} w_m \\ &= \left(c_1 A_{1,\alpha_1} + \cdots + c_{j-1} A_{1,\alpha_{j-1}}\right) w_1 + \cdots + \left(c_1 A_{m,\alpha_1} + \cdots + c_{j-1} A_{m,\alpha_{j-1}}\right) w_m \end{aligned}$$
$\iff \exists\, Tv_{\alpha_j} = c_1 Tv_{\alpha_1} + \cdots + c_{j-1} Tv_{\alpha_{j-1}}$.

(b) Note that each $\mathcal{M}(Tv_{\alpha_i}) = A_{\cdot,\alpha_i}$

$P \Rightarrow Q$: Suppose each $w_i = Iw_i = J_{1,i} Tv_{\alpha_1} + \cdots + J_{k,i} Tv_{\alpha_k}$.
$\forall a \in \mathbf{F}^{m,1}, \exists\, w = a_1 w_1 + \cdots + a_m w_m \in W$, $a = \mathcal{M}(w, B_W)$.
$$\begin{aligned} \text{Because } w &= a_1\left(J_{1,1} Tv_{\alpha_1} + \cdots + J_{k,1} Tv_{\alpha_k}\right) + \cdots + a_m\left(J_{1,m} Tv_{\alpha_1} + \cdots + J_{k,m} Tv_{\alpha_k}\right) \\ &= \left(a_1 J_{1,1} + \cdots + a_m J_{1,m}\right) Tv_{\alpha_1} + \cdots + \left(a_1 J_{k,1} + \cdots + a_m J_{k,m}\right) Tv_{\alpha_k}. \end{aligned}$$
Apply $\mathcal{M}$ to both sides, $a = c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k}$, where each $c_i = a_1 J_{i,1} + \cdots + a_m J_{i,m}$.

$Q \Rightarrow P$: $\forall w \in W, \exists\, a = c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} \in \mathbf{F}^{m,1}$, $\mathcal{M}(w, B_W) = a$
$\Rightarrow w = \left(c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k}\right) w_1 + \cdots + \left(c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k}\right) w_m = c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k}$.

$\neg Q \Rightarrow \neg P$: $\exists\, w \in W, \exists\, a \in \mathbf{F}^{m,1}, \mathcal{M}(w, B_W) = a$, but $\nexists\, c_i \in \mathbf{F}, a = c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k}$
$\Rightarrow \nexists\, c_i \in \mathbf{F}$, $w = c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k}$. □

**COROLLARY:** Let $L = \left(Tv_1, \dots, Tv_n\right)$, $M = \left(A_{\cdot,1}, \cdots, A_{\cdot,n}\right)$.
Then (a*) By [3.B.9, TIPS (4)], $T$ is inje $\iff$ $L$ is linely inde, so is $M$.
And (b*) $T$ is surj $\iff \operatorname{span} L = W \iff \operatorname{span} M = \mathbf{F}^{m,1}$.
**COROLLARY:** $B_{\mathbf{F}^{n,1}} = \left(A_{\cdot,1}, \cdots, A_{\cdot,n}\right) \iff T$ is inje and surj $\iff B_{\mathbf{F}^{1,n}} = \left(A_{\cdot,1}, \cdots, A_{\cdot,n}\right)$.
**COMMENT:** If $T$ is inv. Then by (a*, c) or (b*, d), we have another proof of COROLLARY.
OR. If $m = n$. Then by [3.118] and one of (a*, b*, c, d). Yet another proof.
(c) $T$ surj $\iff T'$ inje $\iff \left(T'(\psi_1), \dots, T'(\psi_m)\right)$ linely inde
$\overset{(a)}{\iff} \left((A^t)_{\cdot,1}, \cdots, (A^t)_{\cdot,m}\right)$ linely inde in $\mathbf{F}^{n,1}$, so is $\left(A_{1,\cdot}, \cdots, A_{m,\cdot}\right)$ in $\mathbf{F}^{1,n}$.
(d) $T$ inje $\iff T'$ surj $\iff V' = \operatorname{span}\left(T'(\psi_1), \dots, T'(\psi_m)\right)$
$\overset{(b)}{\iff} \mathbf{F}^{n,1} = \operatorname{span}\left((A^t)_{\cdot,1}, \cdots, (A^t)_{\cdot,m}\right) \iff \mathbf{F}^{1,n} = \operatorname{span}\left(A_{1,\cdot}, \cdots, A_{m,\cdot}\right)$.

• **Tips 2:** *Suppose $p$ is a poly of $n$ variables in $\mathbf{F}$.*
*Prove that $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big)$.*
*Where the linear maps $T_1,\dots,T_n$ are such that $p(T_1,\dots,T_n)$ makes sense.* See [5.16,17,20].

**Solution:** Suppose the poly $p$ is defined by $p(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^{n} x_i^{k_i}$.

Note that $\mathcal{M}(T^xS^y) = \mathcal{M}(T)^x\mathcal{M}(S)^y$; $\mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$.

Then $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = \mathcal{M}\big(\sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^{n} T_i^{k_i}\big)$

$= \sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^{n}\mathcal{M}(T_i^{k_i}) = p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big)$. □

• **Corollary:** Suppose $\tau$ is an algebraic property. Then $\tau$ holds for linear maps $\iff$ $\tau$ holds for matrices.
Each $\alpha_k \in \{1,\dots,n\}$. Now $p(T_1,\dots,T_n) = p(T_{\alpha_1},\dots,T_{\alpha_n})$
$\iff p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big) = p\big(\mathcal{M}(T_{\alpha_1}),\dots,\mathcal{M}(T_{\alpha_n})\big)$.

---

**13** *Prove that the distr holds for matrix add and matrix multi.*

*Suppose $A,B,C$ are matrices such that $A(B+C)$ make sense, we prove the left distr.*

**Solution:** Suppose $A \in \mathbf{F}^{m,n}$ and $B,C \in \mathbf{F}^{n,p}$.

Note that $\big[A(B+C)\big]_{j,k} = \sum_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum_{r=1}^{n}\big(A_{j,r}B_{r,k} + A_{j,r}C_{r,k}\big) = \big(AB+AC\big)_{j,k}$.

Or. Define $T,S,R$ such that $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$A(B+C) = \mathcal{M}(T(S+R)) \overset{[3.9]}{=\!=} \mathcal{M}(TS+TR) = AB+AC$.

Or. $T(S+R) = TS+TR \Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR) \Rightarrow A(B+C) = AB+AC$. □

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**1** *Suppose $T \in \mathcal{L}(V,W)$. Show that for each pair of $B_V$ and $B_W$,*
*$A = \mathcal{M}(T,B_V,B_W)$ has at least $n = \dim \operatorname{range} T$ nonzero entries.*

**Solution:**

Using [3.B Tips (4)]. Let $U \oplus \operatorname{null} T = V$; $B_U = (v_1,\dots,v_n), B_V = (v_1,\dots,v_m)$.

For each $k \in \{1,\dots,n\}, Tv_k \neq 0 \iff A_{\cdot,k} \neq 0$. Hence every such $A_{\cdot,k}$ has at least one nonzero entry. □

Or. We prove by contradiction. Suppose $A$ has at most $(n-1)$ nonzero entries.

Then by Pigeon Hole Principle, at least one of $A_{\cdot,1},\dots,A_{\cdot,n}$ equals 0.

Thus there are at most $(n-1)$ nonzero vecs in $Tv_1,\dots,Tv_n$.

又 $\operatorname{range} T = \operatorname{span}(Tv_1,\dots,Tv_n) \Rightarrow \dim\operatorname{range} T = \dim\operatorname{span}(Tv_1,\dots,Tv_n) \leqslant n-1$. Contradicts. □

---

**6** *Suppose $V$ and $W$ are finite-dim and $T \in \mathcal{L}(V,W)$.*
*Prove that $\dim\operatorname{range} T = 1 \iff \exists\, B_V, B_W$, all entries of $A = \mathcal{M}\big(T,B_V,B_W\big)$ equal 1.*

**Solution:**

(a) Suppose $B_V = (v_1,\dots,v_n), B_W = (w_1,\dots,w_m)$ are the bases such that all entries of $A$ equal 1.
Then $Tv_i = w_1 + \dots + w_m$ for all $i = 1,\dots,n$. Because $w_1,\dots,w_n$ is linely inde, $w_1 + \dots + w_n \neq 0$.

(b) Suppose $\dim\operatorname{range} T = 1$. Then $\dim\operatorname{null} T = \dim V - 1$.
Let $B_{\operatorname{null} T} = (u_2,\dots,u_n)$. Extend to a basis $(u_1,u_2,\dots,u_n)$ of $V$.
Let $w_1 = Tv_1 - w_2 - \dots - w_m$. Extend to $B_W$. Let $v_1 = u_1,\ v_i = u_1 + u_i$. Extend to $B_V$. □

Or. Suppose $B_{\operatorname{range} T} = (w)$. By [2.C Note For (15)], $\exists\, B_W = (w_1,\dots,w_m),\ w = w_1 + \dots + w_m$.
By [2.C Tips], $\exists$ a basis $(u_1,\dots,u_n)$ of $V$ such that each $u_k \notin \operatorname{null} T$.
Now each $Tu_k \in \operatorname{range} T = \operatorname{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists\, \lambda_k \in \mathbf{F}\backslash\{0\}$.
Let $v_k = \lambda_k^{-1}u_k \neq 0$, so that each $Tv_k = w = w_1 + \dots + w_m$. Thus $B_V = (v_1,\dots,v_n)$ will do. □

---

**3** *Suppose $V$ and $W$ are finite-dim and $T \in \mathcal{L}(V,W)$. Prove that $\exists\, B_V, B_W$ such that* $[$ *letting* $A = \mathcal{M}(T, B_V, B_W)$ $]$ $A_{k,k} = 1, A_{i,j} = 0$, *where* $1 \leqslant k \leqslant \dim \operatorname{range} T, i \neq j$.

**SOLUTION:** Using [3.B TIPS (4)]. Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n), B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$. ☐

**COMMENT:** Let each $Tv_k = w_k$. Extend $B_{\operatorname{range} T}$ to $B_W = (w_1, \dots, w_n, \dots, w_p)$. See [3.D NOTE FOR [3.60]].

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**4** *Suppose $B_V = (v_1, \dots, v_m)$ and $W$ is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
*Prove that $\exists\, B_W = (w_1, \dots, w_n)$, $\mathcal{M}(T, B_V, B_W)_{\cdot,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^t$ or $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}^t$.*

**SOLUTION:** If $Tv_1 = 0$, then we are done. If not then extend $(Tv_1)$ to $B_W$. ☐

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**5** *Suppose $B_W = (w_1, \dots, w_n)$ and $V$ is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
*Prove that $\exists\, B_V = (v_1, \dots, v_m)$, $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$.*

**SOLUTION:**

Let $(u_1, \dots, u_n)$ be a basis of $V$. Denote $\mathcal{M}(T, (u_1, \dots, u_n), B_W)$ by $A$.

If $A_{1,\cdot} = 0$, then $B_V = (u_1, \dots, u_n)$ and we are done. Otherwise, suppose $A_{1,k} \neq 0$.

Let $v_1 = \dfrac{u_k}{A_{1,k}}$, so that $Tv_1 = 1w_1 + \dfrac{A_{2,k}}{A_{1,k}} w_2 + \cdots + \dfrac{A_{n,k}}{A_{1,k}} w_n$.

Let $v_j = u_{j-1} - A_{1,j-1} v_1$ for each $j \in \{2, \dots, k\}$. Let $v_i = u_i - A_{1,i} v_1$ for $i \in \{k+1, \dots, n\}$.

NOTICE that $Tu_i = A_{1,i} w_1 + \cdots + A_{n,i} w_n$. 又 Each $u_i \in \operatorname{span}(v_1, \dots, v_n) = V$. Let $B_V = (v_1, \dots, v_n)$. ☐

OR. Using Problem (4). Let $B_{W'}$ be the $B_V$.

Now $\exists\, B_{V'}$ such that $\mathcal{M}(T', B_{W'}, B_{V'})_{\cdot,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^t$ or $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}^t$.

Which is equiv to $\exists\, B_V$ [Using (3.F.31)] such that $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$. ☐

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**ENDED**

# 3·D

1 2 3 4 5 6 8 9 10 11 12 13 15 16 17 18 19 | 4E: 3 10 15 17 19 20 22 23 24

**2** *Suppose $V$ is finite-dim and* $\dim V > 1$.
*Prove that the set $U$ of non-inv operators on $V$ is not a subsp of $\mathcal{L}(V)$.*
*The set of inv operators is not either. Although multi identity/inv, and commutativity for vec multi hold.*

**SOLUTION:** Let $B_V = (v_1, \dots, v_n)$. [ *If* $\dim V = 1$, *then* $U = \{0\}$ *is a subsp of* $\mathcal{L}(V)$. ]
Define $S, T \in \mathcal{L}(V)$ by $S(a_1 v_1 + \cdots + a_n v_n) = a_1 v_1$, $T(a_1 v_1 + \cdots + a_n v_n) = a_2 v_1 + \cdots + a_n v_n$.
Hence $S, T \in U$ while $S + T \notin U$. ☐

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• **TIPS:** *Suppose $U \oplus X = W \oplus Y$, and $X, Y$ are iso. Prove that $U, W$ are iso.*

**SOLUTION:** Let $\xi$ be an iso of $X$ onto $Y$. That is, $\forall y \in Y, \exists!\, x \in X, \xi(x) = y$.
$\forall u \in U, \exists!\, w \in W, y \in Y, u = w + y \Rightarrow \exists!\, x \in X, u = w + \xi(x)$. Define $\pi : u \mapsto w$.
Now suppose $u_1, u_2 \in U$, then each $u_i = w_i + \xi(x_i), \exists!\, w_i \in W, x_i \in X$.
Linearity: $\forall, \lambda \in \mathbf{F}, \pi(u_1 + \lambda u_2) = w_1 + \lambda w_2 = \pi(u_1) + \lambda \pi(u_2)$.
Injectivity: $\pi(u_1) = \pi(u_2) \Rightarrow w_1 = w_2 \Rightarrow \xi(x_1) = \xi(x_2) \Rightarrow x_1 = x_2 \Rightarrow u_1 = u_2$.
Surjectivity: $\forall w \in W, \pi(w) = w \in \operatorname{range} \pi$. Thus $\pi$ is an iso of $U$ onto $W$. ☐

---

**3** *Suppose $V$ and $W$ are iso, $U$ is a subsp of $V$, and $S \in \mathcal{L}(U, W)$.*
*Prove that $\exists$ inv $T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U \iff S$ is inje.* [ *See also (3.A.11).* ]

**Solution:** (a) $\forall u \in U, u = T^{-1}Su \Rightarrow T^{-1}S = I \in \mathcal{L}(U) \Longrightarrow S$ is inje, by (3.B.20).
Or. $\operatorname{null} S = \operatorname{null} T|_U = \operatorname{null} T \cap U = \{0\}$.

(b) Let $X \oplus U = V$. Because $S : U \to V$ is inje. By (3.B.12), $S : U \to \operatorname{range} S$ is an iso.
Let $Y \oplus \operatorname{range} S = V$. Then by Tips, $X$ and $Y$ are iso. Let $E : X \to Y$ be an iso.
Define $T \in \mathcal{L}(V, W)$ by $Tu = Su, Tw = Ew$ for all $u \in U, w \in X$.

Or. [ *Req $V$ Finite-dim* ] Let $B_U = (u_1, \ldots, u_m)$. Then $S$ inje $\Rightarrow (Su_1, \ldots, Su_m)$ linely inde.
Extend to $B_V = (u_1, \ldots, u_m, v_1, \ldots, v_n), B_W = (Su_1, \ldots, Su_m, w_1, \ldots, w_n)$.
Define $T \in \mathcal{L}(V, W)$ by $T(u_i) = Su_i;\ \ Tv_j = w_j$, for each $u_i$ and $v_j$. □

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**8** *Suppose $T \in \mathcal{L}(V, W)$ is* ***surj****. Prove that $\exists$ subsp $U$ of $V$, $T|_U : U \to W$ is an iso.*

**Solution:** [ *Req* $\operatorname{range} T$ *Finite-dim* ] Let $B_{\operatorname{range} T} = B_W = (Tv_1, \ldots, Tv_m), B_U = (v_1, \ldots, v_m)$. □
Or. By (3.B.12). Note that $\operatorname{range} T = W$. □

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**18** *Show that $V$ and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.*

**Solution:**
Define $\Psi \in \mathcal{L}(V, \mathcal{L}(\mathbf{F}, V))$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.
(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence $\Psi$ is inje.
(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi(T(1))$. Hence $\Psi$ is surj. □
Or. Define $\Phi \in \mathcal{L}(\mathcal{L}(\mathbf{F}, V), V)$ by $\Phi(T) = T(1)$.
(a) Suppose $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus $\Phi$ is inje.
(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus $\Phi$ is surj. □
**Comment:** $\Phi = \Psi^{-1}$.

---

• *Suppose $S, T \in \mathcal{L}(V, W)$.* [ *For Problem (4) and (5), see the* Corollary *in (3.B.24, 25).* ]
**6** *Suppose $V$ and $W$ are finite-dim.* $\dim \operatorname{null} S = \dim \operatorname{null} T = n$.
*Prove that $S = E_2 T E_1, \exists$ inv $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$.*

**Solution:** Define $E_1 :\ \ v_i \mapsto r_i\ ;\ \ u_j \mapsto s_j;$ for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$.
Define $E_2 : Tv_i \mapsto Sr_i\ ;\ \ x_j \mapsto y_j;$ for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. Where:

| | |
|---|---|
| Let $B_{\operatorname{range} T} = (Tv_1, \ldots, Tv_m)$; $B_{\operatorname{range} S} = (Sr_1, \ldots, Sr_m)$.<br>Let $B_W = (Tv_1, \ldots, Tv_m, x_1, \ldots, x_p)$; $B'_W = (Sr_1, \ldots, Sr_m, y_1, \ldots, y_p)$.<br>Let $B_{\operatorname{null} T} = (u_1, \ldots, u_n)$; $B_{\operatorname{null} S} = (s_1, \ldots, s_n)$.<br>Thus $B_V = (v_1, \ldots, v_m, u_1, \ldots, u_n)$; $B'_V = (r_1, \ldots, r_m, s_1, \ldots, s_n)$. | $\therefore E_1, E_2$ are inv and $S = E_2 T E_1$. |

□

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• (a) *Suppose $T = ES$ and $E \in \mathcal{L}(W)$ is inv. Prove that* $\operatorname{null} S = \operatorname{null} T$.
(b) *Suppose $T = SE$ and $E \in \mathcal{L}(V)$ is inv. Prove that* $\operatorname{range} S = \operatorname{range} T$.
(c) *Suppose $T = E_2 S E_1$ and $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$ are inv.*
*Prove that* $\dim \operatorname{null} S = \dim \operatorname{null} T$.

**Solution:** (a) $v \in \operatorname{null} T \iff Tv = 0 = E(Sv) \iff Sv = 0 \iff v \in \operatorname{null} S$.
(b) $w \in \operatorname{range} T \iff \exists v \in V, Tv = S(Ev) \iff \exists u \in V, w = Su \iff w \in \operatorname{range} S$.
(c) Using (3.B.22). $\dim \operatorname{null} E_2 S E_1 \overset{E_2}{\underset{\text{inv}}{=\!=}} \dim \operatorname{null} S E_1 \overset{E_1}{\underset{\text{inv}}{=\!=}} \dim \operatorname{null} S = \dim \operatorname{null} T$. □

---

• **NOTE FOR [3.69]:** Suppose $V, W$ are finite-dim and iso, $T \in \mathcal{L}(V, W)$. Then $T$ inv $\iff$ inje $\iff$ surj.

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**9** [OR 1] *Suppose $U, V, W$ are iso and finite-dim, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Prove that $ST$ is inv $\iff S, T$ are inv.*

**COMMENT:** If any two of $U, V, W$ are not iso or finite-dim, then $S, T$ are inv $\implies ST$ is inv.

**SOLUTION:** Suppose $S, T$ are inv. Then $(ST)(T^{-1}S^{-1}) = I_W, (T^{-1}S^{-1})(ST) = I_U$. Hence $ST$ is inv.
Suppose $ST$ is inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = I_U, (ST)R = I_W$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0. \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S. \end{array}\right| \begin{array}{l} T \text{ is inje, } S \text{ is surj.} \\ \text{又 } \dim U = \dim V = \dim W. \end{array}$$

OR. By (3.B.23), $\dim W = \dim \operatorname{range} ST \leqslant \min\{\operatorname{range} S, \operatorname{range} T\} \Rightarrow S, T$ are surj. ☐

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**13** *Suppose $U, V, W, X$ are iso and finite-dim, $R \in \mathcal{L}(W, X), S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Suppose $RST$ is surj. Prove that $S$ is inje.*

**SOLUTION:** Using Problem (9). Notice that $U, X$ are finite-dim, so that $RST$ is inv.

$$\text{Let } X = (RST)^{-1} \left| \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ is surj.} \end{array} \right\} \Rightarrow S = R^{-1}(RST)T^{-1}.$$ ☐

OR. $(RST)^{-1} = \big((RS)T\big)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. ☐

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**10** *Suppose $V$ is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that $ST = I \iff TS = I$.*

**SOLUTION:** (a) Suppose $ST = I$.

By $(3.\text{B } 20, 21)(\text{a})$, $ST = I \Rightarrow T$ is inje and $S$ is surj. 又 $V$ is finite-dim. $S, T$ are inv.
OR. By Problem (9), $V$ is finite-dim and $ST = I$ is inv $\Rightarrow S, T$ are inv.

Then $\forall v \in V, S\big((TS)v\big) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow TS = I$.
OR. $S^{-1} = T$ 又 $S = S \Rightarrow TS = S^{-1}S = I$.

(b) Reversing the roles of $S$ and $T$, we conclude that $TS = I \Rightarrow ST = I$. ☐

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**11** *Suppose $V$ is finite-dim, $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that $T$ is inv and $T^{-1} = US$.*

**SOLUTION:** Using Problem (9) and (10). *This result can fail without the hypothesis that $V$ is finite-dim.*
$(ST)U = U(ST) = (US)T = I \Rightarrow T^{-1} = US$.
OR. $(ST)U = S(TU) = I \Rightarrow U, S$ are inv $\Rightarrow TU = S^{-1}$. 又 $U^{-1} = U^{-1} \Rightarrow T = S^{-1}U^{-1}$. ☐

**EXAMPLE:** $V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots);\ T(a_1, \dots) = (0, a_1, \dots);\ U = I \Rightarrow STU = I$ but $T$ is not inv.

---

• (4E 3) $T \in \mathcal{L}(V)$, $V$ is finite-dim
$$\left| \begin{array}{l} (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1, \dots, v_n) \text{ of } V \iff T \text{ is surj} \\ (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1, \dots, v_n) \text{ of } V \iff T \text{ is inje} \end{array} \right\} \iff T \text{ is inv.}$$

---

• (4E 15) *Suppose $T \in \mathcal{L}(V)$ and $V = \operatorname{span}(Tv_1, \dots, Tv_m)$. Prove that $V = \operatorname{span}(v_1, \dots, v_m)$.*

**SOLUTION:** Because $V = \operatorname{span}(Tv_1, \dots, Tv_m) \Rightarrow T$ is surj, and therefore is inv $\Rightarrow T^{-1}$ is inv.
$\forall v \in V, \exists a_i \in \mathbf{F}, v = \sum_{i=1}^m a_i Tv_i \Rightarrow T^{-1}v = \sum_{i=1}^m a_i v_i \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \dots, v_m)$.

OR. Reduce the spanning list $(Tv_1, \dots, Tv_m)$ of $V$ to a basis $(Tv_{\alpha_1}, \dots, Tv_{\alpha_k})$ of $V$.
Where $k = \dim V$ and each $\alpha_i \in \{1, \dots, k\}$. Then by Problem (4E 3),
$(v_{\alpha_1}, \dots, v_{\alpha_k})$ is also a basis of $V$, contained in the list $(v_1, \dots, v_m)$. ☐

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**15** *Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multi.*
*In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists\, A \in \mathbf{F}^{m,n}, Tx = Ax,\ \forall x \in \mathbf{F}^{n,1}$.*

**SOLUTION:** Let $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$ be the std bases of $\mathbf{F}^{n,1}, \mathbf{F}^{m,1}$.
$\forall k = 1, \dots, n,\ T(E_k) = A_{1,k}R_1 + \cdots + A_{m,k}R_m, \exists\, A_{j,k} \in \mathbf{F}$, forming $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.
OR. Let $A = \mathcal{M}(T, B_1, B_2)$. Note that $\mathcal{M}(x, B_1) = x, \mathcal{M}(Tx, B_2) = Tx$.
Hence $Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax$, by [3.65]. ☐

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- **NOTE FOR [3.62]:** $\mathcal{M}(v) = \mathcal{M}\big(I, (v), B_V\big)$. Where $I$ is the identity operator restricted to $\operatorname{span}(v)$.
- **NOTE FOR [3.65]:** $\mathcal{M}(Tv) = \mathcal{M}\big(I, (Tv), B_W\big) = \mathcal{M}(T, B_V, B_W)\mathcal{M}\big(I, (v), B_V\big) = \mathcal{M}\big(T, (v), B_W\big)$.
  If $v = 0$, then $\operatorname{span}(v) = \operatorname{span}(\ )$, we replace $(v)$ by $B = (\ )$ ; similar for $Tv = 0$.

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- (4E 23, OR 10.A.4) *Suppose that $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bases of $V$.*
  *Let $T \in \mathcal{L}(V)$ be such that each $T\alpha_k = \beta_k$. Prove that $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)$.*
  *For ease of notation, let $\mathcal{M}\big(T, \alpha \to \beta\big) = \mathcal{M}\big(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)\big)$.*

**SOLUTION:**

Denote $\mathcal{M}(T, \alpha \to \alpha)$ by $A$ and $\mathcal{M}(I, \beta \to \alpha)$ by $B$.
$\forall k \in \{1, \dots, n\}, I\beta_k = \beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. ☐

OR. Note that $\mathcal{M}(T, \alpha \to \beta) = I$. Hence $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)\underbrace{\mathcal{M}(T, \alpha \to \beta)}_{= \mathcal{M}(I, \beta \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. ☐

OR. Note that $\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I$.
$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}\Big(\underbrace{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}_{= \mathcal{M}(T, \alpha \to \beta)}\Big) = \mathcal{M}(I, \beta \to \alpha)$. ☐

**COMMENT:** Let $A' = \mathcal{M}(T, \beta \to \beta)$.
$\beta_k = I\beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n,\ \forall\, k \in \{1, \dots, n\}$.
又 $\quad T\beta_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.
OR. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$.

---

- **TIPS:** When using $\mathcal{M}^{-1}$, you must first declare bases and the purpose for using $\mathcal{M}^{-1}$.
  That is, to declare $B_U, B_V, B_W,\ \mathcal{M} : \mathcal{L}(V, W) \mapsto \mathbf{F}^{m,n}$, or $\mathcal{M} : v \mapsto \mathbf{F}^{n,1}$.
  So that $\mathcal{M}^{-1}(AC, B_U, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(C, B_U, B_V)$;
  Or $\mathcal{M}^{-1}(Ax, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(x, B_V)$. Where everything is well-defined.

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- (4E 22, OR 10.A.1) *Suppose $T \in \mathcal{L}(V)$. Prove that $\mathcal{M}(T, B_V)$ is inv $\Longleftrightarrow$ $T$ itself is inv.*

**SOLUTION:** Notice that $\mathcal{M} : T \mapsto \mathcal{M}(T, B_V)$ is an iso. And that $\mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(TS)$.
(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(I) = \mathcal{M}(T)\mathcal{M}(T^{-1}) \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.
(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I,\ \exists!\, S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$
$\Rightarrow \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = I = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$
$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. ☐

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- (4E 24, OR 10.A.2) *Suppose $A, B \in \mathbf{F}^{n,n}$. Prove that $AB = I \Longleftrightarrow BA = I$.* [*Using Problem (10, 15).*]

**SOLUTION:** Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Now $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.
$AB = I \Leftrightarrow A(Bx) = x \Longleftrightarrow T(Sx) = x \Leftrightarrow TS = I \Longleftrightarrow ST = I \Longleftrightarrow \mathcal{M}(S)\mathcal{M}(T) = BA = I$.
OR. Because $\mathcal{M} : \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1}) \to \mathbf{F}^{n,n}$ is an iso. $\mathcal{M}^{-1}(AB) = TS = ST = \mathcal{M}^{-1}(BA) = I$. ☐

---

• **NOTE FOR [3.60]:** Suppose $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.
Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{i,x} w_j$. **COROLLARY:** $E_{l,k}E_{i,j} = \delta_{j,l}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. And $\left(\mathcal{E}^{(j,i)}\right)_{l,k} = \begin{cases} 1, & \text{if } (i,j) = (l,k); \\ 0, & \text{otherwise.} \end{cases}$

NOTICE that $\mathcal{M} \colon \mathcal{L}(V, W) \to \mathbf{F}^{m,n}$ is an iso. And $E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)} & + & \cdots & + & A_{1,n}\mathcal{E}^{(1,n)} \\ + & & \cdots & & + \\ \vdots & & \ddots & & \vdots \\ + & & \cdots & & + \\ A_{m,1}\mathcal{E}^{(m,1)} & + & \cdots & + & A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff \begin{pmatrix} A_{1,1}E_{1,1} & + & \cdots & + & A_{1,n}E_{n,1} \\ + & & \cdots & & + \\ \vdots & & \ddots & & \vdots \\ + & & \cdots & & + \\ A_{m,1}E_{1,m} & + & \cdots & + & A_{m,n}E_{n,m} \end{pmatrix} = T.$$

$$\text{By [2.42] and [3.61], } B_{\mathcal{L}(V,W)} = \begin{pmatrix} E_{1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \cdots & , E_{n,m} \end{pmatrix}; \quad B_{\mathbf{F}^{m,n}} = \begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & , \mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots & , \mathcal{E}^{(m,n)} \end{pmatrix}.$$

• **TIPS:** Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_p)$, $B_V = (v_1, \dots, v_p, \dots, v_n)$. Let each $w_k = Tv_k$; $B_W = (w_1, \dots, w_p, \dots, w_m)$.
Then $T = E_{1,1} + \cdots + E_{p,p}$, $\mathcal{M}(T, B_V, B_W) = \mathcal{E}^{(1,1)} + \cdots + \mathcal{E}^{(p,p)}$.

---

**17** *Suppose $V$ is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*
*A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}$, $\forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.*

**SOLUTION:** [ See also in (3.A). ] Using NOTE FOR [3.60].
Let $B_V = (v_1, \dots, v_n)$. If $\mathcal{E} = 0$, then we are done. Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.
Then $\forall E_{i,j} \in \mathcal{E}$, by assumption, $\forall x, y \in \{1, \dots, n\}$, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}$, $E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.
Again, $\forall x, x', y, y' \in \{1, \dots, n\}$, $E_{y,x'}, E_{y',x} \in \mathcal{E}$. Thus $\mathcal{E} = \mathcal{L}(V)$. ☐

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• (4E 10) *Suppose $V, W$ are finite-dim, $U$ is a subsp of $V$.*
*Let $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V, W) : T|_U = 0\}$.*
(a) *Show that $\mathcal{E}$ is a subsp of $\mathcal{L}(V, W)$.*
(b) *Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$ and $\dim U$.*
***Hint***: *Define $\Phi : \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. What is $\operatorname{null} \Phi$? What is $\operatorname{range} \Phi$?*

**SOLUTION:**
(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.
(b) Define $\Phi$ as in the hint. $\Phi$ is linear, by [3.A NOTE FOR Restriction].
$\forall T \in \operatorname{null} \Phi, \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$. Thus $\operatorname{null} \Phi = \mathcal{E}$.
$\forall S \in \mathcal{L}(U, W)$, extend to $T \in \mathcal{L}(V, W)$, then $\Phi(T) = S \in \operatorname{range} \Phi$. Thus $\operatorname{range} \Phi = \mathcal{L}(U, W)$.
Thus $\dim \operatorname{null} \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \operatorname{range} \Phi = (\dim V - \dim U) \dim W$. ☐

OR. Let $B_U = (u_1, \dots, u_m)$, $B_V = (u_1, \dots, u_m, v_1, \dots, v_n)$. Let $p = \dim W$. [ See NOTE FOR [3.60]. ]

$$\forall\, T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \underbrace{\operatorname{span} \begin{Bmatrix} E_{1,1}, & \cdots & , E_{m,1}, \\ \vdots & \ddots & \vdots \\ E_{1,p}, & \cdots & , E_{m,p} \end{Bmatrix}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

$$又\ W = \operatorname{span} \begin{Bmatrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{Bmatrix} \subseteq \mathcal{E}.$$

Where $\mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}$.
Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. ☐

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• (4E 17) *Suppose $V$ is finite-dim and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(T) = ST$.*
(a) *Show that* $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Show that* $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

**Solution:** (a) $\forall T \in \mathcal{L}(V), ST = 0 \iff \operatorname{range} T \subseteq \operatorname{null} S$.
Thus $\operatorname{null} \mathcal{A} = \{T \in \mathcal{L}(V) : \operatorname{range} T \subseteq \operatorname{null} S\} = \mathcal{L}(V, \operatorname{null} S)$.
(b) $\forall R \in \mathcal{L}(V), \operatorname{range} R \subseteq \operatorname{range} S \iff \exists\, T \in \mathcal{L}(V), R = ST$, by (3.B 25).
Thus $\operatorname{range} \mathcal{A} = \{R \in \mathcal{L}(V) : \operatorname{range} R \subseteq \operatorname{range} S\} = \mathcal{L}(V, \operatorname{range} S)$. ☐

Or. Using Note For [3.60]. Let $B_{\operatorname{range} S} = (\overbrace{w_1, \dots, w_m}^{Sv_i = w_i}), B_U = (v_1, \dots, v_m)$.
Let $(w_1, \dots, w_n), (v_1, \dots, v_n)$ be bases of $V$. Now $S = E_{1,1} + \dots + E_{m,m}$. $\mathcal{M}(S, v \to w) = \begin{pmatrix} 1 & \ddots & 0 & \ddots & 0 \\ 0 & \ddots & 1 & \ddots & 0 \\ 0 & \ddots & 0 & \ddots & 0 \end{pmatrix}$.
Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j} : w_x \mapsto \delta_{i,x} v_i$. Let $E_{j,k} R_{i,j} = Q_{i,k},\ R_{j,k} E_{i,j} = G_{i,k}$.
Where $E_{i,k} : v_x \mapsto \delta_{i,x} w_k,\ Q_{i,k} : w_x \mapsto \delta_{i,x} w_k$, and $G_{i,k} : v_x \mapsto \delta_{i,x} v_k$.
For any $T \in \mathcal{L}(V), \exists!\, A_{i,j} \in \mathbf{F}, T = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i} \Longrightarrow \mathcal{M}(T, w \to v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} & \cdots & A_{n,m} \end{pmatrix}$.
$\Longrightarrow \mathcal{A}(T) = ST = \Big(\sum_{r=1}^{m} E_{r,r}\Big)\Big(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i}\Big) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j} Q_{j,i}$.

$$\begin{aligned} \mathcal{M}(S, v \to w)\mathcal{M}(T, w \to v) = \mathcal{M}(ST, w) &= \\ \mathcal{M}(\mathcal{A}, R \to Q)\mathcal{M}(T, R) = \mathcal{M}\big(\mathcal{A}(T), Q\big) &= \end{aligned} \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

又 $\mathcal{M}(T, R) = \mathcal{M}(T, w \to v)$.
Let $T = I$, we have
$\mathcal{M}(\mathcal{A}, R \to Q) = \mathcal{M}(S, v \to w)$.

