简介

这是我个人用于复习的笔记，一本习题补注。由于我个人偏好的复习手段的特点，因此我把许多对我个人而言没什么复习价值的习题作了省略。为什么我没有用中文？因为我将来要学习的绝大多数数学课本都是全英的，国内目前的专业翻译速度慢、不全面，所以我只好用英文。但我讨厌英文单词的冗长性，这会让我复习起来很不爽，所以我对许多常用词汇适当地作了简写。这份习题补注的内容范围和标识说明，我已经在README中写得很清楚，不再赘述。这份笔记尚处于缓慢的编撰进度中。

Goto

1	2	3	4	5	6	7	8	9	10
A	A	A	/	A	A	A	A	A	A
B	B	B	/	B	B	B	B	B	B
C	C	C	/	C	C	C	C	/	/
/	/	D	/	/	D	D	D	/	/
/	/	E	/	E*	/	/	/	/	/
/	/	F	/	/	/	F*	/	/	/

Abbreviation Table

def	definition
vec	vector
vecsp	vector space
subsp	subspace
add	addition/additive
multi	multiplication/multiplicative/multiple
assoc	associative/associativity
distr	distributive properties/property
inv	inverse
existns	existence
uniqnes	uniqueness
linely inde	linearly independent/independence
linely dep	linearly dependent/dependence
dim	dimension(al)
inje	injective
surj	surjective
col	column
with resp	with respect
iso	isomorphism/isomorphic
correspd	correspond(ing)
poly	polynomial
eigval	eigenvalue
eigvec	eigenvector
mini poly	minimal polynomial
char poly	characteristic polynomial

1·B

1 *Prove that $\forall v \in V, -(-v) = v$.*

Solution: $\left.\begin{aligned} (-(-v)) + (-v) &= 0 \\ v + (-v) &= 0 \end{aligned}\right\} \Rightarrow$ By the uniqnes of add inv. $\square$

Or. $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v.$ $\square$

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2 *Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.*

Solution: Suppose $a \neq 0$, $\exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0.$ $\square$

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3 *Suppose $v, w \in V$. Explain why $\exists\,!\, x \in V, v + 3x = w$.*

Solution:

[*Existns*] Let $x = \dfrac{1}{3}(w - v)$.

[*Uniqnes*] Suppose $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) $-$ (II) : $3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. $\square$

Or. $\quad v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \frac{1}{3}(w - v).$ $\square$

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5 *Show that in the def of a vecsp, the add inv condition can be replaced by [1.29].*

Suppose V satisfies all conds in the def, except we've replaced the add inv cond with [1.29]. Prove that the add inv is true.

Solution: Using [1.31]. $0v = 0$ for all $v \in V \Longleftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0.$ $\square$

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6 *Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$.*
Define an add and scalar multi on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess.
The operations of real numbers is as usual. While for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

and (I) $t + \infty = \infty + t = \infty + \infty = \infty$,

(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,

(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of add and scalar multi, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vecsp over $\mathbf{R}$? Explain.

Solution: Not a vecpsp, since the add and scalar mult is not assoc and distr.

By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

Or. By Distr: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0.$ $\square$

Ended

1·C

7 *Give a nontrivial example of $U \subseteq \mathbf{R}^2$,*
U is closed under taking add invs and under add, but U is not a subsp of $\mathbf{R}^2$.

Solution: Let $U = \mathbf{Z}^2, (\mathbf{Z}^*)^2, (\mathbf{Q}^*)^2, \mathbf{Q}^2 \setminus \{0\}$, or $\mathbf{R}^2 \setminus \{0\}$.

8 *Give a nontrivial example of $U \subseteq \mathbf{R}^2$,*
U is closed under scalar multi, but U is not a subsp of $\mathbf{R}^2$.

Solution: Let $U = \left\{(x, y) \in \mathbf{R}^2 : x = 0 \vee y = 0\right\}$.

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9 *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if $\exists\, p \in \mathbf{N}^+, f(x) = f(x+p)$ for all $x \in \mathbf{R}$.*
Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution: Denote the set by S.

Suppose $h(x) = \cos(x) + \sin(\sqrt{2}x) \in S$, since $\cos(x), \sin(\sqrt{2}x) \in S$.

Assume $\exists\, p \in \mathbf{N}^+$ such that $h(x) = h(x+p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos(p) + \sin(\sqrt{2}p) = \cos(p) - \sin(\sqrt{2}p)$

$\Rightarrow \sin(\sqrt{2}p) = 0,\ \cos(p) = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Contradiction! □

Or. Because $[\mathrm{I}] : \cos(x) + \sin(\sqrt{2}x) = \cos(x+p) + \sin(\sqrt{2}x + \sqrt{2}p)$. By differentiating twice,

$$[\mathrm{II}] : \cos(x) + 2\sin(\sqrt{2}x) = \cos(x+p) + 2\sin(\sqrt{2}x + \sqrt{2}p).$$

$$\left.\begin{array}{rr} [\mathrm{II}] - [\mathrm{I}] : & \sin(\sqrt{2}x) = \sin(\sqrt{2}x + \sqrt{2}p) \\ 2[\mathrm{I}] - [\mathrm{II}] : & \cos(x) = \cos(x+p) \end{array}\right\} \Rightarrow p = \frac{m\pi}{\sqrt{2}} = 2k\pi, \text{ if } x = 0. \text{ Contradicts.}$$ □

• *Suppose U, W, V_1, V_2, V_3 are subsps of V.*

15 $$U + U \ni u + w \in U.$$ □

16 $$U + W \ni u + w = w + u \in W + U.$$ □

17 $$(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3).$$ □

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18 *Does the add on the subsps of V have an add identity? Which subsps have add invs?*

Solution:

(a) Suppose Ω is the additive identity.
For any subsp U of V. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

(b) Now suppose W is an add inv of $U \Rightarrow U + W = \Omega$.
Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. □

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11 *Prove that the intersection of every collection of subsps of V is a subsp of V.*

Solution: Suppose $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collection of subsps of V; here Γ is an arbitrary index set.

We show that $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vecs that are in U_α for each $\alpha \in \Gamma$, is a subsp of V.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.
(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under add.
(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under scalar multi.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of V that is closed under add and scalar multi. □

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12 *Suppose U, W are subsps of V. Prove that $U \cup W$ is a subsp of $V \iff U \subseteq W$ or $W \subseteq U$.*

SOLUTION:

(a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subsp of V.

(b) Suppose $U \cup W$ is a subsp of V. Suppose $U \not\subseteq W$ and $U \not\supseteq W$ ($U \cup W \neq U$ and W).

Then $\forall a \in U$ but $a \notin W$; $b \in W$ but $b \notin U$. $a + b \in U \cup W$.

$$\left.\begin{array}{l}\text{Consider } a+b \in U \Rightarrow b = (a+b) + (-a) \in U \text{, contradicts!} \\ \text{Consider } a+b \in W \Rightarrow a = (a+b) + (-b) \in W \text{, contradicts!}\end{array}\right\} \Rightarrow U \cup W = U \text{ or } W\text{. Contradicts!}$$

Thus $U \subseteq W$ and $U \supseteq W$. □

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13 *Prove that the union of three subsps of V is a subsp of V if and only if one of the subsps contains the other two.*

This exercise is not true if we replace $\mathbf{F}$ *with a field containing only two elements.*

SOLUTION:

Suppose U_1, U_2, U_3 are subsps of V. Denote $U_1 \cup U_2 \cup U_3$ by $\mathcal{U}$.

(a) Suppose that one of the subsps contains the other two.

Then $\mathcal{U} = U_1, U_2$ or U_3 is a subsp of V.

(b) Suppose that $U_1 \cup U_2 \cup U_3$ is a subsp of V.

By distinct we notice that $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.

Also note that, if $U \cup W = V$ is a vecsp, then in general U and W are not subsps of V.

Hence this literal trick is invalid.

(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.

By applying Problem (12) we conclude that one U_j contains the other two. Thus we are done.

(II) Assume that no U_j is contained in the union of the other two,

and no U_j contains the union of the other two.

Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3$; $v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if $v + \lambda u \in U_1$ then $v + \lambda u - \lambda u = v \in U_1$ while $v \notin U_1$.

又 $W \subseteq U_1 \cup U_2 \cup U_3$. Thus $W \subseteq U_2 \cup U_3$.

$\forall v + \lambda u \in W$, $\exists\, i \in \{2, 3\}, v + \lambda u \in U_i$.

Because U_2, U_3 are subsps and hence have at least one element.

If $U_2 = U_3$, then $\mathcal{U} = U_1 \cup U_2$ and by Problem (12) we are done.

Otherwise, $\exists\, \lambda, \mu \in \mathbf{F}$ with $\lambda \neq \mu$ such that $v + \lambda u, v + \mu u \in U_i$ for some $i \in \{2, 3\}$.

Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Contradicts. □

EXAMPLE: *Suppose $U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}, W = \{(x, x, x, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$. Prove that $U + W = \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$.*

Let T denote $\{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$. By def, $U + W \subseteq T$.

And $T \ni (x, x, y, z) \Rightarrow (0, 0, y - x, y - x) + (x, x, x, -y + x + z) \in U + W$. Hence $T \subseteq U + W$. □

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21 *Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}$. Find a W such that $\mathbf{F}^5 = U \oplus W$.*

SOLUTION:

Let $W = \{(0, 0, z, w, u) \in \mathbf{F}^5 : z, w, u \in \mathbf{F}\}$. Then $U \cap W = \{0\}$.

And $\mathbf{F}^5 \ni (x, y, z, w, u) \Rightarrow (x, y, x + y, x - y, 2x) + (0, 0, z - x - y, w - x - y, u - 2x) \in U + W$. □

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23 *Give an example of vecsps V_1, V_2, U such that $V_1 \oplus U = V_2 \oplus U$, but $V_1 \neq V_2$.*

Solution: $V = \mathbf{F}^2$, $U = \{(x,x) \in \mathbf{F}^2 : x \in \mathbf{F}\}$, $V_1 = \{(x,0) \in \mathbf{F}^2 : x \in \mathbf{F}\}$, $V_2 = \{(0,x) \in \mathbf{F}^2 : x \in \mathbf{F}\}$.

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22 *Suppose $U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5 : x,y \in \mathbf{F}\}$.*
Find three subsps W_1, W_2, W_3 of $\mathbf{F}^5$, none of which equals $\{0\}$,
such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution:

(1) Let $W_1 = \{(0,0,z,0,0) \in \mathbf{F}^5 : z \in \mathbf{F}\}$. Then $W_1 \cap U = \{0\}$.
Let $U_1 = U \oplus W_1$. Then $U_1 = \{(x,y,z,x-y,2x) \in \mathbf{F}^5 : x,y,z \in \mathbf{F}\}$. (Check it!)

(2) Let $W_2 = \{(0,0,0,w,0) \in \mathbf{F}^5 : w \in \mathbf{F}\}$. Then $W_2 \cap U_1 = \{0\}$.
Let $U_2 = U_1 \oplus W_2$. Then $U_2 = \{(x,y,z,w,2x) \in \mathbf{F}^5 : x,y,z,w \in \mathbf{F}\}$.

(3) Let $W_3 = \{(0,0,0,0,u) \in \mathbf{F}^5 : u \in \mathbf{F}\}$. Then $W_3 \cap U_2 = \{0\}$.
Let $U_3 = U_2 \oplus W_3$. Then $U_3 = \{(x,y,z,w,u) \in \mathbf{F}^5 : x,y,z,w,u \in \mathbf{F}\}$.

Thus $\mathbf{F}^5 = ((U \oplus W_1) \oplus W_2) \oplus W_3$. □

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24 *Let $V_E = \{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x)\}$, $V_O = \{f \in \mathbf{R}^{\mathbf{R}} : -f(x) = f(-x)\}$.*
Show that $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$.

Solution:

(a) $V_E \cap V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x)\} = \{0\}$.

(b)
$$\left.\begin{array}{l} f_e \in V_E \Longleftrightarrow f_e(x) = f_e(-x) \Longleftarrow \text{let } f_e(x) = \dfrac{g(x)+g(-x)}{2} \\ f_o \in V_O \Longleftrightarrow f_o(x) = -f_o(-x) \Longleftarrow \text{let } f_o(x) = \dfrac{g(x)-g(-x)}{2} \end{array}\right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}}, g(x) = f_e(x) + f_o(x).$$ □

Ended

2·A

2 (a) $[P]$ *A list (v) of length 1 in V is linely inde $\Longleftrightarrow v \neq 0$.* $[Q]$
(b) $[P]$ *A list (v,w) of length 2 in V is linely inde $\Longleftrightarrow \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$.* $[Q]$

Solution:

(a) $Q \overset{1}{\Rightarrow} P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ linely inde.
$P \overset{2}{\Rightarrow} Q : (v)$ linely inde $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.

Or.
| $\neg Q \overset{3}{\Rightarrow} \neg P : v = 0 \Rightarrow av = 0$ while we can let $a \neq 0 \Rightarrow (v)$ is linely dep.
| $\neg P \overset{4}{\Rightarrow} \neg Q : (v)$ linely dep $\Rightarrow av = 0$ while $a \neq 0 \Rightarrow v = 0$.

Comment: (1) with (3) and (2) with (4) will do as well. □

(b) $P \overset{1}{\Rightarrow} Q : (v,w)$ linely inde $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.
$Q \overset{2}{\Rightarrow} P :$ no scalar multi $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v,w)$ linely inde.

Or.
| $\neg P \overset{3}{\Rightarrow} \neg Q : (v,w)$ linely dep $\Rightarrow$ if $av + bw = 0$, then a or $b \neq 0 \Rightarrow$ scalar multi
| $\neg Q \overset{4}{\Rightarrow} \neg P :$ scalar multi $\Rightarrow$ if $av + bw = 0$, then a or $b \neq 0 \Rightarrow$ linely dep.

Comment: (1) with (3) and (2) with (4) will do as well. □

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1 *Prove that* $[P]$ (v_1, v_2, v_3, v_4) *spans* $V \iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ *also spans* V $[Q]$.

SOLUTION:

Notice that $V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists a_1, \dots, a_n \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n$.

Assume that $\forall v \in V$, $\exists a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbf{F}$, (that is, if $\exists a_i$, then we are to find b_i, vice versa)

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4. \end{aligned}$$

Now we can let $b_i = \sum_{r=1}^{i} a_r$ if we are to prove Q with P already assumed;

or let $a_i = b_i - b_{i-1}$ with $b_{-1} = 0$, if we are to prove P with Q already assumed. $\square$

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6 *Prove that* $[P]$ (v_1, v_2, v_3, v_4) *is linely inde*

$\iff [Q]$ $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ *is linely inde.*

SOLUTION:

$$\begin{aligned} P \Rightarrow Q : & a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0 \\ & \Rightarrow a_1 v_1 + (a_2 - a_1) v_2 + (a_3 - a_2) v_3 + (a_4 - a_3) v_4 = 0 \\ & \Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \end{aligned}$$

$$\begin{aligned} Q \Rightarrow P : & a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0 \\ & \Rightarrow a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \cdots + a_4) v_4 = 0 \\ & \Rightarrow a_1 = a_1 + a_2 = a_1 + a_2 + a_3 = a_1 + \cdots + a_4 = 0. \end{aligned}$$ $\square$

• *Suppose* $(v_1, \dots, v_m)$ *is a list of vecs in* V. *For* $k \in \{1, \dots, m\}$, *let* $w_k = v_1 + \cdots + v_k$.

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

(b) *Show that* $[P]$ $(v_1, \dots, v_m)$ *is linely inde* $\iff (w_1, \dots, w_m)$ *is linely inde* $[Q]$.

SOLUTION:

(a) *let* $a_k = \sum_{j=1}^{k} b_j \Leftarrow \ a_1 v_1 + \cdots + a_m v_m = b_1 w_1 + \cdots + b_m w_m \ \Rightarrow$ *let* $b_1 = a_1,\ b_k = a_k - \sum_{j=1}^{k-1} b_j = \sum_{j=1}^{k} (-1)^{k-j} a_j$.

(b) $P \Rightarrow Q:\ b_1 w_1 + \cdots + b_m w_m = 0 = a_1 v_1 + \cdots + a_m v_m$, where $0 = a_k = \sum_{j=1}^{k} b_k$

$Q \Rightarrow P:\ a_1 v_1 + \cdots + a_m v_m = 0 = b_1 w_1 + \cdots + b_m w_m = 0$, where $0 = b_1 = a_1,\ 0 = b_k = \sum_{j=1}^{k} (-1)^{k-j} a_j$

OR. Because $W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

By [2.21](b), a list of length $(m-1)$ spans W, then by [2.23],

$(w_1, \dots, w_m)$ linely dep $\Rightarrow (v_1, \dots, v_m)$ linely dep. Conversely it is true as well. $\square$

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10 *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* V *and* $w \in V$.

Prove that if $(v_1 + w, \dots, v_m + w)$ *is linely depe, then* $w \in \operatorname{span}(v_1, \dots, v_m)$.

SOLUTION:

Suppose $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0,\ \exists\, a_i \neq 0 \Rightarrow a_1 v_1 + \cdots + a_m v_m = 0 = -(a_1 + \cdots + a_m) w$.

Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1 v_1 + \cdots + a_m v_m = 0$ while $a_i \neq 0$ for some i, contradicts. $\square$

OR. By contrapositive, $w \notin \operatorname{span}(v_1, \dots, v_m)$, similarly. $\square$

OR. $\exists j \in \{1, \dots, m\}, v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and we are done.

If $j \geq 2$, then $\exists a_i \in \mathbf{F}, v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1 v_1 + \cdots + a_{j-1} v_{j-1}$.

Where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, contradicts.

Now $w = \lambda^{-1}(a_1 v_1 + \cdots + a_{j-1} v_{j-1} - v_j) \Rightarrow w \in \operatorname{span}(v_1, \dots, v_m)$. $\square$

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11 *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* V *and* $w \in V$.
Show that $[P]$ $(v_1, \dots, v_m, w)$ *is linely inde* $\Longleftrightarrow w \notin \operatorname{span}(v_1, \dots, v_m)$ $[Q]$.

Solution: $\neg Q \Rightarrow \neg P$: Suppose $w \in \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1, \dots, v_m, w)$ is linely depe.
$\neg P \Rightarrow \neg Q$: Suppose $(v_1, \dots, v_m, w)$ is linely dep. Then by [2.21] $w \in \operatorname{span}(v_1, \dots, v_m)$. □

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14 *Prove that* $[P]$ V *is infinite-dim* $\Longleftrightarrow$ $[Q]$ | *there is a sequence* $(v_1, v_2, \dots)$ *in* V *such that* $(v_1, \dots, v_m)$ *is linely inde for each* $m \in \mathbf{N}^+$.

Solution:

$P \Rightarrow Q$: Suppose V is infinite-dim, so that no list spans V.

Step 1 Pick a $v_1 \neq 0$, (v_1) linely inde.

Step m Pick a $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$, by Problem (10)(b), $(v_1, ..., v_m)$ is linely inde.

This process recursively defines the desired sequence $(v_1, v_2, \dots)$.

$\neg P \Rightarrow \neg Q$: Suppose V is finite-dim and $V = \operatorname{span}(w_1, ..., w_m)$.
Let $(v_1, v_2, \dots)$ be a sequence in V, then $(v_1, v_2, \dots, v_{m+1})$ must be linely dep.

Or. $Q \Rightarrow P$: Suppose there is such a sequence.
Choose an m. Suppose a linely inde list $(v_1, \dots, v_m)$ spans V.
(Similar to [2.16]) Then $\exists\, v_{m+1} \in V \backslash \operatorname{span}(v_1, \dots, v_m)$.
Hence no list spans V. Thus V is infinite-dim. □

---

16 *Prove that the vecsp of all continuous functions in* $\mathbf{R}^{[0,1]}$ *is infinite-dim.*

Solution: Denote the vecsp by U.

Choose an $m \in \mathbf{N}^+$. Suppose $a_0, \dots, a_m \in \mathbf{R}$ are such that $a_0 + a_1 x + \cdots + a_m x^m = 0$, $\forall x \in [0,1]$.
Then the poly has infinitely many roots and hence $a_0 = \cdots = a_m = 0$.
Thus $(1, x, \dots, x^m)$ is linely inde in $\mathbf{R}^{[0,1]}$. Similar to [2.16], U is infinite-dim.

Or. Note that for $a_n = \dfrac{1}{n}$, $\quad a_1 < a_2 < \cdots < a_m$, $\quad \forall m \in \mathbf{N}^+$.

Suppose $f_n = \begin{cases} x - \dfrac{1}{n}, & x \in \left(\dfrac{1}{n}, 1\right) \\ 0 \quad , & x \in \left[0, \dfrac{1}{n}\right] \end{cases}$ | Then for any m, $f_1\left(\dfrac{1}{m}\right) = \cdots = f_m\left(\dfrac{1}{m}\right)$, while $f_{m+1}\left(\dfrac{1}{m}\right) \neq 0$.

Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. Thus by Problem (14), U is infinite-dim.

---

17 *Suppose* $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ *such that* $p_k(2) = 0$ *for each* $k \in \{0, \dots, m\}$.
Prove that $(p_0, p_1, \dots, p_m)$ *is not linely inde in* $\mathcal{P}_m(\mathbf{F})$.

Solution:

Suppose $(p_0, p_1, \dots, p_m)$ is linely inde. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z$ $\forall z \in \mathbf{F}$.
But $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \cdots + a_m p_m(z)$, for if not, let $z = 2$, contradicts. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has length $(m+1)$.
Hence $(p_0, p_1, \dots, p_m)$ is linely depe in $\mathcal{P}_m(\mathbf{F})$.
For if not, because $(1, z, \dots, z^m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$,
thus by [2.23] trivially, $(p_0, p_1, \dots, p_m)$ spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. □

Or. Note that $\mathcal{P}_m(\mathbf{F}) = \operatorname{span} \underbrace{(1, z, \dots, z^m)}_{\text{of length } (m+1)}$ and then $(p_0, p_1, \dots, p_m, x)$ of length $(m+2)$ is linely dep.
(See the above) Now $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$ and hence $(p_0, p_1, \dots, p_m)$ is linely dep. □

Ended

2·B

7 *Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subsp of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then (v_1, v_2) is a basis of U.*

SOLUTION: A counterexample:

Let $V = \mathbf{R}^4$ and e_j be the j^{th} standard basis.

Let $v_1 = e_1, v_2 = e_2, v_3 = e_3 + e_4, v_4 = e_4$. Then $(v_1, \dots, v_4)$ is a basis of $\mathbf{R}^4$.

Let $U = \operatorname{span}(e_1, e_2, e_3) = \operatorname{span}(v_1, v_2, v_3 - v_4)$. Then $v_3 \notin U$ and (v_1, v_2) is not a basis of U. ☐

• **NOTE FOR "$\complement_V U \cap \{0\}$":**

"$\complement_V U \cap \{0\}$" is supposed to be a subsp W such that $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cap \{0\} \\ u \pm w \in \complement_V U \cap \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cap \{0\}$. Contradicts.

To fix this, denote the set $\{W_1, W_2 \dots\}$ by $\mathcal{S}_V U$, where for each W_i, $V = U \oplus W_i$. See also in (1.C.23).

---

1 *Find all vecsps that have exactly one basis.*

SOLUTION: The trivial vecsp $\{0\}$ will do. Indeed, the only basis of $\{0\}$ is the empty list.

Now consider a field containing only the add identity 0 and the multi identity 1,

and we specify that $1 + 1 = 0$. Hence the vecsp $\{0, 1\}$ will do, the list (1) will be the unique basis.

Are there other vecsps? Suppose so.

(I) Consider $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Let $(v_1, \dots, v_m)$ be a basis of $V \neq \{0\}$.

While there are infinitely many bases distinct from this one. Hence we fail.

(II) Consider other $\mathbf{F}$. Note that a field contains at least 0 and 1

By ***some theories or facts*** given in the course of Elementary Abstract Algebra, we fail. ☐

• *Suppose $(v_1, \dots, v_m)$ is a list of vecs in V. For $k \in \{1, \dots, m\}$, let $w_k = v_1 + \cdots + v_k$. Show that $[P]$ $(v_1, \dots, v_m)$ is a basis of $V \iff [Q]$ $(w_1, \dots, w_m)$ is a basis of V.*

SOLUTION: NOTICE that $(u_1, \dots, u_n)$ is a basis of $U \iff \forall u \in U, \exists!\, a_i \in \mathbf{F}, u = a_1 u_1 + \cdots + a_n u_n$.

$P \Rightarrow Q:\ \forall v \in V, \exists!\, a_i \in \mathbf{F},\ v = a_1 v_1 + \cdots + a_m v_m \Rightarrow v = b_1 w_1 + \cdots + b_m v_m,\ \exists!\, b_1 = a_1, b_k = \sum\limits_{j=1}^{k} (-1)^{k-j} a_j.$

$Q \Rightarrow P:\ \forall v \in V, \exists!\, b_i \in \mathbf{F},\ v = b_1 w_1 + \cdots + b_m w_m \Rightarrow v = a_1 v_1 + \cdots + a_m v_m,\ \exists!\, a_k = \sum\limits_{j=1}^{k} b_j.$ ☐

• *Suppose V is finite-dim and U, W are subsps of V such that $V = U + W$. Prove that there exists a basis of V consisting of vecs in $U \cup W$.*

SOLUTION: Let $(u_1, \dots, u_m)$ and $(w_1, \dots, w_n)$ be bases of U and W respectively.

Then $V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.

Hence, by [2.31], we get a basis of V consisting of vecs in U or W. ☐

---

8 *Suppose U and W are subsps of V such that $V = U \oplus W$. Suppose $(u_1, \dots, u_m)$ is a basis of U and $(w_1, \dots, w_n)$ is a basis of W. Prove that $(u_1, \dots, u_m, w_1, \dots, w_n)$ is a basis of V.*

SOLUTION:

$\forall v \in V, \exists!\, u \in U, w \in W, v = u + w = (a_1 u_1 + \cdots + a_m u_m) + (b_1 w_1 + \cdots + b_n w_n), \exists!\, a_i, b_i \in \mathbf{F}$

$\Rightarrow (a_1 u_1 + \cdots + a_m u_m) = -(b_1 w_1 + \cdots + b_n w_n) \in U \cap W = \{0\}$. Thus $a_1 = \cdots = a_m = b_1 = \cdots = b_n$. ☐

• **NOTE FOR** *linely inde sequence and* [2.34]*:*
"$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expression.
If we allow using "infinite list", then we must guarantee that $(v_1, \dots, v_n, \dots)$ is a spanning "list" such that for all $v \in V$, there exists a smallest positive integer n such that $v = a_1 v_1 + \cdots + a_n v_n$,
The key point is, how can we guarantee that such a "list" exists?

ENDED

2·C

1 (COROLLARY for [2.38,39])
Suppose U is a subsp of V such that $\dim V = \dim U$. *Then* $V = U$.

---

9 *Suppose $(v_1, \dots, v_m)$ is linely inde in V and $w \in V$.*
Prove that $\dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$.

SOLUTION: Using the result of Problem (10) and (11) in 2.A.
Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$, for each $i = 1, \dots, m$.
$(v_1, \dots, v_m)$ linely inde $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ linely inde $\Rightarrow (\underbrace{v_2 - v_1, \dots, v_m - v_1}_{\text{of length } (m-1)})$ linely inde.
又 $w \notin \operatorname{span}(v_1, \dots, v_m) \Rightarrow (v_1 + w, \dots, v_m + w)$ is linely inde.
Hence $m \geq \dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$. □

---

10 *Suppose m is a positive integer and $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

SOLUTION:
Using mathematical induction on m.
(i) For p_0, $\deg p_0 = 0 \Rightarrow \operatorname{span}(p_0) = \operatorname{span}(1)$.
(ii) Suppose for $i \geq 1$, $\operatorname{span}(p_0, p_1, \dots, p_i) = \operatorname{span}(1, x, \dots, x^i)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_i, p_{i+1}) \subseteq \operatorname{span}(1, x, \dots, x^i, x^{i+1})$.
又$\deg p_{i+1} = i + 1$, $p_{i+1}(x) = a_{i+1} x^{i+1} + r_{i+1}(x)$; $a_{i+1} \neq 0$, $\deg r_{i+1} \leq i$.
$$\Rightarrow x^{i+1} = \frac{1}{a_{i+1}} (p_{i+1}(x) - r_{i+1}(x)) \in \operatorname{span}(1, x, \dots, x^i, p_{i+1}) = \operatorname{span}(p_0, p_1, \dots, p_i, p_{i+1}).$$
$\therefore x^{i+1} \in \operatorname{span}(p_0, p_1, \dots, p_i, p_{i+1}) \Rightarrow \operatorname{span}(1, x, \dots, x^i, x^{i+1}) \subseteq \operatorname{span}(p_0, p_1, \dots, p_i, p_{i+1})$.
Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m)$. □

OR. 用比较系数法. Denote the coefficient of x^i in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_i(p)$.
Suppose $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.
We use induction on m to show that $a_m = \cdots = a_0 = 0$.
(i) $k = m$, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m, \xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.
Now $L = a_{m-1} p_{m-1}(x) + \cdots + a_0 p_0(x)$.
(ii) $1 \leq k \leq m$, $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k, \xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.
Now $L = a_{k-1} p_{k-1}(x) + \cdots + a_0 p_0(x)$. □

- (4E 2.C.10) *Suppose m is a positive integer. For $0 \leq k \leq m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show that $(p_0, \dots, p_m)$ is a basis of $\mathcal{P}(\mathbf{F})$.*

  The basis in this exercise leads to what are called Bernstein polynomials. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on $[0,1]$.

SOLUTION: Using mathematical induction.

(i) $k = 0, 1, 2$, $p_m(x) = x^m$, $p_{m-1}(x) = x^{m-1} - x^m$, $p_{m-2}(x) = x^{m-2} + x^m - 2x^{m-1}$.

(ii) $k \geq 2$. Suppose for $p_{m-k}(x)$, $\exists\,!\, a_i \in \mathbf{F}$, $x^{m-k} = p_{m-k}(x) + a_m x^m + \cdots + a_{m-k+1}x^{m-k+1}$.

Then for $p_{m-k-1}(x)$, $\exists\,!\, c_i \in \mathbf{F}$,

$$x^{m-k-1} = p_{m-k-1}(x) + C_{k+1}^{1}(-1)^2 x^{m-k} + \cdots + C_{k+1}^{k}(-1)^{k+1}x^{m-1} + (-1)^{k-2}x^m$$

$$\Rightarrow c_{m-i} = C_{k+1}^{k+1-i}(-1)^{k-i}.$$

Thus for each x^i, $\exists\,!\, b_i \in \mathbf{F}, x^i = b_m p_m(x) + \cdots + b_{m-i}p_{m-i}(x)$

$$\Rightarrow \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}\underbrace{(p_m, \dots, p_1, p_0)}_{\text{Basis}}.$$ □

For any $m, k \in \mathbf{N}^+$ such that $k \leq m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k(1-x)^{m-k}$.

Define the statement $S(m)$ by $S(m)$: $\underbrace{(p_{0,m}, \dots, p_{m,m})}_{\dim \mathcal{P}_m(\mathbf{F}) = m+1}$ is linely inde (and therefore is a basis).

We use induction on to show that $S(m)$ holds for all $m \in \mathbf{N}^+$.

(i) $m = 1$. Suppose $a_0(1-x) + a_1 x = 0, \forall x \in \mathbf{F}$. Then $\begin{cases} x = 0 \Rightarrow a_0 = 0; \\ x = 1 \Rightarrow a_1. \end{cases}$

$m = 2$. Suppose $a_0(1-x)^2 + a_1(1-x)x + a_2x^2, \forall x \in \mathbf{F}$. Then $\begin{cases} x = 0 \Rightarrow a_0 + a_1 = 0; \\ x = 1 \Rightarrow a_2 = 0; \\ x = 2 \Rightarrow a_0 + 2a_1 = 0. \end{cases}$

(ii) $2 \leq m$. Assume that $S(m)$ holds.

Suppose $\sum\limits_{k=0}^{m+2} a_k p_{k,m+2}(x) = \sum\limits_{k=0}^{m+2} a_k x^k (1-x)^{m+2-k} = 0, \forall x \in \mathbf{F}$.

While $x = 0 \Rightarrow a_0 = 0$; $x = 1 \Rightarrow a_{m+2} = 0$. Then $\sum\limits_{k=1}^{m+1} a_k x^k (1-x)^{m+2-k} = 0$;

And note that $\sum\limits_{k=1}^{m+1} a_k x^k (1-x)^{m+2-k}$

$$= x(1-x)\sum_{k=1}^{m+1} a_k x^{k-1}(1-x)^{m+1-k}$$

$$= x(1-x)\sum_{k=0}^{m} a_{k+1} x^k (1-x)^{m-k} = x(1-x)\sum_{k=0}^{m} a_{k+1} p_{k,m}(x).$$

Hence $x(1-x)\sum_{k=0}^{m} a_{k+1}p_{k,m}(x) = 0, \forall x \in \mathbf{F} \Rightarrow \sum_{k=0}^{m} a_{k+1}p_{k,m}(x) = 0, \forall x \in \mathbf{F}\backslash\{0,1\}$.

Because $\sum_{k=0}^{m} a_{k+1}p_{k,m}(x)$ has infinitely many zeros. We have $\sum_{k=0}^{m} a_{k+1}p_{k,m}(x) = 0, \forall x \in \mathbf{F}$.

By assumption, $a_1 = \cdots = a_m = 0$, while $a_0 = a_{m+2} = 0$,

and also $a_{m+1} = 0$ (because $\sum_{k=0}^{m} a_{k+1}p_{k,m}(x) = a_{m+1}p_{m,m}(x) = a_{m+1}x^m = 0, \forall x \in \mathbf{F}$.)

Thus $(p_{0,m+2}, \dots, p_{m+2,m+2})$ is linely inde and $S(m+2)$ holds.

Since $S(m) \Rightarrow S(m+2)$ for all $m \in \mathbf{N}^+$. We have $\begin{cases} S(1) \Rightarrow S(3) \Rightarrow \cdots \Rightarrow S(2k+1) \Rightarrow \cdots; \\ S(2) \Rightarrow S(4) \Rightarrow \cdots \Rightarrow S(2k) \Rightarrow \cdots. \end{cases}$

Hence $S(m)$ holds for all $m \in \mathbf{N}^+$. □

7 (a) *Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.*
(b) *Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.*
(c) *Find a subsp W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.*

SOLUTION: Suppose $p(z) = az^4 + bz^3 + cz^2 + dz + e$ such that $p(2) = p(5) = p(6)$.

$$\text{Then}\left|\begin{array}{r} p(2) = 16a + 8b + 4c + 2d + e \ \ \text{(I)} \\ p(5) = 625a + 125b + 25c + 5d + e \ \ \text{(II)} \\ p(6) = 1296a + 216b + 36c + 6d + e \ \ \text{(III)} \end{array}\right\} \Rightarrow \begin{cases} \text{(II)} - \text{(I)} = 0 \\ \text{(III)} - \text{(II)} = 0 \\ \text{(III)} - \text{(I)} = 0 \end{cases}$$

You don't have to compute to know that the dimension of the set of solutions is 3.
(Because $\nexists p \in \mathcal{P}_2(\mathbf{F})$ with $1 \leq \deg p \leq 2, p(2) = p(5) = p(6)$.)
(a) A basis: $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.
(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.
(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. □

• **TIPS:**

(1) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3))$.
(2) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_3) - \dim(V_2 + (V_1 \cap V_3))$.
(3) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim(V_3 + (V_1 \cap V_2))$.

For (1). Because $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim(V_1 + (V_2 \cap V_3))$.
And $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$.

• (4E 2.C.14) *Suppose V is a 10-dim vecsp and V_1, V_2, V_3 are subsps of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*

SOLUTION: By TIPS, $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 - 2\dim V > 0$. □

• (4E 2.C.15) *Suppose V is finite-dim and V_1, V_2, V_3 are subsps of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*

SOLUTION: By TIPS, $\dim(V_1 \cap V_2 \cap V_3) > 2\dim V - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)) \geq 0$. □

• (4E 2.C.16)
Suppose V is finite-dim and U is a subsp of V with $U \neq V$. Let $n = \dim V, m = \dim U$. Prove that there exist $(n-m)$ subsps of V, say $U_1, \dots, U_{n-m}$, each of dimension $(n-1)$, such that $\bigcap_{i=1}^{n-m} U_i = U$.

SOLUTION:

Let $(v_1, \dots, v_m)$ be a basis of U, extend to a basis of V as $(v_1, \dots, v_m, u_1, \dots, v_{n-m})$.
Define $U_i = \operatorname{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m})$ for each i. Then $U \subseteq U_i$ for each i.
And because $\forall v \in \bigcap_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \cdots + b_{n-m} u_{n-m} \in U_i \Rightarrow b_i = 0$ for each $i \Rightarrow v \in U$.
Hence $\bigcap_{i=1}^{n-m} U_i \subseteq U$. □

EXAMPLE: Suppose $\dim V = 6, \dim U = 3$.

$$\underbrace{(\underbrace{v_1, v_2, v_3}_{\text{Basis of U}}, v_4, v_5, v_6)}_{\text{Basis of V}}, \text{define} \left.\begin{array}{l} U_1 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_5, v_6) \\ U_2 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_4, v_6) \\ U_3 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_4, v_5) \end{array}\right\} \Rightarrow \dim U_i = 6 - 1,\ i = \underbrace{1, 2, 3}_{6-3=3}.$$ □

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14 *Suppose that $V_1, \dots, V_m$ are finite-dim subsps of V.*
Prove that $V_1 + \cdots + V_m$ is finite-dim and $\dim(V_1 + \cdots + V_m) \le \dim V_1 + \cdots + \dim V_m$.

