

## 简介

这是我个人用于复习的笔记，一本习题补注。由于我个人的复习特点，我把许多对我个人而言没什么复习价值的习题作了省略。为什么我没有用中文？因为我将来要学习的绝大多数数学课本都是全英的，国内目前的专业翻译速度慢、不全面，况且对于专业学习者来说，直接使用英文不会造成任何困扰，并且我不愿意花费额外的时间去翻译，所以我用英文。但我讨厌英文单词的冗长性，这会让我复习起来很不爽，所以我对许多常用词汇适当地作了简写。这份笔记的内容范围和标识说明，我已经在[README](#)中写得很清楚，不再赘述。这份笔记尚处于缓慢的编撰进度中。

Goto

1	2	3	4	5	6	7	8	9	10
A	A	A	/	A	A	A	A	A	A
B	B	B	/	B <sup>I</sup>	B	B	B	B	B
/	/	/	/	B <sup>II</sup>	/	/	/	/	/
C	C	C	/	C	C	C	C	/	/
/	/	D	/	/	D	D	D	/	/
/	/	E	/	E*	/	/	/	/	/
/	/	F	/	/	/	F*	/	/	/

Abbreviation Table

def	definition
vec	vector
vecsp	vector space
subsp	subspace
add	addition/additive
multi	multiplication/multiplicative/multiple
assoc	associative/associativity
distr	distributive properties/property
inv	inverse
existns	existence
uniques	uniqueness
linely inde	linearly independent/independence
linely dep	linearly dependent/dependence
dim	dimension(al)
inje	injective
surj	surjective
col	column
with resp	with respect
standard basis	std basis
iso	isomorphism/isomorphic
correspd	correspond(ing)
poly	polynomial
eigval	eigenvalue
eigvec	eigenvector
mini poly	minimal polynomial
char poly	characteristic polynomial

# 1.B

1 Prove that  $\forall v \in V, -(-v) = v$ .

SOLUTION:

$$\left. \begin{array}{l} -(-v) + (-v) = 0 \\ v + (-v) = 0 \end{array} \right\} \Rightarrow \text{By the uniqueness of add inv, we are done.}$$

$$\text{OR. } -(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v.$$

□

2 Suppose  $a \in \mathbf{F}, v \in V$ , and  $av = 0$ . Prove that  $a = 0$  or  $v = 0$ .

SOLUTION:

$$\text{Suppose } a \neq 0, \exists a^{-1} \in \mathbf{F}, a^{-1}a = 1, \text{ hence } v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0.$$

□

3 Suppose  $v, w \in V$ . Explain why  $\exists! x \in V, v + 3x = w$ .

SOLUTION:

$$[\text{Existence}] \text{ Let } x = \frac{1}{3}(w - v).$$

$$[\text{Uniqueness}] \text{ Suppose } v + 3x_1 = w, \text{ (I) } v + 3x_2 = w \text{ (II). Then (I) - (II) : } 3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2.$$

□

$$\text{OR. } v + 3x = w \Leftrightarrow 3x = w - v \Leftrightarrow x = \frac{1}{3}(w - v).$$

□

5 Show that in the def of a vecsp, the add inv condition can be replaced by [1.29].

*Hint:* Suppose  $V$  satisfies all conds in the def, except we've replaced the add inv cond with [1.29].

Prove that the add inv is true.

SOLUTION:

$$\text{Using [1.31]. } 0v = 0 \text{ for all } v \in V \Leftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0.$$

□

6 Let  $\infty$  and  $-\infty$  denote two distinct objects, neither of which is in  $\mathbf{R}$ .

Define an add and scalar multi on  $\mathbf{R} \cup \{\infty, -\infty\}$  as you could guess.

The operations of real numbers is as usual. While for  $t \in \mathbf{R}$  define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

$$\text{(I) } t + \infty = \infty + t = \infty + \infty = \infty,$$

$$\text{(II) } t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty,$$

$$\text{(III) } \infty + (-\infty) = (-\infty) + \infty = 0.$$

With these operations of add and scalar multi, is  $\mathbf{R} \cup \{\infty, -\infty\}$  a vecsp over  $\mathbf{R}$ ? Explain.

SOLUTION:

Not a vecsp, since the add and scalar mult is not assoc and distr.

By Assoc:  $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$ .

$$\text{OR. By Distr: } \infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0.$$

□

• **TIPS:** About the Field  $\mathbf{F}$ : Many choices.

**EXAMPLE:**  $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \dots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$ .

# 1.C 7 8 9 11 12 13 15 16 17 18 21 22 23 24

7 Give a nontrivial  $U \subseteq \mathbb{R}^2$ ,

$U$  is closed under taking add invs and under add, but is not a subsp of  $\mathbb{R}^2$ .

SOLUTION:  $(0 \in U; v \in U \Rightarrow -v \in U.)$  Let  $U = \{0, 1\}^2, \mathbb{Z}^2, \mathbb{Q}^2$ .

8 Give a nontrivial  $U \subseteq \mathbb{R}^2$ ,  $U$  is closed under scalar multi, but is not a subsp of  $\mathbb{R}^2$ .

SOLUTION: Let  $U = \{(x, y) \in \mathbb{R}^2 : x = 0 \vee y = 0\}$ .

9 A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called periodic if  $\exists p \in \mathbb{N}^+, f(x) = f(x + p)$  for all  $x \in \mathbb{R}$ .  
Is the set of periodic functions  $\mathbb{R} \rightarrow \mathbb{R}$  a subsp of  $\mathbb{R}^{\mathbb{R}}$ ? Explain.

SOLUTION: Denote the set by  $S$ .

Suppose  $h(x) = \cos x + \sin \sqrt{2}x \in S$ , since  $\cos x, \sin \sqrt{2}x \in S$ .

Assume  $\exists p \in \mathbb{N}^+$  such that  $h(x) = h(x + p), \forall x \in \mathbb{R}$ . Let  $x = 0 \Rightarrow h(0) = h(\pm p) = 1$ .

Thus  $1 = \cos p + \sin \sqrt{2}p = \cos p - \sin \sqrt{2}p$

$\Rightarrow \sin \sqrt{2}p = 0, \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbb{Z}$ , while  $p = \frac{m\pi}{\sqrt{2}}, m \in \mathbb{Z}$ .

Hence  $2k = \frac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \frac{m}{2k} \in \mathbb{Q}$ . Contradiction! □

OR. Because [I] :  $\cos x + \sin \sqrt{2}x = \cos(x + p) + \sin(\sqrt{2}x + \sqrt{2}p)$ . By differentiating twice,

[II] :  $\cos x + 2 \sin \sqrt{2}x = \cos(x + p) + 2 \sin(\sqrt{2}x + \sqrt{2}p)$ .

[II] - [I] :  $\sin \sqrt{2}x = \sin(\sqrt{2}x + \sqrt{2}p)$   
 $2[I] - [II] :$   $\cos x = \cos(x + p)$   $\left\} \Rightarrow \text{Let } x = 0, p = \frac{m\pi}{\sqrt{2}} = 2k\pi. \text{ Contradicts.} \right.$  □

• Suppose  $U, W, V_1, V_2, V_3$  are subsp of  $V$ .

15  $U + U \ni u + w \in U.$  □

16  $U + W \ni u + w = w + u \in W + U.$  □

17  $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3).$  □

18 Does the add on the subsp of  $V$  have an add identity? Which subsp have add invs?

SOLUTION: Suppose  $\Omega$  is the additive identity.

(a) For any subsp  $U$  of  $V$ .  $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$ . Let  $U = \{0\}$ , then  $\Omega = \{0\}$ .

(b) Now suppose  $W$  is an add inv of  $U \Rightarrow U + W = \Omega$ .

Note that  $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$ . Thus  $U = W = \Omega = \{0\}$ . □

11 Prove that the intersection of every collection of subsp of  $V$  is a subsp of  $V$ .

SOLUTION: Suppose  $\{U_\alpha\}_{\alpha \in \Gamma}$  is a collection of subsp of  $V$ ; here  $\Gamma$  is an arbitrary index set.

We show that  $\bigcap_{\alpha \in \Gamma} U_\alpha$ , which equals the set of vecs that are in  $U_\alpha$  for each  $\alpha \in \Gamma$ , is a subsp of  $V$ .

(一)  $0 \in \bigcap_{\alpha \in \Gamma} U_\alpha$ . Nonempty.

(二)  $u, v \in \bigcap_{\alpha \in \Gamma} U_\alpha \Rightarrow u + v \in U_\alpha, \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha \in \Gamma} U_\alpha$ . Closed under add.

(三)  $u \in \bigcap_{\alpha \in \Gamma} U_\alpha, \lambda \in \mathbb{F} \Rightarrow \lambda u \in U_\alpha, \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha \in \Gamma} U_\alpha$ . Closed under scalar multi.

Thus  $\bigcap_{\alpha \in \Gamma} U_\alpha$  is nonempty subset of  $V$  that is closed under add and scalar multi. □

**12** Suppose  $U, W$  are subsp of  $V$ . Prove that  $U \cup W$  is a subsp of  $V \iff U \subseteq W$  or  $W \subseteq U$ .

**SOLUTION:**

(a) Suppose  $U \subseteq W$ . Then  $U \cup W = W$  is a subsp of  $V$ .

(b) Suppose  $U \cup W$  is a subsp of  $V$ . Suppose  $U \not\subseteq W$  and  $U \not\supseteq W$  ( $U \cup W \neq U$  and  $W$ ).

Then  $\forall a \in U \wedge a \notin W, b \in W \wedge b \notin U, a + b \in U \cup W$ .

$\left. \begin{array}{l} \text{If } a + b \in U \Rightarrow b = (a + b) + (-a) \in U, \text{ contradicts!} \\ \text{If } a + b \in W \Rightarrow a = (a + b) + (-b) \in W, \text{ contradicts!} \end{array} \right\} \Rightarrow U \cup W = U \text{ or } W. \text{ Contradicts!}$

Thus  $U \subseteq W$  and  $U \supseteq W$ . □

**13** Prove that the union of three subsp of  $V$  is a subsp of  $V$  if and only if one of the subsp contains the other two.

*This exercise is not true if we replace  $\mathbf{F}$  with a field containing only two elements.*

**SOLUTION:**

Suppose  $U_1, U_2, U_3$  are subsp of  $V$ . Denote  $U_1 \cup U_2 \cup U_3$  by  $\mathcal{U}$ .

(a) Suppose that one of the subsp contains the other two.

Then  $\mathcal{U} = U_1, U_2$  or  $U_3$  is a subsp of  $V$ .

(b) Suppose that  $U_1 \cup U_2 \cup U_3$  is a subsp of  $V$ .

Distinctively notice that  $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$ .

Also note that, if  $U \cup W = V$  is a vecsp, then in general  $U$  and  $W$  are not subsp of  $V$ .

Hence this literal trick is invalid.

(I) If any  $U_j$  is contained in the union of the other two, say  $U_1 \subseteq U_2 \cup U_3$ , then  $\mathcal{U} = U_2 \cup U_3$ .

By applying Problem (12) we conclude that one  $U_j$  contains the other two. Thus we are done.

(II) Assume that no  $U_j$  is contained in the union of the other two,

and no  $U_j$  contains the union of the other two.

Say  $U_1 \not\subseteq U_2 \cup U_3$  and  $U_1 \not\supseteq U_2 \cup U_3$ .

$\exists u \in U_1 \wedge u \notin U_2 \cup U_3; v \in U_2 \cup U_3 \wedge v \notin U_1$ . Let  $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$ .

Note that  $W \cap U_1 = \emptyset$ , for if  $v + \lambda u \in U_1$  then  $v + \lambda u - \lambda u = v \in U_1$ .

又  $W \subseteq U_1 \cup U_2 \cup U_3$ . Thus  $W \subseteq U_2 \cup U_3$ .

$\forall v + \lambda u \in W, \exists i \in \{2, 3\}, v + \lambda u \in U_i$ .

Because  $U_2, U_3$  are subsp and hence have at least one element ( zero ).

If  $U_2 = U_3$ , then  $\mathcal{U} = U_1 \cup U_2$  and by Problem (12) we are done.

( Note that at least one of  $U_2, U_3$  is not  $\{0\}$ , for if not,  $U_1 \supseteq U_2 \cup U_3$ . )

Otherwise,  $\exists$  distinct  $\lambda, \mu \in \mathbf{F}, v + \lambda u, v + \mu u \in U_i$  for some  $i \in \{2, 3\}$ . ( $\Delta$ )

Then  $u \in U_i$  while  $u \notin U_2 \cup U_3$ . Contradicts.

**EXAMPLE:** Let  $\mathbf{F} = \mathbf{Z}_2$ . TODO

□

• **EXAMPLE:** Suppose  $U = \{(x, x, y, y) \in \mathbf{F}^4\}$ ,  $W = \{(x, x, x, y) \in \mathbf{F}^4\}$ .

Prove that  $U + W = \{(x, x, y, z) \in \mathbf{F}^4\}$ .

Let  $T$  denote  $\{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$ . By def,  $U + W \subseteq T$ .

And  $T \ni (x, x, y, z) \Rightarrow (0, 0, y - x, y - x) + (x, x, x, -y + x + z) \in U + W$ . Hence  $T \subseteq U + W$ .  $\square$

**21** Suppose  $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5\}$ . Find a  $W$  such that  $\mathbf{F}^5 = U \oplus W$ .

**SOLUTION:** Let  $W = \{(0, 0, z, w, u) \in \mathbf{F}^5\}$ . Then  $U \cap W = \{0\}$ .

And  $\mathbf{F}^5 \ni (x, y, z, w, u) \Rightarrow (x, y, x + y, x - y, 2x) + (0, 0, z - x - y, w - x - y, u - 2x) \in U + W$ .

**23** Give an example of vecsps  $V_1, V_2, U$  such that  $V_1 \oplus U = V_2 \oplus U$ , but  $V_1 \neq V_2$ .

**SOLUTION:**  $V = \mathbf{F}^2$ ,  $U = \{(x, x) \in \mathbf{F}^2\}$ ,  $V_1 = \{(x, 0) \in \mathbf{F}^2\}$ ,  $V_2 = \{(0, x) \in \mathbf{F}^2\}$ .

• **TIPS:** Suppose  $V_1 \subseteq V_2$  in Exercise (23). Prove or give a counterexample:  $V_1 = V_2$ .

**SOLUTION:**

Because the subset  $V_1$  of vecsp  $V_2$  is closed under add and scalar multi,  $V_1$  is a subspace of  $V_2$ .

Suppose  $W$  is such that  $V_2 = V_1 \oplus W$ . Now  $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$ .

If  $W \neq \{0\}$ , then  $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$ , contradicts. Hence  $W = \{0\}$ ,  $V_1 = V_2$ .  $\square$

Suppose  $V_1 \oplus U_1 = V_2 \oplus U_2$ ,  $V_1 \subseteq V_2$ ,  $U_2 \subseteq U_1$ . Prove or give a counterexample:  $V_1 = V_2$ .

**24** Let  $V_E = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is even}\}$ ,  $V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is odd}\}$ . Show that  $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$ .

**SOLUTION:** (a)  $V_E \cap V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x)\} = \{0\}$ .

(b)  $\left\{ \begin{array}{l} \text{Let } f_e(x) = \frac{1}{2}(g(x) + g(-x)) \Rightarrow f_e \in V_E \\ \text{Let } f_o(x) = \frac{1}{2}(g(x) - g(-x)) \Rightarrow f_o \in V_O \end{array} \right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}}, g(x) = f_e(x) + f_o(x)$ .  $\square$

**ENDED**

## 2.A 1 2 6 10 11 14 16 17 | 4E: 3,14

**2** (a)  $[P]$  A list  $(v)$  of length 1 in  $V$  is linely inde  $\iff v \neq 0$ .  $[Q]$

(b)  $[P]$  A list  $(v, w)$  of length 2 in  $V$  is linely inde  $\iff \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$ .  $[Q]$

**SOLUTION:**

(a)  $Q \xrightarrow{1} P : v \neq 0 \Rightarrow$  if  $av = 0$  then  $a = 0 \Rightarrow (v)$  linely inde.

$P \xrightarrow{2} Q : (v)$  linely inde  $\Rightarrow v \neq 0$ , for if  $v = 0$ , then  $av = 0 \not\Rightarrow a = 0$ .

OR.  $\left\{ \begin{array}{l} \neg Q \xrightarrow{3} \neg P : v = 0 \Rightarrow av = 0 \text{ while we can let } a \neq 0 \Rightarrow (v) \text{ is linely dep.} \\ \neg P \xrightarrow{4} \neg Q : (v) \text{ linely dep} \Rightarrow av = 0 \text{ while } a \neq 0 \Rightarrow v = 0. \end{array} \right.$

**COMMENT:** (1) with (3) and (2) with (4) will do as well.  $\square$

(b)  $P \xrightarrow{1} Q : (v, w)$  linely inde  $\Rightarrow$  if  $av + bw = 0$ , then  $a = b = 0 \Rightarrow$  no scalar multi.

$Q \xrightarrow{2} P : \text{no scalar multi} \Rightarrow$  if  $av + bw = 0$ , then  $a = b = 0 \Rightarrow (v, w)$  linely inde.

OR.  $\left\{ \begin{array}{l} \neg P \xrightarrow{3} \neg Q : (v, w) \text{ linely dep} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{scalar multi} \\ \neg Q \xrightarrow{4} \neg P : \text{scalar multi} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{linely dep.} \end{array} \right.$

**COMMENT:** (1) with (3) and (2) with (4) will do as well.  $\square$

**1** Prove that  $[P] (v_1, v_2, v_3, v_4)$  spans  $V \iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$  also spans  $V [Q]$ .

**SOLUTION:**

Notice that  $V = \text{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists a_1, \dots, a_n \in \mathbb{F}, v = a_1 v_1 + \dots + a_n v_n$ .

Assume that  $\forall v \in V, \exists a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbb{F}$ , ( that is, if  $\exists a_i$ , then we are to find  $b_i$ , vice versa )

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4. \end{aligned}$$

Now we can let  $b_i = \sum_{r=1}^i a_r$  if we are to prove  $Q$  with  $P$  already assumed;

or let  $a_i = b_i - b_{i-1}$  with  $b_0 = 0$ , if we are to prove  $P$  with  $Q$  already assumed.  $\square$

**6** Prove that  $[P] (v_1, v_2, v_3, v_4)$  is linely inde  $\iff [Q] (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$  is linely inde.

**SOLUTION:**

$$\begin{aligned} P \Rightarrow Q : a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 &= 0 \\ \Rightarrow a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 &= 0 \\ \Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \end{aligned}$$

$$\begin{aligned} Q \Rightarrow P : a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 &= 0 \\ \Rightarrow a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \dots + a_4)v_4 &= 0 \\ \Rightarrow a_1 = a_1 + a_2 = a_1 + a_2 + a_3 = a_1 + \dots + a_4 = 0. \end{aligned} \quad \square$$

• Suppose  $(v_1, \dots, v_m)$  is a list of vecs in  $V$ . For each  $k$ , let  $w_k = v_1 + \dots + v_k$ .

(a) Show that  $\text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$ .

(b) Show that  $[P] (v_1, \dots, v_m)$  is linely inde  $\iff (w_1, \dots, w_m)$  is linely inde  $[Q]$ .

**SOLUTION:**

$$(a) \text{ let } a_k = \sum_{j=1}^k b_j \Leftarrow a_1 v_1 + \dots + a_m v_m = b_1 w_1 + \dots + b_m w_m \Rightarrow \text{let } b_1 = a_1, b_k = a_k - \sum_{j=1}^{k-1} b_j = \sum_{j=1}^k (-1)^{k-j} a_j.$$

$$(b) P \Rightarrow Q : b_1 w_1 + \dots + b_m w_m = 0 = a_1 v_1 + \dots + a_m v_m, \text{ where } 0 = a_k = \sum_{j=1}^k b_j.$$

$$Q \Rightarrow P : a_1 v_1 + \dots + a_m v_m = 0 = b_1 w_1 + \dots + b_m w_m = 0, \text{ where } 0 = b_1 = a_1, 0 = b_k = \sum_{j=1}^k (-1)^{k-j} a_j$$

OR. Because  $W = \text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$ .

By [2.21](b), a list of length  $(m-1)$  spans  $W$ , then by [2.23],

$(w_1, \dots, w_m)$  linely dep  $\Rightarrow (v_1, \dots, v_m)$  linely dep. Conversely it is true as well.  $\square$

**10** Suppose  $(v_1, \dots, v_m)$  is linely inde in  $V$  and  $w \in V$ .

Prove that if  $(v_1 + w, \dots, v_m + w)$  is linely depe, then  $w \in \text{span}(v_1, \dots, v_m)$ .

**SOLUTION:**

Suppose  $a_1(v_1 + w) + \dots + a_m(v_m + w) = 0, \exists a_i \neq 0 \Rightarrow a_1 v_1 + \dots + a_m v_m = 0 = -(a_1 + \dots + a_m)w$ .

Then  $a_1 + \dots + a_m \neq 0$ , for if not,  $a_1 v_1 + \dots + a_m v_m = 0$  while  $a_i \neq 0$  for some  $i$ , contradicts.  $\square$

OR. By contrapositive,  $w \notin \text{span}(v_1, \dots, v_m)$ , similarly.  $\square$

OR.  $\exists j \in \{1, \dots, m\}, v_j + w \in \text{span}(v_1 + w, \dots, v_{j-1} + w)$ . If  $j = 1$  then  $v_1 + w = 0$  and we are done.

If  $j \geq 2$ , then  $\exists a_i \in \mathbb{F}, v_j + w = a_1(v_1 + w) + \dots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1 v_1 + \dots + a_{j-1} v_{j-1}$ .

Where  $\lambda = 1 - (a_1 + \dots + a_{j-1})$ . Note that  $\lambda \neq 0$ , for if not,  $v_j + \lambda w = v_j \in \text{span}(v_1, \dots, v_{j-1})$ , contradicts.

Now  $w = \lambda^{-1}(a_1 v_1 + \dots + a_{j-1} v_{j-1} - v_j) \Rightarrow w \in \text{span}(v_1, \dots, v_m)$ .  $\square$

**11** Suppose  $(v_1, \dots, v_m)$  is linely inde in  $V$  and  $w \in V$ .

Show that  $[P] (v_1, \dots, v_m, w)$  is linely inde  $\iff w \notin \text{span}(v_1, \dots, v_m) [Q]$ .

**SOLUTION:**  $\neg Q \Rightarrow \neg P$  : Suppose  $w \in \text{span}(v_1, \dots, v_m)$ . Then  $(v_1, \dots, v_m, w)$  is linely depe.

$\neg P \Rightarrow \neg Q$  : Suppose  $(v_1, \dots, v_m, w)$  is linely dep. Then by [2.21]  $w \in \text{span}(v_1, \dots, v_m)$ . □

**14** Prove that  $[P] V$  is infinite-dim  $\iff [Q] \left| \begin{array}{l} \text{there is a sequence } (v_1, v_2, \dots) \text{ in } V \text{ such that} \\ (v_1, \dots, v_m) \text{ is linely inde for each } m \in \mathbf{N}^+. \end{array} \right.$

**SOLUTION:**

$P \Rightarrow Q$  : Suppose  $V$  is infinite-dim, so that no list spans  $V$ .

Step 1 Pick a  $v_1 \neq 0$ ,  $(v_1)$  linely inde.

Step  $m$  Pick a  $v_m \notin \text{span}(v_1, \dots, v_{m-1})$ , by Problem (10)(b),  $(v_1, \dots, v_m)$  is linely inde.

This process recursively defines the desired sequence  $(v_1, v_2, \dots)$ .

$\neg P \Rightarrow \neg Q$  : Suppose  $V$  is finite-dim and  $V = \text{span}(w_1, \dots, w_m)$ .

Let  $(v_1, v_2, \dots)$  be a sequence in  $V$ , then  $(v_1, v_2, \dots, v_{m+1})$  must be linely dep.

OR.  $Q \Rightarrow P$  : Suppose there is such a sequence.

Choose an  $m$ . Suppose a linely inde list  $(v_1, \dots, v_m)$  spans  $V$ .

( Similar to [2.16] ) Then  $\exists v_{m+1} \in V \setminus \text{span}(v_1, \dots, v_m)$ .

Hence no list spans  $V$ . Thus  $V$  is infinite-dim. □

**16** Prove that the vecsp of all continuous functions in  $\mathbf{R}^{[0,1]}$  is infinite-dim.

**SOLUTION:** Denote the vecsp by  $U$ .

Choose an  $m \in \mathbf{N}^+$ . Suppose  $a_0, \dots, a_m \in \mathbf{R}$  are such that  $a_0 + a_1x + \dots + a_mx^m = 0, \forall x \in [0, 1]$ .

Then the poly has infinitely many roots and hence  $a_0 = \dots = a_m = 0$ .

Thus  $(1, x, \dots, x^m)$  is linely inde in  $\mathbf{R}^{[0,1]}$ . Similar to [2.16],  $U$  is infinite-dim. □

OR. Note that for  $a_n = \frac{1}{n}, a_1 < a_2 < \dots < a_m, \forall m \in \mathbf{N}^+$ .

Suppose  $f_n = \begin{cases} x - \frac{1}{n}, & x \in \left(\frac{1}{n}, 1\right) \\ 0, & x \in \left[0, \frac{1}{n}\right] \end{cases}$  Then for any  $m, f_1\left(\frac{1}{m}\right) = \dots = f_m\left(\frac{1}{m}\right)$ , while  $f_{m+1}\left(\frac{1}{m}\right) \neq 0$ .

Hence  $f_{m+1} \notin \text{span}(f_1, \dots, f_m)$ . Thus by Problem (14),  $U$  is infinite-dim. □

**17** Suppose  $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$  such that  $p_k(2) = 0$  for each  $k \in \{0, \dots, m\}$ .

Prove that  $(p_0, p_1, \dots, p_m)$  is not linely inde in  $\mathcal{P}_m(\mathbf{F})$ .

**SOLUTION:**

Suppose  $(p_0, p_1, \dots, p_m)$  is linely inde. Define  $p \in \mathcal{P}_m(\mathbf{F})$  by  $p(z) = z \forall z \in \mathbf{F}$ .

But  $\forall a_i \in \mathbf{F}, z \neq a_0p_0(z) + \dots + a_mp_m(z)$ , for if not, let  $z = 2$ , contradicts. Thus  $z \notin \text{span}(p_0, p_1, \dots, p_m)$ .

Then  $\text{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$  while the list  $(p_0, p_1, \dots, p_m)$  has length  $(m+1)$ .

Hence  $(p_0, p_1, \dots, p_m)$  is linely depe in  $\mathcal{P}_m(\mathbf{F})$ .

For if not, because  $(1, z, \dots, z^m)$  of length  $(m+1)$  spans  $\mathcal{P}_m(\mathbf{F})$ ,

thus by [2.23] trivially,  $(p_0, p_1, \dots, p_m)$  spans  $\mathcal{P}_m(\mathbf{F})$ . Contradicts. □

OR. Note that  $\mathcal{P}_m(\mathbf{F}) = \text{span}(\underbrace{1, z, \dots, z^m}_{\text{of length } (m+1)})$ .  $(p_0, p_1, \dots, p_m, z)$  of length  $(m+2)$  is linely dep.

( See the above ) Now  $z \notin \text{span}(p_0, p_1, \dots, p_m)$  and hence  $(p_0, p_1, \dots, p_m)$  is linely dep. □

7 Prove or give a counterexample: If  $(v_1, v_2, v_3, v_4)$  is a basis of  $V$  and  $U$  is a subsp of  $V$  such that  $v_1, v_2 \in U$  and  $v_3 \notin U$  and  $v_4 \notin U$ , then  $(v_1, v_2)$  is a basis of  $U$ .

SOLUTION: A counterexample:

Let  $V = \mathbb{R}^4$  and  $e_j$  be the  $j^{\text{th}}$  standard basis.

Let  $v_1 = e_1, v_2 = e_2, v_3 = e_3 + e_4, v_4 = e_4$ . Then  $(v_1, \dots, v_4)$  is a basis of  $\mathbb{R}^4$ .

Let  $U = \text{span}(e_1, e_2, e_3) = \text{span}(v_1, v_2, v_3 - v_4)$ . Then  $v_3 \notin U$  and  $(v_1, v_2)$  is not a basis of  $U$ .  $\square$

• NOTE FOR “ $\mathbb{C}_V U \cap \{0\}$ ”:

“ $\mathbb{C}_V U \cap \{0\}$ ” is supposed to be a subsp  $W$  such that  $V = U \oplus W$ .

But if we let  $u \in U \setminus \{0\}$  and  $w \in W \setminus \{0\}$ , then 
$$\left. \begin{array}{l} w \in \mathbb{C}_V U \cap \{0\} \\ u \pm w \in \mathbb{C}_V U \cap \{0\} \end{array} \right\} \Rightarrow u \in \mathbb{C}_V U \cap \{0\}. \text{ Contradicts.}$$

To fix this, denote the set  $\{W_1, W_2, \dots\}$  by  $\mathcal{S}_V U$ , where for each  $W_i, V = U \oplus W_i$ . See also in (1.C.23).

1 Find all vecsp that have exactly one basis.

SOLUTION: The trivial vecsp  $\{0\}$  will do. Indeed, the only basis of  $\{0\}$  is the empty list.

Now consider a field containing only the add identity 0 and the multi identity 1,

and we specify that  $1 + 1 = 0$ . Hence the vecsp  $\{0, 1\}$  will do, the list (1) will be the unique basis.

And more generally, consider  $\mathbb{F} = \mathbb{Z}_m, \forall m - 1 \in \mathbb{N}^+$ . For each  $s, t \in \{1, \dots, m\}$ ,

$\mathbb{F} = \text{span}(K_s) = \text{span}(K_t)$ . Hence we fail. Are there other vecsp? Suppose so.

(I) Consider  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . Let  $(v_1, \dots, v_m)$  be a basis of  $V \neq \{0\}$ .

While there are infinitely many bases distinct from this one. Hence we fail.

(II) Consider other  $\mathbb{F}$ . Note that a field contains at least 0 and 1

By *some theories or facts* given in the course of Elementary Abstract Algebra, we fail.  $\square$

• Suppose  $(v_1, \dots, v_m)$  is a list of vecs in  $V$ . For  $k \in \{1, \dots, m\}$ , let  $w_k = v_1 + \dots + v_k$ .

Show that  $[P] B_V = (v_1, \dots, v_m) \iff [Q] B_W = (w_1, \dots, w_m)$ .

SOLUTION: NOTICE that  $B_U = (u_1, \dots, u_n) \iff \forall u \in U, \exists! a_i \in \mathbb{F}, u = a_1 u_1 + \dots + a_n u_n$ .

$P \Rightarrow Q: \forall v \in V, \exists! a_i \in \mathbb{F}, v = a_1 v_1 + \dots + a_m v_m \Rightarrow v = b_1 w_1 + \dots + b_m v_m, \exists! b_k = \sum_{j=1}^k (-1)^{k-j} a_j$ .

$Q \Rightarrow P: \forall v \in V, \exists! b_i \in \mathbb{F}, v = b_1 w_1 + \dots + b_m w_m \Rightarrow v = a_1 v_1 + \dots + a_m v_m, \exists! a_k = \sum_{j=1}^k b_j$ .  $\square$

• Suppose  $V$  is finite-dim and  $U, W$  are subsp of  $V$  such that  $V = U + W$ .

Let  $B_U = (u_1, \dots, u_m), B_W = (w_1, \dots, w_n)$ . Prove that  $\exists B_V$  consisting of vecs in  $U \cup W$ .

SOLUTION:

$V = \text{span}(u_1, \dots, u_m) + \text{span}(w_1, \dots, w_n) = \text{span}(u_1, \dots, u_m, w_1, \dots, w_n)$ . By [2.31], we get the basis  $\square$

8 Suppose  $U$  and  $W$  are subsp of  $V$  such that  $V = U \oplus W$ .

Let  $B_U = (u_1, \dots, u_m), B_W = (w_1, \dots, w_n)$ . Prove that  $B_V = (u_1, \dots, u_m, w_1, \dots, w_n)$ .

SOLUTION:

$\forall v \in V, \exists! u \in U, w \in W, v = u + w = (a_1 u_1 + \dots + a_m u_m) + (b_1 w_1 + \dots + b_n w_n), \exists! a_i, b_i \in \mathbb{F}$

$\Rightarrow (a_1 u_1 + \dots + a_m u_m) = -(b_1 w_1 + \dots + b_n w_n) \in U \cap W = \{0\} \Rightarrow a_1 = \dots = a_m = b_1 = \dots = b_n = 0$ .  $\square$



• **NOTE FOR linely inde sequence and [2.34]:**

“ $V = \text{span}(v_1, \dots, v_n, \dots)$ ” is an invalid expression.

If we allow using “infinite list”, then we must guarantee that  $(v_1, \dots, v_n, \dots)$  is a spanning “list” such that for all  $v \in V$ , there exists a smallest positive integer  $n$  such that  $v = a_1v_1 + \dots + a_nv_n$ ,

The key point is, how can we guarantee that such a “list” exists?

---

## 2.C 1 7 9 10 14,16 15 17 | 4E: 10, 14, 15, 16

1 ( COROLLARY for [2.38,39] )

Suppose  $U$  is a subsp of  $V$  such that  $\dim V = \dim U$ . Then  $V = U$ .

---

9 Suppose  $(v_1, \dots, v_m)$  is linely inde in  $V$  and  $w \in V$ .

Prove that  $\dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$ .

SOLUTION: Using the result of Problem (10) and (11) in 2.A.

Note that  $v_i - v_1 = (v_i + w) - (v_1 + w) \in \text{span}(v_1 + w, \dots, v_m + w)$ , for each  $i = 1, \dots, m$ .

$(v_1, \dots, v_m)$  linely inde  $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$  linely inde  $\Rightarrow \underbrace{(v_2 - v_1, \dots, v_m - v_1)}_{\text{of length } (m-1)}$  linely inde.

又  $w \notin \text{span}(v_1, \dots, v_m) \Rightarrow (v_1 + w, \dots, v_m + w)$  is linely inde.

Hence  $m \geq \dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$ . □

---

10 Suppose  $m$  is a positive integer and  $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$  are such that each  $p_k$  has degree  $k$ . Prove that  $(p_0, p_1, \dots, p_m)$  is a basis of  $\mathcal{P}_m(\mathbf{F})$ .

SOLUTION:

Using mathematical induction on  $m$ .

(i) For  $p_0$ ,  $\deg p_0 = 0 \Rightarrow \text{span}(p_0) = \text{span}(1)$ .

(ii) Suppose for  $i \geq 1$ ,  $\text{span}(p_0, p_1, \dots, p_i) = \text{span}(1, x, \dots, x^i)$ .

Then  $\text{span}(p_0, p_1, \dots, p_i, p_{i+1}) \subseteq \text{span}(1, x, \dots, x^i, x^{i+1})$ .

又  $\deg p_{i+1} = i + 1$ ,  $p_{i+1}(x) = a_{i+1}x^{i+1} + r_{i+1}(x)$ ;  $a_{i+1} \neq 0$ ,  $\deg r_{i+1} \leq i$ .

$\Rightarrow x^{i+1} = \frac{1}{a_{i+1}}(p_{i+1}(x) - r_{i+1}(x)) \in \text{span}(1, x, \dots, x^i, p_{i+1}) = \text{span}(p_0, p_1, \dots, p_i, p_{i+1})$ .

$\therefore x^{i+1} \in \text{span}(p_0, p_1, \dots, p_i, p_{i+1}) \Rightarrow \text{span}(1, x, \dots, x^i, x^{i+1}) \subseteq \text{span}(p_0, p_1, \dots, p_i, p_{i+1})$ .

Thus  $\mathcal{P}_m(\mathbf{F}) = \text{span}(1, x, \dots, x^m) = \text{span}(p_0, p_1, \dots, p_m)$ . □

OR. 用比较系数法. Denote the coefficient of  $x^i$  in  $p \in \mathcal{P}(\mathbf{F})$  by  $\xi_i(p)$ .

Suppose  $L = a_m p_m(x) + \dots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \dots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$ .

We use induction on  $m$  to show that  $a_m = \dots = a_0 = 0$ .

(i)  $k = m$ ,  $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$  又  $\deg p_m = m$ ,  $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$ .

Now  $L = a_{m-1} p_{m-1}(x) + \dots + a_0 p_0(x)$ .

(ii)  $1 \leq k \leq m$ ,  $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$  又  $\deg p_k = k$ ,  $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$ .

Now  $L = a_{k-1} p_{k-1}(x) + \dots + a_0 p_0(x)$ . □

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- (4E 2.C.10) Suppose  $m$  is a positive integer. For  $0 \leq k \leq m$ , let  $p_k(x) = x^k(1-x)^{m-k}$ . Show that  $(p_0, \dots, p_m)$  is a basis of  $\mathcal{P}(\mathbb{F})$ .

The basis in this exercise leads to what are called Bernstein polys. You can do a web search to learn how Bernstein polys are used to approximate continuous functions on  $[0, 1]$ .

**SOLUTION:** Using mathematical induction.

- (i)  $k = 0, 1, 2$ ,  $p_m(x) = x^m$ ,  $p_{m-1}(x) = x^{m-1} - x^m$ ,  $p_{m-2}(x) = x^{m-2} + x^m - 2x^{m-1}$ .
- (ii)  $k \geq 2$ . Suppose for  $p_{m-k}(x)$ ,  $\exists ! a_i \in \mathbb{F}$ ,  $x^{m-k} = p_{m-k}(x) + a_m x^m + \dots + a_{m-k+1} x^{m-k+1}$ .

Then for  $p_{m-k-1}(x)$ ,  $\exists ! c_i \in \mathbb{F}$ ,

$$x^{m-k-1} = p_{m-k-1}(x) + C_{k+1}^1(-1)^2 x^{m-k} + \dots + C_{k+1}^k(-1)^{k+1} x^{m-1} + (-1)^{k-2} x^m$$

$$\Rightarrow c_{m-i} = C_{k+1}^{k+1-i}(-1)^{k-i}.$$

Thus for each  $x^i$ ,  $\exists ! b_i \in \mathbb{F}$ ,  $x^i = b_m p_m(x) + \dots + b_{m-i} p_{m-i}(x)$   
 $\Rightarrow \text{span}(x^m, \dots, x, 1) = \text{span}(\underbrace{p_m, \dots, p_1, p_0}_{\text{Basis}}).$  □

OR. For any  $m, k \in \mathbb{N}^+$  such that  $k \leq m$ . Define  $p_{k,m}$  by  $p_{k,m}(x) = x^k(1-x)^{m-k}$ .

Define the statement  $S(m)$  by  $S(m) : \underbrace{(p_{0,m}, \dots, p_{m,m})}_{\dim \mathcal{P}_m(\mathbb{F}) = m+1}$  is linely inde ( and therefore is a basis ).

We use induction on to show that  $S(m)$  holds for all  $m \in \mathbb{N}^+$ .

- (i)  $m = 1$ . Suppose  $a_0(1-x) + a_1x = 0, \forall x \in \mathbb{F}$ . Then  $\begin{cases} x = 0 \Rightarrow a_0 = 0; \\ x = 1 \Rightarrow a_1 = 0. \end{cases}$

$$m = 2. \text{ Suppose } a_0(1-x)^2 + a_1(1-x)x + a_2x^2, \forall x \in \mathbb{F}. \text{ Then } \begin{cases} x = 0 \Rightarrow a_0 + a_1 = 0; \\ x = 1 \Rightarrow a_2 = 0; \\ x = 2 \Rightarrow a_0 + 2a_1 = 0. \end{cases}$$

- (ii)  $2 \leq m$ . Assume that  $S(m)$  holds.

Suppose  $\sum_{k=0}^{m+2} a_k p_{k,m+2}(x) = \sum_{k=0}^{m+2} a_k x^k(1-x)^{m+2-k} = 0, \forall x \in \mathbb{F}$ .

While  $x = 0 \Rightarrow a_0 = 0$ ;  $x = 1 \Rightarrow a_{m+2} = 0$ . Then  $\sum_{k=1}^{m+1} a_k x^k(1-x)^{m+2-k} = 0$ ;

$$\begin{aligned} \text{And note that } \sum_{k=1}^{m+1} a_k x^k(1-x)^{m+2-k} &= x(1-x) \sum_{k=1}^{m+1} a_k x^{k-1}(1-x)^{m+1-k} \\ &= x(1-x) \sum_{k=0}^m a_{k+1} x^k(1-x)^{m-k} = x(1-x) \sum_{k=0}^m a_{k+1} p_{k,m}(x). \end{aligned}$$

Hence  $x(1-x) \sum_{k=0}^m a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbb{F} \Rightarrow \sum_{k=0}^m a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbb{F} \setminus \{0, 1\}$ .

Because  $\sum_{k=0}^m a_{k+1} p_{k,m}(x)$  has infinitely many zeros. We have  $\sum_{k=0}^m a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbb{F}$ .

By assumption,  $a_1 = \dots = a_m = 0$ , while  $a_0 = a_{m+2} = 0$ ,

and also  $a_{m+1} = 0$  ( because  $\sum_{k=0}^m a_{k+1} p_{k,m}(x) = a_{m+1} p_{m,m}(x) = a_{m+1} x^m = 0, \forall x \in \mathbb{F}.$  )

Thus  $(p_{0,m+2}, \dots, p_{m+2,m+2})$  is linely inde and  $S(m+2)$  holds.

Since  $\forall m \in \mathbb{N}^+, S(m) \Rightarrow S(m+2)$ . We have  $\left\{ \begin{array}{l} \forall k \in \mathbb{N}, S(2k+1) \text{ holds} \\ \forall k \in \mathbb{N}^+, S(2k) \text{ holds} \end{array} \right\} \Rightarrow S(m) \text{ holds.}$  □

7 (a) Let  $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$ . Find a basis of  $U$ .

(b) Extend the basis in (b) to a basis of  $\mathcal{P}_4(\mathbf{F})$ .

(c) Find a subsp  $W$  of  $\mathcal{P}_4(\mathbf{F})$  such that  $\mathcal{P}_4(\mathbf{F}) = U \oplus W$ .

**SOLUTION:** Suppose  $p(z) = az^4 + bz^3 + cz^2 + dz + e$  such that  $p(2) = p(5) = p(6)$ .

$$\text{Then } \left\{ \begin{array}{l} p(2) = 16a + 8b + 4c + 2d + e \quad (\text{I}) \\ p(5) = 625a + 125b + 25c + 5d + e \quad (\text{II}) \\ p(6) = 1296a + 216b + 36c + 6d + e \quad (\text{III}) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\text{II}) - (\text{I}) = 0 \\ (\text{III}) - (\text{II}) = 0 \\ (\text{III}) - (\text{I}) = 0 \end{array} \right.$$

You don't have to compute to know that the dimension of the set of solutions is 3.

( Because  $\nexists p \in \mathcal{P}_2(\mathbf{F})$  with  $1 \leq \deg p \leq 2, p(2) = p(5) = p(6)$ . )

(a) A basis:  $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$ .

(b) Extend to a basis of  $\mathcal{P}_4(\mathbf{F})$  as  $1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$ .

(c) Let  $W = \text{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$ , so that  $\mathcal{P}_4(\mathbf{F}) = U \oplus W$ . □

• **TIPS:**

(1)  $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3))$ .

(2)  $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_3) - \dim(V_2 + (V_1 \cap V_3))$ .

(3)  $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim(V_3 + (V_1 \cap V_2))$ .

For (1). Because  $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim(V_1 + (V_2 \cap V_3))$ .

And  $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$ .

• Suppose  $V$  is a 10-dim vecsp and  $V_1, V_2, V_3$  are subsp of  $V$  with

(a)  $\dim V_1 = \dim V_2 = \dim V_3 = 7$ . Prove that  $V_1 \cap V_2 \cap V_3 \neq \{0\}$ .

(b)  $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$ . Prove that  $V_1 \cap V_2 \cap V_3 \neq \{0\}$ .

**SOLUTION:**

(a) By TIPS,  $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 - 2 \dim V > 0$ .

(b) By TIPS,  $\dim(V_1 \cap V_2 \cap V_3) > 2 \dim V - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)) \geq 0$ . □

• (4E 2.C.16)

Suppose  $V$  is finite-dim and  $U$  is a subsp of  $V$  with  $U \neq V$ . Let  $n = \dim V, m = \dim U$ .

Prove that  $\exists (n-m)$  subsp  $U_1, \dots, U_{n-m}$ , each of dim  $(n-1)$ , such that  $\bigcap_{i=1}^{n-m} U_i = U$ .

**SOLUTION:**

Let  $(v_1, \dots, v_m)$  be a basis of  $U$ , extend to a basis of  $V$  as  $(v_1, \dots, v_m, u_1, \dots, u_{n-m})$ .

Define  $U_i = \text{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m})$  for each  $i$ . Then  $U \subseteq U_i$  for each  $i$ .

And because  $\forall v \in \bigcap_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \dots + b_{n-m} u_{n-m} \in U_i \Rightarrow b_i = 0$  for each  $i \Rightarrow v \in U$ .

Hence  $\bigcap_{i=1}^{n-m} U_i \subseteq U$ . □

**EXAMPLE:** Suppose  $\dim V = 6, \dim U = 3$ .

$$\left( \underbrace{(v_1, v_2, v_3, v_4, v_5, v_6)}_{\text{Basis of } V} \right), \text{ define } \left\{ \begin{array}{l} U_1 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_5, v_6) \\ U_2 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_6) \\ U_3 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_5) \end{array} \right\} \Rightarrow \dim U_i = 6-1, i = \underbrace{1, 2, 3}_{6-3=3}.$$

□

**14** Suppose that  $V_1, \dots, V_m$  are finite-dim subsp of  $V$ .

Prove that  $V_1 + \dots + V_m$  is finite-dim and  $\dim(V_1 + \dots + V_m) \leq \dim V_1 + \dots + \dim V_m$ .

**SOLUTION:**

Choose a basis  $\mathcal{E}_i$  of  $V_i \Rightarrow V_1 + \dots + V_m = \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$ ;  $\dim V_i = \text{card } \mathcal{E}_i$ .

Then  $\dim(V_1 + \dots + V_m) = \dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$ .

又  $\dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leq \text{card}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leq \text{card } \mathcal{E}_1 + \dots + \text{card } \mathcal{E}_m$ .

Thus  $\dim(V_1 + \dots + V_m) \leq \dim V_1 + \dots + \dim V_m$ . □

**COMMENT:**  $\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m \iff V_1 + \dots + V_m$  is a direct sum.

For each  $i$ ,  $(V_1 + \dots + V_i) \cap V_{i+1} = \{0\} \iff V_1 + \dots + V_m$  is a direct sum

$\iff (\mathcal{E}_1 \cap \dots \cap \mathcal{E}_{i-1}) \cap \mathcal{E}_i = \emptyset$  for each  $i$  又  $\dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) = \text{card}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$

$\iff \dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) = \text{card } \mathcal{E}_1 + \dots + \text{card } \mathcal{E}_m$

$\iff \dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$ . □

**17** Suppose  $V_1, V_2, V_3$  are subsp of a finite-dim vecsp, then

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3$$

$$- \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

**SOLUTION:**

[Similar to] Given three sets  $A, B$  and  $C$ .

Because  $|X + Y| = |X| + |Y| - |X \cap Y|$ ;  $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$ .

Now  $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$ .

And  $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$ .

Hence  $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$ .

Because  $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$ .

$$\dim(V_1 + V_2 + V_3) = \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \quad (1)$$

$$= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) \quad (2)$$

$$= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) \quad (3)$$

Notice that in general,  $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$ .

For example,  $X = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}, Y = \{(0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}, Z = \{(z, z) \in \mathbb{R}^2 : z \in \mathbb{R}\}$ .

• **COROLLARY:** Suppose  $V_1, V_2$  and  $V_3$  are finite-dim vecsp, then  $\frac{(1) + (2) + (3)}{3}$  :

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3$$

$$- \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3}$$

$$+ \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.$$

The formula above may seem strange because the right side does not look like an integer. □

• **TIPS:** Suppose  $v_1, \dots, v_n \in V, \dim \text{span}(v_1, \dots, v_n) = n$ . Then  $(v_1, \dots, v_n)$  is a basis of  $\text{span}(v_1, \dots, v_n)$ .

Notice that  $(v_1, \dots, v_n)$  is a spanning list of  $\text{span}(v_1, \dots, v_n)$  of length  $n = \dim \text{span}(v_1, \dots, v_n)$ .

**15** Suppose  $V$  is finite-dim and  $\dim V = n \geq 1$ .

Prove that  $\exists$  one-dim subsp $s$   $V_1, \dots, V_n$  of  $V$  such that  $V = V_1 \oplus \dots \oplus V_n$ .

**SOLUTION:**

Suppose  $B_V = (v_1, \dots, v_n)$ . Define  $V_i$  by  $V_i = \text{span}(v_i)$  for each  $i \in \{1, \dots, n\}$ .

Then  $\forall v \in V, \exists! a_i \in \mathbb{F}, v = a_1 v_1 + \dots + a_n v_n$

$$\Rightarrow \exists! u_i \in V_i, v = u_1 + \dots + u_n \Rightarrow V = V_1 \oplus \dots \oplus V_n. \quad \square$$

• **COROLLARY:**

Suppose  $W$  is finite-dim,  $\dim W = m$  and  $w \in W \setminus \{0\}$ .

Prove that  $\exists B_W = (w_1, \dots, w_m)$  such that  $w = w_1 + \dots + w_m$ .

[Proof]

By Problem (15),  $\exists$  one-dim subsp $s$   $W_1, \dots, W_m$  of  $W$  such that  $W = W_1 \oplus \dots \oplus W_m$ .

Note that  $\dim W_i = \dim \text{span}(w_i) = 1 \Rightarrow \forall x_i \in W_i, \exists! c_i \in \mathbb{F}, x_i = c_i w_i$ .

Suppose  $w = x_1 + \dots + x_m$ , where each  $x_i = c_i w_i \in W_i$ . Then  $(x_1, \dots, x_m)$  is also a basis of  $W$ .  $\square$

OR. Note that  $w \neq 0 \Rightarrow m \geq 1$ . If  $m = 1$  then let  $w_1 = w$  and we are done. Suppose  $m > 1$ .

Extend  $(w)$  to a basis  $(w, w_1, \dots, w_{m-1})$  of  $W$ . Let  $w_m = w - w_1 - \dots - w_{m-1}$ .

$\text{span}(w, w_1, \dots, w_{m-1}) = \text{span}(w_1, \dots, w_m)$ . Hence  $(w_1, \dots, w_m)$  is also a basis of  $W$ .  $\square$

• **NEW THEOREM:** Suppose  $V$  is finite-dim with  $\dim V = n$  and  $U$  is a subsp of  $V$  with  $U \neq V$ .

Prove that  $\exists B_V = (v_1, \dots, v_n)$  such that each  $v_k \notin U$ .

Note that  $U \neq V \Rightarrow n \geq 1$ . We will construct  $B_V$  via the following process.

**Step 1.**  $\exists v_1 \in V \setminus U \Rightarrow v_1 \neq 0$ . If  $\text{span}(v_1) = V$  then we stop.

**Step k.** Suppose  $(v_1, \dots, v_{k-1})$  is linely inde in  $V$ , each of which belongs to  $V \setminus U$ .

Note that  $\text{span}(v_1, \dots, v_{k-1}) \neq V$ . And if  $\text{span}(v_1, \dots, v_{k-1}) \cup U = V$ , then by (1.C.12),

( because  $\text{span}(v_1, \dots, v_{k-1}) \not\subseteq U$ , )  $U \subseteq \text{span}(v_1, \dots, v_{k-1}) \Rightarrow \text{span}(v_1, \dots, v_{k-1}) = V$ .

Hence because  $\text{span}(v_1, \dots, v_{k-1}) \neq V$ , it must be case that  $\text{span}(v_1, \dots, v_{k-1}) \cup U \neq V$ .

Thus  $\exists v_k \in V \setminus U$  such that  $v_k \notin \text{span}(v_1, \dots, v_{k-1})$ .

