1·B

- (OR [9.2,9.3]. OR Problem (1) in 9.A)

Suppose V is a real vector space. The complexification of V, denoted by $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.

- *Addition on $V_{\mathbb{C}}$ is defined by*

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for all $u_1, v_1, u_2, v_2 \in V$.

- *Complex scalar multiplication on $V_{\mathbb{C}}$ is defined by*

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions above, $V_{\mathbb{C}}$ is a complex vector space.

Think of V as a subset of $V_{\mathbb{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbb{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.

SOLUTION:

- Commutativity: $(u_1 + \mathrm{i}v_1) + (u_2 + \mathrm{i}v_2) = (u_2 + \mathrm{i}v_2) + (u_1 + \mathrm{i}v_1)$.
- Associativity:

(I) $[(u_1 + \mathrm{i}v_1) + (u_2 + \mathrm{i}v_2)] + (u_3 + \mathrm{i}v_3) = (u_1 + \mathrm{i}v_1) + [(u_2 + \mathrm{i}v_2) + (u_3 + \mathrm{i}v_3)]$.

(II) $\begin{cases} [(a+b\mathrm{i})(c+d\mathrm{i})](u+\mathrm{i}v) = [(ac-bd)+(ad+bc)\mathrm{i}](u+\mathrm{i}v) = [(ac-bd)u-(ad+bc)v] + \mathrm{i}[(ac-bd)v+(ad+bc)u] \\ (a+b\mathrm{i})[(c+d\mathrm{i})(u+\mathrm{i}v)] = (a+b\mathrm{i})[(cu-dv)+\mathrm{i}(cv+du)] = [a(cu-dv)-b(cv+du)] + \mathrm{i}[a(cv+du)+b(cu-dv)] \end{cases}$

- Additive identity.
- Additive inverse: $(u_1 + \mathrm{i}v_1) + (-u_1 + \mathrm{i}(-v_1)) = 0$.
- Multiplication identity.
- Distributive properties:

(I) $\begin{cases} (a+b\mathrm{i})[(u_1+\mathrm{i}v_1)+(u_2+\mathrm{i}v_2)] = (a+b\mathrm{i})[(u_1+u_2)+\mathrm{i}(v_1+v_2)] \\ \qquad = [a(u_1+u_2)-b(v_1+v_2)] + \mathrm{i}[a(v_1+v_2)+b(u_1+u_2)] \\ (a+b\mathrm{i})(u_1+\mathrm{i}v_1)+(a+b\mathrm{i})(u_2+\mathrm{i}v_2) = [(au_1-bv_1)+\mathrm{i}(av_1+bu_1)] + [(au_2-bv_2)+\mathrm{i}(av_2+bu_2)] \end{cases}$

(II) $\begin{cases} [(a+b\mathrm{i})+(c+d\mathrm{i})](u+\mathrm{i}v) = [(a+c)+(b+d)\mathrm{i}](u+\mathrm{i}v) = [(a+c)u-(b+d)v] + \mathrm{i}[(a+c)v+(b+d)u] \\ (a+b\mathrm{i})(u+\mathrm{i}v)+(c+d\mathrm{i})(u+\mathrm{i}v) = [(au-bv)+\mathrm{i}(av+bu)] + [(cu-dv)+\mathrm{i}(cv+du)] \end{cases}$

□

- *Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.*

SOLUTION:

- Addition on V^S is defined by $(f+g)(x) = f(x) + g(x)$ for any $x \in S$ and $f, g \in V^S$.
- Scalar Multiplication on V^S is defined by $(\lambda f)(x) = \lambda f(x)$ for any $x \in S, \lambda \in \mathbf{F}, f \in V^S$.

Commutativity. Associativity.

Additive identity: $0\,(x) = 0$.

Additive inverse: $f(x) + (-f)(x) = 0$.

Multiplication identity: $I(x) = x$.

Distributive properties: $(\lambda(f+g))(x) = \lambda(f(x)+g(x)) = (\lambda f)(x) + (\lambda g)(x)$;

$$((\lambda+\mu)f)(x) = (\lambda+\mu)f(x) = \lambda f(x) + \mu f(x).$$ □

---

1 *Prove that $-(-v) = v$ for every $v \in V$.*

SOLUTION: $\left.\begin{array}{l} (-(-v)) + (-v) = 0 \\ v + (-v) = 0 \end{array}\right\} \Rightarrow$ By the uniqueness of additive inverse. □

OR. $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$ □

---

2 *Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.*

SOLUTION: If $a = 0$, then we are done.

Otherwise, $\exists\ a^{-1} \in \mathbf{F}$, $a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. $\square$

---

3 *Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.*

SOLUTION:

[Existence] Let $x = \dfrac{1}{3}(w - v)$.

[Uniqueness] Suppose $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II).

Then (I) $-$ (II) : $3(x_1 - x_2) = 0 \Rightarrow$ By Problem (2), $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$. $\square$

---

5 *Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that $0v = 0$ for all $v \in V$. Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.*

The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.

SOLUTION: Using [1.31]. $0v = 0$ for all $v \in V \Leftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. $\square$

---

6 *Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$.*
Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

and (I) $t + \infty = \infty + t = \infty + \infty = \infty$,

(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,

(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

SOLUTION: Not a vector space. By Associativity: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

OR By Distributive properties: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. $\square$

ENDED

1·C

2 (1.35)

(b) *The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$*

Denote the set by U. $\forall x \in [0, 1]$ we have $\left.\begin{array}{l} (一)\ 0 \in U;\ f(x) = 0 \Leftrightarrow f = 0 \\ (二)\ \forall f, g \in U,\ (f + g)(x) = f(x) + g(x) \\ (三)\ \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R},\ (\lambda f)(x) = \lambda f(x) \end{array}\right\} \Rightarrow \square$

(c) *The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$*

Denote the set by U. $\left.\begin{array}{l} (一)\ 0 \in U \\ (二)\ \forall f, g \in U,\ (f' + g') = f' + g' \\ (三)\ \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R},\ (\lambda f)' = \lambda (f)' \end{array}\right\} \Rightarrow \square$

(d) *The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.*

Denote the set by U. Suppose $b = 0$. Then

$$\left.\begin{array}{l}(\text{一})\ 0 \in U \\ (\text{二})\ \forall f, g \in U,\ (f+g)'(2) = f'(2) + g'(2) = 0 \\ (\text{三})\ \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R},\ (\lambda f)'(2) = \lambda f'(2)\end{array}\right\} \Rightarrow U \text{ is a subspace of } \mathbf{R}^{(0,3)}.$$

Suppose U is a subspace of $\mathbf{R}^{(0,3)}$. Suppose $f = 0 \Rightarrow f \in U$. Then $f'(2) = 0 = b$. $\square$

(e) *The set of all sequences with limit 0, of complex numbers, is a subspace of* $\mathbf{C}^{\infty}$.

Denote the set by A.

$$\left.\begin{array}{l}(\text{一})\ (0, 0, \dots) \in A \\ (\text{二})\ \forall\ (a_1, a_2, \dots),\ (b_1, b_2, \dots) \in A \iff \lim\limits_{n\to\infty} a_n = 0,\ \lim\limits_{n\to\infty} b_n = 0. \\ \qquad \text{Thus } \lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = 0 \Rightarrow (a_1 + b_1, a_2 + b_2, \dots) \in A \\ (\text{三})\ \forall\ (a_1, a_2, \dots) \in A, \forall \lambda \in \mathbf{F} = \mathbf{C},\ \lim\limits_{n\to\infty}(\lambda a_n) = \lambda \lim\limits_{n\to\infty} a_n = 0 \Rightarrow \lambda(a_1, a_2, \dots) \in A\end{array}\right\} \Rightarrow \square$$

---

4 *Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$*

Solution: Denote the set by V_b.

(a) Suppose V_b is a subspace of $\mathbf{R}^{[0,1]}$, then $\forall f \in V_b$, we have $\int_0^1 f = b$.

Because $kf \in V_b$ for any $k \in \mathbf{R}$. Hence $\int_0^1 (kf) = k\int_0^1 f = kb = b \Leftrightarrow b = 0$.

Or. Because $g = 0 \in V_b \Rightarrow \int_0^1 g = 0 = b$.

(b) Suppose $b = 0$. $\forall f, g \in V_b = V_0, \lambda \in \mathbf{R}$, $\int_0^1 (f + \lambda g) = \int_0^1 f + \int_0^1 g = 0$. $\square$

---

5 *Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?*

Answer: No. Because $\mathbf{R}^2$ is not closed on scalar multiplication on $\mathbf{C}$.

---

6 (a) *Is $\{(a,b,c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?*

Answer: True. As can be easily checked (note that $a^3 = b^3 \Leftrightarrow a = b$).

(b) *Is $\{(a,b,c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of $\mathbf{C}^3$?*

Answer: No. Note that $(\dfrac{-1 \pm \sqrt{3}\mathrm{i}}{2})^3 = 1$.

---

7 *Prove or give a counterexample: If U is a nonempty subset of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbf{R}^2$.*

Solution: Let $U = \mathbf{Z}^2$, $(\mathbf{Z}^*)^2$, $(\mathbf{Q}^*)^2$, $\mathbf{Q}^2 \backslash \{0\}$, or $\mathbf{R}^2 \backslash \{0\}$.

---

8 *Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.*

Solution: $U = \{(x,y) \in \mathbf{R}^2 : x = 0 \vee y = 0\}$.

---

9 *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.*

Solution: Denote the set by S.

Suppose $h(x) = \sin\sqrt{2}x + \cos x \in S$, since both $f(x) = \sin\sqrt{2}$, $g(x) = \cos x$ are periodic functions $\mathbf{R} \to \mathbf{R}$.

Assume $\exists\ p \in \mathbf{N}^+$ such that $h(x) = h(x+p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$

$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Contradiction! $\square$

---

11 *Prove that the intersection of every collection of subspaces of V is a subspace of V.*

SOLUTION:

Suppose $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collection of subspaces of V; here Γ is an arbitrary index set.

We need to prove that $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vectors

that are in U_α for each $\alpha \in \Gamma$, is a subspace of V.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall\alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under addition.

(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall\alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under scalar multiplication.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of V that is closed under addition and scalar multiplication.

Hence $\bigcap_{\alpha\in\Gamma} U_\alpha$ is a subspace of V. $\square$

---

12 *Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.*

SOLUTION: Suppose U and W are subspaces of V.

(a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subspace of V.

(b) Suppose $U \cup W$ is a subspace of V. Suppose $U \not\subseteq W$ and $U \not\supseteq W$ ($U \cup W \neq U$ and W).

Then $\forall a \in U$ but $a \notin W$; $b \in W$ but $b \notin U$. $a + b \in U \cup W$.

$$\left.\begin{array}{l}\text{(1) Consider } a+b \in U \Rightarrow b = (a+b)+(-a) \in U\text{, contradicts!}\\ \text{(2) Consider } a+b \in W \Rightarrow a = (a+b)+(-b) \in W\text{, contradicts!}\end{array}\right\} \Rightarrow U \cup W = U \text{ or } W\text{. Contradicts!}$$

Thus $U \subseteq W$ and $U \supseteq W$. $\square$

---

13 *Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.*

This exercise is surprisingly harder than Exercise 12, possibly because this exercise is not true if we replace **F** *with a field containing only two elements.*

SOLUTION: Suppose A, B, C are subspaces of V.

(a) If any two of them are subsets of the third one, then $A \cup B \cup C = A, B$ or C, which is a subspace of V.

(b)* If $A \cup B \cup C$ is a subspace of V, suppose $\left\{\begin{array}{c} A \not\supseteq B \text{ and } C \\ B \not\supseteq A \text{ and } C \\ C \not\supseteq A \text{ and } B \end{array}\right\} \Longleftrightarrow A \cap B \cap C \neq A, B$ and C.

$\forall a \in A$ but $a \notin B, C$; $\forall b \in B$ but $b \notin A, C$; $\forall c \in C$ but $c \notin A, B$; by assumption, $a+b+c \in A \cup B \cup C$.

(I) $A \cup B$ is a subspace $\Rightarrow$ By Problem (12), $A \subseteq B$ or $A \supseteq B$.

(II) $A \cup C$ is a subspace $\Rightarrow$ By Problem (12), $A \subseteq C$ or $A \supseteq C$.

(III) $B \cup C$ is a subspace $\Rightarrow$ By Problem (12), $B \subseteq C$ or $B \supseteq C$.

Any two of (I), (II) and (III) must be true.

(一). (I) and (II) are true. Then $\left.\begin{array}{r} C \subseteq B \subseteq A \\ \text{or } C \supseteq B \supseteq A \\ \text{or } B \supseteq A, C \\ \text{or } \underbrace{B \subseteq A, C}_{(1)} \end{array}\right\} \Rightarrow \left\{\begin{array}{r} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array}\right.$

(二). (II) and (III) are true. Then $\left.\begin{array}{r} A \subseteq C \subseteq B \\ \text{or } A \supseteq C \supseteq B \\ \text{or } C \supseteq A, B \\ \text{or } \underbrace{C \subseteq A, B}_{(2)} \end{array}\right\} \Rightarrow \left\{\begin{array}{r} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array}\right.$

(三). (I) and (III) are true. Then $\left.\begin{array}{r} B \subseteq A \subseteq C \\ \text{or } B \supseteq A \supseteq C \\ \text{or } A \supseteq B, C \\ \text{or } \underbrace{A \subseteq B, C}_{(3)} \end{array}\right\} \Rightarrow \left\{\begin{array}{r} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array}\right.$

Among these, any two of (1), (2) and (3) must be true. $\left.\begin{array}{l} \left.\begin{array}{l}(1)\\(2)\end{array}\right\} \Rightarrow B \subseteq C \subseteq A \\ \left.\begin{array}{l}(2)\\(3)\end{array}\right\} \Rightarrow C \subseteq A \subseteq B \\ \left.\begin{array}{l}(1)\\(3)\end{array}\right\} \Rightarrow B \subseteq A \subseteq C \end{array}\right\} \Rightarrow \square$

• *Suppose $U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$ and $W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$. Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.*

Solution:

(a) $U + W = \{(x + y, x - y, 2(x + y)) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} = \{(x', y', 2x')) \in \mathbf{F}^3 : x', y' \in \mathbf{F}\}$.

(b) $U + W$ is a plane of which $(1, 0, 2), (0, 1, 0)$ is a basis. $\square$

---

15 *Suppose U is a subspace of V. What is $U + U$?*

Solution: $\left.\begin{array}{r} \forall x, y \in U, x + y \in U \Rightarrow U + U \subseteq U \\ \forall x \in U, 0 \in U, x + 0 \in U + U \Rightarrow U \subseteq U + U \end{array}\right\} \Rightarrow U + U = U. \square$

---

16 *Suppose U and W are subspaces of V. Prove that $U + W = W + U$?*

Solution: $\forall x \in U, y \in W, \left.\begin{array}{r} x + y = y + x \in W + U \Rightarrow U + W \subseteq W + U \\ y + x = x + y \in U + W \Rightarrow W + U \subseteq U + W \end{array}\right\} \Rightarrow U + W = W + U. \square$

---

17 *Suppose V_1, V_2, V_3 are subspaces of V. Prove that $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$.*

Solution:

Let $x \in V_1, y \in V_2, z \in V_3$. Denote $(V_1 + V_2) + V_3$ by L, $V_1 + (V_2 + V_3)$ by R.

$\left.\begin{array}{l} \forall u \in L,\ \exists\, x, y, z,\ u = (x + y) + z = x + (y + z) \in R \Rightarrow L \subseteq R \\ \forall u \in R,\ \exists\, x, y, z,\ u = x + (y + z) = (x + y) + z \in L \Rightarrow R \subseteq L \end{array}\right\} \Rightarrow (V_1 + V_2) + V_3 = V_1 + (V_2 + V_3). \square$

---

18 *Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?*

Solution:

Suppose Ω is the additive identity.

For any subspace U of V. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

Now suppose W is an additive inverse of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. $\square$

---

19 *Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that $V_1 + U = V_2 + U$, then $V_1 = V_2$.*

Solution: A counterexample:

$V = \mathbf{F}^3, U = \{(x, 0, 0) \in \mathbf{F}^3 : x \in \mathbf{F}\}$,

$V_1 = \{(x, x, y)) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}, V_2 = \{(x, y, z)) \in \mathbf{F}^3 : x, y, z \in \mathbf{F}\}$.

EXAMPLE: *Suppose* $U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}, W = \{(x, x, x, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$.
Prove that $U + W = \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$.

SOLUTION: Let T denote $\{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$.

(a) By definition, $U+W = \{(x_1+x_2, x_1+x_2, y_1+x_2, y_1+y_2) \in \mathbf{F}^4 : (x_1, x_1, y_1, y_1) \in U, (x_2, x_2, x_2, y_2) \in W\}$.
$\Rightarrow \forall v \in U + W,\ \exists\, t \in T,\ \ v = t \Rightarrow U + W \subseteq T$.

(b) $\forall x, y, z \in \mathbf{F}$, let $u = (0, 0, y - x, y - x) \in U,\ \ w = (x, x, x, -y + x + z) \in W$
$\Rightarrow (x, x, y, z) = u + w \in U + W$. Hence $\forall t \in T,\ \exists\, u \in U, w \in W,\ \ t = u + w \Rightarrow T \subseteq U + W$. $\square$

---

21 *Suppose* $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}$.
Find a subspace W *of* $\mathbf{F}^5$ *such that* $\mathbf{F}^5 = U \oplus W$.

SOLUTION:

(a) Let $W = \{(0, 0, z, w, u) \in \mathbf{F}^5 : z, w, u \in \mathbf{F}\}$. Then $W \cap U = \{0\}$.

(b) $\forall x, y, z, w, u \in \mathbf{F}$, let $u = (x, y, x + y, x - y, 2x) \in U$,
$$w = (0, 0, z - x - y, w - x - y, u - 2x) \in W$$
$\Rightarrow (x, y, z, w, u) = u + w \Rightarrow \mathbf{F}^5 \subseteq U + W$. $\square$

---

22 *Suppose* $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}$.
Find three subspaces W_1, W_2, W_3 *of* $\mathbf{F}^5$*, none of which equals* $\{0\}$*,*
such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

SOLUTION:

(1) Let $W_1 = \{(0, 0, z, 0, 0) \in \mathbf{F}^5 : z \in \mathbf{F}\}$. Then $W_1 \cap U = \{0\}$.
Let $U_1 = U \oplus W_1$. Then $U_1 = \{(x, y, z, x - y, 2x) \in \mathbf{F}^5 : x, y, z \in \mathbf{F}\}$. (Check it!)

(2) Let $W_2 = \{(0, 0, 0, w, 0) \in \mathbf{F}^5 : w \in \mathbf{F}\}$. Then $W_2 \cap U_1 = \{0\}$.
Let $U_2 = U_1 \oplus W_2$. Then $U_2 = \{(x, y, z, w, 2x) \in \mathbf{F}^5 : x, y, z, w \in \mathbf{F}\}$.

(3) Let $W_3 = \{(0, 0, 0, 0, u) \in \mathbf{F}^5 : u \in \mathbf{F}\}$. Then $W_3 \cap U_2 = \{0\}$.
Let $U_3 = U_2 \oplus W_3$. Then $U_3 = \{(x, y, z, w, u) \in \mathbf{F}^5 : x, y, z, w, u \in \mathbf{F}\}$.

Thus $\mathbf{F}^5 = ((U \oplus W_1) \oplus W_2) \oplus W_3$. $\square$

---

23 *Prove or give a counterexample: If* V_1, V_2, U *are subspaces of* V *such that*
$V = V_1 \oplus U$ *and* $V = V_2 \oplus U$*, then* $V_1 = V_2$.

HINT: *When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in* $\mathbf{F}^2$.

SOLUTION: A counterexample:
$V = \mathbf{F}^2, U = \{(x, x) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_1 = \{(x, 0) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_2 = \{(0, x) \in \mathbf{F}^2 : x \in \mathbf{F}\}$.

---

24 *Let* V_e *denote the set of real-valued even functions on* $\mathbf{R}$
and let V_o *denote the set of real-valued odd functions on* $\mathbf{R}$*. Show that* $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

SOLUTION:

(a) $V_e \cap V_o = \{f : f(x) = f(-x) = -f(-x)\} = \{0\}$.

(b) $$\left.\begin{aligned} f_e \in V_e &\Leftrightarrow f_e(x) = f_e(-x) \Leftarrow \text{let } f_e(x) = \frac{g(x) + g(-x)}{2} \\ f_o \in V_o &\Leftrightarrow f_o(x) = -f_o(-x) \Leftarrow \text{let } f_o(x) = \frac{g(x) - g(-x)}{2} \end{aligned}\right\} \Rightarrow \forall g \in \mathbb{R}^{\mathbb{R}}, g(x) = f_e(x) + f_o(x). \ \square$$

ENDED

2·A

2 (a) *Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.*

(b) *Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.*

SOLUTION:

(a) Suppose $v \neq 0$. Then let $av = 0, a \in \mathbf{F}$. Getting $a = 0$. Thus (v) is linearly independent.
Suppose (v) is linearly independent. $av = 0 \Rightarrow a = 0$. Then $v \neq 0$, for if not, $a \neq 0 \Rightarrow av = 0$. Contradicts.

(b) Denote the list by (v, w), where $v, w \in V$. If (v, w) is linearly independent, suppose $av + bw = 0 \Rightarrow a = b = 0$.
Without loss of generality, suppose $v \neq cw\ \forall c \in \mathbf{F}$. Then let $av + bw = 0$, getting $a = b = 0 \Rightarrow (v, w)$ is linearly independent.

---

1 *Prove that if (v_1, v_2, v_3, v_4) spans V, then the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V.*

SOLUTION: Assume that $\forall v \in V,\ \exists\ a_1, \ldots, a_4 \in \mathbf{F}$,

$$\begin{aligned} v &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 \\ &= b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4, \text{ letting } b_i = \sum_{r=1}^{i} a_r. \end{aligned}$$

Thus $\forall v \in V,\ \exists\ b_i \in \mathbf{F}$, $v = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4$.

Hence the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V. $\square$

---

6 *Suppose (v_1, v_2, v_3, v_4) is linearly independent in V.*
Prove that the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ is also linearly independent.

SOLUTION: $a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4v_4 = 0$

$$\Rightarrow a_1v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0$$

$$\Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \Rightarrow a_1 = \cdots = a_4 = 0 \Rightarrow \quad \square$$

---

7 *Prove that if $(v_1, v_2, \ldots, v_m)$ is a linearly independent list of vectors in V, then $(5v_1 - 4v_2, v_2, v_3, \ldots, v_m)$ is linearly independent.*

SOLUTION: $a_1(5v_1 - 4v_2) + a_2v_2 + a_3v_3 + a_4v_4 = 0$

$$\Rightarrow 5a_1v_1 + (a_2 - 4a_1)v_2 + a_3v_3 + a_4v_4 = 0$$

$$\Rightarrow 5a_1 = a_2 - 4a_1 = a_3 = a_4 = 0 \Rightarrow a_1 = \cdots = a_4 = 0\ \square$$

• *Suppose $(v_1, \ldots, v_m)$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let $w_k = v_1 + \cdots + v_k$.*
(a) *Show that span$(v_1, \ldots, v_m)$ = span$(w_1, \ldots, w_m)$.*
(b) *Show that $(v_1, \ldots, v_m)$ is linearly independent if and only if $(w_1, \ldots, w_m)$ is linearly independent.*

SOLUTION:

(a) Let span$(v_1, \ldots, v_m) = U$. Assume that $\forall v \in U,\ \exists\ a_i \in \mathbf{F}$,

$$v = a_1v_1 + \cdots + a_mv_m = b_1w_1 + \cdots + b_mw_m = \sum_{j=1}^{m}(\sum_{i=j}^{m} b_i)v_j$$

$\Rightarrow b_1 = a_1,\ \ b_i = a_i - \sum_{r=1}^{i-1} b_r$. Thus $\exists\ b_i \in \mathbf{F}$ such that $v = b_1w_1 + \cdots + b_mw_m$.

又 Each $w_i \in U \Rightarrow$ span$(v_1, \ldots, v_m)$ = span$(w_1, \ldots, w_m)$.

(b) $a_1w_1 + \cdots + a_mw_m = 0$

$$\Rightarrow (a_1 + \cdots + a_m)v_1 + \cdots + (a_i + \cdots + a_m)v_i + \cdots + a_mv_m = 0$$

$$\Rightarrow a_m = \cdots = (a_m + \cdots + a_i) = \cdots = (a_m + \cdots + a_1) = 0.\ \square$$

---

10 *Suppose $(v_1, \ldots, v_m)$ is linearly independent in V and $w \in V$.*

(a) *Prove that if $(v_1 + w, \ldots, v_m + w)$ is linearly dependent, then $w \in span(v_1, \ldots, v_m)$.*

(b) *Show that $(v_1, \ldots, v_m, w)$ is linearly independent $\Longleftrightarrow w \notin span(v_1, \ldots, v_m)$.*

SOLUTION:

(a) Suppose $a_1(v_1+w)+\cdots+a_m(v_m+w)=0,\ \exists\ a_i \neq= 0 \Rightarrow a_1v_1+\cdots+a_mv_m = 0 = -(a_1+\cdots+a_m)w$.

Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1v_1 + \cdots + a_mv_m = 0$ while $a_i \neq 0$ for some i, contradicts.

Hence $w \in \text{span}(v_1, \ldots, v_m)$.

(b) Suppose $w \in \text{span}(v_1, \ldots, v_m)$. Then $(v_1, \ldots, v_m, w)$ is linearly dependent.

Thus have we proven the "$\Rightarrow$" by its contrapositive.

Suppose $w \notin \text{span}(v_1, \ldots, v_m)$. Then by [2.23], $(v_1, \ldots, v_m, w)$ is linearly independent. $\square$

---

14 *Prove that V is infinite-dim if and only if there is a sequence $(v_1, v_2, \ldots)$ in V such that $(v_1, \ldots, v_m)$ is linearly independent for every $m \in \mathbf{N}^+$.*

SOLUTION: Similar to [2.16].

Suppose there is a sequence $(v_1, v_2, \ldots)$ in V such that $(v_1, \ldots, v_m)$ is linearly independent for any $m \in \mathbf{N}^+$.

Choose an m. Suppose a linearly independent list $(v_1, \ldots, v_m)$ spans V.

Then there exists $v_{m+1} \in V$ but $v_{m+1} \notin \text{span}(v_1, \ldots, v_m)$. Hence no list spans V. Thus V is infinite-dim.

Conversely it is true as well. For if not, V must be finite-dim, contradicting the assumption. $\square$

---

15 *Prove that $\mathbf{F}^\infty$ is infinite-dim.*

SOLUTION: Let $e_i = (0, \ldots, 0, 1, 0, \ldots) \in \mathbf{F}^\infty$ for every $m \in \mathbf{N}^+$, where '1' is on the ith entry of e_i.

Suppose $\mathbf{F}^\infty$ is finite-dim. Then let $\text{span}(e_1, \ldots, e_m) = V$. But $e_{m+1} \notin \text{span}(e_1, \ldots, e_m)$. Contradicts. $\square$

---

16 *Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dim.*

SOLUTION: Denote the vec-sp by U. Note that for each $m \in \mathbf{N}^+$, $(1, x, \ldots, x^m)$ is linearly independent.

$$\left.\begin{array}{l}\text{Because if } a_0, \ldots, a_m \in \mathbf{R} \text{ are such that } a_0 + a_1x + \cdots + a_mx^m = 0,\ \forall x \in [0,1], \\ \text{then the polynomial has infinitely many roots and hence } a_0 = \cdots = a_m = 0.\end{array}\right\} \text{Similar to [2.16], } U \text{ is infinite-dim.}$$

OR. Note that for $a_n = \dfrac{1}{n}$, $\quad a_1 < a_2 < \cdots < a_m,\ \ \forall m \in \mathbf{N}^+$.

Suppose $f_n = \begin{cases} x - \dfrac{1}{n}, & x \in [\frac{1}{n}, 1) \\ 0 \quad , & x \in [0, \frac{1}{n}) \end{cases}$. Then for any m, $f_1(\frac{1}{m}) = \cdots = f_m(\frac{1}{m})$, while $f_{m+1}(\frac{1}{m}) \neq 0$.

Hence $f_{m+1} \notin \text{span}(f_1, \ldots, f_m)$. Thus by Problem (14), U is infinite-dim.

---

17 *Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \ldots, m\}$. Prove that $(p_0, p_1, \ldots, p_m)$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.*

SOLUTION:

Suppose $(p_0, p_1, \ldots, p_m)$ is linearly independent. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z\ \forall z \in \mathbf{F}$.

But $\forall a_i \in \mathbf{F}, z \neq a_0p_0(z) + \cdots + a_mp_m(z)$, for if not, let $z = 2$, contradicts. Thus $z \notin \text{span}(p_0, p_1, \ldots, p_m)$.

Then $\text{span}(p_0, p_1, \ldots, p_m) \subsetneqq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \ldots, p_m)$ has length $m + 1$.

Hence $(p_0, p_1, \ldots, p_m)$ is linearly dependent in $\mathcal{P}_m(\mathbf{F})$.

For if not, notice that the list $(1, z, \ldots, z^m)$ spans $\mathcal{P}_m(\mathbf{F})$,

thus by [2.23], $(p_0, p_1, \ldots, p_m)$ spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. $\square$

ENDED

2·B

NOTE FOR *linearly independent sequence and [2.34].*

"$V = \text{span}\,(v_1, \ldots, v_n, \ldots)$" is an invalid expression.

If we allow using "infinite list", then we must guarantee that $(v_1, \ldots, v_n, \ldots)$ is a spanning "list" such that for all $v \in V$, there exists a certain positive integer such that $v = a_1 v_{\alpha_1} + \cdots + a_n v_{\alpha_n}$, where $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ is an finite index set. The key point is, how do we find such a "list"?

NOTE FOR "$\complement_V U \cap \{0\}$": "$\complement_V U \cap \{0\}$" is supposed to be "$W$", where $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cap \{0\} \\ u \pm w \in \complement_V U \cap \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cap \{0\}$. Contradicts.

NEW NOTATION: Denote the set $\{W_1, W_2 \ldots\}$ by $\mathcal{S}_V U$, where for each W_i, $V = U \oplus W_i$. See also in (1.C.23).

---

1 *Find all vector spaces that have exactly one basis.* **SOLUTION:** $\mathbf{F} = \mathbf{C}, \mathbf{R}, \mathbf{Q}, \{0,1\}, \mathcal{P}_0(\mathbf{F})$.

---

6 *Suppose (v_1, v_2, v_3, v_4) is a basis of V. Prove that $(v_1+v_2, v_2+v_3, v_3+v_4, v_4)$ is also a basis.*

SOLUTION: $\forall v \in V,\ \exists!\ a_1, \ldots, a_4 \in \mathbf{F},\ v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$.

Assune that $v = b_1(v_1+v_2)+b_2(v_2+v_3)+b_3(v_3+v_4)+b_4v_4$. Then $v = b_1v_1+(b_1+b_2)v_2+(b_2+b_3)v_3+(b_3+b_4)v_4$.

$\Rightarrow\ \exists!\ b_1 = a_1,\ \ b_2 = a_2 - b_1,\ \ b_3 = a_3 - b_2,\ \ b_4 = a_4 - b_3 \in \mathbf{F}$. ☐

---

7 *Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \in U$, then v_1, v_2 is a basis of U.*

SOLUTION: Let $V = \mathbf{F}^4, v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_3 = (0,0,1,1), v_4 = (0,0,0,1)$.

And $U = \{(x,y,z,0) \in \mathbf{R}^4 : x,y,z \in \mathbf{F}\}$. We have a counterexample.

• *Suppose V is finite-dim and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.*

SOLUTION: Let $(u_1, \ldots, u_m)$ and $(w_1, \ldots, w_n)$ be bases of U and W respectively.

Then $V = \text{span}(u_1, \ldots, u_m) + \text{span}(w_1, \ldots, w_n) = \text{span}(u_1, \ldots, u_m, w_1, \ldots, w_n)$.

Hence, by [2.31], we get a basis of V consisting of vectors in U or W. ☐

---

8 *Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $(u_1, \ldots, u_m)$ is a basis of U and $(w_1, \ldots, w_n)$ is a basis of W. Prove that $(u_1, \ldots, u_m, w_1, \ldots, w_n)$ is a basis of V.*

SOLUTION:

$\forall v \in V,\ \exists!\, a_i, b_i \in \mathbf{F},\ v = (a_1u_1 + \cdots + a_mu_m) + (b_1w_1 + \cdots + b_nw_n)$

$\Rightarrow (a_1u_1 + \cdots + a_mu_m) = -(b_1w_1 + \cdots + b_nw_n) \in U \cap W = \{0\}$. Thus $a_1 = \cdots = a_m = b_1 = \cdots = b_n$. ☐

• (OR 9.4) *Suppose V is a real vector space. Show that if $(v_1, \ldots, v_n)$ is a basis of V (as a real vector space), then $(v_1, \ldots, v_n)$ is also a basis of the complexification $V_{\mathbb{C}}$ (as a complex vector space).*

See Section 1B (4e) for the definition of the complexification $V_{\mathbb{C}}$.

SOLUTION:

$\forall u + \mathrm{i}v \in V_{\mathbb{C}},\ \exists!\ u, v \in V, a_i, b_i \in \mathbf{R},$

$u + \mathrm{i}v = (a_1v_1 + \cdots + a_nv_n) + \mathrm{i}(b_1v_1 + \cdots + b_nv_n) = (a_1 + b_1\mathrm{i})v_1 + \cdots + (a_n + b_n\mathrm{i})v_n$

$\Rightarrow\ u + \mathrm{i}v = c_1v_1 + \cdots + c_nv_n,\ \exists!\, c_i = a_i + b_i\mathrm{i} \in \mathbf{C}$

$\Rightarrow$ By the uniqueness of c_i and [2.29], $(v_1, \ldots, v_n)$ is a basis of $V_{\mathbb{C}}$. ☐

ENDED

2·C

1 *Suppose V is finite-dim and U is a subspace of V such that* $\dim V = \dim U$.

Let $(u_1, \dots, u_m)$ be a basis of U. Then $n = \dim U = \dim V$. 又 $u_i \in V$.

Then by [2.39], $(u_1, \dots, u_m)$ is a basis of V. Thus $V = U$.

---

2 *Show that the subspaces of* $\mathbf{R}^2$ *are precisely* $\{0\}$, *all lines in* $\mathbf{R}^2$ *containing the origin, and* $\mathbf{R}^2$.

SOLUTION:

Suppose U is a subspace of $\mathbf{R}^2$. Let $\dim U = n$.

If $n = 0$, then $U = \{0\}$.

If $n = 1$, then $U = \operatorname{span}(v)$, where v is a vector in $\mathbf{R}^2$. Thus U can be any line in $\mathbf{R}^2$ containing the origin.

If $n = 2$, then $U = \operatorname{span}(v, w)$, where v, w are vectors in $\mathbf{R}^2$ and (v, w) is linearly independent $\Rightarrow U = \mathbf{R}^2$. $\square$

---

3 *Show that the subspaces of* $\mathbf{R}^3$ *are precisely* $\{0\}$, *all lines in* $\mathbf{R}^3$ *containing the origin, all planes in* $\mathbf{R}^3$ *containing the origin, and* $\mathbf{R}^3$.

SOLUTION:

Suppose U is a subspace of $\mathbf{R}^3$. Let $\dim U = n$.

If $n = 0$, then $U = \{0\}$.

If $n = 1$, then $U = \operatorname{span}(v)$, where v is a vector in $\mathbf{R}^3$. Thus U can be any line in $\mathbf{R}^3$ containing the origin.

If $n = 2$, then $U = \operatorname{span}(v, w)$, where v, w are vectors in $\mathbf{R}^3$ and (v, w) is linearly independent.

Thus U can be any plane in $\mathbf{R}^3$ containing the origin.

If $n = 3$, then $U = \operatorname{span}(u, v, w)$, where u, v, w are vectors in $\mathbf{R}^3$ and (u, v, w) is linearly independent

$\Rightarrow U = \mathbf{R}^3$. $\square$

---

7 (a) *Let* $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. *Find a basis of* U.

(b) *Extend the basis in* (a) *to a basis of* $\mathcal{P}_4(\mathbf{F})$.

(c) *Find a subspace* W *of* $\mathcal{P}_4(\mathbf{F})$ *such that* $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

SOLUTION:

Suppose $p(z) = az^4 + bz^3 + cz^2 + dz + e$ and $p(2) = p(5) = p(6)$.

$$\text{Then} \begin{cases} p(2) = 16a + 8b + 4c + 2d + e \text{ (I)} \\ p(5) = 625a + 125b + 25c + 5d + e \text{ (II)} \\ p(6) = 1296a + 216b + 36c + 6d + e \text{ (III)} \end{cases}$$

You don't have to compute to know that the dimension of the set of soultions is 3.

(a) A basis: $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. $\square$

---

9 *Suppose* $(v_1, \dots, v_m)$ *is linearly independent in* V *and* $w \in V$.
Prove that $\dim$ *span*$(v_1 + w, \dots, v_m + w) \geq m - 1$.

SOLUTION:

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$, for each $i = 1, \dots, m$.

$(v_1, \dots, v_m)$ is linearly independent $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ is linearly independent

$\Rightarrow (v_2 - v_1, \dots, v_m - v_1)$ is linearly independent of length $m - 1$.

又 By the contrapositive of (2.A.10), $w \notin \operatorname{span}(v_1, \dots, v_m) \Rightarrow (v_1 + w, \dots, v_m + w)$ is linearly independent.

$\therefore\ m \geq \dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$. $\square$

---

10 *Suppose m is a positive integer and $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $(p_0, p_1, \ldots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

SOLUTION: Using mathematical induction on m.

(i) For p_0, $\deg p_0 = 0 \Rightarrow \text{span}\,(p_0) = \text{span}\,(1)$.

(ii) Suppose for $i \geq 1$, $\text{span}\,(p_0, p_1, \ldots, p_i) = \text{span}(1, x, \ldots, x^i)$.

Then $\text{span}(p_0, p_1, \ldots, p_i, p_{i+1}) \subseteq \text{span}\,(1, x, \ldots, x^i, x^{i+1})$.

又 $\deg p_{i+1} = i+1, \quad p_{i+1}(x) = a_{i+1}x^{i+1} + r_{i+1}(x); \quad a_{i+1} \neq 0, \quad \deg r_{i+1} \leq i.$

$\Rightarrow x^{i+1} = \frac{1}{a_{i+1}}(p_{i+1}(x) - r_{i+1}(x)) \in \text{span}\,(1, x, \ldots, x^i, p_{i+1}) = \text{span}\,(p_0, p_1, \ldots, p_i, p_{i+1})$.