SOLUTION:

Choose a basis $\mathcal{E}_i$ of $V_i \Rightarrow V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim V_i = \operatorname{card} \mathcal{E}_i$.
Then $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$.
又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \le \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \le \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.
Thus $\dim(V_1 + \cdots + V_m) \le \dim V_1 + \cdots + \dim V_m$. □

COMMENT: $\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m \iff V_1 + \cdots + V_m$ *is a direct sum.*
For each i, $(V_1 + \cdots + V_i) \cap V_{i+1} = \{0\} \iff V_1 + \cdots + V_m$ is a direct sum
又 $\iff (\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$ for each i 又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$
$\iff \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$
$\iff \dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m$. □

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17 *Suppose V_1, V_2, V_3 are subsps of a finite-dim vecsp, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

SOLUTION:

[*Similar to*] Given three sets A, B and C.
Because $|X \cup Y| = |X| + |Y| - |X \cap Y|$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.
Now $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$.
And $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$.
Hence $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$.

Because $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned}
\dim(V_1 + V_2 + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) && (1)\\
&= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) && (2)\\
&= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) && (3)
\end{aligned}$$

Notice that in general, $(X + Y) \cap Z \ne X \cap Z + Y \cap Z$.
For example, $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

• **COROLLARY:** *Suppose V_1, V_2 and V_3 are finite-dim vecsps, then* $\frac{(1)+(2)+(3)}{3}$:

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.$$

The formula above may seem strange because the right side does not look like an integer. □

ENDED

3·A

• **Tips:** $T : V \to W$ *is linear* $\iff \left| \begin{array}{l} \forall v, u \in V, T(v+u) = Tv + Tu \\ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv) \end{array} \right| \iff T(v + \lambda u) = Tv + \lambda Tu.$

---

3 *Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists\ A_{j,k} \in \mathbf{F}$ such that*

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for any $(x_1, \dots, x_n) \in \mathbf{F}^n$.

Solution:

Let $T(1,0,0,\dots,0,0) = (A_{1,1}, \dots, A_{m,1})$,
$T(0,1,0,\dots,0,0) = (A_{1,2}, \dots, A_{m,2})$,
$\vdots$
$T(0,0,0,\dots,0,1) = (A_{1,n}, \dots, A_{m,n})$.

Note that $(1,0,\dots,0,0), \cdots, (0,0,\dots,0,1)$ is a basis of $\mathbf{F}^n$.
Then by [3.5], we are done. $\square$

---

4 *Suppose $T \in \mathcal{L}(V, W)$ and $(v_1, \dots, v_m)$ is a list of vecs in V such that $(Tv_1, \dots, Tv_m)$ is linely inde in W. Prove that $(v_1, \dots, v_m)$ is linely inde.*

Solution: Suppose $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$. $\square$

---

5 *Prove that $\mathcal{L}(V, W)$ is a vecsp,*

Solution: Note that $\mathcal{L}(V, W)$ is a subsp of W^V. $\square$

---

7 *Show that every linear map from a one-dim vecsp to itself is multi by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.*

Solution:

Let u be a nonzero vec in $V \Rightarrow V = \operatorname{span}(u)$.
Because $Tu \in V \Rightarrow Tu = \lambda u$ for some λ.
Suppose $v \in V \Rightarrow v = au,\ \exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. $\square$

---

8 *Give an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that $\varphi(av) = a\varphi(v)$ for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.*

Solution:

Define $T(x,y) = \begin{cases} x + y, \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, \quad \text{otherwise.} \end{cases}$ Or. Define $T(x,y) = \sqrt[3]{(x^3 + y^3)}$. $\square$

---

9 *Give an example of a function $\varphi : \mathbf{C} \to \mathbf{C}$ such that $\varphi(w + z) = \varphi(w) + \varphi(z)$ for all $w, z \in \mathbf{C}$ but φ is not linear.*
(Here $\mathbf{C}$ is thought of as a complex vecsp.)

Solution:

Suppose $V_{\mathbf{C}}$ is the complexification of a vecsp V. Suppose $\varphi : V_{\mathbf{C}} \to V_{\mathbf{C}}$.
Define $\varphi(u + \mathrm{i}v) = u = \operatorname{Re}(u + \mathrm{i}v)$
Or. Define $\varphi(u + \mathrm{i}v) = v = \operatorname{Im}(u + \mathrm{i}v)$. $\square$

• *Prove or give a counterexample:*
If $q \in \mathcal{P}(\mathbf{R})$ and $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by $Tp = q \circ p$, then T is linear.

Solution: Because in general, $q\,(p_1(x) + \lambda p_2(x)) \neq (q \circ p_1)(x) + \lambda(q \circ p_2)(x)$. $\square$

• OR(3.D.16) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Suppose $ST = TS$ for every $S \in \mathcal{L}(V)$. Prove that T is a scalar multi of the identity.

SOLUTION:

If $V = \{0\}$, then we are done. Now suppose $V \neq \{0\}$.

Assume that (v, Tv) is linely depe for every $v \in V$, then by (2.A.2.(b)), $Tv = \lambda_v v$ for some $\lambda_v \in \mathbf{F}$.

To prove that λ_v is independent of v

(in other words, for any two distinct nonzero vecs v and w in V, we have $\lambda_v \neq \lambda_w$), we discuss in two cases:

$$\left.\begin{array}{l} (-) \text{ If } (v,w) \text{ is linely inde, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = a_v v + a_w w \\ \qquad\qquad\qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=) \text{ Otherwise, suppose } w = cv,\ a_w w = Tw = cTv = ca_v v = a_v w \Rightarrow (a_w - a_v)w \end{array}\right\} \Rightarrow a_w = a_v.$$

Now we prove the assumption by contradiction.

Suppose (v, Tv) is linely inde for every nonzero vec $v \in V$.

Fix one v. Extend to $(v, Tv, u_1, \dots, u_n)$ a basis of V.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Contradicts. ☐

OR. Let $(v_1, \dots, v_m)$ be a basis of V.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$. Let $\lambda = \varphi(Tv_1) \in \mathbf{F}$.

For any $v \in V$, define $S_v \in \mathcal{L}(V)$ by $S_v u = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. ☐

---

10 *Suppose U is a subsp of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$*
(which means that $Su \neq 0$ for some $u \in U$).

Define $T : V \to W$ by $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \backslash U. \end{cases}$ Prove that T is not a linear map on V.

SOLUTION:

Suppose T is a linear map. And $v \in V \backslash U$, $u \in U$ such that $Su \neq 0$.

Then $v + u \in V \backslash U$, (for if not, $v = (v + u) - u \in U$) while $T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$.

Hence we get a contradiction. ☐

---

11 *Suppose V is finite-dim. Prove that every linear map on a subsp of V can be extended to a linear map on V. In other words, show that if U is a subsp of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.*

SOLUTION:

Define $T \in \mathcal{L}(V, W)$ by $T(a_1u_1 + \cdots + a_nu_n + a_{n+1}u_{n+1} + \cdots + a_mu_m) = a_1Su_1 + \cdots + a_nSu_n$.

Where we let $(u_1, \dots, u_n)$ be a basis of U, extend to a basis of V as $(u_1, \dots, u_n, \dots, u_m)$. ☐

---

12 *Suppose V is finite-dim with $\dim V > 0$, and W is infinite-dim.*
Prove that $\mathcal{L}(V, W)$ is infinite-dim.

SOLUTION:

Let $(v_1, \dots, v_n)$ be a basis of V. Let $(w_1, \dots, w_m)$ be linely inde in W for any $m \in \mathbf{N}^+$.

Define $T_{x,y} \in \mathcal{L}(V, W)$ by $T_{x,y}(v_z) = \delta_{zy}w_y$, $\forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{zy} = \begin{cases} 0, & z \neq y, \\ 1, & z = y. \end{cases}$

Suppose $a_1T_{x,1} + \cdots + a_mT_{x,m} = 0$. Then $(a_1T_{x,1} + \cdots + a_mT_{x,m})(v_x) = 0 = a_1w_1 + \cdots + a_mw_m$.

$\Rightarrow a_1 = \cdots = a_m = 0$. 又 m arbitrary.

Thus $(T_{x,1}, \ldots, T_{x,m})$ is a linely inde list in $\mathcal{L}(V, W)$ for any x and length m. Hence by (2.A.14). $\square$

---

13 *Suppose $(v_1, \ldots, v_m)$ is a linely depe list of vecs in V.*
Suppose also that $W \neq \{0\}$. Prove that there exist $(w_1, \ldots, w_m) \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

SOLUTION:

We prove by contradiction. By linear dependence lemma, $\exists\, j \in \{1, \ldots, m\}$ such that $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$.
Fix j. Let $w_j \neq 0$, while $w_1 = \cdots = w_{j-1} = w_{j+1} = w_m = 0$.
Define T by $Tv_k = w_k$ for all k. Suppose $a_1 v_1 + \cdots + a_m v_m = 0$ (where $a_j \neq 0$).
Then $T(a_1 v_1 + \cdots + a_m v_m) = 0 = a_1 w_1 + \cdots + a_m w_m = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. $\square$

OR. We prove the contrapositive:
Suppose for any list $(w_1, \ldots, w_m) \in W$, $\exists\, T \in \mathcal{L}(V, W), Tv_k = w_k$ for each w_k.
(We need to) Prove that $(v_1, \ldots, v_n)$ is linely inde.

Suppose $\exists\, a_i \in \mathbf{F}, a_1 v_1 + \cdots + a_n v_n = 0$. Choose a nonzero $w \in W$.
By assumption, for the list $(\overline{a_1} w, \ldots, \overline{a_m} w)$, $\exists\, T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$ for each v_k.
$0 = T(\sum_{k=1}^{m} a_k v_k) = \sum_{k=1}^{m} a_k T v_k = \sum_{k=1}^{m} a_k \overline{a_k} w = (\sum_{k=1}^{m} |a_k|^2) w$. Hence $\sum_{k=1}^{m} |a_k|^2 = 0 \Rightarrow a_k = 0$. $\square$

• *(4E 3.A.16)*
Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}$, $\forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION:

Let $(v_1, \ldots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.
Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \backslash \{0\}$.
Suppose $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \cdots + a_n v_n$, where $a_k \neq 0$.
Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y}(v_x) = v_y, R_{x,y}(v_z) = 0\,(\,z \neq x\,)$. Then for any $x, y \in \mathbf{N}^+$,
$(R_{k,y} S)(v_i) = a_k v_y \Rightarrow \left((R_{k,y} S) \circ R_{x,i}\right)(v_x) = a_k v_y,\ \left((R_{k,y} S) \circ R_{x,i}\right)(v_z) = 0\,(\,z \neq x\,)$.
Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Denote by $T_{x,y}$.
Getting $(\frac{1}{a_k} T_{1,1} + \cdots + \frac{1}{a_k} T_{n,n}) v_j = v_j \Rightarrow \sum_{r=1}^{n} \frac{1}{a_k} T_{r,r} = I$.
又 By assumption, $T_{x,y} \in \mathcal{E} \Rightarrow I \in \mathcal{E}$.
Hence for any $T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. $\square$

ENDED

3·B

2 *Suppose $S, T \in \mathcal{L}(V)$ are such that* $\operatorname{range} S \subseteq \operatorname{null} T$. *Prove that* $(ST)^2 = 0$.

SOLUTION: $TS = 0 \Rightarrow STST = (ST)^2 = 0$. $\square$

---

3 *Suppose $(v_1, \ldots, v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m$.*
(a) *The surj of T corresponds to $(v_1, \ldots, v_m)$ spanning V.*
(b) *The inje of T corresponds to $(v_1, \ldots, v_m)$ being linely inde.*

7 *Suppose V is finite-dim with $2 \le \dim V$ and also $\dim V \le \dim W$, if W is finite-dim. Show that $U = \{ T \in \mathcal{L}(V,W) : \operatorname{null} T \ne \{0\}\}$ is not a subsp of $\mathcal{L}(V,W)$.*

Solution:

Let $(v_1, \dots, v_n)$ be a basis of V, $(w_1, \dots, w_m)$ be linely inde in W.
(Let $\dim W = m$, if W is finite, otherwise, let $m \in \{n, n+1, \dots\}$; $2 \le n \le m$).
Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 : \quad v_1 \mapsto 0, \qquad v_2 \mapsto w_2, \qquad v_i \mapsto w_i.$
Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 : \quad v_1 \mapsto w_1, \qquad v_2 \mapsto 0, \qquad v_i \mapsto w_i, \qquad i = 3, \dots, n.$
Thus $T_1 + T_2 \notin U$. $\square$

Comment: If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V,W), T$ is inje. Hence $U = \emptyset$.
If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0 v_0 = 0$.
If V is infinite-dim, the result is true as well.

---

8 *Suppose W is finite-dim with $\dim W \ge 2$ and also $\dim V \ge \dim W$, if V is finite-dim. Show that $U = \{ T \in \mathcal{L}(V,W) : \operatorname{range} T \ne W\}$ is not a subsp of $\mathcal{L}(V,W)$.*

Solution:

Let $(v_1, \dots, v_n)$ be linely inde in V, $(w_1, \dots, w_m)$ be a basis of W.
(Let $n = \dim V$, if V is finite, otherwise we choose $n \in \{m, m+1, \dots\}$; $2 \le m \le n$).
Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 : \quad v_1 \mapsto 0, \qquad v_2 \mapsto w_2, \qquad v_j \mapsto w_j, \qquad v_{m+i} \mapsto 0.$
Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 : \quad v_1 \mapsto w_1, \qquad v_2 \mapsto 0, \qquad v_j \mapsto w_j, \qquad v_{m+i} \mapsto 0.$
For each $j = 2, \dots, m;\ i = 1, \dots, n-m$, if V is finite, otherwise let $i \in \mathbf{N}^+$.
Thus $T_1 + T_2 \notin U$. $\square$

Comment: If $\dim W = 0$, then $W = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V,W), T$ is surj. Hence $U = \emptyset$.
If $\dim W = 1$, then $W = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0 v_0 = 0$.
If W is infinite-dim, the result is true as well.

---

9 *Suppose $T \in \mathcal{L}(V,W)$ is inje and $(v_1, \dots, v_n)$ is linely inde in V. Prove that $(Tv_1, \dots, Tv_n)$ is linely inde in W.*

Solution:

$a_1 Tv_1 + \dots + a_n Tv_n = 0 = T(\sum_{i=1}^{n} a_i v_i) \iff \sum_{i=1}^{n} a_i v_i = 0 \iff a_1 = \dots = a_n = 0.$ $\square$

---

10 *Suppose $(v_1, \dots, v_n)$ spans V and $T \in \mathcal{L}(V,W)$. Show that $(Tv_1, \dots, Tv_n)$ spans* $\operatorname{range} T$.

Solution:

(a) $\operatorname{range} T = \{Tv : v \in V\} = \{Tv : v \in \operatorname{span}(v_1, \dots, v_n)\} \Rightarrow Tv_1, \dots, Tv_n \in \operatorname{range} T \Rightarrow$ By [2.7].
Or. $\operatorname{span}(Tv_1, \dots, Tv_n) \ni a_1 Tv_1 + \dots + a_n Tv_n = T(a_1 v_1 + \dots + a_n v_n) \in \operatorname{range} T$.
(b) $\forall\, w \in \operatorname{range} T$, $\exists\, v \in V, w = Tv$. ($\exists\, a_i \in \mathbf{F}, v = a_1 v_1 + \dots + a_n v_n$) $\Rightarrow w = a_1 Tv_1 + \dots + a_n Tv_n \Rightarrow \square$

---

11 *Suppose $S_1, \dots, S_n$ are inje linear maps and $S_1 S_2 \dots S_n$ makes sence. Prove that $S_1 S_2 \dots S_n$ is inje.*

Solution: $S_1 S_2 \dots S_n(v) = 0 \iff S_2 S_3 \dots S_n(v) = 0 \iff \dots \iff S_n(v) = 0 \iff v = 0.$ $\square$

---

12 *Suppose that V is finite-dim and that $T \in \mathcal{L}(V,W)$. Prove that there exists a subsp U of V such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{Tu : u \in U\}$.*

Solution:

By [2.34], there exists a subsp U of V such that $V = U \oplus \operatorname{null} T$.

$\forall v \in V,\ \exists!\ w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w+u) = Tu \in \{Tu : u \in U\} \Rightarrow \square$

Comment: V can be infinite-dim. See the above of [2.34].

---

16 *Suppose there exists a linear map on V whose null space and range are both finite-dim. Prove that V is finite-dim.*

Solution:

Denote the linear map by T. Let $(Tv_1, \dots, Tv_n)$ be a basis of $\operatorname{range} T$, $(u_1, \dots, u_m)$ be a basis of $\operatorname{null} T$.

Then for all $v \in V, T(\underbrace{v - a_1 v_1 - \cdots - a_n v_n}_{u \in \operatorname{null} T}) = 0$, where $Tv = a_1 Tv_1 + \cdots + a_n Tv_n$.

$\Rightarrow u = b_1 u_1 + \cdots + b_m u_m \Rightarrow v = a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m$.

Getting $V \subseteq \operatorname{span}(v_1, \dots, v_n, u_1, \dots, u_m)$. Thus V is finite-dim. $\square$

---

17 *Suppose V and W are both finite-dim. Prove that there exists an inje $T \in \mathcal{L}(V, W)$ if and only if* $\dim V \leq \dim W$.

Solution:

(a) Suppose there exists an inje T. Then $\dim V = \dim \operatorname{range} T \leq \dim W$.

(b) Suppose $\dim V \leq \dim W$, letting $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$ be bases of V and W respectively.

Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i,\ \ i = 1, \dots, n\ (= \dim V)$. $\square$

---

18 *Suppose V and W are both finite-dim. Prove that there exists a surj $T \in \mathcal{L}(V, W)$ if and only if* $\dim V \geq \dim W$.

Solution:

(a) Suppose there exists a surj T. Then $\dim V = \dim W + \dim \operatorname{null} T \Rightarrow \dim W \leq \dim V$.

(b) Suppose $\dim V \geq \dim W$, letting $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$ be bases of V and W respectively.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_m v_m + a_{m+1} v_{m+1} + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m$. $\square$

---

19 *Suppose V and W are finite-dim and that U is a subsp of V.*
Prove that $\exists\ T \in \mathcal{L}(V, W),\ \operatorname{null} T = U \iff \dim U \geq \dim V - \dim W$.

Solution:

(a) Suppose $\exists\ T \in \mathcal{L}(V, W),\ \operatorname{null} T = U$. Then $\dim \operatorname{null} T = \dim U \geq \dim V - \dim W$.

(b) Suppose $\underbrace{\dim U}_{m} \geq \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_{p}\ (\Rightarrow \dim W = p \geq n = \dim V - \dim U)$.

Let $(u_1, \dots, u_m)$ be a basis of U, extend to a basis of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$.

Let $(w_1, \dots, w_p)$ be a basis of W.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m) = a_1 w_1 + \cdots + a_n w_n$. $\square$

• **Tips:** *Suppose $T \in \mathcal{L}(V, W)$ and $R = (Tv_1, \dots, Tv_n)$ is linely inde in* $\operatorname{range} T$.

(*Let* $\dim \operatorname{range} T = n$, *if* $\operatorname{range} T$ *is finite, otherwise let* $n \in \mathbf{N}^+$.)

By (3.A.4), $L = (v_1, \dots, v_n)$ is linely inde in V.

• **New Notation:**

Denote $\mathcal{K}_R$ by $\operatorname{span} L$, *if* $\operatorname{range} T$ *is finite-dim, otherwise, denote it by a vecsp in $\mathcal{S}_V$*$\operatorname{null} T$.

Note that if $\operatorname{range} T$ *is finite-dim, then $\mathcal{K}_{\operatorname{range} T} = \mathcal{K}_R$ for any basis R of* $\operatorname{range} T$.

• **New Theorem:** $\mathcal{K}_R \in \mathcal{S}_V \operatorname{null} T$.

Suppose $\operatorname{range} T$ is finite-dim. Otherwise, we are done immediately.

$$\mathcal{K}_R \oplus \operatorname{null} T = V \Leftarrow \begin{cases} \text{(a) } T\Big(\sum\limits_{i=1}^{n} a_i v_i\Big) = 0 \Rightarrow \sum\limits_{i=1}^{n} a_i T v_i = 0 \Rightarrow a_1 = \cdots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \operatorname{null} T = \{0\}. \\ \text{(b) } \forall v \in V, Tv = \sum\limits_{i=1}^{n} a_i T v_i \Rightarrow Tv - \sum\limits_{i=1}^{n} a_i T v_i = T\Big(v - \sum\limits_{i=1}^{n} a_i v_i\Big) = 0 \\ \qquad \Rightarrow v - \sum\limits_{i=1}^{n} a_i v_i \in \operatorname{null} T \Rightarrow v = \Big(v - \sum\limits_{i=1}^{n} a_i v_i\Big) + \Big(\sum\limits_{i=1}^{n} a_i v_i\Big) \Rightarrow \mathcal{K}_R + \operatorname{null} T = V. \end{cases}$$

• **COMMENT:** $\operatorname{null} T \in \mathcal{S}_V \mathcal{K}_R$.

• (4E 3.B.21) *Suppose V is finite-dim, $T \in \mathcal{L}(V, W)$, and U is a subsp of W.*
Prove that $\mathcal{K}_U = \{ v \in V : Tv \in U \}$ is a subsp of V
and $\dim \mathcal{K}_U = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$.

SOLUTION:

For any $u, w \in \mathcal{K}_U$ and $\lambda \in \mathbf{F}$, $T(u + \lambda w) = Tu + \lambda Tw \in U \Rightarrow \mathcal{K}_U$ is a subsp of V.
Define $S \in \mathcal{L}(\mathcal{K}_U, U)$ as $Rv = Tv$ for all $v \in \mathcal{K}_U$. Hence $\operatorname{range} R = U \cap \operatorname{range} T$.
Suppose $Tv = 0$ for some $v \in V$. 又 $0 \in U \Rightarrow Rv = 0$. Thus $\operatorname{null} T \subseteq \operatorname{null} R$. □

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20 *Suppose $T \in \mathcal{L}(V, W)$. Prove that T is inje $\Longleftrightarrow \exists\ S \in \mathcal{L}(W, V),\ ST = I \in \mathcal{L}(V)$.*

SOLUTION:

(a) Suppose $\exists\ S \in \mathcal{L}(W, V),\ ST = I$. Then if $Tv = 0 \Rightarrow ST(v) = 0 = v$.
(b) Suppose T is inje. Let $R = (Tv_1, \dots, Tv_n)$ be linely inde in $\operatorname{range} T \subseteq W$,
where $n = \dim \operatorname{range} T$ if finite-dim, otherwise $n \in \mathbf{N}^+$.
Then $\mathcal{K}_R \oplus \operatorname{null} T = V$. And supose $U \oplus \operatorname{range} T = W$.
Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$ and $Su = 0,\ u \in U$. Thus $ST = I$. □

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21 *Suppose $T \in \mathcal{L}(V, W)$. Prove that T is surj $\Longleftrightarrow \exists\ S \in \mathcal{L}(W, V),\ TS = I \in \mathcal{L}(W)$.*

SOLUTION:

(a) Suppose $\exists\ S \in \mathcal{L}(W, V),\ TS = I$. Then $\forall\ w \in W, TS(w) = w \in \operatorname{range} T \Rightarrow \operatorname{range} T = W$.
(b) Suppose T is surj. Let $R = (Tv_1, \dots, Tv_n)$ be linely inde in $\operatorname{range} T = W$,
where $n = \dim \operatorname{range} T$ if finite-dim, otherwise $n \in \mathbf{N}^+$.
Then $\mathcal{K}_R \oplus \operatorname{null} T = V$. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$. Then $TS = I$. □

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22 *Suppose U and V are finite-dim vecsps and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$.*
Prove that $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$.

SOLUTION:

Define $R \in \mathcal{L}(\operatorname{null} ST, V)$ by $Ru = Tu$ for all $u \in \operatorname{null} ST \subseteq U$.

$$\left.\begin{array}{r} S(Tu) = 0 = S(Ru) \Rightarrow \operatorname{range} R \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} R \leq \dim \operatorname{null} S \\ Tu = 0 = Ru \Rightarrow \operatorname{null} R \supseteq \operatorname{null} T \Rightarrow \dim \operatorname{null} R = \dim \operatorname{null} T \end{array}\right\} \Rightarrow \square$$

OR. For any $u \in U$, note that $u \in \operatorname{null} ST \Longleftrightarrow S(Tu) = 0 \Longleftrightarrow Tu \in \operatorname{null} S$.
Thus $\operatorname{null} ST = \mathcal{K}_{\operatorname{null} S \cap \operatorname{range} T} = \{u \in U : Tu \in \operatorname{null} S\}$. By Problem (4E 3B.21),
$\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T) \leq \dim \operatorname{null} T + \dim \operatorname{null} S$. □

COROLLARY:

(1) If T is inje, then $\dim \operatorname{null} T = 0 \Rightarrow \dim \operatorname{null} ST \leq \dim \operatorname{null} S$.
(2) If T is surj, then $\operatorname{range} R = \operatorname{null} S \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.
(3) If S is inje, then $\operatorname{range} R = \{0\} \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} R = \dim \operatorname{null} T$.

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23 *Suppose U and V are finite-dim vecsps and $S \in \mathcal{L}(V,W)$ and $T \in \mathcal{L}(U,V)$.*
Prove that $\operatorname{dim}\operatorname{range} ST \leq \min\{\operatorname{dim}\operatorname{range} S, \operatorname{dim}\operatorname{range} T\}$.

SOLUTION:

$\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{span}(Su_1, \dots, Su_{\operatorname{dim}\operatorname{range} T})$,
where $\operatorname{span}(u_1, \dots, u_{\operatorname{dim}\operatorname{range} T}) = \operatorname{range} T$.

$\operatorname{dim}\operatorname{range} ST \leq \operatorname{dim}\operatorname{range} T$ 又 $\operatorname{dim}\operatorname{range} ST \leq \operatorname{dim}\operatorname{range} S \Rightarrow \Box$

OR. Note that $\operatorname{range}(S|_{\operatorname{range} T}) = \operatorname{range} ST$.
Thus $\operatorname{dim}\operatorname{range} ST = \operatorname{dim}\operatorname{range}(S|_{\operatorname{range} T}) = \operatorname{dim}\operatorname{range} T - \operatorname{dim}\operatorname{null}(S|_{\operatorname{range} T}) \leq \operatorname{range} T$. $\Box$

COROLLARY:

(1) If S is inje, then $\operatorname{dim}\operatorname{range} ST = \operatorname{dim}\operatorname{range} T$.
(2) If T is surj, then $\operatorname{dim}\operatorname{range} ST = \operatorname{dim}\operatorname{range} S$.

• (a) *Suppose* $\operatorname{dim} V = 5$ *and* $S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$.
Prove that $\operatorname{dim}\operatorname{range} TS \leq 2$.
(b) *Give an example of* $S, T \in \mathcal{L}(\mathbf{F}^5)$ *with* $ST = 0$ *and* $\operatorname{dim}\operatorname{range} TS = 2$.

SOLUTION:

By Problem (23), $\operatorname{dim}\operatorname{range} TS \leq \min\left\{\overbrace{\operatorname{dim}\operatorname{range} S}^{5-\operatorname{dim}\operatorname{null} T}, \overbrace{\operatorname{dim}\operatorname{range} T}^{5-\operatorname{dim}\operatorname{null} S}\right\}$.

We show that $\operatorname{dim}\operatorname{range} TS \leq 2$ by contradiction. Assume that $\operatorname{dim}\operatorname{range} TS \geq 3$.
Then $\min\{5 - \operatorname{dim}\operatorname{null} T, 5 - \operatorname{dim}\operatorname{null} S\} \geq 3 \Rightarrow \max\{\operatorname{dim}\operatorname{null} T, \operatorname{dim}\operatorname{null} S\} \leq 2$.
又 $\operatorname{dim}\operatorname{null} ST = 5 \leq \operatorname{dim}\operatorname{null} S + \operatorname{dim}\operatorname{null} T \leq 4$. Contradicts. $\Box$

OR. $\left.\begin{array}{l} \operatorname{dim}\operatorname{null} S = 5 - \operatorname{dim}\operatorname{range} S \\ \operatorname{dim}\operatorname{range} TS \leq \operatorname{dim}\operatorname{range} S \end{array}\right\} \Rightarrow \operatorname{dim}\operatorname{null} S \leq 5 - \operatorname{dim}\operatorname{range} TS$.

And $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S \Rightarrow \operatorname{dim}\operatorname{range} TS \leq \operatorname{dim}\operatorname{range} T \leq \operatorname{dim}\operatorname{null} S$.
Thus $\operatorname{dim}\operatorname{range} TS \leq 5 - \operatorname{dim}\operatorname{range} TS \Rightarrow \operatorname{dim}\operatorname{range} TS \leq \frac{5}{2}$. $\Box$

EXAMPLE: Let $(v_1, \dots, v_5)$ be a basis of $\mathbf{F}^5$. Define $S, T \in \mathcal{L}(\mathbf{F}^5)$ by:

$$\begin{array}{llll} T: & v_1 \mapsto 0, & v_2 \mapsto 0, & v_i \mapsto v_i \ ; \\ S: & v_1 \mapsto v_4, & v_2 \mapsto v_5, & v_i \mapsto 0 \ ; \qquad i = 3,4,5. \end{array}$$

• *Suppose* $\operatorname{dim} V = n$ *and* $S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$.
Prove that $\operatorname{dim}\operatorname{range} TS \leq \left\lfloor \frac{n}{2} \right\rfloor$.

SOLUTION:

By Problem (23), $\operatorname{dim}\operatorname{range} TS \leq \min\left\{\overbrace{\operatorname{dim}\operatorname{range} S}^{n-\operatorname{dim}\operatorname{null} T}, \overbrace{\operatorname{dim}\operatorname{range} T}^{n-\operatorname{dim}\operatorname{null} S}\right\}$. We prove by contradiction.

Assume that $\operatorname{dim}\operatorname{range} TS \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Then $\min\{n - \operatorname{dim}\operatorname{null} T, n - \operatorname{dim}\operatorname{null} S\} \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$
$\Rightarrow \max\{\operatorname{dim}\operatorname{null} T, \operatorname{dim}\operatorname{null} S\} \leq n - \left\lfloor \frac{n}{2} \right\rfloor - 1$.

又 $\operatorname{dim}\operatorname{null} ST = n \leq \operatorname{dim}\operatorname{null} S + \operatorname{dim}\operatorname{null} T \leq 2(n - \left\lfloor \frac{n}{2} \right\rfloor - 1)$
$\Rightarrow \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \frac{n}{2}$. Contradicts. Thus $\operatorname{dim}\operatorname{range} TS \leq \left\lfloor \frac{n}{2} \right\rfloor$. $\Box$

OR. $\operatorname{dim}\operatorname{null} S = n - \operatorname{dim}\operatorname{range} S \leq n - \operatorname{dim}\operatorname{range} TS$.

And $ST = 0 \Rightarrow \dim \operatorname{range} TS \leq \dim \operatorname{range} T \leq \dim \operatorname{null} S \leq n - \dim \operatorname{range} TS$

$$\Rightarrow 2 \dim \operatorname{range} TS \leq n \Rightarrow \dim \operatorname{range} TS \leq \frac{n}{2}$$

$$\Rightarrow \dim \operatorname{range} TS \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ (because } \dim \operatorname{range} TS \text{ is an integer).} \square$$

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24 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{null} S \subseteq \operatorname{null} T \iff \exists\, E \in \mathcal{L}(W)$ *such that* $T = ES$.

Solution:

Suppose $\exists\, E \in \mathcal{L}(W)$ such that $T = ES$. Then $\operatorname{null} T = \operatorname{null} ES \supseteq \operatorname{null} S$.

Suppose $\operatorname{null} S \subseteq \operatorname{null} T$. Let $R = (Sv_1, \dots, Sv_n)$ be a basis of $\operatorname{range} S$

$\Rightarrow (v_1, \dots, v_n)$ is linely inde.

Let $\mathcal{K}_R = \operatorname{span}(v_1, \dots, v_n) \Rightarrow V = \mathcal{K}_R \oplus \operatorname{null} S$.

Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i,\ \ Eu = 0;$ for each $i = 1 \dots, n$ and $u \in \operatorname{null} S$.

Hence $\forall v \in V$, $(\exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} S)$, $Tv = a_1 Tv_1 + \dots + a_n Tv_n = E(a_1 Sv_1 + \dots + a_n Sv_n) \Rightarrow T = ES$. $\square$

Or. Extend R to a basis $(Sv_1, \dots, Sv_n, w_1, \dots, w_m)$ of W.

Define $E \in \mathcal{L}(W)$ by $E(Sv_k) = Tv_k,\ \ Ew_j = 0$.

Because $\forall\, v \in V, \exists\, a_i \in \mathbf{F}, Sv = a_1 Sv_1 + \dots + a_n Sv_n$

$$\Rightarrow S\,(v - (a_1 v_1 + \dots + a_n v_n)) = 0$$
$$\Rightarrow v - (a_1 v_1 + \dots + a_n v_n) \in \operatorname{null} S$$
$$\Rightarrow v - (a_1 v_1 + \dots + a_n v_n) \in \operatorname{null} T.$$
$$\Rightarrow T\,(v - (a_1 v_1 + \dots + a_n v_n)) = 0$$

Thus $Tv = a_1 v_1 + \dots + a_n v_n$. Hence $E(Sv) = a_1 E(Sv_1) + \dots + a_n E(Sv_n) = a_1 Tv_1 + \dots + a_n Tv_n = Tv$. $\square$

---

25 *Suppose that V is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{range} S \subseteq \operatorname{range} T \iff \exists\, E \in \mathcal{L}(V)$ *such that* $S = TE$.

Solution:

Suppose $\exists\, E \in \mathcal{L}(V)$ such that $S = TE$. Then $\operatorname{range} S = \operatorname{range} TE \subseteq \operatorname{range} T$.

Suppose $\operatorname{range} S \subseteq \operatorname{range} T$. Let $(v_1, \dots, v_m)$ be a basis of V.

Because $\operatorname{range} S \subseteq \operatorname{range} T \Rightarrow Sv_i \in \operatorname{range} T$ for each i. Suppose $u_i \in V$ for each i such that $Tu_i = Sv_i$.

Thus defining $E \in \mathcal{L}(V)$ by $Ev_i = u_i$ for each $i \Rightarrow S = TE$. $\square$

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26 *Prove that the differentiation map $D \in \mathcal{P}(\mathbf{R})$ is surj.*

Solution:

[*Informal Proof*]

Note that $\deg Dx^n = n - 1$.

Because $\operatorname{span}(Dx, Dx^2, \dots) \subseteq \operatorname{range} D$. 又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, \dots) = \operatorname{span}(1, x, \dots) = \mathcal{P}(\mathbf{R})$. $\square$

[*Proper Proof*]

We will recursively define a sequence of polynomials $(p_k)_{k=0}^{\infty}$ where $Dp_k = x^k$.

Because $\dim Dx = (\deg x) - 1 = 0$, we have $Dx = C \in \mathbf{F}$. Define $p_0 = C^{-1}x$. Then $Dp_0 = C^{-1}Dx = 1$.

Suppose we have defined $p_0, \dots, p_n$ such that $Dp_k = x^k$ for each $k \in \{0, \dots, n\}$.

Because $\deg D(x^{n+2}) = n + 1$, we let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_n x^n + \dots + a_1 x + a_0$, where $a_{n+1} \neq 0$.

Then $a_{n+1}^{-1}D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_nDp_n + \cdots + a_1Dp_1 + a_0Dp_0)$
$\Rightarrow x^{n+1} = D\left(a_{n+1}^{-1}(x^{n+2} - a_np_n - \cdots - a_1p_1 - a_0p_0)\right)$.
Thus defining $p_{n+1} = a_{n+1}^{-1}(x^{n+2} - a_np_n - \cdots - a_1p_1 - a_0p_0)$, we have $Dp_{n+1} = x^{n+1}$.
Hence we get the sequence $(p_k)_{k=0}^{\infty}$ by recursion.
Now it suffices to show that D is surj. Let $p = \sum\limits_{k=0}^{\deg p} a_kx^k \in \mathcal{P}(\mathbf{R})$.
Then $D\left(\sum\limits_{k=0}^{\deg p} a_kp_k\right) = \sum\limits_{k=0}^{\deg p} a_kDp_k = \sum\limits_{k=0}^{\deg p} a_kx^k = p$. $\square$

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27 *Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.*

SOLUTION:

Define $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$.
Note that $\deg Bx^n = n - 1$. Similar to Problem (26), we conclude that B is surj. $\square$

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28 *Suppose $T \in \mathcal{L}(V, W)$ and $(w_1, \ldots, w_m)$ is a basis of* range T. *Prove that $\exists\ \varphi_1, \ldots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ such that for all $v \in V, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.*

SOLUTION:

Suppose $(v_1, \ldots, v_m)$ in V such that $Tv_i = w_i$ for each i.
Then $(v_1, \ldots, v_m)$ is linely inde, extend it to a basis of V as $(v_1, \ldots, v_m, u_1, \ldots, u_n)$.
Note that $\forall v \in V, v = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n,\ \exists!\, a_i, b_i \in \mathbf{F} \Rightarrow Tv = a_1w_1 + \cdots + a_mw_m$.
Define $\varphi_i : V \to \mathbf{F}$ by $\varphi_i(v) = a_iv_i$ for each i. We now check the linearity.
$\forall v, u \in V\ (\ \exists!\, a_i, b_i, c_i, d_i \in \mathbf{F}\ ), \lambda \in \mathbf{F}, \varphi_i(v + \lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda\varphi(u)$. $\square$

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29 *Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})\backslash\{0\}$. Suppose $u \in V\backslash$null φ. Prove that $V =$ null $\varphi \oplus \{au : a \in \mathbf{F}\}$.*

SOLUTION:

(a) $\forall\ v = cu \in \operatorname{null}\varphi \cap \{au : a \in \mathbf{F}\}, \varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$. Hence $\operatorname{null}\varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$.