$\operatorname{range} \mathcal{A} = \operatorname{span} \left\{ \begin{matrix} Q_{1,1}, & \cdots & , Q_{n,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,m}, & \cdots & , Q_{n,m} \end{matrix} \right\}$, $\operatorname{null} \mathcal{A} = \operatorname{span} \left\{ \begin{matrix} R_{1,m+1}, & \cdots & , R_{n,m+1}, \\ \vdots & \ddots & \vdots \\ R_{1,n}, & \cdots & , R_{n,n} \end{matrix} \right\}$.
(a) $\dim \operatorname{null} \mathcal{A} = n \times (n - m)$;
(b) $\dim \operatorname{range} \mathcal{A} = n \times m$. ☐

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• **Note For Problem (4E 17):** *Define $\mathcal{B} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{B}(T) = TS$.*
(a) *Show that* $\dim \operatorname{null} \mathcal{B} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Show that* $\dim \operatorname{range} \mathcal{B} = (\dim V)(\dim \operatorname{range} S)$.

**Solution:** (a) $\forall T \in \mathcal{L}(V), TS = 0 \iff \operatorname{range} S \subseteq \operatorname{null} T$.
Thus $\operatorname{null} \mathcal{B} = \{T \in \mathcal{L}(V) : \operatorname{range} S \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V) : T|_{\operatorname{range} S} = 0\}$.
(b) $\forall R \in \mathcal{L}(V), \operatorname{null} S \subseteq \operatorname{null} R \iff \exists\, T \in \mathcal{L}(V), R = TS$, by (3.B.24).
Thus $\operatorname{range} \mathcal{B} = \{R \in \mathcal{L}(V) : \operatorname{null} S \subseteq \operatorname{null} R\} = \{R \in \mathcal{L}(V) : R|_{\operatorname{null} S} = 0\}$.
Using [3.22] and Problem (4E 10). ☐

Or. Using Note For [3.60] and notation in Problem (4E 17).
$\mathcal{B}(T) = TS = \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i}\Big)\Big(\sum_{r=1}^{m} E_{r,r}\Big) = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{i,j} G_{j,i} \Longrightarrow \mathcal{M}(TS, v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} & \cdots & 0 \end{pmatrix}$.
$\operatorname{range} \mathcal{B} = \operatorname{span} \left\{ \begin{matrix} G_{1,1}, & \cdots & , G_{m,1}, \\ \vdots & \ddots & \vdots \\ G_{1,n}, & \cdots & , G_{m,n} \end{matrix} \right\}$, $\operatorname{null} \mathcal{B} = \operatorname{span} \left\{ \begin{matrix} R_{m+1,1}, & \cdots & , R_{n,1}, \\ \vdots & \ddots & \vdots \\ R_{m+1,n}, & \cdots & , R_{n,n} \end{matrix} \right\}$.
(a) $\dim \operatorname{null} \mathcal{B} = n \times (n - m)$;
(b) $\dim \operatorname{range} \mathcal{B} = n \times m$. ☐

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• (4E 20) *Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.*

**Solution:** Note that $\deg\big[(x^2 + x)p''(x) + 2xp'(x) + p(3)\big] = \deg p$.
Define $T_n \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.
And note that $T_n(p) = 0 \Rightarrow \deg T_n(p) = -\infty = \deg p \Rightarrow p = 0$. Thus $T_n$ is inv.
$\forall q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $n = 0$; if $q \neq 0$, let $n = \deg q$, we have $q \in \mathcal{P}_n(\mathbf{R})$.
Now $\exists\, p \in \mathcal{P}_n(\mathbf{R}), q(x) = T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$. ☐

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**19** *Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is inje. And $\deg Tp \leqslant \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.*
(a) *Prove that $T$ is surj;* (b) *Prove that for every nonzero $p$, $\deg Tp = \deg p$.*

**Solution:** (a) $T$ is inje $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ is inje, so is inv $\iff T$ is surj.
(b) Using mathematical induction.
(i) $\deg p = -\infty \geqslant \deg Tp \iff p = 0 = Tp$. And $\deg p = 0 \geqslant \deg Tp \iff p = C \neq 0$.
(ii) Assume $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$. We show $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$ by contradiction.
Suppose $\exists\, r \in \mathcal{P}_{n+1}(\mathbf{R}),\ \deg Tr \leqslant n < n+1 = \deg r$. Then by (a), $\exists\, s \in \mathcal{P}_n(\mathbf{R}),\ T(s) = (Tr)$.
又 $T$ is inje $\Rightarrow s = r$. While $\deg s = \deg Ts = \deg Tr < \deg r$. Contradicts. ☐

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**16** *Suppose $V$ is finite-dim and $S \in \mathcal{L}(V)$ such that $\forall T \in \mathcal{L}(V), ST = TS$.*
*Prove that $\exists\, \lambda \in \mathbf{F}, S = \lambda I$.* [*Using notation in Problem (4E 17). See also in (3.A).*]

**Solution:** If $S = 0$, we are done. Now suppose $S \neq 0$.
Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, B_U) = \mathcal{M}(I, B_{\operatorname{range} S}, B_U)$. Note that $R_{k,1} : w_x \mapsto \delta_{k,x} v_1$.
Then $\forall k \in \{1, \ldots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $\dim \operatorname{null} S = 0,\ \dim \operatorname{range} S = m = n$.
Notice that $G_{i,j} = R_{i,j}S = SR_{i,j} = Q_{i,j}$. Where $G_{i,j} : v_x \mapsto \delta_{i,x} v_j;\ Q_{i,j} : w_x \mapsto \delta_{i,x} w_j$.
For each $w_i, \exists\,!\, a_{k,i} \in \mathbf{F}, w_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n$. Where $a_{k,i} = \mathcal{M}(I, (w_1, \ldots, w_n), (v_1, \ldots, v_n))_{k,i}$.
Then fix one $i$. Now for each $j \in \{1, \ldots, n\}, Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}\left(\sum_{k=1}^{n} a_{k,i}v_k\right)$.
Let $\lambda = a_{i,i}$. Hence each $w_j = \lambda v_j$. Now fix one $j$, we have $a_{1,1}v_j = \cdots = a_{n,n}v_j$, then all $a_{i,i}$ are equal.
Thus each $w_j = \lambda v_j \Longrightarrow \mathcal{M}(S, B_U) = \mathcal{M}(\lambda I)$. ☐

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• (10.A.3, Or 4E 19) *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.* [*See also in (3.A).*]
*Prove that $\forall B_V \neq B_V',\ \mathcal{M}(T, B_V) = \mathcal{M}(T, B_V') \Longrightarrow T = \lambda I, \exists\, \lambda \in \mathbf{F}$.*

**Solution:** Suppose $\forall B_V \neq B_V',\ \mathcal{M}(T, B_V) = \mathcal{M}(T, B_V')$. If $T = 0$, then we are done.
Suppose $T \neq 0$, and $v \in V \setminus \{0\}$. Assume that $(v, Tv)$ is linely inde.
Extend $(v, Tv)$ to $B_V = (v, Tv, u_3, \ldots, u_n)$. Let $B = \mathcal{M}(T, B_V)$.
$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0,\ \forall i \neq 2$.
By assumption, $A = \mathcal{M}(T, B_V') = B, \forall B_V' = (v, w_2, \ldots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$.
$\Rightarrow Tv = w_2$, which is not true if $w_2 = u_3,\ w_3 = Tv,\ w_j = u_j, \forall j \in \{4, \ldots, n\}$. Contradicts.
Hence $(v, Tv)$ is linely depe $\Rightarrow \forall v \in V, \exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.
Now we show that $\lambda_v$ is independent of $v$, that is, for all distinct $v, w \in V \setminus \{0\}, \lambda_v = \lambda_w$.

$$\left.\begin{array}{l} (v,w) \text{ linely inde} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v,w) \text{ linely depe, } w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \end{array}\right\} \Rightarrow T = \lambda I.$$ ☐

Or. Let $A = \mathcal{M}(T, B_V)$, where $B_V = (u_1, \ldots, u_m)$ is arbitrary.
Fix one $B_V = (v_1, \ldots, v_m)$ and then $(v_1, \ldots, \frac{1}{2}v_k, \ldots, v_m)$ is also a basis for any given $k \in \{1, \ldots, m\}$.
Fix one $k$. Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \cdots + A_{k,k}(\frac{1}{2}v_k) + \cdots + A_{m,k}v_m$
$\Rightarrow Tv_k = 2A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + 2A_{m,k}v_m = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{m,k}v_m$.
Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1, \ldots, m\}$.
Now we show that $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose $j, k$ such that $j \neq k$.
Consider $B_V' = (v_1', \ldots, v_j', \ldots, v_k', \ldots, v_m')$, where $v_j' = v_k,\ v_k' = v_j$ and $v_i' = v_i$ for all $i \in \{1, \ldots, m\} \setminus \{j, k\}$.
Now $T(v_k') = A_{1,k}v_1' + \cdots + A_{k,k}v_k' + \cdots + A_{m,k}v_m' = A_{k,k}v_k' = A_{k,k}v_j$, while $T(v_k') = T(v_j) = A_{j,j}v_j$. ☐

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**Ended**

# 3·E   1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 20 | 4E: 8 14

**1** *A function $T : V \to W$ is linear $\iff$ The graph of $T$ is a subspace of $V \times W$.*

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**2** *Suppose $V_1 \times \cdots \times V_m$ is finite-dim. Prove that each $V_j$ is finite-dim.*

**Solution:**

For any $k \in \{1, \dots, m\}$, define $S_k \in \mathcal{L}(V_1 \times \cdots \times V_m, V_k)$ by $S_k(v_1, \dots, v_m) = v_k$.
Then $S_k$ is linear map. By [3.22], range $S_k = V_k$ is finite-dim. $\square$

Or. Denote $V_1 \times \cdots \times V_m$ by $U$. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by $U_i$.
We show that each $U_i$ is iso to $V_i$. Then $U$ is finite-dim $\Longrightarrow$ its subsp $U_i$ is finite-dim, so is $V_i$.

$$\text{Let } B_U = (v_1, \dots, v_M) \left| \begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U_i) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} R_i S_j|_{U_j} = \delta_{i,j} I_{U_j}, \\ S_i R_j = \delta_{i,j} I_{V_j}. \end{array} \right. \quad \square$$

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**4** *Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.*

**Solution:** Using notation in Problem (2): $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0)$; $S_i : (u_1, \dots, u_m) \mapsto u_i$.

Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$.

$$\left. \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (TR_1, \dots, TR_m). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array} \right\} \Rightarrow \psi = \varphi^{-1}. \quad \square$$

---

**5** *Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.*

**Solution:** Using notation in Problem (2): $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0)$; $S_i : (u_1, \dots, u_m) \mapsto u_i$.

$$\begin{array}{l} \text{Note that } T_i : v \mapsto w_i, \\ T : v \mapsto (w_1, \dots, w_m). \end{array} \left| \left. \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1 T, \dots, S_m T). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = R_1 T_1 + \cdots + R_m T_m. \end{array} \right\} \Rightarrow \psi = \varphi^{-1}. \right. \quad \square$$

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**6** *For $m \in \mathbf{N}^+$, define $V^m$ by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove that $V^m$ and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.*

**Solution:**

Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m$.

(a) Suppose $T(v_1, \dots, v_m) = 0$. Then $\forall (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$
For each $k$, let $a_k = 1, a_j = 0$ for all $j \neq k$. Then each $v_k = 0 \Rightarrow (v_1, \dots, v_m) = 0$. Thus $T$ is inje.

(b) Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be the std basis of $\mathbf{F}^m$. Then $\forall (b_1, \dots, b_n) \in \mathbf{F}^m$,
$$\Big[T\Big(\psi(e_1), \dots, \psi(e_m)\Big)\Big](b_1, \dots, b_m) = b_1 \psi(e_1) + \cdots + b_m \psi(e_m) = \psi\big(b_1 e_1 + \cdots + b_m e_m\big) = \psi(b_1, \dots, b_m).$$
Thus $T\big(\psi(e_1), \dots, \psi(e_m)\big) = \psi$. Hence $T$ is surj. $\square$

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**3** *Give an example of a vecsp $V$ and its two subsps $U_1, U_2$ such that $U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum.* [*$V$ must be infinite-dim.*]

**Solution:** Note *that at least one of $U_1, U_2$ must be infinite-dim. And at least one must be finite-dim??*

Let $V = \mathbf{F}^\infty = U_1$, $U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$. Then $V = U_1 + U_2$ is not a direct sum.

$$\left. \begin{array}{l} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T\big((x_1, x_2, \cdots), (x, 0, \cdots)\big) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = \big((x_1, x_2, \cdots), (x, 0, \cdots)\big) \end{array} \right\} \Rightarrow S = T^{-1}. \quad \square$$

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• **Note For [3.79, 3.83]:** If $U = \{0\}$, then $v + U = v + \{0\} = \{v\}$, $V/U = V/\{0\} = \{\{v\} : v \in V\}$.
If $U = \emptyset$, then $v + U = v + \emptyset = \emptyset$, $V/U = V/\emptyset = \{\emptyset\}$.

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• **Comment:** If $U$ is merely a subset of $V$, then [3.85, 3.86] do not hold, and $V/U$ is not a vecsp.
Because $\big((v-w)+u\big) \in U$ or $u - u' \in U$ needs that $U$ is closed under add.
And because $(v - \hat{v}) + (w - \hat{w}) \in U$ and $\lambda(v - \hat{v}) \in U$ assume that $U$ is a subsp.

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• **Note For [3.85]:** $v + U = w + U \iff v \in w + U,\ w \in v + U$
$\iff v - w \in U \iff (v+U) \cap (w+U) \neq \emptyset$.

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• (4E 8) *Suppose $T \in \mathcal{L}(V, W), w \in \operatorname{range} T$. Prove that $\{u \in V : Tu = w\} = u + \operatorname{null} T$.*

**Solution:** Let $\mathcal{K}_u = \{u \in V : Tu = w\}$. [ Not a vecsp. ] Suppose $u \in \mathcal{K}_u$. Then $u + \operatorname{null} T \subseteq \mathcal{K}_u$.
And $\forall u' \in \mathcal{K}_u,\ u' - u \in \operatorname{null} T \Rightarrow u' \in u + \operatorname{null} T$. Now $\mathcal{K}_u \subseteq u + \operatorname{null} T$. □

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**7** *Suppose $v, x \in V$, and $U, W$ are subsps of $V$. Prove that $v + U = x + W \Rightarrow U = W$.*

**Solution:** (a) $\forall u_1 \in U, \exists w_1 \in W, v + u_1 = x + w_1$. Let $u_1 = 0$, then $v = x + w_1' \Rightarrow v - x \in W$.
(b) $\forall w_2 \in W, \exists u_2 \in U, v + u_2 = x + w_2$. Let $w_2 = 0$, then $x = v + u_2' \Rightarrow x - v \in U$.
Now $x + U = v + U = x + W = v + W$. Thus $\{v + u : u \in U\} = \{v + w : w \in W\} \Rightarrow U = W$.

Or. $\pm(v - x) \in U \cap W \Rightarrow \left\{ \begin{array}{l} u_1 = (x - v) + w_1 \in W \Rightarrow U \subseteq W \\ w_2 = (v - x) + u_2 \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W.$ □

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**8** *Suppose $A$ is a nonempty subset of $V$.*
*Prove that $A$ is a translate of some subsp of $V \iff \lambda v + (1-\lambda)w \in A$ , $\forall v, w \in A, \lambda \in \mathbf{F}$.*

**Solution:**

(a) Suppose $A = a + U$. Then $\lambda(a + u_1) + (1-\lambda)(a + u_2) = a + \big(\lambda(u_1 - u_2) + u_2\big) \in A$.

(b) Suppose $\lambda v + (1-\lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $\underline{a \in A}$ and let $A' = \{x - a : x \in A\}$.
Then $0 \in A'$ and $\forall (v - a), (w - a) \in A', \lambda \in \mathbf{F}$,

(I) $\lambda(v - a) = \big[\lambda v + (1-\lambda)a\big] - a \in A'$.

(II) Because $\lambda(v-a) + (1-\lambda)(w-a) = \big[\lambda v + (1-\lambda)w\big] - a \in A'$.
Let $\lambda = \frac{1}{2}$ here and use (I) above by $\lambda = 2$, we have $(v-a) + (w-a) \in A'$.

Or. Note that $v, a \in A \Rightarrow \lambda v + (1-\lambda)a = 2v - a \in A$. Similarly $2w - a \in A$.
Now $(v - \frac{1}{2}a) + (w - \frac{1}{2}a) = v + w - a \in A \Rightarrow v + w - 2a = (v-a) + (w-a) \in A'$.

Thus $A' = -a + A$ is a subsp of $V$. Hence $a + A' = a + \{x - a : x \in A\} = A$ is a translate. □

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**9** *Suppose $A = v + U$ and $B = x + W$ for some $v, x \in V$ and some subsps $U, W$ of $V$.*
*Prove that $A \cap B$ is either a translate of some subsp of $V$ or is $\emptyset$.*

**Solution:** $\forall v + u, x + w \in A \cap B \neq \emptyset, \lambda \in \mathbf{F}, \lambda(v+u) + (1-\lambda)(x+w) \in A \cap B$. By Problem (8). □
Or. Let $A = v + U,\ B = x + W$. Suppose $\alpha \in (v + U) \cap (x + W) \neq \emptyset$.
Then $\alpha - v \in U \Rightarrow \alpha + U = v + U = A$, and $\alpha - x \in W \Rightarrow \alpha + W = x + W = B$.
We show that $A \cap B = \alpha + (U \cap W)$. Note that $\alpha + (U \cap W) \subseteq A \cap B$.
And $\forall \beta = \alpha + u = \alpha + w \in A \cap B \Rightarrow u = w \in U \cap W \Rightarrow \beta \in \alpha + (U \cap W)$. □

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**10** *Prove that the intersection of any collection of translates of subsps is either a translate of some subsps or $\emptyset$.*

**SOLUTION:** Suppose $\{A_\alpha\}_{\alpha\in\Gamma}$ is a collection of translates of subsps of $V$, where $\Gamma$ is an index set.
$\forall x, y \in \bigcap_{\alpha\in\Gamma} A_\alpha \neq \emptyset, \lambda \in \mathbf{F}, \lambda x + (1-\lambda)y \in A_\alpha$ for each $\alpha$. By Problem (8). $\square$

OR. Let each $A_\alpha = w_\alpha + V_\alpha$. Suppose $x \in \bigcap_{\alpha\in\Gamma}(w_\alpha + V_\alpha) \neq \emptyset$.
Then $x - w_\alpha \in V_\alpha \Longrightarrow x + V_\alpha = w_\alpha + V_\alpha = A_\alpha$, for each $\alpha$.
We show that $\bigcap_{\alpha\in\Gamma} A_\alpha = \bigcap_{\alpha\in\Gamma}(x + V_\alpha) = x + \bigcap_{\alpha\in\Gamma} V_\alpha$.
$y \in \bigcap_{\alpha\in\Gamma} A_\alpha \iff$ for each $\alpha$, $y = x + v_\alpha \in A_\alpha$
$\iff$ each $v_\alpha = y - x \in \bigcap_{\alpha\in\Gamma} V_\alpha \iff y \in x + \bigcap_{\alpha\in\Gamma} V_\alpha$. $\square$

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**11** *Suppose $A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.*
(a) *Prove that $A$ is a translate of some subsp of $V$*
(b) *Prove that if $B$ is a translate of some subsp of $V$ and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.*
(c) *Prove that $A$ is a translate of some subsp of $V$ of dim less than $m$.*

**SOLUTION:**
(a) By Problem (8), $\forall u, w \in A, \lambda \in \mathbf{F}, \lambda u + (1-\lambda)w = \left(\lambda \sum_{i=1}^m a_i + (1-\lambda)\sum_{i=1}^m b_i\right)v_i \in A$. $\square$

(b) Suppose $B = v + U$, where $v \in V$ and $U$ is a subsp of $V$. Let each $v_k = v + u_k \in B, \exists!\, u_k \in U$.
$\forall w \in A,\ w = \sum_{i=1}^m \lambda_i v_i = \sum_{i=1}^m \lambda_i(v + u_i) = \underline{\sum_{i=1}^m \lambda_i v} + \sum_{i=1}^m \lambda_i u_i = v + \sum_{i=1}^m \lambda_i u_i \in v + U = B$. $\square$

OR. Let $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on $m$ by $k$.
(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.
$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By Problem (8), $v \in B$.
(ii) $2 \leqslant k < m$. Assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. $\left(\forall \lambda_i \text{ such that } \sum_{i=1}^k \lambda_i = 1\right)$

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. Fix one $\mu_\iota \neq 1$.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Longrightarrow \left(\sum_{i=1}^{k+1} \dfrac{\mu_i}{1-\mu_\iota}\right) - \dfrac{\mu_\iota}{1-\mu_\iota} = 1$.

Let $w = \underbrace{\dfrac{\mu_1}{1-\mu_\iota}v_1 + \cdots + \dfrac{\mu_{\iota-1}}{1-\mu_\iota}v_{\iota-1} + \dfrac{\mu_{\iota+1}}{1-\mu_\iota}v_{\iota+1} + \cdots + \dfrac{\mu_{k+1}}{1-\mu_\iota}v_{k+1}}_{k \text{ terms}}$.

Let $\lambda_i = \dfrac{\mu_i}{1-\mu_\iota}$ for $i \in \{1, \dots, \iota - 1\}$; $\lambda_j = \dfrac{\mu_{j+1}}{1-\mu_\iota}$ for $j \in \{\iota, \dots, k\}$. Then,

$$\left.\begin{array}{r} \sum_{i=1}^k \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda)v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B.$$ $\square$

(c) If $m = 1$, then let $A = v_1 + \{0\}$ and we are done. Now suppose $m \geqslant 2$. Fix one $k \in \{1, \dots, m\}$.
$A \ni \lambda_1 v_1 + \cdots + \lambda_{k-1}v_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m)v_k + \lambda_{k+1}v_{k+1} + \cdots + \lambda_m v_m$
$= v_k + \lambda_1(v_1 - v_k) + \cdots + \lambda_{k-1}(v_{k-1} - v_k) + \lambda_{k+1}(v_{k+1} - v_k) + \cdots + \lambda_m(v_m - v_k)$
$\in v_k + \operatorname{span}(v_1 - v_k, \dots, v_m - v_k)$. $\square$

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**14** *Suppose* $U = \{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0$ *for only finitely many* $k\}$.

(a) *Show that* $U$ *is a subsp of* $\mathbf{F}^\infty$. [*Do it in your mind*]

(b) *Prove that* $\mathbf{F}^\infty/U$ *is infinite-dim.*

**SOLUTION:** For ease of notation, denote the $p^{\text{th}}$ term of $u = (x_1, \cdots, x_p, \cdots) \in \mathbf{F}^\infty$ by $u[p]$.

For each $r \in \mathbf{N}^+$, let $e_r[k] = \begin{cases} 1, & (k-1) \equiv 0 \ (\operatorname{mod} r) \\ 0, & \text{otherwise} \end{cases}$ simply $e_r = (1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \cdots)$.

For $m \in \mathbf{N}^+$. Let $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 + U \Rightarrow \exists\, u \in U, a_1 e_1 + \cdots + a_m e_m = u$.

Suppose $u = (x_1, \cdots, x_L, 0, \cdots)$, where $L$ is the largest such that $u[L] \neq 0$.

Let $s \in \mathbf{N}^+$ be such that $h = s \cdot m! + 1 > L$, and $e_1[h] = \cdots = e_m[h] = 1$.

NOTICE that for any $p, r \in \{1, \ldots, m\}$, $e_r[s \cdot m! + 1 + p] = e_r[p+1] = 1 \iff p \equiv 0 \ (\operatorname{mod} r) \iff r \mid p$.

Let $1 = p_1 \leqslant \cdots \leqslant p_{\tau(p)} = p$ be the distinct factors of $p$. Moreover, $r \mid p \iff r = p_k$ for some $k$.

Now $u[h+p] = 0 = \left(\sum_{r=1}^{m} a_r e_r\right)[p+1] = \sum_{k=1}^{\tau(p)} a_{p_k}$.

Let $q = p_{\tau(p)-1}$. Then $\tau(q) = \tau(p) - 1$, and each $q_k = p_k$. Again, $\left(\sum_{r=1}^{m} a_r e_r\right)[h+q] = 0 = \sum_{k=1}^{\tau(p)-1} a_{p_k}$.

Thus $a_{p_{\tau(p)}} = a_p = 0$ for all $p \in \{1, \ldots, m\} \Rightarrow (e_1, \ldots, e_m)$ is linely inde in $\mathbf{F}^\infty$.

So is $(e_1 + U, \ldots, e_m + U)$ in $\mathbf{F}^\infty/U$. Because $m$ is arbitrary. By (2.A.14). □

OR. For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, \text{if } 2^r \mid p \\ 0, \text{otherwise} \end{cases}$.

Similarly, let $m \in \mathbf{N}^+$ and $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 \Rightarrow a_1 e_1 + \cdots + a_m e_m = u \in U$.

Suppose $L$ is the largest such that $u[L] \neq 0$. And $l$ is such that $2^{ml} > L$.

Then for each $k \in \{1, \ldots, m\}$, $u[2^{ml} + 2^k] = 0 = \left(\sum_{r=1}^{m} a_r e_r\right)[2^k] = a_1 + \cdots + a_k$.

Thus $a_1 = \cdots = a_m = 0$ and $(e_1, \ldots, e_m)$ is linely inde. Similarly. □

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- **NOTE FOR [3.88, 3.90, 3.91]:** Suppose $W \in \mathcal{S}_V U$. Then $V/U$ is iso to $W$.
  Because $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$. Define $T \in \mathcal{L}(V)$ by $T(v) = w_v$.
  Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$, $\operatorname{range} T \oplus \operatorname{null} T = V$.
  Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, V)$ is defined by $\tilde{T}(v + U) = \tilde{T}(w_v' + U) = T w_v' = w_v$. [See TIPS below]
  Now $\pi \circ \tilde{T} = I_{V/U}$, $\tilde{T} \circ \pi|_W = I_W = T|_W$. Hence $\tilde{T}$ is an iso of $V/U$ onto $W$.
- **TIPS:** Suppose $U$ is a subsp of $V$. Define $S \in \mathcal{L}(V/U, V)$ by $S(v + U) = v$.
  Then $\operatorname{range} S$ is the *purest* in $\mathcal{S}_V U$. Now $\operatorname{null} S = \{0\}$, $U \oplus \operatorname{range} S = V$.
  Let $E = S \circ \pi$. Because $S$ is inje and $\pi$ is surj, $\operatorname{null} E = \operatorname{null} \pi = U$, $\operatorname{range} E = \operatorname{range} S$.
  Then $\operatorname{range} E \oplus \operatorname{null} E = V$. NOTICE that $E : V \to W$ is the *purest eraser*. Now we explain why:

  **EXAMPLE:** Let $V = \mathbf{F}^2, B_U = (e_1), B_W = (e_2 - e_1) \Rightarrow U \oplus W = V$.
  Notice that $T(e_2 - e_1) = (e_2 - e_1)$, while $(e_2 - e_1) + U = e_2 + U$, but
  because $e_2 = e_1 + (e_2 - e_1)$, now still, $\tilde{T}((e_2 - e_1) + U) = e_2 - e_1 = T e_2$.
  In contrast, $S((e_2 - e_1) + U) = S(e_2 + U) = e_2$, $E(e_2 - e_1) = e_2$.
  And $\operatorname{range} E = \operatorname{range} S = \operatorname{span}(e_2)$ is the *purest* in $\mathcal{S}_V U$.

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**12** *Suppose $U$ is a subsp of $V$. Prove that is $V$ is iso to $U \times (V/U)$.*

**SOLUTION:**

[ *Req $V/U$ Finite-dim* ] Let $B_{V/U} = (v_1 + U, \dots, v_n + U)$.

Note that $\forall v \in V, \exists !\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^{n} a_i(v_i + U) = \left( \sum_{i=1}^{n} a_i v_i \right) + U$

$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) \in U \Rightarrow \exists !\, u \in U, v = \sum_{i=1}^{n} a_i v_i + u.$

Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ and $\psi \in \mathcal{L}(U \times (V/U), V)$

by $\varphi(v) = (u,\ v + U)$ and $\psi(u,\ v + U) = v + u$. Then $\psi = \varphi^{-1}$. ☐

OR. Define $S \in \mathcal{L}(V/U, V)$ by $S(v + U) = v$.

By NOTE FOR [3.88, 90, 91], range $S \oplus U = V$. Thus $\forall v \in V, \exists !\, u \in U, w \in \operatorname{range} S, v = u + w$.

Define $T \in \mathcal{L}(U \times (V/U), V)$ by $T(u, v + U) = u + S(v + U) = u + w = v$. Then $T$ is surj.

And $T(u, v + U) = u + S(v + U) = 0 \Longrightarrow \pi\big(T(u, v + U)\big) = v + U = 0$, and $u = -S(v + U) = 0$.

OR. Define $R \in \mathcal{L}(V, U \times (V/U))$ by $R(v) = \big(u, (w + U)\big)$. Now $R \circ T = I_{U \times (V/U)}$, $T \circ R = I_V$. ☐

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• (4E 14) *Suppose $V = U \oplus W$, $B_W = (w_1, \dots, w_m)$. Prove that $B_{V/U} = \big(w_1 + U, \dots, w_m + U\big)$.*

**SOLUTION:** $\forall v \in V, \exists !\, u \in U, w \in W, v = u + w$. 又 $\exists !\, c_i \in \mathbf{F}, w = \sum_{i=1}^{m} c_i w_i \Rightarrow v = \sum_{i=1}^{m} c_i w_i + u$.

Hence $\forall v + U \in V/U, \exists !\, c_i \in \mathbf{F}, v + U = \sum_{i=1}^{m} c_i w_i + U$. ☐

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**13** *Prove that $B_{V/U} = \big(v_1 + U, \dots, v_m + U\big), B_U = (u_1, \dots, u_n) \Rightarrow B_V = \big(v_1, \dots, v_m, u_1, \dots, u_n\big)$.*

**SOLUTION:**

Note that $\forall v \in V, \exists !\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^{m} a_i v_i + U \Rightarrow \exists !\, b_i \in \mathbf{F}, v - \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} b_i u_i \in U$

$\Rightarrow \forall v \in V, \exists !\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^{m} a_i v_i + \sum_{j=1}^{m} b_j u_j.$ ☐

OR. Note that $B = (v_1, \dots, v_m)$ is linely inde. Now we show that span $B \cap U = \{0\}$.

$v \in \operatorname{span} B \cap U \Longleftrightarrow v + U = \sum_{i=1}^{m} a_i(v_i + U) = 0 + U \Longleftrightarrow a_1 = \cdots = a_m = 0 \Longleftrightarrow v = 0.$

Then by Problem (12) and [3.76], $\dim V = \dim\big(U \times (V/U)\big) = n + m$.