By (2.A.11),  $(v_1, \dots, v_k)$  is linely inde in  $V$ . If  $\text{span}(v_1, \dots, v_k) = V$ , then we stop.

Because  $V$  is finite-dim, this process will stop after  $n$  steps.  $\square$

OR. If  $U = \{0\}$  then we are done. Suppose  $\dim U \geq 1$ .

Let  $(u_1, \dots, u_m)$  be a basis of  $U$ , extend to a basis  $(u_1, \dots, u_n)$  of  $V$ .

Then let  $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$ .  $\square$

**ENDED**

### 3.A [3](#) [4](#) [5](#) [7](#) [8](#) [10](#) [11](#) [12](#) [13](#) | [4E: 10](#), [11](#), [16](#)

• **TIPS:**  $T : V \rightarrow W$  is linear  $\iff \left\{ \begin{array}{l} \text{(一)} \forall v, u \in V, T(v + u) = Tv + Tu; \\ \text{(二)} \forall v, u \in V, \lambda \in \mathbb{F}, T(\lambda v) = \lambda(Tv). \end{array} \right. \iff T(v + \lambda u) = Tv + \lambda Tu.$

$$T \in \mathcal{L}(V, W) \iff T \in \mathcal{L}(V, \text{range } T). \text{ And } \{T \in \mathcal{L}(V, W) : \text{range } T \subseteq U\} = \mathcal{L}(V, U).$$

• Suppose  $T \in \mathcal{L}(V, W)$ . Prove that  $Tv \neq 0 \Rightarrow v \neq 0$ .

**SOLUTION:** Assume that  $v = 0$ . Then  $Tv = T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$ .

OR.  $T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$ . Contradicts.  $\square$

- (4E 1.B.7) Suppose  $V \neq \emptyset$  and  $W$  is a vecsp. Let  $W^V = \{f : V \rightarrow W\}$ .

(a) Define a natural add and scalar multi on  $W^V$ .

(b) Prove that  $W^V$  is a vecsp with these definitions.

**SOLUTION:**

(a)  $W^V \ni f + g : x \rightarrow f(x) + g(x)$ ; where  $f(x) + g(x)$  is the vec add on  $W$ .

$W^V \ni \lambda f : x \rightarrow \lambda f(x)$ ; where  $\lambda f(x)$  is the scalar multi on  $W$ .

(b) Commutativity:  $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$ .

Associativity:  $((f + g) + h)(x) = (f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x)) = (f + (g + h))(x)$ .

Additive Identity:  $(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$ .

Additive Inverse:  $(f + g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0 = 0(x)$ .

Distributive Properties:

$(a(f + g))(x) = a(f + g)(x) = a(f(x) + g(x)) = af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x)$ .

Similarly,  $((a + b)f)(x) = (af + bf)(x)$ .

So far, we have used the same properties in  $W$ .

Which means that **if  $W^V$  is a vecsp, then  $W$  must be a vecsp.**

Multiplication Identity:  $(1f)(x) = 1f(x) = f(x)$ . ( NOTICE that the smallest  $F$  is  $\{0, 1\}$ . ) □

**5** Because  $\mathcal{L}(V, W) = \{T : V \rightarrow W \mid T \text{ is linear}\}$  is a subsp of  $W^V$ ,  $\mathcal{L}(V, W)$  is a vecsp.

- Given the fact that  $\mathcal{L}(V, W)$  is a vecsp. Prove or give a counterexample:  $V, W$  are vecsp.

We can guarantee that  $\{0\} \subseteq \mathcal{L}(V, W), \{0\} \subseteq V, \{0\} \subseteq W$ .

And by [3.2], the additivity and homogeneity imply that  $V$  is closed under add and scalar multi.

( We cannot even guarantee that  $W^V$  is a vecsp. )

**SOLUTION:**

(I) If  $W^V = \{0\}$ . Then  $\mathcal{L}(V, W) = \{0\}$ .

And  $W = \{0\}$ , for if not,  $\exists w \in W \setminus \{0\}$ , define a map  $f$  by  $f(x) = w, \forall x \in V$ .

And  $V$  might not be a vecsp. Example:

(II) If  $W^V$  is a nonzero vecsp. Then  $W$  is a vecsp.

(a) If  $\mathcal{L}(V, W) = \{0\}$ , then we cannot guarantee that  $V$  is a vecsp.

Example:

(b) If not, then  $\exists T \in \mathcal{L}(V, W), T \neq 0$ . Which means  $\exists v \in V, Tv \neq 0 \Rightarrow v \neq 0$ .

Then both  $W$  and  $V$  have a nonzero element.

(i) If  $\exists$  inje  $T \in \mathcal{L}(V, W)$ , then  $T(u + v) = (v + u) \Rightarrow u + v = v + u$ . etc. Hence  $V$  is a vecsp.

(ii) If not, then we cannot guarantee that  $V$  is a vecsp.

Example:

(III) If  $W^V$  is not a vecsp, then  $W$  is not a vecsp.

Example:

**TODO**

□

**3** Suppose  $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ . Prove that  $\exists A_{j,k} \in \mathbf{F}$  such that for any  $(x_1, \dots, x_n) \in \mathbf{F}^n$

$$T(x_1, \dots, x_n) = \begin{pmatrix} A_{1,1}x_1 + \dots + A_{1,n}x_n \\ \vdots \quad \ddots \quad \vdots \\ A_{m,1}x_1 + \dots + A_{m,n}x_n \end{pmatrix}$$

**SOLUTION:**

Let  $T(1, 0, 0, \dots, 0, 0) = (A_{1,1}, \dots, A_{m,1})$ , Note that  $(1, 0, \dots, 0, 0), \dots, (0, 0, \dots, 0, 1)$  is a basis of  $\mathbf{F}^n$ .

$T(0, 1, 0, \dots, 0, 0) = (A_{1,2}, \dots, A_{m,2})$ , Then by [3.5], we are done.  $\square$

$\vdots$

$T(0, 0, 0, \dots, 0, 1) = (A_{1,n}, \dots, A_{m,n})$ .

**4** Suppose  $T \in \mathcal{L}(V, W)$ , and  $v_1, \dots, v_m \in V$  such that  $(Tv_1, \dots, Tv_m)$  is linely inde in  $W$ .  
Prove that  $(v_1, \dots, v_m)$  is linely inde.

**SOLUTION:** Suppose  $a_1v_1 + \dots + a_mv_m = 0$ . Then  $a_1Tv_1 + \dots + a_mTv_m = 0$ . Thus  $a_1 = \dots = a_m = 0$ .  $\square$

**7** Show that every linear map from a one-dim vecsp to itself is a multi by some scalar.

More precisely, prove that if  $\dim V = 1$  and  $T \in \mathcal{L}(V)$ , then  $\exists \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$ .

**SOLUTION:**

Let  $u$  be a nonzero vec in  $V \Rightarrow V = \text{span}(u)$ . Because  $Tu \in V \Rightarrow Tu = \lambda u$  for some  $\lambda$ .

Suppose  $v \in V \Rightarrow v = au, \exists! a \in \mathbf{F}$ . Then  $Tv = T(au) = \lambda au = \lambda v$ .  $\square$

**8** Give a function  $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}$  such that  $\forall a \in \mathbf{R}, v \in \mathbf{R}^2, \varphi(av) = a\varphi(v)$  but  $\varphi$  is not linear.

**SOLUTION:** Define  $T(x, y) = \begin{cases} x + y, & \text{if } (x, y) \in \text{span}(3, 1), \\ 0, & \text{otherwise.} \end{cases}$  OR. Define  $T(x, y) = \sqrt[3]{(x^3 + y^3)}$ .  $\square$

**9** Give a function  $\varphi : \mathbf{C} \rightarrow \mathbf{C}$  such that  $\forall w, z \in \mathbf{C}, \varphi(w + z) = \varphi(w) + \varphi(z)$   
but  $\varphi$  is not linear. (Here  $\mathbf{C}$  is thought of as a complex vecsp.)

**SOLUTION:**

Suppose  $V_{\mathbf{C}}$  is the complexification of a vecsp  $V$ . Suppose  $\varphi : V_{\mathbf{C}} \rightarrow V_{\mathbf{C}}$ .

Define  $\varphi(u + iv) = u = \text{Re}(u + iv)$  OR. Define  $\varphi(u + iv) = v = \text{Im}(u + iv)$ .  $\square$

• Prove that if  $q \in \mathcal{P}(\mathbf{R})$  and  $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$  is defined by  $Tp = \underbrace{q \circ p}_{\text{composition}}$ , then  $T$  is not linear.

**SOLUTION:** Composition and product are not the same in  $\mathcal{P}(\mathbf{F})$ .

Because in general,  $q \circ (p_1 + \lambda p_2)(x) = q(p_1(x) + \lambda p_2(x)) \neq (q \circ p_1)(x) + \lambda(q \circ p_2)(x)$ .

**EXAMPLE:** Let  $q$  be defined by  $q(x) = x^2$ , then  $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$ .  $\square$

**10** Suppose  $U$  is a subsp of  $V$  with  $U \neq V$ . Suppose  $S \in \mathcal{L}(U, W)$  with  $S \neq 0$   
(which means that  $\exists u \in U, Su \neq 0$ ).

Define  $T : V \rightarrow W$  by  $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \setminus U. \end{cases}$  Prove that  $T$  is not a linear map on  $V$ .

**SOLUTION:**

Suppose  $T$  is a linear map. And  $v \in V \setminus U, u \in U$  such that  $Su \neq 0$ .

Then  $v + u \in V \setminus U$ , (for if not,  $v = (v + u) - u \in U$ ) while  $T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$ .

Hence we get a contradiction.  $\square$

**11** Suppose  $U$  is a subsp of  $V$  and  $S \in \mathcal{L}(U, W)$ .

Prove that  $\exists T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U$ . ( OR.  $\exists T \in \mathcal{L}(V, W), T|_U = S$  )

In other words, every linear map on a subsp of  $V$  can be extended to a linear map on the entire  $V$ .

**SOLUTION:** Suppose  $W$  is such that  $V = U \oplus W$ . Then  $\forall v \in V, \exists! u_v \in U, w_v \in W, v = u_v + w_v$ .

Define  $T \in \mathcal{L}(V, W)$  by  $T(u_v + w_v) = Su_v$ . □

OR. [Finite-dim Req] Define by  $T\left(\sum_{i=1}^m a_i u_i\right) = \sum_{i=1}^n a_i S u_i$ . Let  $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$ . □

**12** Suppose nonzero  $V$  is finite-dim and  $W$  is infinite-dim. Prove that  $\mathcal{L}(V, W)$  is infinite-dim.

**SOLUTION:**

Let  $(v_1, \dots, v_n)$  be a basis of  $V$ . Let  $(w_1, \dots, w_m)$  be linely inde in  $W$  for any  $m \in \mathbb{N}^+$ .

Define  $T_{x,y} : V \rightarrow W$  by  $T_{x,y}(v_z) = \delta_{z,x} w_y, \forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$ , where  $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$

$\forall v = \sum_{i=1}^n a_i v_i, u = \sum_{i=1}^n b_i v_i, \lambda \in \mathbb{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) v_y = T_{x,y}(v) + \lambda T_{x,y}(u)$ .

Linearity checked. Now suppose  $a_1 T_{x,1} + \dots + a_m T_{x,m} = 0$ .

Then  $(a_1 T_{x,1} + \dots + a_m T_{x,m})(v_x) = 0 = a_1 w_1 + \dots + a_m w_m \Rightarrow a_1 = \dots = a_m = 0$ .  $\forall m$  arbitrary.

Thus  $(T_{x,1}, \dots, T_{x,m})$  is a linely inde list in  $\mathcal{L}(V, W)$  for any  $x$  and length  $m$ . Hence by (2.A.14). □

**13** Suppose  $(v_1, \dots, v_m)$  is linely depe in  $V$  and  $W \neq \{0\}$ .

Prove that  $\exists w_1, \dots, w_m \in W, \nexists T \in \mathcal{L}(V, W)$  such that  $Tv_k = w_k, \forall k = 1, \dots, m$ .

**SOLUTION:**

We prove by contradiction. By linear dependence lemma,  $\exists j \in \{1, \dots, m\}, v_j \in \text{span}(v_1, \dots, v_{j-1})$ .

Fix  $j$ . Let  $w_j \neq 0$ , while  $w_1 = \dots = w_{j-1} = w_{j+1} = \dots = w_m = 0$ .

Define  $T$  by  $Tv_k = w_k$  for all  $k$ . Suppose  $a_1 v_1 + \dots + a_m v_m = 0$  ( where  $a_j \neq 0$  ).

Then  $T(a_1 v_1 + \dots + a_m v_m) = 0 = a_1 w_1 + \dots + a_m w_m = a_j w_j$  while  $a_j \neq 0$  and  $w_j \neq 0$ . Contradicts. □

OR. We prove the contrapositive: Suppose  $\forall w_1, \dots, w_m \in W, \exists T \in \mathcal{L}(V, W), Tv_k = w_k$  for each  $w_k$ .

Now we show that  $(v_1, \dots, v_n)$  is linely inde. Suppose  $\exists a_i \in \mathbb{F}, a_1 v_1 + \dots + a_n v_n = 0$ .

Choose one  $w \in W \setminus \{0\}$ . By assumption, for  $(\overline{a_1} w, \dots, \overline{a_m} w), \exists T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$  for each  $v_k$ .

Now we have  $0 = T\left(\sum_{k=1}^m a_k v_k\right) = \sum_{k=1}^m a_k T v_k = \sum_{k=1}^m a_k \overline{a_k} w = \left(\sum_{k=1}^m |a_k|^2\right) w$ .

Then  $\sum_{k=1}^m |a_k|^2 = 0 \Rightarrow a_k = 0$  for each  $k$ . Hence  $(v_1, \dots, v_n)$  is linely inde. □

• (4E 3.A.17)

Suppose  $V$  is finite-dim. Show that the only two-sided ideals of  $\mathcal{L}(V)$  are  $\{0\}$  and  $\mathcal{L}(V)$ .

A subsp  $\mathcal{E}$  of  $\mathcal{L}(V)$  is called a two-sided ideal of  $\mathcal{L}(V)$  if  $TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$ .

**SOLUTION:** Let  $(v_1, \dots, v_n)$  be a basis of  $V$ . If  $\mathcal{E} = 0$ , then we are done.

Suppose  $\mathcal{E} \neq 0$  and  $\mathcal{E}$  is a two-sided ideal of  $\mathcal{L}(V)$ . Let  $S \in \mathcal{E} \setminus \{0\}$ .

Suppose  $Sv_i \neq 0$  and  $Sv_i = a_1 v_1 + \dots + a_n v_n$ , where  $a_k \neq 0$ .

Define  $R_{x,y} \in \mathcal{L}(V)$  by  $R_{x,y}(v_x) = v_y, R_{x,y}(v_z) = 0 (z \neq x)$ . OR.  $R_{x,y} v_z = \delta_{z,x} v_y$ .

Then  $(R_{1,1} + \dots + R_{n,n})v_j = v_j \Rightarrow \sum_{r=1}^n R_{r,r} = I$ . Assume that each  $R_{x,y} \in \mathcal{E}$ .

Hence  $\forall T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$ . Now we prove the assumption.

Notice that  $\forall x, y \in \mathbb{N}^+, (R_{k,y} S)(v_i) = a_k v_y \Rightarrow ((R_{k,y} S) \circ R_{x,i})(v_z) = \delta_{z,x} (a_k v_y)$ .

Thus  $R_{k,y} S R_{x,i} = a_k R_{x,y}$ . Now  $S \in \mathcal{E} \Rightarrow R_{k,y} S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}$ . □



- (4E 3.B.32) Suppose  $V$  is finite-dim with  $n = \dim V > 1$ .

Show that if  $\varphi : \mathcal{L}(V) \rightarrow \mathbf{F}$  is linear and  $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$ , then  $\varphi = 0$ .

**SOLUTION:**

Using notations in (4E 3.A.16). Using the result in NOTE FOR [3.60].

Suppose  $\varphi \neq 0 \Rightarrow \exists i, j \in \{1, \dots, n\}, \varphi(R_{i,j}) \neq 0$ . Because  $R_{i,j} = R_{x,j} \circ R_{i,x}, \forall x = 1, \dots, n$   
 $\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$  and  $\varphi(R_{i,x}) \neq 0$ .

Again, because  $R_{i,x} = R_{y,x} \circ R_{i,y}, \forall y = 1, \dots, n$ . Thus  $\varphi(R_{y,x}) \neq 0, \forall x, y = 1, \dots, n$ .

Let  $k \neq i, j \neq l$  and then  $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$   
 $\Rightarrow \varphi(R_{l,k}) = 0$  or  $\varphi(R_{i,j}) = 0$ . Contradicts. □

OR. Note that by (4E 3.A.16),  $\exists S, T \in \mathcal{L}(V), ST - TS \neq 0$ .

Then  $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \text{null } \varphi \neq \{0\}$ .

Note that  $\forall E \in \text{null } \varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \text{null } \varphi$ .

Hence  $\text{null } \varphi$  is a nonzero two-sided ideal of  $\mathcal{L}(V)$ . □

- Suppose  $V$  is finite-dim.  $T \in \mathcal{L}(V)$  is such that  $\forall S \in \mathcal{L}(V), ST = TS$ .

Prove that  $\exists \lambda \in \mathbf{F}, T = \lambda I$ .

**SOLUTION:**

If  $V = \{0\}$ , then we are done. Now suppose  $V \neq \{0\}$ .

Assume that  $(v, Tv)$  is linely depe for every  $v \in V$ , then by (2.A.2.(b)),  $Tv = \lambda_v v$  for some  $\lambda_v \in \mathbf{F}$ .

To prove that  $\lambda_v$  is independent of  $v$ , we discuss in two cases:

$$\left. \begin{array}{l} (-) \text{ If } (v, w) \text{ is linely inde, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \\ \quad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=) \text{ Otherwise, suppose } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w \end{array} \right\} \Rightarrow \lambda_w = \lambda_v.$$

Now we show the assumption. Assume that  $(v, Tv)$  is linely inde for some  $v$ . Let  $B_V = (v, Tv, u_1, \dots, u_n)$ .

Define  $S \in \mathcal{L}(V)$  by  $S(av + bTv + c_1 u_1 + \dots + c_n u_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$ . Contradicts. □

OR. Let  $(v_1, \dots, v_m)$  be a basis of  $V$ .

Define  $\varphi \in \mathcal{L}(V, \mathbf{F})$  by  $\varphi(v_1) = \dots = \varphi(v_m) = 1$ . Let  $\lambda = \varphi(Tv_1) \in \mathbf{F}$ .

For any  $v \in V$ , define  $S_v \in \mathcal{L}(V)$  by  $S_v u = \varphi(u)v$ .

Then  $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$ . □

OR. For each  $k \in \{1, \dots, n\}$ , define  $S_k \in \mathcal{L}(V)$  by  $S_k v_j = \begin{cases} v_k, & j = k, \\ 0, & j \neq k. \end{cases}$  OR.  $S_k v_j = \delta_{j,k} v_k$

Note that  $S_k \left( \sum_{i=1}^n a_i v_i \right) = a_k v_k$ . Then  $S_k v = v \iff \exists ! a_k \in \mathbf{F}, v = a_k v_k$ .

Hence  $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$ .

Define  $A^{(j,k)} \in \mathcal{L}(V)$  by  $A^{(j,k)} v_j = v_k, A^{(j,k)} v_k = v_j, A^{(j,k)} v_x = 0, x \neq j, k$ .

Then  $A^{(j,k)} T v_j = T A^{(j,k)} v_j = Tv_k = a_k v_k; A^{(j,k)} T v_j = A^{(j,k)} a_j v_j = a_j A^{(j,k)} v_j = a_j v_k$ .

Hence  $a_k = a_j$ . Thus  $a_k$  is independent of  $v_k$ . □

- Suppose that  $V$  and  $W$  are real vecsps and  $T \in \mathcal{L}(V, W)$ .  
Define  $T_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$  by  $T_{\mathbb{C}}(u + iv) = Tu + iTv$  for all  $u, v \in V$ .  
Show that (a)  $T_{\mathbb{C}}$  is linear, (b)  $T_{\mathbb{C}}$  is inje  $\iff T$  is inje, (c)  $T_{\mathbb{C}}$  is surj  $\iff T$  is surj.

**SOLUTION:**

- (a)  $\forall u_1 + iv_1, u_2 + iv_2 \in V_{\mathbb{C}}, \lambda \in \mathbb{F}$ ,  

$$T((u_1 + iv_1) + \lambda(u_2 + iv_2)) = T((u_1 + \lambda u_2) + i(v_1 + \lambda v_2)) = T(u_1 + \lambda u_2) + iT(v_1 + \lambda v_2)$$

$$= Tu_1 + iTv_1 + \lambda Tu_2 + i\lambda Tv_2 = T(u_1 + iv_1) + \lambda T(u_2 + iv_2).$$
- (b)  $\left\{ \begin{array}{l} \text{Suppose } T_{\mathbb{C}} \text{ is inje. Let } T(u) = 0 \Rightarrow T_{\mathbb{C}}(u + i0) = Tu = 0 \Rightarrow u = 0. \\ \text{Suppose } T \text{ is inje. Let } T_{\mathbb{C}}(u + iv) = Tu + iTv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + iv = 0. \end{array} \right.$
- (c)  $\left\{ \begin{array}{l} \text{Suppose } T_{\mathbb{C}} \text{ is surj. } \forall w \in W, \exists u \in V, T(u + i0) = Tu = w + i0 = w \Rightarrow T \text{ is surj.} \\ \text{Suppose } T \text{ is surj. } \forall w, x \in W, \exists u, v \in V, Tu = w, Tv = x \\ \Rightarrow \forall w + ix \in W_{\mathbb{C}}, \exists u + iv \in V, T(u + iv) = w + ix \Rightarrow T_{\mathbb{C}} \text{ is surj.} \end{array} \right.$

- 3** Suppose  $(v_1, \dots, v_m)$  in  $V$ . Define  $T \in \mathcal{L}(\mathbb{F}^m, V)$  by  $T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m$ .  
 (a) The surj of  $T$  correspds to  $(v_1, \dots, v_m)$  spanning  $V$ .  
 (b) The inje of  $T$  correspds to  $(v_1, \dots, v_m)$  being linely inde.

**COMMENT:** Let  $(e_1, \dots, e_m)$  be the standard basis of  $\mathbb{F}^m$ . Then  $Te_k = v_k$ .

- (a)  $\text{range } T = \text{span}(v_1, \dots, v_m) = V$ ; (b)  $(v_1, \dots, v_m)$  is linely inde  $\iff T$  is inje.

- 7** Suppose  $V$  is finite-dim with  $2 \leq \dim V$ . And  $\dim V \leq \dim W = m$ , if  $W$  is finite-dim.  
 Show that  $U = \{T \in \mathcal{L}(V, W) : \text{null } T \neq \{0\}\}$  is not a subsp of  $\mathcal{L}(V, W)$ .

**SOLUTION:** The set of all inje  $T \in \mathcal{L}(V, W)$  is a not subsp either.

Let  $(v_1, \dots, v_n)$  be a basis of  $V$ ,  $(w_1, \dots, w_m)$  be linely inde in  $W$ . ( $2 \leq n \leq m$ .)  
 Define  $T_1 \in \mathcal{L}(V, W)$  as  $T_1 : v_1 \mapsto 0, v_2 \mapsto w_2, v_i \mapsto w_i$ .  
 Define  $T_2 \in \mathcal{L}(V, W)$  as  $T_2 : v_1 \mapsto w_1, v_2 \mapsto 0, v_i \mapsto w_i, i = 3, \dots, n$ .  $\left| \text{Thus } T_1 + T_2 \notin U. \square \right.$

**COMMENT:** If  $\dim V = 0$ , then  $V = \{0\} = \text{span}(\ )$ .  $\forall T \in \mathcal{L}(V, W)$ ,  $T$  is inje. Hence  $U = \emptyset$ .

If  $\dim V = 1$ , then  $V = \text{span}(v_0)$ . Thus  $U = \text{span}(T_0)$ , where  $T_0 v_0 = 0$ .

- 8** Suppose  $W$  is finite-dim with  $\dim W \geq 2$ . And  $n = \dim V \geq \dim W$ , if  $V$  is finite-dim.  
 Show that  $U = \{T \in \mathcal{L}(V, W) : \text{range } T \neq W\}$  is not a subsp of  $\mathcal{L}(V, W)$ .

**SOLUTION:** The set of all surj  $T \in \mathcal{L}(V, W)$  is not a subspace either.

Let  $(v_1, \dots, v_n)$  be linely inde in  $V$ ,  $(w_1, \dots, w_m)$  be a basis of  $W$ . ( $n \in \{m, m+1, \dots\}; 2 \leq m \leq n$ .)  
 Define  $T_1 \in \mathcal{L}(V, W)$  as  $T_1 : v_1 \mapsto 0, v_2 \mapsto w_2, v_j \mapsto w_j, v_{m+i} \mapsto 0$ .  
 Define  $T_2 \in \mathcal{L}(V, W)$  as  $T_2 : v_1 \mapsto w_1, v_2 \mapsto 0, v_j \mapsto w_j, v_{m+i} \mapsto 0$ .  
 ( For each  $j = 2, \dots, m; i = 1, \dots, n - m$ , if  $V$  is finite, otherwise let  $i \in \mathbb{N}^+$ . ) Thus  $T_1 + T_2 \notin U. \square$

**COMMENT:** If  $\dim W = 0$ , then  $W = \{0\} = \text{span}(\ )$ .  $\forall T \in \mathcal{L}(V, W)$ ,  $T$  is surj. Hence  $U = \emptyset$ .

If  $\dim W = 1$ , then  $W = \text{span}(v_0)$ . Thus  $U = \text{span}(T_0)$ , where  $T_0 v_0 = 0$ .

- 11** Suppose  $S_1, \dots, S_n$  are linear and inje.  $S_1 S_2 \dots S_n$  makes sence. Prove that  $S_1 S_2 \dots S_n$  is inje.

**SOLUTION:**  $S_1 S_2 \dots S_n(v) = 0 \iff S_2 S_3 \dots S_n(v) = 0 \iff \dots \iff S_n(v) = 0 \iff v = 0. \square$

**9** Suppose  $(v_1, \dots, v_n)$  is linely inde. Prove that  $\forall$  inje  $T, (Tv_1, \dots, Tv_n)$  is linely inde.

**SOLUTION:**  $a_1Tv_1 + \dots + a_nTv_n = 0 = T(\sum_{i=1}^n a_iv_i) \iff \sum_{i=1}^n a_iv_i = 0 \iff a_1 = \dots = a_n = 0.$   $\square$

**10** Suppose  $\text{span}(v_1, \dots, v_n) = V$ . Show that  $\text{span}(Tv_1, \dots, Tv_n) = \text{range } T$ .

**SOLUTION:**

(a)  $\text{range } T = \{Tv : v \in V\} = \{Tv : v \in \text{span}(v_1, \dots, v_n)\} \Rightarrow Tv_1, \dots, Tv_n \in \text{range } T \Rightarrow$  By [2.7].

OR.  $\text{span}(Tv_1, \dots, Tv_n) \ni a_1Tv_1 + \dots + a_nTv_n = T(a_1v_1 + \dots + a_nv_n) \in \text{range } T$ .

(b)  $\forall w \in \text{range } T, \exists v \in V, w = Tv. (\exists a_i \in \mathbb{F}, v = a_1v_1 + \dots + a_nv_n) \Rightarrow w = a_1Tv_1 + \dots + a_nTv_n.$   $\square$

**16** Suppose  $\exists T \in \mathcal{L}(V)$  such that  $\text{null } T, \text{range } T$  are finite-dim. Prove that  $V$  is finite-dim.

**SOLUTION:** Let  $B_{\text{range } T} = (Tv_1, \dots, Tv_n), B_{\text{null } T} = (u_1, \dots, u_m).$

$\forall v \in V, T(v - a_1v_1 - \dots - a_nv_n) = 0$ , letting  $Tv = a_1Tv_1 + \dots + a_nTv_n$ .

$\Rightarrow v - a_1v_1 - \dots - a_nv_n = b_1u_1 + \dots + b_mu_m$ . Hence  $V \subseteq \text{span}(v_1, \dots, v_n, u_1, \dots, u_m).$   $\square$

**17** Suppose  $V, W$  are finite-dim. Prove that  $\exists$  inje  $T \in \mathcal{L}(V, W) \iff \dim V \leq \dim W$ .

**SOLUTION:**

(a) Suppose  $\exists$  inje  $T$ . Then  $\dim V = \dim \text{range } T \leq \dim W$ .

(b) Suppose  $\dim V \leq \dim W$ . Let  $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m).$

Define  $T \in \mathcal{L}(V, W)$  by  $Tv_i = w_i, i = 1, \dots, n (= \dim V).$   $\square$

**18** Suppose  $V, W$  are finite-dim. Prove that  $\exists$  surj  $T \in \mathcal{L}(V, W) \iff \dim V \geq \dim W$ .

**SOLUTION:**

(a) Suppose  $\exists$  surj  $T$ . Then  $\dim V = \dim W + \dim \text{null } T \Rightarrow \dim W \leq \dim V$ .

(b) Suppose  $\dim V \geq \dim W$ . Let  $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m).$

Define  $T \in \mathcal{L}(V, W)$  by  $T(a_1v_1 + \dots + a_mv_m + \dots + a_nv_n) = a_1w_1 + \dots + a_mw_m.$   $\square$

**19** Suppose  $V, W$  are finite-dim,  $U$  is a subsp of  $V$ .

Prove that if  $\underbrace{\dim U}_m \geq \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_p$ , then  $\exists T \in \mathcal{L}(V, W), \text{null } T = U$ .

**SOLUTION:**

Let  $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_W = (w_1, \dots, w_p).$

Define  $T \in \mathcal{L}(V, W)$  by  $T(a_1v_1 + \dots + a_nv_n + b_1u_1 + \dots + b_mu_m) = a_1w_1 + \dots + a_nw_n.$   $\square$

• (4E 3.B.21)

Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V, W), U$  is a subsp of  $W$ . Let  $\mathcal{K}_U = \{v \in V : Tv \in U\}$ .

Prove that  $\mathcal{K}_U$  is a subsp of  $V$  and  $\dim \mathcal{K}_U = \dim \text{null } T + \dim(U \cap \text{range } T)$ .

**SOLUTION:**

$\forall u, w \in \mathcal{K}_U, \lambda \in \mathbb{F}, T(u + \lambda w) = Tu + \lambda Tw \in U \Rightarrow \mathcal{K}_U$  is a subsp of  $V$ .

Define  $S \in \mathcal{L}(\mathcal{K}_U, U)$  as  $Rv = Tv$  for all  $v \in \mathcal{K}_U$ . Hence  $\text{range } R = U \cap \text{range } T$ .

Suppose  $\exists v, Tv = 0$ .  $\nexists 0 \in U \Rightarrow Rv = 0$ . Thus  $\text{null } T \subseteq \text{null } R$ .  $\square$

• **TIPS:** Suppose  $U$  is a subsp of  $V$ . Prove that  $\forall T \in \mathcal{L}(V, W), U \cap \text{null } T = \text{null } T|_U$ .

**SOLUTION:** Note that  $U \cap \text{null } T \subseteq \text{null } T|_U$ . On the other hand, suppose  $u \in \text{null } T|_U$ .

Then  $T|_U(u)$  makes sense  $\Rightarrow u \in U$ . And  $T|_U(u) = Tu = 0 \Rightarrow u \in \text{null } T$ .  $\square$

**12** Prove that  $\forall T \in \mathcal{L}(V, W), \exists \text{ subsp } U \text{ of } V \text{ such that}$

$$U \cap \text{null } T = \text{null } T|_U = \{0\}, \quad \text{range } T = \{Tu : u \in U\} = \text{range } T|_U.$$

Which is equivalent to  $T|_U : U \rightarrow \text{range } T$  being an iso.

**SOLUTION:**

By [2.34] (note that  $V$  can be infinite-dim),  $\exists \text{ subsp } U \text{ of } V \text{ such that } V = U \oplus \text{null } T$ .

$\forall v \in V, \exists ! w \in \text{null } T, u \in U, v = w + u$ . Then  $Tv = T(w + u) = Tu \in \{Tu : u \in U\}$ . □

• **NEW NOTATION:**

Suppose  $T \in \mathcal{L}(V, W)$  and  $R = (Tv_1, \dots, Tv_n)$  is linely inde in  $\text{range } T$ .

Where  $n = \dim \text{range } T$  if finite-dim, otherwise  $n \in \mathbb{N}^+$ .

By (3.A.4),  $L = (v_1, \dots, v_n)$  is linely inde in  $V$ .

Denote  $\mathcal{K}_R$  by  $\text{span } L$ , if  $\text{range } T$  is finite-dim, otherwise, denote it by a vecsp in  $\mathcal{S}_V \text{null } T$ .

Note that if  $\text{range } T$  is finite-dim, then  $\mathcal{K}_R = \text{range } T$  for any basis  $R$  of  $\text{range } T$ .

• **COMMENT:**

If  $\text{range } T$  is infinite-dim, we cannot write  $\mathcal{K}_R = \text{range } T$ . For if we do so, we must guarantee that  $\forall Tv \in \text{range } T, \exists ! n \in \mathbb{N}^+, Tv \in \text{span}(Tv_1, \dots, Tv_n)$ , where  $(Tv_k)_{k=1}^\infty$  is linely inde.

So that  $\text{range } T \subseteq \text{span}(Tv_1, \dots, Tv_n, \dots)$ . This would be invalid, as we have shown before.

• **NEW THEOREM:**  $\mathcal{K}_R \in \mathcal{S}_V \text{null } T$ . **COMMENT:**  $\text{null } T \in \mathcal{S}_V \mathcal{K}_R$ .

Suppose  $\text{range } T$  is finite-dim. Otherwise, we are done immediately.

$$(a) T\left(\sum_{i=1}^n a_i v_i\right) = 0 \Rightarrow \sum_{i=1}^n a_i Tv_i = 0 \Rightarrow a_1 = \dots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \text{null } T = \{0\}.$$

$$(b) \forall v \in V, Tv = \sum_{i=1}^n a_i Tv_i \Rightarrow Tv - \sum_{i=1}^n a_i Tv_i = T(v - \sum_{i=1}^n a_i v_i) = 0$$

$$\Rightarrow v - \sum_{i=1}^n a_i v_i \in \text{null } T \Rightarrow v = (v - \sum_{i=1}^n a_i v_i) + (\sum_{i=1}^n a_i v_i) \Rightarrow \mathcal{K}_R + \text{null } T = V. \quad \square$$

• Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V, W)$ ,  $B_{\text{range } T} = (Tv_1, \dots, Tv_n)$ ,  $B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$ .  
Prove or give a counterexample:  $(u_1, \dots, u_m)$  is a basis of  $\text{null } T$ .

**SOLUTION:** A counterexample:

Suppose  $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$ . Then  $\text{span}(Tv_1, Tv_2, Tv_3) = \text{span}(w_1)$ .

Extend  $(v_i)$  to  $(v_1, v_2, v_3)$  for each  $i$ . But none of  $(v_1, v_2), (v_1, v_3), (v_2, v_3)$  is a basis of  $\text{null } T$ . □

**COMMENT:**  $(v_2 - v_1, v_3 - v_1), (v_1 - v_2, v_3 - v_2)$  or  $(v_1 - v_3, v_2 - v_3)$  are all bases of  $\text{null } T$ .

Always notice that  $\mathcal{S}_V \text{span}(v_1, \dots, v_n) = \{U_1, \dots, \text{null } T, \dots, U_n, \dots\}$ .

• Suppose  $V$  is finite-dim,  $X$  is a subsp of  $V$ , and  $Y$  is a finite-dim subsp of  $W$ .

Prove that if  $\dim X + \dim Y = \dim V$ , then  $\exists T \in \mathcal{L}(V, W), \text{null } T = X, \text{range } T = Y$ .

**SOLUTION:**

Suppose  $\dim X + \dim Y = \dim V$ . Let  $B_X = (u_1, \dots, u_n), B_Y = (w_1, \dots, w_m), B_V = (u_1, \dots, u_n, v_1, \dots, v_m)$ .

Define  $T \in \mathcal{L}(V, W)$  by  $Tv_i = w_i, Tu_j = 0$ . Notice that  $\forall v \in V, \exists ! a_i, b_j \in \mathbb{F}, v = \sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i$ .

$$v \in \text{null } T \iff Tv = 0 \iff a_1 = \dots = a_m = 0 \iff v \in X.$$

$$Y \ni w = a_1 w_1 + \dots + a_m w_m = a_1 Tv_1 + \dots + a_m Tv_m \in \text{range } T.$$

$$\text{OR. range } T = \text{span}(Tv_1, \dots, Tv_m, Tu_1, \dots, Tu_n) = \text{span}(Tv_1, \dots, Tv_m) = \text{span}(w_1, \dots, w_m) = Y. \quad \square$$

• OR (5.B.4) Suppose  $P \in \mathcal{L}(V)$  and  $P^2 = P$ . Prove that  $V = \text{null } P \oplus \text{range } P$ .

**SOLUTION:**

(a) If  $v \in \text{null } P \cap \text{range } P \Rightarrow Pv = 0$  and  $\exists u \in V, v = Pu$ . Then  $v = Pu = P^2u = Pv = 0$ .

(b) Note that  $\forall v \in V, v = Pv + (v - Pv)$  and  $P(v - Pv) = 0 \Rightarrow v - Pv \in \text{null } P$ .  $\square$

OR. [Only in Finite-dim] Let  $(P^2v_1, \dots, P^2v_n)$  be a basis of  $\text{range } P^2$ . Then  $(Pv_1, \dots, Pv_n)$  is linely inde.

Let  $\mathcal{K} = \text{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \text{null } P^2$ . While  $\mathcal{K} = \text{range } P = \text{range } P^2$ ;  $\text{null } P = \text{null } P^2$ .  $\square$

**20** Suppose  $W$  is finite-dim. Prove that  $T \in \mathcal{L}(V, W)$  is inje  $\iff \exists S \in \mathcal{L}(W, V), ST = I_V$ .

**SOLUTION:**

(a) Suppose  $\exists S \in \mathcal{L}(W, V), ST = I$ . Then if  $Tv = 0 \Rightarrow ST(v) = 0 = v$ . OR.  $\text{null } T \subseteq \text{null } ST = \{0\}$ .

(b) Suppose  $T$  is inje. Let  $R = B_{\text{range } T} = (Tv_1, \dots, Tv_n)$ . Then  $\mathcal{K}_R \oplus \text{null } T = V$ . Let  $U \oplus \text{range } T = W$ .

Define  $S \in \mathcal{L}(W, V)$  by  $S(Tv_i) = v_i$  and  $Su = 0$ , where  $i \in \{1, \dots, n\}, u \in U$ . Thus  $ST = I$ .

OR. Define  $S \in \mathcal{L}(\text{range } T, V)$  by  $Sw = T^{-1}w$ , where  $T^{-1}$  is the inv of  $T \in \mathcal{L}(V, \text{range } T)$ .

Then extend it to  $S \in \mathcal{L}(W, V)$  by (3.A.11). Now  $\forall v \in V, STv = T^{-1}Tv = v$ .  $\square$

**21** Suppose  $W$  is finite-dim. Prove that  $T \in \mathcal{L}(V, W)$  is surj  $\iff \exists S \in \mathcal{L}(W, V), TS = I_W$ .

**SOLUTION:**

(a) Suppose  $\exists S \in \mathcal{L}(W, V), TS = I$ . Then  $\forall w \in W, TS(w) = w \in \text{range } T \Rightarrow \text{range } T = W$ .

(b) Suppose  $T$  is surj. Let  $R = B_{\text{range } T} = B_W = (Tv_1, \dots, Tv_n)$ . Then  $\mathcal{K}_R \oplus \text{null } T = V$ .

Define  $S \in \mathcal{L}(W, V)$  by  $S(Tv_i) = v_i$ . Then  $TS = I$ .

OR. By Problem (12),  $\exists$  subsp  $U$  of  $V, V = U \oplus \text{null } T, \text{range } T = \{Tu : u \in U\}$ .

Note that  $T|_U : U \rightarrow W$  is an iso. Define  $S = (T|_U)^{-1}$ , where  $(T|_U)^{-1} : W \rightarrow U$ .

Then  $TS = T \circ (T|_U)^{-1} = T|_U \circ (T|_U)^{-1}$ .  $\square$

**24** Suppose  $S, T \in \mathcal{L}(V, W)$ . Prove that  $\text{null } S \subseteq \text{null } T \iff \exists E \in \mathcal{L}(W)$  such that  $T = ES$ .

**SOLUTION:**

Suppose  $\exists E \in \mathcal{L}(W)$  such that  $T = ES$ . Then  $\text{null } T = \text{null } ES \supseteq \text{null } S$ .

Suppose  $\text{null } S \subseteq \text{null } T$ . Let  $W = \text{range } S \oplus U$ .

Define  $E \in \mathcal{L}(W)$  by  $E(Sv + w) = Tv$  for each  $Sv$  and each  $w \in U$ . Now we check that  $E$  is linear.

Because  $\forall w_1, w_2 \in W, \exists! Sv_1, Sv_2 \in \text{range } S, u_1, u_2 \in U, w_1 = Sv_1 + u_1, w_2 = Sv_2 + u_2$ .

Now  $E(w_1 + \lambda w_2) = E((Sv_1 + \lambda Sv_2) + (u_1 + \lambda u_2)) = T(v_1 + \lambda v_2) = Tv_1 + \lambda Tv_2 = Ew_1 + \lambda Ew_2$ .

OR. Let  $V = \mathcal{K} \oplus U$ . Then  $S|_{\mathcal{K}} : \mathcal{K} \rightarrow \text{range } S$  is an iso.

Now extend  $T(S|_{\mathcal{K}})^{-1} \in \mathcal{L}(\text{range } S, W)$  to  $E \in \mathcal{L}(W, W)$ .

OR. [Requires that  $\text{range } S$  is Finite-dim] Let  $R = B_{\text{range } S} = (Sv_1, \dots, Sv_n)$ . Then  $V = \mathcal{K}_R \oplus \text{null } S$ .

Define  $E \in \mathcal{L}(W)$  by  $E(Sv_i) = Tv_i, Eu = 0$ ; for each  $i = 1, \dots, n$  and each  $u \in \text{null } S$ .

Hence  $\forall v \in V, (\exists! a_i \in \mathbb{F}, u \in \text{null } S), Tv = a_1Tv_1 + \dots + a_nTv_n = E(a_1Sv_1 + \dots + a_nSv_n) \Rightarrow T = ES$ .

OR. [Requires that  $W$  is Finite-dim] Extend  $R$  to a basis  $(Sv_1, \dots, Sv_n, w_1, \dots, w_m)$  of  $W$ .

Define  $E \in \mathcal{L}(W)$  by  $E(Sv_k) = Tv_k, Ew_j = 0$ . Because  $\forall v \in V, \exists a_i \in \mathbb{F}, Sv = a_1Sv_1 + \dots + a_nSv_n$ .

Now  $v - (a_1v_1 + \dots + a_nv_n) \in \text{null } S \Rightarrow v - (a_1v_1 + \dots + a_nv_n) \in \text{null } T \Rightarrow T(v - (a_1v_1 + \dots + a_nv_n)) = 0$ .

Thus  $Tv = a_1Tv_1 + \dots + a_nTv_n$ . Hence  $E(Sv) = a_1E(Sv_1) + \dots + a_nE(Sv_n) = a_1Tv_1 + \dots + a_nTv_n = Tv$   $\square$

**25** Suppose  $V$  is finite-dim and  $S, T \in \mathcal{L}(V, W)$ .

Prove that  $\text{range } S \subseteq \text{range } T \iff \exists E \in \mathcal{L}(V)$  such that  $S = TE$ .

**SOLUTION:**

Suppose  $\exists E \in \mathcal{L}(V)$  such that  $S = TE$ . Then  $\text{range } S = \text{range } TE \subseteq \text{range } T$ .

Suppose  $\text{range } S \subseteq \text{range } T$ . Let  $(v_1, \dots, v_m)$  be a basis of  $V$ .

Note that each  $sv_i \in \text{range } T$ . Suppose  $u_i \in V$  such that  $Tu_i = sv_i$ .

Thus defining  $E \in \mathcal{L}(V)$  by  $Ev_i = u_i$  for each  $i \Rightarrow S = TE$ . □

**22** Suppose  $U$  and  $V$  are finite-dim vecsps and  $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$ .

Prove that  $\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T$ .

**SOLUTION:**

Define  $R \in \mathcal{L}(\text{null } ST, V)$  by  $Ru = Tu$  for all  $u \in \text{null } ST \subseteq U$ .

$$\left. \begin{aligned} S(Tu) = 0 = S(Ru) &\Rightarrow \text{range } R \subseteq \text{null } S \Rightarrow \dim \text{range } R \leq \dim \text{null } S \\ Tu = 0 = Ru &\Rightarrow \text{null } R \supseteq \text{null } T \Rightarrow \dim \text{null } R = \dim \text{null } T \end{aligned} \right\} \Rightarrow \text{By [3.22], we are done. } \square$$

OR. For any  $u \in U$ , note that  $u \in \text{null } ST \iff S(Tu) = 0 \iff Tu \in \text{null } S$ .

Thus  $\text{null } ST = \mathcal{K}_{\text{null } S \cap \text{range } T} = \{u \in U : Tu \in \text{null } S\}$ . By Problem (4E 3B.21),

$\dim \text{null } ST = \dim \text{null } T + \dim(\text{null } S \cap \text{range } T) \leq \dim \text{null } T + \dim \text{null } S$ . □

**COROLLARY:** (1) If  $T$  is inje, then  $\dim \text{null } T = 0 \Rightarrow \dim \text{null } ST \leq \dim \text{null } S$ .

(2) If  $T$  is surj, then  $\text{range } R = \text{null } S \Rightarrow \dim \text{null } ST = \dim \text{null } S + \dim \text{null } T$ .

(3) If  $S$  is inje, then  $\text{range } R = \{0\} \Rightarrow \dim \text{null } ST = \dim \text{null } R = \dim \text{null } T$ .

**23** Suppose  $U$  and  $V$  are finite-dim vecsps and  $S \in \mathcal{L}(V, W)$  and  $T \in \mathcal{L}(U, V)$ .

Prove that  $\dim \text{range } ST \leq \min\{\dim \text{range } S, \dim \text{range } T\}$ .

**SOLUTION:**

$\text{range } ST = \{Sv : v \in \text{range } T\} = \text{span}(Su_1, \dots, Su_{\dim \text{range } T})$ , where  $B_{\text{range } T} = (u_1, \dots, u_{\dim \text{range } T})$ .

$\dim \text{range } ST \leq \dim \text{range } T$  and  $\dim \text{range } ST \leq \dim \text{range } S$ . □

OR. Note that  $\text{range } S|_{\text{range } T} = \text{range } ST$ .

Thus  $\dim \text{range } ST = \dim \text{range } S|_{\text{range } T} = \dim \text{range } T - \dim \text{null } S|_{\text{range } T} \leq \dim \text{range } T$ . □

**COROLLARY:** (1) If  $S$  is inje, then  $\dim \text{range } ST = \dim \text{range } T$ .

(2) If  $T$  is surj, then  $\dim \text{range } ST = \dim \text{range } S$ .

• (a) Suppose  $\dim V = 5, S, T \in \mathcal{L}(V)$  are such that  $ST = 0$ . Prove that  $\dim \text{range } TS \leq 2$ .

(b) Let  $\dim V = n$  in (a). Prove that  $\dim \text{range } TS \leq \left\lfloor \frac{n}{2} \right\rfloor$ .

(c) Give an example of  $S, T \in \mathcal{L}(\mathbb{F}^5)$  with  $ST = 0$  and  $\dim \text{range } TS = 2$ .

**SOLUTION:**

(a) By Problem (23),  $\dim \text{range } TS \leq \min\left\{\frac{5 - \dim \text{null } T}{1}, \frac{5 - \dim \text{null } S}{1}\right\}$ .

We show that  $\dim \text{range } TS \leq 2$  by contradiction. Assume that  $\dim \text{range } TS \geq 3$ .

Then  $\min\{5 - \dim \text{null } T, 5 - \dim \text{null } S\} \geq 3 \Rightarrow \max\{\dim \text{null } T, \dim \text{null } S\} \leq 2$ .

and  $\dim \text{null } ST = 5 \leq \dim \text{null } S + \dim \text{null } T \leq 4$ . Contradicts.

OR. 
$$\left. \begin{aligned} \dim \text{null } S &= 5 - \dim \text{range } S \\ \dim \text{range } TS &\leq \dim \text{range } S \end{aligned} \right\} \Rightarrow \dim \text{null } S \leq 5 - \dim \text{range } TS$$

And  $ST = 0 \Rightarrow \text{range } T \subseteq \text{null } S \Rightarrow \dim \text{range } TS \leq \dim \text{range } T \leq \dim \text{null } S$ . □

(b) By Problem (23),  $\dim \text{range } TS \leq \min \left\{ \overbrace{\dim \text{range } S}^{n - \dim \text{null } T}, \overbrace{\dim \text{range } T}^{n - \dim \text{null } S} \right\}$ . We prove by contradiction.

Assume that  $\dim \text{range } TS \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$ .

Then  $\min \{n - \dim \text{null } T, n - \dim \text{null } S\} \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$

$\Rightarrow \max \{ \dim \text{null } T, \dim \text{null } S \} \leq n - \left\lfloor \frac{n}{2} \right\rfloor - 1$ .

又  $\dim \text{null } ST = n \leq \dim \text{null } S + \dim \text{null } T \leq 2(n - \left\lfloor \frac{n}{2} \right\rfloor - 1)$

$\Rightarrow \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \frac{n}{2}$ . Contradicts. Thus  $\dim \text{range } TS \leq \left\lfloor \frac{n}{2} \right\rfloor$ . □

OR.  $\dim \text{null } S = n - \dim \text{range } S \leq n - \dim \text{range } TS$ .

And  $ST = 0 \Rightarrow \dim \text{range } TS \leq \dim \text{range } T \leq \dim \text{null } S \leq n - \dim \text{range } TS$

$\Rightarrow 2 \dim \text{range } TS \leq n \Rightarrow \dim \text{range } TS \leq \frac{n}{2}$

$\Rightarrow \dim \text{range } TS \leq \left\lfloor \frac{n}{2} \right\rfloor$  ( because  $\dim \text{range } TS$  is an integer ). □

(c) Let  $(v_1, \dots, v_5)$  be a basis of  $\mathbf{F}^5$ . Define  $S, T \in \mathcal{L}(\mathbf{F}^5)$  by:

$T : \quad v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i ;$

$S : \quad v_1 \mapsto v_4, \quad v_2 \mapsto v_5, \quad v_i \mapsto 0 ; \quad i = 3, 4, 5.$  □

**26** Suppose  $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$  such that  $\forall p \in \mathcal{P}(\mathbf{R}), \deg(Dp) = (\deg p) - 1$ .

Prove that  $D \in \mathcal{P}(\mathbf{R})$  is surj.

**SOLUTION:**

[Informal Proof]  $\left| \begin{array}{l} \text{Note that } \deg Dx^n = n - 1. \text{ Because } \text{span}(Dx, Dx^2, \dots) \subseteq \text{range } D. \\ \text{又 By (2.C.10), } \text{span}(Dx, Dx^2, \dots) = \text{span}(1, x, \dots) = \mathcal{P}(\mathbf{R}). \end{array} \right.$

[Proper Proof]

We will recursively define a sequence of polys  $(p_k)_{k=0}^\infty$  where  $Dp_k = x^k$ .

(i) Because  $\dim Dx = (\deg x) - 1 = 0$ , we have  $Dx = C \in \mathbf{F}$ .

Define  $p_0 = C^{-1}x$ . Then  $Dp_0 = C^{-1}Dx = 1$ .

(ii) Suppose we have defined  $p_0, \dots, p_n$  such that  $Dp_k = x^k$  for each  $k \in \{0, \dots, n\}$ .