$\therefore\ x^{i+1} \in \text{span}\,(p_0, p_1, \ldots, p_i, p_{i+1})$. $\Rightarrow \text{span}\,(1, x, \ldots, x^i, x^{i+1}) \subseteq \text{span}\,(p_0, p_1, \ldots, p_i, p_{i+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \text{span}\,(1, x, \ldots, x^m) = \text{span}(p_0, p_1, \ldots, p_m)$. $\square$

• *Suppose m is a positive integer. For $0 \leq k \leq m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show that $(p_0, \ldots, p_m)$ is a basis of $\mathcal{P}(\mathbf{F})$.*

The basis in this exercise leads to what are called Bernstein polynomials. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on $[0,1]$.

SOLUTION: Using mathematical induction.

(i) $k = 0, 1, 2$, $p_m(x) = x^m, \quad p_{m-1}(x) = x^{m-1} - x^m, \quad p_{m-2}(x) = x^{m-2} + x^m - 2x^{m-1}$.

(ii) $k \geq 2$. Suppose for $p_{m-k}(x)$, $\exists\,!\ a_i \in \mathbf{F}$, $x^{m-k} = p_{m-k}(x) + a_m x^m + \cdots + a_{m-k+1}x^{m-k+1}$.

Then for $p_{m-k-1}(x)$, $\exists\,!\ c_i \in \mathbf{F}$,

$$x^{m-k-1} = p_{m-k-1}(x) + \mathcal{C}_{k+1}^{1}(-1)^2 x^{m-k} + \cdots + \mathcal{C}_{k+1}^{k}(-1)^{k+1}x^{m-1} + (-1)^{k-2}x^m$$

$$\Rightarrow c_{m-i} = \mathcal{C}_{k+1}^{k+1-i}(-1)^{k-i}.$$

Thus for each x^i, $\exists\,!\ b_i \in \mathbf{F}$, $x^i = b_m p_m(x) + \cdots + b_{m-i}p_{m-i}(x)$.

$\Rightarrow \text{span}(x^m, \ldots, x, 1) = \text{span}\underbrace{(p_m, \ldots, p_1, p_0)}_{\text{Basis}}$. $\square$

• *Suppose V is finite-dim and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*

SOLUTION:
$$\left.\begin{array}{l}\dim V_1 + \dim V_2 > 2\dim V - \dim V_3 \geq \dim V \Rightarrow V_1 \cap V_2 \neq \{0\} \\ \dim V_2 + \dim V_3 > 2\dim V - \dim V_1 \geq \dim V \Rightarrow V_2 \cap V_3 \neq \{0\} \\ \dim V_1 + \dim V_3 > 2\dim V - \dim V_2 \geq \dim V \Rightarrow V_1 \cap V_3 \neq \{0\}\end{array}\right\} \Rightarrow V_1 \cap V_2 \cap V_3 \neq \{0\}. \ \square$$

• *Suppose V is finite-dim and U is a subspace of V with $U \neq V$. Let $n = \dim V$, $m = \dim U$. Prove that there exist $(n-m)$ subspaces of V, say $U_1, \ldots, U_{n-m}$, each of dimension $(n-1)$, such that $\bigcap_{i=1}^{n-m} U_i = U$.*

SOLUTION: Let $(v_1, \ldots, v_m)$ be a basis of U, extend to a basis of V as $(v_1, \ldots, v_m, \ldots, v_n)$.

Define $U_i = \text{span}\,(v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+i-1}, v_{m+i+1}, \ldots, v_n)$ for each i. Thus we are done.

EXAMPLE: Suppose $\dim V = 6, \dim U = 3$.

$$\underbrace{(\underbrace{v_1, v_2, v_3}_{\text{Basis of U}}, v_4, v_5, v_6)}_{\text{Basis of V}}, \text{ define } \left.\begin{array}{l} U_1 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_5, v_6) \\ U_2 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_6) \\ U_3 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_5)\end{array}\right\} \Rightarrow \dim U_i = 6 - 1, \ i = \underbrace{1, 2, 3}_{6-3=3}.$$

$\square$

---

14 *Suppose that $V_1, \dots, V_m$ are finite-dim subspaces of V.*

Prove that $V_1 + \cdots + V_m$ is finite-dim and $\dim(V_1 + \cdots + V_m) \le \dim V_1 + \cdots + \dim V_m$.

SOLUTION:

Choose a basis $\mathcal{E}_i$ of $V_i \Rightarrow V_1 + \cdots + V_m = \text{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim U_i = \text{card}\, \mathcal{E}_i$.

Then $\dim(V_1 + \cdots + V_m) = \dim \text{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$.

又 $\dim \text{span}(\mathcal{E}_1 \cup \cdots \cup E_m) \le \text{card}\,(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \le \text{card}\, \mathcal{E}_1 + \cdots + \text{card}\, \mathcal{E}_m$.

Thus $\dim(V_1 + \cdots + V_m) \le \dim U_1 + \cdots + \dim U_m$.

•*The inequality above is an equality if and only if $V_1 + \cdots + V_m$ is a direct sum.*

For each i, $(V_1 + \cdots + V_i) \cap V_{i+1} = \{0\} \iff V_1 + \cdots + V_m$ is a direct sum $\iff \square$

---

17 *Suppose V_1, V_2, V_3 are subspaces of a finite-dim vector space, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

SOLUTION:

Looks like: given three sets A, B and C.

Note that: $\text{card}\,(X \cup Y) = \text{card}\,(X) + \text{card}\,(Y) - \text{card}\,(X \cap Y)$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

Then: $\text{card}\,((A \cup B) \cup C) = \text{card}\,(A \cup B) + \text{card}\, C - \text{card}\,((A \cup B) \cap C)$.

And: $\text{card}\,((A \cup B) \cap C) = \text{card}\,((A \cap C) \cup (B \cap C)) = \text{card}\,(A \cap C) + \text{card}\,(B \cap C) - \text{card}\,(A \cap B \cap C)$.

Thus: $\text{card}\,((A \cup B) \cup C) = \text{card}\, A + \text{card}\, B + \text{card}\, C + \text{card}\,(A \cap B \cap C) - \text{card}\,(A \cap B) - \text{card}\,(A \cap C) - \text{card}\,(B \cap C)$.

Because $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned}
\dim(V_1 + V_2 + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \quad (1)\\
&= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) \quad (2)\\
&= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) \quad (3)
\end{aligned}$$

Notice that $(X + Y) \cap Z \ne X \cap Z + Y \cap Z$.

For example, $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

• **COROLLARY:** *If V_1, V_2 and V_3 are finite-dim vector spaces, then* $\frac{(1) + (2) + (3)}{3}$:

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.$$

The formula above may seem strange because the right side does not look like an integer. $\square$

ENDED

3·A

2 *Suppose $b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by*

$$Tp = (3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c\sin p(0)).$$

Show that T is linear if and only if $b = c = 0$.

SOLUTION:

(a) Suppose $b = c = 0$, then $\forall p, q \in \mathcal{P}(\mathbf{R}), T(p+q) = (3(p+q)(4) + 5(p+q)'(6), \int_{-1}^{2} x^3 (p+q)(x)\mathrm{d}x)$.

Because $(p+q)(x) = p(x) + q(x), (p+q)'(x) = p'(x) + q'(x)$,

$\int_{-1}^{2} x^3 (p+q)(x)\mathrm{d}x = \int_{-1}^{2} x^3 p(x)\mathrm{d}x + \int_{-1}^{2} x^3 q(x)\mathrm{d}x$.

$\Rightarrow T(p+q) = Tp + Tq$. Similarly, $\forall \lambda \in \mathbf{F}$, $\lambda Tp = T(\lambda p)$. Thus T is linear.

(b) Suppose T is linear, denote the linear map in (a) by $S \Rightarrow (T - S)$ is linear.

$\Rightarrow (T-S)(p) = (bp(1)p(2), c\sin p(0))$ is linear.

Consider $p(x) = q(x) = \frac{\pi}{2},\ \forall x \in \mathbf{R}$.

$\Rightarrow ((T-S)(p+q) = (T-S)(\pi) = (b\pi^2, 0) = (T-S)(\frac{\pi}{2}) + (T-S)(\frac{\pi}{2}) = (b\frac{\pi^2}{2}, 2c) \Rightarrow b = c = 0.$ ☐

• **TIPS:** *$T : V \to W$ is linear* $\iff \begin{cases} \forall v, u \in V, T(v+u) = Tv + Tu \\ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv) \end{cases} \iff T(v + \lambda u) = Tv + \lambda Tu.$

---

3 *Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists\ A_{j,k} \in \mathbf{F}$ such that*

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for any $(x_1, \dots, x_n) \in \mathbf{F}^n$.

SOLUTION:

Let $T(1, 0, 0, \dots, 0, 0) = (A_{1,1}, \dots, A_{m,1})$,
$T(0, 1, 0, \dots, 0, 0) = (A_{1,2}, \dots, A_{m,2})$,
$\vdots$
$T(0, 0, 0, \dots, 0, 1) = (A_{1,n}, \dots, A_{m,n})$.

Note that $(1, 0, \dots, 0, 0), \cdots, (0, 0, \dots, 0, 1)$ is a basis of $\mathbf{F}^n$.
Then by [3.5], we are done. ☐

---

4 *Suppose $T \in \mathcal{L}(V, W)$ and $(v_1, \dots, v_m)$ is a list of vectors in V such that $(Tv_1, \dots, Tv_m)$ is linearly independent in W. Prove that $(v_1, \dots, v_m)$ is linearly independent.*

SOLUTION: Suppose $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$.☐

---

5 *Prove that $\mathcal{L}(V, W)$ is a vector space,* **SOLUTION:** Note that $\mathcal{L}(V, W)$ is a subspace of W^V. ☐

---

7 *Show that every linear map from a one-dim vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.*

SOLUTION:

Let u be a nonzero vector in $V \Rightarrow V = \mathrm{span}(u)$.

Because $Tu \in V \Rightarrow Tu = \lambda u$ for some λ.

Suppose $v \in V \Rightarrow v = au,\ \exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. ☐

---

8 *Give an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that $\varphi(av) = a\varphi(v)$ for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.*

SOLUTION:

Define $T(x, y) = \begin{cases} x + y, \text{if } (x, y) \in \mathrm{span}(3, 1), \\ 0, \qquad \text{otherwise.} \end{cases}$ OR. Define $T(x, y) = \sqrt[3]{(x^3 + y^3)}$. ☐

---

9 *Give an example of a function $\varphi : \mathbf{C} \to \mathbf{C}$ such that $\varphi(w+z) = \varphi(w) + \varphi(z)$ for all $w, z \in \mathbf{C}$ but φ is not linear. (Here $\mathbf{C}$ is thought of as a complex vector space.)*

SOLUTION:

Suppose $V_{\mathbb{C}}$ is the complexification of a vector space V. Suppose $\varphi : V_{\mathbb{C}} \to V_{\mathbb{C}}$.

Define $\varphi(u + \mathrm{i}v) = u = \Re(u + \mathrm{i}v)$

OR. Define $\varphi(u + \mathrm{i}v) = v = \Im(u + \mathrm{i}v)$. $\square$

• OR (3.D.16) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Suppose $ST = TS$ for every $S \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity.*

SOLUTION:

If $V = \{0\}$, then we are done. Now suppose $V \neq \{0\}$.

Assume that (v, Tv) is linearly dependent for every $v \in V$, then by (2.A.2.(b)), $Tv = \lambda_v v$ for some $\lambda_v \in \mathbf{F}$.

To prove that λ_v is independent of v

(in other words, for any two distinct nonzero vectors v and w in V, we have $\lambda_v \neq \lambda_w$), we discuss in two cases:

$$\left.\begin{array}{l} (-)\text{ If } (v,w) \text{ is linearly independent, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = a_v v + a_w w \\ \qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=)\text{ Otherwise, suppose } w = cv,\ a_w w = Tw = cTv = ca_v v = a_v w \Rightarrow (a_w - a_v)w \end{array}\right\} \Rightarrow a_w = a_v.$$

Now we prove the assumption by contradiction.

Suppose (v, Tv) is linearly independent for every nonzero vector $v \in V$.

Fix one v. Extend to $(v, Tv, u_1, \dots, u_n)$ a basis of V.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$.

Hence a contradiction arises. $\square$

OR. Let $(v_1, \dots, v_m)$ be a basis of V.

Define $\varphi \in \mathcal{L}(V,\mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$. Let $\lambda = \varphi(Tv_1) \in \mathbf{F}$.

For any $v \in V$, define $S_v \in \mathcal{L}(V)$ by $S_v u = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. $\square$

---

10 *Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$).*

Define $T : V \to W$ by $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \backslash U. \end{cases}$ Prove that T is not a linear map on V.

SOLUTION:

Suppose T is a linear map. And $v \in V\backslash U,\ u \in U$ such that $Su \neq 0$.

Then $v + u \in V\backslash U$, (for if not, $v = (v+u) - u \in U$) while $T(v+u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$.

Hence we get a contradiction. $\square$

---

11 *Suppose V is finite-dim. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.*

SOLUTION:

Define $T \in \mathcal{L}(V, W)$ by $T(a_1u_1 + \cdots + a_nu_n + a_{n+1}u_{n+1} + \cdots + a_mu_m) = a_1Su_1 + \cdots + a_nSu_n$.

Where we let $(u_1, \dots, u_n)$ be a basis of U, extend to a basis of V as $(u_1, \dots, u_n, \dots, u_m)$. $\square$

---

12 *Suppose V is finite-dim with $\dim V > 0$, and W is infinite-dim. Prove that $\mathcal{L}(V, W)$ is infinite-dim.*

SOLUTION:

Let $(v_1, \ldots, v_n)$ be a basis of V. Let $(w_1, \ldots, w_m)$ be linearly independent in W for any $m \in \mathbf{N}^+$.

Define $T_{x,y} \in \mathcal{L}(V, W)$ by $T_{x,y}(v_z) = \delta_{zy} w_y, \forall x \in \{1, \ldots, n\}, y \in \{1, \ldots, m\}$, where $\delta_{zy} = \begin{cases} 0, & z \neq y, \\ 1, & z = y. \end{cases}$

Suppose $a_1 T_{x,1} + \cdots + a_m T_{x,m} = 0$. Then $(a_1 T_{x,1} + \cdots + a_m T_{x,m})(v_x) = 0 = a_1 w_1 + \cdots + a_m w_m$.

$\Rightarrow a_1 = \cdots = a_m = 0$. 又 m is arbitrarily chosen.

Thus $(T_{x,1}, \ldots, T_{x,m})$ is a linearly independent list in $\mathcal{L}(V, W)$ for any x and length m. Hence by (2.A.14). $\square$

---

13 *Suppose $(v_1, \ldots, v_m)$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $(w_1, \ldots, w_m) \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.*

SOLUTION:

We show it by contradiction. By linear independence lemma, $\exists\, j \in \{1, \ldots, m\}$ such that $v_j \in$ span $(v_1, \ldots, v_{j-1})$.

Fix j. Let $w_j \neq 0$, while $w_1 = \cdots = w_{j-1} = w_{j+1} = w_m = 0$.

Define T by $Tv_k = w_k$ for all k. Suppose $a_1 v_1 + \cdots + a_m v_m = 0$ (where $a_j \neq 0$).

Then $T(a_1 v_1 + \cdots + a_m v_m) = 0 = a_1 w_1 + \cdots + a_m w_m = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. $\square$

OR. We prove the contrapositive:

Suppose for any list $(w_1, \ldots, w_m) \in W, \exists\, T \in \mathcal{L}(V, W), Tv_k = w_k$ for each w_k.

(We need to) Prove that $(v_1, \ldots, v_n)$ is linearly independent.

Suppose $\exists\, a_i \in \mathbf{F}, a_1 v_1 + \cdots + a_n v_n = 0$. Choose a nonzero $w \in W$.

By assumption, for the list $(\overline{a_1} w, \ldots, \overline{a_m} w)$, $\exists\, T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$ for each v_k.

$0 = T(\sum_{k=1}^{m} a_k v_k) = \sum_{k=1}^{m} a_k T v_k = \sum_{k=1}^{m} a_k \overline{a_k} w = (\sum_{k=1}^{m} |a_k|^2) w$. Hence $\sum_{k=1}^{m} |a_k|^2 = 0 \Rightarrow a_k = 0$. $\square$

• *Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION:

Let $(v_1, \ldots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \backslash \{0\}$.

Suppose $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \cdots + a_n v_n$, where $a_k \neq 0$.

Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y}(v_x) = v_y, R_{x,y}(v_z) = 0\,(\, z \neq x\,)$. Then for any $x, y \in \mathbf{N}^+$,

$(R_{k,y} S)(v_i) = a_k v_y \Rightarrow ((R_{k,y} S) \circ R_{x,i})(v_x) = a_k v_y$, and $((R_{k,y} S) \circ R_{x,i})(v_z) = 0$ for $z \neq x$.

Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Denote by $T_{x,y}$.

Getting $(\frac{1}{a_k} T_{1,1} + \cdots + \frac{1}{a_k} T_{n,n}) v_j = v_j \Rightarrow \sum_{r=1}^{n} \frac{1}{a_k} T_{r,r} = I$.

又 By assumption, $T_{x,y} \in \mathcal{E} \Rightarrow I \in \mathcal{E}$.

Hence for any $T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. $\square$

ENDED

3·B

2 *Suppose $S, T \in \mathcal{L}(V)$ are such that range $S \subseteq$ null T. Prove that $(ST)^2 = 0$.*

SOLUTION: $TS = 0 \Rightarrow STST = (ST)^2 = 0$. $\square$

---

3 *Suppose $(v_1, \ldots, v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \ldots, z_m) = z_1v_1 + \cdots + z_mv_m$.*

(a) *What property of T corresponds to $(v_1, \ldots, v_m)$ spanning V?*

(b) *What property of T corresponds to $(v_1, \ldots, v_m)$ being linearly independent?*

ANSWER: (a) Surjectivity; (b) Injectivity. $\square$

---

4 *Show that $U = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \textit{null}\, T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.*

SOLUTION: Let $(v_1, v_2, v_3, v_4, v_5)$ be a basis of $\mathbf{R}^5$, (w_1, w_2, w_3, w_4) be a basis of $\mathbf{R}^4$.

Define $T_1, T_2 \in U$ as $T_1v_1 = 0,\ T_1v_2 = 0,\ T_1v_3 = 0,\ T_1v_4 = w_4,\ T_1v_5 = w_1;$

$T_2v_1 = 0,\ T_2v_2 = 0,\ T_2v_3 = w_3,\ T_2v_4 = 0,\ T_2v_5 = w_4.$ Thus $T_1 + T_2 \notin U$.

For $U' = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \textit{null}\, T > 0\}$,

define $T_1, T_2 \in U'$ as $T_1v_1 = 0,\ T_1v_2 = w_2,\ T_1v_3 = w_3,\ T_1v_4 = w_4,\ T_1v_5 = w_1;$

$T_2v_1 = w_1,\ T_2v_2 = w_2,\ T_2v_3 = 0,\ T_2v_4 = w_3,\ T_2v_5 = w_4.$ Thus $T_1 + T_2 \notin U'$. $\square$

---

7 *Suppose V is finite-dim with $2 \leq \dim V \leq \dim W$, if W is finite-dim.*
Show that $U = \{ T \in \mathcal{L}(V, W) : T$ is not injective $\}$ is not a subspace of $\mathcal{L}(V, W)$.

SOLUTION:

Let $(v_1, \ldots, v_n)$ be a basis of V, $(w_1, \ldots, w_m)$ be linearly independent in W.

(Let $\dim W = m$, if W is finite, otherwise, we choose $m \in \{n, n+1, \ldots\}$ arbitrarily; $2 \leq n \leq m$).

Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 :\ v_1 \mapsto 0,\quad v_2 \mapsto w_2,\quad v_i \mapsto w_i.$

Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 :\ v_1 \mapsto w_1,\quad v_2 \mapsto 0,\quad v_i \mapsto w_i,\quad i = 3, \ldots, n.$

Thus $T_1 + T_2 \notin U$. $\square$

COMMENT: If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V, W), T$ is injective. Hence $U = \varnothing$.

If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0v_0 = 0$.

If V is infinite-dim, the result is true as well.

---

8 *Suppose W is finite-dim with $\dim V \geq \dim W \geq 2$, if V is finite-dim.*
Show that $U = \{ T \in \mathcal{L}(V, W) : T$ is not surjective $\}$ is not a subspace of $\mathcal{L}(V, W)$.

SOLUTION:

Let $(v_1, \ldots, v_n)$ be linearly independent in V, $(w_1, \ldots, w_m)$ be a basis of W.

(Let $n = \dim V$, if V is finite, otherwise we choose $n \in \{m, m+1, \ldots\}$; $2 \leq m \leq n$).

Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 :\ v_1 \mapsto 0,\quad v_2 \mapsto w_2,\quad v_j \mapsto w_j,\quad v_{m+i} \mapsto 0.$

Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 :\ v_1 \mapsto w_1,\quad v_2 \mapsto 0,\quad v_j \mapsto w_j,\quad v_{m+i} \mapsto 0.$

For each $j = 2, \ldots, m;\ i = 1, \ldots, n - m$, if V is finite, otherwise let $i \in \mathbf{N}^+$.

Thus $T_1 + T_2 \notin U$. $\square$

COMMENT: If $\dim W = 0$, then $W = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V, W), T$ is surjective. Hence $U = \varnothing$.

If $\dim W = 1$, then $W = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0v_0 = 0$.

If W is infinite-dim, the result is true as well.

---

9 *Suppose $T \in \mathcal{L}(V, W)$ is injective and $(v_1, \ldots, v_n)$ is linearly independent in V.*
Prove that $(Tv_1, \ldots, Tv_n)$ is linearly independent in W.

SOLUTION:

$a_1Tv_1 + \cdots + a_nTv_n = 0 = T(\sum_{i=1}^{n} a_iv_i) \iff \sum_{i=1}^{n} a_iv_i = 0 \iff a_1 = \cdots = a_n = 0.$ $\square$

---

10 *Suppose* $(v_1, \ldots, v_n)$ *spans* V *and* $T \in \mathcal{L}(V, W)$. *Show that* $(Tv_1, \ldots, Tv_n)$ *spans range* T.

SOLUTION:

(a) range $T = \{ Tv : v \in V \} = \{ Tv : v \in \text{span}\,(v_1, \ldots, v_n)\}$

$\Rightarrow Tv_1, \ldots, Tv_n \in$ range $T \Rightarrow$ By [2.7] and [3.19], span $(Tv_1, \ldots, Tv_n) \subseteq$ range T.

(b) $\forall w \in$ range T, $\exists v \in V, Tv = w$. 又 $\forall v \in V$, $\exists\, a_i \in \mathbf{F}, v = a_1v_1 + \cdots + a_nv_n$

$\Rightarrow w = Tv = a_1Tv_1 + \cdots + a_nTv_n \Rightarrow$ range $T \subseteq$ span $(Tv_1, \ldots, Tv_n)$. $\square$

---

11 *Suppose* $S_1, \ldots, S_n$ *are injective linear maps and* $S_1S_2 \ldots S_n$ *makes sence.*
Prove that $S_1S_2 \ldots S_n$ *is injective.*

SOLUTION: $S_1S_2 \ldots S_n(v) = 0 \iff S_2S_3 \ldots S_n(v) = 0 \iff \cdots \iff S_n(v) = 0 \iff v = 0.$ $\square$

---

12 *Suppose that* V *is finite-dim and that* $T \in \mathcal{L}(V, W)$. *Prove that there exists a subspace* U *of* V *such that* $U \cap$ *null* $T = \{0\}$ *and range* $T = \{ Tu : u \in U \}$.

SOLUTION:

By [2.34], there exists a subspace U of V such that $V = U \oplus$ null T.

$\forall v \in V$, $\exists!\ w \in$ null $T, u \in U, v = w + u$. Then $Tv = T(w + u) = Tu \in \{ Tu : u \in U \} \Rightarrow \square$

COMMENT: V can be infinite-dim. See the above of [2.34].

---

16 *Suppose there exists a linear map on* V *whose null space and range are both finite-dim. Prove that* V *is finite-dim.*

SOLUTION:

Denote the linear map by T. Let $(Tv_1, \ldots, Tv_n)$ be a basis of range T, $(u_1, \ldots, u_m)$ be a basis of null T.

Then for all $v \in V, T(\underbrace{v - a_1v_1 - \cdots - a_nv_n}_{u \,\in\, \text{null}\, T}) = 0$, where $Tv = a_1Tv_1 + \cdots + a_nTv_n$.

$\Rightarrow u = b_1u_1 + \cdots + b_mu_m \Rightarrow v = a_1v_1 + \cdots + a_nv_n + b_1u_1 + \cdots + b_mu_m$.

Getting $V \subseteq$ span $(v_1, \ldots, v_n, u_1, \ldots, u_m)$. Thus V is finite-dim. $\square$

---

17 *Suppose* V *and* W *are both finite-dim. Prove that there exists an injective* $T \in \mathcal{L}(V, W)$ *if and only if* $\dim V \leq \dim W$.

SOLUTION:

(a) Suppose there exists an injective T. Then $\dim V = \dim \text{range}\, T + \dim \text{null}\, T = \dim \text{range}\, T \leq \dim W$.

(b) Suppose $\dim V \leq \dim W$, letting $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ be bases of V and W respectively.

Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i$, $i = 1, \ldots, n\, (= \dim V\,)$. $\square$

---

18 *Suppose* V *and* W *are both finite-dim. Prove that there exists a surjective* $T \in \mathcal{L}(V, W)$ *if and only if* $\dim V \geq \dim W$.

SOLUTION:

(a) Suppose there exists a surjective T. Then $\dim V = \dim \text{range}\, T + \dim \text{null}\, T = \dim W + \dim \text{null}\, T$

$\Rightarrow \dim W = \dim V - \dim \text{null}\, T \leq \dim V$.

(b) Suppose $\dim V \geq \dim W$, letting $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ be bases of V and W respectively.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1v_1 + \cdots + a_mv_m + a_{m+1}v_{m+1} + \cdots + a_nv_n) = a_1w_1 + \cdots + a_mw_m$. $\square$

---

19 *Suppose V and W are finite-dim and that U is a subspace of V.*
Prove that $\exists\ T \in \mathcal{L}(V,W),\ null\,T = U \iff \dim U \geq \dim V - \dim W$.

SOLUTION:

(a) Suppose $\exists\ T \in \mathcal{L}(V,W),\ \text{null}\,T = U$. Then $\dim \text{null}\,T = \dim U \geq \dim V - \dim W$.

(b) Suppose $\underbrace{\dim U}_{m} \geq \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_{p}$ ($\Rightarrow \dim W = p \geq n = \dim V - \dim U$).

Let $(u_1, \ldots, u_m)$ be a basis of U, extend to a basis of V as $(u_1, \ldots, u_m, v_1, \ldots, v_n)$.

Let $(w_1, \ldots, w_p)$ be a basis of W.

Define $T \in \mathcal{L}(V,W)$ by $T(a_1v_1 + \cdots + a_nv_n + b_1u_1 + \cdots + b_mu_m) = a_1w_1 + \cdots + a_nw_n$. $\square$

• **TIPS:** *Suppose $T \in \mathcal{L}(V,W)$ and $R = (Tv_1, \ldots, Tv_n)$ is linearly independent in range T.*

(Let $\dim$ range $T = n$, if range T is finite, otherwise choose n arbitrarily.).

By (3.A.4), $L = (v_1, \ldots, v_n)$ is linearly independent in V.

NEW NOTATION: *Denote $\mathcal{K}_R$ by span L, if range T is finite-dim,*
otherwise, denote it by an vector space in the set $\mathcal{S}_V$ null T.

NEW THEOREM:

$$\mathcal{K}_R \oplus \text{null}\,T = V \Leftarrow \begin{cases} \text{(a) } T(\sum_{i=1}^{n} a_iv_i) = 0 \Rightarrow \sum_{i=1}^{n} a_iTv_i = 0 \Rightarrow a_1 = \cdots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \text{null}\,T = \{0\}. \\ \text{(b) } \forall v \in V, Tv = \sum_{i=1}^{n} a_iTv_i \Rightarrow Tv - \sum_{i=1}^{n} a_iTv_i = T(v - \sum_{i=1}^{n} a_iv_i) = 0 \\ \quad \Rightarrow v - \sum_{i=1}^{n} a_iv_i \in \text{null}\,T \Rightarrow v = (v - \sum_{i=1}^{n} a_iv_i) + (\sum_{i=1}^{n} a_iv_i) \Rightarrow \mathcal{K}_R + \text{null}\,T = V. \end{cases}$$

COMMENT: null $T \in \mathcal{S}_V \mathcal{K}_R$.

• *Suppose V is finite-dim, $T \in \mathcal{L}(V,W)$, and U is a subspace of W.*
Prove that $\mathcal{K}_U = \{\, v \in V : Tv \in U \,\}$ is a subspace of V
and $\dim \mathcal{K}_U = \dim null\,T + \dim(U \cap range\,T)$.

SOLUTION: For any $u, w \in \mathcal{K}_U$ and $\lambda \in \mathbf{F}$, $T(u + \lambda w) = Tu + \lambda Tw \in U \Rightarrow T$ is linear

Define $S \in \mathcal{L}(\mathcal{K}_U, U)$ as $Rv = Tv$ for all $v \in \mathcal{K}_U$. Hence range $R = U \cap$ range T.

Suppose $Tv = 0$ for some $v \in V$. 又 $0 \in U \Rightarrow Rv = 0$. Thus null $T \subseteq$ null R. $\square$

---

20 *Suppose $T \in \mathcal{L}(V,W)$. Prove that T is injective $\iff \exists\ S \in \mathcal{L}(W,V),\ ST = I \in \mathcal{L}(V)$.*

SOLUTION:

(a) Suppose $\exists\ S \in \mathcal{L}(W,V),\ ST = I$. Then if $Tv = 0 \Rightarrow ST(v) = 0 = v$. Hence T is injective.

(b) Suppose T is injective. $\forall w \in$ range T, $\exists!\, v \in V, Tv = w$. (if $w = 0$, then $v = 0$)

Define $S : W \to V$ by $Sw = v$ and $Su = 0,\ u \in U$. Where $W = U \oplus$ range T.

$\Rightarrow S(Tv + \lambda Tu) = S(T(v + \lambda u)) = v + \lambda u$ and $S(x + \nu y) = 0,\ x, y \in U$.

Thus $S|_{\text{range}\,T+U} = S|_W \in \mathcal{L}(W,V)$ and $ST = I$. $\square$

OR. Let $R = (Tv_1, \ldots, Tv_n)$ be linearly independent in range $T \subseteq W$, $(\ldots)$ and then $\mathcal{K}_R \oplus$ null $T = V$.

Supose $W = U \oplus$ range T. Define $S \in \mathcal{L}(W,V)$ by $S(Tv_i) = v_i$ and $Su = 0,\ u \in U$. Thus $ST = I$. $\square$

---

21 *Suppose $T \in \mathcal{L}(V,W)$. Prove that T is surjective $\iff \exists\ S \in \mathcal{L}(W,V),\ TS = I \in \mathcal{L}(W)$.*

SOLUTION:

(a) Suppose $\exists\ S \in \mathcal{L}(W,V),\ TS = I$. Then for any $w \in W, TS(w) = w \in$ range $T \Rightarrow$ range $T = W$. $\square$

(b) Suppose T is surjective. $\forall w \in W,\ \exists v \in V, Tv = w$. Define $S : W \to V$ by $Sw = v$.

But $T(Sv + \lambda Su) = T(Sv) + \lambda T(Su) = v + \lambda u = T(S(v + \lambda u)) \not\Rightarrow Sv + \lambda Su = S(v + \lambda u)$.

So we let $R = (Tv_1, \ldots, Tv_n)$ be linearly independent in range $T = W$, $(\ldots)$ and then $\mathcal{K}_R \oplus \operatorname{null} T = V$. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$. Then $TS = I$. $\square$

---

22 *Suppose U and V are finite-dim vec-sps and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that $\dim \textit{null}\, ST \leq \dim \textit{null}\, S + \dim \textit{null}\, T$.*

SOLUTION: Define $R \in \mathcal{L}(\operatorname{null} ST, V)$ by $Ru = Tu$ for all $u \in \operatorname{null} ST \subseteq U$.

$$\left.\begin{aligned} S(Tu) = 0 = S(Ru) &\Rightarrow \operatorname{range} R \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} R \leq \dim \operatorname{null} S \\ Tu = 0 = Ru &\Rightarrow \operatorname{null} R \supseteq \operatorname{null} T \Rightarrow \dim \operatorname{null} R = \dim \operatorname{null} T \end{aligned}\right\} \Rightarrow \square$$

• **COROLLARY:**

(1) If T is injective, then $\dim \operatorname{null} T = 0 \Rightarrow \dim \operatorname{null} ST \leq \dim \operatorname{null} S$.

(2) If T is surjective, then $\operatorname{range} R = \operatorname{null} S \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.

(3) If S is injective, then $\operatorname{range} R = \{0\} \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} R = \dim \operatorname{null} T$.

---

23 *Suppose U and V are finite-dim vec-sps and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that $\dim \textit{range}\, ST \leq \min\{\dim \textit{range}\, S, \dim \textit{range}\, T\}$.*

SOLUTION:

$\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{span}(Su_1, \ldots, Su_{\dim \operatorname{range} T})$, letting $\operatorname{span}(u_1, \ldots, u_{\dim \operatorname{range} T}) = \operatorname{range} T$.

$\dim \operatorname{range} ST \leq \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leq \dim \operatorname{range} S \Rightarrow \square$

• **COROLLARY:**

(1) If S is injective, then $\dim \operatorname{range} ST = \dim \operatorname{range} T$.

(2) If T is surjective, then $\operatorname{range} ST = \operatorname{range} S$.

• (a) *Suppose $\dim V = 5$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \textit{range}\, TS \leq 2$.*
(b) *Give an example of $S, T \in \mathcal{L}(\mathbf{F}^5)$ with $ST = 0$ and $\dim \textit{range}\, TS = 2$.*

SOLUTION:

By Problem (23), $\dim \operatorname{range} TS \leq \min\{\underbrace{\dim \operatorname{range} S}_{5 - \dim \operatorname{null} T}, \underbrace{\dim \operatorname{range} T}_{5 - \dim \operatorname{null} S}\}$.

Suppose $\dim \operatorname{range} TS \geq 3$. Then $\min\{5 - \dim \operatorname{null} T, 5 - \dim \operatorname{null} S\} \geq 3$

$\Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leq 2$.

又 $\dim \operatorname{null} ST = 5 \leq \dim \operatorname{null} S + \dim \operatorname{null} T \leq 4$. Contradicts. Thus $\dim \operatorname{range} TS \leq 2$. $\square$

EXAMPLE: $V = \operatorname{span}(v_1, \ldots, v_5)$

$$\begin{aligned} T:&\quad v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i \;; \\ S:&\quad v_1 \mapsto v_4, \quad v_2 \mapsto v_5, \quad v_i \mapsto 0 \;; \qquad i = 3, 4, 5 \end{aligned}$$

• *Suppose $\dim V = n$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$.*

Prove that $\dim TS \leq m = \begin{cases} \frac{n}{2}, & \textit{if } 2 \mid n. \\ \frac{n-1}{2}, & \textit{otherwise.} \end{cases}$

SOLUTION:

By Problem (23), $\dim \operatorname{range} TS \leq \min\{\underbrace{\dim \operatorname{range} S}_{n - \dim \operatorname{null} T}, \underbrace{\dim \operatorname{range} T}_{n - \dim \operatorname{null} S}\}$. Suppose $\dim \operatorname{range} TS \geq m + 1$.

Then $\min\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\} \geq m + 1$

$\Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leq n - m - 1$.

又 $\dim \operatorname{null} ST = n \leq \dim \operatorname{null} S + \dim \operatorname{null} T \leq n - m - 1$. Contradicts. Thus $\dim \operatorname{range} TS \leq m$. $\square$

---

24 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$. Prove that $\textit{null}\, S \subseteq \textit{null}\, T \iff \exists\, E \in \mathcal{L}(W)$ such that $T = ES$.*

SOLUTION:

Suppose null $S \subseteq$ null T. Let $R = (Sv_1, \dots, Sv_n)$ be a basis of range $S \Rightarrow (v_1, \dots, v_n)$ is linearly independent.

Let $\mathcal{K}_R =$ span $(v_1, \dots, v_n) \Rightarrow V = \mathcal{K}_R \oplus$ null S.

Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i$, $\ Eu = 0$; for each $i = 1 \ \dots, n$ and $u \in$ null S.

Hence $\forall v \in V$, $(\exists!\, a_i \in \mathbf{F}, u \in \text{null}\, S\,)$, $Tv = a_1Tv_1 + \cdots + a_nTv_n = E(a_1Sv_1 + \cdots + a_nSv_n) \Rightarrow T = ES$.

Suppose $\exists\ E \in \mathcal{L}(W)$ such that $T = ES$. Then null $T =$ null $ES \supseteq$ null S. $\square$

---

25 *Suppose that V is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that range $S \subseteq$ range $T \iff \exists\ E \in \mathcal{L}(V)$ such that $S = TE$.

SOLUTION:

Suppose range $S \subseteq$ range T. Let $(v_1, \dots, v_m)$ be a basis of V.

Because range $S \subseteq$ range $T \Rightarrow Sv_i \in$ range T for each i. Suppose $u_i \in V$ for each i such that $Tu_i = Tv_i$.

Thus defining $E \in \mathcal{L}(V)$ by $Ev_i = u_i$ for each $i \Rightarrow S = TE$.

Suppose $\exists\ E \in \mathcal{L}(V)$ such that $S = TE$. Then range $S =$ range $TE \subseteq$ range T. $\square$

---

26 *Prove that the differentiation map $D \in \mathcal{P}(\mathbf{R})$ is surjective.*

SOLUTION: Note that deg $Dx^n = n - 1$.

Because span $(Dx, Dx^2, \dots) \subseteq$ range D. 又 By (2.A.10), span $(Dx, Dx^2, \dots) =$ span $(1, x, \dots) = \mathcal{P}(\mathbf{R})$. $\square$

---

27 *Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.*

SOLUTION:

Define $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$.

Note that deg $Bx^n = n - 1$. Similar to Problem (26), we conclude that B is surjective.

Hence for any $p \in \mathcal{P}(\mathbf{R})$, there exists $q \in \mathcal{P}(\mathbf{R})$ such that $Bq = p$. $\square$

---

28 *Suppose $T \in \mathcal{L}(V, W)$ and $(w_1, \dots, w_m)$ is a basis of range T. Prove that*
$\exists\ \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ such that for all $v \in V, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.

SOLUTION:

Suppose $(v_1, \dots, v_m)$ in V such that $Tv_i = w_i$ for each i.

