简介

这是我个人用于复习的笔记，一本习题选答与课文补注。由于我个人的复习特点，我把许多对我个人而言没什么复习价值的习题作了省略。为什么我没有用中文？因为使用中文会给我编撰这份笔记带来额外的中英文输入法切换的工作成本，况且包括我在内的数学专业学生将来要学习的绝大多数课文都是全英的，对于专业学习者，特别是数学专业，直接使用英文不会造成任何困扰。但我很讨厌英文词句的冗长性，这会拖慢我复习的效率，所以我对许多常用词汇适当地作了简写。这份笔记的内容范围和标识说明，我已经在自述中写得很清楚，不再赘述。这份笔记尚处于缓慢的编撰进度中。

Goto

1	2	3	4	5	6	7	8	9	10
A	A	A	/	A	A	A	A	A	A
B	B	B	/	B^{I}	B	B	B	B	B
/	/	/	/	B^{II}	/	/	/	/	/
C	C	C	/	C	C	C	C	/	/
/	/	D	/	/	D	D	D	/	/
/	/	E	/	E*	/	/	/	/	/
/	/	F	/	/	/	F*	/	/	/

Abbreviation Table

def	definition
vec	vector
vecsp	vector space
subsp	subspace
add	addition/additive
multi	multiplication/multiplicative/multiple
assoc	associative/associativity
distr	distributive properties/property
inv	inverse
existns	existence
uniqnes	uniqueness
linely inde	linearly independent/independence
linely dep	linearly dependent/dependence
dim	dimension(al)
inje	injective
surj	surjective
col	column
with resp	with respect
standard basis	std basis
iso	isomorphism/isomorphic
correspd	correspond(ing)
poly	polynomial
eigval	eigenvalue
eigvec	eigenvector
mini poly	minimal polynomial
char poly	characteristic polynomial

1·B

1 *Prove that* $\forall v \in V, -(-v) = v$.

Solution:

$$\left.\begin{aligned} -(-v)+(-v)&=0 \\ v+(-v)&=0 \end{aligned}\right\} \Rightarrow \text{By the uniqnes of add inv, we are done.}$$

Or. $-(-v) = (-1)\big((-1)v\big) = \big((-1)(-1)\big)v = 1 \cdot v = v$. ☐

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2 *Suppose* $a \in \mathbf{F}, v \in V$, *and* $av = 0$. *Prove that* $a = 0$ *or* $v = 0$.

Solution:

Suppose $a \neq 0, \exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. ☐

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3 *Suppose* $v, w \in V$. *Explain why* $\exists\,!\, x \in V, v + 3x = w$.

Solution:

[*Existns*] Let $x = \dfrac{1}{3}(w - v)$.

[*Uniqnes*] Suppose $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) − (II) : $3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. ☐

Or. $v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \frac{1}{3}(w - v)$. ☐

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5 *Show that in the def of a vecsp, the add inv condition can be replaced by* [1.29].

Hint: *Suppose* V *satisfies all conds in the def, except we've replaced the add inv cond with* [1.29]. *Prove that the add inv is true.*

Using [1.31]. $0v = 0$ for all $v \in V \iff (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. ☐

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6 *Let* ∞ *and* $-\infty$ *denote two distinct objects, neither of which is in* $\mathbf{R}$.
Define an add and scalar multi on $\mathbf{R} \cup \{\infty, -\infty\}$ *as you could guess.*
The operations of real numbers is as usual. While for $t \in \mathbf{R}$ *define*

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

(I) $t + \infty = \infty + t = \infty + \infty = \infty$,
(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,
(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of add and scalar multi, is $\mathbf{R} \cup \{\infty, -\infty\}$ *a vecsp over* $\mathbf{R}$? *Explain.*

Solution:

Not a vecpsp, since the add and scalar mult is not assoc and distr.
By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.
Or. By Distr: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. ☐

• **Tips:** *About the Field* $\mathbf{F}$*: Many choices.*
Example: $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \dots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$.

Ended

1·C 7 8 9 11 12 13 15 16 17 18 21 22 23 24

7 *Give a nonempty $U \subseteq \mathbf{R}^2$,*
U is closed under taking add invs and under add, but is not a subsp of $\mathbf{R}^2$.

Solution: $\left(0 \in U;\ v \in U \Rightarrow -v \in U.\right)$ Let $U = \{0,1\}^2, \mathbf{Z}^2, \mathbf{Q}^2$.

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8 *Give a nonempty $U \subseteq \mathbf{R}^2$, U is closed under scalar multi, but is not a subsp of $\mathbf{R}^2$.*

Solution: Let $U = \{(x,y) \in \mathbf{R}^2 : x = 0 \vee y = 0\}$.

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9 *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if $\exists\, p \in \mathbf{N}^+$, $f(x) = f(x+p)$ for all $x \in \mathbf{R}$.*
Is the set of periodic functions $\mathbf{R} \to \mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution: Denote the set by S.

Suppose $h(x) = \cos x + \sin\sqrt{2}x \in S$, since $\cos x, \sin\sqrt{2}x \in S$.

Assume $\exists\, p \in \mathbf{N}^+$ such that $h(x) = h(x+p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$

$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Contradiction! $\square$

Or. Because $[\mathrm{I}] : \cos x + \sin\sqrt{2}x = \cos(x+p) + \sin\left(\sqrt{2}x + \sqrt{2}p\right)$. By differentiating twice,

$[\mathrm{II}] : \cos x + 2\sin\sqrt{2}x = \cos(x+p) + 2\sin\left(\sqrt{2}x + \sqrt{2}p\right)$.

$$\left.\begin{array}{ll} [\mathrm{II}] - [\mathrm{I}] : & \sin\sqrt{2}x = \sin\left(\sqrt{2}x + \sqrt{2}p\right) \\ 2[\mathrm{I}] - [\mathrm{II}] : & \cos x = \cos(x+p) \end{array}\right\} \Rightarrow \text{Let } x = 0,\ p = \frac{m\pi}{\sqrt{2}} = 2k\pi. \text{ Contradicts.}$$ $\square$

• *Suppose U, W, V_1, V_2, V_3 are subsps of V.*

15 $U + U \ni u + w \in U.$ $\square$

16 $U + W \ni u + w = w + u \in W + U.$ $\square$

17 $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3).$ $\square$

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18 *Does the add on the subsps of V have an add identity? Which subsps have add invs?*

Solution: Suppose Ω is the additive identity.

(a) For any subsp U of V. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

(b) Now suppose W is an add inv of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. $\square$

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11 *Prove that the intersection of every collection of subsps of V is a subsp of V.*

Solution: Suppose $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collection of subsps of V; here Γ is an arbitrary index set.

We show that $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vecs that are in U_α for each $\alpha \in \Gamma$, is a subsp of V.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under add.

(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under scalar multi.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of V that is closed under add and scalar multi. $\square$

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12 *Suppose U, W are subsps of V. Prove that $U \cup W$ is a subsp of $V \iff U \subseteq W$ or $W \subseteq U$.*

SOLUTION:

(a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subsp of V.

(b) Suppose $U \cup W$ is a subsp of V. Suppose $U \not\subseteq W$ and $U \not\supseteq W$ ($U \cup W \neq U$ and W).

Then $\forall a \in U \wedge a \notin W, b \in W \wedge b \notin U,\ a + b \in U \cup W$.

$$\left.\begin{array}{l}\text{If } a+b \in U \Rightarrow b = (a+b) + (-a) \in U, \text{ contradicts!} \\ \text{If } a+b \in W \Rightarrow a = (a+b) + (-b) \in W, \text{ contradicts!}\end{array}\right\} \Rightarrow U \cup W = U \text{ or } W. \text{ Contradicts!}$$

Thus $U \subseteq W$ and $U \supseteq W$. ☐

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13 *Prove that the union of three subsps of V is a subsp of V if and only if one of the subsps contains the other two.*

This exercise is not true if we replace **F** *with a field containing only two elements.*

SOLUTION:

Suppose U_1, U_2, U_3 are subsps of V. Denote $U_1 \cup U_2 \cup U_3$ by $\mathcal{U}$.

(a) Suppose that one of the subsps contains the other two.

Then $\mathcal{U} = U_1, U_2$ or U_3 is a subsp of V.

(b) Suppose that $U_1 \cup U_2 \cup U_3$ is a subsp of V.

Distinctively notice that $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.

Also note that, if $U \cup W = V$ is a vecsp, then in general U and W are not subsps of V.

Hence this literal trick is invalid.

(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.

By applying Problem (12) we conclude that one U_j contains the other two. Thus we are done.

(II) Assume that no U_j is contained in the union of the other two,

and no U_j contains the union of the other two.

Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3;\ v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if any $v + \lambda u \in W \cap U_1$ then $v + \lambda u - \lambda u = v \in U_1$.

Now $W \subseteq U_1 \cup U_2 \cup U_3 \Rightarrow W \subseteq U_2 \cup U_3$. $\forall v + \lambda u \in W, v + \lambda u \in U_i, i = 2, 3$.

If $U_2 \subseteq U_3$ or $U_2 \supseteq U_3$, then $\mathcal{U} = U_1 \cup U_i, i = 2, 3$. By Problem (12) we are done.

Otherwise, both $U_2, U_3 \neq \{0\}$. Because $W \subseteq U_2 \cup U_3$ *has at least three elements.*

There must be some U_i that contains at least two elements of W.

$\exists$ distinct $\lambda_1, \lambda_2 \in \mathbf{F}, v + \lambda_1 u, v + \lambda_2 u \in U_i, i \in \{2, 3\}$.

Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Contradicts. ☐

EXAMPLE: Let $\mathbf{F} = \mathbf{Z}_2,\ B_V = (v_1, \dots, v_5)$. Then the proof *above* will not work.

• **EXAMPLE:** *Suppose $U = \{(x, x, y, y) \in \mathbf{F}^4\}, W = \{(x, x, x, y) \in \mathbf{F}^4\}$. Prove that $U + W = \{(x, x, y, z) \in \mathbf{F}^4\}$.*

Let T denote $\{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$. By def, $U + W \subseteq T$.

And $T \ni (x, x, y, z) \Rightarrow (0, 0, y - x, y - x) + (x, x, x, -y + x + z) \in U + W$. Hence $T \subseteq U + W$. ☐

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21 *Suppose* $U = \{(x, y, x+y, x-y, 2x) \in \mathbf{F}^5\}$. *Find a* W *such that* $\mathbf{F}^5 = U \oplus W$.

Solution: Let $W = \{(0, 0, z, w, u) \in \mathbf{F}^5\}$. Then $U \cap W = \{0\}$.
And $\mathbf{F}^5 \ni (x, y, z, w, u) \Rightarrow (x, y, x+y, x-y, 2x) + (0, 0, z-x-y, w-x-y, u-2x) \in U + W$.

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23 *Give an example of vecsps* V_1, V_2, U *such that* $V_1 \oplus U = V_2 \oplus U$, *but* $V_1 \neq V_2$.

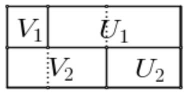
Solution: $V = \mathbf{F}^2$, $U = \{(x, x) \in \mathbf{F}^2\}$, $V_1 = \{(x, 0) \in \mathbf{F}^2\}$, $V_2 = \{(0, x) \in \mathbf{F}^2\}$.

• **Tips:** *Suppose* $V_1 \subseteq V_2$ *in Exercise* (23). *Prove or give a counterexample*: $V_1 = V_2$.

Solution:

Because the subset V_1 of vecsp V_2 is closed under add and scalar multi, V_1 is a subspace of V_2.
Suppose W is such that $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$.
If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, contradicts. Hence $W = \{0\}$, $V_1 = V_2$. □

• *Suppose* V_1, V_2, U_1, U_2 *are vecsps*, $V_1 \oplus U_1 = V_2 \oplus U_2$, $V_1 \subseteq V_2$, $U_2 \subseteq U_1$.
Prove or give a counterexample: $V_1 = V_2$, $U_1 = U_2$.



Solution: A counterexample: [Using notations in Chapter 2.]
Let $V = \mathbf{F}^3$, $B_V = (e_1, e_2, e_3)$, $V_1 = \operatorname{span}(e_1)$, $U_1 = \operatorname{span}(e_2, e_3)$, $V_2 = \operatorname{span}(e_1, e_2)$, $U_2 = \operatorname{span}(e_3)$.
Now $V_1 \subseteq V_2$, $U_2 \subseteq U_1$ and $V_1 \oplus U_1 = V_2 \oplus U_2$. But $V_1 \neq V_2$, $U_1 \neq U_2$. □

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24 *Let* $V_E = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is even}\}$, $V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is odd}\}$. *Show that* $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$.

Solution: (a) $V_E \cap V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x)\} = \{0\}$.

(b) $$\left|\begin{array}{l} \text{Let } f_e(x) = \dfrac{1}{2}[g(x) + g(-x)] \Longrightarrow f_e \in V_E \\ \text{Let } f_o(x) = \dfrac{1}{2}[g(x) - g(-x)] \Longrightarrow f_o \in V_O \end{array}\right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}}, g(x) = f_e(x) + f_o(x).$$ □

Ended

2·A

1 2 6 10 11 14 16 17 | 4E: 3,14

2 (a) $[P]$ *A list* (v) *of length* 1 *in* V *is linely inde* $\Longleftrightarrow v \neq 0$. $[Q]$
(b) $[P]$ *A list* (v, w) *of length* 2 *in* V *is linely inde* $\Longleftrightarrow \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$. $[Q]$

Solution:

(a) $Q \overset{1}{\Rightarrow} P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ linely inde.
$P \overset{2}{\Rightarrow} Q : (v)$ linely inde $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.

Or. $\left|\begin{array}{l} \neg Q \overset{3}{\Rightarrow} \neg P : v = 0 \Rightarrow av = 0 \text{ while we can let } a \neq 0 \Rightarrow (v) \text{ is linely dep.} \\ \neg P \overset{4}{\Rightarrow} \neg Q : (v) \text{ linely dep} \Rightarrow av = 0 \text{ while } a \neq 0 \Rightarrow v = 0. \end{array}\right.$

Comment: (1) with (3) and (2) with (4) will do as well. □

(b) $P \overset{1}{\Rightarrow} Q : (v, w)$ linely inde $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.
$Q \overset{2}{\Rightarrow} P$: no scalar multi $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v, w)$ linely inde.

Or. $\left|\begin{array}{l} \neg P \overset{3}{\Rightarrow} \neg Q : (v, w) \text{ linely dep} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{scalar multi} \\ \neg Q \overset{4}{\Rightarrow} \neg P : \text{scalar multi} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{linely dep.} \end{array}\right.$

Comment: (1) with (3) and (2) with (4) will do as well. □

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1 *Prove that* $[P]$ (v_1, v_2, v_3, v_4) *spans* $V \iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ *also spans* V $[Q]$.

SOLUTION:

Notice that $V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists\, a_1, \dots, a_n \in \mathbf{F}, v = a_1v_1 + \cdots + a_nv_n$.

Assume that $\forall v \in V, \exists\, a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbf{F}$, (that is, if $\exists\, a_i$, then we are to find b_i, vice versa)

$$\begin{aligned} v &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 \\ &= b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4. \end{aligned}$$

Now we can let $b_i = \sum_{r=1}^{i} a_r$ if we are to prove Q with P already assumed;

or let $a_i = b_i - b_{i-1}$ with $b_0 = 0$, if we are to prove P with Q already assumed. $\square$

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6 *Prove that* $[P]$ (v_1, v_2, v_3, v_4) *is linely inde*

$\iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ *is linely inde.* $[Q]$

SOLUTION:

$P \Rightarrow Q : a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4v_4 = 0$

$\Rightarrow a_1v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0 \Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0$

$Q \Rightarrow P : a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$

$\Rightarrow a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \cdots + a_4)v_4 = 0$

$\Rightarrow a_1 = a_1 + a_2 = a_1 + a_2 + a_3 = a_1 + \cdots + a_4 = 0.$ $\square$

• *Suppose* $(v_1, \dots, v_m)$ *is a list of vecs in* V. *For each* k, *let* $w_k = v_1 + \cdots + v_k$.

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

(b) *Show that* $[P]$ $(v_1, \dots, v_m)$ *is linely inde* $\iff (w_1, \dots, w_m)$ *is linely inde* $[Q]$.

SOLUTION:

(a) Assume $a_1v_1 + \cdots + a_mv_m = b_1w_1 + \cdots + b_mw_m = b_1v_1 + \cdots + b_k(v_1 + \cdots + v_k) + \cdots + b_m(v_1 + \cdots + v_m)$.

Then $a_k = b_k + \cdots + b_m;\ \ a_{k+1} = b_{k+1} + \cdots + b_m \Rightarrow b_k = a_k - a_{k+1};\ b_m = a_m$. Similar to Problem (1).

(b) $P \Rightarrow Q$: $\ b_1w_1 + \cdots + b_mw_m = 0 = a_1v_1 + \cdots + a_mv_m$, where $0 = a_k = b_k + \cdots + b_m$.

$Q \Rightarrow P$: $\ a_1v_1 + \cdots + a_mv_m = 0 = b_1w_1 + \cdots + b_mw_m = 0$, where $0 = b_m = a_m$, $0 = b_k = a_k - a_{k+1}$.

OR. Because $W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

By [2.21](b), a list of length $(m - 1)$ spans W, then by [2.23],

$(w_1, \dots, w_m)$ linely dep $\Longrightarrow (v_1, \dots, v_m)$ linely dep. Conversely it is true as well. $\square$

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10 *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* V *and* $w \in V$.

Prove that if $(v_1 + w, \dots, v_m + w)$ *is linely depe, then* $w \in \operatorname{span}(v_1, \dots, v_m)$.

SOLUTION:

Suppose $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0, \exists\, a_i \neq 0 \Rightarrow a_1v_1 + \cdots + a_mv_m = -(a_1 + \cdots + a_m)w$.

Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1v_1 + \cdots + a_mv_m = 0$ while $a_i \neq 0$ for some i, contradicts. $\square$

OR. By contrapositive: Prove that $w \notin \operatorname{span}(v_1, \dots, v_m) \Longrightarrow (v_1 + w, \dots, v_m + w)$ is linely inde.

Suppose $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0 \Rightarrow a_1v_1 + \cdots + a_mv_m = -(a_1 + \cdots + a_m)w$.

Now by assumption, $a_1 + \cdots + a_m = 0$. Then $a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$. $\square$

OR. $\exists\, j \in \{1, \dots, m\}, v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and we are done.

If $j \geqslant 2$, then $\exists\, a_i \in \mathbf{F}, v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1v_1 + \cdots + a_{j-1}v_{j-1}$.

Where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, contradicts.

Now $w = \lambda^{-1}(a_1v_1 + \cdots + a_{j-1}v_{j-1} - v_j) \Rightarrow w \in \operatorname{span}(v_1, \dots, v_m)$. $\square$

---

11 *Suppose $(v_1, \dots, v_m)$ is linely inde in V and $w \in V$.*
Show that $[P]$ $(v_1, \dots, v_m, w)$ is linely inde $\Longleftrightarrow w \notin \operatorname{span}(v_1, \dots, v_m)$ $[Q]$.

Solution: $\neg Q \Rightarrow \neg P$: Suppose $w \in \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1, \dots, v_m, w)$ is linely depe.
$\neg P \Rightarrow \neg Q$: Suppose $(v_1, \dots, v_m, w)$ is linely dep. Then by [2.21](a), $w \in \operatorname{span}(v_1, \dots, v_m)$. □

---

14 *Prove that $[P]$ V is infinite-dim $\Longleftrightarrow [Q]$ there is a sequence $(v_1, v_2, \dots)$ in V such that $(v_1, \dots, v_m)$ is linely inde for each $m \in \mathbf{N}^+$.*

Solution:

$P \Rightarrow Q$: Suppose V is infinite-dim, so that no list spans V.
Step 1 Pick a $v_1 \neq 0$, (v_1) linely inde.
Step m Pick a $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$, by Problem (11), $(v_1, \dots, v_m)$ is linely inde.
This process recursively defines the desired sequence $(v_1, v_2, \dots)$.

$\neg P \Rightarrow \neg Q$: Suppose V is finite-dim and $V = \operatorname{span}(w_1, ..., w_m)$.
Let $(v_1, v_2, \dots)$ be a sequence in V, then $(v_1, v_2, \dots, v_{m+1})$ must be linely dep.

Or. $Q \Rightarrow P$: Suppose there is such a sequence.
Choose an m. Suppose a linely inde list $(v_1, \dots, v_m)$ spans V.
Similar to [2.16]. $\exists\, v_{m+1} \in V \backslash \operatorname{span}(v_1, \dots, v_m)$. Hence no list spans V. □

---

16 *Prove that the vecsp of all continuous functions in $\mathbf{R}^{[0,1]}$ is infinite-dim.*

Solution: Denote the vecsp by U.

Choose one $m \in \mathbf{N}^+$. Suppose $a_0, \dots, a_m \in \mathbf{R}$ are such that $p(x) = a_0 + a_1 x + \cdots + a_m x^m = 0$, $\forall x \in [0,1]$.
Then p has infinitely many roots and hence each $a_k = 0$, otherwise $\deg p \geqslant 0$, contradicts [4.12].
Thus $(1, x, \dots, x^m)$ is linely inde in $\mathbf{R}^{[0,1]}$. Similar to [2.16], U is infinite-dim. □

Or. Note that $\frac{1}{1} > \frac{1}{2} > \cdots > \frac{1}{m}$, $\forall m \in \mathbf{N}^+$. Suppose $f_m = \begin{cases} x - \frac{1}{m}, & x \in \left(\frac{1}{m}, 1\right] \\ 0, & x \in \left[0, \frac{1}{m}\right] \end{cases}$

Then $f_1\left(\frac{1}{m}\right) = \cdots = f_m\left(\frac{1}{m}\right) = 0 \neq f_{m+1}\left(\frac{1}{m}\right)$. Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. By Problem (14). □

---

17 *Suppose $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$.*
Prove that $(p_0, p_1, \dots, p_m)$ is not linely inde in $\mathcal{P}_m(\mathbf{F})$.

Solution:

Suppose $(p_0, p_1, \dots, p_m)$ is linely inde. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z$.
Notice that $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \cdots + a_m p_m(z)$, for if not, let $z = 2$. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has length $(m+1)$.
Hence $(p_0, p_1, \dots, p_m)$ is linely depe in $\mathcal{P}_m(\mathbf{F})$.
For if not, then because $(1, z, \dots, z^m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$,
by the steps in [2.23] trivially, $(p_0, p_1, \dots, p_m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. □

Or. Note that $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(\underbrace{1,\, z,\, \dots,\, z^m}_{\text{of length } (m+1)})$. Then $(p_0, p_1, \dots, p_m, z)$ of length $(m+2)$ is linely dep.
As shown above, $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$. And hence $(p_0, p_1, \dots, p_m)$ is linely dep. □

Ended

2·B 1 7 8 | 4E: 5, 9

7 *Prove or give a counterexample: If (v_1, v_2, v_3, v_4) is a basis of V and U is a subsp of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then (v_1, v_2) is a basis of U.*

Solution: A counterexample:

Let $V = \mathbf{R}^4$ and e_j be the j^{th} standard basis.

Let $v_1 = e_1, v_2 = e_2, v_3 = e_3 + e_4, v_4 = e_4$. Then $(v_1, \dots, v_4)$ is a basis of $\mathbf{R}^4$.

Let $U = \operatorname{span}(e_1, e_2, e_3) = \operatorname{span}(v_1, v_2, v_3 - v_4)$. Then $v_3 \notin U$ and (v_1, v_2) is not a basis of U. $\square$

• **Note For "$\complement_V U \cup \{0\}$":**

"$\complement_V U \cup \{0\}$" is supposed to be a subsp W such that $V = U \oplus W$.

But if we let $u \in U \setminus \{0\}$ and $w \in W \setminus \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cup \{0\} \\ u \pm w \in \complement_V U \cup \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cup \{0\}$. Contradicts.

To fix this, denote the set $\{W_1, W_2 \dots\}$ by $\mathcal{S}_V U$, where for each $W_i, V = U \oplus W_i$. See also in (1.C.23).

---

1 *Find all vecsps that have exactly one basis.*

Solution: The trivial vecsp $\{0\}$ will do. Indeed, the only basis of $\{0\}$ is the empty list.

Now consider a field containing only the add identity 0 and the multi identity 1,

and we specify that $1 + 1 = 0$. Hence the vecsp $\{0, 1\}$ will do, the list (1) will be the unique basis.

And more generally, consider $\mathbf{F} = \mathbf{Z}_m, \forall m - 1 \in \mathbf{N}^+$. For each $s, t \in \{1, \dots, m\}$,

$\mathbf{F} = \operatorname{span}(K_s) = \operatorname{span}(K_t)$. Hence we fail. Are there other vecsps? Suppose so.

(I) Consider $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Let $(v_1, \dots, v_m)$ be a basis of $V \neq \{0\}$.

While there are infinitely many bases distinct from this one. Hence we fail.

(II) Consider other $\mathbf{F}$. Note that a field contains at least 0 and 1

By ***some theories or facts*** given in the course of Elementary Abstract Algebra, we fail. $\square$

• *Suppose $(v_1, \dots, v_m)$ is a list of vecs in V. For $k \in \{1, \dots, m\}$, let $w_k = v_1 + \cdots + v_k$. Show that $[P]\ B_V = (v_1, \dots, v_m) \iff B_W = (w_1, \dots, w_m)$. $[Q]$*

Solution: Notice that $B_U = (u_1, \dots, u_n) \iff \forall u \in U, \exists!\, a_i \in \mathbf{F}, u = a_1 u_1 + \cdots + a_n u_n$.

$P \Rightarrow Q: \forall v \in V, \exists!\, a_i \in \mathbf{F},\ v = a_1 v_1 + \cdots + a_m v_m \Rightarrow v = b_1 w_1 + \cdots + b_m v_m, \exists!\, b_k = a_k - a_{k+1}, b_m = a_m$.

$Q \Rightarrow P: \forall v \in V, \exists!\, b_i \in \mathbf{F},\ v = b_1 w_1 + \cdots + b_m w_m \Rightarrow v = a_1 v_1 + \cdots + a_m v_m, \exists!\, a_k = \sum_{j=k}^{m} b_j$. $\square$

• *Suppose U, W are finite-dim and $V = U + W$. Let $B_U = (u_1, \dots, u_m),\ B_W = (w_1, \dots, w_n)$. Prove that $\exists\, B_V$ consisting of vecs in $U \cup W$.*

Solution: Because $V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.

By [2.10], V is finite-dim. By [2.31], $B_V = (u_1, \dots, u_m, w_1, \dots, w_n)$. $\square$

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8 *Suppose $V = U \oplus W$. Let $B_U = (u_1, \dots, u_m),\ B_W = (w_1, \dots, w_n)$. Prove that $B_V = (u_1, \dots, u_m, w_1, \dots, w_n)$.*

Solution:

$\forall v \in V, \exists!\, u \in U, w \in W \Rightarrow \exists!\, a_i, b_i \in \mathbf{F}, v = u + w = (a_1 u_1 + \cdots + a_m u_m) + (b_1 w_1 + \cdots + b_n w_n)$. $\square$

Or. $V = \operatorname{span}(u_1, \dots, u_m) \oplus \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.

Note that $\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0 \Rightarrow \sum_{i=1}^{m} a_i u_i = -\sum_{i=1}^{n} b_i w_i \in U \cap W = \{0\}$. $\square$

• **Note For** *linely inde sequence and* [2.34]:

"$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expression.

If we allow using "infinite list", then we must guarantee that $(v_1, \dots, v_n, \dots)$ is a spanning "list" such that for all $v \in V$, there exists a smallest positive integer n such that $v = a_1 v_1 + \cdots + a_n v_n$,

The key point is, how can we guarantee that such a "list" exists?

TODO: More details.

Ended

2·C

1 7 9 10 14,16 15 17 | 4E: 10, 14, 15, 16

7 (a) *Let* $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. *Find a basis of* U.

(b) *Extend the basis in* (b) *to a basis of* $\mathcal{P}_4(\mathbf{F})$.

(c) *Find a subsp* W *of* $\mathcal{P}_4(\mathbf{F})$ *such that* $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution: Notice that $\nexists\, p \in \mathcal{P}(\mathbf{F})$ of deg 1 and 2, $p \in U$. Thus $\dim U \leqslant \dim \mathcal{P}_4(\mathbf{F}) - 2 = 3$.

(a) Consider $B = \big(1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

Because $a_0 + a_3(z-2)(z-5)(z-6) + a_4 z(z-2)(z-5)(z-6) = 0 \Rightarrow a_0 = a_3 = a_4$.

The list B is linely inde in U. Now $\dim U \geqslant 3 \Rightarrow \dim U = 3$. Thus $B_U = B$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $\big(1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. □

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9 *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* V *and* $w \in V$.

Prove that $\dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$.

Solution: Using the result of Problem (10) and (11) in 2.A.

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$, for each $i = 1, \dots, m$.

$(v_1, \dots, v_m)$ linely inde $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ linely inde $\Rightarrow \underbrace{(v_2 - v_1, \dots, v_m - v_1)}_{\text{of length } (m-1)}$ linely inde.

又 $w \notin \operatorname{span}(v_1, \dots, v_m) \Rightarrow (v_1 + w, \dots, v_m + w)$ is linely inde.

Hence $m \geqslant \dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$. □

• (4E 2.C.16)

Suppose V *is finite-dim and* U *is a subsp of* V *with* $U \neq V$. *Let* $n = \dim V, m = \dim U$.

Prove that $\exists\, (n-m)$ *subsps* $U_1, \dots, U_{n-m}$, *each of dim* $(n-1)$, *such that* $\bigcap_{i=1}^{n-m} U_i = U$.

Solution:

Let $(v_1, \dots, v_m)$ be a basis of U, extend to a basis of V as $(v_1, \dots, v_m, u_1, \dots, v_{n-m})$.

Define $U_i = \operatorname{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m})$ for each i. Then $U \subseteq U_i$ for each i.

And because $\forall v \in \bigcap_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \cdots + b_{n-m} u_{n-m} \in U_i \Rightarrow b_i = 0$ for each $i \Rightarrow v \in U$.

Hence $\bigcap_{i=1}^{n-m} U_i \subseteq U$. □

Example: Suppose $\dim V = 6, \dim U = 3$.

$$\big(\overbrace{\underbrace{v_1, v_2, v_3}_{\text{Basis of U}}, v_4, v_5, v_6}^{\text{Basis of V}}\big), \text{define} \left|\begin{array}{l} U_1 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_5, v_6) \\ U_2 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_4, v_6) \\ U_3 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_4, v_5) \end{array}\right\} \Rightarrow \dim U_i = 6 - 1,\ i = \underbrace{1, 2, 3}_{6-3=3}.$$ □

---

10 *Suppose $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k.*
Prove that $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Solution: Using mathematical induction on m.

(i) $k = 0, 1.$ $\deg p_0 = 0;$ $\deg p_1 = 1 \Rightarrow \operatorname{span}(p_0, p_1) = \operatorname{span}(1, x).$

(ii) $k \in \{1, \dots, m-1\}$. Assume that $\operatorname{span}(p_0, p_1, \dots, p_k) = \operatorname{span}(1, x, \dots, x^k).$

Then $\operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \subseteq \operatorname{span}(1, x, \dots, x^k, x^{k+1}).$

又 $\deg p_{k+1} = k+1,$ $p_{k+1}(x) = a_{k+1}x^{k+1} + r_{k+1}(x);$ $a_{k+1} \neq 0,$ $\deg r_{k+1} \leqslant k.$

$$\Rightarrow x^{k+1} = \frac{1}{a_{k+1}}\Big(p_{k+1}(x) - r_{k+1}(x)\Big) \in \operatorname{span}(1, x, \dots, x^k, p_{k+1}) = \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}).$$

$\therefore x^{k+1} \in \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \Rightarrow \operatorname{span}(1, x, \dots, x^k, x^{k+1}) \subseteq \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}).$

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m).$ $\square$

Or. 用比较系数法. Denote the coefficient of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Suppose $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}.$

We use induction to show that $a_m = \cdots = a_0 = 0.$

(i) $k = m,$ $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m,$ $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0.$

Now $L = a_{m-1}p_{m-1}(x) + \cdots + a_0 p_0(x).$

(ii) $1 \leqslant k \leqslant m,$ $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k,$ $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0.$

Now $L = a_{k-1}p_{k-1}(x) + \cdots + a_0 p_0(x).$ $\square$

• **Tips:** *Suppose $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ are such that the lowest term of each p_k is of deg k. Prove that $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

Solution: Using mathematical induction on m.

Let each p_k be defined by $p_k(x) = a_{k,k}x^k + \cdots + a_{m,k}x^m$, where $a_{k,k} \neq 0.$

(i) $k = 0, 1.$ $p_m(x) = a_{m,m}x^m;$ $p_{m-1}(x) = a_{m-1,m-1}x^{m-1} + a_{m,m-1}x^m$

$\Rightarrow \operatorname{span}(x^m, x^{m-1}) = \operatorname{span}(p_m, p_{m-1}).$

(ii) $k \in \{1, \dots, m-1\}$. Assume that $\operatorname{span}(x^m, \dots, x^{m-k}) = \operatorname{span}(p_m, \dots, p_{m-k}).$

Then $\operatorname{span}(p_m, \dots, p_{m-(k+1)}) \subseteq \operatorname{span}(x^m, \dots, x^{m-(k+1)}).$

又 $p_{m-(k+1)}$ has the form $a_{m-(k+1),m-(k+1)}x^{m-(k+1)} + r_{m-(k+1)}(x);$

where the lowest term of $r_{m-(k+1)} \in \mathcal{P}_m(\mathbf{F})$ is of deg $(m-k)$.

$$\Rightarrow x^{m-(k+1)} = \frac{1}{a_{m-(k+1),m-(k+1)}}\Big(p_{m-(k+1)}(x) - r_{m-(k+1)}(x)\Big)$$
$$\in \operatorname{span}(x^m, \dots, x^{m-k}, p_{m-(k+1)}) = \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$$

$\therefore x^{m-(k+1)} \in \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$

$\Rightarrow \operatorname{span}(x^m, \dots, x^{m-k}, x^{m-(k+1)}) \subseteq \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}(p_m, \dots, p_1, p_0).$ $\square$

• (4E 2.C.10) *Suppose m is a positive integer. For $0 \leqslant k \leqslant m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show that $(p_0, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

The basis in this exercise leads to what are called Bernstein polys. You can do a web search to learn how Bernstein polys are used to approximate continuous functions on $[0,1]$.

SOLUTION:

Note that each $p_k(x) = \sum_{j=0}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j = \underbrace{(-1)^0 \cdot x^k \cdot 1^0}_{\text{of deg k}} + \underbrace{\sum_{j=1}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j}_{\text{of deg m; denote it by } q_k(x)}$.

And, each $q_k \in \operatorname{span}(x^{k+1}, \dots, x^m)$. Using TIPS above. ☐

OR. Using induction.

(i) $k = 0, 1$. $p_m(x) = x^m$; $p_{m-1}(x) = x^{m-1} - x^m \Longrightarrow x^{m-1} = p_{m-1}(x) + p_m(x)$.

(ii) $k \in \{2, \dots, m-1\}$. Assume that $x^{m-k} = p_{m-k}(x) + a_m x^m + \cdots + a_{m-k+1} x^{m-k+1}$.

Note that $x^{m-k-1} = p_{m-k-1}(x) + \sum_{j=1}^{k+1} C_{k+1}^{j}(-1)^{j+1} x^{j+m-k-1} \Longrightarrow \exists!\, c_{m-i} = C_{k+1}^{k+1-i}(-1)^{k-i}$. (Δ)

Thus each $x^k = b_m p_m(x) + \cdots + b_{m-k} p_{m-k}(x), \exists!\, b_{m-i} \in \mathbf{F}$. $\left[\ (\Delta) \text{ is the core. } \right]$ ☐

COMMENT: Core and context can be inde. Dependence is not a requisite for combination.

OR. For any $m, k \in \mathbf{N}^+$ such that $k \leqslant m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k (1-x)^{m-k}$.

Define the statement $S(m)$ by $S(m) : (p_{0,m}, \dots, p_{m,m})$ is linely inde (and therefore is a basis).

We use induction on to show that $S(m)$ holds for all $m \in \mathbf{N}^+$.

(i) $m = 1$. Let $a_0(1-x) + a_1 x = 0, \forall x \in \mathbf{F}$. Then take $x = 1, x = 0 \Rightarrow a_1 = a_0 = 0$.

$m = 2$. Let $a_0(1-x)^2 + a_1(1-x)x + a_2 x^2, \forall x \in \mathbf{F}$. Then $\begin{cases} x = 0 \Rightarrow a_0 + a_1 = 0; \\ x = 1 \Rightarrow a_2 = 0; \\ x = 2 \Rightarrow a_0 + 2a_1 = 0. \end{cases}$

(ii) $2 \leqslant m$. Assume that $S(m)$ holds.

Suppose $\sum_{k=0}^{m+2} a_k p_{k,m+2}(x) = \sum_{k=0}^{m+2} a_k x^k (1-x)^{m+2-k} = 0, \forall x \in \mathbf{F}$.

While $x = 0 \Rightarrow a_0 = 0$; $x = 1 \Rightarrow a_{m+2} = 0$. Then $\sum_{k=1}^{m+1} a_k x^k (1-x)^{m+2-k} = 0$;

And note that $\sum_{k=1}^{m+1} a_k x^k (1-x)^{m+2-k}$

$$= x(1-x) \sum_{k=1}^{m+1} a_k x^{k-1} (1-x)^{m+1-k}$$
$$= x(1-x) \sum_{k=0}^{m} a_{k+1} x^k (1-x)^{m-k} = x(1-x) \sum_{k=0}^{m} a_{k+1} p_{k,m}(x).$$

Hence $x(1-x) \sum_{k=0}^{m} a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbf{F} \Rightarrow \sum_{k=0}^{m} a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbf{F} \backslash \{0,1\}$.

Because $\sum_{k=0}^{m} a_{k+1} p_{k,m}(x)$ has infinitely many zeros. We have $\sum_{k=0}^{m} a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbf{F}$.

By assumption, $a_1 = \cdots = a_m = 0$, while $a_0 = a_{m+2} = 0$,

and also $a_{m+1} = 0$ (because $\sum_{k=0}^{m} a_{k+1} p_{k,m}(x) = a_{m+1} p_{m,m}(x) = a_{m+1} x^m = 0, \forall x \in \mathbf{F}$.)

Thus $(p_{0,m+2}, \dots, p_{m+2,m+2})$ is linely inde and $S(m+2)$ holds.

Since $\forall m \in \mathbf{N}^+, S(m) \Rightarrow S(m+2)$. We have $\left| \begin{array}{r} \forall k \in \mathbf{N}, S(2k+1) \text{ holds} \\ \forall k \in \mathbf{N}^+, S(2k) \text{ holds} \end{array} \right\} \Rightarrow S(m)$ holds. ☐

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1 [COROLLARY for [2.38,39]] *Suppose U is a subsp of V such that* $\dim V = \dim U$. *Then $V = U$.*

Let $B_U = (u_1, \dots, u_m)$. Then $m = \dim V$. 又 $u_i \in V$. By [2.39], $B_V = (u_1, \dots, u_m)$. ☐

• Let $v_1, \dots, v_n \in V$ and $\dim \operatorname{span}(v_1, \dots, v_n) = n$. Then $(v_1, \dots, v_n)$ is a basis of $\operatorname{span}(v_1, \dots, v_n)$.

Notice that $(v_1, \dots, v_n)$ is a spanning list of $\operatorname{span}(v_1, \dots, v_n)$ *of length* $n = \dim \operatorname{span}(v_1, \dots, v_n)$.

---

14 *Suppose that $V_1, \dots, V_m$ are finite-dim subsps of V.*

Prove that $V_1 + \cdots + V_m$ is finite-dim and $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$.

SOLUTION:

Choose a basis $\mathcal{E}_i$ of $V_i \Rightarrow V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim V_i = \operatorname{card} \mathcal{E}_i$.

Then $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$.

又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.

Thus $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$. □

COMMENT: $\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m \iff V_1 + \cdots + V_m$ *is a direct sum.*

For each i, $(V_1 + \cdots + V_i) \cap V_{i+1} = \{0\} \iff V_1 + \cdots + V_m$ is a direct sum

$\iff (\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$ for each i 又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$

$\iff \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$

$\iff \dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m$. □

---

17 *Suppose V_1, V_2, V_3 are subsps of a finite-dim vecsp, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 \\ -\dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

SOLUTION:

[*Similar to*] Given three sets A, B and C.

Because $|X + Y| = |X| + |Y| - |X \cap Y|$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

Now $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$.

And $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$.

Hence $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$.

Because $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned} \dim(V_1 + V_2 + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim\big((V_1 + V_2) \cap V_3\big) && (1) \\ &= \dim(V_2 + V_3) + \dim(V_1) - \dim\big((V_2 + V_3) \cap V_1\big) && (2) \\ &= \dim(V_1 + V_3) + \dim(V_2) - \dim\big((V_1 + V_3) \cap V_2\big) && (3) \end{aligned}$$

Notice that in general, $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$.

For example, $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

• **COROLLARY:** *Suppose V_1, V_2 and V_3 are finite-dim vecsps, then* $\dfrac{(1) + (2) + (3)}{3}$:

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 \\ - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\ - \frac{\dim\big((V_1 + V_2) \cap V_3\big) + \dim\big((V_1 + V_3) \cap V_2\big) + \dim\big((V_2 + V_3) \cap V_1\big)}{3}.$$

The formula above may seem strange because the right side does not look like an integer. □

• *Suppose V is a 10-dim vecsp and V_1, V_2, V_3 are subsps of V with*

(a) $\dim V_1 = \dim V_2 = \dim V_3 = 7$. *Prove that* $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

By TIPS, $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 \underline{-2\dim V} > 0$.

(b) $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. *Prove that* $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

By TIPS, $\dim(V_1 \cap V_2 \cap V_3) \underline{\geq 2\dim V} - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)) \geq 0$. □

• **TIPS:**

Because $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big)$.

And $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$. We have (1), and (2), (3) similarly.

(1) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 + V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big)$.

(2) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_3) - \dim\big(V_2 + (V_1 \cap V_3)\big)$.

(3) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim\big(V_3 + (V_1 \cap V_2)\big)$.

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15 *Suppose V is finite-dim and $\dim V = n \geqslant 1$.*
Prove that $\exists$ one-dim subsps $V_1, \dots, V_n$ of V such that $V = V_1 \oplus \cdots \oplus V_n$.

SOLUTION: Suppose $B_V = (v_1, \dots, v_n)$. Define V_i by $V_i = \operatorname{span}(v_i)$ for each $i \in \{1, \dots, n\}$.
Then $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n \Rightarrow \exists!\, u_i \in V_i, v = u_1 + \cdots + u_n$ □

• **COROLLARY:** *Suppose W is finite-dim, $\dim W = m$ and $w \in W \setminus \{0\}$.*
Prove that $\exists\, B_W = (w_1, \dots, w_m)$ such that $w = w_1 + \cdots + w_m$.

By Problem (15), $\exists$ one-dim subsps $W_1, \dots, W_m$ of W such that $W = W_1 \oplus \cdots \oplus W_m$.

Note that $\dim W_i = \dim \operatorname{span}(w_i) = 1 \Rightarrow \forall x_i \in W_i, \exists!\, c_i \in \mathbf{F}, x_i = c_i w_i$.

Suppose $w = x_1 + \cdots + x_m$, where each $x_i = c_i w_i \in W_i$. Then $(x_1, \dots, x_m)$ is also a basis of W. □

OR. Note that $w \neq 0 \Rightarrow m \geqslant 1$. If $m = 1$ then let $w_1 = w$ and we are done. Suppose $m > 1$.

Extend (w) to a basis $(w, w_1, \dots, w_{m-1})$ of W. Let $w_m = w - w_1 - \cdots - w_{m-1}$.

又 $\operatorname{span}(w, w_1, \dots, w_{m-1}) = \operatorname{span}(w_1, \dots, w_m)$. Hence $(w_1, \dots, w_m)$ is also a basis of W. □

• **NEW THEOREM:** *Suppose V is finite-dim with $\dim V = n$ and U is a subsp of V with $U \neq V$.*
Prove that $\exists\, B_V = (v_1, \dots, v_n)$ such that each $v_k \notin U$.

Note that $U \neq V \Rightarrow n \geqslant 1$. We will construct B_V via the following process.

Step 1. $\exists\, v_1 \in V \setminus U \Rightarrow v_1 \neq 0$. If $\operatorname{span}(v_1) = V$ then we stop.

Step k. Suppose $(v_1, \dots, v_{k-1})$ is linely inde in V, each of which belongs to $V \setminus U$.
Note that $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$. And if $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U = V$, then by (1.C.12),
(because $\operatorname{span}(v_1, \dots, v_{k-1}) \not\subseteq U$,) $U \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) \Rightarrow \operatorname{span}(v_1, \dots, v_{k-1}) = V$.
Hence because $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$, it must be case that $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U \neq V$.
Thus $\exists\, v_k \in V \setminus U$ such that $v_k \notin \operatorname{span}(v_1, \dots, v_{k-1})$.
By (2.A.11), $(v_1, \dots, v_k)$ is linely inde in V. If $\operatorname{span}(v_1, \dots, v_k) = V$, then we stop.

Because V is finite-dim, this process will stop after n steps. □

OR. If $U = \{0\}$ then we are done. Let $\dim U \geqslant 1$.

Let $(u_1, \dots, u_m)$ be a basis of U. Extend to a basis $(u_1, \dots, u_n)$ of V.