Because  $\deg D(x^{n+2}) = n + 1$ . Let  $D(x^{n+2}) = a_{n+1}x^{n+1} + a_nx^n + \dots + a_1x + a_0$ , where  $a_{n+1} \neq 0$ .

Then  $a_{n+1}^{-1}D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_nDp_n + \dots + a_1Dp_1 + a_0Dp_0)$

$\Rightarrow x^{n+1} = D(a_{n+1}^{-1}(x^{n+2} - a_n p_n - \dots - a_1 p_1 - a_0 p_0))$ .

Thus defining  $p_{n+1} = a_{n+1}^{-1}(x^{n+2} - a_n p_n - \dots - a_1 p_1 - a_0 p_0)$ , we have  $Dp_{n+1} = x^{n+1}$ .

Now we get  $(p_k)_{k=0}^\infty$  by recursion. Hence  $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), \exists q = (\sum_{k=0}^{\deg p} a_k p_k), Dq = p$ . □

OR. Let  $Dx^0 = 0, Dx^k = p_k$  for all  $k \in \mathbf{N}^+$ . For any  $m \in \mathbf{N}^+, (p_1, \dots, p_m)$  is a basis of  $\mathcal{P}_{m-1}(\mathbf{R})$ .

Because  $\forall p' \in \text{range } D, \exists ! m \in \mathbf{N}, \deg p = m - 1 \Rightarrow \exists ! a_k \in \mathbf{R}, p' = a_m p_m + \dots + a_1 p_1$ .

Now  $Dp = p' = a_m p_m + \dots + a_1 p_1 = D(a_m x^m + \dots + a_1 x)$ . Thus  $\exists q \in \mathcal{P}_m(\mathbf{R}), Dq = p$ . □

**27** Suppose  $p \in \mathcal{P}(\mathbf{R})$ . Prove that  $\exists q \in \mathcal{P}(\mathbf{R})$  such that  $5q'' + 3q' = p$ .

**SOLUTION:**

Define  $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$  by  $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$ .

Note that  $\deg Bx^n = n - 1$ . Similar to Problem (26), we conclude that  $B$  is surj. □

**28** Suppose  $T \in \mathcal{L}(V, W)$ ,  $B_{\text{range } T} = (w_1, \dots, w_m)$ .

Prove that  $\exists \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$  such that  $\forall v \in V, Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m$ .

**SOLUTION:**

Suppose  $v_1, \dots, v_m \in V$  such that  $Tv_i = w_i$  for each  $v_i$ . Then  $(v_1, \dots, v_m)$  is linearly inde.

Let  $B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$ . Note that  $\forall v \in V, v = \sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i, \exists! a_i, b_i \in \mathbf{F}$ .

Define  $\varphi_i : V \rightarrow \mathbf{F}$  by  $\varphi_i(v) = a_i v_i$  for each  $i$ . We now check the linearity.

$\forall v, u \in V (\exists! a_i, b_i, c_i, d_i \in \mathbf{F}), \lambda \in \mathbf{F}, \varphi_i(v + \lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda \varphi(u)$ . □

**29** Suppose  $\varphi \in \mathcal{L}(V, \mathbf{F})$ . Suppose  $u \in V \setminus \text{null } \varphi$ . Prove that  $V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$ .

**SOLUTION:** If  $\varphi = 0$  then we are done. Suppose  $\varphi \neq 0$ .

(a)  $\forall v = cu \in \text{null } \varphi \cap \{au : a \in \mathbf{F}\}, \varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$ . Hence  $\text{null } \varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$ .

(b)  $\forall v \in V, v = \left(v - \frac{\varphi(v)}{\varphi(u)}u\right) + \frac{\varphi(v)}{\varphi(u)}u$ .  $\left\{ \begin{array}{l} v - \frac{\varphi(v)}{\varphi(u)}u \in \text{null } \varphi \\ \frac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array} \right\} \Rightarrow V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$ . □

**COMMENT:**  $\varphi \neq 0 \Rightarrow \varphi(v_i) = a_i \neq 0$  for each  $v_i$ , for some linearly inde list  $(v_1, \dots, v_k)$ .

Fix one  $v_k$ . Then  $\forall j \in \{1, \dots, k-1, k+1, \dots, n\}, \text{span}\{a_j v_k - a_k v_j\} \subseteq \text{null } \varphi$ .

Hence every vecsp in  $\mathcal{S}_V \text{null } \varphi$  is one-dim.

**30** Suppose  $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$  and  $\text{null } \varphi_1 = \text{null } \varphi_2 = \text{null } \varphi$ . Prove that  $\exists c \in \mathbf{F}, \varphi_1 = c\varphi_2$

**SOLUTION:**

If  $\text{null } \varphi = V$ , then  $\varphi_1 = \varphi_2 = 0$ , we are done. Suppose  $u \in V \setminus \text{null } \varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$ .

By Problem (29),  $V = \text{null } \varphi \oplus \text{span}(u)$ . Hence for any  $v \in V, v = w + a_v u, \exists! w \in \text{null } \varphi, a_v \in \mathbf{F}$ .

$\varphi_1(v) = a_v \varphi_1(u), \varphi_2(v) = a_v \varphi_2(u) \Rightarrow a_v = \frac{\varphi_1(v)}{\varphi_1(u)} = \frac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \frac{\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}$ . □

**31** Prove that  $\exists T_1, T_2 \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^2), \text{null } T_1 = \text{null } T_2$  and  $T_1 \neq cT_2, \forall c \in \mathbf{F}$ .

**SOLUTION:**

Let  $(v_1, \dots, v_5)$  be a basis of  $\mathbf{R}^5, (w_1, w_2)$  be a basis of  $\mathbf{R}^2$ . Define  $T, S \in \mathcal{L}(V, W)$  by

$\left. \begin{array}{l} Tv_1 = w_1, \quad Tv_2 = w_2, \quad Tv_3 = Tv_4 = Tv_5 = 0 \\ Sv_1 = w_1, \quad Sv_2 = 2w_2, \quad Sv_3 = Sv_4 = Sv_5 = 0 \end{array} \right\} \Rightarrow \text{null } T = \text{null } S$ .

Suppose  $T = \lambda S$ . Then  $w_1 = Tv_1 = \lambda Sv_1 = \lambda w_1 \Rightarrow \lambda = 1$ .

While  $w_2 = Tv_2 = \lambda Sv_2 = 2\lambda w_2 \Rightarrow \lambda = \frac{1}{2}$ . Contradicts. □

• **TIPS:** Suppose  $T \in \mathcal{L}(V, W)$  and  $U$  is a subsp such that  $V = U \oplus \text{null } T$ .

Now  $\forall v \in V, \exists! u_v \in U, w_v \in \text{null } T, v = u_v + w_v$ .

Then  $T = T \circ i$ , where  $i : V \rightarrow U$  is defined by  $i(v) = u_v$ .

Because  $\forall v \in V, T(v) = T(u_v + w_v) = T(u_v) = T(i(v)) = (T \circ i)(v)$ . □

**ENDED**



### 3.C

1 3 4 5 6 9 10 11 12 13 14 15 | 4E: 16, 17

• **NOTE FOR [3.47]:**  $LHS = (AC)_{j,k} = \sum_{r=1}^n A_{j,r} C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r} (C_{\cdot,k})_{r,1} = (A_{j,\cdot} C_{\cdot,k})_{1,1} = A_{j,\cdot} C_{\cdot,k} = RHS.$

• **NOTE FOR [3.48]:**

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} &= \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 5 + 2 \times 8 & 1 \times 6 + 2 \times 9 & 1 \times 7 + 2 \times 10 \\ 3 \times 5 + 4 \times 8 & 3 \times 6 + 4 \times 9 & 3 \times 7 + 4 \times 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix} \end{aligned}$$

• [4E 3.51] Suppose  $C \in \mathbb{F}^{m,c}, R \in \mathbb{F}^{c,p}$ .

(a) For  $k = 1, \dots, p$ ,  $(CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot} R_{\cdot,k} = \sum_{r=1}^c C_{\cdot,r} R_{r,k} = R_{1,k} C_{\cdot,1} + \dots + R_{c,k} C_{\cdot,c}$

Which means that each cols  $CR$  is a linear combination of the cols of  $C$ .

(b) For  $j = 1, \dots, m$ ,  $(CR)_{j,\cdot} = C_{j,\cdot} R = C_{j,\cdot} R_{\cdot,\cdot} = \sum_{r=1}^c C_{j,r} R_{r,\cdot} = C_{j,1} R_{1,\cdot} + \dots + C_{j,c} R_{c,\cdot}$

Which means that each rows  $CR$  is a linear combination of the rows of  $R$ .

• **COLUMN-ROW FACTORIZATION (CR Factorization)** Suppose  $A \in \mathbb{F}^{m,n}, A \neq 0$ .

(a) Let  $S_c = \text{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) \subseteq \mathbb{F}^{m,1}, \dim S_c = c$ , the col rank.

Prove that  $\exists C \in \mathbb{F}^{m,c}, R \in \mathbb{F}^{c,n}, A = CR$ .

(b) Let  $S_r = \text{span}(A_{1,\cdot}, \dots, A_{m,\cdot}) \subseteq \mathbb{F}^{1,n}, \dim S_r = r$ , the row rank.

Prove that  $\exists C \in \mathbb{F}^{m,r}, R \in \mathbb{F}^{r,n}, A = CR$ .

**SOLUTION:** Notice that  $A \neq 0 \Rightarrow c, r \geq 1$ .

(a) Let  $(C_{\cdot,1}, \dots, C_{\cdot,c})$  be a basis of  $S_c$ , forming  $C \in \mathbb{F}^{m,c}$ . Then  $\forall k \in \{1, \dots, n\}$ ,

$$A_{\cdot,k} = R_{1,k} C_{\cdot,1} + \dots + R_{c,k} C_{\cdot,c} = (CR)_{\cdot,k}, \exists! R_{1,k}, \dots, R_{c,k} \in \mathbb{F}, \text{ forming } R \in \mathbb{F}^{c,n}. \text{ Thus } A = CR.$$

(b) Let  $(R_{1,\cdot}, \dots, R_{r,\cdot})$  be a basis of  $S_r$ , forming  $R \in \mathbb{F}^{r,n}$ . Then  $\forall j \in \{1, \dots, m\}$ ,

$$A_{j,\cdot} = C_{j,1} R_{1,\cdot} + \dots + C_{j,r} R_{r,\cdot} = (CR)_{j,\cdot}, \exists! C_{j,1}, \dots, C_{j,r} \in \mathbb{F}, \text{ forming } C \in \mathbb{F}^{m,r}. \text{ Thus } A = CR. \quad \square$$

**EXAMPLE:**

$$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \stackrel{(1)}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \stackrel{(2)}{=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

$$(I) \begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2 \begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix}.$$

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix}$  can be uniquely written as a linear combination of  $(A_{1,\cdot}, A_{2,\cdot})$ .

Hence  $\dim S_r = 2$ .  $(A_{1,\cdot}, A_{2,\cdot})$  is a basis.

$$(II) \begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2 \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}; \quad \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix} - \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix}. \text{ Hence } \dim S_c = 2. (A_{\cdot,2}, A_{\cdot,3}) \text{ is a basis.}$$

• COLUMN RANK EQUALS ROW RANK (Using the notation and result above)

For each  $A_{j,\cdot} \in S_r$ ,  $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$ .

For each  $A_{\cdot,k} \in S_c$ ,  $A_{\cdot,k} = (CR)_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$ .

$\Rightarrow \text{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = S_r = \text{span}(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leq c = \dim S_c$ .

$\Rightarrow \text{span}(A_{\cdot,1}, \dots, A_{\cdot,m}) = S_c = \text{span}(C_{\cdot,1}, \dots, C_{\cdot,r}) \Rightarrow \dim S_c = c \leq r = \dim S_r$ .

OR. Apply the result to  $A^t \in \mathbb{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leq r = \dim S_r = \dim S_c^t$ . □

• [4E 3.C.17, OR 3.F.32] Suppose  $T \in \mathcal{L}(V)$  and  $(u_1, \dots, u_n), (v_1, \dots, v_n)$  are bases of  $V$ . Prove that the following are equi. Here  $A = \mathcal{M}(T)$  means  $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$ .

(a)  $T$  is inje.

(b) The cols of  $\mathcal{M}(T)$  are linely inde in  $\mathbb{F}^{n,1}$ .

(c) The cols of  $\mathcal{M}(T)$  span  $\mathbb{F}^{n,1}$ .

(d) The rows of  $\mathcal{M}(T)$  span  $\mathbb{F}^{1,n}$ .

(e) The rows of  $\mathcal{M}(T)$  are linely inde in  $\mathbb{F}^{1,n}$ .

SOLUTION: Using TIPS in 2.C.

$$T \text{ is inje} \iff \dim V = \dim \text{range } T + \dim \text{null } T = \dim \text{range } T$$

$$\Delta \begin{cases} \iff (Tu_1, \dots, Tu_n) \text{ is a basis of } V; \dim \text{range } T = \dim \text{span}(\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n)) = n \\ \iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n)) \text{ is a basis of } \mathbb{F}^{n,1}, \text{ as well as } (A_{\cdot,1}, \dots, A_{\cdot,n}) \end{cases}$$

$$\left[ \text{又 } \dim S_c = \dim \text{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) = \dim \text{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = \dim S_r = n \right]$$

$$\iff (A_{1,\cdot}, \dots, A_{n,\cdot}) \text{ is a basis of } \mathbb{F}^{1,n}.$$

□

Now we show  $(\Delta)$  properly, that is  $T \text{ is inje} \iff \text{The cols of } \mathcal{M}(T) \text{ are linely inde.}$

(a)  $\Rightarrow$  (b) :

$$\text{Suppose } b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n} = \begin{pmatrix} b_1 A_{1,1} + \cdots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \cdots + b_n A_{n,n} \end{pmatrix} = 0. \text{ Let } u = b_1 u_1 + \cdots + b_n u_n.$$

$$\text{Then } Tu = b_1 Tu_1 + \cdots + b_n Tu_n$$

$$= b_1 (A_{1,1}v_1 + \cdots + A_{n,1}v_n) + \cdots + b_n (A_{1,n}v_1 + \cdots + A_{n,n}v_n)$$

$$= (b_1 A_{1,1} + \cdots + b_n A_{1,n})v_1 + \cdots + (b_1 A_{n,1} + \cdots + b_n A_{n,n})v_n$$

$$= 0v_1 + \cdots + 0v_n = 0$$

$$\Rightarrow b_1 = \cdots = b_n = 0.$$

Thus by (2.39), (b) holds.

(b)  $\Rightarrow$  (a) :

Suppose  $u = b_1 u_1 + \cdots + b_n u_n \in \text{null } T$ .

$$\text{Then } Tu = 0 = (b_1 A_{1,1} + \cdots + b_n A_{1,n})v_1 + \cdots + (b_1 A_{n,1} + \cdots + b_n A_{n,n})v_n.$$

$$\text{Thus } b_1 A_{1,1} + \cdots + b_n A_{1,n} = \cdots = b_1 A_{n,1} + \cdots + b_n A_{n,n} = 0.$$

$$\text{Which is equi to } \begin{pmatrix} b_1 A_{1,1} + \cdots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \cdots + b_n A_{n,n} \end{pmatrix} = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n} = 0 \Rightarrow b_1 = \cdots = b_n = 0.$$

Thus by (2.39), (a) holds. □

- [4E 3.C.16, OR 3.E.11] Suppose  $A$  is an  $m$ -by- $n$  matrix with  $A \neq 0$ .  
Prove that  $\text{rank } A = 1 \iff \exists (c_1, \dots, c_m) \in \mathbf{F}^m, (d_1, \dots, d_n) \in \mathbf{F}^n$   
such that  $A_{j,k} = c_j \cdot d_k$  for every  $j = 1, \dots, m$  and  $k = 1, \dots, n$ .

**SOLUTION:**

Using the notation in CR Factorization.

$$(a) \text{ Suppose } A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \dots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \dots & c_m d_n \end{pmatrix}. \quad (\exists c_j, d_k \in \mathbf{F}, \forall j, k)$$

$$\text{Then } S_c = \left\{ \begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \dots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}.$$

$$\text{OR. } S_r = \text{span} \left\{ \begin{pmatrix} c_1 d_1 & \dots & c_1 d_n \\ c_2 d_1 & \dots & c_2 d_n \\ \vdots & & \vdots \\ c_m d_1 & \dots & c_m d_n \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} \right\}. \quad \text{Hence rank } A = 1.$$

OR. Using also the result in [4E 3.51(a)].

Every col of  $A$  is a scalar multi of  $C$ . Then  $\text{rank } A \leq 1$  又  $\text{rank } A \geq 1$  ( $A \neq 0$ ).

$$(b) \text{ By CR Factorization, } \exists C = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbf{F}^{m,1}, R = \begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} \in \mathbf{F}^{1,n} \text{ such that } A = CR.$$

OR. Not using CR Factorization. Suppose  $\text{rank } A = \dim S_c = \dim S_r = 1$ .

$$\text{Let } c_j = \frac{A_{j,1}}{A_{1,1}} = \frac{A_{j,2}}{A_{1,2}} = \dots = \frac{A_{j,n}}{A_{1,n}}, \quad d'_k = \frac{A_{1,k}}{A_{1,1}} = \frac{A_{2,k}}{A_{2,1}} = \dots = \frac{A_{m,k}}{A_{m,1}}. \quad (\forall j, k)$$

$$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k. \text{ Letting } d_k = d'_k A_{1,1}. \quad \square$$

- 1 Suppose  $T \in \mathcal{L}(V, W)$ . Show that with resp to each choice of bases of  $V$  and  $W$ , the matrix of  $T$  has at least  $\dim \text{range } T$  nonzero entries.

**SOLUTION:**

Let  $B_{\text{null } T} = (v_1, \dots, v_p), B_V = (v_1, \dots, v_n)$ . Let  $B_W = (w_1, \dots, w_m)$ . Denote  $\mathcal{M}(T, B_V, B_W)$  by  $A$ .

Because at most  $p$  of the  $v_k$ 's can belong to  $\text{null } T \iff$  at least  $n - p = q$  of the  $v_k$ 's do not.

For  $v_k \notin \text{null } T, T v_k = A_{1,k} w_1 + \dots + A_{m,k} w_m \neq 0$ . Thus col  $k$  has at least one nonzero entry.

Since there are  $n - p = q$  choices of such  $k$ ,  $A$  has at least  $q = \dim \text{range } T$  nonzero entries.  $\square$

OR. We prove by contradiction.

Suppose  $A$  has at most  $(\dim \text{range } T - 1)$  nonzero entries.

Then by Pigeon Hole Principle, at least one of  $A_{\cdot, p+1}, \dots, A_{\cdot, n}$  equals 0.

Thus there are at most  $(\dim \text{range } T - 1)$  nonzero vecs in  $T v_{p+1}, \dots, T v_n$ .

While  $\text{range } T = \text{span}(T v_{p+1}, \dots, T v_n) \Rightarrow \dim \text{range } T = \dim \text{span}(T v_{p+1}, \dots, T v_n)$ . Contradicts.  $\square$

**3** Suppose  $V$  and  $W$  are finite-dim and  $T \in \mathcal{L}(V, W)$ . Prove that  $\exists B_V, B_W$  such that  
 [ letting  $A = \mathcal{M}(T, B_V, B_W)$  ]  $A_{k,k} = 1, A_{i,j} = 0$ , where  $1 \leq k \leq \dim \text{range } T, i \neq j$ .

**SOLUTION:**

Let  $R = (Tv_1, \dots, Tv_n)$  be a basis of  $\text{range } T$ , extend to  $B_W = (Tv_1, \dots, Tv_n, w_1, \dots, w_p)$ .

Let  $\mathcal{K}_R = \text{span}(v_1, \dots, v_n)$ . Let  $(u_1, \dots, u_m)$  be a basis of  $\text{null } T$ . Then  $B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$ .  $\square$

**4** Suppose  $B_V = (v_1, \dots, v_m)$  and  $W$  is finite-dim. Suppose  $T \in \mathcal{L}(V, W)$ .

Prove that  $\exists B_W = (w_1, \dots, w_n)$ ,  $\mathcal{M}(T, B_V, B_W)_{\cdot,1}^t = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$  or  $\begin{pmatrix} 0 & \dots & 0 \end{pmatrix}$ .

**SOLUTION:** If  $Tv_1 = 0$ , then we are done. If not then extend  $(Tv_1)$ .  $\square$

**5** Suppose  $B_W = (w_1, \dots, w_n)$  and  $V$  is finite-dim. Suppose  $T \in \mathcal{L}(V, W)$ .

Prove that  $\exists B_V = (v_1, \dots, v_m)$ ,  $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}$  or  $\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$ .

**SOLUTION:**

Let  $(u_1, \dots, u_n)$  be a basis of  $V$ . Denote  $\mathcal{M}(T, (u_1, \dots, u_n), B_W)$  by  $A$ .

If  $A_{1,\cdot} = 0$ , then let  $B_V = (u_1, \dots, u_n)$ , we are done.

Otherwise,  $(A_{1,1} \dots A_{1,m}) \neq 0$ , choose one  $A_{1,k} \neq 0$ .

Let  $v_1 = \frac{u_k}{A_{1,k}}$ ;  $v_j = u_{j-1} - A_{1,j-1}v_1$  for  $j = 2, \dots, k$ ;  
 $v_i = u_i - A_{1,i}v_1$  for  $i = k+1, \dots, n$ .

Now because each  $u_k \in \text{span}(v_1, \dots, v_n) \Rightarrow V = \text{span}(v_1, \dots, v_n), B_V = (v_1, \dots, v_n)$ .

And  $Tv_1 = T\left(\frac{u_k}{A_{1,k}}\right) = \frac{1}{A_{1,k}}(A_{1,k}w_1 + \dots + A_{n,k}w_n) = 1w_1 + \dots + \frac{A_{n,k}}{A_{1,k}}w_n$ .

$\forall j \in \{2, \dots, k, \underbrace{k+2, \dots, n+1}_{i \in \{k+1, \dots, n\}}\}$ ,  $Tv_j = T(u_{j-1} - A_{1,j-1}v_1) = Tu_{j-1} - T\left(\frac{A_{1,j-1}u_k}{A_{1,k}}\right)$   
 $= A_{1,j-1}w_1 + \dots + A_{n,j-1}w_n - A_{1,j-1}\left(1w_1 + \dots + \frac{A_{n,k}}{A_{1,k}}w_n\right) = 0w_1 + \dots + \left(A_{n,j-1} - \frac{A_{1,j-1}A_{n,k}}{A_{1,k}}\right)w_n. \square$

**6** Suppose  $V$  and  $W$  are finite-dim and  $T \in \mathcal{L}(V, W)$ .

Prove that  $\dim \text{range } T = 1 \iff \exists B_V, B_W$ , all entries of  $A = \mathcal{M}(T, B_V, B_W)$  equal 1.

**SOLUTION:**

(a) Suppose  $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$  are the bases such that all entries of  $A$  equal 1.

Then  $Tv_i = w_1 + \dots + w_m$  for all  $i = 1, \dots, n$ . Because  $w_1, \dots, w_m$  is linearly inde,  $w_1 + \dots + w_m \neq 0$ .

(b) Suppose  $\dim \text{range } T = 1$ . Then  $\dim \text{null } T = \dim V - 1$ .

Let  $(u_2, \dots, u_n)$  be a basis of  $\text{null } T$ . Extend it to a basis of  $V$  as  $(u_1, u_2, \dots, u_n)$ .

Let  $w_1 = Tv_1 - w_2 - \dots - w_m$ . Extend to a basis of  $W$  and we have  $B_W$ .

Let  $v_1 = u_1, v_i = u_1 + u_i$ . Extend to a basis of  $V$  and we have  $B_V$ .  $\square$

OR. Suppose  $\text{range } T$  has a basis  $(w)$ .

By (2.C.15 [COROLLARY]),  $\exists B_W = (w_1, \dots, w_m)$  such that  $w = w_1 + \dots + w_m$ .

By (2.C [NEW THEOREM]),  $\exists$  a basis  $(u_1, \dots, u_n)$  of  $V$  such that each  $u_k \notin \text{null } T$ .

$\forall k \in \{1, \dots, n\}, Tu_k \in \text{range } T = \text{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists \lambda_k \in \mathbf{F} \setminus \{0\}$ .

Let  $v_k = \lambda_k^{-1}u_k \neq 0 \Rightarrow B_V = (v_1, \dots, v_n)$ . Hence for each  $v_k, Tv_k = w = w_1 + \dots + w_m$ .  $\square$

• **NOTE FOR [3.49]:**  $\cdot[(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$   
 $\cdot(AC)_{\cdot,k} = A_{\cdot,\cdot}C_{\cdot,k} = AC_{\cdot,k}$  □

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• **EXERCISE 10:**  $\cdot[(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$   
 $\cdot(AC)_{j,\cdot} = A_{j,\cdot}C_{\cdot,\cdot} = A_{j,\cdot}C$  □

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• **NOTE FOR [3.52]:**  $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$   
 $\cdot(AC)_{j,1} = \sum_{r=1}^n A_{j,r}c_{r,1} = (\sum_{r=1}^n (A_{\cdot,r}c_{r,1}))_{j,1} = (c_1A_{\cdot,1} + \dots + c_nA_{\cdot,n})_{j,1}$   
 $\cdot Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^n A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \dots + c_nA_{\cdot,n}$  OR. By  $(Ac)_{\cdot,1} = Ac_{\cdot,1}$  Using (a) above. □

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• **EXERCISE 11:**  $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$   
 $\cdot(aC)_{1,k} = \sum_{r=1}^n a_{1,r}C_{r,k} = (\sum_{r=1}^n a_{1,r}(C_{r,\cdot}))_{1,k} = (a_1C_{1,\cdot} + \dots + a_nC_{n,\cdot})_{1,k}$   
 $\cdot aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^n a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \dots + a_nC_{n,\cdot}$  OR. By  $(aC)_{1,\cdot} = a_{1,\cdot}C$  Using (b) above. □

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• Suppose  $p$  is a poly of  $n$  variables in  $\mathbf{F}$ . Prove that  $\mathcal{M}(p(T_1, \dots, T_n)) = p(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n))$ .  
Where the linear maps  $T_1, \dots, T_n$  are such that  $p(T_1, \dots, T_n)$  makes sense. See [5.B.16,17,20].

**SOLUTION:**

Suppose the poly  $p$  is defined by  $p(x_1, \dots, x_n) = \sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n x_i^{k_i}$ .

Note that  $\mathcal{M}(T^x S^y) = \mathcal{M}(T)^x \mathcal{M}(S)^y$ ;  $\mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$ .

Then  $\mathcal{M}(p(T_1, \dots, T_n)) = \mathcal{M}(\sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n T_i^{k_i})$   
 $= \sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n \mathcal{M}(T_i^{k_i}) = p(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n))$ . □

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**13** Prove that the distr holds for matrix add and matrix multi.

Suppose  $A, B, C$  are matrices such that  $A(B + C)$  make sense, we prove the left distr.

**SOLUTION:**

Suppose  $A \in \mathbf{F}^{m,n}$  and  $B, C \in \mathbf{F}^{n,p}$ .

Note that  $[A(B + C)]_{j,k} = \sum_{r=1}^n A_{j,r}(B + C)_{r,k} = \sum_{r=1}^n (A_{j,r}B_{r,k} + A_{j,r}C_{r,k}) = (AB + AC)_{j,k}$ . □

OR. Define  $T, S, R$  such that  $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$ .

$A(B + C) = \mathcal{M}(T(S + R)) \stackrel{[3.9]}{=} \mathcal{M}(TS + TR) = AB + AC$ .

Or  $T(S + R) = TS + TR \Rightarrow \mathcal{M}(T(S + R)) = \mathcal{M}(TS + TR) \Rightarrow A(B + C) = AB + AC$ . □

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**14** Prove that matrix multi is associ.

Suppose  $A, B, C$  are matrices such that  $(AB)C$  makes sense, we prove that  $(AB)C = A(BC)$ .

**SOLUTION:**

Suppose  $A \in \mathbf{F}^{m,n}$  and  $B, C \in \mathbf{F}^{n,p}$ . We will show that  $LHS = [(AB)C]_{j,k} = [A(BC)]_{j,k} = RHS$ .

$LHS = (AB)_{j,\cdot}C_{\cdot,k} = \sum_{s=1}^n (A_{j,s}B_{s,\cdot})C_{\cdot,k} = \sum_{s=1}^n A_{j,s}(B_{s,\cdot}C_{\cdot,k}) = \sum_{s=1}^n A_{j,s}(BC)_{s,k} = RHS$ . □

OR. Define  $T, S, R$  such that  $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$ .

$(AB)C = \mathcal{M}(T(SR)) \stackrel{[3.9]}{=} \mathcal{M}(TSR) \stackrel{[3.9]}{=} \mathcal{M}((TS)R) = A(BC)$ .

OR.  $(TS)R = T(SR) \Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR)) \Rightarrow (AB)C = A(BC)$ . □

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**15** Suppose  $A \in \mathbf{F}^{n,n}$ ,  $j, k \in \{1, \dots, n\}$ . Show that  $(A^3)_{j,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}$ .

**SOLUTION:**  $(AAA)_{j,k} = (AA)_{j,\cdot} A_{\cdot,k} = \sum_{p=1}^n (A_{j,p} A_{p,\cdot}) A_{\cdot,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}$ .

$$\begin{aligned} \text{OR. } (AAA)_{j,k} &= \sum_{r=1}^n (AA)_{j,r} A_{r,k} = \sum_{r=1}^n \left( \sum_{p=1}^n A_{j,p} A_{p,r} \right) A_{r,k} \\ &= \sum_{r=1}^n \left[ A_{j,1} (A_{1,r} A_{r,k}) + \dots + A_{j,n} (A_{n,r} A_{r,k}) \right] \\ &= A_{j,1} \sum_{r=1}^n A_{1,r} A_{r,k} + \dots + A_{j,n} \sum_{r=1}^n A_{n,r} A_{r,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}. \quad \square \end{aligned}$$

• Prove that the commutativity does not hold in  $\mathbf{F}^{m,n}$ .

**SOLUTION:**

Suppose  $\dim V = n, \dim W = m$  and the commutativity holds in  $\mathbf{F}^{n,m}$ .

$$\forall T \in \mathcal{L}(V, W), S \in \mathcal{L}(W, V), \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST).$$

Hence  $ST = TS$ . Which in general is not true. ( See 3.D ) □

• [10.A.3, OR 4E 3.D.19] Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

Prove that  $\forall B_V \neq B'_V, \mathcal{M}(T, B_V) = \mathcal{M}(T, B'_V) \iff T = \lambda \mathcal{M}(I), \exists \lambda \in \mathbf{F}$ .

**SOLUTION:** [ Compare with the first solution of (3.D.16) in 3.A ]

Suppose  $T = \lambda I$  for some  $\lambda \in \mathbf{F}$ . Then  $T = \lambda \mathcal{M}(I)$ .

Suppose  $\forall B_V \neq B'_V, \mathcal{M}(T, B_V) = \mathcal{M}(T, B'_V)$ . If  $T = 0$ , then we are done.

Suppose  $T \neq 0$ , and  $v \in V \setminus \{0\}$ . Assume that  $(v, Tv)$  is linely inde.

Extend  $(v, Tv)$  to  $B_V = (v, Tv, u_3, \dots, u_n)$ . Let  $B = \mathcal{M}() (T, B_V)$ .

$$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \dots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \forall i \neq 2.$$

By assumption,  $A = \mathcal{M}(T, B'_V) = B, \forall B'_V = (v, w_2, \dots, w_n)$ . Then  $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$ .

$\Rightarrow Tv = w_2$ , which is not true if we let  $w_2 = u_3, w_3 = Tv, w_j = u_j, \forall j \in \{4, \dots, n\}$ . Contradicts.

Hence  $(v, Tv)$  is linely depe  $\Rightarrow \forall v \in V, \exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$ .

Now we show that  $\lambda_v$  is independent of  $v$ , that is, to show that for all  $v \neq w \in V \setminus \{0\}, \lambda_v = \lambda_w$ .

$$\left. \begin{aligned} (v, w) \text{ is linely inde} &\Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v, w) \text{ is linely depe, } w = cv &\Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \end{aligned} \right\} \Rightarrow T = \lambda I, \exists \lambda \in \mathbf{F}. \quad \square$$

OR. Conversely, denote  $\mathcal{M}(T, B_V)$  by  $A$ , where  $B_V = (u_1, \dots, u_m)$  is arbitrary.

Fix one  $B_V = (v_1, \dots, v_m)$  and then  $(v_1, \dots, \frac{1}{2}v_k, \dots, v_m)$  is also a basis for any given  $k \in \{1, \dots, m\}$ .

Fix one  $k$ . Now we have  $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \dots + A_{k,k}(\frac{1}{2}v_k) + \dots + A_{m,k}v_m$

$$\Rightarrow Tv_k = 2A_{1,k}v_1 + \dots + A_{k,k}v_k + \dots + 2A_{m,k}v_m = A_{1,k}v_1 + \dots + A_{k,k}v_k + \dots + A_{m,k}v_m.$$

Then  $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$  for all  $j \neq k$ . Thus  $Tv_k = A_{k,k}v_k, \forall k \in \{1, \dots, m\}$ .

Now we show that  $A_{k,k} = A_{j,j}$  for all  $j \neq k$ . Choose  $j, k$  such that  $j \neq k$ .

Consider the basis  $B'_V = (v'_1, \dots, v'_j, \dots, v'_k, \dots, v'_m)$ ,

where  $v'_j = v_k, v'_k = v_j$  and  $v'_i = v_i$  for all  $i \in \{1, \dots, m\} \setminus \{j, k\}$ .

Remember that  $\mathcal{M}(T, B'_V) = \mathcal{M}(T, B_V) = A$ .

Hence  $T(v'_k) = A_{1,k}v'_1 + \dots + A_{k,k}v'_k + \dots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_j$ , while  $T(v'_k) = T(v_j) = A_{j,j}v_j$ .

Thus  $A_{k,k} = A_{j,j}$ . □

### 3.D

1 2 3 4 5 6 8 9 10 11 12 13 15 16 17 18 19 | 4E: 1, 3, 10, 15, 17, 19, 20, 22, 23, 24

• Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

$$\left. \begin{array}{l} (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1, \dots, v_n) \text{ of } V \iff T \text{ is surj} \\ (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1, \dots, v_n) \text{ of } V \iff T \text{ is inje} \end{array} \right\} \iff T \text{ is inv.}$$

• Suppose  $T \in \mathcal{L}(V)$  and  $V = \text{span}(Tv_1, \dots, Tv_m)$ . Prove that  $V = \text{span}(v_1, \dots, v_m)$ .

SOLUTION:

Because  $V = \text{span}(Tv_1, \dots, Tv_m) \Rightarrow T$  is surj,  $\wedge V$  is finite-dim  $\Rightarrow T$  is inv  $\Rightarrow T^{-1}$  is inv.

$$\forall v \in V, \exists a_i \in \mathbb{F}, v = a_1Tv_1 + \dots + a_mTv_m \Rightarrow T^{-1}v = a_1v_1 + \dots + a_mv_m \Rightarrow \text{range } T^{-1} \subseteq \text{span}(v_1, \dots, v_m).$$

OR. Reduce  $(Tv_1, \dots, Tv_m)$  to a basis of  $V$  as  $(Tv_{\alpha_1}, \dots, Tv_{\alpha_k})$ , where  $k = \dim V$  and  $\alpha_i \in \{1, \dots, m\}$ .

Then  $(v_{\alpha_1}, \dots, v_{\alpha_k})$  is linely inde of length  $k$ , hence is a basis of  $V$ , contained in the list  $(v_1, \dots, v_m)$ .  $\square$

• OR (10.A.1) Suppose  $T \in \mathcal{L}(V)$ ,  $B_V = (v_1, \dots, v_n)$ . Prove that  $\mathcal{M}(T, B_V)$  is inv  $\iff T$  is inv.

SOLUTION: Notice that  $\mathcal{M} \in \mathcal{L}(\mathcal{L}(V), \mathbb{F}^{n,n})$  is an iso.

$$(a) T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}.$$

$$(b) \mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I. \quad \exists! S \in \mathcal{L}(V) \text{ such that } \mathcal{M}(T)^{-1} = \mathcal{M}(S)$$

$$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$$

$$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}. \quad \square$$

• Suppose  $T \in \mathcal{L}(V, W)$  is inv. Show that  $T^{-1}$  is inv and  $(T^{-1})^{-1} = T$ .

$$\text{SOLUTION: } \left. \begin{array}{l} TT^{-1} = (T^{-1})^{-1}T^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = T^{-1}(T^{-1})^{-1} = I \in \mathcal{L}(W) \end{array} \right\} \Rightarrow T = (T^{-1})^{-1}, \text{ by the uniqueness of inverse.} \quad \square$$

1 Suppose  $T \in \mathcal{L}(U, V)$ ,  $S \in \mathcal{L}(V, W)$  are inv. Prove that  $ST$  is inv and  $(ST)^{-1} = T^{-1}S^{-1}$ .

$$\text{SOLUTION: } \left. \begin{array}{l} (ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(U) \end{array} \right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}, \text{ by the uniqueness of inv.} \quad \square$$

2 Suppose  $V$  is finite-dim and  $\dim V > 1$ .

Prove that the set of non-inv operators on  $V$  is not a subsp of  $\mathcal{L}(V)$ .

The set of inv operators is not either, although multi identity/inv, and commutativity for vec multi holds.

SOLUTION:

Denote the set by  $U$ . Suppose  $\dim V = n > 1$ . Let  $(v_1, \dots, v_n)$  be a basis of  $V$ . Define  $S, T \in \mathcal{L}(V)$  by

$$S(a_1v_1 + \dots + a_nv_n) = a_1v_1, T(a_1v_1 + \dots + a_nv_n) = a_2v_1 + \dots + a_nv_n. \text{ Hence } S + T = I \text{ is inv.} \quad \square$$

COMMENT: If  $\dim V = 1$ , then  $U = \{0\}$  is a subsp of  $\mathcal{L}(V)$ .

3 Suppose  $V$  is finite-dim,  $U$  is a subsp of  $V$ , and  $S \in \mathcal{L}(U, V)$ .

Prove that  $\exists$  inv  $T \in \mathcal{L}(V)$ ,  $Tu = Su, \forall u \in U \iff S$  is inje. [ Compare this with (3.A.11). ]

SOLUTION:

$$(a) Tu = Su \text{ for every } u \in U \Rightarrow u = T^{-1}Su \Rightarrow S \text{ is inje. OR. } \text{null } S = \text{null } T \cap U = \{0\} \cap U = \{0\}.$$

$$(b) \text{ Suppose } (u_1, \dots, u_m) \text{ be a basis of } U \text{ and } S \text{ is inje} \Rightarrow (Su_1, \dots, Su_m) \text{ is linely inde in } V.$$

Extend these to bases of  $V$  as  $(u_1, \dots, u_m, v_1, \dots, v_n)$  and  $(Su_1, \dots, Su_m, w_1, \dots, w_n)$ .

Define  $T \in \mathcal{L}(V)$  by  $T(u_i) = Su_i; T v_j = w_j$ , for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ .  $\square$

**4** Suppose that  $W$  is finite-dim and  $S, T \in \mathcal{L}(V, W)$ .

Prove that  $\text{null } S = \text{null } T(=U) \iff S = ET, \exists \text{ inv } E \in \mathcal{L}(W)$ .

**SOLUTION:**

Define  $E \in \mathcal{L}(W)$  by  $E(Tv_i) = Sv_i$ ,  $E(w_j) = x_j$ , for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ . Where:

$$\left| \begin{array}{l} \text{Let } B_{\text{range } T} = \mathcal{L}(Tv_1, \dots, Tv_m), \text{ extend to } B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n). \\ \text{Let } \mathcal{K} = \text{span}(v_1, \dots, v_m). \text{ } \mathcal{K} \text{ null } S = \text{null } T \implies V = \mathcal{K} \oplus \text{null } S \Leftrightarrow \mathcal{K} \in \mathcal{S}_V \text{null } S. \\ \implies \text{span}(Sv_1, \dots, Sv_m) = \text{range } S \text{ } \mathcal{K} \text{ dim range } T = \text{dim range } S = m. \\ \text{Hence } B_{\text{range } S} = (Sv_1, \dots, Sv_m). \text{ Thus we let } B'_W = (Sv_1, \dots, Sv_m, x_1, \dots, x_n). \end{array} \right| \begin{array}{l} \therefore E \text{ is inv} \\ \text{and } S = ET. \end{array}$$

Conversely,  $S = ET \Rightarrow \text{null } S = \text{null } ET$ .

Then  $v \in \text{null } ET \iff ET(v) = 0 \iff Tv = 0 \iff v \in \text{null } T$ . Hence  $\text{null } ET = \text{null } T = \text{null } S$ .

**5** Suppose that  $V$  is finite-dim and  $S, T \in \mathcal{L}(V, W)$ .

Prove that  $\text{range } S = \text{range } T_{(=R)} \iff S = TE, \exists \text{ inv } E \in \mathcal{L}(V)$ .

**SOLUTION:**

Define  $E \in \mathcal{L}(V)$  as  $E: v_i \mapsto r_i; u_j \mapsto s_j$ ; for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ . Where:

$$\left| \begin{array}{l} \text{Let } B_R = \mathcal{L}(Tv_1, \dots, Tv_m); B'_R = (Sr_1, \dots, Sr_m) \text{ such that } \forall i, Tv_i = Sr_i. \\ \text{Let } B_{\text{null } T} = (u_1, \dots, u_n); B_{\text{null } S} = (s_1, \dots, s_n). \\ \text{Thus } B_V = (v_1, \dots, v_m, u_1, \dots, u_n); B'_V = (r_1, \dots, r_m, s_1, \dots, s_n). \end{array} \right| \therefore E \text{ is inv and } S = TE.$$

Conversely,  $S = TE \Rightarrow \text{range } S = \text{range } TE$ .

Then  $w \in \text{range } S \iff \exists v \in V, Sv = TE(v) = T(E(v)) = w \in \text{range } T$ . Hence  $\text{range } S = \text{range } T$ .  $\square$

**6** Suppose  $V$  and  $W$  are finite-dim and  $S, T \in \mathcal{L}(V, W)$ .

Prove that  $S = E_2 T E_1, \exists \text{ inv } E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W) \iff \dim \text{null } S = \dim \text{null } T = n$ .

**SOLUTION:**

Define  $E_1: v_i \mapsto r_i; u_j \mapsto s_j$ ; for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ .

Define  $E_2 : Tv_i \mapsto Sr_i$ ;  $x_j \mapsto y_j$ ; for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ . Where:

<p>Let <math>B_{\text{range } T} = \mathcal{L}(Tv_1, \dots, Tv_m)</math>; <math>B_{\text{range } S} = (Sr_1, \dots, Sr_m)</math>.</p> <p>Extend to <math>B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p)</math>; <math>B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p)</math>.</p> <p>Let <math>B_{\text{null } T} = (u_1, \dots, u_n)</math>; <math>B_{\text{null } S} = (s_1, \dots, s_n)</math>.</p> <p>Thus <math>B_V = (v_1, \dots, v_m, u_1, \dots, u_n)</math>; <math>B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)</math>.</p>	<p><math>\therefore E_1, E_2</math> are inv</p> <p>and <math>S = E_2 T E_1</math>.</p>
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Conversely,  $S = E_2TE_1 \Rightarrow \dim \text{null } S = \dim \text{null } E_2TE_1$ .

$v \in \text{null } E_2TE_1 \iff E_2TE_1(v) = 0 \iff TE_1(v) = 0$ . Hence  $\text{null } E_2TE_1 = \text{null } TE_1 = \text{null } S$ .

⌘ By (3.B.22.COROLLARY),  $E$  is inv  $\Rightarrow \dim \text{null } TE_1 = \dim \text{null } T = \dim \text{null } S$ .

**8** Suppose  $V$  is finite-dim and  $T : V \rightarrow W$  is a **surj** linear map of  $V$  onto  $W$ .

Prove that there is a subsp  $U$  of  $V$  such that  $T|_U$  is an iso of  $U$  onto  $W$ .

**SOLUTION:**

Let  $B_{\text{range } T} = B_W = (w_1, \dots, w_m) \Rightarrow \forall w_i, \exists ! v_i \in V, T v_i = w_i$ . Let  $B_{\mathcal{K}} = (v_1, \dots, v_m)$ .

Then  $\dim \mathcal{K} = \dim W$ . Thus  $T|_{\mathcal{K}}$  is an iso of  $\mathcal{K}$  onto  $W$ .

OR. By Problem (12) in (3.B), there is a subsp  $U$  of  $V$  such that

$$U \cap \text{null } T = \{0\} = \text{null } T|_U, \text{range } T = \{Tu : u \in U\} = \text{range } T|_U.$$



**9** Suppose  $V$  is finite-dim and  $S, T \in \mathcal{L}(V)$ . Prove that  $ST$  is inv  $\iff S$  and  $T$  are inv.

**SOLUTION:**

Suppose  $S, T$  are inv. Then  $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$ . Hence  $ST$  is inv.

Suppose  $ST$  is inv. Let  $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$ .

$$\left. \begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \text{range } S \end{array} \right\} \Rightarrow T \text{ is inje, } S \text{ is surj. While } V \text{ is finite-dim.} \quad \square$$

OR. Because by (3.B.23),  $\dim V = \dim \text{range } ST \leq \min\{\text{range } T, \text{range } S\}$ .  $\square$

**10** Suppose  $V$  is finite-dim and  $S, T \in \mathcal{L}(V)$ . Prove that  $ST = I \iff TS = I$ .

**SOLUTION:**

$$\text{Suppose } ST = I. \left. \begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \text{range } S \end{array} \right\} \Rightarrow T \text{ is inje, } S \text{ is surj. While } V \text{ is finite-dim.}$$

OR. By Problem (9),  $V$  is finite-dim and  $ST = I$  is inv  $\Rightarrow S, T$  are inv.

$$S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow S \text{ is inv.}$$

$$\text{OR. } ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T. \text{ } \text{ } S = S \Rightarrow TS = S^{-1}S = I.$$

Reversing the roles of  $S$  and  $T$ , we conclude that  $TS = I \Rightarrow ST = I$ .  $\square$

**11** Suppose  $V$  is finite-dim,  $S, T, U \in \mathcal{L}(V)$  and  $STU = I$ . Show that  $T$  is inv and  $T^{-1} = US$ .

**SOLUTION:** Using Problem (9) and (10). This result can fail without the hypothesis that  $V$  is finite-dim.

$$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I.$$

$$\Rightarrow U^{-1} = ST, \quad T^{-1} = US, \quad S^{-1} = TU. \quad \square$$

**EXAMPLE:**  $V = \mathbb{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots); T(a_1, \dots) = (0, a_1, \dots); U = I \Rightarrow STU = I$  but  $T$  is not inv.

**13** Suppose  $V$  is finite-dim,  $R, S, T \in \mathcal{L}(V)$  are such that  $RST$  is surj. Prove that  $S$  is inje.

**SOLUTION:** By Problem (1) and (9), Notice that  $V$  is finite-dim. Then  $RST$  is inv.

$$\text{Let } X = (RST)^{-1} \left\{ \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje.} \\ \forall v \in V, v = (RST)Xv \in \text{range } R \Rightarrow R \text{ is surj.} \end{array} \right\} \Rightarrow S = R^{-1}(RST)^{-1} \text{ is inv.} \quad \square$$

$$\text{OR. } (RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}. \quad \square$$

**15** Prove that every linear map from  $\mathbb{F}^{n,1}$  to  $\mathbb{F}^{m,1}$  is given by a matrix multi.

In other words, prove that if  $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$ , then  $\exists A \in \mathbb{F}^{m,n}, Tx = Ax, \forall x \in \mathbb{F}^{n,1}$ .

**SOLUTION:**

Let  $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$  be the standard bases of  $\mathbb{F}^{n,1}, \mathbb{F}^{m,1}$ .

$$\forall k = 1, \dots, n, \text{ suppose } T(E_k) = A_{1,k}R_1 + \dots + A_{m,k}R_m, \exists A_{j,k} \in \mathbb{F}, \text{ forming } A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix}. \quad \square$$

OR. Let  $A = \mathcal{M}(T, B_1, B_2)$ . Note that  $\mathcal{M}(x, B_1) = x, \mathcal{M}(y, B_2) = y$ .

$$\text{Hence } Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax, \text{ by [3.65]}. \quad \square$$

• OR (10.A.2) Suppose  $A, B \in \mathbb{F}^{n,n}$ . Prove that  $AB = I \iff BA = I$ .

**SOLUTION:** Using Problem (10) and (15).

Define  $T, S \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{n,1})$  by  $Tx = Ax, Sx = Bx$  for all  $x \in \mathbb{F}^{n,1}$ . Then  $\mathcal{M}(T) = A, \mathcal{M}(S) = B$ .

$$\text{Thus } AB = I \iff A(Bx) = x \iff T(Sx) = x \iff TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I. \quad \square$$

• **NOTE FOR [3.60]:** Suppose  $B_V = (v_1, \dots, v_n)$ ,  $B_W = (w_1, \dots, w_m)$ .

Define  $E_{i,j} \in \mathcal{L}(V, W)$  by  $E_{i,j}(v_x) = \delta_{i,x} w_j$ ; See (3.A.12). **COROLLARY:**  $E_{l,k} E_{i,j} = \delta_{j,l} E_{i,k}$ .

Denote  $\mathcal{M}(E_{i,j})$  by  $\mathcal{E}^{(j,i)}$ . And  $(\mathcal{E}^{(j,i)})_{l,k} = \begin{cases} 0, & i \neq k \vee j \neq l \\ 1, & i = k \wedge j = l \end{cases}$

Because  $\mathcal{L}(V, W)$  and  $\mathbf{F}^{m,n}$  are iso. And  $T = \mathcal{M}^{-1} \mathcal{M}(T)$ ;  $E_{i,j} = \mathcal{M}^{-1} \mathcal{E}^{(j,i)}$ .

Hence  $\forall T \in \mathcal{L}(V, W)$ ,  $\exists! A_{i,j} \in \mathbf{F} \left( \forall i \in \{1, \dots, m\}, j \in \{1, \dots, n\} \right)$ ,  $\mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix}$ .

$$\text{Thus } A = \begin{pmatrix} A_{1,1} \mathcal{E}^{(1,1)} + & \dots & + A_{1,n} \mathcal{E}^{(1,n)} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{m,1} \mathcal{E}^{(m,1)} + & \dots & + A_{m,n} \mathcal{E}^{(m,n)} \end{pmatrix} \iff \begin{pmatrix} A_{1,1} E_{1,1} + & \dots & + A_{1,n} E_{n,1} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{m,1} E_{1,m} + & \dots & + A_{m,n} E_{n,m} \end{pmatrix} = T.$$

$$\therefore \mathcal{L}(V, W) = \text{span} \underbrace{\begin{pmatrix} E_{1,1}, & \dots, & E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \dots, & E_{n,m} \end{pmatrix}}_B; \quad \mathbf{F}^{m,n} = \text{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \dots, & \mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \dots, & \mathcal{E}^{(m,n)} \end{pmatrix}}_{B_{\mathcal{M}}}.$$

Hence by [2.42] and [3.61], we conclude that  $B$  is a basis of  $\mathcal{L}(V, W)$  and that  $B_{\mathcal{M}}$  is a basis of  $\mathbf{F}^{m,n}$ .

• Suppose  $V, W$  are finite-dim,  $U$  is a subsp of  $V$ .

Let  $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\} = \{T \in \mathcal{L}(V, W) : T|_U = 0\}$ .

(a) Show that  $\mathcal{E}$  is a subsp of  $\mathcal{L}(V, W)$ .

(b) Find a formula for  $\dim \mathcal{E}$  in terms of  $\dim V$ ,  $\dim W$  and  $\dim U$ .

**Hint:** Define  $\Phi : \mathcal{L}(V, W) \rightarrow \mathcal{L}(U, W)$  by  $\Phi(T) = T|_U$ . What is  $\text{null } \Phi$ ? What is  $\text{range } \Phi$ ?

**SOLUTION:**

(a)  $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$ .