Then $(v_1, \dots, v_m)$ is linearly independent, extend it to a basis of V as $(v_1, \dots, v_m, u_1, \dots, u_n)$.

Note that $\forall v \in V, v = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n,\ \exists!\, a_i, b_i \in \mathbf{F} \Rightarrow Tv = a_1w_1 + \cdots + a_mw_m$.

Define $\varphi_i : V \to \mathbf{F}$ by $\varphi_i(v) = a_iv_i$ for each i. We now check the linearity.

$\forall v, u \in V\ (\exists!\, a_i, b_i, c_i, d_i \in \mathbf{F}\,), \lambda \in \mathbf{F}, \varphi_i(v + \lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda\varphi(u)$. $\square$

---

29 *Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Suppose $u \in V$ is not in null φ.*
Prove that $V =$ null $\varphi \oplus \{au : a \in \mathbf{F}\}$.

SOLUTION:

(a) Suppose $v = cu \in$ null $\varphi \cap \{au : a \in \mathbf{F}\}$, where $c \in \mathbf{F}$.

Then $\varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$. Hence null $\varphi \cap \{au : a \in \mathbf{F}\}$.

(b) Suppose $v \in V$. Then $v = (v - \dfrac{\varphi(v)}{\varphi(u)}u) + \dfrac{\varphi(v)}{\varphi(u)}u \Rightarrow \varphi(v) = 0$.

$$\left.\begin{array}{l} v - \frac{\varphi(v)}{\varphi(u)}u \in \text{null}\,\varphi \\ \frac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array}\right\} \Rightarrow V = \text{null}\,\varphi \oplus \{au : a \in \mathbf{F}\}. \ \square$$

This may seems strange. Here we explain why.

$\varphi \neq 0 \Rightarrow \exists$ a linearly independent list $(v_1, \ldots, v_n \in V)$ such that $\varphi(v_i) = a_i \neq 0$.

Choose a v_k arbitrarily. Then $\varphi(v_k - \frac{\varphi(v_k)}{\varphi(v_j)} v_j) = 0$ for each $j = 1, \ldots, k-1, k+1, \ldots, n$.

Thus span $\{v_k - \frac{\varphi(v_k)}{\varphi(v_j)} v_j\}_{j \neq k} \subseteq$ null φ. Hence there is only one nonzero vector in every vec-sp in $\mathcal{S}_V$null φ.

---

30 *Suppose $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$ and null $\varphi_1 =$ null $\varphi_2 =$ null φ. Prove that $\exists\ c \in \mathbf{F}, \varphi_1 = c\varphi_2$*

SOLUTION:

If null $\varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $u \in V/\text{null}\,\varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$.

By Problem (29), $V = \text{null}\,\varphi \oplus \text{span}\,(u)$.

Hence for any $v \in V, v = w + a_v u$, $\exists !\, w \in \text{null}\,\varphi, a_v \in \mathbf{F}$.

$\varphi_1(v) = a_v\varphi_1(u),\ \ \varphi_2(v) = a_v\varphi_2(u) \Rightarrow a_v = \dfrac{\varphi_1(v)}{\varphi_1(u)} = \dfrac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \dfrac{\varphi_1(u)}{\varphi_2(u)} = \dfrac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}$.

Thus $\varphi_1 = c\varphi_2$. $\square$

---

31 *Give an example of $T_1, T_2 \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^2)$ such that null $T_1 =$ null T_2 and that T_1 is not a scalar multiple of T_2.*

SOLUTION:

Let $(v_1, \ldots, v_5)$ be a basis of $\mathbf{R}^5$, (w_1, w_2) be a basis of $\mathbf{R}^2$. Define $T, S \in \mathcal{L}(V, W)$ by

$$\left.\begin{array}{lll} Tv_1 = w_1, & Tv_2 = w_2, & Tv_3 = Tv_4 = Tv_5 = 0 \\ Sv_1 = w_1, & Sv_2 = 2w_2, & Sv_3 = Sv_4 = Sv_5 = 0 \end{array}\right\} \Rightarrow \text{null}\,T = \text{null}\,S.$$

Suppose $T = \lambda S$. Then $w_1 = Tv_1 = \lambda S v_1 = \lambda w_1 \Rightarrow \lambda = 1$.

While $w_2 = Tv_2 = \lambda S v_2 = 2\lambda w_2 \Rightarrow \lambda = \frac{1}{2}$. Contradicts. $\square$

• *Suppose V is finite-dim, X is a subspace of V, and Y is a finite-dim subspace of W. Prove that there exists $T \in \mathcal{L}(V, W)$ such that null $T = X$ and range $T = Y$ if and only if* $\dim X + \dim Y = \dim V$.

SOLUTION:

(a) Suppose $\dim X + \dim Y = \dim V$. Let $(u_1, \ldots, u_n)$ be a basis of X, $R = (w_1, \ldots, w_m)$ be a basis of Y.

Extend $(u_1, \ldots, u_n)$ to a basis of V as $(u_1, \ldots, u_n, v_1, \ldots, v_m)$.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1v_1 + \cdots + a_mv_m + b_1v_1 + \cdots + b_nv_n) = a_1w_1 + \cdots + a_mw_m$.

Now we show that null $T = X$ and range $T = Y$

Suppose $v \in V$. Then $\exists !\, a_i, b_j \in \mathbf{F}, v = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n$.

$$\left.\begin{array}{l} v \in \text{null}\,T \Rightarrow Tv = 0 \\ \qquad\qquad\ \ \Rightarrow a_1 = \cdots = a_m = 0 \\ \qquad\qquad\ \ \Rightarrow v \in X \Rightarrow \text{null}\,T \subseteq X. \\ v \in X \Rightarrow v \in \text{null}\,T \Rightarrow \text{null}\,T \supseteq X. \end{array}\right\} \Rightarrow \text{null}\,T = X.$$

$$\left.\begin{array}{r} w \in \text{range}\,T \Rightarrow \exists\ v \in V, Tv = w \Rightarrow \text{let } v = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n \\ \Rightarrow Tv = w = a_1w_1 + \cdots + a_mw_m \Rightarrow w \in Y \Rightarrow \text{range}\,T \subseteq Y. \\ w \in Y \Rightarrow w = a_1Tv_1 + \cdots + a_mTv_m = T(a_1v_1 + \cdots + a_mv_m) \qquad\qquad \\ \Rightarrow w \in \text{range}\,T \Rightarrow \text{range}\,T \supseteq Y. \end{array}\right\} \Rightarrow \text{range}\,T = Y.$$

(b) Conversely it is true as well.

$\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V, W)$. Let $(Tv_1, \ldots, Tv_n)$ be a basis of range T.*

Extend $(v_1, \ldots, v_n)$ *to a basis of* V *as* $(v_1, \ldots, v_n, u_1, \ldots, u_m)$.
Prove or give a counterexample: $(u_1, \ldots, u_m)$ *is a basis of null* T.

SOLUTION: A counterexample:

Suppose $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$. Then span $(Tv_1, Tv_2, Tv_3) =$ span (w_1).
Extend (v_i) to (v_1, v_2, v_3) for each i. But none of $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is a basis of null T. $\square$

• *Suppose* V *is finite-dim and* $T \in \mathcal{L}(V, W)$. *Let* $(u_1, \ldots, u_m)$ *be a basis of null* T.
Extend $(u_1, \ldots, u_m)$ *to a basis of* V *as* $(u_1, \ldots, u_m, v_1, \ldots, v_n)$.
Prove or give a counterexample: $(Tv_1, \ldots, Tv_n)$ *spans range* T.

SOLUTION:

$\forall w \in$ range T, $\exists\, v \in V$, $(\exists!\, a_i, b_i \in \mathbf{F}), Tv = T(a_1v_1 + \cdots + a_nv_n) = w$
$\Rightarrow w \in$ span $(Tv_1, \ldots, Tv_n) \Rightarrow$ range $T \subseteq$ span $(Tv_1, \ldots, Tv_n)$. $\square$

COMMENT: If T is injective, then $(Tv_1, \ldots, Tv_n)$ is a basis of range T.

• (OR (5.B.4)) *Suppose* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove that* $V =$ *null* $P \oplus$ *range* P.

SOLUTION:

Let $P^2v_1, \ldots, P^2v_n$ be a basis of range P^2. Then $(Pv_1, \ldots, Pv_n)$ is linearly independent in V.

$$\left.\begin{array}{l}\text{Let } \mathcal{K} = \text{span}\,(Pv_1, \ldots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \text{null}\, P^2 \\ \text{又 } \mathcal{K} = \text{range}\, P = \text{range}\, P^2;\ \ \text{null}\, P = \text{null}\, P^2\end{array}\right\} \Rightarrow \square$$

OR.

(a) Suppose $v \in$ null $P \cap$ range P.
Then $\exists\, u \in V, v = Pu, Pv = 0 \Rightarrow v = Pu = P^2u = Pv = 0$. Hence null $P \cap$ range $P = \{0\}$.

(b) Note that $v = Pv + (v - Pv), Pv^2 = Pv$ for all $v \in V$.
Then $P(v - Pv) = 0 \Rightarrow v - Pv \in$ null P. Hence $V =$ range $P +$ null P. $\square$

• *Suppose* V *is finite-dim with* $\dim V > 1$. *Show that if* $\varphi : \mathcal{L}(V) \to \mathbf{F}$ *is a linear map such that* $\varphi(ST) = \varphi(S) \cdot \varphi(T)$ *for all* $S, T \in \mathcal{L}(V)$, *then* $\varphi = 0$.

HINT: *The description of the two-sided ideals of* $\mathcal{L}(V)$ *in Section 3A might be useful.*

SOLUTION: Using notations in (3.A.• the last).

Suppose $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \ldots, n\}$, $\varphi(R_{i,j}) \neq 0$.

Because $R_{i,j} = R_{x,j} \circ R_{i,x}$, $\forall x = 1, \ldots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, because $R_{i,x} = R_{y,x} \circ R_{i,y}$, $\forall y = 1, \ldots, n$. Thus $\varphi(R_{y,x}) \neq 0$ for any $x, y = 1, \ldots, n$.

Let $l \neq i, k \neq j$ and then $\varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. $\square$

• *Suppose that* V *and* W *are real vector spaces and* $T \in \mathcal{L}(V, W)$.
Define $T_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ *by* $T_{\mathbb{C}}(u + iv) = Tu + iTv$ *for all* $u, v \in V$.
(a) *Show that* $T_{\mathbb{C}}$ *is a (complex) linear map from* $V_{\mathbb{C}}$ *to* $W_{\mathbb{C}}$.

(b) *Show that* $T_{\mathbb{C}}$ *is injective* $\Longleftrightarrow$ T *is injective.*

(c) *Show that range* $T_{\mathbb{C}} = W_{\mathbb{C}} \Longleftrightarrow$ *range* $T = W$.

See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbb{C}}$.

The linear map $T_{\mathbb{C}}$ *is called the complexification of the linear map* T.

SOLUTION:

(a) $\forall u_1 + \mathrm{i}v_1, u_2 + \mathrm{i}v_2 \in V_{\mathbb{C}}, \lambda \in \mathbf{F}$,

$$T((u_1 + \mathrm{i}v_1) + \lambda(u_2 + \mathrm{i}v_2)) = T((u_1 + \lambda u_2) + \mathrm{i}(v_1 + \lambda v_2)) = T(u_1 + \lambda u_2) + \mathrm{i}T(v_1 + \lambda v_2)$$
$$= Tu_1 + \mathrm{i}Tv_1 + \lambda Tu_2 + \mathrm{i}\lambda Tv_2 = T(u_1 + \mathrm{i}v_1) + \lambda T(u_2 + \mathrm{i}v_2). \quad \square$$

(b)
$$\left.\begin{array}{l} \text{Suppose } T_{\mathbb{C}} \text{ is injective. Let } T(u) = 0 \Rightarrow T_{\mathbb{C}}(u + \mathrm{i}0) = Tu = 0 \Rightarrow u = 0. \\ \text{Suppose } T \text{ is injective. Let } T_{\mathbb{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + \mathrm{i}v = 0. \end{array}\right\} \Rightarrow \square$$

(c)
$$\left.\begin{array}{l} \text{Suppose } T_{\mathbb{C}} \text{ is surjective. } \forall w, x \in W,\ \exists\, u, v \in V, T(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = w + \mathrm{i}x \\ \qquad \Rightarrow Tu = w, Tv = x \Rightarrow \text{T is surjective.} \\ \text{Suppose } T \text{ is surjective. } \forall w, x \in W,\ \exists\, u, v \in V, Tu = w, Tv = x \\ \qquad \Rightarrow \forall w + \mathrm{i}x \in W_{\mathbb{C}},\ \exists\, u + \mathrm{i}v \in V, T(u + \mathrm{i}v) = w + \mathrm{i}x \Rightarrow T_{\mathbb{C}} \text{ is surjective.} \end{array}\right\} \Rightarrow \square$$

ENDED

3·C

• **NOTE FOR [3.47]:** $LHS = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k} = RHS.$

• **NOTE FOR [3.48]:**

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 1 \times 5 + 2 \times 8 & 1 \times 6 + 2 \times 9 & 1 \times 7 + 2 \times 10 \\ 3 \times 5 + 4 \times 8 & 3 \times 6 + 4 \times 9 & 3 \times 7 + 4 \times 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

• **NOTE FOR [3.49]:** $\because [(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$

$\therefore (AC)_{\cdot,k} = A_{\cdot,\cdot}C_{\cdot,k} = AC_{\cdot,k}$

• **EXERCISE 10:** $\because [(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$

$\therefore (AC)_{j,\cdot} = A_{j,\cdot}C_{\cdot,\cdot} = A_{j,\cdot}C.$

• *Suppose* $C \in \mathbf{F}^{m,c}$, $R \in \mathbf{F}^{c,p}$.

(a) For $k = 1, \ldots, p$, $\quad (CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum\limits_{r=1}^{c} C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$

(b) For $j = 1, \ldots, m$, $\quad (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum\limits_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

EXAMPLE:

$$P = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} \in \mathbf{F}^{2,3}.$$

$$P_{\cdot,1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 8 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 21 \\ 47 \end{pmatrix};$$

$$P_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix};$$

$$P_{\cdot,3} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 10 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 27 \\ 61 \end{pmatrix};$$

$$P_{1,\cdot} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = 1 \begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 2 \begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \end{pmatrix};$$

$$P_{2,\cdot} = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = 3 \begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 4 \begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 47 & 54 & 61 \end{pmatrix};$$

• **NOTE FOR [3.52]:** $\quad A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$

$\because (Ac)_{j,1} = \sum\limits_{r=1}^{n} A_{j,r}c_{r,1} = [\sum\limits_{r=1}^{n}(A_{\cdot,r}c_{r,1})]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$

$\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum\limits_{r=1}^{n} A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ OR. By $(Ac)_{\cdot,1} = Ac_{\cdot,1}$ Using (a) above.

• **EXERCISE 10:** $\quad a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$

$\because (aC)_{1,k} = \sum\limits_{r=1}^{n} a_{1,r}C_{r,k} = [\sum\limits_{r=1}^{n} a_{1,r}(C_{r,\cdot})]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$

$\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum\limits_{r=1}^{n} a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ OR. By $(aC)_{1,\cdot} = a_{1,\cdot}C$. Using (b) above.

• COLUMN-ROW FACTORIZATION (CR Factorization)

Suppose $A \in \mathbf{F}^{m,n}$, $A \neq 0$. *Let* $S_c = span\,(A_{\cdot,1}, \ldots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}$, $\dim S_c = c$.

And $S_r = span\,(A_{1,\cdot}, \ldots, A_{n,\cdot}) \subseteq \mathbf{F}^{1,n}$, $\dim S_r = r$.

Prove that $A = CR$. $\exists\ C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,n}$.

SOLUTION: Notice that $A \neq 0 \Rightarrow c, r \geq 1$.

Let $(C_{\cdot,1}, \ldots, C_{\cdot,c})$ be a basis of S_c, forming $C \in \mathbf{F}^{m,c}$.

Then for any $A_{\cdot,k}$, $\quad A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$, $\quad \exists\,!\, R_{1,k}, \ldots, R_{c,k} \in \mathbf{F}$.

Hence, by letting $R = \begin{pmatrix} R_{1,1} & \cdots & R_{1,n} \\ \vdots & \ddots & \vdots \\ R_{c,1} & \cdots & R_{c,n} \end{pmatrix}$, we have $A = CR$.

OR. Let $(R_{1,\cdot}, \ldots, R_{c,\cdot})$ be a basis of S_r, forming $R \in \mathbf{F}^{c,n}$.

For any $A_{j,\cdot}$, $\quad A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot} = (CR)_{j,\cdot}$, $\quad \exists\,!\, C_{j,1}, \ldots, C_{j,c} \in \mathbf{F}$. Similarly. $\square$

EXAMPLE:

$$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

(1) Because $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2 \begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix}$.

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix}$ can be uniquely written as a linear combination of $A_{1,\cdot}, A_{2,\cdot}$.

Hence $\dim S_r = 2$. We choose $(A_{1,\cdot}, A_{2,\cdot})$ as the basis.

(2) Because $\begin{pmatrix}10\\26\\46\end{pmatrix}=2\begin{pmatrix}7\\19\\33\end{pmatrix}-\begin{pmatrix}4\\12\\20\end{pmatrix}$; $\begin{pmatrix}1\\5\\7\end{pmatrix}=2\begin{pmatrix}4\\12\\20\end{pmatrix}-\begin{pmatrix}7\\19\\33\end{pmatrix}$.

Hence $\dim S_c = 2$. We choose $(A_{\cdot,2}, A_{\cdot,3})$ as the basis.

• COLUMN RANK EQUALS ROW RANK (Using the notation above)

For any $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

$\Rightarrow$ span $(A_{1,\cdot}, \dots, A_{m,\cdot}) = S_r =$ span $(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leq c = \dim S_c$.

Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leq r = \dim S_r = \dim S_c^t$. $\square$

• *Suppose $T \in \mathcal{L}(V)$, and $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are bases of V.*
Prove that the following are equivalent.
(a) *T is injective.*
(b) *The columns of $\mathcal{M}(T)$ are linearly independent in* $\mathbf{F}^{n,1}$.
(c) *The columns of $\mathcal{M}(T)$ span* $\mathbf{F}^{n,1}$.
(d) *The rows of $\mathcal{M}(T)$ span* $\mathbf{F}^{1,n}$.
(e) *The rows of $\mathcal{M}(T)$ are linearly independent in* $\mathbf{F}^{1,n}$.
Here $A = \mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.

SOLUTION:

T is injective $\iff \dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim \operatorname{range} T$

$\iff (Tu_1, \dots, Tu_n)$ is linearly independent in V, and therefore is a basis of V

$\iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n))$ is linearly independent, as well as $(A_{\cdot,1}, \dots, A_{\cdot,n})$

$\iff (A_{\cdot,1}, \dots, A_{\cdot,n})$ is a basis of $\mathbf{F}^{n,1}$.

$\Big($ 又 $\dim \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) = \dim \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = n$ $\Big)$

$\iff (A_{1,\cdot}, \dots, A_{n,\cdot})$ is a basis of $\mathbf{F}^{1,n}$. $\square$

• *Suppose A is an m-by-n matrix with $A \neq 0$.*
Prove that the rank of A is 1 if and only if there exist $(c_1, \dots, c_m) \in \mathbf{F}^m$ and $(d_1, \dots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j \cdot d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$.

SOLUTION: Using the notation in CR Factorization.

(a) Suppose $A = \begin{pmatrix}c_1\\\vdots\\c_m\end{pmatrix}\begin{pmatrix}d_1 & \cdots & d_n\end{pmatrix} = \begin{pmatrix}c_1d_1 & \cdots & c_1d_n\\\vdots & \ddots & \vdots\\c_md_1 & \cdots & c_md_n\end{pmatrix}$. $(\ \exists\, c_j, d_k \in \mathbf{F}, \forall j,k\ )$

Then $S_c = \operatorname{span}\left(\begin{pmatrix}c_1d_1\\\vdots\\c_md_1\end{pmatrix}, \begin{pmatrix}c_1d_2\\\vdots\\c_md_2\end{pmatrix}, \dots, \begin{pmatrix}c_1d_n\\\vdots\\c_md_n\end{pmatrix}\right) = \operatorname{span}\left(\begin{pmatrix}c_1\\\vdots\\c_m\end{pmatrix}\right)$.

OR. $S_r = \operatorname{span}\begin{pmatrix}\begin{pmatrix}c_1d_1 & \cdots & c_1d_n\end{pmatrix},\\\begin{pmatrix}c_2d_1 & \cdots & c_2d_n\end{pmatrix},\\\vdots\\\begin{pmatrix}c_md_1 & \cdots & c_md_n\end{pmatrix}\end{pmatrix} = \operatorname{span}\left(\begin{pmatrix}d_1 & \dots & d_n\end{pmatrix}\right)$. Hence the rank of A is 1.

(b) Suppose the rank of A is $\dim S_c = \dim S_r = 1$

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $\quad d_k' = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$. $(\forall j,k\,)$

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$. Letting $d_k = d'_k A_{1,1}$. $\square$

---

1 *Suppose $T \in \mathcal{L}(V,W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least* $\dim$ *range* T *nonzero entries.*

SOLUTION: Let $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$ be bases of V and W respectively. We prove by contradiction.

Suppose $A = \mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$ has at most $(\dim \operatorname{range} T - 1)$ nonzero entries.

Then by Pigeon Hole Principle, at least one of $A_{\cdot,k} = 0$.

Thus there are at most $(\dim \operatorname{range} T - 1)$ nonzero vectors in $Tv_1, \dots, Tv_n$.

While $\operatorname{range} T = \operatorname{span}(Tv_1, \dots, Tv_n) \Rightarrow \dim \operatorname{range} T \leq \dim \operatorname{range} T - 1$. Hence we get a contradiction. $\square$

---

3 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$.*
Prove that there exist a basis of V and a basis of W such that
[letting $A = \mathcal{M}(T)$ with respect to these bases],
$A_{k,k} = 1, A_{i,j} = 0$, *where* $1 \leq k \leq \dim$ *range* $T, i \neq j$.

SOLUTION:

Let $R = (Tv_1, \dots, Tv_n)$ be a basis of range T, extend it to the basis of W as $(Tv_1, \dots, Tv_n, w_1, \dots, w_p)$.

Let $\mathcal{K}_R = \operatorname{span}(v_1, \dots, v_n)$. Let $(u_1, \dots, u_m)$ be a basis of null T.

Then $(v_1, \dots, v_n, u_1, \dots, u_m)$ is the basis of V.

Thus $T(v_k) = Tv_k, T(u_j) = 0 \Rightarrow A_{k,k} = 1, A_{i,j}$ for each $k \in \{1, \dots, \dim \operatorname{range} T\}$ and $j \in \{1, \dots, m\}$. $\square$

---

4 *Suppose $(v_1, \dots, v_m)$ is a basis of V and W is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that there exists a basis $(w_1, \dots, w_n)$ of W such that
all entries in the first column of $A = \mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_n))$ are 0
except for possibly a 1 in the first row, first column.

SOLUTION: If $Tv_1 = 0$, then we are done. Otherwise, extend (Tv_1) to a basis of W, as desired. $\square$

---

5 *Suppose $(w_1, \dots, w_n)$ is a basis of W and V is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that there exists a basis $(v_1, \dots, v_m)$ of V such that
all entries in the first row of $\mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_n))$ are 0
except for possibly a 1 in the first row, first column.

SOLUTION:

Let $(u_1, \dots, u_m)$ be a basis of V. If $A_{1,\cdot} = 0$, then let $v_i = u_i$ for each $i = 1, \dots, n$, we are done.

Otherwise, $\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \end{pmatrix} \neq 0$, choose one $A_{1,k} \neq 0$.

Let $v_1 = \dfrac{u_k}{A_{1,k}}$; $\quad v_j = u_{j-1} - A_{1,j-1} v_1 \quad$ for $j = 2, \dots, k$;
$\qquad v_i = u_i - A_{1,i} v_1 \quad$ for $i = k+1, \dots, n$. $\square$

---

6 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$. Prove that* $\dim$ *range* $T = 1$
if and only if there exist a basis of V and a basis of W such that
with respect to these bases, all entries of $A = \mathcal{M}(T)$ equal 1.

SOLUTION:

Denote the bases of V and W by $B_V = (v_1, \dots, v_n)$ and $B_W = (w_1, \dots, w_m)$ respectively.

(a) Suppose B_V, B_W are the bases such that all entries of A equal 1.
Then $Tv_i = w_1 + \cdots + w_m$ for all $i = 1, \dots, n$. Hence $\dim \operatorname{range} T = 1$.

(b) Suppose $\dim \operatorname{range} T = 1$. Then $\dim \operatorname{null} T = \dim V - 1$.
Let $(u_2, \dots, u_n)$ be a basis of null T. Extend it to a basis of V as $(u_1, u_2, \dots, u_n)$.

Let $w_1 = Tv_1 - w_2 - \cdots - w_m$. Extend it to B_W the basis of W.

Let $v_1 = u_1, v_i = u_1 + u_i$. Extend it to B_V the basis of V. $\square$

---

12 *Give an example of 2-by-2 matrices A and B such that $AB \neq BA$.*

SOLUTION: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

---

13 *Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E and F are matrices whose sizes are such that $A(B + C)$ and $(D + E)F$ make sense. Explain why $AB + AC$ and $DF + EF$ both make sense and prove that.*

SOLUTION: Using [3.36], [3.43].

(a) Left distributive: Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Because $[A(B+C)]_{j,k} = \sum_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum_{r=1}^{n} (A_{j,r}B_{r,k} + A_{j,r}C_{r,k})$.

Hence we conclude that $A(B+C) = AB + AC$.

OR. Let $(e_1, \dots, e_M)$ be the standard basis of $\mathbf{F}^M$, where $M = \max\{m, n, p\}$.

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum_{j=1}^{m} A_{j,k}e_j$ for each $k = 1, \dots, n$. Then $\mathcal{M}(T) = A$.

Similarly, define S, R such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

Thus $T(S+R) = TS + TR$
$$\begin{aligned} &\Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR) \\ &\Rightarrow \mathcal{M}(T)[\mathcal{M}(S) + \mathcal{M}(R)] = \mathcal{M}(T)\mathcal{M}(S) + \mathcal{M}(T)\mathcal{M}(R) \\ &\Rightarrow A(B+C) = AB + AC. \end{aligned}$$

(b) Right distributive: Similarly.

Suppose $\mathcal{M}(T) = D, \mathcal{M}(S) = E, \mathcal{M}(R) = F$.

Then $(T+S)R = TR + SR$
$$\begin{aligned} &\Rightarrow \mathcal{M}((T+S)R) = \mathcal{M}(TR) + \mathcal{M}(SR) \\ &\Rightarrow [\mathcal{M}(T) + \mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)\mathcal{M}(R) + \mathcal{M}(S)\mathcal{M}(R) \\ &\Rightarrow (D+E)F = DF + EF. \ \square \end{aligned}$$

---

14 *Prove that matrix multiplication is associative. In other words, suppose A, B and C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that $(AB)C = A(BC)$.*

Try to find a clean proof that illustrates the following quote from Emil Artin:

"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

SOLUTION:

Because $[(AB)C]_{j,k} = (AB)_{j,\cdot}C_{\cdot,k} = \sum_{s=1}^{n} (A_{j,s}B_{s,\cdot})C_{\cdot,k} = \sum_{s=1}^{n} A_{j,s}(B_{s,\cdot}C_{\cdot,k}) = \sum_{s=1}^{n} A_{j,s}(BC)_{s,k} = A(BC)_{j,k}$

Hence we conclude that $(AB)C = A(BC)$.

OR. Suppose $A \in \mathbf{F}^{m,n}, B \in \mathbf{F}^{n,p}, C \in \mathbf{F}^{p,s}$.

Let $(e_1, \dots, e_M)$ be the standard basis of $\mathbf{F}^M$, where $M = \max\{m, n, p, s\}$.

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum_{j=1}^{m} A_{j,k}e_j$ for each $k = 1, \dots, n$. Then $\mathcal{M}(T) = A$.

Similarly, define S, R such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

Hence $(TS)R = T(SR) \Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR))$
$$\begin{aligned} &\Rightarrow [\mathcal{M}(T)\mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)[\mathcal{M}(S)\mathcal{M}(R)] \\ &\Rightarrow (AB)C = A(BC). \ \square \end{aligned}$$

---

15 *Suppose A is an n-by-n matrix and $1 \leq j, k \leq n$. Show that the entry in row j, column k, of A^3*

$(\textit{which is defined to mean } AAA)$ *is* $\sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.

SOLUTION: $(AAA)_{j,k} = (AA)_{j,\cdot}A_{\cdot,k} = \sum_{p=1}^{n}(A_{j,p}A_{p,\cdot})A_{\cdot,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.

$$OR.\ \ (AAA)_{j,k} = \sum_{r=1}^{n}(AA)_{j,r}A_{r,k} = \sum_{r=1}^{n}(\sum_{p=1}^{n} A_{j,p}A_{p,r})A_{r,k}$$
$$= \sum_{r=1}^{n}(A_{j,1}A_{1,r}A_{r,k} + \cdots + A_{j,n}A_{n,r}A_{r,k})$$
$$= A_{j,1}\sum_{r=1}^{n} A_{1,r}A_{r,k} + \cdots + A_{j,n}\sum_{r=1}^{n} A_{n,r}A_{r,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}. \ \square$$

ENDED

3·D

• *Suppose* $T \in \mathcal{L}(V,W)$ *is invertible. Show that* T^{-1} *is invertible and* $(T^{-1})^{-1} = T$.

SOLUTION:

$$\left.\begin{aligned} TT^{-1} = (T^{-1})^{-1}T^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = T^{-1}(T^{-1})^{-1} = I \in \mathcal{L}(W) \end{aligned}\right\} \Rightarrow T = (T^{-1})^{-1}, \text{ by the uniqueness of inverse.} \quad \square$$

---

1 *Suppose* $T \in \mathcal{L}(U,V)$ *and* $S \in \mathcal{L}(V,W)$ *are both invertible linear maps.*
Prove that $ST \in \mathcal{L}(U,W)$ *is invertible and that* $(ST)^{-1} = T^{-1}S^{-1}$.

SOLUTION:

$$\left.\begin{aligned} (ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(V) \end{aligned}\right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}, \text{ by the uniqueness of inverse.} \ \square$$

---

9 *Suppose* V *is finite-dim and* $S,T \in \mathcal{L}(V)$.
Prove that ST *is invertible* $\iff$ S *and* T *are invertible.*

SOLUTION:

Suppose S,T are invertible. Then $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$. Hence ST is invertible.

Suppose ST is invertible. Let $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$.

$$\left.\begin{aligned} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S \end{aligned}\right\} \Rightarrow T \text{ is injective, } S \text{ is surjective.}$$

Notice that V is finite-dim. Hence S, T are invertible. $\square$

OR. Suppose ST is invertible but S or T is not invertible ($\Rightarrow$ not surjective and injective).

If S is not invertible then $\dim \operatorname{range} S < \dim V$ and by Problem (23) in (3.B),

$\dim \operatorname{range} ST \leq \dim \operatorname{range} S < \dim V$. Thus ST is not surjective. Contradicts.

If T is not invertible then $\dim \operatorname{range} T < 0$. Similarly, ST is not surjective. Contradicts. $\square$

---

10 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that $ST = I \Longleftrightarrow TS = I$.*

SOLUTION:

Suppose $ST = I$. $\left.\begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T$ is injective, S is surjective.

Notice that V is finite-dim. Thus T, S are invertible.

OR. By Problem (9), V is finite-dim and $ST = I$ is invertible $\Rightarrow S, T$ are invertible.

$S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v$ (S is invertible).

OR. $ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T$. 又 $S = S \Rightarrow TS = S^{-1}S = I$.

Reversing the roles of S and T, we conclude that $TS = I \Rightarrow ST = I$. $\square$

---

11 *Suppose V is finite-dim and $S, T, U \in \mathcal{L}(V)$ and $STU = I$.*
Show that T is invertible and that $T^{-1} = US$.

SOLUTION: Using Problem (9) and (10).

$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I$.

$\Rightarrow U^{-1} = ST, \quad T^{-1} = US, \quad S^{-1} = TU$. $\square$

---

12 *Show that the result in Exercise* 11 *can fail without the hypothesis that V is finite-dim.*

SOLUTION:

Let $V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots), T(a_1, \dots) = (0, a_1, \dots), U = I$.

Then $STU = I$ but T^{-1} is not invertible.

---

13 *Suppose V is finite-dim and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective.*
Prove that S is injective.

SOLUTION:

By Problem (1) and (9), Notice that V is finite-dim. Then RST is invertible.

$(RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. $\square$

OR. Let $X = (RST)^{-1}$, $\left| \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is injective, and therefore is invertible.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ is surjective, and therefore is invertible.} \end{array} \right.$

Thus $S = R^{-1}(RST)T^{-1}$ is invertible.

---

15 *Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multiplication.*
In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists\, A \in \mathbf{F}^{m,n}, Tx = Ax, \forall x \in \mathbf{F}^{n,1}$.

SOLUTION:

Let $E_i \in \mathbf{F}^{n,1}$ for each $i = 1, \dots, n$ (where $M = \max\{m, n\}$) be such that $(E_i)_{j,1} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Then $(E_1, \dots, E_n)$ is linearly independent and thus is a basis of $\mathbf{F}^{n,1}$.

Similarly, let $(R_1, \dots, R_m)$ be a basis of $\mathbf{F}^{m,1}$.

Suppose $T(E_i) = A_{1,i}R_1 + \dots + A_{m,i}R_m$ for each $i = 1, \dots, n$. Hence by letting $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$. $\square$

COMMENT: $\mathcal{M}(T) = A$. Conversely it is true as well.

- OR (10.A.2) *Suppose* $A, B \in \mathbf{F}^{n,n}$. *Prove that* $AB = I \iff BA = I$.

SOLUTION: Using Problem (10) and (15).

Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Then $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.

Thus $AB = I \Leftrightarrow A(Bx) = x \iff T(Sx) = x \Leftrightarrow TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I.\Box$

- **NOTE FOR [3.60]:** Suppose $(v_1, \dots, v_n)$ is a basis of V and $(w_1, \dots, w_m)$ is a basis of W.

Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{ix} w_j$; $\quad \delta_{ix} = \begin{cases} 0, & i \neq x \\ 1, & i = x \end{cases}$ COROLLARY: $E_{l,k}E_{i,j} = \delta_{jl} E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. $(\mathcal{E}^{(j,i)})_{l,k} = \begin{cases} 0, & i \neq k \text{ or } j \neq l \\ 1, & i = k \text{ and } j = l \end{cases}$

Because $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$ are isomorphic. And $T = \mathcal{M}^{-1}\mathcal{M}(T), E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

Hence $\forall T \in \mathcal{L}(V, W),\ \exists!\, A_{i,j} \in \mathbf{F}\,(\forall i = \{1, \dots, m\}, j = \{1, \dots, n\}), \mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)}+ & \cdots & +A_{1,n}\mathcal{E}^{(1,n)} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}\mathcal{E}^{(m,1)}+ & \cdots & +A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff T = \begin{pmatrix} A_{1,1}E_{1,1}+ & \cdots & +A_{1,n}E_{n,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}E_{1,m}+ & \cdots & +A_{m,n}E_{n,m} \end{pmatrix}.$$

$$\therefore \mathcal{L}(V, W) = \operatorname{span} \underbrace{\begin{pmatrix} E_{1,1}, & \cdots & , E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,m}, & \cdots & , E_{n,m} \end{pmatrix}}_{B}; \quad \mathbf{F}^{m,n} = \operatorname{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & , \mathcal{E}^{(1,n)}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ \mathcal{E}^{(m,1)}, & \cdots & , \mathcal{E}^{(m,n)} \end{pmatrix}}_{B_M}.$$

Hence by [2.42] and [3.61], we conclude that B is a basis of $\mathcal{L}(V, W)$ and that B_M is a basis of $\mathbf{F}^{m,n}$.

○ *Suppose* V *is finite-dim and* $S \in \mathcal{L}(V)$. *Define* $A \in \mathcal{L}(\mathcal{L}(V))$ *by* $\mathcal{A}(T) = ST$ *for* $T \in \mathcal{L}(V)$.

(a) *Show that* $\dim \textit{null}\, \mathcal{A} = (\dim V)(\dim \textit{null}\, S)$.

(b) *Show that* $\dim \textit{range}\, \mathcal{A} = (\dim V)(\dim \textit{range}\, S)$.

SOLUTION:

(a) For all $T \in \mathcal{L}(V)$, $ST = 0 \iff \operatorname{range} T \subseteq \operatorname{null} S$. Thus $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S) \Rightarrow \Box$

(b) For all $R \in \mathcal{L}(V)$, $\operatorname{range} R \subseteq \operatorname{range} S \iff \exists T \in \mathcal{L}(V), R = ST$. Thus $\operatorname{range} \mathcal{A} = \mathcal{L}(V, \operatorname{range} S) \Rightarrow \Box$

OR. Using NOTE FOR [3.60].

Let $(w_1, \dots, w_m)$ be a basis of range S, extend it to a basis of V as $(w_1, \dots, w_m, \dots, w_n)$.

Let $v_i \in V$ such that $Sv_i = w_i$ for $m = 1, \dots, m$. Extend $(v_1, \dots, v_m)$ to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.

Define $E_{i,j} \in \mathcal{L}(V)$ by $E_{i,j}(v_x) = \delta_{ix} w_i$.

$$\text{Thus } S = E_{1,1} + \cdots + E_{m,m}; \quad \mathcal{M}(S, (v_1, \dots, v_n), (w_1, \dots, w_n)) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j}(w_x) = \delta_{ix} v_i$.