(b) $\forall\ v \in V, v = \left(v - \dfrac{\varphi(v)}{\varphi(u)}u\right) + \dfrac{\varphi(v)}{\varphi(u)}u. \left|\ \begin{array}{l} v - \dfrac{\varphi(v)}{\varphi(u)}u \in \operatorname{null}\varphi \\ \dfrac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array}\right\} \Rightarrow V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}$. $\square$

This may seems strange. Here we explain why.
$\varphi \neq 0 \Rightarrow \varphi(v_i) = a_i \neq 0$ for each v_i, for some linely inde list $(v_1, \ldots, v_k)$.
Fix one v_k. Then $\varphi\left(v_k - \dfrac{a_k}{a_j}v_j\right) = 0$ for each $j = 1, \ldots, k-1, k+1, \ldots, n$.
Thus span $\left\{v_k - \dfrac{a_k}{a_j}v_j\right\}_{j \neq k} \subseteq \operatorname{null}\varphi$. Hence every vecsp in $\mathcal{S}_V\operatorname{null}\varphi$ is one-dim.

---

30 *Suppose $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$ and* null $\varphi_1 =$ null $\varphi_2 =$ null φ. *Prove that $\exists\, c \in \mathbf{F}, \varphi_1 = c\varphi_2$*

SOLUTION:

If $\operatorname{null}\varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $u \in V/\operatorname{null}\varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$.
By Problem (29), $V = \operatorname{null}\varphi \oplus \operatorname{span}(u)$.
Hence for any $v \in V, v = w + a_vu,\ \exists!\, w \in \operatorname{null}\varphi, a_v \in \mathbf{F}$.
$\varphi_1(v) = a_v\varphi_1(u),\ \ \varphi_2(v) = a_v\varphi_2(u) \Rightarrow a_v = \dfrac{\varphi_1(v)}{\varphi_1(u)} = \dfrac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \dfrac{\varphi_1(u)}{\varphi_2(u)} = \dfrac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}$.
Thus $\varphi_1 = c\varphi_2$. $\square$

• *Suppose V is finite-dim, X is a subsp of V, and Y is a finite-dim subsp of W.*

Prove that if $\dim X + \dim Y = \dim V$, *then* $\exists\, T \in \mathcal{L}(V,W), \operatorname{null} T = X$ *and* $\operatorname{range} T = Y$.

SOLUTION:

Suppose $\dim X + \dim Y = \dim V$. Let $(u_1, \dots, u_n)$ be a basis of X, $R = (w_1, \dots, w_m)$ be a basis of Y.
Extend $(u_1, \dots, u_n)$ to a basis of V as $(u_1, \dots, u_n, v_1, \dots, v_m)$.
Define $T \in \mathcal{L}(V,W)$ by $T(\sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i) = \sum_{i=1}^{m} a_i w_i$.
Now we show that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$
Suppose $v \in V$. Then $\exists\,!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i$.

$$\left.\begin{array}{r} v \in \operatorname{null} T \Rightarrow Tv = 0 \Rightarrow a_1 = \cdots = a_m = 0 \Rightarrow v \in X \\ v \in X \Rightarrow v \in \operatorname{null} T \end{array}\right\} \Rightarrow \operatorname{null} T = X.$$

$$\left.\begin{array}{l} w \in \operatorname{range} T \Rightarrow \exists\; v = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i \in V, Tv = w = \sum_{i=1}^{m} a_i w_i \Rightarrow w \in Y \\ w \in Y \Rightarrow w = a_1 T v_1 + \cdots + a_m T v_m = T(a_1 v_1 + \cdots + a_m v_m) \Rightarrow w \in \operatorname{range} T \end{array}\right\} \Rightarrow \operatorname{range} T = Y. \qquad \square$$

- *Suppose* V *is finite-dim and* $T \in \mathcal{L}(V,W)$. *Let* $(Tv_1, \dots, Tv_n)$ *be a basis of* $\operatorname{range} T$.
  Extend $(v_1, \dots, v_n)$ *to a basis of* V *as* $(v_1, \dots, v_n, u_1, \dots, u_m)$.
  Prove or give a counterexample: $(u_1, \dots, u_m)$ *is a basis of* $\operatorname{null} T$.

SOLUTION: A counterexample:

Suppose $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$. Then $\operatorname{span}(Tv_1, Tv_2, Tv_3) = \operatorname{span}(w_1)$.
Extend (v_i) to (v_1, v_2, v_3) for each i. But none of $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is a basis of $\operatorname{null} T$. $\square$

COMMENT: $(v_2 - v_1, v_3 - v_1), (v_1 - v_2, v_3 - v_2)$ or $(v_1 - v_3, v_2 - v_3)$ are all bases of $\operatorname{null} T$.

- *Suppose* V *is finite-dim and* $T \in \mathcal{L}(V,W)$. *Let* $(u_1, \dots, u_m)$ *be a basis of* $\operatorname{null} T$.
  Extend $(u_1, \dots, u_m)$ *to a basis of* V *as* $(u_1, \dots, u_m, v_1, \dots, v_n)$.
  Prove or give a counterexample: $(Tv_1, \dots, Tv_n)$ *spans* $\operatorname{range} T$.

SOLUTION:

$\forall w \in \operatorname{range} T,\ \exists\, v \in V,\ (\exists\,!\, a_i, b_i \in \mathbf{F}), Tv = T(a_1 v_1 + \cdots + a_n v_n) = w$
$\Rightarrow w \in \operatorname{span}(Tv_1, \dots, Tv_n) \Rightarrow \operatorname{range} T \subseteq \operatorname{span}(Tv_1, \dots, Tv_n)$. $\square$

COMMENT: If T is inje, then $(Tv_1, \dots, Tv_n)$ is a basis of $\operatorname{range} T$.

- OR(5.B.4) *Suppose* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove that* $V = \operatorname{null} P \oplus \operatorname{range} P$.

SOLUTION:

Let $(P^2 v_1, \dots, P^2 v_n)$ be a basis of $\operatorname{range} P^2$. Then $(Pv_1, \dots, Pv_n)$ is linely inde in V.

$$\left.\begin{array}{l} \text{Let } \mathcal{K} = \operatorname{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \operatorname{null} P^2 \\ \text{又 } \mathcal{K} = \operatorname{range} P = \operatorname{range} P^2;\ \operatorname{null} P = \operatorname{null} P^2 \end{array}\right\} \Rightarrow \square$$

OR. (a) Suppose $v \in \operatorname{null} P \cap \operatorname{range} P$.
Then $\exists\, u \in V, v = Pu, Pv = 0 \Rightarrow v = Pu = P^2 u = Pv = 0$. Hence $\operatorname{null} P \cap \operatorname{range} P = \{0\}$.
(b) Note that $v = Pv + (v - Pv)$ and $P^2 v = Pv$ for all $v \in V$.
Then $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$. Hence $V = \operatorname{range} P + \operatorname{null} P$. $\square$

- *Suppose* V *is finite-dim with* $\dim V > 1$. *Show that if* $\varphi : \mathcal{L}(V) \to \mathbf{F}$
  is a linear map such that $\varphi(ST) = \varphi(S) \cdot \varphi(T)$ *for all* $S, T \in \mathcal{L}(V)$, *then* $\varphi = 0$.

SOLUTION: Using notations in (4E 3.A.16).

Suppose $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \dots, n\}, \varphi(R_{i,j}) \neq 0$.
Because $R_{i,j} = R_{x,j} \circ R_{i,x},\ \forall x = 1, \dots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, because $R_{i,x} = R_{y,x} \circ R_{i,y}$, $\forall y = 1, \ldots, n$. Thus $\varphi(R_{y,x}) \neq 0$ for any $x, y = 1, \ldots, n$.

Let $l \neq i, k \neq j$ and then $\varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. □

OR. Note that by (4E 3.A.16), $\exists\, S, T \in \mathcal{L}(V), ST - TS \neq 0$.

Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \operatorname{null}\varphi \neq \{0\}$.

Thus $\forall\, E \in \operatorname{null}\varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \operatorname{null}\varphi$.

Hence $\operatorname{null}\varphi$ is a nonzero two-sided ideal of $\mathcal{L}(V)$. □

- *Suppose that V and W are real vecsps and $T \in \mathcal{L}(V, W)$.*
  Define $T_{\mathrm{C}} : V_{\mathrm{C}} \to W_{\mathrm{C}}$ by $T_{\mathrm{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv$ for all $u, v \in V$.
  (a) *Show that T_{C} is a (complex) linear map from V_{C} to W_{C}.*
  (b) *Show that T_{C} is inje $\iff$ T is inje.*
  (c) *Show that* $\operatorname{range} T_{\mathrm{C}} = W_{\mathrm{C}} \iff \operatorname{range} T = W$.

SOLUTION:

(a) $\forall u_1 + \mathrm{i}v_1, u_2 + \mathrm{i}v_2 \in V_{\mathrm{C}}, \lambda \in \mathbf{F}$,

$$T\left((u_1 + \mathrm{i}v_1) + \lambda(u_2 + \mathrm{i}v_2)\right) = T\left((u_1 + \lambda u_2) + \mathrm{i}(v_1 + \lambda v_2)\right) = T(u_1 + \lambda u_2) + \mathrm{i}T(v_1 + \lambda v_2)$$
$$= Tu_1 + \mathrm{i}Tv_1 + \lambda Tu_2 + \mathrm{i}\lambda Tv_2 = T(u_1 + \mathrm{i}v_1) + \lambda T(u_2 + \mathrm{i}v_2).$$

(b) Suppose T_{C} is inje. Let $T(u) = 0 \Rightarrow T_{\mathrm{C}}(u + \mathrm{i}0) = Tu = 0 \Rightarrow u = 0$.
Suppose T is inje. Let $T_{\mathrm{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + \mathrm{i}v = 0$.
$\Rightarrow$ □

(c) Suppose T_{C} is surj. $\forall w \in W$, $\exists\, u \in V, T(u + \mathrm{i}0) = Tu = w + \mathrm{i}0 = w \Rightarrow$ T is surj.
Suppose T is surj. $\forall w, x \in W$, $\exists\, u, v \in V, Tu = w, Tv = x$
$\Rightarrow \forall w + \mathrm{i}x \in W_{\mathrm{C}}$, $\exists\, u + \mathrm{i}v \in V, T(u + \mathrm{i}v) = w + \mathrm{i}x \Rightarrow T_{\mathrm{C}}$ is surj.
$\Rightarrow$ □

ENDED

3·C

- **NOTE FOR [3.47]:** $LHS = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k} = RHS.$

- **NOTE FOR [3.48]:**

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 1\times 5 + 2\times 8 & 1\times 6 + 2\times 9 & 1\times 7 + 2\times 10 \\ 3\times 5 + 4\times 8 & 3\times 6 + 4\times 9 & 3\times 7 + 4\times 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

- **NOTE FOR [3.49]:**

$$\because [(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$$
$$\therefore (AC)_{\cdot,k} = A_{\cdot,\cdot}C_{\cdot,k} = AC_{\cdot,k}$$

- **EXERCISE 10:**

$$\because [(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$$

$$\therefore (AC)_{j,\cdot} = A_{j,\cdot}C_{\cdot,\cdot} = A_{j,\cdot}C.$$

• *Suppose* $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}$.

(a) For $k = 1, \dots, p$, $\quad (CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum\limits_{r=1}^{c} C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$

(b) For $j = 1, \dots, m$, $\quad (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum\limits_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

Example:

$$P = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} \in \mathbf{F}^{2,3}.$$

$$P_{\cdot,1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 8\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 21 \\ 47 \end{pmatrix};$$

$$P_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix};$$

$$P_{\cdot,3} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 10\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 27 \\ 61 \end{pmatrix};$$

$$P_{1,\cdot} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = 1\begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 2\begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \end{pmatrix};$$

$$P_{2,\cdot} = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = 3\begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 4\begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 47 & 54 & 61 \end{pmatrix};$$

• **Note For [3.52]:** $\quad A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$

$\because (Ac)_{j,1} = \sum\limits_{r=1}^{n} A_{j,r}c_{r,1} = [\sum\limits_{r=1}^{n} (A_{\cdot,r}c_{r,1})]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$

$\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum\limits_{r=1}^{n} A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ $\quad$ Or. By $(Ac)_{\cdot,1} = Ac_{\cdot,1}$ Using (a) above.

• **Exercise 11 :** $\quad a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$

$\because (aC)_{1,k} = \sum\limits_{r=1}^{n} a_{1,r}C_{r,k} = [\sum\limits_{r=1}^{n} a_{1,r}(C_{r,\cdot})]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$

$\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum\limits_{r=1}^{n} a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ $\quad$ Or. By $(aC)_{1,\cdot} = a_{1,\cdot}C$. Using (b) above.

• Column-Row Factorization $\qquad$ (CR Factorization)

Suppose $A \in \mathbf{F}^{m,n}, A \neq 0$. *Let* $S_c = \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}, \dim S_c = c$.

And $S_r = \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) \subseteq \mathbf{F}^{1,n}, \dim S_r = r$.

Prove that $A = CR$. $\exists\, C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,n}$.

Solution: Notice that $A \neq 0 \Rightarrow c, r \geq 1$.

Let $(C_{\cdot,1}, \dots, C_{\cdot,c})$ be a basis of S_c, forming $C \in \mathbf{F}^{m,c}$.

Then for any $A_{\cdot,k}$, $\quad A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$, $\quad \exists!\, R_{1,k}, \dots, R_{c,k} \in \mathbf{F}$.

Hence, by letting $R = \begin{pmatrix} R_{1,1} & \cdots & R_{1,n} \\ \vdots & \ddots & \vdots \\ R_{c,1} & \cdots & R_{c,n} \end{pmatrix}$, we have $A = CR$.

Or. Let $(R_{1,\cdot}, \dots, R_{c,\cdot})$ be a basis of S_r, forming $R \in \mathbf{F}^{c,n}$.

For any $A_{j,\cdot}$, $\quad A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot} = (CR)_{j,\cdot}$, $\quad \exists!\, C_{j,1}, \dots, C_{j,c} \in \mathbf{F}$. Similarly. $\square$

Example:

$$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

(I) Because $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2\begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix}$.
Hence $\dim S_r = 2$. We choose $(A_{1,\cdot}, A_{2,\cdot})$ as the basis.

(II) Because $\begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix} - \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix}$.
Hence $\dim S_c = 2$. We choose $(A_{\cdot,2}, A_{\cdot,3})$ as the basis.

• Column Rank Equals Row Rank (Using the notation above)
For any $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
$\Rightarrow \operatorname{span}(A_{1,\cdot}, \dots, A_{m,\cdot}) = S_r = \operatorname{span}(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leq c = \dim S_c$.
Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leq r = \dim S_r = \dim S_c^t$. $\square$

• *Suppose $T \in \mathcal{L}(V)$, and $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are bases of V.*
Prove that the following are equi.Here $A = \mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.
(a) *T is inje.*
(b) *The cols of $\mathcal{M}(T)$ are linely inde in $\mathbf{F}^{n,1}$.*
(c) *The cols of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.*
(d) *The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.*
(e) *The rows of $\mathcal{M}(T)$ are linely inde in $\mathbf{F}^{1,n}$.*

Solution: T is inje $\iff \dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim \operatorname{range} T$
$\iff (Tu_1, \dots, Tu_n)$ is linely inde in V, and therefore is a basis of V
$\iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n))$ is linely inde, as well as $(A_{\cdot,1}, \dots, A_{\cdot,n})$
$\iff (A_{\cdot,1}, \dots, A_{\cdot,n})$ is a basis of $\mathbf{F}^{n,1}$.
(又 $\dim \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) = \dim \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = n$)
$\iff (A_{1,\cdot}, \dots, A_{n,\cdot})$ is a basis of $\mathbf{F}^{1,n}$. $\square$

• *Suppose A is an m-by-n matrix with $A \neq 0$.*
Prove that the rank of A is $1 \iff \exists\, (c_1, \dots, c_m) \in \mathbf{F}^m$ and $(d_1, \dots, d_n) \in \mathbf{F}^n$
such that $A_{j,k} = c_j \cdot d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$.

Solution: Using the notation in CR Factorization.

(a) Suppose $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix}$. ($\exists\, c_j, d_k \in \mathbf{F}, \forall j, k$)

Then $S_c = \operatorname{span}\left(\begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \dots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right) = \operatorname{span}\left(\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right)$.

Or. $S_r = \operatorname{span}\begin{pmatrix} \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \end{pmatrix}, \\ \begin{pmatrix} c_2 d_1 & \cdots & c_2 d_n \end{pmatrix}, \\ \vdots \\ \begin{pmatrix} c_m d_1 & \cdots & c_m d_n \end{pmatrix} \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} \right)$. Hence the rank of A is 1.

(b) Suppose the rank of A is $\dim S_c = \dim S_r = 1$

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}, \qquad d'_k = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}. \quad (\forall j,k)$

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$. Letting $d_k = d'_k A_{1,1}$. ☐

---

1 *Suppose $T \in \mathcal{L}(V,W)$. Show that with resp to each choice of bases of V and W, the matrix of T has at least* $\dim \operatorname{range} T$ *nonzero entries.*

SOLUTION:

Let $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$ be bases of V and W respectively. We prove by contradiction.

Suppose $A = \mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$ has at most $(\dim \operatorname{range} T - 1)$ nonzero entries.

Then by Pigeon Hole Principle, at least one of $A_{\cdot,k} = 0$.

Thus there are at most $(\dim \operatorname{range} T - 1)$ nonzero vecs in $Tv_1, \dots, Tv_n$.

While $\operatorname{range} T = \operatorname{span}(Tv_1, \dots, Tv_n) \Rightarrow \dim \operatorname{range} T \leq \dim \operatorname{range} T - 1$. We get a contradiction. ☐

---

3 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$.*
Prove that there exist a basis of V and a basis of W such that
[letting $A = \mathcal{M}(T)$ with resp to these bases],
$A_{k,k} = 1, A_{i,j} = 0$, *where* $1 \leq k \leq \dim \operatorname{range} T, i \neq j$.

SOLUTION:

Let $R = (Tv_1, \dots, Tv_n)$ be a basis of $\operatorname{range} T$, extend it to the basis of W as $(Tv_1, \dots, Tv_n, w_1, \dots, w_p)$.

Let $\mathcal{K}_R = \operatorname{span}(v_1, \dots, v_n)$. Let $(u_1, \dots, u_m)$ be a basis of $\operatorname{null} T$.

Then $(v_1, \dots, v_n, u_1, \dots, u_m)$ is the basis of V.

Thus $T(v_k) = Tv_k, T(u_j) = 0 \Rightarrow A_{k,k} = 1, A_{i,j}$ for each $k \in \{1, \dots, \dim \operatorname{range} T\}$ and $j \in \{1, \dots, m\}$. ☐

---

4 *Suppose $(v_1, \dots, v_m)$ is a basis of V and W is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that there exists a basis $(w_1, \dots, w_n)$ of W such that
[letting $A = \mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_n))$], $\quad A_{\cdot,1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

SOLUTION: If $Tv_1 = 0$, then we are done. If not then extend (Tv_1). ☐

---

5 *Suppose $(w_1, \dots, w_n)$ is a basis of W and V is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that there exists a basis $(v_1, \dots, v_m)$ of V such that
[letting $A = \mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_n))$], $A_{1,\cdot} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$.

SOLUTION:

Let $(u_1, \dots, u_m)$ be a basis of V. If $A_{1,\cdot} = 0$, then let $v_i = u_i$ for each $i = 1, \dots, n$, we are done.

Otherwise, $\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \end{pmatrix} \neq 0$, choose one $A_{1,k} \neq 0$.

Let $v_1 = \dfrac{u_k}{A_{1,k}}; \qquad v_j = u_{j-1} - A_{1,j-1} v_1 \quad$ for $j = 2, \dots, k$;

$\qquad v_i = u_i - A_{1,i} v_1 \qquad$ for $i = k+1, \dots, n$. ☐

---

6 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$. Prove that* $\dim \operatorname{range} T = 1$ *if and only if there exist a basis of V and a basis of W such that with resp to these bases, all entries of $A = \mathcal{M}(T)$ equal 1.*

SOLUTION: Denote the bases of V and W by $B_V = (v_1, \dots, v_n)$ and $B_W = (w_1, \dots, w_m)$ respectively.

(a) Suppose B_V, B_W are the bases such that all entries of A equal 1.

Then $Tv_i = w_1 + \cdots + w_m$ for all $i = 1, \ldots, n$. Hence $\dim \operatorname{range} T = 1$.

(b) Suppose $\dim \operatorname{range} T = 1$. Then $\dim \operatorname{null} T = \dim V - 1$.

Let $(u_2, \ldots, u_n)$ be a basis of $\operatorname{null} T$. Extend it to a basis of V as $(u_1, u_2, \ldots, u_n)$.

Let $w_1 = Tv_1 - w_2 - \cdots - w_m$. Extend it to B_W the basis of W.

Let $v_1 = u_1, v_i = u_1 + u_i$. Extend it to B_V the basis of V. □

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12 *Give an example of 2-by-2 mtcs A and B such that $AB \neq BA$.*

Solution: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

---

13 *Prove that the distr property holds for matrix add and matrix multi.*
In other words, suppose A, B, C, D, E and F are matrices
whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense.
Explain why $AB + AC$ and $DF + EF$ both make sense and prove that.

Solution: Using [3.36], [3.43].

(a) Left distr: Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Because $[A(B+C)]_{j,k} = \sum_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum_{r=1}^{n}(A_{j,r}B_{r,k} + A_{j,r}C_{r,k})$.

Hence we conclude that $A(B+C) = AB + AC$.

Or. Let $(e_1, \ldots, e_M)$ be the standard basis of $\mathbf{F}^M$, where $M = \max\{m, n, p\}$.

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum_{j=1}^{m} A_{j,k}e_j$ for each $k = 1, \ldots, n$. Then $\mathcal{M}(T) = A$.

Similarly, define S, R such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

Thus $T(S+R) = TS + TR$
$$\begin{aligned} &\Rightarrow \mathcal{M}\left(T(S+R)\right) = \mathcal{M}(TS + TR) \\ &\Rightarrow \mathcal{M}(T)\left[\mathcal{M}(S) + \mathcal{M}(R)\right] = \mathcal{M}(T)\mathcal{M}(S) + \mathcal{M}(T)\mathcal{M}(R) \\ &\Rightarrow A(B+C) = AB + AC. \end{aligned}$$

(b) Right distr: Similarly.

Suppose $\mathcal{M}(T) = D, \mathcal{M}(S) = E, \mathcal{M}(R) = F$.

Then $(T+S)R = TR + SR$
$$\begin{aligned} &\Rightarrow \mathcal{M}\left((T+S)R\right) = \mathcal{M}(TR) + \mathcal{M}(SR) \\ &\Rightarrow \left[\mathcal{M}(T) + \mathcal{M}(S)\right]\mathcal{M}(R) = \mathcal{M}(T)\mathcal{M}(R) + \mathcal{M}(S)\mathcal{M}(R) \\ &\Rightarrow (D+E)F = DF + EF. \end{aligned}$$
□

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14 *Prove that matrix multi is associ. In other words,*
suppose A, B and C are mtcs whose sizes are such that $(AB)C$ makes sense.
Explain why $A(BC)$ makes sense and prove that $(AB)C = A(BC)$.

Solution:

Because $[(AB)C]_{j,k} = (AB)_{j,\cdot}C_{\cdot,k} = \sum_{s=1}^{n}(A_{j,s}B_{s,\cdot})C_{\cdot,k} = \sum_{s=1}^{n} A_{j,s}(B_{s,\cdot}C_{\cdot,k}) = \sum_{s=1}^{n} A_{j,s}(BC)_{s,k} = A(BC)_{j,k}$

Hence we conclude that $(AB)C = A(BC)$.

Or. Suppose $A \in \mathbf{F}^{m,n}, B \in \mathbf{F}^{n,p}, C \in \mathbf{F}^{p,s}$.

Let $(e_1, \ldots, e_M)$ be the standard basis of $\mathbf{F}^M$, where $M = \max\{m, n, p, s\}$.

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum_{j=1}^{m} A_{j,k}e_j$ for each $k = 1, \ldots, n$. Then $\mathcal{M}(T) = A$.

Similarly, define S, R such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

Hence $(TS)R = T(SR) \Rightarrow \mathcal{M}\left((TS)R\right) = \mathcal{M}\left(T(SR)\right)$
$$\begin{aligned} &\Rightarrow \left[\mathcal{M}(T)\mathcal{M}(S)\right]\mathcal{M}(R) = \mathcal{M}(T)\left[\mathcal{M}(S)\mathcal{M}(R)\right] \\ &\Rightarrow (AB)C = A(BC). \end{aligned}$$
□

15 *Suppose A is an n-by-n matrix and $1 \leq j,k \leq n$.*
Show that the entry in row j, col k, of A^3
(which is defined to mean AAA) is $\sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.

SOLUTION:

$$(AAA)_{j,k} = (AA)_{j,\cdot}A_{\cdot,k} = \sum_{p=1}^{n}(A_{j,p}A_{p,\cdot})A_{\cdot,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}.$$

OR.
$$\begin{aligned}(AAA)_{j,k} &= \sum_{r=1}^{n}(AA)_{j,r}A_{r,k} = \sum_{r=1}^{n}(\sum_{p=1}^{n} A_{j,p}A_{p,r})A_{r,k} \\ &= \sum_{r=1}^{n}(A_{j,1}A_{1,r}A_{r,k} + \cdots + A_{j,n}A_{n,r}A_{r,k}) \\ &= A_{j,1}\sum_{r=1}^{n}A_{1,r}A_{r,k} + \cdots + A_{j,n}\sum_{r=1}^{n}A_{n,r}A_{r,k} = \sum_{p=1}^{n}\sum_{r=1}^{n}A_{j,p}A_{p,r}A_{r,k}.\square\end{aligned}$$

ENDED

3·D

• *Suppose $T \in \mathcal{L}(V,W)$ is inv. Show that T^{-1} is inv and $(T^{-1})^{-1} = T$.*

SOLUTION: $\left.\begin{aligned} TT^{-1} = (T^{-1})^{-1}T^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = T^{-1}(T^{-1})^{-1} = I \in \mathcal{L}(W)\end{aligned}\right\} \Rightarrow T = (T^{-1})^{-1}$, by the uniqnes of inverse. □

1 *Suppose $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ are both inv linear maps.*
Prove that $ST \in \mathcal{L}(U,W)$ is inv and that $(ST)^{-1} = T^{-1}S^{-1}$.

SOLUTION: $\left.\begin{aligned} (ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(V)\end{aligned}\right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}$, by the uniqnes of inverse.□

9 *Suppose V is finite-dim and $S,T \in \mathcal{L}(V)$.*
Prove that ST is inv $\Longleftrightarrow$ S and T are inv.

SOLUTION:

Suppose S,T are inv. Then $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$. Hence ST is inv.

Suppose ST is inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$.

$\left.\begin{aligned} &Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ &\forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S\end{aligned}\right\} \Rightarrow T$ is inje, S is surj.

Notice that V is finite-dim. Hence S,T are inv. □

OR. Suppose ST is inv but S or T is not inv ($\Rightarrow$ not surj and inje).

If S is not inv then $\dim \operatorname{range} S < \dim V$ and by Problem (23) in (3.B),

$\dim \operatorname{range} ST \leq \dim \operatorname{range} S < \dim V$. Thus ST is not surj. Contradicts.

If T is not inv then $\dim \operatorname{range} T < 0$. Similarly, ST is not surj. Contradicts. □

10 *Suppose V is finite-dim and $S,T \in \mathcal{L}(V)$. Prove that $ST = I \Longleftrightarrow TS = I$.*

SOLUTION:

Suppose $ST = I$. $\left.\begin{aligned} &Tv = 0 \Rightarrow v = STv = 0 \\ &v \in V \Rightarrow v = S(Tv) \in \operatorname{range} S\end{aligned}\right\} \Rightarrow T$ is inje, S is surj.

Notice that V is finite-dim. Thus T,S are inv.

OR. By Problem (9), V is finite-dim and $ST = I$ is inv $\Rightarrow S,T$ are inv.

$S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow S$ is inv.
OR. $ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T$. 又 $S = S \Rightarrow TS = S^{-1}S = I$.

Reversing the roles of S and T, we conclude that $TS = I \Rightarrow ST = I$. □

---

11 *Suppose V is finite-dim and $S, T, U \in \mathcal{L}(V)$ and $STU = I$.*
Show that T is inv and that $T^{-1} = US$.

SOLUTION: Using Problem (9) and (10).
$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I$.
$\Rightarrow U^{-1} = ST, \quad T^{-1} = US, \quad S^{-1} = TU$. □

---

12 *Show that the result in Exercise* 11 *can fail without the hypothesis that V is finite-dim.*

SOLUTION:
Let $V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots), T(a_1, \dots) = (0, a_1, \dots), U = I$. Then $STU = I$ but T^{-1} is not inv.

---

13 *Suppose V is finite-dim and $R, S, T \in \mathcal{L}(V)$ are such that RST is surj.*
Prove that S is inje.

SOLUTION: By Problem (1) and (9), Notice that V is finite-dim. Then RST is inv.
$(RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$.

OR. Let $X = (RST)^{-1}$ | $Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T$ is inje, and therefore is inv.
$\forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R$ is surj, and therefore is inv.

Thus $S = R^{-1}(RST)T^{-1}$ is inv. □

---

15 *Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multi.*
In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists A \in \mathbf{F}^{m,n}, Tx = Ax, \forall x \in \mathbf{F}^{n,1}$.

SOLUTION:
Let $E_i \in \mathbf{F}^{n,1}$ for each $i = 1, \dots, n$ (where $M = \max\{m, n\}$) be such that $(E_i)_{j,1} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$
Then $(E_1, \dots, E_n)$ is linely inde and thus is a basis of $\mathbf{F}^{n,1}$.
Similarly, let $(R_1, \dots, R_m)$ be a basis of $\mathbf{F}^{m,1}$.

Suppose $T(E_i) = A_{1,i}R_1 + \cdots + A_{m,i}R_m$ for each $i = 1, \dots, n$. Hence by letting $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$. □

COMMENT: $\mathcal{M}(T) = A$. Conversely it is true as well.

• OR(10.A.2) *Suppose $A, B \in \mathbf{F}^{n,n}$. Prove that $AB = I \iff BA = I$.*

SOLUTION: Using Problem (10) and (15).
Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Then $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.
Thus $AB = I \Leftrightarrow A(Bx) = x \iff T(Sx) = x \Leftrightarrow TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I$.□

• **NOTE FOR [3.60]:** Suppose $(v_1, \dots, v_n)$ is a basis of V and $(w_1, \dots, w_m)$ is a basis of W.
Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{ix} w_j; \quad \delta_{ix} = \begin{cases} 0, & i \neq x \\ 1, & i = x \end{cases}$ **COROLLARY:** $E_{l,k}E_{i,j} = \delta_{jl}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. $(\mathcal{E}^{(j,i)})_{l,k} = \begin{cases} 0, & i \neq k \text{ or } j \neq l \\ 1, & i = k \text{ and } j = l \end{cases}$

Because $\mathcal{L}(V,W)$ and $\mathbf{F}^{m,n}$ are iso. And $T=\mathcal{M}^{-1}\mathcal{M}(T), E_{i,j}=\mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

Hence $\forall T\in\mathcal{L}(V,W),\ \exists!\, A_{i,j}\in\mathbf{F}(\ \forall i\in\{1,\dots,m\}, j\in\{1,\dots,n\}\ ), \mathcal{M}(T)=A=\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n}\end{pmatrix}$.

$$\text{Thus } A=\begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)}+ & \cdots & +A_{1,n}\mathcal{E}^{(1,n)} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}\mathcal{E}^{(m,1)}+ & \cdots & +A_{m,n}\mathcal{E}^{(m,n)}\end{pmatrix} \iff T=\begin{pmatrix} A_{1,1}E_{1,1}+ & \cdots & +A_{1,n}E_{n,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}E_{1,m}+ & \cdots & +A_{m,n}E_{n,m}\end{pmatrix}.$$

$$\therefore \mathcal{L}(V,W)=\operatorname{span}\underbrace{\begin{pmatrix} E_{1,1}, & \cdots & ,E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,m}, & \cdots & ,E_{n,m}\end{pmatrix}}_{B};\quad \mathbf{F}^{m,n}=\operatorname{span}\underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & ,\mathcal{E}^{(1,n)}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ \mathcal{E}^{(m,1)}, & \cdots & ,\mathcal{E}^{(m,n)}\end{pmatrix}}_{B_m}.$$

Hence by [2.42] and [3.61], we conclude that B is a basis of $\mathcal{L}(V,W)$ and that B_M is a basis of $\mathbf{F}^{m,n}$.

∘ *Suppose V is finite-dim and $S\in\mathcal{L}(V)$.*
Define $A\in\mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T)=ST$ for $T\in\mathcal{L}(V)$.
(a) *Show that* $\dim\operatorname{null}\mathcal{A}=(\dim V)(\dim\operatorname{null}S)$.
(b) *Show that* $\dim\operatorname{range}\mathcal{A}=(\dim V)(\dim\operatorname{range}S)$.

SOLUTION:

(a) For all $T\in\mathcal{L}(V)$, $ST=0\iff\operatorname{range}T\subseteq\operatorname{null}S$. Thus $\operatorname{null}\mathcal{A}=\mathcal{L}(V,\operatorname{null}S)$.

(b) For all $R\in\mathcal{L}(V)$, $\operatorname{range}R\subseteq\operatorname{range}S\iff\exists\, T\in\mathcal{L}(V), R=ST$. (By Problem (25) in 3.B)
Thus $\operatorname{range}\mathcal{A}=\mathcal{L}(V,\operatorname{range}S)$. □

OR. Using NOTE FOR[3.60].
Let $(w_1,\dots,w_m)$ be a basis of range S, extend it to a basis of V as $(w_1,\dots,w_m,\dots,w_n)$.
Let $v_i\in V$ such that $Sv_i=w_i$ for $m=1,\dots,m$. Extend $(v_1,\dots,v_m)$ to a basis of V as $(v_1,\dots,v_m,\dots,v_n)$.
Define $E_{i,j}\in\mathcal{L}(V)$ by $E_{i,j}(v_x)=\delta_{ix}w_i$.

$$\text{Thus } S=E_{1,1}+\cdots+E_{m,m};\quad \mathcal{M}(S,(v_1,\dots,v_n),(w_1,\dots,w_n))=\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}.$$

Define $R_{i,j}\in\mathcal{L}(V)$ by $R_{i,j}(w_x)=\delta_{ix}v_i$.
Let $E_{j,k}R_{i,j}=Q_{i,k},\quad R_{j,k}E_{i,j}=G_{i,k}$

$$\text{Because } \forall T\in\mathcal{L}(V),\quad \exists!\, A_{i,j}\in\mathbf{F}(\ \forall i,j=1,\dots,n\ ),\quad T=\begin{pmatrix} A_{1,1}R_{1,1}+ & \cdots & +A_{1,m}R_{m,1}+ & \cdots & +A_{1,n}R_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}R_{1,m}+ & \cdots & +A_{m,m}R_{m,m}+ & \cdots & +A_{m,n}R_{n,m} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{n,1}R_{1,n}+ & \cdots & +A_{n,m}R_{m,n}+ & \cdots & +A_{n,n}R_{n,n}\end{pmatrix}.$$

$$\Rightarrow \mathcal{A}(T)=ST=(\sum_{r=1}^{m}E_{r,r})(\sum_{i=1}^{n}\sum_{j=1}^{n}A_{i,j}R_{j,i})=\sum_{i=1}^{m}\sum_{j=1}^{n}A_{i,j}Q_{j,i}=\begin{pmatrix} A_{1,1}Q_{1,1}+ & \cdots & +A_{1,m}Q_{m,1}+ & \cdots & +A_{1,n}Q_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}Q_{1,m}+ & \cdots & +A_{m,m}Q_{m,m}+ & \cdots & +A_{m,n}Q_{n,m}\end{pmatrix}.$$

Thus null $\mathcal{A}$ = span $\begin{pmatrix} R'_{1,m+1}, & \cdots & ,R'_{n,m+1}, \\ \vdots & \ddots & \vdots \\ R'_{1,n}, & \cdots & ,R_{n,n} \end{pmatrix}$, range $\mathcal{A}$ = span $\begin{pmatrix} Q'_{1,1}, & \cdots & ,Q'_{n,1}, \\ \vdots & \ddots & \vdots \\ Q'_{1,m}, & \cdots & ,Q'_{n,m} \end{pmatrix}$.