(b) Define  $\Phi$  as in the hint.

Because  $T \in \text{null } \Phi \iff \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$ .

Hence  $\text{null } \Phi = \mathcal{E}$ .

Because  $S \in \mathcal{L}(U, W) \Rightarrow \exists T \in \mathcal{L}(V, W), \Phi(T) = S$ , by (3.A.11)  $\Rightarrow S \in \text{range } \Phi$ .

Hence  $\text{range } \Phi = \mathcal{L}(U, W)$ .

Thus  $\dim \text{null } \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \text{range } \Phi = (\dim V - \dim U) \dim W$ . □

OR. Extend  $(u_1, \dots, u_m)$  a basis of  $U$  to  $(u_1, \dots, u_m, v_1, \dots, v_n)$  a basis of  $V$ . Let  $p = \dim W$ .

( See NOTE FOR [3.60])

$$\forall T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \text{span} \underbrace{\begin{pmatrix} E_{1,1}, & \dots, & E_{m,1}, \\ \vdots & \ddots & \vdots \\ E_{1,p}, & \dots, & E_{m,p} \end{pmatrix}}_{\text{Denote it by } R} \cap \mathcal{E} = \{0\}.$$

$$\text{又 } W = \text{span} \begin{pmatrix} E_{m+1,1}, & \dots, & E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, & \dots, & E_{n,p} \end{pmatrix} \subseteq \mathcal{E}. \text{ Where } \mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}.$$

Then  $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$ . □

◦ Suppose  $V$  is finite-dim and  $S \in \mathcal{L}(V)$ . Define  $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$  by  $\mathcal{A}(T) = ST$ .

(a) Show that  $\dim \text{null } \mathcal{A} = (\dim V)(\dim \text{null } S)$ .

(b) Show that  $\dim \text{range } \mathcal{A} = (\dim V)(\dim \text{range } S)$ .

**SOLUTION:**

(a)  $\forall T \in \mathcal{L}(V), ST = 0 \iff \text{range } T \subseteq \text{null } S$ .

Thus  $\text{null } \mathcal{A} = \{T \in \mathcal{L}(V) : \text{range } T \subseteq \text{null } S\} = \mathcal{L}(V, \text{null } S)$ .

(b)  $\forall R \in \mathcal{L}(V), \text{range } R \subseteq \text{range } S \iff \exists T \in \mathcal{L}(V), R = ST$ , by (3.B 25).

Thus  $\text{range } \mathcal{A} = \{R \in \mathcal{L}(V) : \text{range } R \subseteq \text{range } S\} = \mathcal{L}(V, \text{range } S)$ .

□

OR. Using NOTE FOR [3.60].

Let  $B_{\text{range } S} = (\underbrace{w_1, \dots, w_m}_{Sv_i=w_i}), B_{\mathcal{K}} = (v_1, \dots, v_m); (w_1, \dots, w_n), (v_1, \dots, v_n)$  are bases of  $V$ .

Define  $E_{i,j} \in \mathcal{L}(V)$  by  $E_{i,j}(v_x) = \delta_{i,x}w_i$ .

Thus  $S = E_{1,1} + \dots + E_{m,m}; \quad \mathcal{M}(S, (v_1, \dots, v_n), (w_1, \dots, w_n)) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$

Define  $R_{i,j} \in \mathcal{L}(V)$  by  $R_{i,j}(w_x) = \delta_{i,x}v_i$ .

Let  $E_{j,k}R_{i,j} = Q_{i,k}, \quad R_{j,k}E_{i,j} = G_{i,k}.$

Because  $\forall T \in \mathcal{L}(V), \exists ! A_{i,j} \in \mathbb{F}, \quad T = \begin{pmatrix} A_{1,1}R_{1,1} + \dots + A_{1,m}R_{m,1} + \dots + A_{1,n}R_{n,1} \\ + \dots + \dots + \dots + \dots + \dots \\ \vdots \quad \ddots \quad \vdots \quad \ddots \quad \vdots \\ + \dots + \dots + \dots + \dots + \dots \\ A_{m,1}R_{1,m} + \dots + A_{m,m}R_{m,m} + \dots + A_{m,n}R_{n,m} \\ + \dots + \dots + \dots + \dots + \dots \\ \vdots \quad \ddots \quad \vdots \quad \ddots \quad \vdots \\ + \dots + \dots + \dots + \dots + \dots \\ A_{n,1}R_{1,n} + \dots + A_{n,m}R_{m,n} + \dots + A_{n,n}R_{n,n} \end{pmatrix}.$

$$\begin{aligned} \Rightarrow \mathcal{A}(T) = ST &= \left( \sum_{r=1}^m E_{r,r} \right) \left( \sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \right) \\ &= \sum_{i=1}^m \sum_{j=1}^n A_{i,j} Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1} + \dots + A_{1,m}Q_{m,1} + \dots + A_{1,n}Q_{n,1} \\ + \dots + \dots + \dots + \dots + \dots \\ \vdots \quad \ddots \quad \vdots \quad \ddots \quad \vdots \\ + \dots + \dots + \dots + \dots + \dots \\ A_{m,1}Q_{1,m} + \dots + A_{m,m}Q_{m,m} + \dots + A_{m,n}Q_{n,m} \end{pmatrix}. \end{aligned}$$

Thus  $\text{null } \mathcal{A} = \text{span} \begin{pmatrix} R_{1,m+1}, & \dots, & R_{n,m+1}, \\ \vdots & \ddots & \vdots \\ R_{1,n}, & \dots, & R_{n,n} \end{pmatrix}, \quad \text{range } \mathcal{A} = \text{span} \begin{pmatrix} Q_{1,1}, & \dots, & Q_{n,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,m}, & \dots, & Q_{n,m} \end{pmatrix}.$

Hence (a)  $\dim \text{null } \mathcal{A} = n \times (n - m); \quad$  (b)  $\dim \text{range } \mathcal{A} = n \times m.$

□

• **COMMENT:** Define  $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$  by  $\mathcal{B}(T) = TS$ . Similarly to Problem (◦),

(a)  $\forall T \in \mathcal{L}(V), TS = 0 \iff \text{range } S \subseteq \text{null } T$ .

Thus  $\text{null } \mathcal{B} = \{T \in \mathcal{L}(V) : \text{range } S \subseteq \text{null } T\} = \{T \in \mathcal{L}(V) : T|_{\text{range } S} = 0\}$ .

(b)  $\forall R \in \mathcal{L}(V), \text{null } S \subseteq \text{null } R \iff \exists T \in \mathcal{L}(V), R = TS$ , by (3.B.24).

Thus  $\text{range } \mathcal{B} = \{R \in \mathcal{L}(V) : \text{null } S \subseteq \text{null } R\} = \{R \in \mathcal{L}(V) : R|_{\text{null } S} = 0\}$ .

Hence  $\dim \text{null } \mathcal{B} = (\dim V - \dim \text{range } S)(\dim V)$ ;

$\dim \text{range } \mathcal{B} = (\dim V - \dim \text{null } S)(\dim V)$ . □

OR. Using NOTE FOR [3.60] and the notation in Problem (◦).

$$\mathcal{B}(T) = TS = \left( \sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \right) \left( \sum_{r=1}^m E_{r,r} \right)$$

$$\text{Thus null } \mathcal{B} = \text{span} \left( \begin{matrix} R_{m+1,1}, & \dots, & R_{n,1} \\ \vdots & \ddots & \vdots \\ R_{m+1,n}, & \dots, & R_{n,n} \end{matrix} \right),$$

$$\text{range } \mathcal{B} = \text{span} \left( \begin{matrix} G_{1,1}, & \dots, & G_{m,1} \\ \vdots & \ddots & \vdots \\ G_{1,n}, & \dots, & G_{m,n} \end{matrix} \right).$$

$$= \sum_{i=1}^n \sum_{j=1}^m A_{i,j} G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1} + & \dots & +A_{1,m}G_{m,1} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{m,1}G_{1,m} + & \dots & +A_{m,m}G_{m,m} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{n,1}G_{1,n} + & \dots & +A_{n,m}G_{m,n} \end{pmatrix}.$$

Hence (a)  $\dim \text{null } \mathcal{B} = n \times (n - m)$ ;  
 (b)  $\dim \text{range } \mathcal{B} = n \times m$ . □

**17** Suppose  $V$  is finite-dim. Show that the only two-sided ideals of  $\mathcal{L}(V)$  are  $\{0\}$  and  $\mathcal{L}(V)$ .

A subsp  $\mathcal{E}$  of  $\mathcal{L}(V)$  is called a two-sided ideal of  $\mathcal{L}(V)$  if  $TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$ .

**SOLUTION:** Using NOTE FOR [3.60]. Let  $(v_1, \dots, v_n)$  be a basis of  $V$ . If  $\mathcal{E} = 0$ , then we are done.

Suppose  $\mathcal{E} \neq 0$  and  $\mathcal{E}$  is a two-sided ideal of  $\mathcal{L}(V)$ .

Then  $\forall E_{i,j} \in \mathcal{E}, (\forall x, y = 1, \dots, n)$ , by assumption,  $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$ .

Again,  $E_{y,x'}, E_{y',x} \in \mathcal{E}$  for all  $x', y', x, y = 1, \dots, n$ . Thus  $\mathcal{E} = \mathcal{L}(V)$ . □

• **OR (10.A.4)** Suppose that  $(\beta_1, \dots, \beta_n)$  and  $(\alpha_1, \dots, \alpha_n)$  are bases of  $V$ .

Let  $T \in \mathcal{L}(V)$  be such that  $T\alpha_k = \beta_k, \forall k$ . Prove that  $\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \beta \rightarrow \alpha)$

For ease of notation, let  $\mathcal{M}(T, \alpha \rightarrow \beta) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n))$ ,  $\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n))$ .

**SOLUTION:**

Denote  $\mathcal{M}(T, \alpha \rightarrow \alpha)$  by  $A$  and  $\mathcal{M}(I, \beta \rightarrow \alpha)$  by  $B$ .

$\forall k \in \{1, \dots, n\}, Iu_k = u_k = B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \dots + A_{n,k}\alpha_n \Rightarrow A = B$ . □

OR. Note that  $\mathcal{M}(T, \alpha \rightarrow \beta) = I$ . Hence  $\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \beta \rightarrow \alpha) \underbrace{\mathcal{M}(T, \alpha \rightarrow \beta)}_{=\mathcal{M}(I, \beta \rightarrow \beta)} = \mathcal{M}(I, \beta \rightarrow \alpha)$ . □

OR. Note that  $\mathcal{M}(T, \beta \rightarrow \beta)\mathcal{M}(I, \alpha \rightarrow \beta) = \mathcal{M}(T, \alpha \rightarrow \beta) = I$ .

$\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \alpha \rightarrow \beta)^{-1} \left( \underbrace{\mathcal{M}(T, \beta \rightarrow \beta)\mathcal{M}(I, \alpha \rightarrow \beta)}_{=\mathcal{M}(T, \alpha \rightarrow \beta)} \right) = \mathcal{M}(I, \beta \rightarrow \alpha)$ . □

**COMMENT:** Denote  $\mathcal{M}(T, \beta \rightarrow \beta)$  by  $A'$ .

$u_k = Iu_k = B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n, \forall k \in \{1, \dots, n\}$ .

又  $Tu_k = T(B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n) = B_{1,k}\beta_1 + \dots + B_{n,k}\beta_n = A'_{1,k}\beta_1 + \dots + A'_{n,k}\beta_n \Rightarrow A' = B$ .

OR.  $\mathcal{M}(T, \beta \rightarrow \beta) = \mathcal{M}(T, \alpha \rightarrow \beta)\mathcal{M}(I, \beta \rightarrow \alpha) = B$ .

**16** Suppose  $V$  is finite-dim and  $S \in \mathcal{L}(V)$  such that  $\forall T \in \mathcal{L}(V), ST = TS$ .

Prove that  $\exists \lambda \in \mathbf{F}, S = \lambda I$ .

**SOLUTION:** Using the notation and result in ( ).

Suppose  $ST = TS$  for every  $T \in \mathcal{L}(V)$ . If  $S = 0$ , we are done. Now suppose  $S \neq 0$ .

Let  $S = E_{1,1} + \dots + E_{m,m} \Rightarrow \mathcal{M}(S, B_{\mathcal{K}}) = \mathcal{M}(I, B_{\text{range } S}, B_{\mathcal{K}})$ .

Then  $\forall k \in \{m+1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$ . Hence  $n = \dim V = \dim \text{range } S = m$ .

NOTICE that  $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$ . Thus  $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \dots + a_{n,i}v_n)$ .

Where  $a_{i,j} = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \dots + a_{n,i}v_n$ ;

And For each  $j$ , for all  $i$ . Thus  $a_{i,i} = a_{k,k} = \lambda, \forall k \neq i$ .

Hence  $w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \mathcal{M}(\lambda I, (v_1, \dots, v_n)) \Rightarrow S = \mathcal{M}^{-1}(\mathcal{M}(\lambda I))\lambda I$ . □

**18** Show that  $V$  and  $\mathcal{L}(\mathbf{F}, V)$  are iso vecsps.

**SOLUTION:**

Define  $\Psi \in \mathcal{L}(V, \mathcal{L}(\mathbf{F}, V))$  by  $\Psi(v) = \Psi_v$ ; where  $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$  and  $\Psi_v(\lambda) = \lambda v$ .

(a)  $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$ . Hence  $\Psi$  is inje.

(b)  $\forall T \in \mathcal{L}(\mathbf{F}, V)$ , let  $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi(T(1))$ . Hence  $\Psi$  is surj. □

OR. Define  $\Phi \in \mathcal{L}(\mathcal{L}(\mathbf{F}, V), V)$  by  $\Phi(T) = T(1)$ .

(a) Suppose  $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$ . Thus  $\Phi$  is inje.

(b) For any  $v \in V$ , define  $T \in \mathcal{L}(\mathbf{F}, V)$  by  $T(\lambda) = \lambda v$ . Then  $\Phi(T) = T(1) = v$ . Thus  $\Phi$  is surj. □

**COMMENT:**  $\Phi = \Psi^{-1}$ .

• Suppose  $q \in \mathcal{P}(\mathbf{R})$ . Prove that  $\exists p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ .

**SOLUTION:**

Note that  $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$ .

Define  $T_n : \mathcal{P}_n(\mathbf{R}) \rightarrow \mathcal{P}_n(\mathbf{R})$  by  $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ . Then  $T_n \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ .

And note that  $T_n(p) = 0 \Rightarrow \deg(T_n p) = -\infty = \deg p \Rightarrow p = 0$ . Thus  $T_n$  is inv.

$\forall q \in \mathcal{P}(\mathbf{R})$ , if  $q = 0$ , let  $m = 0$ ; if  $q \neq 0$ , let  $m = \deg q$ , we have  $q \in \mathcal{P}_m(\mathbf{R})$ .

Hence  $\exists p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$  for all  $x \in \mathbf{R}$ . □

**19** Suppose  $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$  is inje.  $\deg Tp \leq \deg p$  for every nonzero  $p \in \mathcal{P}(\mathbf{R})$ .

(a) Prove that  $T$  is surj; (b) Prove that for every nonzero  $p$ ,  $\deg Tp = \deg p$ .

**SOLUTION:**

(a)  $T$  is inje  $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} : \mathcal{P}_n(\mathbf{R}) \rightarrow \mathcal{P}_n(\mathbf{R})$  is inje and therefore is inv  $\iff T$  is surj.

(b) Using mathematical induction.

(i)  $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$ ;

$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$ .

(ii) Assume that  $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$ .

Suppose  $\exists r \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tr \leq n < \deg r = n + 1$ .

Then by (a),  $\exists s \in \mathcal{P}_n(\mathbf{R}), T(s) = (Tr)$ .

又  $T$  is inje  $\Rightarrow s = r$ . While  $\deg s = \deg Ts = \deg Tr < \deg r$ .

Contradicts. Thus  $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$ . □

### 3.E

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 20 | 4E: 8, 14

1 A function  $T : V \rightarrow W$  is linear  $\iff T$  is a subspace of  $V \times W$ .

2 Suppose  $V_1 \times \cdots \times V_m$  is finite-dim. Prove that each  $V_j$  is finite-dim.

SOLUTION:

For any  $k \in \{1, \dots, m\}$ , define  $p_k : V_1 \times \cdots \times V_m \rightarrow V_k$  by  $p_k(v_1, \dots, v_m) = v_k$ .

Then  $p_k$  is a surj linear map. By [3.22],  $\text{range } p_k = V_k$  is finite-dim. □

OR. Denote  $V_1 \times \cdots \times V_m$  by  $U$ . Denote  $\{0\} \times \cdots \times \{0\} \times V_i \times \{0\} \cdots \times \{0\}$  by  $U_i$ .

Let  $(v_1, \dots, v_m)$  be a basis of  $U$ . Note that  $\forall u_i \in V_i, u_i \in U_i \subseteq U$ , for each  $i$ .

Define  $R_i \in \mathcal{L}(V_i, U)$  by  $R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0)$   $\left. \vphantom{\begin{matrix} \text{Define } R_i \in \mathcal{L}(V_i, U) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \end{matrix}} \right\} \Rightarrow S_i|_{U_i} = R_i^{-1}|_{U_i}$ .

Define  $S_i \in \mathcal{L}(U, V_i)$  by  $S_i(u_1, \dots, u_i, \dots, u_m) = u_i$

Thus  $U_i$  and  $V_i$  are iso.  $\forall U_i$  is a subsp of a finite-dim vecsp  $U$ . □

3 Give an example of a vecsp  $V$  and its two subsp  $U_1, U_2$  such that  $U_1 \times U_2$  and  $U_1 + U_2$  are iso but  $U_1 + U_2$  is not a direct sum.

SOLUTION:  $V$  must be infinite-dim. For if not, both  $U_1$  and  $U_2$  are finite-dim subsp. By [3.76, 3.78].

NOTE that at least one of  $U_1, U_2$  must be infinite-dim. And at least one must be infinite-dim??? TODO

For if not,  $U_1 \times U_2$  is finite-dim and  $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$ .

Let  $V = \mathbb{F}^\infty = U_1$ ,  $U_2 = \{(x, 0, \dots) \in \mathbb{F}^\infty : x \in \mathbb{F}\}$ .

Define  $T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2)$  by  $T((x_1, x_2, \dots), (x, 0, \dots)) = (x, x_1, x_2, \dots)$   $\left. \vphantom{\begin{matrix} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T((x_1, x_2, \dots), (x, 0, \dots)) = (x, x_1, x_2, \dots) \end{matrix}} \right\} \Rightarrow S = T^{-1}$ .

Define  $S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2)$  by  $S(x, x_1, x_2, \dots) = ((x_1, x_2, \dots), (x, 0, \dots))$  □

4 Prove that  $\mathcal{L}(V_1 \times \cdots \times V_m, W)$  and  $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$  are iso.

SOLUTION: Using the notation in Problem (2).

Note that  $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$ .

Define  $\varphi : T \mapsto (T_1, \dots, T_m)$  by  $\varphi(T) = (TR_1, \dots, TR_m)$ .

Define  $\psi : (T_1, \dots, T_m) \mapsto T$  by  $\psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m$ .  $\left. \vphantom{\begin{matrix} \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m \end{matrix}} \right\} \Rightarrow \psi = \varphi^{-1}$ . □

5 Prove that  $\mathcal{L}(V, W_1 \times \cdots \times W_m)$  and  $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$  are iso.

SOLUTION: Using the notation in Problem (2).

Note that  $Tv = (w_1, \dots, w_m)$ . Define  $T_i \in \mathcal{L}(V, W_i)$  by  $T_i(v) = w_i$ .

Define  $\varphi : T \mapsto (T_1, \dots, T_m)$  by  $\varphi(T) = (S_1T, \dots, S_mT)$ .

Define  $\psi : (T_1, \dots, T_m) \mapsto T$  by  $\psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m$ .  $\left. \vphantom{\begin{matrix} \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m \end{matrix}} \right\} \Rightarrow \psi = \varphi^{-1}$ . □

6 For  $m \in \mathbb{N}^+$ , define  $V^m$  by  $\underbrace{V \times \cdots \times V}_{m \text{ times}}$ . Prove that  $V^m$  and  $\mathcal{L}(\mathbb{F}^m, V)$  are iso.

SOLUTION:

Define  $T : (v_1, \dots, v_m) \mapsto \varphi$ , where  $\varphi : (a_1, \dots, a_m) \mapsto v$  is defined by  $\varphi(a_1, \dots, a_m) = a_1v_1 + \cdots + a_mv_m$ .

(a) Suppose  $T(v_1, \dots, v_m) = 0$ . Then  $\forall (a_1, \dots, a_m) \in \mathbb{F}^m$ ,  $\varphi(a_1, \dots, a_m) = a_1v_1 + \cdots + a_mv_m = 0$

$\Rightarrow (v_1, \dots, v_m) = 0 \Rightarrow T$  is inje.

(b) Suppose  $\psi \in \mathcal{L}(\mathbb{F}^m, V)$ . Let  $(e_1, \dots, e_m)$  be the standard basis of  $\mathbb{F}^m$ . Then  $\forall (b_1, \dots, b_m) \in \mathbb{F}^m$ ,

$\left[ T(\psi(e_1), \dots, \psi(e_m)) \right] (b_1, \dots, b_m) = b_1\psi(e_1) + \cdots + b_m\psi(e_m) = \psi(b_1e_1 + \cdots + b_me_m) = \psi(b_1, \dots, b_m)$ .

Thus  $T(\psi(e_1), \dots, \psi(e_m)) = \psi$ . Hence  $T$  is surj. □

**14** Suppose  $U = \{(x_1, x_2, \dots) \in \mathbf{F}^\infty : x_k \neq 0 \text{ for only finitely many } k\}$ .

(a) Show that  $U$  is a subsp of  $\mathbf{F}^\infty$ . [Do it in your mind]

(b) Prove that  $\mathbf{F}^\infty/U$  is infinite-dim.

**SOLUTION:** For ease of notation, denote the  $p^{\text{th}}$  term of  $u = (x_1, \dots, x_p, \dots) \in \mathbf{F}^\infty$  by  $u[p]$ .

For each  $r \in \mathbf{N}^+$ , let  $e_r[p] = \begin{cases} 1, & (p-1) \equiv 0 \pmod{r} \\ 0, & \text{otherwise} \end{cases}$  | simply  $e_r = (1, \underbrace{0, \dots, 0}_{(p-1) \text{ times}}, 1, \underbrace{0, \dots, 0}_{(p-1) \text{ times}}, 1, \dots)$ .

Choose one  $m \in \mathbf{N}^+$ . Let  $a_1(e_1 + U) + \dots + a_m(e_m + U) = 0 + U \Rightarrow \exists u \in U, a_1e_1 + \dots + a_me_m = u$ .

Suppose  $u = (x_1, \dots, x_L, 0, \dots)$ , where  $L$  is the largest such that  $u[L] \neq 0$ .

Let  $s \in \mathbf{N}^+$  be such that  $h = s \cdot m! + 1 > L$  and  $e_1[h] = \dots = e_m[h] = 1$ .

Note that by definition,  $e_r[s \cot m! + 1 + p] = e_r[p + 1] = 1 \iff p \equiv 0 \pmod{r} \iff r | p$ .

Now for any  $p \in \{1, \dots, m\}$ ,  $u[h + p] = \left( \sum_{r=1}^m a_r e_r \right)[p + 1] = \sum_{k=1}^{\tau(p)} a_{p_k} = 0$  (Δ)

where  $1 = p_1 \leq \dots \leq p_{\tau(p)} = p$  are all the distinct factors of  $p$ .

Let  $q = p_{\tau(p)-1}$ . Notice that  $\tau(q) = \tau(p) - 1$  and  $q_k = p_k, \forall k \in \{1, \dots, \tau(q)\}$ .

Again by (Δ),  $\left( \sum_{r=1}^m a_r e_r \right)[h + q] = \sum_{k=1}^{\tau(q)-1} a_{p_k} = 0$ . Thus  $a_{p_{\tau(p)}} = a_p = 0$  for any  $p \in \{1, \dots, m\}$ .

Hence  $\forall m \in \mathbf{N}^+$ ,  $(e_1, \dots, e_m)$  is linely inde in  $\mathbf{F}^\infty$ , so is  $(e_1 + U, \dots, e_m + U)$  in  $\mathbf{F}^\infty/U$ . By (2.A.14).  $\square$

OR. For each  $r \in \mathbf{N}^+$ , let  $e_r[p] = \begin{cases} 1, & \text{if } 2^r | p \\ 0, & \text{otherwise} \end{cases}$ .

Similarly, let  $m \in \mathbf{N}^+$  and  $a_1(e_1 + U) + \dots + a_m(e_m + U) = 0 \Rightarrow a_1e_1 + \dots + a_me_m = u \in U$ .

Suppose  $L$  is the largest such that  $u[L] \neq 0$ . And  $l$  is such that  $2^{ml} > L$ .

Then  $\forall k \in \{1, \dots, m\}, u[2^{ml} + 2^k] = \left( \sum_{r=1}^m a_r e_r \right)[2^k] = a_1 + \dots + a_k = 0$ .

Thus  $a_1 = \dots = a_m = 0$  and  $(e_1, \dots, e_m)$  is linely inde. Similarly.  $\square$

**7** Suppose  $v, x \in V$  and  $U$  and  $W$  are subsp of  $V$ . Prove that  $v + U = x + W \Rightarrow U = W$ .

**SOLUTION:**

(a)  $\forall u_1 \in U, \exists w_1 \in W, v + u_1 = x + w_1$ , let  $u_1 = 0$ , now  $v = x + w'_1 \Rightarrow v - x \in W$ .

(b)  $\forall w_2 \in W, \exists u_2 \in U, v + u_2 = x + w_2$ , let  $w_2 = 0$ , now  $x = v + u'_2 \Rightarrow x - v \in U$ .

Thus  $\pm(v - x) \in U \cap W \Rightarrow \begin{cases} u_1 = (x - v) + w_1 \in W \Rightarrow U \subseteq W \\ w_2 = (v - x) + u_2 \in U \Rightarrow W \subseteq U \end{cases} \Rightarrow U = W$ .  $\square$

• Let  $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$ . Suppose  $A \subseteq \mathbf{R}^3$ .

Then  $A$  is a translate of  $U \iff \exists c \in \mathbf{R}, A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$ .

• Suppose  $T \in \mathcal{L}(V, W)$  and  $c \in W$ . Prove that  $U = \{x \in V : Tx = c\}$  is either  $\emptyset$  or is a translate of  $\text{null } T$ .

**SOLUTION:**

If  $c \in W$  but  $c \notin \text{range } T$ , then  $U = \emptyset$ , we are done. Now suppose  $c \in \text{range } T$  and  $x \in U$ .

$\forall x + y \in x + \text{null } T$  ( $\forall y \in \text{null } T$ ),  $x + y \in U$ . Hence  $x + \text{null } T \subseteq U$ .

$\forall u \in U, u - x \in \text{null } T \Rightarrow u = x + (u - x)x + \text{null } T$ . Hence  $U \subseteq x + \text{null } T$ .  $\square$

**COROLLARY:** The set of solutions to a system of linear equations such as [3.28] is either  $\emptyset$  or a translate.

**8** Suppose  $A$  is a nonempty subset of  $V$ .

Prove that  $A$  is a translate of some subsp of  $V \iff \lambda v + (1 - \lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$ .

**SOLUTION:**

Suppose  $A = a + U$ . Then  $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$ ,  
 $\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + (\lambda(u_1 - u_2) + u_2) \in A$ .

Suppose  $\lambda v + (1 - \lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$ . Suppose  $a \in A$  and let  $A' = \{x - a : x \in A\}$ .

Then  $0 \in A'$  and  $\forall x - a, y - a \in A', (\forall x, y \in A), \lambda \in \mathbf{F}$ ,

(I)  $\lambda(x - a) = [\lambda x + (1 - \lambda)a] - a \in A'$ .

(II)  $\lambda(x - a) + (1 - \lambda)(y - a) = \frac{1}{2}(x - a) + \frac{1}{2}(y - a) = \frac{1}{2}x + (1 - \frac{1}{2})y - a \in A'$ .

OR. By (I),  $2 \times [\frac{1}{2}(x - a) + \frac{1}{2}(y - a)] = (x - a) + (y - a) \in A'$ .

Thus  $A'$  is a subsp of  $V$ . Hence  $a + A' = \{(x - a) + a : x \in A\} = A$  is a translate.  $\square$

OR. Suppose  $x - a, y - a \in A', \lambda \in \mathbf{F}$ .

Note that  $x, a \in A \Rightarrow \lambda x + (1 - \lambda)a = 2x - a \in A$ . Similarly  $2y - a \in A$ .

(I)  $(x - \frac{1}{2}a) + (y - \frac{1}{2}a) = x + y - a \in A \Rightarrow x + y - 2a = (x - a) + (y - a) \in A'$ .

(II)  $\lambda(x - a) = (\lambda x + (1 - \lambda)a) - a \in A'$ .

Thus  $-x + A$  is a subsp of  $V$ . Hence  $A = x + (-x + A)$  is a translate of the subsp  $(-x + A)$ .  $\square$

**9** Suppose  $A_1 = v + U_1$  and  $A_2 = w + U_2$  for some  $v, w \in V$  and some subsp  $U_1, U_2$  of  $V$ .

Prove that the intersection  $A_1 \cap A_2$  is either a translate of some subsp of  $V$  or is  $\emptyset$ .

**SOLUTION:**

Suppose  $v + u_1, w + u_2 \in A_1 \cap A_2 \neq \emptyset$ . By Problem (8),

$\forall \lambda \in \mathbf{F}, \lambda(v + u_1) + (1 - \lambda)(w + u_2) \in A_1 \cap A_2$ . Thus  $A_1 \cap A_2$  is a translate of some subsp of  $V$ .  $\square$

OR. Let  $A_1 = v + U_1, A_2 = w + U_2$ . Suppose  $x \in (v + U_1) \cap (w + U_2) \neq \emptyset$ .

Then  $\exists u_1 \in U_1, x = v + u_1 \Rightarrow x - v \in U_1, \exists u_2 \in U_2, x = w + u_2 \Rightarrow x - w \in U_2$ .

Note that by [3.85],  $A_1 = v + U_1 = x + U_1, A_2 = w + U_2 = x + U_2$ . We show that  $A_1 \cap A_2 = x + (U_1 \cap U_2)$ .

(a)  $y \in A_1 \cap A_2 \Rightarrow \exists u_1 \in U_1, u_2 \in U_2, y = x + u_1 = x + u_2 \Rightarrow u_1 = u_2 \in U_1 \cap U_2 \Rightarrow y \in x + (U_1 \cap U_2)$ .

(b)  $y = x + u \in x + (U_1 \cap U_2) = (x + U_1) \cap (x + U_2) \Rightarrow y \in A_1 \cap A_2$ .  $\square$

**10** Prove that the intersection of any collection of translates of subsp of  $V$  is either a translate of some subsp or  $\emptyset$ .

**SOLUTION:**

Suppose  $\{A_\alpha\}_{\alpha \in \Gamma}$  is a collection of translates of subsp of  $V$ , where  $\Gamma$  is an arbitrary index set.

Suppose  $x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset$ , then by Problem (8),  $\forall \lambda \in \mathbf{F}, \lambda x + (1 - \lambda)y \in A_\alpha$  for every  $\alpha \in \Gamma$ .

Thus  $\bigcap_{\alpha \in \Gamma} A_\alpha$  is a translate of some subsp of  $V$ .  $\square$

OR. Let  $A_\alpha = w_\alpha + V_\alpha$  for each  $\alpha \in \Gamma$ . Suppose  $x \in \bigcap_{\alpha \in \Gamma} (w_\alpha + V_\alpha) \neq \emptyset$ .

Then for each  $A_\alpha, \exists v_\alpha \in V_\alpha, x = w_\alpha + v_\alpha \Rightarrow x - w_\alpha \in V_\alpha \Rightarrow A_\alpha = w_\alpha + V_\alpha = x + V_\alpha$ .

(a)  $y \in \bigcap_{\alpha \in \Gamma} A_\alpha \Rightarrow \forall \alpha \in \Gamma, \exists v_\alpha, y = x + v_\alpha \Rightarrow \forall \alpha, \beta \in \Gamma, v_\alpha = v_\beta \Rightarrow y \in x + \bigcap_{\alpha \in \Gamma} V_\alpha$ .

(b)  $y = x + v \in x + \bigcap_{\alpha \in \Gamma} V_\alpha = \bigcap_{\alpha \in \Gamma} (x + V_\alpha) \Rightarrow y \in \bigcap_{\alpha \in \Gamma} A_\alpha$ . Hence  $\bigcap_{\alpha \in \Gamma} A_\alpha = x + \bigcap_{\alpha \in \Gamma} V_\alpha$ .  $\square$

• **NOTE FOR [3.79, 3.83]:** If  $U = \{0\}$ , then  $v + U = v + \{0\} = \{v\}$ ,  $V/U = V/\{0\} = \{\{v\} : v \in V\}$ .



**11** Suppose  $A = \{\lambda_1 v_1 + \dots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\}$ , where each  $v_i \in V, \lambda_i \in \mathbf{F}$ .

(a) Prove that  $A$  is a translate of some subsp of  $V$

(b) Prove that if  $B$  is a translate of some subsp of  $V$  and  $\{v_1, \dots, v_m\} \subseteq B$ , then  $A \subseteq B$ .

(c) Prove that  $A$  is a translate of some subsp of  $V$  of dim less than  $m$ .

**SOLUTION:**

(a) By Problem (8),  $\forall u, w \in A, \lambda \in \mathbf{F}, \exists a_i, b_i \in \mathbf{F}$ ,

$$\lambda u + (1 - \lambda)w = \left( \lambda \sum_{i=1}^m a_i + (1 - \lambda) \sum_{i=1}^m b_i \right) v_i \in A. \quad \square$$

(b) Suppose  $B = v + U$ , where  $v \in V$  and  $U$  is a subsp of  $V$ . Suppose  $\exists ! u_k \in U, v_k = v + u_k \in B$ .

$$\text{Then for all } v = \sum_{i=1}^m \lambda_i v_i \in A, v = \sum_{i=1}^m \lambda_i (v + u_i) = v + \sum_{i=1}^m \lambda_i u_i \in v + U = B. \quad \square$$

OR. Let  $v = \lambda_1 v_1 + \dots + \lambda_m v_m \in A$ . To show that  $v \in B$ , use induction on  $m$  by  $k$ .

(i)  $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$ .  $\forall v_1 \in B$ . Hence  $v \in B$ .

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$ .  $\forall v_1, v_2 \in B$ . By Problem (8),  $v \in B$ .

(ii)  $2 \leq k \leq m$ , we assume that  $v = \lambda_1 v_1 + \dots + \lambda_k v_k \in A \subseteq B$ . ( $\forall \lambda_i$  such that  $\sum_{i=1}^k \lambda_i = 1$ )

For  $u = \mu_1 v_1 + \dots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$ .  $\forall i = 1, \dots, k, \exists \mu_i \neq 1$ , fix one such  $i$  by  $\iota$ .

$$\text{Then } \sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow \left( \sum_{i=1}^{k+1} \frac{\mu_i}{1 - \mu_\iota} \right) - \frac{\mu_\iota}{1 - \mu_\iota} = 1.$$

$$\text{Let } w = \underbrace{\frac{\mu_1}{1 - \mu_\iota} v_1 + \dots + \frac{\mu_{\iota-1}}{1 - \mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1 - \mu_\iota} v_{\iota+1} + \dots + \frac{\mu_{k+1}}{1 - \mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let  $\lambda_i = \frac{\mu_i}{1 - \mu_\iota}$  for  $i = 1, \dots, \iota - 1$ ;  $\lambda_j = \frac{\mu_{j+1}}{1 - \mu_\iota}$  for  $j = \iota, \dots, k$ . Then,

$$\left. \begin{array}{l} \sum_{i=1}^k \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1 - \lambda) v_\iota \in B \end{array} \right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \quad \square$$

(c) If  $m = 1$ , then let  $A = v_1 + \{0\}$  and we are done.

Choose one  $k \in \{1, \dots, m\}$ . Given  $\lambda_i \in \mathbf{F}$ , where  $i \in \{1, \dots, k - 1, k + 1, \dots, m\}$ .

Let  $\lambda_k = 1 - \lambda_1 - \dots - \lambda_{k-1} - \lambda_{k+1} - \dots - \lambda_m$

Then  $\lambda_1 v_1 + \dots + \lambda_k v_k + \dots + \lambda_m v_m = v_k + \sum_{i=1}^m \lambda_i (v_i - v_k)$ .

Thus  $A = v_k + \text{span}(v_1 - v_k, \dots, v_{k-1} - v_k, v_{k+1} - v_k, \dots, v_m - v_k)$ .  $\square$

**18** Suppose  $T \in \mathcal{L}(V, W)$  and  $U$  is a subsp of  $V$ . Let  $\pi$  denote the quotient map.

Prove that  $\exists S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \subseteq \text{null } T$ .

**SOLUTION:**

(a) Suppose  $U \subseteq \text{null } T$ . Define  $S \in \mathcal{L}(V/U, W)$  by  $S(v + U) = Tv$ . Then  $S \circ \pi = T$ .

Now we show that this map is well-defined.

$$v_1 + U = v_2 + U \iff (v_1 - v_2) \in U \iff S((v_1 - v_2) + U) = T(v_1 - v_2) = 0 \iff Tv_1 = Tv_2.$$

(b) Suppose  $\exists S, T = S \circ \pi$ . Then  $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \text{null } T$ .  $\square$

**20** Define  $\Gamma : \mathcal{L}(V/U, W) \rightarrow \mathcal{L}(V, W)$  by  $\Gamma(S) = S \circ \pi$ . Prove that:

(a)  $\Gamma$  is linear: By [3.9] distr and [3.6].

(b)  $\Gamma$  is inje:  $\Gamma(S) = 0 = S \circ \pi \iff \forall v \in V, S(\pi(v)) = 0 \iff \forall v + U \in V/U, S(v + U) = 0 \iff S = 0$ .

(c)  $\text{range } \Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\}$ : By Problem (18).  $\square$

• **NOTE FOR [3.88, 3.90, 3.91]:** Suppose  $W \in \mathcal{S}_V U$ . Then  $V/U$  and  $W$  are iso.

For any  $W \in \mathcal{S}_V U$ , because  $V = U \oplus W$ ,  $\forall v \in V, \exists! u_v \in U, w_v \in W$  such that  $v = u_v + w_v$ .

Define  $T \in \mathcal{L}(V, W)$  by  $T(v) = w_v$ . Hence  $\text{null } T = U$ ,  $\text{range } T = W$ ,  $\text{range } T \oplus \text{null } T = V$ .

Then  $\tilde{T} \in \mathcal{L}(V/\text{null } T, W)$  is defined by  $\tilde{T}(v + U) = T v = w_v$ .

Now  $\pi \circ \tilde{T} = I_{V/U}$ ,  $\tilde{T} \circ \pi = I_W = T|_W$ . Hence  $\tilde{T}$  is an iso of  $V/U$  onto  $W$ .

• **COMMENT:** Note that  $v = u_v + w_v = (u_v - u') + (w'_v + u')$ , where  $w'_v \notin W \iff u' \neq 0$ .

Define  $S \in \mathcal{L}(V/U, V)$  by  $S(v + U) = v$ . Hence  $\text{null } S = \{0\}$ ,  $\text{range } S \in \mathcal{S}_V U$ ,  $\text{range } S \oplus U = V$ .

Let  $E = S \circ \pi$ . Now  $\text{null } E = \text{null } \pi = U$ . Because  $\pi$  is surj  $\text{range } (S \circ \pi) \subseteq \text{range } S$ .  $\text{range } E = \text{range } S$ .

Then  $\text{range } E \oplus \text{null } E = V$ . NOTICE that  $E : V \rightarrow \text{range } S$  is a pure *eraser*. Now we explain why:

**EXAMPLE:** Suppose  $B_V = (v_1, v_2, v_3)$ ,  $U = \text{span}(v_1)$ . Then it is uniquely fixed that  $\text{range } S = \text{span}(v_2, v_3)$ .

While we might have  $\text{range } T = \text{span}(v_2 - 2v_1, v_3) = W$ , depending on the choice of  $W$ .

Now  $E : v_2 \mapsto v_2$ ;  $v_2 - 2v_1 \mapsto v_2$ . While  $T : v_2 \mapsto v_2 - 2v_1$ ;  $v_2 - 2v_1 \mapsto v_2 - 2v_1$ .

**12** Suppose  $U$  is a subsp of  $V$  such that  $V/U$  is finite-dim. Prove that  $V$  is iso to  $U \times (V/U)$ .

**SOLUTION:**

Let  $(v_1 + U, \dots, v_n + U)$  be a basis of  $V/U$ .

Note that  $\forall v \in V, \exists! a_i \in \mathbb{F}, v + U = \sum_{i=1}^n a_i(v_i + U) = \left( \sum_{i=1}^n a_i v_i \right) + U$

$\Rightarrow (v - a_1 v_1 - \dots - a_n v_n) \in U \Rightarrow \exists! u \in U, v = \sum_{i=1}^n a_i v_i + u$ .

Thus define  $\varphi \in \mathcal{L}(V, U \times (V/U))$  by  $\varphi(v) = (u, v + U)$ ,

and  $\psi \in \mathcal{L}(U \times (V/U), V)$  by  $\psi(u, v + U) = v + u$ , where  $\exists! a_i \in \mathbb{F}, v = \sum_{i=1}^n a_i v_i + U$ .  $\square$

OR. [ $V/U, U$  and  $V$  can be infinite-dim] Define  $S \in \mathcal{L}(V/U, V)$  by  $S(v + U) = v$ .

By the NOTE FOR [3.88, 3.90, 3.91],  $\text{range } S \oplus U = V$ . Thus  $\forall v \in V, \exists! u \in U, w \in \text{range } S, v = u + w$ .

Define  $T \in \mathcal{L}(U \times (V/U), V)$  by  $T(u, v + U) = u + S(v + U) = u + w = v$ . Then  $T$  is surj.

And  $T(u, v + U) = u + S(v + U) = 0 \Rightarrow \pi(T(u, v + U)) = v + U = 0$ , and  $u = -S(v + U) = 0$ .

OR. Define  $R \in \mathcal{L}(V, U \times (V/U))$  by  $R(v) = (u, (w + U))$ . Now  $R \circ T = I_{U \times (V/U)}$ ,  $T \circ R = I_V$ .  $\square$

• (4E 3.E.14) Suppose  $V = U \oplus W$ ,  $(w_1, \dots, w_m)$  is a basis of  $W$ .

Prove that  $(w_1 + U, \dots, w_m + U)$  is a basis of  $V/U$ .

**SOLUTION:**  $\forall v \in V, \exists! u \in U, w \in W, v = u + w$ .  $\text{And } \exists! c_i \in \mathbb{F}, w = \sum_{i=1}^m c_i w_i \Rightarrow v = \sum_{i=1}^m c_i w_i + u$ .

Hence  $\forall v + U \in V/U, \exists! c_i \in \mathbb{F}, v + U = \sum_{i=1}^m c_i w_i + U$ .  $\square$

**13** Suppose  $(v_1 + U, \dots, v_m + U)$  is a basis of  $V/U$  and  $(u_1, \dots, u_n)$  is a basis of  $U$ .

Prove that  $(v_1, \dots, v_m, u_1, \dots, u_n)$  is a basis of  $V$ .

**SOLUTION:** Notice that  $(v_1, \dots, v_m)$  is linely inde.

By Problem (12),  $U$  and  $V/U$  are finite-dim  $\Rightarrow U \times (V/U)$  is finite-dim, so is  $V$ .

$\dim V = \dim(U \times (V/U)) = m + n$ .  $\text{And Each } v_i = S(v_i + U)$ , where we define  $S(v + U) = v$ .

Note that  $\sum_{i=1}^m a_i v_i \in U \iff \left( \sum_{i=1}^m a_i v_i \right) + U = 0 + U \iff a_1 = \dots = a_m = 0$ .

Hence  $\text{span}(v_1, \dots, v_m) \cap U = \{0\} \Rightarrow \text{span}(v_1, \dots, v_m) \oplus U = V$ . By (2.B.8), we are done.  $\square$

OR. Note that  $\forall v \in V, \exists! a_i \in \mathbb{F}, v + U = \sum_{i=1}^m a_i v_i + U \Rightarrow \exists! b_i \in \mathbb{F}, v - \sum_{i=1}^m a_i v_i = \sum_{i=1}^n b_i u_i \in U$

$\Rightarrow \forall v \in V, \exists! a_i, b_j \in \mathbb{F}, v = \sum_{i=1}^m a_i v_i + \sum_{j=1}^n b_j u_j$ .  $\square$

**15** Suppose  $\varphi \in \mathcal{L}(V, \mathbf{F}) \setminus \{0\}$ . Prove that  $\dim V/(\text{null } \varphi) = 1$ .

**SOLUTION:**

By (3.B.29),  $\exists u \in V, V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$ . By (4E 3.E.14),  $(u + \text{null } \varphi)$  is a basis of  $V/\text{null } \varphi$ .

OR. By [3.91] (d),  $\dim \text{range } \varphi = 1 = \dim V/(\text{null } \varphi)$ .  $\square$

**16** Suppose  $\dim V/U = 1$ . Prove that  $\exists \varphi \in \mathcal{L}(V, \mathbf{F})$  such that  $\text{null } \varphi = U$ .

**SOLUTION:**

Suppose  $V_0$  is a subsp of  $V$  such that  $V = U \oplus V_0$ . Then  $V_0$  and  $V/U$  are iso.  $\dim V_0 = 1$ .

Define  $\varphi \in \mathcal{L}(V, \mathbf{F})$  by  $\varphi(v_0) = 1, \varphi(u) = 0$ , where  $v_0 \in V_0, u \in U$ .  $\square$

OR. Let  $(w + U)$  be a basis of  $V/U$ . Then  $\forall v \in V, \exists! a \in \mathbf{F}, v + U = aw + U$ .

Define  $\varphi : V \rightarrow \mathbf{F}$  by  $\varphi(v) = a$ . Assume that  $\varphi$  is linear.

Then  $u \in U \iff u + U = 0w + U \iff \varphi(u) = 0 \iff u \in \text{null } \varphi$ . Thus  $U = \text{null } \varphi$ .  $\square$

Now we prove the assumption.

$\forall x, y \in V, \lambda \in \mathbf{F}, \exists! a, b \in \mathbf{F}, x + U = aw + U, \lambda y + U = \lambda bw + U \Rightarrow (x + \lambda y) + U = (a + \lambda b)w + U$ .

Then  $\varphi(x + \lambda y) = a + \lambda b = \varphi(x) + \lambda \varphi(y)$ .

**17** Suppose  $V/U$  is finite-dim.  $W$  is a subsp of  $V$ .

(a) Show that if  $V = U + W$ , then  $\dim W \geq \dim V/U$ .

(b) Find a  $W$  such that  $\dim W = \dim V/U$  and  $V = U \oplus W$ .

**SOLUTION:** Let  $(w_1, \dots, w_n)$  be a basis of  $W$

(a)  $\forall v \in V, \exists u \in U, w \in W$  such that  $v = u + w \Rightarrow v + U = w + U$

And  $\exists! a_i \in \mathbf{F}, v + U = (a_1 w_1 + \dots + a_n w_n) + U$ . Then  $V/U \subseteq \text{span}(w_1 + U, \dots, w_n + U)$ .

Hence  $\dim V/U = \dim \text{span}(w_1 + U, \dots, w_n + U) \leq \dim W$ .

(b) Let  $W \in \mathcal{S}_V U$ . In other words, reduce  $(w_1 + U, \dots, w_n + U)$

to a basis  $(w_1 + U, \dots, w_m + U)$  of  $V/U$  and let  $W = \text{span}(w_1, \dots, w_m)$ .  $\square$

OR. Let  $(v_1 + U, \dots, v_m + U)$  be a basis of  $V/U$  and define  $\tilde{T} \in \mathcal{L}(V/U, V)$  by  $\tilde{T}(v_k + U) = v_k$ .

Note that  $\pi \circ \tilde{T} = I$ . By (3.B.20),  $\tilde{T}$  is inje. And  $(v_1, \dots, v_m)$  is linely inde.

Let  $W = \text{range } \tilde{T} = \text{span}(v_1, \dots, v_m)$ . Then  $\tilde{T} \in \mathcal{L}(V/U, W)$  is an iso. Thus  $\dim W = \dim V/U$ .

And  $\forall v \in V, \exists! a_i \in \mathbf{F}, v + U = a_1 v_1 + \dots + a_m v_m + U$

$\Rightarrow v - (a_1 v_1 + \dots + a_m v_m) \in U \Rightarrow \exists! w \in W, u \in U, v = w + u$ .  $\square$

**ENDED**

**3.F** [4](#) [5](#) [6](#) [7](#) [8](#) [9](#) [12](#) [13](#) [15](#) [16](#) [17](#) [18](#) [19](#) [20](#) [21](#) [22](#) [23](#) [24](#) [25](#) [26](#)  
[28](#) [29](#) [30](#) [31](#) [33](#) [34](#) [35](#) [36](#) [37](#) | [4E: 5, 6, 8, 17, 23, 24, 25](#)

**20, 21** Suppose  $U$  and  $W$  are subsets of  $V$ . Prove that  $U \subseteq W \iff W^0 \subseteq U^0$ .

**SOLUTION:**

(a) Suppose  $U \subseteq W$ . Then  $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(u) = 0 \Rightarrow \varphi \in U^0$ . Thus  $W^0 \subseteq U^0$ .

(b) Suppose  $W^0 \subseteq U^0$ . Then  $\varphi \in W^0 \Rightarrow \varphi \in U^0$ . Hence  $\text{null } \varphi \supseteq W \Rightarrow \text{null } \varphi \supseteq U$ . Thus  $W \supseteq U$ .

OR. For a subsp  $U$  of  $V$ , let  $A_U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = U$ , by Problem (25).

Suppose  $W^0 \subseteq U^0$ . Then  $\forall \varphi \in W^0, v \in A_U, \varphi(v) = 0 \Rightarrow v \in A_W$ . Thus  $A_U \subseteq A_W$ .  $\square$

**COROLLARY:**  $W^0 = U^0 \iff U = W$ .

**22** Suppose  $U$  and  $W$  are subspaces of  $V$ . Prove that  $(U + W)^0 = U^0 \cap W^0$ .

**SOLUTION:**

$$(a) \left. \begin{array}{l} U \subseteq U + W \\ W \subseteq U + W \end{array} \right\} \Rightarrow \left. \begin{array}{l} (U + W)^0 \subseteq U^0 \\ (U + W)^0 \subseteq W^0 \end{array} \right\} \Rightarrow (U + W)^0 \subseteq U^0 \cap W^0.$$

OR. Suppose  $\varphi \in (U + W)^0$ . Then  $\forall u \in U, w \in W, \varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$ .

(b) Suppose  $\varphi \in U^0 \cap W^0 \subseteq V'$ . Then  $\forall u \in U, w \in W, \varphi(u + w) = 0 \Rightarrow \varphi \in (U + W)^0$ .  $\square$

**23** Suppose  $U$  and  $W$  are subsets of  $V$ . Prove that  $(U \cap W)^0 = U^0 + W^0$ .

**SOLUTION:**

$$(a) \left. \begin{array}{l} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array} \right\} \Rightarrow \left. \begin{array}{l} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array} \right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \quad [\supseteq U^0 \cap W^0 = (U + W)^0.]$$

OR. Suppose  $\varphi = \psi + \beta \in U^0 + W^0$ . Then  $\forall v \in U \cap W, \varphi(v) = (\psi + \beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0$ .

(b) [Only in Finite-dim; Requires that  $U, W$  are subspaces] Using Problem (22).

$$\begin{aligned} \dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\ &= 2 \dim V - \dim U - \dim W - (\dim V - \dim(U + W)) = \dim V - \dim(U \cap W). \end{aligned}$$

OR. Suppose  $\varphi \in (U \cap W)^0$ . Let  $X, Y$  be such that  $V = U \oplus X = W \oplus Y$ .