Then let $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$. □

ENDED

3·A

3 4 5 7 8 10 11 12 13 | 4E: 10, 11, 17

- **Tips 1:** $T: V \to W$ is linear $\iff \left| \begin{array}{l} (一)\ \forall v,u \in V, T(v+u) = Tv + Tu; \\ (二)\ \forall v,u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv). \end{array} \right| \iff T(v+\lambda u) = Tv + \lambda Tu.$

- **Tips 2:** $T \in \mathcal{L}(V,W) \iff T \in \mathcal{L}(V, \operatorname{range} T)$.

- **Tips 3:** If U is a subsp of W, then $\{T \in \mathcal{L}(V,W) : \operatorname{range} T \subseteq U\} = \mathcal{L}(V,U)$.

- (4E 1.B.7) *Suppose $V \neq \emptyset$ and W is a vecsp. Let $W^V = \{f : V \to W\}$.*
  (a) *Define a natural add and scalar multi on W^V.*
  (b) *Prove that W^V is a vecsp with these definitions.*

Solution:

(a) $W^V \ni f + g : x \to f(x) + g(y)$; where $f(x) + g(y)$ is the vec add on W.
$W^V \ni \lambda f : x \to \lambda f(x)$; where $\lambda f(x)$ is the scalar multi on W.

(b) Commutativity: $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$.
Associativity: $((f+g)+h)(x) = (f(x)+g(x)) + h(x)$
$= f(x) + (g(x)+h(x)) = (f + (g+h))(x)$.
Additive Identity: $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$.
Additive Inverse: $(f+g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0 = 0(x)$.
Distributive Properties:
$(a(f+g))(x) = a(f+g)(x) = a(f(x)+g(x))$
$= af(x) + ag(x) = (af)(x) + (ag)(x) = (af+ag)(x)$.
Similarly, $((a+b)f)(x) = (af+bf)(x)$.
So far, we have used the same properties in W.
Which means that ***if** W^V **is a vecsp, then** W **must be a vecsp.***
Multiplication Identity: $(1f)(x) = 1f(x) = f(x)$. (Notice that the smallest $\mathbf{F}$ is $\{0,1\}$.) □

---

5 *Because $\mathcal{L}(V,W) = \{T : V \to W \mid T \text{ is linear}\}$ is a subsp of W^V, $\mathcal{L}(V,W)$ is a vecsp.*

---

3 *Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists A_{j,k} \in \mathbf{F}$ such that for any $(x_1, \dots, x_n) \in \mathbf{F}^n$,*

$$T(x_1,\dots,x_n) = \begin{pmatrix} A_{1,1}x_1 + \cdots + A_{1,n}x_n, \\ \vdots \qquad \ddots \qquad \vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n \end{pmatrix}$$

Solution:

Let $T(1,0,0,\dots,0,0) = (A_{1,1},\dots,A_{m,1})$, Note that $(1,0,\dots,0,0), \cdots, (0,0,\dots,0,1)$ is a basis of $\mathbf{F}^n$.
$T(0,1,0,\dots,0,0) = (A_{1,2},\dots,A_{m,2})$, Then by [3.5], we are done. □
$\vdots$
$T(0,0,0,\dots,0,1) = (A_{1,n},\dots,A_{m,n})$.

---

4 *Suppose $T \in \mathcal{L}(V,W)$, and $v_1,\dots,v_m \in V$ such that $(Tv_1,\dots,Tv_m)$ is linely inde in W. Prove that $(v_1,\dots,v_m)$ is linely inde.*

Solution: Suppose $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$. □

---

7 *Show that every linear map from a one-dim vecsp to itself is a multi by some scalar. More precisely, prove that if* $\dim V = 1$ *and* $T \in \mathcal{L}(V)$*, then* $\exists \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$.

SOLUTION: Let u be a nonzero vec in $V \Rightarrow V = \operatorname{span}(u)$. Because $Tu \in V \Rightarrow Tu = \lambda u$ for some λ.
Suppose $v \in V \Rightarrow v = au$, $\exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. □

---

8 *Give a function* $\varphi : \mathbf{R}^2 \to \mathbf{R}$ *such that* $\forall a \in \mathbf{R}, v \in \mathbf{R}^2, \varphi(av) = a\varphi(v)$ *but* φ *is not linear.*

SOLUTION: Define $T(x,y) = \begin{cases} x+y, & \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, & \text{otherwise.} \end{cases}$ OR. Define $T(x,y) = \sqrt[3]{(x^3+y^3)}$. □

---

9 *Give a function* $\varphi : \mathbf{C} \to \mathbf{C}$ *such that* $\forall w,z \in \mathbf{C}, \varphi(w+z) = \varphi(w) + \varphi(z)$ *but* φ *is not linear.* (*Here* $\mathbf{C}$ *is thought of as a complex vecsp.*)

SOLUTION: Suppose $V_{\mathbf{C}}$ is the complexification of a vecsp V. Suppose $\varphi : V_{\mathbf{C}} \to V_{\mathbf{C}}$.
Define $\varphi(u+iv) = u = \operatorname{Re}(u+iv)$ OR. Define $\varphi(u+iv) = v = \operatorname{Im}(u+iv)$. □

• *Prove that if* $q \in \mathcal{P}(\mathbf{R})$ *and* $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ *is defined by* $\underbrace{Tp = q \circ p}_{\text{composition}}$*, then* T *is not linear.*

SOLUTION: ***Composition and product are not the same in*** $\mathcal{P}(\mathbf{F})$.
Because in general, $q \circ (p_1 + \lambda p_2)(x) = q(p_1(x) + \lambda p_2(x)) \neq (q \circ p_1)(x) + \lambda(q \circ p_2)(x)$.
EXAMPLE: Let q be defined by $q(x) = x^2$, then $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$. □

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10 *Suppose* U *is a subsp of* V *with* $U \neq V$. *Suppose* $S \in \mathcal{L}(U,W)$ *with* $S \neq 0$ (*which means that* $\exists\, u \in U, Su \neq 0$).

Define $T : V \to W$ *by* $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \backslash U. \end{cases}$ *Prove that* T *is not a linear map on* V.

SOLUTION:

Suppose T is a linear map. And $v \in V\backslash U$, $u \in U$ such that $Su \neq 0$.
Then $v+u \in V\backslash U$, (for if not, $v = (v+u)-u \in U$) while $T(v+u) = 0 = Tv+Tu = 0+Su \Rightarrow Su = 0$.
Hence we get a contradiction. □

---

11 *Suppose* U *is a subsp of* V *and* $S \in \mathcal{L}(U,W)$.
Prove that $\exists\, T \in \mathcal{L}(V,W), Tu = Su, \forall u \in U$. (OR. $\exists\, T \in \mathcal{L}(V,W), T|_U = S$.)
In other words, every linear map on a subsp of V *can be extended to a linear map on the entire* V.

SOLUTION: Suppose W is such that $V = U \oplus W$. Then $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.
Define $T \in \mathcal{L}(V,W)$ by $T(u_v + w_v) = Su_v$. □
OR. [*Finite-dim Req*] Define by $T\Big(\sum_{i=1}^{m} a_i u_i\Big) = \sum_{i=1}^{n} a_i S u_i$. Let $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$. □

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12 *Suppose nonzero* V *is finite-dim and* W *is infinite-dim. Prove that* $\mathcal{L}(V,W)$ *is infinite-dim.*

SOLUTION:

Let $(v_1, \dots, v_n)$ be a basis of V. Let $(w_1, \dots, w_m)$ be linely inde in W for any $m \in \mathbf{N}^+$.
Define $T_{x,y} : V \to W$ by $T_{x,y}(v_z) = \delta_{z,x} w_y$, $\forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$
$\forall v = \sum_{i=1}^{n} a_i v_i$, $u = \sum_{i=1}^{n} b_i v_i$, $\lambda \in \mathbf{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) v_y = T_{x,y}(v) + \lambda T_{x,y}(u)$.
Linearity checked. Now suppose $a_1 T_{x,1} + \cdots + a_m T_{x,m} = 0$.
Then $(a_1 T_{x,1} + \cdots + a_m T_{x,m})(v_x) = 0 = a_1 w_1 + \cdots + a_m w_m \Rightarrow a_1 = \cdots = a_m = 0$. 又 m arbitrary.
Thus $(T_{x,1}, \dots, T_{x,m})$ is a linely inde list in $\mathcal{L}(V,W)$ for any x and length m. Hence by (2.A.14). □

13 *Suppose $(v_1,\dots,v_m)$ is linely depe in V and $W\neq\{0\}$.*
Prove that $\exists\, w_1,\dots,w_m\in W, \nexists\, T\in\mathcal{L}(V,W)$ such that $Tv_k=w_k, \forall k=1,\dots,m$.

SOLUTION:

We prove by contradiction. By linear dependence lemma, $\exists\, j\in\{1,\dots,m\}, v_j\in\operatorname{span}(v_1,\dots,v_{j-1})$.
Fix j. Let $w_j\neq 0$, while $w_1=\dots=w_{j-1}=w_{j+1}=w_m=0$.
Define T by $Tv_k=w_k$ for each k. Suppose $a_1v_1+\dots+a_mv_m=0$ (where $a_j\neq 0$).
Then $T(a_1v_1+\dots+a_mv_m)=0=a_1w_1+\dots+a_mw_m=a_jw_j$ while $a_j\neq 0$ and $w_j\neq 0$. Contradicts. □

OR. We prove the contrapositive: Suppose $\forall w_1,\dots,w_m\in W, \exists\, T\in\mathcal{L}(V,W), Tv_k=w_k$ for each w_k.
Now we show that $(v_1,\dots,v_n)$ is linely inde. Suppose $\exists\, a_i\in\mathbf{F}, a_1v_1+\dots+a_nv_n=0$.
Choose one $w\in W\backslash\{0\}$. By assumption, for $(\overline{a_1}w,\dots,\overline{a_m}w), \exists\, T\in\mathcal{L}(V,W), Tv_k=\overline{a_k}w$ for each v_k.
Now we have $0=T\Big(\sum_{k=1}^m a_kv_k\Big)=\sum_{k=1}^m a_kTv_k=\sum_{k=1}^m a_k\overline{a_k}w=\Big(\sum_{k=1}^m|a_k|^2\Big)w$.
Then $\sum_{k=1}^m|a_k|^2=0\Rightarrow a_k=0$ for each k. Hence $(v_1,\dots,v_n)$ is linely inde. □

• (4E 3.A.17)
Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE\in\mathcal{E}, ET\in\mathcal{E}, \ \forall E\in\mathcal{E}, T\in\mathcal{L}(V)$.

SOLUTION: Let $(v_1,\dots,v_n)$ be a basis of V. If $\mathcal{E}=0$, then we are done.
Suppose $\mathcal{E}\neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S\in\mathcal{E}\backslash\{0\}$.
Suppose $Sv_i\neq 0$ and $Sv_i=a_1v_1+\dots+a_nv_n$, where $a_k\neq 0$.
Define $R_{x,y}\in\mathcal{L}(V)$ by $R_{x,y}(v_x)=v_y, R_{x,y}(v_z)=0\ (z\neq x)$. OR. $R_{x,y}v_z=\delta_{z,x}v_y$.
Then $(R_{1,1}+\dots+R_{n,n})v_j=v_j\Rightarrow\sum_{r=1}^n R_{r,r}=I$. Assume that each $R_{x,y}\in\mathcal{E}$.
Hence $\forall T\in\mathcal{L}(V), I\circ T=T\circ I=T\in\mathcal{E}\Rightarrow\mathcal{E}=\mathcal{L}(V)$. Now we prove the assumption.
Notice that $\forall x,y\in\mathbf{N}^+, \ (R_{k,y}S)(v_i)=a_kv_y\Rightarrow\big((R_{k,y}S)\circ R_{x,i}\big)(v_z)=\delta_{z,x}(a_kv_y)$.
Thus $R_{k,y}SR_{x,i}=a_kR_{x,y}$. Now $S\in\mathcal{E}\Rightarrow R_{k,y}S\in\mathcal{E}\Rightarrow R_{x,y}\in\mathcal{E}$. □

• (4E 3.B.32)
Suppose V is finite-dim with $n=\dim V>1$.
Show that if $\varphi:\mathcal{L}(V)\to\mathbf{F}$ is linear and $\forall S,T\in\mathcal{L}(V), \varphi(ST)=\varphi(S)\cdot\varphi(T)$, then $\varphi=0$.

SOLUTION:

Using notations in (4E 3.A.17). Using the result in NOTE FOR [3.60].
Suppose $\varphi\neq 0\Rightarrow\exists\, i,j\in\{1,\dots,n\}, \varphi(R_{i,j})\neq 0$. Because $R_{i,j}=R_{x,j}\circ R_{i,x}, \ \forall x=1,\dots,n$
$\Rightarrow\varphi(R_{i,j})=\varphi(R_{x,j})\cdot\varphi(R_{i,x})\neq 0\Rightarrow\varphi(R_{x,j})\neq 0$ and $\varphi(R_{i,x})\neq 0$.
Again, because $R_{i,x}=R_{y,x}\circ R_{i,y}, \ \forall y=1,\dots,n$. Thus $\varphi(R_{y,x})\neq 0, \ \forall x,y=1,\dots,n$.
Let $k\neq i, j\neq l$ and then $\varphi(R_{i,j}\circ R_{l,k})=\varphi(R_{l,k}\circ R_{i,j})=\varphi(0)=0=\varphi(R_{l,k})\cdot\varphi(R_{i,j})$
$\Rightarrow\varphi(R_{l,k})=0$ or $\varphi(R_{i,j})=0$. Contradicts. □

OR. Note that by (4E 3.A.17), $\exists\, S,T\in\mathcal{L}(V), ST-TS\neq 0$.
Then $\varphi(ST-TS)=\varphi(S)\varphi(T)-\varphi(T)\varphi(S)=0\Rightarrow ST-TS\in\operatorname{null}\varphi\neq\{0\}$.
Note that $\forall\, E\in\operatorname{null}\varphi, T\in\mathcal{L}(V), \varphi(ET)=\varphi(TE)=0\Rightarrow ET, TE\in\operatorname{null}\varphi$.
Hence $\operatorname{null}\varphi$ is a nonzero two-sided ideal of $\mathcal{L}(V)$. □

• *Suppose V is finite-dim. $T \in \mathcal{L}(V)$ is such that $\forall S \in \mathcal{L}(V), ST = TS$. Prove that $\exists \lambda \in \mathbf{F}, T = \lambda I$.*

SOLUTION:

If $V = \{0\}$, then we are done. Now suppose $V \neq \{0\}$.

Assume that (v, Tv) is linely depe for every $v \in V$, then by (2.A.2.(b)), $Tv = \lambda_v v$ for some $\lambda_v \in \mathbf{F}$.

To prove that λ_v is independent of v, we discuss in two cases:

$$\left.\begin{array}{l} (-)\text{ If } (v,w) \text{ is linely inde, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \\ \qquad\qquad\qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=)\text{ Otherwise, suppose } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w \end{array}\right\} \Rightarrow \lambda_w = \lambda_v.$$

Now we show the assumption. Assume that (v, Tv) is linely inde for some v. Let $B_V = (v, Tv, u_1, \dots, u_n)$.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Contradicts. □

OR. Let $(v_1, \dots, v_m)$ be a basis of V.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$. Let $\lambda = \varphi(Tv_1) \in \mathbf{F}$.

For any $v \in V$, define $S_v \in \mathcal{L}(V)$ by $S_v u = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. □

OR. For each $k \in \{1, \dots, n\}$, define $S_k \in \mathcal{L}(V)$ by $S_k v_j = \begin{cases} v_k, & j = k, \\ 0, & j \neq k. \end{cases}$ OR. $S_k v_j = \delta_{j,k} v_k$

Note that $S_k\left(\sum_{i=1}^n a_i v_i \right) = a_k v_k$. Then $S_k v = v \iff \exists !\, a_k \in \mathbf{F}, v = a_k v_k$.

Hence $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$.

Define $A^{(j,k)} \in \mathcal{L}(V)$ by $A^{(j,k)} v_j = v_k, A^{(j,k)} v_k = v_j, A^{(j,k)} v_x = 0, x \neq j, k$.

Then $A^{(j,k)} Tv_j = TA^{(j,k)} v_j = Tv_k = a_k v_k$; $A^{(j,k)} Tv_j = A^{(j,k)} a_j v_j = a_j A^{(j,k)} v_j = a_j v_k$.

Hence $a_k = a_j$. Thus a_k is independent of v_k. □

ENDED

3·B

3 7 8 9 10 11 12 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31
4E: 24, 27, 31, 32, 33

• *Suppose that V and W are real vecsps and $T \in \mathcal{L}(V, W)$.*
Define $T_{\mathrm{C}} : V_{\mathrm{C}} \to W_{\mathrm{C}}$ by $T_{\mathrm{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv$ for all $u, v \in V$.
Show that (a) *T_{C} is linear,* (b) *T_{C} is inje $\Longleftrightarrow$ T is inje,* (c) *T_{C} is surj $\Longleftrightarrow$ T is surj.*

SOLUTION:

(a) $\forall u_1 + \mathrm{i}v_1, u_2 + \mathrm{i}v_2 \in V_{\mathrm{C}}, \lambda \in \mathbf{F}$,

$$T\big((u_1 + \mathrm{i}v_1) + \lambda(u_2 + \mathrm{i}v_2)\big) = T\big((u_1 + \lambda u_2) + \mathrm{i}(v_1 + \lambda v_2)\big) = T(u_1 + \lambda u_2) + \mathrm{i}T(v_1 + \lambda v_2)$$
$$= Tu_1 + \mathrm{i}Tv_1 + \lambda Tu_2 + \mathrm{i}\lambda Tv_2 = T(u_1 + \mathrm{i}v_1) + \lambda T(u_2 + \mathrm{i}v_2).$$

(b) Suppose T_{C} is inje. Let $T(u) = 0 \Rightarrow T_{\mathrm{C}}(u + \mathrm{i}0) = Tu = 0 \Rightarrow u = 0$.
Suppose T is inje. Let $T_{\mathrm{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + \mathrm{i}v = 0$.

(c) Suppose T_{C} is surj. $\forall w \in W, \exists\, u \in V, T(u + \mathrm{i}0) = Tu = w + \mathrm{i}0 = w \Rightarrow$ T is surj.
Suppose T is surj. $\forall w, x \in W, \exists\, u, v \in V, Tu = w, Tv = x$
$\Rightarrow \forall w + \mathrm{i}x \in W_{\mathrm{C}}, \exists\, u + \mathrm{i}v \in V, T(u + \mathrm{i}v) = w + \mathrm{i}x \Rightarrow T_{\mathrm{C}}$ is surj.

---

3 *Suppose $(v_1, \dots, v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1 v_1 + \cdots + z_m v_m$.*
(a) *The surj of T correspds to $(v_1, \dots, v_m)$ spanning V.*
(b) *The inje of T correspds to $(v_1, \dots, v_m)$ being linely inde.*

COMMENT: Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $Te_k = v_k$.
(a) range $T = \operatorname{span}(v_1, \dots, v_m) = V$; (b) $(v_1, \dots, v_m)$ is linely inde $\Longleftrightarrow$ T is inje.

---

7 *Suppose V is finite-dim with $2 \leqslant \dim V$. And $\dim V \leqslant \dim W = m$, if W is finite-dim.*
Show that $U = \big\{T \in \mathcal{L}(V, W) : \operatorname{null} T \neq \{0\}\big\}$ is not a subsp of $\mathcal{L}(V, W)$.

SOLUTION: *The set of all inje $T \in \mathcal{L}(V, W)$ is a not subsp either.*

Let $(v_1, \dots, v_n)$ be a basis of V, $(w_1, \dots, w_m)$ be linely inde in W. $(\, 2 \leqslant n \leqslant m.\,)$
Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 : \ v_1 \mapsto 0, \quad v_2 \mapsto w_2, \quad v_i \mapsto w_i$.
Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 : \ v_1 \mapsto w_1, \quad v_2 \mapsto 0, \quad v_i \mapsto w_i, \quad i = 3, \dots, n$.
Thus $T_1 + T_2 \notin U$. ☐

COMMENT: If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\ T \in \mathcal{L}(V, W), T$ is inje. Hence $U = \emptyset$.
If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0 v_0 = 0$.

---

8 *Suppose W is finite-dim with $\dim W \geqslant 2$. And $n = \dim V \geqslant \dim W$, if V is finite-dim.*
Show that $U = \big\{T \in \mathcal{L}(V, W) : \operatorname{range} T \neq W\big\}$ is not a subsp of $\mathcal{L}(V, W)$.

SOLUTION: *The set of all surj $T \in \mathcal{L}(V, W)$ is not a subspace either.*

Let $(v_1, \dots, v_n)$ be linely inde in V, $(w_1, \dots, w_m)$ be a basis of W. $(\, n \in \{m, m+1, \dots\}; 2 \leqslant m \leqslant n.\,)$
Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 : \ v_1 \mapsto 0, \quad v_2 \mapsto w_2, \quad v_j \mapsto w_j, \quad v_{m+i} \mapsto 0$.
Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 : \ v_1 \mapsto w_1, \quad v_2 \mapsto 0, \quad v_j \mapsto w_j, \quad v_{m+i} \mapsto 0$.
$(\,$For each $j = 2, \dots, m$; $i = 1, \dots, n - m$, if V is finite, otherwise let $i \in \mathbf{N}^+$. $)$ Thus $T_1 + T_2 \notin U$. ☐

COMMENT: If $\dim W = 0$, then $W = \{0\} = \operatorname{span}(\,)$. $\forall\ T \in \mathcal{L}(V, W), T$ is surj. Hence $U = \emptyset$.
If $\dim W = 1$, then $W = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0 v_0 = 0$.

---

11 *Suppose $S_1, \dots, S_n$ are linear and inje. $S_1 S_2 \dots S_n$ makes sence. Prove that $S_1 S_2 \dots S_n$ is inje.*

SOLUTION: $S_1 S_2 \dots S_n(v) = 0 \Longleftrightarrow S_2 S_3 \dots S_n(v) = 0 \Longleftrightarrow \cdots \Longleftrightarrow S_n(v) = 0 \Longleftrightarrow v = 0$. ☐

---

9 *Suppose* $(v_1, \dots, v_n)$ *is linely inde. Prove that* $\forall$ *inje* $T, (Tv_1, \dots, Tv_n)$ *is linely inde.*

SOLUTION: $a_1Tv_1 + \cdots + a_nTv_n = 0 = T\left(\sum_{i=1}^{n} a_i v_i\right) \iff \sum_{i=1}^{n} a_i v_i = 0 \iff a_1 = \cdots = a_n = 0.$ $\square$

---

10 *Suppose* $\operatorname{span}(v_1, \dots, v_n) = V$. *Show that* $\operatorname{span}(Tv_1, \dots, Tv_n) = \operatorname{range} T$.

SOLUTION:

(a) $\operatorname{range} T = \{Tv : v \in V\} = \{Tv : v \in \operatorname{span}(v_1, \dots, v_n)\} \Rightarrow Tv_1, \dots, Tv_n \in \operatorname{range} T \Rightarrow$ By [2.7].
OR. $\operatorname{span}(Tv_1, \dots, Tv_n) \ni a_1Tv_1 + \cdots + a_nTv_n = T(a_1v_1 + \cdots + a_nv_n) \in \operatorname{range} T$.

(b) $\forall w \in \operatorname{range} T, \exists v \in V, w = Tv. (\exists a_i \in \mathbf{F}, v = a_1v_1 + \cdots + a_nv_n) \Rightarrow w = a_1Tv_1 + \cdots + a_nTv_n.$ $\square$

---

16 *Suppose* $\exists T \in \mathcal{L}(V)$ *such that* $\operatorname{null} T, \operatorname{range} T$ *are finite-dim. Prove that* V *is finite-dim.*

SOLUTION: Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n), B_{\operatorname{null} T} = (u_1, \dots, u_m)$.
$\forall v \in V, T(v - a_1v_1 - \cdots - a_nv_n) = 0$, letting $Tv = a_1Tv_1 + \cdots + a_nTv_n$.
$\Rightarrow v - a_1v_1 - \cdots - a_nv_n = b_1u_1 + \cdots + b_mu_m$. Hence $V \subseteq \operatorname{span}(v_1, \dots, v_n, u_1, \dots, u_m)$. $\square$

---

17 *Suppose* V, W *are finite-dim. Prove that* $\exists$ *inje* $T \in \mathcal{L}(V, W) \iff \dim V \leqslant \dim W$.

SOLUTION:

(a) Suppose $\exists$ inje T. Then $\dim V = \dim \operatorname{range} T \leqslant \dim W$.

(b) Suppose $\dim V \leqslant \dim W$. Let $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.
Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i, \ i = 1, \dots, n \ (= \dim V)$. $\square$

---

18 *Suppose* V, W *are finite-dim. Prove that* $\exists$ *surj* $T \in \mathcal{L}(V, W) \iff \dim V \geqslant \dim W$.

SOLUTION:

(a) Suppose $\exists$ surj T. Then $\dim V = \dim W + \dim \operatorname{null} T \Rightarrow \dim W \leqslant \dim V$.

(b) Suppose $\dim V \geqslant \dim W$. Let $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.
Define $T \in \mathcal{L}(V, W)$ by $T(a_1v_1 + \cdots + a_mv_m + \cdots + a_nv_n) = a_1w_1 + \cdots + a_mw_m$. $\square$

---

19 *Suppose* V, W *are finite-dim,* U *is a subsp of* V.
Prove that if $\underbrace{\dim U}_{m} \geqslant \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_{p}$, *then* $\exists T \in \mathcal{L}(V, W), \ \operatorname{null} T = U$.

SOLUTION:

Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_W = (w_1, \dots, w_p)$.
Define $T \in \mathcal{L}(V, W)$ by $T(a_1v_1 + \cdots + a_nv_n + b_1u_1 + \cdots + b_mu_m) = a_1w_1 + \cdots + a_nw_n$. $\square$

• (4E 3.B.21)
Suppose V *is finite-dim,* $T \in \mathcal{L}(V, W)$, U *is a subsp of* W. *Let* $\mathcal{K}_U = \{v \in V : Tv \in U\}$.
Prove that $\mathcal{K}_U$ *is a subsp of* V *and* $\dim \mathcal{K}_U = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$.

SOLUTION:

$\forall u, w \in \mathcal{K}_U, \lambda \in \mathbf{F}, T(u + \lambda w) = Tu + \lambda Tw \in U \Rightarrow \mathcal{K}_U$ is a subsp of V.
Define $S \in \mathcal{L}(\mathcal{K}_U, U)$ as $Rv = Tv$ for all $v \in \mathcal{K}_U$. Hence $\operatorname{range} R = U \cap \operatorname{range} T$.
Suppose $\exists v, Tv = 0$. 又 $0 \in U \Rightarrow Rv = 0$. Thus $\operatorname{null} T \subseteq \operatorname{null} R$. $\square$

• **TIPS:** *Suppose* U *is a subsp of* V. *Prove that* $\forall T \in \mathcal{L}(V, W), U \cap \operatorname{null} T = \operatorname{null} T|_U$.

SOLUTION: Note that $U \cap \operatorname{null} T \subseteq \operatorname{null} T|_U$. On the other hand, suppose $u \in \operatorname{null} T|_U$.
Then $T|_U(u)$ makes sense $\Rightarrow u \in U$. And $T|_U(u) = Tu = 0 \Rightarrow u \in \operatorname{null} T$. $\square$

---

12 *Prove that* $\forall T \in \mathcal{L}(V,W), \exists$ *subsp* U *of* V *such that*
$U \cap \operatorname{null} T = \operatorname{null} T|_U = \{0\}$, $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$.
Which is equivalent to $T|_U : U \to \operatorname{range} T$ *being an iso.*

SOLUTION:
By [2.34] (note that V can be infinite-dim), $\exists$ subsp U of V such that $V = U \oplus \operatorname{null} T$.
$\forall v \in V, \exists!\, w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w+u) = Tu \in \{Tu : u \in U\}$. □

• **NEW NOTATION:**
Suppose $T \in \mathcal{L}(V,W)$ and $R = (Tv_1, \dots, Tv_n)$ is linely inde in range T.
Where $n = \dim \operatorname{range} T$ if finite-dim, otherwise $n \in \mathbf{N}^+$.
By (3.A.4), $L = (v_1, \dots, v_n)$ is linely inde in V.
Denote $\mathcal{K}_R$ by span L, if range T is finite-dim, otherwise, denote it by a vecsp in $\mathcal{S}_V \operatorname{null} T$.
Note that if range T is finite-dim, then $\mathcal{K}_R = \operatorname{range} T$ for any basis R of range T.

• **COMMENT:**
If range T is infinite-dim, we cannot write $\mathcal{K}_R = \operatorname{range} T$. For if we do so, we must guarantee that $\forall Tv \in \operatorname{range} T, \exists!\, n \in \mathbf{N}^+, Tv \in \operatorname{span}(Tv_1, \dots, Tv_n)$, where $(Tv_k)_{k=1}^{\infty}$ is linely inde.
So that range $T \subseteq$ span $(Tv_1, \cdots, Tv_n, \cdots)$. This would be invalid, as we have shown before.

• **NEW THEOREM:** $\mathcal{K}_R \in \mathcal{S}_V \operatorname{null} T$. **COMMENT:** $\operatorname{null} T \in \mathcal{S}_V \mathcal{K}_R$.
Suppose range T is finite-dim. Otherwise, we are done immediately.

(a) $T\left(\sum_{i=1}^{n} a_i v_i\right) = 0 \Rightarrow \sum_{i=1}^{n} a_i T v_i = 0 \Rightarrow a_1 = \cdots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \operatorname{null} T = \{0\}$.

(b) $\forall v \in V, Tv = \sum_{i=1}^{n} a_i T v_i \Rightarrow Tv - \sum_{i=1}^{n} a_i T v_i = T\left(v - \sum_{i=1}^{n} a_i v_i\right) = 0$
$\Rightarrow v - \sum_{i=1}^{n} a_i v_i \in \operatorname{null} T \Rightarrow v = \left(v - \sum_{i=1}^{n} a_i v_i\right) + \left(\sum_{i=1}^{n} a_i v_i\right) \Rightarrow \mathcal{K}_R + \operatorname{null} T = V$. □

• *Suppose* V *is finite-dim,* $T \in \mathcal{L}(V,W), B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n), B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$.
Prove or give a counterexample: $(u_1, \dots, u_m)$ *is a basis of* $\operatorname{null} T$.

SOLUTION: A counterexample:
Suppose $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$. Then $\operatorname{span}(Tv_1, Tv_2, Tv_3) = \operatorname{span}(w_1)$.
Extend (v_i) to (v_1, v_2, v_3) for each i. But none of $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is a basis of null T. □

COMMENT: $(v_2 - v_1, v_3 - v_1), (v_1 - v_2, v_3 - v_2)$ or $(v_1 - v_3, v_2 - v_3)$ are all bases of null T.
Always notice that $\mathcal{S}_V \operatorname{span}(v_1, \dots, v_n) = \{U_1, \cdots, \operatorname{null} T, \cdots, U_n, \cdots\}$.

• *Suppose* V *is finite-dim,* X *is a subsp of* V*, and* Y *is a finite-dim subsp of* W.
Prove that if $\dim X + \dim Y = \dim V$, *then* $\exists\, T \in \mathcal{L}(V,W), \operatorname{null} T = X, \operatorname{range} T = Y$.

SOLUTION:
Suppose $\dim X + \dim Y = \dim V$. Let $B_X = (u_1, \dots, u_n), B_Y = (w_1, \dots, w_m), B_V = (u_1, \dots, u_n, v_1, \dots, v_m)$.
Define $T \in \mathcal{L}(V,W)$ by $Tv_i = w_i, Tu_j = 0$. Notice that $\forall v \in V, \exists!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i$.
$v \in \operatorname{null} T \iff Tv = 0 \iff a_1 = \cdots = a_m = 0 \iff v \in X$.
$Y \ni w = a_1 w_1 + \cdots + a_m w_m = a_1 T v_1 + \cdots + a_m T v_m \in \operatorname{range} T$.
OR. range $T = \operatorname{span}(Tv_1, \dots, Tv_m, Tu_1, \dots, Tu_n) = \operatorname{span}(Tv_1, \dots, Tv_m) = \operatorname{span}(w_1, \dots, w_m) = Y$. □

• OR (5.B.4) *Suppose* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove that* $V = \operatorname{null} P \oplus \operatorname{range} P$.

SOLUTION:

(a) If $v \in \operatorname{null} P \cap \operatorname{range} P \Rightarrow Pv = 0$ and $\exists\, u \in V, v = Pu$. Then $v = Pu = P^2u = Pv = 0$.

(b) Note that $\forall v \in V, v = Pv + (v - Pv)$ and $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$. □

OR. [*Only in Finite-dim*] Let $(P^2v_1, \dots, P^2v_n)$ be a basis of range P^2. Then $(Pv_1, \dots, Pv_n)$ is linely inde.
Let $\mathcal{K} = \operatorname{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \operatorname{null} P^2$. While $\mathcal{K} = \operatorname{range} P = \operatorname{range} P^2$; $\operatorname{null} P = \operatorname{null} P^2$. □

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20 *Suppose W is finite-dim. Prove that* $T \in \mathcal{L}(V, W)$ *is inje* $\iff \exists\, S \in \mathcal{L}(W, V),\ ST = I_V$.

SOLUTION:

(a) Suppose $\exists\, S \in \mathcal{L}(W, V),\ ST = I$. Then if $Tv = 0 \Rightarrow ST(v) = 0 = v$. OR. $\operatorname{null} T \subseteq \operatorname{null} ST = \{0\}$.

(b) Suppose T is inje. Let $R = B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$. Then $\mathcal{K}_R \oplus \operatorname{null} T = V$. Let $U \oplus \operatorname{range} T = W$.
Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$ and $Su = 0$, where $i \in \{1, \dots, n\}, u \in U$. Thus $ST = I$.

OR. Define $S \in \mathcal{L}(\operatorname{range} T, V)$ by $Sw = T^{-1}w$, *where* T^{-1} *is the inv of* $T \in \mathcal{L}(V, \operatorname{range} T)$.
Then extend it to $S \in \mathcal{L}(W, V)$ by (3.A.11). Now $\forall v \in V, STv = T^{-1}Tv = v$. □

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21 *Suppose W is finite-dim. Prove that* $T \in \mathcal{L}(V, W)$ *is surj* $\iff \exists\, S \in \mathcal{L}(W, V),\ TS = I_W$.

SOLUTION:

(a) Suppose $\exists\, S \in \mathcal{L}(W, V),\ TS = I$. Then $\forall\, w \in W, TS(w) = w \in \operatorname{range} T \Rightarrow \operatorname{range} T = W$.

(b) Suppose T is surj. Let $R = B_{\operatorname{range} T} = B_W = (Tv_1, \dots, Tv_n)$. Then $\mathcal{K}_R \oplus \operatorname{null} T = V$.
Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$. Then $TS = I$.

OR. By Problem (12), $\exists$ subsp U of $V, V = U \oplus \operatorname{null} T, \operatorname{range} T = \{Tu : u \in U\}$.
Note that $T|_U : U \to W$ is an iso. Define $S = (T|_U)^{-1}$, where $(T|_U)^{-1} : W \to U$.
Then $TS = T \circ (T|_U)^{-1} = T|_U \circ (T|_U)^{-1}$. □

---

24 *Suppose* $S, T \in \mathcal{L}(V, W)$. *Prove that* $\operatorname{null} S \subseteq \operatorname{null} T \iff \exists\, E \in \mathcal{L}(W)$ *such that* $T = ES$.

SOLUTION:

Suppose $\exists\, E \in \mathcal{L}(W)$ such that $T = ES$. Then $\operatorname{null} T = \operatorname{null} ES \supseteq \operatorname{null} S$.
Suppose $\operatorname{null} S \subseteq \operatorname{null} T$. Let $W = \operatorname{range} S \oplus U$.
Define $E \in \mathcal{L}(W)$ by $E(Sv + w) = Tv$ for each Sv and each $w \in U$. Now we check that E is linear.
Because $\forall w_1, w_2 \in W, \exists!\, Sv_1, Sv_2 \in \operatorname{range} S, u_1, u_2 \in U, w_1 = Sv_1 + u_1, w_2 = Sv_2 + u_2$.
Now $E(w_1 + \lambda w_2) = E\big((Sv_1 + \lambda Sv_2) + (u_1 + \lambda u_2)\big) = T(v_1 + \lambda v_2) = Tv_1 + \lambda Tv_2 = Ew_1 + \lambda Ew_2$.

OR. Let $V = \mathcal{K} \oplus U$. Then $S|_{\mathcal{K}} : \mathcal{K} \to \operatorname{range} S$ is an iso.
Now extend $T(S|_{\mathcal{K}})^{-1} \in \mathcal{L}(\operatorname{range} S, W)$ to $E \in \mathcal{L}(W, W)$.

OR. [*Requires that* range S *is Finite-dim*] Let $R = B_{\operatorname{range} S} = (Sv_1, \dots, Sv_n)$. Then $V = \mathcal{K}_R \oplus \operatorname{null} S$.
Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i,\ Eu = 0$; for each $i = 1 \dots, n$ and each $u \in \operatorname{null} S$.
Hence $\forall v \in V,\ (\exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} S),\ Tv = a_1Tv_1 + \dots + a_nTv_n = E(a_1Sv_1 + \dots + a_nSv_n) \Rightarrow T = ES$.

OR. [*Requires that W is Finite-dim*] Extend R to a basis $(Sv_1, \dots, Sv_n, w_1, \dots, w_m)$ of W.
Define $E \in \mathcal{L}(W)$ by $E(Sv_k) = Tv_k,\ Ew_j = 0$. Because $\forall\, v \in V, \exists\, a_i \in \mathbf{F}, Sv = a_1Sv_1 + \dots + a_nSv_n$.
Now $v - (a_1v_1 + \dots + a_nv_n) \in \operatorname{null} S \Rightarrow v - (a_1v_1 + \dots + a_nv_n) \in \operatorname{null} T \Rightarrow T\big(v - (a_1v_1 + \dots + a_nv_n)\big) = 0$.
Thus $Tv = a_1v_1 + \dots + a_nv_n$. Hence $E(Sv) = a_1E(Sv_1) + \dots + a_nE(Sv_n) = a_1Tv_1 + \dots + a_nTv_n = Tv$. □

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25 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{range} S \subseteq \operatorname{range} T \iff \exists\, E \in \mathcal{L}(V)$ *such that* $S = TE$.

Solution:

Suppose $\exists\, E \in \mathcal{L}(V)$ such that $S = TE$. Then $\operatorname{range} S = \operatorname{range} TE \subseteq \operatorname{range} T$.
Suppose $\operatorname{range} S \subseteq \operatorname{range} T$. Let $(v_1, \dots, v_m)$ be a basis of V.
Note that each $Sv_i \in \operatorname{range} T$. Suppose $u_i \in V$ such that $Tu_i = Sv_i$.
Thus defining $E \in \mathcal{L}(V)$ by $Ev_i = u_i$ for each $i \Rightarrow S = TE$. □

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22 *Suppose U and V are finite-dim vecsps and $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.*
Prove that $\dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$.

Solution:

Define $R \in \mathcal{L}(\operatorname{null} ST, V)$ by $Ru = Tu$ for all $u \in \operatorname{null} ST \subseteq U$.

$$\left.\begin{aligned} S(Tu) = 0 = S(Ru) \Rightarrow \operatorname{range} R \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} R \leqslant \dim \operatorname{null} S \\ Tu = 0 = Ru \Rightarrow \operatorname{null} R \supseteq \operatorname{null} T \Rightarrow \dim \operatorname{null} R = \dim \operatorname{null} T \end{aligned}\right\} \Rightarrow \text{By [3.22], we are done.} \ \square$$

Or. For any $u \in U$, note that $u \in \operatorname{null} ST \iff S(Tu) = 0 \iff Tu \in \operatorname{null} S$.
Thus $\operatorname{null} ST = \mathcal{K}_{\operatorname{null} S \cap \operatorname{range} T} = \{u \in U : Tu \in \operatorname{null} S\}$. By Problem (4E 3B.21),
$\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T) \leqslant \dim \operatorname{null} T + \dim \operatorname{null} S$. □

Corollary: (1) If T is inje, then $\dim \operatorname{null} T = 0 \Rightarrow \dim \operatorname{null} ST \leqslant \dim \operatorname{null} S$.
(2) If T is surj, then $\operatorname{range} R = \operatorname{null} S \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.
(3) If S is inje, then $\operatorname{range} R = \{0\} \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} R = \dim \operatorname{null} T$.

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23 *Suppose U and V are finite-dim vecsps and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$.*
Prove that $\dim \operatorname{range} ST \leqslant \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.

Solution:

$\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{span}(Su_1, \dots, Su_{\dim \operatorname{range} T})$, where $B_{\operatorname{range} T} = (u_1, \dots, u_{\dim \operatorname{range} T})$.
$\dim \operatorname{range} ST \leqslant \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leqslant \dim \operatorname{range} S$. □

Or. Note that $\operatorname{range} S|_{\operatorname{range} T} = \operatorname{range} ST$.
Thus $\dim \operatorname{range} ST = \dim \operatorname{range} S|_{\operatorname{range} T} = \dim \operatorname{range} T - \dim \operatorname{null} S|_{\operatorname{range} T} \leqslant \operatorname{range} T$. □

Corollary: (1) If S is inje, then $\dim \operatorname{range} ST = \dim \operatorname{range} T$.
(2) If T is surj, then $\dim \operatorname{range} ST = \dim \operatorname{range} S$.

• (a) *Suppose* $\dim V = 5$, $S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$. *Prove that* $\dim \operatorname{range} TS \leqslant 2$.
(b) *Let* $\dim V = n$ *in* (a). *Prove that* $\dim \operatorname{range} TS \leqslant \left\lfloor \dfrac{n}{2} \right\rfloor$.
(c) *Give an example of $S, T \in \mathcal{L}(\mathbf{F}^5)$ with $ST = 0$ and* $\dim \operatorname{range} TS = 2$.

Solution:

(a) By Problem (23), $\dim \operatorname{range} TS \leqslant \min\{\overbrace{\dim \operatorname{range} S}^{5-\dim \operatorname{null} T}, \overbrace{\dim \operatorname{range} T}^{5-\dim \operatorname{null} S}\}$.
We show that $\dim \operatorname{range} TS \leqslant 2$ by contradiction. Assume that $\dim \operatorname{range} TS \geqslant 3$.
Then $\min\{5 - \dim \operatorname{null} T, 5 - \dim \operatorname{null} S\} \geq 3 \Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant 2$.
又 $\dim \operatorname{null} ST = 5 \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \leqslant 4$. Contradicts.

Or. $$\left.\begin{aligned} \dim \operatorname{null} S = 5 - \dim \operatorname{range} S \\ \dim \operatorname{range} TS \leqslant \dim \operatorname{range} S \end{aligned}\right\} \Rightarrow \dim \operatorname{null} S \leqslant 5 - \dim \operatorname{range} TS.$$

And $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S$. □

(b) By Problem (23), $\dim \operatorname{range} TS \leqslant \min\{ \overbrace{\dim \operatorname{range} S}^{n-\dim \operatorname{null} T}, \overbrace{\dim \operatorname{range} T}^{n-\dim \operatorname{null} S} \}$. We prove by contradiction.

Assume that $\dim \operatorname{range} TS \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Then $\min\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\} \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$

$\Rightarrow \max\{ \dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant n - \left\lfloor \frac{n}{2} \right\rfloor - 1$.

又 $\dim \operatorname{null} ST = n \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \leqslant 2(n - \left\lfloor \frac{n}{2} \right\rfloor - 1)$

$\Rightarrow \left\lfloor \frac{n}{2} \right\rfloor + 1 \leqslant \frac{n}{2}$. Contradicts. Thus $\dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$. □

OR. $\dim \operatorname{null} S = n - \dim \operatorname{range} S \leqslant n - \dim \operatorname{range} TS$.

And $ST = 0 \Rightarrow \dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S \leqslant n - \dim \operatorname{range} TS$

$\Rightarrow 2 \dim \operatorname{range} TS \leqslant n \Rightarrow \dim \operatorname{range} TS \leqslant \frac{n}{2}$

$\Rightarrow \dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$ (because $\dim \operatorname{range} TS$ is an integer). □

(c) Let $(v_1, \dots, v_5)$ be a basis of $\mathbf{F}^5$. Define $S, T \in \mathcal{L}(\mathbf{F}^5)$ by:

$T: \quad v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i \ ;$

$S: \quad v_1 \mapsto v_4, \quad v_2 \mapsto v_5, \quad v_i \mapsto 0 \ ; \qquad i = 3, 4, 5.$ □

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26 *Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ such that $\forall p \in \mathcal{P}(\mathbf{R}), \deg(Dp) = (\deg p) - 1$. Prove that $D \in \mathcal{P}(\mathbf{R})$ is surj.*

SOLUTION:

[*Informal Proof*] | Note that $\deg Dx^n = n - 1$. Because $\operatorname{span}(Dx, Dx^2, \dots) \subseteq \operatorname{range} D$.
又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, \dots) = \operatorname{span}(1, x, \dots) = \mathcal{P}(\mathbf{R})$.

[*Proper Proof*]

We will recursively define a sequence of polys $(p_k)_{k=0}^{\infty}$ where $Dp_k = x^k$.

(i) Because $\dim Dx = (\deg x) - 1 = 0$, we have $Dx = C \in \mathbf{F}$.
Define $p_0 = C^{-1}x$. Then $Dp_0 = C^{-1}Dx = 1$.