Hence (a) $\dim \operatorname{null} \mathcal{A} = n \times (n-m)$; (b) $\dim \operatorname{range} \mathcal{A} = n \times m$. □

COMMENT: Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(T) = TS$ for $T \in \mathcal{L}(V)$.

Similarly, $\mathcal{B}(T) = TS = (\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i})(\sum_{r=1}^{m} E_{r,r}) = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{i,j}G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1}+ & \cdots & +A_{1,m}G_{m,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}G_{1,m}+ & \cdots & +A_{m,m}G_{m,m} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{n,1}G_{1,n}+ & \cdots & +A_{n,m}G_{m,n} \end{pmatrix}$.

• OR(10.A.1) *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V.*
Prove that $\mathcal{M}(T, (v_1, \ldots, v_n))$ is inv $\Longleftrightarrow$ T is inv.

SOLUTION:

Notice that $\mathcal{M}$ is an iso of $\mathcal{L}(V)$ onto $\mathbf{F}^{n,n}$.

(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I$. $\exists!\, S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$

$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$

$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. □

• OR(10.A.4) *Suppose that $(\beta_1, \ldots, \beta_n)$ and $(\alpha_1, \ldots, \alpha_n)$ are bases of V.*
Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1, \ldots, n$.
Prove that $\mathcal{M}(T, (\alpha_1, \ldots, \alpha_n)) = \mathcal{M}(I, (\beta_1, \ldots, \beta_n), (\alpha_1, \ldots, \alpha_n))$.

SOLUTION:

For ease of notation, let $\mathcal{M}(T, \alpha \to \beta) = \mathcal{M}(T, (\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))$, $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(T, (\alpha_1, \ldots, \alpha_n))$

Denote $\mathcal{M}(T, \alpha \to \alpha)$ by A and $\mathcal{M}(I, \beta \to \alpha)$ by B.

$\forall k \in \{1, \ldots, n\}, Iu_k = u_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. □

OR. Note that $\mathcal{M}(T, \alpha \to \beta)$ is the identity matrix.

$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)\underbrace{\mathcal{M}(T, \alpha \to \beta)}_{=\mathcal{M}(I, \beta \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. □

OR. Note that $\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I$.

$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}[\underbrace{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}_{\mathcal{M}(T, \alpha \to \beta)}] = \mathcal{M}(I, \beta \to \alpha)$. □

COMMENT: Denote $\mathcal{M}(T, \beta \to \beta)$ by A'.

$u_k = Iu_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n$, $\forall\, k \in \{1, \ldots, n\}$.

又 $Tu_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = B_{1,k}\beta_1 + \cdots + B_{n,k}\beta_n = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.

OR. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$.

---

16 *Suppose V is finite-dim and $S \in \mathcal{L}(V)$.*
Prove that $\exists\, \lambda \in \mathbf{F}, S = \lambda I \Longleftrightarrow ST = TS$ for every $T \in \mathcal{L}(V)$.

SOLUTION: Using the notation and result in ($\circ$).

Suppose $S = \lambda I$. Then $ST = TS = \lambda T$ for every $T \in \mathcal{L}(V)$. Conversely, if $S = 0$, then we are done.

Suppose $S \neq 0, ST = TS, \forall T \in \mathcal{L}(V)$.

Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, (v_1, \dots, v_1)) = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))$.

Then $\forall k \in \{m+1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $n = \dim V = \dim \operatorname{range} S = m$.

Note that $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$. Thus $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \cdots + a_{n,i}v_n)$. Where:

$a_{i,j} = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n$;

For each j, for all i. Thus $a_{i,i} = a_{k,k} = \lambda, \forall\, k \neq i$.

Hence $w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} = \mathcal{M}(\lambda I, (v_1, \dots, v_n)) \Rightarrow S = \mathcal{M}^{-1}(\mathcal{M}(\lambda I)) = \lambda I$. □

• OR(10.A.3) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that T has the same matrix with resp to every basis of V
if and only if T is a scalar multi of the identity operator.

SOLUTION: [Compare with the first solution of Problem (16) in (3.A)]

Suppose $T = \lambda I$ for some $\lambda \in \mathbf{F}$. Then T has the same matrix with resp to every basis of V.

Conversely, if $T = 0$, then we are done; Suppose $T \neq 0$. And v is a nonzero vec in V.

Assume that (v, Tv) is linely inde.

Extend (v, Tv) to a basis of V as $(v, Tv, u_3, \dots, u_n)$. Let $B = \mathcal{M}(T, (v, Tv, u_3, \dots, u_n))$.

$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \ \forall i \neq 2$.

By assumption, $A = \mathcal{M}(T, (v, w_2, \dots, w_n)) = B$ for any basis $(v, w_2, \dots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0\,(\cdots)$.

$\Rightarrow Tv = w_2$, which is not true if we let $w_2 = u_3, w_3 = Tv, w_j = u_j\ (j = 4, \dots, n)$. Contradicts.

Hence (v, Tv) is linely depe $\Rightarrow \forall v \in V, \exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show that λ_v is independent of v, that is,

to show that for any two nonzero distinct vecs $v, w \in V, \lambda_v = \lambda_w$. Thus $T = \lambda I, \exists \lambda \in \mathbf{F}$.

$$\left.\begin{array}{l} (v,w) \text{ is linely inde} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_{v+w}v + \lambda_{v+w}w \\ \qquad\qquad\qquad\qquad = \lambda_v v + \lambda_w w = Tv + Tw \Rightarrow \lambda_{v+w} = \lambda_v = \lambda_w \\ (v,w) \text{ is linely depe, } w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \Rightarrow \lambda_v = \lambda_w \end{array}\right\} \Rightarrow \square$$

OR. Conversely, denote $\mathcal{M}(T, (u_1, \dots, u_m))$ by A, where the basis $(u_1, \dots, u_m)$ is arbitrary.

Fix one basis $(v_1, \dots, v_m)$ and then $(v_1, \dots, \frac{1}{2}v_k, \dots, v_m)$ is also a basis for any given $k \in \{1, \dots, m\}$.

Fix one k. Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \cdots + A_{k,k}(\frac{1}{2}v_k) + \cdots + A_{m,k}v_m$

$\Rightarrow Tv_k = 2A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + 2A_{m,k}v_m = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{m,k}v_m$.

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1, \dots, m\}$.

Now we show that $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j, k such that $j \neq k$.

Consider the basis $(v'_1, \dots, v'_j, \dots, v'_k, \dots, v'_m)$,

where $v'_j = v_k,\ v_k{}' = v_j$ and $v'_i = v_i$ for all $i \in \{1, \dots, m\} \backslash \{j, k\}$.

Remember that $\mathcal{M}(T, (v'_1, \dots, v'_m)) = \mathcal{M}(T, (v_1, \dots, v_m)) = A$.

Hence $T(v'_k) = A_{1,k}v'_1 + \cdots + A_{k,k}v'_k + \cdots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_j$, while $T(v'_k) = T(v_j) = A_{j,j}v_j$.

Thus $A_{k,k} = A_{j,j}$. □

---

17 *Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Using NOTE FOR[3.60]. Let $(v_1, \dots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.

Then for any $E_{i,j} \in \mathcal{E}$, ($\forall x, y = 1, \dots, n$), by assumption, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.

Again, $E_{y,x'}, E_{y',x} \in \mathcal{E}$ for all $x', y', x, y = 1, \dots, n$. Thus $\mathcal{E} = \mathcal{L}(V)$. $\square$

---

18 *Show that V and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.*

SOLUTION:

Define $\Psi \in \mathcal{L}\big(V, \mathcal{L}(\mathbf{F}, V)\big)$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.

(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is inje.

(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi\left(T(1)\right)$. Hence Ψ is surj. $\square$

OR. Define $\Phi \in \mathcal{L}(\mathcal{L}(\mathbf{F}, V), V)$ by $\Phi(T) = T(1)$.

(a) Suppose $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is inje.

(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surj. $\square$

COMMENT: $\Phi = \Psi^{-1}$.

• *Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3), \forall x \in \mathbf{R}$.*

SOLUTION:

Note that $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$.

Define $T_n : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.

As can be easily checked, T_n is an operator.

Because $\deg(T_n p) = \deg p$. If $T_n(p) = 0 \Rightarrow \deg(T_n p) = -\infty$, then $\deg p = -\infty \Rightarrow p = 0$.

Hence T_n is inje and therefore is surj.

For all $q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $m = 0$; if $q \neq 0$, let $m = \deg q$. We have $q \in \mathcal{P}_m(\mathbf{R})$.

Hence $\exists\, p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$.

---

19 *Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is inje. $\deg Tp \leq \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.*

(a) *Prove that T is surj.*

(b) *Prove that for every nonzero p, $\deg Tp = \deg p$.*

SOLUTION:

(a) T is inje $\iff T|_{\mathcal{P}_n(\mathbf{R})} : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ is inje for any $n \in \mathbf{N}^+$

$\iff T|_{\mathcal{P}_n(\mathbf{R})}$ is surj for any $n \in \mathbf{N}^+ \iff T$ is surj.

(b) Using mathematical induction.

(i) $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$.

$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$.

(ii) Suppose $\deg f = \deg Tf$ for all $f \in \mathcal{P}_n(\mathbf{R})$. Then suppose $\deg g = n + 1, g \in \mathcal{P}_{n+1}(\mathbf{R})$.

Assume that $\deg Tg < \deg g$ ($\Rightarrow \deg Tg \leq n, Tg \in \mathcal{P}_n(\mathbf{R})$).

Then by (a), $\exists f \in \mathcal{P}_n(\mathbf{R}), T(f) = (Tg)$. 又 T is inje $\Rightarrow f = g$.

While $\deg f = \deg Tf = \deg Tg < \deg g$. Contradicts the assumption.

Hence $\deg Tp = \deg p$ for all $p \in \mathcal{P}_{n+1}(\mathbf{R})$.

Thus $\deg Tp = \deg p$ for all $p \in \mathcal{P}(\mathbf{R})$. $\square$

• *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \dots, v_m)$ is a list in V such that $(Tv_1, \dots, Tv_m)$ spans V. Prove that $(v_1, \dots, v_m)$ spans V.*

SOLUTION:

Because $V = \operatorname{span}(Tv_1, \dots, Tv_m) \Rightarrow T$ is surj, 又 V is finite-dim $\Rightarrow T$ is inv $\Rightarrow T^{-1}$ is inv.

$\forall v \in V,\ \exists\, a_i \in \mathbf{F}, v = a_1 Tv_1 + \cdots + a_n Tv_n \Rightarrow T^{-1}v = a_1 v_1 + \cdots + a_n v_n \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \dots, v_n)$. $\square$

OR. Reduce $(Tv_1, \dots, Tv_n)$ to a basis of V as $(Tv_{\alpha_1}, \dots, Tv_{\alpha_m})$, where $m = \dim V$ and $\alpha_i \in \{1, \dots, m\}$.
Then $(v_{\alpha_1}, \dots, v_{\alpha_m})$ is linely inde of length m, therefore is a basis of V, contained in the list $(v_1, \dots, v_m)$.☐

---

2 *Suppose V is finite-dim and* $\dim V > 1$.
Prove that the set of non-inv operators on V is not a subsp of $\mathcal{L}(V)$.

SOLUTION: Denote the set by U.

Suppose $\dim V = n > 1$. Let $(v_1, \dots, v_n)$ be a basis of V.
Define $S, T \in \mathcal{L}(V)$ by $S(a_1v_1 + \cdots + a_nv_n) = a_1v_1$ and $T(a_1v_1 + \cdots + a_nv_n) = a_2v_1 + \cdots + a_nv_n$.
Hence $S + T = I$ is inv.
Thus U is not closed under add and therefore is not a subsp. ☐

COMMENT: If $\dim V = 1$, then $U = \{0\}$ is a subsp of $\mathcal{L}(V)$.

---

3 *Suppose V is finite-dim, U is a subsp of V, and $S \in \mathcal{L}(U, V)$.
Prove that there exists an inv $T \in \mathcal{L}(V, V)$ such that
$Tu = Su$ for every $u \in U$ if and only if S is inje.* [*Compare this with* (3.A.11).]

SOLUTION:

(a) $Tu = Su$ for every $u \in U \Rightarrow u = T^{-1}Su \Rightarrow S$ is inje. OR. $\operatorname{null} S = \operatorname{null} T \cap U = \{0\} \cap U = \{0\}$.

(b) Suppose $(u_1, \dots, u_m)$ be a basis of U and S is inje $\Rightarrow (Su_1, \dots, Su_m)$ is linely inde in V.
Extend these to bases of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$ and $(Su_1, \dots, Su_m, w_1, \dots, w_n)$.
Define $T \in \mathcal{L}(V)$ by $T(u_i) = Su_i$; $Tv_j = w_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.

---

4 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.
Prove that* $\operatorname{null} S = \operatorname{null} T\,(=U) \iff S = ET,\ \exists$ *inv* $E \in \mathcal{L}(W)$.

SOLUTION:

Define $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i$, $E(w_j) = x_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $(Tv_1, \dots, Tv_m)$ be a basis of $\operatorname{range} T$, extend it to a basis of W as $(Tv_1, \dots, Tv_m, w_1, \dots, w_n)$.
Let $(u_1, \dots, u_n)$ be a basis of U. Then by (3.B.TIPS), $(v_1, \dots, v_m, u_1, \dots, u_n)$ is a basis of V.
又 $\operatorname{null} S = \operatorname{null} T \Rightarrow V = \operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} S \Rightarrow \operatorname{span}(Sv_1, \dots, Sv_m) = \operatorname{range} S$.
And $\dim \operatorname{range} T = \dim \operatorname{range} S = \dim V - \operatorname{null} U = m$. Hence $(Sv_1, \dots, Sv_m)$ is a basis of $\operatorname{range} S$.
Thus we let $(Sv_1, \dots, Sv_m, x_1, \dots, x_n)$ be a basis of W.

$\therefore E$ is inv and $S = ET$.

Conversely, $S = ET \Rightarrow \operatorname{null} S = \operatorname{null} ET$.
Then $v \in \operatorname{null} ET \iff ET(v) = 0 \iff Tv = 0 \iff v \in \operatorname{null} T$. Hence $\operatorname{null} ET = \operatorname{null} T = \operatorname{null} S$. ☐

---

5 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.
Prove that* $\operatorname{range} S = \operatorname{range} T\,(=R) \iff S = TE,\ \exists$ *inv* $E \in \mathcal{L}(V)$.

SOLUTION:

Define $E \in \mathcal{L}(V)$ as $E: \ v_i \mapsto r_i$; $u_j \mapsto s_j$; for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $(Tv_1, \dots, Tv_m)$ and $(Sr_1, \dots, Sr_m)$ be bases of R such that $\forall\, i, Tv_i = Sr_i$.
Let $(u_1, \dots, u_n)$ and $(s_1, \dots, s_n)$ be bases of $\operatorname{null} T$ and $\operatorname{null} S$ respectively.
Thus $(v_1, \dots, v_m, u_1, \dots, u_n)$ and $(r_1, \dots, r_m, s_1, \dots, s_n)$ are bases of V.

$\therefore E$ is inv and $S = TE$.

Conversely, $S = TE \Rightarrow \operatorname{range} S = \operatorname{range} TE$.
Then $w \in \operatorname{range} S \iff \exists\, v \in V, Sv = TE(v) = T\,(E(v)) = w \in \operatorname{range} T$. Hence $\operatorname{range} S = \operatorname{range} T$. ☐

---

6 *Suppose V and W are finite-dim and $S, T \in \mathcal{L}(V, W)$.* $[\dim \operatorname{null} S = \dim \operatorname{null} T = n]$
Prove that $S = E_2 T E_1$, $\exists$ inv $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W) \Longleftrightarrow \dim \operatorname{null} S = \dim \operatorname{null} T$.

SOLUTION:

Define $E_1 : \quad v_i \mapsto r_i \ ; \quad u_j \mapsto s_j; \quad$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.
Define $E_2 : Tv_i \mapsto Sr_i \ ; \ x_j \mapsto y_j; \quad$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $(Tv_1, \dots, Tv_m)$ and $(Sr_1, \dots, Sr_m)$ be bases of range T and range S. Let $(u_1, \dots, u_n)$ and $(s_1, \dots, s_n)$ be bases of null T and null S respectively. Thus $(v_1, \dots, v_m, u_1, \dots, u_n)$ and $(r_1, \dots, r_m, s_1, \dots, s_n)$ are bases of V. Extend $(Tv_1, \dots, Tv_m)$ and $(Sr_1, \dots, Sr_m)$ to bases of W as $(Tv_1, \dots, Tv_m, x_1, \dots, x_p)$ and $(Sr_1, \dots, Sr_m, y_1, \dots, y_p)$.	$\therefore E_1, E_2$ are inv and $S = E_2 T E_1$.

Conversely, $S = E_2 T E_1 \Rightarrow \dim \operatorname{null} S = \dim \operatorname{null} E_2 T E_1$.
$v \in \operatorname{null} E_2 T E_1 \Longleftrightarrow E_2 T E_1(v) = 0 \Longleftrightarrow T E_1(v) = 0$. Hence $\operatorname{null} E_2 T E_1 = \operatorname{null} T E_1 = \operatorname{null} S$.
又 By (3.B.22.COROLLARY), E is inv $\Rightarrow \dim \operatorname{null} T E_1 = \dim \operatorname{null} T = \dim \operatorname{null} S$. □

---

8 *Suppose V is finite-dim and $T : V \to W$ is a surj linear map of V onto W.*
Prove that there is a subsp U of V such that $T|_U$ is an iso of U onto W.
$T|_U$ is the function whose domain is U, with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.

SOLUTION:

T is surj $\Rightarrow$ range $T = W \Rightarrow \dim \operatorname{range} T = \dim W = \dim V - \dim \operatorname{null} T$.
Let $(w_1, \dots, w_m)$ be a basis of range $T = W \Rightarrow \forall w_i, \exists!\, v_i \in V, Tv_i = w_i$.
$\Rightarrow (v_1, \dots, v_m)$ is a basis of $\mathcal{K}$. Thus $\dim \mathcal{K} = \dim W$.
Thus $T|_{\mathcal{K}}$ maps a basis of $\mathcal{K}$ to a basis of range $T = W$. Denote $\mathcal{K}$ by U. □

OR. By Problem (12) in (3.B), there is a subsp U of V such that
$U \cap \operatorname{null} T = \{0\} = \operatorname{null} T|_U$, range $T = \{Tu : u \in U\} = \operatorname{range} T|_U$. □

• *Suppose V and W are finite-dim and U is a subsp of V.*
Let $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}$.
(a) *Show that $\mathcal{E}$ is a subsp of $\mathcal{L}(V, W)$.*
(b) *Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$ and $\dim U$.*
Hint: Define $\Phi : \mathcal{L}(V, W) \to L(U, W)$ by $\Phi(T) = T|_U$. What is null Φ? What is range Φ?

SOLUTION:

(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.
(b) Define Φ as in the hint.
Because $T \in \operatorname{null} \Phi \Longleftrightarrow \Phi(T) = 0 \Longleftrightarrow \forall u \in U, Tu = 0 \Longleftrightarrow T \in \mathcal{E}$.
Hence null $\Phi = \mathcal{E}$.
Because $S \in \mathcal{L}(U, W) \Rightarrow \exists\, T \in \mathcal{L}(V, W), \Phi(T) = S$, by (3.B.11) $\Rightarrow S \in$ range T.
Hence range $\Phi = \mathcal{L}(U, W)$.
Thus $\dim \operatorname{null} \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \operatorname{range} \Phi = (\dim V - \dim U) \dim W$. □

OR. Extend $(u_1, \dots, u_m)$ a basis of U to $(u_1, \dots, u_m, v_1, \dots, v_n)$ a basis of V. Let $p = \dim W$.
(See NOTE FOR[3.60])

$\forall\, T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \underbrace{\operatorname{span}\begin{Bmatrix} E_{1,1}, & \cdots & , E_{m,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,p}, & \cdots & , E_{m,p} \end{Bmatrix}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$

又 $W = \operatorname{span}\begin{Bmatrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{Bmatrix} \subseteq \mathcal{E}$. Where $\mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}$.

Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. □

ENDED

3·E

2 *Suppose $V_1, \dots, V_m$ are vecsps such that $V_1 \times \cdots \times V_m$ is finite-dim. Prove that every V_j is finite-dim.*

SOLUTION: Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.

Let $(v_1, \dots, v_M)$ be a basis of U. Note that $\forall u_i \in V_i, \in U_i \subseteq U$, for each i.

Define $R_i \in \mathcal{L}(V_i, U)$ by $R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0)$.
Define $S_i \in \mathcal{L}(U, V_i)$ by $S_i(u_1, \dots, u_i, \dots, u_m) = u_i$
$\Big\} \Rightarrow S_i|_{U_i} = R_i^{-1}|_{U_i}$.

Thus U_i and V_i are iso. 又 U_i is a subsp of a finite-dim vecsp U. □

---

3 *Give an example of a vecsp V and its two subsps U_1, U_2 such that $U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum.*

SOLUTION:

NOTE*that at least one of U_1, U_2 must be infinite-dim.*

For if not, $U_1 \times U_2$ is finite-dim and $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$.

And V must be infinite-dim. For if not, both U_1 and U_2 are finite-dim subsps.

Let $V = \mathbf{F}^\infty = U_1, U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$.

Define $T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2)$ by $T\left((x_1, x_2, \cdots), (x, 0, \cdots)\right) = (x, x_1, x_2, \cdots)$
Define $S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2)$ by $S(x, x_1, x_2, \cdots) = \left((x_1, x_2, \cdots), (x, 0, \cdots)\right)$
$\Big\} \Rightarrow S = T^{-1}$. □

---

4 *Suppose $V_1, \dots, V_m$ are vecsps. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.*

SOLUTION: Using the notations in Problem (2).

Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$.

Define $\varphi : T \mapsto (T_1, \dots, T_m)$ by $\varphi(T) = (TR_1, \dots, TR_m)$.
Define $\psi : (T_1, \dots, T_m) \mapsto T$ by $\psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m$.
$\Big\} \Rightarrow \psi = \varphi^{-1}$. □

---

5 *Suppose $W_1, \dots, W_m$ are vecsps. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.*

SOLUTION: Using the notations in Problem (2).

Note that $Tv = (w_1, \dots, w_m)$. Define $T_i \in \mathcal{L}(V, W_i)$ by $T_i(v) = w_i$.

Define $\varphi : T \mapsto (T_1, \dots, T_m)$ by $\varphi(T) = (S_1 T, \dots, S_m T)$.
Define $\psi : (T_1, \dots, T_m) \mapsto T$ by $\psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m$.
$\Big\} \Rightarrow \psi = \varphi^{-1}$. □

---

6 *For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.*

SOLUTION:

Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m$.

(a) Suppose $T(v_1, \dots, v_m) = 0$. Then $\forall\ (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$

$\Rightarrow (v_1, \dots, v_m) = 0 \Rightarrow T$ is inje.

(b) Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $\forall\ (b_1, \dots, b_n) \in \mathbf{F}^m$,

$[T\,(\psi(e_1), \dots, \psi(e_m))]\,(b_1, \dots, b_m) = b_1\psi(e_1) + \cdots + b_m\psi(e_m) = \psi(b_1 e_1 + \cdots + b_m e_m) = \psi(b_1, \dots, b_m)$.

Thus $T\,(\psi(e_1), \dots, \psi(e_m)) = \psi$. Hence T is surj. $\square$

---

7 *Suppose $v, x \in V$ (arbitrary) and U and W are subsps of V.*
Suppose $v + U = x + W$. Prove that $U = W$.

SOLUTION:

(a) $\forall u \in U,\ \exists\, w \in W, v + u = x + w$, let $u = 0$, now $v = x + w \Rightarrow v - x \in W$.

(b) $\forall w \in W,\ \exists\, u \in U, v + u = x + w$, let $w = 0$, now $x = v + u \Rightarrow x - v \in U$.

Thus $\pm(v - x) \in U \cap W \Rightarrow \begin{cases} u = (x - v) + w \in W \Rightarrow U \subseteq W \\ w = (v - x) + u \in U \Rightarrow W \subseteq U \end{cases} \Rightarrow U = W.$ $\square$

• *Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$.*
Prove that A is a translate of $U \iff \exists\, c \in \mathbf{R}, A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$.
[Do it in your mind.]

• *Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $U = \{x \in V : Tx = c\}$ is either $\emptyset$*
or is a translate of null T.

SOLUTION:

If $c \in W$ but $c \notin \operatorname{range} T$, then $U = \emptyset$ and we are done.

Suppose $c \in \operatorname{range} T$, then $\ \exists\, u \in V, Tu = c \Rightarrow u \in U$.

Suppose $y \in \operatorname{null} T \Rightarrow y + u \in U \iff T(y + u) = Ty + c = c$.

Thus $u + \operatorname{null} T \subseteq U$. Hence $u + \operatorname{null} T = U$,

for if not, suppose $z \notin u + \operatorname{null} T$ but $Tz = c (\Leftrightarrow z \in U)$,

then $\ \forall w \in \operatorname{null} T, z \neq u + w \Leftrightarrow z - u \notin \operatorname{null} T$.

又 $\tilde{T}(z + \operatorname{null} T) = \tilde{T}(u + \operatorname{null} T) \Rightarrow z + \operatorname{null} T = u + \operatorname{null} T \Rightarrow z - u \in \operatorname{null} T$, contradicts. $\square$

COROLLARY: *The set of solutions to a system of linear equations such as [3.28]*
is either $\emptyset$ or a translate of the null subsp.

---

8 *Suppose A is a nonempty subset of V.*
Prove that A is a translate of some subsp of $V \iff \lambda v + (1 - \lambda) w \in A\ , \forall v, w \in A, \lambda \in \mathbf{F}$.

SOLUTION:

Suppose $A = a + U$, where U is a subsp of V. $\ \forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$,

$\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + [\lambda(u_1 - u_2) + u_2] \in A$.

Suppose $\lambda v + (1 - \lambda) w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $a \in A$ and let $A' = \{x - a : x \in A\}$.

Then $\ \forall x - a, y - a \in A', \lambda \in \mathbf{F}$,

(I) $\lambda(x - a) = [\lambda x + (1 - \lambda) a] - a \in A'$. Then let $\lambda = 2$.

(II) $\lambda(x - a) + (1 - \lambda)(y - a) = \frac{1}{2}(x - a) + \frac{1}{2}(y - a) = \frac{1}{2}x + (1 - \frac{1}{2})(y) - a \in A'$.

By (I), $2 \times [\frac{1}{2}(x - a) + \frac{1}{2}(y - a)] = (x - a) + (y - a) \in A'$.

Thus A' is a subsp of V. Hence $a + A' = \{(x - a) + a : x \in A\} = A$ is a translate. $\square$

---

9 *Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subsps U_1, U_2 of V. Prove that the intersection $A_1 \cap A_2$ is either a translate of some subsp of V or is $\emptyset$.*

SOLUTION:

Suppose $v + u_1, w + u_2 \in A_1 \cap A_2 \neq \emptyset$. By Problem (8),
$\forall \lambda \in \mathbf{F}, \lambda(v + u_1) + (1 - \lambda)(w + u_2) \in A_1$ and A_2. Thus $A_1 \cap A_2$ is a translate of some subsp of V. □

---

10 *Prove that the intersection of any collection of translates of subsps of V is either a translate of some subsp or $\emptyset$.*

SOLUTION:

Suppose $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collection of translates of subsps of V, where Γ is an arbitrary index set.
Suppose $x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset$, then by Problem (18), $\forall \lambda \in \mathbf{F}, \lambda x + (1 - \lambda)y \in A_\alpha$ for every $\alpha \in \Gamma$.
Thus $\bigcap_{\alpha \in \Gamma} A_\alpha$ is a translate of some subsp of V. □

---

11 *Suppose $A = \left\{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum\limits_{i=1}^{m} \lambda_i = 1\right\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.*

(a) *Prove that A is a translate of some subsp of V*

(b) *Prove that if B is a translate of some subsp of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.*

(c) *Prove that A is a translate of some subsp of V and $\dim V < m$.*

SOLUTION:

(a) By Problem (8), $\forall u, w \in A, \lambda \in \mathbf{F}$, $\exists\, a_i, b_i \mathbf{F}, \lambda u + (1 - \lambda)w = \left(\lambda \sum\limits_{i=1}^{m} a_i + (1 - \lambda) \sum\limits_{i=1}^{m} b_i\right) v_i \in A$. □

(b) Let $v = \lambda_1 v_+ \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on m by k.

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.
$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By problem (8), $v \in B$.

(ii) $2 \leq k \leq m$, we assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. ($\forall \lambda_i$ such that $\sum\limits_{i=1}^{k} \lambda_i = 1$)
For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. $\forall i = 1, \dots, k$, $\exists\, \mu_i \neq 1$, fix one such i by ι.
Then $\sum\limits_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow (\sum\limits_{i=1}^{k+1} \dfrac{\mu_i}{1 - \mu_\iota}) - \dfrac{\mu_\iota}{1 - \mu_\iota} = 1$.
Let $w = \underbrace{\dfrac{\mu_1}{1 - \mu_\iota} v_1 + \cdots + \dfrac{\mu_{\iota-1}}{1 - \mu_\iota} v_{\iota-1} + \dfrac{\mu_{\iota+1}}{1 - \mu_\iota} v_{\iota+1} + \cdots + \dfrac{\mu_{k+1}}{1 - \mu_\iota} v_{k+1}}_{k \text{ terms}}$.
Let $\lambda_i = \dfrac{\mu_i}{1 - \mu_\iota}$ for $i = 1, \dots, \iota - 1$; $\lambda_j = \dfrac{\mu_{j+1}}{1 - \mu_\iota}$ for $j = \iota, \dots, k$. Then,

$$\left.\begin{array}{r} \sum\limits_{i=1}^{k} \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1 - \lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B.$$ □

(c) Fix a $k \in \{1, \dots, m\}$. Given $\lambda_i \in \mathbf{F}$ ($i \in \{1, \dots, m\} \backslash \{k\}$).
Let $\lambda_k = 1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m$
Then $\lambda_1 v_1 + \cdots + \lambda_k v_k + \cdots + \lambda_m v_m = v_k + \sum\limits_{i=1}^{m} \lambda_i (v_i - v_k)$.
Thus $A = v_k + \operatorname{span}(v_1 - v_k, \dots, v_{k-1} - v_k, v_{k+1} - v_k, \dots, v_m - v_k)$. □

---

12 *Suppose U is a subsp of V such that V/U is finite-dim. Prove that is V is iso to $U \times (V/U)$.*

SOLUTION:

Let $(v_1 + U, \dots, v_n + U)$ be a basis of V/U. Note that
$\forall v \in V$, $\exists\,!\, a_1, \dots, a_n \in \mathbf{F}, v + U = \sum\limits_{i=1}^{n} a_i (v_i + U) = (\sum\limits_{i=1}^{n} a_i v_i) + U$

$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) = u \in U$ for some $u; v = \sum_{i=1}^{n} a_i v_i + u$.

Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ by $\varphi(v) = (u, \sum_{i=1}^{n} a_i v_i + U)$

and $\psi \in \mathcal{L}(U \times (V/U), V)$ by $\psi(u, w + U) = u + w; w = \sum_{i=1}^{n} b_i v_i + U$.

So that $\psi = \varphi^{-1}$. □

• *Suppose $V = U \oplus W$, $(w_1, \ldots, w_m)$ is a basis of W.*
Prove that $(w_1 + U, \ldots, w_m + U)$ is a basis of V/U.

SOLUTION:

Note that $\forall v \in V, \exists!\, u \in U, w \in W, v = u + w$ 又 $\exists!\, c_i \in \mathbf{F}$ such that $w = \sum_{i=1}^{m} c_i w_i \Rightarrow v = u + \sum_{i=1}^{m} c_i w_i$.

Thus $v + U = \sum_{i=1}^{m} c_i w_i + U \Rightarrow v + U \in \operatorname{span}(w_1 + U, \ldots, w_m + U) \Rightarrow V/U \subseteq \operatorname{span}(w_1 + U, \ldots, w_m + U)$.

Now suppose $a_1(w_1 + U) + \cdots + a_m(w_m + U) = 0 + U \Rightarrow \sum_{i=1}^{m} a_i w_i \in U$ while $U \cap W = \{0\}$.

Then $\sum_{i=1}^{m} a_i w_i = 0 \Rightarrow a_1 = \cdots = a_m = 0$. □

---

13 *Suppose $(v_1 + U, \ldots, v_m + U)$ is a basis of V/U and $(u_1, \ldots, u_n)$ is a basis of U.*
Prove that $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ is a basis of V.

SOLUTION:

By Problem (12), U and V/U are finite-dim $\Rightarrow U \times (V/U)$ is finite-dim, so is V.

$\dim V = \dim(U \times (V/U)) = \dim U + \dim V/U = m + n$.

OR. Note that $\forall v \in V, v + U = \sum_{i=1}^{m} a_i v_i + U, \exists!\, a_i \in \mathbf{F} \Rightarrow U \ni v - \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} b_i v_i, \exists!\, b_i \in \mathbf{F}$.

$\Rightarrow v \in \operatorname{span}(v_1, \ldots, v_m, u_1, \ldots, u_n)$.

又 Notice that $(\sum_{i=1}^{m} a_i v_i) + U = 0 + U (\Longleftrightarrow \sum_{i=1}^{m} a_i v_i \in U) \Longleftrightarrow a_1 = \cdots = a_m = 0$.

Hence $\operatorname{span}(v_1, \ldots, v_m) \cap U = \{0\} \Rightarrow \operatorname{span}(v_1, \ldots, v_m) \oplus U = V$

Thus $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ is linely inde, so is a basis of V. □

---

14 *Suppose $U = \{(x_1, x_2, \ldots) \in \mathbf{F}^\infty : x_k \neq 0$ for only finitely many $k\}$.*
(a) *Show that U is a subsp of $\mathbf{F}^\infty$.* [*Do it in your mind*]
(b) *Prove that $\mathbf{F}^\infty/U$ is infinite-dim.*

SOLUTION:

For $u = (x_1, \ldots, x_p, \ldots) \in \mathbf{F}^\infty$, denote x_p by $u[p]$. For each $r \in \mathbf{N}^+$.

Define $e_r[p] = \begin{cases} 1, (p-1) \equiv 0 \pmod r \\ 0, \text{otherwise} \end{cases}$, simply $e_r = (1, \underbrace{0, \ldots, 0}_{(p-1)\ times}, 1, \underbrace{0, \ldots, 0}_{(p-1)\ times}, 1, \ldots) \in \mathbf{F}^\infty$.

Choose $m \in \mathbf{N}^+$ arbitrarily.

Suppose $a_1(e_1 + U) + \cdots + a_m(e_m + U) = (a_1 e_1 + \cdots + a_m e_m) + U = 0 + U = 0$.

$\Rightarrow a_1 e_1 + \cdots + a_m e_m = u$ for some $u \in U$.

Then suppose $u = (x_1, \ldots, x_t, 0, \ldots) \Rightarrow u[t + i] = 0, \forall i \in \mathbf{N}^+$,

then let $j = s \cdot m! + 1 \geq t$ $(\exists\, s \in \mathbf{N}^+)$ so that $e_1[j] = \cdots = e_m[j] = 1,\ u[j + i] = 0$.

Now we have: $u[j + i] = (\Sigma_{r=1}^{m} a_r e_r)[j + i] = \Sigma_{r=1}^{m} a_r e_r[s \cdot m! + 1 + i] = a_{i_1} + \cdots + a_{i_{\tau(i)}} = 0$,

$\Rightarrow (\Sigma_{r=1}^{m} a_r e_r)[j + i] = a_{i_1} + \cdots + a_{i_{\tau(i)}} = 0.\ (\Delta)$

where $i_1, \ldots, i_{\tau(i)}$ are distinct ordered factors of i $(1 = i_1 \leq \cdots \leq i_{\tau(i)} = i)$.

$\Big($ Note that by definition, $e_r[s \cdot m! + 1 + i] = 1 \Longleftrightarrow s \cdot m! + i \equiv i \equiv 0 \pmod r \Longleftrightarrow r \mid i.$ $\Big)$

Let $i' = i_{\tau(i)-1}$. Notice that $i'_l = i_l, \forall l \in \{1, \dots, \tau(i')\}$; and $\tau(i') = \tau(i) - 1$.

Again by (Δ), $(\Sigma_{r=1}^{m} a_r e_r)[j + i'] = a_{i'_1} + \dots + a_{i'_{\tau(i')}} = a_{i_1} + \dots + a_{i_{\tau(i)-1}} = 0$.