Let $E_{j,k}R_{i,j} = Q_{i,k}, \quad R_{j,k}E_{i,j} = G_{i,k}$

Because $\forall T \in \mathcal{L}(V), \quad \exists!\, A_{i,j} \in \mathbf{F}\,(\,\forall i,j = 1,\dots,n\,), \quad T = \begin{pmatrix} A_{1,1}R_{1,1}+ & \cdots & +A_{1,m}R_{m,1}+ & \cdots & +A_{1,n}R_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}R_{1,m}+ & \cdots & +A_{m,m}R_{m,m}+ & \cdots & +A_{m,n}R_{n,m} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{n,1}R_{1,n}+ & \cdots & +A_{n,m}R_{m,n}+ & \cdots & +A_{n,n}R_{n,n} \end{pmatrix}.$

$$\Rightarrow \mathcal{A}(T) = ST = (\sum_{r=1}^{m} E_{r,r})(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i}) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{i,j}Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1}+ & \cdots & +A_{1,m}Q_{m,1}+ & \cdots & +A_{1,n}Q_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}Q_{1,m}+ & \cdots & +A_{m,m}Q_{m,m}+ & \cdots & +A_{m,n}Q_{n,m} \end{pmatrix}.$$

Thus $\operatorname{null}\mathcal{A} = \operatorname{span}\begin{pmatrix} R_{1,m+1}, & \cdots & ,R_{n,m+1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ R_{1,n}, & \cdots & ,R_{n,n} \end{pmatrix}, \quad \operatorname{range}\mathcal{A} = \operatorname{span}\begin{pmatrix} Q_{1,1}, & \cdots & ,Q_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ Q_{1,m}, & \cdots & ,Q_{n,m} \end{pmatrix}.$

Hence (a) $\dim\operatorname{null}\mathcal{A} = n \times (n-m)$; (b) $\dim\operatorname{range}\mathcal{A} = n \times m$. $\square$

• COMMENT: Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(T) = TS$ for $T \in \mathcal{L}(V)$.

Similarly, $\mathcal{B}(T) = TS = (\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i})(\sum_{r=1}^{m} E_{r,r}) = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{i,j}G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1}+ & \cdots & +A_{1,m}G_{m,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}G_{1,m}+ & \cdots & +A_{m,m}G_{m,m} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{n,1}G_{1,n}+ & \cdots & +A_{n,m}G_{m,n} \end{pmatrix}.$

• OR (10.A.1) *Suppose $T \in \mathcal{L}(V)$ and $(v_1,\dots,v_n)$ is a basis of V.*
Prove that $\mathcal{M}(T,(v_1,\dots,v_n))$ is invertible $\Longleftrightarrow$ T is invertible.

SOLUTION: Notice that $\mathcal{M}$ is an isomorphism of $\mathcal{L}(V)$ onto $\mathbf{F}^{n,n}$.

(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I$. $\exists!\, S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$
$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$
$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. $\square$

• OR (10.A.4) *Suppose that $(\beta_1,\dots,\beta_n)$ and $(\alpha_1,\dots,\alpha_n)$ are bases of V.*
Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1,\dots,n$.
Prove that $\mathcal{M}(T,(\alpha_1,\dots,\alpha_n)) = \mathcal{M}(I,(\beta_1,\dots,\beta_n),(\alpha_1,\dots,\alpha_n))$.

SOLUTION:

For ease of notation, write $\mathcal{M}(T,\alpha\to\beta) = \mathcal{M}(T,(\alpha_1,\dots,\alpha_n),(\beta_1,\dots,\beta_n))$
and $\mathcal{M}(T,\alpha\to\alpha) = \mathcal{M}(T,(\alpha_1,\dots,\alpha_n))$.

Denote $\mathcal{M}(T,\alpha\to\alpha)$ by A and $\mathcal{M}(I,\beta\to\alpha)$ by B.

$\forall\, k \in \{1,\dots,n\}, Iu_k = u_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. $\square$

OR. Note that $\mathcal{M}(T,\alpha\to\beta)$ is the identity matrix.

$$\mathcal{M}(T,\alpha\to\alpha) = \mathcal{M}(I,\beta\to\alpha)\underbrace{\mathcal{M}(T,\alpha\to\beta)}_{=\mathcal{M}(I,\beta\to\beta)} = \mathcal{M}(I,\beta\to\alpha). \quad \square$$

OR. Note that $\mathcal{M}(T,\beta\to\beta)\mathcal{M}(I,\alpha\to\beta) = \mathcal{M}(T,\alpha\to\beta) = I$.

$$\mathcal{M}(T,\alpha\to\alpha) = \mathcal{M}(I,\alpha\to\beta)^{-1}[\underbrace{\mathcal{M}(T,\beta\to\beta)\mathcal{M}(I,\alpha\to\beta)}_{\mathcal{M}(T,\alpha\to\beta)}] = \mathcal{M}(I,\beta\to\alpha). \quad \square$$

• COMMENT: Denote $\mathcal{M}(T,\beta\to\beta)$ by A'.

$u_k = Iu_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n, \ \forall\, k \in \{1, \ldots, n\}.$

又 $Tu_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = B_{1,k}\beta_1 + \cdots + B_{n,k}\beta_n = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B.$

OR. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B.$

---

16 *Suppose V is finite-dim and $S \in \mathcal{L}(V)$.*
Prove that $\exists\, \lambda \in \mathbf{F}, S = \lambda I \iff ST = TS$ for every $T \in \mathcal{L}(V)$.

Solution: Using the notation and result in $(\circ)$.

Suppose $S = \lambda I$. Then $ST = TS = \lambda T$ for every $T \in \mathcal{L}(V)$. Conversely, if $S = 0$, then we are done.

Suppose $S \neq 0, ST = TS, \forall T \in \mathcal{L}(V)$.

Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, (v_1, \ldots, v_1)) = \mathcal{M}(I, (w_1, \ldots, w_n), (v_1, \ldots, v_n))$.

Then $\forall k \in \{m+1, \ldots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $n = \dim V = \dim \operatorname{range} S = m$.

Note that $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$. Thus $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \cdots + a_{n,i}v_n)$. Where:

$a_{i,j} = \mathcal{M}(I, (w_1, \ldots, w_n), (v_1, \ldots, v_n))_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n;$

For each j, for all i. Thus $a_{i,i} = a_{k,k} = \lambda, \forall\, k \neq i$.

Hence $w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} = \mathcal{M}(\lambda I, (v_1, \ldots, v_n)) \Rightarrow S = \mathcal{M}^{-1}(\mathcal{M}(\lambda I)) = \lambda I.$ $\square$

• OR (10.A.3) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that T has the same matrix with respect to every basis of V if and only if T is a scalar multiple of the identity operator.

Solution: [Compare with the first solution of Problem (16) in (3.A)]

Suppose $T = \lambda I$ for some $\lambda \in \mathbf{F}$. Then T has the same matrix with respect to every basis of V.

Conversely, if $T = 0$, then we are done; Suppose $T \neq 0$. And v is a nonzero vector in V.

Assume that (v, Tv) is linearly independent.

Extend (v, Tv) to a basis of V as $(v, Tv, u_3, \ldots, u_n)$. Let $B = \mathcal{M}(T, (v, Tv, u_3, \ldots, u_n))$.

$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \forall i \neq 2.$

By assumption, $A = \mathcal{M}(T, (v, w_2, \ldots, w_n)) = B$ for any basis $(v, w_2, \ldots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0\,(\cdots)$.

$\Rightarrow Tv = w_2$, which is not true if we let $w_2 = u_3, w_3 = Tv, w_j = u_j\ (\,j = 4, \ldots, n\,)$. Contradicts.

Hence (v, Tv) is linearly dependent $\Rightarrow \forall v \in V, \exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show that λ_v is independent of v, that is,

to show that for any two nonzero distinct vectors $v, w \in V, \lambda_v = \lambda_w$. Thus $T = \lambda I$ for some $\lambda \in \mathbf{F}$.

$$\left.\begin{aligned} &(v, w) \text{ is linearly independent} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) \\ &\qquad = \lambda_{v+w}v + \lambda_{v+w}w \\ &\qquad = \lambda_v v + \lambda_w w = Tv + Tw \Rightarrow \lambda_{v+w} = \lambda_v = \lambda_w \\ &(v, w) \text{ is linearly dependent, } w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \Rightarrow \lambda_v = \lambda_w \end{aligned}\right\} \Rightarrow \square$$

OR. Conversely, denote $\mathcal{M}(T, (u_1, \ldots, u_m))$ by A, where the basis $(u_1, \ldots, u_m)$ is arbitrarily chosen.

Fix one basis $(v_1, \ldots, v_m)$ and then $(v_1, \ldots, \frac{1}{2}v_k, \ldots, v_m)$ is also a basis for any given $k \in \{1, \ldots, m\}$.

Fix one k. Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \cdots + A_{k,k}(\frac{1}{2}v_k) + \cdots + A_{m,k}v_m$

$\Rightarrow Tv_k = 2A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + 2A_{m,k}v_m = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{m,k}v_m.$

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1, \ldots, m\}$.

Now we show that $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j, k arbitrarily but $j \neq k$.

Consider the basis $(v'_1, \ldots, v'_j, \ldots, v'_k, \ldots, v'_m)$, where $v'_j = v_k,\ v'_k = v_j$ and $v'_i = v_i$ for all $i \in \{1, \ldots, m\}\backslash\{j, k\}$.

Remember that $\mathcal{M}(T, (v'_1, \ldots, v'_m)) = \mathcal{M}(T, (v_1, \ldots, v_m)) = A$.

Hence $T(v'_k) = A_{1,k}v'_1 + \cdots + A_{k,k}v'_k + \cdots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_j$, while $T(v'_k) = T(v_j) = A_{j,j}v_j$.

Thus $A_{k,k} = A_{j,j}$. $\square$

---

17 *Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Using NOTE FOR [3.60]. Let $(v_1, \dots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.

Then for any $E_{i,j} \in \mathcal{E}$, ($\forall x, y = 1, \dots, n$), by assumption, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.

Again, $E_{y,x'}, E_{y',x} \in \mathcal{E}$ for all $x', y', x, y = 1, \dots, n$. Thus $\mathcal{E} = \mathcal{L}(V)$. $\square$

---

18 *Show that V and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces.*

SOLUTION:

Define $\Psi \in \mathcal{L}(V, \mathcal{L}(\mathbf{F}, V))$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.

(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is injective.

(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi(T(1))$. Hence Ψ is surjective.$\square$

OR.

Define $\Phi \in \mathcal{L}(\mathcal{L}(\mathbf{F}, V), V)$ by $\Phi(T) = T(1)$.

(a) Suppose $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is injective.

(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surjective. $\square$

COMMENT: $\Phi = \Psi^{-1}$.

• *Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that $q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$.*

SOLUTION:

Note that $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$.

Define $T_n : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.

As can be easily checked, T_n is an operator.

Because $\deg(T_n p) = \deg p$. If $T_n(p) = 0 \Rightarrow \deg(T_n p) = -\infty$, then $\deg p = -\infty \Rightarrow p = 0$.

Hence T_n is injective and therefore is surjective.

For all $q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $m = 0$; if $q \neq 0$, let $m = \deg q$. We have $q \in \mathcal{P}_m(\mathbf{R})$.

Hence $\exists\, p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$.

---

19 *Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is injective.* $\deg Tp \leq \deg p$ *for every nonzero $p \in \mathcal{P}(\mathbf{R})$.*

(a) *Prove that T is surjective.*

(b) *Prove that for every nonzero p,* $\deg Tp = \deg p$.

SOLUTION:

(a) T is injective $\iff T|_{\mathcal{P}_n(\mathbb{R})} : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ is injective for any $n \in \mathbf{N}^+$

$\iff T|_{\mathcal{P}_n(\mathbb{R})}$ is surjective for any $n \in \mathbf{N}^+ \iff T$ is surjective.

(b) Using mathematical induction.

(i) $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$.

$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$.

(ii) Suppose $\deg f = \deg Tf$ for all $f \in \mathcal{P}_n(\mathbf{R})$. Then suppose $\deg g = n + 1, g \in \mathcal{P}_{n+1}(\mathbf{R})$.

Assume that $\deg Tg < \deg g$ ($\Rightarrow \deg Tg \leq n, Tg \in \mathcal{P}_n(\mathbf{R})$).

Then by (a), $\exists\, f \in \mathcal{P}_n(\mathbf{R}), T(f) = (Tg)$. 又 T is injective $\Rightarrow f = g$.

While $\deg f = \deg Tf = \deg Tg < \deg g$. Contradicts the assumption.

Hence $\deg Tp = \deg p$ for all $p \in \mathcal{P}_{n+1}(\mathbf{R})$.

Thus $\deg Tp = \deg p$ for all $p \in \mathcal{P}(\mathbf{R})$. $\square$

• *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_m)$ is a list in V such that $(Tv_1, \ldots, Tv_m)$ spans V. Prove that $(v_1, \ldots, v_m)$ spans V.*

SOLUTION:

$V = \operatorname{span}(Tv_1, \ldots, Tv_m) \Rightarrow T$ is surjective, 又 V is finite-dim $\Rightarrow T$ is invertible $\Rightarrow T^{-1}$ is invertible.

$\forall v \in V,\ \exists\, a_i \in \mathbf{F}, v = a_1 Tv_1 + \cdots + a_n Tv_n$

$\Rightarrow T^{-1}v = a_1 v_1 + \cdots + a_n v_n$

$\Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \ldots, v_n)$ 又 $\operatorname{range} T^{-1} = V$. $\square$

OR. Reduce $(Tv_1, \ldots, Tv_n)$ to a basis of V as $(Tv_{\alpha_1}, \ldots, Tv_{\alpha_m})$, where $m = \dim V$ and $\alpha_i \in \{1, \ldots, m\}$.
Then $(v_{\alpha_1}, \ldots, v_{\alpha_m})$ is linearly independet of length m, therefore is a basis of V, contained in the list $(v_1, \ldots, v_m)$. $\square$

---

2 *Suppose V is finite-dim and $\dim V > 1$.*
Prove that the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.

SOLUTION:

Suppose $\dim V = n > 1$. Let $(v_1, \ldots, v_n)$ be a basis of V.
Define $S, T \in \mathcal{L}(V)$ by $S(a_1 v_1 + \cdots + a_n v_n) = a_1 v_1$ and $T(a_1 v_1 + \cdots + a_n v_n) = a_2 v_1 + \cdots + a_n v_n$.
Hence $S + T = I$ is invertible.
Thus the set of noninvertible linear maps in $\mathcal{L}(V)$ is not closed under addition and therefore is not a subspace. $\square$
COMMENT: If $\dim V = 1$, then the set of noninvertible operators on V equals $\{0\}$, which is a subspace of $\mathcal{L}(V)$.

---

3 *Suppose V is finite-dim, U is a subspace of V, and $S \in \mathcal{L}(U, V)$.*
Prove that there exists an invertible $T \in \mathcal{L}(V, V)$ such that
$Tu = Su$ for every $u \in U$ if and only if S is injective.

SOLUTION: [*Compare this with (3.A.11).*]

(a) $Tu = Su$ for every $u \in U \Rightarrow u = T^{-1}Su \Rightarrow S$ is injective. OR. $\operatorname{null} S = \operatorname{null} T \cap U = \{0\} \cap U = \{0\}$.

(b) Suppose $(u_1, \ldots, u_m)$ be a basis of U and S is injective $\Rightarrow (Su_1, \ldots, Su_m)$ is linearly independent in V.
Extend these to bases of V as $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ and $(Su_1, \ldots, Su_m, w_1, \ldots, w_n)$.
Define $T \in \mathcal{L}(V)$ by $T(u_i) = Su_i;\ \ Tv_j = w_j$, for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$.

---

4 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that null S = null T $(= U) \Longleftrightarrow S = ET,\ \exists$ invertible $E \in \mathcal{L}(W)$.

SOLUTION:

Define $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i,\ \ E(w_j) = x_j$, for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. Where:

Let $(Tv_1, \ldots, Tv_m)$ be a basis of $\operatorname{range} T$, extend it to a basis of W as $(Tv_1, \ldots, Tv_m, w_1, \ldots, w_n)$.
Let $(u_1, \ldots, u_n)$ be a basis of U. Then by (3.B.TIPS), $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ is a basis of V.
又 $\operatorname{null} S = \operatorname{null} T \Rightarrow V = \operatorname{span}(v_1, \ldots, v_m) \oplus \operatorname{null} S \Rightarrow \operatorname{span}(Sv_1, \ldots, Sv_m) = \operatorname{range} S$.
And $\dim \operatorname{range} T = \dim \operatorname{range} S = \dim V - \operatorname{null} U = m$. Hence $(Sv_1, \ldots, Sv_m)$ is a basis of $\operatorname{range} S$.
Thus we let $(Sv_1, \ldots, Sv_m, x_1, \ldots, x_n)$ be a basis of W.

Hence E is invertible and $S = ET$.

Conversely, $S = ET \Rightarrow \operatorname{null} S = \operatorname{null} ET$.
Then $v \in \operatorname{null} ET \Longleftrightarrow ET(v) = 0 \Longleftrightarrow Tv = 0 \Longleftrightarrow v \in \operatorname{null} T$. Hence $\operatorname{null} ET = \operatorname{null} T = \operatorname{null} S$. $\square$

---

5 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that range S = range T $(= R) \Longleftrightarrow S = TE,\ \exists$ invertible $E \in \mathcal{L}(V)$.

SOLUTION:

Define $E \in \mathcal{L}(V)$ as $E: \quad v_i \mapsto r_i \ ; \quad u_j \mapsto s_j;$ for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. Where:

Let $(Tv_1, \ldots, Tv_m)$ and $(Sr_1, \ldots, Sr_m)$ be bases of R such that $\forall\, i, Tv_i = Sr_i$.
Let $(u_1, \ldots, u_n)$ and $(s_1, \ldots, s_n)$ be bases of null T and null S respectively.
Thus $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ and $(r_1, \ldots, r_m, s_1, \ldots, s_n)$ are bases of V.

Hence E is invertible and $S = TE$.

Conversely, $S = TE \Rightarrow$ range $S =$ range TE.
Then $w \in$ range $S \iff \exists\, v \in V, Sv = TE(v) = T(E(v)) = w \in$ range T. Hence range $S =$ range T. $\square$

---

6 *Suppose V and W are finite-dim and $S, T \in \mathcal{L}(V, W)$.* [$\dim \operatorname{null} S = \dim \operatorname{null} T = n$]
Prove that $S = E_2 T E_1$, $\exists$ invertible $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W) \iff \dim$ null $S = \dim$ null T.

SOLUTION:

Define $E_1: \quad v_i \mapsto r_i \ ; \quad u_j \mapsto s_j;$ for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$.
Define $E_2: Tv_i \mapsto Sr_i \ ; \ x_j \mapsto y_j;$ for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. Where:

Let $(Tv_1, \ldots, Tv_m)$ and $(Sr_1, \ldots, Sr_m)$ be bases of range T and range S.
Let $(u_1, \ldots, u_n)$ and $(s_1, \ldots, s_n)$ be bases of null T and null S respectively.
Thus $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ and $(r_1, \ldots, r_m, s_1, \ldots, s_n)$ are bases of V.
Extend $(Tv_1, \ldots, Tv_m)$ and $(Sr_1, \ldots, Sr_m)$ to bases of W as
$(Tv_1, \ldots, Tv_m, x_1, \ldots, x_p)$ and $(Sr_1, \ldots, Sr_m, y_1, \ldots, y_p)$.

Thus E_1, E_2 are invertible and $S = E_2 T E_1$.

Conversely, $S = E_2 T E_1 \Rightarrow \dim$ null $S = \dim$ null $E_2 T E_1$.
$v \in$ null $E_2 T E_1 \iff E_2 T E_1(v) = 0 \iff T E_1(v) = 0$. Hence null $E_2 T E_1 =$ null $T E_1 =$ null S.
又 By (3.B.22.COROLLARY), E is invertible $\Rightarrow \dim$ null $T E_1 = \dim$ null $T = \dim$ null S. $\square$

---

8 *Suppose V is finite-dim and $T: V \to W$ is a surjective linear map of V onto W.*
Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.

$T|_U$ is the function whose domain is U, with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.

SOLUTION:

T is surjective $\Rightarrow$ range $T = W \Rightarrow \dim$ range $T = \dim W = \dim V - \dim$ null T.
Let $(w_1, \ldots, w_m)$ be a basis of range $T = W \Rightarrow \forall w_i, \exists!\, v_i \in V, Tv_i = w_i$.
$\Rightarrow (v_1, \ldots, v_m)$ is a basis of $\mathcal{K}$. Thus $\dim \mathcal{K} = \dim W$.
Thus $T|_{\mathcal{K}}$ maps a basis of $\mathcal{K}$ to a basis of range $T = W$. Denote $\mathcal{K}$ by U. $\square$
OR. By Problem (12) in (3.B), there is a subspace U of V such that
$U \cap$ null $T = \{0\} =$ null $T|_U$, range $T = \{Tu : u \in U\} =$ range $T|_U$. $\square$

• *Suppose V and W are finite-dim and U is a subspace of V.*
Let $\mathcal{E} = \{\, T \in \mathcal{L}(V, W) : U \subseteq$ null $T \,\}$.
(a) *Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.*
(b) *Find a formula for* $\dim \mathcal{E}$ *in terms of* $\dim V$, $\dim W$ *and* $\dim U$.

Hint: Define $\Phi: \mathcal{L}(V, W) \to L(U, W)$ by $\Phi(T) = T|_U$. What is null Φ? What is range Φ?

SOLUTION:

(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.
(b) Define Φ as in the hint.
$T \in$ null $\Phi \iff \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$. Hence null $\Phi = \mathcal{E}$.
$S \in \mathcal{L}(U, W) \Rightarrow \exists T \in \mathcal{L}(V, W), \Phi(T) = S$, by (3.B.11) $\Rightarrow S \in$ range T. Hence range $\Phi = \mathcal{L}(U, W)$.
Thus $\dim$ null $\Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim$ range $\Phi = (\dim V - \dim U) \dim W$. $\square$
OR. Extend $(u_1, \ldots, u_m)$ a basis of U to $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ a basis of V. Let $p = \dim W$.

(See NOTE FOR [3.60])

$$\forall T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \underbrace{\text{span}\left\{\begin{matrix} E_{1,1}, & \cdots & , E_{m,1}, \\ \vdots & \ddots & \vdots \\ E_{1,p}, & \cdots & , E_{m,p} \end{matrix}\right\}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

又 $W = \text{span}\left\{\begin{matrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{matrix}\right\} \subseteq \mathcal{E}$. Where $\mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}$.

Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. □

ENDED

3·E

2 *Suppose $V_1, \dots, V_m$ are vec-sps such that $V_1 \times \cdots \times V_m$ is finite-dim. Prove that every V_j is finite-dim.*

SOLUTION: Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.

Let $(v_1, \dots, v_M)$ be a basis of U. Note that $\forall u_i \in V_i, \in U_i \subseteq U$, for each i.

$$\left.\begin{matrix} \text{Define } R_i \in \mathcal{L}(V_i, U) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0). \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{matrix}\right\} \Rightarrow S_i|_{U_i} = R_i^{-1}|_{U_i}.$$

Thus U_i and V_i are isomorphic. 又 U_i is a subspace of a finite-dim vec-sp U. □

---

3 *Give an example of a vec-sp V and its two subspaces U_1, U_2 such that $U_1 \times U_2$ and $U_1 + U_2$ are isomorphic but $U_1 + U_2$ is not a direct sum.*

SOLUTION:

NOTE *that at least one of U_1, U_2 must be infinite-dim.*

For if not, $U_1 \times U_2$ is finite-dim and $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$.

And V must be infinite-dim. For if not, both U_1 and U_2 are finite-dim subspaces.

Let $V = \mathbf{F}^\infty = U_1, U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$.

$$\left.\begin{matrix} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T((x_1, x_2, \cdots), (x, 0, \cdots)) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = ((x_1, x_2, \cdots), (x, 0, \cdots)) \end{matrix}\right\} \Rightarrow S = T^{-1}. \ \square$$

---

4 *Suppose $V_1, \dots, V_m$ are vec-sps. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic.*

SOLUTION: Using the notations in Problem (2). Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$.

$$\left.\begin{matrix} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (TR_1, \dots, TR_m). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m. \end{matrix}\right\} \Rightarrow \psi = \varphi^{-1}. \square$$

---

5 *Suppose $W_1, \dots, W_m$ are vec-sps. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic.*

SOLUTION: Using the notations in Problem (2).

Note that $Tv = (w_1, \dots, w_m)$. Define $T_i \in \mathcal{L}(V, W_i)$ by $T_i(v) = w_i$.

$$\left.\begin{matrix} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1T, \dots, S_mT). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m. \end{matrix}\right\} \Rightarrow \psi = \varphi^{-1}. \square$$

---

6 *For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are isomorphic.*

SOLUTION:

Define $T : (v_1, \ldots, v_m) \to \varphi$, where $\varphi : (a_1, \ldots, a_m) \mapsto v$ is defined by $\varphi(a_1, \ldots, a_m) = a_1v_1 + \cdots + a_mv_m$.
Suppose $T(v_1, \ldots, v_m) = 0$. Then $\forall\, (a_1, \ldots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \ldots, a_m) = a_1v_1 + \cdots + a_mv_m = 0$
$\Rightarrow (v_1, \ldots, v_m) = 0 \Rightarrow T$ is injective.
Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \ldots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $\forall\, (b_1, \ldots, b_n) \in \mathbf{F}^m$,
$(\,T(\psi(e_1), \ldots, \psi(e_m))\,)(b_1, \ldots, b_m) = b_1\psi(e_1) + \cdots + b_m\psi(e_m) = \psi(b_1e_1 + \cdots + b_me_m) = \psi(b_1, \ldots, b_m)$.
Thus $T(\psi(e_1), \ldots, \psi(e_m)) = \psi$. Hence T is surjective. $\square$

---

7 *Suppose $v, x \in V$ (chosen arbitrarily) of which U and W are subspaces.*
Suppose $v + U = x + W$. Prove that $U = W$.

SOLUTION:

(a) $\forall u \in U,\ \exists\, w \in W, v + u = x + w$, let $u = 0$, getting $v = x + w \Rightarrow v - x \in W$.
(b) $\forall w \in W,\ \exists\, u \in U, v + u = x + w$, let $w = 0$, getting $x = v + u \Rightarrow x - v \in U$.
Thus $\pm(v - x) \in U \cap W \Rightarrow \left\{ \begin{array}{c} u = (x - v) + w \in W \Rightarrow U \subseteq W \\ w = (v - x) + u \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W.$ $\square$

• *Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$.*
Prove that A is a translate of $U \iff \exists\, c \in \mathbf{R},\ A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$.
[*Do it in your mind.*]

• *Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $U = \{x \in V : Tx = c\}$ is either $\varnothing$ or is a translate of null T.*

SOLUTION:

If $c \in W$ but $c \notin$ range T, then $U = \varnothing$ and we are done.
Suppose $c \in$ range T, then $\exists\ u \in V, Tu = c \Rightarrow u \in U$.
Suppose $y \in$ null $T \Rightarrow y + u \in U \iff T(y + u) = Ty + c = c$. Thus $u +$ null $T \subseteq U$. Hence $u +$ null $T = U$,
for if not, suppose $z \notin u +$ null$T$ but $Tz = c (\Leftrightarrow z \in U)$, then $\forall w \in$ null $T, z \neq u + w \Leftrightarrow z - u \notin$ null$T$.
又 $\tilde{T}(z +$ null $T) = \tilde{T}(u +$ null $T) \Rightarrow z +$ null $T = u +$ null $T \Rightarrow z - u \in$ null T, contradicts. $\square$

• **COROLLARY:** *The set of solutions to a system of linear equations such as [3.28] is either $\varnothing$ or a translate of the null subspace.*

---

8 *Prove that a nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$.*

SOLUTION:

Suppose $A = a + U$, where U is a subspace of V. $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$,
$\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + [\lambda(u_1 - u_2) + u_2] \in A$.
Suppose $\lambda v + (1 - \lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $a \in A$ and let $A' = \{x - a : x \in A\}$.
Then $\forall x - a, y - a \in A', \lambda \in \mathbf{F}$,
(I) $\lambda(x - a) = [\lambda x + (1 - \lambda)a] - a \in A'$. Then let $\lambda = 2$.
(II) $\lambda(x - a) + (1 - \lambda)(y - a) = \frac{1}{2}(x - a) + \frac{1}{2}(y - a) = \frac{1}{2}x + (1 - \frac{1}{2})(y) - a \in A'$.
By (I), $2 \times [\frac{1}{2}(x - a) + \frac{1}{2}(y - a)] = (x - a) + (y - a) \in A'$.
Thus A' is a subspace of V. Hence $a + A' = \{(x - a) + a : x \in A\} = A$ is a translate. $\square$

---

9 *Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V.*
Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or is $\varnothing$.

SOLUTION: Suppose $v + u_1, w + u_2 \in A_1 \cap A_2 \neq \varnothing$. By Problem (8),
$\forall \lambda \in \mathbf{F}, \lambda(v + u_1) + (1 - \lambda)(w + u_2) \in A_1$ and A_2. Thus $A_1 \cap A_2$ is a translate of some subspace of V. $\square$

---

10 *Prove that the intersection of any collection of translates of subspaces of V is either a translate of some subspace or $\varnothing$.*

SOLUTION: Suppose $\{A_\alpha\}_{\alpha\in\Gamma}$ is a collection of translates of subspaces of V, where Γ is an arbitrary index set.

Suppose $x, y \in \bigcap_{\alpha\in\Gamma} A_\alpha \neq \varnothing$, then by Problem (18), $\forall \lambda \in \mathbf{F}$, $\lambda x + (1-\lambda)y \in A_\alpha$ for every $\alpha \in \Gamma$.

Thus $\bigcap_{\alpha\in\Gamma} A_\alpha$ is a translate of some subspace of V. $\square$

---

11 *Suppose* $A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^{m} \lambda_i = 1\}$, *where each* $v_i \in V$, $\lambda_i \in \mathbf{F}$.

(a) *Prove that A is a translate of some subspace of V: By Problem (8),*

$$\forall \sum_{i=1}^{m} a_i v_i, \sum_{i=1}^{m} b_i v_i \in A, \lambda \in \mathbf{F}, \quad \lambda \sum_{i=1}^{m} a_i v_i + (1-\lambda) \sum_{i=1}^{m} b_i v_i = (\lambda \sum_{i=1}^{m} a_i + (1-\lambda) \sum_{i=1}^{m} b_i) v_i \in A. \ \square$$

(b) *Prove that if B is a translate of some subspace of V and $\{v_1, \ldots, v_m\} \subseteq B$, then $A \subseteq B$.*

(c) *Prove that A is a translate of some subspace of V and* $\dim V < m$.

SOLUTION:

(b) Let $v = \lambda_1 v_+ \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on m by k.

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By problem (8), $v \in B$.

(ii) $2 \leq k \leq m$, we assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. ($\forall \lambda_i$ such that $\sum_{i=1}^{k} \lambda_i = 1$)

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. $\forall i = 1, \ldots, k$, $\exists\, \mu_i \neq 1$, fix one such i by ι.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow (\sum_{i=1}^{k+1} \frac{\mu_i}{1-\mu_\iota}) - \frac{\mu_\iota}{1-\mu_\iota} = 1$.

Let $w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k\, terms}$.

Let $\lambda_i = \frac{\mu_i}{1-\mu_\iota}$ for $i = 1, \ldots, \iota - 1$; $\lambda_j = \frac{\mu_{j+1}}{1-\mu_\iota}$ for $j = \iota, \ldots, k$. Then,

$$\left.\begin{array}{l} \sum_{i=1}^{k} \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \ \square$$

(c) $\forall k = 1, \ldots, m,\ \forall \lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_m$, let $\lambda_k = 1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m$

$\Rightarrow \lambda_1 v_1 + \cdots + \lambda_m v_m$

$= \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m) v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_m v_m$

$= v_k + \lambda_1 (v_1 - v_k) + \cdots + \lambda_{k-1}(v_{k-1} - v_k) + \lambda_{k+1}(v_{k+1} - v_k) + \cdots + \lambda_m (v_m - v_k)$.

Thus $A = v_k +$ span$(v_1 - v_k, \ldots, v_{k-1} - v_k, v_{k+1} - v_k, \ldots, v_m - v_k)$. $\square$

---

12 *Suppose U is a subspace of V such that V/U is finite-dim. Prove that is V is isomorphic to $U \times (V/U)$.*

SOLUTION: Let $(v_1 + U, \ldots, v_n + U)$ be a basis of V/U. Note that

$\forall v \in V,\ \exists\,!\ a_1, \ldots, a_n \in \mathbf{F}, v + U = \sum_{i=1}^{n} a_i (v_i + U) = (\sum_{i=1}^{n} a_i v_i) + U$

$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) = u \in U$ for some u; $v = \sum_{i=1}^{n} a_i v_i + u$.

Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ by $\varphi(v) = (u, \sum_{i=1}^{n} a_i v_i + U)$

and $\psi \in \mathcal{L}(U \times (V/U), V)$ by $\psi(u, w + U) = u + w$; $w = \sum_{i=1}^{n} b_i v_i + U$.

So that $\psi = \varphi^{-1}$. $\square$

• *Suppose $V = U \oplus W$, $(w_1, \ldots, w_m)$ is a basis of W. Prove that $(w_1 + U, \ldots, w_m + U)$ is a basis of V/U.*

Solution: Note that for any $v \in V$,

$\exists ! \ u \in U, w \in W, v = u + w$ 又 $\exists ! \ c_i \in \mathbf{F}$ such that $w = \sum_{i=1}^{m} c_i w_i \Rightarrow v = u + \sum_{i=1}^{m} c_i w_i$.

Thus $v + U = \sum_{i=1}^{m} c_i w_i + U \Rightarrow v + U \in \operatorname{span}(w_1 + U, \dots, w_m + U) \Rightarrow V/U \subseteq \operatorname{span}(w_1 + U, \dots, w_m + U)$.

Now suppose $a_1(w_1 + U) + \cdots + a_m(w_m + U) = 0 + U \Rightarrow \sum_{i=1}^{m} a_i w_i \in U$ while $U \cap W = \{0\}$.

Then $\sum_{i=1}^{m} a_i w_i = 0 \Rightarrow a_1 = \cdots = a_m = 0$. $\square$

---

13 *Suppose $(v_1 + U, \dots, v_m + U)$ is a basis of V/U and $(u_1, \dots, u_n)$ is a basis of U. Prove that $(v_1, \dots, v_m, u_1, \dots, u_n)$ is a basis of V.*

Solution: By Problem (12), U and V/U are finite-dim$\Rightarrow$ $U \times (V/U)$ is finite-dim, so is V.

$\dim V = \dim(U \times (V/U)) = \dim U + \dim V/U = m + n$.

Or. Note that for any $v \in V, v + U = \sum_{i=1}^{m} a_i v_i + U, \ \exists ! \ a_i \in \mathbf{F} \Rightarrow v = \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} b_i v_i, \ \exists ! \ b_i \in \mathbf{F}$.

$\Rightarrow v \in \operatorname{span}(v_1, \dots, v_m, u_1, \dots, u_n) \supseteq V$.

又 Notice that $(\sum_{i=1}^{m} a_i v_i) + U = 0 + U (\Rightarrow \sum_{i=1}^{m} a_i v_i \in U) \iff a_1 = \cdots = a_m = 0$.

Hence $\operatorname{span}(v_1, \dots, v_m) \cap U = \{0\} \Rightarrow \operatorname{span}(v_1, \dots, v_m) \oplus U = V$

Thus $(v_1, \dots, v_m, u_1, \dots, u_n)$ is linearly independent, so is a basis of V. $\square$

---

14 *Suppose $U = \{(x_1, x_2, \dots) \in \mathbf{F}^\infty : x_k \neq 0$ for only finitely many $k\}$.*

(a) *Show that U is a subspace of $\mathbf{F}^\infty$. [Do it in your mind]*

(b) *Prove that $\mathbf{F}^\infty/U$ is infinite-dim.*

Solution:

For $u = (x_1, \dots, x_p, \dots) \in \mathbf{F}^\infty$, denote x_p by $u[p]$. For each $r \in \mathbf{N}^+$.

Define $e_r[p] = \begin{cases} 1, (p-1) \equiv 0 \ (\operatorname{mod} r) \\ 0, \text{otherwise} \end{cases}$, simply $e_r = (1, \underbrace{0, \dots, 0}_{(p-1) \ times}, 1, \underbrace{0, \dots, 0}_{(p-1) \ times}, 1, \dots) \in \mathbf{F}^\infty$.

Choose $m \in \mathbf{N}^+$ arbitrarily.

Suppose $a_1(e_1 + U) + \cdots + a_m(e_m + U) = (a_1 e_1 + \cdots + a_m e_m) + U = 0 + U = 0$.

$\Rightarrow a_1 e_1 + \cdots + a_m e_m = u$ for some $u \in U$.

Then suppose $u = (x_1, \dots, x_t, 0, \dots) \Rightarrow u[t + i] = 0, \forall i \in \mathbf{N}^+$,

then let $j = s \cdot m! + 1 \geq t$ ($\exists \ s \in \mathbf{N}^+$) so that $e_1[j] = \cdots = e_m[j] = 1, \ u[j + i] = 0$.

Now we have: $u[j + i] = (\Sigma_{r=1}^m a_r e_r)[j + i] = \Sigma_{r=1}^m a_r e_r[s \cdot m! + 1 + i] = a_{i_1} + \cdots + a_{i_{\tau(i)}} = 0$,

$\Rightarrow (\Sigma_{r=1}^m a_r e_r)[j + i] = a_{i_1} + \cdots + a_{i_{\tau(i)}} = 0$. (Δ)

where $i_1, \dots, i_{\tau(i)}$ are distinct ordered factors of i ($1 = i_1 \leq \cdots \leq i_{\tau(i)} = i$).

$\Big($ Note that by definition, $e_r[s \cdot m! + 1 + i] = 1 \iff s \cdot m! + i \equiv i \equiv 0 \ (\operatorname{mod} r) \iff r \mid i$. $\Big)$

Let $i' = i_{\tau(i)-1}$. Notice that $i'_l = i_l, \forall l \in \{1, \dots, \tau(i')\}$; and $\tau(i') = \tau(i) - 1$.

Again by (Δ), $(\Sigma_{r=1}^m a_r e_r)[j + i'] = a_{i'_1} + \cdots + a_{i'_{\tau(i')}} = a_{i_1} + \cdots + a_{i_{\tau(i)-1}} = 0$.

Thus $a_{i_{\tau(i)}} = a_i = 0$ for any $i \in \{1, \dots, m\}$.

Hence $(e_1, \dots, e_m)$ is linearly independent in $\mathbf{F}^\infty$, so is $(e_1, \dots, e_m, \dots)$, since $m \in \mathbf{N}^+$.

又 $e_i \notin U \Rightarrow (e_1 + U, e_2 + U, \dots)$ is linearly independent in $\mathbf{F}^\infty/U$. By [2.B.14]. $\square$

---

15 *Suppose $\varphi \in \mathcal{L}(V, \mathbf{F}) \setminus \{0\}$. Prove that* $\dim V/(\textit{null } \varphi) = 1$.

Solution: By [3.91] (d), $\dim \operatorname{range} \varphi = 1 = \dim V/(\operatorname{null} \varphi)$. $\square$

NOTE FOR [3.88, 3.90, 3.91]

For any $W \in \mathcal{S}_V U$, because $V = U \oplus W$. $\forall v \in V$, $\exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.

Define $T \in \mathcal{L}(V, W)$ by $T(v) = w_v$. Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$.

Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, W)$ is defined as $\tilde{T}(v + U) = Tv = w_v$.

Thus $\tilde{T}$ is injective (by [3.91(b)]) and surjective ($\operatorname{range} \tilde{T} = \operatorname{range} T = W$),

and therefore is an isomorphism. We conclude that V/U and W, namely any vec-sp in $\mathcal{S}_V$, are isomorphic.

---

16 *Suppose* $\dim V/U = 1$. *Prove that* $\exists\, \varphi \in \mathcal{L}(V, \mathbf{F})$ *such that null* $\varphi = U$.

SOLUTION:

Suppose V_0 is a subspace of V such that $V = U \oplus V_0$. Then V_0 and V/U are isomorphic. $\dim V_0 = 1$.

Define a linear map $\varphi : v \mapsto \lambda$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. $\square$

---

17 *Suppose* V/U *is finite-dim.* W *is a subspace of* V.

(a) *Show that if* $V = U + W$, *then* $\dim W \geq \dim V/U$.

(b) *Suppose* $\dim W = \dim V/U$ *and* $V = U \oplus W$. *Find such* W.

SOLUTION:

Let $(w_1, \dots, w_n)$ be a basis of W

(a) $\forall v \in V$, $\exists\, u \in U, w \in W$ such that $v = u + w \Rightarrow v + U = w + U$

Then $V/U \subseteq \operatorname{span}(w_1 + U, \dots, w_n + U) \Rightarrow V/U = \operatorname{span}(w_1 + U, \dots, w_n + U)$.

Hence $\dim V/U = \dim \operatorname{span}(w_1 + U, \dots, w_n + U) \leq \dim W$.

(b) Let $W \in \mathcal{S}_V U$. In other words,

reduce $(w_1{+}U, \dots, w_n{+}U)$ to a basis of V/U as $(w_{\alpha_1}{+}U, \dots, w_{\alpha_m}{+}U)$ and let $W = \operatorname{span}(w_{\alpha_1}, \dots, w_{\alpha_m})$. $\square$

---

18 *Suppose* $T \in \mathcal{L}(V, W)$ *and* U *is a subspace of* V. *Let* π *denote the quotient map.*

Prove that $\exists\, S \in \mathcal{L}(V/U, W)$ *such that* $T = S \circ \pi$ *if and only if* $U \subseteq$ *null* T.

SOLUTION:

(a) Define $S \in \mathcal{L}(V/U, W)$ by $S(v + U) = Tv$. We have to check it is *well-defined*.

Suppose $v_1 + U = v_2 + U$, while $v_1 \neq v_2$.

Then $(v_1 - v_2) \in U \Rightarrow S((v_1 - v_2) + U) = T(v_1 - v_2) = 0 \Rightarrow Tv_1 = Tv_2$. Checked. $\square$

(b) Suppose $\exists\, S \in \mathcal{L}(V/U, W)$, $T = S \circ \pi$. Then $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null} T$.$\square$

---

20 *Define* $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ *by* $\Gamma(S) = S \circ \pi (= \pi'(S))$.

(a) *Prove that* Γ *is linear:* By [3.9] distributive properties and [3.6]. $\square$

(b) *Prove that* Γ *is injective:*

$$
\begin{aligned}
&\Gamma(S) = 0 \\
\iff &\forall v \in V, S(\pi(v)) = 0 \\
\iff &\forall v + U \in V/U, S(v + U) = 0 \\
\iff &S = 0. \ \square
\end{aligned}
$$

(c) *Prove that range* Γ $($= *range* π' $) = \{T \in \mathcal{L}(V, W) : U \subseteq$ *null* $T\}$*:*

By Problem (18). $\square$

ENDED

3·F

- By (18) in (3.D) *we know that* $\varphi : V \to \mathcal{L}(\mathbf{F}, V)$ *is an isomorphism. Now we prove that*

$(v_1, \ldots, v_m)$ is linearly independent $\Longleftrightarrow$ $(\varphi(v_1), \ldots, \varphi(v_m))$ is linearly independent.

SOLUTION:

(a) Suppose $(v_1, \ldots, v_m)$ is linearly independent and $\vartheta \in \operatorname{span}(\varphi(v_1), \ldots, \varphi(v_m))$.

Let $\vartheta = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m)$. Then $\vartheta(1) = 0 = a_1v_1 + \cdots + a_mv_m \Rightarrow a_1 = \cdots = a_m = 0$.

OR Because φ is injective. Suppose $a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) = 0 = \varphi(a_1v_1 + \cdots + a_mv_m)$.

Then $a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

Thus $(\varphi(v_1), \ldots, \varphi(v_m))$ is linearly independent.

(b) Suppose $(\varphi(v_1), \ldots, \varphi(v_m))$ is linearly independent and $v \in \operatorname{span}(v_1, \ldots, v_m)$.

Let $v = 0 = a_1v_1 + \cdots + a_mv_m$. Then $\varphi(v) = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

Thus $v_1, \ldots, v_m$ is linearly independent. $\square$

---

1 *Explain why each linear functional is surjective or is the zero map.*

SOLUTION:
$$\left.\begin{array}{r}\text{For any } \varphi \in V' \text{ and } \varphi \neq 0,\ \exists\, v \in V, \text{ such that } \varphi(v) \neq 0. \text{ (a)}\\ \dim \operatorname{range} \varphi = \dim \mathbf{F} = 1. \text{ (b)}\end{array}\right\} \Rightarrow \square$$

---

4 *Suppose V is finite-dim and U is a subspace of V such that $U \neq V$.*
Prove that $\exists\, \varphi \in V'$ and $\varphi \neq 0$ such that $\varphi(u) = 0$ for every $u \in U$.

SOLUTION:

Let $(u_1, \ldots, u_m)$ be a basis of U, extend to $(u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+n})$ a basis of V.

Choose $k \in \{1, \ldots, n\}$ arbitrarily. Define $\varphi \in V'$ by $\varphi(u_i) = \begin{cases} 1, & \text{if } i = m+k. \\ 0, & \text{otherwise.} \end{cases}$

OR: Equivalent to proving that $U^0 \neq \{0\}$. By [3.106], $\dim U^0 = \dim V - \dim U > 0$. $\square$

• *Suppose $T \in \mathcal{L}(V, W)$ and $(w_1, \ldots, w_m)$ is a basis of range$T$.*
Hence $\forall v \in V$, $Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$, $\exists!\, \varphi_1(v), \ldots, \varphi_m(v)$,
thus defining functions $\varphi_1, \ldots, \varphi_m$ from V to $\mathbf{F}$. Show that each $\varphi_i \in V'$.

SOLUTION:

For each w_i, $\exists\, v_i \in V$, $Tv_i = w_i$, getting a linearly independent list $(v_1, \ldots, v_m)$.

Now we have $Tv = a_1Tv_1 + \cdots + a_mTv_m$, $\forall v \in V$, $\exists!\, a_i \in \mathbf{F}$.

Let $(\psi_1, \ldots, \psi_m)$ be the dual basis of range$T$. Then $(T'(\psi_i))(v) = \psi_i \circ T(v) = a_i$.

Thus letting $\varphi_i = \psi_i \circ T$. $\square$

• *Suppose φ, $\beta \in V'$. Prove that null$\varphi \subseteq$ nullβ if and only if $\beta = c\varphi$. $\exists\, c \in \mathbf{F}$.*

SOLUTION: Using (3.B.29, 30)

(a) Suppose null$\varphi \subseteq$ nullβ. Choose a $u \notin$ nullβ. $V =$ null$\beta \oplus \{au : a \in \mathbf{F}\}$.

If null$\varphi =$ nullβ, then let $c = \frac{\beta(u)}{\varphi(u)}$, we are done.

Otherwise, suppose $u' \in$ nullβ, but $u' \notin$ nullφ, then $V =$ null$\varphi \oplus \{bu' : b \in \mathbf{F}\}$.

$\forall v \in V, v = w + au = w' + bu'$, $\exists!\, w, w' \in$ nullφ, $a, b \in \mathbf{F}$.

Thus $\beta(v) = a\beta(u)$, $\varphi(v) = b\varphi(u')$. Let $c = \frac{a\beta(u)}{b\varphi(u')}$. We are done

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$.

If $c = 0$, then null$\beta = V \supseteq$ nullφ, we are done.

Otherwise, $\left.\begin{array}{l}\forall v \in \text{null}\varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \text{null}\varphi \subseteq \text{null}\beta.\\ \forall v \in \text{null}\beta, \beta(v) = 0 = \varphi(v) \Rightarrow \text{null}\beta \subseteq \text{null}\varphi.\end{array}\right\} \Rightarrow \text{null}\varphi = \text{null}\beta$

$\Rightarrow$ null$\varphi \subseteq$ nullβ. $\square$

---

5 *Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic.*

SOLUTION: Using notations in (3.E.2).

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to V_1' \times \cdots \times V_m' \\ \quad \text{by } \varphi(T) = (T \circ R_1, \ldots, T \circ R_m) = (R_1'(T), \ldots, R_m'(T)). \\ \text{Define } \psi : V_1' \times \cdots \times V_m' \to (V_1 \times \cdots \times V_m)' \\ \quad \text{by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m = S_1'(T_1) + \cdots + S_m'(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}. \ \square$$

• *Suppose $(v_1, \ldots, v_n)$ is a basis of V and $(\varphi_1, \ldots, \varphi_n)$ is the dual basis of V'.*

$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = (\varphi_1(v), \ldots, \varphi_n(v)). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

---

35 *Prove that $(\mathcal{P}(\mathbf{R}))'$ and $\mathbf{R}^\infty$ are isomorphic.*

SOLUTION:

Define $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{R}))', \mathbf{R}^\infty)$ by $\theta(\varphi) = (\varphi(1), \varphi(x), \ldots, \varphi(x^n), \ldots)$.

Injectivity: $\theta(\varphi) = 0 \Rightarrow \forall x^k$ in the basis $(1, x, \ldots, x^n, \ldots)$ of $\mathcal{P}_n(\mathbf{R})$ for any n, $\varphi(x^k) = 0 \Rightarrow \varphi = 0$.

Surjectivity: $\forall\, (a_0, a_1, \ldots, a_n, \ldots) \in \mathbf{F}^\infty$, let ψ be such that $\psi(x^k) = a_k$ and thus $\theta(\psi) = (a_0, a_1, \ldots, a_n, \ldots)$.

Hence θ is an isomorphism from $(\mathcal{P}(\mathbf{R}))'$ onto $\mathbf{R}^\infty$. $\square$

---

7 *Suppose m is a positive integer. Show that the dual basis of the basis $(1, x, \ldots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where $\varphi_k = \dfrac{p^{(k)}(0)}{k!}$. Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.*

SOLUTION:

For each j and k, $(x^j)^{(k)} = \begin{cases} j(j-1)\ldots(j-k+1) \cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\ldots(j-j+1) = j!\,, & j = k. \\ 0\,, & j \leq k. \end{cases}$ Then $(x^j)^{(k)}(0) = \begin{cases} 0\,, & j \neq k. \\ k!\,, & j = k. \end{cases}$

Thus $\varphi_k = \psi_k$, where $\psi_1, \ldots, \psi_m$ is the dual basis of $(1, x, \ldots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$. $\square$

---

8 *Suppose m is a positive integer.*

(a) By [2.C.10], $B = (1, x-5, \ldots, (x-5)^m)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) Let $\varphi_k = \frac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \ldots, m$. Then $(\varphi_0, \varphi_1, \ldots, \varphi_m)$ is the dual basis of B.

---

9 *Suppose $(v_1, \ldots, v_n)$ is a basis of V and $(\varphi_1, \cdots, \varphi_n)$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.*

SOLUTION: $\psi(v) = \psi(\sum\limits_{i=1}^{n} a_i v_i) = \sum\limits_{i=1}^{n} a_i \psi(v_i) = \sum\limits_{i=1}^{n} \psi(v_i)\varphi_i(v) = [\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n](v) \Rightarrow \square$

COMMENT: For any other basis $(u_1, \ldots, u_n)$ of V and the corresponding dual basis of $(\rho_1, \ldots, \rho_n)$,

$\psi = \rho(u_1)\rho_1 + \cdots + \rho(u_n)\rho_n$.

---

12 *Show that the dual map of the identity operator on V is the identity operator on V'.*

SOLUTION: $I'(\varphi) = \varphi \circ I = \varphi, \ \forall \varphi \in V'$. $\square$

• *Suppose W is finite-dim and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \Longleftrightarrow T = 0$.*

SOLUTION: $T = 0 \Leftrightarrow T'(\varphi) = \varphi \circ T = 0$ for all $\varphi \in V' \Leftrightarrow T' = 0$. $\square$

---

13 *Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ denote the dual basis of the standard basis of $\mathbf{R}^2$ and $\mathbf{R}^3$.*

(a) *Describe the linear functionals $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$*

For any $(x, y, z) \in \mathbf{R}^3$, $(T'(\varphi_1))(x, y, z) = 4x + 5y + 6z$, $(T'(\varphi_2))(x, y, z) = 7x + 8y + 9z$.

(b) *Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.*

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$. $\square$

---

14 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for each $x \in \mathbf{R}$.*

(a) *Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe $T'(\varphi) \in \mathcal{P}(\mathbf{R})'$.*

$(T'(\varphi))(p) = [x^2p(x) + p''(x)]'(4) = [2xp(x) + x^2p'(x) + p'''(x)](4) = 8p(4) + 16p'(4) + p'''(4).$

(b) *Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\varphi))(x^3)$.*

$(T'(\varphi))(x^3) = \int_0^1 (x^5 + 6x)\mathrm{d}x = \int_0^1 (\frac{1}{6}x^6 + 3x^2)'\mathrm{d}x = \frac{6}{19}.$ $\square$

• *Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$.*
Prove that T is invertible if and only if $T' \in \mathcal{L}(W', V')$ is invertible.

SOLUTION: By [3.108] and [3.110]. $\square$

---

16 *Suppose V and W are finite-dim. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(L, W)$.*
Prove that Γ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

SOLUTION:

V, W are finite-dim $\Rightarrow$ dim $\mathcal{L}(V, W) =$ dim $\mathcal{L}(W', V')$. And by [3.101], Γ is linear.

又 Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus T is injective $\Rightarrow T$ is invertible. $\square$

---

17 *Suppose $U \subseteq V$. Explain why $U^0 = \{\varphi \in V' : U \subseteq null\varphi\}$.*

SOLUTION: Because for $\varphi \in V'$, $U \subseteq$ null$\varphi \iff \forall u \in U, \varphi(u) = 0$. By definition in [3.102]. $\square$

---

18 *$U \subseteq V$. We have $U = \{0\} \iff \forall\varphi \in V', U \subseteq null\varphi \iff U^0 = V'$.*

19 *U is a subspace of V. Prove that $U = V \iff U^0_V = \{0\} = V^0_V$.*

SOLUTION:

Suppose $U^0_V = \{0\}$. Then $U = V$.

Conversely, suppose $U = V$, then $U^0_V = \{\varphi \in V' : V \subseteq \text{null}\,\varphi\}$, therefore $U^0_V = \{0\}$.

20, 21 *Suppose U and W are subsets of V. Prove that $U \subseteq W \iff W^0 \subseteq U^0$.*

SOLUTION:

(a) $U \subseteq W \Rightarrow \forall w \in W, u \in U \cap W = U,\ \ \forall\varphi \in W^0, \varphi(w) = 0 = \varphi(u) \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.

(b) $W^0 \subseteq U^0 \Rightarrow \forall w \in W, u \in U, \varphi(w) = 0 \Rightarrow \varphi(u) = 0$. Then null$\varphi \supseteq W \Rightarrow$ null$\varphi \supseteq U$. Thus $W \supseteq U$. $\square$.

• **COROLLARY:** $W^0 = U^0 \iff U = W$.

---

22 *Prove that $(U + W)^0 = U^0 \cap W^0$.*

SOLUTION:

(a) $\left.\begin{array}{c} U \subseteq U + W \\ W \subseteq U + W \end{array}\right\} \Rightarrow \left.\begin{array}{c} (U + W)^0 \subseteq U^0 \\ (U + W)^0 \subseteq W^0 \end{array}\right\} \Rightarrow (U + W)^0 \subseteq U^0 \cap W^0.$

(b) $\forall\varphi \in U^0 \cap W^0, \varphi(u + w) = 0$, where $u \in U, w \in W \Rightarrow \varphi \in (U + W)^0$. Thus $(U + W)^0 \supseteq U^0 \cap W^0$. $\square$

---

23 *Prove that $(U \cap W)^0 = U^0 + W^0$.*

SOLUTION:

(a) $\left.\begin{array}{c} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array}\right\} \Rightarrow \left.\begin{array}{c} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array}\right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \supseteq U^0 \cap W^0.$

(b) $\forall\varphi \in U^0, \psi \in W^0$ and $\forall v \in U \cap W$, $(\varphi + \psi)(v) = \varphi(v) + \psi(v) = 0$. Thus $U^0 + W^0 \subseteq (U \cap W)^0$. $\square$

• **COROLLARY:**

Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subspaces of V.
Then $(\sum_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \bigcap_{\alpha_i \in \Gamma}(V^0_{\alpha_i})$;
And $(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \sum_{\alpha_i \in \Gamma}(V^0_{\alpha_i})$.

---

24 *Suppose V is finite-dim and U is a subspace of V.*

Prove, using the pattern of [3.104], that dimU+ dimU^0 = dimV.

SOLUTION:

Let $(u_1, \ldots, u_m)$ be a basis of U, extend to a basis of V as $(u_1, \ldots, u_m, \ldots, u_n)$, and let $(\varphi_1, \ldots, \varphi_m, \ldots, \varphi_n)$ be the dual basis.

(a) Suppose $\varphi \in \text{span}(\varphi_{m+1}, \ldots, \varphi_n)$, then $\exists\, a_i \in \mathbf{F}$, $\varphi = a_{m+1}\varphi_{m+1} + \cdots + a_n\varphi_n$.
For all $u \in U$, $\varphi(u) = 0$. Thus $\varphi \in U^0$, getting $\text{span}(\varphi_{m+1}, \ldots, \varphi_n) \subseteq U^0$.

(b) Suppose $\varphi \in U^0$, then $\exists\, a_i \in \mathbf{F}$, $\varphi = a_1\varphi_1 + \cdots + a_m\varphi_m + \cdots + a_n\varphi_n$.
For all $u_i \in U$, $0 = \varphi(u_i) = \sum_{i=1}^{n} \varphi(u_i) = a_i$. Then $\varphi = a_{m+1}\varphi_{m+1} + \cdots + a_n\varphi_n$.
Thus $\varphi \in \text{span}(\varphi_{m+1}, \ldots, \varphi_n)$, getting $\text{span}(\varphi_{m+1}, \ldots, \varphi_n) \supseteq U^0$.

Hence $\text{span}(\varphi_{m+1}, \ldots, \varphi_n) = U^0$, $\dim U^0 = n - m = \dim V - \dim U$. $\square$

---

25 *Suppose U is a subspace of V. Explain why $U = \{v \in V : \varphi(v) = 0$ for every $\varphi \in U^0\}$.*

SOLUTION: Note that $U = \{v \in V : v \in U\}$ is a subspace of V and $\varphi(v) = 0$ for every $\varphi \in U^0 \iff v \in U$. $\square$

---

26 *Suppose V is finite-dim and Ω is a subspace of V'.*
Prove that $\Omega = \{v \in V : \forall\varphi \in \Omega, \varphi(v) = 0\}^0$.

SOLUTION: Using the corollary in Problem (20, 21).

Suppose $U = \{v \in V : \forall\varphi \in \Omega, \varphi(v) = 0\}$.

Getting $U^0 = \{v \in V : \forall\varphi \in \Omega, \varphi(v) = 0\}^0$. We need to show that $\Omega = U^0$.

$$\left.\begin{array}{l} \text{(a) } \forall\varphi \in \Omega, v \in U, \varphi(v) = 0 \Rightarrow \varphi \in U^0 \Rightarrow \Omega \subseteq U^0. \\ \text{(b) } v \in U \Leftrightarrow \left\{\begin{array}{l} \forall\varphi \in \Omega, \varphi(v) = 0 \\ \forall\psi \in U^0, \psi(v) = 0 \end{array}\right. \text{ Thus } \Omega \supseteq U^0. \end{array}\right\} \Rightarrow \square$$

---

27 *Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ and nullT$' = $ span(φ), where φ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that rangeT $= \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$.*

SOLUTION: By Problem (26), $\text{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \psi(p) = 0, \forall\psi \in \text{span}(\varphi)\}^0$,

Hence $\text{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \varphi(p) = p(8) = 0\}^0$, 又 $\text{span}(\varphi) = \text{null}T' = (\text{range}T)^0$.

By the corollary in Problem (20, 21), $\text{range}T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$. $\square$

28, 29 *Suppose V, W are finite-dim, $T \in \mathcal{L}(V, W)$.*

(a) *Suppose $\exists\,\varphi \in W'$ such that nullT$' = $ span(φ). Prove that rangeT $=$ nullφ.*

(b) *Suppose $\exists\,\varphi \in V'$ such that rangeT$' = $ span(φ). Prove that nullT $=$ nullφ.*

SOLUTION: Using Problem (26), [3.107] and [3.109].

Because $\text{span}(\varphi) = \{v \in V : \forall\psi \in \text{span}(\varphi), \psi(v) = 0\}^0 = \{v \in V : \varphi(v) = 0\}^0 = (\text{null}\varphi)^0$.

$$\left.\begin{array}{l} \text{(a) } (\text{range}T)^0 = \text{null}T' = \text{span}(\varphi) = (\text{null}\varphi)^0 \iff \text{range}T = \text{null}\varphi. \\ \text{(b) } (\text{null}T)^0 = \text{range}T' = \text{span}(\varphi) = (\text{null}\varphi)^0 \iff \text{null}T = \text{null}\varphi. \end{array}\right\} \Rightarrow \square$$

---

31 *Suppose V is finite-dim and $(\varphi_1, \ldots, \varphi_n)$ is a basis of V'.*
Show that there exists a basis of V whose dual basis is $(\varphi_1, \ldots, \varphi_n)$.

SOLUTION: Using (3.B.29,30).

For each φ_i, $\text{null}\varphi_i \oplus \{au_i : a \in \mathbf{F}\} = V$.

Because $\varphi_1, \ldots, \varphi_m$ is linearly independent. $\text{null}\varphi_i \neq \text{null}\varphi_j$ for all $i, j \in \mathbf{N}^+$ such that $i \neq j$.

Thus $(u_1, \ldots, u_m)$ is linearly independent, for if not, then $\exists\, i, j$ such that $\text{null}\varphi_i = \text{null}\varphi_j$, contradicts.

又 $\dim V' = m = \dim V$. Then $(u_1, \ldots, u_m)$ is a basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$. $\square$.

• *Suppose dim and $\varphi_1, \ldots, \varphi_m \in V'$. Prove that the following three sets are equal to each other.*

(a) $span(\varphi_1, \ldots, \varphi_m)$
(b) $((null\varphi_1) \cap \cdots \cap (null\varphi_m))^0$
(c) $\{\varphi \in V' : (\ null\varphi_1) \cap \cdots \cap (null\varphi_m) \subseteq null\varphi\}$

SOLUTION: By Problem (17), (b) and (c) are equivalent. By Problem (26) and the corollary in Problem (23),

$$\left.\begin{array}{l} ((\text{null}\varphi_1) \cap \cdots \cap (\text{null}\varphi_m))^0 = (\text{null}\varphi_1)^0 + \cdots + (\text{null}\varphi_m)^0. \\ \text{又 span}(\varphi_i) = \{v \in V : \forall \psi \in \text{span}(\varphi_i), \psi(v) = 0\}^0 = (\text{null}\varphi_i)^0. \end{array}\right\} \Rightarrow (a) = (b). \quad \square$$

30 OR COROLLARY:
Suppose V is finite-dim and $\varphi_1, \ldots, \varphi_m$ is a linearly independent list in V'.
Then $dim((null\varphi_1) \cap \cdots \cap (null\varphi_m)) = (dimV) - m$.

---

6 *Define $\Gamma : V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m))$, where $v_1, \ldots, v_m \in V$.*
(a) *Show that $span(v_1, \ldots, v_m) = V \iff \Gamma$ is injective.*
(b) *Show that $(v_1, \ldots, v_m)$ is linearly independent $\iff \Gamma$ is surjective.*

SOLUTION:

(a) Suppose Γ is injective. Then let $\Gamma(\varphi) = 0$, getting $\varphi = 0 \Leftrightarrow \text{null}\varphi = V = \text{span}(v_1, \ldots, v_m)$.
Suppose $\text{span}(v_1, \ldots, v_m) = V$. Then let $\Gamma(\varphi) = 0$, getting $\varphi(v_i) = 0$ for each i,
$\text{null}\varphi = \text{span}(v_1, \ldots, v_m) = V$, thus $\varphi = 0$, Γ is injective.

(b) Suppose Γ is surjective. Then let $\Gamma(\varphi_i) = e_i$ for each i, where $(e_1, \ldots, e_m)$ is the standard basis of $\mathbf{F}^m$.
Then $(\varphi_1, \ldots, \varphi_m)$ is linearly independent, suppose $a_1v_1 + \cdots + a_mv_m = 0$,
then for each i, we have $\varphi_i(a_1v_1 + \cdots + a_mv_m) = a_i = 0$. Thus $v_1, \ldots, v_n$ is linearly independent.
Suppose $(v_1, \ldots, v_m)$ is linearly independent. Let $(\varphi_1, \ldots, \varphi_m)$ be the dual basis of $\text{span}(v_1, \ldots, v_m)$.
Thus for each $(a_1, \ldots, a_m) \in \mathbf{F}^m$, we have $\varphi = a_1\varphi_1 + \cdots + a_m\varphi_m$ so that $\Gamma(\varphi) = (a_1, \ldots, a_m)$. $\square$

• *Define $\Gamma : V \to \mathbf{F}^m$ by $\Gamma(v) = (\varphi_1(v), \ldots, \varphi_m(v))$, where $\varphi_1, \ldots, \varphi_m \in V'$.*
(c) *Show that $span(\varphi_1, \ldots, \varphi_m) = V' \iff \Gamma$ is injective.*
(d) *Show that $(\varphi_1, \ldots, \varphi_m)$ is linearly independent $\iff \Gamma$ is surjective.*

SOLUTION:

(c) Suppose Γ is injective. Then $\Gamma(v) = 0 \Leftrightarrow \forall i, \varphi_i(v) = 0 \Leftrightarrow v \in (\text{null}\varphi_1) \cap \cdots \cap (\text{null}\,\varphi_m) \Leftrightarrow v = 0$.
Getting $(\text{null}\,\varphi_1) \cap \cdots \cap (\text{null}\varphi_m) = \{0\}$. By Problem (•) above, $\text{span}\,(\varphi_1, \ldots, \varphi_m) = V'$
Suppose $\text{span}\,(\varphi_1, \ldots, \varphi_m) = V'$. Again by Problem (•), $(\text{null}\,\varphi_1) \cap \cdots \cap (\text{null}\,\varphi_m) = \{0\}$.
Thus $\Gamma(v) = 0 \Rightarrow \forall i, \varphi_i(v) = 0 \Rightarrow v = 0$.

(d) Suppose $(\varphi_1, \ldots, \varphi_m)$ is linearly independent. Then by Problem (31), $(v_1, \ldots, v_m)$ is linearly independent.
Thus for any $(a_1, \ldots, a_m) \in \mathbf{F}$, by letting $v = \sum_{i=1}^{m} a_iv_i$, then $\varphi_i(v) = a_i \Rightarrow \Gamma(v) = (a_1, \ldots, a_m)$.
Suppose Γ is surjective. Let $e_1, \ldots, e_m$ be a basis of $\mathbf{F}^m$.
For every e_i, $\exists\, v_i \in V$ such that $\Gamma(v_i) = (\varphi_1(v_i), \ldots, \varphi_m(v)) = e_i$,
fix v_i ($\Rightarrow (v_1, \ldots, v_m)$ is linearly independent). Thus $\varphi_i(v_i) = 1, \varphi_i(v_j) = 0$.
Hence $(\varphi_1, \ldots, \varphi_m)$ is the dual basis of the basis $v_1, \ldots, \varphi_m$ of $\text{span}\,(v_1, \ldots, v_m)$. $\square$

---

33 *Suppose $A \in \mathbf{F}^{m,n}$. Define $T : A \to A^t$. Prove that T is an isomorphism of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$*

SOLUTION: By [3.111], T is linear. Note that $(A^t)^t = A$.

(a) For any $B \in \mathbf{F}^{n,m}$, let $A = B^t$ so that $T(A) = B$. Thus T is surjective.
(b) If $T(A) = 0$ for some $A \in \mathbf{F}^{n,m}$, then $A = 0$. Thus T is injective.
for if not, $\exists\ j, k \in \mathbf{N}^+$ such that $A_{j,k} \neq 0$, then $T(A)_{k,j} \neq 0$, contradicts.
$\Rightarrow \square$

---

32 *Suppose $T \in \mathcal{L}(V)$, and $(u_1, \ldots, u_m)$ and $(v_1, \ldots, v_m)$ are bases of V. Prove that*

T is invertible $\iff$ The rows of $\mathcal{M}(T,(u_1,\dots,u_m),(v_1,\dots,v_m))$ form a basis of $\mathbf{F}^{1,n}$.

Solution: Note that T is inveritlbe $\Rightarrow T'$ is invertible. And $\mathcal{M}(T') = \mathcal{M}(T)^t = A^t$, denote it by B.

Let $(\varphi_1,\dots,\varphi_m)$ be the dual basis of $(v_1,\dots,v_m)$, $(\psi_1,\dots,\psi_m)$ be the dual basis of $(u_1,\dots,u_m)$.

(a) Suppose T is invertible, so is T'. Because $T'(\varphi_1),\dots,T'(\varphi_m)$ is linearly independent.

Noticing that $T'(\varphi_i) = B_{1,i}\psi_1 + \cdots + B_{m,i}\psi_m$.

Thus the columns of B, namely the rows of A, are linearly independent (check it by contradiction).

(b) Suppose the rows of A are linearly independent, so are the columns of B.

Then $(T'(\varphi_1),\dots,T'(\varphi_m))$ is a basis of range T', namely V'. Thus T' is surjective.

Hence T' is invertible, so is T. $\square$

---

34 *The double dual space of V, denoted by V'', is defined to be the dual space of V'.*
In other words, $V'' = \mathcal{L}(V',\mathbf{F})$. Define $\Lambda : V \to V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.

(a) *Show that Λ is a linear map from V to V''.*

(b) *Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*

(c) *Show that if V is finite-dim, then Λ is an isomorphism from V onto V''.*

Suppose V is finite-dim. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

Solution:

(a) $\forall \varphi \in V'$, $\forall v, w \in V, a \in \mathbf{F}$, $(\Lambda(v+aw))(\varphi) = \varphi(v+aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.

Thus $\Lambda(v+aw) = \Lambda v + a\Lambda w$. Hence Λ is linear.

(b) $(T''(\Lambda v))(\varphi) = ((\Lambda v)\circ(T'))(\varphi) = (\Lambda v)(T'(\varphi)) = (T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = (\Lambda(Tv))(\varphi)$.

Hence $T''(\Lambda v) = (\Lambda(Tv))$, getting $T'' \circ \Lambda = \Lambda \circ T$.

(c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is injective.

又 Because V is finite-dim. $\dim V = \dim V' = \dim V''$. Hence Λ is an isomorphism. $\square$

---

36 *Suppose U is a subspace of V. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*

(a) *Show that null $i' = U^0$: null $i' =$ (range $i)^0 = U^0 \Leftarrow$ range $i = U$.* $\square$

(b) *Prove that if V is finite-dim, then range $i' = U'$: range $i' =$ (null $i)^0_U = (\{0\})^0_U = U'$.* $\square$

(c) *Prove that if V is finite-dim, then $\tilde{i'}$ is an isomorphism from V'/U^0 onto U':*

Note that $\tilde{i'} : V'/\text{null } i' \to \text{range } i' \Rightarrow \tilde{i'} : V'/U^0 \to U'$. By (a), (b) and [3.91(d)]. $\square$

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

---

37 *Suppose U is a subspace of V and π is the quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.*

(a) *Show that π' is injective: Because π is surjective. Use [3.108].* $\square$

(b) *Show that range $\pi' = U^0$.*

(c) *Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.*

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

In fact, there is no assumption here that any of these vector spaces are finite-dim.

Solution: [3.109] is not available. Using (3.E.18), also see (3.E.20).

(b) $\psi \in \text{range } \pi' \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \text{null } \psi \supseteq U \iff \psi \in U^0$. Hence range $\pi' = U^0$.

(c) $\psi \in U^0 \iff \text{null } \psi \supseteq U \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi = \pi(\varphi)$. Thus π' is surjective. And by (a). $\square$

Ended

- **NOTE FOR [4.8]:** *division algorithm for polynomials*

  Suppose $p, s \in \mathcal{P}(\mathbf{F})$, with $s \neq 0$. Then $\exists!\, q, r \in \mathcal{P}(\mathbf{F})$ such that $p = sq + r$ and $\deg r < \deg s$. *Another Proof:*

  Suppose $\deg p \geq \deg s$. Then $(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of length } \deg s}, \underbrace{s, zs, \cdots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)})$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.

  Because $q \in \mathcal{P}(\mathbf{F})$, $\exists!\, a_i, b_j \in \mathbf{F}$,

  $$\begin{aligned} q &= a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s \\ &= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1}}_{r} + s \underbrace{\left(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s}\right)}_{q}. \end{aligned}$$

  With r, q as defined uniquely above, we are done. $\square$

- **NOTE FOR [4.11]:** *each zero of a polynomial corresponds to a degree-one factor;* *Another Proof:*

  First suppose $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, $\exists!\, a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.

  Then $p(z) = p(z) - p(\lambda) = a_1(z - \lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.

  Hence $\forall k \in \{1, \dots, m\}$, $z^k - \lambda^k = (z - \lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.

  Thus $p(z) = \sum\limits_{j=1}^{m} a_j (z - \lambda) \sum\limits_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) \sum\limits_{j=1}^{m} a_j \sum\limits_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) q(z)$.

- **NOTE FOR [4.13]:** *fundamental theorem of algebra, first version*

  Every nonconstant polynomial with complex coefficients has a zero in $\mathbf{C}$. *Another Proof:*

  De Moivre's theorem (which you can prove using induction on k and the addition formulas for cosine and sine), states that if $k \in \mathbf{N}^+, \theta \in \mathbf{R}$, then $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$.

  Suppose $w \in \mathbf{C}, k \in \mathbf{N}^+$ and using polar coordinates. $\exists\, r \geq 0, \theta \in \mathbf{R}$ such that $r(\cos\theta + \mathrm{i}\sin\theta) = w$.

  Hence $(r^{1/k}(\cos\frac{\theta}{k} + \mathrm{i}\sin\frac{\theta}{k}))^k = w$. Thus every complex number has a k^{th} *root*, a fact that we will soon use.

  Suppose a nonconstant $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.

  Then $|p(z)| \to \infty$ as $|z| \to \infty$ (because $\dfrac{|p(z)|}{|z_m|} \to |c_m|$ as $|z| \to \infty$).

  Thus the continuous function $z \to |p(z)|$ has a global minimum at some point $\zeta \in \mathbf{C}$.

  To show that $p(\zeta) = 0$, suppose that $p(\zeta) \neq 0$.

  Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \dfrac{p(z + \zeta)}{p(\zeta)}$.

  The function $z \to |q(z)|$ has a global minimum value of 1 at $z = 0$.

  Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where k is the smallest positive integer such that $a_k \neq 0$.

  Let $\beta \in \mathbf{C}$ be such that $\beta^k = -\dfrac{1}{a_k}$.

  There is a constant $c > 1$ such that if $t \in (0, 1)$,

  then $|q(t\beta)| \leq |1 + a_k t^k \beta^k| + t^{k+1} c = 1 - t^k(1 - tc)$.

  Thus taking t to be $1/(2c)$ in the inequality above, we have $|q(t\beta)| < 1$,

  which contradicts the assumption that the global minimum of $z \to |q(z)|$ is 1.

  Hence $p(\zeta) = 0$, as desired. $\square$

- *Prove that if $w, z \in \mathbf{C}$, then $||w| - |z|| \leq |w - z|$.* *The inequality here is called the* **reverse triangle inequality**.

SOLUTION:

$$\begin{aligned}|w-z|^2 &= (w-z)(\overline{w}-\overline{z})\\ &= |w|^2+|z|^2-(w\overline{z}+\overline{w}z)\\ &= |w|^2+|z|^2-(\overline{\overline{w}z}+\overline{w}z)\\ &= |w|^2+|z|^2-2Re(\overline{w}z)\\ &\geq |w|^2+|z|^2-2|\overline{w}z|\\ &= |w|^2+|z|^2-2|w||z| = ||w|-|z||^2. \qquad \square\end{aligned}$$

Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides.

• *Suppose V is a complex vector space and $\varphi \in V'$.*
Define $: V \to \mathbf{R}$ by $\sigma(v) = \Re\varphi(v)$ for each $v \in V$.
Show that $\varphi(v) = \sigma(v) - i\sigma(iv)$ for all $v \in V$.

SOLUTION:

Notice that $\varphi(v) = \Re\varphi(v) + \mathrm{i}\Im\varphi(v) = \sigma(v) + \mathrm{i}\Im\varphi(v)$. 又 $\Re\varphi(\mathrm{i}v) = \Re[\mathrm{i}\varphi(v)] = -\Im\varphi(v) = \sigma(\mathrm{i}v)$.
Hence $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$. $\square$

---

2 *Suppose m is a positive integer. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subspace of $\mathcal{P}(\mathbf{F})$?*

SOLUTION:

$x^m, x^m + x^{m-1} \in U$ but $\deg[(x^m + x^{m-1}) - (x^m)] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$.
Hence U is not closed under addition, and therefore is not a subspace. $\square$

---

3 *Suppose m is a positive integer. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p \text{ is even }\}$ a subspace of $\mathcal{P}(\mathbf{F})$?*

SOLUTION:

$x^2, x^2 + x \in U$ but $\deg[(x^2 + x) - (x^2)]$ is odd and hence $(x^2 + x) - (x^2) \notin U$.
Thus U is not closed under addition, and therefore is not a subspace. $\square$

---

4 *Suppose that m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{F})$ such that $\deg p = n$, the zeros of p are $\lambda_1, \ldots, \lambda_m$.*

SOLUTION: Let $p(z) = (z-\lambda_1)^{n-(m-1)}(z-\lambda_2)\cdots(z-\lambda_m)$. $\square$

---

5 *Suppose that $m \in \mathbf{N}$, $z_1, \ldots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \ldots, w_{m+1} \in \mathbf{F}$. Prove that $\exists!\, p \in \mathcal{P}_m(\mathbf{F})$ such that $p(z_k) = w_k$ for each $k = 1, \ldots, m+1$.*

This result can be proved without using linear algebra. However, try to find the clearer, shorter proof that uses some linear algebra.