(ii) Suppose we have defined $p_0, \dots, p_n$ such that $Dp_k = x^k$ for each $k \in \{0, \dots, n\}$.
Because $\deg D(x^{n+2}) = n + 1$. Let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$, where $a_{n+1} \neq 0$.
Then $a_{n+1}^{-1} D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_n Dp_n + \cdots + a_1 Dp_1 + a_0 Dp_0)$
$\Rightarrow x^{n+1} = D\big(a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)\big)$.
Thus defining $p_{n+1} = a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)$, we have $Dp_{n+1} = x^{n+1}$.

Now we get $(p_k)_{k=0}^{\infty}$ by recursion. Hence $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), \exists\, q = \big(\sum_{k=0}^{\deg p} a_k p_k\big), Dq = p$. □

OR. Let $Dx^0 = 0, Dx^k = p_k$ for all $k \in \mathbf{N}^+$. For any $m \in \mathbf{N}^+$, $(p_1, \dots, p_m)$ is a basis of $\mathcal{P}_{m-1}(\mathbf{R})$.
Because $\forall p' \in \operatorname{range} D, \exists!\, m \in \mathbf{N}, \deg p = m - 1 \Rightarrow \exists!\, a_k \in \mathbf{R}, p' = a_m p_m + \cdots + a_1 p_1$.
Now $Dp = p' = a_m p_m + \cdots + a_1 p_1 = D(a_m x^m + \cdots + a_1 x)$. Thus $\exists\, q \in \mathcal{P}_m(\mathbf{R}), Dq = p$. □

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27 *Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.*

SOLUTION:

Define $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$.
Note that $\deg Bx^n = n - 1$. Similar to Problem (26), we conclude that B is surj. □

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28 *Suppose* $T \in \mathcal{L}(V,W), B_{\operatorname{range} T} = (w_1, \dots, w_m)$.
Prove that $\exists\, \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ *such that* $\forall v \in V, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.

SOLUTION:

Suppose $v_1, \dots, v_m \in V$ such that $Tv_i = w_i$ for each v_i. Then $(v_1, \dots, v_m)$ is linely inde.
Let $B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$. Note that $\forall v \in V, v = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i, \exists\,!\, a_i, b_i \in \mathbf{F}$.
Define $\varphi_i : V \to \mathbf{F}$ by $\varphi_i(v) = a_i v_i$ for each i. We now check the linearity.
$\forall v, u \in V\ (\exists\,!\, a_i, b_i, c_i, d_i \in \mathbf{F}\,), \lambda \in \mathbf{F}, \varphi_i(v + \lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda\varphi(u)$. □

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29 *Suppose* $\varphi \in \mathcal{L}(V, \mathbf{F})$. *Suppose* $u \in V \backslash \operatorname{null}\varphi$. *Prove that* $V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}$.

SOLUTION: If $\varphi = 0$ then we are done. Suppose $\varphi \neq 0$.

(a) $\forall\, v = cu \in \operatorname{null}\varphi \cap \{au : a \in \mathbf{F}\}, \varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$. Hence $\operatorname{null}\varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$.

(b) $$\forall\, v \in V, v = \left(v - \frac{\varphi(v)}{\varphi(u)}u\right) + \frac{\varphi(v)}{\varphi(u)}u. \left|\ \begin{array}{l} v - \dfrac{\varphi(v)}{\varphi(u)}u \in \operatorname{null}\varphi \\[2ex] \dfrac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array} \right\} \Rightarrow V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}.$$ □

COMMENT: $\varphi \neq 0 \Rightarrow \varphi(v_i) = a_i \neq 0$ for each v_i, for some linely inde list $(v_1, \dots, v_k)$.
Fix one v_k. Then $\forall j \in \{1, \dots, k-1, k+1, \dots, n\}, \operatorname{span}\{a_j v_k - a_k v_j\} \subseteq \operatorname{null}\varphi$.
Hence every vecsp in $\mathcal{S}_V \operatorname{null}\varphi$ is one-dim.

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30 *Suppose* $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$ *and* $\operatorname{null}\varphi_1 = \operatorname{null}\varphi_2 = \operatorname{null}\varphi$. *Prove that* $\exists\, c \in \mathbf{F}, \varphi_1 = c\varphi_2$

SOLUTION:

If $\operatorname{null}\varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $u \in V \backslash \operatorname{null}\varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$.
By Problem (29), $V = \operatorname{null}\varphi \oplus \operatorname{span}(u)$. Hence for any $v \in V, v = w + a_v u, \exists\,!\, w \in \operatorname{null}\varphi, a_v \in \mathbf{F}$.
$$\varphi_1(v) = a_v\varphi_1(u),\ \ \varphi_2(v) = a_v\varphi_2(u) \Rightarrow a_v = \frac{\varphi_1(v)}{\varphi_1(u)} = \frac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \frac{\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}.$$ □

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31 *Prove that* $\exists\, T_1, T_2 \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^2), \operatorname{null} T_1 = \operatorname{null} T_2$ *and* $T_1 \neq cT_2, \forall c \in \mathbf{F}$.

SOLUTION:

Let $(v_1, \dots, v_5)$ be a basis of $\mathbf{R}^5$, (w_1, w_2) be a basis of $\mathbf{R}^2$. Define $T, S \in \mathcal{L}(V, W)$ by
$$\left.\begin{array}{lll} Tv_1 = w_1, & Tv_2 = w_2, & Tv_3 = Tv_4 = Tv_5 = 0 \\ Sv_1 = w_1, & Sv_2 = 2w_2, & Sv_3 = Sv_4 = Sv_5 = 0 \end{array}\right\} \Rightarrow \operatorname{null} T = \operatorname{null} S.$$
Suppose $T = \lambda S$. Then $w_1 = Tv_1 = \lambda S v_1 = \lambda w_1 \Rightarrow \lambda = 1$.
While $w_2 = Tv_2 = \lambda S v_2 = 2\lambda w_2 \Rightarrow \lambda = \frac{1}{2}$. Contradicts. □

• **TIPS:** *Suppose* $T \in \mathcal{L}(V,W)$ *and* U *is a subsp such that* $V = U \oplus \operatorname{null} T$.
Now $\forall v \in V, \exists\,!\, u_v \in U, w_v \in \operatorname{null} T, v = u_v + w_v$.
Then $T = T \circ i$, *where* $i : V \to U$ *is defined by* $i(v) = u_v$.
Because $\forall v \in V, T(v) = T(u_v + w_v) = T(u_v) = T(i(v)) = (T \circ i)(v)$. □

ENDED

3·C

1 3 4 5 6 9 10 11 12 13 14 15 | 4E: 16, 17

• **Note For [3.47]:** *LHS* $= (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k} =$ *RHS*.

• **Note For [3.48]:**

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6&7\\8&9&10\end{pmatrix} = \begin{pmatrix}\begin{pmatrix}1&2\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} & \begin{pmatrix}1&2\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & \begin{pmatrix}1&2\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix}\\ \begin{pmatrix}3&4\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} & \begin{pmatrix}3&4\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & \begin{pmatrix}3&4\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}1\times5+2\times8 & 1\times6+2\times9 & 1\times7+2\times10\\ 3\times5+4\times8 & 3\times6+4\times9 & 3\times7+4\times10\end{pmatrix} = \begin{pmatrix}21&24&27\\47&54&61\end{pmatrix}$$

• [4E 3.51] *Suppose* $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}$.

(a) For $k = 1, \dots, p$, $\quad (CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^{c} C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$

Which means that each cols CR is a linear combination of the cols of C.

(b) For $j = 1, \dots, m$, $\quad (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

Which means that each rows CR is a linear combination of the rows of R.

• **Column-Row Factorization** (CR Factorization) *Suppose* $A \in \mathbf{F}^{m,n}, A \neq 0$.

(a) *Let* $S_c = \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}, \dim S_c = c$, *the col rank.*

Prove that $\exists\, C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,n}, A = CR$.

(b) *Let* $S_r = \operatorname{span}(A_{1,\cdot}, \dots, A_{m,\cdot}) \subseteq \mathbf{F}^{1,n}, \dim S_r = r$, *the row rank.*

Prove that $\exists\, C \in \mathbf{F}^{m,r}, R \in \mathbf{F}^{r,n}, A = CR$.

Solution: Notice that $A \neq 0 \Rightarrow c, r \geqslant 1$.

(a) Let $(C_{\cdot,1}, \dots, C_{\cdot,c})$ be a basis of S_c, forming $C \in \mathbf{F}^{m,c}$. Then $\forall k \in \{1, \dots, n\}$,

$A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}, \exists!\, R_{1,k}, \dots, R_{c,k} \in \mathbf{F}$, forming $R \in \mathbf{F}^{c,n}$. Thus $A = CR$.

(b) Let $(R_{1,\cdot}, \dots, R_{r,\cdot})$ be a basis of S_r, forming $R \in \mathbf{F}^{r,n}$. Then $\forall j \in \{1, \dots, m\}$,

$A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,r}R_{r,\cdot} = (CR)_{j,\cdot}, \exists!\, C_{j,1}, \dots, C_{j,r} \in \mathbf{F}$, forming $C \in \mathbf{F}^{m,r}$. Thus $A = CR$. ☐

Example:

$$A = \begin{pmatrix}10&7&4&1\\26&19&12&5\\46&33&20&7\end{pmatrix} \overset{(1)}{=} \begin{pmatrix}1&0\\0&1\\2&1\end{pmatrix}\begin{pmatrix}10&7&4&1\\26&19&12&5\end{pmatrix} \overset{(2)}{=} \begin{pmatrix}7&4\\19&12\\33&20\end{pmatrix}\begin{pmatrix}2&-1\\1&0\\0&1\\-1&2\end{pmatrix}$$

(I) $\begin{pmatrix}46&33&20&7\end{pmatrix} = 2\begin{pmatrix}10&7&4&1\end{pmatrix} + \begin{pmatrix}26&19&12&5\end{pmatrix}$.

$\begin{pmatrix}46&33&20&7\end{pmatrix}$ can be uniquely written as a linear combination of $(A_{1,\cdot}, A_{2,\cdot})$.

Hence $\dim S_r = 2$. $(A_{1,\cdot}, A_{2,\cdot})$ is a basis.

(II) $\begin{pmatrix}10\\26\\46\end{pmatrix} = 2\begin{pmatrix}7\\19\\33\end{pmatrix} - \begin{pmatrix}4\\12\\20\end{pmatrix}$; $\quad \begin{pmatrix}1\\5\\7\end{pmatrix} = 2\begin{pmatrix}4\\12\\20\end{pmatrix} - \begin{pmatrix}7\\19\\33\end{pmatrix}$. Hence $\dim S_c = 2$. $(A_{\cdot,2}, A_{\cdot,3})$ is a basis.

• Column Rank Equals Row Rank (Using the notation and result above)
For each $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
For each $A_{\cdot,k} \in S_c$, $A_{\cdot,k} = (CR)_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$.
$\Rightarrow \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = S_r = \operatorname{span}(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leqslant c = \dim S_c$.
$\Rightarrow \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,m}) = S_r = \operatorname{span}(C_{\cdot,1}, \dots, C_{\cdot,r}) \Rightarrow \dim S_c = c \leqslant r = \dim S_r$.
Or. Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leqslant r = \dim S_r = \dim S_c^t$. □

• [4E 3.C.17, Or 3.F.32] *Suppose $T \in \mathcal{L}(V)$ and $(u_1, \dots, u_n), (v_1, \dots, v_n)$ are bases of V. Prove that the following are equi.* *Here $A = \mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.*
(a) *T is inje.*
(b) *The cols of $\mathcal{M}(T)$ are linely inde in $\mathbf{F}^{n,1}$.*
(c) *The cols of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.*
(d) *The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.*
(e) *The rows of $\mathcal{M}(T)$ are linely inde in $\mathbf{F}^{1,n}$.*

Solution: Using Tips in 2.C.

$$T \text{ is inje} \iff \dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim \operatorname{range} T$$

$$\Delta \begin{cases} \iff (Tu_1, \dots, Tu_n) \text{ is a basis of } V; \ \dim \operatorname{range} T = \dim \operatorname{span}(\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n)) = n \\ \iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n)) \text{ is a basis of } \mathbf{F}^{n,1}, \text{ as well as } (A_{\cdot,1}, \dots, A_{\cdot,n}) \end{cases}$$

$$\left[\text{又} \dim S_c = \dim \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) = \dim \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = \dim S_r = n \right]$$

$$\iff (A_{1,\cdot}, \dots, A_{n,\cdot}) \text{ is a basis of } \mathbf{F}^{1,n}.$$ □

Now we show (Δ) properly, that is T is inje $\iff$ The cols of $\mathcal{M}(T)$ are linely inde.

$(a) \Rightarrow (b)$:

Suppose $b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n} = \begin{pmatrix} b_1 A_{1,1} + \cdots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \cdots + b_n A_{n,n} \end{pmatrix} = 0$. Let $u = b_1 u_1 + \cdots + b_n u_n$.

$$\begin{aligned} \text{Then } Tu &= b_1 Tu_1 + \cdots + b_n Tu_n \\ &= b_1(A_{1,1}v_1 + \cdots + A_{n,1}v_n) + \cdots + b_n(A_{1,n}v_1 + \cdots + A_{n,n}v_n) \\ &= (b_1 A_{1,1} + \cdots + b_n A_{1,n})v_1 + \cdots + (b_1 A_{n,1} + \cdots + b_n A_{n,n})v_n \\ &= 0v_1 + \cdots + 0v_n = 0 \\ &\Rightarrow b_1 = \cdots = b_n = 0. \end{aligned}$$

Thus by (2.39), (b) holds.

$(b) \Rightarrow (a)$:

Suppose $u = b_1 u_1 + \cdots + b_n u_n \in \operatorname{null} T$.
Then $Tu = 0 = (b_1 A_{1,1} + \cdots + b_n A_{1,n})v_1 + \cdots + (b_1 A_{n,1} + \cdots + b_n A_{n,n})v_n$.
Thus $b_1 A_{1,1} + \cdots + b_n A_{1,n} = \cdots = b_1 A_{n,1} + \cdots + b_n A_{n,n} = 0$.

Which is equi to $\begin{pmatrix} b_1 A_{1,1} + \cdots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \cdots + b_n A_{n,n} \end{pmatrix} = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n} = 0 \Rightarrow b_1 = \cdots = b_n = 0$.

Thus by (2.39), (a) holds. □

• $[$4E 3.C.16, OR 3.E.11$]$ *Suppose A is an m-by-n matrix with $A \neq 0$.*
Prove that $\operatorname{rank} A = 1 \iff \exists\, (c_1, \dots, c_m) \in \mathbf{F}^m, (d_1, \dots, d_n) \in \mathbf{F}^n$
such that $A_{j,k} = c_j \cdot d_k$ *for every* $j = 1, \dots, m$ *and* $k = 1, \dots, n$.

SOLUTION:

Using the notation in CR Factorization.

(a) Suppose $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix}$. $(\ \exists\, c_j, d_k \in \mathbf{F}, \forall j, k\ )$

Then $S_c = \left\{ \begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \dots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right\} = \operatorname{span} \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$.

OR. $S_r = \operatorname{span} \left\{ \begin{array}{c} \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \end{pmatrix}, \\ \begin{pmatrix} c_2 d_1 & \cdots & c_2 d_n \end{pmatrix}, \\ \vdots \\ \begin{pmatrix} c_m d_1 & \cdots & c_m d_n \end{pmatrix} \end{array} \right\} = \operatorname{span} \left\{ \begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} \right\}$. Hence $\operatorname{rank} A = 1$.

OR. Using also the result in $[$4E 3.51(a)$]$.

Every col of A is a scalar multi of C. Then $\operatorname{rank} A \leqslant 1$ 又 $\operatorname{rank} A \geqslant 1$ $(\ A \neq 0\ )$.

(b) By CR Factorization, $\exists\, C = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbf{F}^{m,1}, R = \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} \in \mathbf{F}^{1,n}$ such that $A = CR$.

OR. Not using CR Factorization. Suppose $\operatorname{rank} A = \dim S_c = \dim S_r = 1$.

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $\qquad d'_k = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$. $\quad (\ \forall j, k\ )$

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$. Letting $d_k = d'_k A_{1,1}$. $\square$

---

1 *Suppose $T \in \mathcal{L}(V, W)$. Show that with resp to each choice of bases of V and W, the matrix of T has at least* $\dim \operatorname{range} T$ *nonzero entries.*

SOLUTION:

Let $B_{\operatorname{null} T} = (v_1, \dots, v_p), B_V = (v_1, \dots, v_n)$. Let $B_W = (w_1, \dots, w_m)$. Denote $\mathcal{M}(T, B_V, B_W)$ by A.

Because at most p of the v_k's can belong to $\operatorname{null} T \iff$ at least $n - p = q$ of the v_k's do not.

For $v_k \notin \operatorname{null} T, Tv_k = A_{1,k} w_1 + \cdots + A_{m,k} w_m \neq 0$. Thus col k has at least one nonzero entry.

Since there are $(n - p) = q$ choices of such k, A has at least $q = \dim \operatorname{range} T$ nonzero entries. $\square$

OR. We prove by contradiction.

Suppose A has at most $(\ \dim \operatorname{range} T - 1)$ nonzero entries.

Then by Pigeon Hole Principle, at least one of $A_{\cdot,p+1}, \dots, A_{\cdot,n}$ equals 0.

Thus there are at most $(\ \dim \operatorname{range} T - 1)$ nonzero vecs in $Tv_{p+1}, \dots, Tv_n$.

While $\operatorname{range} T = \operatorname{span}(Tv_{p+1}, \dots, Tv_n) \Rightarrow \dim \operatorname{range} T = \dim \operatorname{span}(Tv_{p+1}, \dots, Tv_n)$. Contradicts. $\square$

---

3 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$. Prove that $\exists\, B_V, B_W$ such that* $[$ *letting* $A = \mathcal{M}(T, B_V, B_W)$ $]$ $A_{k,k} = 1, A_{i,j} = 0$, *where* $1 \leqslant k \leqslant \dim \operatorname{range} T, i \neq j$.

SOLUTION:

Let $R = (Tv_1, \dots, Tv_n)$ be a basis of range T, extend to $B_W = (Tv_1, \dots, Tv_n, w_1, \dots, w_p)$.
Let $\mathcal{K}_R = \operatorname{span}(v_1, \dots, v_n)$. Let $(u_1, \dots, u_m)$ be a basis of null T. Then $B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$. $\square$

---

4 *Suppose $B_V = (v_1, \dots, v_m)$ and W is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that $\exists\, B_W = (w_1, \dots, w_n)$, $\mathcal{M}(T, B_V, B_W)^t_{\cdot,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$.

SOLUTION: If $Tv_1 = 0$, then we are done. If not then extend (Tv_1). $\square$

---

5 *Suppose $B_W = (w_1, \dots, w_n)$ and V is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that $\exists\, B_V = (v_1, \dots, v_m)$, $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$.

SOLUTION:

Let $(u_1, \dots, u_n)$ be a basis of V. Denote $\mathcal{M}\big(T, (u_1, \dots, u_n), B_W\big)$ by A.
If $A_{1,\cdot} = 0$, then let $B_V = (u_1, \dots, u_n)$, we are done.
Otherwise, $\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \end{pmatrix} \neq 0$, choose one $A_{1,k} \neq 0$.

$$\text{Let } v_1 = \frac{u_k}{A_{1,k}}; \quad v_j = u_{j-1} - A_{1,j-1} v_1 \quad \text{for } j = 2, \dots, k;$$
$$v_i = u_i - A_{1,i} v_1 \quad \text{for } i = k+1, \dots, n.$$

Now because each $u_k \in \operatorname{span}(v_1, \dots, v_n) \Rightarrow V = \operatorname{span}(v_1, \dots, v_n), B_V = (v_1, \dots, v_n)$.

And $Tv_1 = T(\dfrac{u_k}{A_{1,k}}) = \dfrac{1}{A_{1,k}}(A_{1,k} w_1 + \cdots + A_{n,k} w_n) = 1 w_1 + \cdots + \dfrac{A_{n,k}}{A_{1,k}} w_n$.

$$\forall j \in \{2, \dots, k, \underbrace{k+2, \dots, n+1}_{i \in \{k+1, \dots, n\}}\},\ Tv_j = T(u_{j-1} - A_{1,j-1} v_1) = Tu_{j-1} - T(\frac{A_{1,j-1} u_k}{A_{1,k}})$$
$$= A_{1,j-1} w_1 + \cdots + A_{n,j-1} w_n - A_{1,j-1}(1 w_1 + \cdots + \frac{A_{n,k}}{A_{1,k}} w_n) = 0 w_1 + \cdots + (A_{n,j-1} - \frac{A_{1,j-1} A_{n,k}}{A_{1,k}}) w_n.$$ $\square$

---

6 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$.*
Prove that $\dim \operatorname{range} T = 1 \iff \exists\, B_V, B_W$, *all entries of* $A = \mathcal{M}(T, B_V, B_W)$ *equal* 1.

SOLUTION:

(a) Suppose $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$ are the bases such that all entries of A equal 1.

Then $Tv_i = w_1 + \cdots + w_m$ for all $i = 1, \dots, n$. Because $w_1, \dots, w_n$ is linely inde, $w_1 + \cdots + w_n \neq 0$.

(b) Suppose $\dim \operatorname{range} T = 1$. Then $\dim \operatorname{null} T = \dim V - 1$.
Let $(u_2, \dots, u_n)$ be a basis of null T. Extend it to a basis of V as $(u_1, u_2, \dots, u_n)$.
Let $w_1 = Tv_1 - w_2 - \cdots - w_m$. Extend to a basis of W and we have B_W.
Let $v_1 = u_1, v_i = u_1 + u_i$. Extend to a basis of V and we have B_V. $\square$

OR. Suppose range T has a basis (w).
By (2.C.15 [COROLLARY]), $\exists\, B_W = (w_1, \dots, w_m)$ such that $w = w_1 + \cdots + w_m$.
By (2.C [NEW THEOREM]), $\exists$ a basis $(u_1, \dots, u_n)$ of V such that each $u_k \notin$ null T.
$\forall k \in \{1, \dots, n\}, Tu_k \in \operatorname{range} T = \operatorname{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists\, \lambda_k \in \mathbf{F} \backslash \{0\}$.
Let $v_k = \lambda_k^{-1} u_k \neq 0 \Rightarrow B_V = (v_1, \dots, v_n)$. Hence for each $v_k, Tv_k = w = w_1 + \cdots + w_m$. $\square$

• **NOTE FOR [3.49]:** $\because [(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$

$\therefore (AC)_{\cdot,k} = A_{\cdot,\cdot}C_{\cdot,k} = AC_{\cdot,k}$ ☐

• **EXERCISE 10:** $\because [(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$

$\therefore (AC)_{j,\cdot} = A_{j,\cdot}C_{\cdot,\cdot} = A_{j,\cdot}C.$ ☐

• **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$

$\because (Ac)_{j,1} = \sum_{r=1}^{n} A_{j,r}c_{r,1} = \left(\sum_{r=1}^{n} (A_{\cdot,r}c_{r,1})\right)_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$

$\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^{n} A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ OR. By $(Ac)_{\cdot,1} = Ac_{\cdot,1}$ Using (a) above. ☐

• **EXERCISE 11:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$

$\because (aC)_{1,k} = \sum_{r=1}^{n} a_{1,r}C_{r,k} = \left(\sum_{r=1}^{n} a_{1,r}(C_{r,\cdot})\right)_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$

$\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^{n} a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ OR. By $(aC)_{1,\cdot} = a_{1,\cdot}C$. Using (b) above. ☐

• *Suppose p is a poly of n variables in $\mathbf{F}$. Prove that $\mathcal{M}\big(p(T_1, \dots, T_n)\big) = p\big(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n)\big)$.*

Where the linear maps $T_1, \dots, T_n$ are such that $p(T_1, \dots, T_n)$ makes sense. See [5.B.16,17,20].

SOLUTION:

Suppose the poly p is defined by $p(x_1, \dots, x_n) = \sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} \prod_{i=1}^{n} x_i^{k_i}$.

Note that $\mathcal{M}(T^xS^y) = \mathcal{M}(T)^x\mathcal{M}(S)^y$; $\mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$.

Then $\mathcal{M}\big(p(T_1, \dots, T_n)\big) = \mathcal{M}\Big(\sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} \prod_{i=1}^{n} T_i^{k_i}\Big)$

$= \sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} \prod_{i=1}^{n} \mathcal{M}(T_i^{k_i}) = p\big(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n)\big).$ ☐

---

13 *Prove that the distr holds for matrix add and matrix multi.*

Suppose A, B, C are matrices such that $A(B + C)$ make sense, we prove the left distr.

SOLUTION:

Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Note that $[A(B + C)]_{j,k} = \sum_{r=1}^{n} A_{j,r}(B + C)_{r,k} = \sum_{r=1}^{n} (A_{j,r}B_{r,k} + A_{j,r}C_{r,k}) = (AB + AC)_{j,k}$. ☐

OR. Define T, S, R such that $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$A(B + C) = \mathcal{M}(T(S + R)) \overset{[3.9]}{=} \mathcal{M}(TS + TR) = AB + AC$.

Or $T(S + R) = TS + TR \Rightarrow \mathcal{M}(T(S + R)) = \mathcal{M}(TS + TR) \Rightarrow A(B + C) = AB + AC$. ☐

---

14 *Prove that matrix multi is associ.*

Suppose A, B, C are matrices such that $(AB)C$ makes sense, we prove that $(AB)C = A(BC)$.

SOLUTION:

Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$. We will show that $LHS = [(AB)C]_{j,k} = [A(BC)]_{j,k} = RHS$.

$LHS = (AB)_{j,\cdot}C_{\cdot,k} = \sum_{s=1}^{n} (A_{j,s}B_{s,\cdot})C_{\cdot,k} = \sum_{s=1}^{n} A_{j,s}(B_{s,\cdot}C_{\cdot,k}) = \sum_{s=1}^{n} A_{j,s}(BC)_{s,k} = RHS.$ ☐

OR. Define T, S, R such that $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$(AB)C = \mathcal{M}(T(SR)) \overset{[3.9]}{=} \mathcal{M}(TSR) \overset{[3.9]}{=} \mathcal{M}((TS)R) = A(BC)$.

OR. $(TS)R = T(SR) \Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR)) \Rightarrow (AB)C = A(BC)$. ☐

---

15 *Suppose $A \in \mathbf{F}^{n,n}, j,k \in \{1,\dots,n\}$. Show that $(A^3)_{j,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.*

SOLUTION: $(AAA)_{j,k} = (AA)_{j,\cdot}A_{\cdot,k} = \sum_{p=1}^{n}(A_{j,p}A_{p,\cdot})A_{\cdot,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.

OR. $(AAA)_{j,k} = \sum_{r=1}^{n}(AA)_{j,r}A_{r,k} = \sum_{r=1}^{n}\left(\sum_{p=1}^{n} A_{j,p}A_{p,r}\right)A_{r,k}$

$= \sum_{r=1}^{n}\left[A_{j,1}(A_{1,r}A_{r,k}) + \cdots + A_{j,n}(A_{n,r}A_{r,k})\right]$

$= A_{j,1}\sum_{r=1}^{n} A_{1,r}A_{r,k} + \cdots + A_{j,n}\sum_{r=1}^{n} A_{n,r}A_{r,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$. □

• *Prove that the commutativity does not hold in $\mathbf{F}^{m,n}$.*

SOLUTION:

Suppose $\dim V = n, \dim W = m$ and the commutativity holds in $\mathbf{F}^{n,m}$.

$\forall T \in \mathcal{L}(V,W), S \in \mathcal{L}(W,V), \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$.

Hence $ST = TS$. Which in general is not true. (See 3.D) □

• [10.A.3, OR 4E 3.D.19] *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that $\forall B_V \neq B_V^{'}$, $\mathcal{M}(T,B_V) = \mathcal{M}(T,B_V^{'}) \iff T = \lambda\mathcal{M}(I), \exists\lambda \in \mathbf{F}$.

SOLUTION: [Compare with the first solution of (3.D.16) in 3.A]

Suppose $T = \lambda I$ for some $\lambda \in \mathbf{F}$. Then $T = \lambda\mathcal{M}(I)$.

Suppose $\forall B_V \neq B_V'$, $\mathcal{M}(T,B_V) = \mathcal{M}(T,B_V')$. If $T = 0$, then we are done.

Suppose $T \neq 0$, and $v \in V\backslash\{0\}$. Assume that (v,Tv) is linely inde.

Extend (v,Tv) to $B_V = (v,Tv,u_3,\dots,u_n)$. Let $B = \mathcal{M}()(T,B_V)$.

$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \forall i \neq 2$.

By assumption, $A = \mathcal{M}(T,B_V') = B, \forall B_V' = (v,w_2,\dots,w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$.

$\Rightarrow Tv = w_2$, which is not true if we let $w_2 = u_3,\ w_3 = Tv,\ w_j = u_j, \forall j \in \{4,\dots,n\}$. Contradicts.

Hence (v,Tv) is linely depe $\Rightarrow \forall v \in V, \exists\lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show that λ_v is independent of v, that is, to show that for all $v \neq w \in V\backslash 0, \lambda_v = \lambda_w$.

$$\left.\begin{array}{l}(v,w) \text{ is linely inde} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v,w) \text{ is linely depe, } w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv)\end{array}\right\} \Rightarrow T = \lambda I, \exists\lambda \in \mathbf{F}.$$ □

OR. Conversely, denote $\mathcal{M}(T,B_V)$ by A, where $B_V = (u_1,\dots,u_m)$ is arbitrary.

Fix one $B_V = (v_1,\dots,v_m)$ and then $(v_1,\dots,\frac{1}{2}v_k,\dots,v_m)$ is also a basis for any given $k \in \{1,\dots,m\}$.

Fix one k. Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \cdots + A_{k,k}(\frac{1}{2}v_k) + \cdots + A_{m,k}v_m$

$\Rightarrow Tv_k = 2A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + 2A_{m,k}v_m = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{m,k}v_m$.

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1,\dots,m\}$.

Now we show that $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j,k such that $j \neq k$.

Consider the basis $B_V' = ({v'}_1,\dots,{v'}_j,\dots,{v'}_k,\dots,{v'}_m)$,

where ${v'}_j = v_k,\ {v_k}' = v_j$ and ${v'}_i = v_i$ for all $i \in \{1,\dots,m\} \setminus \{j,k\}$.

Remember that $\mathcal{M}(T,B_V') = \mathcal{M}(T,B_V) = A$.

Hence $T({v'}_k) = A_{1,k}{v'}_1 + \cdots + A_{k,k}{v'}_k + \cdots + A_{m,k}{v'}_m = A_{k,k}{v'}_k = A_{k,k}v_j$, while $T({v'}_k) = T(v_j) = A_{j,j}v_j$.

Thus $A_{k,k} = A_{j,j}$. □

ENDED

3·D 1 2 3 4 5 6 8 9 10 11 12 13 15 16 17 18 19 | 4E: 1, 3, 10, 15, 17, 19, 20, 22, 23, 24

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*

$$\left.\begin{array}{l}(Tv_1,\dots,Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1,\dots,v_n) \text{ of } V \iff T \text{ is surj} \\ (Tv_1,\dots,Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1,\dots,v_n) \text{ of } V \iff T \text{ is inje}\end{array}\right\} \iff T \text{ is inv.}$$

• *Suppose $T \in \mathcal{L}(V)$ and $V = \operatorname{span}(Tv_1,\dots,Tv_m)$. Prove that $V = \operatorname{span}(v_1,\dots,v_m)$.*

Solution:

Because $V = \operatorname{span}(Tv_1,\dots,Tv_m) \Rightarrow T$ is surj, 又 V is finite-dim $\Rightarrow T$ is inv $\Rightarrow T^{-1}$ is inv.
$\forall v \in V, \exists a_i \in \mathbf{F}, v = a_1Tv_1 + \dots + a_mTv_m \Rightarrow T^{-1}v = a_1v_1 + \dots + a_mv_m \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1,\dots,v_m)$.

Or. Reduce $(Tv_1,\dots,Tv_m)$ to a basis of V as $(Tv_{\alpha_1},\dots,Tv_{\alpha_k})$, where $k = \dim V$ and $\alpha_i \in \{1,\dots,k\}$.
Then $(v_{\alpha_1},\dots,v_{\alpha_k})$ is linely inde of length k, hence is a basis of V, contained in the list $(v_1,\dots,v_m)$. □

• Or (10.A.1) *Suppose $T \in \mathcal{L}(V), B_V = (v_1,\dots,v_n)$. Prove that $\mathcal{M}(T,B_V)$ is inv $\iff T$ is inv.*

Solution: Notice that $\mathcal{M} \in \mathcal{L}(\mathcal{L}(V),\mathbf{F}^{n,n})$ is an iso.

(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.
(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I$. $\exists!\, S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$
$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$
$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. □

• *Suppose $T \in \mathcal{L}(V,W)$ is inv. Show that T^{-1} is inv and $(T^{-1})^{-1} = T$.*

Solution: $\left.\begin{array}{l} TT^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = I \in \mathcal{L}(W)\end{array}\right\} \Rightarrow T = (T^{-1})^{-1}$, by the uniqnes of inverse. □

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1 *Suppose $T \in \mathcal{L}(U,V), S \in \mathcal{L}(V,W)$ are inv. Prove that ST is inv and $(ST)^{-1} = T^{-1}S^{-1}$.*

Solution: $\left.\begin{array}{l} (ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(V)\end{array}\right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}$, by the uniqnes of inv. □

---

2 *Suppose V is finite-dim and* $\dim V > 1$.
Prove that the set of non-inv operators on V is not a subsp of $\mathcal{L}(V)$.
The set of inv operators is not either, although multi identity/inv, and commutativity for vec multi holds.

Solution:

Denote the set by U. Suppose $\dim V = n > 1$. Let $(v_1,\dots,v_n)$ be a basis of V. Define $S,T \in \mathcal{L}(V)$ by
$S(a_1v_1 + \dots + a_nv_n) = a_1v_1, T(a_1v_1 + \dots + a_nv_n) = a_2v_1 + \dots + a_nv_n$. Hence $S + T = I$ is inv. □

Comment: If $\dim V = 1$, then $U = \{0\}$ is a subsp of $\mathcal{L}(V)$.

---

3 *Suppose V is finite-dim, U is a subsp of V, and $S \in \mathcal{L}(U,V)$.*
Prove that $\exists$ inv $T \in \mathcal{L}(V), Tu = Su, \forall u \in U \iff S$ is inje. [*Compare this with (3.A.11).*]

Solution:

(a) $Tu = Su$ for every $u \in U \Rightarrow u = T^{-1}Su \Rightarrow S$ is inje. Or. $\operatorname{null} S = \operatorname{null} T \cap U = \{0\} \cap U = \{0\}$.
(b) Suppose $(u_1,\dots,u_m)$ be a basis of U and S is inje $\Rightarrow (Su_1,\dots,Su_m)$ is linely inde in V.
Extend these to bases of V as $(u_1,\dots,u_m,v_1,\dots,v_n)$ and $(Su_1,\dots,Su_m,w_1,\dots,w_n)$.
Define $T \in \mathcal{L}(V)$ by $T(u_i) = Su_i$; $Tv_j = w_j$, for each $i \in \{1,\dots,m\}, j \in \{1,\dots,n\}$. □

---

4 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{null} S = \operatorname{null} T\,(=U) \iff S = ET,\ \exists$ *inv* $E \in \mathcal{L}(W)$.

SOLUTION:

Define $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i,\ \ E(w_j) = x_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m)$, extend to $B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n)$. Let $\mathcal{K} = \operatorname{span}(v_1, \dots, v_m)$. 又 $\operatorname{null} S = \operatorname{null} T \Longrightarrow V = \mathcal{K} \oplus \operatorname{null} S \Leftrightarrow \mathcal{K} \in \mathcal{S}_V \operatorname{null} S$. $\Rightarrow \operatorname{span}(Sv_1, \dots, Sv_m) = \operatorname{range} S$ 又 $\dim \operatorname{range} T = \dim \operatorname{range} S = m$. Hence $B_{\operatorname{range} S} = (Sv_1, \dots, Sv_m)$. Thus we let $B'_W = (Sv_1, \dots, Sv_m, x_1, \dots, x_n)$.	$\therefore E$ is inv and $S = ET$.

Conversely, $S = ET \Rightarrow \operatorname{null} S = \operatorname{null} ET$.

Then $v \in \operatorname{null} ET \iff ET(v) = 0 \iff Tv = 0 \iff v \in \operatorname{null} T$. Hence $\operatorname{null} ET = \operatorname{null} T = \operatorname{null} S$. ☐

---

5 *Suppose that V is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{range} S = \operatorname{range} T\,(=R) \iff S = TE,\ \exists$ *inv* $E \in \mathcal{L}(V)$.

SOLUTION:

Define $E \in \mathcal{L}(V)$ as $E:\ \ v_i \mapsto r_i\ ;\ \ \ u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $B_R = (Tv_1, \dots, Tv_m);\ B'_R = (Sr_1, \dots, Sr_m)$ such that $\forall\, i, Tv_i = Sr_i$. Let $B_{\operatorname{null} T} = (u_1, \dots, u_n);\ B_{\operatorname{null} S} = (s_1, \dots, s_n)$. Thus $B_V = (v_1, \dots, v_m, u_1, \dots, u_n);\ B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)$.	$\therefore E$ is inv and $S = TE$.

Conversely, $S = TE \Rightarrow \operatorname{range} S = \operatorname{range} TE$.

Then $w \in \operatorname{range} S \iff \exists\, v \in V, Sv = TE(v) = T\big(E(v)\big) = w \in \operatorname{range} T$. Hence $\operatorname{range} S = \operatorname{range} T$. ☐

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6 *Suppose V and W are finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $S = E_2 T E_1, \exists$ inv $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$ $\iff \dim \operatorname{null} S = \dim \operatorname{null} T = n$.

SOLUTION:

Define $E_1:\ \ v_i \mapsto r_i\ ;\ \ \ u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.

Define $E_2: Tv_i \mapsto Sr_i\ ;\ \ x_j \mapsto y_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m);\ B_{\operatorname{range} S} = (Sr_1, \dots, Sr_m)$. Extend to $B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p);\ B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p)$. Let $B_{\operatorname{null} T} = (u_1, \dots, u_n);\ B_{\operatorname{null} S} = (s_1, \dots, s_n)$. Thus $B_V = (v_1, \dots, v_m, u_1, \dots, u_n);\ B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)$.	$\therefore E_1, E_2$ are inv and $S = E_2 T E_1$.

Conversely, $S = E_2 T E_1 \Rightarrow \dim \operatorname{null} S = \dim \operatorname{null} E_2 T E_1$.

$v \in \operatorname{null} E_2 T E_1 \iff E_2 T E_1(v) = 0 \iff TE_1(v) = 0$. Hence $\operatorname{null} E_2 T E_1 = \operatorname{null} TE_1 = \operatorname{null} S$.

又 By (3.B.22.COROLLARY), E is inv $\Rightarrow \dim \operatorname{null} TE_1 = \dim \operatorname{null} T = \dim \operatorname{null} S$. ☐

---

8 *Suppose V is finite-dim and $T : V \to W$ is a **surj** linear map of V onto W.*
Prove that there is a subsp U of V such that $T|_U$ is an iso of U onto W.

SOLUTION:

Let $B_{\operatorname{range} T} = B_W = (w_1, \dots, w_m) \Rightarrow \forall w_i, \exists!\, v_i \in V, Tv_i = w_i$. Let $B_{\mathcal{K}} = (v_1, \dots, v_m)$.

Then $\dim \mathcal{K} = \dim W$. Thus $T|_{\mathcal{K}}$ is an iso of $\mathcal{K}$ onto W. ☐

OR. By (3.B.12), there is a subsp U of V such that

$U \cap \operatorname{null} T = \{0\} = \operatorname{null} T|_U, \operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$. ☐

---

9 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that ST is inv $\Longleftrightarrow$ S and T are inv.*

Solution:

Suppose S, T are inv. Then $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$. Hence ST is inv.

Suppose ST is inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T \text{ is inje, } S \text{ is surj. While } V \text{ is finite-dim.} \qquad \square$$

Or. Because by (3.B.23), $\dim V = \dim \operatorname{range} ST \leqslant \min\{\operatorname{range} T, \operatorname{range} S\}$. $\square$

---

10 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that $ST = I \Longleftrightarrow TS = I$.*

Solution:

Suppose $ST = I$. $\left.\begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T$ is inje, S is surj. While V is finite-dim.

Or. By Problem (9), V is finite-dim and $ST = I$ is inv $\Rightarrow S, T$ are inv.

$S\big((TS)v\big) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow S$ is inv.

Or. $ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T$. 又 $S = S \Rightarrow TS = S^{-1}S = I$.

Reversing the roles of S and T, we conclude that $TS = I \Rightarrow ST = I$. $\square$

---

11 *Suppose V is finite-dim, $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is inv and $T^{-1} = US$.*

Solution: Using Problem (9) and (10). *This result can fail without the hypothesis that V is finite-dim.*

$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I$.

$\Rightarrow U^{-1} = ST, \qquad T^{-1} = US, \qquad S^{-1} = TU.$ $\square$

Example: $V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots);\ T(a_1, \dots) = (0, a_1, \dots);\ U = I \Rightarrow STU = I$ but T is not inv.

---

13 *Suppose V is finite-dim, $R, S, T \in \mathcal{L}(V)$ are such that RST is surj. Prove that S is inje.*

Solution: By Problem (1) and (9), Notice that V is finite-dim. Then RST is inv.

$$\text{Let } X = (RST)^{-1} \left| \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ is surj.} \end{array}\right\} \Rightarrow S = R^{-1}(RST)T^{-1} \text{ is inv.} \qquad \square$$

Or. $(RST)^{-1} = \big((RS)T\big)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. $\square$

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15 *Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multi. In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists A \in \mathbf{F}^{m,n}, Tx = Ax,\ \forall x \in \mathbf{F}^{n,1}$.*

Solution:

Let $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$ be the standard bases of $\mathbf{F}^{n,1}, \mathbf{F}^{m,1}$.

$\forall k = 1, \dots, n$, suppose $T(E_k) = A_{1,k}R_1 + \cdots + A_{m,k}R_m, \exists A_{j,k} \in \mathbf{F}$, forming $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$. $\square$

Or. Let $A = \mathcal{M}(T, B_1, B_2)$. Note that $\mathcal{M}(x, B_1) = x, \mathcal{M}(y, B_2) = y$.

Hence $Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax$, by [3.65]. $\square$

• Or (10.A.2) *Suppose $A, B \in \mathbf{F}^{n,n}$. Prove that $AB = I \Longleftrightarrow BA = I$.*

Solution: Using Problem (10) and (15).

Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Then $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.

Thus $AB = I \Leftrightarrow A(Bx) = x \Longleftrightarrow T(Sx) = x \Leftrightarrow TS = I \Longleftrightarrow ST = I \Longleftrightarrow \mathcal{M}(S)\mathcal{M}(T) = BA = I$. $\square$

• **Note For [3.60]:** Suppose $B_V = (v_1,\dots,v_n)$, $B_W = (w_1,\dots,w_m)$.

Define $E_{i,j} \in \mathcal{L}(V,W)$ by $E_{i,j}(v_x) = \delta_{i,x} w_j$; See (3.A.12). **Corollary:** $E_{l,k}E_{i,j} = \delta_{j,l}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. And $\left(\mathcal{E}^{(j,i)}\right)_{l,k} = \begin{cases} 0, & i \neq k \vee j \neq l \\ 1, & i = k \wedge j = l \end{cases}$

Because $\mathcal{L}(V,W)$ and $\mathbf{F}^{m,n}$ are iso. And $T = \mathcal{M}^{-1}\mathcal{M}(T)$; $E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

Hence $\forall T \in \mathcal{L}(V,W), \exists!\, A_{i,j} \in \mathbf{F} \ \big(\ \forall i \in \{1,\dots,m\}, j \in \{1,\dots,n\}\ \big), \mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)} + & \cdots & +A_{1,n}\mathcal{E}^{(1,n)} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}\mathcal{E}^{(m,1)} + & \cdots & +A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff \begin{pmatrix} A_{1,1}E_{1,1} + & \cdots & +A_{1,n}E_{n,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}E_{1,m} + & \cdots & +A_{m,n}E_{n,m} \end{pmatrix} = T.$$

$$\therefore \mathcal{L}(V,W) = \operatorname{span} \underbrace{\begin{pmatrix} E_{1,1}, & \cdots & ,E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,m}, & \cdots & ,E_{n,m} \end{pmatrix}}_{B}; \quad \mathbf{F}^{m,n} = \operatorname{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & ,\mathcal{E}^{(1,n)}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ \mathcal{E}^{(m,1)}, & \cdots & ,\mathcal{E}^{(m,n)} \end{pmatrix}}_{B_{\mathcal{M}}}.$$

Hence by [2.42] and [3.61], we conclude that B is a basis of $\mathcal{L}(V,W)$ and that $B_{\mathcal{M}}$ is a basis of $\mathbf{F}^{m,n}$.

• *Suppose V, W are finite-dim, U is a subsp of V.*
Let $\mathcal{E} = \{T \in \mathcal{L}(V,W) : U \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V,W) : T|_U = 0\}$.
(a) *Show that $\mathcal{E}$ is a subsp of $\mathcal{L}(V,W)$.*
(b) *Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$ and $\dim U$.*
Hint: *Define $\Phi : \mathcal{L}(V,W) \to \mathcal{L}(U,W)$ by $\Phi(T) = T|_U$. What is $\operatorname{null}\Phi$? What is $\operatorname{range}\Phi$?*

Solution:

(a) $\forall S,T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S+\lambda T)u = 0 \Rightarrow (S+\lambda T) \in \mathcal{E}$.

(b) Define Φ as in the hint.

Because $T \in \operatorname{null}\Phi \iff \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$.