Define  $\psi \in U^0, \beta \in W^0$  by  $\psi(u + x) = \frac{1}{2}\varphi(x), \beta(w + y) = \frac{1}{2}\varphi(y)$ .

$\forall v = u + x = w + y \in V, \varphi(v) = \varphi(x) = \varphi(y)$ . Now  $\varphi(v) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y) = \psi(v) + \beta(v)$ .

Hence  $\varphi \in U^0 + W^0$ . Now  $(U \cap W)^0 \subseteq U^0 + W^0$ .  $\square$

• **COROLLARY:**

(a) Suppose  $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$  is a collection of subsets of  $V$ . Then  $\left(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \sum_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$ .

(b) Suppose  $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$  is a collection of subspaces of  $V$ . Then  $\left(\sum_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \bigcap_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$ .

(c) Suppose  $V = U \oplus W$ . Then  $V' = U^0 \oplus W^0$ . And  $U'_V = W^0, W'_V = U^0$ .

Where  $U'_V = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$ . And  $\iota \in \mathcal{L}(V, U)$  is defined by  $\iota(u_v + w_v) = u_v$ .

• (4E 3.F.23) Suppose  $\varphi_1, \dots, \varphi_m \in V'$ . Prove that the following sets are the same.

(a)  $\text{span}(\varphi_1, \dots, \varphi_m)$

(b)  $((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m))^0 \stackrel{(c)}{=} \{\varphi \in V' : (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) \subseteq \text{null } \varphi\}$

**SOLUTION:** By Problem (17), (c) holds.

By Problem (26) [May require finite-dim] and the COROLLARY in Problem (23),

$$\left. \begin{array}{l} ((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m))^0 = (\text{null } \varphi_1)^0 + \dots + (\text{null } \varphi_m)^0 \\ \text{span}(\varphi_i) = \{v \in V : \forall \psi \in \text{span}(\varphi_i), \psi(v) = 0\}^0 = (\text{null } \varphi_i)^0 \end{array} \right\} \Rightarrow (a) = (b). \quad \square$$

OR. Note that by COROLLARY in Problem (4E 6), for each  $\varphi_i$ , we have

$\forall c \in \mathbf{F} \setminus \{0\}, \psi = c\varphi_i \in \text{span}(\varphi_i) \iff \text{null } \psi = \text{null } \varphi_i \iff \psi \in (\text{null } \psi)^0 = (\text{null } \varphi_i)^0$ .

And  $0 \in \text{span}(\varphi_i), 0 \in (\text{null } \varphi_i)^0$ . Hence  $\text{span}(\varphi_i) = (\text{null } \varphi_i)^0$ . Similarly.  $\square$

OR. [Only in Finite-dim] Suppose  $\varphi \in V'$ . Note that  $\dim(\text{null } \varphi)^0 = \dim \text{range } \varphi = \dim \text{span}(\varphi)$ .

And because  $\forall c \in \mathbf{F}, v \in \text{null } \varphi, c\varphi(v) = 0 \Rightarrow \text{span}(\varphi) \subseteq (\text{null } \varphi)^0$ . Similarly.  $\square$

**COROLLARY: 30** Suppose  $V$  is finite-dim and  $\varphi_1, \dots, \varphi_m$  is a linearly inde list in  $V'$ .

Then  $\dim((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)) = (\dim V) - m$ .

**31** Suppose  $V$  is finite-dim and  $B_V = (\varphi_1, \dots, \varphi_n)$ . Show that the correspd  $B_V$  exists.

**SOLUTION:**

Using (3.B.29). Let  $\varphi_i(u_i) = 1$  and then  $V = \text{null } \varphi_i \oplus \text{span}(u_i)$  for each  $\varphi_i$ .

Suppose  $a_1 u_1 + \dots + a_n u_n = 0$ . Then  $0 = \varphi_i(a_1 u_1 + \dots + a_n u_n) = a_i$  for each  $i$ .

Thus  $B_V = (\varphi_1, \dots, \varphi_n)$ . And  $\varphi_i(u_x) = \delta_{i,x}$ . □

OR. For each  $k \in \{1, \dots, n\}$ , define  $\Gamma_k = \{1, \dots, k-1, k+1, \dots, n\}$  and  $U_k = \bigcap_{j \in \Gamma_k} \text{null } \varphi_j$ .

By Problem (30) OR (4E 2.C.16),  $\dim U_k = 1$ . Thus  $\exists u_k \in V, U_k = \text{span}(u_k) \neq 0$ .

又 By Problem (30),  $(\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_n) = \{0\} = U \cap \text{null } \varphi_k$ .

Then if  $\varphi_k(u_k) = 0 \Rightarrow u_k \in \text{null } \varphi_k$  while  $u_k \in U \Rightarrow u_k \in \{0\}$ , contradicts.

Thus  $\varphi_k(u_k) \neq 0$ . Let  $v_k = (\varphi_k(u_k))^{-1} u_k \Rightarrow \varphi_k(v_k) = 1$ . Now for  $j \neq k, u_k \in \text{null } \varphi_j \Rightarrow \varphi_j(v_k) = 0$ .

Similarly, suppose  $a_1 v_1 + \dots + a_n v_n = 0 \Rightarrow a_1 = \dots = a_n = 0$ .  $B_V = (v_1, \dots, v_n)$ . And  $\varphi_j(v_k) = \delta_{j,k}$ . □

**25** Suppose  $U$  is a subsp of  $V$ . Explain why  $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$ .

**SOLUTION:** Note that  $U = \{v \in V : v \in U\}$  is a subsp of  $V$ ; And  $v \in U \iff \varphi(v) = 0, \forall \varphi \in U^0$ . □

**COROLLARY:**  $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}^0$ .

**COMMENT:**  $\{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = ((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) \cap \dots)$ , where  $\varphi_k \in U^0$ , always remains a subsp, whether the subset  $U$  is a subsp or not.

**26** Suppose  $\Omega$  is a subsp of  $V'$ . Prove that  $\Omega = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega^0\}$ .

**SOLUTION:**

Suppose  $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}$ , which is the set of vecs that each  $\varphi \in \Omega$  sends to zero in common.

Then  $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}^0$ . 又  $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}^0$ .

Immediately by the COROLLARY in Problem (20,21), we may conclude that  $\Omega = U^0$ . □

OR. [Requires  $\Omega$  finite-dim] Let  $(\varphi_1, \dots, \varphi_m)$  be a basis of  $\Omega$ . Then by def,  $U \subseteq (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ .

$\forall \varphi \in \Omega, \exists ! a_i \in \mathbb{F}, \varphi = a_1 \varphi_1 + \dots + a_m \varphi_m \Rightarrow \forall v \in (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m), \varphi(v) = 0 \Rightarrow v \in U$ .

Hence  $(\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) = U$ . 又  $\text{span}(\varphi_1, \dots, \varphi_m) = \Omega$ . By Problem (23), we are done. □

**COROLLARY:** For every subsp  $\Omega$  of  $V'$ ,  $\exists !$  subsp  $U$  of  $V$  such that  $\Omega = U^0$ .

**COMMENT:** [Only in Finite-dim] Using Problem (31) and the COROLLARY(c) in Problem (22, 23).

Let  $B_\Omega = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n), B_V = (v_1, \dots, v_m, \dots, v_n)$ .

$V' = \text{span}(\varphi_1, \dots, \varphi_m) \oplus \text{span}(\varphi_{m+1}, \dots, \varphi_n) \stackrel{(I)}{=} \text{span}(v_{m+1}, \dots, v_n)^0 \oplus \text{span}(v_1, \dots, v_m)^0$ .

$\Omega = \text{span}(\varphi_1, \dots, \varphi_m) \stackrel{(II)}{=} \text{span}(v_{m+1}, \dots, v_n)^0 = U^0; \text{span}(\varphi_{m+1}, \dots, \varphi_n) \stackrel{(III)}{=} \text{span}(v_1, \dots, v_m)^0$ .

$\iff U = \text{span}(v_{m+1}, \dots, v_n) = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ . [Another proof of [3.106] OR. Problem (24)]

(I) Using the COROLLARY(c), immediately.

(II) NOTICE that each  $\text{null } \varphi_k = \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n) = U_k; \dim U_k = \dim V - 1$ .

By (4E 2.C.16),  $U = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) = \bigcap_{k=1}^m U_k = \text{span}(v_{m+1}, \dots, v_n)$ .

Hence  $\text{span}(v_{m+1}, \dots, v_n)^0 = U^0 = \Omega = \text{span}(\varphi_1, \dots, \varphi_m)$ .

(III) NOTICE that  $V' = \Omega \oplus \text{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0 \oplus \text{span}(v_1, \dots, v_m)^0$ .

And that  $\text{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq \text{span}(v_1, \dots, v_m)^0$ .

By the TIPS in (1.C),  $\text{span}(\varphi_{m+1}, \dots, \varphi_n) = \text{span}(v_1, \dots, v_m)$ .

OR. Similar to (II), let  $\Omega = \text{span}(\varphi_{m+1}, \dots, \varphi_n)$ , immediately. □

• Suppose  $T \in \mathcal{L}(V, W)$ ,  $\varphi_k \in V'$ ,  $\psi_k \in W'$ .

**28** Prove that  $\text{null } T' = \text{span}(\psi_1, \dots, \psi_m) \iff \text{range } T = (\text{null } \psi_1) \cap \dots \cap (\text{null } \psi_m)$ .

**29** Prove that  $\text{range } T' = \text{span}(\varphi_1, \dots, \varphi_m) \iff \text{null } T = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ .

**SOLUTION:** Using [3.107], [3.109], Problem (23) and the COROLLARY in Problem (20, 21).

$$(28) (\text{range } T)^0 = \text{null } T' = \text{span}(\psi_1, \dots, \psi_m) = ((\text{null } \psi_1) \cap \dots \cap (\text{null } \psi_m))^0.$$

$$(29) (\text{null } T)^0 = \text{range } T' = \text{span}(\varphi_1, \dots, \varphi_m) = ((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m))^0. \quad \square$$

**COROLLARY:** Using the COMMENT in Problem (26).

$$\text{null } T = \text{span}(v_1, \dots, v_m) \iff \text{null } T = (\text{null } \varphi_{m+1}) \cap \dots \cap (\text{null } \varphi_n) \iff \text{range } T' = \text{span}(\varphi_{m+1}, \dots, \varphi_n).$$

$$\text{---Where } B_V = (v_1, \dots, v_m, \dots, v_n) \iff B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n).$$

$$\text{range } T = \text{span}(w_1, \dots, w_m) \iff \text{range } T = (\text{null } \psi_{m+1}) \cap \dots \cap (\text{null } \psi_n) \iff \text{null } T' = \text{span}(\psi_{m+1}, \dots, \psi_n).$$

$$\text{---Where } B_W = (w_1, \dots, w_m, \dots, w_n) \iff B_{W'} = (\psi_1, \dots, \psi_m, \dots, \psi_n).$$

**9** Let  $B_V = (v_1, \dots, v_n)$ ,  $B_{V'} = (\varphi_1, \dots, \varphi_n)$ . Then  $\forall \psi \in V'$ ,  $\psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n$ .

**COROLLARY:** For other  $B'_V = (u_1, \dots, u_n)$ ,  $B'_{V'} = (\rho_1, \dots, \rho_n)$ ,  $\forall \psi \in V'$ ,  $\psi = \psi(u_1)\rho_1 + \dots + \psi(u_n)\rho_n$ .

**SOLUTION:**

$$\psi(v) = \psi\left(\sum_{i=1}^n a_i v_i\right) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i) \varphi_i(v) = [\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n](v).$$

$$\text{OR. } [\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n]\left(\sum_{i=1}^n a_i v_i\right) = \psi(v_1)\varphi_1\left(\sum_{i=1}^n a_i v_i\right) + \dots + \psi(v_n)\varphi_n\left(\sum_{i=1}^n a_i v_i\right). \quad \square$$

**13** Define  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  by  $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$ .

Let  $(\varphi_1, \varphi_2)$ ,  $(\psi_1, \psi_2, \psi_3)$  denote the dual basis of the standard basis of  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .

(a) Describe the linear functionals  $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$

$$\text{For any } (x, y, z) \in \mathbb{R}^3, (T'(\varphi_1))(x, y, z) = 4x + 5y + 6z, (T'(\varphi_2))(x, y, z) = 7x + 8y + 9z.$$

(b) Write  $T'(\varphi_1)$  and  $T'(\varphi_2)$  as linear combinations of  $\psi_1, \psi_2, \psi_3$ .

$$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3, \quad T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3.$$

(c) What is  $\text{null } T'$ ? What is  $\text{range } T'$ ?

$$T(x, y, z) = 0 \iff \begin{cases} 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases} \iff \begin{cases} x + y + z = 0 \\ y = 2z = 0 \end{cases} \iff (x, y, z) \in \text{span}(e_1 - 2e_2 + e_3).$$

Where  $(e_1, e_2, e_3)$  is standard basis of  $\mathbb{R}^3$ .

Let  $(e_1 - 2e_2 + e_3, -2e_2, e_3)$  be a basis, with the correspond dual basis  $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ .

$$\text{Thus } \text{span}(e_1 - 2e_2 + e_3) = \text{null } T \Rightarrow \text{span}(e_1 - 2e_2 + e_3)^0 = \text{span}(\varepsilon_2, \varepsilon_3) = \text{range } T'.$$

$$\text{Note that } \varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3.$$

$$\text{And } \begin{cases} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{cases}$$

$$\text{Hence } \varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2, \quad \varepsilon_3 = -\psi_1 + \psi_3. \text{ Now } \text{range } T' = \text{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3).$$

$$\text{OR. } \text{range } T' = \text{span}(T'(\varphi_1), T'(\varphi_2)) = \text{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3).$$

$$\text{Suppose } T'(x\varphi_1 + y\varphi_2) = (4x + 7y)\varphi_1 + (5x + 8y)\varphi_2 + (6x + 9y)\varphi_3 = 0.$$

$$\text{Then } x + y = 4x + 7y = x = y = 0. \text{ Hence } \text{null } T' = \{0\}.$$

$$\text{OR. } \text{null } T = \text{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \text{span}(-2e_2, e_3) \oplus \text{null } T.$$

$$\Rightarrow \text{range } T = \{Tx : x \in \text{span}(-2e_2, e_3)\} = \text{span}(T(-2e_2), T(e_3))$$

$$= \text{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \text{span}(f_1, f_2) = \mathbb{R}^2. \text{ Now } \text{null } T' = (\text{range } T)^0 = \{0\}. \quad \square$$

**24** Suppose  $V$  is finite-dim and  $U$  is a subsp of  $V$ .

Prove, using the pattern of [3.104], that  $\dim U + \dim U^0 = \dim V$ .

**SOLUTION:**

By Problem (31) and the COMMENT in Problem (26),  $B_U = (v_1, \dots, v_m) \iff B_{U^0} = (\varphi_{m+1}, \dots, \varphi_n)$ .  $\square$

**37** Suppose  $U$  is a subsp of  $V$  and  $\pi$  is the quotient map. Thus  $\pi' \in \mathcal{L}((V/U)', V')$ .

(a) Show that  $\pi'$  is inje: Because  $\pi$  is surj. Use [3.108].

(b) Show that  $\text{range } \pi' = U^0$ : By [3.109](b),  $\text{range } \pi' = (\text{null } \pi)^0 = U^0$ .

(c) Conclude that  $\pi'$  is an iso from  $(V/U)'$  onto  $U^0$ : Immediately.

**SOLUTION:** OR. Using (3.E.18), also see (3.E.20).

(a)  $\pi'(\varphi) = 0 \iff \forall v \in V (\forall v + U \in V), \varphi(\pi(v)) = \varphi(v + U) = 0 \iff \varphi = 0$ .

(b)  $\psi \in \text{range } \pi' \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \text{null } \psi \supseteq U \iff \psi \in U^0$ . Hence  $\text{range } \pi' = U^0$ .  $\square$

• Suppose  $U$  is a subsp of  $V$ . Prove that  $(V/U)'$  and  $U^0$  are iso. [Another proof of [3.106]]

**SOLUTION:**

Define  $\xi : U^0 \rightarrow (V/U)'$  by  $\xi(\varphi) = \tilde{\varphi}$ , where  $\tilde{\varphi} \in (V/U)'$  is defined by  $\tilde{\varphi}(v + U) = \varphi(v)$ .

We show that  $\xi$  is inje and surj.

Inje:  $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V (\forall v + U \in V/U), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$ .

Surj:  $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \text{null } (\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$ .  $\square$

OR. Define  $\nu : (V/U)' \rightarrow U^0$  by  $\nu(\Phi) = \Phi \circ \pi$ . Now  $\nu \circ \xi = I_{U^0}$ ,  $\xi \circ \nu = I_{(V/U)'}$ ,  $\Rightarrow \xi = \nu^{-1}$ .  $\square$

**4** Suppose  $U$  is a subsp of  $V$  and  $U \neq V$ . Prove that  $\exists \varphi \in V' \setminus \{0\}, \varphi(u) = 0$  for all  $u \in U$ .

**SOLUTION:**  $\iff U_V^0 \neq \{0\}$ .

Let  $X$  be such that  $V = U \oplus X$ . Then  $X \neq \{0\}$ . Suppose  $s \in X$  and  $s \neq 0$ .

Let  $Y$  be such that  $X = \text{span}(s) \oplus Y$ . Now  $V = U \oplus (\text{span}(s) \oplus Y)$ .

Define  $\varphi \in V'$  by  $\varphi(u + \lambda s + y) = \lambda$ . Hence  $\varphi \neq 0$  and  $\varphi(u) = 0$  for all  $u \in U$ .  $\square$

OR. [Requires that  $V$  is finite-dim] By [3.106],  $\dim U^0 = \dim V - \dim U > 0$ . Then  $U^0 \neq \{0\}$ .

OR. Let  $B_V = (\underbrace{u_1, \dots, u_m}_{B_U}, v_1, \dots, v_n)$  with  $n \geq 1$ . Let  $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$ . Let  $\varphi = \varphi_i$ .

OR. Define  $\varphi \in V'$  by  $\varphi(u_1) = \dots = \varphi(u_m) = 0$  and  $\varphi(v_1) = \dots = \varphi(v_n) = 1$ .  $\square$

**COMMENT:** [Another proof of [3.108]]:  $T$  is surj  $\iff T'$  is inje.

(a) Suppose  $T'$  is inje. Note that  $T'(\psi) = 0 \Rightarrow \psi = 0$ .

Then  $\nexists \psi \in W' \setminus \{0\}, (T'(\psi))(v) = \psi(Tv) = 0$  for all  $w \in \text{range } T (\forall v \in V)$ .

Thus if we assume that  $\text{range } T \neq W$  then contradicts. Hence  $\text{range } T = W$ .

(b) Suppose  $T$  is surj. Then  $(\text{range } T)^0 = W_W^0 = \{0\} = \text{null } T'$ .  $\square$

• Suppose  $V$  is a vecsp and  $U$  is a subsp of  $V$ .

**17**  $U^0 = \{\varphi \in V' : U \subseteq \text{null } \varphi\}$ . Noticing  $\varphi \in V', U \subseteq \text{null } \varphi \iff \forall u \in U, \varphi(u) = 0$ .

**18**  $U^0 = V' \iff \forall \varphi \in V', U \subseteq \text{null } \varphi \iff U = \{0\}$ . [Which means  $\{0\}_V^0 = V'$ .]

OR.  $U^0 = V' \iff \dim U^0 = \dim V' = \dim V \iff \dim U = 0 \iff U = \{0\}$ .

**19**  $U_V^0 = \{0\} = V_V^0 \iff U = V$ . By the inverse and contrapositive of Problem (4). OR. By [3.106].

- Suppose  $V = U \oplus W$ . Define  $\iota : V \rightarrow U$  by  $\iota(u + w) = u$ . Thus  $\iota' \in \mathcal{L}(U', V')$ .
  - (a) Show that  $\text{null } \iota' = U_U^0 = \{0\}$ :  $\text{null } \iota' = (\text{range } \iota)_U^0 = U_U^0 = \{0\}$ .
  - (b) Prove that  $\text{range } \iota' = W_V^0$ :  $\text{range } \iota' = (\text{null } \iota)_V^0 = W_V^0$ .
  - (c) Prove that  $\tilde{\iota}'$  is an iso from  $U'/\{0\}$  onto  $W^0$ : By (a), (b) and [3.91](d).

**SOLUTION:**

- (a)  $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \text{null } \psi$ .
- (b) Note that  $W = \text{null } (\iota) \subseteq \text{null } (\psi \circ \iota)$ . Then  $\psi \circ \iota \in W^0 \Rightarrow \text{range } \iota' \in W^0$ .  
Suppose  $\varphi \in W^0$ . Because  $\text{null } \iota = W \subseteq \text{null } \varphi$ . By TIPS in (3.B),  $\varphi = \varphi \circ \iota = \iota'(\varphi)$ . □

**36** Suppose  $U$  is a subsp of  $V$ . Define  $i : U \rightarrow V$  by  $i(u) = u$ . Thus  $i' \in \mathcal{L}(V', U')$ .

- (a) Show that  $\text{null } i' = U^0$ :  $\text{null } i' = (\text{range } i)^0 = U^0 \Leftarrow \text{range } i = U$ .
- (b) Prove that  $\text{range } i' = U'$ :  $\text{range } i' = (\text{null } i)_U^0 = \{0\}_U^0 = U'$ .
- (c) Prove that  $\tilde{i}'$  is an iso from  $V'/U^0$  onto  $U'$ : By (a), (b) and [3.91](d).

**SOLUTION:**

- (a)  $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$ . Thus  $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$ .
- (b) Suppose  $\psi \in U'$ . By (3.A.11),  $\exists \varphi \in V', \varphi|_U = \psi$ . Then  $i'(\varphi) = \psi$ . □

• Suppose  $T \in \mathcal{L}(V, W)$ . Prove that  $\text{range } T' = (\text{null } T)^0$ . [Another proof of [3.109](b)]

**SOLUTION:**

Suppose  $\Phi \in (\text{null } T)^0$ . Because by (3.B.12),  $T|_U : U \rightarrow \text{range } T$  is an iso;  $V = U \oplus \text{null } T$ .  
And  $\forall v \in V, \exists! u_v \in U, w_v \in \text{null } T, v = u_v + w_v$ . Define  $\iota \in \mathcal{L}(V, U)$  by  $\iota(v) = u_v$ .  
Let  $\psi = \Phi \circ (T|_{\text{range } T}^{-1})$ . Then  $T'(\psi) = \psi \circ T = \Phi \circ (T^{-1}|_{\text{range } T} \circ T|_V)$ .  
Where  $T^{-1}|_{\text{range } T} : \text{range } T \rightarrow U$ ;  $T : V \rightarrow \text{range } T$ . Note that  $T^{-1}|_{\text{range } T} \circ T|_V = \iota$ .  
By TIPS in (3.B),  $\Phi = \Phi \circ \iota$ . Thus  $T'(\psi) = \psi \circ T = \Phi \circ \iota = \Phi$ . □

• Suppose  $T \in \mathcal{L}(V, W)$ . Using [3.108], [3.110].

$$\text{Now } T \text{ is inv} \iff \left| \begin{array}{l} \text{null } T = \{0\} \iff (\text{null } T)^0 = V' = \text{range } T' \\ \text{range } T = W \iff (\text{range } T)^0 = \{0\} = \text{null } T' \end{array} \right| \iff T' \text{ is inv.}$$

**15** Suppose  $T \in \mathcal{L}(V, W)$ . Prove that  $T' = 0 \iff T = 0$ .

**SOLUTION:**

Suppose  $T = 0$ . Then  $\forall \varphi \in W', T'(\varphi) = \varphi \circ T = 0$ . Hence  $T' = 0$ .

Suppose  $T' = 0$ . Then  $\text{null } T' = W' = (\text{range } T)^0$ , by [3.107](a).

[ $W$  can be infinite-dim] By Problem (25),

$$\text{range } T = \{w \in W : \varphi(w) = 0, \forall \varphi \in (\text{range } T)^0\} = \{w \in W : \varphi(w) = 0, \forall \varphi \in W'\}.$$

Now we prove that if  $\forall \varphi \in W', \varphi(w) = 0$ , then  $w = 0$ . So that  $\text{range } T = \{0\}$  and we are done.

Assume that  $w \neq 0$ . Then let  $U$  be such that  $W = U \oplus \text{span}(w)$ .

Define  $\psi \in W'$  by  $\psi(u + \lambda w) = \lambda$ . So that  $\psi(w) = 1 \neq 0$ . □

OR. [Only if  $W$  is finite-dim] By [3.106],  $\dim \text{range } T = \dim W - \dim(\text{range } T)^0 = 0$ . □

**12** NOTICE that  $I_V : V' \rightarrow V'$ . Now  $\forall \varphi \in V', I_V(\varphi) = \varphi = \varphi \circ I_V = I_V'(\varphi)$ . Thus  $I_V = I_V'$ .



**16** Suppose  $V, W$  are finite-dim. Define  $\Gamma$  by  $\Gamma(T) = T'$  for any  $T \in \mathcal{L}(V, W)$ .

Prove that  $\Gamma$  is an iso of  $\mathcal{L}(V, W)$  onto  $\mathcal{L}(W', V')$ .

**SOLUTION:** By [3.101],  $\Gamma$  is linear.

Suppose  $\Gamma(T) = T' = 0$ . By Problem (15),  $T = 0$ . Thus  $\Gamma$  is inje.

Because  $V, W$  are finite-dim.  $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$ . Now  $\Gamma$  inje  $\Rightarrow$  inv. □

**COMMENT:** Let  $X = \{T \in \mathcal{L}(V, W) : \text{range } T \text{ is finite-dim}\}$ .

Let  $Y = \{\mathcal{T} \in \mathcal{L}(W', V') : \text{range } \mathcal{T} \text{ is finite-dim}\}$ .

Then  $\Gamma|_X$  is an iso of  $X$  onto  $Y$ , even if  $V$  and  $W$  are infinite-dim.

The inje of  $\Gamma|_X$  is equiv to the inje of  $\Gamma$ , as shown before.

Now we show that  $\Gamma|_X$  is surj without the cond that  $V$  or  $W$  is finite-dim.

Suppose  $\mathcal{T} \in Y$ . Let  $B_{\text{range } \mathcal{T}} = (\varphi_1, \dots, \varphi_m)$ , with the correspd  $(v_1, \dots, v_m)$ .

Let  $\varphi_k = \mathcal{T}(\psi_k)$ . Let  $\mathcal{K}$  be such that  $W' = \mathcal{K} \oplus \text{null } \mathcal{T}$ .

Let  $B_{\mathcal{K}} = (\psi_1, \dots, \psi_m)$ , with the correspd  $(w_1, \dots, w_m)$ .

Define  $T \in \mathcal{L}(V, W)$  by  $Tv_k = w_k, Tu = 0; k \in \{1, \dots, m\}, u \in U$ .

$\forall \psi \in \text{null } \mathcal{T}, [T'(\psi)](v) = \psi(Tv) = \psi(a_1w_1 + \dots + a_pw_p) = 0 = [\mathcal{T}(\psi)](v)$ .

$\forall k \in \{1, \dots, m\}, [T'(\psi_k)](v) = \psi_k(Tv) = \psi_k(a_1w_1 + \dots + a_mw_m) = a_k = \varphi_k(v) = [\mathcal{T}(\psi)](v)$ . □

**COMMENT:** This is another proof of [3.109(a)]:  $\dim \text{range } T = \dim \text{range } T'$ .

• (4E 3.F.6) Suppose  $\varphi, \beta \in V'$ . Prove that  $\text{null } \varphi \subseteq \text{null } \beta \iff \beta = c\varphi, \exists c \in \mathbf{F}$ .

**COROLLARY:**  $\text{null } \varphi = \text{null } \beta \iff \beta = c\varphi, \exists c \in \mathbf{F} \setminus \{0\}$ .

**SOLUTION:**

Using (3.B.29, 30).

(a) Suppose  $\text{null } \varphi \subseteq \text{null } \beta$ . Suppose  $u \notin \text{null } \beta$ , then  $u \notin \text{null } \varphi$ .

Now  $V = \text{null } \beta \oplus \text{span}(u) = \text{null } \varphi \oplus \text{span}(u)$ .

Then by TIPS in (1.C),  $\text{null } \beta = \text{null } \varphi$ . Hence let  $c = \frac{\beta(u)}{\varphi(u)}$ .

OR. We discuss in two cases. If  $\text{null } \varphi = \text{null } \beta$ , then we are done.

Otherwise,  $\text{null } \beta \neq \text{null } \varphi$ . Suppose  $u' \in \text{null } \beta \setminus \text{null } \varphi$ , then  $V = \text{null } \varphi \oplus \text{span}(u')$ .

$\forall v \in V, v = w + au = w' + bu', \exists ! w, w' \in \text{null } \varphi, a, b \in \mathbf{F}$ .

Thus  $\beta(v) = a\beta(u), \varphi(v) = b\varphi(u')$ . Let  $c = \frac{a\beta(u)}{b\varphi(u')}$ . We are done.

NOTICE that by (b) below, we have  $\text{null } \beta \subseteq \text{null } \varphi$ , thus contradicts the assumption.

(b) Suppose  $\beta = c\varphi$  for some  $c \in \mathbf{F}$ .

If  $c = 0$ , then  $\text{null } \beta = V \supseteq \text{null } \varphi$ , we are done.

Otherwise,  $\left. \begin{array}{l} \forall v \in \text{null } \varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \text{null } \varphi \subseteq \text{null } \beta. \\ \forall v \in \text{null } \beta, \beta(v) = 0 = \varphi(v) \Rightarrow \text{null } \beta \subseteq \text{null } \varphi. \end{array} \right\} \Rightarrow \text{null } \varphi = \text{null } \beta$   
 $\Rightarrow \text{null } \varphi \subseteq \text{null } \beta$ . □

OR. By (3.B.24),  $\text{null } \varphi \subseteq \text{null } \beta \iff \exists E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi$ . ( if  $E$  is inv, then  $\text{null } \varphi = \text{null } \beta$  )

Now we show that  $[P] \exists E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi \iff \exists c \in \mathbf{F}, \beta = c\varphi$ . [Q].

$[P] \Rightarrow [Q]$ : Let  $c = E(1)$ . Then  $\forall v \in V, \beta(v) = E(\varphi(v)) = \varphi(v)E(1) = c\varphi(v)$ . (  $E(1) \neq 0$  )

$[Q] \Rightarrow [P]$ : Define  $E \in \mathcal{L}(\mathbf{F})$  by  $E(x) = cx$ . Then  $\forall v \in V, \beta(v) = c\varphi(v) = E(\varphi(v))$ . (  $c \neq 0$  ) □

- In (3.D.18),  $\varphi : V \rightarrow \mathcal{L}(\mathbf{F}, V)$  is an iso. Now we prove that  
 $[P] (v_1, \dots, v_m) \text{ is linely inde} \iff (\varphi(v_1), \dots, \varphi(v_m)) \text{ is linely inde. } [Q]$

**SOLUTION:**

$[P] \Rightarrow [Q]$  : Notice that  $\varphi$  is inje and by (3.B.9).

OR. Suppose  $\vartheta \in \text{span}(\varphi(v_1), \dots, \varphi(v_m))$ . Let  $\vartheta = 0 = a_1\varphi(v_1) + \dots + a_m\varphi(v_m)$ .

Then  $\vartheta(1) = 0 = a_1v_1 + \dots + a_mv_m \Rightarrow a_1 = \dots = a_m = 0$ .

$[Q] \Rightarrow [P]$  : Suppose  $v \in \text{span}(v_1, \dots, v_m)$ . Let  $v = 0 = a_1v_1 + \dots + a_mv_m$ .

Then  $\varphi(v) = 0 = a_1\varphi(v_1) + \dots + a_m\varphi(v_m) \Rightarrow a_1 = \dots = a_m = 0$ . □

**33** Suppose  $A \in \mathbf{F}^{m,n}$ . Define  $T : A \rightarrow A^t$ . Prove that  $T$  is an iso of  $\mathbf{F}^{m,n}$  onto  $\mathbf{F}^{n,m}$

**SOLUTION:** By [3.111],  $T$  is linear. Note that  $(A^t)^t = A$ .

Inje: If  $T(A) = 0$  for some  $A \in \mathbf{F}^{n,m}$ , then  $A = (T(A))^t = 0$ .

Surj: For any  $B \in \mathbf{F}^{n,m}$ , let  $A = B^t$  so that  $T(A) = B$ . □

**32** Let  $B_\alpha = (\alpha_1, \dots, \alpha_m)$ ,  $B_\alpha' = (\varphi_1, \dots, \varphi_m)$ ,  $B_\beta = (v_1, \dots, v_m)$ ,  $B_\beta' = (\psi_1, \dots, \psi_m)$ .

Prove that  $\forall T \in \mathcal{L}(V)$ ,  $T$  is inv  $\iff$  the rows of  $A = \mathcal{M}(T, B_\alpha, B_\beta)$  form a basis of  $\mathbf{F}^{1,n}$ .

**SOLUTION:** Note that  $T$  is invertible  $\iff T'$  is inv. And  $A^t = \mathcal{M}(T', B_\beta', B_\alpha')$ .

(a) Suppose  $T$  is inv, so is  $T'$ . Because  $(T'(\varphi_1), \dots, T'(\varphi_m))$  is linely inde.

NOTICE that  $T'(\varphi_i) = A_{1,i}^t\psi_1 + \dots + A_{m,i}^t\psi_m$ . By the  $(\Delta)$  part in (4E 3.C.17),

the cols of  $A^t$ , namely the rows of  $A$ , are linely inde.

(b) Suppose the rows of  $A$  are linely inde, so are the cols of  $A^t$ . NOTICE that  $A^t$  has  $\dim V'$  cols.

Then  $B_{\text{range } T'} = B_{V'} = (T'(\varphi_1), \dots, T'(\varphi_m))$ . Thus  $T'$  is surj. Hence  $T'$  is inv, so is  $T$ . □

**5** Prove that  $(V_1 \times \dots \times V_m)'$  and  $V'_1 \times \dots \times V'_m$  are iso.

[Using notations in (3.E.2).]

Define  $\varphi : (V_1 \times \dots \times V_m)' \rightarrow V'_1 \times \dots \times V'_m$

by  $\varphi(T) = (T \circ R_1, \dots, T \circ R_m) = (R'_1(T), \dots, R'_m(T))$ .

Define  $\psi : V'_1 \times \dots \times V'_m \rightarrow (V_1 \times \dots \times V_m)'$

by  $\psi(T_1, \dots, T_m) = T_1S_1 + \dots + T_mS_m = S'_1(T_1) + \dots + S'_m(T_m)$ .

$\Rightarrow \psi = \varphi^{-1}$ . □

• (4E 3.F.8) Suppose  $B_V = (v_1, \dots, v_n)$ ,  $B_{V'} = (\varphi_1, \dots, \varphi_n)$ .

Define  $\Gamma : V \rightarrow \mathbf{F}^n$  by  $\Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v))$ .

Define  $\Lambda : \mathbf{F}^n \rightarrow V$  by  $\Lambda(a_1, \dots, a_n) = a_1v_1 + \dots + a_nv_n$ .

$\Rightarrow \Lambda = \Gamma^{-1}$ .

• (4E 3.F.5) Suppose  $T \in \mathcal{L}(V, W)$ .  $B_{\text{range } T} = (w_1, \dots, w_m)$ .

Hence  $\forall v \in V$ ,  $Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m$ ,  $\exists! \varphi_1(v), \dots, \varphi_m(v)$ ,

thus defining  $\varphi_i : V \rightarrow \mathbf{F}$  for each  $i \in \{1, \dots, m\}$ . Show that each  $\varphi_i \in V'$ .

**SOLUTION:**

$\forall u, v \in V, \lambda \in \mathbf{F}, T(u + \lambda v) = \sum_{i=1}^m \varphi_i(u + \lambda v)w_i$

$= Tu + \lambda Tv = \left( \sum_{i=1}^m \varphi_i(u)w_i \right) + \lambda \left( \sum_{i=1}^m \varphi_i(v)w_i \right) = \sum_{i=1}^m (\varphi_i(u) + \lambda \varphi_i(v))w_i$ . □

OR. For each  $w_i, \exists v_i \in V$ ,  $Tv_i = w_i$ , then  $(v_1, \dots, v_m)$  is linely inde.

Now we have  $Tv = a_1Tv_1 + \dots + a_mTv_m, \forall v \in V, \exists! a_i \in \mathbf{F}$ . Let  $B_{(\text{range } T)'} = (\psi_1, \dots, \psi_m)$ .

Then  $(T'(\psi_i))(v) = \psi_i \circ T(v) = a_i$ . Where  $T : V \rightarrow \text{range } T; T' : (\text{range } T)' \rightarrow V'$ .

Thus for each  $i \in \{1, \dots, m\}, \varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$ . □

6 Define  $\Gamma : V' \rightarrow \mathbf{F}^m$  by  $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$ , where  $v_1, \dots, v_m \in V$ .

(a) Show that  $\text{span}(v_1, \dots, v_m) = V \iff \Gamma$  is inje.

(b) Show that  $(v_1, \dots, v_m)$  is linely inde  $\iff \Gamma$  is surj.

**SOLUTION:**

(a) NOTICE that  $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \dots = \varphi(v_m) = 0 \iff \text{null } \varphi = \text{span}(v_1, \dots, v_m)$ .

If  $\Gamma$  is inje, then  $\Gamma(\varphi) = 0 \iff V = \text{null } \varphi = \text{span}(v_1, \dots, v_m)$ .

If  $V = \text{span}(v_1, \dots, v_m)$ , then  $\Gamma(\varphi) = 0 \iff \text{null } \varphi = \text{span}(v_1, \dots, v_m)$ , thus  $\Gamma$  is inje.

(b) Suppose  $\Gamma$  is surj. Then let  $\Gamma(\varphi_i) = e_i$  for each  $i$ , where  $(e_1, \dots, e_m)$  is the standard basis of  $\mathbf{F}^m$ .

Then by (3.A.4),  $(\varphi_1, \dots, \varphi_m)$  is linely inde.

Now  $a_1 v_1 + \dots + a_m v_m = 0 \Rightarrow 0 = \varphi_i(a_1 v_1 + \dots + a_m v_m) = a_i$  for each  $i$ .

Suppose  $(v_1, \dots, v_m)$  is linely inde. Let  $U = \text{span}(\varphi_1, \dots, \varphi_m)$ ,  $B_{U'} = (\varphi_1, \dots, \varphi_m)$ .

Thus  $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists! \varphi = a_1 \varphi_1 + \dots + a_m \varphi_m$ .

Let  $W$  be such that  $V = U \oplus W$ . Now  $\forall v \in V, \exists! u_v \in U, w_v \in W, v = u_v + w_v$ .

Define  $\iota \in \mathcal{L}(V, U)$  by  $\iota(v) = u_v$ . So that  $\Gamma(\varphi \circ \iota -) = (a_1, \dots, a_m)$ . □

OR. Let  $(e_1, \dots, e_m)$  be the standard basis of  $\mathbf{F}^m$  and let  $(\psi_1, \dots, \psi_m)$  be the correspd dual basis.

Define  $\Psi : \mathbf{F}^m \rightarrow (\mathbf{F}^m)'$  by  $\Psi(e_k) = \psi_k$ . Then  $\Psi$  is an iso.

Define  $T \in \mathcal{L}(\mathbf{F}^m, V)$  by  $T e_k = v_k$ . Now  $T(x_1, \dots, x_m) = T(x_1 e_1 + \dots + x_m e_m) = x_1 v_1 + \dots + x_m v_m$ .

$\forall \varphi \in V', k \in \{1, \dots, m\}, [T'(\varphi)](e_k) = \varphi(T e_k) = \varphi(v_k) = [\varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m](e_k)$

Now  $T'(\varphi) = \varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m = \Psi(\varphi(v_1), \dots, \varphi(v_m)) = \Psi(\Gamma(\varphi))$ . Hence  $T' = \Psi \circ \Gamma$ .

By (3.B.3), (a)  $\text{range } T = \text{span}(v_1, \dots, v_m) = V \iff T' = \Psi \circ \Gamma$  inje  $\iff \Gamma$  inje.

(b)  $(v_1, \dots, v_m)$  is linely inde  $\iff T$  is inje  $\iff T' = \Psi \circ \Gamma$  surj  $\iff \Gamma$  surj. □

• (4E 3.F.25) Define  $\Gamma : V \rightarrow \mathbf{F}^m$  by  $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$ , where  $\varphi_1, \dots, \varphi_m \in V'$ .

(c) Show that  $\text{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$  is inje.

(d) Show that  $(\varphi_1, \dots, \varphi_m)$  is linely inde  $\iff \Gamma$  is surj.

**SOLUTION:**

(c) NOTICE that  $\Gamma(v) = 0 \iff \varphi_1(v) = \dots = \varphi_m(v) = 0 \iff v \in (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ .

By Problem (4E 23) and (18),  $\text{span}(\varphi_1, \dots, \varphi_m) = V' \iff (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) = \{0\}$ .

And  $\text{null } \Gamma = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ . Hence  $\Gamma$  inje  $\iff \text{null } \Gamma = \{0\} \iff \text{span}(\varphi_1, \dots, \varphi_m) = V'$ .

(d) Suppose  $(\varphi_1, \dots, \varphi_m)$  is linely inde. Then by Problem (31),  $(v_1, \dots, v_m)$  is linely inde.

Thus  $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists! v = \sum_{i=1}^m a_i v_i \in V \Rightarrow \varphi_i(v) = a_i, \Gamma(v) = (a_1, \dots, a_m)$ . Hence  $\Gamma$  is surj.

Suppose  $\Gamma$  is surj. Let  $(e_1, \dots, e_m)$  be the standard basis of  $\mathbf{F}^m$ .

Suppose  $v_i \in V$  such that  $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i$ , for each  $i$ .

Then  $(v_1, \dots, v_m)$  is linely inde. And  $\varphi_j(v_k) = \delta_{j,k}$ .

Now  $a_1 \varphi_1 + \dots + a_m \varphi_m = 0 \Rightarrow 0(v_i) = a_i$  for each  $i$ . Hence  $(\varphi_1, \dots, \varphi_m)$  is linely inde.

OR. Let  $\text{span}(v_1, \dots, v_m) = U$ . Then  $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U)$ . Hence  $(\varphi_1, \dots, \varphi_m)$  is linely inde. □

OR. Similar to Problem (6), we get  $(e_1, \dots, e_m), (\psi_1, \dots, \psi_m)$  and the iso  $\Psi$ .

$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'(\Psi(x_1, \dots, x_m)) = \Gamma'(\Psi(x_1 e_1 + \dots + x_m e_m)) = (x_1 \psi_1 + \dots + x_m \psi_m) \circ \Gamma$ .

$\forall v \in V, [\Gamma'(\Psi(x_1, \dots, x_m))](v) = [x_1 \psi_1 + \dots + x_m \psi_m](\Gamma(v)) = [x_1 \varphi_1 + \dots + x_m \varphi_m](v)$ .

Now  $\Gamma'(\Psi(x_1, \dots, x_m)) = x_1 \varphi_1 + \dots + x_m \varphi_m$ .

Define  $\Phi : \mathbf{F}^m \rightarrow (\mathbf{F}^m)'$  by  $\Phi = \Psi \circ \Gamma$ .  $\Phi(x_1, \dots, x_m) = x_1 \varphi_1 + \dots + x_m \varphi_m$ . Thus by (4E 3.B.3),

(c) the inje of  $\Phi$  correspds to  $(\varphi_1, \dots, \varphi_m)$  spanning  $V'$ ; 又  $\Phi = \Psi \circ \Gamma$  inje  $\iff \Gamma$  inje.

(d) the surj of  $\Phi$  correspds to  $(\varphi_1, \dots, \varphi_m)$  being linely inde; 又  $\Phi = \Psi \circ \Gamma$  surj  $\iff \Gamma$  surj. □

**35** Prove that  $(\mathcal{P}(\mathbf{F}))'$  and  $\mathbf{F}^\infty$  are iso.

**SOLUTION:**

Define  $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{F}))', \mathbf{F}^\infty)$  by  $\theta(\varphi) = (\varphi(1), \varphi(z), \dots, \varphi(z^n), \dots)$ .

Inje:  $\theta(\varphi) = 0 \Rightarrow \forall z^k$  in the basis  $(1, z, \dots, z^n)$  of  $\mathcal{P}_n(\mathbf{F})$  ( $\forall n$ ),  $\varphi(z^k) = 0 \Rightarrow \varphi = 0$ .

[ NOTICE that  $\forall p \in \mathcal{P}(\mathbf{R}), \exists ! a_i \in \mathbf{F}, m = \deg p, p = a_0 z + a_1 z^2 + \dots + a_m z^m \in \mathcal{P}_m(\mathbf{F})$ . ]

Surj:  $\forall (a_k)_{k=1}^\infty \in \mathbf{F}^\infty$ , let  $\psi$  be such that  $\forall k, \psi(z^k) = a_k$  [ by [3.5] ] and thus  $\theta(\psi) = (a_k)_{k=1}^\infty$ .  $\square$

**COMMENT:** NOTICE that  $\mathcal{P}(\mathbf{F})$  and  $\mathbf{F}^\infty$  are not iso, so are  $\mathcal{P}(\mathbf{F})$  and  $(\mathcal{P}(\mathbf{F}))'$

But if we let  $\mathbf{F}^\infty = \{ (a_1, \dots, a_n, \underbrace{0, \dots, 0}_{\text{all zero}}) \in \mathbf{F}^\infty \mid \exists ! n \in \mathbf{N}^+ \}$ . Then  $\mathcal{P}(\mathbf{F})$  and  $\mathbf{F}^\infty$  are iso.

**7** Show that the dual basis of  $(1, x, \dots, x^m)$  of  $\mathcal{P}_m(\mathbf{R})$  is  $(\varphi_0, \varphi_1, \dots, \varphi_m)$ , where  $\varphi_k(p) = \frac{p^{(k)}(0)}{k!}$ .

Here  $p^{(k)}$  denotes the  $k^{\text{th}}$  derivative of  $p$ , with the understanding that the  $0^{\text{th}}$  derivative of  $p$  is  $p$ .

**SOLUTION:**

$$\forall j, k \in \mathbf{N}, (x^j)^{(k)} = \begin{cases} j(j-1) \dots (j-k+1) \cdot x^{(j-k)}, & j \geq k. \\ j(j-1) \dots (j-j+1) = j! & j = k. \\ 0, & j \leq k. \end{cases} \quad \text{Then } (x^j)^{(k)}(0) = \begin{cases} 0, & j \neq k. \\ k!, & j = k. \end{cases} \quad \square$$

OR. Because  $\forall j, k \in \{1, \dots, m\}$  such that  $j \neq k$ ,  $\varphi_k(x^j) = \frac{(x^j)^{(k)}(0)}{k!} = \frac{0}{k!} = 0$ ;  $\varphi_k(x^k) = \frac{(x^k)^{(k)}(0)}{k!} = 1$ .

Thus  $\frac{p^{(k)}(0)}{k!}$  act exactly the same as  $\varphi_k$  on the same basis  $(1, \dots, x^m)$ , hence is just another def of  $\varphi_k$   $\square$

**EXAMPLE:** Suppose  $m \in \mathbf{N}^+$ . By [2.C.10],  $B = (1, x-5, \dots, (x-5)^m)$  is a basis of  $\mathcal{P}_m(\mathbf{R})$ .

Let  $\varphi_k = \frac{p^{(k)}(5)}{k!}$  for each  $k = 0, 1, \dots, m$ . Then  $(\varphi_0, \varphi_1, \dots, \varphi_m)$  is the dual basis of  $B$ .

**34** The double dual space of  $V$ , denoted by  $V''$ , is defined to be the dual space of  $V'$ .

In other words,  $V'' = \mathcal{L}(V', \mathbf{F})$ . Define  $\Lambda : V \rightarrow V''$  by  $(\Lambda v)(\varphi) = \varphi(v)$ .

(a) Show that  $\Lambda$  is a linear map from  $V$  to  $V''$ .

(b) Show that if  $T \in \mathcal{L}(V)$ , then  $T'' \circ \Lambda = \Lambda \circ T$ , where  $T'' = (T')'$ .

(c) Show that if  $V$  is finite-dim, then  $\Lambda$  is an iso from  $V$  onto  $V''$ .

Suppose  $V$  is finite-dim. Then  $V$  and  $V'$  are iso, and finding an iso from  $V$  onto  $V'$  generally requires choosing a basis of  $V$ . In contrast, the iso  $\Lambda$  from  $V$  onto  $V''$  does not require a choice of basis and thus is considered more natural.

**SOLUTION:**

(a)  $\forall \varphi \in V', v, w \in V, a \in \mathbf{F}, (\Lambda(v+aw))(\varphi) = \varphi(v+aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$ .

Thus  $\Lambda(v+aw) = \Lambda v + a\Lambda w$ . Hence  $\Lambda$  is linear.

(b)  $(T''(\Lambda v))(\varphi) = ((\Lambda v) \circ T')(\varphi) = (\Lambda v)(T'(\varphi))$   
 $= (T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = (\Lambda(Tv))(\varphi)$ .

Hence  $T''(\Lambda v) = (\Lambda(Tv)) \Rightarrow T'' \circ \Lambda = \Lambda \circ T$ .

(c) Suppose  $\Lambda v = 0$ . Then  $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$ . Thus  $\Lambda$  is inje.

又 Because  $V$  is finite-dim.  $\dim V = \dim V' = \dim V''$ . Hence  $\Lambda$  is an iso.  $\square$

**ENDED**

- **TIPS:** Suppose  $p \in \mathcal{P}(\mathbf{F})$ ,  $\deg p \leq m$  and  $p$  has at least  $(m+1)$  distinct zeros.

Then by the contrapositive of [4.12], 又  $\deg p = m$ , we conclude that  $m < 0$ . Hence  $p = 0$ .

OR. We show that if  $p$  has at least  $m$  distinct zeros, then either  $p = 0$  or  $\deg p \geq m$ .

If  $p = 0$  then we are done. If not, then suppose  $p$  has exactly  $n$  distinct zeros  $\lambda_1, \dots, \lambda_n$ .

Because  $\exists! \alpha_i \geq 1, q \in \mathcal{P}(\mathbf{F})$ , and  $q \neq 0$ , such that  $p(z) = [(z - \lambda_1)^{\alpha_1} \dots (z - \lambda_n)^{\alpha_n}] q(z)$ .  $\square$

- **COMMENT:** NOTICE that by [4.17], some term of the poly factorization might not in the form  $(x - \lambda_k)^{\alpha_k}$ .

- **NOTE FOR [4.7]:** the uniqueness of coeffs of polys

[Another proof]

If a poly had two different sets of coeffs, then subtracting the two representations would give a poly with some nonzero coeffs but infinitely many zeros. By TIPS.  $\square$

- **NOTE FOR [4.8]:** division algorithm for polys

[Another proof]

Suppose  $\deg p \geq \deg s$ . Then  $\left( \underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of length } \deg s}, \underbrace{s, zs, \dots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)} \right)$  is a basis of  $\mathcal{P}_{\deg p}(\mathbf{F})$ .

Because  $q \in \mathcal{P}(\mathbf{F})$ ,  $\exists! a_i, b_j \in \mathbf{F}$ ,

$$\begin{aligned} q &= a_0 + a_1 z + \dots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \dots + b_{\deg p - \deg s} z^{\deg p - \deg s} s \\ &= \underbrace{a_0 + a_1 z + \dots + a_{\deg s - 1} z^{\deg s - 1}}_r + \underbrace{(b_0 + b_1 z + \dots + b_{\deg p - \deg s} z^{\deg p - \deg s}) s}_q. \end{aligned}$$

Note that  $r, q$  are unique.  $\square$

- **NOTE FOR [4.11]:** each zero of a poly corresponds to a degree-one factor;

[Another proof]

First suppose  $p(\lambda) = 0$ . Write  $p(z) = a_0 + a_1 z + \dots + a_m z^m$ ,  $\exists! a_0, a_1, \dots, a_m \in \mathbf{F}$  for all  $z \in \mathbf{F}$ .