SOLUTION:

Define $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by $Tq = (q(z_1), \ldots, q(z_m), q(z_{m+1}))$. As can be easily checked, T is linear.
We need to show that T is surjective, so that such p exists; and that T is injective, so that such p is unique.

$$\begin{aligned}Tq = 0 &\iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0\\ &\iff q \in \mathcal{P}_m(\mathbf{F}) \text{ is the zero polynomial, for if not,}\end{aligned}$$

q has at least $m+1$ distinct roots, while $\deg q = m$. Contradicts (by [4.12]). Hence T is injective.

$\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m + 1 = \dim \mathbf{F}^{m+1}$. 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1}$. Hence T is surjective. $\square$

---

6 *Suppose $p \in \mathcal{P}_m(\mathbf{C})$ has degree m. Prove that*

p has m distinct zeros $\Longleftrightarrow$ p and its derivative p' have no zeros in common.

SOLUTION:

(a) Suppose p has m distinct zeros. By [4.14] and $\deg p = m$, let $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$, $\exists!\, c, \lambda_i \in \mathbf{C}$.

For each $j \in \{1, \dots, m\}$, let $\dfrac{p(z)}{(z-\lambda_j)} = q_j \in \mathcal{P}_{m-1}(\mathbf{C})$, then $p(z) = (z-\lambda_j)q_j(z)$ and $q_j(\lambda_j) \neq 0$.

$p'(z) = (z-\lambda_j)q_j'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

(b) To prove the implication on the other direction, we prove the contrapositive:

Suppose p has less than m distinct roots.

We must show that p and its derivative p' have at least one zero in common.

Let λ be a zero of p, then write $p(z) = (z-\lambda)^n q(z)$, $\exists!\, n \in \mathbf{N}^+, q \in \mathcal{P}_{m-n}(\mathbf{C})$.

$p'(z) = (z-\lambda)^n q'(z) + n(z-\lambda)^{n-1}q(z) \Rightarrow p'(\lambda) = 0$, λ is a common root of p' and p. $\square$

---

7 *Prove that every polynomial of odd degree with real coefficients has a real zero.*

SOLUTION:

Using the notation proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow$ m is odd. Hence λ_1 exists. $\square$

OR. Using calculus but not using [4.17].

Suppose $p \in \mathcal{P}_m(\mathbf{F})$, $\deg p = m$, m is odd.

Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1}a_m$ by δ

Write $p(x) = x^m(\dfrac{a_0}{x^m} + \dfrac{a_1}{x^{m-1}} + \cdots + \dfrac{a_{m-1}}{x} + a_m)$.

Thus $p(x)$ is continuous, and $\lim\limits_{x\to-\infty} p(x) = -\delta\infty$; $\lim\limits_{x\to\infty} p(x) = \delta\infty$.

Hence we conclude that p has at least one real zero. $\square$

---

8 *For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \to \mathbf{R}$ by* $(Tp)(x) = \begin{cases} \dfrac{p(x)-p(3)}{x-3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3 \end{cases}$ *for all $x \in \mathbf{R}$.*

Show that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$ and that $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is a linear map.

SOLUTION:

For $x \neq 3$, $T(x^n) = \dfrac{x^n - 3^n}{x-3} = \sum\limits_{i=1}^{n} 3^{i-1}x^{n-i}$.

For $x = 3$, $T(x^n) = 3^{n-1}\cdot n$. Note that if $x = 3$, then $\sum\limits_{i=1}^{n} 3^{i-1}x^{n-i} = \sum\limits_{i=1}^{n} 3^{n-1} = 3^{n-1}\cdot n$.

Hence for all $x \in \mathbf{R}$ and for all $n \in \mathbf{N}$, $T(x^n) = \sum\limits_{i=1}^{n} 3^{i-1}x^{n-i} \in \mathcal{P}(\mathbf{R})$.

Because T is linear, we conclude that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$.

Now we show that T is linear:

$\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, T(p+\lambda q)(x) = \begin{cases} \dfrac{(p+\lambda q)(x) - (p+\lambda q)(3)}{x-3} & \text{if } x \neq 3, \\ (p+\lambda q)'(3) & \text{if } x = 3 \end{cases}$ for all $x \in \mathbf{R}$.

Notice that $(p+\lambda q)(x) - (p+\lambda q)(3) = (p(x) - p(3)) + (\lambda q(x) - \lambda q(3))$;

$(p+\lambda q)'(3) = p'(3) + \lambda q'(3)$.

Thus $T(p+\lambda q)(x) = (T(p) + \lambda T(q))(x)$ for all $x \in \mathbf{R}$. $\square$

---

9 *Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\overline{z})}$.*

Prove that q is a polynomial with real coefficients.

SOLUTION:

$p(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow p(\overline{z}) = \underline{a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0} \Rightarrow \overline{p(\overline{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\overline{z})} = \overline{p(\overline{z})} p(z) = \overline{p(\overline{z})\overline{p(\overline{\overline{z}})}} = \overline{q(\overline{z})}$.

Hence letting $q(z) = c_m x^m + \cdots + c_1 x + c_0 \ \Rightarrow\ \overline{c_k} = c_k$, $c_k \in \mathbf{R}$ for each k. $\square$

---

10 *Suppose $m \in \mathbf{N}$ and $p \in \mathcal{P}_m(\mathbf{C})$ is such that there are $(m+1)$ distinct real numbers $x_0, x_1, \ldots, x_m$ with $p(x_k) \in \mathbf{R}$ for each x_k. Prove that all coefficients of p are real.*

SOLUTION: Let $p(x_k) = y_k$ for each k. By Problem (5), $\exists!\, q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_k) = y_k$. Hence $p = q$. $\square$

OR. Using the Lagrange Interpolating Polynomial.

Define $q(x) = \displaystyle\sum_{j=0}^{m} \frac{(x - x_0)(x - x_1)\cdots(x - x_{j-1})(x - x_{j+1})\cdots(x - x_m)}{(x_j - x_0)(x_j - x_1)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_m)} p(x_j)$.

又 For each j, $x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R}) \subseteq \mathcal{P}_m(\mathbf{C})$.

Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q - p)(x_k) = 0$ for each $k \in \{0, 1, \ldots, m\}$.

Then $(q - p)$ has $(m+1)$ distinct zeros, while $(q - p) \in \mathcal{P}_m(\mathbf{C})$. Hence by [4.12], $q - p = 0 \Rightarrow p = q$. $\square$

---

11 *Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*

(a) *Show that* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) *Find a basis of $\mathcal{P}(\mathbf{F})/U$.*

SOLUTION:

U is a subspace of $\mathcal{P}(\mathbf{F})$ because $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, pf + \lambda pg = p(f + \lambda g) \in U$.

NOTE: Define $P :\to \mathcal{P}(\mathbf{F})$ by $(Pq)(x) = p(q(x)) = (p \circ q)(x)\ (\neq p(x)q(x)\ )$. P is not linear.

(a) By [4.8], $\forall f \in \mathcal{P}(\mathbf{F})$, $\exists!\, q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r)$; $\deg r < \deg p$.

Hence $\forall f \in \mathcal{P}(\mathbf{F})$, $\exists!\, pq \in U, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), f = (pq) + (r); r \notin U$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p - 1}(\mathbf{F})$. Therefore $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p-1}(\mathbf{F})$ are isomorphic.

OR. $\forall f \in \mathcal{P}(\mathbf{F})$, $\exists!\, q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r)$; $\deg r < \deg p$.

Define $R : \mathcal{P}(\mathbf{F}) \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $(Rf)(z) = r(z)$ for each $z \in \mathbf{F}$.

$\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, R(f + \lambda g)(z) = R(f) + \lambda R(g)$.

BECAUSE: $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}$,

$\exists!\, q_1, r_1 \in \mathcal{P}(\mathbf{F}), f = (p)q_1 + (r_1)$, $\deg r_1 < \deg p$;

$\exists!\, q_2, r_2 \in \mathcal{P}(\mathbf{F}), g = (p)q_2 + (r_2)$, $\deg r_2 < \deg p$;

$\exists!\, q_3, r_3 \in \mathcal{P}(\mathbf{F}), \lambda g = (p)q_3 + (r_3) = (p)(\lambda q_2) + (\lambda r_2)$, $\deg r_3 < \deg p$ and $\deg \lambda r_2 < \deg p$.

$\Rightarrow q_3 = \lambda q_2, r_3 = \lambda r_2$.

$\exists!\, q_0, r_0 \in \mathcal{P}(\mathbf{F}), (f + \lambda g) = (p)q_0 + (r_0)$

$= (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2)$, $\deg r_0 < \deg p$ and $\deg(r_1 + \lambda r_2) < \deg p$.

$\Rightarrow q_1 + \lambda q_2 = q_0;\ \ r_1 + \lambda r_2 = r_0$.

Hence R is linear.

$R(f) = 0 \iff f = pq$, $\exists!\, q \in \mathcal{P}(\mathbf{F})$. Thus $\operatorname{null} R = U$.

$\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F})$, let $f = p + r$, then $R(f) = r$. Thus $\operatorname{range} R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Finally, by [3.91(d)], $\mathcal{P}(\mathbf{F})/\operatorname{null} R$, namely $\mathcal{P}(\mathbf{F})/U$, and $\operatorname{range} R$, namely $\mathcal{P}_{\deg p-1}(\mathbf{F})$, are isomorphic.

(b) $(1 + U, x + U, \ldots, x^{\deg p - 1}) + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. $\square$

• *Suppose nonconstant $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p$, $n = \deg q$.*

Use (a)—(c) *below to prove that* $\exists\,!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ *such that* $rp + sq = 1$.

(a) *Define* $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ *by* $T(r, s) = rp + sq$. *Show that the linear map* T *is injective.*

(b) *Show that the linear map* T *in* (a) *is surjective.*

(c) *Use* (b) *to conclude that* $\exists\,!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ *such that* $rp + sq = 1$.

SOLUTION:

(a) T is linear because $\forall\, r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,

$$T((r_1, s_1) + \lambda(r_2, s_2)) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1, s_1) + \lambda T(r_2, s_2).$$

Suppose $T(r, s) = rp + sq = 0$. Notice that p, q have no zeros in common.

Then $r = s = 0$, for if not, write $\dfrac{q(z)}{r(z)} = \dfrac{p(z)}{s(z)}$, while for any zero λ of q, $\dfrac{q(\lambda)}{r(z)} = 0 \neq \dfrac{p(\lambda)}{s(z)}$. Hence $\square$

(b) $\dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) = \dim \mathcal{P}_{n-1}(\mathbf{C}) + \dim \mathcal{P}_{m-1}(\mathbf{C}) = n + m = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.

又 T is injective. Hence $\dim \operatorname{range} T = \dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) - \dim \operatorname{null} T = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.

Thus $\operatorname{range} T = \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$ is surjective, and therefore is an isomorphism. $\square$

(c) Immediately. $\square$

ENDED

5.A

[1]: 31; [2]: 1, 2, 3, 15, 21; [3]: 23, (2E Ch5.20), (4E.5.A.37), 4, 5; [4]: 6, (4E.5.A.17, 18) OR 16, (4E.5.A.15); [5]: 7, 8, (4E.5.A.8), 22, 9, 10; [6]: 11, 12, 14, 30, 13, (4E.5.A.11); [7]: 17, (4E.5.A.16), 18; [8]: 19, 20, 24; [9]: 24', 25, 26, 27, 28; [10]: (4E.5.A.39), 29; [11]: 32, (4E.5.A.35), (4E.5.A.38) OR 35, 36; [12] 32, 34.

• **NOTE FOR [5.6]:**

More generally, suppose we do not know whether V is finite-dim. Then $(a) \iff (b)$.

Suppose (a) λ is an eigenvalue of T with an eigenvector v. Then $(T - \lambda I)v = 0$.

Hence we get (b), $(T - \lambda I)$ is not injective. And then (d), $(T - \lambda I)$ is not invertible.

But $(d) \not\Rightarrow (b)$ (because S is not invertible $\iff$ S is not injective *or* S is not surjective).

• **NOTE FOR [5.10]:** linearly independent eigenvectors *Suppose* $T \in \mathcal{L}(V)$. *Prove that every list of eigenvectors of* T *corresponding to distinct eigenvalues is linearly independent.*

SOLUTION: OR Another Proof.

Suppose the desired result is false. Then ($m \neq 1$ because eigenvectors are nonzero)

$\exists$ smallest $\mathbf{N}^+ \ni m > 1$ such that $\exists\,(v_1, \dots, v_m)$ of eigenvectors of T linearly dependent

corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_m$ of T.

Suppose $a_1 v_1 + \cdots + a_{m-1} v_{m-1} + a_m v_m = 0$. Then each a_j is zero, for if not, contradicts the minimality of m.

Apply $T - \lambda_m I$ to both sides, getting $a_1(\lambda_1 - \lambda_m)v_1 + \cdots + a_{m-1}(\lambda_{m-1} - \lambda_m)v_{m-1} = 0$.

Because the eigenvalues $\lambda_1, \dots, \lambda_m$ are distinct, and $a_j \neq 0$ for all a_j.

Thus $(v_1, \dots, v_{m-1})$ of length $(m - 1)$ is linearly dependent corresponding to distinct eigenvalues.

Contradicts the minimality of m. $\square$

---

31 *Suppose* V *is finite-dim and* $v_1, \dots, v_m \in V$. *Prove that* $(v_1, \dots, v_m)$ *is linearly independent* $\iff \exists\, T \in \mathcal{L}(V)$, $v_1, \dots, v_m$ *are eigenvectors of* T *corresponding to distinct eigenvalues.*

SOLUTION: Suppose $(v_1, \dots, v_m)$ is linearly independent, extend it to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.

Define $T \in \mathcal{L}(V)$ by $Tv_k = kv_k$ for each $k \in \{1, \dots, m, \dots, n\}$. Conversely by [5.10]. $\square$

---

1 *Suppose* $T \in \mathcal{L}(V)$ *and* U *is a subspace of* V.

(a) *Prove that if $U \subseteq$ null T, then U is invariant under T.* $\forall u \in U \subseteq \text{null}\, T, Tu = 0 \in U.$ $\square$

(b) *Prove that if range $T \subseteq U$, then U is invariant under T.* $\forall u \in U, Tu \in \text{range}\, T \subseteq U.$ $\square$

• *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*

(a) *Prove that null $(T - \lambda I)$ is invariant under S, where λ is chosen arbitrarily.*

(b) *Prove that range $(T - \lambda I)$ is invariant under S, where λ is chosen arbitrarily.*

SOLUTION:

Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.

(a) Suppose $v \in \text{null}\,(T - \lambda I)$, then $(T - \lambda I)(Sv) = S(T - \lambda I)v = S(0) = 0$.

Hence $Sv \in \text{null}\,(T - \lambda I)$ and therefore null $(T - \lambda I)$ is invariant under S.

(b) Suppose $v \in \text{range}\,(T - \lambda I)$, therefore $\exists\, u \in V, (T - \lambda I)u = v$.

Then $Sv = S(T - \lambda I)u = (T - \lambda I)(Su) \in \text{range}\,(T - \lambda I)$.

Hence $Sv \in \text{range}\,(T - \lambda I)$ and therefore range $(T - \lambda I)$ is invariant under S. $\square$

COMMENT: Reversing the roles of S and T, letting $\lambda = 0$, we can conclude that null S and range S is invariant under T, which is what we will prove in Problem (2) and (3) below.

• *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*

2 *Show that $W =$ null S is invariant under T.* $\forall u \in W, Su = 0 \Rightarrow TSu = 0 = STu \Rightarrow Tu \in W.$ $\square$

3 *Show that $U =$ range S is invariant under T.* $\forall w \in U, \exists\, v \in V, Sv = w, STv = TSv = Tw \in U.$ $\square$

---

15 *Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.*

(a) *Prove that T and $S^{-1}TS$ have the same eigenvalues.*

(b) *What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?*

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector v.

Then $S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v$.

Thus λ is also an eigenvalue of $S^{-1}TS$ with an eigenvector $S^{-1}v$.

Suppose λ is an eigenvalue of $S^{-1}TS$ with an eigenvector v.

Then $S(S^{-1}TS)v = TSv = \lambda Sv$.

Thus λ is also an eigenvalue of T with an eigenvector Sv. $\square$

OR. Note that $S(S^{-1}TS)S^{-1} = T$. Hence every eigenvalue of $S^{-1}TS$ is an eigenvalue of $S(S^{-1}TS)S^{-1} = T$.

And every eigenvector v of $S^{-1}TS$ is $S^{-1}v$, every eigenvector u of T is Su. $\square$

---

21 *Suppose $T \in \mathcal{L}(V)$ is invertible.*

(a) *Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$.*

Prove that λ is an eigenvalue of $T \iff \frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) *Prove that T and T^{-1} have the same eigenvectors.*

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector v.

Then $T^{-1}Tv = \lambda T^{-1}v = v \Rightarrow T^{-1}v = \frac{1}{\lambda}v$. Hence $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Suppose $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} with an eigenvector v.

Then $TT^{-1}v = v = \frac{1}{\lambda}Tv \Rightarrow Tv = \lambda v$. Hence λ is an eigenvalue of T.

OR. Note that $(T^{-1})^{-1} = T$ and $\frac{1}{\frac{1}{\lambda}} = \lambda$. $\square$

---

23 *Suppose V is finite-dim, $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.*

SOLUTION:

Suppose λ is an eigenvalue of ST with an eigenvector v. Then $T(STv) = \lambda Tv = TS(Tv)$.

If $Tv \neq 0$, then λ is an eigenvalue of TS.

Otherwise, $\lambda = 0$, $(\ v \neq 0, \lambda v = 0 = STv\ )$, then T is not invertible

$\Rightarrow TS$ is not invertible $\Rightarrow (TS - 0I)$ is not invertible $\Rightarrow \lambda = 0$ is an eigenvalue of TS.

Reversing the roles of T and S, we conclude that ST and TS have the same eigenvalues. $\square$

• (2E Ch5.20)

Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and $S \in \mathcal{L}(V)$ has the same eigenvectors (but might not with the same eigenvalues). Prove that $ST = TS$.

SOLUTION:

Let $n = \dim V$. For each $j \in \{1, \dots, n\}$, let v_j be an eigenvector with eigenvalue λ_j of T and α_j of S.

Then $(v_1, \dots, v_n)$ is a basis of V. Because $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$ for each j. Hence $ST = TS$. $\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(S) = TS$ for each $S \in \mathcal{L}(V)$.
Prove that the set of eigenvalues of T equals the set of eigenvalues of $\mathcal{A}$.

SOLUTION:

(a) Suppose $v_1, \dots, v_m$ are all linearly independent eigenvectors of T

with corresponding eigenvalues $\lambda_1, \dots, \lambda_m$ respectively (possibly with repetitions).

Extend to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.

Then for each $k \in \{1, \dots, m\}$, span $(v_k) \subseteq$ null $(T - \lambda_k I)$.

Define $S_k \in \mathcal{L}(V)$ by $S_k(v_j) = v_k$ for each $j \in \{1, \dots, n\}$,

so that range $S_k =$ span (v_k) for each $k \in \{1, \dots, m\}$, then $\mathcal{A}(S_k) = TS_k = \lambda_k S_k$.

Thus the eigenvalues of T are eigenvalues of $\mathcal{A}$.

(b) Suppose $\lambda_1, \dots, \lambda_m$ are all eigenvalues of $\mathcal{A}$ with eigenvectors $S_1, \dots, S_m$ respectively.

Then for each $k \in \{1, \dots, m\}$, $\exists\, v \in V, 0 \neq u = S_k(v) \in V \Rightarrow Tu = (TS_k)v = (\lambda_k S_k)v = \lambda_k u$.

Thus the eigenvalues of $\mathcal{A}$ are eigenvalues of T. $\square$

OR.

(a) Suppose λ is an eigenvalue of T with an eigenvector v.

Let $v_1 = v$ and extend to a basis $(v_1, \dots, v_m)$ of V.

Define $S \in \mathcal{L}(V)$ by $Sv_1 = v_1,\ Sv_k = 0$ for $k \geq 2$.

Then $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$.

Hence $(T - \lambda I)S = 0 \Rightarrow TS = \lambda S$ while $S \neq 0$. Thus λ is also an eigenvalue of $\mathcal{A}$.

(b) Suppose λ is an eigenvalue of $\mathcal{A}$ with an eigenvector S. Then $(T - \lambda I)S = 0$ while $S \neq 0$.

Hence $(T - \lambda I)$ is not injective. Thus λ is also an eigenvalue of T. $\square$

• COMMENT: Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(S) = ST, \forall S \in \mathcal{L}(V)$. Then the eigenvalues of $\mathcal{B}$ are not the eigenvalues of T.

---

4 *Suppose $T \in \mathcal{L}(V)$ and $V_1, \dots, V_m$ are subspaces of V invariant under T.*
Prove that $V_1 + \dots + V_m$ is invariant under T.

SOLUTION:

For each $i = 1, \dots, m$, $\forall v_i \in V_i, Tv_i \in V_i$

Hence $\forall v = v_1 + \dots + v_m \in V_1 + \dots + V_m, Tv = Tv_1 + \dots + Tv_m \in V_1 + \dots + V_m$. $\square$

---

5 *Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection* *of subspaces of V invariant under T is invariant under T.*

SOLUTION:

Suppose $\{V_\alpha\}_{\alpha\in\Gamma}$ is a collection of subspaces of V invariant under T; here Γ is an arbitrary index set.

We need to prove that $\bigcap_{\alpha\in\Gamma} V_\alpha$, which equals the set of vectors that are in V_α for each $\alpha \in \Gamma$, is invariant under T.

For each $\alpha \in \Gamma, \forall v_\alpha \in V_\alpha, Tv_\alpha \in V_i$.

Hence $\forall v \in \bigcap_{\alpha\in\Gamma} V_\alpha, Tv \in V_\alpha, \forall \alpha \in \Gamma \Rightarrow Tv \in \bigcap_{\alpha\in\Gamma} V_\alpha$. Thus $\bigcap_{\alpha\in\Gamma} V_\alpha$ is invariant under T. $\square$

---

6 *Prove or give a counterexample:*
If V is finite-dim and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

SOLUTION:

Notice that V might be $\{0\}$. In this case we are done. Suppose $\dim V \geq 1$. We prove by contrapositive:

Suppose $U \neq \{0\}$ and $U \neq V$, then $\exists T \in \mathcal{L}(V)$ such that U is not invariant under T.

Let W be such that $V = U \oplus W$.

Let $(u_1, \dots, u_m)$ be a basis of U and $(w_1, \dots, w_n)$ be a basis of W.

Hence $(u_1, \dots, u_m, w_1, \dots, w_n)$ is a basis of V.

Define $T \in \mathcal{L}(V)$ by $T(a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n) = b_1w_1 + \cdots + b_nw_n$. $\square$

- *Suppose* $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.

(a) (OR (9.11)) $\lambda \in \mathbf{R}$. *Prove that λ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of $T_\mathbb{C}$.*

(b) (OR Problem (16)) $\lambda \in \mathbf{C}$. *Prove that λ is an eigenvalue of $T_\mathbb{C} \iff \overline{\lambda}$ is an eigenvalue of $T_\mathbb{C}$.*

SOLUTION:

(a) Suppose $v \in V$ is an eigenvector corresponding to the eigenvalue λ.

Then $Tv = \lambda v \Rightarrow T_\mathbb{C}(v + \mathrm{i}0) = Tv + \mathrm{i}T0 = \lambda v$.

Thus λ is an eigenvalue of T.

Suppose $v + \mathrm{i}u \in V_\mathbb{C}$ is an eigenvector corresponding to the eigenvalue λ.

Then $T_\mathbb{C}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \Rightarrow Tv = \lambda v, Tu = \lambda u$. (Note that v or u might be zero).

Thus λ is an eigenvalue of $T_\mathbb{C}$.

(b) Suppose λ is an eigenvalue of $T_\mathbb{C}$ with an eigenvector $v + \mathrm{i}u$.

Let $(v_1, \dots, v_n)$ be a basis of V. Write $v = \sum_{i=1}^{n} a_i v_i, u = \sum_{i=1}^{n} b_i v_i$, where $a_i, b_i \in \mathbf{R}$.

Then $T_\mathbb{C}(v + \mathrm{i}u) = Tv + \mathrm{i}Tu = \lambda v + \mathrm{i}\lambda u = \lambda \sum_{i=1}^{n} (a_i + \mathrm{i}b_i)v_i$. Conjugating two sides, we have:

$$\overline{T_\mathbb{C}(v + \mathrm{i}u)} = \overline{Tv + \mathrm{i}Tu} = \overline{Tv} - \mathrm{i}\overline{Tu} = Tv - \mathrm{i}Tu = T_\mathbb{C}(\overline{v + \mathrm{i}u}) = \overline{\lambda \sum_{i=1}^{n} (a_i + \mathrm{i}b_i)v_i} = \overline{\lambda} \sum_{i=1}^{n} (a_i - \mathrm{i}b_i)v_i.$$

Hence $\overline{\lambda}$ is an eigenvalue of $T_\mathbb{C}$. To prove the other direction, notice that $\overline{\overline{\lambda}} = \lambda$. $\square$

- *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*
Show that λ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector v.

Then $(T - \lambda I_V)$ is not invertible. 又 V is finite-dim.

Thus by [3.108, 110], [3.101] and Problem (12) in (3.F), $(T - \lambda I_V)' = T' - \lambda I_{V'}$ is not invertible.

Hence λ is an eigenvalue of T'.

(b) Suppose λ is an eigenvalue T' with an eigenvector ψ. Then $T'(\psi) = \psi \circ T = \lambda\psi$.

又 $\psi \neq 0 \Rightarrow \exists v \in V$ such that $\psi(v) \neq 0$. Note that $\psi(Tv) = \lambda\psi(v)$.

Thus $\lambda = \dfrac{\psi(Tv)}{\psi(v)} \Rightarrow Tv = \dfrac{\psi(Tv)}{\psi(v)} v = \lambda v$. Hence λ is an eigenvalue of T. $\square$

---

7 *Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.*

SOLUTION:

Suppose $\lambda \in \mathbf{R}$ and $(x, y) \in \mathbf{R}^2 \backslash \{0\}$ such that $T(x, y) = (-3y, x) = \lambda(x, y)$. Then $-3y = \lambda x$ and $x = \lambda y$.

Thus $-3y = \lambda^2 y \Rightarrow \lambda^2 = -3$, ignoring the possibility of $y = 0$ (because if $y = 0$, then $x = 0$).

Hence the set of solution for this equation is $\varnothing$, and therefore T has no eigenvalues in $\mathbf{R}$. $\square$

---

8 *Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.*

SOLUTION:

Suppose $\lambda \in \mathbf{F}$ and $(w, z) \in \mathbf{F}^2$ such that $T(w, z) = (z, w) = \lambda(w, z)$. Then $z = \lambda w$ and $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, ignoring the possibility of $z = 0$ ($z = 0 \Rightarrow w = 0$).

Hence $\lambda_1 = -1$ and $\lambda_2 = 1$ are all eigenvalues of T. For $\lambda_1 = -1, z = -w, w = -z$; For $\lambda_2 = 1, z = w$.

Thus the set of all eigenvectors is $\{(z, -z), (z, z) : z \in \mathbf{F} \wedge z \neq 0\}$. $\square$

• *Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$.*
Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.

SOLUTION: (See also at (3.B), just below Problem (25), where (5.B.4) is answered.)

Suppose λ is an eigenvalue with an eigenvector v. Then $P(Pv) = Pv \Rightarrow \lambda^2 v = \lambda v$. Thus $\lambda = 1$ or 0. $\square$

---

22 *Suppose $T \in \mathcal{L}(V)$ and $\exists$ nonzero vectors u, w in V such that $Tu = 3w$ and $Tw = 3u$.*
Prove that 3 or -3 is an eigenvalue of T.

SOLUTION: COMMENT: $Tu = 3w, Tw = 3u \Rightarrow T(Tu) = 9u \Rightarrow T^2$ has an eigenvalue 9.

$Tu = 3w, Tw = 3u \Rightarrow T(u + w) = 3(u + w), T(u - w) = 3(w - u) = -3(u - w)$. $\square$

---

9 *Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$.*
Find all eigenvalues and eigenvectors of T.

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector $(z_1, z_2, z_3) \in \mathbf{F}^3$.

Then $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$. Thus $2z_2 = \lambda z_1, \quad 0 = \lambda z_2, \quad 5z_3 = \lambda z_3$.

We discuss in two cases:

For $\lambda = 0,\ z_2 = z_3 = 0$ and z_1 can be arbitrary ($z_1 \neq 0$).

For $\lambda \neq 0,\ z_2 = 0 = z_1$, and z_3 can be arbitrary ($z_3 \neq 0$), then $\lambda = 5$.

The set of all eigenvectors is $\{(0, 0, z), (z, 0, 0) : z \in \mathbf{F} \wedge z \neq 0\}$. $\square$

---

10 *Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$*

(a) *Find all eigenvalues and eigenvectors of T.*

(b) *Find all invariant subspaces of V under T.*

SOLUTION:

(a) Suppose $v = (x_1, x_2, x_3, \ldots, x_n)$ is an eigenvector of T with an eigenvalue λ.

Then $Tv = \lambda v = (x_1, 2x_2, 3x_3, \ldots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots, \lambda x_n)$.

Hence $1, \ldots, n$ are eigenvalues of T.

And $\{(0, \ldots, 0, x_\lambda, 0, \ldots, 0) \in \mathbf{F}^n : \lambda = 1, \ldots, n,\ x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$ is the set of all eigenvectors of T.

(b) Let $V_\lambda = \{(0, \ldots, 0, x_\lambda, 0, \ldots, 0) \in \mathbf{F}^n : x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$. Then $V_1, \ldots, V_n$ are invariant under T.

Hence by Problem (4), every sum of $V_1, \ldots, V_n$ is a invariant subspace of V under T. $\square$

---

11 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.*

SOLUTION:

Note that in general, $\deg p' < \deg p$ ($\deg 0 = -\infty$).

Suppose λ is an eigenvalue of T with an eigenvector p.

Suppose $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p \neq p'$. Contradicts. Thus $\lambda = 0$.

Therefore $\deg \lambda p = -\infty = \deg p \Rightarrow p$ is a nonzero constant polynomial.

Hence the set of all eigenvectors is $\{C : C \in \mathbf{R} \wedge C \neq 0\} = \mathcal{P}_0(\mathbf{R}) \backslash \{0\}$. $\square$

---

12 *Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$.*
Find all eigenvalues and eigenvectors of T.

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector p, then $(Tp)(x) = xp'(x) = \lambda p(x)$.

Let $p = a_0 + a_1 x + \cdots + a_n x^n$.

Then $xp'(x) = a_1 x + 2a_2 x^2 + \cdots + na_n x^n = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n$.

Similar to Problem (10), $0, 1, \ldots, n$ are eigenvalues of T.

The set of all eigenvectors of T is $\{cx^\lambda : \lambda = 0, 1, \ldots, n,\ c \in \mathbf{F} \wedge c \neq 0\}$. $\square$

---

30 *Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigenvalues of T.*
Prove that $\exists\, x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

SOLUTION: Because 9 is not an eigenvalue. Hence $(T - 9I)$ is surjective. $\square$

---

14 *Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V.*
Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.
Find all eigenvalues and eigenvectors of P.

SOLUTION:

Suppose λ is an eigenvalue of P with an eigenvector $(u + w)$.

Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$. By [1.44] and $V = U \oplus W$, $(\lambda - 1)u = \lambda w = 0$.

Thus if $\lambda = 1$, then $w = 0$; if $\lambda = 0$, then $u = 0$.

Hence the eigenvalues of P are 0 and 1, the set of all eigenvectors in P is $U \cup W$. $\square$

---

13 *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*
Prove that $\exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}$ and $(T - \alpha I)$ is invertible.

SOLUTION:

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \dfrac{1}{1000 + k}$ for each $k = 1, \ldots, \dim V + 1$.

Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigenvalues.

Hence $\exists\, k = 1, \ldots, \dim V + 1$ such that α_k is not an eigenvalue of T and therefore $(T - \alpha_k I)$ is invertible. $\square$

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*
Prove that $\exists\, \delta > 0$ such that $(T - \alpha I)$ is invertible for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.

SOLUTION:

If T has no eigenvalues, then $(T - \alpha I)$ is injective for all $\alpha \in \mathbf{F}$ and we are done.

Let $\delta > 0$ be such that, for each eigenvalue λ_k, $\lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.

So that for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$, $(T - \alpha I)$ is not injective. $\square$

---

17 *Give an example of an operator on* $\mathbf{R}^4$ *that has no (real) eigenvalues.*

SOLUTION:

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $\mathcal{M}(T,(e_1,e_2,e_3,e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$. Where (e_1,e_2,e_3,e_4) is the standard basis of $\mathbf{R}^4$.

Suppose λ is an eigenvalue of T with an eigenvector (x,y,z,w).

Then $T(x,y,z,w) = \lambda(x,y,z,w) \Rightarrow \begin{cases} (1-\lambda)x + y + z + w = 0 \\ -x + (1-\lambda)y - z - w = 0 \\ 3x + 8y + (11-\lambda)z + 5w = 0 \\ 3x - 8y - 11z + (5-\lambda)w = 0 \end{cases}$

This linear equation has no solutions.

(You can type it on `https://zh.numberempire.com/equationsolver.php` to check.)

OR. Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(x_1,x_2,x_3,x_4) = (-x_2,x_1,-x_4,x_3)$.

Suppose λ is an eigenvalue of T with an eigenvector (x,y,z,w).

Then $T(x,y,z,w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y,x,-w,z) \Rightarrow \begin{cases} -y = \lambda x \\ x = \lambda y \\ -w = \lambda z \\ z = \lambda w \end{cases} \Rightarrow \begin{cases} -xy = \lambda^2 xy \\ -zw = \lambda^2 zw \end{cases}$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Otherwise, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, contradicts.

Similarly, $y = z = w = 0$. Then we fail.

Thus T has no eigenvalues. $\square$

• TODO *Suppose* $(v_1,\dots,v_n)$ *is a basis of* V *and* $T \in \mathcal{L}(V)$, $\mathcal{M}(T,(v_1,\dots,v_n)) = A$. *Prove that if* λ *is an eigenvalue of* T, *then* $|\lambda| \leq n \max\{|A_{j,k}| : 1 \leq j,k \leq n\}$.

SOLUTION:

First we show that $|\lambda| = n \max\{|A_{j,k}| : 1 \leq j,k \leq n\}$ for some cases.

Consider $A = \begin{pmatrix} k & \cdots & k \\ \vdots & \ddots & \vdots \\ k & \cdots & k \end{pmatrix}$. Then nk is an eigenvalue of T with an eigenvector $v_1 + \cdots + v_n$.

Now we show that if $|\lambda| \neq n \max\{|A_{j,k}| : 1 \leq j,k \leq n\}$, then $|\lambda| < n \max\{|A_{j,k}| : 1 \leq j,k \leq n\}$.

---

18 *Show that the forward shift operator* $T \in \mathcal{L}(\mathbf{F}^\infty)$ *defined by* $T(z_1,z_2,\dots) = (0,z_1,z_2,\dots)$ *has no eigenvalues.*

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector $(z_1,z_2,\dots)$.

Then $T(z_1,z_2,\dots) = (\lambda z_1, \lambda z_2, \dots) = (0,z_1,z_2,\dots)$.

Thus $\lambda z_1 = 0, \lambda z_2 = z_1, \dots, \lambda z_k = z_{k-1}, \dots$.

Let $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \lambda z_k = z_{k-1}$, therefore $(z_1,z_2,\dots) = 0$ is not an eigenvector.

Suppose $\lambda \neq 0$. Then $\lambda z_1 = 0 \Rightarrow z_1 = 0 \Rightarrow z_2 = 0 = z_k$ for all $k \in \mathbf{N}^+$.

And then $(z_1, z_2, \dots) = 0$ is not an eigenvector. Hence T has no eigenvalues. $\square$

---

19 *Suppose $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by*

$$T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n).$$

In other words, the entries of $\mathcal{M}(T)$ with respect to the standard basis are all 1's.

Find all eigenvalues and eigenvectors of T.

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector $(x_1, \dots, x_n)$.

Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$.

Thus $\lambda x_1 = \dots = \lambda x_n = x_1 + \dots + x_n$.

For $\lambda = 0$, $x_1 + \dots + x_n = 0$.

For $\lambda \neq 0$, $x_1 = \dots = x_n$ and then $\lambda x_k = n x_k$ for each k.

Hence $0, n$ are eigenvectors of T.

And the set of all eigenvectors of T is $\{(x_1, \dots, x_n) \in \mathbf{F}^n : x_1 + \dots + x_n = 0 \vee x_1 = \dots = x_n\}$. $\square$

---

20 *Define the backward shift operator $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.*

(a) *Show that every element of $\mathbf{F}$ is an eigenvalue of S.*

(b) *Find all eigenvectors of S.*

SOLUTION:

Suppose λ is an eigenvalue of S with an eigenvector $(z_1, z_2, \dots)$.

Then $S(z_1, z_2, z_3 \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$.

Thus $\lambda z_1 = z_2, \lambda z_2 = z_3, \dots, \lambda z_k = z_{k+1}, \dots$.

For $\lambda = 0$, $\lambda z_1 = z_2 = 0 = \lambda z_2 = z_3 = \dots = z_k$ for all k.

While z_1 can be arbitrary, so that $(z_1, 0, \dots)$ is an eigenvector with $z_1 \neq 0$.

For $\lambda \neq 0$, $\lambda^k z_1 = \lambda^{k-1} z_2 = \dots = \lambda z_k = z_{k+1}$ for all k.

Then $(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots)$ is an eigenvector with $z_1 \neq 0$.

Hence (a) each element of $\lambda \in \mathbf{F}$ is an eigenvalue of T.

And (b) the set of all eigenvectors of T is $\{(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots) \in \mathbf{F}^\infty : \lambda \in \mathbf{F}, z_1 \neq 0\}$ $\square$

---

24 *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = Ax$,*

where elements of $\mathbf{F}^n$ are thought of as n-by-1 column vectors.

(a) *Suppose the sum of the entries in each row of A equals 1.*

Prove that 1 is an eigenvalue of T.

(b) *Suppose the sum of the entries in each column of A equals 1.*

Prove that 1 is an eigenvalue of T.