Hence $\operatorname{null}\Phi = \mathcal{E}$.

Because $S \in \mathcal{L}(U,W) \Rightarrow \exists\, T \in \mathcal{L}(V,W), \Phi(T) = S$, by (3.A.11) $\Rightarrow S \in \operatorname{range} T$.

Hence $\operatorname{range}\Phi = \mathcal{L}(U,W)$.

Thus $\dim\operatorname{null}\Phi = \dim\mathcal{E} = \dim\mathcal{L}(V,W) - \dim\operatorname{range}\Phi = (\dim V - \dim U)\dim W$. ☐

Or. Extend $(u_1,\dots,u_m)$ a basis of U to $(u_1,\dots,u_m,v_1,\dots,v_n)$ a basis of V. Let $p = \dim W$.
(See Note For [3.60])

$$\forall\, T \in \mathcal{E}, k \in \{1,\dots,m\}, TE_{k,k} = 0 \Rightarrow \underbrace{\operatorname{span}\left\{\begin{matrix} E_{1,1}, & \cdots & ,E_{m,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,p}, & \cdots & ,E_{m,p} \end{matrix}\right\}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

$$\text{又 } W = \operatorname{span}\left\{\begin{matrix} E_{m+1,1}, & \cdots & ,E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{m+1,p}, & \cdots & ,E_{n,p} \end{matrix}\right\} \subseteq \mathcal{E}. \text{ Where } \mathcal{L}(V,W) = R \oplus W \Rightarrow \mathcal{L}(V,W) = R + \mathcal{E}.$$

Then $\dim\mathcal{E} = \dim\mathcal{L}(V,W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U)\dim W$. ☐

∘ *Suppose V is finite-dim and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(T) = ST$.*
(a) *Show that* $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Show that* $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

SOLUTION:

(a) $\forall T \in \mathcal{L}(V)$, $ST = 0 \Longleftrightarrow \operatorname{range} T \subseteq \operatorname{null} S$.
Thus $\operatorname{null} \mathcal{A} = \big\{T \in \mathcal{L}(V) : \operatorname{range} T \subseteq \operatorname{null} S\big\} = \mathcal{L}(V, \operatorname{null} S)$.

(b) $\forall R \in \mathcal{L}(V)$, $\operatorname{range} R \subseteq \operatorname{range} S \Longleftrightarrow \exists\, T \in \mathcal{L}(V), R = ST$, by (3.B 25).
Thus $\operatorname{range} \mathcal{A} = \big\{R \in \mathcal{L}(V) : \operatorname{range} R \subseteq \operatorname{range} S\big\} = \mathcal{L}(V, \operatorname{range} S)$.

□

OR. Using NOTE FOR [3.60].

Let $B_{\operatorname{range} S} = (\underbrace{w_1, \dots, w_m}_{Sv_i = w_i}), B_{\mathcal{K}} = (v_1, \dots, v_m); (w_1, \dots, w_n), (v_1, \dots, v_n)$ are bases of V.

Define $E_{i,j} \in \mathcal{L}(V)$ by $E_{i,j}(v_x) = \delta_{i,x} w_i$.

Thus $S = E_{1,1} + \cdots + E_{m,m}$; $\quad \mathcal{M}\big(S, (v_1, \dots, v_n), (w_1, \dots, w_n)\big) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$.

Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j}(w_x) = \delta_{i,x} v_i$.
Let $E_{j,k} R_{i,j} = Q_{i,k}$, $\quad R_{j,k} E_{i,j} = G_{i,k}$.

Because $\forall T \in \mathcal{L}(V)$, $\exists\,!\, A_{i,j} \in \mathbf{F}$, $\quad T = \begin{pmatrix} A_{1,1}R_{1,1}+ & \cdots & +A_{1,m}R_{m,1}+ & \cdots & +A_{1,n}R_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}R_{1,m}+ & \cdots & +A_{m,m}R_{m,m}+ & \cdots & +A_{m,n}R_{n,m} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{n,1}R_{1,n}+ & \cdots & +A_{n,m}R_{m,n}+ & \cdots & +A_{n,n}R_{n,n} \end{pmatrix}$.

$$\Rightarrow \mathcal{A}(T) = ST = \Big(\sum_{r=1}^{m} E_{r,r}\Big)\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j} R_{j,i}\Big)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} A_{i,j} Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1}+ & \cdots & +A_{1,m}Q_{m,1}+ & \cdots & +A_{1,n}Q_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}Q_{1,m}+ & \cdots & +A_{m,m}Q_{m,m}+ & \cdots & +A_{m,n}Q_{n,m} \end{pmatrix}.$$

Thus $\operatorname{null} \mathcal{A} = \operatorname{span} \begin{pmatrix} R_{1,m+1}, & \cdots & ,R_{n,m+1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ R'_{1,n}, & \cdots & ,R'_{n,n} \end{pmatrix}$, $\quad \operatorname{range} \mathcal{A} = \operatorname{span} \begin{pmatrix} Q_{1,1}, & \cdots & ,Q_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ Q'_{1,m}, & \cdots & ,Q'_{n,m} \end{pmatrix}$.

Hence (a) $\dim \operatorname{null} \mathcal{A} = n \times (n - m)$; $\quad$ (b) $\dim \operatorname{range} \mathcal{A} = n \times m$. □

• **COMMENT:** Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(T) = TS$. Similarly to Problem $(\circ)$,

(a) $\forall T \in \mathcal{L}(V), TS = 0 \Longleftrightarrow \operatorname{range} S \subseteq \operatorname{null} T$.
Thus $\operatorname{null} \mathcal{B} = \{T \in \mathcal{L}(V) : \operatorname{range} S \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V) : T|_{\operatorname{range} S} = 0\}$.

(b) $\forall R \in \mathcal{L}(V), \operatorname{null} S \subseteq \operatorname{null} R \Longleftrightarrow \exists T \in \mathcal{L}(V), R = TS$, by (3.B.24).
Thus $\operatorname{range} \mathcal{B} = \{R \in \mathcal{L}(V) : \operatorname{null} S \subseteq \operatorname{null} R\} = \{R \in \mathcal{L}(V) : R|_{\operatorname{null} S} = 0\}$.

Hence $\dim \operatorname{null} \mathcal{B} = (\dim V - \dim \operatorname{range} S)(\dim V)$;
$\dim \operatorname{range} \mathcal{B} = (\dim V - \dim \operatorname{null} S)(\dim V)$. □

OR. Using NOTE FOR [3.60] and the notation in Problem $(\circ)$.

$$\mathcal{B}(T) = TS = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i}\right)\left(\sum_{r=1}^{m} E_{r,r}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{i,j}G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1}+ & \cdots & +A_{1,m}G_{m,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}G_{1,m}+ & \cdots & +A_{m,m}G_{m,m} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{n,1}G_{1,n}+ & \cdots & +A_{n,m}G_{m,n} \end{pmatrix}.$$

Thus $\operatorname{null} \mathcal{B} = \operatorname{span}\begin{pmatrix} R_{m+1,1}, & \cdots & ,R_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ R_{m+1,n}, & \cdots & ,R_{n,n} \end{pmatrix}$,

$\operatorname{range} \mathcal{B} = \operatorname{span}\begin{pmatrix} G_{1,1}, & \cdots & ,G_{m,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ G_{1,n}, & \cdots & ,G_{m,n} \end{pmatrix}$. Hence (a) $\dim \operatorname{null} \mathcal{B} = n \times (n-m)$;
(b) $\dim \operatorname{range} \mathcal{B} = n \times m$. □

---

17 *Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Using NOTE FOR [3.60]. Let $(v_1, \dots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.
Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.
Then $\forall E_{i,j} \in \mathcal{E}$, $(\forall x, y = 1, \dots, n)$, by assumption, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.
Again, $E_{y,x'}, E_{y',x} \in \mathcal{E}$ for all $x', y', x, y = 1, \dots, n$. Thus $\mathcal{E} = \mathcal{L}(V)$. □

• OR (10.A.4) *Suppose that $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bases of V.*
Let $T \in \mathcal{L}(V)$ be such that $T\alpha_k = \beta_k, \forall k$. Prove that $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)$
For ease of notation, let $\mathcal{M}(T, \alpha \to \beta) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n))$, $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n))$.

SOLUTION:

Denote $\mathcal{M}(T, \alpha \to \alpha)$ by A and $\mathcal{M}(I, \beta \to \alpha)$ by B.
$\forall k \in \{1, \dots, n\}, Iu_k = u_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. □

OR. Note that $\mathcal{M}(T, \alpha \to \beta) = I$. Hence $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha) \underbrace{\mathcal{M}(T, \alpha \to \beta)}_{=\mathcal{M}(I, \beta \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. □

OR. Note that $\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I$.
$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}\Big(\underbrace{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}_{=\mathcal{M}(T, \alpha \to \beta)} \Big) = \mathcal{M}(I, \beta \to \alpha)$. □

COMMENT: Denote $\mathcal{M}(T, \beta \to \beta)$ by A'.
$u_k = Iu_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n, \ \forall \ k \in \{1, \dots, n\}$.
又 $Tu_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = B_{1,k}\beta_1 + \cdots + B_{n,k}\beta_n = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.
OR. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$.

---

16 *Suppose V is finite-dim and $S \in \mathcal{L}(V)$ such that $\forall T \in \mathcal{L}(V), ST = TS$.*
Prove that $\exists \lambda \in \mathbf{F}, S = \lambda I$.

SOLUTION: Using the notation and result in ($\circ$).

Suppose $ST = TS$ for every $T \in \mathcal{L}(V)$. If $S = 0$, we are done. Now suppose $S \neq 0$.

Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, B_{\mathcal{K}}) = \mathcal{M}(I, B_{\text{range}\,S}, B_{\mathcal{K}})$.

Then $\forall k \in \{m+1, \ldots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $n = \dim V = \dim \text{range}\, S = m$.

NOTICE that $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$. Thus $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \cdots + a_{n,i}v_n)$.

Where $a_{i,j} = \mathcal{M}\big(I, (w_1, \ldots, w_n), (v_1, \ldots, v_n)\big)_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n$;

And For each j, for all i. Thus $a_{i,i} = a_{k,k} = \lambda, \forall\, k \neq i$.

Hence $w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \mathcal{M}\big(\lambda I, (v_1, \ldots, v_n)\big) \Rightarrow S = \mathcal{M}^{-1}(\mathcal{M}(\lambda I))\lambda I.$ □

---

18 *Show that V and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.*

SOLUTION:

Define $\Psi \in \mathcal{L}\big(V, \mathcal{L}(\mathbf{F}, V)\big)$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.

(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is inje.

(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi\big(T(1)\big)$. Hence Ψ is surj. □

OR. Define $\Phi \in \mathcal{L}\big(\mathcal{L}(\mathbf{F}, V), V\big)$ by $\Phi(T) = T(1)$.

(a) Suppose $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is inje.

(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surj. □

COMMENT: $\Phi = \Psi^{-1}$.

• *Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.*

SOLUTION:

Note that $\deg\big[(x^2 + x)p''(x) + 2xp'(x) + p(3)\big] = \deg p$.

Define $T_n : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$. Then $T_n \in \mathcal{L}\big(\mathcal{P}_n(\mathbf{R})\big)$.

And note that $T_n(p) = 0 \Rightarrow \deg(T_n p) = -\infty = \deg p \Rightarrow p = 0$. Thus T_n is inv.

$\forall q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $m = 0$; if $q \neq 0$, let $m = \deg q$, we have $q \in \mathcal{P}_m(\mathbf{R})$.

Hence $\exists\, p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$. □

---

19 *Suppose $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$ is inje. $\deg Tp \leqslant \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.*
(a) *Prove that T is surj;* (b) *Prove that for every nonzero p, $\deg Tp = \deg p$.*

SOLUTION:

(a) T is inje $\iff \forall n \in \mathbf{N}^+, T\,|_{\mathcal{P}_n(\mathbf{R})} : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ is inje and therefore is inv $\iff T$ is surj.

(b) Using mathematical induction.

(i) $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$;
$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$.

(ii) Assume that $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$.
Suppose $\exists\, r \in \mathcal{P}_{n+1}(\mathbf{R}),\ \deg Tr \leqslant n < \deg r = n + 1$.
Then by (a), $\exists\, s \in \mathcal{P}_n(\mathbf{R}),\ T(s) = (Tr)$.
又 T is inje $\Rightarrow s = r$. While $\deg s = \deg Ts = \deg Tr < \deg r$.
Contradicts. Thus $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$. □

ENDED

3·E

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 20 | 4E: 8, 14

1 *A function $T : V \to W$ is linear $\iff$ T is a subspace of $V \times W$.*

---

2 *Suppose $V_1 \times \cdots \times V_m$ is finite-dim. Prove that each V_j is finite-dim.*

Solution:

For any $k \in \{1, \dots, m\}$, define $p_k : V_1 \times \cdots \times V_m \to V_k$ by $p_k(v_1, \dots, v_m) = v_k$.
Then p_k is a surj linear map. By [3.22], range $p_k = V_k$ is finite-dim. ☐

Or. Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.
Let $(v_1, \dots, v_M)$ be a basis of U. Note that $\forall u_i \in V_i, \in U_i \subseteq U$, for each i.

$$\left.\begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{array}\right\} \Rightarrow S_i |_{U_i} = R_i^{-1} |_{U_i}.$$

Thus U_i and V_i are iso. 又 U_i is a subsp of a finite-dim vecsp U. ☐

---

3 *Give an example of a vecsp V and its two subsps U_1, U_2 such that $U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum.*

Solution: V must be infinite-dim. For if not, both U_1 and U_2 are finite-dim subsps. By [3.76, 3.78].

Note *that at least one of U_1, U_2 must be infinite-dim. And at least one must be infinite-dim???* TODO
For if not, $U_1 \times U_2$ is finite-dim and $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$.

Let $V = \mathbf{F}^\infty = U_1$, $U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$.

$$\left.\begin{array}{l} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T((x_1, x_2, \cdots), (x, 0, \cdots)) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = ((x_1, x_2, \cdots), (x, 0, \cdots)) \end{array}\right\} \Rightarrow S = T^{-1}.$$ ☐

---

4 *Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.*

Solution: Using the notation in Problem (2).

Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$.

$$\left.\begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (TR_1, \dots, TR_m). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ ☐

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5 *Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.*

Solution: Using the notation in Problem (2).

Note that $Tv = (w_1, \dots, w_m)$. Define $T_i \in \mathcal{L}(V, W_i)$ by $T_i(v) = w_i$.

$$\left.\begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1 T, \dots, S_m T). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ ☐

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6 *For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.*

Solution:

Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m$.

(a) Suppose $T(v_1, \dots, v_m) = 0$. Then $\forall\, (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$
$\Rightarrow (v_1, \dots, v_m) = 0 \Rightarrow T$ is inje.

(b) Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $\forall\, (b_1, \dots, b_n) \in \mathbf{F}^m$,
$\Big[T(\psi(e_1), \dots, \psi(e_m))\Big](b_1, \dots, b_m) = b_1 \psi(e_1) + \cdots + b_m \psi(e_m) = \psi(b_1 e_1 + \cdots + b_m e_m) = \psi(b_1, \dots, b_m)$.
Thus $T(\psi(e_1), \dots, \psi(e_m)) = \psi$. Hence T is surj. ☐

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14 *Suppose* $U = \{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0$ *for only finitely many* $k\}$.
(a) *Show that* U *is a subsp of* $\mathbf{F}^\infty$. [*Do it in your mind*]
(b) *Prove that* $\mathbf{F}^\infty/U$ *is infinite-dim.*

SOLUTION: For ease of notation, denote the p^{th} term of $u = (x_1, \cdots, x_p, \cdots) \in \mathbf{F}^\infty$ by $u[p]$.

For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, (p-1) \equiv 0 \pmod r \\ 0, \text{otherwise} \end{cases}$ | simply $e_r = (1, \underbrace{0, \ \cdots, \ 0}_{(p-1)\ times}, 1, \underbrace{0, \ \cdots, \ 0}_{(p-1)\ times}, 1, \cdots)$.

Choose one $m \in \mathbf{N}^+$. Let $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 + U \Rightarrow \exists\, u \in U, a_1 e_1 + \cdots + a_m e_m = u$.

Suppose $u = (x_1, \cdots, x_L, 0, \cdots)$, where L is the largest such that $u[L] \neq 0$.

Let $s \in \mathbf{N}^+$ be such that $h = s \cdot m! + 1 > L$ and $e_1[h] = \cdots = e_m[h] = 1$.

Note that by definition, $e_r[s \cot m! + 1 + p] = e_r[p+1] = 1 \iff p \equiv 0 \ (\bmod r) \iff r \mid p$.

Now for any $p \in \{1, \ldots, m\}$, $u[h+p] = \left(\sum_{r=1}^{m} a_r e_r\right)[p+1] = \sum_{k=1}^{\tau(p)} a_{p_k} = 0$ $\quad (\Delta)$

where $1 = p_1 \leqslant \cdots \leqslant p_{\tau(p)} = p$ are all the distinct factors of p.

Let $q = p_{\tau(p)-1}$. Notice that $\tau(q) = \tau(p) - 1$ and $q_k = p_k, \forall k \in \{1, \ldots, \tau(q)\}$.

Again by (Δ), $\left(\sum_{r=1}^{m} a_r e_r\right)[h+q] = \sum_{k=1}^{\tau(p)-1} a_{p_k} = 0$. Thus $a_{p_{\tau(p)}} = a_p = 0$ for any $p \in \{1, \ldots, m\}$.

Hence $\forall m \in \mathbf{N}^+$, $(e_1, \ldots, e_m)$ is linely inde in $\mathbf{F}^\infty$, so is $(e_1 + U, \ldots, e_m + U)$ in $\mathbf{F}^\infty/U$. By (2.A.14). □

OR. For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, \text{if } 2^r \mid p \\ 0, \text{otherwise} \end{cases}$.

Similarly, let $m \in \mathbf{N}^+$ and $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 \Rightarrow a_1 e_1 + \cdots + a_m e_m = u \in U$.

Suppose L is the largest such that $u[L] \neq 0$. And l is such that $2^{ml} > L$.

Then $\forall k \in \{1, \ldots, m\}, u[2^{ml} + 2^k] = \left(\sum_{r=1}^{m} a_r e_r\right)[2^k] = a_1 + \cdots + a_k = 0$.

Thus $a_1 = \cdots = a_m = 0$ and $(e_1, \ldots, e_m)$ is linely inde. Similarly. □

---

7 *Suppose* $v, x \in V$ *and* U *and* W *are subsps of* V. *Prove that* $v + U = x + W \Rightarrow U = W$.

SOLUTION:

(a) $\forall u_1 \in U, \exists\, w_1 \in W, v + u_1 = x + w_1$, let $u_1 = 0$, now $v = x + w_1' \Rightarrow v - x \in W$.
(b) $\forall w_2 \in W, \exists\, u_2 \in U, v + u_2 = x + w_2$, let $w_2 = 0$, now $x = v + u_2' \Rightarrow x - v \in U$.

Thus $\pm(v - x) \in U \cap W \Rightarrow \left\{ \begin{array}{l} u_1 = (x - v) + w_1 \in W \Rightarrow U \subseteq W \\ w_2 = (v - x) + u_2 \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W$. □

• Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$.
Then A is a translate of $U \iff \exists\, c \in \mathbf{R}, A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$.

• *Suppose* $T \in \mathcal{L}(V, W)$ *and* $c \in W$. *Prove that* $U = \{x \in V : Tx = c\}$ *is either* $\emptyset$ *or is a translate of* null T.

SOLUTION:

If $c \in W$ but $c \notin \operatorname{range} T$, then $U = \emptyset$, we are done. Now suppose $c \in \operatorname{range} T$ and $x \in U$.

$\forall x + y \in x + \operatorname{null} T \ (\ \forall y \in \operatorname{null} T\ ), x + y \in U$. Hence $x + \operatorname{null} T \subseteq U$.

$\forall u \in U, u - x \in \operatorname{null} T \Rightarrow u = x + (u - x)x + \operatorname{null} T$. Hence $U \subseteq x + \operatorname{null} T$. □

COROLLARY: *The set of solutions to a system of linear equations such as* [3.28] *is either* $\emptyset$ *or a translate.*

---

8 *Suppose A is a nonempty subset of V.*
Prove that A is a translate of some subsp of $V \iff \lambda v + (1-\lambda)w \in A$, $\forall v, w \in A, \lambda \in \mathbf{F}$.

SOLUTION:

Suppose $A = a + U$. Then $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$,
$\lambda(a+u_1) + (1-\lambda)(a+u_2) = a + \big(\lambda(u_1 - u_2) + u_2\big) \in A$.

Suppose $\lambda v + (1-\lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $a \in A$ and let $A' = \{x - a : x \in A\}$.

Then $0 \in A'$ and $\forall x - a, y - a \in A'$, $(\ \forall x, y \in A\ )$, $\lambda \in \mathbf{F}$,
(I) $\lambda(x-a) = [\lambda x + (1-\lambda)a] - a \in A'$.
(II) $\lambda(x-a) + (1-\lambda)(y-a) = \frac{1}{2}(x-a) + \frac{1}{2}(y-a) = \frac{1}{2}x + (1-\frac{1}{2})y - a \in A'$.
OR. By (I), $2 \times [\frac{1}{2}(x-a) + \frac{1}{2}(y-a)] = (x-a) + (y-a) \in A'$.
Thus A' is a subsp of V. Hence $a + A' = \{(x-a) + a : x \in A\} = A$ is a translate. $\square$

OR. Suppose $x - a, y - a \in A', \lambda \in \mathbf{F}$.
Note that $x, a \in A \Rightarrow \lambda x + (1-\lambda)a = 2x - a \in A$. Similarly $2y - a \in A$.
(I) $(x - \frac{1}{2}a) + (y - \frac{1}{2}a) = x + y - a \in A \Rightarrow x + y - 2a = (x-a) + (y-a) \in A'$.
(II) $\lambda(x-a) = (\lambda x + (1-\lambda)a) - a \in A'$.
Thus $-x + A$ is a subsp of V. Hence $A = x + (-x + A)$ is a translate of the subsp $(-x + A)$. $\square$

---

9 *Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subsps U_1, U_2 of V.*
Prove that the intersection $A_1 \cap A_2$ is either a translate of some subsp of V or is $\emptyset$.

SOLUTION:

Suppose $v + u_1, w + u_2 \in A_1 \cap A_2 \neq \emptyset$. By Problem (8),
$\forall \lambda \in \mathbf{F}, \lambda(v+u_1) + (1-\lambda)(w+u_2) \in A_1 \cap A_2$. Thus $A_1 \cap A_2$ is a translate of some subsp of V. $\square$

OR. Let $A_1 = v + U_1, A_2 = w + U_2$. Suppose $x \in (v + U_1) \cap (w + U_2) \neq \emptyset$.
Then $\exists\, u_1 \in U_1, x = v + u_1 \Rightarrow x - v \in U_1$, $\exists\, u_2 \in U_2, x = w + u_2 \Rightarrow x - w \in U_2$.
Note that by [3.85], $A_1 = v + U_1 = x + U_1, A_2 = w + U_2 = x + U_2$. We show that $A_1 \cap A_2 = x + (U_1 \cap U_2)$.
(a) $y \in A_1 \cap A_2 \Rightarrow \exists\, u_1 \in U_1, u_2 \in U_2, y = x + u_1 = x + u_2 \Rightarrow u_1 = u_2 \in U_1 \cap U_2 \Rightarrow y \in x + (U_1 \cap U_2)$.
(b) $y = x + u \in x + (U_1 \cap U_2) = (x + U_1) \cap (x + U_2) \Rightarrow y \in A_1 \cap A_2$. $\square$

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10 *Prove that the intersection of any collection of translates of subsps of V is either a translate of some subsp or $\emptyset$.*

SOLUTION:

Suppose $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collection of translates of subsps of V, where Γ is an arbitrary index set.

Suppose $x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset$, then by Problem (8), $\forall \lambda \in \mathbf{F}, \lambda x + (1-\lambda)y \in A_\alpha$ for every $\alpha \in \Gamma$.
Thus $\bigcap_{\alpha \in \Gamma} A_\alpha$ is a translate of some subsp of V. $\square$

OR. Let $A_\alpha = w_\alpha + V_\alpha$ for each $\alpha \in \Gamma$. Suppose $x \in \bigcap_{\alpha \in \Gamma}(w_\alpha + V_\alpha) \neq \emptyset$.
Then for each $A_\alpha, \exists\, v_\alpha \in V_\alpha, x = w_\alpha + v_\alpha \Rightarrow x - w_\alpha \in V_\alpha \Rightarrow A_\alpha = w_\alpha + V_\alpha = x + V_\alpha$.
(a) $y \in \bigcap_{\alpha \in \Gamma} A_\alpha \Rightarrow \forall \alpha \in \Gamma, \exists\, v_\alpha, y = x + v_\alpha \Rightarrow \forall \alpha, \beta \in \Gamma, v_\alpha = v_\beta \Rightarrow y \in x + \bigcap_{\alpha \in \Gamma} V_\alpha$.
(b) $y = x + v \in x + \bigcap_{\alpha \in \Gamma} V_\alpha = \bigcap_{\alpha \in \Gamma}(x + V_\alpha) \Rightarrow y \in \bigcap_{\alpha \in \Gamma} A_\alpha$. Hence $\bigcap_{\alpha \in \Gamma} A_\alpha = x + \bigcap_{\alpha \in \Gamma} V_\alpha$. $\square$

• **NOTE FOR [3.79, 3.83]:** If $U = \{0\}$, then $v + U = v + \{0\} = \{v\}$, $V/U = V/\{0\} = \{\{v\} : v \in V\}$.

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11 *Suppose* $A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^{m} \lambda_i = 1\}$, *where each* $v_i \in V, \lambda_i \in \mathbf{F}$.

(a) *Prove that A is a translate of some subsp of V*

(b) *Prove that if B is a translate of some subsp of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.*

(c) *Prove that A is a translate of some subsp of V of dim less than m.*

SOLUTION:

(a) By Problem (8), $\forall u, w \in A, \lambda \in \mathbf{F}, \exists\, a_i, b_i \mathbf{F}$,
$\lambda u + (1-\lambda) w = \left(\lambda \sum_{i=1}^{m} a_i + (1-\lambda) \sum_{i=1}^{m} b_i\right) v_i \in A$. ☐

(b) Suppose $B = v + U$, where $v \in V$ and U is a subsp of V. Suppose $\exists\,! u_k \in U, v_k = v + u_k \in B$.
Then for all $v = \sum_{i=1}^{m} \lambda_i v_i \in A$, $v = \sum_{i=1}^{m} \lambda_i (v + u_i) = v + \sum_{i=1}^{m} \lambda_i u_i \in v + U = B$. ☐

OR. Let $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on m by k.

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By Problem (8), $v \in B$.

(ii) $2 \leqslant k \leqslant m$, we assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. $\left(\forall \lambda_i \text{ such that } \sum_{i=1}^{k} \lambda_i = 1\right)$

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. $\forall i = 1, \dots, k$, $\exists\, \mu_i \neq 1$, fix one such i by ι.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow \left(\sum_{i=1}^{k+1} \frac{\mu_i}{1-\mu_\iota}\right) - \frac{\mu_\iota}{1-\mu_\iota} = 1$.

$$\text{Let } w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let $\lambda_i = \frac{\mu_i}{1-\mu_\iota}$ for $i = 1, \dots, \iota - 1$; $\lambda_j = \frac{\mu_{j+1}}{1-\mu_\iota}$ for $j = \iota, \dots, k$. Then,

$$\left.\begin{array}{r} \sum_{i=1}^{k} \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B.$$ ☐

(c) If $m = 1$, then let $A = v_1 + \{0\}$ and we are done.
Choose one $k \in \{1, \dots, m\}$. Given $\lambda_i \in \mathbf{F}$, where $i \in \{1, \dots, k-1, k+1, \dots, m\}$.
Let $\lambda_k = 1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m$
Then $\lambda_1 v_1 + \cdots + \lambda_k v_k + \cdots + \lambda_m v_m = v_k + \sum_{i=1}^{m} \lambda_i (v_i - v_k)$.
Thus $A = v_k + \operatorname{span}(v_1 - v_k, \dots, v_{k-1} - v_k, v_{k+1} - v_k, \dots, v_m - v_k)$. ☐

---

18 *Suppose $T \in \mathcal{L}(V, W)$ and U is a subsp of V. Let π denote the quotient map.*
Prove that $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \subseteq \operatorname{null} T$.

SOLUTION:

(a) Suppose $U \subseteq \operatorname{null} T$. Define $S \in \mathcal{L}(V/U, W)$ by $S(v + U) = Tv$. Then $S \circ \pi = T$.
Now we show that this map is *well-defined*.
$v_1 + U = v_2 + U \iff (v_1 - v_2) \in U \iff S\big((v_1 - v_2) + U\big) = T(v_1 - v_2) = 0 \iff Tv_1 = Tv_2$.

(b) Suppose $\exists\, S, T = S \circ \pi$. Then $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null} T$. ☐

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20 *Define* $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ *by* $\Gamma(S) = S \circ \pi$. *Prove that*:

(a) *Γ is linear*: By [3.9] distr and [3.6].

(b) *Γ is inje*: $\Gamma(S) = 0 = S \circ \pi \iff \forall v \in V, S(\pi(v)) = 0 \iff \forall v + U \in V/U, S(v + U) = 0 \iff S = 0$.

(c) $\operatorname{range} \Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}$: By Problem (18). ☐

• **NOTE FOR [3.88, 3.90, 3.91]:** Suppose $W \in \mathcal{S}_V U$. Then V/U and W are iso.

For any $W \in \mathcal{S}_V U$, because $V = U \oplus W$, $\forall v \in V, \exists!\, u_v \in U, w_v \in W$ such that $v = u_v + w_v$.

Define $T \in \mathcal{L}(V, W)$ by $T(v) = w_v$. Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$, $\operatorname{range} T \oplus \operatorname{null} T = V$.

Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, W)$ is defined by $\tilde{T}(v + U) = Tv = w_v$.

Now $\pi \circ \tilde{T} = I_{V/U}$, $\tilde{T} \circ \pi = I_W = T\,|_W$ Hence $\tilde{T}$ is an iso of V/U onto W.

• **COMMENT:** Note that $v = u_v + w_v = (u_v - u') + (w'_v + u')$, where $w'_v \notin W \iff u' \neq 0$.

Define $S \in \mathcal{L}(V/U, V)$ by $S(v + U) = v$. Hence $\operatorname{null} S = \{0\}$, $\operatorname{range} S \in \mathcal{S}_V U$, $\operatorname{range} S \oplus U = V$.

Let $E = S \circ \pi$. Now $\operatorname{null} E = \operatorname{null} \pi = U$. Because π is surj 又 $\operatorname{range}(S \circ \pi) \subseteq \operatorname{range} S$. $\operatorname{range} E = \operatorname{range} S$.

Then $\operatorname{range} E \oplus \operatorname{null} E = V$. NOTICE that $E : V \to \operatorname{range} S$ is a pure *eraser*. Now we explain why:

EXAMPLE: Suppose $B_V = (v_1, v_2, v_3)$, $U = \operatorname{span}(v_1)$. Then it is uniquely fixed that $\operatorname{range} S = \operatorname{span}(v_2, v_3)$.

While we might have $\operatorname{range} T = \operatorname{span}(v_2 - 2v_1, v_3) = W$, depending on the choice of W.

Now $E : v_2 \mapsto v_2;\ \ v_2 - 2v_1 \mapsto v_2$. While $T : v_2 \mapsto v_2 - 2v_1;\ \ v_2 - 2v_1 \mapsto v_2 - 2v_1$.

---

12 *Suppose U is a subsp of V such that V/U is finite-dim. Prove that is V is iso to $U \times (V/U)$.*

SOLUTION:

Let $(v_1 + U, \dots, v_n + U)$ be a basis of V/U.

Note that $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^n a_i(v_i + U) = \left(\sum_{i=1}^n a_i v_i\right) + U$

$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) \in U \Rightarrow \exists!\, u \in U, v = \sum_{i=1}^n a_i v_i + u$.

Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ by $\varphi(v) = (u, v + U)$,

and $\psi \in \mathcal{L}(U \times (V/U), V)$ by $\psi(u, v + U) = v + u$, where $\exists!\, a_i \in \mathbf{F}, v = \sum_{i=1}^n a_i v_i + U$. ☐

OR. [*V/U, U and V can be infinite-dim*] Define $S \in \mathcal{L}(V/U, V)$ by $S(v + U) = v$.

By the NOTE FOR [3.88,3.90,3.91], $\operatorname{range} S \oplus U = V$. Thus $\forall v \in V, \exists!\, u \in U, w \in \operatorname{range} S, v = u + w$.

Define $T \in \mathcal{L}(U \times (V/U), V)$ by $T(u, v + U) = u + S(v + U) = u + w = v$. Then T is surj.

And $T(u, v + U) = u + S(v + U) = 0 \Longrightarrow \pi(T(u, v + U)) = v + U = 0$, and $u = -S(v + U) = 0$.

OR. Define $R \in \mathcal{L}(V, U \times (V/U))$ by $R(v) = (u, (w + U))$. Now $R \circ T = I_{U \times (V/U)}$, $T \circ R = I_V$. ☐

• (4E 3.E.14) *Suppose $V = U \oplus W$, $(w_1, \dots, w_m)$ is a basis of W.*
Prove that $(w_1 + U, \dots, w_m + U)$ is a basis of V/U.

SOLUTION: $\forall v \in V, \exists!\, u \in U, w \in W, v = u + w$. 又 $\exists!\, c_i \in \mathbf{F}, w = \sum_{i=1}^m c_i w_i \Rightarrow v = \sum_{i=1}^m c_i w_i + u$.

Hence $\forall v + U \in V/U, \exists!\, c_i \in \mathbf{F}, v + U = \sum_{i=1}^m c_i w_i + U$. ☐

---

13 *Suppose $(v_1 + U, \dots, v_m + U)$ is a basis of V/U and $(u_1, \dots, u_n)$ is a basis of U.*
Prove that $(v_1, \dots, v_m, u_1, \dots, u_n)$ is a basis of V.

SOLUTION: Notice that $(v_1, \dots, v_m)$ is linely inde.

By Problem (12), U and V/U are finite-dim $\Longrightarrow U \times (V/U)$ is finite-dim, so is V.

$\dim V = \dim(U \times (V/U)) = m + n$. 又 Each $v_i = S(v_i + U)$, where we define $S(v + U) = v$.

Note that $\sum_{i=1}^m a_i v_i \in U \iff \left(\sum_{i=1}^m a_i v_i\right) + U = 0 + U \iff a_1 = \cdots = a_m = 0$.

Hence $\operatorname{span}(v_1, \dots, v_m) \cap U = \{0\} \Rightarrow \operatorname{span}(v_1, \dots, v_m) \oplus U = V$. By (2.B.8), we are done. ☐

OR. Note that $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^m a_i v_i + U \Rightarrow \exists!\, b_i \in \mathbf{F}, v - \sum_{i=1}^m a_i v_i = \sum_{i=1}^m b_i u_i \in U$

$\Rightarrow \forall v \in V, \exists!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^m a_i v_i + \sum_{j=1}^m b_j u_j$. ☐

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15 *Suppose* $\varphi \in \mathcal{L}(V, \mathbf{F}) \backslash \{0\}$. *Prove that* $\dim V/(\operatorname{null}\varphi) = 1$.

SOLUTION:

By (3.B.29), $\exists u \in V, V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}$. By (4E 3.E.14), $(u + \operatorname{null}\varphi)$ is a basis of $V/\operatorname{null}\varphi$.

OR. By [3.91] (d), $\dim \operatorname{range}\varphi = 1 = \dim V/(\operatorname{null}\varphi)$. ☐

---

16 *Suppose* $\dim V/U = 1$. *Prove that* $\exists \varphi \in \mathcal{L}(V, \mathbf{F})$ *such that null* $\varphi = U$.

SOLUTION:

Suppose V_0 is a subsp of V such that $V = U \oplus V_0$. Then V_0 and V/U are iso. $\dim V_0 = 1$.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. ☐

OR. Let $(w + U)$ be a basis of V/U. Then $\forall v \in V, \exists! a \in \mathbf{F}, v + U = aw + U$.

Define $\varphi : V \to \mathbf{F}$ by $\varphi(v) = a$. Assume that φ is linear.

Then $u \in U \iff u + U = 0w + U \iff \varphi(u) = 0 \iff u \in \operatorname{null}\varphi$. Thus $U = \operatorname{null}\varphi$. ☐

Now we prove the assumption.

$\forall x, y \in V, \lambda \in \mathbf{F}, \exists! a, b \in \mathbf{F}, x + U = aw + U, \lambda y + U = \lambda bw + U \Rightarrow (x + \lambda y) + U = (a + \lambda b)w + U$.

Then $\varphi(x + \lambda y) = a + \lambda b = \varphi(x) + \lambda\varphi(y)$.

---

17 *Suppose* V/U *is finite-dim.* W *is a subsp of* V.

(a) *Show that if* $V = U + W$, *then* $\dim W \geqslant \dim V/U$.

(b) *Find a* W *such that* $\dim W = \dim V/U$ *and* $V = U \oplus W$.

SOLUTION: Let $(w_1, \dots, w_n)$ be a basis of W

(a) $\forall v \in V, \exists u \in U, w \in W$ such that $v = u + w \Rightarrow v + U = w + U$

And $\exists! a_i \in \mathbf{F}, v + U = (a_1 w_1 + \cdots + a_n w_n) + U$. Then $V/U \subseteq \operatorname{span}(w_1 + U, \dots, w_n + U)$.

Hence $\dim V/U = \dim \operatorname{span}(w_1 + U, \dots, w_n + U) \leqslant \dim W$.

(b) Let $W \in \mathcal{S}_V U$. In other words, reduce $(w_1 + U, \dots, w_n + U)$

to a basis $(w_1 + U, \dots, w_m + U)$ of V/U and let $W = \operatorname{span}(w_1, \dots, w_m)$. ☐

OR. Let $(v_1 + U, \dots, v_m + U)$ be a basis of V/U and define $\tilde{T} \in \mathcal{L}(V/U, V)$ by $\tilde{T}(v_k + U) = v_k$.

Note that $\pi \circ \tilde{T} = I$. By (3.B.20), $\tilde{T}$ is inje. And $(v_1, \dots, v_m)$ is linely inde.

Let $W = \operatorname{range}\tilde{T} = \operatorname{span}(v_1, \dots, v_m)$. Then $\tilde{T} \in \mathcal{L}(V/U, W)$ is an iso. Thus $\dim W = \dim V/U$.

And $\forall v \in V, \exists! a_i \in \mathbf{F}, v + U = a_1 v_1 + \cdots + a_m v_m + U$

$\Rightarrow v - (a_1 v_1 + \cdots + a_m v_m) \in U \Rightarrow \exists! w \in W, u \in U, v = w + u$. ☐

ENDED

3·F

4 5 6 7 8 9 12 13 15 16 17 18 19 20 21 22 23 24 25 26
28 29 30 31 32 33 34 35 36 37 | 4E: 5, 6, 8, 17, 23, 24, 25

20, 21 *Suppose* U *and* W *are subsets of* V. *Prove that* $U \subseteq W \iff W^0 \subseteq U^0$.

SOLUTION:

(a) Suppose $U \subseteq W$. Then $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(w) = 0 \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.

(b) Suppose $W^0 \subseteq U^0$. Then $\varphi \in W^0 \Rightarrow \varphi \in U^0$. Hence $\operatorname{null}\varphi \supseteq W \Rightarrow \operatorname{null}\varphi \supseteq U$. Thus $W \supseteq U$.

OR. For a subsp U of V, let $A_U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = U$, by Problem (25).

Suppose $W^0 \subseteq U^0$. Then $\forall \varphi \in W^0, v \in A_U, \varphi(v) = 0 \Rightarrow v \in A_W$. Thus $A_U \subseteq A_W$. ☐

COROLLARY: $W^0 = U^0 \iff U = W$.

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22 *Suppose U and W are subsps of V. Prove that $(U+W)^0 = U^0 \cap W^0$.*

SOLUTION:

(a) $\left.\begin{array}{c} U \subseteq U+W \\ W \subseteq U+W \end{array}\right\} \Rightarrow \left.\begin{array}{c} (U+W)^0 \subseteq U^0 \\ (U+W)^0 \subseteq W^0 \end{array}\right\} \Rightarrow (U+W)^0 \subseteq U^0 \cap W^0.$

OR. Suppose $\varphi \in (U+W)^0$. Then $\forall u \in U, w \in W, \varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$.

(b) Suppose $\varphi \in U^0 \cap W^0 \subseteq V'$. Then $\forall u \in U, w \in W, \varphi(u+w) = 0 \Rightarrow \varphi \in (U+W)^0$. □

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23 *Suppose U and W are subsets of V. Prove that $(U \cap W)^0 = U^0 + W^0$.*

SOLUTION:

(a) $\left.\begin{array}{c} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array}\right\} \Rightarrow \left.\begin{array}{c} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array}\right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \ [\supseteq U^0 \cap W^0 = (U+W)^0.]$

OR. Suppose $\varphi = \psi + \beta \in U^0 + W^0$. Then $\forall v \in U \cap W$, $\varphi(v) = (\psi+\beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0$.

(b) [*Only in Finite-dim; Requires that U, W are subsps*] Using Problem (22).

$$\begin{aligned} \dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\ &= 2\dim V - \dim U - \dim W - \big(\dim V - \dim(U+W)\big) = \dim V - \dim(U \cap W). \end{aligned}$$

OR. Suppose $\varphi \in (U \cap W)^0$. Let X, Y be such that $V = U \oplus X = W \oplus Y$.
Define $\psi \in U^0, \beta \in W^0$ by $\psi(u+x) = \frac{1}{2}\varphi(x), \beta(w+y) = \frac{1}{2}\varphi(y)$.
$\forall v = u+x = w+y \in V, \varphi(v) = \varphi(x) = \varphi(y)$. Now $\varphi(v) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y) = \psi(v) + \beta(v)$.
Hence $\varphi \in U^0 + W^0$. Now $(U \cap W)^0 \subseteq U^0 + W^0$. □

• **COROLLARY:**

(a) *Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subsets of V. Then $\left(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \sum_{\alpha_i \in \Gamma} \left(V_{\alpha_i}^0\right)$.*

(b) *Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subsps of V. Then $\left(\sum_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \bigcap_{\alpha_i \in \Gamma} \left(V_{\alpha_i}^0\right)$.*

(c) *Suppose $V = U \oplus W$. Then $V' = U^0 \oplus W^0$. And $U_V' = W^0$, $W_V' = U^0$.*

Where $U_V' = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$. And $\iota \in \mathcal{L}(V, U)$ is defined by $\iota(u_v + w_v) = u_v$.

• (4E 3.F.23) *Suppose $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following sets are the same.*

(a) $\operatorname{span}(\varphi_1, \dots, \varphi_m)$

(b) $\big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big)^0 \overset{(c)}{=\!=} \{\varphi \in V' : (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) \subseteq \operatorname{null}\varphi\}$

SOLUTION: By Problem (17), (c) holds.

By Problem (26) [*May require finite-dim*] and the COROLLARY in Problem (23),

$$\left.\begin{array}{r} \big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big)^0 = (\operatorname{null}\varphi_1)^0 + \cdots + (\operatorname{null}\varphi_m)^0 \\ \operatorname{span}(\varphi_i) = \{v \in V : \forall \psi \in \operatorname{span}(\varphi_i), \psi(v) = 0\}^0 = (\operatorname{null}\varphi_i)^0 \end{array}\right\} \Rightarrow \text{(a)} = \text{(b)}.$$

□

OR. Note that by COROLLARY in Problem (4E 6), for each φ_i, we have
$\forall c \in \mathbf{F}\backslash\{0\}, \psi = c\varphi_i \in \operatorname{span}(\varphi_i) \iff \operatorname{null}\psi = \operatorname{null}\varphi_i \iff \psi \in (\operatorname{null}\psi)^0 = (\operatorname{null}\varphi_i)^0$.
And $0 \in \operatorname{span}(\varphi_i), 0 \in (\operatorname{null}\varphi_i)^0$. Hence $\operatorname{span}(\varphi_i) = (\operatorname{null}\varphi_i)^0$. Similarly. □

OR. [*Only in Finite-dim*] Suppose $\varphi \in V'$. Note that $\dim(\operatorname{null}\varphi)^0 = \dim\operatorname{range}\varphi = \dim\operatorname{span}(\varphi)$.
And because $\forall c \in \mathbf{F}, v \in \operatorname{null}\varphi, c\varphi(v) = 0 \Rightarrow \operatorname{span}(\varphi) \subseteq (\operatorname{null}\varphi)^0$. Similarly. □

COROLLARY: **30** Suppose V is finite-dim and $\varphi_1, \dots, \varphi_m$ is a linely inde list in V'.
Then $\dim\big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big) = (\dim V) - m$.

---

31 *Suppose V is finite-dim and $B_{V'} = (\varphi_1, \dots, \varphi_n)$. Show that the correspd B_V exists.*

SOLUTION:

Using (3.B.29). Let $\varphi_i(u_i) = 1$ and then $V = \operatorname{null}\varphi_i \oplus \operatorname{span}(u_i)$ for each φ_i.

Suppose $a_1u_1 + \cdots + a_nu_n = 0$. Then $0 = \varphi_i(a_1u_1 + \cdots + a_nu_n) = a_i$ for each i.

Thus $B_V = (\varphi_1, \dots, \varphi_n)$. And $\varphi_i(u_x) = \delta_{i,x}$. $\square$

OR. For each $k \in \{1, \dots, n\}$, define $\Gamma_k = \{1, \dots, k-1, k+1, \dots, n\}$ and $U_k = \bigcap_{j\in\Gamma_k} \operatorname{null}\varphi_j$.