While $\dim\big[\operatorname{span}(v_1, \dots, v_m) \oplus U\big] = m + n$ and $\big[\operatorname{span}(v_1, \dots, v_m) \oplus U\big] \subseteq V$. Hence by (2.B.8). ☐

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• **NOTE FOR Problem (13) and (4E 14):** Let $U \oplus W = V$. Define $S(w + U) = w$. [ See also the TIPS. ]

(a) Let $B_W = (w_1, \dots, w_m) \Rightarrow B_{V/U} = \big(w_1 + U, \dots, w_m + U\big)$. Then $S(w_k + U)$ might not equal $w_k$.

(b) Let $B_{V/U} = \big(w_1 + U, \dots, w_m + U\big)$, then let $B_W = (w_1, \dots, w_m)$. Now each $S(w_k + U) = w_k$.

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**15** *Suppose $\varphi \in \mathcal{L}(V, \mathbf{F}) \backslash \{0\}$. Prove that* $\dim V/(\operatorname{null} \varphi) = 1$.

**SOLUTION:** By [3.91] (d), $\dim \operatorname{range} \varphi = 1 = \dim V/(\operatorname{null} \varphi)$.

OR. By (3.B.29), $\exists\, u$, $\operatorname{span}(u) \oplus \operatorname{null} \varphi = V$. Then $B_{V/\operatorname{null} \varphi} = (u + \operatorname{null} \varphi)$. ☐

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**16** *Suppose* $\dim V/U = 1$. *Prove that $\exists\, \varphi \in \mathcal{L}(V, \mathbf{F})$,* $\operatorname{null} \varphi = U$.

**SOLUTION:** Suppose $V_0 \oplus U = V$. Then $V_0$ is iso to $V/U$. $\dim V_0 = 1$.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. ☐

OR. Let $B_{V/U} = (w + U)$. Then $\forall v \in V, \exists !\, a \in \mathbf{F}, v + U = aw + U$.

Define $\varphi : V \to \mathbf{F}$ by $\varphi(v) = a$. Then $\varphi(v_1 + \lambda v_2) = a_1 + \lambda a_2 = \varphi(v_1) + \lambda \varphi(v_2)$.

Now $u \in U \Longleftrightarrow u + U = 0w + U \Longleftrightarrow \varphi(u) = 0$. ☐

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**17** *Suppose $V/U$ is finite-dim, $W$ is a subsp of $V$.*

(a) *Show that if $V = U + W$, then $\dim W \geqslant \dim V/U$.*

(b) *Show that $\exists\, W \in \mathcal{S}_V U, \dim W = \dim V/U$.*

**Solution:** Let $B_W = (w_1, \dots, w_n)$.

(a) $\forall v \in V, \exists\, u \in U, w \in W,\ v = u + w \Longrightarrow v + U = w + U = (a_1 w_1 + \cdots + a_n w_n) + U, \exists\,!\, a_i \in \mathbf{F}$.
Then $V/U \subseteq \operatorname{span}(w_1 + U, \dots, w_n + U)$. Hence $\dim V/U \leqslant \dim \operatorname{span}(w_1 + U, \dots, w_n + U)$.

(b) Reduce $(w_1 + U, \dots, w_n + U)$ to $B_{V/U} = (w_1 + U, \dots, w_m + U)$, and let $W = \operatorname{span}(w_1, \dots, w_m)$. □

Or. Let $B_{V/U} = (v_1 + U, \dots, v_m + U)$ and define $\tilde{T} \in \mathcal{L}(V/U, V)$ by $\tilde{T}(v_k + U) = v_k$.
Note that $\pi \circ \tilde{T} = I$. By (3.B.20), $\tilde{T}$ is inje. And $(v_1, \dots, v_m)$ is linely inde.
Let $W = \operatorname{range} \tilde{T} = \operatorname{span}(v_1, \dots, v_m)$. Then $\tilde{T} \in \mathcal{L}(V/U, W)$ is an iso. Thus $\dim W = \dim V/U$.
And $\forall v \in V, \exists\,!\, a_i \in \mathbf{F}, v + U = a_1 v_1 + \cdots + a_m v_m + U \Rightarrow \exists\,!\, w \in W, u \in U, v = w + u$. □

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**18** *Suppose $T \in \mathcal{L}(V, W)$ and $U$ is a subsp of $V$. Let $\pi : V \to V/U$ be the quotient map. Prove that $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi \Longleftrightarrow U \subseteq \operatorname{null} T$.*

**Solution:**

(a) Suppose $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi$. Then $U = \operatorname{null} \pi \subseteq \operatorname{null}(S \circ \pi) = \operatorname{null} T$.

(b) Suppose $U = \operatorname{null} \pi \subseteq \operatorname{null} T$. By (3.B.24), we are done. Or. Define $S : (v + U) \mapsto Tv$.
$v_1 + U = v_2 + U \Longleftrightarrow v_1 - v_2 \in \operatorname{null} T \Longleftrightarrow Tv_1 = Tv_2$. Thus $S$ is well-defined. Hence $S \circ \pi = T$. □

**Corollary:** Define $\Gamma : S \mapsto S \circ \pi$. Then $\Gamma$ is inje, $\operatorname{range} \Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}$.

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**Ended**

# 3·F

4 5 6 7 8 9 12 13 15 16 17 18 19 20 21 22 23 24 25 26
28 29 30 31 32 33 34 35 36 37 | 4E: 5 6 8 17 23 24 25

**20, 21** *Suppose $U$ and $W$ are subsets of $V$. Prove that $U \subseteq W \Longleftrightarrow W^0 \subseteq U^0$.*

**Solution:**

(a) Suppose $U \subseteq W$. Then $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(w) = 0 \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.

(b) Suppose $W^0 \subseteq U^0$. Then $\varphi \in W^0 \Rightarrow \varphi \in U^0$. Hence $\operatorname{null} \varphi \supseteq W \Rightarrow \operatorname{null} \varphi \supseteq U$. Thus $W \supseteq U$.

Or. For a subsp $U$ of $V$, let $A_U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = U$, by Problem (25).
Suppose $W^0 \subseteq U^0$. Then $\forall \varphi \in W^0, v \in A_U, \varphi(v) = 0 \Rightarrow v \in A_W$. Thus $A_U \subseteq A_W$. □

**Corollary:** $W^0 = U^0 \Longleftrightarrow U = W$.

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**22** *Suppose $U$ and $W$ are subsps of $V$. Prove that $(U+W)^0 = U^0 \cap W^0$.*

**SOLUTION:**

(a) $\left.\begin{array}{r} U \subseteq U+W \\ W \subseteq U+W \end{array}\right\} \Rightarrow \left.\begin{array}{r} (U+W)^0 \subseteq U^0 \\ (U+W)^0 \subseteq W^0 \end{array}\right\} \Rightarrow (U+W)^0 \subseteq U^0 \cap W^0.$

OR. Suppose $\varphi \in (U+W)^0$. Then $\forall u \in U, w \in W, \varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$.

(b) Suppose $\varphi \in U^0 \cap W^0 \subseteq V'$. Then $\forall u \in U, w \in W, \varphi(u+w) = 0 \Rightarrow \varphi \in (U+W)^0$. ☐

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**23** *Suppose $U$ and $W$ are subsets of $V$. Prove that $(U \cap W)^0 = U^0 + W^0$.*

**SOLUTION:**

(a) $\left.\begin{array}{r} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array}\right\} \Rightarrow \left.\begin{array}{r} (U\cap W)^0 \supseteq U^0 \\ (U\cap W)^0 \supseteq W^0 \end{array}\right\} \Rightarrow (U\cap W)^0 \supseteq U^0 + W^0 \ \big[ \supseteq U^0 \cap W^0 = (U+W)^0. \big]$

OR. Suppose $\varphi = \psi + \beta \in U^0 + W^0$. Then $\forall v \in U \cap W, \varphi(v) = (\psi+\beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0$.

(b) $\big[$ *Only in Finite-dim; Req $U, W$ are subsps* $\big]$ Using Problem (22).

$\dim(U^0 + W^0) = \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0)$

$= 2\dim V - \dim U - \dim W - \big(\dim V - \dim(U+W)\big) = \dim V - \dim(U \cap W)$.

OR. Suppose $\varphi \in (U \cap W)^0$. Let $X, Y$ be such that $V = U \oplus X = W \oplus Y$.

Define $\psi \in U^0, \beta \in W^0$ by $\psi(u+x) = \frac{1}{2}\varphi(x), \beta(w+y) = \frac{1}{2}\varphi(y)$.

$\forall v = u + x = w + y \in V, \varphi(v) = \varphi(x) = \varphi(y)$. Now $\varphi(v) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y) = \psi(v) + \beta(v)$.

Hence $\varphi \in U^0 + W^0$. Now $(U \cap W)^0 \subseteq U^0 + W^0$. ☐

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• **COROLLARY:**

(a) *Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subsets of $V$. Then $\left(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \sum_{\alpha_i \in \Gamma} \left(V_{\alpha_i}^0\right)$.*

(b) *Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subsps of $V$. Then $\left(\sum_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \bigcap_{\alpha_i \in \Gamma} \left(V_{\alpha_i}^0\right)$.*

(c) *Suppose $V = U \oplus W$. Then $V' = U^0 \oplus W^0$. And $U_V' = W^0$, $W_V' = U^0$.*

Where $U_V' = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$. And $\iota \in \mathcal{L}(V, U)$ is defined by $\iota(u_v + w_v) = u_v$.

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• (4E 3.F.23) *Suppose $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following sets are the same.*

(a) $\operatorname{span}(\varphi_1, \dots, \varphi_m)$

(b) $\big((\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)\big)^0 \overset{(c)}{=\!=} \{\varphi \in V' : (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) \subseteq \operatorname{null}\varphi\}$

**SOLUTION:** By Problem (17), (c) holds.

By Problem (26) $\big[$ *May req Finite-dim* $\big]$ and the COROLLARY in Problem (23),

$\left.\begin{array}{r} \big((\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)\big)^0 = (\operatorname{null}\varphi_1)^0 + \dots + (\operatorname{null}\varphi_m)^0 \\ \operatorname{span}(\varphi_i) = \{v \in V : \forall \psi \in \operatorname{span}(\varphi_i), \psi(v) = 0\}^0 = (\operatorname{null}\varphi_i)^0 \end{array}\right\} \Rightarrow \text{(a)} = \text{(b)}.$ ☐

OR. Note that by COROLLARY in Problem (4E 6), for each $\varphi_i$, we have

$\forall c \in \mathbf{F}\backslash\{0\}, \psi = c\varphi_i \in \operatorname{span}(\varphi_i) \iff \operatorname{null}\psi = \operatorname{null}\varphi_i \iff \psi \in (\operatorname{null}\psi)^0 = (\operatorname{null}\varphi_i)^0$.

And $0 \in \operatorname{span}(\varphi_i), 0 \in (\operatorname{null}\varphi_i)^0$. Hence $\operatorname{span}(\varphi_i) = (\operatorname{null}\varphi_i)^0$. Similarly. ☐

OR. $\big[$ *Only in Finite-dim* $\big]$ Suppose $\varphi \in V'$. Note that $\dim(\operatorname{null}\varphi)^0 = \dim \operatorname{range}\varphi = \dim \operatorname{span}(\varphi)$.

And because $\forall c \in \mathbf{F}, v \in \operatorname{null}\varphi, c\varphi(v) = 0 \Rightarrow \operatorname{span}(\varphi) \subseteq (\operatorname{null}\varphi)^0$. Similarly. ☐

**COROLLARY:** **30** Suppose $V$ is finite-dim and $\varphi_1, \dots, \varphi_m$ is a linely inde list in $V'$.

Then $\dim\big((\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)\big) = (\dim V) - m$.

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**31** *Suppose $V$ is finite-dim and $B_{V'} = (\varphi_1, \dots, \varphi_n)$. Show that the correspd $B_V$ exists.*

**SOLUTION:**

Using (3.B.29). Let $\varphi_i(u_i) = 1$ and then $V = \operatorname{null}\varphi_i \oplus \operatorname{span}(u_i)$ for each $\varphi_i$.
Suppose $a_1u_1 + \cdots + a_nu_n = 0$. Then $0 = \varphi_i(a_1u_1 + \cdots + a_nu_n) = a_i$ for each $i$.
Thus $B_V = (\varphi_1, \dots, \varphi_n)$. And $\varphi_i(u_x) = \delta_{i,x}$. $\square$

OR. For each $k \in \{1, \dots, n\}$, define $\Gamma_k = \{1, \dots, k-1, k+1, \dots, n\}$ and $U_k = \bigcap_{j \in \Gamma_k} \operatorname{null}\varphi_j$.
By Problem (30) OR (4E 2.C.16), $\dim U_k = 1$. Thus $\exists\, u_k \in V, U_k = \operatorname{span}(u_k) \neq 0$.
又 By Problem (30), $(\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_n) = \{0\} = U \cap \operatorname{null}\varphi_k$.
Then if $\varphi_k(u_k) = 0 \Rightarrow u_k \in \operatorname{null}\varphi_k$ while $u_k \in U \Rightarrow u_k \in \{0\}$, contradicts.
Thus $\varphi_k(u_k) \neq 0$. Let $v_k = (\varphi_k(u_k))^{-1}u_k \Rightarrow \varphi_k(v_k) = 1$. Now for $j \neq k, u_k \in \operatorname{null}\varphi_j \Rightarrow \varphi_j(v_k) = 0$.
Similarly, suppose $a_1v_1 + \cdots + a_nv_n = 0 \Rightarrow a_1 = \cdots = a_n = 0$. $B_V = (v_1, \dots, v_n)$. And $\varphi_j(v_k) = \delta_{j,k}$. $\square$

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**25** *Suppose $U$ is a subsp of $V$. Explain why $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.*

**SOLUTION:** Note that $U = \{v \in V : v \in U\}$ is a subsp of $V$; And $v \in U \iff \varphi(v) = 0, \forall \varphi \in U^0$. $\square$

**COROLLARY:** $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}^0$.

**COMMENT:** $\{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = ((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) \cap \cdots)$, where $\varphi_k \in U^0$, always remains a subsp, whether the subset $U$ is a subsp or not.

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**26** *Suppose $\Omega$ is a subsp of $V'$. Prove that $\Omega = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}^0$.*

**SOLUTION:**

Suppose $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}$, *which is the set of vecs that each $\varphi \in \Omega$ sends to zero in common.*
Then $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}^0$. 又 $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}^0$.
Immediately by the COROLLARY in Problem (20,21), we may conclude that $\Omega = U^0$. $\square$

OR. [ *Req $\Omega$ finite-dim* ] Let $(\varphi_1, \dots, \varphi_m)$ be a basis of $\Omega$. Then by def, $U \subseteq (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.
$\forall \varphi \in \Omega, \exists!\, a_i \in \mathbf{F}, \varphi = a_1\varphi_1 + \cdots + a_m\varphi_m \Rightarrow \forall v \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m), \varphi(v) = 0 \Rightarrow v \in U$.
Hence $(\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = U$. 又 $\operatorname{span}(\varphi_1, \dots, \varphi_m) = \Omega$. By Problem (23), we are done. $\square$

**COROLLARY:** For every subsp $\Omega$ of $V'$, $\exists!$ subsp $U$ of $V$ such that $\Omega = U^0$.

**COMMENT:** [ *Only in Finite-dim* ] Using Problem (31) and the COROLLARY(c) in Problem (22, 23).

Let $B_\Omega = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n), B_V = (v_1, \dots, v_m, \dots, v_n)$.
$V' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \overset{\text{(I)}}{=\!=} \operatorname{span}(v_{m+1}, \dots, v_n)^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.
$\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m) \overset{\text{(II)}}{=\!=} \operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0$; $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \overset{\text{(III)}}{=\!=} \operatorname{span}(v_1, \dots, v_m)^0$.
$\iff U = \operatorname{span}(v_{m+1}, \dots, v_n) = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$. [ ***Another proof of* [3.106]** OR. Problem (24) ]

(I) Using the COROLLARY(c), immediately.

(II) NOTICE that each $\operatorname{null}\varphi_k = \operatorname{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n) = U_k$; $\dim U_k = \dim V - 1$.
By (4E 2.C.16), $U = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = \bigcap_{k=1}^{m} U_k = \operatorname{span}(v_{m+1}, \dots, v_n)$.
Hence $\operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0 = \Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$.

(III) NOTICE that $V' = \Omega \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.
And that $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq \operatorname{span}(v_1, \dots, v_m)^0$.
By (1.C TIPS), $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = \operatorname{span}(v_1, \dots, v_m)$.
OR. Similar to (II), let $\Omega = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, immediately. $\square$

---

• *Suppose* $T \in \mathcal{L}(V,W), \varphi_k \in V', \psi_k \in W'$.

**28** *Prove that* $\operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) \iff \operatorname{range} T = (\operatorname{null}\psi_1) \cap \dots \cap (\operatorname{null}\psi_m)$.

**29** *Prove that* $\operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \iff \operatorname{null} T = (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)$.

**SOLUTION:** Using [3.107], [3.109], Problem (23) and the COROLLARY in Problem (20, 21).

(28) $(\operatorname{range} T)^0 = \operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) = \big((\operatorname{null}\psi_1) \cap \dots \cap (\operatorname{null}\psi_m)\big)^0$.

(29) $(\operatorname{null} T)^0 = \operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) = \big((\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)\big)^0$. □

**COROLLARY:** Using the COMMENT in Problem (26).

$\operatorname{null} T = \operatorname{span}(v_1, \dots, v_m) \iff \operatorname{null} T = (\operatorname{null}\varphi_{m+1}) \cap \dots \cap (\operatorname{null}\varphi_n) \iff \operatorname{range} T' = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$.

——Where $B_V = (v_1, \dots, v_m, \dots, v_n) \iff B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$.

$\operatorname{range} T = \operatorname{span}(w_1, \dots, w_m) \iff \operatorname{range} T = (\operatorname{null}\psi_{m+1}) \cap \dots \cap (\operatorname{null}\psi_n) \iff \operatorname{null} T' = \operatorname{span}(\psi_{m+1}, \dots, \psi_n)$.

——Where $B_W = (w_1, \dots, w_m, \dots, w_n) \iff B_{W'} = (\psi_1, \dots, \psi_m, \dots, \psi_n)$.

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**9** *Let* $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \cdots, \varphi_n)$. *Then* $\forall \psi \in V', \psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n$.

**COROLLARY:** *For other* $B'_V = (u_1, \dots, u_n), B'_{V'} = (\rho_1, \dots, \rho_n), \forall \psi \in V', \psi = \psi(u_1)\rho_1 + \dots + \psi(u_n)\rho_n$.

**SOLUTION:**

$\psi(v) = \psi\Big(\sum_{i=1}^n a_i v_i\Big) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i)\varphi_i(v) = \big[\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n\big](v)$.

OR. $\big[\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n\big]\Big(\sum_{i=1}^n a_i v_i\Big) = \psi(v_1)\varphi_1\Big(\sum_{i=1}^n a_i v_i\Big) + \dots + \psi(v_n)\varphi_n\Big(\sum_{i=1}^n a_i v_i\Big)$. □

---

**13** *Define* $T : \mathbf{R}^3 \to \mathbf{R}^2$ *by* $T(x,y,z) = (4x + 5y + 6z, 7x + 8y + 9z)$.

*Let* $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ *denote the dual basis of the std basis of* $\mathbf{R}^2$ *and* $\mathbf{R}^3$.

(a) *Describe the linear functionals* $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$

For any $(x,y,z) \in \mathbf{R}^3$, $\big(T'(\varphi_1)\big)(x,y,z) = 4x + 5y + 6z$, $\big(T'(\varphi_2)\big)(x,y,z) = 7x + 8y + 9z$.

(b) *Write* $T'(\varphi_1)$ *and* $T'(\varphi_2)$ *as linear combinations of* $\psi_1, \psi_2, \psi_3$.

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$.

(c) *What is* $\operatorname{null} T'$? *What is* $\operatorname{range} T'$?

$$T(x,y,z) = 0 \iff \begin{cases} 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases} \iff \begin{cases} x + y + z = 0 \\ y = 2z = 0 \end{cases} \iff (x,y,z) \in \operatorname{span}(e_1 - 2e_2 + e_3).$$

Where $(e_1, e_2, e_3)$ is std basis of $\mathbf{R}^3$.

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a basis, with the correspd dual basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Thus $\operatorname{span}(e_1 - 2e_2 + e_3) = \operatorname{null} T \Rightarrow \operatorname{span}(e_1 - 2e_2 + e_3)^0 = \operatorname{span}(\varepsilon_2, \varepsilon_3) = \operatorname{range} T'$.

Note that $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$.

And $\left| \begin{array}{l} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{array} \right.$

Hence $\varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2$, $\varepsilon_3 = -\psi_1 + \psi_3$. Now $\operatorname{range} T' = \operatorname{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3)$.

OR. $\operatorname{range} T' = \operatorname{span}(T'(\varphi_1), T'(\varphi_2)) = \operatorname{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3)$.

Suppose $T'(x\varphi_1 + y\varphi_2) = (4x + 7y)\varphi_1 + (5x + 8y)\varphi_2 + (6x + 9y)\varphi_3 = 0$.

Then $x + y = 4x + 7y = x = y = 0$. Hence $\operatorname{null} T' = \{0\}$.

OR. $\operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \operatorname{span}(-2e_2, e_3) \oplus \operatorname{null} T$.

$\Rightarrow \operatorname{range} T = \{Tx : x \in \operatorname{span}(-2e_2, e_3)\} = \operatorname{span}(T(-2e_2), T(e_3))$

$= \operatorname{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \operatorname{span}(f_1, f_2) = \mathbf{R}^2$. Now $\operatorname{null} T' = (\operatorname{range} T)^0 = \{0\}$. □

---

**24** *Suppose $V$ is finite-dim and $U$ is a subsp of $V$.*
*Prove, using the pattern of [3.104], that* $\dim U + \dim U^0 = \dim V$.

**SOLUTION:**

By Problem (31) and the COMMENT in Problem (26), $B_U = (v_1, \dots, v_m) \iff B_{U^0} = (\varphi_{m+1}, \dots, \varphi_n)$. $\square$

---

**37** *Suppose $U$ is a subsp of $V$ and $\pi$ is the quotient map. Thus $\pi' \in \mathcal{L}\big((V/U)', V'\big)$.*

(a) *Show that $\pi'$ is inje*: Because $\pi$ is surj. Use [3.108].

(b) *Show that* $\operatorname{range} \pi' = U^0$: By [3.109](b), $\operatorname{range} \pi' = (\operatorname{null} \pi)^0 = U^0$.

(c) *Conclude that $\pi'$ is an iso from $(V/U)'$ onto $U^0$*: Immediately.

**SOLUTION:** OR. Using (3.E.18), also see (3.E.20).

(a) $\pi'(\varphi) = 0 \iff \forall v \in V\ (\ \forall v + U \in V\ ), \varphi(\pi(v)) = \varphi(v + U) = 0 \iff \varphi = 0$.

(b) $\psi \in \operatorname{range} \pi' \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null} \psi \supseteq U \iff \psi \in U^0$. Hence $\operatorname{range} \pi' = U^0$. $\square$

---

• *Suppose $U$ is a subsp of $V$. Prove that $(V/U)'$ is iso to $U^0$.* $\Big[$ ***Another proof of* [3.106]** $\Big]$

**SOLUTION:**

Define $\xi : U^0 \to (V/U)'$ by $\xi(\varphi) = \tilde{\varphi}$, where $\tilde{\varphi} \in (V/U)'$ is defined by $\tilde{\varphi}(v + U) = \varphi(v)$.

We show that $\xi$ is inje and surj.

Inje: $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V\ (\ \forall v + U \in V/U\ ), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$.

Surj: $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null}(\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$. $\square$

OR. Define $\nu : (V/U)' \to U^0$ by $\nu(\Phi) = \Phi \circ \pi$. Now $\nu \circ \xi = I_{U^0},\ \xi \circ \nu = I_{(V/U)'} \Rightarrow \xi = \nu^{-1}$. $\square$

---

**4** *Suppose $U$ is a subsp of $V$ and $U \neq V$. Prove that* $\underbrace{\exists\, \varphi \in V' \backslash \{0\}, \varphi(u) = 0 \textit{ for all } u \in U.}_{\iff U^0_V \neq \{0\}.}$

**SOLUTION:**

Let $X$ be such that $V = U \oplus X$. Then $X \neq \{0\}$. Suppose $s \in X$ and $x \neq 0$.

Let $Y$ be such that $X = \operatorname{span}(s) \oplus Y$. Now $V = U \oplus \big(\operatorname{span}(s) \oplus Y\big)$.

Define $\varphi \in V'$ by $\varphi(u + \lambda s + y) = \lambda$. Hence $\varphi \neq 0$ and $\varphi(u) = 0$ for all $u \in U$. $\square$

OR. [ *Req V Finite-dim* ] By [3.106], $\dim U^0 = \dim V - \dim U > 0$. Then $U^0 \neq \{0\}$.

OR. Let $B_V = (\underbrace{u_1, \dots, u_m}_{B_U}, v_1, \dots, v_n)$ with $n \geqslant 1$. Let $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$. Let $\varphi = \varphi_i$.

OR. Define $\varphi \in V'$ by $\varphi(u_1) = \cdots = \varphi(u_m) = 0$ and $\varphi(v_1) = \cdots = \varphi(v_n) = 1$. $\square$

**COMMENT:** ***Another proof of* [3.108]**: $T$ is surj $\iff T'$ is inje.

(a) Suppose $T'$ is inje. Note that $T'(\psi) = 0 \Rightarrow \psi = 0$.

Then $\nexists\, \psi \in W' \backslash \{0\}, \big(T'(\psi)\big)(v) = \psi(Tv) = 0$ for all $w \in \operatorname{range} T\ (\ \forall v \in V\ )$.

Thus if we assume that $\operatorname{range} T \neq W$ then contradicts. Hence $\operatorname{range} T = W$.

(b) Suppose $T$ is surj. Then $(\operatorname{range} T)^0 = W^0_W = \{0\} = \operatorname{null} T'$. $\square$

---

• *Suppose $V$ is a vecsp and $U$ is a subsp of $V$.*

**17** $U^0 = \{\varphi \in V' : U \subseteq \textit{null}\varphi\}$. Noticing $\varphi \in V', U \subseteq \operatorname{null} \varphi \iff \forall u \in U, \varphi(u) = 0$.

**18** $U^0 = V' \iff \forall \varphi \in V', U \subseteq \operatorname{null} \varphi \iff U = \{0\}$. [ *Which means* $\{0\}^0_V = V'$. ]

OR. $U^0 = V' \iff \dim U^0 = \dim V' = \dim V \iff \dim U = 0 \iff U = \{0\}$.

**19** $U^0_V = \{0\} = V^0_V \iff U = V$. By the inverse and contrapositive of Problem (4). OR. By [3.106].

---

• *Suppose $V = U \oplus W$. Define $\iota : V \to U$ by $\iota(u+w) = u$. Thus $\iota' \in \mathcal{L}(U', V')$.*

(a) *Show that* $\operatorname{null}\iota' = U_U^0 = \{0\}$: $\operatorname{null}\iota' = (\operatorname{range}\iota)_U^0 = U_U^0 = \{0\}$.

(b) *Prove that* $\operatorname{range}\iota' = W_V^0$: $\operatorname{range}\iota' = (\operatorname{null}\iota)_V^0 = W_V^0$.

(c) *Prove that $\widetilde{\iota'}$ is an iso from $U'/\{0\}$ onto $W^0$*: By (a), (b) and [3.91](d).

**SOLUTION:**

(a) $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \operatorname{null}\psi$.

(b) Note that $W = \operatorname{null}(\iota) \subseteq \operatorname{null}(\psi \circ \iota)$. Then $\psi \circ \iota \in W^0 \Rightarrow \operatorname{range}\iota' \in W^0$.

Suppose $\varphi \in W^0$. Because $\operatorname{null}\iota = W \subseteq \operatorname{null}\varphi$. By $\big[$3.B TIPS (3)$\big]$, $\varphi = \varphi \circ \iota = \iota'(\varphi)$. □

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**36** *Suppose $U$ is a subsp of $V$. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*

(a) *Show that* $\operatorname{null} i' = U^0$: $\operatorname{null} i' = (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$.

(b) *Prove that* $\operatorname{range} i' = U'$: $\operatorname{range} i' = (\operatorname{null} i)_U^0 = \{0\}_U^0 = U'$.

(c) *Prove that $\widetilde{i'}$ is an iso from $V'/U^0$ onto $U'$*: By (a), (b) and [3.91](d).

**SOLUTION:**

(a) $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$. Thus $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$.

(b) Suppose $\psi \in U'$. By (3.A.11), $\exists\, \varphi \in V', \varphi|_U = \psi$. Then $i'(\varphi) = \psi$. □

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• *Suppose $T \in \mathcal{L}(V, W)$. Prove that* $\operatorname{range} T' = \big(\operatorname{null} T\big)^0$. $\Big[$ ***Another proof of* [3.109](b)** $\Big]$

**SOLUTION:**

Suppose $\Phi \in \big(\operatorname{null} T\big)^0$. Because by (3.B.12), $T|_U : U \to \operatorname{range} T$ is an iso; $V = U \oplus \operatorname{null} T$.

And $\forall v \in V, \exists!\, u_v \in U, w_v \in \operatorname{null} T, v = u_v + w_v$. Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$.

Let $\psi = \Phi \circ \big(T^{-1}|_{\operatorname{range} T}\big)$. Then $T'(\psi) = \psi \circ T = \Phi \circ \big(T^{-1}|_{\operatorname{range} T} \circ T|_V\big)$.

Where $T^{-1}|_{\operatorname{range} T} : \operatorname{range} T \to U$; $T : V \to \operatorname{range} T$. Note that $T^{-1}|_{\operatorname{range} T} \circ T|_V = \iota$.

By $\big[$3.B TIPS (3)$\big]$, $\Phi = \Phi \circ \iota$. Thus $T'(\psi) = \psi \circ T = \Phi \circ \iota = \Phi$. □

---

• *Suppose $T \in \mathcal{L}(V, W)$. Using [3.108], [3.110].*

$$\textit{Now } T \textit{ is inv} \iff \left| \begin{array}{c} \operatorname{null} T = \{0\} \iff (\operatorname{null} T)^0 = V' = \operatorname{range} T' \\ \operatorname{range} T = W \iff (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T' \end{array} \right| \iff T' \textit{ is inv.}$$

---

**15** *Suppose $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \iff T = 0$.*

**SOLUTION:**

Suppose $T = 0$. Then $\forall \varphi \in W', T'(\varphi) = \varphi \circ T = 0$. Hence $T' = 0$.

Suppose $T' = 0$. Then $\operatorname{null} T' = W' = \big(\operatorname{range} T\big)^0$, by [3.107](a).

$\big[$ *$W$ can be infinite-dim* $\big]$ By Problem (25),

$\operatorname{range} T = \big\{w \in W : \varphi(w) = 0, \forall \varphi \in \big(\operatorname{range} T\big)^0\big\} = \big\{w \in W : \varphi(w) = 0, \forall \varphi \in W'\big\}$.

Now we prove that if $\forall \varphi \in W', \varphi(w) = 0$, then $w = 0$. So that $\operatorname{range} T = \{0\}$ and we are done.

Assume that $w \neq 0$. Then let $U$ be such that $W = U \oplus \operatorname{span}(w)$.

Define $\psi \in W'$ by $\psi(u + \lambda w) = \lambda$. So that $\psi(w) = 1 \neq 0$. □

OR. $\big[$ *Only if $W$ is finite-dim* $\big]$ By [3.106], $\dim \operatorname{range} T = \dim W - \dim(\operatorname{range} T)^0 = 0$. □

---

**12** NOTICE that $I_{V'} : V' \to V'$. *Now $\forall \varphi \in V', I_{V'}(\varphi) = \varphi = \varphi \circ I_V = {I_V}'(\varphi)$. Thus $I_{V'} = {I_V}'$.*

---

**16** *Suppose $V, W$ are finite-dim. Define $\Gamma$ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(V, W)$.*
*Prove that $\Gamma$ is an iso of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.*

**SOLUTION:** By [3.101], $\Gamma$ is linear.

Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus $\Gamma$ is inje.

Because $V, W$ are finite-dim. $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$. Now $\Gamma$ inje $\Rightarrow$ inv. □

**COMMENT:** Let $X = \{T \in \mathcal{L}(V, W) : \operatorname{range} T \text{ is finite-dim}\}$.
Let $Y = \{\mathcal{T} \in \mathcal{L}(W', V') : \operatorname{range} \mathcal{T} \text{ is finite-dim}\}$.
Then $\Gamma|_X$ is an iso of $X$ onto $Y$, even if $V$ and $W$ are infinite-dim.

*The inje of $\Gamma|_X$ is equiv to the inje of $\Gamma$, as shown before.*

*Now we show that $\Gamma|_X$ is surj without the cond that $V$ or $W$ is finite-dim.*

Suppose $\mathcal{T} \in Y$. Let $B_{\operatorname{range}\mathcal{T}} = (\varphi_1, \dots, \varphi_m)$, with the correspd $(v_1, \dots, v_m)$. Let $\varphi_k = \mathcal{T}(\psi_k)$.

Let $\mathcal{K}$ be such that $W' = \mathcal{K} \oplus \operatorname{null} \mathcal{T}$. Let $B_{\mathcal{K}} = (\psi_1, \dots, \psi_m)$, with the correspd $(w_1, \dots, w_m)$.

Define $T \in \mathcal{L}(V, W)$ by $Tv_k = w_k, Tu = 0;\ k \in \{1, \dots, m\}, u \in U$.