Thus $a_{i_{\tau(i)}} = a_i = 0$ for any $i \in \{1, \dots, m\}$.
Hence $(e_1, \dots, e_m)$ is linely inde in$\mathbf{F}^\infty$, so is $(e_1, \dots, e_m, \dots)$, since $m \in \mathbf{N}^+$.
又 $e_i \notin U \Rightarrow (e_1 + U, e_2 + U, \dots)$ is linely inde in $\mathbf{F}^\infty / U$. By [2.B.14]. □

---

15 *Suppose* $\varphi \in \mathcal{L}(V, \mathbf{F}) \backslash \{0\}$. *Prove that* $\dim V/(\operatorname{null} \varphi) = 1$.

Solution: By [3.91] (d), $\dim \operatorname{range} \varphi = 1 = \dim V/(\operatorname{null} \varphi)$. □

• **Note For [3.88, 3.90, 3.91]:**
For any $W \in \mathcal{S}_V U$, because $V = U \oplus W$. $\forall v \in V$, $\exists !\, u_v \in U, w_v \in W, v = u_v + w_v$.
Define $T \in \mathcal{L}(V, W)$ by $T(v) = w_v$. Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$.
Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, W)$ is defined as $\tilde{T}(v + U) = Tv = w_v$.
Thus $\tilde{T}$ is inje (by [3.91(b)]) and surj ($\operatorname{range} \tilde{T} = \operatorname{range} T = W$),
and therefore is an iso. We conclude that V/U and W, namely any vecsp in $\mathcal{S}_V$, are iso.

---

16 *Suppose* $\dim V/U = 1$. *Prove that* $\exists\, \varphi \in \mathcal{L}(V, \mathbf{F})$ *such that null* $\varphi = U$.

Solution:

Suppose V_0 is a subsp of V such that $V = U \oplus V_0$. Then V_0 and V/U are iso. $\dim V_0 = 1$.
Define a linear map $\varphi : v \mapsto \lambda$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. □

---

17 *Suppose* V/U *is finite-dim.* W *is a subsp of* V.
(a) *Show that if* $V = U + W$, *then* $\dim W \geq \dim V/U$.
(b) *Suppose* $\dim W = \dim V/U$ *and* $V = U \oplus W$. *Find such* W.

Solution: Let $(w_1, \dots, w_n)$ be a basis of W

(a) $\forall v \in V$, $\exists\, u \in U, w \in W$ such that $v = u + w \Rightarrow v + U = w + U$
Then $V/U \subseteq \operatorname{span}(w_1 + U, \dots, w_n + U) \Rightarrow V/U = \operatorname{span}(w_1 + U, \dots, w_n + U)$.
Hence $\dim V/U = \dim \operatorname{span}(w_1 + U, \dots, w_n + U) \leq \dim W$.

(b) Let $W \in \mathcal{S}_V U$. In other words,
reduce $(w_1 + U, \dots, w_n + U)$ to a basis of V/U as $(w_1 + U, \dots, w_m + U)$ and let $W = \operatorname{span}(w_1, \dots, w_m)$. □

---

18 *Suppose* $T \in \mathcal{L}(V, W)$ *and* U *is a subsp of* V. *Let* π *denote the quotient map.*
Prove that $\exists\, S \in \mathcal{L}(V/U, W)$ *such that* $T = S \circ \pi$ *if and only if* $U \subseteq \operatorname{null} T$.

Solution:

(a) Define $S \in \mathcal{L}(V/U, W)$ by $S(v + U) = Tv$. We have to check it is *well-defined*.
Suppose $v_1 + U = v_2 + U$, while $v_1 \neq v_2$.
Then $(v_1 - v_2) \in U \Rightarrow S((v_1 - v_2) + U) = T(v_1 - v_2) = 0 \Rightarrow Tv_1 = Tv_2$. Checked.

(b) Suppose $\exists\, S \in \mathcal{L}(V/U, W)$, $T = S \circ \pi$.
Then $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null} T$. □

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20 *Define* $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ *by* $\Gamma(S) = S \circ \pi$ $(= \pi'(S))$.
(a) *Prove that* Γ *is linear:* By [3.9] distr properties and [3.6].
(b) *Prove that* Γ *is inje:*
$\Gamma(S) = 0 = S \circ \pi \iff \forall v \in V, S(\pi(v)) = 0 \iff \forall v + U \in V/U, S(v + U) = 0 \iff S = 0$.

(c) *Prove that* range Γ (= range π') $= \{T \in \mathcal{L}(V,W) : U \subseteq \text{null}\, T\}$*:* By Problem (18). $\square$

Ended

3·F

• By (18) in (3.D), $\varphi : V \to \mathcal{L}(\mathbf{F}, V)$ *is an iso. Now we prove that* $(v_1, \dots, v_m)$ *is linely inde* $\iff (\varphi(v_1), \dots, \varphi(v_m))$ *is linely inde.*

Solution:

(a) Suppose $(v_1, \dots, v_m)$ is linely inde and $\vartheta \in$ span $(\varphi(v_1), \dots, \varphi(v_m))$.

Let $\vartheta = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m)$. Then $\vartheta(1) = 0 = a_1 v_1 + \cdots + a_m v_m \Rightarrow a_1 = \cdots = a_m = 0$.

Or. Because φ is inje. Suppose $a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) = 0 = \varphi(a_1 v_1 + \cdots + a_m v_m)$.

Then $a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

Thus $(\varphi(v_1), \dots, \varphi(v_m))$ is linely inde.

(b) Suppose $(\varphi(v_1), \dots, \varphi(v_m))$ is linely inde and $v \in$ span $(v_1, \dots, v_m)$.

Let $v = 0 = a_1 v_1 + \cdots + a_m v_m$. Then $\varphi(v) = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

Thus $v_1, \dots, v_m$ is linely inde. $\square$

• *Suppose* $T \in \mathcal{L}(V,W)$ *and* $(w_1, \dots, w_m)$ *is a basis of* range T.
Hence $\forall v \in V$, $Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$, $\exists!\, \varphi_1(v), \dots, \varphi_m(v)$,
thus defining functions $\varphi_1, \dots, \varphi_m$ *from* V *to* $\mathbf{F}$. *Show that each* $\varphi_i \in V'$.

Solution:

For each $w_i, \exists\, v_i \in V$, $Tv_i = w_i$, getting a linely inde list $(v_1, \dots, v_m)$.

Now we have $Tv = a_1 Tv_1 + \cdots + a_m Tv_m$, $\forall v \in V, \exists!\, a_i \in \mathbf{F}$.

Let $(\psi_1, \dots, \psi_m)$ be the dual basis of range T. Then $(T'(\psi_i))\,(v) = \psi_i \circ T(v) = a_i$.

Thus letting $\varphi_i = \psi_i \circ T$. $\square$

• *Suppose* $\varphi, \beta \in V'$. *Prove that* null $\varphi \subseteq$ null $\beta \iff \beta = c\varphi$. $\exists\, c \in \mathbf{F}$.

Solution: Using (3.B.29, 30)

(a) Suppose null $\varphi \subseteq$ null β. Choose a $u \notin$ null β. $V = \text{null}\,\beta \oplus \{au : a \in \mathbf{F}\}$.

If null φ = null β, then let $c = \frac{\beta(u)}{\varphi(u)}$, we are done.

Otherwise, suppose $u' \in$ null β, but $u' \notin$ null φ, then $V = \text{null}\,\varphi \oplus \{bu' : b \in \mathbf{F}\}$.

$\forall v \in V, v = w + au = w' + bu'$, $\exists!\, w, w' \in \text{null}\,\varphi, a, b \in \mathbf{F}$.

Thus $\beta(v) = a\beta(u)$, $\varphi(v) = b\varphi(u')$. Let $c = \frac{a\beta(u)}{b\varphi(u')}$. We are done

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$.

If $c = 0$, then null $\beta = V \supseteq$ null φ, we are done.

$$\text{Otherwise,} \quad \left.\begin{array}{l} \forall v \in \text{null}\,\varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \text{null}\,\varphi \subseteq \text{null}\,\beta. \\ \forall v \in \text{null}\,\beta, \beta(v) = 0 = \varphi(v) \Rightarrow \text{null}\,\beta \subseteq \text{null}\,\varphi. \end{array}\right\} \Rightarrow \text{null}\,\varphi = \text{null}\,\beta$$

$\Rightarrow$ null $\varphi \subseteq$ null β. $\square$

---

5 *Prove that* $(V_1 \times \cdots \times V_m)'$ *and* ${V'}_1 \times \cdots \times {V'}_m$ *are iso.*

Solution: Using notations in (3.E.2).

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to {V'}_1 \times \cdots \times {V'}_m \\ \quad \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = ({R'}_1(T), \dots, {R'}_m(T)). \\ \text{Define } \psi : {V'}_1 \times \cdots \times {V'}_m \to (V_1 \times \cdots \times V_m)' \\ \quad \text{by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m = {S'}_1(T_1) + \cdots + {S'}_m(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$

$\square$

- *Suppose $(v_1, \dots, v_n)$ is a basis of V and $(\varphi_1, \dots, \varphi_n)$ is the dual basis of V'.*

$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

---

9 *Suppose $(v_1, \dots, v_n)$ is a basis of V and $(\varphi_1, \cdots, \varphi_n)$ is the correspd dual basis of V'. Suppose $\psi \in V'$. Prove that $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.*

SOLUTION: $\psi(v) = \psi(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i \psi(v_i) = \sum_{i=1}^{n} \psi(v_i)\varphi_i(v) = [\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n](v)$. □

COMMENT: For other basis $(u_1, \dots, u_n)$ and the dual basis $(\rho_1, \dots, \rho_n)$, $\psi = \psi(u_1)\rho_1 + \cdots + \psi(u_n)\rho_n$.

---

35 *Prove that $(\mathcal{P}(\mathbf{R}))'$ and $\mathbf{R}^\infty$ are iso.*

SOLUTION:

Define $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{R}))', \mathbf{R}^\infty)$ by $\theta(\varphi) = (\varphi(1), \varphi(x), \dots, \varphi(x^n), \dots)$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall x^k$ in the basis $(1, x, \dots, x^n, \dots)$ of $\mathcal{P}_n(\mathbf{R})$ for any n, $\varphi(x^k) = 0 \Rightarrow \varphi = 0$.

Surj: $\forall\ (a_0, a_1, \dots, a_n, \dots) \in \mathbf{F}^\infty$, let ψ be such that $\psi(x^k) = a_k$ and thus $\theta(\psi) = (a_0, a_1, \dots, a_n, \dots)$.

Hence θ is an iso from $(\mathcal{P}(\mathbf{R}))'$ onto $\mathbf{R}^\infty$. □

---

7 *Suppose $m \in \mathbf{N}^+$. Show that the dual basis of the basis $(1, x, \dots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \dots, \varphi_m$, where $\varphi_k = \dfrac{p^{(k)}(0)}{k!}$.*

Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

SOLUTION:

For each j and k, $(x^j)^{(k)} = \begin{cases} j(j-1)\dots(j-k+1)\cdot x^{(j-k)}\ , & j \geq k. \\ j(j-1)\dots(j-j+1) = j!\ , & j = k. \\ 0\ , & j \leq k. \end{cases}$ Then $(x^j)^{(k)}(0) = \begin{cases} 0\ ,\ j \neq k. \\ k!\ ,\ j = k. \end{cases}$

Thus $\varphi_k = \psi_k$, where $\psi_1, \dots, \psi_m$ is the dual basis of $(1, x, \dots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$. □

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8 *Suppose $m \in \mathbf{N}^+$.*

(a) By [2.C.10], $B = (1, x-5, \dots, (x-5)^m)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) $\varphi_k = \dfrac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \dots, m$. Then $(\varphi_0, \varphi_1, \dots, \varphi_m)$ is the dual basis of B.

---

13 *Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ denote the dual basis of the standard basis of $\mathbf{R}^2$ and $\mathbf{R}^3$.*

(a) *Describe the linear functionals $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$*

For any $(x, y, z) \in \mathbf{R}^3$, $(T'(\varphi_1))(x, y, z) = 4x + 5y + 6z$, $(T'(\varphi_2))(x, y, z) = 7x + 8y + 9z$.

(b) *Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.*

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$. □

---

14 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for each $x \in \mathbf{R}$.*

(a) *Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe $T'(\varphi) \in \mathcal{P}(\mathbf{R})'$.*

$(T'(\varphi))(p) = [x^2 p(x) + p''(x)]'(4) = [2xp(x) + x^2 p'(x) + p'''(x)](4) = 8p(4) + 16p'(4) + p'''(4)$.

(b) *Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p(x)\mathrm{d}x$. Evaluate $(T'(\varphi))(x^3)$.*

$(T'(\varphi))(x^3) = \int_0^1 (x^5 + 6x)\mathrm{d}x = \int_0^1 (\frac{1}{6}x^6 + 3x^2)'\mathrm{d}x = \frac{6}{19}$. □

---

12 *Show that the dual map of the identity operator on V is the identity operator on V'.*

Solution: $I'(\varphi) = \varphi \circ I = \varphi, \ \forall \varphi \in V'$. $\square$

• *Suppose W is finite-dim and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \iff T = 0$.*

Solution: $T = 0 \iff T'(\varphi) = \varphi \circ T = 0$ for all $\varphi \in V' \iff T' = 0$. $\square$

• *Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$. Prove that T is inv $\iff T'$ is inv.*

Solution: By [3.108] and [3.110]. $\square$

---

16 *Suppose V and W are finite-dim. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(V, W)$. Prove that Γ is an iso of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.*

Solution:

V, W are finite-dim $\Rightarrow \dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$. And by [3.101], Γ is linear.

又 Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus T is inje $\Rightarrow T$ is inv. $\square$

---

4 *Suppose V is finite-dim and U is a subsp of $V, U \neq V$. Prove that $\exists\, \varphi \in V' \backslash \{0\}, \varphi(u) = 0$ for all $u \in U$.*

Solution:

Let $(u_1, \dots, u_m)$ be a basis of U, extend to $(u_1, \dots, u_m, u_{m+1}, \dots, u_{m+n})$ a basis of V.

Choose a $k \in \{1, \dots, n\}$. Define $\varphi \in V'$ by $\varphi(u_i) = \begin{cases} 1, & \text{if } i = m + k. \\ 0, & \text{otherwise.} \end{cases}$

Or. Equivalent to proving that $U^0 \neq \{0\}$. By [3.106], $\dim U^0 = \dim V - \dim U > 0$. $\square$

• *Suppose V is a vecsp and $U \subseteq V$.*

17 $U^0 = \{\varphi \in V' : U \subseteq \mathit{null}\varphi\}$. Noticing $\varphi \in V', U \subseteq \operatorname{null} \varphi \iff \forall u \in U, \varphi(u) = 0$.

18 $U = \{0\} \iff \forall \varphi \in V', U \subseteq \operatorname{null} \varphi \iff U^0 = V'$.

19 $U = V \iff U^0_V = \{0\} = V^0_V$. By the inverse and contrapositive of Problem (4).

20, 21 *Suppose U and W are subsets of V. Prove that $U \subseteq W \iff W^0 \subseteq U^0$.*

Solution:

(a) Suppose $U \subseteq W$. Then $\forall w \in W, u \in U, \varphi \in W^0, \varphi(w) = 0 = \varphi(u) \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.

(b) Suppose $W^0 \subseteq U^0$. Then $\varphi \in W^0 \Rightarrow \varphi \in U^0$. Hence $\operatorname{null} \varphi \supseteq W \Rightarrow \operatorname{null} \varphi \supseteq U$. Thus $W \supseteq U$. $\square$

Corollary: $W^0 = U^0 \iff U = W$.

---

22 *Suppose U and W are subsps of V. Prove that $(U + W)^0 = U^0 \cap W^0$.*

Solution:

(a) $\left.\begin{array}{l} U \subseteq U + W \\ W \subseteq U + W \end{array}\right\} \Rightarrow \left.\begin{array}{l} (U + W)^0 \subseteq U^0 \\ (U + W)^0 \subseteq W^0 \end{array}\right\} \Rightarrow (U + W)^0 \subseteq U^0 \cap W^0.$

(b) $\forall \varphi \in U^0 \cap W^0, \varphi(u + w) = 0$, where $u \in U, w \in W \Rightarrow \varphi \in (U + W)^0$. Thus $(U + W)^0 \supseteq U^0 \cap W^0$. $\square$

---

23 *Suppose U and W are subsets of V. Prove that $(U \cap W)^0 = U^0 + W^0$.*

Solution:

(a) $\left.\begin{array}{l} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array}\right\} \Rightarrow \left.\begin{array}{l} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array}\right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \supseteq U^0 \cap W^0.$

(b) $\forall \varphi \in U^0, \psi \in W^0$ and $\forall v \in U \cap W$, $(\varphi + \psi)(v) = \varphi(v) + \psi(v) = 0$. Thus $U^0 + W^0 \subseteq (U \cap W)^0$. $\square$

• **COROLLARY:** *Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subsps of V.*
Then $\left(\sum_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \bigcap_{\alpha_i \in \Gamma} \left(V_{\alpha_i}^0\right)$; And $\left(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \sum_{\alpha_i \in \Gamma} \left(V_{\alpha_i}^0\right)$.

---

24 *Suppose V is finite-dim and U is a subsp of V.*
Prove, using the pattern of [3.104], that $\dim U + \dim U^0 = \dim V$.

SOLUTION:

Let $(u_1, \dots, u_m)$ be a basis of U, extend to a basis of V as $(u_1, \dots, u_m, \dots, u_n)$,
and let $(\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$ be the dual basis.

(a) Suppose $\varphi \in \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, then $\exists\, a_i \in \mathbf{F}, \varphi = a_{m+1}\varphi_{m+1} + \cdots + a_n\varphi_n$.
For all $u \in U$, $\varphi(u) = 0$. Thus $\varphi \in U^0$, getting $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq U^0$.

(b) Suppose $\varphi \in U^0$, then $\exists\, a_i \in \mathbf{F}, \varphi = a_1\varphi_1 + \cdots + a_m\varphi_m + \cdots + a_n\varphi_n$.
For all $u_i \in U$, $0 = \varphi(u_i) = \sum_{i=1}^{n} \varphi(u_i) = a_i$. Then $\varphi = a_{m+1}\varphi_{m+1} + \cdots + a_n\varphi_n$.
Thus $\varphi \in \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, getting $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \supseteq U^0$.

Hence $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0$, $\dim U^0 = n - m = \dim V - \dim U$. $\square$

---

25 *Suppose U is a subsp of V. Explain why $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.*

SOLUTION: Note that $U = \{v \in V : v \in U\}$ is a subsp of V and $\varphi(v) = 0$ for every $\varphi \in U^0 \iff v \in U$. $\square$

---

26 *Suppose V is finite-dim, Ω is a subsp of V'. Prove that $\Omega = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}^0$.*

SOLUTION: Using the corollary in Problem (20, 21).

Suppose $U = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}$.
Getting $U^0 = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}^0$. We need to show that $\Omega = U^0$.

$$\left.\begin{array}{l} \text{(a) } \forall \varphi \in \Omega, v \in U, \varphi(v) = 0 \Rightarrow \varphi \in U^0 \Rightarrow \Omega \subseteq U^0. \\ \text{(b) } v \in U \Leftrightarrow \left\{\begin{array}{l} \forall \varphi \in \Omega, \varphi(v) = 0 \\ \forall \psi \in U^0, \psi(v) = 0 \end{array}\right. \quad \text{Thus } \Omega \supseteq U^0. \end{array}\right\} \Rightarrow \square$$

---

27 *Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that $\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$.*

SOLUTION:

By Problem (26), $\operatorname{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \psi(p) = 0, \forall \psi \in \operatorname{span}(\varphi)\}^0$,
Hence $\operatorname{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \varphi(p) = p(8) = 0\}^0$, 又 $\operatorname{span}(\varphi) = \operatorname{null} T' = (\operatorname{range} T)^0$.
By the corollary in Problem (20, 21), $\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$. $\square$

28, 29 *Suppose V, W are finite-dim, $T \in \mathcal{L}(V, W)$.*

(a) *Suppose $\exists\, \varphi \in W'$, $\operatorname{null} T' = \operatorname{span}(\varphi)$. Prove that $\operatorname{range} T = \operatorname{null} \varphi$.*
(b) *Suppose $\exists\, \varphi \in V'$, $\operatorname{range} T' = \operatorname{span}(\varphi)$. Prove that $\operatorname{null} T = \operatorname{null} \varphi$.*

SOLUTION: Using Problem (26), [3.107] and [3.109].

Because $\operatorname{span}(\varphi) = \{v \in V : \forall \psi \in \operatorname{span}(\varphi), \psi(v) = 0\}^0 = \{v \in V : \varphi(v) = 0\}^0 = (\operatorname{null} \varphi)^0$.

(a) $(\operatorname{range} T)^0 = \operatorname{null} T' = \operatorname{span}(\varphi) = (\operatorname{null} \varphi)^0 \iff \operatorname{range} T = \operatorname{null} \varphi$.
(b) $(\operatorname{null} T)^0 = \operatorname{range} T' = \operatorname{span}(\varphi) = (\operatorname{null} \varphi)^0 \iff \operatorname{null} T = \operatorname{null} \varphi$. $\square$

---

31 *Suppose V is finite-dim and $(\varphi_1, \dots, \varphi_n)$ is a basis of V'.*

Show that there exists a basis of V whose dual basis is $(\varphi_1, \dots, \varphi_n)$.

SOLUTION: Using Problem (29) and (30) in (3,B).

$\forall \varphi_i, \operatorname{null}\varphi_i \oplus \{au_i : a \in \mathbf{F}\} = V$.

Because $\varphi_1, \dots, \varphi_m$ is linely inde. $\operatorname{null}\varphi_i \neq \operatorname{null}\varphi_j$ for each $i, j \in \mathbf{N}^+$ such that $i \neq j$.

Thus $(u_1, \dots, u_m)$ is linely inde, for if not, then $\exists\, i, j$ such that $\operatorname{null}\varphi_i = \operatorname{null}\varphi_j$, contradicts.

又 $\dim V' = m = \dim V$. Then $(u_1, \dots, u_m)$ is a basis of V whose dual basis is $(\varphi_1, \dots, \varphi_n)$. $\square$.

• *Suppose V is finite-dim and $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following sets are the same.*

(a) $\operatorname{span}(\varphi_1, \dots, \varphi_m)$

(b) $\big((\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)\big)^0$

(c) $\{\varphi \in V' : (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) \subseteq \operatorname{null}\varphi\}$

SOLUTION: By Problem (17), (b) and (c) are equi. By Problem (26) and the corollary in Problem (23),

$$\left.\begin{aligned} \big((\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)\big)^0 &= (\operatorname{null}\varphi_1)^0 + \dots + (\operatorname{null}\varphi_m)^0. \\ \text{又 } \operatorname{span}(\varphi_i) = \{v \in V : \forall \psi \in \operatorname{span}(\varphi_i), \psi(v) = 0\}^0 &= (\operatorname{null}\varphi_i)^0. \end{aligned}\right\} \Rightarrow (a) = (b). \qquad \square$$

COROLLARY: 30 *Suppose V is finite-dim and $\varphi_1, \dots, \varphi_m$ is a linely inde list in V'.*
Then $\dim\big((\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)\big) = (\dim V) - m$.

---

6 *Define $\Gamma : V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$, where $v_1, \dots, v_m \in V$.*

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = V \iff \Gamma$ *is inje.*

(b) *Show that $(v_1, \dots, v_m)$ is linely inde $\iff \Gamma$ is surj.*

SOLUTION:

(a)
- Suppose Γ is inje. Then let $\Gamma(\varphi) = 0$, getting $\varphi = 0 \Leftrightarrow \operatorname{null}\varphi = V = \operatorname{span}(v_1, \dots, v_m)$.
- Suppose $\operatorname{span}(v_1, \dots, v_m) = V$. Then let $\Gamma(\varphi) = 0$, getting $\varphi(v_i) = 0$ for each i,
  $\operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m) = V$, thus $\varphi = 0$, Γ is inje.

(b)
- Suppose Γ is surj. Then let $\Gamma(\varphi_i) = e_i$ for each i, where $(e_1, \dots, e_m)$ is the standard basis of $\mathbf{F}^m$.
  Then $(\varphi_1, \dots, \varphi_m)$ is linely inde, suppose $a_1 v_1 + \dots + a_m v_m = 0$,
  then for each i, we have $\varphi_i(a_1 v_1 + \dots + a_m v_m) = a_i = 0$. Thus $v_1, \dots, v_n$ is linely inde.
- Suppose $(v_1, \dots, v_m)$ is linely inde. Let $(\varphi_1, \dots, \varphi_m)$ be the dual basis of $\operatorname{span}(v_1, \dots, v_m)$.
  Thus for each $(a_1, \dots, a_m) \in \mathbf{F}^m$, we have $\varphi = a_1\varphi_1 + \dots + a_m\varphi_m$ so that $\Gamma(\varphi) = (a_1, \dots, a_m)$. $\square$

• *Define $\Gamma : V \to \mathbf{F}^m$ by $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$, where $\varphi_1, \dots, \varphi_m \in V'$.*

(c) *Show that* $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ *is inje.*

(d) *Show that $(\varphi_1, \dots, \varphi_m)$ is linely inde $\iff \Gamma$ is surj.*

SOLUTION:

(c)
- Suppose Γ is inje. Then $\Gamma(v) = 0 \Leftrightarrow \forall i, \varphi_i(v) = 0 \Leftrightarrow v \in (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) \Leftrightarrow v = 0$.
  Getting $(\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) = \{0\}$. By Problem (•) above, $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V'$
- Suppose $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V'$. Again by Problem (•), $(\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) = \{0\}$.
  Thus $\Gamma(v) = 0 \Rightarrow \forall i, \varphi_i(v) = 0 \Rightarrow v = 0$.

(d)
- Suppose $(\varphi_1, \dots, \varphi_m)$ is linely inde. Then by Problem (31), $(v_1, \dots, v_m)$ is linely inde.
  Thus for any $(a_1, \dots, a_m) \in \mathbf{F}$, by letting $v = \sum_{i=1}^{m} a_i v_i$, then $\varphi_i(v) = a_i \Rightarrow \Gamma(v) = (a_1, \dots, a_m)$.
- Suppose Γ is surj. Let $e_1, \dots, e_m$ be a basis of $\mathbf{F}^m$.
  For every e_i, $\exists\, v_i \in V$ such that $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v)) = e_i$,
  fix v_i ($\Rightarrow (v_1, \dots, v_m)$ is linely inde). Thus $\varphi_i(v_i) = 1, \varphi_i(v_j) = 0$.
  Hence $(\varphi_1, \dots, \varphi_m)$ is the dual basis of the basis $v_1, \dots, \varphi_m$ of $\operatorname{span}(v_1, \dots, v_m)$. $\square$

---

33 *Suppose $A \in \mathbf{F}^{m,n}$. Define $T : A \to A^t$. Prove that T is an iso of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$*

Solution: By [3.111], T is linear. Note that $(A^t)^t = A$.

(a) For any $B \in \mathbf{F}^{n,m}$, let $A = B^t$ so that $T(A) = B$. Thus T is surj.

(b) If $T(A) = 0$ for some $A \in \mathbf{F}^{n,m}$, then $A = 0$. Thus T is inje,

for if not, $\exists\ j, k \in \mathbf{N}^+$ such that $A_{j,k} \neq 0$, then $T(A)_{k,j} \neq 0$, contradicts. $\square$

---

32 *Suppose $T \in \mathcal{L}(V)$, and $(u_1, \dots, u_m), (v_1, \dots, v_m)$ are bases of V. Prove that T is inv $\iff$ the rows of $\mathcal{M}\left(T, (u_1, \dots, u_m), (v_1, \dots, v_m)\right)$ form a basis of $\mathbf{F}^{1,n}$.*

Solution: Note that T is inveritlbe $\iff T'$ is inv. And $\mathcal{M}(T') = \mathcal{M}(T)^t = A^t$, denote it by B.

Let $(\varphi_1, \dots, \varphi_m)$ be the dual basis of $(v_1, \dots, v_m)$, $(\psi_1, \dots, \psi_m)$ be the dual basis of $(u_1, \dots, u_m)$.

(a) Suppose T is inv, so is T'. Because $T'(\varphi_1), \dots, T'(\varphi_m)$ is linely inde.

Noticing that $T'(\varphi_i) = B_{1,i}\psi_1 + \cdots + B_{m,i}\psi_m$.

Thus the cols of B, namely the rows of A, are linely inde (check it by contradiction).

(b) Suppose the rows of A are linely inde, so are the cols of B.

Then $(T'(\varphi_1), \dots, T'(\varphi_m))$ is a basis of range T', namely V'. Thus T' is surj.

Hence T' is inv, so is T. $\square$

---

34 *The double dual space of V, denoted by V'', is defined to be the dual space of V'. In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \to V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.*

(a) *Show that Λ is a linear map from V to V''.*

(b) *Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*

(c) *Show that if V is finite-dim, then Λ is an iso from V onto V''.*

Suppose V is finite-dim. Then V and V' are iso, but finding an iso from V onto V' generally requires choosing a basis of V. In contrast, the iso Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

Solution:

(a) $\forall \varphi \in V'$, $\forall v, w \in V, a \in \mathbf{F}$, $\left(\Lambda(v + aw)\right)(\varphi) = \varphi(v + aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.

Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence Λ is linear.

(b) $\left(T''(\Lambda v)\right)(\varphi) = \left((\Lambda v) \circ (T')\right)(\varphi) = (\Lambda v)\left(T'(\varphi)\right) = \left(T'(\varphi)\right)(v) = (\varphi \circ T)(v) = \varphi(Tv) = \left(\Lambda(Tv)\right)(\varphi)$.

Hence $T''(\Lambda v) = (\Lambda(Tv))$, getting $T'' \circ \Lambda = \Lambda \circ T$.

(c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is inje.

又 Because V is finite-dim. $\dim V = \dim V' = \dim V''$. Hence Λ is an iso. $\square$

---

36 *Suppose U is a subsp of V. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*

(a) *Show that null $i' = U^0$:* *null i'* $= (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$. $\square$

(b) *Prove that if V is finite-dim, then* range $i' = U'$: $\operatorname{range} i' = (\operatorname{null} i)^0_U = (\{0\})^0_U = U'$. $\square$

(c) *Prove that if V is finite-dim, then $\tilde{i'}$ is an iso from V'/U^0 onto U':*

The iso in (c) is natural in that it does not depend on a choice of basis in either vecsp.

Solution: Note that $\tilde{i'} : V'/\operatorname{null} i' \to \operatorname{range} i' \ \Rightarrow\ \tilde{i'} : V'/U^0 \to U'$. By (a), (b) and [3.91(d)]. $\square$

---

37 *Suppose U is a subsp of V and π is the quotient map. Thus $\pi' \in \mathcal{L}\left((V/U)', V'\right)$.*

(a) *Show that π' is inje:* *Because π is surj. Use [3.108].* $\square$

(b) *Show that $\pi' = U^0$.*

(c) *Conclude that π' is an iso from $(V/U)'$ onto U^0.*

The iso in (c) is natural in that it does not depend on a choice of basis in either vecsp.

In fact, there is no assumption here that any of these vecsps are finite-dim.

Solution: [3.109] is not available. Using (3.E.18), also see (3.E.20).

(b) $\psi \in \operatorname{range} \pi' \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null} \psi \supseteq U \iff \psi \in U^0$. Hence $\operatorname{range} \pi' = U^0$.

(c) $\psi \in U^0 \iff \operatorname{null} \psi \supseteq U \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi = \pi(\varphi)$. Thus π' is surj. And by (a). □

Ended

4

• **Note For [4.8]:** *division algorithm for polynomials*

Suppose $p, s \in \mathcal{P}(\mathbf{F})$, with $s \neq 0$. Then $\exists\,!\, q, r \in \mathcal{P}(\mathbf{F})$ such that $p = sq + r$ and $\deg r < \deg s$. *Another Proof:*

Suppose $\deg p \geq \deg s$. Then $(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of length } \deg s}, \underbrace{s, zs, \cdots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)})$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.

Because $q \in \mathcal{P}(\mathbf{F})$, $\exists\,!\, a_i, b_j \in \mathbf{F}$,

$q = a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s$

$= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1}}_{r} + s \underbrace{(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s})}_{q}$.

With r, q as defined uniquely above, we are done. □

• **Note For [4.11]:** *each zero of a poly corresponds to a degree-one factor;* *Another Proof:*

First suppose $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, $\exists\,!\, a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.

Then $p(z) = p(z) - p(\lambda) = a_1(z - \lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.

Hence $\forall k \in \{1, \dots, m\}, z^k - \lambda^k = (z - \lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.

Thus $p(z) = \sum\limits_{j=1}^{m} a_j (z - \lambda) \sum\limits_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) \sum\limits_{j=1}^{m} a_j \sum\limits_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) q(z)$.

• **Note For [4.13]:** *fundamental theorem of algebra, first version*

Every nonconst poly with complex coefficients has a zero in $\mathbf{C}$. *Another Proof:*

For any $w \in \mathbf{C}, k \in \mathbf{N}^+$, by polar coordinates, $\exists\, r \geq 0, \theta \in \mathbf{R}, r(\cos\theta + \mathrm{i}\sin\theta) = w$.

By De Moivre' theorem, $w^k = [r(\cos\theta + \mathrm{i}\sin\theta)]^k = r^k(\cos k\theta + \mathrm{i}\sin k\theta)$.

Hence $\left(r^{1/k}(\cos\frac{\theta}{k} + \mathrm{i}\sin\frac{\theta}{k})\right)^k = w$. Thus every complex number has a k^{th} *root*.

Suppose a nonconst $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.

Then $|p(z)| \to \infty$ as $|z| \to \infty$ (because $\dfrac{|p(z)|}{|z_m|} \to |c_m|$ as $|z| \to \infty$).

Thus the continuous function $z \to |p(z)|$ has a global minimum at some point $\zeta \in \mathbf{C}$.

To show that $p(\zeta) = 0$, assume $p(\zeta) \neq 0$. Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \dfrac{p(z + \zeta)}{p(\zeta)}$.

The function $z \to |q(z)|$ has a global minimum value of 1 at $z = 0$.

Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where $k \in \mathbf{N}^+$ is the smallest such that $a_k \neq 0$.

Let $\beta \in \mathbf{C}$ be such that $\beta^k = -\dfrac{1}{a_k}$.

There is a const $c > 1$ so that if $t \in (0, 1)$, then $|q(t\beta)| \leq \left|1 + a_k t^k \beta^k\right| + t^{k+1}c = 1 - t^k(1 - tc)$.

Now letting $t = 1/(2c)$, we get $|q(t\beta)| < 1$. Contradicts. Hence $p(\zeta) = 0$, as desired. □

• *Prove that if $w, z \in \mathbf{C}$, then $||w| - |z|| \leq |w - z|$.*

Solution:

$$
\begin{aligned}
|w-z|^2 &= (w-z)(\overline{w}-\overline{z}) \\
&= |w|^2+|z|^2-(w\overline{z}+\overline{w}z) \\
&= |w|^2+|z|^2-(\overline{\overline{w}z}+\overline{w}z) \\
&= |w|^2+|z|^2-2Re(\overline{w}z) \\
&\geq |w|^2+|z|^2-2|\overline{w}z| \\
&= |w|^2+|z|^2-2|w||z| = \big||w|-|z|\big|^2 .
\end{aligned}
$$

Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides. □

• *Suppose V is on $\mathbf{C}$ and $\varphi \in V'$. Define $\sigma : V \to \mathbf{R}$ by $\sigma(v) = \operatorname{Re}\varphi(v)$ for each $v \in V$. Show that $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$ for all $v \in V$.*

SOLUTION:

Notice that $\varphi(v) = \operatorname{Re}\varphi(v) + \mathrm{i}\operatorname{Im}\varphi(v) = \sigma(v) + \mathrm{i}\operatorname{Im}\varphi(v)$.

又 $\operatorname{Re}\varphi(\mathrm{i}v) = \operatorname{Re}[\mathrm{i}\varphi(v)] = -\operatorname{Im}\varphi(v) = \sigma(\mathrm{i}v)$.

Hence $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$. □

---

2 *Suppose $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

SOLUTION:

$x^m, x^m + x^{m-1} \in U$ but $\deg\left[(x^m + x^{m-1}) - (x^m)\right] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$.

Hence U is not closed under add, and therefore is not a subsp. □

---

3 *Suppose $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : 2 \mid \deg p\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

SOLUTION:

$x^2, x^2 + x \in U$ but $\deg[(x^2+x)-(x^2)]$ is odd and hence $(x^2+x)-(x^2) \notin U$.

Thus U is not closed under add, and therefore is not a subsp. □

---

5 *Suppose that $m \in \mathbf{N}, z_1, \dots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$. Prove that $\exists\,!\, p \in \mathcal{P}_m(\mathbf{F})$ such that $p(z_k) = w_k$ for each $k = 1, \dots, m+1$.*

SOLUTION:

Define $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by $Tq = (q(z_1), \dots, q(z_m), q(z_{m+1}))$. As can be easily checked, T is linear.

We need to show that T is surj, so that such p exists; and that T is inje, so that such p is unique.

$Tq = 0 \iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0$

$\iff q = 0 \in \mathcal{P}_m(\mathbf{F})$, for if not, q of deg m has at least $m+1$ distinct roots. Contradicts [4.12].