By Problem (30) OR (4E 2.C.16), $\dim U_k = 1$. Thus $\exists\, u_k \in V, U_k = \operatorname{span}(u_k) \neq 0$.

又 By Problem (30), $(\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_n) = \{0\} = U \cap \operatorname{null}\varphi_k$.

Then if $\varphi_k(u_k) = 0 \Rightarrow u_k \in \operatorname{null}\varphi_k$ while $u_k \in U \Rightarrow u_k \in \{0\}$, contradicts.

Thus $\varphi_k(u_k) \neq 0$. Let $v_k = (\varphi_k(u_k))^{-1}u_k \Rightarrow \varphi_k(v_k) = 1$. Now for $j \neq k, u_k \in \operatorname{null}\varphi_j \Rightarrow \varphi_j(v_k) = 0$.

Similarly, suppose $a_1v_1 + \cdots + a_nv_n = 0 \Rightarrow a_1 = \cdots = a_n = 0$. $B_V = (v_1, \dots, v_n)$. And $\varphi_j(v_k) = \delta_{j,k}$. $\square$

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25 *Suppose U is a subsp of V. Explain why $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.*

SOLUTION: Note that $U = \{v \in V : v \in U\}$ is a subsp of V; And $v \in U \iff \varphi(v) = 0, \forall\varphi \in U^0$. $\square$

COROLLARY: $U^0 = \{v \in V : \varphi(v) = 0, \forall\varphi \in U^0\}^0$.

COMMENT: $\{v \in V : \varphi(v) = 0, \forall\varphi \in U^0\} = ((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) \cap \cdots)$, where $\varphi_k \in U^0$, always remains a subsp, whether the subset U is a subsp or not.

---

26 *Suppose Ω is a subsp of V'. Prove that $\Omega = \{v \in V : \varphi(v) = 0, \forall\varphi \in \Omega\}^0$.*

SOLUTION:

Suppose $U = \{v \in V : \varphi(v) = 0, \forall\varphi \in \Omega\}$, *which is the set of vecs that each $\varphi \in \Omega$ sends to zero in common.*

Then $U^0 = \{v \in V : \varphi(v) = 0, \forall\varphi \in \Omega\}^0$. 又 $U^0 = \{v \in V : \varphi(v) = 0, \forall\varphi \in U^0\}^0$.

Immediately by the COROLLARY in Problem (20,21), we may conclude that $\Omega = U^0$. $\square$

OR. [*Requires Ω finite-dim*] Let $(\varphi_1, \dots, \varphi_m)$ be a basis of Ω. Then by def, $U \subseteq (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.

$\forall\varphi \in \Omega, \exists!\, a_i \in \mathbf{F}, \varphi = a_1\varphi_1 + \cdots + a_m\varphi_m \Rightarrow \forall v \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m), \varphi(v) = 0 \Rightarrow v \in U$.

Hence $(\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = U$. 又 $\operatorname{span}(\varphi_1, \dots, \varphi_m) = \Omega$. By Problem (23), we are done. $\square$

COROLLARY: For every subsp Ω of V', $\exists!$ subsp U of V such that $\Omega = U^0$.

COMMENT: [*Only in Finite-dim*] Using Problem (31) and the COROLLARY(c) in Problem (22, 23).

Let $B_\Omega = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n), B_V = (v_1, \dots, v_m, \dots, v_n)$.

$V' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \overset{\text{(I)}}{=\!=} \operatorname{span}(v_{m+1}, \dots, v_n)^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.

$\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m) \overset{\text{(II)}}{=\!=} \operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0$; $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \overset{\text{(III)}}{=\!=} \operatorname{span}(v_1, \dots, v_m)^0$.

$\iff U = \operatorname{span}(v_{m+1}, \dots, v_n) = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$. [***Another proof of*** [*3.106*] OR. Problem (24)]

(I) Using the COROLLARY(c), immediately.

(II) NOTICE that each $\operatorname{null}\varphi_k = \operatorname{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n) = U_k$; $\dim U_k = \dim V - 1$.

By (4E 2.C.16), $U = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = \bigcap_{k=1}^{m} U_k = \operatorname{span}(v_{m+1}, \dots, v_n)$.

Hence $\operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0 = \Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$.

(III) NOTICE that $V' = \Omega \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.

And that $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq \operatorname{span}(v_1, \dots, v_m)^0$.

By the TIPS in (1.C), $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = \operatorname{span}(v_1, \dots, v_m)$.

OR. Similar to (II), let $\Omega = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, immediately. $\square$

• *Suppose* $T \in \mathcal{L}(V, W), \varphi_k \in V', \psi_k \in W'$.

28 *Prove that* $\operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) \iff \operatorname{range} T = (\operatorname{null} \psi_1) \cap \cdots \cap (\operatorname{null} \psi_m)$.

29 *Prove that* $\operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \iff \operatorname{null} T = (\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)$.

SOLUTION: Using [3.107], [3.109], Problem (23) and the COROLLARY in Problem (20, 21).

(28) $(\operatorname{range} T)^0 = \operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) = \big((\operatorname{null} \psi_1) \cap \cdots \cap (\operatorname{null} \psi_m)\big)^0$.

(29) $(\operatorname{null} T)^0 = \operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) = \big((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)\big)^0$. □

COROLLARY: Using the COMMENT in Problem (26).

$\operatorname{null} T = \operatorname{span}(v_1, \dots, v_m) \iff \operatorname{null} T = (\operatorname{null} \varphi_{m+1}) \cap \cdots \cap (\operatorname{null} \varphi_n) \iff \operatorname{range} T' = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$.

——Where $B_V = (v_1, \dots, v_m, \dots, v_n) \iff B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$.

$\operatorname{range} T = \operatorname{span}(w_1, \dots, w_m) \iff \operatorname{range} T = (\operatorname{null} \psi_{m+1}) \cap \cdots \cap (\operatorname{null} \psi_n) \iff \operatorname{null} T' = \operatorname{span}(\psi_{m+1}, \dots, \psi_n)$.

——Where $B_W = (w_1, \dots, w_m, \dots, w_n) \iff B_{W'} = (\psi_1, \dots, \psi_m, \dots, \psi_n)$.

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9 *Let* $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \cdots, \varphi_n)$. *Then* $\forall \psi \in V', \psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.

COROLLARY: *For other* $B'_V = (u_1, \dots, u_n), B'_{V'} = (\rho_1, \dots, \rho_n), \forall \psi \in V', \psi = \psi(u_1)\rho_1 + \cdots + \psi(u_n)\rho_n$.

SOLUTION:

$\psi(v) = \psi\Big(\sum_{i=1}^n a_i v_i \Big) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i)\varphi_i(v) = \big[\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n\big](v)$.

OR. $\big[\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n\big]\Big(\sum_{i=1}^n a_i v_i \Big) = \psi(v_1)\varphi_1\Big(\sum_{i=1}^n a_i v_i \Big) + \cdots + \psi(v_n)\varphi_n\Big(\sum_{i=1}^n a_i v_i \Big)$. □

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13 *Define* $T : \mathbf{R}^3 \to \mathbf{R}^2$ *by* $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$.

Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ *denote the dual basis of the standard basis of* $\mathbf{R}^2$ *and* $\mathbf{R}^3$.

(a) *Describe the linear functionals* $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$

For any $(x, y, z) \in \mathbf{R}^3$, $\big(T'(\varphi_1)\big)(x, y, z) = 4x + 5y + 6z$, $\big(T'(\varphi_2)\big)(x, y, z) = 7x + 8y + 9z$.

(b) *Write* $T'(\varphi_1)$ *and* $T'(\varphi_2)$ *as linear combinations of* ψ_1, ψ_2, ψ_3.

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$.

(c) *What is* $\operatorname{null} T'$? *What is* $\operatorname{range} T'$?

$$T(x, y, z) = 0 \iff \begin{cases} 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases} \iff \begin{cases} x + y + z = 0 \\ y = 2z = 0 \end{cases} \iff (x, y, z) \in \operatorname{span}(e_1 - 2e_2 + e_3).$$

Where (e_1, e_2, e_3) is standard basis of $\mathbf{R}^3$.

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a basis, with the correspd dual basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Thus $\operatorname{span}(e_1 - 2e_2 + e_3) = \operatorname{null} T \Rightarrow \operatorname{span}(e_1 - 2e_2 + e_3)^0 = \operatorname{span}(\varepsilon_2, \varepsilon_3) = \operatorname{range} T'$.

Note that $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$.

And $\left| \begin{array}{l} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{array} \right.$

Hence $\varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2$, $\varepsilon_3 = -\psi_1 + \psi_3$. Now $\operatorname{range} T' = \operatorname{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3)$.

OR. $\operatorname{range} T' = \operatorname{span}(T'(\varphi_1), T'(\varphi_2)) = \operatorname{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3)$.

Suppose $T'(x\varphi_1 + y\varphi_2) = (4x + 7y)\varphi_1 + (5x + 8y)\varphi_2 + (6x + 9y)\varphi_3 = 0$.

Then $x + y = 4x + 7y = x = y = 0$. Hence $\operatorname{null} T' = \{0\}$.

OR. $\operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \operatorname{span}(-2e_2, e_3) \oplus \operatorname{null} T$.

$\Rightarrow \operatorname{range} T = \{Tx : x \in \operatorname{span}(-2e_2, e_3)\} = \operatorname{span}(T(-2e_2), T(e_3))$

$= \operatorname{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \operatorname{span}(f_1, f_2) = \mathbf{R}^2$. Now $\operatorname{null} T' = (\operatorname{range} T)^0 = \{0\}$. □

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24 *Suppose V is finite-dim and U is a subsp of V.*
Prove, using the pattern of [3.104], that $\dim U + \dim U^0 = \dim V$.

SOLUTION:

By Problem (31) and the COMMENT in Problem (26), $B_U = (v_1, \dots, v_m) \iff B_{U^0} = (\varphi_{m+1}, \dots, \varphi_n)$. $\square$

---

37 *Suppose U is a subsp of V and π is the quotient map. Thus $\pi' \in \mathcal{L}\big((V/U)', V'\big)$.*

(a) *Show that π' is inje*: Because π is surj. Use [3.108].

(b) *Show that* range $\pi' = U^0$: By [3.109](b), range $\pi' = (\operatorname{null} \pi)^0 = U^0$.

(c) *Conclude that π' is an iso from $(V/U)'$ onto U^0*: Immediately.

SOLUTION: OR. Using (3.E.18), also see (3.E.20).

(a) $\pi'(\varphi) = 0 \iff \forall v \in V\ (\ \forall v + U \in V\ ), \varphi\big(\pi(v)\big) = \varphi(v + U) = 0 \iff \varphi = 0$.

(b) $\psi \in \operatorname{range} \pi' \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null} \psi \supseteq U \iff \psi \in U^0$. Hence range $\pi' = U^0$. $\square$

• *Suppose U is a subsp of V. Prove that $(V/U)'$ and U^0 are iso.* [***Another proof of* [3.106]**]

SOLUTION:

Define $\xi : U^0 \to (V/U)'$ by $\xi(\varphi) = \tilde{\varphi}$, where $\tilde{\varphi} \in (V/U)'$ is defined by $\tilde{\varphi}(v + U) = \varphi(v)$.

We show that ξ is inje and surj.

Inje: $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V\ (\ \forall v + U \in V/U\ ), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$.

Surj: $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null}(\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$. $\square$

OR. Define $\nu : (V/U)' \to U^0$ by $\nu(\Phi) = \Phi \circ \pi$. Now $\nu \circ \xi = I_{U^0},\ \xi \circ \nu = I_{(V/U)'} \Rightarrow \xi = \nu^{-1}$. $\square$

---

4 *Suppose U is a subsp of V and $U \neq V$. Prove that* $\underbrace{\exists\, \varphi \in V' \backslash \{0\}, \varphi(u) = 0 \textit{ for all } u \in U}_{\iff U^0_V \neq \{0\}}$.

SOLUTION:

Let X be such that $V = U \oplus X$. Then $X \neq \{0\}$. Suppose $s \in X$ and $x \neq 0$.

Let Y be such that $X = \operatorname{span}(s) \oplus Y$. Now $V = U \oplus \big(\operatorname{span}(s) \oplus Y\big)$.

Define $\varphi \in V'$ by $\varphi(u + \lambda s + y) = \lambda$. Hence $\varphi \neq 0$ and $\varphi(u) = 0$ for all $u \in U$. $\square$

OR. [*Requires that V is finite-dim*] By [3.106], $\dim U^0 = \dim V - \dim U > 0$. Then $U^0 \neq \{0\}$.

OR. Let $B_V = (\underbrace{u_1, \dots, u_m}_{B_U}, v_1, \dots, v_n)$ with $n \geqslant 1$. Let $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$. Let $\varphi = \varphi_i$.

OR. Define $\varphi \in V'$ by $\varphi(u_1) = \cdots = \varphi(u_m) = 0$ and $\varphi(v_1) = \cdots = \varphi(v_n) = 1$. $\square$

COMMENT: [***Another proof of* [3.108]**]: T is surj $\iff T'$ is inje.

(a) Suppose T' is inje. Note that $T'(\psi) = 0 \Rightarrow \psi = 0$.

Then $\nexists\, \psi \in W' \backslash \{0\}, \big(T'(\psi)\big)(v) = \psi(Tv) = 0$ for all $w \in \operatorname{range} T\ (\ \forall v \in V\ )$.

Thus if we assume that range $T \neq W$ then contradicts. Hence range $T = W$.

(b) Suppose T is surj. Then $(\operatorname{range} T)^0 = W^0_W = \{0\} = \operatorname{null} T'$. $\square$

• *Suppose V is a vecsp and U is a subsp of V.*

17 $U^0 = \{\varphi \in V' : U \subseteq \textit{null}\varphi\}$. Noticing $\varphi \in V', U \subseteq \operatorname{null} \varphi \iff \forall u \in U, \varphi(u) = 0$.

18 $U^0 = V' \iff \forall \varphi \in V', U \subseteq \operatorname{null} \varphi \iff U = \{0\}$. [*Which means* $\{0\}^0_V = V'$.]

OR. $U^0 = V' \iff \dim U^0 = \dim V' = \dim V \iff \dim U = 0 \iff U = \{0\}$.

19 $U^0_V = \{0\} = V^0_V \iff U = V$. By the inverse and contrapositive of Problem (4). OR. By [3.106].

• *Suppose $V = U \oplus W$. Define $\iota : V \to U$ by $\iota(u+w) = u$. Thus $\iota' \in \mathcal{L}(U', V')$.*

(a) *Show that* $\operatorname{null} \iota' = U_U^0 = \{0\}$: $\operatorname{null} \iota' = (\operatorname{range} \iota)_U^0 = U_U^0 = \{0\}$.

(b) *Prove that* $\operatorname{range} \iota' = W_V^0$: $\operatorname{range} \iota' = (\operatorname{null} \iota)_V^0 = W_V^0$.

(c) *Prove that $\widetilde{\iota'}$ is an iso from $U'/\{0\}$ onto W^0*: By (a), (b) and [3.91](d).

SOLUTION:

(a) $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \operatorname{null} \psi$.

(b) Note that $W = \operatorname{null}(\iota) \subseteq \operatorname{null}(\psi \circ \iota)$. Then $\psi \circ \iota \in W^0 \Rightarrow \operatorname{range} \iota' \in W^0$.

Suppose $\varphi \in W^0$. Because $\operatorname{null} \iota = W \subseteq \operatorname{null} \varphi$. By TIPS in (3.B), $\varphi = \varphi \circ \iota = \iota'(\varphi)$. □

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36 *Suppose U is a subsp of V. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*

(a) *Show that* $\operatorname{null} i' = U^0$: $\operatorname{null} i' = (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$.

(b) *Prove that* $\operatorname{range} i' = U'$: $\operatorname{range} i' = (\operatorname{null} i)_U^0 = \{0\}_U^0 = U'$.

(c) *Prove that $\widetilde{i'}$ is an iso from V'/U^0 onto U'*: By (a), (b) and [3.91](d).

SOLUTION:

(a) $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$. Thus $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$.

(b) Suppose $\psi \in U'$. By (3.A.11), $\exists\, \varphi \in V', \varphi|_U = \psi$. Then $i'(\varphi) = \psi$. □

• *Suppose $T \in \mathcal{L}(V, W)$. Prove that* $\operatorname{range} T' = (\operatorname{null} T)^0$. [***Another proof of* [3.109](b)**]

SOLUTION:

Suppose $\Phi \in (\operatorname{null} T)^0$. Because by (3.B.12), $T|_U : U \to \operatorname{range} T$ is an iso; $V = U \oplus \operatorname{null} T$.

And $\forall v \in V, \exists!\, u_v \in U, w_v \in \operatorname{null} T, v = u_v + w_v$. Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$.

Let $\psi = \Phi \circ (T|_{\operatorname{range} T}^{-1})$. Then $T'(\psi) = \psi \circ T = \Phi \circ (T^{-1}|_{\operatorname{range} T} \circ T|_V)$.

Where $T^{-1}|_{\operatorname{range} T} : \operatorname{range} T \to U$; $T : V \to \operatorname{range} T$. Note that $T^{-1}|_{\operatorname{range} T} \circ T|_V = \iota$.

By TIPS in (3.B), $\Phi = \Phi \circ \iota$. Thus $T'(\psi) = \psi \circ T = \Phi \circ \iota = \Phi$. □

• *Suppose $T \in \mathcal{L}(V, W)$. Using* [3.108], [3.110].

$$\textit{Now } T \textit{ is inv} \iff \left| \begin{array}{c} \operatorname{null} T = \{0\} \iff (\operatorname{null} T)^0 = V' = \operatorname{range} T' \\ \operatorname{range} T = W \iff (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T' \end{array} \right| \iff T' \textit{ is inv.}$$

---

15 *Suppose $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \iff T = 0$.*

SOLUTION:

Suppose $T = 0$. Then $\forall \varphi \in W', T'(\varphi) = \varphi \circ T = 0$. Hence $T' = 0$.

Suppose $T' = 0$. Then $\operatorname{null} T' = W' = (\operatorname{range} T)^0$, by [3.107](a).

[*W can be infinite-dim*] By Problem (25),

$\operatorname{range} T = \{w \in W : \varphi(w) = 0, \forall \varphi \in (\operatorname{range} T)^0\} = \{w \in W : \varphi(w) = 0, \forall \varphi \in W'\}$.

Now we prove that if $\forall \varphi \in W', \varphi(w) = 0$, then $w = 0$. So that $\operatorname{range} T = \{0\}$ and we are done.

Assume that $w \neq 0$. Then let U be such that $W = U \oplus \operatorname{span}(w)$.

Define $\psi \in W'$ by $\psi(u + \lambda w) = \lambda$. So that $\psi(w) = 1 \neq 0$. □

OR. [*Only if W is finite-dim*] By [3.106], $\dim \operatorname{range} T = \dim W - \dim(\operatorname{range} T)^0 = 0$. □

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12 NOTICE that $I_{V'} : V' \to V'$. *Now $\forall \varphi \in V', I_{V'}(\varphi) = \varphi = \varphi \circ I_V = I_V{}'(\varphi)$. Thus $I_{V'} = I_V{}'$.*

---

16 *Suppose V, W are finite-dim. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(V, W)$.*
Prove that Γ is an iso of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

Solution: By [3.101], Γ is linear.

Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus Γ is inje.

Because V, W are finite-dim. $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$. Now Γ inje $\Rightarrow$ inv. □

Comment: Let $X = \{T \in \mathcal{L}(V, W) : \operatorname{range} T \text{ is finite-dim}\}$.
Let $Y = \{\mathcal{T} \in \mathcal{L}(W', V') : \operatorname{range} \mathcal{T} \text{ is finite-dim}\}$.
Then $\Gamma|_X$ is an iso of X onto Y, even if V and W are infinite-dim.

The inje of $\Gamma|_X$ is equiv to the inje of Γ, as shown before.

Now we show that $\Gamma|_X$ is surj without the cond that V or W is finite-dim.

Suppose $\mathcal{T} \in Y$. Let $B_{\operatorname{range} \mathcal{T}} = (\varphi_1, \dots, \varphi_m)$, with the correspd $(v_1, \dots, v_m)$. Let $\varphi_k = \mathcal{T}(\psi_k)$.

Let $\mathcal{K}$ be such that $W' = \mathcal{K} \oplus \operatorname{null} \mathcal{T}$. Let $B_{\mathcal{K}} = (\psi_1, \dots, \psi_m)$, with the correspd $(w_1, \dots, w_m)$.

Define $T \in \mathcal{L}(V, W)$ by $Tv_k = w_k, Tu = 0;\ k \in \{1, \dots, m\}, u \in U$.

$\forall \psi \in \operatorname{null} \mathcal{T}, [T'(\psi)](v) = \psi(Tv) = \psi(a_1 w_1 + \cdots + a_p w_p) = 0 = [\mathcal{T}(\psi)](v)$.

$\forall k \in \{1, \dots, m\}, [T'(\psi_k)](v) = \psi_k(Tv) = \psi_k(a_1 w_1 + \cdots + a_m w_m) = a_k = \varphi_k(v) = [\mathcal{T}(\psi)](v)$. □

Comment: This is another proof of [3.109(a)]: $\dim \operatorname{range} T = \dim \operatorname{range} T'$.

• (4E 3.F.6) *Suppose $\varphi, \beta \in V'$. Prove that $\operatorname{null} \varphi \subseteq \operatorname{null} \beta \iff \beta = c\varphi, \exists c \in \mathbf{F}$.*

Corollary: $\operatorname{null} \varphi = \operatorname{null} \beta \iff \beta = c\varphi, \exists c \in \mathbf{F} \backslash \{0\}$.

Solution:

Using (3.B.29, 30).

(a) Suppose $\operatorname{null} \varphi \subseteq \operatorname{null} \beta$. Suppose $u \notin \operatorname{null} \beta$, then $u \notin \operatorname{null} \varphi$.

Now $V = \operatorname{null} \beta \oplus \operatorname{span}(u) = \operatorname{null} \varphi \oplus \operatorname{span}(u)$. By Tips in (1.C), $\operatorname{null} \beta = \operatorname{null} \varphi$. Let $c = \dfrac{\beta(u)}{\varphi(u)}$.

Or. We discuss in two cases. If $\operatorname{null} \varphi = \operatorname{null} \beta$, then we are done.

Otherwise, $\operatorname{null} \beta \neq \operatorname{null} \varphi$. Then $\exists u' \in \operatorname{null} \beta \backslash \operatorname{null} \varphi$.

Now $V = \operatorname{null} \varphi \oplus \operatorname{span}(u') = \operatorname{null} \varphi \oplus \operatorname{span}(u)$. $\forall v \in V, v = w + au = w' + bu', \exists!\, w, w' \in \operatorname{null} \varphi$.

Thus $\beta(v) = a\beta(u),\ \varphi(v) = b\varphi(u')$. Let $c = \dfrac{a\beta(u)}{b\varphi(u')}$. We are done.

Notice that by (b) below, we have $\operatorname{null} \beta \subseteq \operatorname{null} \varphi,\ u = u'$. Thus contradicts the assumption.

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$. If $c = 0$, then $\operatorname{null} \beta = V \supseteq \operatorname{null} \varphi$, we are done.

$$\text{Otherwise, } \left.\begin{array}{l} \forall v \in \operatorname{null} \varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \operatorname{null} \varphi \subseteq \operatorname{null} \beta \\ \forall v \in \operatorname{null} \beta, \beta(v) = 0 = \varphi(v) \Rightarrow \operatorname{null} \beta \subseteq \operatorname{null} \varphi \end{array}\right\} \Rightarrow \operatorname{null} \varphi = \operatorname{null} \beta.$$ □

Or. By (3.B.24), $\operatorname{null} \varphi \subseteq \operatorname{null} \beta \iff \exists E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi$. (if ***E* is inv**, then $\operatorname{null} \varphi = \operatorname{null} \beta$)

Now we show that [P] $\exists E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi \iff \exists c \in \mathbf{F}, \beta = c\varphi$. [Q].

$[P] \Rightarrow [Q]$: Let $c = E(1)$. Then $\forall v \in V, \beta(v) = E(\varphi(v)) = \varphi(v)E(1) = c\varphi(v)$. ($\boldsymbol{E(1) \neq 0}$)

$[Q] \Rightarrow [P]$: Define $E \in \mathcal{L}(\mathbf{F})$ by $E(x) = cx$. Then $\forall v \in V, \beta(v) = c\varphi(v) = E(\varphi(v))$. ($\boldsymbol{c \neq 0}$) □

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5 *Prove that $(V_1 \times \cdots \times V_m)'$ and $V'_1 \times \cdots \times V'_m$ are iso.* [Using notations in (3.E.2).]

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to V'_1 \times \cdots \times V'_m \\ \quad \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = (R'_1(T), \dots, R'_m(T)). \\ \text{Define } \psi : V'_1 \times \cdots \times V'_m \to (V_1 \times \cdots \times V_m)' \\ \quad \text{by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m = S'_1(T_1) + \cdots + S'_m(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ □

• *In* (3.D.18), $\varphi : V \to \mathcal{L}(\mathbf{F}, V)$ *is an iso. Now we prove that*
$[P]$ $(v_1, \dots, v_m)$ *is linely inde* $\iff \big(\varphi(v_1), \dots, \varphi(v_m)\big)$ *is linely inde.* $[Q]$

Solution:

$[P] \Rightarrow [Q]$: Notice that φ is inje and by (3.B.9).
Or. Suppose $\vartheta \in \operatorname{span}\big(\varphi(v_1), \dots, \varphi(v_m)\big)$. Let $\vartheta = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m)$.
Then $\vartheta(1) = 0 = a_1v_1 + \cdots + a_mv_m \Rightarrow a_1 = \cdots = a_m = 0$.

$[Q] \Rightarrow [P]$: Suppose $v \in \operatorname{span}(v_1, \dots, v_m)$. Let $v = 0 = a_1v_1 + \cdots + a_mv_m$.
Then $\varphi(v) = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) \Rightarrow a_1 = \cdots = a_m = 0$. □

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32 *Let* $B_\alpha = (\alpha_1, \dots, \alpha_m), B_\alpha{}' = (\varphi_1, \dots, \varphi_m),\ B_\beta = (v_1, \dots, v_m), B_\beta{}' = (\psi_1, \dots, \psi_m)$.
Prove that $\forall T \in \mathcal{L}(V), T$ *is inv* $\iff$ *the rows of* $A = \mathcal{M}\big(T, B_\alpha, B_\beta\big)$ *form a basis of* $\mathbf{F}^{1,n}$.

Solution: Note that T is inveritlbe $\iff T'$ is inv. And $A^t = \mathcal{M}(T', B_\beta{}', B_\alpha{}')$.

(a) Suppose T is inv, so is T'. Because $\big(T'(\varphi_1), \dots, T'(\varphi_m)\big)$ is linely inde.
Notice that $T'(\varphi_i) = A^t_{1,i}\psi_1 + \cdots + A^t_{m,i}\psi_m$. By the (Δ) part in (4E 3.C.17),
the cols of A^t, namely the rows of A, are linely inde.

(b) Suppose the rows of A are linely inde, so are the cols of A^t. Notice that A^t has $\dim V'$ cols.
Then $B_{\operatorname{range} T'} = B_{V'} = \big(T'(\varphi_1), \dots, T'(\varphi_m)\big)$. Thus T' is surj. Hence T' is inv, so is T. □

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33 *Suppose* $A \in \mathbf{F}^{m,n}$. *Define* $T : A \to A^t$. *Prove that* T *is an iso of* $\mathbf{F}^{m,n}$ *onto* $\mathbf{F}^{n,m}$

Solution: By [3.111], T is linear. Note that $(A^t)^t = A,\ T \circ T = I$. □

• Define $T \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Tx = xA$, where $A \in \mathbf{F}^{n,n}$, for all $x \in \mathbf{F}^{1,n}$.
Let $B_e = (e_1, \dots, e_n)$ be the standard basis of $\mathbf{F}^{1,n}$, with the dual basis $B_\varphi = (\varphi_1, \dots, \varphi_n)$.
What is $\mathcal{M}(T)$? Because $Te_k = e_kA = \sum_{j=1}^n A_{k,j}e_j = \sum_{j=1}^n A^t_{j,k}e_j$. Now $\mathcal{M}(T) = A^t$.

Note that $A = \mathcal{M}(A, B_e) \in \mathbf{F}^{n,n},\ \mathcal{M}(Te_k) = \mathcal{M}(Te_k, B_e) \in \mathbf{F}^{n,1}$,
$$\mathcal{M}(e_k) = \mathcal{M}(e_k, B_e) \in \mathbf{F}^{n,1},\ \mathcal{M}(e_kA) = \mathcal{M}(e_kA, B_e) \in \mathbf{F}^{n,1}.$$
Now $\mathcal{M}(Te_k) = \mathcal{M}(T)_{\cdot,k} = \mathcal{M}(e_kA) = A^t_{\cdot,k} \Longrightarrow \mathcal{M}(T)\mathcal{M}(e_k) = \mathcal{M}(T)_{\cdot,k} = \mathcal{M}(e_k)\mathcal{M}(A)$.
Then $\mathcal{M}(e_k)\mathcal{M}(A)$ does not make sense. And now??? FIXME: BASIS NOT AGREED

• (4E 3.F.8) *Suppose* $B_V = (v_1, \dots, v_n),\ B_{V'} = (\varphi_1, \dots, \varphi_n)$.
$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = \big(\varphi_1(v), \dots, \varphi_n(v)\big). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \dots, a_n) = a_1v_1 + \cdots + a_nv_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

• (4E 3.F.5) *Suppose* $T \in \mathcal{L}(V, W)$. $B_{\operatorname{range} T} = (w_1, \dots, w_m)$.
Hence $\forall v \in V,\ Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,\ \exists!\, \varphi_1(v), \dots, \varphi_m(v)$,
thus defining $\varphi_i : V \to \mathbf{F}$ *for each* $i \in \{1, \dots, m\}$. *Show that each* $\varphi_i \in V'$.

Solution:
$$\begin{aligned} \forall u, v \in V, \lambda \in \mathbf{F}, T(u + \lambda v) &= \textstyle\sum_{i=1}^m \varphi_i(u + \lambda v)w_i \\ &= Tu + \lambda Tv = \Big(\textstyle\sum_{i=1}^m \varphi_i(u)w_i\Big) + \lambda\Big(\textstyle\sum_{i=1}^m \varphi_i(v)w_i\Big) = \textstyle\sum_{i=1}^m \big(\varphi_i(u) + \lambda\varphi_i(v)\big)w_i. \end{aligned}$$ □

Or. For each $w_i, \exists\, v_i \in V,\ Tv_i = w_i$, then $(v_1, \dots, v_m)$ is linely inde.
Now we have $Tv = a_1Tv_1 + \cdots + a_mTv_m,\ \forall v \in V, \exists!\, a_i \in \mathbf{F}$. Let $B_{(\operatorname{range} T)'} = (\psi_1, \dots, \psi_m)$.
Then $\big(T'(\psi_i)\big)(v) = \psi_i \circ T(v) = a_i$. Where $T : V \to \operatorname{range} T;\ \ T' : (\operatorname{range} T)' \to V'$.
Thus for each $i \in \{1, \dots, m\}, \varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$. □

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6 *Define* $\Gamma : V' \to \mathbf{F}^m$ *by* $\Gamma(\varphi) = \big(\varphi(v_1), \dots, \varphi(v_m)\big)$, *where* $v_1, \dots, v_m \in V$.

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = V \iff \Gamma$ *is inje.*

(b) *Show that* $(v_1, \dots, v_m)$ *is linely inde* $\iff \Gamma$ *is surj.*

SOLUTION:

(a) NOTICE that $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \cdots = \varphi(v_m) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.
If Γ is inje, then $\Gamma(\varphi) = 0 \iff V = \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.
If $V = \operatorname{span}(v_1, \dots, v_m)$, then $\Gamma(\varphi) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$, thus Γ is inje.

(b) Suppose Γ is surj. Then let $\Gamma(\varphi_i) = e_i$ for each i, where $(e_1, \dots, e_m)$ is the standard basis of $\mathbf{F}^m$.
Then by (3.A.4), $(\varphi_1, \dots, \varphi_m)$ is linely inde.
Now $a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow 0 = \varphi_i(a_1 v_1 + \cdots + a_m v_m) = a_i$ for each i.
Suppose $(v_1, \dots, v_m)$ is linely inde. Let $U = \operatorname{span}(\varphi_1, \dots, \varphi_m), B_{U'} = (\varphi_1, \dots, \varphi_m)$.
Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists!\, \varphi = a_1\varphi_1 + \cdots + a_m\varphi_m$.
Let W be such that $V = U \oplus W$. Now $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.
Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$. So that $\Gamma(\varphi \circ i -) = (a_1, \dots, a_m)$. □

OR. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$ and let $(\psi_1, \dots, \psi_m)$ be the correspd dual basis.
Define $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$. Then Ψ is an iso.
Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $Te_k = v_k$. Now $T(x_1, \dots, x_m) = T(x_1 e_1 + \cdots + x_m e_m) = x_1 v_1 + \cdots + x_m v_m$.
$\forall \varphi \in V', k \in \{1, \dots, m\}, [T'(\varphi)](e_k) = \varphi(Te_k) = \varphi(v_k) = [\varphi(v_1) \circ \psi_1 + \cdots + \varphi(v_m) \circ \psi_m](e_k)$
Now $T'(\varphi) = \varphi(v_1) \circ \psi_1 + \cdots + \varphi(v_m) \circ \psi_m = \Psi\big(\varphi(v_1), \dots, \varphi(v_m)\big) = \Psi\big(\Gamma(\varphi)\big)$. Hence $T' = \Psi \circ \Gamma$.
By (3.B.3), (a) $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V \iff T' = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.
(b) $(v_1, \dots, v_m)$ is linely inde $\iff T$ is inje $\iff T' = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

• (4E 3.F.25) *Define* $\Gamma : V \to \mathbf{F}^m$ *by* $\Gamma(v) = \big(\varphi_1(v), \dots, \varphi_m(v)\big)$, *where* $\varphi_1, \dots, \varphi_m \in V'$.

(c) *Show that* $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ *is inje.*

(d) *Show that* $(\varphi_1, \dots, \varphi_m)$ *is linely inde* $\iff \Gamma$ *is surj.*

SOLUTION:

(c) NOTICE that $\Gamma(v) = 0 \iff \varphi_1(v) = \cdots = \varphi_m(v) = 0 \iff v \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.
By Problem (4E 23) and (18), $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = \{0\}$.
And $\operatorname{null}\Gamma = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$. Hence Γ inje $\iff \operatorname{null}\Gamma = \{0\} \iff \operatorname{span}(\varphi_1, \dots, \varphi_m) = V'$.

(d) Suppose $(\varphi_1, \dots, \varphi_m)$ is linely inde. Then by Problem (31), $(v_1, \dots, v_m)$ is linely inde.
Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}, \exists!\, v = \sum_{i=1}^{m} a_i v_i \in V \Rightarrow \varphi_i(v) = a_i, \Gamma(v) = (a_1, \dots, a_m)$. Hence Γ is surj.
Suppose Γ is surj. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$.
Suppose $v_i \in V$ such that $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i$, for each i.
Then $(v_1, \dots, v_m)$ is linely inde. And $\varphi_j(v_k) = \delta_{j,k}$.
Now $a_1\varphi_1 + \cdots + a_m\varphi_m = 0 \Rightarrow 0(v_i) = a_i$ for each i. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde.
OR. Let $\operatorname{span}(v_1, \dots, v_m) = U$. Then $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U)$. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde. □

OR. Similar to Problem (6), we get $(e_1, \dots, e_m), (\psi_1, \dots, \psi_m)$ and the iso Ψ.
$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'\big(\Psi(x_1, \dots, x_m)\big) = \Gamma'\big(\Psi(x_1 e_1 + \cdots + x_m e_m)\big) = (x_1\psi_1 + \cdots + x_m\psi_m) \circ \Gamma$.
$\forall v \in V, \big[\Gamma'\big(\Psi(x_1, \dots, x_m)\big)\big](v) = [x_1\psi_1 + \cdots + x_m\psi_m]\big(\Gamma(v)\big) = [x_1\varphi_1 + \cdots + x_m\varphi_m](v)$.
Now $\Gamma'\big(\Psi(x_1, \dots, x_m)\big) = x_1\varphi_1 + \cdots + x_m\varphi_m$.
Define $\Phi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Phi = \Psi \circ \Gamma$. $\Phi(x_1, \dots, x_m) = x_1\varphi_1 + \cdots + x_m\varphi_m$. Thus by (4E 3.B.3),
(c) the inje of Φ correspds to $(\varphi_1, \dots, \varphi_m)$ spanning V'; 又 $\Phi = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.
(d) the surj of Φ correspds to $(\varphi_1, \dots, \varphi_m)$ being linely inde; 又 $\Phi = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

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35 *Prove that $\big(\mathcal{P}(\mathbf{F})\big)'$ and $\mathbf{F}^\infty$ are iso.*

SOLUTION:

Define $\theta \in \mathcal{L}\big(\big(\mathcal{P}(\mathbf{F})\big)', \mathbf{F}^\infty\big)$ by $\theta(\varphi) = \big(\varphi(1), \varphi(z), \cdots, \varphi(z^n), \cdots\big)$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall z^k$ in the basis $\big(1, z, \ldots, z^n\big)$ of $\mathcal{P}_n(\mathbf{F})$ $\big(\,\forall n\,\big)$, $\varphi(z^k) = 0 \Rightarrow \varphi = 0$.

$\big[$ NOTICE that $\forall p \in \mathcal{P}(\mathbf{R}), \exists\,!\, a_i \in \mathbf{F}, m = \deg p,\ p = a_0 z + a_1 z + \cdots + a_m z^m \in \mathcal{P}_m(\mathbf{F})$. $\big]$

Surj: $\forall \big(a_k\big)_{k=1}^\infty \in \mathbf{F}^\infty$, let ψ be such that $\forall k, \psi\big(z^k\big) = a_k$ $\big[$ by [3.5] $\big]$ and thus $\theta(\psi) = \big(a_k\big)_{k=1}^\infty$. $\square$

COMMENT: NOTICE that $\mathcal{P}(\mathbf{F})$ and $\mathbf{F}^\infty$ are not iso, so are $\mathcal{P}(\mathbf{F})$ and $\big(\mathcal{P}(\mathbf{F})\big)'$

But if we let $\mathbf{F}^\infty = \big\{(a_1, \cdots, a_n, \underbrace{0, \cdots, 0, \cdots}_{\text{all zero}}) \in \mathbf{F}^\infty \;\big|\; \exists\,!\, n \in \mathbf{N}^+\big\}$. Then $\mathcal{P}(\mathbf{F})$ and $\mathbf{F}^\infty$ are iso.

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7 *Show that the dual basis of $(1, x, \ldots, x^m)$ of $\mathcal{P}_m(\mathbf{R})$ is $(\varphi_0, \varphi_1, \ldots, \varphi_m)$, where $\varphi_k(p) = \dfrac{p^{(k)}(0)}{k!}$.*

Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

SOLUTION:

$$\forall j, k \in \mathbf{N},\ \big(x^j\big)^{(k)} = \begin{cases} j(j-1)\ldots(j-k+1)\cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\ldots(j-j+1) = j! & j = k. \\ 0, & j \leqslant k. \end{cases} \Bigg| \quad \text{Then } \big(x^j\big)^{(k)}(0) = \begin{cases} 0\ , & j \neq k. \\ k!\ , & j = k. \end{cases} \quad \square$$

OR. Because $\forall j, k \in \{1, \ldots, m\}$ such that $j \neq k$, $\varphi_k\big(x^j\big) = \frac{(x^j)^{(k)}(0)}{k!} = \frac{0}{k!} = 0$; $\varphi_k\big(x^k\big) = \frac{(x^k)^{(k)}(0)}{k!} = 1$.

Thus $\frac{p^{(k)}(0)}{k!}$ act exactly the same as φ_k on the same basis $\big(1, \ldots, x^m\big)$, hence is just another def of φ_k. $\square$

EXAMPLE: Suppose $m \in \mathbf{N}^+$. By [2.C.10], $B = \big(1, x-5, \ldots, (x-5)^m\big)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

Let $\varphi_k = \dfrac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \ldots, m$. Then $(\varphi_0, \varphi_1, \ldots, \varphi_m)$ is the dual basis of B.

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34 *The double dual space of V, denoted by V'', is defined to be the dual space of V'.*
In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \to V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.

(a) *Show that Λ is a linear map from V to V''.*

(b) *Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*

(c) *Show that if V is finite-dim, then Λ* ***is an iso from V onto V''****.*

Suppose V is finite-dim. Then V and V' are iso, and finding an iso from V onto V' generally requires choosing a basis of V. In contrast, the iso Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

SOLUTION:

(a) $\forall \varphi \in V', v, w \in V, a \in \mathbf{F}$, $\big(\Lambda(v + aw)\big)(\varphi) = \varphi(v + aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.

Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence Λ is linear.

(b)
$$\begin{aligned}\big(T''(\Lambda v)\big)(\varphi) &= \big((\Lambda v) \circ T'\big)(\varphi) = (\Lambda v)\big(T'(\varphi)\big) \\ &= \big(T'(\varphi)\big)(v) = (\varphi \circ T)(v) = \varphi(Tv) = \big(\Lambda(Tv)\big)(\varphi).\end{aligned}$$

Hence $T''(\Lambda v) = \big(\Lambda(Tv)\big) \Rightarrow T'' \circ \Lambda = \Lambda \circ T$.

(c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V'$, $(\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is inje.

又 Because V is finite-dim. $\dim V = \dim V' = \dim V''$. Hence Λ is an iso. $\square$

ENDED

- **Tips:** *Suppose $p \in \mathcal{P}(\mathbf{F})$, $\deg p \leqslant m$ and p has at least $(m+1)$ distinct zeros.*
  Then by the contrapositive of [4.12], 又 $\deg p = m$, we conclude that $m < 0$. Hence $p = 0$.

  Or. We show that if p has at least m distinct zeros, then either $p = 0$ or $\deg p \geqslant m$.
  If $p = 0$ then we are done. If not, then suppose p has exactly n distinct zeros $\lambda_1, \dots, \lambda_n$.
  Because $\exists!\, \alpha_i \geqslant 1, q \in \mathcal{P}(\mathbf{F})$, and $q \neq 0$, such that $p(z) = \left[(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_n)^{\alpha_n}\right] q(z)$. □
- **Comment:** Notice *that by [4.17], some term of the poly factorization might not be in the form $(x-\lambda_k)^{\alpha_k}$.*

- **Note For [4.7]:** *the uniqness of coeffs of polys* [***Another proof***]

  If a poly had two different sets of coeffs, then subtracting the two representations would give a poly with some nonzero coeffs but infinitely many zeros. By Tips . □

- **Note For [4.8]:** *division algorithm for polys* [***Another proof***]

  Suppose $\deg p \geqslant \deg s$. Then $\Big(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of length } \deg s}, \underbrace{s, zs, \cdots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)} \Big)$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.

  Because $q \in \mathcal{P}(\mathbf{F}), \exists!\, a_i, b_j \in \mathbf{F}$,
  $q = a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s$
  $= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s-1} z^{\deg s - 1}}_{r} + s \underbrace{\left(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} \right)}_{q}$. Note that r, q are unique. □

- **Note For [4.11]:** *each zero of a poly corresponds to a degree-one factor;* [***Another proof***]

  First suppose $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, $\exists!\, a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.
  Then $p(z) = p(z) - p(\lambda) = a_1(z-\lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.
  Hence $\forall k \in \{1, \dots, m\}, z^k - \lambda^k = (z-\lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.
  Thus $p(z) = \sum_{j=1}^{m} a_j (z-\lambda) \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z-\lambda) \sum_{j=1}^{m} a_j \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z-\lambda) q(z)$. □

- **Note For [4.13]:** *Every nonconst poly with complex coefficients has a zero in* $\mathbf{C}$. [***Another proof***]

  For any $w \in \mathbf{C}, k \in \mathbf{N}^+$, by polar coordinates, $\exists\, r \geqslant 0, \theta \in \mathbf{R}, r(\cos\theta + \mathrm{i}\sin\theta) = w$.
  By De Moivre' theorem, $w^k = [r(\cos\theta + \mathrm{i}\sin\theta)]^k = r^k(\cos k\theta + \mathrm{i}\sin k\theta)$.
  Hence $\left(r^{1/k}(\cos\frac{\theta}{k} + \mathrm{i}\sin\frac{\theta}{k})\right)^k = w$. Thus every complex number has a k^{th} *root*.