$\forall \psi \in \operatorname{null} \mathcal{T}, [T'(\psi)](v) = \psi(Tv) = \psi(a_1 w_1 + \cdots + a_p w_p) = 0 = [\mathcal{T}(\psi)](v)$.

$\forall k \in \{1, \dots, m\}, [T'(\psi_k)](v) = \psi_k(Tv) = \psi_k(a_1 w_1 + \cdots + a_m w_m) = a_k = \varphi_k(v) = [\mathcal{T}(\psi)](v)$. □

**COMMENT:** This is another proof of [3.109(a)]: $\dim \operatorname{range} T = \dim \operatorname{range} T'$.

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- (4E 3.F.6) *Suppose $\varphi, \beta \in V'$. Prove that* $\operatorname{null} \varphi \subseteq \operatorname{null} \beta \iff \beta = c\varphi, \exists c \in \mathbf{F}$.

**COROLLARY:** $\operatorname{null} \varphi = \operatorname{null} \beta \iff \beta = c\varphi, \exists c \in \mathbf{F} \setminus \{0\}$.

**SOLUTION:**

Using (3.B.29, 30).

(a) Suppose $\operatorname{null} \varphi \subseteq \operatorname{null} \beta$. Suppose $u \notin \operatorname{null} \beta$, then $u \notin \operatorname{null} \varphi$.
Now $V = \operatorname{null} \beta \oplus \operatorname{span}(u) = \operatorname{null} \varphi \oplus \operatorname{span}(u)$. By (1.C TIPS), $\operatorname{null} \beta = \operatorname{null} \varphi$. Let $c = \dfrac{\beta(u)}{\varphi(u)}$.

OR. We discuss in two cases. If $\operatorname{null} \varphi = \operatorname{null} \beta$, then we are done.
Otherwise, $\operatorname{null} \beta \neq \operatorname{null} \varphi$. Then $\exists\, u' \in \operatorname{null} \beta \setminus \operatorname{null} \varphi$.
Now $V = \operatorname{null} \varphi \oplus \operatorname{span}(u') = \operatorname{null} \varphi \oplus \operatorname{span}(u)$. $\forall v \in V, v = w + au = w' + bu', \exists!\, w, w' \in \operatorname{null} \varphi$.
Thus $\beta(v) = a\beta(u),\ \varphi(v) = b\varphi(u')$. Let $c = \dfrac{a\beta(u)}{b\varphi(u')}$. We are done.

NOTICE that by (b) below, we have $\operatorname{null} \beta \subseteq \operatorname{null} \varphi,\ u = u'$. Thus contradicts the assumption.

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$. If $c = 0$, then $\operatorname{null} \beta = V \supseteq \operatorname{null} \varphi$, we are done.

Otherwise, $\left.\begin{array}{l} \forall v \in \operatorname{null} \varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \operatorname{null} \varphi \subseteq \operatorname{null} \beta \\ \forall v \in \operatorname{null} \beta, \beta(v) = 0 = \varphi(v) \Rightarrow \operatorname{null} \beta \subseteq \operatorname{null} \varphi \end{array}\right\} \Rightarrow \operatorname{null} \varphi = \operatorname{null} \beta$. □

OR. By (3.B.24), $\operatorname{null} \varphi \subseteq \operatorname{null} \beta \iff \exists\, E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi$. ( if ***E* is *inv***, then $\operatorname{null} \varphi = \operatorname{null} \beta$ )

Now we show that [P] $\exists\, E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi \iff \exists\, c \in \mathbf{F}, \beta = c\varphi$. [Q].

$[P] \Rightarrow [Q]:$ Let $c = E(1)$. Then $\forall v \in V, \beta(v) = E(\varphi(v)) = \varphi(v)E(1) = c\varphi(v)$. ( $\boldsymbol{E(1) \neq 0}$ )

$[Q] \Rightarrow [P]:$ Define $E \in \mathcal{L}(\mathbf{F})$ by $E(x) = cx$. Then $\forall v \in V, \beta(v) = c\varphi(v) = E(\varphi(v))$. ( $\boldsymbol{c \neq 0}$ ) □

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**5** *Prove that $(V_1 \times \cdots \times V_m)'$ and ${V'}_1 \times \cdots \times {V'}_m$ are iso.* [ Using notations in (3.E.2). ]

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to {V'}_1 \times \cdots \times {V'}_m \\ \qquad \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = ({R'}_1(T), \dots, {R'}_m(T)). \\ \text{Define } \psi : {V'}_1 \times \cdots \times {V'}_m \to (V_1 \times \cdots \times V_m)' \\ \qquad \text{by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m = {S'}_1(T_1) + \cdots + {S'}_m(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$

□

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- (4E 3.F.8) *Suppose* $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \dots, \varphi_n)$.

$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

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- (4E 3.F.5) *Suppose* $T \in \mathcal{L}(V, W)$. $B_{\operatorname{range} T} = (w_1, \dots, w_m)$.

*Hence* $\forall v \in V$, $Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$, $\exists!\, \varphi_1(v), \dots, \varphi_m(v)$, *thus defining* $\varphi_i : V \to \mathbf{F}$ *for each* $i \in \{1, \dots, m\}$. *Show that each* $\varphi_i \in V'$.

**SOLUTION:**

$$\begin{aligned} \forall u, v \in V, \lambda \in \mathbf{F}, T(u + \lambda v) &= \textstyle\sum_{i=1}^m \varphi_i(u + \lambda v) w_i \\ &= Tu + \lambda Tv = \left( \textstyle\sum_{i=1}^m \varphi_i(u) w_i \right) + \lambda \left( \textstyle\sum_{i=1}^m \varphi_i(v) w_i \right) = \textstyle\sum_{i=1}^m \big(\varphi_i(u) + \lambda \varphi_i(v)\big) w_i. \end{aligned} \quad \square$$

OR. For each $w_i$, $\exists\, v_i \in V$, $Tv_i = w_i$, then $(v_1, \dots, v_m)$ is linely inde.
Now we have $Tv = a_1 T v_1 + \cdots + a_m T v_m$, $\forall v \in V$, $\exists!\, a_i \in \mathbf{F}$. Let $B_{(\operatorname{range} T)'} = (\psi_1, \dots, \psi_m)$.
Then $\big(T'(\psi_i)\big)(v) = \psi_i \circ T(v) = a_i$. Where $T : V \to \operatorname{range} T$; $T' : (\operatorname{range} T)' \to V'$.
Thus for each $i \in \{1, \dots, m\}$, $\varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$. $\square$

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**6** *Define* $\Gamma : V' \to \mathbf{F}^m$ *by* $\Gamma(\varphi) = \big(\varphi(v_1), \dots, \varphi(v_m)\big)$, *where* $v_1, \dots, v_m \in V$.

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = V \iff \Gamma$ *is inje.*

(b) *Show that* $(v_1, \dots, v_m)$ *is linely inde* $\iff \Gamma$ *is surj.*

**SOLUTION:**

(a) NOTICE that $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \cdots = \varphi(v_m) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.
If $\Gamma$ is inje, then $\Gamma(\varphi) = 0 \iff V = \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.
If $V = \operatorname{span}(v_1, \dots, v_m)$, then $\Gamma(\varphi) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$, thus $\Gamma$ is inje.

(b) Suppose $\Gamma$ is surj. Then let $\Gamma(\varphi_i) = e_i$ for each $i$, where $(e_1, \dots, e_m)$ is the std basis of $\mathbf{F}^m$.
Then by (3.A.4), $(\varphi_1, \dots, \varphi_m)$ is linely inde.
Now $a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow 0 = \varphi_i(a_1 v_1 + \cdots + a_m v_m) = a_i$ for each $i$.
Suppose $(v_1, \dots, v_m)$ is linely inde. Let $U = \operatorname{span}(\varphi_1, \dots, \varphi_m), B_{U'} = (\varphi_1, \dots, \varphi_m)$.
Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists!\, \varphi = a_1\varphi_1 + \cdots + a_m\varphi_m$.
Let $W$ be such that $V = U \oplus W$. Now $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.
Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$. So that $\Gamma(\varphi \circ i - ) = (a_1, \dots, a_m)$. □

OR. Let $(e_1, \dots, e_m)$ be the std basis of $\mathbf{F}^m$ and let $(\psi_1, \dots, \psi_m)$ be the correspd dual basis.
Define $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$. Then $\Psi$ is an iso.
Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $Te_k = v_k$. Now $T(x_1, \dots, x_m) = T(x_1 e_1 + \cdots + x_m e_m) = x_1 v_1 + \cdots + x_m v_m$.
$\forall \varphi \in V', k \in \{1, \dots, m\}, \big[T'(\varphi)\big](e_k) = \varphi(Te_k) = \varphi(v_k) = \big[\varphi(v_1) \circ \psi_1 + \cdots + \varphi(v_m) \circ \psi_m\big](e_k)$
Now $T'(\varphi) = \varphi(v_1) \circ \psi_1 + \cdots + \varphi(v_m) \circ \psi_m = \Psi\big(\varphi(v_1), \dots, \varphi(v_m)\big) = \Psi\big(\Gamma(\varphi)\big)$. Hence $T' = \Psi \circ \Gamma$.
By (3.B.3), (a) $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V \iff T' = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.
(b) $(v_1, \dots, v_m)$ is linely inde $\iff T$ is inje $\iff T' = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

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• (4E 3.F.25) *Define* $\Gamma : V \to \mathbf{F}^m$ *by* $\Gamma(v) = \big(\varphi_1(v), \dots, \varphi_m(v)\big)$, *where* $\varphi_1, \dots, \varphi_m \in V'$.

(c) *Show that* $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ *is inje.*

(d) *Show that* $(\varphi_1, \dots, \varphi_m)$ *is linely inde* $\iff \Gamma$ *is surj.*

**SOLUTION:**

(c) NOTICE that $\Gamma(v) = 0 \iff \varphi_1(v) = \cdots = \varphi_m(v) = 0 \iff v \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.
By Problem (4E 23) and (18), $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = \{0\}$.
And $\operatorname{null}\Gamma = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$. Hence $\Gamma$ inje $\iff \operatorname{null}\Gamma = \{0\} \iff \operatorname{span}(\varphi_1, \dots, \varphi_m) = V'$.

(d) Suppose $(\varphi_1, \dots, \varphi_m)$ is linely inde. Then by Problem (31), $(v_1, \dots, v_m)$ is linely inde.
Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}, \exists!\, v = \sum_{i=1}^m a_i v_i \in V \Rightarrow \varphi_i(v) = a_i, \Gamma(v) = (a_1, \dots, a_m)$. Hence $\Gamma$ is surj.
Suppose $\Gamma$ is surj. Let $(e_1, \dots, e_m)$ be the std basis of $\mathbf{F}^m$.
Suppose $v_i \in V$ such that $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i$, for each $i$.
Then $(v_1, \dots, v_m)$ is linely inde. And $\varphi_j(v_k) = \delta_{j,k}$.
Now $a_1\varphi_1 + \cdots + a_m\varphi_m = 0 \Rightarrow 0(v_i) = a_i$ for each $i$. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde.
OR. Let $\operatorname{span}(v_1, \dots, v_m) = U$. Then $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U)$. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde. □

OR. Similar to Problem (6), we get $(e_1, \dots, e_m), (\psi_1, \dots, \psi_m)$ and the iso $\Psi$.
$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'\big(\Psi(x_1, \dots, x_m)\big) = \Gamma'\big(\Psi(x_1 e_1 + \cdots + x_m e_m)\big) = (x_1\psi_1 + \cdots + x_m\psi_m) \circ \Gamma$.
$\forall v \in V, \big[\Gamma'\big(\Psi(x_1, \dots, x_m)\big)\big](v) = \big[x_1\psi_1 + \cdots + x_m\psi_m\big]\big(\Gamma(v)\big) = \big[x_1\varphi_1 + \cdots + x_m\varphi_m\big](v)$.
Now $\Gamma'\big(\Psi(x_1, \dots, x_m)\big) = x_1\varphi_1 + \cdots + x_m\varphi_m$.
Define $\Phi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Phi = \Psi \circ \Gamma$. $\Phi(x_1, \dots, x_m) = x_1\varphi_1 + \cdots + x_m\varphi_m$. Thus by (4E 3.B.3),
(c) the inje of $\Phi$ correspds to $(\varphi_1, \dots, \varphi_m)$ spanning $V'$; 又 $\Phi = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.
(d) the surj of $\Phi$ correspds to $(\varphi_1, \dots, \varphi_m)$ being linely inde; 又 $\Phi = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

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**35** *Prove that $\big(\mathcal{P}(\mathbf{F})\big)'$ is iso to $\mathbf{F}^\infty$.*

**SOLUTION:**

Define $\theta \in \mathcal{L}\Big(\big(\mathcal{P}(\mathbf{F})\big)', \mathbf{F}^\infty\Big)$ by $\theta(\varphi) = \big(\varphi(1), \varphi(z), \cdots, \varphi(z^n), \cdots\big)$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall z^k$ in the basis $(1, z, \dots, z^n)$ of $\mathcal{P}_n(\mathbf{F})$ $(\ \forall n\ )$, $\varphi(z^k) = 0 \Rightarrow \varphi = 0$.

$\big[$ NOTICE that $\forall p \in \mathcal{P}(\mathbf{R}), \exists\,! a_i \in \mathbf{F}, m = \deg p,\ p = a_0 z + a_1 z + \cdots + a_m z^m \in \mathcal{P}_m(\mathbf{F})$. $\big]$

Surj: $\forall \big(a_k\big)_{k=1}^\infty \in \mathbf{F}^\infty$, let $\psi$ be such that $\forall k, \psi(z^k) = a_k$ $\big[$ by [3.5] $\big]$ and thus $\theta(\psi) = \big(a_k\big)_{k=1}^\infty$. $\square$

**COMMENT:** NOTICE that $\mathcal{P}(\mathbf{F})$ is not iso to $\mathbf{F}^\infty$, so is $\mathcal{P}(\mathbf{F})$ to $\big(\mathcal{P}(\mathbf{F})\big)'$

But if we let $\mathbf{F}^\infty = \Big\{(a_1, \cdots, a_n, \underbrace{0, \cdots, 0, \cdots}_{\text{all zero}}) \in \mathbf{F}^\infty \ \Big|\ \exists\,! n \in \mathbf{N}^+\Big\}$. Then $\mathcal{P}(\mathbf{F})$ is iso to $\mathbf{F}^\infty$.

---

**7** *Show that the dual basis of $(1, x, \dots, x^m)$ of $\mathcal{P}_m(\mathbf{R})$ is $(\varphi_0, \varphi_1, \dots, \varphi_m)$, where $\varphi_k(p) = \dfrac{p^{(k)}(0)}{k!}$.*

*Here $p^{(k)}$ denotes the $k^{th}$ derivative of $p$, with the understanding that the $0^{th}$ derivative of $p$ is $p$.*

**SOLUTION:**

$$\forall j, k \in \mathbf{N},\ (x^j)^{(k)} = \begin{cases} j(j-1)\dots(j-k+1)\cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\dots(j-j+1) = j! & j = k. \\ 0, & j \leqslant k. \end{cases} \Bigg|\ \text{Then } (x^j)^{(k)}(0) = \begin{cases} 0\ , & j \neq k. \\ k!\ , & j = k. \end{cases} \quad \square$$

OR. Because $\forall j, k \in \{1, \dots, m\}$ such that $j \neq k$, $\varphi_k(x^j) = \frac{(x^j)^{(k)}(0)}{k!} = \frac{0}{k!} = 0$; $\varphi_k(x^k) = \frac{(x^k)^{(k)}(0)}{k!} = 1$.

Thus $\frac{p^{(k)}(0)}{k!}$ act exactly the same as $\varphi_k$ on the same basis $(1, \dots, x^m)$, hence is just another def of $\varphi_k$. $\square$

**EXAMPLE:** Suppose $m \in \mathbf{N}^+$. By [2.C.10], $B = \big(1, x-5, \dots, (x-5)^m\big)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

Let $\varphi_k = \dfrac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \dots, m$. Then $(\varphi_0, \varphi_1, \dots, \varphi_m)$ is the dual basis of $B$.

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**34** *The double dual space of $V$, denoted by $V''$, is defined to be the dual space of $V'$. In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \to V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.*

(a) *Show that $\Lambda$ is a linear map from $V$ to $V''$.*

(b) *Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*

(c) *Show that if $V$ is finite-dim, then $\Lambda$ **is an iso from $V$ onto $V''$**.*

*Suppose $V$ is finite-dim. Then $V$ and $V'$ are iso, and finding an iso from $V$ onto $V'$ generally requires choosing a basis of $V$. In contrast, the iso $\Lambda$ from $V$ onto $V''$ does not require a choice of basis and thus is considered more natural.*

**SOLUTION:**

(a) $\forall \varphi \in V', v, w \in V, a \in \mathbf{F}, \big(\Lambda(v+aw)\big)(\varphi) = \varphi(v+aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.

Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence $\Lambda$ is linear.

(b) $\big(T''(\Lambda v)\big)(\varphi) = \big((\Lambda v) \circ T'\big)(\varphi) = (\Lambda v)\big(T'(\varphi)\big)$

$$= \big(T'(\varphi)\big)(v) = (\varphi \circ T)(v) = \varphi(Tv) = \big(\Lambda(Tv)\big)(\varphi).$$

Hence $T''(\Lambda v) = \big(\Lambda(Tv)\big) \Rightarrow T'' \circ \Lambda = \Lambda \circ T$.

(c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus $\Lambda$ is inje.

又 Because $V$ is finite-dim. $\dim V = \dim V' = \dim V''$. Hence $\Lambda$ is an iso. $\square$

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**ENDED**

- **TIPS:** *Suppose $p \in \mathcal{P}(\mathbf{F})$, $\deg p \leqslant m$ and $p$ has at least $(m+1)$ distinct zeros.*
  *Then by the contrapositive of [4.12],* 又 $\deg p = m$, *we conclude that $m < 0$. Hence $p = 0$.*

  OR. We show that if $p$ has at least $m$ distinct zeros, then either $p = 0$ or $\deg p \geqslant m$.
  If $p = 0$ then we are done. If not, then suppose $p$ has exactly $n$ distinct zeros $\lambda_1, \dots, \lambda_n$.
  Because $\exists!\, \alpha_i \geqslant 1, q \in \mathcal{P}(\mathbf{F})$, and $q \neq 0$, such that $p(z) = \left[(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_n)^{\alpha_n}\right] q(z)$. □
- **COMMENT:** NOTICE *that by [4.17], some term of the poly factorization might not be in the form $(x-\lambda_k)^{\alpha_k}$.*

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- **NOTE FOR [4.7]:** *the uniqness of coeffs of polys* [*Another proof*]

  If a poly had two different sets of coeffs, then subtracting the two representations would give a poly with some nonzero coeffs but infinitely many zeros. By TIPS. □

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- **NOTE FOR [4.8]:** *division algorithm for polys* [*Another proof*]

  Suppose $\deg p \geqslant \deg s$. Then $\Big(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of length } \deg s}, \underbrace{s, zs, \cdots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)}\Big)$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.

  Because $q \in \mathcal{P}(\mathbf{F})$, $\exists!\, a_i, b_j \in \mathbf{F}$,
  $q = a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s$
  $= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1}}_{r} + s \underbrace{\left(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s}\right)}_{q}$. Note that $r, q$ are unique. □

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- **NOTE FOR [4.11]:** *each zero of a poly corresponds to a degree-one factor;* [*Another proof*]

  First suppose $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, $\exists!\, a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.
  Then $p(z) = p(z) - p(\lambda) = a_1(z-\lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.
  Hence $\forall k \in \{1, \dots, m\}$, $z^k - \lambda^k = (z-\lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.
  Thus $p(z) = \sum_{j=1}^{m} a_j (z-\lambda) \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z-\lambda) \sum_{j=1}^{m} a_j \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z-\lambda) q(z)$. □

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- **NOTE FOR [4.13]:** *Every nonconst poly with complex coeffs has a zero in* $\mathbf{C}$. [*Another proof*]

  For any $w \in \mathbf{C}, k \in \mathbf{N}^+$, by polar coordinates, $\exists\, r \geqslant 0, \theta \in \mathbf{R}$, $r(\cos\theta + \mathrm{i}\sin\theta) = w$.
  By De Moivre' theorem, $w^k = \left[r(\cos\theta + \mathrm{i}\sin\theta)\right]^k = r^k(\cos k\theta + \mathrm{i}\sin k\theta)$.
  Hence $\left(r^{1/k}(\cos\frac{\theta}{k} + \mathrm{i}\sin\frac{\theta}{k})\right)^k = w$. Thus every complex number has a $k^{th}$ *root*.

  Suppose a nonconst $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.

  Then $|p(z)| \to \infty$ as $|z| \to \infty$ $\left(\text{because } \dfrac{|p(z)|}{|z_m|} \to |c_m| \text{ as } |z| \to \infty\right)$.

  Thus the continuous function $z \to |p(z)|$ has a global minimum at some point $\zeta \in \mathbf{C}$.

  To show that $p(\zeta) = 0$, assume $p(\zeta) \neq 0$. Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \dfrac{p(z+\zeta)}{p(\zeta)}$.

  The function $z \to |q(z)|$ has a global minimum value of 1 at $z = 0$.

  Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where $k \in \mathbf{N}^+$ is the smallest such that $a_k \neq 0$.

  Let $\beta \in \mathbf{C}$ be such that $\beta^k = -\dfrac{1}{a_k}$.

  There is a const $c > 1$ so that if $t \in (0,1)$, then $|q(t\beta)| \leqslant |1 + a_k t^k \beta^k| + t^{k+1}c = 1 - t^k(1 - tc)$.

  Now letting $t = 1/(2c)$, we get $|q(t\beta)| < 1$. Contradicts. Hence $p(\zeta) = 0$, as desired. □

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• (4E 4.2) *Prove that if $w, z \in \mathbf{C}$, then $||w| - |z|| \leqslant |w - z|$.*

**SOLUTION:**

$$\begin{aligned} |w-z|^2 &= (w-z)(\overline{w}-\overline{z}) \\ &= |w|^2 + |z|^2 - (w\overline{z} + \overline{w}z) \\ &= |w|^2 + |z|^2 - (\overline{\overline{w}z} + \overline{w}z) \\ &= |w|^2 + |z|^2 - 2Re(\overline{w}z) \\ &\geqslant |w|^2 + |z|^2 - 2|\overline{w}z| \\ &= |w|^2 + |z|^2 - 2|w||z| = ||w| - |z||^2. \end{aligned}$$

OR. $\left.\begin{array}{l} |w| = |w-z+z| \leqslant |w-z| + |z| \Rightarrow |w| - |z| \leqslant |w-z| \\ |z| = |z-w+w| \leqslant |z-w| + |w| \Rightarrow |z| - |w| \leqslant |w-z| \end{array}\right\}$

*Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides.*

□

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• (4E 4.3) *Suppose $\mathbf{F} = \mathbf{C}, \varphi \in V'$. Define $\sigma : V \to \mathbf{R}$ by $\sigma(v) = \operatorname{Re}\varphi(v)$ for each $v \in V$. Show that $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$ for all $v \in V$.*

**SOLUTION:** Notice that $\varphi(v) = \operatorname{Re}\varphi(v) + \mathrm{i}\,\mathfrak{Im}\,\varphi(v) = \sigma(v) + \mathrm{i}\,\mathfrak{Im}\,\varphi(v)$.
又 $\operatorname{Re}\varphi(\mathrm{i}\,v) = \operatorname{Re}\big(\mathrm{i}\,\varphi(v)\big) = -\mathfrak{Im}\,\varphi(v) = \sigma(\mathrm{i}\,v)$. Hence $\varphi(v) = \sigma(v) - \mathrm{i}\,\sigma(\mathrm{i}\,v)$. □

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**4** *Suppose $m, n \in \mathbf{N}^+$ with $m \leqslant n$, $\lambda_1, \dots, \lambda_m \in \mathbf{F}$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{F}), \deg p = n$, the zeros of $p$ are $\lambda_1, \dots, \lambda_m$.*

**SOLUTION:** Let $p(z) = (z-\lambda_1)^{n-(m-1)}(z-\lambda_2)\cdots(z-\lambda_m)$. □

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**5** *Suppose $m \in \mathbf{N}$, and $z_1, \dots, z_{m+1}$ are distinct in $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$. Prove that $\exists!\, p \in \mathcal{P}_m(\mathbf{F}), p(z_k) = w_k$ for each $k \in \{1, \dots, m+1\}$.*

**SOLUTION:**

Define $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by $Tq = \big(q(z_1), \dots, q(z_m), q(z_{m+1})\big)$. Moreover, $T$ is linear.

We now show that $T$ is surj, so that such $p$ exists; and that $T$ is inje, so that such $p$ is unique.
Inje: $Tq = 0 \Longleftrightarrow q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0 \Longleftrightarrow q = 0$, by TIPS.
Surj: $\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m+1 = \dim \mathbf{F}^{m+1}$ 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1} \Rightarrow T$ is surj. □

OR. Let $p_1 = 1$, $p_k(z) = \prod_{i=1}^{k-1}(z-z_i) = (z-z_1)\cdots(z-z_{k-1})$ for each $k \in \{2, \dots, m+1\}$.
By (2.C.10), $B_p = (p_1, \dots, p_{m+1})$ is a basis of $\mathcal{P}_m(\mathbf{F})$. Let $B_e = (e_1, \dots, e_{m+1})$ be the std basis of $\mathbf{F}^{m+1}$.
NOTICE that $Tp_1 = (1, \dots, 1)$, $Tp_k = \Big(\prod_{i=1}^{k-1}(z_1 - z_i), \dots, \underbrace{\prod_{i=1}^{k-1}(z_j - z_i)}_{j^{th}\text{ entry}}, \dots, \prod_{i=1}^{k-1}(z_{m+1} - z_i)\Big)$.

And that $\prod_{i=1}^{k-1}(z_j - z_i) = 0 \Longleftrightarrow j \leqslant k-1$, because $z_1, \dots, z_{m+1}$ are distinct.

$$\text{Thus } \mathcal{M}(T, B_p, B_e) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & A_{2,2} & 0 & \cdots & 0 \\ 1 & A_{3,2} & A_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{m+1,2} & A_{m+1,3} & \cdots & A_{m+1,m+1} \end{pmatrix}.$$

Where $A_{j,k} = \prod_{i=1}^{k-1}(z_j - z_i) \neq 0$ for all $j > k-1 \geqslant 1$. The rows of $\mathcal{M}(T)$ is linely inde.
By (4E 3.C.17) 又 $\dim \mathcal{P}_m(\mathbf{F}) = \dim \mathbf{F}^{m+1}$; OR By (3.F.32); $T$ is inv. □

---

**2** *Suppose $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

**SOLUTION:** $x^m, x^m + x^{m-1} \in U$ but $\deg\big[(x^m + x^{m-1}) - (x^m)\big] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$. □

---

**3** *Suppose $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : 2 \mid \deg p\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

**SOLUTION:** $x^2, x^2 + x \in U$ but $\deg[(x^2+x)-(x^2)]$ is odd and hence $(x^2+x)-(x^2) \notin U$. □

---

**6** *Suppose nonzero $p \in \mathcal{P}_m(\mathbf{F})$ has degree $m$. Prove that*
*$[P]$ $p$ has $m$ distinct zeros $\iff$ $p$ and its derivative $p'$ have no zeros in common $[Q]$.*

**SOLUTION:**

(a) Suppose $p$ has $m$ distinct zeros. And $\deg p = m$. By [4.14], $\exists!\, c, \lambda_i \in \mathbf{R}, p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$.
If $m = 0$, then $p = c \neq 0 \Rightarrow p$ has no zeros, and $p' = 0$, we are done.
If $m = 1$, then $p(z) = c(z-\lambda_1)$, and $p' = c$ has no zeros, we are done.
For each $j \in \{1, \dots, m\}$, let $q_j \in \mathcal{P}_{m-1}(\mathbf{F})$ be such that $p(z) = (z-\lambda_j)q_j \Rightarrow q_j(\lambda_j) \neq 0$.
Now $p'(z) = (z-\lambda_j)q_j{}'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

OR. To prove $[P] \Rightarrow [Q]$, we prove $\neg[Q] \Rightarrow \neg[P]$:
Suppose $p(z) = (z-\lambda)q(z),\ p'(z) = (z-\lambda)r(z)$. 又 $p'(z) = (z-\lambda)q'(z) + q(z)$.
Now $p'(\lambda) = q(\lambda) = 0 \Rightarrow q(z) = (z-\lambda)s(z), p(z) = (z-\lambda)^2 s(z)$.
Hence $p$ has strictly less than $m$ distinct zeros.

(b) To prove $[Q] \Rightarrow [P]$, we prove $\neg[P] \Rightarrow \neg[Q]$:
Because nonzero $p \in \mathcal{P}_m(\mathbf{F})$, we suppose $\lambda_1, \dots, \lambda_M$ are all the distinct zeros of $p$, where $M < m$.
By Pigeon Hole Principle, $\exists \lambda_k$ such that $p(z) = (z-\lambda_k)^2 q(z)$ for some $q \in \mathcal{P}(\mathbf{F})$.
Hence $p'(z) = 2(z-\lambda_k)q(z) + (z-\lambda_k)^2 q'(z) \Rightarrow p'(\lambda_k) = 0 = p(\lambda_k)$. □

---

**7** *Prove that every $p \in \mathcal{P}(\mathbf{R})$ of odd degree has a zero.*

**SOLUTION:**

Using the notation and proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence $\lambda_1$ exists. □

OR. Using calculus only. Suppose $p \in \mathcal{P}_m(\mathbf{F}), \deg p = m, m$ is odd.
Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1} a_m$ by $\delta$.
Write $p(x) = x^m \left( \dfrac{a_0}{x^m} + \dfrac{a_1}{x^{m-1}} + \cdots + \dfrac{a_{m-1}}{x} + a_m \right)$.
Thus $p(x)$ is continuous, and $\lim\limits_{x \to -\infty} p(x) = -\delta\infty$; $\lim\limits_{x \to \infty} p(x) = \delta\infty$.
Hence we conclude that $p$ has at least one real zero. □

---

**9** *Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\overline{z})}$. Prove that $q \in \mathcal{P}(\mathbf{R})$.*

**SOLUTION:**

NOTICE that by [4.5], $\overline{z}^n = \overline{z^n}$.
Suppose $q(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow q(\overline{z}) = a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0 \Rightarrow \overline{q(\overline{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\overline{z})} = \overline{p(\overline{z})}p(z) = \overline{p(\overline{z})\overline{\overline{p(\overline{\overline{z}})}}} = \overline{q(\overline{z})}$. Hence for each $a_k, \overline{a_k} = a_k \Rightarrow a_k \in \mathbf{R}$. □

OR. Suppose $p(z) = a_m z^m + \cdots + a_1 z + a_0$. Now $\overline{p(\overline{z})} = \overline{a_m} z^m + \cdots + \overline{a_1} z + \overline{a_0}$.
NOTICE that $q(z) = p(z)\overline{p(\overline{z})} = \sum_{k=0}^{2m} \left( \sum_{i+j=k} a_i \overline{a_j} \right) z^k$.
NOTICE that by [4.5], $z - \overline{z} = 2(\mathfrak{Im}\, z) \Rightarrow z = \overline{z} + 2(\mathfrak{Im}\, z)$. So that $z = \overline{z} \iff \mathfrak{Im}\, z = 0 \iff z \in \mathbf{R}$.
Now for each $k \in \{0, \dots, 2m\}$, $\overline{\sum_{i+j=k} a_i \overline{a_j}} = \sum_{i+j=k} \overline{a_i \overline{a_j}} = \sum_{i+j=k} a_j \overline{a_i} = \sum_{i+j=k} a_i \overline{a_j} \in \mathbf{R}$. □

---

**8** *For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \to \mathbf{R}$ by* $(Tp)(x) = \begin{cases} \dfrac{p(x) - p(3)}{x - 3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}$

*Show that* (a) $Tp \in \mathcal{P}(\mathbf{R})$ *for all* $p \in \mathcal{P}(\mathbf{R})$ *and that* (b) $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ *is linear.*

**SOLUTION:**

(a) For $x \neq 3, T(x^n) = \dfrac{x^n - 3^n}{x - 3} = \sum_{i=1}^{n} 3^{i-1} x^{n-i}$. For $x = 3, T(x^n) = 3^{n-1} \cdot n$.

Note that if $x = 3$, then $\sum_{i=1}^{n} 3^{i-1} x^{n-i} = \sum_{i=1}^{n} 3^{n-1} = 3^{n-1} \cdot n$.

Hence $T(x^n) = \sum_{i=1}^{n} 3^{i-1} x^{n-i} \Rightarrow T(x^n) \in \mathcal{P}(\mathbf{R})$.

(b) Now we show that $T$ is linear: $\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}$,

$$T(p + \lambda q)(x) = \left\{ \begin{array}{ll} \dfrac{(p+\lambda q)(x) - (p + \lambda q)(3)}{x - 3}, & \text{if } x \neq 3, \\ (p + \lambda q)'(3), & \text{if } x = 3 \end{array} \right| = \left[ T(p) + \lambda T(q) \right](x) \text{ for all } x \in \mathbf{R}.$$

□

OR. (a) Note that $\exists!\, q \in \mathcal{P}(\mathbf{R}), p(x) - p(3) = (x - 3)q(z) \Rightarrow q(x) = \dfrac{p(x) - p(3)}{x - 3}$.