Then  $p(z) = p(z) - p(\lambda) = a_1(z - \lambda) + \dots + a_m(z^m - \lambda^m)$  for all  $z \in \mathbf{F}$ .

Hence  $\forall k \in \{1, \dots, m\}, z^k - \lambda^k = (z - \lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \dots + z^{k-(j+1)}\lambda^j + \dots + z\lambda^{k-2} + z^0\lambda^{k-1})$ .

Thus  $p(z) = \sum_{j=1}^m a_j(z - \lambda) \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z - \lambda) \sum_{j=1}^m a_j \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z - \lambda)q(z)$ .  $\square$

- **NOTE FOR [4.13]:** Every nonconst poly with complex coefficients has a zero in  $\mathbf{C}$ .

[Another proof]

For any  $w \in \mathbf{C}, k \in \mathbf{N}^+$ , by polar coordinates,  $\exists r \geq 0, \theta \in \mathbf{R}, r(\cos \theta + i \sin \theta) = w$ .

By De Moivre' theorem,  $w^k = [r(\cos \theta + i \sin \theta)]^k = r^k(\cos k\theta + i \sin k\theta)$ .

Hence  $(r^{1/k}(\cos \frac{\theta}{k} + i \sin \frac{\theta}{k}))^k = w$ . Thus every complex number has a  $k^{\text{th}}$  root.

Suppose a nonconst  $p \in \mathcal{P}(\mathbf{C})$  with highest-order nonzero term  $c_m z^m$ .

Then  $|p(z)| \rightarrow \infty$  as  $|z| \rightarrow \infty$  (because  $\frac{|p(z)|}{|z_m|} \rightarrow |c_m|$  as  $|z| \rightarrow \infty$ ).

Thus the continuous function  $z \rightarrow |p(z)|$  has a global minimum at some point  $\zeta \in \mathbf{C}$ .

To show that  $p(\zeta) = 0$ , assume  $p(\zeta) \neq 0$ . Define  $q \in \mathcal{P}(\mathbf{C})$  by  $q(z) = \frac{p(z + \zeta)}{p(\zeta)}$ .

The function  $z \rightarrow |q(z)|$  has a global minimum value of 1 at  $z = 0$ .

Write  $q(z) = 1 + a_k z^k + \dots + a_m z^m$ , where  $k \in \mathbf{N}^+$  is the smallest such that  $a_k \neq 0$ .

Let  $\beta \in \mathbf{C}$  be such that  $\beta^k = -\frac{1}{a_k}$ .

There is a const  $c > 1$  so that if  $t \in (0, 1)$ , then  $|q(t\beta)| \leq |1 + a_k t^k \beta^k| + t^{k+1}c = 1 - t^k(1 - tc)$ .

Now letting  $t = 1/(2c)$ , we get  $|q(t\beta)| < 1$ . Contradicts. Hence  $p(\zeta) = 0$ , as desired.  $\square$

- (4E 4 2) Prove that if  $w, z \in \mathbb{C}$ , then  $||w| - |z|| \leq |w - z|$ .

**SOLUTION:**

$$\begin{aligned} |w - z|^2 &= (w - z)(\bar{w} - \bar{z}) \\ &= |w|^2 + |z|^2 - (w\bar{z} + \bar{w}z) \\ &= |w|^2 + |z|^2 - (\overline{wz} + \overline{wz}) \\ &= |w|^2 + |z|^2 - 2\operatorname{Re}(\overline{wz}) \\ &\geq |w|^2 + |z|^2 - 2|\overline{wz}| \\ &= |w|^2 + |z|^2 - 2|w||z| = ||w| - |z||^2. \end{aligned} \quad \left| \begin{array}{l} \text{OR. } \begin{cases} |w| = |w - z + z| \leq |w - z| + |z| \Rightarrow |w| - |z| \leq |w - z| \\ |z| = |z - w + w| \leq |z - w| + |w| \Rightarrow |z| - |w| \leq |w - z| \end{cases} \\ \text{Geometric interpretation: The length of each side of a triangle} \\ \text{is greater than or equal to the difference of the lengths of the two other sides.} \end{array} \right|$$

□

- (4E 4 3) Suppose  $\mathbf{F} = \mathbb{C}$ ,  $\varphi \in V'$ . Define  $\sigma : V \rightarrow \mathbb{R}$  by  $\sigma(v) = \operatorname{Re} \varphi(v)$  for each  $v \in V$ .

Show that  $\varphi(v) = \sigma(v) - i\sigma(iv)$  for all  $v \in V$ .

**SOLUTION:** Notice that  $\varphi(v) = \operatorname{Re} \varphi(v) + i\operatorname{Im} \varphi(v) = \sigma(v) + i\operatorname{Im} \varphi(v)$ .

又  $\operatorname{Re} \varphi(iv) = \operatorname{Re}(i\varphi(v)) = -\operatorname{Im} \varphi(v) = \sigma(iv)$ . Hence  $\varphi(v) = \sigma(v) - i\sigma(iv)$ . □

- 4 Suppose  $m, n \in \mathbb{N}^+$  with  $m \leq n$ ,  $\lambda_1, \dots, \lambda_m \in \mathbf{F}$ .

Prove that  $\exists p \in \mathcal{P}(\mathbf{F})$ ,  $\deg p = n$ , the zeros of  $p$  are  $\lambda_1, \dots, \lambda_m$ .

**SOLUTION:** Let  $p(z) = (z - \lambda_1)^{n-(m-1)}(z - \lambda_2) \cdots (z - \lambda_m)$ . □

- 5 Suppose  $m \in \mathbb{N}$ , and  $z_1, \dots, z_{m+1}$  are distinct in  $\mathbf{F}$ , and  $w_1, \dots, w_{m+1} \in \mathbf{F}$ .

Prove that  $\exists! p \in \mathcal{P}_m(\mathbf{F})$ ,  $p(z_k) = w_k$  for each  $k \in \{1, \dots, m+1\}$ .

**SOLUTION:**

Define  $T : \mathcal{P}_m(\mathbf{F}) \rightarrow \mathbf{F}^{m+1}$  by  $Tq = (q(z_1), \dots, q(z_m), q(z_{m+1}))$ . Moreover,  $T$  is linear.

We now show that  $T$  is surj, so that such  $p$  exists; and that  $T$  is inje, so that such  $p$  is unique.

Inje:  $Tq = 0 \iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0 \iff q = 0$ , by TIPS.

Surj:  $\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m + 1 = \dim \mathbf{F}^{m+1}$  又  $\operatorname{range} T \subseteq \mathbf{F}^{m+1} \Rightarrow T$  is surj. □

OR. Let  $p_1 = 1, p_k(z) = \prod_{i=1}^{k-1} (z - z_i) = (z - z_1) \cdots (z - z_{k-1})$  for each  $k \in \{2, \dots, m+1\}$ .

By (2.C.10),  $B_p = (p_1, \dots, p_{m+1})$  is a basis of  $\mathcal{P}_m(\mathbf{F})$ . Let  $B_e = (e_1, \dots, e_{m+1})$  be the std basis of  $\mathbf{F}^{m+1}$ .

NOTICE that  $Tp_1 = (1, \dots, 1), Tp_k = \left( \prod_{i=1}^{k-1} (z_1 - z_i), \dots, \underbrace{\prod_{i=1}^{k-1} (z_j - z_i)}_{j^{\text{th}} \text{ entry}}, \dots, \prod_{i=1}^{k-1} (z_{m+1} - z_i) \right)$ .

And that  $\prod_{i=1}^{k-1} (z_j - z_i) = 0 \iff j \leq k - 1$ , because  $z_1, \dots, z_{m+1}$  are distinct.

$$\text{Thus } \mathcal{M}(T, B_p, B_e) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & A_{2,2} & 0 & \cdots & 0 \\ 1 & A_{3,2} & A_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{m+1,2} & A_{m+1,3} & \cdots & A_{m+1,m+1} \end{pmatrix}.$$

Where  $A_{j,k} = \prod_{i=1}^{k-1} (z_j - z_i) \neq 0$  for all  $j > k - 1 \geq 1$ . The rows of  $\mathcal{M}(T)$  is linely inde.

By (4E 3.C.17) 又  $\dim \mathcal{P}_m(\mathbf{F}) = \dim \mathbf{F}^{m+1}$ ; OR By (3.F.32);  $T$  is inv. □

- 2 Suppose  $m \in \mathbb{N}^+$ . Is the set  $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$  a subsp of  $\mathcal{P}(\mathbf{F})$ ?

**SOLUTION:**  $x^m, x^m + x^{m-1} \in U$  but  $\deg[(x^m + x^{m-1}) - (x^m)] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$ . □

3 Suppose  $m \in \mathbb{N}^+$ . Is the set  $U = \{0\} \cup \{p \in \mathcal{P}(\mathbb{F}) : 2 \mid \deg p\}$  a subsp of  $\mathcal{P}(\mathbb{F})$ ?

SOLUTION:  $x^2, x^2 + x \in U$  but  $\deg[(x^2 + x) - (x^2)]$  is odd and hence  $(x^2 + x) - (x^2) \notin U$ .  $\square$

6 Suppose nonzero  $p \in \mathcal{P}_m(\mathbb{F})$  has degree  $m$ . Prove that

$[P] \ p \text{ has } m \text{ distinct zeros} \iff p \text{ and its derivative } p' \text{ have no zeros in common } [Q].$

SOLUTION:

(a) Suppose  $p$  has  $m$  distinct zeros. And  $\deg p = m$ . By [4.14],  $\exists! c, \lambda_i \in \mathbb{R}, p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$ .

If  $m = 0$ , then  $p = c \neq 0 \Rightarrow p$  has no zeros, and  $p' = 0$ , we are done.

If  $m = 1$ , then  $p(z) = c(z - \lambda_1)$ , and  $p' = c$  has no zeros, we are done.

For each  $j \in \{1, \dots, m\}$ , let  $q_j \in \mathcal{P}_{m-1}(\mathbb{F})$  be such that  $p(z) = (z - \lambda_j)q_j \Rightarrow q_j(\lambda_j) \neq 0$ .

Now  $p'(z) = (z - \lambda_j)q_j'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$ , as desired.

OR. To prove  $[P] \Rightarrow [Q]$ , we prove  $\neg[Q] \Rightarrow \neg[P]$ :

Suppose  $p(z) = (z - \lambda)q(z)$ ,  $p'(z) = (z - \lambda)r(z)$ .

又  $p'(z) = (z - \lambda)q'(z) + q(z)$ . Now  $p'(\lambda) = q(\lambda) = 0 \Rightarrow p(z) = (z - \lambda)^2 s(z)$ .

Hence  $p$  has strictly less than  $m$  distinct zeros.

(b) To prove  $[Q] \Rightarrow [P]$ , we prove  $\neg[P] \Rightarrow \neg[Q]$ :

Because nonzero  $p \in \mathcal{P}_m(\mathbb{F})$ , we suppose  $\lambda_1, \dots, \lambda_M$  are the distinct zeros of  $p$ , where  $M < m$ .

By Pigeon Hole Principle,  $\exists \lambda_k$  such that  $p(z) = (z - \lambda_k)^2 q(z)$  for some  $q \in \mathcal{P}(\mathbb{F})$ .

Hence  $p'(z) = 2(z - \lambda_k)q(z) + (z - \lambda_k)^2 q'(z) \Rightarrow p'(\lambda_k) = 0 = p(\lambda_k)$ .  $\square$

7 Prove that every  $p \in \mathcal{P}(\mathbb{R})$  of odd degree has a zero.

SOLUTION:

Using the notation and proof of [4.17].  $\deg p = 2M + m$  is odd  $\Rightarrow m$  is odd. Hence  $\lambda_1$  exists.  $\square$

OR. Using calculus only.

Suppose  $p \in \mathcal{P}_m(\mathbb{F})$ ,  $\deg p = m$ ,  $m$  is odd.

Let  $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ . Then  $a_m \neq 0$ . Denote  $|a_m|^{-1} a_m$  by  $\delta$

Write  $p(x) = x^m \left( \frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \cdots + \frac{a_{m-1}}{x} + a_m \right)$ .

Thus  $p(x)$  is continuous, and  $\lim_{x \rightarrow -\infty} p(x) = -\delta\infty$ ;  $\lim_{x \rightarrow \infty} p(x) = \delta\infty$ .

Hence we conclude that  $p$  has at least one real zero.  $\square$

9 Suppose  $p \in \mathcal{P}(\mathbb{C})$ . Define  $q : \mathbb{C} \rightarrow \mathbb{C}$  by  $q(z) = p(z)\overline{p(\bar{z})}$ . Prove that  $q \in \mathcal{P}(\mathbb{R})$ .

SOLUTION:

NOTICE that by [4.5],  $\bar{\bar{z}}^n = \bar{z}^n$ .

Suppose  $q(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow q(\bar{z}) = a_n \bar{z}^n + \cdots + a_1 \bar{z} + a_0 \Rightarrow \overline{q(\bar{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$ .

Note that  $q(z) = p(z)\overline{p(\bar{z})} = \overline{\overline{p(\bar{z})p(z)}} = \overline{p(\bar{z})\overline{p(z)}} = \overline{p(\bar{z})p(\bar{\bar{z}})} = \overline{q(\bar{z})}$ . Hence for each  $a_k, \overline{a_k} = a_k \Rightarrow a_k \in \mathbb{R}$ .  $\square$

OR. Suppose  $p(z) = a_m z^m + \cdots + a_1 z + a_0$ . Now  $\overline{p(\bar{z})} = \overline{a_m} z^m + \cdots + \overline{a_1} z + \overline{a_0}$ .

NOTICE that  $q(z) = p(z)\overline{p(\bar{z})} = \sum_{k=0}^2 m \left( \sum_{i+j=k} a_i \overline{a_j} \right) z^k$ .

NOTICE that by [4.5],  $z - \bar{z} = 2(\text{Im } z) \Rightarrow z = \bar{z} + 2(\text{Im } z)$ . So that  $z = \bar{z} \iff \text{Im } z = 0 \iff z \in \mathbb{R}$ .

Now for each  $k \in \{0, \dots, 2m\}$ ,  $\overline{\sum_{i+j=k} a_i \overline{a_j}} = \sum_{i+j=k} \overline{a_i} a_j = \sum_{i+j=k} a_j \overline{a_i} = \sum_{i+j=k} a_i \overline{a_j} \in \mathbb{R}$ .  $\square$

**8** For  $p \in \mathcal{P}(\mathbf{R})$ , define  $Tp : \mathbf{R} \rightarrow \mathbf{R}$  by  $(Tp)(x) = \begin{cases} \frac{p(x) - p(3)}{x - 3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}$

Show that (a)  $Tp \in \mathcal{P}(\mathbf{R})$  for all  $p \in \mathcal{P}(\mathbf{R})$  and that (b)  $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$  is linear.

**SOLUTION:**

$$(a) \text{ For } x \neq 3, T(x^n) = \frac{x^n - 3^n}{x - 3} = \sum_{i=1}^n 3^{i-1} x^{n-i}. \text{ For } x = 3, T(x^n) = 3^{n-1} \cdot n.$$

Note that if  $x = 3$ , then  $\sum_{i=1}^n 3^{i-1} x^{n-i} = \sum_{i=1}^n 3^{n-1} = 3^{n-1} \cdot n$ .

Hence  $T(x^n) = \sum_{i=1}^n 3^{i-1} x^{n-i} \Rightarrow T(x^n) \in \mathcal{P}(\mathbf{R})$ .

(b) Now we show that  $T$  is linear:  $\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}$ ,

$$T(p + \lambda q)(x) = \begin{cases} \frac{(p + \lambda q)(x) - (p + \lambda q)(3)}{x - 3}, & \text{if } x \neq 3, \\ (p + \lambda q)'(3), & \text{if } x = 3 \end{cases} = [T(p) + \lambda T(q)](x) \text{ for all } x \in \mathbf{R}. \quad \square$$

OR. (a) Note that  $\exists! q \in \mathcal{P}(\mathbf{R}), p(x) - p(3) = (x - 3)q(x) \Rightarrow q(x) = \frac{p(x) - p(3)}{x - 3}$ .

$$p'(x) = (p(x) - p(3))' = ((x - 3)q(x))' = q(x) + (x - 3)q'(x).$$

Hence  $p'(3) = q(3)$ . Now  $Tp = q \in \mathcal{P}(\mathbf{R})$ .

(b)  $\forall p_1, p_2 \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, \exists! q_1, q_2 \in \mathcal{P}(\mathbf{R})$ ,

$$p_1(x) - p_1(3) = (x - 3)q_1(x) \text{ and } p_2(x) - p_2(3) = (x - 3)q_2(x).$$

By (a),  $Tp_1 = q_1, Tp_2 = q_2$ . Note that  $(p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x - 3)(q_1 + \lambda q_2)(x)$ .

Hence by the uniqueness of  $q_1 + \lambda q_2$  for  $p_1 + \lambda p_2$ , we must have  $T(p_1 + \lambda p_2) = q_1 + \lambda q_2$ .  $\square$

**11** Suppose  $p \in \mathcal{P}(\mathbf{F})$  with  $p \neq 0$ . Let  $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$ .

(a) Show that  $\dim \mathcal{P}(\mathbf{F})/U = \deg p$ .

(b) Find a basis of  $\mathcal{P}(\mathbf{F})/U$ .

**SOLUTION:** NOTICE that  $pq \neq p \circ q$ , see (4E 3.A.10).

$U$  is a subsp of  $\mathcal{P}(\mathbf{F})$  because  $\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, ps_1 + \lambda ps_2 = p(s_1 + \lambda s_2) \in U$ .

If  $\deg p = 0$ , then  $U = \mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F})/U = \{0\}$ , with the unique basis  $()$ . Suppose  $\deg p \geq 1$ .

(a) By [4.8],  $\forall s \in \mathcal{P}(\mathbf{F}), \exists! r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}) [ \exists! pq \in U ], s = (p)q + (r)$ .

Thus  $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F})$ . By the NOTE FOR [3.91] in (3.E),  $\mathcal{P}(\mathbf{F})/U$  and  $\mathcal{P}_{\deg p-1}(\mathbf{F})$  are iso.

OR. Define  $R : \mathcal{P}(\mathbf{F}) \rightarrow \mathcal{P}_{\deg p-1}(\mathbf{F})$  by  $R(s) = r$  for all  $s \in \mathcal{P}(\mathbf{F})$ . We show that  $R$  is linear.

$$\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, \exists! r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q_1, q_2 \in \mathcal{P}(\mathbf{F}), s_1 = (p)q_1 + (r_1); s_2 = (p)q_2 + (r_2).$$

$$\text{又 } \exists! r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}), (s_1 + \lambda s_2) = (p)q + (r) = (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2).$$

Note that  $r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}) \Rightarrow r_1 + \lambda r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F})$ .

OR Note that  $\deg(r_1 + \lambda r_2) \leq \max\{\deg r_1, \deg(\lambda r_2)\} \leq \max\{\deg r_1, \deg r_2\} < \deg p$ .

By the uniqueness part of [4.8],  $s = s_1 + \lambda s_2; r = r_1 + \lambda r_2$ . Thus  $R(s_1 + \lambda s_2) = R(s_1) + \lambda R(s_2)$ .

Because  $Rs = 0 \iff s = pq, \exists! q \in \mathcal{P}(\mathbf{F}) \iff s \in U$ . And  $\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), Rr = r$ .

Now  $\text{null } R = U, \text{ range } R = \mathcal{P}_{\deg p-1}(\mathbf{F})$ .

Hence  $\tilde{R} : \mathcal{P}(\mathbf{F})/U \rightarrow \mathcal{P}_{\deg p-1}(\mathbf{F})$  is defined by  $\tilde{R}(s + U) = Rs$ . By [3.91(d)],  $\tilde{R}$  is an iso.

(b) For each  $k \in \{0, 1, \dots, \deg p - 1\}, \tilde{R}(z^k + U) = R(z^k) = z^k \Rightarrow \tilde{R}^{-1}(z^k) = z^k + U$ .

Thus  $(1 + U, z + U, \dots, z^{\deg p-1} + U)$  can be a basis of  $\mathcal{P}(\mathbf{F})/U$ .  $\square$



**10** Suppose  $m \in \mathbf{N}$ ,  $p \in \mathcal{P}_m(\mathbf{C})$  is such that  $p(x_k) \in \mathbf{R}$  for each of distinct  $x_0, x_1, \dots, x_m \in \mathbf{R}$ . Prove that  $p \in \mathcal{P}(\mathbf{R})$ .

**SOLUTION:**

By TIPS and Problem (5),  $\exists ! q \in \mathcal{P}_m(\mathbf{R})$  such that  $q(x_k) = p(x_k)$ . Hence  $p = q$ .  $\square$

OR. Using the Lagrange Interpolating Polynomial.

Define  $q(x) = \sum_{j=0}^m \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j)$ .

又 Each  $x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R})$ . Notice that  $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$  for each  $x_k$ .

Then  $(q-p)$  has  $(m+1)$  zeros, while  $(q-p) \in \mathcal{P}_m(\mathbf{C})$ . By TIPS,  $q-p=0 \Rightarrow p=q \in \mathcal{P}(\mathbf{R})$ .  $\square$

• (4E413) Suppose nonconst  $p, q \in \mathcal{P}(\mathbf{C})$  have no zeros in common. Let  $m = \deg p, n = \deg q$ . Define  $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \rightarrow \mathcal{P}_{m+n-1}(\mathbf{C})$  by  $T(r, s) = rp + sq$ . Prove that  $T$  is an iso.

**COROLLARY:**  $\exists ! r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$  such that  $rp + sq = 1$ .

**SOLUTION:**

$T$  is linear because  $\forall r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$ ,

$$T((r_1, s_1) + \lambda(r_2, s_2)) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1, s_1) + \lambda T(r_2, s_2).$$

Let  $\lambda_1, \dots, \lambda_M$  and  $\mu_1, \dots, \mu_N$  be the distinct zeros of  $p$  and  $q$  respectively. NOTICE that  $M \leq m, N \leq n$ .

Note that the contrapositive of [4.13],  $M = 0 \Leftrightarrow m = 0 \Rightarrow s = 0 \Leftrightarrow r = 0 \Leftarrow n = 0 \Leftrightarrow N = 0$ .

Now suppose  $M, N \geq 1$ . We show that  $s = 0$ . Showing  $r = 0$  is almost the same.

Write  $p(z) = a(z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_M)^{\alpha_M}$ . ( $\exists ! \alpha_j \geq 1, a \in \mathbf{F}$ .) Let  $\max\{\alpha_1, \dots, \alpha_M\} = A$ .

For each  $D \in \{0, 1, \dots, A-1\}$ , let  $I_{D, \alpha} = \{\gamma_{D,1}, \dots, \gamma_{D,J}\}$  be such that each  $\alpha_{\gamma_{D,j}} \geq D+1$ .

Note that  $I_{A-1, \alpha} \subseteq \cdots \subseteq I_{0, \alpha} = \{1, \dots, M\}$ . Because  $rp + sq = 0 \Rightarrow (rp + sq)^{(k)} = 0$  for all  $k \in \mathbf{N}^+$ .

We use induction by  $D$  to show that  $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$  for each  $D \in \{0, \dots, A-1\}$ .

NOTICE that  $p^{(D)}(\lambda_{\gamma}) = 0$  for each  $D \in \{0, \dots, A-1\}$  and each  $\lambda_{\gamma} \in I_{D, \alpha}$ .  $(\Delta)$

(i)  $D = 0$ .  $(rp + sq)(\lambda_{\gamma_{0,j}}) = (sq)(\lambda_{\gamma_{0,j}}) = s(\lambda_{\gamma_{0,j}}) = 0$ .

$D = 1$ .  $(rp + sq)'(\lambda_{\gamma_{1,j}}) = (r'p + rp')(\lambda_{\gamma_{1,j}}) + (s'q + sq')(\lambda_{\gamma_{1,j}}) = (s'q)(\lambda_{\gamma_{1,j}}) = s'(\lambda_{\gamma_{1,j}}) = 0$ .

(ii)  $2 \leq D \leq A-1$ . Assume that  $s^{(d)}(\lambda_{\gamma_{d,j}}) = 0$  for each  $d \in \{1, \dots, D-1\}$  and each  $\lambda_{\gamma_{d,j}} \in I_{d, \alpha}$ .

(Because  $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}$ .)  $(\Delta)$

$$\begin{aligned} \text{Now } [rp + sq]^{(D)}(\lambda_{\gamma_{D,j}}) &= [C_D^D r^{(D)} p^{(0)} + \cdots + C_D^d r^{(d)} p^{(D-d)} + \cdots + C_D^0 r^{(0)} p^{(D)}](\lambda_{\gamma_{D,j}}) \\ &\quad + [C_D^D s^{(D)} q^{(0)} + \cdots + C_D^d s^{(d)} q^{(D-d)} + \cdots + C_D^0 s^{(0)} q^{(D)}](\lambda_{\gamma_{D,j}}) \\ &= [C_D^D s^{(D)} q^{(0)}](\lambda_{\gamma_{D,j}}). \text{ Where each } \lambda_{\gamma_{D,j}} \in I_{D, \alpha} \subseteq I_{D-1, \alpha}. \end{aligned}$$

Hence  $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$ . The assumption holds for all  $D \in \{0, \dots, A-1\}$ .

NOTICE that  $\forall k = \{0, \dots, A-2\}, s^{(k)}$  and  $s^{(k+1)}$  have zeros  $\{\lambda_{\gamma_{k+1,1}}, \dots, \lambda_{\gamma_{k+1,J}}\}$  in common.

Now  $\forall D \in \{1, A-1\}, s = s^{(0)}, \dots, s^{(D)}$  have zeros  $\{\lambda_{\gamma_{D,1}}, \dots, \lambda_{\gamma_{D,J}}\}$  in common.

Thus  $\forall D \in \{0, A-1\}, s(z)$  is divisible by  $(z - \lambda_{\gamma_{D,1}})^{\alpha_{\gamma_{D,1}}} \cdots (z - \lambda_{\gamma_{D,J}})^{\alpha_{\gamma_{D,J}}}$ .

Hence we write  $s(z) = \left( (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_M)^{\alpha_M} \right) s_0(z)$ , while  $\deg s \leq m-1 < m = \alpha_1 + \cdots + \alpha_M$ .

Thus by TIPS,  $s = 0$ . Following the same pattern, we conclude that  $r = 0$ .

Hence  $T$  is inje. And  $\dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) = \dim \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$  is surj. Thus  $T$  is an iso.  $\square$

COMMENT: We now prove the statement that marked by  $(\Delta)$  above.

**L1:** Prove that  $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \dots + C_k^j p^{(j)} q^{(k-j)} + \dots + C_k^0 p^{(0)} q^{(k)}$ .

SOLUTION:

We use induction by  $k \in \mathbf{N}^+$ .

(i)  $k = 1$ .  $(pq)^{(1)} = pq = C_1^1 p^{(1)} q^{(0)} + C_1^0 p^{(0)} q^{(1)}$ .

(ii)  $k \geq 2$ . Assume that for  $(pq)^{(k-1)} = C_{k-1}^{k-1} p^{(k-1)} q^{(0)} + \dots + C_{k-1}^j p^{(j)} q^{(k-1-j)} + \dots + C_{k-1}^0 p^{(0)} q^{(k-1)}$ .

$$\begin{aligned} \text{Now } (pq)^{(k)} &= ((pq)^{(k-1)})' = \left( \sum_{j=0}^{k-1} C_{k-1}^j p^{(j)} q^{(k-1-j)} \right)' = \sum_{j=0}^{k-1} \left[ C_{k-1}^j \left( p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)} \right) \right] \\ &= \left[ C_{k-1}^0 \left( \underbrace{p^{(1)} q^{(k-1)}}_{\text{}} + \boxed{p^{(0)} q^{(k)}} \right) \right] + \left[ C_{k-1}^1 \left( p^{(2)} q^{(k-2)} + \underbrace{p^{(1)} q^{(k-1)}}_{\text{}} \right) \right] \\ &\quad + \dots + \left[ C_{k-1}^{j-2} \left( \underbrace{p^{(j-1)} q^{(k-j+1)}}_{\text{}} + p^{(j-2)} q^{(k-j+2)} \right) \right] + \left[ C_{k-1}^{j-1} \left( \underbrace{p^{(j)} q^{(k-j)}}_{\text{}} + \underbrace{p^{(j-1)} q^{(k-j+1)}}_{\text{}} \right) \right] \\ &\quad + \left[ C_{k-1}^j \left( \underbrace{p^{(j+1)} q^{(k-j-1)}}_{\text{}} + \underbrace{p^{(j)} q^{(k-j)}}_{\text{}} \right) \right] + \left[ C_{k-1}^{j+1} \left( p^{(j+2)} q^{(k-j-2)} + \underbrace{p^{(j+1)} q^{(k-j-1)}}_{\text{}} \right) \right] \\ &\quad + \dots + \left[ C_{k-1}^{k-2} \left( \underbrace{p^{(k-1)} q^{(1)}}_{\text{}} + p^{(k-2)} q^{(2)} \right) \right] + \left[ C_{k-1}^{k-1} \left( \boxed{p^{(k)} q^{(0)}} + \underbrace{p^{(k-1)} q^{(1)}}_{\text{}} \right) \right]. \end{aligned}$$

$$\text{Hence } (pq)^{(k)} = C_k^0 p^{(0)} q^{(k)} + \dots + \left[ C_{k-1}^j + C_{k-1}^{j-1} \right] p^{(j)} q^{(k-j)} + \dots + C_k^k p^{(k)} q^{(0)}. \quad \square$$

**L2:** Suppose  $p(z) = (z - \lambda)^\alpha q(z)$  and  $\alpha \in \mathbf{N}^+$ . Prove that  $p^{(\alpha-1)}(\lambda) = 0$ .

SOLUTION:

Suppose  $p \in \mathcal{P}(\mathbf{F})$ . Write  $p(z) = (z - \lambda)^A q(z)$ , where  $A \in \mathbf{N}^+, q(\lambda) \neq 0$ .

We use induction to show that for all  $\alpha \in \{1, \dots, A\}, p^{(\alpha-1)}(\lambda) = 0$ .

(i)  $\alpha = 1$ .  $p^{(0)}(\lambda) = 0$ .

(ii)  $2 \leq \alpha \leq A$ . Assume that  $p^{(a-2)}(\lambda) = 0$  for all  $a \in \{1, \dots, \alpha\}$ .

NOTICE that  $p(z) = (z - \lambda)^{\alpha-1} q_{\alpha-1}(z) = (z - \lambda)^\alpha q_\alpha(z)$ , where  $q_\alpha(z) = (z - \lambda) q_{\alpha-1}(z)$ .

$$\begin{aligned} \text{Because } p^{(\alpha-1)}(z) &= \left[ C_{\alpha-1}^{\alpha-1} (z - \lambda)^0 q_{\alpha-1}(z) + \dots + C_{\alpha-1}^k (z - \lambda)^{\alpha-1-k} q_{\alpha-1-k}(z) \right. \\ &\quad \left. + \dots + C_{\alpha-1}^0 (z - \lambda)^{\alpha-1} q_{\alpha-1}^{(\alpha-1)}(z) \right]. \text{ Now } p^{(\alpha-1)}(\lambda) = C_{\alpha-1}^{\alpha-1} q_{\alpha-1}(\lambda) = 0. \quad \square \end{aligned}$$

ENDED

**5.A** [1](#) [2](#) [3](#) [4](#) [5](#) [6](#) [7](#) [8](#) [9](#) [10](#) [11](#) [12](#) [13](#) [14](#) [15](#) [16](#) [17](#) [18](#) [19](#) [20](#) [22](#) [23](#) [24](#) [25](#) [26](#) [27](#) [28](#)  
[29](#) [30](#) [31](#) [32](#) [33](#) [34](#) [35](#) [36](#) | 2E: Ch5.20 | 4E: 8, 11, 15, 16, 17, 36, 37, 38, 39

• NOTE FOR [5.6]:

More generally, suppose we do not know whether  $V$  is finite-dim. We show that  $(a) \iff (b)$ .

Suppose (a)  $\lambda$  is an eigval of  $T$  with an eigvec  $v$ . Then  $(T - \lambda I)v = 0$ .

Hence we get (b),  $(T - \lambda I)$  is not inje. And then (d),  $(T - \lambda I)$  is not inv.

But  $(d) \Rightarrow (b)$  fails, because  $S$  is not inv  $\iff S$  is not inje OR  $S$  is not surj.

• TIPS: For  $T_1, \dots, T_m \in \mathcal{L}(V)$ :

(a) Suppose  $T_1, \dots, T_m$  are all inje. Then  $(T_1 \circ \dots \circ T_m)$  is inje.

(b) Suppose  $(T_1 \circ \dots \circ T_m)$  is not inje. Then at least one of  $T_1, \dots, T_m$  is not inje.

(c) At least one of  $T_1, \dots, T_m$  is not inje  $\nRightarrow (T_1 \circ \dots \circ T_m)$  is not inje.

EXAMPLE: In infinite-dim only. Let  $V = \mathbf{F}^\infty$ .

Let  $S$  be the backward shift ( surj but not inje )  
 Let  $T$  be the forward shift ( inje but not surj )  $\Bigg\} \Rightarrow \text{Then } ST = I.$

$\square$

• **NOTE FOR [5.2]:** Suppose  $T \in \mathcal{L}(V)$ . Then  $U$  is an invar subsp of  $V$  under  $T \iff \text{range } T|_U \subseteq U$ .

• Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ , and  $U$  is an invar subsp of  $V$  under  $T$ .  
Prove that there exists an invar subsp  $W$  of dimension  $\dim V - \dim U$ .

**SOLUTION:**

Using the NOTE FOR [3.88,90,91]. Define the eraser  $S$ . Now  $V = \text{range } S \oplus U$ .

Define  $E_1$  by  $E_1(u + w) = u$ . Define  $E_2$  by  $E_2(u + w) = w$ . ( $E_2 = S \circ \pi$ .)

Note that  $T - TE_1 = T(I - E_1) = TE_2$ . And  $\text{null } TE_2 = \text{null } T \oplus U$ ,  $\text{range } T = \text{range } TE_2 \oplus U$ .

Because  $\dim \text{null } TE_2 \geq \dim U \Rightarrow \dim \text{range } TE_2 \leq \dim V - \dim U$ .

Let  $B_U = (u_1, \dots, u_n)$ ,  $B_{\text{range } TE_2} = (v_1, \dots, v_m) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n, \dots, u_p)$ .

Let  $X = \text{span}(v_1, \dots, v_m, u_{\alpha_1}, \dots, u_{\alpha_{p-\dim U}})$ . Where  $\alpha_1, \dots, \alpha_{p-\dim U} \in \{1, \dots, p\}$  are distinct.

Then  $\dim X = \dim V - \dim U$ . [ $\text{range } TE_2 \subseteq$ ]  $X$  is invar under  $TE_2$ , by Problem (1)(b).

We have  $x \in X \Rightarrow TE_2(x) \in X \Rightarrow Tx - TE_1(x) \in X \Rightarrow Tx \in X$ . Hence  $X$  is invar under  $T$ . □

(Note that  $E_1(x) \in \text{span}(u_{\beta_1}, \dots, u_{\beta_t})$ , where  $\beta_1, \dots, \beta_t \in \{\alpha_1, \dots, \alpha_{p-\dim U}\}$  and each  $u_{\beta_i} \in U$ .)

**COMMENT:** Conversely, by reversing the roles of  $U$  and  $W$ , we conclude that it is true as well.

• Suppose  $T \in \mathcal{L}(V)$  and  $U$  is an invar subsp of  $V$  under  $T$ .

Suppose  $\lambda_1, \dots, \lambda_m$  are the distinct eigvals of  $T$  correspd eigvecs  $v_1, \dots, v_m$ .

• **TIPS 1:** Prove that  $v_1 + \dots + v_m \in U \iff$  each  $v_k \in U$ .

**SOLUTION:**

Suppose each  $v_k \in U$ . Then because  $U$  is a subsp,  $v_1 + \dots + v_m \in U$ .

Define the statement  $P(k)$  : if  $v_1 + \dots + v_k \in U$ , then each  $v_j \in U$ . We use induction on  $m$ .

(i) For  $k = 1$ ,  $v_1 \in U$ .

(ii) For  $2 \leq k \leq m$ . Assume that  $P(k-1)$  holds. Suppose  $v = v_1 + \dots + v_k \in U$ .

Then  $Tv = \lambda_1 v_1 + \dots + \lambda_k v_k \in U \implies Tv - \lambda_k v = (\lambda_1 - \lambda_k)v_1 + \dots + (\lambda_{k-1} - \lambda_k)v_{k-1} \in U$ .

For each  $j \in \{1, \dots, k-1\}$ ,  $\lambda_j - \lambda_k \neq 0 \Rightarrow (\lambda_j - \lambda_k)v_j = v'_j$  is an eigvec of  $T$  correspd  $\lambda_j$ .

By assumption, each  $v'_j \in U$ . Thus  $v_1, \dots, v_{k-1} \in U$ . So that  $v_k = v - v_1 - \dots - v_{k-1} \in U$ . □

• **TIPS 2:** If  $\dim V = m$ . Prove that  $U = (U \cap E_1) \oplus \dots \oplus (U \cap E_m)$ , where  $E_k = \text{span}(v_k)$ .

**SOLUTION:**

Because  $V = E_1 \oplus \dots \oplus E_m$ .  $\forall u \in U, \exists ! e_j \in E_j, u = e_1 + \dots + e_m$ .

If  $e_j \neq 0$ , then  $e_j$  is an eigvec correspd  $\lambda_j$ . Otherwise  $e_j = 0 \in U$ . By (TIPS 1), each nonzero  $e_j \in U$ .

Thus  $u \in (U \cap E_1) + \dots + (U \cap E_m) = U$ . Because each  $(U \cap E_j) \subseteq E_j$ .

For each  $k \in \{2, \dots, n\}$ ,  $((U \cap E_1) + \dots + (U \cap E_{k-1})) \cap (U \cap E_k) \subseteq (E_1 + \dots + E_{k-1}) \cap E_k = \{0\}$  □

• **TIPS 3:** Suppose  $W$  is a nonzero invar subsp of  $V$  under  $T$ . If  $\dim V = m \geq 1$ .

Prove that  $W = \text{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$  for some distinct  $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$ .

**SOLUTION:**

Each  $\text{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$  is invar under  $T$ .

By (TIPS 2),  $U = (U \cap E_1) \oplus \dots \oplus (U \cap E_m)$ . Because each  $\dim E_k = 1$ ,  $U \cap E_k = \{0\}$  or  $E_k$ .

There must be at least one  $k$  such that  $E_k = U \cap E_k$ , for if not,  $U = \{0\}$  since  $V = E_1 \oplus \dots \oplus E_m$ .

Let  $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$  be all the distinct indices for which  $E_k = U \cap E_k$ .

Thus  $U = (U \cap E_1) \oplus \dots \oplus (U \cap E_m) = E_{\alpha_1} \oplus \dots \oplus E_{\alpha_A} = \text{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ . □

1 Suppose  $T \in \mathcal{L}(V)$  and  $U$  is a subsp of  $V$ .

(a) If  $U \subseteq \text{null } T$ , then  $U$  is invar under  $T$ .  $\forall u \in U \subseteq \text{null } T, Tu = 0 \in U$ . □

(b) If  $\text{range } T \subseteq U$ , then  $U$  is invar under  $T$ .  $\forall u \in U, Tu \in \text{range } T \subseteq U$ . □

• Suppose  $S, T \in \mathcal{L}(V)$  are such that  $ST = TS$ .

(a) Prove that  $\text{null } (T - \lambda I)$  is invar under  $S$  for any  $\lambda \in \mathbf{F}$ .

(b) Prove that  $\text{range } (T - \lambda I)$  is invar under  $S$  for any  $\lambda \in \mathbf{F}$ .

**SOLUTION:**

Note that  $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$ .

(a)  $(T - \lambda I)(v) = 0 \Rightarrow (T - \lambda I)(Sv) = (S(T - \lambda I))(v) = 0$ .

(b)  $(T - \lambda I)(u) = v \in \text{range } (T - \lambda I) \Rightarrow Sv = (S(T - \lambda I))(u) = (T - \lambda I)(Su) \in \text{range } (T - \lambda I)$ . □

• Suppose  $S, T \in \mathcal{L}(V)$  are such that  $ST = TS$ .

2 Show that  $W = \text{null } T$  is invar under  $S$ .  $\forall u \in W, Tu = 0 \Rightarrow STu = 0 = TSu \Rightarrow Su \in W$ . □

3 Show that  $U = \text{range } T$  is invar under  $S$ .  $\forall w \in U, \exists v \in V, Tv = w, TSv = STv = Sw \in U$ . □

• Suppose  $T \in \mathcal{L}(V)$  and  $V_1, \dots, V_m$  are invar subsp of  $V$  under  $T$ .

4  $\forall v_i \in V_i, Tv_i \in V_i \Rightarrow \forall v = v_1 + \dots + v_m \in V_1 + \dots + V_m, Tv = Tv_1 + \dots + Tv_m \in V_1 + \dots + V_m$ . □

5  $\forall v \in \bigcap_{i=1}^m V_i, Tv \in V_i, \forall i \in \{1, \dots, m\} \Rightarrow Tv \in \bigcap_{i=1}^m V_i$ . Thus  $\bigcap_{i=1}^m V_i$  is invar under  $T$ . □

6 Suppose  $U$  is an invar subsp of  $V$  under each  $T \in \mathcal{L}(V)$ . Show that  $U = \{0\}$  or  $U = V$ .

**SOLUTION:** If  $V = \{0\}$ . Then we are done. Suppose  $V \neq \{0\}$ . We show the contrapositive:

Suppose  $U \neq \{0\}$  and  $U \neq V$ . Prove that  $\exists T \in \mathcal{L}(V)$  such that  $U$  is not invar under  $T$ .

Let  $W$  be such that  $V = U \oplus W$ . Define  $T \in \mathcal{L}(V)$  by  $T(u + w) = w$ . □

• **TIPS:** Suppose  $T \in \mathcal{L}(\mathbf{R}^2)$  is the counterclockwise rotation by the angle  $\theta \in \mathbf{R}$ .

Define  $\mathcal{C} \in \mathcal{L}(\mathbf{R}^2, \mathbf{C})$  by  $\mathcal{C}(a, b) = a + ib = r(\cos \alpha + i \sin \alpha) \Rightarrow a = r \cos \alpha, b = r \sin \alpha$ , where  $r = a^2 + b^2$ .

Then  $(\cos \theta + i \sin \theta)(a + ib) = r(\cos(\alpha + \theta) + i \sin(\alpha + \theta)) = \mathcal{C}^{-1}T(a, b)$ .

Hence  $T(a, b) = (a \cos \theta - b \sin \theta, a \sin \theta + b \cos \theta)$ . Now  $\mathcal{M}(T) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ .

**EXAMPLE:** OR 7 Suppose  $T \in \mathcal{L}(\mathbf{R}^2)$  is defined by  $T(x, y) = (-3y, x)$ . Find all eigvals of  $T$ .

NOTICE that  $\mathcal{M}(T) = \begin{pmatrix} \cos 90^\circ & -3 \sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$ . By [5.8](a), we conclude that  $T$  has no eigvals.

OR. Suppose  $\lambda$  is an eigval with an eigvec  $(x, y)$ . Then  $(\lambda x, \lambda y) = (-3y, x) \Rightarrow -3y = \lambda^2 y \Rightarrow \lambda^2 = -3$ .

[ Ignoring the possibility of  $y = 0$ , because  $x = 0 \Leftrightarrow y = 0$ . ] □

8 Define  $T \in \mathcal{L}(\mathbf{F}^2)$  by  $T(w, z) = (z, w)$ . Find all eigvals and eigvecs.

**SOLUTION:** Suppose  $\lambda$  is an eigval with an eigvec  $(w, z)$ . Then  $z = \lambda w$  and  $w = \lambda z$ .

Thus  $z = \lambda^2 z \Rightarrow \lambda^2 = 1$ , ignoring the possibility of  $z = 0$  ( $z = 0 \Leftrightarrow w = 0$ ).

Hence  $\lambda_1 = -1$  and  $\lambda_2 = 1$  are all the eigvals of  $T$ . And  $T(z, z) = (z, z), T(z, -z) = (-z, z)$ .

又  $\dim \mathbf{F}^2 = 2$ . Thus the set of all eigvecs is  $\{(z, z), (z, -z) : z \neq 0\}$ . □

**9** Define  $T \in \mathcal{L}(\mathbf{F}^3)$  by  $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$ . Find all eigvals and eigvecs.

**SOLUTION:** Suppose  $\lambda$  is an eigval with an eigvec  $(z_1, z_2, z_3)$ .

Then  $(2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$ . We discuss in two cases:

For  $\lambda = 0$ ,  $z_2 = z_3 = 0$  and  $z_1$  can be arbitrary ( $z_1 \neq 0$ ).

For  $\lambda \neq 0$ ,  $z_2 = 0 = z_1$ , and  $z_3$  can be arbitrary ( $z_3 \neq 0$ ), then  $\lambda = 5$ .

The set of all eigvecs is  $\{(0, 0, w), (w, 0, 0) : w \neq 0\}$ . □

**10** Define  $T \in \mathcal{L}(\mathbf{F}^n)$  by  $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$

(a) Find all eigvals and eigvecs; (b) Find all invar subsp of  $V$  under  $T$ .

**SOLUTION:**

(a) Suppose  $x = (x_1, x_2, x_3, \dots, x_n)$  is an eigvec with an eigval  $\lambda$ .

Then  $Tx = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$ .

Hence  $1, \dots, n$  of length  $\dim \mathbf{F}^n$  are all the eigvals.

And  $\{(0, \dots, 0, x_k, 0, \dots, 0) \in \mathbf{F}^n : x_k \neq 0, k = 1, \dots, n\}$  is the set of all eigvecs.

(b) Let  $(e_1, \dots, e_n)$  be the standard basis of  $\mathbf{F}^n$ . Let  $V_k = \text{span}(e_k)$ . Then  $V_1, \dots, V_n$  are invar under  $T$ .

Hence by (TIPS 3), every sum of  $V_1, \dots, V_n$  is a invar subsp of  $V$  under  $T$ . □

**18** Define the forward shift operator  $T \in \mathcal{L}(\mathbf{F}^\infty)$  by  $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$ .

Show that  $T$  has no eigvals.

**SOLUTION:** Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $(z_1, z_2, \dots)$ .

Then  $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$ . Thus  $\lambda z_1 = 0, \lambda z_k = z_{k-1}$ .

If  $\lambda = 0$ , then  $\lambda z_2 = z_1 = 0 = \dots = z_k \Rightarrow (z_1, z_2, \dots) = 0 \Rightarrow 0$  is not an eigval.

If  $\lambda \neq 0$ , then  $\lambda z_1 = 0 \Rightarrow z_1 = \dots = z_k = 0 \Rightarrow \lambda$  is not an eigval. Now no  $\lambda \in \mathbf{F}$  is an eigval. □

**19** Suppose  $n \in \mathbf{N}^+$ . Define  $T \in \mathcal{L}(\mathbf{F}^n)$  by  $T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$ .

In other words, the entries of  $\mathcal{M}(T)$  with resp to the standard basis are all 1's.

Find all eigvals and eigvecs of  $T$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $(x_1, \dots, x_n)$ .

Then  $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$ .

Thus  $\lambda x_1 = \dots = \lambda x_n = x_1 + \dots + x_n$ .

For  $\lambda = 0$ ,  $x_1 + \dots + x_n = 0$  }  $\Rightarrow 0, n$  are the eigvals of  $T$ .

For  $\lambda \neq 0$ ,  $x_1 = \dots = x_n \Rightarrow \lambda x_k = nx_k$  }

And the set of all eigvecs of  $T$  is  $\{(x_1, \dots, x_n) \in \mathbf{F}^n \setminus \{0\} : x_1 + \dots + x_n = 0 \vee x_1 = \dots = x_n\}$ . □

**20** Define the backward shift operator  $S \in \mathcal{L}(\mathbf{F}^\infty)$  by  $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$ .

(a) Show that every element of  $\mathbf{F}$  is an eigval of  $S$ ; (b) Find all eigvecs of  $S$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigval of  $S$  with an eigvec  $(z_1, z_2, \dots)$ .

Then  $S(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$ . Thus for each  $k \in \mathbf{N}^+$ ,  $\lambda z_k = z_{k+1}$ .

If  $\lambda = 0$ , then  $\lambda z_1 = z_2 = \dots = z_k = 0$  for all  $k$ , while  $z_1$  can be nonzero. Thus 0 is an eigval.

If  $\lambda \neq 0$ , then  $\lambda^k z_1 = \lambda^{k-1} z_2 = \dots = \lambda z_k = z_{k+1}$ , let  $z_1 \neq 0 \Rightarrow (1, \lambda, \lambda^2, \dots, \lambda^k, \dots)$  is an eigvec.

Now each  $\lambda \in \mathbf{F}$  is an eigval of  $T$ , with the corresp eigvecs in  $\text{span}((1, \lambda, \lambda^2, \dots, \lambda^k, \dots))$ . □

**11** Define  $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$  by  $Tp = p'$ . Find all eigvals and eigvecs.

**SOLUTION:**

Note that  $\forall p \in \mathcal{P}(\mathbf{R}) \setminus \{0\}, \deg p' < \deg p$ . And  $\deg 0 = -\infty$ . Suppose  $\lambda$  is an eigval with an eigvec  $p$ . Assume that  $\lambda \neq 0$ . Then  $\deg \lambda p > \deg p'$  while  $\lambda p = p'$ . Contradicts. Thus  $\lambda = 0$ .  
Therefore  $\deg \lambda p = -\infty = \deg p' \Rightarrow p \in \mathcal{P}_0(\mathbf{R})$ . Hence the eigvecs are all the nonzero consts.  $\square$

**12** Define  $T \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$  by  $(Tp)(x) = xp'(x)$  for all  $x \in \mathbf{R}$ . Find all eigvals and eigvecs.

**SOLUTION:**

Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $p$ , then  $(Tp)(x) = xp'(x) = \lambda p(x)$ .  
Let  $p = a_0 + a_1x + \dots + a_nx^n$ . Then  $xp'(x) = a_1x + 2a_2x^2 + \dots + na_nx^n = \lambda a_0 + \lambda a_1x + \lambda a_2x^2 + \dots + \lambda a_nx^n$ .  
Define  $S \in \mathcal{L}(\mathbf{F}^{n+1}, \mathcal{P}_n(\mathbf{R}))$  by  $S(a_0, a_1, \dots, a_n) = a_0 + a_1x + \dots + a_nx^n$ .  
Then  $(S^{-1}TS)(a_0, a_1, \dots, a_n) = (0 \cdot a_0, 1 \cdot a_1, 2 \cdot a_2, \dots, n \cdot a_n)$ . Thus  $0, 1, \dots, n$  are the eigvals of  $S^{-1}TS$ .  
By Problem (15),  $0, 1, \dots, n$  are the eigvals of  $T$ . The set of all eigvecs is  $\{cx^\lambda : c \neq 0, \lambda = 0, 1, \dots, n\}$ .  $\square$

• Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$ .

**13** Prove that  $\forall \lambda \in \mathbf{F}, \exists \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}, (T - \alpha I)$  is inv.

**SOLUTION:**

Let  $\alpha_k \in \mathbf{F}$  be such that  $|\alpha_k - \lambda| = \frac{1}{1000+k}$  for each  $k = 1, \dots, \dim V + 1$ .  
Note that each  $T \in \mathcal{L}(V)$  has at most  $\dim V$  distinct eigvals.  
Hence  $\exists k = 1, \dots, \dim V + 1$  such that  $\alpha_k$  is not an eigval of  $T$  and therefore  $(T - \alpha_k I)$  is inv.  $\square$

• (4E 5.A.11) Prove that  $\exists \delta > 0$  such that  $(T - \alpha I)$  is inv for all  $\alpha \in \mathbf{F}$  such that  $0 < |\alpha - \lambda| < \delta$ .