  Suppose a nonconst $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.
  Then $|p(z)| \to \infty$ as $|z| \to \infty$ $\left(\text{because } \dfrac{|p(z)|}{|z_m|} \to |c_m| \text{ as } |z| \to \infty \right)$.
  Thus the continuous function $z \to |p(z)|$ has a global minimum at some point $\zeta \in \mathbf{C}$.
  To show that $p(\zeta) = 0$, assume $p(\zeta) \neq 0$. Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \dfrac{p(z+\zeta)}{p(\zeta)}$.
  The function $z \to |q(z)|$ has a global minimum value of 1 at $z = 0$.
  Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where $k \in \mathbf{N}^+$ is the smallest such that $a_k \neq 0$.
  Let $\beta \in \mathbf{C}$ be such that $\beta^k = -\dfrac{1}{a_k}$.
  There is a const $c > 1$ so that if $t \in (0,1)$, then $|q(t\beta)| \leqslant |1 + a_k t^k \beta^k| + t^{k+1} c = 1 - t^k(1 - tc)$.
  Now letting $t = 1/(2c)$, we get $|q(t\beta)| < 1$. Contradicts. Hence $p(\zeta) = 0$, as desired. □

• (4E 4 2) *Prove that if $w, z \in \mathbf{C}$, then $||w|-|z|| \leqslant |w-z|$.*

SOLUTION:

$$\begin{aligned} |w-z|^2 &= (w-z)(\overline{w}-\overline{z}) \\ &= |w|^2+|z|^2-(w\overline{z}+\overline{w}z) \\ &= |w|^2+|z|^2-(\overline{\overline{w}z}+\overline{w}z) \\ &= |w|^2+|z|^2-2Re(\overline{w}z) \\ &\geqslant |w|^2+|z|^2-2|\overline{w}z| \\ &= |w|^2+|z|^2-2|w||z| = ||w|-|z||^2. \end{aligned}$$

OR. $\left.\begin{array}{l} |w| = |w-z+z| \leqslant |w-z|+|z| \Rightarrow |w|-|z| \leqslant |w-z| \\ |z| = |z-w+w| \leqslant |z-w|+|w| \Rightarrow |z|-|w| \leqslant |w-z| \end{array}\right\}$

Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides.

□

• (4E 4 3) *Suppose $\mathbf{F} = \mathbf{C}, \varphi \in V'$. Define $\sigma : V \to \mathbf{R}$ by $\sigma(v) = \operatorname{Re}\varphi(v)$ for each $v \in V$. Show that $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$ for all $v \in V$.*

SOLUTION: Notice that $\varphi(v) = \operatorname{Re}\varphi(v) + \mathrm{i}\operatorname{Im}\varphi(v) = \sigma(v) + \mathrm{i}\operatorname{Im}\varphi(v)$.
又 $\operatorname{Re}\varphi(\mathrm{i}v) = \operatorname{Re}(\mathrm{i}\varphi(v)) = -\operatorname{Im}\varphi(v) = \sigma(\mathrm{i}v)$. Hence $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$. □

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4 *Suppose $m, n \in \mathbf{N}^+$ with $m \leqslant n$, $\lambda_1, \dots, \lambda_m \in \mathbf{F}$.
Prove that $\exists\, p \in \mathcal{P}(\mathbf{F}), \deg p = n$, the zeros of p are $\lambda_1, \dots, \lambda_m$.*

SOLUTION: Let $p(z) = (z-\lambda_1)^{n-(m-1)}(z-\lambda_2)\cdots(z-\lambda_m)$. □

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5 *Suppose $m \in \mathbf{N}$, and $z_1, \dots, z_{m+1}$ are distinct in $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$.
Prove that $\exists!\, p \in \mathcal{P}_m(\mathbf{F}), p(z_k) = w_k$ for each $k \in \{1, \dots, m+1\}$.*

SOLUTION:

Define $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by $Tq = \big(q(z_1), \dots, q(z_m), q(z_{m+1})\big)$. Moreover, T is linear.

We now show that T is surj, so that such p exists; and that T is inje, so that such p is unique.
Inje: $Tq = 0 \Longleftrightarrow q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0 \Longleftrightarrow q = 0$, by TIPS .
Surj: $\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m+1 = \dim \mathbf{F}^{m+1}$ 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1} \Rightarrow T$ is surj. □

OR. Let $p_1 = 1,\ p_k(z) = \prod_{i=1}^{k-1}(z-z_i) = (z-z_1)\cdots(z-z_{k-1})$ for each $k \in \{2, \dots, m+1\}$.
By (2.C.10), $B_p = (p_1, \dots, p_{m+1})$ is a basis of $\mathcal{P}_m(\mathbf{F})$. Let $B_e = (e_1, \dots, e_{m+1})$ be the std basis of $\mathbf{F}^{m+1}$.
NOTICE that $Tp_1 = (1, \dots, 1),\ Tp_k = \Big(\prod_{i=1}^{k-1}(z_1-z_i), \dots, \underbrace{\prod_{i=1}^{k-1}(z_j-z_i)}_{j^{th}\text{ entry}}, \dots, \prod_{i=1}^{k-1}(z_{m+1}-z_i)\Big)$.

And that $\prod_{i=1}^{k-1}(z_j-z_i) = 0 \Longleftrightarrow j \leqslant k-1$, because $z_1, \dots, z_{m+1}$ are distinct.

$$\text{Thus } \mathcal{M}(T, B_p, B_e) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & A_{2,2} & 0 & \cdots & 0 \\ 1 & A_{3,2} & A_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{m+1,2} & A_{m+1,3} & \cdots & A_{m+1,m+1} \end{pmatrix}.$$

Where $A_{j,k} = \prod_{i=1}^{k-1}(z_j-z_i) \neq 0$ for all $j > k-1 \geqslant 1$. The rows of $\mathcal{M}(T)$ is linely inde.
By (4E 3.C.17) 又 $\dim \mathcal{P}_m(\mathbf{F}) = \dim \mathbf{F}^{m+1}$; OR By (3.F.32); T is inv. □

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2 *Suppose $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

SOLUTION: $x^m, x^m + x^{m-1} \in U$ but $\deg[(x^m+x^{m-1})-(x^m)] \neq m \Rightarrow (x^m+x^{m-1})-(x^m) \notin U$. □

---

3 *Suppose $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : 2 \mid \deg p\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

SOLUTION: $x^2, x^2 + x \in U$ but $\deg[(x^2 + x) - (x^2)]$ is odd and hence $(x^2 + x) - (x^2) \notin U$. □

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6 *Suppose nonzero $p \in \mathcal{P}_m(\mathbf{F})$ has degree m. Prove that*
$[P]$ p has m distinct zeros $\Longleftrightarrow$ p and its derivative p' have no zeros in common $[Q]$.

SOLUTION:

(a) Suppose p has m distinct zeros. And $\deg p = m$. By [4.14], $\exists!\, c, \lambda_i \in \mathbf{R}, p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$.

If $m = 0$, then $p = c \neq 0 \Rightarrow p$ has no zeros, and $p' = 0$, we are done.

If $m = 1$, then $p(z) = c(z - \lambda_1)$, and $p' = c$ has no zeros, we are done.

For each $j \in \{1, \dots, m\}$, let $q_j \in \mathcal{P}_{m-1}(\mathbf{F})$ be such that $p(z) = (z - \lambda_j)q_j \Rightarrow q_j(\lambda_j) \neq 0$.

Now $p'(z) = (z - \lambda_j){q_j}'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

OR. To prove $[P] \Rightarrow [Q]$, we prove $\neg[Q] \Rightarrow \neg[P]$:

Suppose $p(z) = (z - \lambda)q(z),\ p'(z) = (z - \lambda)r(z)$. 又 $p'(z) = (z - \lambda)q'(z) + q(z)$.

Now $p'(\lambda) = q(\lambda) = 0 \Rightarrow q(z) = (z - \lambda)s(z), p(z) = (z - \lambda)^2 s(z)$.

Hence p has strictly less than m distinct zeros.

(b) To prove $[Q] \Rightarrow [P]$, we prove $\neg[P] \Rightarrow \neg[Q]$:

Because nonzero $p \in \mathcal{P}_m(\mathbf{F})$, we suppose $\lambda_1, \dots, \lambda_M$ are all the distinct zeros of p, where $M < m$.

By Pigeon Hole Principle, $\exists \lambda_k$ such that $p(z) = (z - \lambda_k)^2 q(z)$ for some $q \in \mathcal{P}(\mathbf{F})$.

Hence $p'(z) = 2(z - \lambda_k)q(z) + (z - \lambda_k)^2 q'(z) \Rightarrow p'(\lambda_k) = 0 = p(\lambda_k)$. □

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7 *Prove that every $p \in \mathcal{P}(\mathbf{R})$ of odd degree has a zero.*

SOLUTION:

Using the notation and proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence λ_1 exists. □

OR. Using calculus only. Suppose $p \in \mathcal{P}_m(\mathbf{F}), \deg p = m, m$ is odd.

Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1} a_m$ by δ.

Write $p(x) = x^m \left(\dfrac{a_0}{x^m} + \dfrac{a_1}{x^{m-1}} + \cdots + \dfrac{a_{m-1}}{x} + a_m \right)$.

Thus $p(x)$ is continuous, and $\lim\limits_{x \to -\infty} p(x) = -\delta\infty$; $\lim\limits_{x \to \infty} p(x) = \delta\infty$.

Hence we conclude that p has at least one real zero. □

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9 *Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\overline{z})}$. Prove that $q \in \mathcal{P}(\mathbf{R})$.*

SOLUTION:

NOTICE that by [4.5], $\overline{z}^n = \overline{z^n}$.

Suppose $q(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow q(\overline{z}) = a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0 \Rightarrow \overline{q(\overline{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\overline{z})} = \overline{p(\overline{z})}p(z) = \overline{p(\overline{z})\overline{\overline{p(\overline{\overline{z}})}}} = \overline{q(\overline{z})}$. Hence for each $a_k, \overline{a_k} = a_k \Rightarrow a_k \in \mathbf{R}$. □

OR. Suppose $p(z) = a_m z^m + \cdots + a_1 z + a_0$. Now $\overline{p(\overline{z})} = \overline{a_m} z^m + \cdots + \overline{a_1} z + \overline{a_0}$.

NOTICE that $q(z) = p(z)\overline{p(\overline{z})} = \sum_{k=0}^{2} m \left(\sum_{i+j=k} a_i \overline{a_j} \right) z^k$.

NOTICE that by [4.5], $z - \overline{z} = 2(\operatorname{Im} z) \Rightarrow z = \overline{z} + 2(\operatorname{Im} z)$. So that $z = \overline{z} \Longleftrightarrow \operatorname{Im} z = 0 \Longleftrightarrow z \in \mathbf{R}$.

Now for each $k \in \{0, \dots, 2m\}$, $\overline{\sum_{i+j=k} a_i \overline{a_j}} = \sum_{i+j=k} \overline{a_i \overline{a_j}} = \sum_{i+j=k} a_j \overline{a_i} = \sum_{i+j=k} a_i \overline{a_j} \in \mathbf{R}$. □

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8 *For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \to \mathbf{R}$ by* $(Tp)(x) = \begin{cases} \dfrac{p(x) - p(3)}{x-3} & \textit{if } x \neq 3, \\ p'(3) & \textit{if } x = 3. \end{cases}$

Show that (a) $Tp \in \mathcal{P}(\mathbf{R})$ *for all* $p \in \mathcal{P}(\mathbf{R})$ *and that* (b) $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ *is linear.*

SOLUTION:

(a) For $x \neq 3, T(x^n) = \dfrac{x^n - 3^n}{x-3} = \sum\limits_{i=1}^{n} 3^{i-1}x^{n-i}$. For $x = 3, T(x^n) = 3^{n-1} \cdot n$.

Note that if $x = 3$, then $\sum_{i=1}^{n} 3^{i-1}x^{n-i} = \sum_{i=1}^{n} 3^{n-1} = 3^{n-1} \cdot n$.

Hence $T(x^n) = \sum_{i=1}^{n} 3^{i-1}x^{n-i} \Rightarrow T(x^n) \in \mathcal{P}(\mathbf{R})$.

(b) Now we show that T is linear: $\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}$,

$$T(p+\lambda q)(x) = \left\{ \begin{array}{ll} \dfrac{(p+\lambda q)(x) - (p+\lambda q)(3)}{x-3}, & \text{if } x \neq 3, \\ (p+\lambda q)'(3), & \text{if } x = 3 \end{array} \right| = \big[T(p) + \lambda T(q)\big](x) \text{ for all } x \in \mathbf{R}.$$ $\square$

OR. (a) Note that $\exists!\, q \in \mathcal{P}(\mathbf{R}), p(x) - p(3) = (x-3)q(z) \Rightarrow q(x) = \dfrac{p(x) - p(3)}{x-3}$.

$p'(x) = \big(p(x) - p(3)\big)' = \big((x-3)q(x)\big)' = q(x) + (x-3)q'(x)$.

Hence $p'(3) = q(3)$. Now $Tp = q \in \mathcal{P}(\mathbf{R})$.

(b) $\forall p_1, p_2 \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, \exists!\, q_1, q_2 \in \mathcal{P}(\mathbf{R})$,

$p_1(x) - p_1(3) = (x-3)q_1(x)$ and $p_2(x) - p_2(3) = (x-3)q_2(x)$.

By (a), $Tp_1 = q_1, Tp_2 = q_2$. Note that $(p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x-3)(q_1 + \lambda q_2)(x)$.

Hence by the uniqnes of $q_1 + \lambda q_2$ for $p_1 + \lambda p_2$, we must have $T(p_1 + \lambda p_2) = q_1 + \lambda q_2$. $\square$

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11 *Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*

(a) *Show that* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) *Find a basis of* $\mathcal{P}(\mathbf{F})/U$.

SOLUTION: NOTICE that $pq \neq p \circ q$, see (4E 3.A.10).

U is a subsp of $\mathcal{P}(\mathbf{F})$ because $\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, ps_1 + \lambda ps_2 = p(s_1 + \lambda s_2) \in U$.

If $\deg p = 0$, then $U = \mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F})/U = \{0\}$, with the unique basis $()$. Suppose $\deg p \geqslant 1$.

(a) By [4.8], $\forall s \in \mathcal{P}(\mathbf{F}), \exists!\, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F})\ [\,\exists!\, pq \in U\,], s = (p)q + (r)$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F})$. By the NOTE FOR [3.91] in (3.E), $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p-1}(\mathbf{F})$ are iso.

OR. Define $R : \mathcal{P}(\mathbf{F}) \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $R(s) = r$ for all $s \in \mathcal{P}(\mathbf{F})$ We show that R is linear.

$\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, \exists!\, r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q_1, q_2 \in \mathcal{P}(\mathbf{F}), s_1 = (p)q_1 + (r_1);\ s_2 = (p)q_2 + (r_2)$.

又 $\exists!\, r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}), (s_1 + \lambda s_2) = (p)q + (r) = (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2)$.

Note that $r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}) \Rightarrow r_1 + \lambda r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F})$.

OR Note that $\deg(r_1 + \lambda r_2) \leqslant \max\{\deg r_1, \deg(\lambda r_2)\} \leqslant \max\{\deg r_1, \deg r_2\} < \deg p$.

By the uniqnes part of [4.8], $s = s_1 + \lambda s_2;\ r = r_1 + \lambda r_2$. Thus $R(s_1 + \lambda s_2) = R(s_1) + \lambda R(s_2)$.

Because $Rs = 0 \Longleftrightarrow s = pq, \exists!\, q \in \mathcal{P}(\mathbf{F}) \Longleftrightarrow s \in U$. And $\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), Rr = r$.

Now $\operatorname{null} R = U,\ \operatorname{range} R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Hence $\tilde{R} : \mathcal{P}(\mathbf{F})/U \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ is defined by $\tilde{R}(s+U) = Rs$. By [3.91(d)], $\tilde{R}$ is an iso.

(b) For each $k \in \{0, 1, \dots, \deg p - 1\}, \tilde{R}(z^k + U) = R(z^k) = z^k \Rightarrow \tilde{R}^{-1}(z^k) = z^k + U$.

Thus $(1 + U, z + U, \dots, z^{\deg p-1} + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. $\square$

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10 *Suppose $m \in \mathbf{N}, p \in \mathcal{P}_m(\mathbf{C})$ is such that $p(x_k) \in \mathbf{R}$ for each of distinct $x_0, x_1, \dots, x_m \in \mathbf{R}$. Prove that $p \in \mathcal{P}(\mathbf{R})$.*

SOLUTION:

By TIPS and Problem (5), $\exists !\, q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_k) = p(x_k)$. Hence $p = q$. □

OR. Using the Lagrange Interpolating Polynomial.

Define $q(x) = \displaystyle\sum_{j=0}^{m} \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j)$.

又 Each x_j, $p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R})$. Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$ for each x_k.
Then $(q-p)$ has $(m+1)$ zeros, while $(q-p) \in \mathcal{P}_m(\mathbf{C})$. By TIPS , $q - p = 0 \Rightarrow p = q \in \mathcal{P}(\mathbf{R})$. □

• (4E 4 13) *Suppose nonconst $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p$, $n = \deg q$. Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r,s) = rp + sq$. Prove that T is an iso.*
COROLLARY: *$\exists !\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*

SOLUTION:

T is linear because $\forall\, r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,
$T\big((r_1,s_1) + \lambda(r_2,s_2)\big) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1,s_1) + \lambda T(r_2,s_2)$.

Let $\lambda_1, \dots, \lambda_M$ and $\mu_1, \dots, \mu_N$ be the distinct zeros of p and q respectively. NOTICE that $M \leqslant m, N \leqslant n$.
Note that the contrapositive of [4.13], $M = 0 \iff m = 0 \Rightarrow s = 0 \iff r = 0 \Leftarrow n = 0 \iff N = 0$.
Now suppose $M, N \geqslant 1$. We show that $s = 0$. Showing $r = 0$ is almost the same.
Write $p(z) = a(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}$. $\big(\, \exists !\, \alpha_j \geqslant 1, a \in \mathbf{F}.\, \big)$ Let $\max\{\alpha_1, \dots, \alpha_M\} = A$.
For each $D \in \{0, 1, \dots, A-1\}$, let $I_{D,\alpha} = \{\gamma_{D,1}, \dots, \gamma_{D,J}\}$ be such that each $\alpha_{\gamma_{D,j}} \geqslant D+1$.
Note that $I_{A-1,\alpha} \subseteq \cdots \subseteq I_{0,\alpha} = \{1, \dots, M\}$. Because $rp + sq = 0 \Rightarrow (rp+sq)^{(k)} = 0$ for all $k \in \mathbf{N}^+$.
We use induction by D to show that $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$ for each $D \in \{0, \dots, A-1\}$.
NOTICE that $p^{(D)}(\lambda_\gamma) = 0$ for each $D \in \{0, \dots, A-1\}$ and each $\lambda_\gamma \in I_{D,\alpha}$. (Δ)

(i) $D = 0$. $(rp+sq)(\lambda_{\gamma_{0,j}}) = (sq)(\lambda_{\gamma_{0,j}}) = s(\lambda_{\gamma_{0,j}}) = 0$.
$D = 1$. $(rp+sq)'(\lambda_{\gamma_{1,j}}) = (r'p + rp')(\lambda_{\gamma_{1,j}}) + (s'q + sq')(\lambda_{\gamma_{1,j}}) = (s'q)(\lambda_{\gamma_{1,j}}) = s'(\lambda_{\gamma_{1,j}}) = 0$.

(ii) $2 \leqslant D \leqslant A-1$. Assume that $s^{(d)}(\lambda_{\gamma_{d,j}}) = 0$ for each $d \in \{1, \dots, D-1\}$ and each $\lambda_{\gamma_{d,j}} \in I_{d,\alpha}$.
$\Big($ Because $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}.\, \Big)$ (Δ)

$$\begin{aligned} \text{Now } [rp+sq]^{(D)}(\lambda_{\gamma_{D,j}}) &= \big[C_D^D r^{(D)} p^{(0)} + \cdots + C_D^d r^{(d)} p^{(D-d)} + \cdots + C_D^0 r^{(0)} p^{(D)}\big](\lambda_{\gamma_{D,j}}) \\ &\quad + \big[C_D^D s^{(D)} q^{(0)} + \cdots + C_D^d s^{(d)} q^{(D-d)} + \cdots + C_D^0 s^{(0)} q^{(D)}\big](\lambda_{\gamma_{D,j}}) \\ &= \big[C_D^D s^{(D)} q^{(0)}\big](\lambda_{\gamma_{D,j}}). \text{ Where each } \lambda_{\gamma_{D,j}} \in I_{D,\alpha} \subseteq I_{D-1,\alpha}. \end{aligned}$$

Hence $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$. The assumption holds for all $D \in \{0, \dots, A-1\}$.

NOTICE that $\forall k = \{0, \dots, A-2\}, s^{(k)}$ and $s^{(k+1)}$ have zeros $\{\lambda_{\gamma_{k+1,1}}, \dots, \lambda_{\gamma_{k+1,J}}\}$ in common.
Now $\forall D \in \{1, A-1\}, s = s^{(0)}, \dots, s^{(D)}$ have zeros $\{\lambda_{\gamma_{D,1}}, \dots, \lambda_{\gamma_{D,J}}\}$ in common.
Thus $\forall D \in \{0, A-1\}$, $s(z)$ is divisible by $(z - \lambda_{\gamma_{D,1}})^{\alpha_{\gamma_{D,1}}} \cdots (z - \lambda_{\gamma_{D,J}})^{\alpha_{\gamma_{D,J}}}$.
Hence we write $s(z) = \Big((z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}\Big) s_0(z)$, while $\deg s \leqslant m - 1 < m = \alpha_1 + \cdots + \alpha_M$.
Thus by TIPS , $s = 0$. Following the same pattern, we conclude that $r = 0$.
Hence T is inje. And $\dim\big(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})\big) = \dim \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$ is surj. Thus T is an iso. □

Comment: We now prove the statement that marked by (Δ) above.

L1: *Prove that* $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}$.

Solution:

We use induction by $k \in \mathbf{N}^+$.

(i) $k = 1$. $(pq)^{(1)} = pq = C_1^1 p^{(1)} q^{(0)} + C_1^0 p^{(0)} q^{(1)}$.

(ii) $k \geqslant 2$. Assume that for $(pq)^{(k-1)} = C_{k-1}^{k-1} p^{(k-1)} q^{(0)} + \cdots + C_{k-1}^{j} p^{(j)} q^{(k-1-j)} + \cdots + C_{k-1}^{0} p^{(0)} q^{(k-1)}$.

Now $(pq)^{(k)} = \left((pq)^{(k-1)}\right)' = \left(\sum_{j=0}^{k-1} C_{k-1}^{j} p^{(j)} q^{(k-j-1)} \right)' = \sum_{j=0}^{k-1} \left[C_{k-1}^{j} \left(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)} \right) \right]$.

$$\begin{aligned}
&= \left[C_{k-1}^{0} \left(p^{(1)} q^{(k-1)} + \boxed{p^{(0)} q^{(k)}} \right) \right] + \left[C_{k-1}^{1} \left(p^{(2)} q^{(k-2)} + p^{(1)} q^{(k-1)} \right) \right] \\
&\quad + \cdots + \left[C_{k-1}^{j-2} \left(p^{(j-1)} q^{(k-j+1)} + p^{(j-2)} q^{(k-j+2)} \right) \right] + \left[C_{k-1}^{j-1} \left(p^{(j)} q^{(k-j)} + p^{(j-1)} q^{(k-j+1)} \right) \right] \\
&\quad + \left[C_{k-1}^{j} \left(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)} \right) \right] + \left[C_{k-1}^{j+1} \left(p^{(j+2)} q^{(k-j-2)} + p^{(j+1)} q^{(k-j-1)} \right) \right] \\
&\quad + \cdots + \left[C_{k-1}^{k-2} \left(p^{(k-1)} q^{(1)} + p^{(k-2)} q^{(2)} \right) \right] + \left[C_{k-1}^{k-1} \left(\boxed{p^{(k)} q^{(0)}} + p^{(k-1)} q^{(1)} \right) \right].
\end{aligned}$$

Hence $(pq)^{(k)} = C_k^0 p^{(0)} q^{(k)} + \cdots + \left[C_{k-1}^{j} + C_{k-1}^{j-1} \right] \left(p^{(j)} q^{(k-j)} \right) + \cdots + C_k^k p^{(k)} q^{(0)}$. □

L2: *Suppose* $p(z) = (z-\lambda)^{\alpha} q(z)$ *and* $\alpha \in \mathbf{N}^+$. *Prove that* $p^{(\alpha-1)}(\lambda) = 0$.

Solution:

Suppose $p \in \mathcal{P}(\mathbf{F})$. Write $p(z) = (z-\lambda)^{A} q(z)$, where $A \in \mathbf{N}^+, q(\lambda) \neq 0$.

We use induction to show that for all $\alpha \in \{1, \ldots, A\}, p^{(\alpha-1)}(\lambda) = 0$.

(i) $\alpha = 1$. $p^{(0)}(\lambda) = 0$.

(ii) $2 \leqslant \alpha \leqslant A$. Assume that $p^{(a-2)}(\lambda) = 0$ for all $a \in \{1, \ldots, \alpha\}$.

Notice that $p(z) = (z-\lambda)^{\alpha-1} q_{\alpha-1}(z) = (z-\lambda)^{\alpha} q_{\alpha}(z)$, where $q_{\alpha}(z) = (z-\lambda) q_{\alpha-1}(z)$.

Because $p^{(\alpha-1)}(z) = \Big[C_{\alpha-1}^{\alpha-1} (z-\lambda)^0 q_{\alpha-1}(z) + \cdots + C_{\alpha-1}^{k} (z-\lambda)^{\alpha-1-k} q_{\alpha-1-k}(z)$
$+ \cdots + C_{\alpha-1}^{0} (z-\lambda)^{\alpha-1} {q_{\alpha-1}}^{(\alpha-1)}(z) \Big]$. Now $p^{(\alpha-1)}(\lambda) = C_{\alpha-1}^{\alpha-1} q_{\alpha-1}(\lambda) = 0$. □

Ended

5.A

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28
29 30 31 32 33 34 35 36 | 2E: Ch5.20 | 4E: 8, 11, 15, 16, 17, 36, 37, 38, 39

- **Note For [5.6]:**

More generally, suppose we do not know whether V is finite-dim. We show that $(a) \iff (b)$.
Suppose (a) λ is an eigval of T with an eigvec v. Then $(T - \lambda I)v = 0$.
Hence we get (b), $(T - \lambda I)$ is not inje. And then (d), $(T - \lambda I)$ is not inv.
But (d) $\Rightarrow$ (b) fails, because S is not inv $\iff$ S is not inje Or S is not surj.

- **Tips:** For $T_1, \ldots, T_m \in \mathcal{L}(V)$:

(a) Suppose $T_1, \ldots, T_m$ are all inje. Then $(T_1 \circ \cdots \circ T_m)$ is inje.
(b) Suppose $(T_1 \circ \cdots \circ T_m)$ is not inje. Then at least one of $T_1, \ldots, T_m$ is not inje.
(c) At least one of $T_1, \ldots, T_m$ is not inje $\not\Rightarrow$ $(T_1 \circ \cdots \circ T_m)$ is not inje.

Example: In infinite-dim only. Let $V = \mathbf{F}^{\infty}$.

$\left.\begin{array}{l} \text{Let } S \text{ be the backward shift (surj but not inje)} \\ \text{Let } T \text{ be the forward shift (inje but not surj)} \end{array}\right\} \Rightarrow$ Then $ST = I$. □

• **NOTE FOR [5.2]:** Suppose $T \in \mathcal{L}(V)$. Then U is an invar subsp of V under $T \iff \operatorname{range} T|_U \subseteq U$.

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invar subsp of V under T. Prove that there exists an invar subsp W of dimension $\dim V - dimU$.*

SOLUTION:

Using the NOTE FOR [3.88,90,91]. Define the eraser S. Now $V = \operatorname{range} S \oplus U$.

Define E_1 by $E_1(u+w) = u$. Define E_2 by $E_2(u+w) = w$. ($E_2 = S \circ \pi$.)

Note that $T - TE_1 = T(I - E_1) = TE_2$. And $\operatorname{null} TE_2 = \operatorname{null} T \oplus U$, $\operatorname{range} T = \operatorname{range} TE_2 \oplus U$.

Because $\dim \operatorname{null} TE_2 \geqslant \dim U \iff \dim \operatorname{range} TE_2 \leqslant \dim V - \dim U$.

Let $B_U = (u_1, \dots, u_n)$, $B_{\operatorname{range} TE_2} = (v_1, \dots, v_m) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n, \dots, u_p)$.

Let $X = \operatorname{span}(v_1, \dots, v_m, u_{\alpha_1}, \dots, u_{\alpha_{p-\dim U}})$. Where $\alpha_1, \dots, \alpha_{p-\dim U} \in \{1, \dots, p\}$ are distinct.

Then $\dim X = \dim V - \dim U$. $\left[\operatorname{range} TE_2 \subseteq \right] X$ is invar under TE_2, by Problem (1)(b).

We have $x \in X \Rightarrow TE_2(x) \in X \Rightarrow Tx - TE_1(x) \in X \Rightarrow Tx \in X$. Hence X is invar under T. $\square$

(Note that $E_1(x) \in \operatorname{span}(u_{\beta_1}, \dots, u_{\beta_t})$, where $\beta_1, \dots, \beta_t \in \{\alpha_1, \dots, \alpha_{p-\dim U}\}$ and each $u_{\beta_i} \in U$.)

COMMENT: Conversely, by reversing the roles of U and W, we conclude that it is true as well.

• *Suppose $T \in \mathcal{L}(V)$ and U is an invar subsp of V under T. Suppose $\lambda_1, \dots, \lambda_m$ are the distinct eigvals of T correspd eigvecs $v_1, \dots, v_m$.*

• **TIPS 1:** *Prove that $v_1 + \cdots + v_m \in U \iff$ each $v_k \in U$.*

SOLUTION:

Suppose each $v_k \in U$. Then because U is a subsp, $v_1 + \cdots + v_m \in U$.

Define the statement $P(k)$: if $v_1 + \cdots + v_k \in U$, then each $v_j \in U$. We use induction on m.

(i) For $k = 1$, $v_1 \in U$.

(ii) For $2 \leqslant k \leqslant m$. Assume that $P(k-1)$ holds. Suppose $v = v_1 + \cdots + v_k \in U$.

Then $Tv = \lambda_1 v_1 + \cdots + \lambda_k v_k \in U \Longrightarrow Tv - \lambda_k v = (\lambda_1 - \lambda_k)v_1 + \cdots + (\lambda_{k-1} - \lambda_k)v_{k-1} \in U$.

For each $j \in \{1, \dots, k-1\}, \lambda_j - \lambda_k \neq 0 \Rightarrow (\lambda_j - \lambda_k)v_j = v_j'$ is an eigvec of T correspd λ_j.

By assumption, each $v_j' \in U$. Thus $v_1, \dots, v_{k-1} \in U$. So that $v_k = v - v_1 - \cdots - v_{k-1} \in U$. $\square$

• **TIPS 2:** *If $\dim V = m$. Prove that $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m)$, where $E_k = \operatorname{span}(v_k)$.*

SOLUTION:

Because $V = E_1 \oplus \cdots \oplus E_m$. $\forall u \in U, \exists!\, e_j \in E_j, u = e_1 + \cdots + e_m$.

If $e_j \neq 0$, then e_j is an eigvec correspd λ_j. Otherwise $e_j = 0 \in U$. By (TIPS 1), each nonzero $e_j \in U$.

Thus $u \in (U \cap E_1) + \cdots + (U \cap E_m) = U$. Because each $(U \cap E_j) \subseteq E_j$.

For each $k \in \{2, \dots, n\}$, $\big((U \cap E_1) + \cdots + (U \cap E_{k-1})\big) \bigcap (U \cap E_k) \subseteq (E_1 + \cdots + E_{k-1}) \cap E_k = \{0\}$. $\square$

• **TIPS 3:** *Suppose W is a nonzero invar subsp of V under T. If $\dim V = m \geqslant 1$. Prove that $W = \operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ for some distinct $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$.*

SOLUTION:

Each $\operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ is invar under T.

By (TIPS 2), $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m)$. Because each $\dim E_k = 1$, $U \cap E_k = \{0\}$ or E_k.

There must be at least one k such that $E_k = U \cap E_k$, for if not, $U = \{0\}$ since $V = E_1 \oplus \cdots \oplus E_m$.

Let $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$ be all the distinct indices for which $E_k = U \cap E_k$.

Thus $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m) = E_{\alpha_1} \oplus \cdots E_{\alpha_A} = \operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$. $\square$

---

1 *Suppose $T \in \mathcal{L}(V)$ and U is a subsp of V.*

(a) *If $U \subseteq \operatorname{null} T$, then U is invar under T.* $\forall u \in U \subseteq \operatorname{null} T, Tu = 0 \in U$. $\square$

(b) *If $\operatorname{range} T \subseteq U$, then U is invar under T.* $\forall u \in U, Tu \in \operatorname{range} T \subseteq U$. $\square$

• *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*

(a) *Prove that $\operatorname{null}(T - \lambda I)$ is invar under S for any $\lambda \in \mathbf{F}$.*

(b) *Prove that $\operatorname{range}(T - \lambda I)$ is invar under S for any $\lambda \in \mathbf{F}$.*

SOLUTION:

Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.

(a) $(T - \lambda I)(v) = 0 \Rightarrow (T - \lambda I)(Sv) = \big(S(T - \lambda I)\big)(v) = 0$.

(b) $(T - \lambda I)(u) = v \in \operatorname{range}(T - \lambda I) \Rightarrow Sv = \big(S(T - \lambda I)\big)(u) = (T - \lambda I)(Su) \in \operatorname{range}(T - \lambda I)$. $\square$

• *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*

2 *Show that $W = \operatorname{null} T$ is invar under S.* $\forall u \in W, Tu = 0 \Rightarrow STu = 0 = TSu \Rightarrow Su \in W$. $\square$

3 *Show that $U = \operatorname{range} T$ is invar under S.* $\forall w \in U, \exists v \in V, Tv = w, TSv = STv = Sw \in U$. $\square$

• *Suppose $T \in \mathcal{L}(V)$ and $V_1, \dots, V_m$ are invar subsps of V under T.*

4 $\forall v_i \in V_i, Tv_i \in V_i \Rightarrow \forall v = v_1 + \cdots + v_m \in V_1 + \cdots + V_m, Tv = Tv_1 + \cdots + Tv_m \in V_1 + \cdots + V_m$. $\square$

5 $\forall v \in \bigcap\limits_{i=1}^{m} V_i, Tv \in V_i, \forall i \in \{1, \dots, m\} \Rightarrow Tv \in \bigcap\limits_{i=1}^{m} V_i$. Thus $\bigcap\limits_{i=1}^{m} V_i$ is invar under T. $\square$

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6 *Suppose U is an invar subsp of V under each $T \in \mathcal{L}(V)$. Show that $U = \{0\}$ or $U = V$.*

SOLUTION: If $V = \{0\}$. Then we are done. Suppose $V \neq \{0\}$. We show the contrapositive:

Suppose $U \neq \{0\}$ and $U \neq V$. Prove that $\exists T \in \mathcal{L}(V)$ such that U is not invar under T.

Let W be such that $V = U \oplus W$. Define $T \in \mathcal{L}(V)$ by $T(u + w) = w$. $\square$

• **TIPS:** Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is the counterclockwise rotation by the angle $\theta \in \mathbf{R}$.

Define $\mathcal{C} \in \mathcal{L}(\mathbf{R}^2, \mathbf{C})$ by $\mathcal{C}(a, b) = a + ib = r(\cos\alpha + \mathrm{i}\sin\alpha) \Rightarrow a = r\cos\alpha, b = r\sin\alpha$, where $r = a^2 + b^2$.

Then $(\cos\theta + \mathrm{i}\sin\theta)(a + ib) = r\big(\cos(\alpha + \theta) + \mathrm{i}\sin(\alpha + \theta)\big) = \mathcal{C}^{-1}T(a, b)$.

Hence $T(a, b) = \big(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta\big)$. Now $\mathcal{M}(T) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

EXAMPLE: OR **7** Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find all eigvals of T.

NOTICE that $\mathcal{M}(T) = \begin{pmatrix} \cos 90^\circ & -3\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$. By [5.8](a), we conclude that T has no eigvals.

OR. Suppose λ is an eigval with an eigvec (x, y). Then $(\lambda x, \lambda y) = (-3y, x) \Rightarrow -3y = \lambda^2 y \Rightarrow \lambda^2 = -3$.

$\big[$ Ignoring the possibility of $y = 0$, because $x = 0 \iff y = 0$. $\big]$ $\square$

---

8 *Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigvals and eigvecs.*

SOLUTION: Suppose λ is an eigval with an eigvec (w, z). Then $z = \lambda w$ and $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, ignoring the possibility of $z = 0$ ($z = 0 \iff w = 0$).

Hence $\lambda_1 = -1$ and $\lambda_2 = 1$ are all the eigvals of T. And $T(z, z) = (z, z), T(z, -z) = (-z, z)$.

又 $\dim \mathbf{F}^2 = 2$. Thus the set of all eigvecs is $\{(z, z), (z, -z) : z \neq 0\}$. $\square$

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9 *Define* $T \in \mathcal{L}(\mathbf{F}^3)$ *by* $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. *Find all eigvals and eigvecs.*

SOLUTION: Suppose λ is an eigval with an eigvec (z_1, z_2, z_3).
Then $(2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$. We discuss in two cases:
For $\lambda = 0$, $z_2 = z_3 = 0$ and z_1 can be arbitrary ($z_1 \neq 0$).
For $\lambda \neq 0$, $z_2 = 0 = z_1$, and z_3 can be arbitrary ($z_3 \neq 0$), then $\lambda = 5$.
The set of all eigvecs is $\{(0,0,w), (w,0,0) : w \neq 0\}$. □

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10 *Define* $T \in \mathcal{L}(\mathbf{F}^n)$ *by* $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$
(a) *Find all eigvals and eigvecs;* (b) *Find all invar subsps of* V *under* T.

SOLUTION:

(a) Suppose $x = (x_1, x_2, x_3, \dots, x_n)$ is an eigvec with an eigval λ.
Then $Tx = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$.
Hence $1, \dots, n$ of length $\dim \mathbf{F}^n$ are all the eigvals.
And $\{(0, \dots, 0, x_k, 0, \dots, 0) \in \mathbf{F}^n : x_k \neq 0, k = 1, \dots, n\}$ is the set of all eigvecs.

(b) Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbf{F}^n$. Let $V_k = \operatorname{span}(e_k)$. Then $V_1, \dots, V_n$ are invar under T.
Hence by (TIPS 3), every sum of $V_1, \dots, V_n$ is a invar subsp of V under T. □

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18 *Define the forward shift operator* $T \in \mathcal{L}(\mathbf{F}^\infty)$ *by* $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$.
Show that T *has no eigvals.*

SOLUTION: Suppose λ is an eigval of T with an eigvec $(z_1, z_2, \dots)$.
Then $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$. Thus $\lambda z_1 = 0, \lambda z_k = z_{k-1}$.
If $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \cdots = z_k \Rightarrow (z_1, z_2, \dots) = 0 \Longrightarrow 0$ is not an eigval.
If $\lambda \neq 0$, then $\lambda z_1 = 0 \Rightarrow z_1 = \cdots = z_k = 0 \Longrightarrow \lambda$ is not an eigval. Now no $\lambda \in \mathbf{F}$ is an eigval. □

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19 *Suppose* $n \in \mathbf{N}^+$. *Define* $T \in \mathcal{L}(\mathbf{F}^n)$ *by* $T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.
In other words, the entries of $\mathcal{M}(T)$ *with resp to the standard basis are all* 1*'s.*
Find all eigvals and eigvecs of T.

SOLUTION:

Suppose λ is an eigval of T with an eigvec $(x_1, \dots, x_n)$.
Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.
Thus $\lambda x_1 = \cdots = \lambda x_n = x_1 + \cdots + x_n$.

$$\left.\begin{array}{r} \text{For } \lambda = 0, x_1 + \cdots + x_n = 0 \\ \text{For } \lambda \neq 0, x_1 = \cdots = x_n \Longrightarrow \lambda x_k = n x_k \end{array}\right\} \Rightarrow 0, n \text{ are the eigvals of } T.$$

And the set of all eigvecs of T is $\{(x_1, \dots, x_n) \in \mathbf{F}^n \backslash \{0\} : x_1 + \cdots + x_n = 0 \vee x_1 = \cdots = x_n\}$. □

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20 *Define the backward shift operator* $S \in \mathcal{L}(\mathbf{F}^\infty)$ *by* $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.
(a) *Show that every element of* $\mathbf{F}$ *is an eigval of* S; (b) *Find all eigvecs of* S.

SOLUTION:

Suppose λ is an eigval of S with an eigvec $(z_1, z_2, \dots)$.
Then $S(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$. Thus for each $k \in \mathbf{N}^+, \lambda z_k = z_{k+1}$.
If $\lambda = 0$, then $\lambda z_1 = z_2 = \cdots = z_k = 0$ for all k, while z_1 can be nonzero. Thus 0 is an eigval.
If $\lambda \neq 0$, then $\lambda^k z_1 = \lambda^{k-1} z_2 = \cdots = \lambda z_k = z_{k+1}$, let $z_1 \neq 0 \Longrightarrow (1, \lambda, \lambda^2, \dots, \lambda^k, \dots)$ is an eigvec.
Now each $\lambda \in \mathbf{F}$ is an eigval of T, with the correspd eigvecs in $\operatorname{span}((1, \lambda, \lambda^2, \dots, \lambda^k, \dots))$. □

---

11 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigvals and eigvecs.*

Solution:

Note that $\forall p \in \mathcal{P}(\mathbf{R}) \backslash \{0\}, \deg p' < \deg p$. And $\deg 0 = -\infty$. Suppose λ is an eigval with an eigvec p.
Assume that $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p = p'$. Contradicts. Thus $\lambda = 0$.
Therefore $\deg \lambda p = -\infty = \deg p' \Rightarrow p \in \mathcal{P}_0(\mathbf{R})$. Hence the eigvecs are all the nonzero consts. ☐

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12 *Define $T \in \mathcal{L}\big(\mathcal{P}_n(\mathbf{R})\big)$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigvals and eigvecs.*

Solution:

Suppose λ is an eigval of T with an eigvec p, then $(Tp)(x) = xp'(x) = \lambda p(x)$.
Let $p = a_0 + a_1 x + \cdots + a_n x^n$. Then $xp'(x) = a_1 x + 2a_2 x^2 + \cdots + n a_n x^n = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n$.
Define $S \in \mathcal{L}(\mathbf{F}^{n+1}, \mathcal{P}_n(\mathbf{R}))$ by $S(a_0, a_1, \dots, a_n) = a_0 + a_1 x + \cdots + a_n x^n$.
Then $(S^{-1}TS)(a_0, a_1, \dots, a_n) = (0 \cdot a_0, 1 \cdot a_1, 2 \cdot a_2, \dots, n \cdot a_n)$. Thus $0, 1, \dots, n$ are the eigvals of $S^{-1}TS$.
By Problem (15), $0, 1, \dots, n$ are the eigvals of T. The set of all eigvecs is $\{cx^\lambda : c \neq 0, \lambda = 0, 1, \dots, n\}$.☐

• *Suppose V is finite-dim, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$.*

13 *Prove that $\forall \lambda \in \mathbf{F}, \exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}$, $(T - \alpha I)$ is inv.*

Solution:

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \frac{1}{1000+k}$ for each $k = 1, \dots, \dim V + 1$.
Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigvals.
Hence $\exists\, k = 1, \dots, \dim V + 1$ such that α_k is not an eigval of T and therefore $(T - \alpha_k I)$ is inv. ☐

• (4E 5.A.11) *Prove that $\exists\, \delta > 0$ such that $(T - \alpha I)$ is inv for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.*

Solution:

If T has no eigvals, then $(T - \alpha I)$ is inje for all $\alpha \in \mathbf{F}$ and we are done.
Suppose $\lambda_1, \dots, \lambda_m$ are all the distinct eigvals of T.
Let $\delta > 0$ be such that, for each eigval λ_k, $\lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.
So that for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$, $(T - \alpha I)$ is not inje. ☐

Or. Let $\delta = \min\{|\lambda - \lambda_k| : k \in \{1, \dots, m\}, \lambda_k \neq \lambda\}$.
Then $\delta > 0$ and each $\lambda_k \neq \alpha$ $\big[\iff (T - \alpha I)$ is inv $\big]$ for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$. ☐

• (5.B.4 Or 4E 3.B.27) *Suppose λ is an eigval of $P \in \mathcal{L}(V), P^2 = P$. Prove that $\lambda = 0$ or $\lambda = 1$.*

Solution: Suppose λ is an eigval with an eigvec v. Then $P(Pv) = Pv \Rightarrow \lambda^2 v = \lambda v$. Thus $\lambda = 1$ or 0. ☐

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14 *Suppose $V = U \oplus W$, where U and W are nonzero subsps of V.*
Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.
Find all eigvals and eigvecs of P.

Solution:

Suppose λ is an eigval of P with an eigvec $(u + w)$.
Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$.
Or. Note that $P|_{\operatorname{range} P} = I|_{\operatorname{range} P} \iff P^2 = P$. By (4E 5.A.8), 1 and 0 are the eigvals.
By [1.44], $(\lambda - 1)u = \lambda w = 0$, hence $\lambda = 0 \iff u = 0$, and $\lambda = 1 \iff w = 0$.
Thus $Pu = u, Pw = 0$. Hence the eigvals are 0 and 1, the set of all eigvecs of P is $U \cup W$. ☐

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15 *Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is inv.*

(a) *Prove that T and $S^{-1}TS$ have the same eigvals.*

(b) *What is the relationship between the eigvecs of T and the eigvecs of $S^{-1}TS$?*

Solution:

(a) λ is an eigval of T with an eigvec $v \Rightarrow S^{-1}TS(\underline{S^{-1}v}) = S^{-1}Tv = S^{-1}(\lambda v) = \underline{\lambda S^{-1}v}$.

λ is an eigval of $S^{-1}TS$ with an eigvec $v \Rightarrow S(S^{-1}TS)v = T\underline{Sv} = \underline{\lambda Sv}$.

Or. Note that $S(S^{-1}TS)S^{-1} = T$. Hence every eigval of $S^{-1}TS$ is an eigval of $S(S^{-1}TS)S^{-1} = T$.

Or. $Tv = \lambda v \iff (TS)(u) = \lambda Su \iff (S^{-1}TS)(u) = \lambda u$. Where $v = Su$.