$\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m+1 = \dim \mathbf{F}^{m+1}$. 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1}$. Hence T is surj. □

---

6 *Suppose $p \in \mathcal{P}_m(\mathbf{C})$ has degree m. Prove that p has m distinct zeros $\iff$ p and its derivative p' have no zeros in common.*

SOLUTION:

(a) Suppose p has m distinct zeros. By [4.14] and $\deg p = m$, let $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$, $\exists\,!\, c, \lambda_i \in \mathbf{C}$.

For each $j \in \{1, \dots, m\}$, let $\dfrac{p(z)}{(z-\lambda_j)} = q_j \in \mathcal{P}_{m-1}(\mathbf{C})$, then $p(z) = (z-\lambda_j)q_j(z)$ and $q_j(\lambda_j) \neq 0$.

$p'(z) = (z-\lambda_j){q_j}'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

(b) To prove the implication on the other direction, we prove the contrapositive:

Suppose p has less than m distinct roots.

We must show that p and its derivative p' have at least one zero in common.

Let λ be a zero of p, then write $p(z) = (z-\lambda)^n q(z)$, $\exists!\, n \in \mathbf{N}^+, q \in \mathcal{P}_{m-n}(\mathbf{C})$.

$p'(z) = (z-\lambda)^n q'(z) + n(z-\lambda)^{n-1} q(z) \Rightarrow p'(\lambda) = 0, \lambda$ is a common root of p' and p. $\square$

---

7 *Prove that every $p \in \mathcal{P}(\mathbf{R})$ of odd degree has a zero.*

SOLUTION:

Using the notation and proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence λ_1 exists. $\square$

OR. Using calculus only.

Suppose $p \in \mathcal{P}_m(\mathbf{F}), \deg p = m, m$ is odd.

Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1} a_m$ by δ

Write $p(x) = x^m \left(\dfrac{a_0}{x^m} + \dfrac{a_1}{x^{m-1}} + \cdots + \dfrac{a_{m-1}}{x} + a_m \right)$.

Thus $p(x)$ is continuous, and $\lim\limits_{x \to -\infty} p(x) = -\delta\infty$; $\lim\limits_{x \to \infty} p(x) = \delta\infty$.

Hence we conclude that p has at least one real zero. $\square$

---

8 *For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \to \mathbf{R}$ by $(Tp)(x) = \begin{cases} \dfrac{p(x)-p(3)}{x-3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3 \end{cases}$ for all $x \in \mathbf{R}$.*

Show that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$ and that $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is a linear map.

SOLUTION:

For $x \neq 3, T(x^n) = \dfrac{x^n - 3^n}{x-3} = \sum\limits_{i=1}^{n} 3^{i-1} x^{n-i}$.

For $x = 3, T(x^n) = 3^{n-1} \cdot n$. Note that if $x = 3$, then $\sum\limits_{i=1}^{n} 3^{i-1} x^{n-i} = \sum\limits_{i=1}^{n} 3^{n-1} = 3^{n-1} \cdot n$.

Hence for all $x \in \mathbf{R}$ and for all $n \in \mathbf{N}, T(x^n) = \sum\limits_{i=1}^{n} 3^{i-1} x^{n-i} \in \mathcal{P}(\mathbf{R})$.

Because T is linear, we conclude that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$.

Now we show that T is linear:

$\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, T(p+\lambda q)(x) = \begin{cases} \dfrac{(p+\lambda q)(x) - (p+\lambda q)(3)}{x-3} & \text{if } x \neq 3, \\ (p+\lambda q)'(3) & \text{if } x = 3 \end{cases}$ for all $x \in \mathbf{R}$.

Notice that $\begin{cases} (p+\lambda q)(x) - (p+\lambda q)(3) = (p(x) - p(3)) + (\lambda q(x) - \lambda q(3)). \\ (p+\lambda q)'(3) = p'(3) + \lambda q'(3). \end{cases}$

Thus $T(p+\lambda q)(x) = (T(p) + \lambda T(q))(x)$ for all $x \in \mathbf{R}$. $\square$

---

9 *Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\bar{z})}$. Prove that $q \in \mathcal{P}(\mathbf{R})$.*

SOLUTION:

$p(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow p(\bar{z}) = a_n \bar{z}^n + \cdots + a_1 \bar{z} + a_0 \Rightarrow \overline{p(\bar{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\bar{z})} = \overline{p(\bar{z})} p(z) = \overline{p(\bar{z}) \overline{p(\bar{\bar{z}})}} = \overline{q(\bar{z})}$.

Hence letting $q(z) = c_m x^m + \cdots + c_1 x + c_0 \Rightarrow \overline{c_k} = c_k, c_k \in \mathbf{R}$ for each k. $\square$

---

10 *Suppose $m \in \mathbf{N}$ and $p \in \mathcal{P}_m(\mathbf{C})$ such that $p(x_k) \in \mathbf{R}$ for each x_k, where $x_0, x_1, \ldots, x_m \in \mathbf{R}$ are distinct. Prove that $p \in \mathcal{P}(\mathbf{R})$.*

SOLUTION:

Let $p(x_k) = y_k$ for each k. By Problem (5), $\exists!\, q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_k) = y_k$. Hence $p = q$. $\square$

OR. Using the Lagrange Interpolating Polynomial.

Define $q(x) = \sum_{j=0}^{m} \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j)$.

又 For each j, $x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R}) \subseteq \mathcal{P}_m(\mathbf{C})$.

Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$ for each $k \in \{0, 1, \dots, m\}$.

Then $(q-p)$ has $(m+1)$ distinct zeros, while $(q-p) \in \mathcal{P}_m(\mathbf{C})$. Hence by [4.12], $q - p = 0 \Rightarrow p = q$. □

---

11 *Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*

(a) *Show that* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) *Find a basis of* $\mathcal{P}(\mathbf{F})/U$.

SOLUTION:

U is a subsp of $\mathcal{P}(\mathbf{F})$ because $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, pf + \lambda pg = p(f + \lambda g) \in U$.

NOTE: Define $P :\to \mathcal{P}(\mathbf{F})$ by $(Pq)(x) = p\left(q(x)\right) = (p \circ q)(x)$ ($\neq p(x)q(x)$). P is not linear.

(a) By [4.8], $\forall f \in \mathcal{P}(\mathbf{F}), \exists!\, q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r)$; $\deg r < \deg p$.

Hence $\forall f \in \mathcal{P}(\mathbf{F}), \exists!\, pq \in U, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), f = (pq) + (r); r \notin U$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F})$. Therefore $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p-1}(\mathbf{F})$ are iso.

OR. $\forall f \in \mathcal{P}(\mathbf{F}), \exists!\, q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r)$; $\deg r < \deg p$.

Define $R : \mathcal{P}(\mathbf{F}) \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $(Rf)(z) = r(z)$ for each $z \in \mathbf{F}$.

$\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, R(f + \lambda g)(z) = R(f) + \lambda R(g)$.

BECAUSE: $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}$,

$\exists!\, q_1, r_1 \in \mathcal{P}(\mathbf{F}), f = (p)q_1 + (r_1)$, $\deg r_1 < \deg p$;

$\exists!\, q_2, r_2 \in \mathcal{P}(\mathbf{F}), g = (p)q_2 + (r_2)$, $\deg r_2 < \deg p$;

$\exists!\, q_3, r_3 \in \mathcal{P}(\mathbf{F}), \lambda g = (p)q_3 + (r_3) = (p)(\lambda q_2) + (\lambda r_2)$, $\deg r_3 < \deg p$ and $\deg \lambda r_2 < \deg p$.

$\Rightarrow q_3 = \lambda q_2, r_3 = \lambda r_2$.

$\exists!\, q_0, r_0 \in \mathcal{P}(\mathbf{F}), (f + \lambda g) = (p)q_0 + (r_0)$

$= (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2)$, $\deg r_0 < \deg p$ and $\deg(r_1 + \lambda r_2) < \deg p$.

$\Rightarrow q_1 + \lambda q_2 = q_0$; $r_1 + \lambda r_2 = r_0$.

Hence R is linear.

$R(f) = 0 \iff f = pq, \exists!\, q \in \mathcal{P}(\mathbf{F})$. Thus $\operatorname{null} R = U$.

$\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F})$, let $f = p + r$, then $R(f) = r$. Thus $\operatorname{range} R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Finally, by [3.91(d)], $\mathcal{P}(\mathbf{F})/\operatorname{null} R$, namely $\mathcal{P}(\mathbf{F})/U$, and $\operatorname{range} R$, namely $\mathcal{P}_{\deg p-1}(\mathbf{F})$, are iso.

(b) $(1 + U, x + U, \dots, x^{\deg p - 1}) + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. □

• *Suppose nonconst $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p$, $n = \deg q$. Use* (a)–(c) *below to prove that $\exists!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*

(a) *Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r, s) = rp + sq$.*
Show that the linear map T is inje.

(b) *Show that the linear map T in* (a) *is surj.*

(c) *Use* (b) *to conclude that $\exists!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*

SOLUTION:

(a) T is linear because $\forall\, r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,

$T\left((r_1, s_1) + \lambda(r_2, s_2)\right) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1, s_1) + \lambda T(r_2, s_2)$.

Suppose $T(r, s) = rp + sq = 0$. Notice that p, q have no zeros in common.

Then $r = s = 0$, for if not, write $\frac{q(z)}{r(z)} = \frac{p(z)}{s(z)}$, while for any zero λ of q, $\frac{q(\lambda)}{r(z)} = 0 \neq \frac{p(\lambda)}{s(z)}$. □

(b) $\dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) = \dim \mathcal{P}_{n-1}(\mathbf{C}) + \dim \mathcal{P}_{m-1}(\mathbf{C}) = n + m = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.
又 T is inje. Hence $\dim \operatorname{range} T = \dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) - \dim \operatorname{null} T = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.
Thus $\operatorname{range} T = \mathcal{P}m + n - 1 \Rightarrow T$ is surj, and therefore is an iso.
(c) Immediately. $\square$

ENDED

5.A

[1]: 31; [2]: 1, 2, 3, 15, 21; [3]: 23, (2E Ch5.20), (4E.5.A.37), 4, 5; [4]: 6, (4E.5.A.17, 18) OR16, (4E.5.A.15); [5]: 7, 8, (4E.5.A.8), 22, 9, 10; [6]: 11, 12, 14, 30, 13, (4E.5.A.11); [7]: 17, (4E.5.A.16), 18; [8]: 19, 20, 24; [9]: 24', 25, 26, 27, 28; [10]: (4E.5.A.39), 29; [11]: 32, (4E.5.A.35), (4E.5.A.38) OR35, 36; [12] 32, 34.

- **NOTE FOR [5.6]:**
More generally, suppose we do not know whether V is finite-dim. Then $(a) \iff (b)$.
Suppose (a) λ is an eigval of T with an eigvec v. Then $(T - \lambda I)v = 0$.
Hence we get (b), $(T - \lambda I)$ is not inje. And then (d), $(T - \lambda I)$ is not inv.
But $(d) \Rightarrow (b)$ fails (because S is not inv $\iff$ S is not inje *or* S is not surj).

---

31 *Suppose V is finite-dim and $v_1, \dots, v_m \in V$. Prove that $(v_1, \dots, v_m)$ is linely inde $\iff \exists\, T \in \mathcal{L}(V)$, $v_1, \dots, v_m$ are eigvecs of T correspd to distinct eigvals.*

SOLUTION:
Suppose $(v_1, \dots, v_m)$ is linely inde, extend it to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.
Define $T \in \mathcal{L}(V)$ by $Tv_k = kv_k$ for each $k \in \{1, \dots, m, \dots, n\}$. Conversely by [5.10]. $\square$

---

1 *Suppose $T \in \mathcal{L}(V)$ and U is a subsp of V.*
(a) *Prove that if $U \subseteq \operatorname{null} T$, then U is invar under T.* $\forall u \in U \subseteq \operatorname{null} T, Tu = 0 \in U$. $\square$
(b) *Prove that if $\operatorname{range} T \subseteq U$, then U is invar under T.* $\forall u \in U, Tu \in \operatorname{range} T \subseteq U$. $\square$

- *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*
(a) *Prove that $\operatorname{null}(T - \lambda I)$ is invar under S for any $\lambda \in \mathbf{F}$.*
(b) *Prove that $\operatorname{range}(T - \lambda I)$ is invar under S for any $\lambda \in \mathbf{F}$.*

SOLUTION: Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.
(a) Suppose $v \in \operatorname{null}(T - \lambda I)$, then $(T - \lambda I)(Sv) = S(T - \lambda I)v = S(0) = 0$.
Hence $Sv \in \operatorname{null}(T - \lambda I)$ and therefore $\operatorname{null}(T - \lambda I)$ is invar under S.
(b) Suppose $v \in \operatorname{range}(T - \lambda I)$, therefore $\exists\, u \in V, (T - \lambda I)u = v$.
Then $Sv = S(T - \lambda I)u = (T - \lambda I)(Su) \in \operatorname{range}(T - \lambda I)$.
Hence $Sv \in \operatorname{range}(T - \lambda I)$ and therefore $\operatorname{range}(T - \lambda I)$ is invar under S. $\square$

- *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*
2 *Show that $W = \operatorname{null} T$ is invar under S.* $\forall u \in W, Tu = 0 \Rightarrow STu = 0 = TSu \Rightarrow Su \in W$. $\square$
3 *Show that $U = \operatorname{range} T$ is invar under S.* $\forall w \in U, \exists\, v \in V, Tv = w, TSv = STv = Sw \in U$. $\square$

---

15 *Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is inv.*
(a) *Prove that T and $S^{-1}TS$ have the same eigvals.*
(b) *What is the relationship between the eigvecs of T and the eigvecs of $S^{-1}TS$?*

SOLUTION:

Suppose λ is an eigval of T with an eigvec v.
Then $S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v$.
Thus λ is also an eigval of $S^{-1}TS$ with an eigvec $S^{-1}v$.

Suppose λ is an eigval of $S^{-1}TS$ with an eigvec v.
Then $S(S^{-1}TS)v = TSv = \lambda Sv$.
Thus λ is also an eigval of T with an eigvec Sv. $\square$

OR. Note that $S(S^{-1}TS)S^{-1} = T$. Hence every eigval of $S^{-1}TS$ is an eigval of $S(S^{-1}TS)S^{-1} = T$.
And every eigvec v of $S^{-1}TS$ is $S^{-1}v$, every eigvec u of T is Su. $\square$

---

21 *Suppose $T \in \mathcal{L}(V)$ is inv.*
(a) *Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that λ is an eigval of $T \iff \frac{1}{\lambda}$ is an eigval of T^{-1}.*
(b) *Prove that T and T^{-1} have the same eigvecs.*

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec v.
Then $T^{-1}Tv = \lambda T^{-1}v = v \Rightarrow T^{-1}v = \frac{1}{\lambda}v$. Hence $\frac{1}{\lambda}$ is an eigval of T^{-1}.
(b) Suppose $\frac{1}{\lambda}$ is an eigval of T^{-1} with an eigvec v.
Then $TT^{-1}v = v = \frac{1}{\lambda}Tv \Rightarrow Tv = \lambda v$. Hence λ is an eigval of T.
OR. Note that $(T^{-1})^{-1} = T$ and $1/(\frac{1}{\lambda}) = \lambda$. $\square$

---

23 *Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigvals.*

SOLUTION:

Suppose λ is an eigval of ST with an eigvec v. Then $T(STv) = \lambda Tv = TS(Tv)$.
If $Tv = 0$ (while $v \neq 0$), then T is not inje $\Rightarrow (TS - 0I)$ and $(ST - 0I)$ are not inje.
Thus $\lambda = 0$ is an eigval of ST and TS with the same eigvec v.
Otherwise, $Tv \neq 0$, then λ is an eigval of TS. Reversing the roles of T and S. $\square$

• (2E Ch5.20)
Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigvals and $S \in \mathcal{L}(V)$ has the same eigvecs (but might not with the same eigvals). Prove that $ST = TS$.

SOLUTION:

Let $n = \dim V$. For each $j \in \{1, \dots, n\}$, let v_j be an eigvec with eigval λ_j of T and α_j of S.
Then $(v_1, \dots, v_n)$ is a basis of V. Because $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$ for each j. Hence $ST = TS$. $\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(S) = TS$ for each $S \in \mathcal{L}(V)$.
Prove that the set of eigvals of T equals the set of eigvals of $\mathcal{A}$.

SOLUTION:

(a) Suppose $v_1, \dots, v_m$ are all linely inde eigvecs of T
with correspd eigvals $\lambda_1, \dots, \lambda_m$ respectively (possibly with repetitions).
Extend to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.
Then for each $k \in \{1, \dots, m\}$, $\operatorname{span}(v_k) \subseteq \operatorname{null}(T - \lambda_k I)$.
Define $S_k \in \mathcal{L}(V)$ by $S_k(v_j) = v_k$ for each $j \in \{1, \dots, n\}$,

so that range $S_k = \operatorname{span}(v_k)$ for each $k \in \{1, \dots, m\}$, then $\mathcal{A}(S_k) = TS_k = \lambda_k S_k$.

Thus the eigvals of T are eigvals of $\mathcal{A}$.

(b) Suppose $\lambda_1, \dots, \lambda_m$ are all eigvals of $\mathcal{A}$ with eigvecs $S_1, \dots, S_m$ respectively.

Then for each $k \in \{1, \dots, m\}$, $\exists\, v \in V, 0 \neq u = S_k(v) \in V \Rightarrow Tu = (TS_k)v = (\lambda_k S_k)v = \lambda_k u$.

Thus the eigvals of $\mathcal{A}$ are eigvals of T. □

OR.

(a) Suppose λ is an eigval of T with an eigvec v.

Let $v_1 = v$ and extend to a basis $(v_1, \dots, v_m)$ of V.

Define $S \in \mathcal{L}(V)$ by $Sv_1 = v_1,\ Sv_k = 0$ for $k \geq 2$.

Then $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$.

Hence $(T - \lambda I)S = 0 \Rightarrow TS = \lambda S$ while $S \neq 0$. Thus λ is also an eigval of $\mathcal{A}$.

(b) Suppose λ is an eigval of $\mathcal{A}$ with an eigvec S. Then $(T - \lambda I)S = 0$ while $S \neq 0$.

Hence $(T - \lambda I)$ is not inje. Thus λ is also an eigval of T. □

COMMENT: Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(S) = ST$, $\forall S \in \mathcal{L}(V)$. Then the eigvals of $\mathcal{B}$ are not the eigvals of T.

---

4 *Suppose $T \in \mathcal{L}(V)$ and $V_1, \dots, V_m$ are invar subsps of V under T. Prove that $V_1 + \cdots + V_m$ is invar under T.*

SOLUTION: For each $i = 1, \dots, m$, $\forall v_i \in V_i, Tv_i \in V_i$

Hence $\forall v = v_1 + \cdots + v_m \in V_1 + \cdots + V_m, Tv = Tv_1 + \cdots + Tv_m \in V_1 + \cdots + V_m$. □

---

6 *Prove or give a counterexample:*
If V is finite-dim and U is a subsp of V that is invar under every operator on V, then $U = \{0\}$ or $U = V$.

SOLUTION:

Notice that V might be $\{0\}$. In this case we are done. Suppose $\dim V \geq 1$. We prove by contrapositive:

Suppose $U \neq \{0\}$ and $U \neq V$. Prove that $\exists\, T \in \mathcal{L}(V)$ such that U is not invar under T.

Let W be such that $V = U \oplus W$.

Let $(u_1, \dots, u_m)$ be a basis of U and $(w_1, \dots, w_n)$ be a basis of W.

Hence $(u_1, \dots, u_m, w_1, \dots, w_n)$ is a basis of V.

Define $T \in \mathcal{L}(V)$ by $T(a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n) = b_1w_1 + \cdots + b_nw_n$. □

• *Suppose* $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.

(a) (OR (9.11)) $\lambda \in \mathbf{R}$. *Prove that λ is an eigval of $T \Longleftrightarrow \lambda$ is an eigval of T_{C}.*

(b) (OR Problem (16)) $\lambda \in \mathbf{C}$. *Prove that λ is an eigval of $T_{\mathrm{C}} \Longleftrightarrow \overline{\lambda}$ is an eigval of T_{C}.*

SOLUTION:

(a) Suppose $v \in V$ is an eigvec correspd to the eigval λ.

Then $Tv = \lambda v \Rightarrow T_{\mathrm{C}}(v + \mathrm{i}0) = Tv + \mathrm{i}T0 = \lambda v$.

Thus λ is an eigval of T.

Suppose $v + \mathrm{i}u \in V_{\mathrm{C}}$ is an eigvec correspd to the eigval λ.

Then $T_{\mathrm{C}}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \Rightarrow Tv = \lambda v, Tu = \lambda u$. (Note that v or u might be zero).

Thus λ is an eigval of T_{C}.

(b) Suppose λ is an eigval of T_{C} with an eigvec $v + \mathrm{i}u$.

Let $(v_1, \dots, v_n)$ be a basis of V. Write $v = \sum\limits_{i=1}^{n} a_i v_i, u = \sum\limits_{i=1}^{n} b_i v_i$, where $a_i, b_i \in \mathbf{R}$.

Then $T_{\mathbf{C}}(v+\mathrm{i}u)=Tv+\mathrm{i}Tu=\lambda v+\mathrm{i}\lambda u=\lambda\sum_{i=1}^{n}(a_i+\mathrm{i}b_i)v_i$. Conjugating two sides, we have:

$\overline{T_{\mathbf{C}}(v+\mathrm{i}u)}=\overline{Tv+\mathrm{i}Tu}=\overline{Tv}-\mathrm{i}\overline{Tu}=Tv-\mathrm{i}Tu=T_{\mathbf{C}}(\overline{v+\mathrm{i}u})=\overline{\lambda\sum_{i=1}^{n}(a_i+\mathrm{i}b_i)v_i}=\overline{\lambda}\sum_{i=1}^{n}(a_i-\mathrm{i}b_i)v_i$.

Hence $\overline{\lambda}$ is an eigval of $T_{\mathbf{C}}$. To prove the other direction, notice that $\overline{\left(\overline{\lambda}\right)}=\lambda$. ☐

• *Suppose V is finite-dim, $T\in\mathcal{L}(V)$, and $\lambda\in\mathbf{F}$.*
Show that λ is an eigval of $T\iff\lambda$ is an eigval of the dual operator $T'\in\mathcal{L}(V')$.

Solution:

(a) Suppose λ is an eigval of T with an eigvec v.
Then $(T-\lambda I_V)$ is not inv. 又 V is finite-dim.
Thus by [3.108, 110], [3.101] and Problem (12) in (3.F), $(T-\lambda I_V)'=T'-\lambda I_{V'}$ is not inv.
Hence λ is an eigval of T'.

(b) Suppose λ is an eigval T' with an eigvec ψ. Then $T'(\psi)=\psi\circ T=\lambda\psi$.
又 $\psi\neq 0\Rightarrow\exists\, v\in V$ such that $\psi(v)\neq 0$. Note that $\psi(Tv)=\lambda\psi(v)$.
Thus $\lambda=\dfrac{\psi(Tv)}{\psi(v)}\Rightarrow Tv=\dfrac{\psi(Tv)}{\psi(v)}v=\lambda v$. Hence λ is an eigval of T. ☐

---

7 *Suppose $T\in\mathcal{L}(\mathbf{R}^2)$ is defined by $T(x,y)=(-3y,x)$. Find the eigvals of T.*

Solution:

Suppose $\lambda\in\mathbf{R}$ and $(x,y)\in\mathbf{R}^2\backslash\{0\}$ such that $T(x,y)=(-3y,x)=\lambda(x,y)$. Then $-3y=\lambda x$ and $x=\lambda y$.
Thus $-3y=\lambda^2 y\Rightarrow\lambda^2=-3$, ignoring the possibility of $y=0$ (because if $y=0$, then $x=0$).
Hence the set of solution for this equation is $\emptyset$, and therefore T has no eigvals in $\mathbf{R}$. ☐

---

8 *Define $T\in\mathcal{L}(\mathbf{F}^2)$ by $T(w,z)=(z,w)$. Find all eigvals and eigvecs of T.*

Solution:

Suppose $\lambda\in\mathbf{F}$ and $(w,z)\in\mathbf{F}^2$ such that $T(w,z)=(z,w)=\lambda(w,z)$. Then $z=\lambda w$ and $w=\lambda z$.
Thus $z=\lambda^2 z\Rightarrow\lambda^2=1$, ignoring the possibility of $z=0$ ($z=0\Rightarrow w=0$).
Hence $\lambda_1=-1$ and $\lambda_2=1$ are all eigvals of T. For $\lambda_1=-1, z=-w, w=-z$; For $\lambda_2=1, z=w$.
Thus the set of all eigvecs is $\{(z,-z),(z,z): z\in\mathbf{F}\ \wedge\ z\neq 0\}$. ☐

• *Suppose $P\in\mathcal{L}(V)$ is such that $P^2=P$.*
Prove that if λ is an eigval of P, then $\lambda=0$ or $\lambda=1$.

Solution: (See also at (3.B), just below Problem (25), where (5.B.4) was answered.)

Suppose λ is an eigval with an eigvec v. Then $P(Pv)=Pv\Rightarrow\lambda^2 v=\lambda v$. Thus $\lambda=1$ or 0. ☐

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22 *Suppose $T\in\mathcal{L}(V)$ and $\exists$ nonzero vecs u,w in V such that $Tu=3w$ and $Tw=3u$.*
Prove that 3 or -3 is an eigval of T.

Solution: **Comment:** $Tu=3w, Tw=3u\Rightarrow T(Tu)=9u\Rightarrow T^2$ has an eigval 9.

$Tu=3w, Tw=3u\Rightarrow T(u+w)=3(u+w), T(u-w)=3(w-u)=-3(u-w)$. ☐

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9 *Define $T\in\mathcal{L}(\mathbf{F}^3)$ by $T(z_1,z_2,z_3)=(2z_2,0,5z_3)$.*
Find all eigvals and eigvecs of T.

Solution:

Suppose λ is an eigval of T with an eigvec $(z_1,z_2,z_3)\in\mathbf{F}^3$.

Then $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$. Thus $2z_2 = \lambda z_1, \quad 0 = \lambda z_2, \quad 5z_3 = \lambda z_3$.
We discuss in two cases:
For $\lambda = 0$, $z_2 = z_3 = 0$ and z_1 can be arbitrary ($z_1 \neq 0$).
For $\lambda \neq 0$, $z_2 = 0 = z_1$, and z_3 can be arbitrary ($z_3 \neq 0$), then $\lambda = 5$.
The set of all eigvecs is $\{(0,0,z), (z,0,0) : z \in \mathbf{F} \wedge z \neq 0\}$. □

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10 *Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$*
(a) *Find all eigvals and eigvecs of T.*
(b) *Find all invar subsps of V under T.*

SOLUTION:
(a) Suppose $v = (x_1, x_2, x_3, \dots, x_n)$ is an eigvec of T with an eigval λ.
Then $Tv = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$.
Hence $1, \dots, n$ are eigvals of T.
And $\{(0, \dots, 0, x_\lambda, 0, \dots, 0) \in \mathbf{F}^n : \lambda = 1, \dots, n,\ x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$ is the set of all eigvecs of T.
(b) Let $V_\lambda = \{(0, \dots, 0, x_\lambda, 0, \dots, 0) \in \mathbf{F}^n : x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$. Then $V_1, \dots, V_n$ are invar under T.
Hence by Problem (4), every sum of $V_1, \dots, V_n$ is a invar subsp of V under T. □

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11 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigvals and eigvecs of T.*

SOLUTION:
Note that in general, $\deg p' < \deg p$ ($\deg 0 = -\infty$).
Suppose λ is an eigval of T with an eigvec p.
Suppose $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p \neq p'$. Contradicts. Thus $\lambda = 0$.
Therefore $\deg \lambda p = -\infty = \deg p \Rightarrow p$ is a nonzero const poly.
Hence the set of all eigvecs is $\{C : C \in \mathbf{R} \wedge C \neq 0\} = \mathcal{P}_0(\mathbf{R}) \backslash \{0\}$. □

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12 *Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$.*
Find all eigvals and eigvecs of T.

SOLUTION:
Suppose λ is an eigval of T with an eigvec p, then $(Tp)(x) = xp'(x) = \lambda p(x)$.
Let $p = a_0 + a_1 x + \cdots + a_n x^n$.
Then $xp'(x) = a_1 x + 2a_2 x^2 + \cdots + na_n x^n = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n$.
Similar to Problem (10), $0, 1, \dots, n$ are eigvals of T.
The set of all eigvecs of T is $\{cx^\lambda : \lambda = 0, 1, \dots, n,\ c \in \mathbf{F} \wedge c \neq 0\}$. □

---

30 *Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigvals of T.*
Prove that $\exists\, x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

SOLUTION: Because 9 is not an eigval. Hence $(T - 9I)$ is surj. □

---

14 *Suppose $V = U \oplus W$, where U and W are nonzero subsps of V.*
Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.
Find all eigvals and eigvecs of P.

SOLUTION:
Suppose λ is an eigval of P with an eigvec $(u + w)$.
Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$. By [1.44] and $V = U \oplus W$, $(\lambda - 1)u = \lambda w = 0$.

Thus if $\lambda = 1$, then $w = 0$; if $\lambda = 0$, then $u = 0$.

Hence the eigvals of P are 0 and 1, the set of all eigvecs in P is $U \cup W$. $\square$

---

13 *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*
Prove that $\exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}$ and $(T - \alpha I)$ is inv.

Solution:

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \dfrac{1}{1000 + k}$ for each $k = 1, \ldots, \dim V + 1$.

Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigvals.

Hence $\exists\, k = 1, \ldots, \dim V + 1$ such that α_k is not an eigval of T and therefore $(T - \alpha_k I)$ is inv. $\square$

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*
Prove that $\exists\, \delta > 0$ such that $(T - \alpha I)$ is inv for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.

Solution:

If T has no eigvals, then $(T - \alpha I)$ is inje for all $\alpha \in \mathbf{F}$ and we are done.

Let $\delta > 0$ be such that, for each eigval λ_k, $\lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.

So that for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$, $(T - \alpha I)$ is not inje. $\square$

---

17 *Give an example of an operator on $\mathbf{R}^4$ that has no (real) eigvals.*

Solution: Where (e_1, e_2, e_3, e_4) is the standard basis of $\mathbf{R}^4$.

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $\mathcal{M}\left(T, (e_1, e_2, e_3, e_4)\right) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$.

Suppose λ is an eigval of T with an eigvec (x, y, z, w).

Then $T(x, y, z, w) = \lambda(x, y, z, w) \Rightarrow \begin{cases} (1-\lambda)x + y + z + w = 0 \\ -x + (1-\lambda)y - z - w = 0 \\ 3x + 8y + (11-\lambda)z + 5w = 0 \\ 3x - 8y - 11z + (5-\lambda)w = 0 \end{cases}$

This linear equation has no solutions.

(You can type it on `https://zh.numberempire.com/equationsolver.php` to check.)

Or. Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Suppose λ is an eigval of T with an eigvec (x, y, z, w).

Then $T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \Rightarrow \begin{cases} -y = \lambda x \\ x = \lambda y \\ -w = \lambda z \\ z = \lambda w \end{cases} \Rightarrow \begin{cases} -xy = \lambda^2 xy \\ -zw = \lambda^2 zw \end{cases}$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Otherwise, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, contradicts.

Similarly, $y = z = w = 0$. Then we fail. Thus T has no eigvals. $\square$

• *Suppose $(v_1, \ldots, v_n)$ is a basis of V and $T \in \mathcal{L}(V)$, $\mathcal{M}\left(T, (v_1, \ldots, v_n)\right) = A$.*
Prove that if λ is an eigval of T, then $|\lambda| \leq n \max\left\{\left|A_{j,k}\right| : 1 \leq j, k \leq n\right\}$.

Solution:

First we show that $|\lambda| = n \max\left\{\left|A_{j,k}\right| : 1 \leq j, k \leq n\right\}$ for some cases.

Consider $A = \begin{pmatrix} k & \cdots & k \\ \vdots & \ddots & \vdots \\ k & \cdots & k \end{pmatrix}$. Then nk is an eigval of T with an eigvec $v_1 + \cdots + v_n$.

Now we show that if $|\lambda| \neq n \max\left\{\left|A_{j,k}\right| : 1 \leq j,k \leq n\right\}$, then $|\lambda| < n \max\left\{\left|A_{j,k}\right| : 1 \leq j,k \leq n\right\}$.

---

18 *Show that the forward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$ has no eigvals.*

Solution:

Suppose λ is an eigval of T with an eigvec $(z_1, z_2, \dots)$.
Then $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$.
Thus $\lambda z_1 = 0, \lambda z_2 = z_1, \dots, \lambda z_k = z_{k-1}, \dots$.
Let $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \lambda z_k = z_{k-1}$, therefore $(z_1, z_2, \dots) = 0$ is not an eigvec.
Suppose $\lambda \neq 0$. Then $\lambda z_1 = 0 \Rightarrow z_1 = 0 \Rightarrow z_2 = 0 = z_k$ for all $k \in \mathbf{N}^+$.
And then $(z_1, z_2, \dots) = 0$ is not an eigvec. Hence T has no eigvals. □

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19 *Suppose $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by*

$$T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n).$$

In other words, the entries of $\mathcal{M}(T)$ with resp to the standard basis are all 1's. Find all eigvals and eigvecs of T.

Solution:

Suppose λ is an eigval of T with an eigvec $(x_1, \dots, x_n)$.
Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$.
Thus $\lambda x_1 = \dots = \lambda x_n = x_1 + \dots + x_n$.
For $\lambda = 0$, $x_1 + \dots + x_n = 0$.
For $\lambda \neq 0$, $x_1 = \dots = x_n$ and then $\lambda x_k = n x_k$ for each k.
Hence $0, n$ are eigvecs of T.
And the set of all eigvecs of T is $\{(x_1, \dots, x_n) \in \mathbf{F}^n : x_1 + \dots + x_n = 0 \vee x_1 = \dots = x_n\}$. □

---

20 *Define the backward shift operator $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.*
(a) *Show that every element of $\mathbf{F}$ is an eigval of S.*
(b) *Find all eigvecs of S.*

Solution:

Suppose λ is an eigval of S with an eigvec $(z_1, z_2, \dots)$.
Then $S(z_1, z_2, z_3 \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$.
Thus $\lambda z_1 = z_2, \lambda z_2 = z_3, \dots, \lambda z_k = z_{k+1}, \dots$.
For $\lambda = 0$, $\lambda z_1 = z_2 = 0 = \lambda z_2 = z_3 = \dots = z_k$ for all k.
While z_1 can be arbitrary, so that $(z_1, 0, \dots)$ is an eigvec with $z_1 \neq 0$.
For $\lambda \neq 0$, $\lambda^k z_1 = \lambda^{k-1} z_2 = \dots = \lambda z_k = z_{k+1}$ for all k.
Then $(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots)$ is an eigvec with $z_1 \neq 0$.
Hence (a) each element of $\lambda \in \mathbf{F}$ is an eigval of T.
And (b) the set of all eigvecs of T is $\left\{(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots) \in \mathbf{F}^\infty : \lambda \in \mathbf{F}, z_1 \neq 0\right\}$ □

---

24 *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = Ax$, where elements of $\mathbf{F}^n$ are thought of as n-by-1 col vecs.*
(a) *Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigval of T.*

(b) *Suppose the sum of the entries in each col of A equals 1.*
Prove that 1 is an eigval of T.

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $Tx = Ax = \begin{pmatrix} \sum_{c=1}^{n} A_{1,c}x_c \\ \vdots \\ \sum_{c=1}^{n} A_{n,c}x_c \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. While $\sum_{r=1}^{n} A_{R,c} = 1$ for each $R = 1, \dots, n$.

Thus if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, and hence is an eigval of T.

(b) Suppose λ is an eigval of T with an eigvec $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $Tx = Ax = \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}x_r \\ \vdots \\ \sum_{r=1}^{n} A_{n,r}x_r \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. While $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C = 1, \dots, n$.

Thus $\sum_{r=1}^{n} (Ax)_{r,\cdot} = \sum_{r=1}^{n} (Ax)_{r,1}$

$$= \sum_{c=1}^{n} (A_{1,c} + \cdots + A_{n,c})x_c = \sum_{c=1}^{n} x_c = \lambda \begin{pmatrix} x_1 \\ + \\ \vdots \\ + \\ x_n \end{pmatrix}.$$

Hence $\lambda = 1$, for all x such that $\sum_{c=1}^{n} x_{c,1} \neq 0$. □

OR. Prove that $(T - I)$ is not inv, so that we can conclude $\lambda = 1$ is an eigval.

Because $(T - I)x = (A - \mathcal{M}(I))\, x = \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^{n} A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

Then $y_1 + \cdots + y_n = \sum_{r=1}^{n} \sum_{c=1}^{n} (A_{r,c}x_c - x_r) = \sum_{c=1}^{n} x_c \sum_{r=1}^{n} A_{r,c} - \sum_{r=1}^{n} x_r = 0$.

Thus range $(T - I) \subseteq \{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbf{F}^n : y_1 + \cdots + y_n = 0 \}$. Hence $(T - I)$ is not surj. □

- *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = xA$, where elements of $\mathbf{F}^n$ are thought of as 1-by-n row vecs.*

  (a) *Suppose the sum of the entries in each col of A equals 1.*
  Prove that 1 is an eigval of T.