**SOLUTION:**

If  $T$  has no eigvals, then  $(T - \alpha I)$  is inje for all  $\alpha \in \mathbf{F}$  and we are done.  
Suppose  $\lambda_1, \dots, \lambda_m$  are all the distinct eigvals of  $T$ .  
Let  $\delta > 0$  be such that, for each eigval  $\lambda_k, \lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$ .  
So that for all  $\alpha \in \mathbf{F}$  such that  $0 < |\alpha - \lambda| < \delta, (T - \alpha I)$  is not inje.  $\square$   
OR. Let  $\delta = \min\{|\lambda - \lambda_k| : k \in \{1, \dots, m\}, \lambda_k \neq \lambda\}$ .  
Then  $\delta > 0$  and each  $\lambda_k \neq \alpha$  [  $\Leftrightarrow (T - \alpha I)$  is inv ] for all  $\alpha \in \mathbf{F}$  such that  $0 < |\alpha - \lambda| < \delta$ .  $\square$

• (5.B.4 OR 4E 3.B.27) Suppose  $\lambda$  is an eigval of  $P \in \mathcal{L}(V), P^2 = P$ . Prove that  $\lambda = 0$  or  $\lambda = 1$ .

**SOLUTION:** Suppose  $\lambda$  is an eigval with an eigvec  $v$ . Then  $P(Pv) = Pv \Rightarrow \lambda^2 v = \lambda v$ . Thus  $\lambda = 1$  or  $0$ .  $\square$

**14** Suppose  $V = U \oplus W$ , where  $U$  and  $W$  are nonzero subsp of  $V$ .

Define  $P \in \mathcal{L}(V)$  by  $P(u + w) = u$  for each  $u \in U$  and each  $w \in W$ .

Find all eigvals and eigvecs of  $P$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigval of  $P$  with an eigvec  $(u + w)$ .  
Then  $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$ .  
OR. Note that  $P|_{\text{range } P} = I|_{\text{range } P} \Leftrightarrow P^2 = P$ . By (4E 5.A.8), 1 and 0 are the eigvals.  
By [1.44],  $(\lambda - 1)u = \lambda w = 0$ , hence  $\lambda = 0 \Leftrightarrow u = 0$ , and  $\lambda = 1 \Leftrightarrow w = 0$ .  
Thus  $Pu = u, Pw = 0$ . Hence the eigvals are 0 and 1, the set of all eigvecs of  $P$  is  $U \cup W$ .  $\square$

**15** Suppose  $T \in \mathcal{L}(V)$ . Suppose  $S \in \mathcal{L}(V)$  is inv.

(a) Prove that  $T$  and  $S^{-1}TS$  have the same eigvals.

(b) What is the relationship between the eigvecs of  $T$  and the eigvecs of  $S^{-1}TS$ ?

**SOLUTION:**

(a)  $\lambda$  is an eigval of  $T$  with an eigvec  $v \Rightarrow S^{-1}TS(\underline{S^{-1}v}) = S^{-1}Tv = S^{-1}(\lambda v) = \underline{\lambda S^{-1}v}$ .

$\lambda$  is an eigval of  $S^{-1}TS$  with an eigvec  $v \Rightarrow S(S^{-1}TS)v = T\underline{Sv} = \underline{\lambda Sv}$ .

OR. Note that  $S(S^{-1}TS)S^{-1} = T$ . Hence every eigval of  $S^{-1}TS$  is an eigval of  $S(S^{-1}TS)S^{-1} = T$ .

OR.  $Tv = \lambda v \Leftrightarrow (TS)(u) = \lambda Su \Leftrightarrow (S^{-1}TS)(u) = \lambda u$ . Where  $v = Su$ .

$(S^{-1}TS)(u) = \lambda u \Leftrightarrow (S^{-1}T)(v) = \lambda S^{-1}v \Leftrightarrow Tv = \lambda v$ . Where  $u = S^{-1}v$ .

(b) Because  $\lambda$  is an eigval of  $T \Leftrightarrow \lambda$  is an eigval of  $S^{-1}TS$ .

( See [5.36]. ) Now  $E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\}; E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}$ .  $\square$

**17** Give an example of an operator on  $\mathbb{R}^4$  that has no real eigvals.

**SOLUTION:**

Let  $(e_1, e_2, e_3, e_4)$  be the standard basis of  $\mathbb{R}^4$ .

Define  $T \in \mathcal{L}(\mathbb{R}^4)$  by  $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$ .

Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $(x, y, z, w)$ . Then we get 
$$\begin{cases} (1 - \lambda)x + y + z + w = 0, \\ -x + (1 - \lambda)y - z - w = 0, \\ 3x + 8y + (11 - \lambda)z + 5w = 0, \\ 3x - 8y - 11z + (5 - \lambda)w = 0. \end{cases}$$

This set of linear equations has no solutions.

[ You can type it on <https://zh.numberempire.com/equationsolver.php> to check. ]

OR. Define  $T \in \mathcal{L}(\mathbb{R}^4)$  by  $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$ .

Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $(x, y, z, w)$ .

Then  $T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \Rightarrow \begin{cases} -y = \lambda x, x = \lambda y \Rightarrow -xy = \lambda^2 xy \\ -w = \lambda z, z = \lambda w \Rightarrow -zw = \lambda^2 zw \end{cases}$

If  $xy \neq 0$  or  $zw \neq 0$ , then  $\lambda^2 = -1$ , we fail.

Otherwise,  $xy = 0 \Rightarrow x = y = 0$ , for if  $x \neq 0$ , then  $\lambda = 0 \Rightarrow x = 0$ , contradicts.

Similarly,  $y = z = w = 0$ . Then we fail. Thus  $T$  has no eigvals.  $\square$

• (4E 5.A.16) Suppose  $B_V = (v_1, \dots, v_n)$ ,  $T \in \mathcal{L}(V)$ ,  $\mathcal{M}(T, (v_1, \dots, v_n)) = A$ .  
Prove that if  $\lambda$  is an eigval of  $T$ , then  $|\lambda| \leq n \max\{|A_{j,k}| : 1 \leq j, k \leq n\}$ .

**SOLUTION:**

Suppose  $v$  is an eigval of  $T$  correspd to  $\lambda$ . Let  $v = c_1 v_1 + \dots + c_n v_n$ .

Because  $\lambda c_1 v_1 + \dots + \lambda c_n v_n = c_1 T v_1 + \dots + c_n T v_n = \sum_{k=1}^n c_k (\sum_{j=1}^n A_{j,k} v_j)$ .

We have  $\lambda c_j = \sum_{k=1}^n c_k A_{j,k} \Rightarrow |\lambda| |c_j| = \sum_{k=1}^n |c_k| |A_{j,k}|$  for each  $j \in \{1, \dots, n\}$

Let  $|c_j| = \max\{|c_1|, \dots, |c_n|\}$ . Note that  $|c_j| \neq 0$ , for if not,  $c_1 = \dots = c_n = 0 \Rightarrow v = 0$ , contradicts.

Let  $M = \max\{|A_{j,k}| : 1 \leq j, k \leq n\}$ . Note that for each  $j$ ,  $\sum_{k=1}^n |A_{j,k}| \leq \sum_{k=1}^n M = nM$ .

Thus  $|\lambda| |c_j| = \sum_{k=1}^n |c_k| |A_{j,k}| \Rightarrow |\lambda| \leq \sum_{k=1}^n |A_{j,k}| \frac{|c_k|}{|c_j|} \leq \sum_{k=1}^n |A_{j,k}| \leq nM$ .  $\square$

- (4E 5.A.15) Suppose  $T \in \mathcal{L}(V)$ , and  $\lambda \in \mathbb{F}$ .

Show that  $\lambda$  is an eigval of  $T \iff \lambda$  is an eigval of the dual operator  $T' \in \mathcal{L}(V')$ .

**SOLUTION:**

- (a) Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $v$ .

Let  $U$  be invar such that  $V = \text{span}(v) \oplus U$  [ by (4E 5.A.39) ].

Define  $\psi \in V'$  by  $\psi(cv + u) = c$ .

Now  $[T'(\psi)](cv + u) = \psi(cv + Tu) = \lambda cv = \lambda\psi(cv + u)$ . Hence  $T'(\psi) = \lambda\psi$ .

- (b) Suppose  $\lambda$  is an eigval  $T'$  with an eigvec  $\psi$ . Then  $T'(\psi) = \psi \circ T = \lambda\psi$ .

Note that  $\psi \neq 0, \psi(Tv) = \lambda\psi(v)$  Thus  $\exists v \in V \setminus \{0\}, Tv = \frac{\psi(Tv)}{\psi(v)}v = \lambda v$ . □

OR. [Only in Finite-dim] Using [5.6], (4E 3.F.17), [3.101] and (3.F.12).

$\lambda$  is an eigval of  $T \iff (T - \lambda I_V)$  is not inv

$\iff (T - \lambda I_V)' = T' - \lambda I_{V'},$  is not inv  $\iff \lambda$  is an eigval of  $T'$ . □

**24** Suppose  $A \in \mathbb{F}^{n,n}$ . Define  $T \in \mathcal{L}(\mathbb{F}^{n,1})$  by  $Tx = Ax$ .

- (a) Suppose the sum of the entries in each row of  $A$  equals 1. Prove that 1 is an eigval of  $T$ .

- (b) Suppose the sum of the entries in each col of  $A$  equals 1. Prove that 1 is an eigval of  $T$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $x$ . Then  $Tx = Ax = \begin{pmatrix} \sum_{k=1}^n A_{1,k}x_k \\ \vdots \\ \sum_{k=1}^n A_{n,k}x_k \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ .

- (a) Suppose  $\sum_{r=1}^n A_{R,c} = 1$  for each  $R \in \{1, \dots, n\}$ .

Then if we let  $x_1 = \dots = x_n$ , then  $\lambda = 1$ , and hence is an eigval of  $T$ .

- (b) Suppose  $\sum_{r=1}^n A_{r,C} = 1$  for each  $C \in \{1, \dots, n\}$ .

Then  $\sum_{r=1}^n (Ax)_{r,\cdot} = \sum_{r=1}^n (Ax)_{r,1} = \sum_{c=1}^n (A_{1,c} + \dots + A_{n,c})x_c = \sum_{c=1}^n x_c = \lambda(x_1 + \dots + x_n)$ .

Hence  $\lambda = 1$  for all  $x \in \mathbb{F}^{n,1}$  such that  $\sum_{c=1}^n x_{c,1} \neq 0$ . □

OR. We show that  $(T - I)$  is not inv, so that  $\lambda = 1$  is an eigval.

Because  $(T - I)x = (A - I)x = \begin{pmatrix} \sum_{r=1}^n A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^n A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ .

Then  $y_1 + \dots + y_n = \sum_{r=1}^n \sum_{c=1}^n (A_{r,c}x_c - x_r) = \sum_{c=1}^n x_c \sum_{r=1}^n A_{r,c} - \sum_{r=1}^n x_r = 0$ .

Thus  $\text{range}(T - I) \subseteq \left\{ \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix}^t \in \mathbb{F}^{n,1} : y_1 + \dots + y_n = 0 \right\}$ . Hence  $(T - I)$  is not surj. □

OR. Let  $(e_1, \dots, e_n)$  be the standard basis of  $\mathbb{F}^{n,1}$ . Define  $\psi \in (\mathbb{F}^{n,1})'$  by  $\psi(e_k) = 1$ .

Thus  $(\psi \circ (T - I))(e_k) = \psi\left(\left(\sum_{j=1}^n A_{j,k}e_j\right) - e_k\right) = \left(\sum_{j=1}^n A_{j,k}\right) - 1 = 0$ .

Which means that  $\psi \circ (T - I) = 0$ . 又  $\psi \neq 0$ . Hence  $(T - I)$  is not inje. □

OR. Define  $S \in \mathcal{L}(\mathbb{F}^{n,1})$  by  $Sx = A^t x$ . Because the rows of  $A^t$  are the cols of  $A$ .

Now by (a), 1 is an eigval of  $S$ . Let  $(\varphi_1, \dots, \varphi_n)$  be the dual basis of  $(e_1, \dots, e_n)$ .

Define  $\Phi \in \mathcal{L}(\mathbb{F}^{n,1}, (\mathbb{F}^{1,n})')$  by  $\Phi(e_k) = \varphi_k$ . Note that  $\mathcal{M}(T') = A^t$ .

Now  $(\Phi^{-1}T'\Phi)(e_k) = (\Phi^{-1}T')(\varphi_k) = \Phi^{-1}\left(\sum_{j=1}^n A_{k,j}\varphi_j\right) = \sum_{j=1}^n A_{k,j}e_j = A^t e_k = S e_k$ .

Thus 1 is an eigval of  $S = \Phi^{-1}T'\Phi$ , so of  $T'$ , [ by Problem (15) ], so of  $T$ , [ by (4E 5.A.15) ]. □



• Suppose  $A \in \mathbf{F}^{n,n}$ . Define  $T \in \mathcal{L}(\mathbf{F}^{1,n})$  by  $Tx = xA$ .

- (a) Suppose the sum of the entries in each col of  $A$  equals 1. Prove that 1 is an eigval of  $T$ .  
(b) Suppose the sum of the entries in each row of  $A$  equals 1. Prove that 1 is an eigval of  $T$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigval with an eigvec  $x$ . Then  $(\sum_{r=1}^n x_r A_{r,1} \quad \cdots \quad \sum_{r=1}^n x_r A_{r,n}) = \lambda(x_1 \quad \cdots \quad x_n)$ .

(a) Suppose  $\sum_{r=1}^n A_{r,C} = 1$  for each  $C \in \{1, \dots, n\}$ .

Thus if  $x_1 = \cdots = x_n$ , then  $\lambda = 1$ , hence 1 is an eigval of  $T$ .

(b) Suppose  $\sum_{c=1}^n A_{R,c} = 1$  for each  $R \in \{1, \dots, n\}$ .

Thus  $\sum_{c=1}^n (xA)_{.,c} = \sum_{c=1}^n (A_{c,1} + \cdots + A_{c,n})x_c = \sum_{c=1}^n x_c = \lambda(x_1 + \cdots + x_n)$ .

Hence  $\lambda = 1$ , for all  $x$  such that  $\sum_{r=1}^n x_{1,r} \neq 0$ . □

OR. We show that  $(T - I)$  is not inv, so that  $\lambda = 1$  is an eigval.

Because  $(T - I)x = x(A - \mathcal{M}(I)) = (\sum_{c=1}^n x_c A_{c,1} - x_1 \quad \cdots \quad \sum_{c=1}^n x_c A_{c,n} - x_n) = (y_1 \quad \cdots \quad y_n)$ .

Then  $y_1 + \cdots + y_n = \sum_{c=1}^n \sum_{r=1}^n (x_r A_{r,c} - x_c) = \sum_{r=1}^n x_r \sum_{c=1}^n A_{r,c} - \sum_{c=1}^n x_c = 0$ .

Thus  $\text{range}(T - I) \subseteq \{(y_1 \quad \cdots \quad y_n) \in \mathbf{F}^{1,n} : y_1 + \cdots + y_n = 0\}$ . Hence  $(T - I)$  is not surj. □

OR. Let  $(e_1, \dots, e_n)$  be the standard basis of  $\mathbf{F}^{1,n}$ . Define  $\psi \in (\mathbf{F}^{n,1})'$  by  $\psi(e_k) = 1$ .

Because  $Te_k = e_k A = (A_{k,1} \quad \cdots \quad A_{k,n}) = \sum_{i=1}^n A_{k,i} e_i$ . **COROLLARY:**  $\mathcal{M}(T) = A^t$ .

$(\psi \circ (T - I))(e_k) = (\sum_{i=1}^n A_{k,i}) - 1 = 0$ . Then  $\psi \circ (T - I) = 0$ .  $\nexists \psi \neq 0$ .  $(T - I)$  is not inje. □

OR. Define  $S \in \mathcal{L}(\mathbf{F}^{1,n})$  by  $Sx = xA^t$ . Because the rows of  $A$  are the cols of  $A^t$ .

Now by (a), 1 is an eigval of  $S$ . Let  $(\varphi_1, \dots, \varphi_n)$  be the dual basis of  $(e_1, \dots, e_n)$ .

Define  $\Phi \in \mathcal{L}(\mathbf{F}^{1,n}, (\mathbf{F}^{1,n})')$  by  $\Phi(e_k) = \varphi_k$ . Because  $[T'(\varphi_k)](e_j) = \varphi_k(\sum_{i=1}^n A_{j,i} e_i) = A_{j,k}$ .

By (3.F.9),  $T'(\varphi_k) = \sum_{j=1}^n A_{j,k} \varphi_j$ . **COROLLARY:**  $\mathcal{M}(T') = A = \mathcal{M}(T)^t$ .

Now  $(\Phi^{-1}T'\Phi)(e_k) = (\Phi^{-1}T')(\varphi_k) = \Phi^{-1}(\sum_{j=1}^n A_{j,k} \varphi_j) = \sum_{j=1}^n A_{j,k} e_j = e_k A^t = S e_k$ .

Thus 1 is an eigval of  $S = \Phi^{-1}T'\Phi$ , so of  $T'$ , [ by Problem (15) ], so of  $T$ , [ by (4E 5.A.15) ]. □

• Suppose  $\mathbf{F} = \mathbf{R}$ ,  $T \in \mathcal{L}(V)$ .

(a) [OR (9.11)]  $\lambda \in \mathbf{R}$ . Prove that  $\lambda$  is an eigval of  $T \iff \lambda$  is an eigval of  $T_{\mathbf{C}}$ .

(b) [OR 16 OR [9.16]]  $\lambda \in \mathbf{C}$ . Prove that  $\lambda$  is an eigval of  $T_{\mathbf{C}} \iff \bar{\lambda}$  is an eigval of  $T_{\mathbf{C}}$ .

**SOLUTION:**

(a) Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $v$ .

Then  $Tv = \lambda v \implies T_{\mathbf{C}}(v + i0) = Tv + iT0 = \lambda v$ . Thus  $\lambda$  is an eigval of  $T_{\mathbf{C}}$ .

Suppose  $\lambda$  is an eigval of  $T_{\mathbf{C}}$  with an eigvec  $v + iu$ .

Then  $T_{\mathbf{C}}(v + iu) = \lambda v + i\lambda u \implies Tv = \lambda v, Tu = \lambda u$ . Thus  $\lambda$  is an eigval of  $T$ .

( Note that  $v + iu$  is nonzero  $\iff$  at least one of  $v, u$  is nonzero ).

(b) Suppose  $\lambda$  is an eigval of  $T_{\mathbf{C}}$  with an eigvec  $v + iu$ . Then  $T_{\mathbf{C}}(v + iu) = Tv + iTu = \lambda(v + iu)$ .

Note that  $\overline{T_{\mathbf{C}}(v + iu)} = \overline{Tv + iTu} = Tv - iTu = T_{\mathbf{C}}(v - iu) = T_{\mathbf{C}}(\overline{v + iu})$ .

And that  $\lambda(v + iu) = \bar{\lambda}v - i\bar{\lambda}u = \bar{\lambda}(v - iu) = \bar{\lambda}(\overline{v + iu})$ .

Hence  $\bar{\lambda}$  is an eigval of  $T_{\mathbf{C}}$ . To prove the other direction, notice that  $\overline{\bar{\lambda}} = \lambda$ . □

OR. Suppose  $\lambda = a + ib$  is an eigval of  $T_{\mathbf{C}}$  with an eigvec  $v + iu$ .

Because  $T_{\mathbf{C}}(v + iu) = \lambda(v + iu) = (av - bu) + i(au + bv) = Tv + iTu \implies Tv = av - bu, Tu = au + bv$ .

Now  $T_{\mathbf{C}}(\overline{v + iu}) = Tv - iTu = (av - bu) - i(au + bv) = (a - ib)(v - iu) = \bar{\lambda}(\overline{v + iu})$ . Similarly □

**21** Suppose  $T \in \mathcal{L}(V)$  is inv.

(a) Suppose  $\lambda \in \mathbb{F}$  with  $\lambda \neq 0$ . Prove that  $\lambda$  is an eigval of  $T \iff \lambda^{-1}$  is an eigval of  $T^{-1}$ .

(b) Prove that  $T$  and  $T^{-1}$  have the same eigvecs.

**SOLUTION:** (a)  $Tv = \lambda v \iff v = \lambda T^{-1}v \iff \lambda^{-1}v = T^{-1}v$ . Where  $v \neq 0$ .

(b) NOTICE that  $T$  is inv  $\implies 0$  is not an eigval of  $T$  or  $T^{-1}$ . By (a), immediately.  $\square$

**22** Suppose  $T \in \mathcal{L}(V)$  and  $\exists$  nonzero vecs  $u, w$  in  $V$  such that  $Tu = 3w$ ,  $Tw = 3u$ .

Prove that 3 or  $-3$  is an eigval of  $T$ .

**SOLUTION:**  $T(u+w) = 3(u+w)$ ,  $T(u-w) = 3(w-u) = -3(u-w)$ . Note that  $u-w \neq 0$  or  $u+w \neq 0$ .

OR.  $T(Tu) = 9u \implies T^2 - 9 = (T-3I)(T+3I)$  is not injective  $\implies 3$  or  $-3$  is an eigval.  $\square$

**23** Suppose  $S, T \in \mathcal{L}(V)$ . Prove that  $ST$  and  $TS$  have the same eigvals.

**SOLUTION:** Suppose  $\lambda$  is an eigval of  $ST$  with an eigvec  $v$ . Then  $T(STv) = \lambda Tv = TS(Tv)$ .

If  $Tv = 0$  ( while  $v \neq 0$  ), then  $T$  is not inje  $\implies (TS - 0I)$  and  $(ST - 0I)$  are not inje.

Thus  $\lambda = 0$  is an eigval of  $ST$  and  $TS$  with the same eigvec  $v$ .

Otherwise,  $Tv \neq 0$ , then  $\lambda$  is an eigval of  $TS$ . Reversing the roles of  $T$  and  $S$ .  $\square$

• (2E 20) Suppose  $T \in \mathcal{L}(V)$  has  $\dim V$  distinct eigvals and  $S \in \mathcal{L}(V)$  has the same eigvecs ( but might not with the same eigvals ). Prove that  $ST = TS$ .

**SOLUTION:** Let  $n = \dim V$ . For each  $j \in \{1, \dots, n\}$ , let  $v_j$  be an eigvec with eigval  $\lambda_j$  of  $T$  and  $\alpha_j$  of  $S$ .

Then  $B_V = (v_1, \dots, v_n)$ . Because  $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$  for each  $j$ . Hence  $ST = TS$ .  $\square$

• (4E 5.A.37) Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

Define  $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$  by  $\mathcal{A}(S) = TS$  for each  $S \in \mathcal{L}(V)$ .

Prove that the set of eigvals of  $T$  equals the set of eigvals of  $\mathcal{A}$ .

**SOLUTION:**

(a) Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $v = v_1$ . Let  $B_V = (v_1, \dots, v_m, \dots, v_n)$ .

Note that  $\text{span}(v) \subseteq \text{null}(T - \lambda I)$ . Define  $S \in \mathcal{L}(V)$  by  $S(v_j) = v$  for each  $j \in \{1, \dots, n\}$ .

OR. Define  $S \in \mathcal{L}(V)$  by  $Sv_1 = v_1$ ,  $Sv_j = 0$  for  $j \geq 2$ . Then  $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$ .

Then  $(T - \lambda I)S = 0$ . Thus  $\mathcal{A}(S) = TS = \lambda S$  while  $S \neq 0$ . Hence  $\lambda$  is an eigval of  $\mathcal{A}$ .

(b) Suppose  $\lambda$  is an eigval of  $\mathcal{A}$  with an eigvec  $S$ .

Then  $\exists v \in V, 0 \neq u = S(v) \in V \implies Tu = (TS)v = (\lambda S)v = \lambda u$ . Thus  $\lambda$  is an eigval  $T$ .

OR. Because  $TS - \lambda S = (T - \lambda I)S = 0 \implies \{0\} \subsetneq \text{range } S \subseteq \text{null}(T - \lambda I)$ .  $(T - \lambda I)$  is not inje.  $\square$

**COMMENT:** If  $\mathcal{A}(S) = ST, \forall S \in \mathcal{L}(V)$ . Then the eigvals of  $\mathcal{A}$  are not the eigvals of  $T$ .

**25** Suppose  $T \in \mathcal{L}(V)$  and  $u, w$  are eigvecs of  $T$  such that  $u + w$  is also an eigvec of  $T$ .

Prove that  $u$  and  $w$  correspd to the same eigval.

**SOLUTION:** Suppose  $\lambda_1, \lambda_2, \lambda_0$  are eigvals of  $T$  with eigvecs to  $u, w, u + w$  respectively.

Then  $T(u + w) = \lambda_0(u + w) = Tu + Tw = \lambda_1 u + \lambda_2 w \implies (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$ .

If  $(u, w)$  is linely depe, then let  $w = cu$ , therefore  $\lambda_2 cu = Tw = cTu = \lambda_1 cu \implies \lambda_2 = \lambda_1$ .

Otherwise,  $(u, w)$  is linely inde. Then  $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \implies \lambda_1 = \lambda_2 = \lambda_0$ .  $\square$

OR. Assume that  $\lambda_1 \neq \lambda_2$ . Then  $(u, w)$  is linely inde. Thus  $\lambda_0 - \lambda_1 = \lambda_0 - \lambda_2$ . Contradicts  $\square$

**26** Suppose  $T \in \mathcal{L}(V)$  is such that every nonzero vec in  $V$  is an eigvec of  $T$ .

Prove that  $T$  is a scalar multi of the identity operator.

**SOLUTION:** If  $\dim V = 0, 1$  then we are done. Suppose  $\dim V \geq 2$ .

Because  $\forall v \in V, \exists! \lambda_v \in \mathbb{F}, Tv = \lambda_v v$ . For any two distinct nonzero vecs  $v, w \in V$ ,  
 $T(v + w) = \lambda_{v+w}(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$ . □

OR. For any two nonzero vecs  $u, v \in V, u, v$  are eigvecs.

If  $u + v \neq 0$ , then  $u + v$  is also an eigvec. Otherwise,  $u + v = 0$ , then  $Tu = -Tv = \lambda u = -\lambda v$ .

Thus by Problem (25),  $\forall u, v \in V, Tu = \lambda u, Tv = \lambda v \Rightarrow \forall v \in V, Tv = \lambda v$ . □

**27, 28** Suppose  $V$  is finite-dim and  $k \in \{1, \dots, \dim V - 1\}$ .

Suppose  $T \in \mathcal{L}(V)$  is such that every subsp of  $V$  of dim  $k$  is invar under  $T$ .

Prove that  $T$  is a scalar multi of the identity operator.

**SOLUTION:** If  $\dim V \leq 1$  then we are done. Suppose  $\dim V \geq 2$ .

We prove the contrapositive: If  $T$  is not a scalar multi of  $I$ . Then  $\exists$  subsp  $U$  of dim  $k$  not invar under  $T$ .

By Problem (26),  $\exists v \in V$  and  $v \neq 0$  such that  $v$  is not an eigvec of  $T$ .

Thus  $(v, Tv)$  is linely inde. Extend to  $B_V = (v, Tv, u_1, \dots, u_n)$ .

Let  $U = \text{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$  is not an invar subsp of  $V$  under  $T$ . □

OR. Suppose  $0 \neq v = v_1 \in V$ . Extend to  $B_V = (v_1, \dots, v_n)$ . Suppose  $Tv_1 = c_1 v_1 + \dots + c_n v_n, \exists! c_i \in \mathbb{F}$ .

Consider a  $k$ -dim subsp  $U = \text{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$ . Where  $\alpha_1, \dots, \alpha_{k-1} \in \{2, \dots, n\}$  are distinct.

Because every subsp such  $U$  is invar.  $Tv_1 = c_1 v_1 + \dots + c_n v_n \in U \Rightarrow c_2 = \dots = c_n = 0$ .

For if not,  $\exists c_i \neq 0$ , let  $W = \text{span}(v_1, v_{\beta_1}, \dots, v_{\beta_{k-1}})$ , where each  $\beta_j \in \{2, \dots, i-1, i+1, \dots, n\}$ .

Hence  $Tv_1 = c_1 v_1$ . Because  $v_1 = v \in V$  is arbitrary. We conclude that  $T = \lambda I$  for some  $\lambda \in \mathbb{F}$ . □

OR. For each  $k \in \{1, \dots, \dim V - 1\}$ , define  $P(k)$  : if every subsp of dim  $k$  is invar, then  $T = \lambda I$ .

(i) If every subsp of dim 1 is invar, then by Problem (26),  $T = \lambda I$ . Thus  $P(1)$  holds.

(ii) Assume that  $P(k)$  holds for  $k \in \{1, \dots, \dim V - 1\}$ . And every subsp of dim  $k + 1$  is invar.

Let  $U$  be a subsp of dim  $k$ . If  $\dim U = \dim V - 1$  then extend  $B_U$  to  $B_V$  and we are done.

Suppose  $\dim U \in \{1, \dots, \dim V - 2\}$ . Choose two linely inde vecs  $v, w \notin U$ .

Because  $U \oplus \text{span}(v)$  and  $U \oplus \text{span}(w)$  of dim  $k + 1$  are invar.

Suppose  $u \in U$ . Let  $Tu = a_1 u_1 + bv = a_2 u_2 + cw, \exists! u_1, u_2 \in U, a_1, a_2, b, c \in \mathbb{F}$ .

Now  $a_1 u_1 - a_2 u_2 = cw - bv \in U \cap \text{span}(v) = \{0\} \Rightarrow b = c = 0$ . Thus  $Tu \in U$ .

Because  $P(k)$  holds, we conclude that  $T = \lambda I$ . Thus  $P(k + 1)$  holds. □

**29** Suppose  $T \in \mathcal{L}(V)$  and range  $T$  is finite-dim.

Prove that  $T$  has at most  $1 + \dim \text{range } T$  distinct eigvals.

**SOLUTION:**

Let  $\lambda_1, \dots, \lambda_m$  be the distinct eigvals of  $T$  with correspd eigvecs  $v_1, \dots, v_m$ .

( Because range  $T$  is finite-dim. The correspd eigvals are finite. )

Then  $(v_1, \dots, v_m)$  linely inde  $\Rightarrow (\lambda_1 v_1, \dots, \lambda_m v_m)$  linely inde, if each  $\lambda_k \neq 0$ .

Otherwise,  $\exists! \lambda_k = 0$ . Now  $(\lambda_1 v_1, \dots, \lambda_{k-1} v_{k-1}, \lambda_{k+1} v_{k+1}, \dots, \lambda_m v_m)$  is linely inde.

Hence, by [2.23],  $m - 1 \leq \dim \text{range } T$ . □

**30** Suppose  $T \in \mathcal{L}(\mathbb{R}^3)$  and  $-4, 5, \sqrt{7}$  are eigvals. Prove that  $\exists x, Tx - 9x = (-4, 5, \sqrt{7})$ .

**SOLUTION:**  $T$  has  $\dim \mathbb{R}^3$  eigvals not including 9  $\Rightarrow (T - 9I)$  is inv.  $x = (T - 9I)^{-1}(-4, 5, \sqrt{7})$ . □

**31** Suppose  $V$  is finite-dim, and  $v_1, \dots, v_m \in V$ . Prove that

$(v_1, \dots, v_m)$  is linely inde  $\iff v_1, \dots, v_m$  are eigvecs of some  $T$  correspd to distinct eigvals.

**SOLUTION:** Suppose  $(v_1, \dots, v_m)$  is linely inde. Let  $B_V = (v_1, \dots, v_m, \dots, v_n)$ .

Define  $T \in \mathcal{L}(V)$  by  $Tv_k = k \cdot v_k$  for each  $k \in \{1, \dots, m, \dots, n\}$ . Conversely by [5.10].  $\square$

• Suppose  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$  are distinct.

(a) **32** Prove that  $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$  is linely inde in  $\mathbb{R}^{\mathbb{R}}$ .

**HINT:** Let  $V = \text{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ . Define  $D \in \mathcal{L}(V)$  by  $Df = f'$ . Find eigvals and eigvecs of  $D$ .

(b) [4E 36] Show that  $(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$  is linely inde in  $\mathbb{R}^{\mathbb{R}}$ .

**SOLUTION:**

(a) Define  $V$  and  $D \in \mathcal{L}(V)$  as in HINT. Then because for each  $k$ ,  $D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$ .

Thus  $\lambda_1, \dots, \lambda_n$  are distinct eigvals of  $D$ . By [5.10],  $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$  is linely inde in  $\mathbb{R}^{\mathbb{R}}$ .  $\square$

(b) Let  $V = \text{span}(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$ . Define  $D \in \mathcal{L}(V)$  by  $Df = f'$ .

Then because  $D(\cos(\lambda_k x)) = -\lambda_k \sin(\lambda_k x)$ .  $\times D(\sin(\lambda_k x)) = \lambda_k \cos(\lambda_k x)$ .

Thus  $D^2(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$ .

Notice that  $\lambda_1, \dots, \lambda_n$  are distinct  $\implies -\lambda_1^2, \dots, -\lambda_n^2$  are distinct. And  $\dim V = n$ .

Hence  $-\lambda_1^2, \dots, -\lambda_n^2$  are all the eigvals of  $D^2$  with correspd eigvecs  $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ .

And then  $(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$  is linely inde in  $\mathbb{R}^{\mathbb{R}}$ .  $\square$

**33** Suppose  $T \in \mathcal{L}(V)$ . Prove that  $T/(\text{range } T) = 0$ .

**SOLUTION:**  $v + \text{range } T \in V/\text{range } T \implies v + \text{range } T \in \text{null}(T/(\text{range } T))$ . Hence  $T/(\text{range } T) = 0$ .  $\square$

**34** Suppose  $T \in \mathcal{L}(V)$ . Prove that  $T/(\text{null } T)$  is inje  $\iff (\text{null } T) \cap (\text{range } T) = \{0\}$ .

**SOLUTION:** NOTICE that  $(T/(\text{null } T))(u + \text{null } T) = Tu + \text{null } T = 0 \iff Tu \in (\text{null } T) \cap (\text{range } T)$ .

Now  $T/(\text{null } T)$  is inje  $\iff u + \text{null } T = 0 \iff Tu = 0 \iff (\text{null } T) \cap (\text{range } T) = \{0\}$ .  $\square$

• Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ , and  $U$  is an invar subsp of  $V$  under  $T$ .

Define  $T/U : V/U \rightarrow V/U$  by  $(T/U)(v + U) = Tv + U$  for each  $v \in V$ .

(a) Show that  $T/U$  is well-defined and is linear. Requires that  $U$  is invar under  $T$ .

(b) [OR 35] Show that each eigval of  $T/U$  is an eigval of  $T$ .

**SOLUTION:**

(a)  $v + U = w + U \iff v - w \in U \implies T(v - w) \in U \iff Tv + U = Tw + U$ .

Hence  $T/U$  is well-defined. Now we show that  $T/U$  is linear.

$(T/U)((v + U) + \lambda(w + U)) = T(v + \lambda w) + U = (T/U)(v + U) + \lambda(T/U)(w + U)$ . Checked.

(b) Suppose  $\lambda$  is an eigval of  $T/U$  with an eigvec  $v + U$ . Then  $Tv + U = \lambda v + U \implies (T - \lambda I)v = u \in U$ .

If  $u = 0 \implies Tv = \lambda v$ , then we are done. Otherwise, we discuss in two cases.

If  $(T - \lambda I)|_U$  is inv. Then  $\exists! w \in U, (T - \lambda I)(w) = u = (T - \lambda I)v \implies T(v + w) = \lambda(v + w)$ .

Note that  $v + w \neq 0$ , for if not,  $v \in U \implies v + U = 0$ , contradicts. Thus  $\lambda$  is an eigval of  $T$ .

If  $(T - \lambda I)|_U$  is not inv. Then because  $V$  is finite-dim,  $(T - \lambda I)|_U$  is not inje,

so that  $\exists w \in \text{null}(T - \lambda I)|_U, w \neq 0, (T - \lambda I)w = 0 \implies Tw = \lambda w$ .  $\square$

OR. Let  $B_U = (u_1, \dots, u_m)$ . Then  $((T - \lambda I)v, (T - \lambda I)u_1, \dots, (T - \lambda I)u_m)$  is linely inde in  $U$ .

So that  $a_0(T - \lambda I)v + a_1(T - \lambda I)u_1 + \dots + a_m(T - \lambda I)u_m = 0, \exists a_0, a_1, \dots, a_m \in \mathbb{F}$  with some  $a_i \neq 0$ .

Let  $w = a_0 v + a_1 u_1 + \dots + a_m u_m \implies Tw = \lambda w$ . Note that  $w \neq 0$ , for if not,  $a_0 v \in U$ , each  $a_i = 0$ .  $\square$

**36** Prove or give a counterexample: The result in Exercise 35 is still true if  $V$  is infinite-dim.

**SOLUTION:** A counterexample:

Consider  $V = \{f \in \mathbb{R}^{\mathbb{R}} : \exists! m \in \mathbb{N}, f \in \text{span}(1, e^x, \dots, e^{mx})\}$ . Note that  $V$  is infinite-dim.

And a subsp  $U = \{f \in \mathbb{R}^{\mathbb{R}} : \exists! m \in \mathbb{N}^+, f \in \text{span}(e^x, \dots, e^{mx})\}$ .

Define  $T \in \mathcal{L}(V)$  by  $Tf = e^x f$ . Then  $\text{range } T = U$  is invar under  $T$ .

Consider  $(T/U)(1 + U) = e^x + U = 0 \implies 0$  is an eigval of  $T/U$  but is not an eigval of  $T$ .

[  $\text{null } T = \{0\}$ , for if not,  $\exists f \in V \setminus \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbb{R} \implies f = 0$ , contradicts. ]  $\square$

• (4E 5.A.39) Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

Prove that  $T$  has an eigval  $\iff \exists$  an invar subsp  $U$  under  $T$  of dimension  $\dim V - 1$ .

**SOLUTION:**

(a) Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $v$ . ( If  $\dim V = 1$ , then  $U = \{0\}$  and we are done. )

Extend  $v_1 = v$  to  $B_V = (v_1, v_2, \dots, v_n)$ .

**Step 1.** If  $\exists w_1 \in \text{span}(v_2, \dots, v_n)$  such that  $0 \neq Tw_1 \in \text{span}(v_1)$ .

Then extend  $w_1 = \alpha_{1,2}$  to a basis of  $\text{span}(v_2, \dots, v_n)$  as  $(\alpha_{1,2}, \dots, \alpha_{1,n})$ .

Otherwise, we stop at step 1.

**Step 2.** If  $\exists w_2 \in \text{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$  such that  $0 \neq Tw_2 \in \text{span}(v_1, w_1)$ .

Then extend  $w_2 = \alpha_{2,3}$  to a basis of  $\text{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$  as  $(\alpha_{2,3}, \dots, \alpha_{2,n})$ .

Otherwise, we stop at step 2.

**Step k.** If  $\exists w_k \in \text{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$  such that  $0 \neq Tw_k \in \text{span}(v_1, w_1, \dots, w_{k-1})$ ,

Then extend  $w_k = \alpha_{k,k+1}$  to a basis of  $\text{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$  as  $(\alpha_{k,k+1}, \dots, \alpha_{k,n})$ .

Otherwise, we stop at step  $k$ .

Finally, we stop at step  $m$ , thus we get  $(v_1, w_1, \dots, w_{m-1})$  and  $(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})$ ,

$\text{range } T|_{\text{span}(w_1, \dots, w_{m-1})} = \text{span}(v_1, w_1, \dots, w_{m-2}) \implies \dim \text{null } T|_{\text{span}(w_1, \dots, w_{m-1})} = 0$ ,

$\underbrace{\text{span}(v_1, w_1, \dots, w_{m-1})}_{\dim m}$  and  $\underbrace{\text{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})}_{\dim(n-m)}$  are invar under  $T$ .

Let  $U = \text{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n}) \oplus \text{span}(v_1, w_1, \dots, w_{m-2})$  and we are done.  $\square$

**COMMENT:** Both  $\text{span}(v_2, \dots, v_n)$  and  $U \oplus \text{span}(w_{m-1})$  are in  $\mathcal{S}_V \text{span}(v_1)$ .

If  $T|_U$  is inv, then by the similar algorithm, we can extend  $U$  to an invar subsp.

OR. Note that  $\dim \text{null } (T - \lambda I) \geq 1$ . And  $\dim \text{range } (T - \lambda I) \leq \dim V - 1$ .

Let  $B_{\text{range } (T - \lambda I)} = (w_1, \dots, w_m)$ ,  $B_V = (w_1, \dots, w_m, u_1, \dots, u_n)$ .

If  $m = \dim V - 1$ . [  $\iff n = 0$ . ] Then  $\text{range } (T - \lambda I)$  is an invar subsp of  $\dim \dim V - 1$ .

Otherwise, choose  $k \in \{1, \dots, n\}$  and then let  $U = \text{span}(w_1, \dots, w_m, u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n)$ .

By Problem (1)(b),  $U$  is invar under  $(T - \lambda I)$ . Now  $u \in U \implies (T - \lambda I)(u) \in U \implies Tu \in U$ .

(b) Suppose  $U$  is an invar subsp under  $T$  of  $\dim m = \dim V - 1$ . ( If  $m = 0$ , then we are done. )

Let  $B_U = (u_1, \dots, u_m)$ ,  $B_V = (u_0, u_1, \dots, u_m)$ . We discuss in cases:

(I) If  $Tu_0 \in U$ , then  $\text{range } T = U$  so that  $T$  is not surj  $\iff \text{null } T \neq \{0\} \iff 0$  is an eigval of  $T$ .

(II) If  $Tu_0 \notin U$ , then  $Tu_0 = a_0 u_0 + a_1 u_1 + \dots + a_m u_m$ .

If  $\text{range } T|_U = U \iff a_1 = \dots = a_m = 0 \iff Tu_0 \in \text{span}(u_0)$  then we are done.

Otherwise,  $T|_U : U \rightarrow U$  is not surj, so is not inje. Thus  $0$  is an eigval of  $T|_U$ , so of  $T$ .  $\square$

OR. Consider  $T/U \in \mathcal{L}(V/U)$ . Because  $\dim V/U = 1$ .  $\exists \lambda \in \mathbb{F}, T/U = \lambda I$ . By Problem (35).  $\square$

## 5.B: I [ See 5.B: II below. ]

**COMMENT:** 下面, 为了照顾原书 5.B 节两版过大的差距, 特别将此节补注分成 I 和 II 两部分。又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本质征值」(相当一部分是从原第 3 版 8.C 节挪过来的) 是对原第 3 版「多项式作用于算子」与「本征值的存在性」(也即第 3 版 5.B 前半部分) 的极大扩充, 这一扩充也大大改变了原第 3 版后半部分的「上三角矩阵」这一小节, 故而将第 4 版 5.B 节放在第 3 版前面。

I 部分除了覆盖第 4 版 5.B 节全部和第 3 版 5.B 节前半部分与之相关的所有习题, 还会覆盖第 4 版 5.A 节末。

II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节, 还会覆盖第 4 版 5.C 节; 并且, 下面 5.C 还会覆盖第 4 版 5.D 节。

[ 注: [8.40] OR (4E 5.22) — mini poly;  
 [8.44,8.45] OR (4E 5.25,5.26) — how to find the mini poly;  
 [8.49] OR (4E 5.27) — eigvals are the zeros of the mini poly;  
 [8.46] OR (4E 5.29) —  $q(T) = 0 \Leftrightarrow q$  is a poly multi of the mini poly.]

1 2 3 5 6 7 8 10 11 12 13 18 19 | 2E Ch5.24  
 4E: 5.A.32, 5.A.33; 3, 7, 8, 9, 10, 11, 12, 13, 14, 15,  
 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29.

- (4E 5.A.33) Suppose  $T \in \mathcal{L}(V)$  and  $m$  is a positive integer.
  - (a) Prove that  $T$  is inje  $\Leftrightarrow T^m$  is inje.
  - (b) Prove that  $T$  is surj  $\Leftrightarrow T^m$  is surj.

**SOLUTION:**

(a) Suppose  $T^m$  is inje. Then  $Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$ .

Suppose  $T$  is inje. Then  $T^m v = T^{m-1}v = \dots = T^2 v = Tv = v = 0$ .

(b) Suppose  $T^m$  is surj.  $\forall u \in V, \exists v \in V, T^m v = u = Tw$ , let  $w = T^{m-1}v$ .

Suppose  $T$  is surj. Then  $\forall u \in V, \exists v_1, \dots, v_m \in V, T(v_1) = T^2 v_2 = \dots = T^m v_m = u$ . □

### • NOTE FOR [5.17]:

Suppose  $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{F})$ . Prove that  $\text{null } p(T)$  and  $\text{range } p(T)$  are invar under  $T$ .

**SOLUTION:** Using the commutativity in [5.10].

(a) Suppose  $u \in \text{null } p(T)$ . Then  $p(T)u = 0$ .

Thus  $p(T)(Tu) = (p(T)T)u(Tp(T))u = T(p(T)u) = 0$ . Hence  $Tu \in \text{null } p(T)$ . □

(b) Suppose  $u \in \text{range } p(T)$ . Then  $\exists v \in V$  such that  $u = p(T)v$ .

Thus  $Tu = T(p(T)v) = p(T)(Tv) \in \text{range } p(T)$ . □

### • NOTE FOR [5.21]: Every operator on a finite-dim nonzero complex vecsp has an eigval.

Suppose  $V$  is a finite-dim complex vecsp of  $\dim n > 0$  and  $T \in \mathcal{L}(V)$ .

Choose a nonzero  $v \in V$ .  $(v, Tv, T^2 v, \dots, T^n v)$  of length  $n + 1$  is linely depe.

Suppose  $a_0 I + a_1 T + \dots + a_n T^n = 0$ . Then  $\exists a_j \neq 0$ .

Thus  $\exists$  nonconst  $p$  of smallest degree ( $\deg p > 0$ ) such that  $p(T)v = 0$ .

Because  $\exists \lambda \in \mathbf{C}$  such that  $p(\lambda) = 0 \Rightarrow \exists q \in \mathcal{P}(\mathbf{C}), p(z) = (z - \lambda)q(z), \forall z \in \mathbf{C}$ .

Thus  $0 = p(T)v = (T - \lambda I)(q(T)v)$ . By the minimality of  $\deg p$  and  $\deg q < \deg p, q(T)v \neq 0$ .

Then  $(T - \lambda I)$  is not inje. Thus  $\lambda$  is an eigval of  $T$  with eigvec  $q(T)v$ .

### • EXAMPLE: an operator on a complex vecsp with no eigvals

Define  $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$  by  $(Tp)(z) = zp(z)$ .

Suppose  $p \in \mathcal{P}(\mathbb{C})$  is a nonzero poly. Then  $\deg Tp = \deg p + 1$ , and thus  $Tp \neq \lambda p, \forall \lambda \in \mathbb{C}$ .  
Hence  $T$  has no eigvals.

**13** Suppose  $V$  is a complex vecsp and  $T \in \mathcal{L}(V)$  has no eigvals.

Prove that every subsp of  $V$  invar under  $T$  is either  $\{0\}$  or infinite-dim.

**SOLUTION:** Suppose  $U$  is a finite-dim nonzero invar subsp on  $\mathbb{C}$ . Then by [5.21],  $T|_U$  has an eigval.  $\square$

**16** Suppose  $0 \neq v \in V$ . Define  $S \in \mathcal{L}(\mathcal{P}_{\dim V}(\mathbb{C}), V)$  by  $S(p) = p(T)v$ . Prove [5.21].

**SOLUTION:**

Because  $\dim \mathcal{P}_{\dim V}(\mathbb{C}) = \dim V + 1$ . Then  $S$  is not inje. Hence  $\exists 0 \neq p \in \mathcal{P}_{\dim V}(\mathbb{C}), p(T)v = 0$ .

Using [4.14], write  $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$ . Apply  $T$  to both sides:  $p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$ .

Thus at least one of  $(T - \lambda_j I)$  is not inje ( because  $p(T)$  is not inje ).  $\square$

**17** Suppose  $0 \neq v \in V$ . Define  $S \in \mathcal{L}(\mathcal{P}_{(\dim V)^2}(\mathbb{C}), \mathcal{L}(V))$  by  $S(p) = p(T)$ . Prove [5.21].

**SOLUTION:**

Because  $\dim \mathcal{P}_{(\dim V)^2}(\mathbb{C}) = (\dim V)^2 + 1$ . Then  $S$  is not inje.

Hence  $\exists p \in \mathcal{P}_{(\dim V)^2}(\mathbb{C}) \setminus \{0\}, 0 = S(p) = p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$ , where  $c \neq 0$ .

Thus  $(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0 \implies \exists j, (T - \lambda_j I)$  is not inje.  $\square$

**COMMENT:**  $\exists$  monic  $q \in \text{null } S \neq \{0\}$  of smallest degree,  $S(q) = q(T) = 0$ , then  $q$  is the *mini poly*.

• **NOTE FOR [8.40]:** def for *mini poly*

Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

Suppose  $M_T^0 = \{p_j\}_{j \in \Gamma}$  is the set of all monic poly that give 0 whenever  $T$  is applied.

Prove that  $\exists! p_k \in M_T^0, \deg p_k = \min\{\deg p_j\}_{j \in \Gamma} \leq \dim V$ .

**SOLUTION:** OR. Another Proof :

[ Existns Part ] We use induction on  $\dim V$ .

(i) If  $\dim V = 0$ , then  $I = 0 \in \mathcal{L}(V)$  and let  $p = 1$ , we are done.

(ii) Suppose  $\dim V \geq 1$ .

Assume that  $\dim V > 0$  and that the desired result is true for all operators on all vecsp of smaller dim.

Let  $u \in V, u \neq 0$ . The list  $(u, Tu, \dots, T^{\dim V} u)$  of length  $(1 + \dim V)$  is linely depe.

Then  $\exists! T^m$  of smallest degree such that  $T^m u \in \text{span}(u, Tu, \dots, T^{m-1} u)$ .

Thus  $\exists c_j \in \mathbb{F}, c_0 u + c_1 Tu + \dots + c_{m-1} T^{m-1} u + T^m u = 0$ .

Define  $q$  by  $q(z) = c_0 + c_1 z + \dots + c_{m-1} z^{m-1} + z^m$ .

Then  $0 = T^k(q(T)u) = q(T)(T^k u), \forall k \in \{1, \dots, m-1\} \subseteq \mathbb{N}$ .

Because  $(u, Tu, \dots, T^{m-1} u)$  is linely inde.

Thus  $\dim \text{null } q(T) \geq m \implies \dim \text{range } q(T) = \dim V - \dim \text{null } q(T) \leq \dim V - m$ .

Let  $W = \text{range } q(T)$ .

By assumption,  $\exists s \in M_T^0$  of smallest degree ( and  $\deg s \leq \dim W$ , ) so that  $s(T|_W) = 0$ .

Hence  $\forall v \in V, ((sq)(T))(v) = s(T)(q(T)v) = 0$ .

Thus  $sq \in M_T^0$  and  $\deg sq \leq \dim V$ .

[ Uniques Part ]

Suppose  $p, q \in M_T^0$  are of the smallest degree. Then  $(p-q)(T) = 0$ . 又  $\deg(p-q) = m < \min\{\deg p_j\}_{j \in \Gamma}$ .

Hence  $p - q = 0$ , for if not,  $\exists! c \in \mathbb{F}, c(p - q) \in M_T^0$ . Contradicts.  $\square$

- (4E 5.31, 4E 5.B.25 and 26) *mini poly of restriction operator and mini poly of quotient operator*  
Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ , and  $U$  is an invar subsp of  $V$  under  $T$ .

Let  $p$  be the mini poly of  $T$ .

- Prove that  $p$  is a poly multi of the mini poly of  $T|_U$ .
- Prove that  $p$  is a poly multi of the mini poly of  $T/U$ .
- Prove that ( mini poly of  $T|_U$  )  $\times$  ( mini poly of  $T/U$  ) is a poly multi of  $p$ .
- Prove that the set of eigvals of  $T$  equals  
the union of the set of eigvals of  $T|_U$  and the set of eigvals of  $T/U$ .