$(S^{-1}TS)(u) = \lambda u \iff (S^{-1}T)(v) = \lambda S^{-1}v \iff Tv = \lambda v$. Where $u = S^{-1}v$.

(b) Because λ is an eigval of $T \iff \lambda$ is an eigval of $S^{-1}TS$.

(See [5.36].) Now $E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\}$; $E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}$. □

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17 *Give an example of an operator on $\mathbf{R}^4$ that has no real eigvals.*

Solution:

Let (e_1, e_2, e_3, e_4) be the standard basis of $\mathbf{R}^4$.

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$.

Suppose λ is an eigval of T with an eigvec (x, y, z, w). Then we get $\begin{cases} (1-\lambda)x + y + z + w = 0, \\ -x + (1-\lambda)y - z - w = 0, \\ 3x + 8y + (11-\lambda)z + 5w = 0, \\ 3x - 8y - 11z + (5-\lambda)w = 0. \end{cases}$

This set of linear equations has no solutions.

[You can type it on https://zh.numberempire.com/equationsolver.php to check.]

Or. Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Suppose λ is an eigval of T with an eigvec (x, y, z, w).

Then $T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \implies \begin{cases} -y = \lambda x, x = \lambda y \implies -xy = \lambda^2 xy \\ -w = \lambda z, z = \lambda w \implies -zw = \lambda^2 zw \end{cases}$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Otherwise, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, contradicts.

Similarly, $y = z = w = 0$. Then we fail. Thus T has no eigvals. □

• (4E 5.A.16) *Suppose $B_V = (v_1, \dots, v_n)$, $T \in \mathcal{L}(V)$, $\mathcal{M}(T, (v_1, \dots, v_n)) = A$. Prove that if λ is an eigval of T, then $|\lambda| \leqslant n \max\{|A_{j,k}| : 1 \leqslant j, k \leqslant n\}$.*

Solution:

Suppose v is an eigval of T correspd to λ. Let $v = c_1v_1 + \dots + c_nv_n$.

Because $\lambda c_1 v_1 + \dots + \lambda c_n v_n = c_1 Tv_1 + \dots + c_n Tv_n = \sum_k^n c_k \left(\sum_j^n A_{j,k} v_j \right)$.

We have $\lambda c_j = \sum_{k=1}^n c_k A_{j,k} \implies |\lambda||c_j| = \sum_{k=1}^n |c_k||A_{j,k}|$ for each $j \in \{1, \dots, n\}$

Let $|c_j| = \max\{|c_1|, \dots, |c_n|\}$. Note that $|c_j| \neq 0$, for if not, $c_1 = \dots = c_n = 0 \Rightarrow v = 0$, contradicts.

Let $M = \max\{|A_{j,k}| : 1 \leqslant j, k \leqslant n\}$. Note that for each j, $\sum_{k=1}^n |A_{j,k}| \leqslant \sum_{k=1}^n M = nM$.

Thus $|\lambda||c_j| = \sum_{k=1}^n |c_k||A_{j,k}| \implies |\lambda| \leqslant \sum_{k=1}^n |A_{j,k}| \dfrac{|c_k|}{|c_j|} \leqslant \sum_{k=1}^n |A_{j,k}| \leqslant nM$. □

• (4E 5.A.15) *Suppose $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*
Show that λ is an eigval of $T \Longleftrightarrow \lambda$ is an eigval of the dual operator $T' \in \mathcal{L}(V')$.

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec v.
Let U be invar such that $V = \operatorname{span}(v) \oplus U$ $\big[$ by (4E 5.A.39) $\big]$.
Define $\psi \in V'$ by $\psi(cv+u) = c$.
Now $\big[T'(\psi)\big](cv+u) = \psi\big(cv+Tu\big) = \lambda cv = \lambda\psi(cv+u)$. Hence $T'(\psi) = \lambda\psi$.

(b) Suppose λ is an eigval T' with an eigvec ψ. Then $T'(\psi) = \psi \circ T = \lambda\psi$.
Note that $\psi \neq 0, \psi(Tv) = \lambda\psi(v)$ Thus $\exists\, v \in V\backslash\{0\}, Tv = \frac{\psi(Tv)}{\psi(v)}v = \lambda v$. □

OR. [*Only in Finite-dim*] Using [5.6], (4E 3.F.17), [3.101] and (3.F.12).
λ is an eigval of $T \Longleftrightarrow (T - \lambda I_V)$ is not inv
$\Longleftrightarrow (T-\lambda I_V)' = T' - \lambda I_{V'}$ is not inv $\Longleftrightarrow \lambda$ is an eigval of T'. □

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24 *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Tx = Ax$.*
(a) *Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigval of T.*
(b) *Suppose the sum of the entries in each col of A equals 1. Prove that 1 is an eigval of T.*

SOLUTION:

Suppose λ is an eigval of T with an eigvec x. Then $Tx = Ax = \begin{pmatrix} \sum_{k=1}^n A_{1,k}x_k \\ \vdots \\ \sum_{k=1}^n A_{n,k}x_k \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

(a) Suppose $\sum_{r=1}^n A_{R,c} = 1$ for each $R \in \{1, \dots, n\}$.
Then if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, and hence is an eigval of T.

(b) Suppose $\sum_{r=1}^n A_{r,C} = 1$ for each $C \in \{1, \dots, n\}$.
Then $\sum_{r=1}^n (Ax)_{r,\cdot} = \sum_{r=1}^n (Ax)_{r,1} = \sum_{c=1}^n (A_{1,c} + \cdots + A_{n,c})x_c = \sum_{c=1}^n x_c = \lambda(x_1 + \cdots + x_n)$.
Hence $\lambda = 1$ for all $x \in \mathbf{F}^{n,1}$ such that $\sum_{c=1}^n x_{c,1} \neq 0$. □

OR. We show that $(T - I)$ is not inv, so that $\lambda = 1$ is an eigval.
Because $(T-I)x = (A-I)x = \begin{pmatrix} \sum_{r=1}^n A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^n A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.
Then $y_1 + \cdots + y_n = \sum_{r=1}^n \sum_{c=1}^n (A_{r,c}x_c - x_r) = \sum_{c=1}^n x_c \sum_{r=1}^n A_{r,c} - \sum_{r=1}^n x_r = 0$.
Thus range $(T-I) \subseteq \left\{ \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}^t \in \mathbf{F}^{n,1} : y_1 + \cdots + y_n = 0 \right\}$. Hence $(T-I)$ is not surj. □

OR. Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbf{F}^{n,1}$. Define $\psi \in (\mathbf{F}^{n,1})'$ by $\psi(e_k) = 1$.
Thus $\big(\psi \circ (T-I)\big)(e_k) = \psi\Big(\big(\sum_{j=1}^n A_{j,k}e_j\big) - e_k\Big) = \big(\sum_{j=1}^n A_{j,k}\big) - 1 = 0$.
Which means that $\psi \circ (T-I) = 0$. 又 $\psi \neq 0$. Hence $(T-I)$ is not inje. □

OR. Define $S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Sx = A^t x$. Because the rows of A^t are the cols of A.
Now by (a), 1 is an eigval of S. Let $(\varphi_1, \dots, \varphi_n)$ be the dual basis of $(e_1, \dots, e_n)$.
Define $\Phi \in \mathcal{L}\Big(\mathbf{F}^{n,1}, (\mathbf{F}^{1,n})'\Big)$ by $\Phi(e_k) = \varphi_k$. Note that $\mathcal{M}(T') = A^t$.
Now $(\Phi^{-1}T'\Phi)(e_k) = (\Phi^{-1}T')(\varphi_k) = \Phi^{-1}\big(\sum_{j=1}^n A_{k,j}\varphi_j\big) = \sum_{j=1}^n A_{k,j}e_j = A^t e_k = Se_k$.
Thus 1 is an eigval of $S = \Phi^{-1}T'\Phi$, so of T', $\big[$ by Problem (15) $\big]$, so of T, $\big[$ by (4E 5.A.15) $\big]$. □

• *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Tx = xA$.*
(a) *Suppose the sum of the entries in each col of A equals 1. Prove that 1 is an eigval of T.*
(b) *Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigval of T.*

SOLUTION:

Suppose λ is an eigval with an eigvec x. Then $\left(\sum_{r=1}^{n} x_r A_{r,1} \quad \cdots \quad \sum_{r=1}^{n} x_r A_{r,n}\right) = \lambda\left(x_1 \quad \cdots \quad x_n\right)$.

(a) Suppose $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C \in \{1, \dots, n\}$.
Thus if $x_1 = \cdots = x_n$, then $\lambda = 1$, hence is an eigval of T.

(b) Suppose $\sum_{c=1}^{n} A_{R,c} = 1$ for each $R \in \{1, \dots, n\}$.
Thus $\sum_{c=1}^{n} (xA)_{\cdot,c} = \sum_{c=1}^{n} (A_{c,1} + \cdots + A_{c,n}) x_c = \sum_{c=1}^{n} x_c = \lambda(x_1 + \cdots + x_n)$.
Hence $\lambda = 1$, for all x such that $\sum_{r=1}^{n} x_{1,r} \neq 0$. □

OR. We show that $(T - I)$ is not inv, so that $\lambda = 1$ is an eigval.
Because $(T - I)x = x(A - \mathcal{M}(I)) = \left(\sum_{c=1}^{n} x_c A_{c,1} - x_1 \quad \cdots \quad \sum_{c=1}^{n} x_c A_{c,n} - x_n\right) = \left(y_1 \quad \cdots \quad y_n\right)$.
Then $y_1 + \cdots + y_n = \sum_{c=1}^{n} \sum_{r=1}^{n} (x_r A_{r,c} - x_c) = \sum_{r=1}^{n} x_r \sum_{c=1}^{n} A_{r,c} - \sum_{c=1}^{n} x_c = 0$.
Thus range $(T - I) \subseteq \{ (y_1 \quad \cdots \quad y_n) \in \mathbf{F}^{1,n} : y_1 + \cdots + y_n = 0 \}$. Hence $(T - I)$ is not surj. □

OR. Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbf{F}^{1,n}$. Define $\psi \in (\mathbf{F}^{n,1})'$ by $\psi(e_k) = 1$.
Because $Te_k = e_k A = \left(A_{k,1} \quad \cdots \quad A_{k,n}\right) = \sum_{i=1}^{n} A_{k,i} e_i$. **COROLLARY:** $\mathcal{M}(T) = A^t$.
$(\psi \circ (T - I))(e_k) = \left(\sum_{i=1}^{n} A_{k,i}\right) - 1 = 0$. Then $\psi \circ (T - I) = 0$. 又 $\psi \neq 0$. $(T - I)$ is not inje. □

OR. Define $S \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Sx = xA^t$. Because the rows of A are the cols of A^t.
Now by (a), 1 is an eigval of S. Let $(\varphi_1, \dots, \varphi_n)$ be the dual basis of $(e_1, \dots, e_n)$.
Define $\Phi \in \mathcal{L}\left(\mathbf{F}^{1,n}, (\mathbf{F}^{1,n})'\right)$ by $\Phi(e_k) = \varphi_k$. Because $[T'(\varphi_k)](e_j) = \varphi_k\left(\sum_{i=1}^{n} A_{j,i} e_i\right) = A_{j,k}$.
By (3.F.9), $T'(\varphi_k) = \sum_{j=1}^{n} A_{j,k} \varphi_j$. **COROLLARY:** $\mathcal{M}(T') = A = \mathcal{M}(T)^t$. FIXME: $\mathcal{M}(T) e_k = A^t e_k = e_k A$
Now $(\Phi^{-1} T' \Phi)(e_k) = (\Phi^{-1} T')(\varphi_k) = \Phi^{-1}\left(\sum_{j=1}^{n} A_{j,k} \varphi_j\right) = \sum_{j=1}^{n} A_{j,k} e_j = e_k A^t = Se_k$.
Thus 1 is an eigval of $S = \Phi^{-1} T' \Phi$, so of T', [by Problem (15)], so of T, [by (4E 5.A.15)]. □

• *Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.*
(a) [OR (9.11)] *$\lambda \in \mathbf{R}$. Prove that λ is an eigval of $T \iff \lambda$ is an eigval of T_{C}.*
(b) [OR **16** OR [9.16]] *$\lambda \in \mathbf{C}$. Prove that λ is an eigval of $T_{\mathrm{C}} \iff \overline{\lambda}$ is an eigval of T_{C}.*

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec v.
Then $Tv = \lambda v \Longrightarrow T_{\mathrm{C}}(v + \mathrm{i}0) = Tv + \mathrm{i}T0 = \lambda v$. Thus λ is an eigval of T_{C}.
Suppose λ is an eigval of T_{C} with an eigvec $v + \mathrm{i}u$.
Then $T_{\mathrm{C}}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \Longrightarrow Tv = \lambda v, Tu = \lambda u$. Thus λ is an eigval of T.
(Note that $v + \mathrm{i}u$ is nonzero $\iff$ at least one of v, u is nonzero).

(b) Suppose λ is an eigval of T_{C} with an eigvec $v + \mathrm{i}u$. Then $T_{\mathrm{C}}(v + \mathrm{i}u) = Tv + \mathrm{i}Tu = \lambda(v + \mathrm{i}u)$.
Note that $\overline{T_{\mathrm{C}}(v + \mathrm{i}u)} = \overline{Tv + \mathrm{i}Tu} = Tv - \mathrm{i}Tu = T_{\mathrm{C}}(v - \mathrm{i}u) = T_{\mathrm{C}}(\overline{v + \mathrm{i}u})$.
And that $\overline{\lambda(v + \mathrm{i}u)} = \overline{\lambda} v - \mathrm{i}\overline{\lambda} u = \overline{\lambda}(v - \mathrm{i}u) = \overline{\lambda}(\overline{v + \mathrm{i}u})$.
Hence $\overline{\lambda}$ is an eigval of T_{C}. To prove the other direction, notice that $\overline{\overline{\lambda}} = \lambda$. □

OR. Suppose $\lambda = a + \mathrm{i}b$ is an eigval of T_{C} with an eigvec $v + \mathrm{i}u$.
Because $T_{\mathrm{C}}(v + \mathrm{i}u) = \lambda(v + \mathrm{i}u) = (av - bu) + \mathrm{i}(au + bv) = Tv + \mathrm{i}Tu \Longrightarrow Tv = av - bu, Tu = au + bv$.
Now $T_{\mathrm{C}}(\overline{v + \mathrm{i}u}) = Tv - \mathrm{i}Tu = (av - bu) - \mathrm{i}(au + bv) = (a - \mathrm{i}b)(v - \mathrm{i}u) = \overline{\lambda}(\overline{v - \mathrm{i}u})$. Similarly □

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21 *Suppose $T \in \mathcal{L}(V)$ is inv.*

(a) *Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that λ is an eigval of $T \Longleftrightarrow \lambda^{-1}$ is an eigval of T^{-1}.*

(b) *Prove that T and T^{-1} have the same eigvecs.*

SOLUTION: (a) $Tv = \lambda v \Longleftrightarrow v = \lambda T^{-1} v \Longleftrightarrow \lambda^{-1} v = T^{-1} v$. Where $v \neq 0$.

(b) NOTICE that T is inv $\Longrightarrow 0$ is not an eigval of T or T^{-1}. By (a), immediately. ☐

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22 *Suppose $T \in \mathcal{L}(V)$ and $\exists$ nonzero vecs u, w in V such that $Tu = 3w,\ Tw = 3u$.*
Prove that 3 or -3 is an eigval of T.

SOLUTION: $T(u+w) = 3(u+w),\ T(u-w) = 3(w-u) = -3(u-w)$. Note that $u - w \neq 0$ or $u + w \neq 0$.
OR. $T(Tu) = 9u \Rightarrow T^2 - 9 = (T - 3I)(T + 3I)$ is not injective $\Rightarrow 3$ or -3 is an eigval. ☐

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23 *Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigvals.*

SOLUTION: Suppose λ is an eigval of ST with an eigvec v. Then $T(STv) = \lambda Tv = TS(Tv)$.
If $Tv = 0$ (while $v \neq 0$), then T is not inje $\Rightarrow (TS - 0I)$ and $(ST - 0I)$ are not inje.
Thus $\lambda = 0$ is an eigval of ST and TS with the same eigvec v.
Otherwise, $Tv \neq 0$, then λ is an eigval of TS. Reversing the roles of T and S. ☐

• (2E 20) *Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigvals and $S \in \mathcal{L}(V)$ has the same eigvecs (but might not with the same eigvals). Prove that $ST = TS$.*

SOLUTION: Let $n = \dim V$. For each $j \in \{1, \dots, n\}$, let v_j be an eigvec with eigval λ_j of T and α_j of S.
Then $B_V = (v_1, \dots, v_n)$. Because $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$ for each j. Hence $ST = TS$. ☐

• (4E 5.A.37) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(S) = TS$ for each $S \in \mathcal{L}(V)$.
Prove that the set of eigvals of T equals the set of eigvals of $\mathcal{A}$.

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec $v = v_1$. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.
Note that $\operatorname{span}(v) \subseteq \operatorname{null}(T - \lambda I)$. Define $S \in \mathcal{L}(V)$ by $S(v_j) = v$ for each $j \in \{1, \dots, n\}$.
OR. Define $S \in \mathcal{L}(V)$ by $Sv_1 = v_1,\ Sv_j = 0$ for $j \geqslant 2$. Then $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$.
Then $(T - \lambda I)S = 0$. Thus $\mathcal{A}(S) = TS = \lambda S$ while $S \neq 0$. Hence λ is an eigval of $\mathcal{A}$.

(b) Suppose λ is an eigval of $\mathcal{A}$ with an eigvec S.
Then $\exists\, v \in V, 0 \neq u = S(v) \in V \Rightarrow Tu = (TS)v = (\lambda S)v = \lambda u$. Thus λ is an eigval T.
OR. Because $TS - \lambda S = (T - \lambda I)S = 0 \Rightarrow \{0\} \subsetneq \operatorname{range} S \subseteq \operatorname{null}(T - \lambda I)$. $(T - \lambda I)$ is not inje. ☐

COMMENT: If $\mathcal{A}(S) = ST, \forall S \in \mathcal{L}(V)$. Then the eigvals of $\mathcal{A}$ are not the eigvals of T.

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25 *Suppose $T \in \mathcal{L}(V)$ and u, w are eigvecs of T such that $u + w$ is also an eigvec of T.*
Prove that u and w correspd to the same eigval.

SOLUTION: Suppose $\lambda_1, \lambda_2, \lambda_0$ are eigvals of T with eigvecs to $u, w, u + w$ respectively.
Then $T(u+w) = \lambda_0(u+w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$.
If (u, w) is linely depe, then let $w = cu$, therefore $\lambda_2 cu = Tw = cTu = \lambda_1 cu \Rightarrow \lambda_2 = \lambda_1$.
Otherwise, (u, w) is linely inde. Then $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. ☐
OR. Assume that $\lambda_1 \neq \lambda_2$. Then (u, w) is linely inde. Thus $\lambda_0 - \lambda_1 = \lambda_0 - \lambda_2$. Contradicts. ☐

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26 *Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vec in V is an eigvec of T.*
Prove that T is a scalar multi of the identity operator.

Solution: If $\dim V = 0, 1$ then we are done. Suppose $\dim V \geqslant 2$.

Because $\forall v \in V, \exists ! \lambda_v \in \mathbf{F}, Tv = \lambda_v v$. For any two distinct nonzero vecs $v, w \in V$,
$T(v+w) = \lambda_{v+w}(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w.$ □

Or. For any two nonzero vecs $u, v \in V$, u, v are eigvecs.
If $u + v \neq 0$, then $u + v$ is also an eigvec. Otherwise, $u + v = 0$, then $Tu = -Tv = \lambda u = -\lambda v$.
Thus by Problem (25), $\forall u, v \in V, Tu = \lambda u, Tv = \lambda v \Rightarrow \forall v \in V, Tv = \lambda v$. □

27, 28 *Suppose V is finite-dim and $k \in \{1, \dots, \dim V - 1\}$.*
Suppose $T \in \mathcal{L}(V)$ is such that every subsp of V of dim k is invar under T.
Prove that T is a scalar multi of the identity operator.

Solution: If $\dim V \leqslant 1$ then we are done. Suppose $\dim V \geqslant 2$.

We prove the contrapositive: *If T is not a scalar multi of I. Then $\exists$ subsp U of dim k not invar under T.*
By Problem (26), $\exists\, v \in V$ and $v \neq 0$ such that v is not an eigvec of T.
Thus (v, Tv) is linely inde. Extend to $B_V = (v, Tv, u_1, \dots, u_n)$.
Let $U = \operatorname{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$ is not an invar subsp of V under T. □

Or. Suppose $0 \neq v = v_1 \in V$. Extend to $B_V = (v_1, \dots, v_n)$. Suppose $Tv_1 = c_1 v_1 + \cdots + c_n v_n, \exists !\, c_i \in \mathbf{F}$.
Consider a k-dim subsp $U = \operatorname{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$. Where $\alpha_1, \dots, \alpha_{k-1} \in \{2, \dots, n\}$ are distinct.
Because every subsp such U is invar. $Tv_1 = c_1 v_1 + \cdots + c_n v_n \in U \Longrightarrow c_2 = \cdots = c_n = 0$.
For if not, $\exists\, c_i \neq 0$, let $W = \operatorname{span}(v_1, v_{\beta_1}, \dots, v_{\beta_{k-1}})$, where each $\beta_j \in \{2, \dots, i-1, i+1, \dots, n\}$.
Hence $Tv_1 = c_1 v_1$. Because $v_1 = v \in V$ is arbitrary. We conclude that $T = \lambda I$ for some $\lambda \in \mathbf{F}$. □

Or. For each $k \in \{1, \dots, \dim V - 1\}$, define $P(k)$: if every subsp of dim k is invar, then $T = \lambda I$.
(i) If every subsp of dim 1 is invar, then by Problem (26), $T = \lambda I$. Thus $P(1)$ holds.
(ii) Assume that $P(k)$ holds for $k \in \{1, \dots, \dim V - 1\}$. And every subsp of dim $k+1$ is invar.
Let U be a subsp of dim k. If $\dim U = \dim V - 1$ then extend B_U to B_V and we are done.
Suppose $\dim U \in \{1, \dots, \dim V - 2\}$. Choose two linely inde vecs $v, w \notin U$.
Because $U \oplus \operatorname{span}(v)$ and $U \oplus \operatorname{span}(w)$ of dim $k+1$ are invar.
Suppose $u \in U$. Let $Tu = a_1 u_1 + bv = a_2 u_2 + cw, \exists !\, u_1, u_2 \in U, a_1, a_2, b, c \in \mathbf{F}$.
Now $a_1 u_1 - a_2 u_2 = cw - bv \in U \cap \operatorname{span}(v) = \{0\} \Rightarrow b = c = 0$. Thus $Tu \in U$.
Because $P(k)$ holds, we conclude that $T = \lambda I$. Thus $P(k+1)$ holds. □

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29 *Suppose $T \in \mathcal{L}(V)$ and* range T *is finite-dim.*
Prove that T has at most $1 + \dim \operatorname{range} T$ distinct eigvals.

Solution:

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigvals of T with correspd eigvecs $v_1, \dots, v_m$.
(Because range T is finite-dim. The correspd eigvals are finite.)
Then $(v_1, \dots, v_m)$ linely inde $\Longrightarrow (\lambda_1 v_1, \dots, \lambda_m v_m)$ linely inde, if each $\lambda_k \neq 0$.
Otherwise, $\exists !\, \lambda_k = 0$. Now $(\lambda_1 v_1, \dots, \lambda_{k-1} v_{k-1}, \lambda_{k+1} v_{k+1}, \dots, \lambda_m v_m)$ is linely inde.
Hence, by [2.23], $m - 1 \leqslant \dim \operatorname{range} T$. □

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30 *Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigvals. Prove that $\exists\, x, Tx - 9x = (-4, 5, \sqrt{7})$.*

Solution: T has $\dim \mathbf{R}^3$ eigvals not including $9 \Rightarrow (T - 9I)$ is inv. $x = (T - 9I)^{-1}(-4, 5, \sqrt{7})$. □

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31 *Suppose V is finite-dim, and $v_1, \dots, v_m \in V$. Prove that*
$(v_1, \dots, v_m)$ is linely inde $\iff v_1, \dots, v_m$ are eigvecs of some T correspd to distinct eigvals.

Solution: Suppose $(v_1, \dots, v_m)$ is linely inde. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.
Define $T \in \mathcal{L}(V)$ by $Tv_k = k \cdot v_k$ for each $k \in \{1, \dots, m, \dots, n\}$. Conversely by [5.10]. $\square$

• *Suppose $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ are distinct.*
(a) **32** *Prove that $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linely inde in $\mathbf{R}^{\mathbf{R}}$.*
Hint: Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$. Define $D \in \mathcal{L}(V)$ by $Df = f'$. Find eigvals and eigvecs of D.
(b) [4E 36] *Show that $\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$ is linely inde in $\mathbf{R}^{\mathbf{R}}$.*

Solution:
(a) Define V and $D \in \mathcal{L}(V)$ as in Hint. Then because for each k, $D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.
Thus $\lambda_1, \dots, \lambda_n$ are distinct eigvals of D. By [5.10], $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linely inde in $\mathbf{R}^{\mathbf{R}}$. $\square$
(b) Let $V = \operatorname{span}\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.
Then because $D\big(\cos(\lambda_k x)\big) = -\lambda_k \sin(\lambda_k x)$. 又 $D\big(\sin(\lambda_k x)\big) = \lambda_k \cos(\lambda_k x)$.
Thus $D^2\big(\cos(\lambda_k x)\big) = -\lambda_k^2 \cos(\lambda_k x)$.
Notice that $\lambda_1, \dots, \lambda_n$ are distinct $\Longrightarrow -\lambda_1^2, \dots, -\lambda_n^2$ are distinct. And $\dim V = n$.
Hence $-\lambda_1^2, \dots, -\lambda_n^2$ are all the eigvals of D^2 with correspd eigvecs $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$.
And then $\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$ is linely inde in $\mathbf{R}^{\mathbf{R}}$. $\square$

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33 *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{range} T) = 0$.*

Solution: $v + \operatorname{range} T \in V/\operatorname{range} T \Longrightarrow v + \operatorname{range} T \in \operatorname{null}\big(T/(\operatorname{range} T)\big)$. Hence $T/(\operatorname{range} T) = 0$. $\square$

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34 *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{null} T)$ is inje $\iff (\operatorname{null} T) \bigcap (\operatorname{range} T) = \{0\}$.*

Solution: Notice that $\big(T/(\operatorname{null} T)\big)(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0 \iff Tu \in (\operatorname{null} T) \bigcap (\operatorname{range} T)$.
Now $T/(\operatorname{null} T)$ is inje $\iff u + \operatorname{null} T = 0 \iff Tu = 0 \iff (\operatorname{null} T) \bigcap (\operatorname{range} T) = \{0\}$. $\square$

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invar subsp of V under T.*
Define $T/U : V/U \to V/U$ by $(T/U)(v + U) = Tv + U$ for each $v \in V$.
(a) *Show that T/U is well-defined and is linear.* *Requires that U is invar under T.*
(b) [Or **35**] *Show that each eigval of T/U is an eigval of T.*

Solution:
(a) $v + U = w + U \iff v - w \in U \Longrightarrow T(v - w) \in U \iff Tv + U = Tw + U$.
Hence T/U is well-defined. Now we show that T/U is linear.
$(T/U)\big((v + U) + \lambda(w + U)\big) = T(v + \lambda w) + U = (T/U)(v + U) + \lambda(T/U)(w)$. Checked.
(b) Suppose λ is an eigval of T/U with an eigvec $v + U$. Then $Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v = u \in U$.
If $u = 0 \Rightarrow Tv = \lambda v$, then we are done. Otherwise, we discuss in two cases.
If $(T - \lambda I)\,|_U$ is inv. Then $\exists!\, w \in U, (T - \lambda I)(w) = u = (T - \lambda I)v \Rightarrow T(v + w) = \lambda(v + w)$.
Note that $v + w \neq 0$, for if not, $v \in U \Rightarrow v + U = 0$, contradicts. Thus λ is an eigval of T.
If $(T - \lambda I)\,|_U$ is not inv. Then because V is finite-dim, $(T - \lambda I)\,|_U$ is not inje,
so that $\exists\, w \in \operatorname{null}(T - \lambda I)\,|_U, w \neq 0, (T - \lambda I)w = 0 \Rightarrow Tw = \lambda w$. $\square$

Or. Let $B_U = (u_1, \dots, u_m)$. Then $\big((T - \lambda I)v, (T - \lambda I)u_1, \dots, (T - \lambda I)u_m\big)$ is linely inde in U.
So that $a_0(T - \lambda I)v + a_1(T - \lambda I)u_1 + \cdots + a_m(T - \lambda I)u_m = 0, \exists\, a_0, a_1, \dots, a_m \in \mathbf{F}$ with some $a_i \neq 0$.
Let $w = a_0 v + a_1 u_1 + \cdots + a_m u_m \Longrightarrow Tw = \lambda w$. Note that $w \neq 0$, for if not, $a_0 v \in U$, each $a_i = 0$. $\square$

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36 *Prove or give a counterexample: The result in Exercise 35 is still true if V is infinite-dim.*

Solution: A counterexample:

Consider $V = \{f \in \mathbf{R}^{\mathbf{R}} : \exists!\, m \in \mathbf{N}, f \in \operatorname{span}(1, e^x, \dots, e^{mx})\}$. Note that V is infinite-dim.

And a subsp $U = \{f \in \mathbf{R}^{\mathbf{R}} : \exists!\, m \in \mathbf{N}^+, f \in \operatorname{span}(e^x, \dots, e^{mx})\}$.

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then range $T = U$ is invar under T.

Consider $(T/U)(1+U) = e^x + U = 0 \Longrightarrow 0$ is an eigval of T/U but is not an eigval of T.

$\big[$ null $T = \{0\}$, for if not, $\exists f \in V \backslash \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, contradicts. $\big]$ □

• (4E 5.A.39) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that T has an eigval $\Longleftrightarrow \exists$ an invar subsp U under T of dimension $\dim V - 1$.

Solution:

(a) Suppose λ is an eigval of T with an eigvec v. (If $\dim V = 1$, then $U = \{0\}$ and we are done.)

Extend $v_1 = v$ to $B_V = (v_1, v_2 \dots, v_n)$.

Step 1. If $\exists\, w_1 \in \operatorname{span}(v_2, \dots, v_n)$ such that $0 \neq Tw_1 \in \operatorname{span}(v_1)$.
Then extend $w_1 = \alpha_{1,2}$ to a basis of $\operatorname{span}(v_2, \dots, v_n)$ as $(\alpha_{1,2}, \dots, \alpha_{1,n})$.
Otherwise, we stop at step 1.

Step 2. If $\exists\, w_2 \in \operatorname{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$ such that $0 \neq Tw_2 \in \operatorname{span}(v_1, w_1)$.
Then extend $w_2 = \alpha_{2,3}$ to a basis of $\operatorname{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$ as $(\alpha_{2,3}, \dots, \alpha_{2,n})$.
Otherwise, we stop at step 2.

Step k. If $\exists\, w_k \in \operatorname{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$ such that $0 \neq Tw_k \in \operatorname{span}(v_1, w_1, \dots, w_{k-1})$,
Then extend $w_k = \alpha_{k,k+1}$ to a basis of $\operatorname{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$ as $(\alpha_{k,k+1}, \dots, \alpha_{k,n})$.
Otherwise, we stop at step k.

Finally, we stop at step m, thus we get $(v_1, w_1, \dots, w_{m-1})$ and $(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})$,
range $T|_{\operatorname{span}(w_1,\dots,w_{m-1})} = \operatorname{span}(v_1, w_1, \dots, w_{m-2}) \Rightarrow \dim \operatorname{null} T|_{\operatorname{span}(w_1,\dots,w_{m-1})} = 0$,
$\underbrace{\operatorname{span}(v_1, w_1, \dots, w_{m-1})}_{\dim m}$ and $\underbrace{\operatorname{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})}_{\dim (n-m)}$ are invar under T.

Let $U = \operatorname{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n}) \oplus \operatorname{span}(v_1, w_1, \dots, w_{m-2})$ and we are done. □

Comment: Both $\operatorname{span}(v_2, \dots, v_n)$ and $U \oplus \operatorname{span}(w_{m-1})$ are in $\mathcal{S}_V \operatorname{span}(v_1)$.
If $T|_U$ is inv, then by the similar algorithm, we can extend U to an invar subsp.

Or. Note that $\dim \operatorname{null}(T - \lambda I) \geqslant 1$. And $\dim \operatorname{range}(T - \lambda I) \leqslant \dim V - 1$.
Let $B_{\operatorname{range}(T-\lambda I)} = (w_1, \dots, w_m)$, $B_V = (w_1, \dots, w_m, u_1, \dots, u_n)$.
If $m = \dim V - 1$. $[\Longleftrightarrow n = 0.]$ Then range $(T - \lambda I)$ is an invar subsp of dim $\dim V - 1$.
Otherwise, choose $k \in \{1, \dots, n\}$ and then let $U = \operatorname{span}(w_1, \dots, w_m, u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n)$.
By Problem (1)(b), U is invar under $(T - \lambda I)$. Now $u \in U \Rightarrow (T - \lambda I)(u) \in U \Rightarrow Tu \in U$.

(b) Suppose U is an invar subsp under T of dim $m = \dim V - 1$. (If $m = 0$, then we are done.)
Let $B_U = (u_1, \dots, u_m), B_V = (u_0, u_1, \dots, u_m)$. We discuss in cases:
(I) If $Tu_0 \in U$, then range $T = U$ so that T is not surj $\Longleftrightarrow$ null $T \neq \{0\} \Longleftrightarrow 0$ is an eigval of T.
(II) If $Tu_0 \notin U$, then $Tu_0 = a_0 u_0 + a_1 u_1 + \dots + a_m u_m$.
If range $T|_U = U \Longleftrightarrow a_1 = \dots = a_m = 0 \Longleftrightarrow Tu_0 \in \operatorname{span}(u_0)$ then we are done.
Otherwise, $T|_U \colon U \to U$ is not surj, so is not inje. Thus 0 is an eigval of $T|_U$, so of T. □

Or. Consider $T/U \in \mathcal{L}(V/U)$. Because $\dim V/U = 1$. $\exists \lambda \in \mathbf{F}, T/U = \lambda I$. By Problem (35). □

Ended

5.B: I [*See 5.B: II below.*]

Comment: 下面，为了照顾原书 5.B 节两版过大的差距，特别将此节补注分成 I 和 II 两部分。
又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本征值」
(相当一部分是从原第 3 版 8.C 节挪过来的) 是对原第 3 版「多项式作用于算子」与
「本征值的存在性」(也即第 3 版 5.B 前半部分) 的极大扩充，这一扩充也大大改变了
原第 3 版后半部分的「上三角矩阵」这一小节，故而将第 4 版 5.B 节放在第 3 版前面。

I 部分除了覆盖第 4 版 5.B 节全部和第 3 版 5.B 节前半部分与之相关的所有习题，
还会覆盖第 4 版 5.A 节末。
II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节，还会覆盖第 4 版 5.C 节；
并且，下面 5.C 还会覆盖第 4 版 5.D 节。
「注：[8.40] Or (4E 5.22) —— mini poly;
[8.44,8.45] Or (4E 5.25,5.26) —— how to find the mini poly;
[8.49] Or (4E 5.27) —— eigvals are the zeros of the mini poly;
[8.46] Or (4E 5.29) —— $q(T) = 0 \Leftrightarrow q$ is a poly multi of the mini poly.」

1 2 3 5 6 7 8 10 11 12 13 18 19 | 2E Ch5.24
4E: 5.A.32, 5.A.33; 3, 7, 8, 9, 10, 11, 12, 13, 14, 15,
16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29.

- (4E 5.A.33) *Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.*
  (a) *Prove that T is inje $\iff T^m$ is inje.*
  (b) *Prove that T is surj $\iff T^m$ is surj.*

Solution:
(a) Suppose T^m is inje. Then $Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$.
Suppose T is inje. Then $T^m v = T^{m-1}v = \cdots = T^2 v = Tv = v = 0$.

(b) Suppose T^m is surj. $\forall u \in V, \exists v \in V, T^m v = u = Tw$, let $w = T^{m-1}v$.
Suppose T is surj. Then $\forall u \in V, \exists v_1, \dots, v_m \in V, T(v_1) = T^2 v_2 = \cdots = T^m v_m = u$. □

- **Note For [5.17]:**
Suppose $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{F})$. Prove that null $p(T)$ *and* range $p(T)$ *are invar under T.*
Solution: Using the commutativity in [5.10].
(a) Suppose $u \in \operatorname{null} p(T)$. Then $p(T)u = 0$.
Thus $p(T)(Tu) = \big(p(T)T\big)u\big(Tp(T)\big)u = T\big(p(T)u\big) = 0$. Hence $Tu \in \operatorname{null} p(T)$. □
(b) Suppose $u \in \operatorname{range} p(T)$. Then $\exists v \in V$ such that $u = p(T)v$.
Thus $Tu = T\big(p(T)v\big) = p(T)(Tv) \in \operatorname{range} p(T)$. □

- **Note For [5.21]:** *Every operator on a finite-dim nonzero complex vecsp has an eigval.*
Suppose V is a finite-dim complex vecsp of dim $n > 0$ and $T \in \mathcal{L}(V)$.
Choose a nonzero $v \in V$. $(v, Tv, T^2v, \dots, T^n v)$ of length $n+1$ is linely depe.
Suppose $a_0 I + a_1 T + \cdots + a_n T^n = 0$. Then $\exists a_j \neq 0$.
Thus $\exists$ nonconst p of smallest degree ($\deg p > 0$) *such that* $p(T)v = 0$.
Because $\exists \lambda \in \mathbf{C}$ such that $p(\lambda) = 0 \Rightarrow \exists q \in \mathcal{P}(\mathbf{C}), p(z) = (z-\lambda)q(z), \forall z \in \mathbf{C}$.
Thus $0 = p(T)v = (T - \lambda I)\big(q(T)v\big)$. By the minimality of $\deg p$ and $\deg q < \deg p$, $q(T)v \neq 0$.
Then $(T - \lambda I)$ is not inje. Thus λ is an eigval of T with eigvec $q(T)v$.
- **Example:** *an operator on a complex vecsp with no eigvals*
Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by $(Tp)(z) = zp(z)$.

Suppose $p \in \mathcal{P}(\mathbf{C})$ is a nonzero poly. Then $\deg Tp = \deg p + 1$, and thus $Tp \neq \lambda p$, $\forall \lambda \in \mathbf{C}$.
Hence T has no eigvals.

---

13 *Suppose V is a complex vecsp and $T \in \mathcal{L}(V)$ has no eigvals.*
Prove that every subsp of V invar under T is either $\{0\}$ or infinite-dim.

SOLUTION: Suppose U is a finite-dim nonzero invar subsp on $\mathbf{C}$. Then by [5.21], $T|_U$ has an eigval. $\square$

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16 *Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}\big(\mathcal{P}_{\dim V}(\mathbf{C}), V\big)$ by $S(p) = p(T)v$. Prove [5.21].*

SOLUTION:

Because $\dim \mathcal{P}_{\dim V}(\mathbf{C}) = \dim V + 1$. Then S is not inje. Hence $\exists\, 0 \neq p \in \mathcal{P}_{\dim V}(\mathbf{C}), p(T)v = 0$.
Using [4.14], write $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$. Apply T to both sides: $p(T) = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$.
Thus at least one of $(T - \lambda_j I)$ is not inje (because $p(T)$ is not inje). $\square$

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17 *Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}\big(\mathcal{P}_{(\dim V)^2}(\mathbf{C}), \mathcal{L}(V)\big)$ by $S(p) = p(T)$. Prove [5.21].*

SOLUTION:

Because $\dim \mathcal{P}_{(\dim V)^2}(\mathbf{C}) = (\dim V)^2 + 1$. Then S is not inje.
Hence $\exists\, p \in \mathcal{P}_{(\dim V)^2}(\mathbf{C}) \backslash \{0\}, 0 = S(p) = p(T) = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$, where $c \neq 0$.
Thus $(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0 \Longrightarrow \exists\, j, (T-\lambda_j)$ is not inje. $\square$

COMMENT: $\exists$ monic $q \in \operatorname{null} S \neq \{0\}$ of smallest degree, $S(q) = q(T) = 0$, then q is the *mini poly.*

• **NOTE FOR [8.40]:** *def for mini poly*
Suppose V is finite-dim and $T \in \mathcal{L}(V)$.
Suppose $M_T^0 = \{p_j\}_{j\in\Gamma}$ is the set of all monic poly that give 0 whenever T is applied.
Prove that $\exists\,!\, p_k \in M_T^0, \deg p_k = \min\{\deg p_j\}_{j\in\Gamma} \leqslant \dim V$.

SOLUTION: OR. Another Proof :

[*Existns Part*] We use induction on $\dim V$.

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, we are done.

(ii) Suppose $\dim V \geqslant 1$.

Assume that $\dim V > 0$ *and that the desired result is true for all operators on all vecsps of smaller dim.*
Let $u \in V, u \neq 0$. The list $(u, Tu, \dots, T^{\dim V}u)$ of length $(1 + \dim V)$ is linely depe.
Then $\exists\,!\, T^m$ of smallest degree such that $T^m u \in \operatorname{span}(u, Tu, \dots, T^{m-1}u)$.
Thus $\exists\, c_j \in \mathbf{F}, c_0 u + c_1 Tu + \cdots + c_{m-1}T^{m-1}u + T^m u = 0$.
Define q by $q(z) = c_0 + c_1 z + \cdots + c_{m-1}z^{m-1} + z^m$.
Then $0 = T^k\big(q(T)u\big) = q(T)(T^k u), \forall k \in \{1, \dots, m-1\} \subseteq \mathbf{N}$.
Because $(u, Tu, \dots, T^{m-1}u)$ is linely inde.
Thus $\dim \operatorname{null} q(T) \geq m \Rightarrow \dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leqslant \dim V - m$.

Let $W = \operatorname{range} q(T)$.
By *assumption*, $\exists\, s \in M_T^0$ of smallest degree (and $\deg s \leqslant \dim W$,) so that $s(T|_W) = 0$.
Hence $\forall v \in V, \big((sq)(T)\big)(v) = s(T)\big(q(T)v\big) = 0$.
Thus $sq \in M_T^0$ and $\deg sq \leqslant \dim V$.

[*Uniqnes Part*]

Suppose $p, q \in M_T^0$ are of the smallest degree. Then $(p-q)(T) = 0$. 又 $\deg(p-q) = m < \min\{\deg p_j\}_{j\in\Gamma}$.
Hence $p - q = 0$, for if not, $\exists\,!\, c \in \mathbf{F}, c(p-q) \in M_T^0$. Contradicts. $\square$

• (4E 5.31, 4E 5.B.25 and 26) *mini poly of restriction operator and mini poly of quotient operator*
Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invar subsp of V under T.
Let p be the mini poly of T.
(a) *Prove that p is a poly multi of the mini poly of $T|_U$.*
(b) *Prove that p is a poly multi of the mini poly of T/U.*
(c) *Prove that (mini poly of $T|_U$) $\times$ (mini poly of T/U) is a poly multi of p.*
(d) *Prove that the set of eigvals of T equals*
the union of the set of eigvals of $T|_U$ and the set of eigvals of T/U.

SOLUTION:

(a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46]. □

(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v+U) = p(T)v + U = 0$. □

(c) Suppose r is the mini poly of $T|_U$, s is the mini poly of T/U.
Because $\forall v \in V, s(T/U)(v+U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.
又 $\forall u \in U, r(T|_U)u = r(T)u = 0$.
Thus $\forall v \in V$ but $v \notin U, (rs)(T)v = r(s(T)v) = 0$.
And $\forall u \in U, (rs)(T)u = r(s(T)u) = 0$ (because $s(T)u = s(T|_U)u \in U$).
Hence $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0$. □

(d) By [8.49], immediately. □

• (4E 5.B.27) *Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, and $T \in \mathcal{L}(V)$.*
Prove that the mini poly p of T_C equals the mini poly q of T.

SOLUTION:

(a) $\forall u + \mathrm{i}0 \in V_C, p(T_C)(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p$ is a poly multi of q.

(b) $q(T) = 0 \Rightarrow \forall u + \mathrm{i}v \in V_C, q(T_C)(u+\mathrm{i}v) = q(T)u + \mathrm{i}q(T)v = 0 \Rightarrow q$ is a poly multi of p. □

• (4E 5.B.28) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that the mini poly p of $T' \in \mathcal{L}(V')$ equals the mini poly q of T.

SOLUTION:

(a) $\forall \varphi \in V', p(T')\varphi = \varphi \circ (p(T)) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \operatorname{null} \varphi \Rightarrow p(T) = 0, p$ is a poly multi of q.

(b) $q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ (q(T)) = q(T')\varphi = 0 \Rightarrow q(T) = 0, q$ is a poly multi of p. □

• (4E 5.32) *Suppose $T \in \mathcal{L}(V)$ and p is the mini poly.*
Prove that T is not inje $\Longleftrightarrow$ the const term of p is 0.

SOLUTION:

T is not inje $\Longleftrightarrow$ 0 is an eigval of $T \Longleftrightarrow$ 0 is a zero of $p \Longleftrightarrow$ the const term of p is 0. □

OR. Because $p(0) = (z-0)(z-\lambda_1)\cdots(z-\lambda_m) = 0 \Rightarrow T(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0$
又 p is the mini poly $\Rightarrow q$ define by $q(z) = (z-\lambda_1)\cdots(z-\lambda_m)$ is such that $q(T) \neq 0$.
Hence $0 = p(T) = Tq(T) \Rightarrow T$ is not inje.
Conversely, suppose $(T - 0I)$ is not inje, then 0 is a zero of p, so that the const term is 0. □

• (4E 5.B.22)
Suppose V is finite-dim, $T \in \mathcal{L}(V)$. Prove that T is inv $\Longleftrightarrow I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$.