$p'(x) = \big(p(x) - p(3)\big)' = \big((x - 3)q(x)\big)' = q(x) + (x - 3)q'(x)$.

Hence $p'(3) = q(3)$. Now $Tp = q \in \mathcal{P}(\mathbf{R})$.

(b) $\forall p_1, p_2 \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, \exists!\, q_1, q_2 \in \mathcal{P}(\mathbf{R})$,

$p_1(x) - p_1(3) = (x - 3)q_1(x)$ and $p_2(x) - p_2(3) = (x - 3)q_2(x)$.

By (a), $Tp_1 = q_1, Tp_2 = q_2$. Note that $(p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x - 3)(q_1 + \lambda q_2)(x)$.

Hence by the uniqnes of $q_1 + \lambda q_2$ for $p_1 + \lambda p_2$, we must have $T(p_1 + \lambda p_2) = q_1 + \lambda q_2$. □

---

**11** *Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*

(a) *Show that* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) *Find a basis of* $\mathcal{P}(\mathbf{F})/U$.

**SOLUTION:** NOTICE that $pq \neq p \circ q$, see (4E 3.A.10).

$U$ is a subsp of $\mathcal{P}(\mathbf{F})$ because $\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, ps_1 + \lambda ps_2 = p(s_1 + \lambda s_2) \in U$.

If $\deg p = 0$, then $U = \mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F})/U = \{0\}$, with the unique basis $()$. Suppose $\deg p \geqslant 1$.

(a) By [4.8], $\forall s \in \mathcal{P}(\mathbf{F}), \exists!\, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F})\ [\,\exists!\, pq \in U\,], s = (p)q + (r)$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p - 1}(\mathbf{F})$. By the NOTE FOR [3.91] in (3.E), $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p - 1}(\mathbf{F})$ are iso.

OR. Define $R : \mathcal{P}(\mathbf{F}) \to \mathcal{P}_{\deg p - 1}(\mathbf{F})$ by $R(s) = r$ for all $s \in \mathcal{P}(\mathbf{F})$ We show that $R$ is linear.

$\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, \exists!\, r_1, r_2 \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), q_1, q_2 \in \mathcal{P}(\mathbf{F}), s_1 = (p)q_1 + (r_1);\ s_2 = (p)q_2 + (r_2)$.

又 $\exists!\, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}), (s_1 + \lambda s_2) = (p)q + (r) = (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2)$.

Note that $r_1, r_2 \in \mathcal{P}_{\deg p - 1}(\mathbf{F}) \Rightarrow r_1 + \lambda r_2 \in \mathcal{P}_{\deg p - 1}(\mathbf{F})$.

OR Note that $\deg(r_1 + \lambda r_2) \leqslant \max\{\deg r_1, \deg(\lambda r_2)\} \leqslant \max\{\deg r_1, \deg r_2\} < \deg p$.

By the uniqnes part of [4.8], $s = s_1 + \lambda s_2;\ r = r_1 + \lambda r_2$. Thus $R(s_1 + \lambda s_2) = R(s_1) + \lambda R(s_2)$.

Because $Rs = 0 \iff s = pq, \exists!\, q \in \mathcal{P}(\mathbf{F}) \iff s \in U$. And $\forall r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), Rr = r$.

Now $\operatorname{null} R = U$, $\operatorname{range} R = \mathcal{P}_{\deg p - 1}(\mathbf{F})$.

Hence $\tilde{R} : \mathcal{P}(\mathbf{F})/U \to \mathcal{P}_{\deg p - 1}(\mathbf{F})$ is defined by $\tilde{R}(s + U) = Rs$. By [3.91(d)], $\tilde{R}$ is an iso.

(b) For each $k \in \{0, 1, \dots, \deg p - 1\}, \tilde{R}(z^k + U) = R(z^k) = z^k \Rightarrow \tilde{R}^{-1}(z^k) = z^k + U$.

Thus $(1 + U, z + U, \dots, z^{\deg p - 1} + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. □

---

**10** *Suppose $m \in \mathbf{N}, p \in \mathcal{P}_m(\mathbf{C})$ is such that $p(x_k) \in \mathbf{R}$ for each of distinct $x_0, x_1, \dots, x_m \in \mathbf{R}$. Prove that $p \in \mathcal{P}(\mathbf{R})$.*

**SOLUTION:**

By TIPS and Problem (5), $\exists\,!\, q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_k) = p(x_k)$. Hence $p = q$. $\square$

OR. Using the Lagrange Interpolating Polynomial.

Define $q(x) = \sum_{j=0}^{m} \dfrac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j)$.

又 Each $x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R})$. Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$ for each $x_k$.
Then $(q-p)$ has $(m+1)$ zeros, while $(q-p) \in \mathcal{P}_m(\mathbf{C})$. By TIPS, $q - p = 0 \Rightarrow p = q \in \mathcal{P}(\mathbf{R})$. $\square$

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- (4E 4 13) *Suppose nonconst $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p, n = \deg q$. Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r,s) = rp + sq$. Prove that $T$ is an iso.*
  **COROLLARY:** *$\exists\,!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*

**SOLUTION:**

$T$ is linear because $\forall\, r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,
$T\big((r_1,s_1) + \lambda(r_2,s_2)\big) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1,s_1) + \lambda T(r_2,s_2)$.

Let $\lambda_1, \dots, \lambda_M$ and $\mu_1, \dots, \mu_N$ be the distinct zeros of $p$ and $q$ respectively. NOTICE that $M \leqslant m, N \leqslant n$.

Note that the contrapositive of [4.13], $M = 0 \iff m = 0 \Rightarrow s = 0 \iff r = 0 \Leftarrow n = 0 \iff N = 0$.

Now suppose $M, N \geqslant 1$. We show that $s = 0$. Showing $r = 0$ is almost the same.

Write $p(z) = a(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}$. $(\,\exists\,!\,\alpha_j \geqslant 1, a \in \mathbf{F}.\,)$ Let $\max\{\alpha_1, \dots, \alpha_M\} = A$.

For each $D \in \{0, 1, \dots, A-1\}$, let $I_{D,\alpha} = \{\gamma_{D,1}, \dots, \gamma_{D,J}\}$ be such that each $\alpha_{\gamma_{D,j}} \geqslant D + 1$.

Note that $I_{A-1,\alpha} \subseteq \cdots \subseteq I_{0,\alpha} = \{1, \dots, M\}$. Because $rp + sq = 0 \Rightarrow (rp+sq)^{(k)} = 0$ for all $k \in \mathbf{N}^+$.

We use induction by $D$ to show that $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$ for each $D \in \{0, \dots, A-1\}$.

NOTICE that $p^{(D)}(\lambda_\gamma) = 0$ for each $D \in \{0, \dots, A-1\}$ and each $\lambda_\gamma \in I_{D,\alpha}$. $(\Delta)$

(i) $D = 0$. $(rp+sq)(\lambda_{\gamma_{0,j}}) = (sq)(\lambda_{\gamma_{0,j}}) = s(\lambda_{\gamma_{0,j}}) = 0$.
$D = 1$. $(rp+sq)'(\lambda_{\gamma_{1,j}}) = (r'p + rp')(\lambda_{\gamma_{1,j}}) + (s'q + sq')(\lambda_{\gamma_{1,j}}) = (s'q)(\lambda_{\gamma_{1,j}}) = s'(\lambda_{\gamma_{1,j}}) = 0$.

(ii) $2 \leqslant D \leqslant A-1$. Assume that $s^{(d)}(\lambda_{\gamma_{d,j}}) = 0$ for each $d \in \{1, \dots, D-1\}$ and each $\lambda_{\gamma_{d,j}} \in I_{d,\alpha}$.
$\Big($ Because $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}.\Big)$ $(\Delta)$

$$\begin{aligned} \text{Now } [rp+sq]^{(D)}(\lambda_{\gamma_{D,j}}) &= \big[C_D^D r^{(D)} p^{(0)} + \cdots + C_D^d r^{(d)} p^{(D-d)} + \cdots + C_D^0 r^{(0)} p^{(D)}\big](\lambda_{\gamma_{D,j}}) \\ &\quad + \big[C_D^D s^{(D)} q^{(0)} + \cdots + C_D^d s^{(d)} q^{(D-d)} + \cdots + C_D^0 s^{(0)} q^{(D)}\big](\lambda_{\gamma_{D,j}}) \\ &= \big[C_D^D s^{(D)} q^{(0)}\big](\lambda_{\gamma_{D,j}}). \text{ Where each } \lambda_{\gamma_{D,j}} \in I_{D,\alpha} \subseteq I_{D-1,\alpha}. \end{aligned}$$

Hence $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$. The assumption holds for all $D \in \{0, \dots, A-1\}$.

NOTICE that $\forall k = \{0, \dots, A-2\}, s^{(k)}$ and $s^{(k+1)}$ have zeros $\{\lambda_{\gamma_{k+1,1}}, \dots, \lambda_{\gamma_{k+1,J}}\}$ in common.

Now $\forall D \in \{1, A-1\}, s = s^{(0)}, \dots, s^{(D)}$ have zeros $\{\lambda_{\gamma_{D,1}}, \dots, \lambda_{\gamma_{D,J}}\}$ in common.

Thus $\forall D \in \{0, A-1\}$, $s(z)$ is divisible by $(z - \lambda_{\gamma_{D,1}})^{\alpha_{\gamma_{D,1}}} \cdots (z - \lambda_{\gamma_{D,J}})^{\alpha_{\gamma_{D,J}}}$.

Hence we write $s(z) = \Big((z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}\Big) s_0(z)$, while $\deg s \leqslant m - 1 < m = \alpha_1 + \cdots + \alpha_M$.

Thus by TIPS, $s = 0$. Following the same pattern, we conclude that $r = 0$.

Hence $T$ is inje. And $\dim\big(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})\big) = \dim \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$ is surj. Thus $T$ is an iso. $\square$

---

**COMMENT:** We now prove the statement that marked by $(\Delta)$ above.

**L1:** *Prove that* $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+$, $(pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}$.

**SOLUTION:**

We use induction by $k \in \mathbf{N}^+$.

(i) $k = 1$. $(pq)^{(1)} = pq = C_1^1 p^{(1)} q^{(0)} + C_1^0 p^{(0)} q^{(1)}$.

(ii) $k \geqslant 2$. Assume that for $(pq)^{(k-1)} = C_{k-1}^{k-1} p^{(k-1)} q^{(0)} + \cdots + C_{k-1}^j p^{(j)} q^{(k-1-j)} + \cdots + C_{k-1}^0 p^{(0)} q^{(k-1)}$.

Now $(pq)^{(k)} = \big((pq)^{(k-1)}\big)' = \Big(\sum_{j=0}^{k-1} C_{k-1}^j p^{(j)} q^{(k-j-1)}\Big)' = \sum_{j=0}^{k-1} \Big[C_{k-1}^j \Big(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)}\Big)\Big]$.

$$\begin{aligned}
&= \Big[C_{k-1}^0 \Big(p^{(1)} q^{(k-1)} + \boxed{p^{(0)} q^{(k)}}\Big)\Big] + \Big[C_{k-1}^1 \Big(p^{(2)} q^{(k-2)} + p^{(1)} q^{(k-1)}\Big)\Big] \\
&\quad + \cdots + \Big[C_{k-1}^{j-2} \Big(\underline{p^{(j-1)} q^{(k-j+1)}} + p^{(j-2)} q^{(k-j+2)}\Big)\Big] + \Big[C_{k-1}^{j-1} \Big(\underline{\underline{p^{(j)} q^{(k-j)}}} + \underline{p^{(j-1)} q^{(k-j+1)}}\Big)\Big] \\
&\quad + \Big[C_{k-1}^{j} \Big(p^{(j+1)} q^{(k-j-1)} + \underline{\underline{p^{(j)} q^{(k-j)}}}\Big)\Big] + \Big[C_{k-1}^{j+1} \Big(p^{(j+2)} q^{(k-j-2)} + p^{(j+1)} q^{(k-j-1)}\Big)\Big] \\
&\quad + \cdots + \Big[C_{k-1}^{k-2} \Big(\underline{p^{(k-1)} q^{(1)}} + p^{(k-2)} q^{(2)}\Big)\Big] + \Big[C_{k-1}^{k-1} \Big(\boxed{p^{(k)} q^{(0)}} + \underline{p^{(k-1)} q^{(1)}}\Big)\Big].
\end{aligned}$$

Hence $(pq)^{(k)} = C_k^0 p^{(0)} q^{(k)} + \cdots + \Big[C_{k-1}^j + C_{k-1}^{j-1}\Big]\big(p^{(j)} q^{(k-j)}\big) + \cdots + C_k^k p^{(k)} q^{(0)}$. □

---

**L2:** *Suppose* $p(z) = (z-\lambda)^\alpha q(z)$ *and* $\alpha \in \mathbf{N}^+$. *Prove that* $p^{(\alpha-1)}(\lambda) = 0$.

**SOLUTION:**

Suppose $p \in \mathcal{P}(\mathbf{F})$. Write $p(z) = (z-\lambda)^A q(z)$, where $A \in \mathbf{N}^+, q(\lambda) \neq 0$.

We use induction to show that for all $\alpha \in \{1, \dots, A\}, p^{(\alpha-1)}(\lambda) = 0$.

(i) $\alpha = 1$. $p^{(0)}(\lambda) = 0$.

(ii) $2 \leqslant \alpha \leqslant A$. Assume that $p^{(a-2)}(\lambda) = 0$ for all $a \in \{1, \dots, \alpha\}$.

NOTICE that $p(z) = (z-\lambda)^{\alpha-1} q_{\alpha-1}(z) = (z-\lambda)^\alpha q_\alpha(z)$, where $q_\alpha(z) = (z-\lambda) q_{\alpha-1}(z)$.

Because $p^{(\alpha-1)}(z) = \Big[C_{\alpha-1}^{\alpha-1}(z-\lambda)^0 q_{\alpha-1}(z) + \cdots + C_{\alpha-1}^k (z-\lambda)^{\alpha-1-k} q_{\alpha-1-k}(z)$

$+ \cdots + C_{\alpha-1}^0 (z-\lambda)^{\alpha-1} {q_{\alpha-1}}^{(\alpha-1)}(z)\Big]$. Now $p^{(\alpha-1)}(\lambda) = C_{\alpha-1}^{\alpha-1} q_{\alpha-1}(\lambda) = 0$. □

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**ENDED**

# 5.A

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28
29 30 31 32 33 34 35 36 | 2E: Ch5.20 | 4E: 8 11 15 16 17 36 37 38 39

- **NOTE FOR [5.6]:**

More generally, suppose we do not know whether $V$ is finite-dim. We show that $(a) \iff (b)$.

Suppose (a) $\lambda$ is an eigval of $T$ with an eigvec $v$. Then $(T - \lambda I)v = 0$.

Hence we get (b), $(T - \lambda I)$ is not inje. And then (d), $(T - \lambda I)$ is not inv.

But (d) $\Rightarrow$ (b) fails, because $S$ is not inv $\iff$ $S$ is not inje OR $S$ is not surj.

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- **TIPS:** For $T_1, \dots, T_m \in \mathcal{L}(V)$:

(a) Suppose $T_1, \dots, T_m$ are all inje. Then $(T_1 \circ \cdots \circ T_m)$ is inje.

(b) Suppose $(T_1 \circ \cdots \circ T_m)$ is not inje. Then at least one of $T_1, \dots, T_m$ is not inje.

(c) At least one of $T_1, \dots, T_m$ is not inje $\not\Rightarrow$ $(T_1 \circ \cdots \circ T_m)$ is not inje.

**EXAMPLE:** In infinite-dim only. Let $V = \mathbf{F}^\infty$.

$\left.\begin{array}{l}\text{Let } S \text{ be the backward shift ( surj but not inje )} \\ \text{Let } T \text{ be the forward shift ( inje but not surj )}\end{array}\right\} \Rightarrow$ Then $ST = I$. □

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• **NOTE FOR [5.2]:** Suppose $T \in \mathcal{L}(V)$. Then $U$ is an invar subsp of $V$ under $T \iff \operatorname{range} T|_U \subseteq U$.

---

• *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$, and $U$ is an invar subsp of $V$ under $T$.*
*Prove that there exists an invar subsp $W$ of dimension $\dim V - dimU$.*

**SOLUTION:**

Using the NOTE FOR [3.88,90,91]. Define the eraser $S$. Now $V = \operatorname{range} S \oplus U$.

Define $E_1$ by $E_1(u+w) = u$. Define $E_2$ by $E_2(u+w) = w$. $(\ E_2 = S \circ \pi.\ )$

Note that $T - TE_1 = T(I - E_1) = TE_2$. And $\operatorname{null} TE_2 = \operatorname{null} T \oplus U,\ \operatorname{range} T = \operatorname{range} TE_2 \oplus U$.

Because $\dim \operatorname{null} TE_2 \geqslant \dim U \iff \dim \operatorname{range} TE_2 \leqslant \dim V - \dim U$.

Let $B_U = (u_1, \dots, u_n),\ B_{\operatorname{range} TE_2} = (v_1, \dots, v_m) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n, \dots, u_p)$.

Let $X = \operatorname{span}(v_1, \dots, v_m, u_{\alpha_1}, \dots, u_{\alpha_{p-\dim U}})$. Where $\alpha_1, \dots, \alpha_{p-\dim U} \in \{1, \dots, p\}$ are distinct.

Then $\dim X = \dim V - \dim U$. $\left[\ \operatorname{range} TE_2 \subseteq\ \right]$ $X$ is invar under $TE_2$, by Problem (1)(b).

We have $x \in X \Rightarrow TE_2(x) \in X \Rightarrow Tx - TE_1(x) \in X \Rightarrow Tx \in X$. Hence $X$ is invar under $T$. $\square$

$\big($ Note that $E_1(x) \in \operatorname{span}(u_{\beta_1}, \dots, u_{\beta_t})$, where $\beta_1, \dots, \beta_t \in \{\alpha_1, \dots, \alpha_{p-\dim U}\}$ and each $u_{\beta_i} \in U$. $\big)$

**COMMENT:** Conversely, by reversing the roles of $U$ and $W$, we conclude that it is true as well.

---

• *Suppose $T \in \mathcal{L}(V)$ and $U$ is an invar subsp of $V$ under $T$.*
*Suppose $\lambda_1, \dots, \lambda_m$ are the distinct eigvals of $T$ correspd eigvecs $v_1, \dots, v_m$.*

• **TIPS 1:** *Prove that $v_1 + \dots + v_m \in U \iff$ each $v_k \in U$.*

**SOLUTION:**

Suppose each $v_k \in U$. Then because $U$ is a subsp, $v_1 + \dots + v_m \in U$.

Define the statement $P(k)$ : if $v_1 + \dots + v_k \in U$, then each $v_j \in U$. We use induction on $m$.

(i) For $k = 1$, $v_1 \in U$.

(ii) For $2 \leqslant k \leqslant m$. Assume that $P(k-1)$ holds. Suppose $v = v_1 + \dots + v_k \in U$.

Then $Tv = \lambda_1 v_1 + \dots + \lambda_k v_k \in U \Longrightarrow Tv - \lambda_k v = (\lambda_1 - \lambda_k)v_1 + \dots + (\lambda_{k-1} - \lambda_k)v_{k-1} \in U$.

For each $j \in \{1, \dots, k-1\}, \lambda_j - \lambda_k \neq 0 \Rightarrow (\lambda_j - \lambda_k)v_j = v_j'$ is an eigvec of $T$ correspd $\lambda_j$.

By assumption, each $v_j' \in U$. Thus $v_1, \dots, v_{k-1} \in U$. So that $v_k = v - v_1 - \dots - v_{k-1} \in U$. $\square$

---

• **TIPS 2:** *If $\dim V = m$. Prove that $U = (U \cap E_1) \oplus \dots \oplus (U \cap E_m)$, where $E_k = \operatorname{span}(v_k)$.*

**SOLUTION:**

Because $V = E_1 \oplus \dots \oplus E_m$. $\forall u \in U, \exists!\, e_j \in E_j, u = e_1 + \dots + e_m$.

If $e_j \neq 0$, then $e_j$ is an eigvec correspd $\lambda_j$. Otherwise $e_j = 0 \in U$. By TIPS (1), each nonzero $e_j \in U$.

Thus $u \in (U \cap E_1) + \dots + (U \cap E_m) = U$. Because each $(U \cap E_j) \subseteq E_j$.

For each $k \in \{2, \dots, n\}$, $\big((U \cap E_1) + \dots + (U \cap E_{k-1})\big) \bigcap (U \cap E_k) \subseteq (E_1 + \dots + E_{k-1}) \cap E_k = \{0\}$. $\square$

---

• **TIPS 3:** *Suppose $W$ is a nonzero invar subsp of $V$ under $T$. If $\dim V = m \geqslant 1$.*
*Prove that $W = \operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ for some distinct $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$.*

**SOLUTION:**

Each $\operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ is invar under $T$.

By TIPS (2), $U = (U \cap E_1) \oplus \dots \oplus (U \cap E_m)$. Because each $\dim E_k = 1$, $U \cap E_k = \{0\}$ or $E_k$.

There must be at least one $k$ such that $E_k = U \cap E_k$, for if not, $U = \{0\}$ since $V = E_1 \oplus \dots \oplus E_m$.

Let $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$ be all the distinct indices for which $E_k = U \cap E_k$.

Thus $U = (U \cap E_1) \oplus \dots \oplus (U \cap E_m) = E_{\alpha_1} \oplus \cdots E_{\alpha_A} = \operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$. $\square$

---

**1** *Suppose $T \in \mathcal{L}(V)$ and $U$ is a subsp of $V$.*

(a) *If $U \subseteq \operatorname{null} T$, then $U$ is invar under $T$.* $\forall u \in U \subseteq \operatorname{null} T, Tu = 0 \in U$. □

(b) *If $\operatorname{range} T \subseteq U$, then $U$ is invar under $T$.* $\forall u \in U, Tu \in \operatorname{range} T \subseteq U$. □

---

• *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*

(a) *Prove that $\operatorname{null}(T - \lambda I)$ is invar under $S$ for any $\lambda \in \mathbf{F}$.*

(b) *Prove that $\operatorname{range}(T - \lambda I)$ is invar under $S$ for any $\lambda \in \mathbf{F}$.*

**SOLUTION:**

Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.

(a) $(T - \lambda I)(v) = 0 \Rightarrow (T - \lambda I)(Sv) = \big(S(T - \lambda I)\big)(v) = 0$.

(b) $(T - \lambda I)(u) = v \in \operatorname{range}(T - \lambda I) \Rightarrow Sv = \big(S(T - \lambda I)\big)(u) = (T - \lambda I)(Su) \in \operatorname{range}(T - \lambda I)$. □

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• *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*

**2** *Show that $W = \operatorname{null} T$ is invar under $S$.* $\forall u \in W, Tu = 0 \Rightarrow STu = 0 = TSu \Rightarrow Su \in W$. □

**3** *Show that $U = \operatorname{range} T$ is invar under $S$.* $\forall w \in U, \exists v \in V, Tv = w, TSv = STv = Sw \in U$. □

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• *Suppose $T \in \mathcal{L}(V)$ and $V_1, \dots, V_m$ are invar subsps of $V$ under $T$.*

**4** $\forall v_i \in V_i, Tv_i \in V_i \Rightarrow \forall v = v_1 + \cdots + v_m \in V_1 + \cdots + V_m, Tv = Tv_1 + \cdots + Tv_m \in V_1 + \cdots + V_m$. □

**5** $\forall v \in \bigcap_{i=1}^{m} V_i, Tv \in V_i, \forall i \in \{1, \dots, m\} \Rightarrow Tv \in \bigcap_{i=1}^{m} V_i$. Thus $\bigcap_{i=1}^{m} V_i$ is invar under $T$. □

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**6** *Suppose $U$ is an invar subsp of $V$ under each $T \in \mathcal{L}(V)$. Show that $U = \{0\}$ or $U = V$.*

**SOLUTION:** If $V = \{0\}$. Then we are done. Suppose $V \neq \{0\}$. We show the contrapositive:

*Suppose $U \neq \{0\}$ and $U \neq V$. Prove that $\exists T \in \mathcal{L}(V)$ such that $U$ is not invar under $T$.*

Let $W$ be such that $V = U \oplus W$. Define $T \in \mathcal{L}(V)$ by $T(u + w) = w$. □

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• **TIPS:** Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is the counterclockwise rotation by the angle $\theta \in \mathbf{R}$.

Define $\mathcal{C} \in \mathcal{L}(\mathbf{R}^2, \mathbf{C})$ by $\mathcal{C}(a, b) = a + ib = r(\cos\alpha + \mathrm{i}\sin\alpha) \Rightarrow a = r\cos\alpha, b = r\sin\alpha$, where $r = a^2 + b^2$.

Then $(\cos\theta + \mathrm{i}\sin\theta)(a + \mathrm{i}b) = r\big(\cos(\alpha + \theta) + \mathrm{i}\sin(\alpha + \theta)\big) = \mathcal{C}^{-1}T(a, b)$.

Hence $T(a, b) = \big(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta\big)$. Now $\mathcal{M}(T) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

**EXAMPLE:** OR **7** Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find all eigvals of $T$.

NOTICE that $\mathcal{M}(T) = \begin{pmatrix} \cos 90^\circ & -3\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$. By [5.8](a), we conclude that $T$ has no eigvals.

OR. Suppose $\lambda$ is an eigval with an eigvec $(x, y)$. Then $(\lambda x, \lambda y) = (-3y, x) \Rightarrow -3y = \lambda^2 y \Rightarrow \lambda^2 = -3$.

$\big[$ Ignoring the possibility of $y = 0$, because $x = 0 \Longleftrightarrow y = 0$. $\big]$ □

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**8** *Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigvals and eigvecs.*

**SOLUTION:** Suppose $\lambda$ is an eigval with an eigvec $(w, z)$. Then $z = \lambda w$ and $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, ignoring the possibility of $z = 0$ ( $z = 0 \Longleftrightarrow w = 0$ ).

Hence $\lambda_1 = -1$ and $\lambda_2 = 1$ are all the eigvals of $T$. And $T(z, z) = (z, z), T(z, -z) = (-z, z)$.

又 $\dim \mathbf{F}^2 = 2$. Thus the set of all eigvecs is $\{(z, z), (z, -z) : z \neq 0\}$. □

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**9** *Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigvals and eigvecs.*

**Solution:** Suppose $\lambda$ is an eigval with an eigvec $(z_1, z_2, z_3)$.
Then $(2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$. We discuss in two cases:
For $\lambda = 0$, $z_2 = z_3 = 0$ and $z_1$ can be arbitrary ( $z_1 \neq 0$ ).
For $\lambda \neq 0$, $z_2 = 0 = z_1$, and $z_3$ can be arbitrary ( $z_3 \neq 0$ ), then $\lambda = 5$.
The set of all eigvecs is $\{(0,0,w), (w,0,0) : w \neq 0\}$. ☐

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**10** *Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$*
(a) *Find all eigvals and eigvecs;* (b) *Find all invar subsps of $V$ under $T$.*

**Solution:**
(a) Suppose $x = (x_1, x_2, x_3, \dots, x_n)$ is an eigvec with an eigval $\lambda$.
Then $Tx = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$.
Hence $1, \dots, n$ of length $\dim \mathbf{F}^n$ are all the eigvals.
And $\{(0, \dots, 0, x_k, 0, \dots, 0) \in \mathbf{F}^n : x_k \neq 0, k = 1, \dots, n\}$ is the set of all eigvecs.
(b) Let $(e_1, \dots, e_n)$ be the std basis of $\mathbf{F}^n$. Let $V_k = \operatorname{span}(e_k)$. Then $V_1, \dots, V_n$ are invar under $T$.
Hence by Tips (3), every sum of $V_1, \dots, V_n$ is a invar subsp of $V$ under $T$. ☐

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**18** *Define the forward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ by $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$.*
*Show that T has no eigvals.*

**Solution:** Suppose $\lambda$ is an eigval of $T$ with an eigvec $(z_1, z_2, \dots)$.
Then $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$. Thus $\lambda z_1 = 0, \lambda z_k = z_{k-1}$.
If $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \cdots = z_k \Rightarrow (z_1, z_2, \dots) = 0 \Longrightarrow 0$ is not an eigval.
If $\lambda \neq 0$, then $\lambda z_1 = 0 \Rightarrow z_1 = \cdots = z_k = 0 \Longrightarrow \lambda$ is not an eigval. Now no $\lambda \in \mathbf{F}$ is an eigval. ☐

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**19** *Suppose $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.*
*In other words, the entries of $\mathcal{M}(T)$ with resp to the std basis are all 1's.*
*Find all eigvals and eigvecs of $T$.*

**Solution:**
Suppose $\lambda$ is an eigval of $T$ with an eigvec $(x_1, \dots, x_n)$.
Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.
Thus $\lambda x_1 = \cdots = \lambda x_n = x_1 + \cdots + x_n$.

$$\left.\begin{array}{r} \text{For } \lambda = 0, x_1 + \cdots + x_n = 0 \\ \text{For } \lambda \neq 0, x_1 = \cdots = x_n \Longrightarrow \lambda x_k = n x_k \end{array}\right\} \Rightarrow 0, n \text{ are the eigvals of } T.$$

And the set of all eigvecs of $T$ is $\{(x_1, \dots, x_n) \in \mathbf{F}^n \backslash \{0\} : x_1 + \cdots + x_n = 0 \vee x_1 = \cdots = x_n\}$. ☐

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**20** *Define the backward shift operator $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.*
(a) *Show that every element of $\mathbf{F}$ is an eigval of $S$;* (b) *Find all eigvecs of $S$.*

**Solution:**
Suppose $\lambda$ is an eigval of $S$ with an eigvec $(z_1, z_2, \dots)$.
Then $S(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$. Thus for each $k \in \mathbf{N}^+, \lambda z_k = z_{k+1}$.
If $\lambda = 0$, then $\lambda z_1 = z_2 = \cdots = z_k = 0$ for all $k$, while $z_1$ can be nonzero. Thus 0 is an eigval.
If $\lambda \neq 0$, then $\lambda^k z_1 = \lambda^{k-1} z_2 = \cdots = \lambda z_k = z_{k+1}$, let $z_1 \neq 0 \Longrightarrow (1, \lambda, \lambda^2, \dots, \lambda^k, \dots)$ is an eigvec.
Now each $\lambda \in \mathbf{F}$ is an eigval of $T$, with the correspd eigvecs in $\operatorname{span}((1, \lambda, \lambda^2, \dots, \lambda^k, \dots))$. ☐

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**11** *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigvals and eigvecs.*

**SOLUTION:**

Note that $\forall p \in \mathcal{P}(\mathbf{R}) \backslash \{0\}, \deg p' < \deg p$. And $\deg 0 = -\infty$. Suppose $\lambda$ is an eigval with an eigvec $p$.

Assume that $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p = p'$. Contradicts. Thus $\lambda = 0$.

Therefore $\deg \lambda p = -\infty = \deg p' \Rightarrow p \in \mathcal{P}_0(\mathbf{R})$. Hence the eigvecs are all the nonzero consts. $\square$

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**12** *Define $T \in \mathcal{L}\big(\mathcal{P}_n(\mathbf{R})\big)$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigvals and eigvecs.*

**SOLUTION:**

Suppose $\lambda$ is an eigval of $T$ with an eigvec $p$, then $(Tp)(x) = xp'(x) = \lambda p(x)$.

Let $p = a_0 + a_1 x + \cdots + a_n x^n$. Then $xp'(x) = a_1 x + 2a_2 x^2 + \cdots + na_n x^n = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n$.

Define $S \in \mathcal{L}(\mathbf{F}^{n+1}, \mathcal{P}_n(\mathbf{R}))$ by $S(a_0, a_1, \ldots, a_n) = a_0 + a_1 x + \cdots + a_n x^n$.

Then $(S^{-1}TS)(a_0, a_1, \ldots, a_n) = (0 \cdot a_0, 1 \cdot a_1, 2 \cdot a_2, \ldots, n \cdot a_n)$. Thus $0, 1, \ldots, n$ are the eigvals of $S^{-1}TS$.

By Problem (15), $0, 1, \ldots, n$ are the eigvals of $T$. The set of all eigvecs is $\{cx^\lambda : c \neq 0, \lambda = 0, 1, \ldots, n\}$. $\square$

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• *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$.*

**13** *Prove that $\forall \lambda \in \mathbf{F}, \exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}, (T - \alpha I)$ is inv.*

**SOLUTION:**

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \frac{1}{1000+k}$ for each $k = 1, \ldots, \dim V + 1$.

Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigvals.

Hence $\exists\, k = 1, \ldots, \dim V + 1$ such that $\alpha_k$ is not an eigval of $T$ and therefore $(T - \alpha_k I)$ is inv. $\square$

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• (4E 5.A.11) *Prove that $\exists\, \delta > 0$ such that $(T - \alpha I)$ is inv for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.*

**SOLUTION:**

If $T$ has no eigvals, then $(T - \alpha I)$ is inje for all $\alpha \in \mathbf{F}$ and we are done.

Suppose $\lambda_1, \ldots, \lambda_m$ are all the distinct eigvals of $T$.

Let $\delta > 0$ be such that, for each eigval $\lambda_k, \lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.

So that for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta, (T - \alpha I)$ is not inje. $\square$

OR. Let $\delta = \min\{|\lambda - \lambda_k| : k \in \{1, \ldots, m\}, \lambda_k \neq \lambda\}$.