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $Tx = Ax = \begin{pmatrix} \sum\limits_{c=1}^{n} A_{1,c} x_c \\ \vdots \\ \sum\limits_{c=1}^{n} A_{n,c} x_c \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. While $\sum\limits_{r=1}^{n} A_{R,c} = 1$ for each $R = 1, \dots, n$.

Thus if we let $x_1 = \dots = x_n$, then $\lambda = 1$, and hence is an eigenvalue of T.

(b) Suppose λ is an eigenvalue of T with an eigenvector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $Tx = Ax = \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}x_r \\ \vdots \\ \sum_{r=1}^{n} A_{n,r}x_r \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. While $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C = 1, \dots, n$.

Thus $\sum_{r=1}^{n} (Ax)_{r,\cdot} = \sum_{r=1}^{n} (Ax)_{r,1}$

$$= \sum_{c=1}^{n} (A_{1,c} + \cdots + A_{n,c})x_c = \sum_{c=1}^{n} x_c = \lambda \begin{pmatrix} x_1 \\ + \\ \vdots \\ + \\ x_n \end{pmatrix}.$$

Hence $\lambda = 1$, for all x such that $\sum_{c=1}^{n} x_{c,1} \neq 0$. $\square$

OR. Prove that $(T - I)$ is not invertible, so that we can conclude $\lambda = 1$ is an eigenvalue.

Because $(T - I)x = (A - \mathcal{M}(I))x = \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^{n} A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

Then $y_1 + \cdots + y_n = \sum_{r=1}^{n}\sum_{c=1}^{n} (A_{r,c}x_c - x_r) = \sum_{c=1}^{n} x_c \sum_{r=1}^{n} A_{r,c} - \sum_{r=1}^{n} x_r = 0.$

Thus range $(T - I) \subseteq \{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbf{F}^n : y_1 + \cdots + y_n = 0\}$. Hence $(T - I)$ is not surjective. $\square$

- *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = xA$, where elements of $\mathbf{F}^n$ are thought of as 1-by-n row vectors.*
  (a) *Suppose the sum of the entries in each column of A equals 1. Prove that 1 is an eigenvalue of T.*
  (b) *Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.*

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.

Then $Tx = xA = \begin{pmatrix} \sum_{r=1}^{n} x_r A_{r,1} & \cdots & \sum_{r=1}^{n} x_r A_{r,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. While $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C = 1, \dots, n$.

Thus if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, hence is an eigenvalue of T.

(b) Suppose λ is an eigenvalue of T with an eigenvector $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.

Then $Tx = xA = \begin{pmatrix} \sum_{c=1}^{n} x_c A_{c,1} & \cdots & \sum_{c=1}^{n} x_c A_{c,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. While $\sum_{c=1}^{n} A_{R,c} = 1$ for each $R = 1, \dots, n$.

Thus $\sum_{c=1}^{n} (xA)_{\cdot,c} = \sum_{c=1}^{n} (xA)_{1,c} = \sum_{c=1}^{n} (A_{c,1} + \cdots + A_{c,n})x_c = \sum_{c=1}^{n} x_c = \lambda \begin{pmatrix} x_1 & + & \cdots & + & x_n \end{pmatrix}$.

Hence $\lambda = 1$, for all x such that $\sum_{r=1}^{n} x_{1,r} \neq 0$. $\square$

OR. Prove that $(T - I)$ is not invertible, so that we can conclude $\lambda = 1$ is an eigenvalue.

Because $(T - I)x = x(A - \mathcal{M}(I)) == \begin{pmatrix} \sum_{c=1}^{n} x_c A_{c,1} - x_1 & \cdots & \sum_{c=1}^{n} x_c A_{c,n} - x_n \end{pmatrix} = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}$.

Then $y_1 + \cdots + y_n = \sum_{c=1}^{n}\sum_{r=1}^{n} (x_r A_{r,c} - x_c) = \sum_{r=1}^{n} x_r \sum_{c=1}^{n} A_{r,c} - \sum_{c=1}^{n} x_c = 0.$

Thus range $(T - I) \subseteq \{ \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}. \in \mathbf{F}^n : y_1 + \cdots + y_n = 0\}$. Hence $(T - I)$ is not surjective. $\square$

25 *Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T.*
Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

SOLUTION:

Suppose $\lambda_1, \lambda_2, \lambda_0$ are eigenvalues of T corresponding to $u, w, u + w$ respectively.

Then $T(u+w) = \lambda_0(u+w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$.

Notice that $u, w, u + w$ are nonzero.

If (u, w) is linearly dependent, then let $w = cu$, therefore

$$\lambda_2 cu = Tw = cTu = \lambda_1 cu \qquad \Rightarrow \lambda_2 = \lambda_1.$$
$$\lambda_0(u+w) = T(u+w) = \lambda_1 u + \lambda_1 cu = \lambda_1(u+w) \quad \Rightarrow \lambda_0 = \lambda_1.$$

Otherwise, $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. $\square$

---

26 *Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T.*
Prove that T is a scalar multiple of the identity operator.

SOLUTION:

Because $\forall v \in V, \exists!\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Then for any two distinct nonzero vectors $v, w \in V$,

$T(v+w) = \lambda_{v+w}(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$.

If (v, w) is linearly independent, then let $w = cv$, therefore

$$\lambda_v cv = cTv = Tw = \lambda_w w \qquad \Rightarrow \lambda_w = \lambda_v.$$
$$\lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v(v+cv) \Rightarrow \lambda_{v+w} = \lambda_v.$$

Otherwise, $\lambda_v = \lambda_{v+w} = \lambda_w$. $\square$

27, 28 *Suppose V is finite-dim and $k \in \{1, \ldots, \dim V - 1\}$.*
Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V of dim k is invariant under T.
Prove that T is a scalar multiple of the identity operator.

SOLUTION:

We prove the contrapositive:

If $T \neq \lambda I, \forall \lambda \in \mathbf{F}$, then $\exists$ a subspace U of V such that $\dim U = k$, and U is invariant under T.

By Problem (26), $\exists\, v \in V$ and $v \neq 0$ such that v is not an eigenvector of T.

Thus (v, Tv) is linearly independent. Extend to a basis of V as $(v, Tv, u_1, \ldots, u_n)$.

Let $U = \text{span}\,(v, u_1, \ldots, u_{k-1}) \Rightarrow U$ is not an invariant subspace of V under T.

OR. Suppose $0 \neq v = v_1 \in V$ and extend to a basis of V as $(v_1, \ldots, v_n)$.

Suppose $Tv_1 = c_1 v_1 + \cdots + c_n v_n$, $\exists!\, c_i \in \mathbf{F}$.

Consider a k - dim subspace $U = \text{span}\,(v_1, v_{\alpha_1}, \ldots, v_{\alpha_{k-1}})$,

where $\alpha_j \in \{2, \ldots, n\}$ for each j, and $\alpha_1, \ldots, \alpha_{k-1}$ are distinct and are chosen arbitrarily.

Because every subspace such U is invariant.

Thus $Tv_1 = c_1 v_1 + \cdots + c_n v_n \in U$

$\Rightarrow c_2 = \cdots = c_n = 0$,

for if not, for each $c_i \neq 0$, choose U_i such that $\alpha_j \in \{\underbrace{2, \ldots, i-1, i+1, \ldots, n}_{\text{length } (n-2)}\}$ for each j,

hence for $Tv_1 = c_1 v_1 + \cdots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \cdots + c_n v_n \in U_i$, we conclude that $c_i = 0$.

$\Rightarrow Tv_1 = c_1 v_1$, 又 $v_1 = v \in V$ is arbitrarily chosen $\Rightarrow T = \lambda I$ for some λ. $\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Prove that*
T has an eigenvalue $\Longleftrightarrow \exists$ a subspace U of V
such that $\dim U = \dim V - 1$, U is invariant under T.

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector v.

(If $\dim V = 1$, then $U = \{0\}$ and we are done.)

Extend $v_1 = v$ to a basis of V as $(v_1, v_2 \ldots, v_n)$.

Step 1 If $\exists\, w_1 \in \operatorname{span}(v_2, \ldots, v_n)$ such that $0 \neq Tw_1 \in \operatorname{span}(v_1)$,
then extend $w_1 = \alpha_{1,1}$ to a basis of $\operatorname{span}(v_2, \ldots, v_n)$ as $(\alpha_{1,1}, \ldots, \alpha_{1,n-1})$.
Otherwise, we stop at step 1.

⋮

Step k If $\exists\, w_k \in \operatorname{span}(\alpha_{k-1,2}, \ldots, \alpha_{k-1,n-k+1})$ such that $0 \neq Tw_k \in \operatorname{span}(v_1, w_1, \ldots, w_{k-1})$,
then extend $w_k = \alpha_{k,1}$ to a basis of $\operatorname{span}(\alpha_{k-1,2}, \ldots, \alpha_{k-1,n-k+1})$ as $(\alpha_{k,1}, \ldots, \alpha_{k,n-k})$.
Otherwise, we stop at step k.

⋮

Finally, we stop at step m, thus we get $(v_1, w_1, \ldots, w_{m-1})$ and $(\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1})$,

$\operatorname{range} T|_{\operatorname{span}(w_1,\ldots,w_{m-1})} = \operatorname{span}(v_1, w_1, \ldots, w_{m-2}) \Rightarrow \dim \operatorname{null} T|_{\operatorname{span}(w_1,\ldots,w_{m-1})} = 0$,

$\operatorname{span}\underbrace{(v_1, w_1, \ldots, w_{m-1})}_{\text{length }\dim m}$ and $\operatorname{span}\underbrace{(\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1})}_{\text{length }\dim(n-m)}$ are invariant under T.

Let $U = \operatorname{span}(\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1}) \oplus \operatorname{span}(v_1, w_1, \ldots, w_{m-2})$ and we are done. ☐

COMMENT: Both $\operatorname{span}(v_2, \ldots, v_n)$ and $\operatorname{span}(\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1}) \oplus \operatorname{span}(w_1, \ldots, w_{m-1})$ are in $\mathcal{S}_V \operatorname{span}(v_1)$.

(b) Suppose U is an invariant subpsace of V under T with $\dim U = m = \dim V - 1$.

(If $m = 0$, then $\dim V = 1$ and we are done).

Let $(u_1, \ldots, u_m)$ be a basis of U, extend to a basis of V as $(u_0, u_1, \ldots, u_m)$.

We discuss in cases:

For $Tu_0 \in U$, then $\operatorname{range} T = U$ so that T is not surjective $\Longleftrightarrow \operatorname{null} T \neq \{0\} \Longleftrightarrow 0$ is an eigenvalue of T.

For $Tu_0 \notin U$, then $Tu_0 = a_0 u_0 + a_1 u_1 + \cdots + a_m u_m$.

(1) If $Tu_0 \in \operatorname{span}(u_0)$, then we are done.

(2) Otherwise, if $\operatorname{range} T|_U = U$, then $Tu_0 = a_0 u_0$ and we are done;
otherwise, $T|_U : U \to U$ is not surjective ($\Rightarrow$ not injective), suppose $\operatorname{range} T|_U \neq \{0\}$
(Suppose $\operatorname{range} T|_U = \{0\}$. If $\dim U = 0$ then we are done.
Otherwise $\exists\, u \in U \backslash \{0\}, Tu = 0$ and we are done.)
then $\exists\, u \in U \backslash \{0\}, Tu = 0$, we are done. ☐

---

29 *Suppose $T \in \mathcal{L}(V)$ and range T is finite-dim.*
Prove that T has at most $1 + \dim$ range T distinct eigenvalues.

SOLUTION:

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T and let $v_1, \ldots, v_m$ be the corresponding eigenvectors.

(Because range T is finite-dim. Let $(v_1, \ldots, v_n)$ be a list of all the linearly independent eigvecs of T,
so that the corresponding eigvals are finite.)

For every $\lambda_k \neq 0, T(\frac{1}{\lambda_k} v_k) = v_k$. And if $T = T - 0I$ is not injective, then $\exists!\, \lambda_A = 0$ and $Tv_A = \lambda_A v_A = 0$.

Thus for $\lambda_k \neq 0, \forall\, k$, $(Tv_1, \ldots, Tv_m)$ is a linearly independent list of length m in range T.

And for $\lambda_A = 0$, there is a linearly independent list of length at most $(m-1)$ in range T.

Hence, by [2.23], $m \leq \dim \operatorname{range} T + 1$. ☐

32 *Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct real numbers.*
Prove that $(e^{\lambda_1}x, \ldots, e^{\lambda_n}x)$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

HINT: *Let $V = \text{span}\,(e^{\lambda_1}x, \ldots, e^{\lambda_n}x)$, and define an operator $D \in \mathcal{L}(V)$ by $Df = f'$.*
Find eigenvalues and eigenvectors of D.

SOLUTION:

Define V and $D \in \mathcal{L}(V)$ as in HINT. Then because for each k, $D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.
Thus $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of D. By [5.10], $(e^{\lambda_1}x, \ldots, e^{\lambda_n}x)$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$. $\square$

• *Suppose $\lambda_1, \ldots, \lambda_n$ are distinct positive numbers.*
Prove that $(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

SOLUTION:

Let $V = \text{span}\,(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.
Then because $D(\cos(\lambda_k x)) = -\lambda_k \sin(\lambda_k x)$. 又 $D(\sin(\lambda_k x)) = \lambda_k \cos(\lambda_k x)$.
Thus $D^2(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$.
Notice that $\lambda_1, \ldots, \lambda_n$ are distinct $\Rightarrow -\lambda_1^2, \ldots, -\lambda_n^2$ are distinct.
Hence $-\lambda_1^2, \ldots, -\lambda_n^2$ are distinct eigenvalues of D^2
with the corresponding eigenvectors $\cos(\lambda_1 x), \ldots, \cos(\lambda_n x)$ respectively.
And then $(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$. $\square$

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T.*
The quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U \text{ for each } v \in V.$$

(a) *Show that the definition of T/U makes sense*
(which requires using the condition that U is invariant under T)
and show that T/U is an operator on V/U.

(b) (OR Problem 35) *Show that each eigenvalue of T/U is an eigenvalue of T.*

SOLUTION:

(a) Suppose $v + U = w + U$ ($\Longleftrightarrow v - w \in U$).
Then because U is invariant under T, $T(v - w) \in U \Longleftrightarrow Tv + U = Tw + U$.
Hence the definition of T/U makes sense.
Now we show that T/U is linear.
$\forall v + U, w + U \in V/U, \lambda \in \mathbf{F}, (T/U)((v + U) + \lambda(w + U))$

$$= T(v + \lambda w) + U = (Tv + U) + \lambda(Tw + U)$$
$$= (T/U)(v + U) + \lambda(T/U)(w).$$

(b) Suppose λ is an eigenvalue of T/U with an eigenvector $v + U$.
Then $(T/U)(v + U) = \lambda(v + U) = Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v \in U$.
If $(T - \lambda I)v = 0 \Rightarrow Tv = \lambda v$, then we are done.
Otherwise, then $(T|_U - \lambda I) : U \to U$ is invertible,
hence $\exists!\, w \in U, (T|_U - \lambda I)(w) = (T - \lambda I)v \Rightarrow T(v - w) = \lambda(v - w)$.
Note that $v - w \neq 0$ (for if not, $v \in U \Rightarrow v + U = 0 + U$ is not an eigenvector). $\square$

---

36 *Prove or give a counterexample:*

The result of (b) *in Exercise 35 is still true if V is infinite-dim.*

SOLUTION: A counterexample:

Consider $V = \operatorname{span}(1, e^x, e^{2x}, \dots)$ in $\mathbb{R}^{\mathbb{R}}$, and a subspace $U = \operatorname{span}(e^x, e^{2x}, \dots)$ of V.

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then range $T = U$ is invariant under T.

Consider $(T/U)(1+U) = e^x + U = 0$

$\Rightarrow 0$ is an eigenvalue of T/U but is not an eigenvalue of T

(null $T = \{0\}$, for if not, $\exists\, f \in V \backslash \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, contradicts). $\square$

---

33 *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{range}\, T) = 0$.*

SOLUTION:

$\forall v + \text{range}\, T \in V/\text{range}\, T, v + \text{range}\, T \in \text{null}\,(T/(\text{range}\, T))$

$\Rightarrow \text{null}\,(T/(\text{range}\, T)) = V/\text{range}\, T \Rightarrow T/(\text{range}\, T)$ is a zero map. $\square$

---

34 *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null}\, T)$ is injective $\iff (\text{null}\, T) \cap (\text{range}\, T) = \{0\}$.*

SOLUTION:

(a) Suppose $T/(\text{null}\, T)$ is injective.

Then $(T/(\text{null}\, T))(u + \text{null}\, T) = Tu + \text{null}\, T = 0$

$\iff Tu \in \text{null}\, T$ 又 $Tu \in \text{range}\, T \iff u + \text{null}\, T = 0 \iff u \in \text{null}\, T \iff Tu = 0$.

Thus $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$.

(b) Suppose $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$.

Then $(T/(\text{null}\, T))(u + \text{null}\, T) = Tu + \text{null}\, T = 0$

$\iff Tu \in \text{null}\, T$ 又 $Tu \in \text{range}\, T \iff Tu = 0 \iff u \in \text{null}\, T \iff u + \text{null}\, T = 0$.

Thus $T/(\text{null}\, T)$ is injective. $\square$

ENDED

5.B: I [*See* **5.B: II** *below.*]

COMMENT: 下面是第 5 章 B 节。为了照顾 5.B 节两版过大的差距，特别将 5.B 补注分成 **I** 和 **II** 两部分。又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本征值」(相当一部分是从原第 3 版 8.C 节挪过来的) 是对原第 3 版「多项式作用于算子」与「本征值的存在性」(也即第 3 版 5.B 前半部分) 的极大扩充，这一扩充也大大改变了原第 3 版后半部分的「上三角矩阵」这一小节，故而将第 4 版 5.B 节放在第 3 版 5.B 节前面。

I 部分除了覆盖第 4 版 5.B 节和第 3 版 5.B 节前半部分与之相关的所有习题，还会覆盖第 4 版 5.A 节末。

II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节，还会覆盖第 4 版 5.C 节；并且，下面 **5.C** 还会覆盖第 4 版 5.D 节。

「注：[8.40] OR (4E 5.22) —— minimal polynomial;

[8.44,8.45] OR (4E 5.25,5.26) —— how to find the minimal polynomial;

[8.49] OR (4E 5.27) —— eigenvalues are the zeros of the minimal polynomial;

[8.46] OR (4E 5.29) —— $q(T) = 0 \Leftrightarrow q$ is a poly multiple of the mini poly.」

[1]: (4E.5.A.33), 13; [2]: (4E.5.B.25, 26, 27, 28, 22); [3]: 6, (4E.5.B.10, 23, 21), 19; [4]: (4E 5.B.13, 14);
[5]: (4E.5.B.20, 24), 10; [6]: 1, 2, 7, 3, (4E.5.A.32); [7]: 8, (4E 5.B.12, 3, 8); [8]: (4E.5.B.11), 5, (4E.5.B.7);
[9]: 11, 12, (4E.5.B.17, 18); [10]: 18 OR (4E.5.B.15), (4E.5.B.9), (4E.5.B.16); [11]: (2E Ch5.24), (4E.5.B.29).

• *Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.*
(a) *Prove that T is injective $\iff T^m$ is injective.*
(b) *Prove that T is surjective $\iff T^m$ is surjective.*

SOLUTION:

(a) Suppose T^m is injective. Then $Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$. $\square$

Suppose T is injective.

Then $T^m v = T(T^{m-1}v) = 0$

$\Rightarrow T^{m-1}v = 0 = T(T^{m-2}v) \Rightarrow \cdots$

$\Rightarrow T^2 v = TTv = 0$

$\Rightarrow Tv = 0 \Rightarrow v = 0$. $\square$

(b) Suppose T^m is surjective. $\forall u \in V$, $\exists\, v \in V, T^m v = u = Tw$, let $w = T^{m-1}v$. $\square$

Suppose T is surjective.

Then $\forall u \in V$, $\exists\, v \in V, T(\underline{v}) = u$

$\Rightarrow \exists\, v_2 \in V, Tv_2 = \underline{v}, T^2(\underline{v_2}) = u$

$\vdots$

$\Rightarrow \exists\, v_k \in V, Tv_k = \underline{v_{k-1}}, T^k(\underline{v_k}) = u$

$\vdots$

$\Rightarrow \exists\, v_{m-1} \in V, Tv_{m-1} = \underline{v_{m-2}}, T^{m-1}(\underline{v_{m-1}}) = u$

$\Rightarrow \exists\, v_m \in V, Tv_m = \underline{v_{m-1}}, T^{m-1}(Tv_m) = u$. $\square$

• **NOTE FOR [5.17]:** *Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$.*
Prove that null $p(T)$ and range $p(T)$ are invariant under T.

SOLUTION: Using the commutativity in [5.10].

(a) Suppose $u \in \text{null}\, p(T)$. Then $p(T)u = 0$.

Thus $p(T)(Tu) = (p(T)T)u(Tp(T))u = T(p(T)u) = 0$. Hence $Tu \in \text{null}\, p(T)$. $\square$

(b) Suppose $u \in \text{range}\, p(T)$. Then $\exists\, v \in V$ such that $u = p(T)v$.

Thus $Tu = T(p(T)v) = p(T)(Tv) \in \text{range}\, p(T)$. $\square$

• **NOTE FOR [5.21]:** *Every operator on a finite-dim nonzero complex vector space has an eigval.*

Suppose V is a finite-dim complex vector space of dim $n > 0$ and $T \in \mathcal{L}(V)$.

Choose a nonzero $v \in V$. $(v, Tv, T^2 v, \dots, T^n v)$ of length $n+1$ is linearly dependent.

Suppose $a_0 I + a_1 T + \cdots + a_n T^n = 0$. Then $\exists\, a_j \neq 0$.

Thus $\exists$ nonconst p of smallest degree ($\deg p > 0$) *such that* $p(T)v = 0$.

Because $\exists\, \lambda \in \mathbf{C}$ such that $p(\lambda) = 0 \Rightarrow \exists\, q \in \mathcal{P}(\mathbf{C}), p(z) = (z - \lambda)q(z), \forall z \in \mathbf{C}$.

Thus $0 = p(T)v = (T - \lambda I)(q(T)v)$. By the minimality of $\deg p$ and $\deg q < \deg p$, $q(T)v \neq 0$.

Then $(T - \lambda I)$ is not injective. Thus λ is an eigval of T with eigvec $q(T)v$.

• **EXAMPLE:** *an operator on a complex vector space with no eigvals*

Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by $(Tp)(z) = zp(z)$.

Suppose $p \in \mathcal{P}(\mathbf{C})$ is a nonzero poly. Then $\deg Tp = \deg p + 1$, and thus $Tp \neq \lambda p$, $\forall \lambda \in \mathbf{C}$.

Hence T has no eigvals. Because $\mathcal{P}(\mathbf{C})$ is infinite-dim, this example does not contradict the result above.

---

13 *Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ has no eigvals.*
Prove that every subspace of V invariant under T is either $\{0\}$ or infinite-dim.

SOLUTION: Suppose U is a finite-dim nonzero invariant subspace on $\mathbf{C}$. Then by [5.21], $T|_U$ has an eigval. $\square$

• **TIPS:** *For $T_1, \dots, T_m \in \mathcal{L}(V)$:*

(a) Suppose $T_1, \dots, T_m$ are all injective. Then $(T_1 \circ \cdots \circ T_m)$ is injective.

(b) Suppose $(T_1 \circ \cdots \circ T_m)$ is not injective. Then at least one of $T_1, \dots, T_m$ is not injective.

(c) At least one of $T_1, \dots, T_m$ is not injective $\not\Rightarrow$ $(T_1 \circ \cdots \circ T_m)$ is not injective.

EXAMPLE: On infinite-dim only. Let $V = \mathbf{F}^\infty$.

Let S be the backward shift (surj but not inje), T be the forward shift (inje but not surj). Then $ST = I$. $\square$

---

16 *Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}(\mathcal{P}_{\dim V}(\mathbf{C}), V)$ by $S(p) = p(T)v$. Prove [5.21].*

SOLUTION:

Because $\dim \mathcal{P}_{\dim V}(\mathbf{C}) = \dim V + 1$. Then S is not injective. Hence $\exists\, 0 \neq p \in \mathcal{P}_{\dim V}(\mathbf{C}), p(T)v = 0$.

Using [4.14], write $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$. Apply T to both sides: $p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.

Thus at least one of $(T - \lambda_j I)$ is not injective (because $p(T)$ is not injective). $\square$

---

17 *Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}(\mathcal{P}_{(\dim V)^2}(\mathbf{C}), \mathcal{L}(V))$ by $S(p) = p(T)$. Prove [5.21].*

SOLUTION:

Because $\dim \mathcal{P}_{(\dim V)^2}(\mathbf{C}) = (\dim V)^2 + 1$. Then S is not injective. Hence $\exists\, 0 \neq p \in \mathcal{P}_{(\dim V)^2}(\mathbf{C}), p(T) = 0$.

Using [4.14], write $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$. Applying T, we have $0 = p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.

Thus $(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0 \Rightarrow \exists\, j, (T - \lambda_j)$ is not injective. $\square$

COMMENT: $\exists$ monic $q \in \operatorname{null} S \neq \{0\}$ of smallest degree, $S(q) = q(T) = 0$, then q is the *minimal polynomial.*

• **NOTE FOR [8.40]:** *definition for minimal polynomial*

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that $\exists\,!$ monic poly $p \in \mathcal{P}(\mathbf{F})$ of smallest degree, $p(T) = 0$. Moreover, $\deg p \leq \dim V$.

SOLUTION OR Another Proof **:**

[*Existence Part*] We use induction on $\dim V$.

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, we are done.

(ii) Suppose $\dim V \geq 1$.

Assume that $\dim V > 0$ *and that the desired result is true for all operators on all vecsps of smaller dim.*

Let $u \in V, u \neq 0$. The list $(u, Tu, \dots, T^{\dim V} u)$ of length $(1 + \dim V)$ is linearly dependent.

Then $\exists\,!\, T^m$ of smallest degree such that $T^m u \in \operatorname{span}(u, Tu, \dots, T^{m-1} u)$.

Thus $\exists\, c_j \in \mathbf{F}, c_0 u + c_1 Tu + \cdots + c_{m-1} T^{m-1} u + T^m u = 0$.

Define q by $q(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m$.

Then $0 = T^k(q(T)u) = q(T)(T^k u), \forall k \in \{1, \dots, m-1\} \subseteq \mathbf{N}$.

Because $(u, Tu, \dots, T^{m-1} u)$ is linearly independent.

Thus $\dim \operatorname{null} q(T) \geq m \Rightarrow \dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leq \dim V - m$.

Let $W = \operatorname{range} q(T)$.

By *assumption*, $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ and $\deg s \leq \dim W$, so that $s(T|_W) = 0$.

Hence $\forall v \in V, ((sq)(T))(v) = s(T)(q(T)v) = 0$.

Thus sq is a monic poly such that $\deg sq \leq \dim V$ and $(sq)(T) = 0$.

[*Uniqueness Part*]

Let $p, q \in \mathcal{P}(\mathbf{F})$ be monic polys of smallest degree such that $p(T) = q(T) = 0$

$\Rightarrow (p - q)(T) = 0$ 又 $\deg(p - q) < \deg p$.

If $p - q = a_m z^m + \cdots + a_1 z_1 + a_0 \neq 0$, then $\dfrac{1}{a_m}(p - q)$ is a monic poly of smaller degree than p.

Hence contradicts the minimality of $\deg p$. Thus $p - q = 0$ and we are done. $\square$

• (4E 5.31, 4E 5.B.25 and 26) *mini poly of restriction operator and mini poly of quotient operator*

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invariant subspace of V under T.
Let p be the mini poly of T.
(a) *Prove that p is a polynomial multiple of the mini poly of $T|_U$.*
(b) *Prove that p is a polynomial multiple of the mini poly of T/U.*
(c) *Prove that (mini poly of $T|_U$) $\times$ (mini poly of T/U) is a polynomial multiple of p.*
(d) *Prove that the set of eigvals of T equals the union of the set of eigvals of $T|_U$ and the set of eigvals of T/U.*

SOLUTION:
(a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46].☐
(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v+U) = p(T)v + U = 0.$ ☐
(c) Suppose r is the mini poly of $T|_U$, s is the mini poly of T/U.
Because $\forall v \in V, s(T/U)(v+U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.
又 $\forall u \in U, r(T|_U)u = r(T)u = 0$.
Thus $\forall v \in V$ but $v \notin U, (rs)(T)v = r(s(T)v) = 0$.
And $\forall u \in U, (rs)(T)u = r(s(T)u) = 0$ (because $s(T)u = s(T|_U)u \in U$).
Hence $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0.$ ☐
(d) By [8.49], immediately. ☐

• (4E 5.B.27)

Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, and $T \in \mathcal{L}(V)$.
Prove that the mini poly p of $T_{\mathbb{C}}$ equals the mini poly q of T.

SOLUTION:
$$\left.\begin{array}{l} \forall u + \mathrm{i}0 \in V_{\mathbb{C}}, p(T_{\mathbb{C}})(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p \text{ is a polynomial multiple of } q. \\ q(T) = 0 \Rightarrow \forall u + \mathrm{i}v \in V_{\mathbb{C}}, q(T_{\mathbb{C}})(u+\mathrm{i}v) = q(T)u + \mathrm{i}q(T)v = 0 \Rightarrow q \text{ is a polynomial multiple of } p. \end{array}\right\} \Rightarrow \square$$

• (4E 5.B.28)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.
Prove that the mini poly p of $T' \in \mathcal{L}(V')$ equals the mini poly q of T.

SOLUTION:
$$\left.\begin{array}{l} \forall \varphi \in V', p(T')\varphi = \varphi \circ (p(T)) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \operatorname{null} \varphi \Rightarrow p(T) = 0 \\ \qquad \Rightarrow p(T) = 0 \Rightarrow p \text{ is a polynomial multiple of } q. \\ q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ (q(T)) = q(T')\varphi = 0 \Rightarrow q \text{ is a polynomial multiple of } p. \end{array}\right\} \Rightarrow \square$$

• (4E 5.32) *Suppose $T \in \mathcal{L}(V)$ and p is the mini poly.*
Prove that T is not injective $\iff$ the const term of p is 0.

SOLUTION: T is not injective $\iff$ 0 is an eigval of T $\iff$ 0 is a zero of p $\iff$ the const term of p is 0. ☐

OR. Because $p(0) = (z-0)(z-\lambda_1)\cdots(z-\lambda_m) = 0 \Rightarrow T(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0$
又 p is the mini poly $\Rightarrow q$ define by $q(z) = (z-\lambda_1)\cdots(z-\lambda_m)$ is such that $q(T) \neq 0$.
Hence $0 = p(T) = Tq(T) \Rightarrow T$ is not injective.
Conversely, suppose $(T - 0I)$ is not injective, then 0 is a zero of p, so that the const term is 0. ☐

• (4E 5.B.22)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$. Prove that T is invertible $\iff$ $I \in$ span $(T, T^2, \ldots, T^{\dim V})$.

SOLUTION: Denote the mini poly by p, where for all $z \in \mathbf{F}, p(z) = a_0 + a_1 z + \cdots + z^m$.
Notice that V is finite-dim. T is invertible $\iff$ T is injective $\iff$ $p(0) \neq 0$.
Hence $p(T) = 0 = a_0 I + a_1 T + \cdots + T^m$, where $a_0 \neq 0$ and $m \leq \dim V$. ☐

---

6 *Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T.*
Prove that U is invariant under $p(T)$ for every poly $p \in \mathcal{P}(\mathbf{F})$.

SOLUTION:

$\forall u \in U, Tu \in U \Rightarrow \forall u \in U, Iu, Tu, T(Tu), \ldots, T^m u \in U \Rightarrow \forall u \in U, (a_0 I + a_1 T + \cdots + a_m T^m)u \in U.$ $\square$

• (4E 5.B.10, 5.B.23)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$ and p is the mini poly with degree m. Suppose $v \in V$.
(a) *Prove that span $(v, Tv, \ldots, T^{m-1}v) =$ span $(v, Tv, \ldots, T^{j-1}v)$ for some $j \leq m$.*
(b) *Prove that span $(v, Tv, \ldots, T^{m-1}v) =$ span $(v, Tv, \ldots, T^{m-1}v, \ldots, T^n v)$.*

COMMENT: By NOTE FOR [8.40], j has an upper bound $m-1$, m has an upper bound $\dim V$.

SOLUTION: Write $p(z) = a_0 + a_1 z + \cdots + z^m$ ($m \leq \dim V$). If $v = 0$, then we are done. Suppose $v \neq 0$.

(a) Suppose $j \in \mathbf{N}^+$ is the smallest such that $T^j v \in \operatorname{span}(v, Tv, \ldots, T^{j-1}v) = U_0$. Then $j \leq m$.
Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1} T^{j-1} v$. And because $T(T^k v) = T^{k+1} \in U_0$. U_0 is invariant under T.
By Problem (6), $\forall k \in \mathbf{N}$, $T^{j+k} v = T^k(T^j v) \in U_0$.
Thus $U_0 = \operatorname{span}(v, Tv, \ldots, T^{j-1}v, \ldots, T^n v)$ for all $n \geq j-1$. Let $n = m-1$ and we are done.

(b) Let $U = \operatorname{span}(v, Tv, \ldots, T^{m-1}v)$.
By (a), $U = U_0 = \operatorname{span}(v, Tv, \ldots, T^{j-1}, \ldots, T^{m-1}, \ldots, T^n)$ for all $n \geq m-1$. $\square$

• (4E 5.B.21)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.
Prove that the mini poly p has degree at most $1 + \dim$ range T.
If $\dim$ range $T < \dim V - 1$, then this result gives a better upper bound for the degree of mini poly.

SOLUTION:

If T is injective, then range $T = V$ and we are done. Now choose $0 \neq v \in \operatorname{null} T$, then $Tv + 0 \cdot v = 0$.
1 is the smallest positive integer such that $T^1 v \in \operatorname{span}(v, \ldots, T^0 v)$. Define q by $q(z) = z \Rightarrow q(T)v = 0$.
Let $W = \operatorname{range} q(T) = \operatorname{range} T$. $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ of smallest degree ($\deg s \leq \dim W$), $s(T|_W) = 0$.
Hence sq is the mini poly (see NOTE FOR [8.40]) and $\deg(sq) = \deg s + \deg q \leq \dim \operatorname{range} T + 1$. $\square$

---

19 *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.*
Prove that $\dim \mathcal{E}$ equals the degree of the minimal polynomial of T.

SOLUTION:

Because the list $(I, T, \ldots, T^{(\dim V)^2})$ of length $\dim \mathcal{L}(V) + 1$ is linearly dependent in $\dim \mathcal{L}(V)$.
Suppose $m \in \mathbf{N}^+$ is the smallest such that $T^m = a_0 I + \cdots + a_{m-1} T^{m-1}$.
Then q defined by $q(z) = z^m - a_{m-1} z^{m-1} - \cdots - a_0$ is the mini poly (see [8.40]).
For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k(T^m) \in \operatorname{span}(I, T, \ldots, T^{m-1}) = U$.
Hence $\operatorname{span}(I, T, \ldots, T^{(\dim V)^2}) = \operatorname{span}(I, T, \ldots, T^{(\dim V)^2 - 1}) = U$.
Note that by the minimality of m, the list $(I, T, \ldots, T^{m-1})$ is linearly independent.
Thus $\dim U = m = \dim \operatorname{span}(I, T, \ldots, T^{(\dim V)^2 - 1}) = \dim \operatorname{span}(I, T, \ldots, T^n)$ for all $m < n \in \mathbf{N}^+$.

Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.
(a) Suppose $p(T) = 0$. 又 $\deg p \leq m - 1 \Rightarrow p = 0$. Then φ is injective.
(b) $\forall S = a_0 I + a_1 T + \cdots + a_{m-1} T^{m-1} \in \mathcal{E}$, define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by
$p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} \Rightarrow \varphi(p) = S$. Then φ is surjective.

Hence $\mathcal{E}$ and $\mathcal{P}_{m-1}(\mathbf{F})$ are isomorphic. 又 $\dim \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. $\square$

• (4E 5.B.13)
Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$ is defined by

$$q(z) = a_0 + a_1 z + \cdots + a_n z^n, \text{ where } a_n \neq 0, \text{for all } z \in \mathbf{F}.$$

Denote the mini poly of T by p defined by

$$p(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m \text{ for all } z \in \mathbf{F}.$$

Prove that $\exists!\, r \in \mathcal{P}(\mathbf{F})$ such that $q(T) = r(T), \deg r < \deg p$.

Solution:

If $\deg q < \deg p$, then we are done.

If $\deg q = \deg p$, notice that $p(T) = 0 = c_0 I + c_1 T + \cdots + c_{m-1} T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0 I - c_1 T - \cdots - c_{m-1} T^{m-1},$$

define r by $r(z) = q(z) + [-a_m z^m + a_m(-c_0 - c_1 z - \cdots - c_{m-1} z^{m-1})]$

$$= (a_0 - a_m c_0) + (a_1 - a_m c_1) z + \cdots + (a_{m-1} - a_m c_{m-1}) z^{m-1},$$

hence $r(T) = 0, \deg r < m$ and we are done.

Now suppose $\deg q \geq \deg p$. We use induction on $\deg q$.

(i) $\deg q = \deg p$, then the desired result is true, as shown above.

(ii) $\deg q > \deg p$, assume that the desired result is true for $\deg q = n$.

Suppose $f \in \mathcal{P}(\mathbf{F})$ such that $f(z) = b_0 + b_1 z + \cdots + b_n z^n + b_{n+1} z^{n+1}$.

Apply the assumption to g defined by $g(z) = b_0 + b_1 z + \cdots + b_n z^n$,

getting s defined by $s(z) = d_0 + d_1 z + \cdots + d_{m-1} z^{m-1}$.

Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1} T^{n+1} = s(T) + b_{n+1} T^{n+1}$.

Apply the assumption to t defined by $t(z) = z^n$,

getting δ defined by $\delta(z) = c'_0 + c'_1 z + \cdots + c'_{m-1} z^{m-1}$.

Thus $t(T) = T^n = c'_0 + c'_1 z + \cdots + c'_{m-1} z^{m-1} = \delta(T)$.

又 span $(v, Tv, \ldots, T^{m-1} v)$ is invariant under T.

Hence $\exists!\, k_j \in \mathbf{F}, T^{n+1} = T(T^n) = k_0 + k_1 z + \cdots + k_{m-1} z^{m-1}$.

And $f(T) = s(T) + b_{n+1}(k_0 + k_1 T + \cdots + k_{m-1} T^{m-1})$

$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1) z + \cdots + (d_{m-1} + k_{m-1}) z^{m-1} = h(T)$, thus defining h. $\square$

• (4E 5.B.14) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has mini poly p defined by $p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m$, $a_0 \neq 0$.*
Find the mini poly of T^{-1}.