SOLUTION: Denote the mini poly by p, where for all $z \in \mathbf{F}, p(z) = a_0 + a_1 z + \cdots + z^m$.
Notice that V is finite-dim. T is inv $\Longleftrightarrow T$ is inje $\Longleftrightarrow p(0) \neq 0$.

Hence $p(T) = 0 = a_0 I + a_1 T + \cdots + T^m$, where $a_0 \neq 0$ and $m \leqslant \dim V$. $\square$

---

6 *Suppose $T \in \mathcal{L}(V)$ and U is a subsp of V invar under T.*
Prove that U is invar under $p(T)$ for every poly $p \in \mathcal{P}(\mathbf{F})$.

SOLUTION:

$\forall u \in U, Tu \in U \Rightarrow Iu, Tu, T(Tu), \dots, T^m u \in U \Longrightarrow \forall a_k \in \mathbf{F}, (a_0 I + a_1 T + \cdots + a_m T^m)u \in U.$ $\square$

• (4E 5.B.10, 23) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$ and p is the mini poly with degree m. Suppose $v \in V$.*
(a) *Prove that* $\operatorname{span}(v, Tv, \dots, T^{m-1}v) = \operatorname{span}(v, Tv, \dots, T^{j-1}v)$ *for some $j \leqslant m$.*
(b) *Prove that* $\operatorname{span}(v, Tv, \dots, T^{m-1}v) = \operatorname{span}(v, Tv, \dots, T^{m-1}v, \dots, T^n v)$.

SOLUTION:

COMMENT: By NOTE FOR[8.40], j has an upper bound $m-1$, m has an upper bound $\dim V$.

Write $p(z) = a_0 + a_1 z + \cdots + z^m$ ($m \leqslant \dim V$). If $v = 0$, then we are done. Suppose $v \neq 0$.

(a) Suppose $j \in \mathbf{N}^+$ is the smallest such that $T^j v \in \operatorname{span}(v, Tv, \dots, T^{j-1}v) = U_0$. Then $j \leqslant m$.
Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1} T^{j-1} v$. And because $T(T^k v) = T^{k+1} \in U_0$. U_0 is invar under T.
By Problem (6), $\forall k \in \mathbf{N}$, $T^{j+k} v = T^k(T^j v) \in U_0$.
Thus $U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}v, \dots, T^n v)$ for all $n \geqslant j-1$. Let $n = m-1$ and we are done.

(b) Let $U = \operatorname{span}(v, Tv, \dots, T^{m-1}v)$.
By (a), $U = U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}, \dots, T^{m-1}, \dots, T^n)$ for all $n \geqslant m-1$. $\square$

• (4E 5.B.21) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that the mini poly p has degree at most $1 + \dim \operatorname{range} T$.
If $\dim \operatorname{range} T < \dim V - 1$, *then this result gives a better upper bound for the degree of mini poly.*

SOLUTION:

If T is inje, then range $T = V$ and we are done. Now choose $0 \neq v \in \operatorname{null} T$, then $Tv + 0 \cdot v = 0$.

1 is the smallest positive integer such that $T^1 v \in \operatorname{span}(v, \dots, T^0 v)$. Define q by $q(z) = z \Rightarrow q(T)v = 0$.

Let $W = \operatorname{range} q(T) = \operatorname{range} T$. $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ of smallest degree ($\deg s \leqslant \dim W$), $s(T|_W) = 0$.

Hence sq is the mini poly (see NOTE FOR[8.40]) and $\deg(sq) = \deg s + \deg q \leqslant \dim \operatorname{range} T + 1$. $\square$

---

19 *Suppose V is finite-dim,* $\dim V > 1, T \in \mathcal{L}(V)$. *Prove that* $\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V)$.

SOLUTION: If $\forall S \in \mathcal{L}(V), \exists p \in \mathcal{P}(\mathbf{F}), S = p(T)$. Then by [5.20], $\forall S_1, S_2 \in \mathcal{L}(V), S_1 S_2 = S_2 S_1$.
Note that dim $\geqslant 2$. By (3.A.14), $\exists S_1, S_2 \in \mathcal{L}(V), S_1 S_2 \neq S_2 S_1$. Contradicts. $\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.*
Prove that $\dim \mathcal{E}$ *equals the degree of the mini poly of T.*

SOLUTION:

Because the list $(I, T, \dots, T^{(\dim V)^2})$ of length $\dim \mathcal{L}(V) + 1$ is linely depe in $\dim \mathcal{L}(V)$.

Suppose $m \in \mathbf{N}^+$ is the smallest such that $T^m = a_0 I + \cdots + a_{m-1} T^{m-1}$.

Then q defined by $q(z) = z^m - a_{m-1} z^{m-1} - \cdots - a_0$ is the mini poly (see [8.40]).

For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k(T^m) \in \operatorname{span}(I, T, \dots, T^{m-1}) = U$.

Hence $\operatorname{span}(I, T, \dots, T^{(\dim V)^2}) = \operatorname{span}(I, T, \dots, T^{(\dim V)^2 - 1}) = U$.

Note that by the minimality of m, $(I, T, \dots, T^{m-1})$ is linely inde.

Thus $\dim U = m = \dim \operatorname{span}(I, T, \dots, T^{(\dim V)^2-1}) = \dim \operatorname{span}(I, T, \dots, T^n)$ for all $m < n \in \mathbf{N}^+$.

Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.

(a) Suppose $p(T) = 0$. 又 $\deg p \leqslant m-1 \Rightarrow p = 0$. Then φ is inje.

(b) $\forall S = a_0 I + a_1 T + \cdots + a_{m-1} T^{m-1} \in \mathcal{E}$, define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by
$p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} \Rightarrow \varphi(p) = S$. Then φ is surj.

Hence $\mathcal{E}$ and $\mathcal{P}_{m-1}(\mathbf{F})$ are iso. 又 $\dim \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. □

• (4E 5.B.13) *Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$ is defined by*

$$q(z) = a_0 + a_1 z + \cdots + a_n z^n, \text{ where } a_n \neq 0, \text{ for all } z \in \mathbf{F}.$$

Denote the mini poly of T by p defined by

$$p(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m \text{ for all } z \in \mathbf{F}.$$

Prove that $\exists\, ! \, r \in \mathcal{P}(\mathbf{F})$ such that $q(T) = r(T), \deg r < \deg p$.

Solution:

If $\deg q < \deg p$, then we are done.

If $\deg q = \deg p$, notice that $p(T) = 0 = c_0 I + c_1 T + \cdots + c_{m-1} T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0 I - c_1 T - \cdots - c_{m-1} T^{m-1},$$

define r by $r(z) = q(z) + \left[-a_m z^m + a_m(-c_0 - c_1 z - \cdots - c_{m-1} z^{m-1}) \right]$

$$= (a_0 - a_m c_0) + (a_1 - a_m c_1) z + \cdots + (a_{m-1} - a_m c_{m-1}) z^{m-1},$$

hence $r(T) = 0, \deg r < m$ and we are done.

Now suppose $\deg q \geqslant \deg p$. We use induction on $\deg q$.

(i) $\deg q = \deg p$, then the desired result is true, as shown above.

(ii) $\deg q > \deg p$, assume that the desired result is true for $\deg q = n$.

Suppose $f \in \mathcal{P}(\mathbf{F})$ such that $f(z) = b_0 + b_1 z + \cdots + b_n z^n + b_{n+1} z^{n+1}$.

Apply the assumption to g defined by $g(z) = b_0 + b_1 z + \cdots + b_n z^n$,

getting s defined by $s(z) = d_0 + d_1 z + \cdots + d_{m-1} z^{m-1}$.

Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1} T^{n+1} = s(T) + b_{n+1} T^{n+1}$.

Apply the assumption to t defined by $t(z) = z^n$,

getting δ defined by $\delta(z) = c_0{}' + c_1{}' z + \cdots + c_{m-1}{}' z^{m-1}$.

Thus $t(T) = T^n = c_0{}' + c_1{}' z + \cdots + c_{m-1}{}' z^{m-1} = \delta(T)$.

又 $\operatorname{span}(v, Tv, \dots, T^{m-1} v)$ is invar under T.

Hence $\exists\, ! \, k_j \in \mathbf{F}, T^{n+1} = T(T^n) = k_0 + k_1 z + \cdots + k_{m-1} z^{m-1}$.

And $f(T) = s(T) + b_{n+1}(k_0 + k_1 T + \cdots + k_{m-1} T^{m-1})$

$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1) z + \cdots + (d_{m-1} + k_{m-1}) z^{m-1} = h(T)$, thus defining h. □

• (4E 5.B.14) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has mini poly p*

$$\text{defined by } p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m, a_0 \neq 0.$$

Find the mini poly of T^{-1}.

Solution:

Notice that V is finite-dim. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is inv.

Then $p(T) = a_0 I + a_1 T + \cdots + T^m = 0$. Apply T^{-m} to both sides,

$a_0 (T^{-1})^m + a_1 (T^{-1})^{m-1} + \cdots + a_{m-1} T^{-1} + I = 0$.

Define q by $q(z) = z^m + \dfrac{a_1}{a_0} z^{m-1} + \cdots + \dfrac{a_{m-1}}{a_0} z + \dfrac{1}{a_0}$ for all $z \in \mathbf{F}$.

We now show that $(T^{-1})^k \notin \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$

for every $k \in \{1, \dots, m-1\}$ by contradiction, so that q is exactly the mini poly of T^{-1}.

Suppose $(T^{-1})^k \in \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$.

Then let $(T^{-1})^k = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. Apply T^k to both sides,

getting $I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T$, hence $T^k \in \operatorname{span}(I, T, \dots, T^{k-1})$.

Thus f defined by $f(z) = z^k + \frac{b_1}{b_0} z^{k-1} + \cdots + \frac{b_{k-1}}{b_0} z - \frac{1}{b_0}$ is a poly multi of p.

While $\deg f < \deg p$. Contradicts. □

• **Note For [8.49]:**

Suppose V is a finite-dim complex vecsp and $T \in \mathcal{L}(V)$.

By [4.14], the mini poly has the form $(z - \lambda_1) \cdots (z - \lambda_m)$,

*where $\lambda_1, \dots, \lambda_m$ are all the eigvals of T, **possibly with repetitions**.*

• **Comment:**

A nonzero poly has at most as many distinct zeros as its degree (see [4.12]).

Thus by the upper bound for the deg of mini poly given in Note For*[8.40], and by [8.49,]*

we can ***give an alternative proof of* [5.13]**.

• Notice (See also 4E 5.B.20,24)

Suppose $\alpha_1, \dots, \alpha_n$ are all the distinct eigvals of T,

and therefore are all the distinct zeros of the mini poly.

Also, the mini poly of T is a poly multi of, but not equal to, $(z - \alpha_1) \cdots (z - \alpha_n)$.

If we define q by $q(z) = (z - \alpha_1)^{\dim V - (n-1)} \cdots (z - \alpha_n)^{\dim V - (n-1)}$,

then q is a poly multi of the char poly (see [8.34] and [8.26])

(Because $\dim V > n$ and $n - 1 > 0$, $n\big[\dim V - (n-1)\big] > \dim V$.)

The char poly has the form $(z - \alpha_1)^{\gamma_1} \cdots (z - \alpha_n)^{\gamma_n}$, where $\gamma_1 + \cdots + \gamma_n = \dim V$.

The mini poly has the form $(z - \alpha_1)^{\delta_1} \cdots (z - \alpha_n)^{\delta_n}$, where $0 \leqslant \delta_1 + \cdots + \delta_n \leqslant \dim V$.

---

10 *Suppose $T \in \mathcal{L}(V)$, λ is an eigval of T with an eigvec v.*

Prove that for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.

Solution:

Suppose p is defined by $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbf{F}$. Because for any $n \in \mathbf{N}^+$, $T^n v = \lambda^n v$.

Thus $p(T)v = a_0 v + a_1 T v + \cdots + a_m T^m v = a_0 v + a_1 \lambda v + \cdots + a_m \lambda^m v = p(\lambda)v$. □

Comment: For any $p \in \mathcal{P}(\mathbf{F})$ such that $p(z) = (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_m)^{\alpha_m}$, the result is true as well.

Now we prove that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v = (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v$.

Define q_i by $q_i(z) = (z - \lambda_i)^{\alpha_i}$ for all $z \in \mathbf{F}$.

Because $(a + b)^n = a^n + C_n^1 a^{n-1} b + \cdots + C_n^k a^{n-k} b^k + \cdots + C_n^n b^n$.

Let $a = z, b = \lambda_i, n = \alpha_i$, so we can write $q_i(z)$ in the form $a_0 + a_1 z + \cdots + a_m z^m$.

Hence $q_i(T)v = q_i(\lambda)v \Rightarrow (T - \lambda_i I)^{\alpha_i} v = (\lambda - \lambda_i)^{\alpha_i} v$.

Then for each $k \in \{2, \dots, m\}$, $(T - \lambda_{k-1} I)^{\alpha_{k-1}} (T - \lambda_k I)^{\alpha_k} v$

$$
\begin{aligned}
&= q_{k-1}(T)\big(q_k(T)v\big) \\
&= q_{k-1}(T)\big(q_k(\lambda)v\big) \\
&= q_{k-1}(\lambda)\big(q_k(\lambda)v\big) \\
&= (\lambda - \lambda_{k-1})^{\alpha_{k-1}} (\lambda - \lambda_k)^{\alpha_k} v.
\end{aligned}
$$

So that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v$

$$
\begin{aligned}
&= q_1(T)\Big(q_2(T)\Big(\dots(q_m(T)v)\dots\Big)\Big)\\
&= q_1(\lambda)(q_2(\lambda)(\dots(q_m(\lambda)v)\dots))\\
&= (\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_m)^{\alpha_m}v.
\end{aligned}
$$

□

1 *Suppose $T \in \mathcal{L}(V)$ and $\exists\, n \in \mathbf{N}^+$ such that $T^n = 0$.*
Prove that $(I - T)$ is inv and $(I - T)^{-1} = I + T + \cdots + T^{n-1}$.

Solution: Note that $1 - x^n = (1 - x)(1 + x + \cdots + x^{n-1})$.

$$\left.\begin{array}{l}(I - T)(1 + T + \cdots + T^{n-1}) = I - T^n = I \\ (1 + T + \cdots + T^{n-1})(I - T) = I - T^n = I\end{array}\right\} \Rightarrow (I - T)^{-1} = 1 + T + \cdots + T^{n-1}.$$ □

---

2 *Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$.*
Suppose λ is an eigval of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Solution:

Suppose v is an eigvec correspd to λ. Then for any $p \in \mathcal{P}(\mathbf{F}), p(T)v = p(\lambda)v$.
Hence $0 = (T - 2I)(T - 3I)(T - 4I)v = (\lambda - 2)(\lambda - 3)(\lambda - 4)v$ while $v \neq 0 \Rightarrow \lambda = 2, 3$ or 4. □

Comment: Note that $(T - 2I)(T - 3I)(T - 4I) = 0$ is not inje, so that $2, 3, 4$ are eigvals of T.
But it doesn't mean that all the eigvals of T are exactly $2, 3, 4$.

---

7 [See 5.A.22] *Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigval of $T^2 \iff 3$ or -3 is an eigval of T.*

Solution:

(a) Suppose λ is an eigval of T with an eigvec v.
Then $(T - 3I)(T + 3I)v = (\lambda - 3)(\lambda + 3)v = 0 \Rightarrow \lambda = \pm 3$.

(b) Suppose 3 or -3 is an eigval of T with an eigvec v. Then $Tv = \pm 3v \Rightarrow T^2 v = T(Tv) = 9v$ □

Or. 9 is an eigval of $T^2 \iff (T^2 - 9I) = (T - 3I)(T + 3I)$ is not inje $\iff \pm 3$ is an eigval. □

---

3 *Suppose $T \in \mathcal{L}(V), T^2 = I$ and -1 is not an eigval of T. Prove that $T = I$.*

Solution:

$T^2 - I = (T + I)(T - I)$ is not inje, 又 -1 is not an eigval of $T \Longrightarrow$ By Tips . □

Or. Note that $\forall v \in V, v = [\frac{1}{2}(I - T)v] + [\frac{1}{2}(I + T)v]$.

$$\left.\begin{array}{l}(I + T)\big((I - T)v\big) = 0 \Longrightarrow (I - T)v \in \operatorname{null}(I + T) \\ (I - T)\big((I + T)v\big) = 0 \Longrightarrow (I + T)v \in \operatorname{null}(I - T)\end{array}\right\} \Rightarrow V = \operatorname{null}(I + T) + \operatorname{null}(I - T).$$

又 -1 is not an eigval of $T \iff (I + T)$ is inje $\iff \operatorname{null}(I + T) = \{0\}$.
Hence $V = \operatorname{null}(I - T) \Rightarrow \operatorname{range}(I - T) = \{0\}$. Thus $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$. □

• (4E 5.A.32) *Suppose $T \in \mathcal{L}(V)$ has no eigvals and $T^4 = I$. Prove that $T^2 = -I$.*

Solution:

Because $T^4 - I = (T^2 - I)(T^2 + I) = 0$ is not inje $\Rightarrow (T^2 - I)$ or $(T^2 + I)$ is not inje.
又 T has no eigvals $\Rightarrow (T^2 - I) = (T - I)(T + I)$ is inje. Hence $T^2 + I = 0 \in \mathcal{L}(V)$, for if not,
$\exists\, v \in V, (T^2 + I)v \neq 0$ while $(T^2 - I)\big((T^2 + I)v\big) = 0$ but $(T^2 - I)$ is inje. Contradicts.
Or. $\forall v \in V, 0 = (T^2 - I)(T^2 + I)v \iff 0 = (T^2 + I)v$. Hence $T^2 + I = 0$. □

Or. Note that $\forall v \in V, v = \big[\frac{1}{2}(I - T^2)v\big] + \big[\frac{1}{2}(I + T^2)v\big]$.

$$\left.\begin{array}{l}(I + T^2)\big((I - T^2)v\big) = 0 \Longrightarrow (I - T^2)v \in \operatorname{null}(I + T^2) \\ (I - T^2)\big((I + T^2)v\big) = 0 \Longrightarrow (I + T^2)v \in \operatorname{null}(I - T^2)\end{array}\right\} \Rightarrow V = \operatorname{null}(I + T^2) + \operatorname{null}(I - T^2).$$

又 T has no eigvals $\iff (I - T^2)$ is inje $\iff \operatorname{null}(I - T^2) = \{0\}$.
Hence $V = \operatorname{null}(I + T^2) \Rightarrow \operatorname{range}(I + T^2) = \{0\}$. Thus $I + T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$. □

---

8 [OR (4E 5.A.31)] *Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$.*

SOLUTION:

Define $\mathrm{i} \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y) = (-y, x)$. Just like $\mathrm{i} : \mathbf{C} \to \mathbf{C}$ defined by $\mathrm{i}(x + \mathrm{i}y) = -y + \mathrm{i}x$.

Define $\mathrm{i}^n \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y) = \left(\operatorname{Re}\left(\mathrm{i}^n x + \mathrm{i}^{n+1} y\right), \operatorname{Im}\left(\mathrm{i}^n x + \mathrm{i}^{n+1} y\right)\right)$.

$T^4 + I = (T^2 + \mathrm{i}I)(T^2 - \mathrm{i}I) = (T + \mathrm{i}^{1/2} I)(T - \mathrm{i}^{1/2} I)(T - (-\mathrm{i})^{1/2} I)(T + (-\mathrm{i})^{1/2} I)$.

Note that $\mathrm{i}^{1/2} = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}$, $(-\mathrm{i})^{1/2} = \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}$. Hence $T = \pm(\pm\mathrm{i})^{1/2} I$.

Let $T = \mathrm{i}^{1/2} I$ defined by $\mathrm{i}^{1/2}(x,y) = \left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)$. □

OR. Because $\mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix}$. Using $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}$.

We define $T \in \mathcal{L}(\mathbf{R}^2)$ such that $\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix}$. □

• (4E 5.B.12) *Find the mini poly of T defined in (5.A.10).*

SOLUTION: By (5.A.9) and [8.40, 8.49], $1, 2, \dots, n$ are all the zeros of the mini poly of T. □

• (4E 5.B.3) *Find the mini poly of T defined in (5.A.19).*

SOLUTION:

If $n = 1$ then 1 is the only eigval of T, and $(z-1)$ is the mini poly.

Because n and 0 are all the eigvals of T, 又 $\forall k \in \{1, \dots, n\}, Te_k = e_1 + \cdots + e_n$; $T^2 e_k = n(e_1 + \cdots + e_n)$.

Hence $T^2 e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T-n) = 0$. Thus $\left(z(z-n)\right)$ is the mini poly. □

• (4E 5.B.8) *Find the mini poly of T. Where $T \in \mathcal{L}(\mathbf{R}^2)$ is the operator of counterclockwise rotation by θ, where $\theta \in \mathbf{R}^+$.*

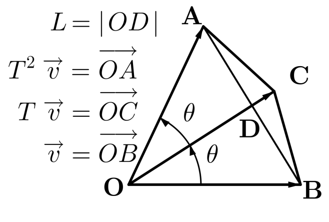
SOLUTION:

If $\theta = \pi + 2k\pi$, then $T(w,z) = (-w, -z), T^2 = I$ and the mini poly is $z + 1$.

If $\theta = 2k\pi$, then $T = I$ and the mini poly is $z - 1$.

Otherwise (v, Tv) is linely inde. Then $\operatorname{span}(v, Tv) = \mathbf{R}^2$. Note that $\nexists b \in \mathbf{F}, T - bI = 0$.

Thus suppose the mini poly p is defined by $p(z) = z^2 + bz + c$ for all $z \in \mathbf{R}$.

Because 

$$Tv = \frac{|\vec{v}|}{2L}(T^2 v + v) \Rightarrow T = \frac{|\vec{v}|}{2L}(T^2 + I)$$

$$L = |\vec{v}|\cos\theta \Rightarrow \frac{|\vec{v}|}{2L} = \frac{1}{2\cos\theta}$$

Hence $p(T) = T^2 - 2\cos\theta\, T + I = 0$ and $z^2 - 2\cos\theta\, z + 1$ is the mini poly of T. □

OR. Let (e_1, e_2) be the standard basis of $\mathbf{R}^2$. We use the pattern shown in [8.44].

Because $Te_1 = \cos\theta\, e_1 + \sin\theta\, e_2$, $T^2 e_1 = \cos 2\theta\, e_1 + \sin 2\theta\, e_2$.

Thus $ce_1 + bTe_1 = -T^2 e_1 \iff \begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix}\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$. Now $\det = \sin\theta \neq 0, c = 1, b = 2\cos\theta$. □

OR. $\mathcal{M}(T, (e_1, e_2)) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. By (4E 5.B.11), the mini poly is $(z \pm 1)$ or $(z^2 - 2\cos\theta\, z + 1)$. □

• (4E 5.B.11) *Suppose V is a two-dim vecsp, $T \in \mathcal{L}(V)$, and the matrix of T with resp to some basis of V is* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) *Show that $T^2 - (a+d)T + (ad-bc)I = 0$.*

(b) *Show that the mini poly of T equals*

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad-bc) & \text{otherwise.} \end{cases}$$

SOLUTION:

(a) Suppose the basis is (v, w). Because $\begin{cases} Tv = av + bw \Rightarrow (T - aI)v = bw, \text{ then apply } (T - dI) \text{ to both sid} \\ Tw = cv + dw \Rightarrow (T - dI)w = cv, \text{ then apply } (T - aI) \text{ to both sic} \end{cases}$

Hence $(T - aI)(T - dI) = bcI \Rightarrow T^2 - (a+d)T + (ad-bc)I = 0$.

(b) If $b = c = 0$ and $a = d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the mini poly is $z - a$.

Otherwise, by (a), $z^2 - (a+d)z + (ad-bc)$ is a poly multi of the mini poly.

Now we prove that $T \notin \operatorname{span}(I)$, so that then the mini poly of T has exactly degree 2.

(At least one of the assumption of (I),(II) below is true.)

(I) Suppose $a = d$, then $Tv = av + bw \notin \operatorname{span}(v)$, $Tw = cv + aw \notin \operatorname{span}(w)$.

(II) Suppose at most one of b, c is not 0. If $b = 0$, then $Tw \notin \operatorname{span}(w)$; If $c = 0$, then $Tv \notin \operatorname{span}(v)$. □

• *Suppose $S, T \in \mathcal{L}(V)$, S is inv, and $p \in \mathcal{P}(\mathbf{F})$. Prove that $Sp(TS) = p(ST)S$.*

SOLUTION:

We prove $S(TS)^m = (ST)^m S$ for each $m \in \mathbf{N}$ by induction.

(i) If $m = 0, 1$. Then $S(TS)^0 = I = (ST)^0 S$; $S(TS)^1 = (ST)S$.

(ii) If $m > 1$. Assume that $S(TS)^m = (ST)^m S$.

Then $S(TS)^{m+1} = S(TS)^m(TS) = (ST)^m STS = (ST)^{m+1}S$.

Hence $\forall p \in \mathcal{P}(\mathbf{F})$, $Sp(TS) = \sum_{k=1}^{m} a_k S(TS)^k = \sum_{k=1}^{m} a_k p(ST)^k S = \left[\sum_{k=1}^{m} a_k (TS)^k\right] S$. □

COMMENT: $p(TS) = S^{-1}p(ST)S$, $p(ST) = Sp(TS)S^{-1}$.

COROLLARY: **5** Because S is inv, $T \in \mathcal{L}(V)$ is arbitrary $\Longleftrightarrow R = ST$ is arbitrary.

Hence $\forall R \in \mathcal{L}(V)$, inv $S \in \mathcal{L}(V)$, $p(S^{-1}RS) = S^{-1}p(R)S$.

• (4E 5.B.7) *Suppose $S, T \in \mathcal{L}(V)$. Let p, q be the mini polys of ST, TS respectively.*

(a) *If $V = \mathbf{F}^2$. Give an example such that $p \neq q$;* (b) *If S or T is inv. Prove that $p = q$.*

SOLUTION:

(a) Define S by $S(x, y) = (x, x)$. Define T by $T(x, y) = (0, y)$.

Then $ST(x, y) = 0$, $TS(x, y) = (0, x)$ for all $(x, y) \in \mathbf{F}^2$. Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.

Hence the mini poly of ST does not equal to the mini poly of TS.

(b) Suppose S is inv. Because p, q are monic.

$$\left.\begin{array}{l} p(ST) = 0 = Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a poly multi of } q \\ q(TS) = 0 = S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a poly multi of } p \end{array}\right\} \Rightarrow p = q.$$

Reversing the roles of S and T, we conclude that if T is inv, then $p = q$ as well. □

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11 *Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$, and $\alpha \in \mathbf{C}$.*

Prove that α is an eigval of $p(T) \Longleftrightarrow \alpha = p(\lambda)$ for some eigval λ of T.

SOLUTION:

(a) Suppose α is an eigval of $p(T) \Leftrightarrow \big(p(T) - \alpha I\big)$ is not inje.

Write $p(z)-\alpha = c(z-\lambda_1)\cdots(z-\lambda_m) \Rightarrow p(T)-\alpha I = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$.
By TIPS , $\exists\,(T-\lambda_j I)$ not inje. Thus $p(\lambda_j)-\alpha=0$.

(b) Suppose $\alpha = p(\lambda)$ and λ is an eigval of T with an eigvec v. Then $p(T)v = p(\lambda)v = \alpha v$. $\square$

OR. Define q by $q(z)=p(z)-\alpha$. λ is a zero of q.
Because $q(T)v = \big(p(T)-\alpha I\big)v = q(\lambda)v = \big(p(\lambda)-\alpha\big)v = 0$.
Hence $q(T)$ is not inje $\Rightarrow \big(p(T)-\alpha I\big)$ is not inje. $\square$

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12 [OR (4E.5.B.6)] *Give an example of an operator on* $\mathbf{R}^2$ *that shows the result above does not hold if* $\mathbf{C}$ *is replaced with* $\mathbf{R}$.

SOLUTION:

Define $T\in\mathcal{L}(\mathbf{R}^2)$ by $T(w,z)=(-z,w)$.
By Problem (4E 5.B.11), $\mathcal{M}\big(T,((1,0),(0,1))\big)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\Rightarrow$ the mini poly of T is z^2+1.
Define p by $p(z)=z^2$. Then $p(T)=T^2=-I$. Thus $p(T)$ has eigval -1.
While $\nexists\,\lambda\in\mathbf{R}$ such that $-1=p(\lambda)=\lambda^2$. $\square$

• (4E 5.B.17) *Suppose V is finite-dim, $T\in\mathcal{L}(V)$, $\lambda\in\mathbf{F}$, and p is the mini poly of T. Show that the mini poly of $(T-\lambda I)$ is the poly q defined by $q(z)=p(z+\lambda)$.*

SOLUTION:

$q(T-\lambda I)=0\Rightarrow q$ is poly multi of the mini poly of $(T-\lambda I)$.
Suppose the degree of the mini poly of $(T-\lambda I)$ is n, and the degree of the mini poly of T is m.
By definition of mini poly,
n is the smallest such that $(T-\lambda I)^n\in\operatorname{span}\big(I,(T-\lambda I),\dots,(T-\lambda I)^{n-1}\big)$;
m is the smallest such that $T^m\in\operatorname{span}(I,T,\dots,T^{m-1})$.
又 $T^k\in\operatorname{span}(I,T,\dots,T^{k-1})\iff(T-\lambda)^k\in\operatorname{span}\big(I,(T-\lambda I),\dots,(T-\lambda I)^{k-1}\big)$.
Thus $n=m$. 又 q is monic. By the uniqnes of mini poly. $\square$

• (4E 5.B.18) *Suppose V is finite-dim, $T\in\mathcal{L}(V)$, $\lambda\in\mathbf{F}\backslash\{0\}$, and p is the mini poly of T. Show that the mini poly of λT is the poly q defined by $q(z)=\lambda^{\deg p}p(\frac{z}{\lambda})$.*

SOLUTION:

$q(\lambda T)=\lambda^{\deg p}p(T)=0\Rightarrow q$ is a poly multi of the mini poly of λT.
Suppose the degree of the mini poly of λT is n, and the degree of the mini poly of T is m.
By definition of mini poly,
n is the smallest such that $(\lambda T)^n\in\operatorname{span}\big(\lambda I,\lambda T,\dots,(\lambda T)^{n-1}\big)$;
m is the smallest such that $T^m\in\operatorname{span}(I,T,\dots,T^{m-1})$.
又 $(\lambda T)^k\in\operatorname{span}\big(\lambda I,\lambda T,\dots,(\lambda T)^{k-1}\big)\iff T^k\in\operatorname{span}(I,T\dots,T^{k-1})$.
Thus $n=m$. 又 q is monic. By the uniqnes of mini poly. $\square$

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18 [OR (4E 5.B.15)] *Suppose V is a finite-dim complex vecsp with* $\dim V>0$ *and* $T\in\mathcal{L}(V)$.
Define $f:\mathbf{C}\to\mathbf{R}$ by $f(\lambda)=\dim\operatorname{range}(T-\lambda I)$.
Prove that f is not a continuous function.

SOLUTION: Note that V is finite-dim.

Let λ_0 be an eigval of T. Then $(T-\lambda_0 I)$ is not surj. Hence $\dim\operatorname{range}(T-\lambda_0 I)<\dim V$.
Because T has finitely many eigvals. There exist a sequence of number $\{\lambda_n\}$ such that $\lim\limits_{n\to\infty}\lambda_n=\lambda_0$.

And λ_n is not an eigval of T for each $n \Rightarrow \dim \operatorname{range}(T-\lambda_n I) = \dim V \neq \dim \operatorname{range}(T-\lambda_0 I)$.
Thus $f(\lambda_0) \neq \lim_{n\to\infty} f(\lambda_n)$. □

- (4E 5.B.9) *Suppose $T \in \mathcal{L}(V)$ is such that with resp to some basis of V, all entries of the matrix of T are rational numbers. Explain why all coefficients of the mini poly of T are rational numbers.*

Solution:

Let $(v_1,\dots,v_n)$ denote the basis such that $\mathcal{M}(T,(v_1,\dots,v_n))_{j,k} = A_{j,k} \in \mathbf{Q}$ for all $j,k = 1,\dots,n$.
Denote $\mathcal{M}(v_j,(v_1,\dots,v_n))$ by x_j for each v_j.
Suppose p is the mini poly of T and $p(z) = z^m + \cdots + c_1 z + c_0$. Now we show that each $c_j \in \mathbf{Q}$.
Note that $\forall s \in \mathbf{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbf{Q}^{n,n}$ and $T^s v_k = A^s_{1,k}v_1 + \cdots + A^s_{n,k}v_n$ for all $k \in \{1,\dots,n\}$.

$$\text{Thus } \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \cdots + c_1 A + c_0 I)x_1 = \sum_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,1} x_j = 0; \\ \qquad\vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \cdots + c_1 A + c_0 I)x_n = \sum_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly, } \begin{cases} (A^m + \cdots + c_1 A + c_0 I)_{1,1} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,1} = 0; \\ \qquad\vdots \quad \ddots \quad \vdots \\ (A^m + \cdots + c_1 A + c_0 I)_{1,n} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,n} = 0; \end{cases}$$

Hence we get a system of n^2 linear equations in m unknowns $c_0, c_1, \dots, c_{m-1}$.
We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbf{Q}$. □

- [Or (4E 5.B.16), Or (8.C.18)] *Suppose $a_0,\dots,a_{n-1} \in \mathbf{F}$. Let T be the operator on $\mathbf{F}^n$ such that*

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ with resp to the standard basis } (e_1,\dots,e_n).$$

Show that the mini poly of T is p defined by $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$.

*$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the mini poly of some operator. Hence a formula or an algorithm that could produce exact eigvals for each operator on each $\mathbf{F}^n$ could then produce exact zeros for each poly [by 8.36(b)]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an operator.*

Solution: Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ is linely inde. 又 The deg of mini poly is at most n.
$T^n e_1 = \cdots = T^{n-k}e_{1+k} = \cdots = Te_n = -a_0 e_1 - a_1 e_2 - a_2 e_3 - \cdots - a_{n-1}e_n$
$= (-a_0 I - a_1 T - a_2 T^2 - \cdots - a_{n-1}T^{n-1})e_1$. Thus $p(T)e_1 = 0 = p(T)e_j$ for each $e_j = T^{j-1}e_1$. □

- Eigenvalues On Odd-Dimensional Real Vector Spaces
- Even-Dimensional Null Space

Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$. Prove that $\dim \operatorname{null}(T^2 + bT + cI)$ is an even number.

Solution:

Denote $\operatorname{null}(T^2 + bT + cI)$ by R. Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.
Suppose λ is an eigval of T_R with an eigvec $v \in R$.
Then $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \left((\lambda + b)^2 + c - \frac{b^2}{4}\right)v$.

Because $c - \frac{b^2}{4} > 0$ and we have $v = 0$. Thus T_R has no eigvals.

Let U be an invar subsp of R that has the largest, even dim among all invar subsps.

Assume that $U \neq R$. Then $\exists\, w \in R$ but $w \notin U$. Let W be such that $(w, T|_R\, w)$ is a basis of W.

Because $T|_R^2\, w = -bT|_R\, w - cw \in W$. Hence W is an invar subsp of dim 2.

Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,

for if not, because $w \notin U, T|_R\, w \in U$,

$U \cap W$ is invar under $T|_R$ of one dim (impossible because $T|_R$ has no eigvecs).

Hence $U + W$ is even-dim invar subsp under $T|_R$, contradicting the maximality of $\dim U$.

Thus the assumption was incorrect. Hence $R = \operatorname{null}(T^2 + bT + cI) = U$ has even dim. ☐

• Operators On Odd-Dimensional Vector Spaces Have Eigenvalues

(a) *Suppose* $\mathbf{F} = \mathbf{C}$. Then by [5.21], we are done.

(b) *Suppose* $\mathbf{F} = \mathbf{R}$, *V is finite-dim, and* $\dim V = n$ *is an odd number.*
Let $T \in \mathcal{L}(V)$ and the mini poly is p. Prove that T has an eigval.

Solution:

(i) If $n = 1$, then we are done.

(ii) Suppose $n \geqslant 3$. Assume that every operator, on odd-dim vecsps of dim less than n, has an eigval.

If p is a poly multi of $(x - \lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] λ is an eigval of T and we are done.

Now suppose $b, c \in \mathbf{R}$ such that $b^2 < 4c$ and p is a poly multi of $x^2 + bx + c$ (see [4.17]).

Then $\exists\, q \in \mathcal{P}(\mathbf{R})$ such that $p(x) = q(x)(x^2 + bx + c)$ for all $x \in \mathbf{R}$.

Now $0 = p(T) = \big(q(T)\big)(T^2 + bT + cI)$, which means that $q(T)|_{\operatorname{range}(T^2+bT+cI)} = 0$.

Because $\deg q < \deg p$ and p is the mini poly of T, hence $\operatorname{range}(T^2 + bT + cI) \neq V$.

又 $\dim V$ is odd and $\dim \operatorname{null}(T^2 + bT + cI)$ is even (by our previous result).

Thus $\dim V - \dim \operatorname{null}(T^2 + bT + cI) = \dim \operatorname{range}(T^2 + bT + cI)$ is odd.

By [5.18], $\operatorname{range}(T^2 + bT + cI)$ is an invar subsp of V under T that has odd dim less than n.

Our induction hypothesis now implies that $T|_{\operatorname{range}(T^2+bT+cI)}$ has an eigval.

By mathematical induction. ☐

• (2E Ch5.24) *Suppose* $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$ *has no eigvals.*
Prove that every invar subsp of V under T is even-dim.

Solution:

Suppose U is such a subsp. Then $T|_U \in \mathcal{L}(U)$. We prove by contradiction.

If $\dim U$ is odd, then $T|_U$ has an eigval and so is T, so that $\exists$ invar subsp of 1 dim, contradicts. ☐

• (4E 5.B.29) *Show that every operator on a finite-dim vecsp of dim* ≥ 2 *has a* 2*-dim invar subsp.*

Solution:

Using induction on $\dim V$.

(i) $\dim V = 2$, we are done.

(ii) $\dim V > 2$. Assume that the desired result is true for vecsp of smaller dim.

Suppose p is the mini poly of degree m and $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.

If $T = \lambda I$ ($\Leftrightarrow m = 1 \vee m = -\infty$), then we are done. ($m \neq 0$ because $\dim V \neq 0$.)

Now define a q by $q(z) = (z - \lambda_1)(z - \lambda_2)$.

By assumption, $T|_{\operatorname{null} q(T)}$ has an invar subsp of dim 2. ☐

5.B: II

9 14 15 20 | 4E: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 11, 12, 13, 14

• (4E 5.C.1) *Prove or give a counterexample*:
If $T \in \mathcal{L}(V)$ and T^2 has an upper-trig matrix, then T has an upper-trig matrix.

SOLUTION:

• (4E 5.C.2) *Suppose A and B are upper-trig matrices of the same size, with $\alpha_1, \dots, \alpha_n$ on the diag of A and $\beta_1, \dots, \beta_n$ on the diag of B.*
(a) *Show that $A + B$ is an upper-trig matrix with $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ on the diag.*
(b) *Show that AB is an upper-trig matrix with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diag.*

SOLUTION:

• (4E 5.C.3)
Suppose $T \in \mathcal{L}(V)$ is inv and $B = (v_1, \dots, v_n)$ is a basis of V such that $\mathcal{M}(T, B) = A$ is upper trig, with $\lambda_1, \dots, \lambda_n$ on the diag.
Show that the matrix of $\mathcal{M}(T^{-1}, B) = A^{-1}$ is also upper trig, with $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ on the diag.

SOLUTION:

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9 [4E 5.C.7] *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $v \in V$.*
(a) *Prove that $\exists\,!$ monic poly p_v of smallest degree such that $p_v(T)v = 0$.*
(b) *Prove that the mini poly of T is a poly multi of p_v.*

SOLUTION:

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14 [OR (4E 5.C.4)] *Give an operator T such that with resp to some basis, $\mathcal{M}(T)_{k,k} = 0$ for each k, while T is inv.*

SOLUTION:

---

15 [OR (4E 5.C.5)] *Give an operator T such that with resp to some basis, $\mathcal{M}(T)_{k,k} \neq 0$ for each k, while T is not inv.*

SOLUTION:

---

20 [OR (OR 4E 5.C.6)]
Suppose $\mathbf{F} = \mathbf{C}$, V is finite-dim, and $T \in \mathcal{L}(V)$.
Prove that if $k \in \{1, \dots, \dim V\}$, then V has a k dim subsp invar under T.

SOLUTION:

• (4E 5.C.8) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\exists\, v \in V \setminus \{0\}$ such that $T^2v + 2Tv = -2v$.*
(a) *Prove that if $\mathbf{F} = \mathbf{R}$, then $\nexists$ a basis of V with resp to which T has an upper-trig matrix.*
(b) *Prove that if $\mathbf{F} = \mathbf{C}$ and A is an upper-trig matrix that equals the matrix of T with resp to some basis of V, then $-1 + \mathrm{i}$ or $-1 - \mathrm{i}$ appears on the diag of A.*

SOLUTION:

• (4E 5.C.9) *Suppose $B \in \mathbf{F}^{n,n}$ with complex entries.*

Prove that $\exists$ inv $A \in \mathbf{F}^{n,n}$ with complex entries such that $A^{-1}BA$ is an upper-trig matrix.

SOLUTION:

- (4E 5.C.10) *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \dots, v_n)$ is a basis of V.*
  Show that the following are equi.
  (a) *The matrix of T with resp to $(v_1, \dots, v_n)$ is lower trig.*
  (b) $\operatorname{span}(v_k, \dots, v_n)$ *is invar under T for each $k = 1, \dots, n$.*
  (c) $Tv_k \in \operatorname{span}(v_k, \dots, v_n)$ *for each $k = 1, \dots, n$.*

SOLUTION:

- (4E 5.C.11) *Suppose $\mathbf{F} = \mathbf{C}$ and V is finite-dim.*
  Prove that if $T \in \mathcal{L}(V)$, then T has a lower-trig matrix with resp to some basis.

SOLUTION:

- (4E 5.C.12)
  Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has an upper-trig matrix with resp to some basis, and U is a subsp of V that is invar under T.
  (a) *Prove that $T|_U$ has an upper-trig matrix with resp to some basis of U.*
  (b) *Prove that T/U has an upper-trig matrix with resp to some basis of V/U.*

SOLUTION:

- (4E 5.C.13) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$. Suppose U is an invar subsp of V under T such that $T|_U$ has an upper-trig matrix and also T/U has an upper-trig matrix.*
  Prove that T has an upper-trig matrix.

SOLUTION:

- (4E 5.C.14) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
  Prove that T has an upper-trig matrix $\Longleftrightarrow$ T' has an upper-trig matrix.

SOLUTION:

ENDED

5.C

XXXX

ENDED

5.E* (4E)

1 2 3 4 5 6 7 8 9 10

1 *Give an example of two commuting operators $S, T \in \mathbf{F}^4$ such that there is an invar subsp of $\mathbf{F}^4$ under S but not under T and an invar subsp of $\mathbf{F}^4$ under T but not under S.*

SOLUTION:

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2 *Suppose $\mathcal{E}$ is a subset of $\mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagable.*
Prove that $\exists$ a basis of V with resp to which

every element of $\mathcal{E}$ has a diag matrix $\Longleftrightarrow$ every pair of elements of $\mathcal{E}$ commutes.

This exercise extends [5.76], which considers the case in which $\mathcal{E}$ contains only two elements.

For this exercise, $\mathcal{E}$ may contain any number of elements, and $\mathcal{E}$ may even be an infinite set.

SOLUTION:

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3 *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.*

(a) *Prove that* $\operatorname{null} p(S)$ *is invar under T.*

(b) *Prove that* $\operatorname{range} p(S)$ *is invar under T.*

See NOTE FOR[5.17] *for the special case $S = T$.*

SOLUTION:

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4 *Prove or give a counterexample*:

A diag matrix A and an upper-trig matrix B of the same size commute.

SOLUTION:

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5 *Prove that a pair of operators on a finite-dim vecsp commute $\Longleftrightarrow$ their dual operators commute.*

SOLUTION:

---

6 *Suppose V is a finite-dim complex vecsp and $S, T \in \mathcal{L}(V)$ commute.*

Prove that $\exists\, \alpha, \lambda \in \mathbf{C}$ such that $\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V$.

SOLUTION:

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7 *Suppose V is a complex vecsp, $S \in \mathcal{L}(V)$ is diagable, and T commutes with S.*

Prove that $\exists$ basis B of V such that S has a diag matrix with resp to B

and T has an upper-trig matrix with resp to B.

SOLUTION:

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8 *Suppose $m = 3$ in Example [5.72]*

and D_x, D_y are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbf{R}^2)$ from that example.

Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ with resp to which D_x and D_y each have an upper-trig matrix.

SOLUTION:

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9 *Suppose V is a finite-dim nonzero complex vecsp.*

Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.

(a) *Prove that $\exists\, v \in V$ is an eigvec for every element of $\mathcal{E}$.*

(b) *Prove that $\exists$ a basis of V with resp to which every element of $\mathcal{E}$ has an upper-trig matrix.*

SOLUTION:

---

10 *Give an example of two commuting operators S, T on a finite-dim real vecsp such that*

$S + T$ has a eigval that does not equal an eigval of S plus an eigval of T

and ST has a eigval that does not equal an eigval of S times an eigval of T.

SOLUTION:

ENDED