Then $\delta > 0$ and each $\lambda_k \neq \alpha \ \big[ \iff (T - \alpha I)$ is inv $\big]$ for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$. $\square$

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• (5.B.4 OR 4E 3.B.27) *Suppose $\lambda$ is an eigval of $P \in \mathcal{L}(V), P^2 = P$. Prove that $\lambda = 0$ or $\lambda = 1$.*

**SOLUTION:** Suppose $\lambda$ is an eigval with an eigvec $v$. Then $P(Pv) = Pv \Rightarrow \lambda^2 v = \lambda v$. Thus $\lambda = 1$ or $0$. $\square$

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**14** *Suppose $V = U \oplus W$, where $U$ and $W$ are nonzero subsps of $V$.*
*Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.*
*Find all eigvals and eigvecs of $P$.*

**SOLUTION:**

Suppose $\lambda$ is an eigval of $P$ with an eigvec $(u + w)$.

Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$.

OR. Note that $P|_{\operatorname{range} P} = I|_{\operatorname{range} P} \iff P^2 = P$. By (4E 5.A.8), 1 and 0 are the eigvals.

By [1.44], $(\lambda - 1)u = \lambda w = 0$, hence $\lambda = 0 \iff u = 0$, and $\lambda = 1 \iff w = 0$.

Thus $Pu = u, Pw = 0$. Hence the eigvals are 0 and 1, the set of all eigvecs of $P$ is $U \cup W$. $\square$

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**15** *Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is inv.*

(a) *Prove that $T$ and $S^{-1}TS$ have the same eigvals.*

(b) *What is the relationship between the eigvecs of $T$ and the eigvecs of $S^{-1}TS$?*

**Solution:**

(a) $\lambda$ is an eigval of $T$ with an eigvec $v \Rightarrow S^{-1}TS(\underline{S^{-1}v}) = S^{-1}Tv = S^{-1}(\lambda v) = \underline{\lambda S^{-1}v}$.

$\lambda$ is an eigval of $S^{-1}TS$ with an eigvec $v \Rightarrow S(S^{-1}TS)v = T\underline{Sv} = \underline{\lambda Sv}$.

Or. Note that $S(S^{-1}TS)S^{-1} = T$. Hence every eigval of $S^{-1}TS$ is an eigval of $S(S^{-1}TS)S^{-1} = T$.

Or. $Tv = \lambda v \iff (TS)(u) = \lambda Su \iff (S^{-1}TS)(u) = \lambda u$. Where $v = Su$.

$(S^{-1}TS)(u) = \lambda u \iff (S^{-1}T)(v) = \lambda S^{-1}v \iff Tv = \lambda v$. Where $u = S^{-1}v$.

(b) Because $\lambda$ is an eigval of $T \iff \lambda$ is an eigval of $S^{-1}TS$.

$\big($ See [5.36]. $\big)$ Now $E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\}$; $E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}$. □

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**17** *Give an example of an operator on $\mathbf{R}^4$ that has no real eigvals.*

**Solution:**

Let $(e_1, e_2, e_3, e_4)$ be the std basis of $\mathbf{R}^4$.

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$.

Suppose $\lambda$ is an eigval of $T$ with an eigvec $(x, y, z, w)$. Then we get $\begin{cases} (1-\lambda)x + y + z + w = 0, \\ -x + (1-\lambda)y - z - w = 0, \\ 3x + 8y + (11-\lambda)z + 5w = 0, \\ 3x - 8y - 11z + (5-\lambda)w = 0. \end{cases}$

This set of linear equations has no solutions.

[ You can type it on https://zh.numberempire.com/equationsolver.php to check. ]

Or. Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Suppose $\lambda$ is an eigval of $T$ with an eigvec $(x, y, z, w)$.

Then $T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \implies \begin{cases} -y = \lambda x, x = \lambda y \implies -xy = \lambda^2 xy \\ -w = \lambda z, z = \lambda w \implies -zw = \lambda^2 zw \end{cases}$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Otherwise, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, contradicts.

Similarly, $y = z = w = 0$. Then we fail. Thus $T$ has no eigvals. □

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• (4E 5.A.16) *Suppose $B_V = (v_1, \dots, v_n)$, $T \in \mathcal{L}(V)$, $\mathcal{M}(T, (v_1, \dots, v_n)) = A$. Prove that if $\lambda$ is an eigval of $T$, then $|\lambda| \leqslant n \max\{|A_{j,k}| : 1 \leqslant j, k \leqslant n\}$.*

**Solution:**

Suppose $v$ is an eigval of $T$ correspd to $\lambda$. Let $v = c_1 v_1 + \dots + c_n v_n$.

Because $\lambda c_1 v_1 + \dots + \lambda c_n v_n = c_1 T v_1 + \dots + c_n T v_n = \sum_k^n c_k \big( \sum_j^n A_{j,k} v_j \big)$.

We have $\lambda c_j = \sum_{k=1}^n c_k A_{j,k} \implies |\lambda||c_j| = \sum_{k=1}^n |c_k||A_{j,k}|$ for each $j \in \{1, \dots, n\}$

Let $|c_j| = \max\{|c_1|, \dots, |c_n|\}$. Note that $|c_j| \neq 0$, for if not, $c_1 = \dots = c_n = 0 \Rightarrow v = 0$, contradicts.

Let $M = \max\{|A_{j,k}| : 1 \leqslant j, k \leqslant n\}$. Note that for each $j$, $\sum_{k=1}^n |A_{j,k}| \leqslant \sum_{k=1}^n M = nM$.

Thus $|\lambda||c_j| = \sum_{k=1}^n |c_k||A_{j,k}| \implies |\lambda| \leqslant \sum_{k=1}^n |A_{j,k}| \dfrac{|c_k|}{|c_j|} \leqslant \sum_{k=1}^n |A_{j,k}| \leqslant nM$. □

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• (4E 5.A.15) *Suppose $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*
*Show that $\lambda$ is an eigval of $T \iff \lambda$ is an eigval of the dual operator $T' \in \mathcal{L}(V')$.*

**Solution:**

(a) Suppose $\lambda$ is an eigval of $T$ with an eigvec $v$.
Let $U$ be invar such that $V = \operatorname{span}(v) \oplus U$ $\big[$ by (4E 5.A.39) $\big]$.
Define $\psi \in V'$ by $\psi(cv+u) = c$.
Now $\big[T'(\psi)\big](cv+u) = \psi\big(cv+Tu\big) = \lambda cv = \lambda\psi(cv+u)$. Hence $T'(\psi) = \lambda\psi$.

(b) Suppose $\lambda$ is an eigval $T'$ with an eigvec $\psi$. Then $T'(\psi) = \psi \circ T = \lambda\psi$.
Note that $\psi \neq 0, \psi(Tv) = \lambda\psi(v)$ Thus $\exists\, v \in V\backslash\{0\}, Tv = \frac{\psi(Tv)}{\psi(v)}v = \lambda v$. ☐

Or. [*Only in Finite-dim* ] Using [5.6], (4E 3.F.17), [3.101] and (3.F.12).
$\lambda$ is an eigval of $T \iff (T - \lambda I_V)$ is not inv
$\iff (T-\lambda I_V)' = T' - \lambda I_{V'}$ is not inv $\iff \lambda$ is an eigval of $T'$. ☐

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**24** *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Tx = Ax$.*
(a) *Suppose the sum of the entries in each row of $A$ equals 1. Prove that 1 is an eigval of $T$.*
(b) *Suppose the sum of the entries in each col of $A$ equals 1. Prove that 1 is an eigval of $T$.*

**Solution:**

Suppose $\lambda$ is an eigval of $T$ with an eigvec $x$. Then $Tx = Ax = \begin{pmatrix} \sum_{k=1}^{n} A_{1,k}x_k \\ \vdots \\ \sum_{k=1}^{n} A_{n,k}x_k \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

(a) Suppose $\sum_{r=1}^{n} A_{R,c} = 1$ for each $R \in \{1, \dots, n\}$.
Then if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, and hence is an eigval of $T$.

(b) Suppose $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C \in \{1, \dots, n\}$.
Then $\sum_{r=1}^{n} (Ax)_{r,\cdot} = \sum_{r=1}^{n} (Ax)_{r,1} = \sum_{c=1}^{n} (A_{1,c} + \cdots + A_{n,c})x_c = \sum_{c=1}^{n} x_c = \lambda(x_1 + \cdots + x_n)$.
Hence $\lambda = 1$ for all $x \in \mathbf{F}^{n,1}$ such that $\sum_{c=1}^{n} x_{c,1} \neq 0$. ☐

Or. We show that $(T - I)$ is not inv, so that $\lambda = 1$ is an eigval.
Because $(T-I)x = (A-I)x = \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^{n} A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.
Then $y_1 + \cdots + y_n = \sum_{r=1}^{n}\sum_{c=1}^{n} (A_{r,c}x_c - x_r) = \sum_{c=1}^{n} x_c \sum_{r=1}^{n} A_{r,c} - \sum_{r=1}^{n} x_r = 0$.
Thus range $(T-I) \subseteq \left\{ \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}^t \in \mathbf{F}^{n,1} : y_1 + \cdots + y_n = 0 \right\}$. Hence $(T-I)$ is not surj. ☐

Or. Let $(e_1, \dots, e_n)$ be the std basis of $\mathbf{F}^{n,1}$. Define $\psi \in (\mathbf{F}^{n,1})'$ by $\psi(e_k) = 1$.
Thus $\big(\psi \circ (T-I)\big)(e_k) = \psi\Big(\big(\sum_{j=1}^{n} A_{j,k}e_j\big) - e_k\Big) = \big(\sum_{j=1}^{n} A_{j,k}\big) - 1 = 0$.
Which means that $\psi \circ (T-I) = 0$. 又 $\psi \neq 0$. Hence $(T-I)$ is not inje. ☐

Or. Define $S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Sx = A^t x$. Because the rows of $A^t$ are the cols of $A$.
Now by (a), 1 is an eigval of $S$. Let $(\varphi_1, \dots, \varphi_n)$ be the dual basis of $(e_1, \dots, e_n)$.
Define $\Phi \in \mathcal{L}\Big(\mathbf{F}^{n,1}, (\mathbf{F}^{1,n})'\Big)$ by $\Phi(e_k) = \varphi_k$. Note that $\mathcal{M}(T') = A^t$.
Now $(\Phi^{-1}T'\Phi)(e_k) = (\Phi^{-1}T')(\varphi_k) = \Phi^{-1}\big(\sum_{j=1}^{n} A_{k,j}\varphi_j\big) = \sum_{j=1}^{n} A_{k,j}e_j = A^t e_k = Se_k$.
Thus 1 is an eigval of $S = \Phi^{-1}T'\Phi$, so of $T'$, $\big[$ by Problem (15) $\big]$, so of $T$, $\big[$ by (4E 5.A.15) $\big]$. ☐

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• *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Tx = xA$.*
 (a) *Suppose the sum of the entries in each col of $A$ equals 1. Prove that 1 is an eigval of $T$.*
 (b) *Suppose the sum of the entries in each row of $A$ equals 1. Prove that 1 is an eigval of $T$.*

**SOLUTION:**

Suppose $\lambda$ is an eigval with an eigvec $x$. Then $\left(\sum_{r=1}^{n} x_r A_{r,1} \quad \cdots \quad \sum_{r=1}^{n} x_r A_{r,n}\right) = \lambda\left(x_1 \quad \cdots \quad x_n\right)$.

(a) Suppose $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C \in \{1, \dots, n\}$.
 Thus if $x_1 = \cdots = x_n$, then $\lambda = 1$, hence is an eigval of $T$.

(b) Suppose $\sum_{c=1}^{n} A_{R,c} = 1$ for each $R \in \{1, \dots, n\}$.
 Thus $\sum_{c=1}^{n} (xA)_{\cdot,c} = \sum_{c=1}^{n} (A_{c,1} + \cdots + A_{c,n}) x_c = \sum_{c=1}^{n} x_c = \lambda(x_1 + \cdots + x_n)$.
 Hence $\lambda = 1$, for all $x$ such that $\sum_{r=1}^{n} x_{1,r} \neq 0$. ☐

OR. We show that $(T - I)$ is not inv, so that $\lambda = 1$ is an eigval.
Because $(T - I)x = x(A - \mathcal{M}(I)) = \left(\sum_{c=1}^{n} x_c A_{c,1} - x_1 \quad \cdots \quad \sum_{c=1}^{n} x_c A_{c,n} - x_n\right) = \left(y_1 \quad \cdots \quad y_n\right)$.
Then $y_1 + \cdots + y_n = \sum_{c=1}^{n} \sum_{r=1}^{n} (x_r A_{r,c} - x_c) = \sum_{r=1}^{n} x_r \sum_{c=1}^{n} A_{r,c} - \sum_{c=1}^{n} x_c = 0$.
Thus range $(T - I) \subseteq \left\{ \left(y_1 \quad \cdots \quad y_n\right) \in \mathbf{F}^{1,n} : y_1 + \cdots + y_n = 0 \right\}$. Hence $(T - I)$ is not surj. ☐

OR. Let $(e_1, \dots, e_n)$ be the std basis of $\mathbf{F}^{1,n}$. Define $\psi \in (\mathbf{F}^{n,1})'$ by $\psi(e_k) = 1$.
Because $Te_k = e_k A = \left(A_{k,1} \quad \cdots \quad A_{k,n}\right) = \sum_{i=1}^{n} A_{k,i} e_i$. **COROLLARY:** $\mathcal{M}(T) = A^t$.
$(\psi \circ (T - I))(e_k) = \left(\sum_{i=1}^{n} A_{k,i}\right) - 1 = 0$. Then $\psi \circ (T - I) = 0$. 又 $\psi \neq 0$. $(T - I)$ is not inje. ☐

OR. Define $S \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Sx = xA^t$. Because the rows of $A$ are the cols of $A^t$.
Now by (a), 1 is an eigval of $S$. Let $(\varphi_1, \dots, \varphi_n)$ be the dual basis of $(e_1, \dots, e_n)$.
Define $\Phi \in \mathcal{L}\left(\mathbf{F}^{1,n}, (\mathbf{F}^{1,n})'\right)$ by $\Phi(e_k) = \varphi_k$. Because $[T'(\varphi_k)](e_j) = \varphi_k\left(\sum_{i=1}^{n} A_{j,i} e_i\right) = A_{j,k}$.
By (3.F.9), $T'(\varphi_k) = \sum_{j=1}^{n} A_{j,k} \varphi_j$. **COROLLARY:** $\mathcal{M}(T') = A = \mathcal{M}(T)^t$. FIXME: $\mathcal{M}(T) e_k = A^t e_k = e_k A$
Now $(\Phi^{-1} T' \Phi)(e_k) = (\Phi^{-1} T')(\varphi_k) = \Phi^{-1}\left(\sum_{j=1}^{n} A_{j,k} \varphi_j\right) = \sum_{j=1}^{n} A_{j,k} e_j = e_k A^t = Se_k$.
Thus 1 is an eigval of $S = \Phi^{-1} T' \Phi$, so of $T'$, [ by Problem (15) ], so of $T$, [ by (4E 5.A.15) ]. ☐

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• *Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.*
 (a) [OR (9.11)] *$\lambda \in \mathbf{R}$. Prove that $\lambda$ is an eigval of $T \iff \lambda$ is an eigval of $T_{\mathrm{C}}$.*
 (b) [OR **16** OR [9.16] ] *$\lambda \in \mathbf{C}$. Prove that $\lambda$ is an eigval of $T_{\mathrm{C}} \iff \overline{\lambda}$ is an eigval of $T_{\mathrm{C}}$.*

**SOLUTION:**

(a) Suppose $\lambda$ is an eigval of $T$ with an eigvec $v$.
 Then $Tv = \lambda v \Longrightarrow T_{\mathrm{C}}(v + \mathrm{i}0) = Tv + \mathrm{i}T0 = \lambda v$. Thus $\lambda$ is an eigval of $T_{\mathrm{C}}$.
 Suppose $\lambda$ is an eigval of $T_{\mathrm{C}}$ with an eigvec $v + \mathrm{i}u$.
 Then $T_{\mathrm{C}}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \Longrightarrow Tv = \lambda v, Tu = \lambda u$. Thus $\lambda$ is an eigval of $T$.
 ( Note that $v + \mathrm{i}u$ is nonzero $\iff$ at least one of $v, u$ is nonzero ).

(b) Suppose $\lambda$ is an eigval of $T_{\mathrm{C}}$ with an eigvec $v + \mathrm{i}u$. Then $T_{\mathrm{C}}(v + \mathrm{i}u) = Tv + \mathrm{i}Tu = \lambda(v + \mathrm{i}u)$.
 Note that $\overline{T_{\mathrm{C}}(v + \mathrm{i}u)} = \overline{Tv + \mathrm{i}Tu} = Tv - \mathrm{i}Tu = T_{\mathrm{C}}(v - \mathrm{i}u) = T_{\mathrm{C}}(\overline{v + \mathrm{i}u})$.
 And that $\overline{\lambda(v + \mathrm{i}u)} = \overline{\lambda}v - \mathrm{i}\overline{\lambda}u = \overline{\lambda}(v - \mathrm{i}u) = \overline{\lambda}(\overline{v + \mathrm{i}u})$.
 Hence $\overline{\lambda}$ is an eigval of $T_{\mathrm{C}}$. To prove the other direction, notice that $\overline{\overline{\lambda}} = \lambda$. ☐

OR. Suppose $\lambda = a + \mathrm{i}b$ is an eigval of $T_{\mathrm{C}}$ with an eigvec $v + \mathrm{i}u$.
Because $T_{\mathrm{C}}(v + \mathrm{i}u) = \lambda(v + \mathrm{i}u) = (av - bu) + \mathrm{i}(au + bv) = Tv + \mathrm{i}Tu \Longrightarrow Tv = av - bu, Tu = au + bv$.
Now $T_{\mathrm{C}}(\overline{v + \mathrm{i}u}) = Tv - \mathrm{i}Tu = (av - bu) - \mathrm{i}(au + bv) = (a - \mathrm{i}b)(v - \mathrm{i}u) = \overline{\lambda}(\overline{v - \mathrm{i}u})$. Similarly. ☐

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**21** *Suppose $T \in \mathcal{L}(V)$ is inv.*

(a) *Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that $\lambda$ is an eigval of $T \iff \lambda^{-1}$ is an eigval of $T^{-1}$.*

(b) *Prove that $T$ and $T^{-1}$ have the same eigvecs.*

**SOLUTION:** (a) $Tv = \lambda v \iff v = \lambda T^{-1} v \iff \lambda^{-1} v = T^{-1} v$. Where $v \neq 0$.

(b) NOTICE that $T$ is inv $\Longrightarrow 0$ is not an eigval of $T$ or $T^{-1}$. By (a), immediately. $\square$

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**22** *Suppose $T \in \mathcal{L}(V)$ and $\exists$ nonzero vecs $u, w$ in $V$ such that $Tu = 3w,\ Tw = 3u$. Prove that $3$ or $-3$ is an eigval of $T$.*

**SOLUTION:** $T(u+w) = 3(u+w),\ T(u-w) = 3(w-u) = -3(u-w)$. Note that $u - w \neq 0$ or $u + w \neq 0$.

OR. $T(Tu) = 9u \Rightarrow T^2 - 9 = (T-3I)(T+3I)$ is not injective $\Rightarrow 3$ or $-3$ is an eigval. $\square$

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**23** *Suppose $S, T \in \mathcal{L}(V)$. Prove that $ST$ and $TS$ have the same eigvals.*

**SOLUTION:** Suppose $\lambda$ is an eigval of $ST$ with an eigvec $v$. Then $T(STv) = \lambda Tv = TS(Tv)$.

If $Tv = 0$ ( while $v \neq 0$ ), then $T$ is not inje $\Rightarrow (TS - 0I)$ and $(ST - 0I)$ are not inje.

Thus $\lambda = 0$ is an eigval of $ST$ and $TS$ with the same eigvec $v$.

Otherwise, $Tv \neq 0$, then $\lambda$ is an eigval of $TS$. Reversing the roles of $T$ and $S$. $\square$

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• (2E 20) *Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigvals and $S \in \mathcal{L}(V)$ has the same eigvecs ( but might not with the same eigvals ). Prove that $ST = TS$.*

**SOLUTION:** Let $n = \dim V$. For each $j \in \{1, \dots, n\}$, let $v_j$ be an eigvec with eigval $\lambda_j$ of $T$ and $\alpha_j$ of $S$.

Then $B_V = (v_1, \dots, v_n)$. Because $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$ for each $j$. Hence $ST = TS$. $\square$

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• (4E 5.A.37) *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.*
*Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(S) = TS$ for each $S \in \mathcal{L}(V)$.*
*Prove that the set of eigvals of $T$ equals the set of eigvals of $\mathcal{A}$.*

**SOLUTION:**

(a) Suppose $\lambda$ is an eigval of $T$ with an eigvec $v = v_1$. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.

Note that $\operatorname{span}(v) \subseteq \operatorname{null}(T - \lambda I)$. Define $S \in \mathcal{L}(V)$ by $S(v_j) = v$ for each $j \in \{1, \dots, n\}$.

OR. Define $S \in \mathcal{L}(V)$ by $Sv_1 = v_1,\ Sv_j = 0$ for $j \geqslant 2$. Then $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$.

Then $(T - \lambda I)S = 0$. Thus $\mathcal{A}(S) = TS = \lambda S$ while $S \neq 0$. Hence $\lambda$ is an eigval of $\mathcal{A}$.

(b) Suppose $\lambda$ is an eigval of $\mathcal{A}$ with an eigvec $S$.

Then $\exists\, v \in V, 0 \neq u = S(v) \in V \Rightarrow Tu = (TS)v = (\lambda S)v = \lambda u$. Thus $\lambda$ is an eigval $T$.

OR. Because $TS - \lambda S = (T - \lambda I)S = 0 \Rightarrow \{0\} \subsetneq \operatorname{range} S \subseteq \operatorname{null}(T - \lambda I)$. $(T - \lambda I)$ is not inje. $\square$

**COMMENT:** If $\mathcal{A}(S) = ST, \forall S \in \mathcal{L}(V)$. Then the eigvals of $\mathcal{A}$ are not the eigvals of $T$.

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**25** *Suppose $T \in \mathcal{L}(V)$ and $u, w$ are eigvecs of $T$ such that $u + w$ is also an eigvec of $T$. Prove that $u$ and $w$ correspd to the same eigval.*

**SOLUTION:** Suppose $\lambda_1, \lambda_2, \lambda_0$ are eigvals of $T$ with eigvecs to $u, w, u + w$ respectively.

Then $T(u+w) = \lambda_0(u+w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$.

If $(u, w)$ is linely depe, then let $w = cu$, therefore $\lambda_2 cu = Tw = cTu = \lambda_1 cu \Rightarrow \lambda_2 = \lambda_1$.

Otherwise, $(u, w)$ is linely inde. Then $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \ \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. $\square$

OR. Assume that $\lambda_1 \neq \lambda_2$. Then $(u, w)$ is linely inde. Thus $\lambda_0 - \lambda_1 = \lambda_0 - \lambda_2$. Contradicts. $\square$

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**26** *Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vec in $V$ is an eigvec of $T$.*
*Prove that $T$ is a scalar multi of the identity operator.*

**Solution:** If $\dim V = 0, 1$ then we are done. Suppose $\dim V \geqslant 2$.

Because $\forall v \in V, \exists!\lambda_v \in \mathbf{F}, Tv = \lambda_v v$. For any two distinct nonzero vecs $v, w \in V$,
$T(v+w) = \lambda_{v+w}(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$. □

Or. For any two nonzero vecs $u, v \in V$, $u, v$ are eigvecs.
If $u + v \neq 0$, then $u + v$ is also an eigvec. Otherwise, $u + v = 0$, then $Tu = -Tv = \lambda u = -\lambda v$.
Thus by Problem (25), $\forall u, v \in V, Tu = \lambda u, Tv = \lambda v \Rightarrow \forall v \in V, Tv = \lambda v$. □

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**27, 28** *Suppose $V$ is finite-dim and $k \in \{1, \dots, \dim V - 1\}$.*
*Suppose $T \in \mathcal{L}(V)$ is such that every subsp of $V$ of dim $k$ is invar under $T$.*
*Prove that $T$ is a scalar multi of the identity operator.*

**Solution:** If $\dim V \leqslant 1$ then we are done. Suppose $\dim V \geqslant 2$.

We prove the contrapositive: *If $T$ is not a scalar multi of $I$. Then $\exists$ subsp $U$ of dim $k$ not invar under $T$.*
By Problem (26), $\exists\, v \in V$ and $v \neq 0$ such that $v$ is not an eigvec of $T$.
Thus $(v, Tv)$ is linely inde. Extend to $B_V = (v, Tv, u_1, \dots, u_n)$.
Let $U = \operatorname{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$ is not an invar subsp of $V$ under $T$. □

Or. Suppose $0 \neq v = v_1 \in V$. Extend to $B_V = (v_1, \dots, v_n)$. Suppose $Tv_1 = c_1 v_1 + \dots + c_n v_n, \exists!\, c_i \in \mathbf{F}$.
Consider a $k$-dim subsp $U = \operatorname{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$. Where $\alpha_1, \dots, \alpha_{k-1} \in \{2, \dots, n\}$ are distinct.
Because every subsp such $U$ is invar. $Tv_1 = c_1 v_1 + \dots + c_n v_n \in U \Longrightarrow c_2 = \dots = c_n = 0$.
For if not, $\exists\, c_i \neq 0$, let $W = \operatorname{span}(v_1, v_{\beta_1}, \dots, v_{\beta_{k-1}})$, where each $\beta_j \in \{2, \dots, i-1, i+1, \dots, n\}$.
Hence $Tv_1 = c_1 v_1$. Because $v_1 = v \in V$ is arbitrary. We conclude that $T = \lambda I$ for some $\lambda \in \mathbf{F}$. □

Or. For each $k \in \{1, \dots, \dim V - 1\}$, define $P(k)$ : if every subsp of dim $k$ is invar, then $T = \lambda I$.
(i) If every subsp of dim 1 is invar, then by Problem (26), $T = \lambda I$. Thus $P(1)$ holds.
(ii) Assume that $P(k)$ holds for $k \in \{1, \dots, \dim V - 1\}$. And every subsp of dim $k+1$ is invar.
Let $U$ be a subsp of dim $k$. If $\dim U = \dim V - 1$ then extend $B_U$ to $B_V$ and we are done.
Suppose $\dim U \in \{1, \dots, \dim V - 2\}$. Choose two linely inde vecs $v, w \notin U$.
Because $U \oplus \operatorname{span}(v)$ and $U \oplus \operatorname{span}(w)$ of dim $k+1$ are invar.
Suppose $u \in U$. Let $Tu = a_1 u_1 + bv = a_2 u_2 + cw, \exists!\, u_1, u_2 \in U, a_1, a_2, b, c \in \mathbf{F}$.
Now $a_1 u_1 - a_2 u_2 = cw - bv \in U \cap \operatorname{span}(v) = \{0\} \Rightarrow b = c = 0$. Thus $Tu \in U$.
Because $P(k)$ holds, we conclude that $T = \lambda I$. Thus $P(k+1)$ holds. □

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**29** *Suppose $T \in \mathcal{L}(V)$ and* range $T$ *is finite-dim.*
*Prove that $T$ has at most* $1 + \dim \operatorname{range} T$ *distinct eigvals.*

**Solution:**

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigvals of $T$ with correspd eigvecs $v_1, \dots, v_m$.
( Because range $T$ is finite-dim. The correspd eigvals are finite. )
Then $(v_1, \dots, v_m)$ linely inde $\Longrightarrow (\lambda_1 v_1, \dots, \lambda_m v_m)$ linely inde, if each $\lambda_k \neq 0$.
Otherwise, $\exists!\lambda_k = 0$. Now $(\lambda_1 v_1, \dots, \lambda_{k-1} v_{k-1}, \lambda_{k+1} v_{k+1}, \dots, \lambda_m v_m)$ is linely inde.
Hence, by [2.23], $m - 1 \leqslant \dim \operatorname{range} T$. □

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**30** *Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigvals. Prove that $\exists\, x, Tx - 9x = (-4, 5, \sqrt{7})$.*

**Solution:** $T$ has $\dim \mathbf{R}^3$ eigvals not including $9 \Rightarrow (T - 9I)$ is inv. $x = (T - 9I)^{-1}(-4, 5, \sqrt{7})$. □

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**31** *Suppose $V$ is finite-dim, and $v_1, \dots, v_m \in V$. Prove that*
*$(v_1, \dots, v_m)$ is linely inde $\Longleftrightarrow$ $v_1, \dots, v_m$ are eigvecs of some $T$ correspd to distinct eigvals.*

**SOLUTION:** Suppose $(v_1, \dots, v_m)$ is linely inde. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.
Define $T \in \mathcal{L}(V)$ by $Tv_k = k \cdot v_k$ for each $k \in \{1, \dots, m, \dots, n\}$. Conversely by [5.10]. $\square$

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• *Suppose $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ are distinct.*
(a) **32** *Prove that $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linely inde in $\mathbf{R}^{\mathbf{R}}$.*
***HINT***: *Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$. Define $D \in \mathcal{L}(V)$ by $Df = f'$. Find eigvals and eigvecs of $D$.*
(b) [4E 36] *Show that $\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$ is linely inde in $\mathbf{R}^{\mathbf{R}}$.*

**SOLUTION:**
(a) Define $V$ and $D \in \mathcal{L}(V)$ as in HINT. Then because for each $k$, $D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.
Thus $\lambda_1, \dots, \lambda_n$ are distinct eigvals of $D$. By [5.10], $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linely inde in $\mathbf{R}^{\mathbf{R}}$. $\square$
(b) Let $V = \operatorname{span}\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.
Then because $D\big(\cos(\lambda_k x)\big) = -\lambda_k \sin(\lambda_k x)$. 又 $D\big(\sin(\lambda_k x)\big) = \lambda_k \cos(\lambda_k x)$.
Thus $D^2\big(\cos(\lambda_k x)\big) = -\lambda_k^2 \cos(\lambda_k x)$.
Notice that $\lambda_1, \dots, \lambda_n$ are distinct $\Longrightarrow -\lambda_1^2, \dots, -\lambda_n^2$ are distinct. And $\dim V = n$.
Hence $-\lambda_1^2, \dots, -\lambda_n^2$ are all the eigvals of $D^2$ with correspd eigvecs $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$.
And then $\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$ is linely inde in $\mathbf{R}^{\mathbf{R}}$. $\square$

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**33** *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{range} T) = 0$.*

**SOLUTION:** $v + \operatorname{range} T \in V/\operatorname{range} T \Longrightarrow v + \operatorname{range} T \in \operatorname{null}\big(T/(\operatorname{range} T)\big)$. Hence $T/(\operatorname{range} T) = 0$. $\square$

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**34** *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{null} T)$ is inje $\Longleftrightarrow$ $(\operatorname{null} T) \bigcap (\operatorname{range} T) = \{0\}$.*

**SOLUTION:** NOTICE that $\big(T/(\operatorname{null} T)\big)(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0 \Longleftrightarrow Tu \in (\operatorname{null} T) \bigcap (\operatorname{range} T)$.
Now $T/(\operatorname{null} T)$ is inje $\Longleftrightarrow u + \operatorname{null} T = 0 \Longleftrightarrow Tu = 0 \Longleftrightarrow (\operatorname{null} T) \bigcap (\operatorname{range} T) = \{0\}$. $\square$

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• *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$, and $U$ is an invar subsp of $V$ under $T$.*
*Define $T/U : V/U \to V/U$ by $(T/U)(v + U) = Tv + U$ for each $v \in V$.*
(a) *Show that $T/U$ is well-defined and is linear.* *Requires that $U$ is invar under $T$.*
(b) [OR **35**] *Show that each eigval of $T/U$ is an eigval of $T$.*

**SOLUTION:**
(a) $v + U = w + U \Longleftrightarrow v - w \in U \Longrightarrow T(v - w) \in U \Longleftrightarrow Tv + U = Tw + U$.
Hence $T/U$ is well-defined. Now we show that $T/U$ is linear.
$(T/U)\big((v + U) + \lambda(w + U)\big) = T(v + \lambda w) + U = (T/U)(v + U) + \lambda(T/U)(w)$. Checked.
(b) Suppose $\lambda$ is an eigval of $T/U$ with an eigvec $v + U$. Then $Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v = u \in U$.
If $u = 0 \Rightarrow Tv = \lambda v$, then we are done. Otherwise, we discuss in two cases.
If $(T - \lambda I)|_U$ is inv. Then $\exists !\, w \in U, (T - \lambda I)(w) = u = (T - \lambda I)v \Rightarrow T(v + w) = \lambda(v + w)$.
Note that $v + w \neq 0$, for if not, $v \in U \Rightarrow v + U = 0$, contradicts. Thus $\lambda$ is an eigval of $T$.
If $(T - \lambda I)|_U$ is not inv. Then because $V$ is finite-dim, $(T - \lambda I)|_U$ is not inje,
so that $\exists\, w \in \operatorname{null}(T - \lambda I)|_U, w \neq 0, (T - \lambda I)w = 0 \Rightarrow Tw = \lambda w$. $\square$

OR. Let $B_U = (u_1, \dots, u_m)$. Then $\big((T - \lambda I)v, (T - \lambda I)u_1, \dots, (T - \lambda I)u_m\big)$ is linely inde in $U$.
So that $a_0(T - \lambda I)v + a_1(T - \lambda I)u_1 + \cdots + a_m(T - \lambda I)u_m = 0, \exists\, a_0, a_1, \dots, a_m \in \mathbf{F}$ with some $a_i \neq 0$.
Let $w = a_0 v + a_1 u_1 + \cdots + a_m u_m \Longrightarrow Tw = \lambda w$. Note that $w \neq 0$, for if not, $a_0 v \in U$, each $a_i = 0$. $\square$

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**36** *Prove or give a counterexample: The result in Exercise 35 is still true if $V$ is infinite-dim.*

**SOLUTION:** A counterexample:

Consider $V = \{f \in \mathbf{R}^{\mathbf{R}} : \exists!\, m \in \mathbf{N}, f \in \operatorname{span}(1, e^x, \dots, e^{mx})\}$. Note that $V$ is infinite-dim.