Solution: Notice that V is finite-dim. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is invertible.

Then $p(T) = a_0 I + a_1 T + \cdots + T^m = 0$. Apply T^{-m} to both sides,

$a_0 (T^{-1})^m + a_1 (T^{-1})^{m-1} + \cdots + a_{m-1} T^{-1} + I = 0$.

Define q by $q(z) = z^m + \dfrac{a_1}{a_0} z^{m-1} + \cdots + \dfrac{a_{m-1}}{a_0} z + \dfrac{1}{a_0}$ for all $z \in \mathbf{F}$.

We now show that $(T^{-1})^k \notin$ span $(I, T^{-1}, \ldots, (T^{-1})^{k-1})$

for every $k \in \{1, \ldots, m-1\}$ by contradiction, so that q is exactly the mini poly of T^{-1}.

Suppose $(T^{-1})^k \in$ span $(I, T^{-1}, \ldots, (T^{-1})^{k-1})$.

Then let $(T^{-1})^k = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. Apply T^k to both sides,

getting $I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T$, hence $T^k \in$ span $(I, T, \ldots, T^{k-1})$.

Thus f defined by $f(z) = z^k + \dfrac{b_1}{b_0} z^{k-1} + \cdots + \dfrac{b_{k-1}}{b_0} z - \dfrac{1}{b_0}$ is a polynomial multiple of p.

While $\deg f < \deg p$. Contradicts. $\square$

• **NOTE FOR [8.49]:** Suppose V is a finite-dim complex vecsp, $T \in \mathcal{L}(V)$.

By [4.14], the mini poly has the form $(z-\lambda_1)\cdots(z-\lambda_m)$, where $\lambda_1,\ldots,\lambda_m$ is a list of all eigvals of T, possibly with repetitions.

• **COMMENT**: A nonzero poly has at most as many distinct zeros as its degree (see [4.12]). Thus by the upper bound for the deg of mini poly given in NOTE FOR [8.40], and by [8.49,] we can ***give an alternative proof of*** [***5.13***].

• **NOTICE**: (See also 4E 5.B.20,24)

Suppose $\alpha_1,\ldots,\alpha_n$ are all the distinct eigvals of T,

and therefore are all the distinct zeros of the mini poly.

Also, the mini poly of T is a polynomial multiple of, but not equal to, $(z-\alpha_1)\cdots(z-\alpha_n)$.

If we define q by $q(z)=(z-\alpha_1)^{\dim V-(n-1)}\cdots(z-\alpha_n)^{\dim V-(n-1)}$,

then q is a polynomial multiple of the char poly (see [8.34] and [8.26])

(Because $\dim V>n$ and $n-1>0$, $n[\dim V-(n-1)]>\dim V$.)

The char poly has the form $(z-\alpha_1)^{\gamma_1}\cdots(z-\alpha_n)^{\gamma_n}$, where $\gamma_1+\cdots+\gamma_n=\dim V$.

The mini poly has the form $(z-\alpha_1)^{\delta_1}\cdots(z-\alpha_n)^{\delta_n}$, where $0\le\delta_1+\cdots+\delta_n\le\dim V$.

---

10 *Suppose $T\in\mathcal{L}(V)$, λ is an eigval of T with an eigvec v.*
Prove that for any $p\in\mathcal{P}(\mathbf{F})$, $p(T)v=p(\lambda)v$.

SOLUTION:

Suppose p is defined by $p(z)=a_0+a_1z+\cdots+a_mz^m$ for all $z\in\mathbf{F}$. Because for any $n\in\mathbf{N}^+$, $T^nv=\lambda^nv$.
Thus $p(T)v=a_0v+a_1Tv+\cdots+a_mT^mv=a_0v+a_1\lambda v+\cdots+a_m\lambda^mv=p(\lambda)v$. $\square$

• **COMMENT:** For any $p\in\mathcal{P}(\mathbf{F})$ such that $p(z)=(z-\lambda_1)^{\alpha_1}\cdots(z-\lambda_m)^{\alpha_m}$, the result is true as well.

Now we prove that $(T-\lambda_1I)^{\alpha_1}\cdots(T-\lambda_mI)^{\alpha_m}v=(\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_m)^{\alpha_m}v$.

Define q_i by $q_i(z)=(z-\lambda_i)^{\alpha_i}$ for all $z\in\mathbf{F}$.

Because $(a+b)^n=a^n+\mathcal{C}_n^1a^{n-1}b+\cdots+\mathcal{C}_n^ka^{n-k}b^k+\cdots+\mathcal{C}_n^nb^n$.

Let $a=z, b=\lambda_i, n=\alpha_i$, so we can write $q_i(z)$ in the form $a_0+a_1z+\cdots+a_mz^m$.

Hence $q_i(T)v=q_i(\lambda)v\Rightarrow(T-\lambda_iI)^{\alpha_i}v=(\lambda-\lambda_i)^{\alpha_i}v$.

Then for each $k\in\{2,\ldots,m\}$, $(T-\lambda_{k-1}I)^{\alpha_{k-1}}(T-\lambda_kI)^{\alpha_k}v$

$$
\begin{aligned}
&= q_{k-1}(T)(q_k(T)v)\\
&= q_{k-1}(T)(q_k(\lambda)v)\\
&= q_{k-1}(\lambda)(q_k(\lambda)v)\\
&= (\lambda-\lambda_{k-1})^{\alpha_{k-1}}(\lambda-\lambda_k)^{\alpha_k}v.
\end{aligned}
$$

So that $(T-\lambda_1I)^{\alpha_1}\cdots(T-\lambda_mI)^{\alpha_m}v$

$$
\begin{aligned}
&= q_1(T)(q_2(T)(\ldots(q_m(T)v)\ldots))\\
&= q_1(\lambda)(q_2(\lambda)(\ldots(q_m(\lambda)v)\ldots))\\
&= (\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_m)^{\alpha_m}v. \quad \square
\end{aligned}
$$

---

1 *Suppose* $T \in \mathcal{L}(V)$ *and* $\exists\, n \in \mathbf{N}^+$ *such that* $T^n = 0$.
Prove that $(I - T)$ *is invertible and* $(I - T)^{-1} = I + T + \cdots + T^{n-1}$.

SOLUTION: Note that $1 - x^n = (1 - x)(1 + x + \cdots + x^{n-1})$.

$$\left.\begin{array}{l}(I - T)(1 + T + \cdots + T^{n-1}) = I - T^n = I \\ (1 + T + \cdots + T^{n-1})(I - T) = I - T^n = I\end{array}\right\} \Rightarrow (I - T)^{-1} = 1 + T + \cdots + T^{n-1}. \quad \square$$

---

2 *Suppose* $T \in \mathcal{L}(V)$ *and* $(T - 2I)(T - 3I)(T - 4I) = 0$.
Suppose λ *is an eigval of* T. *Prove that* $\lambda = 2$ *or* $\lambda = 3$ *or* $\lambda = 4$.

SOLUTION:

Suppose v is an eigvec corresponding to λ. Then for any $p \in \mathcal{P}(\mathbf{F}), p(T)v = p(\lambda)v$.

Hence $0 = (T-2I)(T-3I)(T-4I)v = (\lambda-2)(\lambda-3)(\lambda-4)v$ while $v \neq 0 \Rightarrow \lambda = 2$ or $\lambda = 3$ or $\lambda = 4$. $\square$

OR. Because $(T - 2I)(T - 3I)(T - 4I) = 0$ is not injective. By TIPS. $\square$

---

7 *Suppose* $T \in \mathcal{L}(V)$. *Prove that* 9 *is an eigval of* $T^2 \iff 3$ *or* -3 *is an eigval of* T.

SOLUTION: COMMENT: Note that V can be infinite-dim. See also in (5.A.22).

(a) Suppose 9 is an eigval of T^2. Then $(T^2 - 9I)v = (T - 3I)(T + 3I)v = 0$ for some v. By TIPS.

(b) Suppose 3 or -3 is an eigval of T with an eigenvector v. Then $Tv = \pm 3v \Rightarrow T^2 v = T(Tv) = 9v$ $\square$

---

3 *Suppose* $T \in \mathcal{L}(V), T^2 = I$ *and* -1 *is not an eigval of* T. *Prove that* $T = I$.

SOLUTION:

$T^2 - I = (T + I)(T - I)$ is not injective, 又 -1 is not an eigval of $T \Rightarrow$ By TIPS. $\square$

OR. Note that $v = [\frac{1}{2}(I - T)v] + [\frac{1}{2}(I + T)v]$ for all $v \in V$.
And $(I - T^2)v = (I - T)(I + T)v = 0$ for all $v \in V$,

$$\left.\begin{array}{l}(I + T)(\frac{1}{2}(I - T)v) = \frac{1}{2}(I - T^2)v = 0 \Rightarrow \frac{1}{2}(I - T)v \in \operatorname{null}(I + T) \\ (I - T)(\frac{1}{2}(I + T)v) = \frac{1}{2}(I - T^2)v = 0 \Rightarrow \frac{1}{2}(I + T)v \in \operatorname{null}(I - T)\end{array}\right\} \Rightarrow V = \operatorname{null}(I+T) + \operatorname{null}(I-T).$$

又 -1 is not an eigval of $T \Rightarrow (I + T)$ is injective $\Rightarrow \operatorname{null}(I + T) = \{0\}$.
Hence $V = \operatorname{null}(I - T) \Rightarrow \operatorname{range}(I - T) = \{0\}$. Thus $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$. $\square$

• (4E 5.A.32) *Suppose* $T \in \mathcal{L}(V)$ *has no eigvals and* $T^4 = I$. *Prove that* $T^2 = -I$.

SOLUTION:

Because $T^4 - I = (T^2 - I)(T^2 + I) = 0$ is not injective $\Rightarrow (T^2 - I)$ or $(T^2 + I)$ is not injective.
又 T has no eigvals $\Rightarrow (T^2 - I) = (T - I)(T + I)$ is injective,
for if not, $(T - I)$ or $(T + I)$ is not injective. Contradicts.
Hence $T^2 + I = 0 \in \mathcal{L}(V)$, for if not, $\exists\, v \in V, (T^2 + I)v \neq 0$ while $(T^2 - I)((T^2 + I)v) = 0$. Contradicts.$\square$

OR. Note that $v = [\frac{1}{2}(I - T^2)v] + [\frac{1}{2}(I + T^2)v]$ for all $v \in V$.
And $(I - T^4)v = (I - T^2)(I + T^2)v = 0$ for all $v \in V$,

$$\left.\begin{array}{l}(I + T^2)(\frac{1}{2}(I - T^2)v) = 0 \Rightarrow \frac{1}{2}(I - T^2)v \in \operatorname{null}(I + T^2) \\ (I - T^2)(\frac{1}{2}(I + T^2)v) = 0 \Rightarrow \frac{1}{2}(I + T^2)v \in \operatorname{null}(I - T^2)\end{array}\right\} \Rightarrow V = \operatorname{null}(I + T^2) + \operatorname{null}(I - T^2).$$

又 T has no eigvals $\Rightarrow (I - T^2)$ is injective $\Rightarrow \operatorname{null}(I - T^2) = \{0\}$.
Hence $V = \operatorname{null}(I + T^2) \Rightarrow \operatorname{range}(I + T^2) = \{0\}$. Thus $I + T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$. $\square$

---

8 (OR 4E 5.A.31) *Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$.*

SOLUTION:

Simply by computing: $p(z) = z^4 + 1 = (z^2 + \mathrm{i})(z^2 - \mathrm{i}) = (z + \mathrm{i}^{1/2})(z - \mathrm{i}^{1/2})(z - (-\mathrm{i})^{1/2})(z + (-\mathrm{i})^{1/2})$.

Note that $\mathrm{i}^{1/2} = \dfrac{\sqrt{2}}{2} + \mathrm{i}\dfrac{\sqrt{2}}{2}$, $(-\mathrm{i})^{1/2} = \dfrac{\sqrt{2}}{2} - \mathrm{i}\dfrac{\sqrt{2}}{2}$.

Hence $T = \pm(\pm\mathrm{i})^{1/2}$.

Define T by $T(x, y) = (\dfrac{\sqrt{2}}{2}x - \dfrac{\sqrt{2}}{2}y, \dfrac{\sqrt{2}}{2}x + \dfrac{\sqrt{2}}{2}y)$.

$\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix} \Rightarrow \mathcal{M}(T)^4 = \mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathcal{M}(I)$. ☐

$\left(\text{Using } \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}. \right)$

• (4E 5.B.12 See also at 5.A.9)

Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$. Find the mini poly.

SOLUTION:

$T(x_1, \ldots, 0) =$ By (5.A.9) and [8.49], $1, 2, \ldots, n$ are zeros of the mini poly of T.

(又 Each eigvals of T corresponds to exact one-dim subspace of $\mathbf{F}^n$.)

Define a poly q by $q(z) = (z-1)(z-2)\cdots(z-n)$, for all $z \in \mathbf{F}$. (Then q is the char poly of T.)

Because $q(T)e_j = [\,(T - I)\cdots(T - (j-1)I)(T - (j+1)I)\cdots(T - nI)\,]\,(T - jI)e_j = 0$ for each j, where $(e_1, \ldots, e_n)$ is the standard basis. Thus $\forall v \in \mathbf{F}^n, q(T)v = 0$. Hence q is the mini poly of T. ☐

• *Suppose $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n)$.*

[*See also at* (5.A.19)] *Find the mini poly of T.*

SOLUTION:

Because n and 0 are all eigvals of T, 又 For all $e_k, Te_k = e_1 + \cdots + e_n$; $\quad T^2e_k = n(e_1 + \cdots + e_n)$.

Hence $T^2e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T - n)$. Thus $z(z-n)$ is the mini poly of T. ☐

• (4E 5.B.8)

Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is the operator of counterclockwise rotation by the angel θ, where $x \in \mathbf{R}^+$. Find the minimal polynomial of T.

SOLUTION:

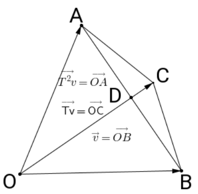
If $\theta = \pi$, then $T(w, z) = (-w, -z), T^2 = I$ and the mini poly is $z + 1$.

If $2\pi | \theta$, then $T = I$ and the mini poly is $z - 1$.

Now suppose (v, Tv) is linearly independent.

Then span $(v, Tv) = \mathbf{R}^2$.

Suppose the mini poly p is defined by $p(z) = z^2 + bz + c$ for all $z \in \mathbf{R}$.

Because 

$$\begin{cases} Tv = \dfrac{|\vec{v}|}{2L}(T^2v + v) \Rightarrow T = \dfrac{|\vec{v}|}{2L}(T^2 + I) \\[2ex] L = |\vec{v}|\cos\theta \Rightarrow \dfrac{|\vec{v}|}{2L} = \dfrac{1}{2\cos\theta} \end{cases}$$

Hence $p(T) = T^2 - 2\cos\theta T + I = 0$. $z^2 - 2\cos\theta z + 1$ is the mini poly of T. ☐

• (4E 5.B.11)
Suppose V is a two-dim vecsp, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.
(a) *Show that $T^2 - (a+d)T + (ad-bc)I = 0$.*
(b) *Show that the mini poly of T equals*

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad-bc) & \text{otherwise.} \end{cases}$$

SOLUTION:

(a) Suppose the basis is (v, w). Because $\begin{cases} Tv = av + bw \Rightarrow (T - aI)v = bw, \text{ then apply } (T - dI) \text{ to both sides.} \\ Tw = cv + dw \Rightarrow (T - dI)w = cv, \text{ then apply } (T - aI) \text{ to both sides.} \end{cases}$

Hence $(T - aI)(T - dI) = bcI \Rightarrow T^2 - (a+d)T + (ad-bc)I = 0$.

(b) If $b = c = 0$ and $a = d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the mini poly is $z - a$.

Otherwise, by (a), $z^2 - (a+d)z + (ad-bc)$ is a polynomial multiple of the mini poly.

Now we prove that $T \notin \operatorname{span}(I)$, so that then the mini poly of T has exactly degree 2.

(At least one of the assumption of (I),(II) below is true.)

(I) Suppose $a = d$, then $Tv = av + bw \notin \operatorname{span}(v), Tw = cv + aw \notin \operatorname{span}(w)$.

(II) Suppose at most one of b, c is not 0. If $b = 0$, then $Tw \notin \operatorname{span}(w)$; If $c = 0$, then $Tv \notin \operatorname{span}(v)$. □

---

5 *Suppose $S, T \in \mathcal{L}(V)$, S is invertible, and $p \in \mathcal{P}(\mathbf{F})$. Prove that $p(TS) = S^{-1}p(ST)S$.*

SOLUTION: We prove $(TS)^m = S^{-1}(ST)^m S$ for each $m \in \mathbf{N}$ by induction.

(i) $m = 0, 1$. $TS^0 = I = S^{-1}(ST)^0 S$; $TS = S^{-1}(ST)S$.

(ii) $m > 1$. Assume that $(TS)^m = S^{-1}(ST)^m S$.

Then $(TS)^{m+1} = (TS)^m(TS) = S^{-1}(ST)^m STS = S^{-1}(ST)^{m+1}S$.

Hence $\forall p \in \mathcal{P}(\mathbf{F}), p(TS) = a_0(TS)^0 + a_1(TS) + \cdots + a_m(TS)^m$

$$= a_0[S^{-1}(ST)^0 S] + a_1[S^{-1}(ST)S] + \cdots + a_m[S^{-1}(ST)^m S]$$
$$= S^{-1}[a_0(ST)^0 + a_1(ST) + \cdots + a_m(ST)^m]S = S^{-1}p(ST)S. \quad \square$$

• (4E 5.B.7)
(a) *Give an example of $S, T \in \mathcal{L}(\mathbf{F}^2)$ such that the mini poly of ST does not equal the mini poly of TS.*
(b) *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that if S or T is invertible, then the mini poly of ST equals the mini poly of TS.*

SOLUTION:

(a) Define S by $S(x, y) = (x, x)$. Define T by $T(x, y) = (0, y)$.

Then $ST(x, y) = 0$, $TS(x, y) = (0, x)$ for all $(x, y) \in \mathbf{F}^2$.

Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.

Hence the mini poly of ST does not equal to the mini poly of TS.

(b) Denote the mini poly of ST by p, and the mini poly TS by q.

Suppose S is invertible.

$$\left.\begin{array}{l} p(ST) = 0 = Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a polynomial multiple of } q. \\ q(TS) = 0 = S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a polynomial multiple of } p. \end{array}\right\} \Rightarrow p = q.$$

Reversing the roles of S and T, we conclude that if T is invertible, then $p = q$ as well. □

---

11 *Suppose* $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{C})$, *and* $\alpha \in \mathbf{C}$.
Prove that α *is an eigval of* $p(T) \iff \alpha = p(\lambda)$ *for some eigval* λ *of* T.

SOLUTION:

(a) Suppose α is an eigval of $p(T) \Leftrightarrow (p(T) - \alpha I)$ is not injective.
Write $p(z) - \alpha = c(z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow p(T) - \alpha I = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.
By TIPS, $\exists\, (T - \lambda_j I)$ not injective. Thus $p(\lambda_j) - \alpha = 0$. $\square$

(b) Suppose $\alpha = p(\lambda)$ and λ is an eigval of T with an eigvec v. Then $p(T)v = p(\lambda)v = \alpha v$. $\square$
OR. Define q by $q(z) = p(z) - \alpha$. λ is a zero of q.
Because $q(T)v = (p(T) - \alpha I)v = q(\lambda)v = (p(\lambda) - \alpha)v = 0$.
Hence $q(T)$ is not injective $\Rightarrow (p(T) - \alpha I)$ is not injective. $\square$

---

12 (OR 4E.5.B.6) *Give an example of an operator on* $\mathbf{R}^2$
that shows the result above does not hold if $\mathbf{C}$ *is replaced with* $\mathbf{R}$.

SOLUTION:

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(w, z) = (-z, w)$.
By Problem (4E 5.B.11), $\mathcal{M}(T, ((1,0),(0,1))) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ the mini poly of T is $z^2 + 1$.
Define p by $p(z) = z^2$. Then $p(T) = T^2 = -I$. Thus $p(T)$ has eigval -1.
While $\nexists\, \lambda \in \mathbf{R}$ such that $-1 = p(\lambda) = \lambda^2$. $\square$

• (4E 5.B.17)
Suppose V *is finite-dim,* $T \in \mathcal{L}(V)$, *and* p *is the mini poly of* T. *Suppose* $\lambda \in \mathbf{F}$.
Show that the mini poly of $(T - \lambda I)$ *is the polynomial* q *defined by* $q(z) = p(z + \lambda)$.

SOLUTION:

$q(T - \lambda I) = 0 \Rightarrow q$ is polynomial multiple of the mini poly of $(T - \lambda I)$.
Suppose the degree of the mini poly of $(T - \lambda I)$ is n, and the degree of the mini poly of T is m.
By definition of mini poly,
n is the smallest such that $(T - \lambda I)^n \in \operatorname{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{n-1})$;
m is the smallest such that $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.
又 $T^k \in \operatorname{span}(I, T, \dots, T^{k-1}) \iff (T - \lambda)^k \in \operatorname{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{k-1})$.
Thus $n = m$. 又 q is monic. By the uniqueness of mini poly. $\square$

• (4E 5.B.18)
Suppose V *is finite-dim,* $T \in \mathcal{L}(V)$, *and* p *is the mini poly of* T. *Suppose* $\lambda \in \mathbf{F} \backslash \{0\}$.
Show that the mini poly of λT *is the polynomial* q *defined by* $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.

SOLUTION:

$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow q$ is a polynomial multiple of the mini poly of λT.
Suppose the degree of the mini poly of λT is n, and the degree of the mini poly of T is m.
By definition of mini poly,
n is the smallest such that $(\lambda T)^n \in \operatorname{span}(\lambda I, \lambda T, \dots, (\lambda T)^{n-1})$;
m is the smallest such that $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.
又 $(\lambda T)^k \in \operatorname{span}(\lambda I, \lambda T, \dots, (\lambda T)^{k-1}) \iff T^k \in \operatorname{span}(I, T \dots, T^{k-1})$.
Thus $n = m$. 又 q is monic. By the uniqueness of mini poly. $\square$

---

18 (OR 4E 5.B.15)

Suppose V is a finite-dim complex vector space with $\dim V > 0$ and $T \in \mathcal{L}(V)$.
Define $f : \mathbf{C} \to \mathbf{R}$ by $f(\lambda) = \dim \operatorname{range}(T - \lambda I)$. Prove that f is not a continuous function.

SOLUTION: Note that V is finite-dim.

Let λ_0 be an eigval of T. Then $(T - \lambda_0 I)$ is not surjective. Hence $\dim \operatorname{range}(T - \lambda_0 I) < \dim V$.
Because T has finitely many eigvals. There exist a sequence of number $\{\lambda_n\}$ such that $\lim_{n\to\infty} \lambda_n = \lambda_0$.
And λ_n is not an eigval of T for each $n \Rightarrow \dim \operatorname{range}(T - \lambda_n I) = \dim V \neq \dim \operatorname{range}(T - \lambda_0 I)$.
Thus $f(\lambda_0) \neq \lim_{n\to\infty} f(\lambda_n)$. $\square$

• (4E 5.B.9)

Suppose $T \in \mathcal{L}(V)$ is such that with respect to some basis of V,
all entries of the matrix of T are rational numbers.
Explain why all coefficients of the mini poly of T are rational numbers.

SOLUTION:

Let $(v_1, \dots, v_n)$ denote the basis such that $\mathcal{M}(T, (v_1, \dots, v_n))_{j,k} = A_{j,k} \in \mathbf{Q}$ for all $j, k = 1, \dots, n$.
Denote $\mathcal{M}(v_j, (v_1, \dots, v_n))$ by x_j for each v_j.
Suppose p is the mini poly of T and $p(z) = z^m + \cdots + c_1 z + c_0$. Now we show that each $c_j \in \mathbf{Q}$.
Note that $\forall s \in \mathbf{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbf{Q}^{n,n}$ and $T^s v_k = A^s_{1,k} v_1 + \cdots + A^s_{n,k} v_n$ for all $k \in \{1, \dots, n\}$.

$$\text{Thus } \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \cdots + c_1 A + c_0 I)x_1 = \sum_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,1} x_j = 0; \\ \quad\vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \cdots + c_1 A + c_0 I)x_n = \sum_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly, } \begin{cases} (A^m + \cdots + c_1 A + c_0 I)_{1,1} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,1} = 0; \\ \qquad\qquad\vdots \quad \ddots \qquad\qquad \vdots \\ (A^m + \cdots + c_1 A + c_0 I)_{1,n} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,n} = 0; \end{cases}$$

Hence we get a system of n^2 linear equations in m unknowns $c_0, c_1, \dots, c_{m-1}$.
We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbf{Q}$. $\square$

• OR (4E 5.B.16), OR (8.C.18)

Suppose $a_0, \dots, a_{n-1} \in \mathbf{F}$. Let T be the operator on $\mathbf{F}^n$ such that

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ with respect to the standard basis } (e_1, \dots, e_n).$$

Show that the mini poly of T is p defined by $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$.

*$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the mini poly of some operator. Hence a formula or an algorithm that could produce exact eigvals for each operator on each $\mathbf{F}^n$ could then produce exact zeros for each poly [by 8.36(b)]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an operator.*

SOLUTION: Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ is linearly independent. 又 The deg of mini poly is at most n.

$T^n e_1 = \cdots = T^{n-k} e_{1+k} = \cdots = T e_n = -a_0 e_1 - a_1 e_2 - a_2 e_3 - \cdots - a_{n-1} e_n$
$= (-a_0 I - a_1 T - a_2 T^2 - \cdots - a_{n-1} T^{n-1}) e_1$. Thus $p(T)e_1 = 0 = p(T)e_j$ for each $e_j = T^{j-1} e_1$. $\square$

• EIGENVALUES ON ODD-DIMENSIONAL REAL VECTOR SPACES (Eigvals on Odd-dim Real Vecsps)

• EVEN-DIMENSIONAL NULL SPACE

Suppose $\mathbf{F}=\mathbf{R}$, V *is finite-dim,* $T \in \mathcal{L}(V)$ *and* $b, c \in \mathbf{R}$ *with* $b^2 < 4c$.
Prove that $\dim \textit{null}\,(T^2 + bT + cI)$ *is an even number.*

SOLUTION:

Denote null $(T^2 + bT + cI)$ by R. Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.

Suppose λ is an eigval of T_R with an eigvec $v \in R$.

Then $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \left((\lambda + b)^2 + c - \dfrac{b^2}{4}\right) v.$

Because $c - \dfrac{b^2}{4} > 0$ and we have $v = 0$. Thus T_R has no eigvals.

Let U be an invariant subspace of R that has the largest, even dim among all invariant subspaces.

Assume that $U \neq R$. Then $\exists\, w \in R$ but $w \notin U$. Let W be such that $(w, T|_R w)$ is a basis of W.

Because $T|_R^2 w = -bT|_R w - cw \in W$. Hence W is an invariant subspace of dim 2.

Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,

for if not, because $w \notin U, T|_R w \in U$,

$U \cap W$ is invariant under $T|_R$ of one dim (impossible because $T|_R$ has no eigvecs).

Hence $U + W$ is even-dim invariant subspace under $T|_R$, contradicting the maximality of $\dim U$.

Thus the assumption was incorrect. Hence $R = \text{null}\,(T^2 + bT + cI) = U$ has even dim. $\square$

• OPERATORS ON ODD-DIMENSIONAL VECTOR SPACES HAVE EIGENVALUES

(a) Suppose $\mathbf{F}=\mathbf{C}$. Then by [5.21], we are done.

(b) *Suppose* $\mathbf{F}=\mathbf{R}$, V *is finite-dim, and* $\dim V = n \neq 0$ *is an odd number.*
Let $T \in \mathcal{L}(V)$ *and the mini poly is* p. *Prove that* T *has an eigval.*

SOLUTION:

(i) If $n = 1$, then we are done.

(ii) Suppose $n \geq 3$. Assume that every operator, on odd-dim vecsps of dim less than n, has an eigval.

If p is a polynomial multiple of $(x - \lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] λ is an eigval of T and we are done.

Now suppose $b, c \in \mathbf{R}$ such that $b^2 < 4c$ and p is a polynomial multiple of $x^2 + bx + c$ (see [4.17]).

Then $\exists\, q \in \mathcal{P}(\mathbf{R})$ such that $p(x) = q(x)(x^2 + bx + c)$ for all $x \in \mathbf{R}$.

Now $0 = p(T) = (q(T))(T^2 + bT + cI)$, which means that $q(T)|_{\text{range}\,(T^2+bT+cI)} = 0$.

Because $\deg q < \deg p$ and p is the mini poly of T, hence range $(T^2 + bT + cI) \neq V$.

又 $\dim V$ is odd and $\dim \text{null}\,(T^2 + bT + cI)$ is even (by our previous result).

Thus $\dim V - \dim \text{null}\,(T^2 + bT + cI) = \dim \text{range}\,(T^2 + bT + cI)$ is odd.

By [5.18], range $(T^2 + bT + cI)$ is an invariant subspace of V under T that has odd dim less than n.

Our induction hypothesis now implies that $T|_{\text{range}\,(T^2+bT+cI)}$ has an eigenvalue.

By mathematical induction. $\square$

• (2E Ch5.24)

Suppose $\mathbf{F}=\mathbf{R}, T \in \mathcal{L}(V)$ *has no eigvals.*
Prove that every invariant subspace of V *under* T *is even-dim.*

SOLUTION:

Suppose U is such a subspace. Then $T|_U \in \mathcal{L}(U)$. We prove by contradiction.

If $\dim U$ is odd, then $T|_U$ has an eigval and so is T, so that $\exists$ invariant subspace of 1 dim, contradicts. $\square$

• (4E 5.B.29)

Show that every operator on a finite-dim vecsp of dim ≥ 2
has an invariant subspace of dim 2.

Exercise (4E 5.C.6) will give an improvement of this result when $\mathbf{F} = \mathbf{C}$.

SOLUTION:

Using induction on $\dim V$.

(i) $\dim V = 2$, we are done.

(ii) $\dim V > 2$. Assume that the desired result is true for vecsp of smaller dim.

Suppose p is the mini poly of degree m and $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.

If $T = \lambda I$ ($\Leftrightarrow m = 1 \vee m = -\infty$), then we are done. ($m \neq 0$ because $\dim V \neq 0$.)

Now define a q by $q(z) = (z - \lambda_1)(z - \lambda_2)$.

By assumption, $T|_{\operatorname{null} q(T)}$ has an invariant subspace of dim 2. $\square$

ENDED

5.B: II

• **NOTE FOR []:**

• **NOTE FOR []:**

• **NOTE FOR []:**

• (4E 5.C.1)

Prove or give a counterexample:
If $T \in \mathcal{L}(V)$ *and* T^2 *has an upper-trig matrix with respect to some basis of* V,
then T *has an upper-trig matrix with respect to some basis of* V.

SOLUTION:

• (4E 5.C.2)

Suppose A *and* B *are upper-trig matrices of the same size,*
with $\alpha_1, \ldots, \alpha_n$ *on the diagonal of* A *and* $\beta_1, \ldots, \beta_n$ *on the diagonal of* B.
(a) *Show that* $A + B$ *is an upper-trig matrix with* $\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n$ *on the diagonal.*
(b) *Show that* AB *is an upper-trig matrix with* $\alpha_1\beta_1, \ldots, \alpha_n\beta_n$ *on the diagonal.*

SOLUTION:

• (4E 5.C.3)

Suppose $T \in \mathcal{L}(V)$ *is invertible and* $(v_1, \ldots, v_n)$ *is a basis of* V *with respect*
to which the matrix of T *is upper trig, with* $\lambda_1, \ldots, \lambda_n$ *on the diagonal.*
Show that the matrix of T^{-1} *is also upper trig with respect to the same basis,*
with $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$ *on the diagonal.*

SOLUTION:

---

9 (4E 5.C.7)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $v \in V$.

(a) *Prove that $\exists!$ monic poly p_v of smallest degree such that $p_v(T)v = 0$.*

(b) *Prove that the mini poly of T is a polynomial multiple of p_v.*

SOLUTION:

---

14 (OR 4E 5.C.4)

Give an operator T such that with respect to some basis, $\mathcal{M}(T)_{k,k} = 0$ for each k, while T is invertible.

SOLUTION:

---

15 (OR 4E 5.C.5)

Give an operator T such that with respect to some basis, $\mathcal{M}(T)_{k,k} \neq 0$ for each k, while T is not invertible.

SOLUTION:

---

20 (OR 4E 5.C.6)

Suppose $\mathbf{F} = \mathbf{C}$, V is finite-dim, and $T \in \mathcal{L}(V)$.

Prove that if $k \in \{1, \dots, \dim V\}$, then V has a k dim subspace invariant under T.

SOLUTION:

• (4E 5.C.8)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\exists\, v \in V \backslash \{0\}$ such that $T^2 v + 2Tv = -2v$.

(a) *Prove that if $\mathbf{F} = \mathbf{R}$, then $\nexists$ a basis of V with respect to which T has an upper-trig matrix.*

(b) *Prove that if $\mathbf{F} = \mathbf{C}$ and A is an upper-trig matrix that equals the matrix of T with respect to some basis of V, then $-1 + i$ or $-1 - i$ appears on the diagonal of A.*

SOLUTION:

• (4E 5.C.9)

Suppose $B \in \mathbf{F}^{n,n}$ with complex entries.

Prove that $\exists$ invertible $A \in \mathbf{F}^{n,n}$ with complex entries such that $A^{-1}BA$ is an upper-trig matrix.

SOLUTION:

• (4E 5.C.10)

Suppose $T \in \mathcal{L}(V)$ and $(v_1, \dots, v_n)$ is a basis of V.

Show that the following are equivalent.

(a) *The matrix of T with respect to $(v_1, \dots, v_n)$ is lower trig.*

(b) *span $(v_k, \dots, v_n)$ is invariant under T for each $k = 1, \dots, n$.*

(c) $Tv_k \in$ *span* $(v_k, \ldots, v_n)$ *for each* $k = 1, \ldots, n$.

A square matrix is called lower trig if all entries above the diagonal are 0.

SOLUTION:

• (4E 5.C.11)

Suppose $\mathbf{F} = \mathbf{C}$ *and* V *is finite-dim.*

Prove that if $T \in \mathcal{L}(V)$, *then* T *has a lower-trig matrix with respect to some basis.*

SOLUTION:

• (4E 5.C.12)

Suppose V *is finite-dim,* $T \in \mathcal{L}(V)$ *has an upper-trig matrix with respect to some basis, and* U *is a subspace of* V *that is invariant under* T.

(a) *Prove that* $T|_U$ *has an upper-trig matrix with respect to some basis of* U.

(b) *Prove that* T/U *has an upper-trig matrix with respect to some basis of* V/U.

SOLUTION:

• (4E 5.C.13)

Suppose V *is finite-dim and* $T \in \mathcal{L}(V)$.

Suppose $\exists U$ *of* V *that is invariant under* T *such that*

$T|_U$ *has an upper-trig matrix with respect to some basis of* U

and also T/U *has an upper-trig matrix with respect to some basis of* V/U.

Prove that T *has an upper-trig matrix with respect to some basis of* V.

SOLUTION:

• (4E 5.C.14)

Suppose V *is finite-dim and* $T \in \mathcal{L}(V)$.

Prove that T *has an upper-trig matrix with respect to some basis of* V

$\iff$ T' *has an upper-trig matrix with respect to some basis of* V'.

SOLUTION:

ENDED

5.C

• **N**

SOLUTION:

• **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

- **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

• **N**

SOLUTION:

ENDED

5.E* (4E)

1 *Give an example of two commuting operators* $S, T \in \mathbf{F}^4$ *such that there is a subspace of* $\mathbf{F}^4$ *that is invariant under* S *but not under* T *and there is a subspace of* $\mathbf{F}^4$ *that is invariant under* T *but not under* S.

SOLUTION:

---

2 *Suppose* $\mathcal{E}$ *is a subset of* $\mathcal{L}(V)$ *and every element of* $\mathcal{E}$ *is diagonalizable.*
Prove that $\exists$ *a basis of* V *with respect to which*
every element of $\mathcal{E}$ *has a diagonal matrix* $\Longleftrightarrow$ *every pair of elements of* $\mathcal{E}$ *commutes.*

This exercise extends [5.76], which considers the case in which $\mathcal{E}$ *contains only two elements.*

For this exercise, $\mathcal{E}$ *may contain any number of elements, and* $\mathcal{E}$ *may even be an infinite set.*

SOLUTION:

---

3 *Suppose* $S, T \in \mathcal{L}(V)$ *are such that* $ST = TS$. *Suppose* $p \in \mathcal{P}(\mathbf{F})$.
(a) *Prove that null* $p(S)$ *is invariant under* T.
(b) *Prove that range* $p(S)$ *is invariant under* T.
See NOTE FOR [5.17] for the special case $S = T$.

SOLUTION:

---

4 *Prove or give a counterexample:*
If A is a diagonal matrix and B is an upper-trig matrix of the same size as A, then A and B commute.

SOLUTION:

---

5 *Prove that a pair of operators on a finite-dim vecsp commute $\iff$ their dual operators commute.*

SOLUTION:

---

6 *Suppose V is a finite-dim complex vecsp and $S, T \in \mathcal{L}(V)$ commute.*
Prove that $\exists\, \alpha, \lambda \in \mathbf{C}$ such that range $(S - \alpha I)$ + range $(T - \lambda I) \neq V$.

SOLUTION:

---

7 *Suppose V is a complex vecsp, $S \in \mathcal{L}(V)$ is diagonalizable, and T commutes with S.*
Prove that $\exists$ basis B of V such that S has a diagonal matrix with respect to B and T has an upper-trig matrix with respect to B.

SOLUTION:

---

8 *Suppose $m = 3$ in Example [5.72]*
and D_x, D_y are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbf{R}^2)$ from that example.
Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ with respect to which D_x and D_y each have an upper-trig matrix.

SOLUTION:

---

9 *Suppose V is a finite-dim nonzero complex vecsp.*
Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.
(a) *Prove that $\exists\, v \in V$ is an eigvec for every element of $\mathcal{E}$.*
(b) *Prove that $\exists$ a basis of V with respect to which every element of $\mathcal{E}$ has an upper-trig matrix.*

SOLUTION:

---

10 *Give an example of two commuting operators S, T on a finite-dim real vecsp such that $S + T$ has a eigval that does not equal an eigval of S plus an eigval of T and ST has a eigval that does not equal an eigval of S times an eigval of T.*

SOLUTION:

- **N**

SOLUTION:

- **NOTE FOR []:**