  (b) *Suppose the sum of the entries in each row of A equals 1.*
  Prove that 1 is an eigval of T.

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.

Then $Tx = xA = \begin{pmatrix} \sum_{r=1}^{n} x_r A_{r,1} & \cdots & \sum_{r=1}^{n} x_r A_{r,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. While $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C = 1, \dots, n$.

Thus if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, hence is an eigval of T.

(b) Suppose λ is an eigval of T with an eigvec $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.

Then $Tx = xA = \begin{pmatrix} \sum_{c=1}^{n} x_c A_{c,1} & \cdots & \sum_{c=1}^{n} x_c A_{c,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. While $\sum_{c=1}^{n} A_{R,c} = 1$ for each $R = 1, \dots, n$.

Thus $\sum_{c=1}^{n} (xA)_{\cdot,c} = \sum_{c=1}^{n} (xA)_{1,c} = \sum_{c=1}^{n} (A_{c,1} + \cdots + A_{c,n}) x_c = \sum_{c=1}^{n} x_c = \lambda \big(x_1 \ + \ \cdots \ + \ x_n \big)$.

Hence $\lambda = 1$, for all x such that $\sum_{r=1}^{n} x_{1,r} \neq 0$. ☐

OR. Prove that $(T - I)$ is not inv, so that we can conclude $\lambda = 1$ is an eigval.

Because $(T-I)x = x\,(A - \mathcal{M}(I)) == \begin{pmatrix} \sum_{c=1}^{n} x_c A_{c,1} - x_1 & \cdots & \sum_{c=1}^{n} x_c A_{c,n} - x_n \end{pmatrix} = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}$.

Then $y_1 + \cdots + y_n = \sum_{c=1}^{n} \sum_{r=1}^{n} (x_r A_{r,c} - x_c) = \sum_{r=1}^{n} x_r \sum_{c=1}^{n} A_{r,c} - \sum_{c=1}^{n} x_c = 0$.

Thus range $(T - I) \subseteq \{ \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}. \in \mathbf{F}^n : y_1 + \cdots + y_n = 0 \}$. Hence $(T - I)$ is not surj. ☐

---

25 *Suppose $T \in \mathcal{L}(V)$ and u, w are eigvecs of T such that $u + w$ is also an eigvec of T. Prove that u and w are eigvecs of T correspd to the same eigval.*

SOLUTION:

Suppose $\lambda_1, \lambda_2, \lambda_0$ are eigvals of T correspd to $u, w, u + w$ respectively.

Then $T(u + w) = \lambda_0 (u + w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1) u = (\lambda_2 - \lambda_0) w$.

Notice that $u, w, u + w$ are nonzero.

If (u, w) is linely depe, then let $w = cu$, therefore

$$\lambda_2 cu = Tw = cTu = \lambda_1 cu \qquad \Rightarrow \lambda_2 = \lambda_1.$$
$$\lambda_0 (u + w) = T(u + w) = \lambda_1 u + \lambda_1 cu = \lambda_1 (u + w) \qquad \Rightarrow \lambda_0 = \lambda_1.$$

Otherwise, $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. ☐

---

26 *Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vec in V is an eigvec of T. Prove that T is a scalar multi of the identity operator.*

SOLUTION:

Because $\forall v \in V, \exists!\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$. For any two distinct nonzero vecs $v, w \in V$,

$T(v + w) = \lambda_{v+w}(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v) v = (\lambda_w - \lambda_{v+w}) w$.

If (v, w) is linely inde, then let $w = cv$, therefore

$$\lambda_v cv = cTv = Tw = \lambda_w w \qquad \Rightarrow \lambda_w = \lambda_v.$$
$$\lambda_{v+w}(v + w) = T(v + w) = Tv + Tw = \lambda_v (v + cv) \ \Rightarrow \lambda_{v+w} = \lambda_v.$$

Otherwise, $\lambda_v = \lambda_{v+w} = \lambda_w$. ☐

27, 28 *Suppose V is finite-dim and $k \in \{1, \dots, \dim V - 1\}$. Suppose $T \in \mathcal{L}(V)$ is such that every subsp of V of dim k is invar under T. Prove that T is a scalar multi of the identity operator.*

SOLUTION: We prove the contrapositive:

Suppose T is not a scalar multi of I. Prove that $\exists$ an invar subsp U of V under T such that $\dim U = k$.

By Problem (26), $\exists\, v \in V$ and $v \neq 0$ such that v is not an eigvec of T.

Thus (v, Tv) is linely inde. Extend to a basis of V as $(v, Tv, u_1, \dots, u_n)$.

Let $U = \operatorname{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$ is not an invar subsp of V under T.

OR. Suppose $0 \neq v = v_1 \in V$ and extend to a basis of V as $(v_1, \dots, v_n)$.

Suppose $Tv_1 = c_1 v_1 + \cdots + c_n v_n$, $\exists!\, c_i \in \mathbf{F}$.

Consider a k-dim subsp $U = \operatorname{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$,

where $\alpha_j \in \{2, \dots, n\}$ for each j, and $\alpha_1, \dots, \alpha_{k-1}$ are distinct.

Because every subsp such U is invar.

Thus $Tv_1 = c_1v_1 + \cdots + c_nv_n \in U \Rightarrow c_2 = \cdots = c_n = 0$,

for if not, for each $c_i \neq 0$, choose U_i such that $\alpha_j \in \{\overbrace{2, \ldots, i-1, i+1, \ldots, n}^{\text{length } (n-2)}\}$ for each j,

hence for $Tv_1 = c_1v_1 + \cdots + c_{i-1}v_{i-1} + c_{i+1}v_{i+1} + \cdots + c_nv_n \in U_i$, we conclude that $c_i = 0$.

$\Rightarrow Tv_1 = c_1v_1$, 又 $v_1 = v \in V$ is arbitrary $\Rightarrow T = \lambda I$ for some λ. $\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Prove that T has an eigval $\Longleftrightarrow \exists$ an invar subsp U of V under T such that* $\dim U = \dim V - 1$.

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec v.

(If $\dim V = 1$, then $U = \{0\}$ and we are done.)

Extend $v_1 = v$ to a basis of V as $(v_1, v_2 \ldots, v_n)$.

Step 1 If $\exists\, w_1 \in \operatorname{span}(v_2, \ldots, v_n)$ such that $0 \neq Tw_1 \in \operatorname{span}(v_1)$,

then extend $w_1 = \alpha_{1,1}$ to a basis of $\operatorname{span}(v_2, \ldots, v_n)$ as $(\alpha_{1,1}, \ldots, \alpha_{1,n-1})$.

Otherwise, we stop at step 1.

$\vdots$

Step k If $\exists\, w_k \in \operatorname{span}(\alpha_{k-1,2}, \ldots, \alpha_{k-1,n-k+1})$ such that $0 \neq Tw_k \in \operatorname{span}(v_1, w_1, \ldots, w_{k-1})$,

then extend $w_k = \alpha_{k,1}$ to a basis of $\operatorname{span}(\alpha_{k-1,2}, \ldots, \alpha_{k-1,n-k+1})$ as $(\alpha_{k,1}, \ldots, \alpha_{k,n-k})$.

Otherwise, we stop at step k.

$\vdots$

Finally, we stop at step m, thus we get $(v_1, w_1, \ldots, w_{m-1})$ and $(\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1})$,

$\operatorname{range} T|_{\operatorname{span}(w_1,\ldots,w_{m-1})} = \operatorname{span}(v_1, w_1, \ldots, w_{m-2}) \Rightarrow \dim \operatorname{null} T|_{\operatorname{span}(w_1,\ldots,w_{m-1})} = 0$,

$\operatorname{span}(\underbrace{v_1, w_1, \ldots, w_{m-1}}_{\text{length dim } m})$ and $\operatorname{span}(\underbrace{\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1}}_{\text{length dim}(n-m)})$ are invar under T.

Let $U = \operatorname{span}(\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1}) \oplus \operatorname{span}(v_1, w_1, \ldots, w_{m-2})$ and we are done. $\square$

COMMENT: Both $\operatorname{span}(v_2, \ldots, v_n)$ and $\operatorname{span}(\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1}) \oplus \operatorname{span}(w_1, \ldots, w_{m-1})$ are in $\mathcal{S}_V \operatorname{span}(v_1)$.

(b) Suppose U is an invar subpsace of V under T with $\dim U = m = \dim V - 1$.

(If $m = 0$, then $\dim V = 1$ and we are done.)

Let $(u_1, \ldots, u_m)$ be a basis of U, extend to a basis of V as $(u_0, u_1, \ldots, u_m)$.

We discuss in cases:

For $Tu_0 \in U$, then $\operatorname{range} T = U$ so that T is not surj $\Longleftrightarrow \operatorname{null} T \neq \{0\} \Longleftrightarrow 0$ is an eigval of T.

For $Tu_0 \notin U$, then $Tu_0 = a_0u_0 + a_1u_1 + \cdots + a_mu_m$.

(1) If $Tu_0 \in \operatorname{span}(u_0)$, then we are done.

(2) Otherwise, if $\operatorname{range} T|_U = U$, then $Tu_0 = a_0u_0$ and we are done;

otherwise, $T|_U : U \to U$ is not surj ($\Rightarrow$ not inje), suppose $\operatorname{range} T|_U \neq \{0\}$

(Suppose $\operatorname{range} T|_U = \{0\}$. If $\dim U = 0$ then we are done.

Otherwise $\exists\, u \in U \backslash \{0\}, Tu = 0$ and we are done.)

then $\exists\, u \in U \backslash \{0\}, Tu = 0$, we are done. $\square$

---

29 *Suppose $T \in \mathcal{L}(V)$ and* $\operatorname{range} T$ *is finite-dim.*
Prove that T has at most $1 + \dim \operatorname{range} T$ *distinct eigvals.*

SOLUTION:

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigvals of T and let $v_1, \ldots, v_m$ be the correspd eigvecs.

(Because range T is finite-dim. Let $(v_1, \ldots, v_n)$ be a list of all the linely inde eigvecs of T, so that the correspd eigvals are finite.)

For every $\lambda_k \neq 0, T(\frac{1}{\lambda_k} v_k) = v_k$. And if $T = T - 0I$ is not inje, then $\exists\,! \lambda_A = 0$ and $Tv_A = \lambda_A v_A = 0$.

Thus for $\lambda_k \neq 0, \forall\, k$, $(Tv_1, \ldots, Tv_m)$ is a linely inde list of length m in range T.

And for $\lambda_A = 0$, there is a linely inde list of length at most $(m-1)$ in range T.

Hence, by [2.23], $m \leq \dim \operatorname{range} T + 1$. ☐

---

32 *Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct real numbers.*
Prove that $(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$ is linely inde in $\mathbf{R}^{\mathbf{R}}$.
HINT: *Let $V = \operatorname{span}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$, and define an operator $D \in \mathcal{L}(V)$ by $Df = f'$.*
Find eigvals and eigvecs of D.

SOLUTION:

Define V and $D \in \mathcal{L}(V)$ as in HINT. Then because for each $k, D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.

Thus $\lambda_1, \ldots, \lambda_n$ are distinct eigvals of D. By [5.10], $(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$ is linely inde in $\mathbf{R}^{\mathbf{R}}$. ☐

- *Suppose $\lambda_1, \ldots, \lambda_n$ are distinct positive numbers.*
Prove that $(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$ is linely inde in $\mathbf{R}^{\mathbf{R}}$.

SOLUTION:

Let $V = \operatorname{span}\ (\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.

Then because $D\,(\cos(\lambda_k x)) = -\lambda_k \sin(\lambda_k x)$. 又 $D\,(\sin(\lambda_k x)) = \lambda_k \cos(\lambda_k x)$.

Thus $D^2\,(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$.

Notice that $\lambda_1, \ldots, \lambda_n$ are distinct $\Rightarrow -\lambda_1^2, \ldots, -\lambda_n^2$ are distinct.

Hence $-\lambda_1^2, \ldots, -\lambda_n^2$ are distinct eigvals of D^2
with the correspd eigvecs $\cos(\lambda_1 x), \ldots, \cos(\lambda_n x)$ respectively.

And then $(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$ is linely inde in $\mathbf{R}^{\mathbf{R}}$. ☐

- *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is a subsp of V invar under T.*
The quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by
$$(T/U)(v+U) = Tv + U \text{ for each } v \in V.$$
(a) *Show that the definition of T/U makes sense*
(which requires using the condition that U is invar under T)
and show that T/U is an operator on V/U.
(b) (OR Problem 35) *Show that each eigval of T/U is an eigval of T.*

SOLUTION:

(a) Suppose $v + U = w + U$ ($\Longleftrightarrow v - w \in U$).

Then because U is invar under T, $T(v-w) \in U \Longleftrightarrow Tv + U = Tw + U$.

Hence the definition of T/U makes sense.

Now we show that T/U is linear.

$$\forall v+U, w+U \in V/U, \lambda \in \mathbf{F}, (T/U)\left((v+U) + \lambda(w+U)\right)$$
$$= T(v + \lambda w) + U = (Tv + U) + \lambda(Tw + U)$$
$$= (T/U)(v+U) + \lambda(T/U)(w).$$

(b) Suppose λ is an eigval of T/U with an eigvec $v + U$.

Then $(T/U)(v+U) = \lambda(v+U) = Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v \in U$.

If $(T - \lambda I)v = 0 \Rightarrow Tv = \lambda v$, then we are done.

Otherwise, then $(T|_U - \lambda I) : U \to U$ is inv,

hence $\exists!\, w \in U, (T|_U - \lambda I)(w) = (T - \lambda I)v \Rightarrow T(v-w) = \lambda(v-w)$.

Note that $v - w \neq 0$ (for if not, $v \in U \Rightarrow v + U = 0 + U$ is not an eigvec). ☐

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36 *Prove or give a counterexample:*
The result of (*b*) *in Exercise 35 is still true if* V *is infinite-dim.*

Solution: A counterexample:

Consider $V = \operatorname{span}(1, e^x, e^{2x}, \dots)$ in $\mathbf{R}^{\mathbf{R}}$, and a subsp $U = \operatorname{span}(e^x, e^{2x}, \dots)$ of V.

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then range $T = U$ is invar under T.

Consider $(T/U)(1+U) = e^x + U = 0$

$\Rightarrow 0$ is an eigval of T/U but is not an eigval of T

($\operatorname{null} T = \{0\}$, for if not, $\exists f \in V \backslash \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, contradicts). ☐

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33 *Suppose* $T \in \mathcal{L}(V)$. *Prove that* $T/(\operatorname{range} T) = 0$.

Solution:

$\forall v + \operatorname{range} T \in V/\operatorname{range} T, v + \operatorname{range} T \in \operatorname{null}\,(T/(\operatorname{range} T))$

$\Rightarrow \operatorname{null}\,(T/(\operatorname{range} T)) = V/\operatorname{range} T \Rightarrow T/(\operatorname{range} T)$ is a zero map. ☐

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34 *Suppose* $T \in \mathcal{L}(V)$. *Prove that* $T/(\operatorname{null} T)$ *is inje* $\iff (\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$.

Solution:

(a) Suppose $T/(\operatorname{null} T)$ is inje.

Then $(T/(\operatorname{null} T))\,(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0$

$\iff Tu \in \operatorname{null} T$ 又 $Tu \in \operatorname{range} T \iff u + \operatorname{null} T = 0 \iff u \in \operatorname{null} T \iff Tu = 0$.

Thus $(\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$.

(b) Suppose $(\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$.

Then $(T/(\operatorname{null} T))\,(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0$

$\iff Tu \in \operatorname{null} T$ 又 $Tu \in \operatorname{range} T \iff Tu = 0 \iff u \in \operatorname{null} T \iff u + \operatorname{null} T = 0$.

Thus $T/(\operatorname{null} T)$ is inje. ☐

Ended

5.B: I [*See* 5.B: II *below.*]

Comment: 下面，为了照顾原书 5.B 节两版过大的差距，特别将此节补注分成 I 和 II 两部分。
又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本征值」
(相当一部分是从原第 3 版 8.C 节挪过来的) 是对原第 3 版「多项式作用于算子」与
「本征值的存在性」(也即第 3 版 5.B 前半部分) 的极大扩充，这一扩充也大大改变了
原第 3 版后半部分的「上三角矩阵」这一小节，故而将第 4 版 5.B 节放在第 3 版前面。

I 部分除了覆盖第 4 版 5.B 节全部和第 3 版 5.B 节前半部分与之相关的所有习题，
还会覆盖第 4 版 5.A 节末。
II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节，还会覆盖第 4 版 5.C 节；
并且，下面 5.C 还会覆盖第 4 版 5.D 节。

「注：[8.40] Or(4E 5.22) —— mini poly;
[8.44,8.45] Or(4E 5.25,5.26) —— how to find the mini poly;

[8.49] $\quad$ Or(4E 5.27) —— eigvals are the zeros of the mini poly;

[8.46] $\quad$ Or(4E 5.29) —— $q(T) = 0 \Leftrightarrow q$ is a poly multi of the mini poly.⌋

[1]: (4E.5.A.33), 13; [2]: (4E.5.B.25, 26, 27, 28, 22); [3]: 6, (4E.5.B.10, 23, 21), 19; [4]: (4E 5.B.13, 14); [5]: (4E.5.B.20, 24), 10; [6]: 1, 2, 7, 3, (4E.5.A.32); [7]: 8, (4E 5.B.12, 3, 8); [8]: (4E.5.B.11), 5, (4E.5.B.7); [9]: 11, 12, (4E.5.B.17, 18); [10]: 18 Or(4E.5.B.15), (4E.5.B.9), (4E.5.B.16); [11]: (2E Ch5.24), (4E.5.B.29).

• *Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.*
(a) *Prove that T is inje $\Longleftrightarrow T^m$ is inje.*
(b) *Prove that T is surj $\Longleftrightarrow T^m$ is surj.*

Solution:
(a) Suppose T^m is inje. Then $Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$.
Suppose T is inje. Then $T^m v = T^{m-1}v = \cdots = T^2 v = Tv = v = 0$.
(b) Suppose T^m is surj. $\forall u \in V, \exists v \in V, T^m v = u = Tw$, let $w = T^{m-1}v$. ☐
Suppose T is surj. Then $\forall u \in V, \exists v_1, \dots, v_m \in V, T(v_1) = T^2 v_2 = \cdots = T^m v_m = u$. ☐

• **Note For [5.17]:**
Suppose $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{F})$. Prove that $\operatorname{null} p(T)$ *and* $\operatorname{range} p(T)$ *are invar under T.*
Solution: Using the commutativity in [5.10].
(a) Suppose $u \in \operatorname{null} p(T)$. Then $p(T)u = 0$.
Thus $p(T)(Tu) = (p(T)T)u\, (Tp(T))\, u = T(p(T)u) = 0$. Hence $Tu \in \operatorname{null} p(T)$. ☐
(b) Suppose $u \in \operatorname{range} p(T)$. Then $\exists v \in V$ such that $u = p(T)v$.
Thus $Tu = T(p(T)v) = p(T)(Tv) \in \operatorname{range} p(T)$. ☐

• **Note For [5.21]:** *Every operator on a finite-dim nonzero complex vecsp has an eigval.*
Suppose V is a finite-dim complex vecsp of dim $n > 0$ and $T \in \mathcal{L}(V)$.
Choose a nonzero $v \in V$. $(v, Tv, T^2v, \dots, T^n v)$ of length $n+1$ is linely depe.
Suppose $a_0 I + a_1 T + \cdots + a_n T^n = 0$. Then $\exists a_j \neq 0$.
Thus $\exists$ nonconst p of smallest degree ($\deg p > 0$) *such that* $p(T)v = 0$.
Because $\exists \lambda \in \mathbf{C}$ such that $p(\lambda) = 0 \Rightarrow \exists q \in \mathcal{P}(\mathbf{C}), p(z) = (z - \lambda)q(z), \forall z \in \mathbf{C}$.
Thus $0 = p(T)v = (T - \lambda I)(q(T)v)$. By the minimality of $\deg p$ and $\deg q < \deg p$, $q(T)v \neq 0$.
Then $(T - \lambda I)$ is not inje. Thus λ is an eigval of T with eigvec $q(T)v$.

• **Example:** *an operator on a complex vecsp with no eigvals*
Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by $(Tp)(z) = zp(z)$.
Suppose $p \in \mathcal{P}(\mathbf{C})$ is a nonzero poly. Then $\deg Tp = \deg p + 1$, and thus $Tp \neq \lambda p$, $\forall \lambda \in \mathbf{C}$.
Hence T has no eigvals.

---

13 *Suppose V is a complex vecsp and $T \in \mathcal{L}(V)$ has no eigvals.*
Prove that every subsp of V invar under T is either $\{0\}$ or infinite-dim.

Solution: Suppose U is a finite-dim nonzero invar subsp on $\mathbf{C}$. Then by [5.21], $T|_U$ has an eigval. ☐

• **Tips:** For $T_1, \dots, T_m \in \mathcal{L}(V)$:
(a) Suppose $T_1, \dots, T_m$ are all inje. Then $(T_1 \circ \cdots \circ T_m)$ is inje.
(b) Suppose $(T_1 \circ \cdots \circ T_m)$ is not inje. Then at least one of $T_1, \dots, T_m$ is not inje.
(c) At least one of $T_1, \dots, T_m$ is not inje $\nRightarrow (T_1 \circ \cdots \circ T_m)$ is not inje.

EXAMPLE: On infinite-dim only. Let $V = \mathbf{F}^\infty$.

$\left.\begin{array}{l}\text{Let } S \text{ be the backward shift (surj but not inje)} \\ \text{Let } T \text{ be the forward shift (inje but not surj)}\end{array}\right\} \Rightarrow$ Then $ST = I$. □

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16 *Suppose* $0 \neq v \in V$. *Define* $S \in \mathcal{L}(\mathcal{P}_{\dim V}(\mathbf{C})), V)$ *by* $S(p) = p(T)v$. *Prove* [5.21].

SOLUTION:

Because $\dim \mathcal{P}_{\dim V}(\mathbf{C})) = \dim V + 1$. Then S is not inje. Hence $\exists\, 0 \neq p \in \mathcal{P}_{\dim V}(\mathbf{C})), p(T)v = 0$.
Using [4.14], write $p(z) = c(z - \lambda_1)\cdots(z - \lambda_m)$. Apply T to both sides: $p(T) = c(T - \lambda_1 I)\cdots(T - \lambda_m I)$.
Thus at least one of $(T - \lambda_j I)$ is not inje (because $p(T)$ is not inje). □

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17 *Suppose* $0 \neq v \in V$. *Define* $S \in \mathcal{L}\left(\mathcal{P}_{(\dim V)^2}(\mathbf{C})), \mathcal{L}(V)\right)$ *by* $S(p) = p(T)$. *Prove* [5.21].

SOLUTION:

Because $\dim \mathcal{P}_{(\dim V)^2}(\mathbf{C})) = (\dim V)^2 + 1$. Then S is not inje. Hence $\exists\, 0 \neq p \in \mathcal{P}_{(\dim V)^2}(\mathbf{C})), p(T) = 0$.
Using [4.14], write $p(z) = c(z - \lambda_1)\cdots(z - \lambda_m)$. Applying T, we have $0 = p(T) = c(T - \lambda_1 I)\cdots(T - \lambda_m I)$.
Thus $(T - \lambda_1 I)\cdots(T - \lambda_m I) = 0 \Rightarrow \exists\, j, (T - \lambda_j)$ is not inje. □

COMMENT: $\exists$ monic $q \in \operatorname{null} S \neq \{0\}$ of smallest degree, $S(q) = q(T) = 0$, then q is the *mini poly.*

• **NOTE FOR [8.40]:** *def for mini poly*

Suppose V *is finite-dim and* $T \in \mathcal{L}(V)$.
Suppose $M_T^0 = \{p_j\}_{j \in \Gamma}$ *is the set of all monic poly that give 0 whenever* T *is applied.*
Prove that $\exists!\, p_k \in M_T^0, \deg p_k = \min\{\deg p_j\}_{j \in \Gamma} \leq \dim V$.

SOLUTION: OR. Another Proof :

[*Existns Part*] We use induction on $\dim V$.

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, we are done.

(ii) Suppose $\dim V \geq 1$.

Assume that $\dim V > 0$ *and that the desired result is true for all operators on all vecsps of smaller dim.*
Let $u \in V, u \neq 0$. The list $(u, Tu, \ldots, T^{\dim V} u)$ of length $(1 + \dim V)$ is linely depe.
Then $\exists!\, T^m$ of smallest degree such that $T^m u \in \operatorname{span}(u, Tu, \ldots, T^{m-1}u)$.
Thus $\exists\, c_j \in \mathbf{F}, c_0 u + c_1 Tu + \cdots + c_{m-1} T^{m-1} u + T^m u = 0$.
Define q by $q(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m$.
Then $0 = T^k(q(T)u) = q(T)(T^k u), \forall k \in \{1, \ldots, m-1\} \subseteq \mathbf{N}$.
Because $(u, Tu, \ldots, T^{m-1}u)$ is linely inde.
Thus $\dim \operatorname{null} q(T) \geq m \Rightarrow \dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leq \dim V - m$.

Let $W = \operatorname{range} q(T)$.
By *assumption*, $\exists\, s \in M_T^0$ of smallest degree (and $\deg s \leq \dim W$,) so that $s(T|_W) = 0$.
Hence $\forall v \in V, ((sq)(T))\,(v) = s(T)(q(T)v) = 0$.
Thus $sq \in M_T^0$ and $\deg sq \leq \dim V$.

[*Uniqnes Part*]

Suppose $p, q \in M_T^0$ are of the smallest degree. Then $(p - q)(T) = 0$. 又 $\deg(p - q) = m < \min\{\deg p_j\}_{j \in \Gamma}$.
Hence $p - q = 0$, for if not, $\exists!\, c \in \mathbf{F}, c(p - q) \in M_T^0$. Contradicts. □

• (4E 5.31, 4E 5.B.25 and 26) *mini poly of restriction operator and mini poly of quotient operator*

Suppose V *is finite-dim,* $T \in \mathcal{L}(V)$, *and* U *is an invar subsp of* V *under* T.
Let p *be the mini poly of* T.

(a) *Prove that* p *is a poly multi of the mini poly of* $T|_U$.
(b) *Prove that* p *is a poly multi of the mini poly of* T/U.
(c) *Prove that* (*mini poly of* $T|_U$) $\times$ (*mini poly of* T/U) *is a poly multi of* p.
(d) *Prove that the set of eigvals of* T *equals the union of the set of eigvals of* $T|_U$ *and the set of eigvals of* T/U.

Solution:

(a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46]. ☐

(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v + U) = p(T)v + U = 0$. ☐

(c) Suppose r is the mini poly of $T|_U$, s is the mini poly of T/U.
Because $\forall v \in V, s(T/U)(v + U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.
又 $\forall u \in U, r(T|_U)u = r(T)u = 0$.
Thus $\forall v \in V$ but $v \notin U, (rs)(T)v = r(s(T)v) = 0$.
And $\forall u \in U, (rs)(T)u = r(s(T)u) = 0$ (because $s(T)u = s(T|_U)u \in U$).
Hence $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0$. ☐

(d) By [8.49], immediately. ☐

• (4E 5.B.27) *Suppose* $\mathbf{F} = \mathbf{R}$, V *is finite-dim, and* $T \in \mathcal{L}(V)$.
Prove that the mini poly p *of* T_C *equals the mini poly* q *of* T.

Solution:

(a) $\forall u + \mathrm{i}0 \in V_C, p(T_C)(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p$ is a poly multi of q.

(b) $q(T) = 0 \Rightarrow \forall u + \mathrm{i}v \in V_C, q(T_C)(u + \mathrm{i}v) = q(T)u + \mathrm{i}q(T)v = 0 \Rightarrow q$ is a poly multi of p. ☐

• (4E 5.B.28) *Suppose* V *is finite-dim and* $T \in \mathcal{L}(V)$.
Prove that the mini poly p *of* $T' \in \mathcal{L}(V')$ *equals the mini poly* q *of* T.

Solution:

(a) $\forall \varphi \in V', p(T')\varphi = \varphi \circ (p(T)) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \operatorname{null} \varphi \Rightarrow p(T) = 0, p$ is a poly multi of q.

(b) $q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ (q(T)) = q(T')\varphi = 0 \Rightarrow q(T) = 0, q$ is a poly multi of p. ☐

• (4E 5.32) *Suppose* $T \in \mathcal{L}(V)$ *and* p *is the mini poly.*
Prove that T *is not inje* $\Longleftrightarrow$ *the const term of* p *is* 0.

Solution:

T is not inje $\Longleftrightarrow$ 0 is an eigval of $T \Longleftrightarrow$ 0 is a zero of $p \Longleftrightarrow$ the const term of p is 0. ☐

Or. Because $p(0) = (z - 0)(z - \lambda_1) \cdots (z - \lambda_m) = 0 \Rightarrow T(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0$
又 p is the mini poly $\Rightarrow q$ define by $q(z) = (z - \lambda_1) \cdots (z - \lambda_m)$ is such that $q(T) \neq 0$.
Hence $0 = p(T) = Tq(T) \Rightarrow T$ is not inje.
Conversely, suppose $(T - 0I)$ is not inje, then 0 is a zero of p, so that the const term is 0. ☐

• (4E 5.B.22)
Suppose V *is finite-dim,* $T \in \mathcal{L}(V)$. *Prove that* T *is inv* $\Longleftrightarrow I \in \operatorname{span}(T, T^2, \ldots, T^{\dim V})$.

Solution: Denote the mini poly by p, where for all $z \in \mathbf{F}, p(z) = a_0 + a_1 z + \cdots + z^m$.
Notice that V is finite-dim. T is inv $\Longleftrightarrow T$ is inje $\Longleftrightarrow p(0) \neq 0$.
Hence $p(T) = 0 = a_0 I + a_1 T + \cdots + T^m$, where $a_0 \neq 0$ and $m \leq \dim V$. ☐

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6 *Suppose* $T \in \mathcal{L}(V)$ *and* U *is a subsp of* V *invar under* T.

Prove that U is invar under $p(T)$ for every poly $p \in \mathcal{P}(\mathbf{F})$.

SOLUTION:

$\forall u \in U, Tu \in U \Rightarrow \forall u \in U, Iu, Tu, T(Tu), \dots, T^m u \in U \Rightarrow \forall u \in U, (a_0 I + a_1 T + \cdots + a_m T^m)u \in U.\square$

• (4E 5.B.10, 5.B.23) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$ and p is the mini poly with degree m. Suppose $v \in V$.*

(a) *Prove that* $\operatorname{span}(v, Tv, \dots, T^{m-1}v) = \operatorname{span}(v, Tv, \dots, T^{j-1}v)$ *for some $j \leq m$.*

(b) *Prove that* $\operatorname{span}(v, Tv, \dots, T^{m-1}v) = \operatorname{span}(v, Tv, \dots, T^{m-1}v, \dots, T^n v)$.

SOLUTION:

COMMENT: By NOTE FOR[8.40], j has an upper bound $m-1$, m has an upper bound $\dim V$.

Write $p(z) = a_0 + a_1 z + \cdots + z^m$ ($m \leq \dim V$). If $v = 0$, then we are done. Suppose $v \neq 0$.

(a) Suppose $j \in \mathbf{N}^+$ is the smallest such that $T^j v \in \operatorname{span}(v, Tv, \dots, T^{j-1}v) = U_0$. Then $j \leq m$.

Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1}T^{j-1}v$. And because $T(T^k v) = T^{k+1} \in U_0$. U_0 is invar under T.

By Problem (6), $\forall k \in \mathbf{N},\ T^{j+k}v = T^k(T^j v) \in U_0$.

Thus $U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}v, \dots, T^n v)$ for all $n \geq j-1$. Let $n = m-1$ and we are done.

(b) Let $U = \operatorname{span}(v, Tv, \dots, T^{m-1}v)$.

By (a), $U = U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}, \dots, T^{m-1}, \dots, T^n)$ for all $n \geq m-1$. $\square$

• (4E 5.B.21) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*

Prove that the mini poly p has degree at most $1 + \dim \operatorname{range} T$.

If $\dim \operatorname{range} T < \dim V - 1$, then this result gives a better upper bound for the degree of mini poly.

SOLUTION:

If T is inje, then $\operatorname{range} T = V$ and we are done. Now choose $0 \neq v \in \operatorname{null} T$, then $Tv + 0 \cdot v = 0$.

1 is the smallest positive integer such that $T^1 v \in \operatorname{span}(v, \dots, T^0 v)$. Define q by $q(z) = z \Rightarrow q(T)v = 0$.

Let $W = \operatorname{range} q(T) = \operatorname{range} T$. $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ of smallest degree ($\deg s \leq \dim W$), $s(T|_W) = 0$.

Hence sq is the mini poly (see NOTE FOR[8.40]) and $\deg(sq) = \deg s + \deg q \leq \dim \operatorname{range} T + 1$. $\square$

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19 *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.*

Prove that $\dim \mathcal{E}$ equals the degree of the mini poly of T.

SOLUTION:

Because the list $(I, T, \dots, T^{(\dim V)^2})$ of length $\dim \mathcal{L}(V) + 1$ is linely depe in $\dim \mathcal{L}(V)$.

Suppose $m \in \mathbf{N}^+$ is the smallest such that $T^m = a_0 I + \cdots + a_{m-1}T^{m-1}$.

Then q defined by $q(z) = z^m - a_{m-1}z^{m-1} - \cdots - a_0$ is the mini poly (see [8.40]).

For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k(T^m) \in \operatorname{span}(I, T, \dots, T^{m-1}) = U$.

Hence $\operatorname{span}(I, T, \dots, T^{(\dim V)^2}) = \operatorname{span}(I, T, \dots, T^{(\dim V)^2 - 1}) = U$.

Note that by the minimality of m, the list $(I, T, \dots, T^{m-1})$ is linely inde.

Thus $\dim U = m = \dim \operatorname{span}(I, T, \dots, T^{(\dim V)^2-1}) = \dim \operatorname{span}(I, T, \dots, T^n)$ for all $m < n \in \mathbf{N}^+$.

Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.

(a) Suppose $p(T) = 0$. 又 $\deg p \leq m-1 \Rightarrow p = 0$. Then φ is inje.

(b) $\forall S = a_0 I + a_1 T + \cdots + a_{m-1}T^{m-1} \in \mathcal{E}$, define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by

$p(z) = a_0 + a_1 z + \cdots + a_{m-1}z^{m-1} \Rightarrow \varphi(p) = S$. Then φ is surj.

Hence $\mathcal{E}$ and $\mathcal{P}_{m-1}(\mathbf{F})$ are iso. 又 $\dim \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. $\square$

• (4E 5.B.13) *Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$ is defined by*

$$q(z) = a_0 + a_1 z + \cdots + a_n z^n, \text{ where } a_n \neq 0, \text{ for all } z \in \mathbf{F}.$$

Denote the mini poly of T by p defined by

$$p(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m \text{ for all } z \in \mathbf{F}.$$

Prove that $\exists!\, r \in \mathcal{P}(\mathbf{F})$ such that $q(T) = r(T), \deg r < \deg p$.

Solution:

If $\deg q < \deg p$, then we are done.

If $\deg q = \deg p$, notice that $p(T) = 0 = c_0 I + c_1 T + \cdots + c_{m-1} T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0 I - c_1 T - \cdots - c_{m-1} T^{m-1},$$

define r by $r(z) = q(z) + [-a_m z^m + a_m(-c_0 - c_1 z - \cdots - c_{m-1} z^{m-1})]$

$$= (a_0 - a_m c_0) + (a_1 - a_m c_1) z + \cdots + (a_{m-1} - a_m c_{m-1}) z^{m-1},$$

hence $r(T) = 0, \deg r < m$ and we are done.

Now suppose $\deg q \geq \deg p$. We use induction on $\deg q$.

(i) $\deg q = \deg p$, then the desired result is true, as shown above.

(ii) $\deg q > \deg p$, assume that the desired result is true for $\deg q = n$.

Suppose $f \in \mathcal{P}(\mathbf{F})$ such that $f(z) = b_0 + b_1 z + \cdots + b_n z^n + b_{n+1} z^{n+1}$.

Apply the assumption to g defined by $g(z) = b_0 + b_1 z + \cdots + b_n z^n$,

getting s defined by $s(z) = d_0 + d_1 z + \cdots + d_{m-1} z^{m-1}$.

Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1} T^{n+1} = s(T) + b_{n+1} T^{n+1}$.

Apply the assumption to t defined by $t(z) = z^n$,

getting δ defined by $\delta(z) = c_0{}' + c_1{}' z + \cdots + c_{m-1}{}' z^{m-1}$.

Thus $t(T) = T^n = c_0{}' + c_1{}' z + \cdots + c_{m-1}{}' z^{m-1} = \delta(T)$.

又 span $(v, Tv, \dots, T^{m-1} v)$ is invar under T.

Hence $\exists!\, k_j \in \mathbf{F}, T^{n+1} = T(T^n) = k_0 + k_1 z + \cdots + k_{m-1} z^{m-1}$.

And $f(T) = s(T) + b_{n+1}(k_0 + k_1 T + \cdots + k_{m-1} T^{m-1})$

$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1) z + \cdots + (d_{m-1} + k_{m-1}) z^{m-1} = h(T)$, thus defining h. □

• (4E 5.B.14) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has mini poly p*

defined by $p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m, a_0 \neq 0$.