And a subsp $U = \{f \in \mathbf{R}^{\mathbf{R}} : \exists!\, m \in \mathbf{N}^+, f \in \operatorname{span}(e^x, \dots, e^{mx})\}$.

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then $\operatorname{range} T = U$ is invar under $T$.

Consider $(T/U)(1+U) = e^x + U = 0 \Longrightarrow 0$ is an eigval of $T/U$ but is not an eigval of $T$.

$\big[$ $\operatorname{null} T = \{0\}$, for if not, $\exists f \in V \backslash \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, contradicts. $\big]$ $\square$

---

• (4E 5.A.39) *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.*
*Prove that $T$ has an eigval $\Longleftrightarrow \exists$ an invar subsp $U$ under $T$ of dimension* $\dim V - 1$.

**SOLUTION:**

(a) Suppose $\lambda$ is an eigval of $T$ with an eigvec $v$. ( If $\dim V = 1$, then $U = \{0\}$ and we are done. )

Extend $v_1 = v$ to $B_V = (v_1, v_2 \dots, v_n)$.

**Step 1.** If $\exists\, w_1 \in \operatorname{span}(v_2, \dots, v_n)$ such that $0 \neq Tw_1 \in \operatorname{span}(v_1)$.

Then extend $w_1 = \alpha_{1,2}$ to a basis of $\operatorname{span}(v_2, \dots, v_n)$ as $(\alpha_{1,2}, \dots, \alpha_{1,n})$.

Otherwise, we stop at step 1.

**Step 2.** If $\exists\, w_2 \in \operatorname{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$ such that $0 \neq Tw_2 \in \operatorname{span}(v_1, w_1)$.

Then extend $w_2 = \alpha_{2,3}$ to a basis of $\operatorname{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$ as $(\alpha_{2,3}, \dots, \alpha_{2,n})$.

Otherwise, we stop at step 2.

**Step k.** If $\exists\, w_k \in \operatorname{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$ such that $0 \neq Tw_k \in \operatorname{span}(v_1, w_1, \dots, w_{k-1})$,

Then extend $w_k = \alpha_{k,k+1}$ to a basis of $\operatorname{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$ as $(\alpha_{k,k+1}, \dots, \alpha_{k,n})$.

Otherwise, we stop at step $k$.

Finally, we stop at step $m$, thus we get $(v_1, w_1, \dots, w_{m-1})$ and $(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})$,

$\operatorname{range} T|_{\operatorname{span}(w_1,\dots,w_{m-1})} = \operatorname{span}(v_1, w_1, \dots, w_{m-2}) \Rightarrow \dim \operatorname{null} T|_{\operatorname{span}(w_1,\dots,w_{m-1})} = 0$,

$\underbrace{\operatorname{span}(v_1, w_1, \dots, w_{m-1})}_{\dim m}$ and $\underbrace{\operatorname{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})}_{\dim (n-m)}$ are invar under $T$.

Let $U = \operatorname{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n}) \oplus \operatorname{span}(v_1, w_1, \dots, w_{m-2})$ and we are done. $\square$

**COMMENT:** Both $\operatorname{span}(v_2, \dots, v_n)$ and $U \oplus \operatorname{span}(w_{m-1})$ are in $\mathcal{S}_V \operatorname{span}(v_1)$.

*If $T|_U$ is inv, then by the similar algorithm, we can extend $U$ to an invar subsp.*

OR. Note that $\dim \operatorname{null}(T - \lambda I) \geqslant 1$. And $\dim \operatorname{range}(T - \lambda I) \leqslant \dim V - 1$.

Let $B_{\operatorname{range}(T-\lambda I)} = (w_1, \dots, w_m)$, $B_V = (w_1, \dots, w_m, u_1, \dots, u_n)$.

If $m = \dim V - 1$. $\big[ \Longleftrightarrow n = 0. \big]$ Then $\operatorname{range}(T - \lambda I)$ is an invar subsp of dim $\dim V - 1$.

Otherwise, choose $k \in \{1, \dots, n\}$ and then let $U = \operatorname{span}(w_1, \dots, w_m, u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n)$.

By Problem (1)(b), $U$ is invar under $(T - \lambda I)$. Now $u \in U \Rightarrow (T - \lambda I)(u) \in U \Rightarrow Tu \in U$.

(b) Suppose $U$ is an invar subsp under $T$ of dim $m = \dim V - 1$. ( If $m = 0$, then we are done. )

Let $B_U = (u_1, \dots, u_m), B_V = (u_0, u_1, \dots, u_m)$. We discuss in cases:

(I) If $Tu_0 \in U$, then $\operatorname{range} T = U$ so that $T$ is not surj $\Longleftrightarrow \operatorname{null} T \neq \{0\} \Longleftrightarrow 0$ is an eigval of $T$.

(II) If $Tu_0 \notin U$, then $Tu_0 = a_0 u_0 + a_1 u_1 + \dots + a_m u_m$.

If $\operatorname{range} T|_U = U \Longleftrightarrow a_1 = \dots = a_m = 0 \Longleftrightarrow Tu_0 \in \operatorname{span}(u_0)$ then we are done.

Otherwise, $T|_U : U \to U$ is not surj, so is not inje. Thus 0 is an eigval of $T|_U$, so of $T$. $\square$

OR. Consider $T/U \in \mathcal{L}(V/U)$. Because $\dim V/U = 1$. $\exists \lambda \in \mathbf{F}, T/U = \lambda I$. By Problem (35). $\square$

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**ENDED**

# 5.B: I [ *See 5.B: II below.* ]

**Comment:** 下面，为了照顾原书 5.B 节两版过大的差距，特别将此节补注分成 I 和 II 两部分。
又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本征值」
(相当一部分是从原第 3 版 8.C 节挪过来的) 是对原第 3 版「多项式作用于算子」与
「本征值的存在性」(也即第 3 版 5.B 前半部分) 的极大扩充，这一扩充也大大改变了
原第 3 版后半部分的「上三角矩阵」这一小节，故而将第 4 版 5.B 节放在第 3 版前面。

I 部分除了覆盖第 4 版 5.B 节全部和第 3 版 5.B 节前半部分与之相关的所有习题，
还会覆盖第 4 版 5.A 节末。
II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节，还会覆盖第 4 版 5.C 节；
并且，下面 5.C 还会覆盖第 4 版 5.D 节。

「注：[8.40] Or (4E 5.22) —— mini poly;
[8.44,8.45] Or (4E 5.25,5.26) —— how to find the mini poly;
[8.49] Or (4E 5.27) —— eigvals are the zeros of the mini poly;
[8.46] Or (4E 5.29) —— $q(T) = 0 \Leftrightarrow q$ is a poly multi of the mini poly.」

1 2 3 5 6 7 8 10 11 12 13 18 19 | 2E: Ch5.24
4E: 5.A.32 5.A.33 3 7 8 9 10 11 12 13 14 15
16 17 18 19 20 21 22 23 24 25 26 27 28 29

• (4E 5.A.33) *Suppose $T \in \mathcal{L}(V)$ and $m$ is a positive integer.*
(a) *Prove that $T$ is inje $\Longleftrightarrow T^m$ is inje.*
(b) *Prove that $T$ is surj $\Longleftrightarrow T^m$ is surj.*

**Solution:**
(a) Suppose $T^m$ is inje. Then $Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$.
Suppose $T$ is inje. Then $T^m v = T^{m-1}v = \cdots = T^2 v = Tv = v = 0$.
(b) Suppose $T^m$ is surj. $\forall u \in V, \exists v \in V, T^m v = u = Tw$, let $w = T^{m-1}v$.
Suppose $T$ is surj. Then $\forall u \in V, \exists v_1, \ldots, v_m \in V, T(v_1) = T^2 v_2 = \cdots = T^m v_m = u$. □

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• **Note For [5.17]:**
*Suppose $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{F})$. Prove that* $\operatorname{null} p(T)$ *and* $\operatorname{range} p(T)$ *are invar under $T$.*
**Solution:** Using the commutativity in [5.10].
(a) Suppose $u \in \operatorname{null} p(T)$. Then $p(T)u = 0$.
Thus $p(T)(Tu) = (p(T)T)u(Tp(T))u = T(p(T)u) = 0$. Hence $Tu \in \operatorname{null} p(T)$. □
(b) Suppose $u \in \operatorname{range} p(T)$. Then $\exists v \in V$ such that $u = p(T)v$.
Thus $Tu = T(p(T)v) = p(T)(Tv) \in \operatorname{range} p(T)$. □

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• **Note For [5.21]:** *Every operator on a finite-dim nonzero complex vecsp has an eigval.*
Suppose $V$ is a finite-dim complex vecsp of dim $n > 0$ and $T \in \mathcal{L}(V)$.
Choose a nonzero $v \in V$. $(v, Tv, T^2 v, \ldots, T^n v)$ of length $n+1$ is linely depe.
Suppose $a_0 I + a_1 T + \cdots + a_n T^n = 0$. Then $\exists a_j \neq 0$.
*Thus $\exists$ nonconst $p$ of smallest degree* ( $\deg p > 0$ ) *such that* $p(T)v = 0$.
Because $\exists \lambda \in \mathbf{C}$ such that $p(\lambda) = 0 \Rightarrow \exists q \in \mathcal{P}(\mathbf{C}), p(z) = (z - \lambda)q(z), \forall z \in \mathbf{C}$.
Thus $0 = p(T)v = (T - \lambda I)(q(T)v)$. By the minimality of $\deg p$ and $\deg q < \deg p$, $q(T)v \neq 0$.
Then $(T - \lambda I)$ is not inje. Thus $\lambda$ is an eigval of $T$ with eigvec $q(T)v$.

• **Example:** *an operator on a complex vecsp with no eigvals*
Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by $(Tp)(z) = zp(z)$.

Suppose $p \in \mathcal{P}(\mathbf{C})$ is a nonzero poly. Then $\deg Tp = \deg p + 1$, and thus $Tp \neq \lambda p$, $\forall \lambda \in \mathbf{C}$.
Hence $T$ has no eigvals.

---

**13** *Suppose $V$ is a complex vecsp and $T \in \mathcal{L}(V)$ has no eigvals.*
*Prove that every subsp of $V$ invar under $T$ is either $\{0\}$ or infinite-dim.*

**Solution:** Suppose $U$ is a finite-dim nonzero invar subsp on $\mathbf{C}$. Then by [5.21], $T|_U$ has an eigval. □

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**16** *Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}\big(\mathcal{P}_{\dim V}(\mathbf{C}), V\big)$ by $S(p) = p(T)v$. Prove [ 5.21 ].*

**Solution:**

Because $\dim \mathcal{P}_{\dim V}(\mathbf{C}) = \dim V + 1$. Then $S$ is not inje. Hence $\exists\, 0 \neq p \in \mathcal{P}_{\dim V}(\mathbf{C}), p(T)v = 0$.
Using [4.14], write $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$. Apply $T$ to both sides: $p(T) = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$.
Thus at least one of $(T - \lambda_j I)$ is not inje ( because $p(T)$ is not inje ). □

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**17** *Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}\big(\mathcal{P}_{(\dim V)^2}(\mathbf{C}), \mathcal{L}(V)\big)$ by $S(p) = p(T)$. Prove [5.21].*

**Solution:**

Because $\dim \mathcal{P}_{(\dim V)^2}(\mathbf{C}) = (\dim V)^2 + 1$. Then $S$ is not inje.
Hence $\exists\, p \in \mathcal{P}_{(\dim V)^2}(\mathbf{C}) \backslash \{0\}, 0 = S(p) = p(T) = c(T - \lambda_1 I)\cdots(T - \lambda_m I)$, where $c \neq 0$.
Thus $(T - \lambda_1 I)\cdots(T - \lambda_m I) = 0 \Longrightarrow \exists\, j, (T - \lambda_j)$ is not inje. □

**Comment:** $\exists$ monic $q \in \operatorname{null} S \neq \{0\}$ of smallest degree, $S(q) = q(T) = 0$, then $q$ is the *mini poly.*

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- **Note For [8.40]:** *def for mini poly*

*Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.*
*Suppose $M_T^0 = \{p_j\}_{j\in\Gamma}$ is the set of all monic poly that give 0 whenever $T$ is applied.*
*Prove that $\exists\,!\, p_k \in M_T^0, \deg p_k = \min\{\deg p_j\}_{j\in\Gamma} \leqslant \dim V$.*

**Solution:** Or. Another Proof :

[ *Existns Part* ] We use induction on $\dim V$.

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, we are done.

(ii) Suppose $\dim V \geqslant 1$.

*Assume that* $\dim V > 0$ *and that the desired result is true for all operators on all vecsps of smaller dim.*
Let $u \in V, u \neq 0$. The list $(u, Tu, \dots, T^{\dim V}u)$ of length $(1 + \dim V)$ is linely depe.
Then $\exists\,!\, T^m$ of smallest degree such that $T^m u \in \operatorname{span}(u, Tu, \dots, T^{m-1}u)$.
Thus $\exists\, c_j \in \mathbf{F}, c_0 u + c_1 Tu + \cdots + c_{m-1}T^{m-1}u + T^m u = 0$.
Define $q$ by $q(z) = c_0 + c_1 z + \cdots + c_{m-1}z^{m-1} + z^m$.
Then $0 = T^k\big(q(T)u\big) = q(T)(T^k u), \forall k \in \{1, \dots, m-1\} \subseteq \mathbf{N}$.
Because $(u, Tu, \dots, T^{m-1}u)$ is linely inde.
Thus $\dim \operatorname{null} q(T) \geq m \Rightarrow \dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leqslant \dim V - m$.

Let $W = \operatorname{range} q(T)$.
By *assumption*, $\exists\, s \in M_T^0$ of smallest degree ( and $\deg s \leqslant \dim W$, ) so that $s(T|_W) = 0$.
Hence $\forall v \in V, \big((sq)(T)\big)(v) = s(T)\big(q(T)v\big) = 0$.
Thus $sq \in M_T^0$ and $\deg sq \leqslant \dim V$.

[ *Uniqnes Part* ]

Suppose $p, q \in M_T^0$ are of the smallest degree. Then $(p-q)(T) = 0$. 又 $\deg(p-q) = m < \min\{\deg p_j\}_{j\in\Gamma}$.
Hence $p - q = 0$, for if not, $\exists\,!\, c \in \mathbf{F}, c(p-q) \in M_T^0$. Contradicts. □

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• (4E 5.31, 4E 5.B.25 and 26) *mini poly of restriction operator and mini poly of quotient operator*
*Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$, and $U$ is an invar subsp of $V$ under $T$.*
*Let $p$ be the mini poly of $T$.*
(a) *Prove that $p$ is a poly multi of the mini poly of $T|_U$.*
(b) *Prove that $p$ is a poly multi of the mini poly of $T/U$.*
(c) *Prove that ( mini poly of $T|_U$ ) $\times$ ( mini poly of $T/U$ ) is a poly multi of $p$.*
(d) *Prove that the set of eigvals of $T$ equals the union of the set of eigvals of $T|_U$ and the set of eigvals of $T/U$.*

**SOLUTION:**
(a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46]. □
(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v+U) = p(T)v + U = 0$. □
(c) Suppose $r$ is the mini poly of $T|_U$, $s$ is the mini poly of $T/U$.
Because $\forall v \in V, s(T/U)(v+U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.
又 $\forall u \in U, r(T|_U)u = r(T)u = 0$.
Thus $\forall v \in V$ but $v \notin U, (rs)(T)v = r\big(s(T)v\big) = 0$.
And $\forall u \in U, (rs)(T)u = r\big(s(T)u\big) = 0$ ( because $s(T)u = s(T|_U)u \in U$ ).
Hence $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0$. □
(d) By [8.49], immediately. □

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• (4E 5.B.27) *Suppose $\mathbf{F} = \mathbf{R}$, $V$ is finite-dim, and $T \in \mathcal{L}(V)$.*
*Prove that the mini poly $p$ of $T_C$ equals the mini poly $q$ of $T$.*

**SOLUTION:**
(a) $\forall u + \mathrm{i}0 \in V_C, p(T_C)(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p$ is a poly multi of $q$.
(b) $q(T) = 0 \Rightarrow \forall u + \mathrm{i}v \in V_C, q(T_C)(u + \mathrm{i}v) = q(T)u + \mathrm{i}q(T)v = 0 \Rightarrow q$ is a poly multi of $p$. □

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• (4E 5.B.28) *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.*
*Prove that the mini poly $p$ of $T' \in \mathcal{L}(V')$ equals the mini poly $q$ of $T$.*

**SOLUTION:**
(a) $\forall \varphi \in V', p(T')\varphi = \varphi \circ \big(p(T)\big) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \operatorname{null} \varphi \Rightarrow p(T) = 0, p$ is a poly multi of $q$.
(b) $q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ \big(q(T)\big) = q(T')\varphi = 0 \Rightarrow q(T) = 0, q$ is a poly multi of $p$. □

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• (4E 5.32) *Suppose $T \in \mathcal{L}(V)$ and $p$ is the mini poly.*
*Prove that $T$ is not inje $\Longleftrightarrow$ the const term of $p$ is $0$.*

**SOLUTION:**
$T$ is not inje $\Longleftrightarrow$ 0 is an eigval of $T \Longleftrightarrow$ 0 is a zero of $p \Longleftrightarrow$ the const term of $p$ is 0. □

OR. Because $p(0) = (z-0)(z-\lambda_1)\cdots(z-\lambda_m) = 0 \Rightarrow T(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0$
又 $p$ is the mini poly $\Rightarrow q$ define by $q(z) = (z-\lambda_1)\cdots(z-\lambda_m)$ is such that $q(T) \neq 0$.
Hence $0 = p(T) = Tq(T) \Rightarrow T$ is not inje.
Conversely, suppose $(T - 0I)$ is not inje, then 0 is a zero of $p$, so that the const term is 0. □

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• (4E 5.B.22)
*Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$. Prove that $T$ is inv $\Longleftrightarrow I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$.*

**SOLUTION:** Denote the mini poly by $p$, where for all $z \in \mathbf{F}, p(z) = a_0 + a_1 z + \cdots + z^m$.
Notice that $V$ is finite-dim. $T$ is inv $\Longleftrightarrow T$ is inje $\Longleftrightarrow p(0) \neq 0$.

Hence $p(T) = 0 = a_0 I + a_1 T + \cdots + T^m$, where $a_0 \neq 0$ and $m \leqslant \dim V$. $\square$

---

**6** *Suppose $T \in \mathcal{L}(V)$ and $U$ is a subsp of $V$ invar under $T$.*
*Prove that $U$ is invar under $p(T)$ for every poly $p \in \mathcal{P}(\mathbf{F})$.*

**SOLUTION:**

$\forall u \in U, Tu \in U \Rightarrow Iu, Tu, T(Tu), \dots, T^m u \in U \Longrightarrow \forall a_k \in \mathbf{F}, (a_0 I + a_1 T + \cdots + a_m T^m)u \in U$. $\square$

---

• (4E 5.B.10, 23) *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$ and $p$ is the mini poly with degree $m$. Suppose $v \in V$.*
(a) *Prove that* $\operatorname{span}(v, Tv, \dots, T^{m-1}v) = \operatorname{span}(v, Tv, \dots, T^{j-1}v)$ *for some $j \leqslant m$.*
(b) *Prove that* $\operatorname{span}(v, Tv, \dots, T^{m-1}v) = \operatorname{span}(v, Tv, \dots, T^{m-1}v, \dots, T^n v)$.

**SOLUTION:**

**COMMENT:** By NOTE FOR[8.40], $j$ has an upper bound $m-1$, $m$ has an upper bound $\dim V$.

Write $p(z) = a_0 + a_1 z + \cdots + z^m$ ( $m \leqslant \dim V$ ). If $v = 0$, then we are done. Suppose $v \neq 0$.

(a) Suppose $j \in \mathbf{N}^+$ is the smallest such that $T^j v \in \operatorname{span}(v, Tv, \dots, T^{j-1}v) = U_0$. Then $j \leqslant m$.
Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1} T^{j-1} v$. And because $T(T^k v) = T^{k+1} \in U_0$. $U_0$ is invar under $T$.
By Problem (6), $\forall k \in \mathbf{N}$, $T^{j+k} v = T^k(T^j v) \in U_0$.
Thus $U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}v, \dots, T^n v)$ for all $n \geqslant j-1$. Let $n = m-1$ and we are done.

(b) Let $U = \operatorname{span}(v, Tv, \dots, T^{m-1}v)$.
By (a), $U = U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}, \dots, T^{m-1}, \dots, T^n)$ for all $n \geqslant m-1$. $\square$

---

• (4E 5.B.21) *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.*
*Prove that the mini poly $p$ has degree at most $1 + \dim \operatorname{range} T$.*
*If $\dim \operatorname{range} T < \dim V - 1$, then this result gives a better upper bound for the degree of mini poly.*

**SOLUTION:**

If $T$ is inje, then range $T = V$ and we are done. Now choose $0 \neq v \in \operatorname{null} T$, then $Tv + 0 \cdot v = 0$.
1 is the smallest positive integer such that $T^1 v \in \operatorname{span}(v, \dots, T^0 v)$. Define $q$ by $q(z) = z \Rightarrow q(T)v = 0$.
Let $W = \operatorname{range} q(T) = \operatorname{range} T$. $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ of smallest degree ( $\deg s \leqslant \dim W$ ), $s(T|_W) = 0$.
Hence $sq$ is the mini poly ( see NOTE FOR[8.40] ) and $\deg(sq) = \deg s + \deg q \leqslant \dim \operatorname{range} T + 1$. $\square$

---

**19** *Suppose $V$ is finite-dim,* $\dim V > 1, T \in \mathcal{L}(V)$. *Prove that* $\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V)$.

**SOLUTION:** If $\forall S \in \mathcal{L}(V), \exists p \in \mathcal{P}(\mathbf{F}), S = p(T)$. Then by [5.20], $\forall S_1, S_2 \in \mathcal{L}(V), S_1 S_2 = S_2 S_1$.
Note that $\dim \geqslant 2$. By (3.A.14), $\exists S_1, S_2 \in \mathcal{L}(V), S_1 S_2 \neq S_2 S_1$. Contradicts. $\square$

---

• *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.*
*Prove that* $\dim \mathcal{E}$ *equals the degree of the mini poly of $T$.*

**SOLUTION:**

Because the list $(I, T, \dots, T^{(\dim V)^2})$ of length $\dim \mathcal{L}(V) + 1$ is linely depe in $\dim \mathcal{L}(V)$.
Suppose $m \in \mathbf{N}^+$ is the smallest such that $T^m = a_0 I + \cdots + a_{m-1} T^{m-1}$.
Then $q$ defined by $q(z) = z^m - a_{m-1} z^{m-1} - \cdots - a_0$ is the mini poly ( see [8.40] ).
For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k(T^m) \in \operatorname{span}(I, T, \dots, T^{m-1}) = U$.
Hence $\operatorname{span}(I, T, \dots, T^{(\dim V)^2}) = \operatorname{span}(I, T, \dots, T^{(\dim V)^2 - 1}) = U$.
Note that by the minimality of $m$, $(I, T, \dots, T^{m-1})$ is linely inde.

Thus $\dim U = m = \dim \operatorname{span}(I, T, \dots, T^{(\dim V)^2-1}) = \dim \operatorname{span}(I, T, \dots, T^n)$ for all $m < n \in \mathbf{N}^+$.

Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.

(a) Suppose $p(T) = 0$. 又 $\deg p \leqslant m-1 \Rightarrow p = 0$. Then $\varphi$ is inje.

(b) $\forall S = a_0 I + a_1 T + \cdots + a_{m-1} T^{m-1} \in \mathcal{E}$, define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by
$p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} \Rightarrow \varphi(p) = S$. Then $\varphi$ is surj.

Hence $\mathcal{E}$ and $\mathcal{P}_{m-1}(\mathbf{F})$ are iso. 又 $\dim \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. □

---

• (4E 5.B.13) *Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$ is defined by*
$$q(z) = a_0 + a_1 z + \cdots + a_n z^n, \text{ where } a_n \neq 0, \text{ for all } z \in \mathbf{F}.$$
*Denote the mini poly of $T$ by $p$ defined by*
$$p(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m \text{ for all } z \in \mathbf{F}.$$
*Prove that $\exists!\, r \in \mathcal{P}(\mathbf{F})$ such that $q(T) = r(T), \deg r < \deg p$.*

**Solution:**

If $\deg q < \deg p$, then we are done.

If $\deg q = \deg p$, notice that $p(T) = 0 = c_0 I + c_1 T + \cdots + c_{m-1} T^{m-1} + T^m$
$$\Rightarrow T^m = -c_0 I - c_1 T - \cdots - c_{m-1} T^{m-1},$$
define $r$ by $r(z) = q(z) + \left[ -a_m z^m + a_m \left( -c_0 - c_1 z - \cdots - c_{m-1} z^{m-1} \right) \right]$
$$= (a_0 - a_m c_0) + (a_1 - a_m c_1) z + \cdots + (a_{m-1} - a_m c_{m-1}) z^{m-1},$$
hence $r(T) = 0, \deg r < m$ and we are done.

Now suppose $\deg q \geqslant \deg p$. We use induction on $\deg q$.

(i) $\deg q = \deg p$, then the desired result is true, as shown above.

(ii) $\deg q > \deg p$, assume that the desired result is true for $\deg q = n$.

Suppose $f \in \mathcal{P}(\mathbf{F})$ such that $f(z) = b_0 + b_1 z + \cdots + b_n z^n + b_{n+1} z^{n+1}$.

Apply the assumption to $g$ defined by $g(z) = b_0 + b_1 z + \cdots + b_n z^n$,
getting $s$ defined by $s(z) = d_0 + d_1 z + \cdots + d_{m-1} z^{m-1}$.

Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1} T^{n+1} = s(T) + b_{n+1} T^{n+1}$.

Apply the assumption to $t$ defined by $t(z) = z^n$,
getting $\delta$ defined by $\delta(z) = c_0{}' + c_1{}' z + \cdots + c_{m-1}{}' z^{m-1}$.

Thus $t(T) = T^n = c_0{}' + c_1{}' z + \cdots + c_{m-1}{}' z^{m-1} = \delta(T)$.

又 $\operatorname{span}(v, Tv, \dots, T^{m-1} v)$ is invar under $T$.

Hence $\exists!\, k_j \in \mathbf{F}, T^{n+1} = T(T^n) = k_0 + k_1 z + \cdots + k_{m-1} z^{m-1}$.

And $f(T) = s(T) + b_{n+1}(k_0 + k_1 T + \cdots + k_{m-1} T^{m-1})$

$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1) z + \cdots + (d_{m-1} + k_{m-1}) z^{m-1} = h(T)$, thus defining $h$. □

---

• (4E 5.B.14) *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$ has mini poly $p$*
$$\text{defined by } p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m, a_0 \neq 0.$$
*Find the mini poly of $T^{-1}$.*

**Solution:**

Notice that $V$ is finite-dim. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is inv.

Then $p(T) = a_0 I + a_1 T + \cdots + T^m = 0$. Apply $T^{-m}$ to both sides,

$a_0 (T^{-1})^m + a_1 (T^{-1})^{m-1} + \cdots + a_{m-1} T^{-1} + I = 0$.

Define $q$ by $q(z) = z^m + \dfrac{a_1}{a_0} z^{m-1} + \cdots + \dfrac{a_{m-1}}{a_0} z + \dfrac{1}{a_0}$ for all $z \in \mathbf{F}$.

We now show that $(T^{-1})^k \notin \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$

for every $k \in \{1, \dots, m-1\}$ by contradiction, so that $q$ is exactly the mini poly of $T^{-1}$.

Suppose $\left(T^{-1}\right)^k \in \operatorname{span}\left(I, T^{-1}, \dots, \left(T^{-1}\right)^{k-1}\right)$.

Then let $\left(T^{-1}\right)^k = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. Apply $T^k$ to both sides,

getting $I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T$, hence $T^k \in \operatorname{span}(I, T, \dots, T^{k-1})$.

Thus $f$ defined by $f(z) = z^k + \dfrac{b_1}{b_0} z^{k-1} + \cdots + \dfrac{b_{k-1}}{b_0} z - \dfrac{1}{b_0}$ is a poly multi of $p$.

While $\deg f < \deg p$. Contradicts. □

---

- **NOTE FOR [8.49]:**

*Suppose $V$ is a finite-dim complex vecsp and $T \in \mathcal{L}(V)$.*

*By [4.14], the mini poly has the form $(z - \lambda_1) \cdots (z - \lambda_m)$,*

*where $\lambda_1, \dots, \lambda_m$ are all the eigvals of $T$, **possibly with repetitions**.*

- **COMMENT:**

*A nonzero poly has at most as many distinct zeros as its degree ( see [4.12] ).*

*Thus by the upper bound for the deg of mini poly given in* NOTE FOR*[8.40], and by [8.49,]*

we can ***give an alternative proof of*** **[5.13]**.

---

- NOTICE ( See also 4E 5.B.20,24 )

Suppose $\alpha_1, \dots, \alpha_n$ are all the distinct eigvals of $T$,

and therefore are all the distinct zeros of the mini poly.

Also, the mini poly of $T$ is a poly multi of, but not equal to, $(z - \alpha_1) \cdots (z - \alpha_n)$.

If we define $q$ by $q(z) = (z - \alpha_1)^{\dim V - (n-1)} \cdots (z - \alpha_n)^{\dim V - (n-1)}$,

then $q$ is a poly multi of the char poly ( see [8.34] and [8.26] )

( Because $\dim V > n$ and $n - 1 > 0$, $n\left[\dim V - (n-1)\right] > \dim V$. )

The char poly has the form $(z - \alpha_1)^{\gamma_1} \cdots (z - \alpha_n)^{\gamma_n}$, where $\gamma_1 + \cdots + \gamma_n = \dim V$.

*The mini poly has the form $(z - \alpha_1)^{\delta_1} \cdots (z - \alpha_n)^{\delta_n}$, where $0 \leqslant \delta_1 + \cdots + \delta_n \leqslant \dim V$.*

---

**10** *Suppose $T \in \mathcal{L}(V), \lambda$ is an eigval of $T$ with an eigvec $v$.*
*Prove that for any $p \in \mathcal{P}(\mathbf{F}), p(T)v = p(\lambda)v$.*

**SOLUTION:**

Suppose $p$ is defined by $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbf{F}$. Because for any $n \in \mathbf{N}^+, T^n v = \lambda^n v$.

Thus $p(T)v = a_0 v + a_1 T v + \cdots + a_m T^m v = a_0 v + a_1 \lambda v + \cdots + a_m \lambda^m v = p(\lambda)v$. □

**COMMENT:** For any $p \in \mathcal{P}(\mathbf{F})$ such that $p(z) = (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_m)^{\alpha_m}$, the result is true as well.

Now we prove that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v = (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v$.

Define $q_i$ by $q_i(z) = (z - \lambda_i)^{\alpha_i}$ for all $z \in \mathbf{F}$.

Because $(a+b)^n = a^n + C_n^1 a^{n-1} b + \cdots + C_n^k a^{n-k} b^k + \cdots + C_n^n b^n$.

Let $a = z, b = \lambda_i, n = \alpha_i$, so we can write $q_i(z)$ in the form $a_0 + a_1 z + \cdots + a_m z^m$.

Hence $q_i(T)v = q_i(\lambda)v \Rightarrow (T - \lambda_i I)^{\alpha_i} v = (\lambda - \lambda_i)^{\alpha_i} v$.

Then for each $k \in \{2, \dots, m\}$, $(T - \lambda_{k-1} I)^{\alpha_{k-1}} (T - \lambda_k I)^{\alpha_k} v$

$$
\begin{aligned}
&= q_{k-1}(T)\left(q_k(T)v\right) \\
&= q_{k-1}(T)\left(q_k(\lambda)v\right) \\
&= q_{k-1}(\lambda)\left(q_k(\lambda)v\right) \\
&= (\lambda - \lambda_{k-1})^{\alpha_{k-1}} (\lambda - \lambda_k)^{\alpha_k} v.
\end{aligned}
$$

So that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v$

$$
\begin{aligned}
&= q_1(T)\Big(q_2(T)\Big(\dots(q_m(T)v)\dots\Big)\Big) \\
&= q_1(\lambda)(q_2(\lambda)(\dots(q_m(\lambda)v)\dots)) \\
&= (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v.
\end{aligned}
$$

□

---

**1** *Suppose $T \in \mathcal{L}(V)$ and $\exists\, n \in \mathbf{N}^+$ such that $T^n = 0$.*
*Prove that $(I - T)$ is inv and $(I-T)^{-1} = I + T + \cdots + T^{n-1}$.*

**Solution:** Note that $1 - x^n = (1-x)(1 + x + \cdots + x^{n-1})$.

$$\left.\begin{array}{l}(I-T)(1+T+\cdots+T^{n-1}) = I - T^n = I\\(1+T+\cdots+T^{n-1})(I-T) = I - T^n = I\end{array}\right\} \Rightarrow (I-T)^{-1} = 1 + T + \cdots + T^{n-1}. \qquad \square$$

---

**2** *Suppose $T \in \mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I) = 0$.*
*Suppose $\lambda$ is an eigval of $T$. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.*

**Solution:**

Suppose $v$ is an eigvec correspd to $\lambda$. Then for any $p \in \mathcal{P}(\mathbf{F}), p(T)v = p(\lambda)v$.
Hence $0 = (T-2I)(T-3I)(T-4I)v = (\lambda-2)(\lambda-3)(\lambda-4)v$ while $v \neq 0 \Rightarrow \lambda = 2, 3$ or $4$. $\square$

**Comment:** Note that $(T-2I)(T-3I)(T-4I) = 0$ is not inje, so that $2, 3, 4$ are eigvals of $T$.
But it doesn't mean that all the eigvals of $T$ are exactly $2, 3, 4$.