Find the mini poly of T^{-1}.

Solution:

Notice that V is finite-dim. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is inv.

Then $p(T) = a_0 I + a_1 T + \cdots + T^m = 0$. Apply T^{-m} to both sides,

$a_0 (T^{-1})^m + a_1 (T^{-1})^{m-1} + \cdots + a_{m-1} T^{-1} + I = 0$.

Define q by $q(z) = z^m + \dfrac{a_1}{a_0} z^{m-1} + \cdots + \dfrac{a_{m-1}}{a_0} z + \dfrac{1}{a_0}$ for all $z \in \mathbf{F}$.

We now show that $(T^{-1})^k \notin \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$

for every $k \in \{1, \dots, m-1\}$ by contradiction, so that q is exactly the mini poly of T^{-1}.

Suppose $(T^{-1})^k \in \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$.

Then let $(T^{-1})^k = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. Apply T^k to both sides,

getting $I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T$, hence $T^k \in \operatorname{span}(I, T, \dots, T^{k-1})$.

Thus f defined by $f(z) = z^k + \dfrac{b_1}{b_0} z^{k-1} + \cdots + \dfrac{b_{k-1}}{b_0} z - \dfrac{1}{b_0}$ is a poly multi of p.

While $\deg f < \deg p$. Contradicts. □

• **Note For [8.49]:**

Suppose V is a finite-dim complex vecsp and $T \in \mathcal{L}(V)$.
By [4.14], the mini poly has the form $(z-\lambda_1)\cdots(z-\lambda_m)$,
*where $\lambda_1, \dots, \lambda_m$ is a list of all eigvals of T, **possibly with repetitions**.*

- **COMMENT:**

A nonzero poly has at most as many distinct zeros as its degree (see [4.12]).
Thus by the upper bound for the deg of mini poly given in NOTE FOR*[8.40], and by [8.49,]*
*we can **give an alternative proof of [5.13]***

- NOTICE(See also 4E 5.B.20,24)

Suppose $\alpha_1, \dots, \alpha_n$ are all the distinct eigvals of T,
and therefore are all the distinct zeros of the mini poly.
Also, the mini poly of T is a poly multi of, but not equal to, $(z-\alpha_1)\cdots(z-\alpha_n)$.
If we define q by $q(z) = (z-\alpha_1)^{\dim V-(n-1)} \cdots (z-\alpha_n)^{\dim V-(n-1)}$,
then q is a poly multi of the char poly (see [8.34] and [8.26])
(Because $\dim V > n$ and $n-1>0$, $n[\dim V-(n-1)] > \dim V$.)
The char poly has the form $(z-\alpha_1)^{\gamma_1}\cdots(z-\alpha_n)^{\gamma_n}$, where $\gamma_1+\cdots+\gamma_n = \dim V$.
The mini poly has the form $(z-\alpha_1)^{\delta_1}\cdots(z-\alpha_n)^{\delta_n}$, where $0 \le \delta_1+\cdots+\delta_n \le \dim V$.

---

10 *Suppose $T \in \mathcal{L}(V), \lambda$ is an eigval of T with an eigvec v.*
Prove that for any $p \in \mathcal{P}(\mathbf{F}), p(T)v = p(\lambda)v$.

SOLUTION:

Suppose p is defined by $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbf{F}$. Because for any $n \in \mathbf{N}^+, T^n v = \lambda^n v$.
Thus $p(T)v = a_0 v + a_1 T v + \cdots + a_m T^m v = a_0 v + a_1 \lambda v + \cdots + a_m \lambda^m v = p(\lambda)v$. □

COMMENT: For any $p \in \mathcal{P}(\mathbf{F})$ such that $p(z) = (z-\lambda_1)^{\alpha_1}\cdots(z-\lambda_m)^{\alpha_m}$, the result is true as well.
Now we prove that $(T-\lambda_1 I)^{\alpha_1}\cdots(T-\lambda_m I)^{\alpha_m} v = (\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_m)^{\alpha_m} v$.
Define q_i by $q_i(z) = (z-\lambda_i)^{\alpha_i}$ for all $z \in \mathbf{F}$.
Because $(a+b)^n = a^n + C_n^1 a^{n-1} b + \cdots + C_n^k a^{n-k} b^k + \cdots + C_n^n b^n$.
Let $a = z, b = \lambda_i, n = \alpha_i$, so we can write $q_i(z)$ in the form $a_0 + a_1 z + \cdots + a_m z^m$.
Hence $q_i(T)v = q_i(\lambda)v \Rightarrow (T-\lambda_i I)^{\alpha_i} v = (\lambda-\lambda_i)^{\alpha_i} v$.
Then for each $k \in \{2, \dots, m\}$, $(T-\lambda_{k-1} I)^{\alpha_{k-1}}(T-\lambda_k I)^{\alpha_k} v$

$$\begin{aligned}
&= q_{k-1}(T)(q_k(T)v) \\
&= q_{k-1}(T)(q_k(\lambda)v) \\
&= q_{k-1}(\lambda)(q_k(\lambda)v) \\
&= (\lambda-\lambda_{k-1})^{\alpha_{k-1}}(\lambda-\lambda_k)^{\alpha_k} v.
\end{aligned}$$

So that $(T-\lambda_1 I)^{\alpha_1}\cdots(T-\lambda_m I)^{\alpha_m} v$

$$\begin{aligned}
&= q_1(T)\,(q_2(T)(\dots(q_m(T)v)\dots)) \\
&= q_1(\lambda)(q_2(\lambda)\,(\dots(q_m(\lambda)v)\dots)) \\
&= (\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_m)^{\alpha_m} v.
\end{aligned}$$

□

---

1 *Suppose $T \in \mathcal{L}(V)$ and $\exists\, n \in \mathbf{N}^+$ such that $T^n = 0$.*
Prove that $(I-T)$ is inv and $(I-T)^{-1} = I + T + \cdots + T^{n-1}$.

SOLUTION:

Note that $1 - x^n = (1-x)(1+x+\cdots+x^{n-1})$.

$$\left.\begin{aligned}(I-T)(1+T+\cdots+T^{n-1})=I-T^n=I\\(1+T+\cdots+T^{n-1})(I-T)=I-T^n=I\end{aligned}\right\}\Rightarrow (I-T)^{-1}=1+T+\cdots+T^{n-1}.\qquad\square$$

---

2 *Suppose $T\in\mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I)=0$.*
Suppose λ is an eigval of T. Prove that $\lambda=2$ or $\lambda=3$ or $\lambda=4$.

SOLUTION:

Suppose v is an eigvec correspd to λ. Then for any $p\in\mathcal{P}(\mathbf{F}), p(T)v=p(\lambda)v$.
Hence $0=(T-2I)(T-3I)(T-4I)v=(\lambda-2)(\lambda-3)(\lambda-4)v$ while $v\neq0\Rightarrow\lambda=2$ or $\lambda=3$ or $\lambda=4$.□
OR. Because $(T-2I)(T-3I)(T-4I)=0$ is not inje. By TIPS. □

---

7 (See 5.A.22) *Suppose $T\in\mathcal{L}(V)$. Prove that 9 is an eigval of $T^2\iff 3$ or -3 is an eigval of T.*

SOLUTION:

(a) Suppose 9 is an eigval of T^2. Then $(T^2-9I)v=(T-3I)(T+3I)v=0$ for some v. By TIPS.
OR. Suppose λ is an eigval with an eigvec v. Then $(T-3I)(T+3I)v=(\lambda-3)(\lambda+3)v=0\Rightarrow\lambda=\pm3$.
(b) Suppose 3 or -3 is an eigval of T with an eigvec v. Then $Tv=\pm3v\Rightarrow T^2v=T(Tv)=9v$ □

---

3 *Suppose $T\in\mathcal{L}(V), T^2=I$ and -1 is not an eigval of T. Prove that $T=I$.*

SOLUTION:

$T^2-I=(T+I)(T-I)$ is not inje, 又 -1 is not an eigval of $T\Rightarrow$ By TIPS. □

OR. Note that $v=[\frac{1}{2}(I-T)v]+[\frac{1}{2}(I+T)v]$ for all $v\in V$.
And $(I-T^2)v=(I-T)(I+T)v=0$ for all $v\in V$,

$$\left.\begin{aligned}(I+T)(\tfrac{1}{2}(I-T)v)=\tfrac{1}{2}(I-T^2)v=0\Rightarrow\tfrac{1}{2}(I-T)v\in\operatorname{null}(I+T)\\(I-T)(\tfrac{1}{2}(I+T)v)=\tfrac{1}{2}(I-T^2)v=0\Rightarrow\tfrac{1}{2}(I+T)v\in\operatorname{null}(I-T)\end{aligned}\right\}\Rightarrow V=\operatorname{null}(I+T)+\operatorname{null}(I-T).$$

又 -1 is not an eigval of $T\Rightarrow(I+T)$ is inje $\Rightarrow\operatorname{null}(I+T)=\{0\}$.
Hence $V=\operatorname{null}(I-T)\Rightarrow\operatorname{range}(I-T)=\{0\}$. Thus $I-T=0\in\mathcal{L}(V)\Rightarrow T=I$. □

• (4E 5.A.32) *Suppose $T\in\mathcal{L}(V)$ has no eigvals and $T^4=I$. Prove that $T^2=-I$.*

SOLUTION:

Because $T^4-I=(T^2-I)(T^2+I)=0$ is not inje $\Rightarrow(T^2-I)$ or (T^2+I) is not inje.
又 T has no eigvals $\Rightarrow(T^2-I)=(T-I)(T+I)$ is inje.
Hence $T^2+I=0\in\mathcal{L}(V)$, for if not,
$\exists\, v\in V,(T^2+I)v\neq0$ while $(T^2-I)\big((T^2+I)v\big)=0$ but (T^2-I) is inje. Contradicts.□

OR. Note that $v=[\frac{1}{2}(I-T^2)v]+[\frac{1}{2}(I+T^2)v]$ for all $v\in V$.
And $(I-T^4)v=(I-T^2)(I+T^2)v=0$ for all $v\in V$,

$$\left.\begin{aligned}(I+T^2)(\tfrac{1}{2}(I-T^2)v)=0\Rightarrow\tfrac{1}{2}(I-T^2)v\in\operatorname{null}(I+T^2)\\(I-T^2)(\tfrac{1}{2}(I+T^2)v)=0\Rightarrow\tfrac{1}{2}(I+T^2)v\in\operatorname{null}(I-T^2)\end{aligned}\right\}\Rightarrow V=\operatorname{null}(I+T^2)+\operatorname{null}(I-T^2).$$

又 T has no eigvals $\Rightarrow(I-T^2)$ is inje $\Rightarrow\operatorname{null}(I-T^2)=\{0\}$.
Hence $V=\operatorname{null}(I+T^2)\Rightarrow\operatorname{range}(I+T^2)=\{0\}$. Thus $I+T^2=0\in\mathcal{L}(V)\Rightarrow T^2=-I$. □

---

8 (OR4E 5.A.31) *Give an example of $T\in\mathcal{L}(\mathbf{R}^2)$ such that $T^4=-I$.*

SOLUTION:

$T^4+1=(T^2+\mathrm{i}I)(T^2-\mathrm{i}I)=(T+\mathrm{i}^{1/2}I)(T-\mathrm{i}^{1/2}I)(T-(-\mathrm{i})^{1/2}I)(T+(-\mathrm{i})^{1/2}I)$.
Note that $\mathrm{i}^{1/2}=\dfrac{\sqrt{2}}{2}+\mathrm{i}\dfrac{\sqrt{2}}{2},\ (-\mathrm{i})^{1/2}=\dfrac{\sqrt{2}}{2}-\mathrm{i}\dfrac{\sqrt{2}}{2}$. Hence $T=\pm(\pm\mathrm{i})^{1/2}I$.

Define T by $T(x,y) = (\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y)$.

$$\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix} \Rightarrow \mathcal{M}(T)^4 = \mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathcal{M}(I). \quad \square$$

$$\left(\text{Using } \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}. \right)$$

- (4E 5.B.12 See also at 5.A.9)

  Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$. Find the mini poly.

SOLUTION:

$T(x_1, \dots, 0) =$ By (5.A.9) and [8.49], $1, 2, \dots, n$ are zeros of the mini poly of T.

(又 Each eigvals of T corresponds to exact one-dim subsp of $\mathbf{F}^n$.)

Define a poly q by $q(z) = (z-1)(z-2)\cdots(z-n)$, for all $z \in \mathbf{F}$. (Then q is the char poly of T.)

Because $q(T)e_j = [(T-I)\cdots(T-(j-1)I)(T-(j+1)I)\cdots(T-nI)](T-jI)e_j = 0$ for each j,

where $(e_1, \dots, e_n)$ is the standard basis. Thus $\forall v \in \mathbf{F}^n, q(T)v = 0$. Hence q is the mini poly of T. $\square$

- *Suppose $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.*

  [*See also at* (5.A.19)] *Find the mini poly of T.*

SOLUTION:

Because n and 0 are all eigvals of T, 又 For all $e_k, Te_k = e_1 + \cdots + e_n$; $\quad T^2 e_k = n(e_1 + \cdots + e_n)$.

Hence $T^2 e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T-n)$. Thus $z(z-n)$ is the mini poly of T. $\square$

- (4E 5.B.8)

  Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is the operator of counterclockwise rotation by the angel θ, where $\theta \in \mathbf{R}^+$. Find the mini poly of T.

SOLUTION:

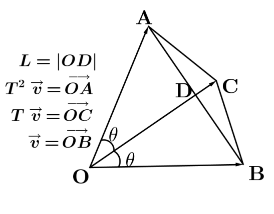
If $\theta = \pi + 2k\pi$, then $T(w,z) = (-w,-z), T^2 = I$ and the mini poly is $z+1$.

If $\theta = 2k\pi$, then $T = I$ and the mini poly is $z - 1$.

Now suppose (v, Tv) is linely inde. Then span $(v, Tv) = \mathbf{R}^2$.

Suppose the mini poly p is defined by $p(z) = z^2 + bz + c$ for all $z \in \mathbf{R}$.

Because



$$Tv = \frac{|\vec{v}|}{2L}(T^2 v + v) \Rightarrow T = \frac{|\vec{v}|}{2L}(T^2 + I)$$

$$L = |\vec{v}|\cos\theta \Rightarrow \frac{|\vec{v}|}{2L} = \frac{1}{2\cos\theta}$$

Hence $p(T) = T^2 - 2\cos\theta\, T + I = 0$ and $z^2 - 2\cos\theta\, z + 1$ is the mini poly of T. $\square$

OR. By (4E 5.B.11), $\mathcal{M}(T, (e_1, e_2)) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Hence the mini poly is $z \pm 1$ or $z^2 - 2\cos\theta\, z + 1$. $\square$

- (4E 5.B.11) *Suppose V is a two-dim vecsp, $T \in \mathcal{L}(V)$, and the matrix of T with resp to some basis of V is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.*

  (a) *Show that $T^2 - (a+d)T + (ad - bc)I = 0$.*

  (b) *Show that the mini poly of T equals*

$$\begin{cases} z-a & \textit{if } b=c=0 \textit{ and } a=d, \\ z^2-(a+d)z+(ad-bc) & \textit{otherwise.} \end{cases}$$

SOLUTION:

(a) Suppose the basis is (v,w). Because $\begin{cases} Tv = av+bw \Rightarrow (T-aI)v = bw, \text{ then apply } (T-dI) \text{ to both sides.} \\ Tw = cv+dw \Rightarrow (T-dI)w = cv, \text{ then apply } (T-aI) \text{ to both sides.} \end{cases}$

Hence $(T-aI)(T-dI) = bcI \Rightarrow T^2-(a+d)T+(ad-bc)I = 0$.

(b) If $b=c=0$ and $a=d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the mini poly is $z-a$.

Otherwise, by (a), $z^2-(a+d)z+(ad-bc)$ is a poly multi of the mini poly.

Now we prove that $T \notin \operatorname{span}(I)$, so that then the mini poly of T has exactly degree 2.

(At least one of the assumption of (I),(II) below is true.)

(I) Suppose $a=d$, then $Tv = av+bw \notin \operatorname{span}(v)$, $Tw = cv+aw \notin \operatorname{span}(w)$.

(II) Suppose at most one of b,c is not 0. If $b=0$, then $Tw \notin \operatorname{span}(w)$; If $c=0$, then $Tv \notin \operatorname{span}(v)$.□

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5 *Suppose $S,T \in \mathcal{L}(V)$, S is inv, and $p \in \mathcal{P}(\mathbf{F})$. Prove that $p(TS) = S^{-1}p(ST)S$.*

SOLUTION:

We prove $(TS)^m = S^{-1}(ST)^mS$ for each $m \in \mathbf{N}$ by induction.

(i) $m=0,1$. $TS^0 = I = S^{-1}(ST)^0S$; $TS = S^{-1}(ST)S$.

(ii) $m>1$. Assume that $(TS)^m = S^{-1}(ST)^mS$.

Then $(TS)^{m+1} = (TS)^m(TS) = S^{-1}(ST)^mSTS = S^{-1}(ST)^{m+1}S$.

Hence $\forall p \in \mathcal{P}(\mathbf{F})$,
$$\begin{aligned} p(TS) &= a_0(TS)^0 + a_1(TS) + \cdots + a_m(TS)^m \\ &= a_0[S^{-1}(ST)^0S] + a_1[S^{-1}(ST)S] + \cdots + a_m[S^{-1}(ST)^mS] \\ &= S^{-1}[a_0(ST)^0 + a_1(ST) + \cdots + a_m(ST)^m]S \\ &= S^{-1}p(ST)S. \end{aligned}$$
□

• (4E 5.B.7)

(a) *Give an example of $S,T \in \mathcal{L}(\mathbf{F}^2)$ such that the mini poly of ST does not equal the mini poly of TS.*

(b) *Suppose V is finite-dim and $S,T \in \mathcal{L}(V)$. Prove that if S or T is inv, then the mini poly of ST equals the mini poly of TS.*

SOLUTION:

(a) Define S by $S(x,y) = (x,x)$. Define T by $T(x,y) = (0,y)$.

Then $ST(x,y) = 0$, $TS(x,y) = (0,x)$ for all $(x,y) \in \mathbf{F}^2$.

Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.

Hence the mini poly of ST does not equal to the mini poly of TS.

(b) Denote the mini poly of ST by p, and the mini poly TS by q.

Suppose S is inv.

$$\left.\begin{aligned} &p(ST) = 0 = Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a poly multi of } q. \\ &q(TS) = 0 = S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a poly multi of } p. \end{aligned}\right\} \Rightarrow p = q.$$

Reversing the roles of S and T, we conclude that if T is inv, then $p=q$ as well. □

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11 *Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$, and $\alpha \in \mathbf{C}$. Prove that α is an eigval of $p(T) \iff \alpha = p(\lambda)$ for some eigval λ of T.*

SOLUTION:

(a) Suppose α is an eigval of $p(T) \Leftrightarrow (p(T)-\alpha I)$ is not inje.

Write $p(z) - \alpha = c(z-\lambda_1)\cdots(z-\lambda_m) \Rightarrow p(T) - \alpha I = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$.

By TIPS, $\exists\, (T-\lambda_j I)$ not inje. Thus $p(\lambda_j) - \alpha = 0$.

(b) Suppose $\alpha = p(\lambda)$ and λ is an eigval of T with an eigvec v. Then $p(T)v = p(\lambda)v = \alpha v$. ☐

OR. Define q by $q(z) = p(z) - \alpha$. λ is a zero of q.

Because $q(T)v = (p(T) - \alpha I)v = q(\lambda)v = (p(\lambda) - \alpha)v = 0$.

Hence $q(T)$ is not inje $\Rightarrow (p(T) - \alpha I)$ is not inje. ☐

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12 (OR4E.5.B.6) *Give an example of an operator on* $\mathbf{R}^2$ *that shows the result above does not hold if* $\mathbf{C}$ *is replaced with* $\mathbf{R}$.

SOLUTION:

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(w,z) = (-z,w)$.

By Problem (4E 5.B.11), $\mathcal{M}(T,((1,0),(0,1))) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ the mini poly of T is z^2+1.

Define p by $p(z) = z^2$. Then $p(T) = T^2 = -I$. Thus $p(T)$ has eigval -1.

While $\nexists\, \lambda \in \mathbf{R}$ such that $-1 = p(\lambda) = \lambda^2$. ☐

• (4E 5.B.17) *Suppose V is finite-dim, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$, and p is the mini poly of T. Show that the mini poly of $(T-\lambda I)$ is the poly q defined by $q(z) = p(z+\lambda)$.*

SOLUTION:

$q(T-\lambda I) = 0 \Rightarrow q$ is poly multi of the mini poly of $(T-\lambda I)$.

Suppose the degree of the mini poly of $(T-\lambda I)$ is n, and the degree of the mini poly of T is m.

By definition of mini poly,

n is the smallest such that $(T-\lambda I)^n \in \operatorname{span}(I,(T-\lambda I),\dots,(T-\lambda I)^{n-1})$;

m is the smallest such that $T^m \in \operatorname{span}(I,T,\dots,T^{m-1})$.

又 $T^k \in \operatorname{span}(I,T,\dots,T^{k-1}) \iff (T-\lambda)^k \in \operatorname{span}(I,(T-\lambda I),\dots,(T-\lambda I)^{k-1})$.

Thus $n = m$. 又 q is monic. By the uniqnes of mini poly. ☐

• (4E 5.B.18) *Suppose V is finite-dim, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}\backslash\{0\}$, and p is the mini poly of T. Show that the mini poly of λT is the poly q defined by $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.*

SOLUTION:

$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow q$ is a poly multi of the mini poly of λT.

Suppose the degree of the mini poly of λT is n, and the degree of the mini poly of T is m.

By definition of mini poly,

n is the smallest such that $(\lambda T)^n \in \operatorname{span}(\lambda I,\lambda T,\dots,(\lambda T)^{n-1})$;

m is the smallest such that $T^m \in \operatorname{span}(I,T,\dots,T^{m-1})$.

又 $(\lambda T)^k \in \operatorname{span}(\lambda I,\lambda T,\dots,(\lambda T)^{k-1}) \iff T^k \in \operatorname{span}(I,T\dots,T^{k-1})$.

Thus $n = m$. 又 q is monic. By the uniqnes of mini poly. ☐

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18 (OR4E 5.B.15) *Suppose V is a finite-dim complex vecsp with* $\dim V > 0$ *and* $T \in \mathcal{L}(V)$. *Define $f: \mathbf{C} \to \mathbf{R}$ by $f(\lambda) = \dim \operatorname{range}(T-\lambda I)$. Prove that f is not a continuous function.*

SOLUTION: Note that V is finite-dim.

Let λ_0 be an eigval of T. Then $(T-\lambda_0 I)$ is not surj. Hence $\dim \operatorname{range}(T-\lambda_0 I) < \dim V$.

Because T has finitely many eigvals. There exist a sequence of number $\{\lambda_n\}$ such that $\lim\limits_{n\to\infty} \lambda_n = \lambda_0$.

And λ_n is not an eigval of T for each $n \Rightarrow \dim \operatorname{range}(T-\lambda_n I) = \dim V \neq \dim \operatorname{range}(T-\lambda_0 I)$.

Thus $f(\lambda_0) \neq \lim_{n\to\infty} f(\lambda_n)$. $\square$

- (4E 5.B.9) *Suppose $T \in \mathcal{L}(V)$ is such that with resp to some basis of V, all entries of the matrix of T are rational numbers. Explain why all coefficients of the mini poly of T are rational numbers.*

Solution:

Let $(v_1, \dots, v_n)$ denote the basis such that $\mathcal{M}(T, (v_1, \dots, v_n))_{j,k} = A_{j,k} \in \mathbf{Q}$ for all $j, k = 1, \dots, n$.
Denote $\mathcal{M}\left(v_j, (v_1, \dots, v_n)\right)$ by x_j for each v_j.
Suppose p is the mini poly of T and $p(z) = z^m + \cdots + c_1 z + c_0$. Now we show that each $c_j \in \mathbf{Q}$.
Note that $\forall s \in \mathbf{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbf{Q}^{n,n}$ and $T^s v_k = A^s_{1,k} v_1 + \cdots + A^s_{n,k} v_n$ for all $k \in \{1, \dots, n\}$.

$$\text{Thus} \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \cdots + c_1 A + c_0 I)x_1 = \sum_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,1} x_j = 0; \\ \qquad \vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \cdots + c_1 A + c_0 I)x_n = \sum_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly,} \begin{cases} (A^m + \cdots + c_1 A + c_0 I)_{1,1} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,1} = 0; \\ \qquad\qquad \vdots \quad \ddots \quad \vdots \\ (A^m + \cdots + c_1 A + c_0 I)_{1,n} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,n} = 0; \end{cases}$$

Hence we get a system of n^2 linear equations in m unknowns $c_0, c_1, \dots, c_{m-1}$.
We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbf{Q}$. $\square$

- Or(4E 5.B.16), Or(8.C.18) *Suppose $a_0, \dots, a_{n-1} \in \mathbf{F}$. Let T be the operator on $\mathbf{F}^n$ such that*

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ with resp to the standard basis } (e_1, \dots, e_n).$$

Show that the mini poly of T is p defined by $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$.

*$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the mini poly of some operator. Hence a formula or an algorithm that could produce exact eigvals for each operator on each $\mathbf{F}^n$ could then produce exact zeros for each poly [by 8.36(b)]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an operator.*

Solution: Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ is linely inde. 又 The deg of mini poly is at most n.

$T^n e_1 = \cdots = T^{n-k} e_{1+k} = \cdots = T e_n = -a_0 e_1 - a_1 e_2 - a_2 e_3 - \cdots - a_{n-1} e_n$
$= (-a_0 I - a_1 T - a_2 T^2 - \cdots - a_{n-1} T^{n-1}) e_1$. Thus $p(T)e_1 = 0 = p(T)e_j$ for each $e_j = T^{j-1} e_1$. $\square$

- Eigenvalues On Odd-Dimensional Real Vector Spaces
- Even-Dimensional Null Space

Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$. Prove that $\dim \operatorname{null}(T^2 + bT + cI)$ is an even number.

Solution:

Denote $\operatorname{null}(T^2 + bT + cI)$ by R. Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.
Suppose λ is an eigval of T_R with an eigvec $v \in R$.
Then $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \left((\lambda + b)^2 + c - \dfrac{b^2}{4} \right) v$.

Because $c - \frac{b^2}{4} > 0$ and we have $v = 0$. Thus T_R has no eigvals.

Let U be an invar subsp of R that has the largest, even dim among all invar subsps.

Assume that $U \neq R$. Then $\exists\, w \in R$ but $w \notin U$. Let W be such that $(w, T|_R w)$ is a basis of W.

Because $T|_R^2 w = -bT|_R w - cw \in W$. Hence W is an invar subsp of dim 2.

Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,

for if not, because $w \notin U, T|_R w \in U$,

$U \cap W$ is invar under $T|_R$ of one dim (impossible because $T|_R$ has no eigvecs).

Hence $U + W$ is even-dim invar subsp under $T|_R$, contradicting the maximality of $\dim U$.

Thus the assumption was incorrect. Hence $R = \operatorname{null}(T^2 + bT + cI) = U$ has even dim. □

• Operators On Odd-Dimensional Vector Spaces Have Eigenvalues

(a) *Suppose* $\mathbf{F} = \mathbf{C}$. Then by [5.21], we are done.

(b) *Suppose* $\mathbf{F} = \mathbf{R}$, *V is finite-dim, and* $\dim V = n$ *is an odd number.*
Let $T \in \mathcal{L}(V)$ and the mini poly is p. Prove that T has an eigval.

Solution:

(i) If $n = 1$, then we are done.

(ii) Suppose $n \geq 3$. Assume that every operator, on odd-dim vecsps of dim less than n, has an eigval.

If p is a poly multi of $(x - \lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] λ is an eigval of T and we are done.

Now suppose $b, c \in \mathbf{R}$ such that $b^2 < 4c$ and p is a poly multi of $x^2 + bx + c$ (see [4.17]).

Then $\exists\, q \in \mathcal{P}(\mathbf{R})$ such that $p(x) = q(x)(x^2 + bx + c)$ for all $x \in \mathbf{R}$.

Now $0 = p(T) = (q(T))\,(T^2 + bT + cI)$, which means that $q(T)|_{\operatorname{range}(T^2+bT+cI)} = 0$.

Because $\deg q < \deg p$ and p is the mini poly of T, hence $\operatorname{range}(T^2 + bT + cI) \neq V$.

又 $\dim V$ is odd and $\dim \operatorname{null}(T^2 + bT + cI)$ is even (by our previous result).

Thus $\dim V - \dim \operatorname{null}(T^2 + bT + cI) = \dim \operatorname{range}(T^2 + bT + cI)$ is odd.

By [5.18], $\operatorname{range}(T^2 + bT + cI)$ is an invar subsp of V under T that has odd dim less than n.

Our induction hypothesis now implies that $T|_{\operatorname{range}(T^2+bT+cI)}$ has an eigval.

By mathematical induction. □

• (2E Ch5.24) *Suppose* $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$ *has no eigvals.*
Prove that every invar subsp of V under T is even-dim.

Solution:

Suppose U is such a subsp. Then $T|_U \in \mathcal{L}(U)$. We prove by contradiction.

If $\dim U$ is odd, then $T|_U$ has an eigval and so is T, so that $\exists$ invar subsp of 1 dim, contradicts. □

• (4E 5.B.29) *Show that every operator on a finite-dim vecsp of dim* ≥ 2 *has a 2-dim invar subsp.*

Solution:

Using induction on $\dim V$.

(i) $\dim V = 2$, we are done.

(ii) $\dim V > 2$. Assume that the desired result is true for vecsp of smaller dim.

Suppose p is the mini poly of degree m and $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.

If $T = \lambda I$ ($\Leftrightarrow m = 1 \vee m = -\infty$), then we are done. ($m \neq 0$ because $\dim V \neq 0$.)

Now define a q by $q(z) = (z - \lambda_1)(z - \lambda_2)$.

By assumption, $T|_{\operatorname{null} q(T)}$ has an invar subsp of dim 2. □

5.B: II

• (4E 5.C.1) *Prove or give a counterexample:*
If $T \in \mathcal{L}(V)$ and T^2 has an upper-trig matrix, then T has an upper-trig matrix.

Solution:

• (4E 5.C.2) *Suppose A and B are upper-trig mtcs of the same size,*
with $\alpha_1, \ldots, \alpha_n$ on the diag of A and $\beta_1, \ldots, \beta_n$ on the diag of B.
(a) *Show that $A + B$ is an upper-trig matrix with $\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n$ on the diag.*
(b) *Show that AB is an upper-trig matrix with $\alpha_1\beta_1, \ldots, \alpha_n\beta_n$ on the diag.*

Solution:

• (4E 5.C.3)
Suppose $T \in \mathcal{L}(V)$ is inv and $B = (v_1, \ldots, v_n)$ is a basis of V such that
$\mathcal{M}(T, B) = A$ is upper trig, with $\lambda_1, \ldots, \lambda_n$ on the diag.
Show that the matrix of $\mathcal{M}(T^{-1}, B) = A^{-1}$ is also upper trig, with $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$ on the diag.

Solution:

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9 (4E 5.C.7)
Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $v \in V$.
(a) *Prove that $\exists\,!$ monic poly p_v of smallest degree such that $p_v(T)v = 0$.*
(b) *Prove that the mini poly of T is a poly multi of p_v.*

Solution:

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14 (OR4E 5.C.4) *Give an operator T such that with resp to some basis,*
$\mathcal{M}(T)_{k,k} = 0$ for each k, while T is inv.

Solution:

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15 (OR4E 5.C.5) *Give an operator T such that with resp to some basis,*
$\mathcal{M}(T)_{k,k} \neq 0$ for each k, while T is not inv.

Solution:

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20 (OR4E 5.C.6)
Suppose $\mathbf{F} = \mathbf{C}$, V is finite-dim, and $T \in \mathcal{L}(V)$.
Prove that if $k \in \{1, \ldots, \dim V\}$, then V has a k dim subsp invar under T.

Solution:

• (4E 5.C.8) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\exists\, v \in V \backslash \{0\}$ such that $T^2 v + 2Tv = -2v$.*
(a) *Prove that if $\mathbf{F} = \mathbf{R}$, then $\nexists$ a basis of V with resp to which T has an upper-trig matrix.*
(b) *Prove that if $\mathbf{F} = \mathbf{C}$ and A is an upper-trig matrix that equals the matrix of T*
with resp to some basis of V, then $-1 + \mathrm{i}$ or $-1 - \mathrm{i}$ appears on the diag of A.

Solution:

- (4E 5.C.9) *Suppose* $B \in \mathbf{F}^{n,n}$ *with complex entries.*
  Prove that $\exists$ *inv* $A \in \mathbf{F}^{n,n}$ *with complex entries such that* $A^{-1}BA$ *is an upper-trig matrix.*

Solution:

- (4E 5.C.10) *Suppose* $T \in \mathcal{L}(V)$ *and* $(v_1, \dots, v_n)$ *is a basis of* V.
  Show that the following are equi.
  (a) *The matrix of* T *with resp to* $(v_1, \dots, v_n)$ *is lower trig.*
  (b) span $(v_k, \dots, v_n)$ *is invar under* T *for each* $k = 1, \dots, n$.
  (c) $Tv_k \in$ span $(v_k, \dots, v_n)$ *for each* $k = 1, \dots, n$.

Solution:

- (4E 5.C.11) *Suppose* $\mathbf{F} = \mathbf{C}$ *and* V *is finite-dim.*
  Prove that if $T \in \mathcal{L}(V)$, *then* T *has a lower-trig matrix with resp to some basis.*

Solution:

- (4E 5.C.12)
  Suppose V *is finite-dim,* $T \in \mathcal{L}(V)$ *has an upper-trig matrix with resp to some basis, and* U *is a subsp of* V *that is invar under* T.
  (a) *Prove that* $T|_U$ *has an upper-trig matrix with resp to some basis of* U.
  (b) *Prove that* T/U *has an upper-trig matrix with resp to some basis of* V/U.

Solution:

- (4E 5.C.13) *Suppose* V *is finite-dim,* $T \in \mathcal{L}(V)$. *Suppose* U *is an invar subsp of* V *under* T *such that* $T|_U$ *has an upper-trig matrix and also* T/U *has an upper-trig matrix.*
  Prove that T *has an upper-trig matrix.*

Solution:

- (4E 5.C.14) *Suppose* V *is finite-dim and* $T \in \mathcal{L}(V)$.
  Prove that T *has an upper-trig matrix* $\Longleftrightarrow$ T' *has an upper-trig matrix.*

Solution:

Ended

5.C

Ended

5.E* (4E)

1 *Give an example of two commuting operators* $S, T \in \mathbf{F}^4$ *such that there is an invar subsp of* $\mathbf{F}^4$ *under* S *but not under* T *and an invar subsp of* $\mathbf{F}^4$ *under* T *but not under* S.

Solution:

---

2 *Suppose* $\mathcal{E}$ *is a subset of* $\mathcal{L}(V)$ *and every element of* $\mathcal{E}$ *is diagable.*
Prove that $\exists$ *a basis of* V *with resp to which*

every element of $\mathcal{E}$ has a diag matrix $\Longleftrightarrow$ every pair of elements of $\mathcal{E}$ commutes.

This exercise extends [5.76], which considers the case in which $\mathcal{E}$ contains only two elements.

For this exercise, $\mathcal{E}$ may contain any number of elements, and $\mathcal{E}$ may even be an infinite set.

Solution:

---

3 *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.*

(a) *Prove that* $\operatorname{null} p(S)$ *is invar under T.*

(b) *Prove that* $\operatorname{range} p(S)$ *is invar under T.*

See Note For *[5.17] for the special case $S = T$.*

Solution:

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4 *Prove or give a counterexample:*

A diag matrix A and an upper-trig matrix B of the same size commute.

Solution:

---

5 *Prove that a pair of operators on a finite-dim vecsp commute $\Longleftrightarrow$ their dual operators commute.*

Solution:

---

6 *Suppose V is a finite-dim complex vecsp and $S, T \in \mathcal{L}(V)$ commute.*

Prove that $\exists\, \alpha, \lambda \in \mathbf{C}$ such that $\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V$.

Solution:

---

7 *Suppose V is a complex vecsp, $S \in \mathcal{L}(V)$ is diagable, and T commutes with S.*

Prove that $\exists$ basis B of V such that S has a diag matrix with resp to B

and T has an upper-trig matrix with resp to B.

Solution:

---

8 *Suppose $m = 3$ in Example [5.72]*

and D_x, D_y are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbf{R}^2)$ from that example.

Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ with resp to which D_x and D_y each have an upper-trig matrix.

Solution:

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9 *Suppose V is a finite-dim nonzero complex vecsp.*

Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.

(a) *Prove that $\exists\, v \in V$ is an eigvec for every element of $\mathcal{E}$.*

(b) *Prove that $\exists$ a basis of V with resp to which every element of $\mathcal{E}$ has an upper-trig matrix.*

Solution:

---

10 *Give an example of two commuting operators S, T on a finite-dim real vecsp such that*

$S + T$ has a eigval that does not equal an eigval of S plus an eigval of T

and ST has a eigval that does not equal an eigval of S times an eigval of T.

Solution:

Ended