---

**7** [See 5.A.22] *Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigval of $T^2 \Longleftrightarrow 3$ or $-3$ is an eigval of $T$.*

**Solution:**

(a) Suppose $\lambda$ is an eigval of $T$ with an eigvec $v$.
Then $(T-3I)(T+3I)v = (\lambda-3)(\lambda+3)v = 0 \Rightarrow \lambda = \pm 3$.

(b) Suppose 3 or $-3$ is an eigval of $T$ with an eigvec $v$. Then $Tv = \pm 3v \Rightarrow T^2 v = T(Tv) = 9v$ $\square$

Or. 9 is an eigval of $T^2 \Longleftrightarrow (T^2 - 9I) = (T-3I)(T+3I)$ is not inje $\Longleftrightarrow \pm 3$ is an eigval. $\square$

---

**3** *Suppose $T \in \mathcal{L}(V), T^2 = I$ and $-1$ is not an eigval of $T$. Prove that $T = I$.*

**Solution:**

$T^2 - I = (T+I)(T-I)$ is not inje, 又 $-1$ is not an eigval of $T \Longrightarrow$ By Tips. $\square$

Or. Note that $\forall v \in V, v = [\frac{1}{2}(I-T)v] + [\frac{1}{2}(I+T)v]$.

$$\left.\begin{array}{l}(I+T)\big((I-T)v\big) = 0 \Longrightarrow (I-T)v \in \operatorname{null}(I+T)\\(I-T)\big((I+T)v\big) = 0 \Longrightarrow (I+T)v \in \operatorname{null}(I-T)\end{array}\right\} \Rightarrow V = \operatorname{null}(I+T) + \operatorname{null}(I-T).$$

又 $-1$ is not an eigval of $T \Longleftrightarrow (I+T)$ is inje $\Longleftrightarrow \operatorname{null}(I+T) = \{0\}$.
Hence $V = \operatorname{null}(I-T) \Rightarrow \operatorname{range}(I-T) = \{0\}$. Thus $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$. $\square$

---

• (4E 5.A.32) *Suppose $T \in \mathcal{L}(V)$ has no eigvals and $T^4 = I$. Prove that $T^2 = -I$.*

**Solution:**

Because $T^4 - I = (T^2 - I)(T^2 + I) = 0$ is not inje $\Rightarrow (T^2 - I)$ or $(T^2 + I)$ is not inje.
又 $T$ has no eigvals $\Rightarrow (T^2 - I) = (T-I)(T+I)$ is inje. Hence $T^2 + I = 0 \in \mathcal{L}(V)$, for if not, $\exists\, v \in V, (T^2+I)v \neq 0$ while $(T^2-I)\big((T^2+I)v\big) = 0$ but $(T^2 - I)$ is inje. Contradicts.
Or. $\forall v \in V, 0 = (T^2 - I)(T^2 + I)v \Longleftrightarrow 0 = (T^2+I)v$. Hence $T^2 + I = 0$. $\square$

Or. Note that $\forall v \in V, v = \big[\frac{1}{2}(I - T^2)v\big] + \big[\frac{1}{2}(I+T^2)v\big]$.

$$\left.\begin{array}{l}(I+T^2)\big((I-T^2)v\big) = 0 \Longrightarrow (I-T^2)v \in \operatorname{null}(I+T^2)\\(I-T^2)\big((I+T^2)v\big) = 0 \Longrightarrow (I+T^2)v \in \operatorname{null}(I-T^2)\end{array}\right\} \Rightarrow V = \operatorname{null}(I+T^2) + \operatorname{null}(I-T^2).$$

又 $T$ has no eigvals $\Longleftrightarrow (I - T^2)$ is inje $\Longleftrightarrow \operatorname{null}(I-T^2) = \{0\}$.
Hence $V = \operatorname{null}(I+T^2) \Rightarrow \operatorname{range}(I+T^2) = \{0\}$. Thus $I + T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$. $\square$

---

**8** [OR (4E 5.A.31)] *Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$.*

**SOLUTION:**

Define $\mathrm{i} \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y) = (-y, x)$. Just like $\mathrm{i} : \mathbf{C} \to \mathbf{C}$ defined by $\mathrm{i}(x+\mathrm{i}y) = -y + \mathrm{i}x$.

Define $\mathrm{i}^n \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y) = \Big(\mathrm{Re}\big(\mathrm{i}^n x + \mathrm{i}^{n+1} y\big), \mathfrak{Im}\big(\mathrm{i}^n x + \mathrm{i}^{n+1} y\big)\Big)$.

$T^4 + I = (T^2 + \mathrm{i}I)(T^2 - \mathrm{i}I) = (T + \mathrm{i}^{1/2} I)(T - \mathrm{i}^{1/2} I)(T - (-\mathrm{i})^{1/2} I)(T + (-\mathrm{i})^{1/2} I)$.

Note that $\mathrm{i}^{1/2} = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}$, $(-\mathrm{i})^{1/2} = \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}$. Hence $T = \pm(\pm\mathrm{i})^{1/2} I$.

Let $T = \mathrm{i}^{1/2} I$ defined by $\mathrm{i}^{1/2}(x,y) = \Big(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\Big)$. ☐

OR. Because $\mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix}$. Using $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}$.

We define $T \in \mathcal{L}(\mathbf{R}^2)$ such that $\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix}$. ☐

---

• (4E 5.B.12) *Find the mini poly of $T$ defined in (5.A.10).*

**SOLUTION:** By (5.A.9) and [8.40, 8.49], $1, 2, \dots, n$ are all the zeros of the mini poly of $T$. ☐

---

• (4E 5.B.3) *Find the mini poly of $T$ defined in (5.A.19).*

**SOLUTION:**

If $n = 1$ then 1 is the only eigval of $T$, and $(z-1)$ is the mini poly.

Because $n$ and 0 are all the eigvals of $T$, 又 $\forall k \in \{1, \dots, n\}, Te_k = e_1 + \cdots + e_n$; $T^2 e_k = n(e_1 + \cdots + e_n)$.

Hence $T^2 e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T-n) = 0$. Thus $\big(z(z-n)\big)$ is the mini poly. ☐

---

• (4E 5.B.8) *Find the mini poly of $T$. Where $T \in \mathcal{L}(\mathbf{R}^2)$ is the operator of counterclockwise rotation by $\theta$, where $\theta \in \mathbf{R}^+$.*

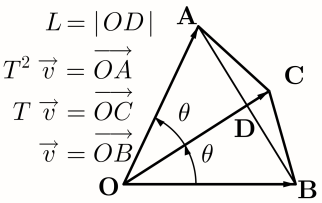
**SOLUTION:**

If $\theta = \pi + 2k\pi$, then $T(w,z) = (-w, -z), T^2 = I$ and the mini poly is $z+1$.

If $\theta = 2k\pi$, then $T = I$ and the mini poly is $z - 1$.

Otherwise $(v, Tv)$ is linely inde. Then $\mathrm{span}(v, Tv) = \mathbf{R}^2$. Note that $\nexists b \in \mathbf{F}, T - bI = 0$.

Thus suppose the mini poly $p$ is defined by $p(z) = z^2 + bz + c$ for all $z \in \mathbf{R}$.

Because 


$Tv = \frac{|\vec{v}|}{2L}(T^2 v + v) \Rightarrow T = \frac{|\vec{v}|}{2L}(T^2 + I)$

$L = |\vec{v}| \cos\theta \Rightarrow \frac{|\vec{v}|}{2L} = \frac{1}{2\cos\theta}$

Hence $p(T) = T^2 - 2\cos\theta\, T + I = 0$ and $z^2 - 2\cos\theta\, z + 1$ is the mini poly of $T$. ☐

OR. Let $(e_1, e_2)$ be the std basis of $\mathbf{R}^2$. We use the pattern shown in [8.44].

Because $Te_1 = \cos\theta\, e_1 + \sin\theta\, e_2$, $T^2 e_1 = \cos 2\theta\, e_1 + \sin 2\theta\, e_2$.

Thus $ce_1 + bTe_1 = -T^2 e_1 \iff \begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix}\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$. Now $\det = \sin\theta \neq 0, c = 1, b = 2\cos\theta$. ☐

OR. $\mathcal{M}(T, (e_1, e_2)) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. By (4E 5.B.11), the mini poly is $(z \pm 1)$ or $(z^2 - 2\cos\theta\, z + 1)$. ☐

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• (4E 5.B.11) *Suppose $V$ is a two-dim vecsp, $T \in \mathcal{L}(V)$, and the matrix of $T$ with resp to some basis of $V$ is* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) *Show that $T^2 - (a+d)T + (ad-bc)I = 0$.*
(b) *Show that the mini poly of $T$ equals*

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad-bc) & \text{otherwise.} \end{cases}$$

**SOLUTION:**

(a) Suppose the basis is $(v, w)$. Because $\begin{cases} Tv = av + bw \Rightarrow (T-aI)v = bw, \text{ then apply } (T-dI) \text{ to both sid} \\ Tw = cv + dw \Rightarrow (T-dI)w = cv, \text{ then apply } (T-aI) \text{ to both sid} \end{cases}$

Hence $(T-aI)(T-dI) = bcI \Rightarrow T^2 - (a+d)T + (ad-bc)I = 0$.

(b) If $b = c = 0$ and $a = d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the mini poly is $z - a$.

Otherwise, by (a), $z^2 - (a+d)z + (ad-bc)$ is a poly multi of the mini poly.

Now we prove that $T \notin \operatorname{span}(I)$, so that then the mini poly of $T$ has exactly degree 2.

( At least one of the assumption of (I),(II) below is true. )

(I) Suppose $a = d$, then $Tv = av + bw \notin \operatorname{span}(v)$, $Tw = cv + aw \notin \operatorname{span}(w)$.

(II) Suppose at most one of $b, c$ is not 0. If $b = 0$, then $Tw \notin \operatorname{span}(w)$; If $c = 0$, then $Tv \notin \operatorname{span}(v)$. □

---

• *Suppose $S, T \in \mathcal{L}(V)$, $S$ is inv, and $p \in \mathcal{P}(\mathbf{F})$. Prove that $Sp(TS) = p(ST)S$.*

**SOLUTION:**

We prove $S(TS)^m = (ST)^m S$ for each $m \in \mathbf{N}$ by induction.

(i) If $m = 0, 1$. Then $S(TS)^0 = I = (ST)^0 S$; $S(TS)^1 = (ST)S$.

(ii) If $m > 1$. Assume that $S(TS)^m = (ST)^m S$.

Then $S(TS)^{m+1} = S(TS)^m(TS) = (ST)^m STS = (ST)^{m+1}S$.

Hence $\forall p \in \mathcal{P}(\mathbf{F})$, $Sp(TS) = \sum_{k=1}^{m} a_k S(TS)^k = \sum_{k=1}^{m} a_k p(ST)^k S = \left[\sum_{k=1}^{m} a_k (TS)^k\right] S$. □

**COMMENT:** $p(TS) = S^{-1}p(ST)S$, $p(ST) = Sp(TS)S^{-1}$.

**COROLLARY: 5** Because $S$ is inv, $T \in \mathcal{L}(V)$ is arbitrary $\Longleftrightarrow R = ST$ is arbitrary.

Hence $\forall R \in \mathcal{L}(V)$, inv $S \in \mathcal{L}(V)$, $p(S^{-1}RS) = S^{-1}p(R)S$.

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• (4E 5.B.7) *Suppose $S, T \in \mathcal{L}(V)$. Let $p, q$ be the mini polys of $ST, TS$ respectively.*
(a) *If $V = \mathbf{F}^2$. Give an example such that $p \neq q$;* (b) *If $S$ or $T$ is inv. Prove that $p = q$.*

**SOLUTION:**

(a) Define $S$ by $S(x, y) = (x, x)$. Define $T$ by $T(x, y) = (0, y)$.

Then $ST(x, y) = 0$, $TS(x, y) = (0, x)$ for all $(x, y) \in \mathbf{F}^2$. Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.

Hence the mini poly of $ST$ does not equal to the mini poly of $TS$.

(b) Suppose $S$ is inv. Because $p, q$ are monic.

$$\left.\begin{array}{l} p(ST) = 0 = Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a poly multi of } q \\ q(TS) = 0 = S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a poly multi of } p \end{array}\right\} \Rightarrow p = q.$$

Reversing the roles of $S$ and $T$, we conclude that if $T$ is inv, then $p = q$ as well. □

---

**11** *Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$, and $\alpha \in \mathbf{C}$.*
*Prove that $\alpha$ is an eigval of $p(T) \Longleftrightarrow \alpha = p(\lambda)$ for some eigval $\lambda$ of $T$.*

**SOLUTION:**

(a) Suppose $\alpha$ is an eigval of $p(T) \Leftrightarrow \big(p(T) - \alpha I\big)$ is not inje.

Write $p(z) - \alpha = c(z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow p(T) - \alpha I = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.
By TIPS, $\exists\ (T - \lambda_j I)$ not inje. Thus $p(\lambda_j) - \alpha = 0$.

(b) Suppose $\alpha = p(\lambda)$ and $\lambda$ is an eigval of $T$ with an eigvec $v$. Then $p(T)v = p(\lambda)v = \alpha v$. $\square$

OR. Define $q$ by $q(z) = p(z) - \alpha$. $\lambda$ is a zero of $q$.
Because $q(T)v = \big(p(T) - \alpha I\big)v = q(\lambda)v = \big(p(\lambda) - \alpha\big)v = 0$.
Hence $q(T)$ is not inje $\Rightarrow \big(p(T) - \alpha I\big)$ is not inje. $\square$

---

**12** [OR (4E.5.B.6)] *Give an example of an operator on* $\mathbf{R}^2$ *that shows the result above does not hold if* $\mathbf{C}$ *is replaced with* $\mathbf{R}$.

**SOLUTION:**

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(w,z) = (-z, w)$.

By Problem (4E 5.B.11), $\mathcal{M}\big(T, ((1,0),(0,1))\big) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ the mini poly of $T$ is $z^2 + 1$.

Define $p$ by $p(z) = z^2$. Then $p(T) = T^2 = -I$. Thus $p(T)$ has eigval $-1$.

While $\nexists\, \lambda \in \mathbf{R}$ such that $-1 = p(\lambda) = \lambda^2$. $\square$

---

• (4E 5.B.17) *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$, and $p$ is the mini poly of $T$. Show that the mini poly of $(T - \lambda I)$ is the poly $q$ defined by $q(z) = p(z + \lambda)$.*

**SOLUTION:**

$q(T - \lambda I) = 0 \Rightarrow q$ is poly multi of the mini poly of $(T - \lambda I)$.
Suppose the degree of the mini poly of $(T - \lambda I)$ is $n$, and the degree of the mini poly of $T$ is $m$.
By definition of mini poly,
$n$ is the smallest such that $(T - \lambda I)^n \in \operatorname{span}\big(I, (T - \lambda I), \dots, (T - \lambda I)^{n-1}\big)$;
$m$ is the smallest such that $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.
又 $T^k \in \operatorname{span}(I, T, \dots, T^{k-1}) \iff (T - \lambda)^k \in \operatorname{span}\big(I, (T - \lambda I), \dots, (T - \lambda I)^{k-1}\big)$.
Thus $n = m$. 又 $q$ is monic. By the uniqnes of mini poly. $\square$

---

• (4E 5.B.18) *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V), \lambda \in \mathbf{F} \backslash \{0\}$, and $p$ is the mini poly of $T$. Show that the mini poly of $\lambda T$ is the poly $q$ defined by $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.*

**SOLUTION:**

$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow q$ is a poly multi of the mini poly of $\lambda T$.
Suppose the degree of the mini poly of $\lambda T$ is $n$, and the degree of the mini poly of $T$ is $m$.
By definition of mini poly,
$n$ is the smallest such that $(\lambda T)^n \in \operatorname{span}\big(\lambda I, \lambda T, \dots, (\lambda T)^{n-1}\big)$;
$m$ is the smallest such that $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.
又 $(\lambda T)^k \in \operatorname{span}\big(\lambda I, \lambda T, \dots, (\lambda T)^{k-1}\big) \iff T^k \in \operatorname{span}(I, T \dots, T^{k-1})$.
Thus $n = m$. 又 $q$ is monic. By the uniqnes of mini poly. $\square$

---

**18** [OR (4E 5.B.15)] *Suppose $V$ is a finite-dim complex vecsp with* $\dim V > 0$ *and* $T \in \mathcal{L}(V)$. *Define $f : \mathbf{C} \to \mathbf{R}$ by $f(\lambda) = \dim \operatorname{range}(T - \lambda I)$. Prove that $f$ is not a continuous function.*

**SOLUTION:** Note that $V$ is finite-dim.

Let $\lambda_0$ be an eigval of $T$. Then $(T - \lambda_0 I)$ is not surj. Hence $\dim \operatorname{range}(T - \lambda_0 I) < \dim V$.
Because $T$ has finitely many eigvals. There exist a sequence of number $\{\lambda_n\}$ such that $\lim\limits_{n \to \infty} \lambda_n = \lambda_0$.

And $\lambda_n$ is not an eigval of $T$ for each $n \Rightarrow \dim \operatorname{range}(T-\lambda_n I) = \dim V \neq \dim \operatorname{range}(T-\lambda_0 I)$.
Thus $f(\lambda_0) \neq \lim_{n\to\infty} f(\lambda_n)$. $\square$

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- (4E 5.B.9) *Suppose $T \in \mathcal{L}(V)$ is such that with resp to some basis of $V$, all entries of the matrix of $T$ are rational numbers. Explain why all coeffs of the mini poly of $T$ are rational numbers.*

**Solution:**

Let $(v_1,\dots,v_n)$ denote the basis such that $\mathcal{M}\big(T,(v_1,\dots,v_n)\big)_{j,k} = A_{j,k} \in \mathbf{Q}$ for all $j,k = 1,\dots,n$.
Denote $\mathcal{M}\big(v_j,(v_1,\dots,v_n)\big)$ by $x_j$ for each $v_j$.
Suppose $p$ is the mini poly of $T$ and $p(z) = z^m + \cdots + c_1 z + c_0$. Now we show that each $c_j \in \mathbf{Q}$.
Note that $\forall s \in \mathbf{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbf{Q}^{n,n}$ and $T^s v_k = A^s_{1,k}v_1 + \cdots + A^s_{n,k}v_n$ for all $k \in \{1,\dots,n\}$.

$$\text{Thus } \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \cdots + c_1 A + c_0 I)x_1 = \sum\limits_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,1} x_j = 0; \\ \quad\vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \cdots + c_1 A + c_0 I)x_n = \sum\limits_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly, } \begin{cases} (A^m + \cdots + c_1 A + c_0 I)_{1,1} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,1} = 0; \\ \qquad\qquad \vdots \quad \ddots \quad \vdots \\ (A^m + \cdots + c_1 A + c_0 I)_{1,n} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,n} = 0; \end{cases}$$

Hence we get a system of $n^2$ linear equations in $m$ unknowns $c_0, c_1, \dots, c_{m-1}$.
We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbf{Q}$. $\square$

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- [Or (4E 5.B.16), Or (8.C.18)] *Suppose $a_0,\dots,a_{n-1} \in \mathbf{F}$. Let $T$ be the operator on $\mathbf{F}^n$ such that*

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ with resp to the std basis } (e_1,\dots,e_n).$$

*Show that the mini poly of $T$ is $p$ defined by $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$.*

*$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the mini poly of some operator. Hence a formula or an algorithm that could produce exact eigvals for each operator on each $\mathbf{F}^n$ could then produce exact zeros for each poly [ by 8.36(b) ]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an operator.*

**Solution:** Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ is linely inde. 又 The deg of mini poly is at most $n$.
$T^n e_1 = \cdots = T^{n-k}e_{1+k} = \cdots = Te_n = -a_0 e_1 - a_1 e_2 - a_2 e_3 - \cdots - a_{n-1}e_n$
$= (-a_0 I - a_1 T - a_2 T^2 - \cdots - a_{n-1}T^{n-1})e_1$. Thus $p(T)e_1 = 0 = p(T)e_j$ for each $e_j = T^{j-1}e_1$. $\square$

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- Eigenvalues On Odd-Dimensional Real Vector Spaces
- Even-Dimensional Null Space

*Suppose $\mathbf{F} = \mathbf{R}$, $V$ is finite-dim, $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$. Prove that $\dim \operatorname{null}(T^2 + bT + cI)$ is an even number.*

**Solution:**

Denote $\operatorname{null}(T^2 + bT + cI)$ by $R$. Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.
Suppose $\lambda$ is an eigval of $T_R$ with an eigvec $v \in R$.
Then $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \big((\lambda + b)^2 + c - \frac{b^2}{4}\big)v$.

Because $c - \dfrac{b^2}{4} > 0$ and we have $v = 0$. Thus $T_R$ has no eigvals.

Let $U$ be an invar subsp of $R$ that has the largest, even dim among all invar subsps.

Assume that $U \neq R$. Then $\exists\, w \in R$ but $w \notin U$. Let $W$ be such that $(w, T|_R w)$ is a basis of $W$.

Because $T|_R^2 w = -bT|_R w - cw \in W$. Hence $W$ is an invar subsp of dim 2.

Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,

for if not, because $w \notin U, T|_R w \in U$,

$U \cap W$ is invar under $T|_R$ of one dim ( impossible because $T|_R$ has no eigvecs ).

Hence $U + W$ is even-dim invar subsp under $T|_R$, contradicting the maximality of $\dim U$.

Thus the assumption was incorrect. Hence $R = \operatorname{null}(T^2 + bT + cI) = U$ has even dim. □

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• Operators On Odd-Dimensional Vector Spaces Have Eigenvalues

(a) *Suppose* $\mathbf{F} = \mathbf{C}$. Then by [5.21], we are done.

(b) *Suppose* $\mathbf{F} = \mathbf{R}$, *$V$ is finite-dim, and* $\dim V = n$ *is an odd number.*
*Let $T \in \mathcal{L}(V)$ and the mini poly is $p$. Prove that $T$ has an eigval.*

**Solution:**

(i) If $n = 1$, then we are done.

(ii) Suppose $n \geqslant 3$. Assume that every operator, on odd-dim vecsps of dim less than $n$, has an eigval.

If $p$ is a poly multi of $(x - \lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] $\lambda$ is an eigval of $T$ and we are done.

Now suppose $b, c \in \mathbf{R}$ such that $b^2 < 4c$ and $p$ is a poly multi of $x^2 + bx + c$ (see [4.17]).

Then $\exists\, q \in \mathcal{P}(\mathbf{R})$ such that $p(x) = q(x)(x^2 + bx + c)$ for all $x \in \mathbf{R}$.

Now $0 = p(T) = \big(q(T)\big)(T^2 + bT + cI)$, which means that $q(T)|_{\operatorname{range}(T^2+bT+cI)} = 0$.

Because $\deg q < \deg p$ and $p$ is the mini poly of $T$, hence $\operatorname{range}(T^2 + bT + cI) \neq V$.

又 $\dim V$ is odd and $\dim \operatorname{null}(T^2 + bT + cI)$ is even ( by our previous result ).

Thus $\dim V - \dim \operatorname{null}(T^2 + bT + cI) = \dim \operatorname{range}(T^2 + bT + cI)$ is odd.

By [5.18], $\operatorname{range}(T^2 + bT + cI)$ is an invar subsp of $V$ under $T$ that has odd dim less than $n$.

Our induction hypothesis now implies that $T|_{\operatorname{range}(T^2+bT+cI)}$ has an eigval.

By mathematical induction. □

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• (2E Ch5.24) *Suppose* $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$ *has no eigvals.*
*Prove that every invar subsp of $V$ under $T$ is even-dim.*

**Solution:**

Suppose $U$ is such a subsp. Then $T|_U \in \mathcal{L}(U)$. We prove by contradiction.

If $\dim U$ is odd, then $T|_U$ has an eigval and so is $T$, so that $\exists$ invar subsp of 1 dim, contradicts. □

---

• (4E 5.B.29) *Show that every operator on a finite-dim vecsp of dim* $\geq 2$ *has a 2-dim invar subsp.*

**Solution:**

Using induction on $\dim V$.

(i) $\dim V = 2$, we are done.

(ii) $\dim V > 2$. Assume that the desired result is true for vecsp of smaller dim.

Suppose $p$ is the mini poly of degree $m$ and $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.

If $T = \lambda I$ ( $\Leftrightarrow m = 1 \lor m = -\infty$ ), then we are done. ( $m \neq 0$ because $\dim V \neq 0$. )

Now define a $q$ by $q(z) = (z - \lambda_1)(z - \lambda_2)$.

By assumption, $T|_{\operatorname{null} q(T)}$ has an invar subsp of dim 2. □

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# 5.B: II

9 14 15 20 | 4E: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 11, 12, 13, 14

- (4E 5.C.1) *Prove or give a counterexample*:
  *If $T \in \mathcal{L}(V)$ and $T^2$ has an upper-trig matrix, then $T$ has an upper-trig matrix.*

**Solution:**

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- (4E 5.C.2) *Suppose $A$ and $B$ are upper-trig matrices of the same size, with $\alpha_1, \dots, \alpha_n$ on the diag of $A$ and $\beta_1, \dots, \beta_n$ on the diag of $B$.*
  (a) *Show that $A + B$ is an upper-trig matrix with $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ on the diag.*
  (b) *Show that $AB$ is an upper-trig matrix with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diag.*

**Solution:**

---

- (4E 5.C.3)
  *Suppose $T \in \mathcal{L}(V)$ is inv and $B = (v_1, \dots, v_n)$ is a basis of $V$ such that $\mathcal{M}(T, B) = A$ is upper trig, with $\lambda_1, \dots, \lambda_n$ on the diag.*
  *Show that the matrix of $\mathcal{M}(T^{-1}, B) = A^{-1}$ is also upper trig, with $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ on the diag.*

**Solution:**

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**9** [4E 5.C.7] *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$, and $v \in V$.*
(a) *Prove that $\exists\,!$ monic poly $p_v$ of smallest degree such that $p_v(T)v = 0$.*
(b) *Prove that the mini poly of $T$ is a poly multi of $p_v$.*

**Solution:**

---

**14** [Or (4E 5.C.4)] *Give an operator $T$ such that with resp to some basis, $\mathcal{M}(T)_{k,k} = 0$ for each $k$, while $T$ is inv.*

**Solution:**

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**15** [Or (4E 5.C.5)] *Give an operator $T$ such that with resp to some basis, $\mathcal{M}(T)_{k,k} \neq 0$ for each $k$, while $T$ is not inv.*

**Solution:**

---

**20** [Or (Or 4E 5.C.6)]
*Suppose $\mathbf{F} = \mathbf{C}$, $V$ is finite-dim, and $T \in \mathcal{L}(V)$.*
*Prove that if $k \in \{1, \dots, \dim V\}$, then $V$ has a $k$ dim subsp invar under $T$.*

**Solution:**

---

- (4E 5.C.8) *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$, and $\exists\, v \in V \setminus \{0\}$ such that $T^2v + 2Tv = -2v$.*
  (a) *Prove that if $\mathbf{F} = \mathbf{R}$, then $\nexists$ a basis of $V$ with resp to which $T$ has an upper-trig matrix.*
  (b) *Prove that if $\mathbf{F} = \mathbf{C}$ and $A$ is an upper-trig matrix that equals the matrix of $T$ with resp to some basis of $V$, then $-1 + \mathrm{i}$ or $-1 - \mathrm{i}$ appears on the diag of $A$.*

**Solution:**

---

- (4E 5.C.9) *Suppose $B \in \mathbf{F}^{n,n}$ with complex entries.*

*Prove that ∃ inv $A \in \mathbf{F}^{n,n}$ with complex entries such that $A^{-1}BA$ is an upper-trig matrix.*

**Solution:**

---

- (4E 5.C.10) *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of $V$.*
  *Show that the following are equi.*
  (a) *The matrix of $T$ with resp to $(v_1, \ldots, v_n)$ is lower trig.*
  (b) $\operatorname{span}(v_k, \ldots, v_n)$ *is invar under $T$ for each $k = 1, \ldots, n$.*
  (c) $Tv_k \in \operatorname{span}(v_k, \ldots, v_n)$ *for each $k = 1, \ldots, n$.*

**Solution:**

---

- (4E 5.C.11) *Suppose $\mathbf{F} = \mathbf{C}$ and $V$ is finite-dim.*
  *Prove that if $T \in \mathcal{L}(V)$, then $T$ has a lower-trig matrix with resp to some basis.*

**Solution:**

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- (4E 5.C.12)
  *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$ has an upper-trig matrix with resp to some basis, and $U$ is a subsp of $V$ that is invar under $T$.*
  (a) *Prove that $T|_U$ has an upper-trig matrix with resp to some basis of $U$.*
  (b) *Prove that $T/U$ has an upper-trig matrix with resp to some basis of $V/U$.*

**Solution:**

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- (4E 5.C.13) *Suppose $V$ is finite-dim, $T \in \mathcal{L}(V)$. Suppose $U$ is an invar subsp of $V$ under $T$ such that $T|_U$ has an upper-trig matrix and also $T/U$ has an upper-trig matrix.*
  *Prove that $T$ has an upper-trig matrix.*

**Solution:**

---

- (4E 5.C.14) *Suppose $V$ is finite-dim and $T \in \mathcal{L}(V)$.*
  *Prove that $T$ has an upper-trig matrix $\iff$ $T'$ has an upper-trig matrix.*

**Solution:**

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**Ended**

# 5.C

xxxx

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**Ended**

# 5.E* (4E)

1 2 3 4 5 6 7 8 9 10

**1** *Give an example of two commuting operators $S, T \in \mathbf{F}^4$ such that there is an invar subsp of $\mathbf{F}^4$ under $S$ but not under $T$ and an invar subsp of $\mathbf{F}^4$ under $T$ but not under $S$.*

**Solution:**

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**2** *Suppose $\mathcal{E}$ is a subset of $\mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagable.*
*Prove that ∃ a basis of $V$ with resp to which*

*every element of $\mathcal{E}$ has a diag matrix $\iff$ every pair of elements of $\mathcal{E}$ commutes.*

*This exercise extends [5.76], which considers the case in which $\mathcal{E}$ contains only two elements.*

*For this exercise, $\mathcal{E}$ may contain any number of elements, and $\mathcal{E}$ may even be an infinite set.*

**SOLUTION:**

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**3** *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.*

(a) *Prove that* $\operatorname{null} p(S)$ *is invar under $T$.*

(b) *Prove that* $\operatorname{range} p(S)$ *is invar under $T$.*

*See* NOTE FOR[5.17] *for the special case $S = T$.*

**SOLUTION:**

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**4** *Prove or give a counterexample:*

*A diag matrix $A$ and an upper-trig matrix $B$ of the same size commute.*

**SOLUTION:**

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**5** *Prove that a pair of operators on a finite-dim vecsp commute $\iff$ their dual operators commute.*

**SOLUTION:**

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**6** *Suppose $V$ is a finite-dim complex vecsp and $S, T \in \mathcal{L}(V)$ commute.*

*Prove that $\exists\, \alpha, \lambda \in \mathbf{C}$ such that* $\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V$.

**SOLUTION:**

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**7** *Suppose $V$ is a complex vecsp, $S \in \mathcal{L}(V)$ is diagable, and $T$ commutes with $S$.*

*Prove that $\exists$ basis $B$ of $V$ such that $S$ has a diag matrix with resp to $B$*

*and $T$ has an upper-trig matrix with resp to $B$.*

**SOLUTION:**

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**8** *Suppose $m = 3$ in Example [5.72]*

*and $D_x, D_y$ are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbf{R}^2)$ from that example.*

*Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ with resp to which $D_x$ and $D_y$ each have an upper-trig matrix.*

**SOLUTION:**

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**9** *Suppose $V$ is a finite-dim nonzero complex vecsp.*

*Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that $S$ and $T$ commute for all $S, T \in \mathcal{E}$.*

(a) *Prove that $\exists\, v \in V$ is an eigvec for every element of $\mathcal{E}$.*

(b) *Prove that $\exists$ a basis of $V$ with resp to which every element of $\mathcal{E}$ has an upper-trig matrix.*

**SOLUTION:**

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**10** *Give an example of two commuting operators $S, T$ on a finite-dim real vecsp such that*

*$S + T$ has a eigval that does not equal an eigval of $S$ plus an eigval of $T$*

*and $ST$ has a eigval that does not equal an eigval of $S$ times an eigval of $T$.*

**SOLUTION:**

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**ENDED**