**SOLUTION:**

- $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$  By [8.46]. □
- $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v + U) = p(T)v + U = 0$ . □
- Suppose  $r$  is the mini poly of  $T|_U$ ,  $s$  is the mini poly of  $T/U$ .  
Because  $\forall v \in V, s(T/U)(v + U) = s(T)v + U = 0$ . So that  $\forall v \in V$  but  $v \notin U, s(T)v \in U$ .  
 $\wedge \forall u \in U, r(T|_U)u = r(T)u = 0$ .  
Thus  $\forall v \in V$  but  $v \notin U, (rs)(T)v = r(s(T)v) = 0$ .  
And  $\forall u \in U, (rs)(T)u = r(s(T)u) = 0$  ( because  $s(T)u = s(T|_U)u \in U$  ).  
Hence  $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0$ . □
- By [8.49], immediately. □

- (4E 5.B.27) Suppose  $\mathbf{F} = \mathbf{R}$ ,  $V$  is finite-dim, and  $T \in \mathcal{L}(V)$ .  
Prove that the mini poly  $p$  of  $T_C$  equals the mini poly  $q$  of  $T$ .

**SOLUTION:**

- $\forall u + i0 \in V_C, p(T_C)(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p$  is a poly multi of  $q$ .
- $q(T) = 0 \Rightarrow \forall u + iv \in V_C, q(T_C)(u + iv) = q(T)u + iq(T)v = 0 \Rightarrow q$  is a poly multi of  $p$ . □

- (4E 5.B.28) Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .  
Prove that the mini poly  $p$  of  $T' \in \mathcal{L}(V')$  equals the mini poly  $q$  of  $T$ .

**SOLUTION:**

- $\forall \varphi \in V', p(T')\varphi = \varphi \circ (p(T)) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \text{null } \varphi \Rightarrow p(T) = 0, p$  is a poly multi of  $q$ .
- $q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ (q(T)) = q(T')\varphi = 0 \Rightarrow q(T) = 0, q$  is a poly multi of  $p$ . □

- (4E 5.32) Suppose  $T \in \mathcal{L}(V)$  and  $p$  is the mini poly.  
Prove that  $T$  is not inje  $\iff$  the const term of  $p$  is 0.

**SOLUTION:**

- $T$  is not inje  $\iff 0$  is an eigval of  $T \iff 0$  is a zero of  $p \iff$  the const term of  $p$  is 0. □
- OR. Because  $p(0) = (z - 0)(z - \lambda_1) \cdots (z - \lambda_m) = 0 \Rightarrow T(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0$   
 $\wedge p$  is the mini poly  $\Rightarrow q$  define by  $q(z) = (z - \lambda_1) \cdots (z - \lambda_m)$  is such that  $q(T) \neq 0$ .  
Hence  $0 = p(T) = Tq(T) \Rightarrow T$  is not inje.  
Conversely, suppose  $(T - 0I)$  is not inje, then  $0$  is a zero of  $p$ , so that the const term is 0. □

- (4E 5.B.22)  
Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ . Prove that  $T$  is inv  $\iff I \in \text{span}(T, T^2, \dots, T^{\dim V})$ .

**SOLUTION:** Denote the mini poly by  $p$ , where for all  $z \in \mathbf{F}, p(z) = a_0 + a_1 z + \cdots + z^m$ .

Notice that  $V$  is finite-dim.  $T$  is inv  $\iff T$  is inje  $\iff p(0) \neq 0$ .



Hence  $p(T) = 0 = a_0I + a_1T + \dots + T^m$ , where  $a_0 \neq 0$  and  $m \leq \dim V$ . □

**6** Suppose  $T \in \mathcal{L}(V)$  and  $U$  is a subsp of  $V$  invar under  $T$ .

Prove that  $U$  is invar under  $p(T)$  for every poly  $p \in \mathcal{P}(\mathbf{F})$ .

**SOLUTION:**

$\forall u \in U, Tu \in U \Rightarrow Iu, Tu, T(Tu), \dots, T^m u \in U \Rightarrow \forall a_k \in \mathbf{F}, (a_0I + a_1T + \dots + a_m T^m)u \in U$ . □

• (4E 5.B.10, 23) Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$  and  $p$  is the mini poly with degree  $m$ . Suppose  $v \in V$ .

(a) Prove that  $\text{span}(v, Tv, \dots, T^{m-1}v) = \text{span}(v, Tv, \dots, T^{j-1}v)$  for some  $j \leq m$ .

(b) Prove that  $\text{span}(v, Tv, \dots, T^{m-1}v) = \text{span}(v, Tv, \dots, T^{m-1}v, \dots, T^n v)$ .

**SOLUTION:**

**COMMENT:** By NOTE FOR[8.40],  $j$  has an upper bound  $m - 1$ ,  $m$  has an upper bound  $\dim V$ .

Write  $p(z) = a_0 + a_1z + \dots + z^m$  ( $m \leq \dim V$ ). If  $v = 0$ , then we are done. Suppose  $v \neq 0$ .

(a) Suppose  $j \in \mathbf{N}^+$  is the smallest such that  $T^j v \in \text{span}(v, Tv, \dots, T^{j-1}v) = U_0$ . Then  $j \leq m$ .

Write  $T^j v = c_0 v + c_1 Tv + \dots + c_{j-1} T^{j-1}v$ . And because  $T(T^k v) = T^{k+1}v \in U_0$ .  $U_0$  is invar under  $T$ .

By Problem (6),  $\forall k \in \mathbf{N}$ ,  $T^{j+k}v = T^k(T^j v) \in U_0$ .

Thus  $U_0 = \text{span}(v, Tv, \dots, T^{j-1}v, \dots, T^n v)$  for all  $n \geq j - 1$ . Let  $n = m - 1$  and we are done.

(b) Let  $U = \text{span}(v, Tv, \dots, T^{m-1}v)$ .

By (a),  $U = U_0 = \text{span}(v, Tv, \dots, T^{j-1}v, \dots, T^{m-1}v, \dots, T^n v)$  for all  $n \geq m - 1$ . □

• (4E 5.B.21) Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

Prove that the mini poly  $p$  has degree at most  $1 + \dim \text{range } T$ .

If  $\dim \text{range } T < \dim V - 1$ , then this result gives a better upper bound for the degree of mini poly.

**SOLUTION:**

If  $T$  is inje, then  $\text{range } T = V$  and we are done. Now choose  $0 \neq v \in \text{null } T$ , then  $Tv + 0 \cdot v = 0$ .

1 is the smallest positive integer such that  $T^1 v \in \text{span}(v, \dots, T^0 v)$ . Define  $q$  by  $q(z) = z \Rightarrow q(T)v = 0$ .

Let  $W = \text{range } q(T) = \text{range } T$ .  $\exists$  monic  $s \in \mathcal{P}(\mathbf{F})$  of smallest degree ( $\deg s \leq \dim W$ ),  $s(T|_W) = 0$ .

Hence  $sq$  is the mini poly ( see NOTE FOR[8.40] ) and  $\deg(sq) = \deg s + \deg q \leq \dim \text{range } T + 1$ . □

**19** Suppose  $V$  is finite-dim,  $\dim V > 1$ ,  $T \in \mathcal{L}(V)$ . Prove that  $\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V)$ .

**SOLUTION:** If  $\forall S \in \mathcal{L}(V), \exists p \in \mathcal{P}(\mathbf{F}), S = p(T)$ . Then by [5.20],  $\forall S_1, S_2 \in \mathcal{L}(V), S_1 S_2 = S_2 S_1$ .

Note that  $\dim \geq 2$ . By (3.A.14),  $\exists S_1, S_2 \in \mathcal{L}(V), S_1 S_2 \neq S_2 S_1$ . Contradicts. □

• Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ . Let  $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$ .

Prove that  $\dim \mathcal{E}$  equals the degree of the mini poly of  $T$ .

**SOLUTION:**

Because the list  $(I, T, \dots, T^{(\dim V)^2})$  of length  $\dim \mathcal{L}(V) + 1$  is linely depe in  $\dim \mathcal{L}(V)$ .

Suppose  $m \in \mathbf{N}^+$  is the smallest such that  $T^m = a_0 I + \dots + a_{m-1} T^{m-1}$ .

Then  $q$  defined by  $q(z) = z^m - a_{m-1} z^{m-1} - \dots - a_0$  is the mini poly ( see [8.40] ).

For any  $k \in \mathbf{N}^+$ ,  $T^{m+k} = T^k(T^m) \in \text{span}(I, T, \dots, T^{m-1}) = U$ .

Hence  $\text{span}(I, T, \dots, T^{(\dim V)^2}) = \text{span}(I, T, \dots, T^{(\dim V)^2-1}) = U$ .

Note that by the minimality of  $m$ ,  $(I, T, \dots, T^{m-1})$  is linely inde.

Thus  $\dim U = m = \dim \text{span}(I, T, \dots, T^{(\dim V)^2 - 1}) = \dim \text{span}(I, T, \dots, T^n)$  for all  $m < n \in \mathbb{N}^+$ .

Define  $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbb{F}), \mathcal{E})$  by  $\varphi(p) = p(T)$ .

(a) Suppose  $p(T) = 0$ . 又  $\deg p \leq m - 1 \Rightarrow p = 0$ . Then  $\varphi$  is inje.

(b)  $\forall S = a_0I + a_1T + \dots + a_{m-1}T^{m-1} \in \mathcal{E}$ , define  $p \in \mathcal{P}_{m-1}(\mathbb{F})$  by

$$p(z) = a_0 + a_1z + \dots + a_{m-1}z^{m-1} \Rightarrow \varphi(p) = S. \text{ Then } \varphi \text{ is surj.}$$

Hence  $\mathcal{E}$  and  $\mathcal{P}_{m-1}(\mathbb{F})$  are iso. 又  $\dim \mathcal{P}_{m-1}(\mathbb{F}) = m = \dim U$ . □

• (4E 5.B.13) Suppose  $T \in \mathcal{L}(V)$  and  $q \in \mathcal{P}(\mathbb{F})$  is defined by

$$q(z) = a_0 + a_1z + \dots + a_nz^n, \text{ where } a_n \neq 0, \text{ for all } z \in \mathbb{F}.$$

Denote the mini poly of  $T$  by  $p$  defined by

$$p(z) = c_0 + c_1z + \dots + c_{m-1}z^{m-1} + z^m \text{ for all } z \in \mathbb{F}.$$

Prove that  $\exists ! r \in \mathcal{P}(\mathbb{F})$  such that  $q(T) = r(T)$ ,  $\deg r < \deg p$ .

**SOLUTION:**

If  $\deg q < \deg p$ , then we are done.

If  $\deg q = \deg p$ , notice that  $p(T) = 0 = c_0I + c_1T + \dots + c_{m-1}T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0I - c_1T - \dots - c_{m-1}T^{m-1},$$

$$\text{define } r \text{ by } r(z) = q(z) + [-a_mz^m + a_m(-c_0 - c_1z - \dots - c_{m-1}z^{m-1})]$$

$$= (a_0 - a_m c_0) + (a_1 - a_m c_1)z + \dots + (a_{m-1} - a_m c_{m-1})z^{m-1},$$

hence  $r(T) = 0$ ,  $\deg r < m$  and we are done.

Now suppose  $\deg q \geq \deg p$ . We use induction on  $\deg q$ .

(i)  $\deg q = \deg p$ , then the desired result is true, as shown above.

(ii)  $\deg q > \deg p$ , assume that the desired result is true for  $\deg q = n$ .

Suppose  $f \in \mathcal{P}(\mathbb{F})$  such that  $f(z) = b_0 + b_1z + \dots + b_nz^n + b_{n+1}z^{n+1}$ .

Apply the assumption to  $g$  defined by  $g(z) = b_0 + b_1z + \dots + b_nz^n$ ,

getting  $s$  defined by  $s(z) = d_0 + d_1z + \dots + d_{m-1}z^{m-1}$ .

Thus  $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1}T^{n+1} = s(T) + b_{n+1}T^{n+1}$ .

Apply the assumption to  $t$  defined by  $t(z) = z^n$ ,

getting  $\delta$  defined by  $\delta(z) = c_0' + c_1'z + \dots + c_{m-1}'z^{m-1}$ .

Thus  $t(T) = T^n = c_0' + c_1'T + \dots + c_{m-1}'T^{m-1} = \delta(T)$ .

又  $\text{span}(v, Tv, \dots, T^{m-1}v)$  is invar under  $T$ .

Hence  $\exists ! k_j \in \mathbb{F}$ ,  $T^{n+1} = T(T^n) = k_0 + k_1T + \dots + k_{m-1}T^{m-1}$ .

And  $f(T) = s(T) + b_{n+1}(k_0 + k_1T + \dots + k_{m-1}T^{m-1})$

$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1)z + \dots + (d_{m-1} + k_{m-1})z^{m-1} = h(T)$ , thus defining  $h$ . □

• (4E 5.B.14) Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$  has mini poly  $p$

$$\text{defined by } p(z) = a_0 + a_1z + \dots + a_{m-1}z^{m-1} + z^m, a_0 \neq 0.$$

Find the mini poly of  $T^{-1}$ .

**SOLUTION:**

Notice that  $V$  is finite-dim. Then  $p(0) = a_0 \neq 0 \Rightarrow 0$  is not a zero of  $p \Rightarrow T - 0I = T$  is inv.

Then  $p(T) = a_0I + a_1T + \dots + T^m = 0$ . Apply  $T^{-m}$  to both sides,

$$a_0(T^{-1})^m + a_1(T^{-1})^{m-1} + \dots + a_{m-1}T^{-1} + I = 0.$$

Define  $q$  by  $q(z) = z^m + \frac{a_1}{a_0}z^{m-1} + \dots + \frac{a_{m-1}}{a_0}z + \frac{1}{a_0}$  for all  $z \in \mathbb{F}$ .

We now show that  $(T^{-1})^k \notin \text{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$

for every  $k \in \{1, \dots, m-1\}$  by contradiction, so that  $q$  is exactly the mini poly of  $T^{-1}$ .  
 Suppose  $(T^{-1})^k \in \text{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$ .  
 Then let  $(T^{-1})^k = b_0 I + b_1 T^{-1} + \dots + b_{k-1} T^{k-1}$ . Apply  $T^k$  to both sides,  
 getting  $I = b_0 T^k + b_1 T^{k-1} + \dots + b_{k-1} T$ , hence  $T^k \in \text{span}(I, T, \dots, T^{k-1})$ .  
 Thus  $f$  defined by  $f(z) = z^k + \frac{b_1}{b_0} z^{k-1} + \dots + \frac{b_{k-1}}{b_0} z - \frac{1}{b_0}$  is a poly multi of  $p$ .  
 While  $\deg f < \deg p$ . Contradicts. □

• **NOTE FOR [8.49]:**

Suppose  $V$  is a finite-dim complex vecsp and  $T \in \mathcal{L}(V)$ .  
 By [4.14], the mini poly has the form  $(z - \lambda_1) \cdots (z - \lambda_m)$ ,  
 where  $\lambda_1, \dots, \lambda_m$  are all the eigvals of  $T$ , possibly with repetitions.

• **COMMENT:**

A nonzero poly has at most as many distinct zeros as its degree ( see [4.12] ).  
 Thus by the upper bound for the deg of mini poly given in NOTE FOR[8.40], and by [8.49,]  
 we can give an alternative proof of [5.13].

• **NOTICE** ( See also 4E 5.B.20,24 )

Suppose  $\alpha_1, \dots, \alpha_n$  are all the distinct eigvals of  $T$ ,  
 and therefore are all the distinct zeros of the mini poly.  
 Also, the mini poly of  $T$  is a poly multi of, but not equal to,  $(z - \alpha_1) \cdots (z - \alpha_n)$ .  
 If we define  $q$  by  $q(z) = (z - \alpha_1)^{\dim V - (n-1)} \cdots (z - \alpha_n)^{\dim V - (n-1)}$ ,  
 then  $q$  is a poly multi of the char poly ( see [8.34] and [8.26] )  
 ( Because  $\dim V > n$  and  $n - 1 > 0$ ,  $n[\dim V - (n - 1)] > \dim V$ . )  
 The char poly has the form  $(z - \alpha_1)^{\gamma_1} \cdots (z - \alpha_n)^{\gamma_n}$ , where  $\gamma_1 + \dots + \gamma_n = \dim V$ .  
 The mini poly has the form  $(z - \alpha_1)^{\delta_1} \cdots (z - \alpha_n)^{\delta_n}$ , where  $0 \leq \delta_1 + \dots + \delta_n \leq \dim V$ .

**10** Suppose  $T \in \mathcal{L}(V)$ ,  $\lambda$  is an eigval of  $T$  with an eigvec  $v$ .

Prove that for any  $p \in \mathcal{P}(\mathbb{F})$ ,  $p(T)v = p(\lambda)v$ .

**SOLUTION:**

Suppose  $p$  is defined by  $p(z) = a_0 + a_1 z + \dots + a_m z^m$  for all  $z \in \mathbb{F}$ . Because for any  $n \in \mathbb{N}^+$ ,  $T^n v = \lambda^n v$ .  
 Thus  $p(T)v = a_0 v + a_1 T v + \dots + a_m T^m v = a_0 v + a_1 \lambda v + \dots + a_m \lambda^m v = p(\lambda)v$ . □

**COMMENT:** For any  $p \in \mathcal{P}(\mathbb{F})$  such that  $p(z) = (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_m)^{\alpha_m}$ , the result is true as well.

Now we prove that  $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v = (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v$ .

Define  $q_i$  by  $q_i(z) = (z - \lambda_i)^{\alpha_i}$  for all  $z \in \mathbb{F}$ .

Because  $(a + b)^n = a^n + C_n^1 a^{n-1} b + \dots + C_n^k a^{n-k} b^k + \dots + C_n^n b^n$ .

Let  $a = z, b = \lambda_i, n = \alpha_i$ , so we can write  $q_i(z)$  in the form  $a_0 + a_1 z + \dots + a_m z^m$ .

Hence  $q_i(T)v = q_i(\lambda)v \Rightarrow (T - \lambda_i I)^{\alpha_i} v = (\lambda - \lambda_i)^{\alpha_i} v$ .

Then for each  $k \in \{2, \dots, m\}$ ,  $(T - \lambda_{k-1} I)^{\alpha_{k-1}} (T - \lambda_k I)^{\alpha_k} v$   

$$= q_{k-1}(T)(q_k(T)v)$$
  

$$= q_{k-1}(T)(q_k(\lambda)v)$$
  

$$= q_{k-1}(\lambda)(q_k(\lambda)v)$$
  

$$= (\lambda - \lambda_{k-1})^{\alpha_{k-1}} (\lambda - \lambda_k)^{\alpha_k} v.$$

So that  $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v$

$$\begin{aligned}
&= q_1(T)\Big(q_2(T)\Big(\dots(q_m(T)v)\dots\Big)\Big) \\
&= q_1(\lambda)(q_2(\lambda)(\dots(q_m(\lambda)v)\dots)) \\
&= (\lambda-\lambda_1)^{\alpha_1}\dots(\lambda-\lambda_m)^{\alpha_m}v.
\end{aligned}$$

□

---

**1** Suppose  $T \in \mathcal{L}(V)$  and  $\exists n \in \mathbb{N}^+$  such that  $T^n = 0$ .

Prove that  $(I - T)$  is inv and  $(I - T)^{-1} = I + T + \dots + T^{n-1}$ .

**SOLUTION:** Note that  $1 - x^n = (1 - x)(1 + x + \dots + x^{n-1})$ .

$$\left. \begin{aligned} (I - T)(1 + T + \dots + T^{n-1}) &= I - T^n = I \\ (1 + T + \dots + T^{n-1})(I - T) &= I - T^n = I \end{aligned} \right\} \Rightarrow (I - T)^{-1} = 1 + T + \dots + T^{n-1}. \quad \square$$

**2** Suppose  $T \in \mathcal{L}(V)$  and  $(T - 2I)(T - 3I)(T - 4I) = 0$ .

Suppose  $\lambda$  is an eigval of  $T$ . Prove that  $\lambda = 2$  or  $\lambda = 3$  or  $\lambda = 4$ .

**SOLUTION:**

Suppose  $v$  is an eigvec correspd to  $\lambda$ . Then for any  $p \in \mathcal{P}(\mathbb{F})$ ,  $p(T)v = p(\lambda)v$ .

Hence  $0 = (T - 2I)(T - 3I)(T - 4I)v = (\lambda - 2)(\lambda - 3)(\lambda - 4)v$  while  $v \neq 0 \Rightarrow \lambda = 2, 3$  or  $4$ .  $\square$

**COMMENT:** Note that  $(T - 2I)(T - 3I)(T - 4I) = 0$  is not inje, so that  $2, 3, 4$  are eigvals of  $T$ .

But it doesn't mean that all the eigvals of  $T$  are exactly  $2, 3, 4$ .

**7** [See 5.A.22] Suppose  $T \in \mathcal{L}(V)$ . Prove that  $9$  is an eigval of  $T^2 \iff 3$  or  $-3$  is an eigval of  $T$ .

**SOLUTION:**

(a) Suppose  $\lambda$  is an eigval of  $T$  with an eigvec  $v$ .

Then  $(T - 3I)(T + 3I)v = (\lambda - 3)(\lambda + 3)v = 0 \Rightarrow \lambda = \pm 3$ .

(b) Suppose  $3$  or  $-3$  is an eigval of  $T$  with an eigvec  $v$ . Then  $Tv = \pm 3v \Rightarrow T^2v = T(Tv) = 9v$   $\square$

OR.  $9$  is an eigval of  $T^2 \iff (T^2 - 9I) = (T - 3I)(T + 3I)$  is not inje  $\iff \pm 3$  is an eigval.  $\square$

**3** Suppose  $T \in \mathcal{L}(V)$ ,  $T^2 = I$  and  $-1$  is not an eigval of  $T$ . Prove that  $T = I$ .

**SOLUTION:**

$T^2 - I = (T + I)(T - I)$  is not inje,  $\nexists -1$  is not an eigval of  $T \Rightarrow$  By TIPS.  $\square$

OR. Note that  $\forall v \in V, v = [\frac{1}{2}(I - T)v] + [\frac{1}{2}(I + T)v]$ .

$$\left. \begin{aligned} (I + T)((I - T)v) &= 0 \Rightarrow (I - T)v \in \text{null}(I + T) \\ (I - T)((I + T)v) &= 0 \Rightarrow (I + T)v \in \text{null}(I - T) \end{aligned} \right\} \Rightarrow V = \text{null}(I + T) + \text{null}(I - T).$$

$\nexists -1$  is not an eigval of  $T \iff (I + T)$  is inje  $\iff \text{null}(I + T) = \{0\}$ .

Hence  $V = \text{null}(I - T) \Rightarrow \text{range}(I - T) = \{0\}$ . Thus  $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$ .  $\square$

• (4E 5.A.32) Suppose  $T \in \mathcal{L}(V)$  has no eigvals and  $T^4 = I$ . Prove that  $T^2 = -I$ .

**SOLUTION:**

Because  $T^4 - I = (T^2 - I)(T^2 + I) = 0$  is not inje  $\Rightarrow (T^2 - I)$  or  $(T^2 + I)$  is not inje.

$\nexists T$  has no eigvals  $\Rightarrow (T^2 - I) = (T - I)(T + I)$  is inje. Hence  $T^2 + I = 0 \in \mathcal{L}(V)$ , for if not,  $\exists v \in V, (T^2 + I)v \neq 0$  while  $(T^2 - I)((T^2 + I)v) = 0$  but  $(T^2 - I)$  is inje. Contradicts.

OR.  $\forall v \in V, 0 = (T^2 - I)(T^2 + I)v \iff 0 = (T^2 + I)v$ . Hence  $T^2 + I = 0$ .  $\square$

OR. Note that  $\forall v \in V, v = [\frac{1}{2}(I - T^2)v] + [\frac{1}{2}(I + T^2)v]$ .

$$\left. \begin{aligned} (I + T^2)((I - T^2)v) &= 0 \Rightarrow (I - T^2)v \in \text{null}(I + T^2) \\ (I - T^2)((I + T^2)v) &= 0 \Rightarrow (I + T^2)v \in \text{null}(I - T^2) \end{aligned} \right\} \Rightarrow V = \text{null}(I + T^2) + \text{null}(I - T^2).$$

$\nexists T$  has no eigvals  $\iff (I - T^2)$  is inje  $\iff \text{null}(I - T^2) = \{0\}$ .

Hence  $V = \text{null}(I + T^2) \Rightarrow \text{range}(I + T^2) = \{0\}$ . Thus  $I + T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$ .  $\square$

8 [OR (4E 5.A.31)] Give an example of  $T \in \mathcal{L}(\mathbb{R}^2)$  such that  $T^4 = -I$ .

SOLUTION:

Define  $i \in \mathcal{L}(\mathbb{R}^2)$  by  $i(x, y) = (-y, x)$ . Just like  $i : \mathbb{C} \rightarrow \mathbb{C}$  defined by  $i(x + iy) = -y + ix$ .

Define  $i^n \in \mathcal{L}(\mathbb{R}^2)$  by  $i^n(x, y) = (\operatorname{Re}(i^n x + i^{n+1} y), \operatorname{Im}(i^n x + i^{n+1} y))$ .

$$T^4 + I = (T^2 + iI)(T^2 - iI) = (T + i^{1/2}I)(T - i^{1/2}I)(T - (-i)^{1/2}I)(T + (-i)^{1/2}I).$$

Note that  $i^{1/2} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ ,  $(-i)^{1/2} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ . Hence  $T = \pm(\pm i)^{1/2}I$ .

Let  $T = i^{1/2}I$  defined by  $i^{1/2}(x, y) = (\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y)$ . □

OR. Because  $\mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix}$ . Using  $\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}$ .

We define  $T \in \mathcal{L}(\mathbb{R}^2)$  such that  $\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix}$ . □

• (4E 5.B.12) Find the mini poly of  $T$  defined in (5.A.10).

SOLUTION: By (5.A.9) and [8.40, 8.49],  $1, 2, \dots, n$  are all the zeros of the mini poly of  $T$ . □

• (4E 5.B.3) Find the mini poly of  $T$  defined in (5.A.19).

SOLUTION:

If  $n = 1$  then 1 is the only eigval of  $T$ , and  $(z - 1)$  is the mini poly.

Because  $n$  and 0 are all the eigvals of  $T$ ,  $\forall k \in \{1, \dots, n\}, Te_k = e_1 + \dots + e_n; T^2e_k = n(e_1 + \dots + e_n)$ .

Hence  $T^2e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T - n) = 0$ . Thus  $(z(z - n))$  is the mini poly. □

• (4E 5.B.8) Find the mini poly of  $T$ . Where  $T \in \mathcal{L}(\mathbb{R}^2)$  is the operator of counterclockwise rotation by  $\theta$ , where  $\theta \in \mathbb{R}^+$ .

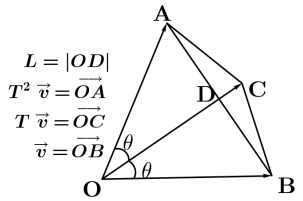
SOLUTION:

If  $\theta = \pi + 2k\pi$ , then  $T(w, z) = (-w, -z), T^2 = I$  and the mini poly is  $z + 1$ .

If  $\theta = 2k\pi$ , then  $T = I$  and the mini poly is  $z - 1$ .

Otherwise  $(v, Tv)$  is linely inde. Then  $\operatorname{span}(v, Tv) = \mathbb{R}^2$ . Note that  $\nexists b \in \mathbb{F}, T - bI = 0$ .

Thus suppose the mini poly  $p$  is defined by  $p(z) = z^2 + bz + c$  for all  $z \in \mathbb{R}$ .

Because   $\left\{ \begin{array}{l} L = |OD| \\ T^2 \vec{v} = \vec{OA} \\ T \vec{v} = \vec{OC} \\ \vec{v} = \vec{OB} \end{array} \right. \quad \left\{ \begin{array}{l} Tv = \frac{|\vec{v}|}{2L}(T^2v + v) \Rightarrow T = \frac{|\vec{v}|}{2L}(T^2 + I) \\ L = |\vec{v}| \cos \theta \Rightarrow \frac{|\vec{v}|}{2L} = \frac{1}{2 \cos \theta} \end{array} \right.$

Hence  $p(T) = T^2 - 2 \cos \theta T + I = 0$  and  $z^2 - 2 \cos \theta z + 1$  is the mini poly of  $T$ . □

OR. Let  $(e_1, e_2)$  be the standard basis of  $\mathbb{R}^2$ . We use the pattern shown in [8.44].

Because  $Te_1 = \cos \theta e_1 + \sin \theta e_2, T^2e_1 = \cos 2\theta e_1 + \sin 2\theta e_2$ .

Thus  $ce_1 + bTe_1 = -T^2e_1 \Leftrightarrow \begin{pmatrix} 1 & \cos \theta \\ 0 & \sin \theta \end{pmatrix} \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$ . Now  $\det = \sin \theta \neq 0, c = 1, b = 2 \cos \theta$ . □

OR.  $\mathcal{M}(T, (e_1, e_2)) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ . By (4E 5.B.11), the mini poly is  $(z \pm 1)$  or  $(z^2 - 2 \cos \theta z + 1)$ . □

- (4E 5.B.11) Suppose  $V$  is a two-dim vecsp,  $T \in \mathcal{L}(V)$ , and the matrix of  $T$  with resp to some basis of  $V$  is  $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ .

(a) Show that  $T^2 - (a + d)T + (ad - bc)I = 0$ .

(b) Show that the mini poly of  $T$  equals

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a + d)z + (ad - bc) & \text{otherwise.} \end{cases}$$

**SOLUTION:**

(a) Suppose the basis is  $(v, w)$ . Because  $\begin{cases} Tv = av + bw \Rightarrow (T - aI)v = bw, \text{ then apply } (T - dI) \text{ to both sides} \\ Tw = cv + dw \Rightarrow (T - dI)w = cv, \text{ then apply } (T - aI) \text{ to both sides} \end{cases}$

Hence  $(T - aI)(T - dI) = bcI \Rightarrow T^2 - (a + d)T + (ad - bc)I = 0$ .

(b) If  $b = c = 0$  and  $a = d$ . Then  $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$ . Thus  $T = aI$ . Hence the mini poly is  $z - a$ .

Otherwise, by (a),  $z^2 - (a + d)z + (ad - bc)$  is a poly multi of the mini poly.

Now we prove that  $T \notin \text{span}(I)$ , so that then the mini poly of  $T$  has exactly degree 2.

( At least one of the assumption of (I),(II) below is true. )

(I) Suppose  $a = d$ , then  $Tv = av + bw \notin \text{span}(v)$ ,  $Tw = cv + aw \notin \text{span}(w)$ .

(II) Suppose at most one of  $b, c$  is not 0. If  $b = 0$ , then  $Tw \notin \text{span}(w)$ ; If  $c = 0$ , then  $Tv \notin \text{span}(v)$   $\square$

- Suppose  $S, T \in \mathcal{L}(V)$ ,  $S$  is inv, and  $p \in \mathcal{P}(\mathbf{F})$ . Prove that  $Sp(TS) = p(ST)S$ .

**SOLUTION:**

We prove  $S(TS)^m = (ST)^mS$  for each  $m \in \mathbf{N}$  by induction.

(i) If  $m = 0, 1$ . Then  $S(TS)^0 = I = (ST)^0S$ ;  $S(TS)^1 = (ST)S$ .

(ii) If  $m > 1$ . Assume that  $S(TS)^m = (ST)^mS$ .

Then  $S(TS)^{m+1} = S(TS)^m(TS) = (ST)^mSTS = (ST)^{m+1}S$ .

Hence  $\forall p \in \mathcal{P}(\mathbf{F}), Sp(TS) = \sum_{k=1}^m a_k S(TS)^k = \sum_{k=1}^m a_k p(ST)^k S = [\sum_{k=1}^m a_k (TS)^k] S$ .  $\square$

**COMMENT:**  $p(TS) = S^{-1}p(ST)S$ ,  $p(ST) = Sp(TS)S^{-1}$ .

**COROLLARY: 5** Because  $S$  is inv,  $T \in \mathcal{L}(V)$  is arbitrary  $\iff R = ST$  is arbitrary.

Hence  $\forall R \in \mathcal{L}(V)$ , inv  $S \in \mathcal{L}(V)$ ,  $p(S^{-1}RS) = S^{-1}p(R)S$ .

- (4E 5.B.7) Suppose  $S, T \in \mathcal{L}(V)$ . Let  $p, q$  be the mini polys of  $ST, TS$  respectively.

(a) If  $V = \mathbf{F}^2$ . Give an example such that  $p \neq q$ ; (b) If  $S$  or  $T$  is inv. Prove that  $p = q$ .

**SOLUTION:**

(a) Define  $S$  by  $S(x, y) = (x, x)$ . Define  $T$  by  $T(x, y) = (0, y)$ .

Then  $ST(x, y) = 0$ ,  $TS(x, y) = (0, x)$  for all  $(x, y) \in \mathbf{F}^2$ . Thus  $ST = 0 \neq TS$  and  $(TS)^2 = 0$ .

Hence the mini poly of  $ST$  does not equal to the mini poly of  $TS$ .

(b) Suppose  $S$  is inv. Because  $p, q$  are monic.

$$\left. \begin{aligned} p(ST) = 0 &= Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a poly multi of } q \\ q(TS) = 0 &= S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a poly multi of } p \end{aligned} \right\} \Rightarrow p = q.$$

Reversing the roles of  $S$  and  $T$ , we conclude that if  $T$  is inv, then  $p = q$  as well.  $\square$

**11** Suppose  $\mathbf{F} = \mathbf{C}$ ,  $T \in \mathcal{L}(V)$ ,  $p \in \mathcal{P}(\mathbf{C})$ , and  $\alpha \in \mathbf{C}$ .

Prove that  $\alpha$  is an eigval of  $p(T) \iff \alpha = p(\lambda)$  for some eigval  $\lambda$  of  $T$ .

**SOLUTION:**

(a) Suppose  $\alpha$  is an eigval of  $p(T) \iff (p(T) - \alpha I)$  is not inje.

Write  $p(z) - \alpha = c(z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow p(T) - \alpha I = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$ .

By TIPS,  $\exists (T - \lambda_j I)$  not inje. Thus  $p(\lambda_j) - \alpha = 0$ .

(b) Suppose  $\alpha = p(\lambda)$  and  $\lambda$  is an eigval of  $T$  with an eigvec  $v$ . Then  $p(T)v = p(\lambda)v = \alpha v$ . □

OR. Define  $q$  by  $q(z) = p(z) - \alpha$ .  $\lambda$  is a zero of  $q$ .

Because  $q(T)v = (p(T) - \alpha I)v = q(\lambda)v = (p(\lambda) - \alpha)v = 0$ .

Hence  $q(T)$  is not inje  $\Rightarrow (p(T) - \alpha I)$  is not inje. □

**12** [OR (4E.5.B.6)] *Give an example of an operator on  $\mathbf{R}^2$  that shows the result above does not hold if  $\mathbf{C}$  is replaced with  $\mathbf{R}$ .*

**SOLUTION:**

Define  $T \in \mathcal{L}(\mathbf{R}^2)$  by  $T(w, z) = (-z, w)$ .

By Problem (4E 5.B.11),  $\mathcal{M}(T, ((1, 0), (0, 1))) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$  the mini poly of  $T$  is  $z^2 + 1$ .

Define  $p$  by  $p(z) = z^2$ . Then  $p(T) = T^2 = -I$ . Thus  $p(T)$  has eigval  $-1$ .

While  $\nexists \lambda \in \mathbf{R}$  such that  $-1 = p(\lambda) = \lambda^2$ . □

• (4E 5.B.17) *Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ ,  $\lambda \in \mathbf{F}$ , and  $p$  is the mini poly of  $T$ . Show that the mini poly of  $(T - \lambda I)$  is the poly  $q$  defined by  $q(z) = p(z + \lambda)$ .*

**SOLUTION:**

$q(T - \lambda I) = 0 \Rightarrow q$  is poly multi of the mini poly of  $(T - \lambda I)$ .

Suppose the degree of the mini poly of  $(T - \lambda I)$  is  $n$ , and the degree of the mini poly of  $T$  is  $m$ .

By definition of mini poly,

$n$  is the smallest such that  $(T - \lambda I)^n \in \text{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{n-1})$ ;

$m$  is the smallest such that  $T^m \in \text{span}(I, T, \dots, T^{m-1})$ .

$\text{又 } T^k \in \text{span}(I, T, \dots, T^{k-1}) \iff (T - \lambda I)^k \in \text{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{k-1})$ .

Thus  $n = m$ . 又  $q$  is monic. By the uniqueness of mini poly. □

• (4E 5.B.18) *Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ ,  $\lambda \in \mathbf{F} \setminus \{0\}$ , and  $p$  is the mini poly of  $T$ . Show that the mini poly of  $\lambda T$  is the poly  $q$  defined by  $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$ .*

**SOLUTION:**

$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow q$  is a poly multi of the mini poly of  $\lambda T$ .

Suppose the degree of the mini poly of  $\lambda T$  is  $n$ , and the degree of the mini poly of  $T$  is  $m$ .

By definition of mini poly,

$n$  is the smallest such that  $(\lambda T)^n \in \text{span}(\lambda I, \lambda T, \dots, (\lambda T)^{n-1})$ ;

$m$  is the smallest such that  $T^m \in \text{span}(I, T, \dots, T^{m-1})$ .

$\text{又 } (\lambda T)^k \in \text{span}(\lambda I, \lambda T, \dots, (\lambda T)^{k-1}) \iff T^k \in \text{span}(I, T, \dots, T^{k-1})$ .

Thus  $n = m$ . 又  $q$  is monic. By the uniqueness of mini poly. □

**18** [OR (4E 5.B.15)] *Suppose  $V$  is a finite-dim complex vecsp with  $\dim V > 0$  and  $T \in \mathcal{L}(V)$ . Define  $f : \mathbf{C} \rightarrow \mathbf{R}$  by  $f(\lambda) = \dim \text{range}(T - \lambda I)$ . Prove that  $f$  is not a continuous function.*

**SOLUTION:** Note that  $V$  is finite-dim.

Let  $\lambda_0$  be an eigval of  $T$ . Then  $(T - \lambda_0 I)$  is not surj. Hence  $\dim \text{range}(T - \lambda_0 I) < \dim V$ .

Because  $T$  has finitely many eigvals. There exist a sequence of number  $\{\lambda_n\}$  such that  $\lim_{n \rightarrow \infty} \lambda_n = \lambda_0$ .



And  $\lambda_n$  is not an eigval of  $T$  for each  $n \Rightarrow \dim \text{range}(T - \lambda_n I) = \dim V \neq \dim \text{range}(T - \lambda_0 I)$ .

Thus  $f(\lambda_0) \neq \lim_{n \rightarrow \infty} f(\lambda_n)$ . □

- (4E 5.B.9) Suppose  $T \in \mathcal{L}(V)$  is such that with resp to some basis of  $V$ , all entries of the matrix of  $T$  are rational numbers.

Explain why all coefficients of the mini poly of  $T$  are rational numbers.

**SOLUTION:**

Let  $(v_1, \dots, v_n)$  denote the basis such that  $\mathcal{M}(T, (v_1, \dots, v_n))_{j,k} = A_{j,k} \in \mathbb{Q}$  for all  $j, k = 1, \dots, n$ .

Denote  $\mathcal{M}(v_j, (v_1, \dots, v_n))$  by  $x_j$  for each  $v_j$ .

Suppose  $p$  is the mini poly of  $T$  and  $p(z) = z^m + \dots + c_1 z + c_0$ . Now we show that each  $c_j \in \mathbb{Q}$ .

Note that  $\forall s \in \mathbb{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbb{Q}^{n,n}$  and  $T^s v_k = A_{1,k}^s v_1 + \dots + A_{n,k}^s v_n$  for all  $k \in \{1, \dots, n\}$ .

$$\text{Thus } \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \dots + c_1 A + c_0 I)x_1 = \sum_{j=1}^n (A^m + \dots + c_1 A + c_0 I)_{j,1} x_j = 0; \\ \vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \dots + c_1 A + c_0 I)x_n = \sum_{j=1}^n (A^m + \dots + c_1 A + c_0 I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly, } \begin{cases} (A^m + \dots + c_1 A + c_0 I)_{1,1} = \dots = (A^m + \dots + c_1 A + c_0 I)_{n,1} = 0; \\ \vdots \quad \ddots \quad \vdots \\ (A^m + \dots + c_1 A + c_0 I)_{1,n} = \dots = (A^m + \dots + c_1 A + c_0 I)_{n,n} = 0; \end{cases}$$

Hence we get a system of  $n^2$  linear equations in  $m$  unknowns  $c_0, c_1, \dots, c_{m-1}$ .

We conclude that  $c_0, c_1, \dots, c_{m-1} \in \mathbb{Q}$ . □

- [OR (4E 5.B.16), OR (8.C.18)] Suppose  $a_0, \dots, a_{n-1} \in \mathbb{F}$ . Let  $T$  be the operator on  $\mathbb{F}^n$  such that

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & -a_0 \\ 1 & 0 & & -a_1 \\ & 1 & \ddots & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ with resp to the standard basis } (e_1, \dots, e_n).$$

Show that the mini poly of  $T$  is  $p$  defined by  $p(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + z^n$ .

$\mathcal{M}(T)$  is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the mini poly of some operator. Hence a formula or an algorithm that could produce exact eigvals for each operator on each  $\mathbb{F}^n$  could then produce exact zeros for each poly [ by 8.36(b) ]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an operator.

**SOLUTION:** Note that  $(e_1, Te_1, \dots, T^{n-1}e_1)$  is linely inde.  $\times$  The deg of mini poly is at most  $n$ .

$$\begin{aligned} T^n e_1 &= \dots = T^{n-k} e_{1+k} = \dots = T e_n = -a_0 e_1 - a_1 e_2 - a_2 e_3 - \dots - a_{n-1} e_n \\ &= (-a_0 I - a_1 T - a_2 T^2 - \dots - a_{n-1} T^{n-1}) e_1. \text{ Thus } p(T)e_1 = 0 = p(T)e_j \text{ for each } e_j = T^{j-1}e_1. \end{aligned} \quad \square$$

## • EIGENVALUES ON ODD-DIMENSIONAL REAL VECTOR SPACES

### • EVEN-DIMENSIONAL NULL SPACE

Suppose  $\mathbb{F} = \mathbb{R}$ ,  $V$  is finite-dim,  $T \in \mathcal{L}(V)$  and  $b, c \in \mathbb{R}$  with  $b^2 < 4c$ .

Prove that  $\dim \text{null}(T^2 + bT + cI)$  is an even number.

**SOLUTION:**

Denote  $\text{null}(T^2 + bT + cI)$  by  $R$ . Then  $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$ .

Suppose  $\lambda$  is an eigval of  $T_R$  with an eigvec  $v \in R$ .

Then  $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = ((\lambda + b)^2 + c - \frac{b^2}{4})v$ .

Because  $c - \frac{b^2}{4} > 0$  and we have  $v = 0$ . Thus  $T_R$  has no eigvals.

Let  $U$  be an invar subsp of  $R$  that has the largest, even dim among all invar subsp.

Assume that  $U \neq R$ . Then  $\exists w \in R$  but  $w \notin U$ . Let  $W$  be such that  $(w, T|_R w)$  is a basis of  $W$ .

Because  $T|_R^2 w = -bT|_R w - cw \in W$ . Hence  $W$  is an invar subsp of dim 2.

Thus  $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$ , where  $U \cap W = \{0\}$ ,

for if not, because  $w \notin U, T|_R w \in U$ ,

$U \cap W$  is invar under  $T|_R$  of one dim ( impossible because  $T|_R$  has no eigvecs ).

Hence  $U + W$  is even-dim invar subsp under  $T|_R$ , contradicting the maximality of  $\dim U$ .

Thus the assumption was incorrect. Hence  $R = \text{null}(T^2 + bT + cI) = U$  has even dim.  $\square$

• OPERATORS ON ODD-DIMENSIONAL VECTOR SPACES HAVE EIGENVALUES

(a) Suppose  $\mathbf{F} = \mathbf{C}$ . Then by [5.21], we are done.

(b) Suppose  $\mathbf{F} = \mathbf{R}$ ,  $V$  is finite-dim, and  $\dim V = n$  is an odd number.

Let  $T \in \mathcal{L}(V)$  and the mini poly is  $p$ . Prove that  $T$  has an eigval.

SOLUTION:

(i) If  $n = 1$ , then we are done.

(ii) Suppose  $n \geq 3$ . Assume that every operator, on odd-dim vecsp of dim less than  $n$ , has an eigval.

If  $p$  is a poly multi of  $(x - \lambda)$  for some  $\lambda \in \mathbf{R}$ , then by [8.49]  $\lambda$  is an eigval of  $T$  and we are done.

Now suppose  $b, c \in \mathbf{R}$  such that  $b^2 < 4c$  and  $p$  is a poly multi of  $x^2 + bx + c$  (see [4.17]).

Then  $\exists q \in \mathcal{P}(\mathbf{R})$  such that  $p(x) = q(x)(x^2 + bx + c)$  for all  $x \in \mathbf{R}$ .

Now  $0 = p(T) = (q(T))(T^2 + bT + cI)$ , which means that  $q(T)|_{\text{range}(T^2 + bT + cI)} = 0$ .

Because  $\deg q < \deg p$  and  $p$  is the mini poly of  $T$ , hence  $\text{range}(T^2 + bT + cI) \neq V$ .

$\nexists \dim V$  is odd and  $\dim \text{null}(T^2 + bT + cI)$  is even ( by our previous result ).

Thus  $\dim V - \dim \text{null}(T^2 + bT + cI) = \dim \text{range}(T^2 + bT + cI)$  is odd.

By [5.18],  $\text{range}(T^2 + bT + cI)$  is an invar subsp of  $V$  under  $T$  that has odd dim less than  $n$ .

Our induction hypothesis now implies that  $T|_{\text{range}(T^2 + bT + cI)}$  has an eigval.

By mathematical induction.  $\square$

• (2E Ch5.24) Suppose  $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$  has no eigvals.

Prove that every invar subsp of  $V$  under  $T$  is even-dim.

SOLUTION:

Suppose  $U$  is such a subsp. Then  $T|_U \in \mathcal{L}(U)$ . We prove by contradiction.

If  $\dim U$  is odd, then  $T|_U$  has an eigval and so is  $T$ , so that  $\exists$  invar subsp of 1 dim, contradicts.  $\square$

• (4E 5.B.29) Show that every operator on a finite-dim vecsp of  $\dim \geq 2$  has a 2-dim invar subsp.

SOLUTION:

Using induction on  $\dim V$ .

(i)  $\dim V = 2$ , we are done.

(ii)  $\dim V > 2$ . Assume that the desired result is true for vecsp of smaller dim.

Suppose  $p$  is the mini poly of degree  $m$  and  $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$ .

If  $T = \lambda I$  (  $\Leftrightarrow m = 1 \vee m = -\infty$  ), then we are done. (  $m \neq 0$  because  $\dim V \neq 0$  ).

Now define a  $q$  by  $q(z) = (z - \lambda_1)(z - \lambda_2)$ .

By assumption,  $T|_{\text{null } q(T)}$  has an invar subsp of dim 2. □

ENDED

## 5.B: II

9 14 15 20 | 4E: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 11, 12, 13, 14

- (4E 5.C.1) *Prove or give a counterexample:*

*If  $T \in \mathcal{L}(V)$  and  $T^2$  has an upper-trig matrix, then  $T$  has an upper-trig matrix.*

SOLUTION:

- (4E 5.C.2) *Suppose  $A$  and  $B$  are upper-trig matrices of the same size, with  $\alpha_1, \dots, \alpha_n$  on the diag of  $A$  and  $\beta_1, \dots, \beta_n$  on the diag of  $B$ .*

(a) *Show that  $A + B$  is an upper-trig matrix with  $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$  on the diag.*

(b) *Show that  $AB$  is an upper-trig matrix with  $\alpha_1\beta_1, \dots, \alpha_n\beta_n$  on the diag.*

SOLUTION:

- (4E 5.C.3)

*Suppose  $T \in \mathcal{L}(V)$  is inv and  $B = (v_1, \dots, v_n)$  is a basis of  $V$  such that  $\mathcal{M}(T, B) = A$  is upper trig, with  $\lambda_1, \dots, \lambda_n$  on the diag.*

*Show that the matrix of  $\mathcal{M}(T^{-1}, B) = A^{-1}$  is also upper trig, with  $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$  on the diag.*

SOLUTION:

- 9 [4E 5.C.7] *Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ , and  $v \in V$ .*

(a) *Prove that  $\exists!$  monic poly  $p_v$  of smallest degree such that  $p_v(T)v = 0$ .*

(b) *Prove that the mini poly of  $T$  is a poly multi of  $p_v$ .*

SOLUTION:

- 14 [OR (4E 5.C.4)] *Give an operator  $T$  such that with resp to some basis,  $\mathcal{M}(T)_{k,k} = 0$  for each  $k$ , while  $T$  is inv.*

SOLUTION:

- 15 [OR (4E 5.C.5)] *Give an operator  $T$  such that with resp to some basis,  $\mathcal{M}(T)_{k,k} \neq 0$  for each  $k$ , while  $T$  is not inv.*

SOLUTION:

- 20 [OR (OR 4E 5.C.6)]

*Suppose  $\mathbf{F} = \mathbf{C}$ ,  $V$  is finite-dim, and  $T \in \mathcal{L}(V)$ .*

*Prove that if  $k \in \{1, \dots, \dim V\}$ , then  $V$  has a  $k$  dim subsp invar under  $T$ .*

SOLUTION:

- (4E 5.C.8) *Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ , and  $\exists v \in V \setminus \{0\}$  such that  $T^2v + 2Tv = -2v$ .*

(a) *Prove that if  $\mathbf{F} = \mathbf{R}$ , then  $\nexists$  a basis of  $V$  with resp to which  $T$  has an upper-trig matrix.*

(b) *Prove that if  $\mathbf{F} = \mathbf{C}$  and  $A$  is an upper-trig matrix that equals the matrix of  $T$  with resp to some basis of  $V$ , then  $-1 + i$  or  $-1 - i$  appears on the diag of  $A$ .*

SOLUTION:

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- (4E 5.C.9) Suppose  $B \in \mathbf{F}^{n,n}$  with complex entries.  
Prove that  $\exists$  inv  $A \in \mathbf{F}^{n,n}$  with complex entries such that  $A^{-1}BA$  is an upper-trig matrix.

SOLUTION:

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- (4E 5.C.10) Suppose  $T \in \mathcal{L}(V)$  and  $(v_1, \dots, v_n)$  is a basis of  $V$ .  
Show that the following are equi.
  - (a) The matrix of  $T$  with resp to  $(v_1, \dots, v_n)$  is lower trig.
  - (b)  $\text{span}(v_k, \dots, v_n)$  is invar under  $T$  for each  $k = 1, \dots, n$ .
  - (c)  $Tv_k \in \text{span}(v_k, \dots, v_n)$  for each  $k = 1, \dots, n$ .

SOLUTION:

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- (4E 5.C.11) Suppose  $\mathbf{F} = \mathbf{C}$  and  $V$  is finite-dim.  
Prove that if  $T \in \mathcal{L}(V)$ , then  $T$  has a lower-trig matrix with resp to some basis.

SOLUTION:

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- (4E 5.C.12)  
Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$  has an upper-trig matrix with resp to some basis, and  $U$  is a subsp of  $V$  that is invar under  $T$ .
  - (a) Prove that  $T|_U$  has an upper-trig matrix with resp to some basis of  $U$ .
  - (b) Prove that  $T/U$  has an upper-trig matrix with resp to some basis of  $V/U$ .

SOLUTION:

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- (4E 5.C.13) Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ . Suppose  $U$  is an invar subsp of  $V$  under  $T$  such that  $T|_U$  has an upper-trig matrix and also  $T/U$  has an upper-trig matrix.  
Prove that  $T$  has an upper-trig matrix.

SOLUTION:

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- (4E 5.C.14) Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .  
Prove that  $T$  has an upper-trig matrix  $\iff T'$  has an upper-trig matrix.

SOLUTION:

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ENDED

## 5.C

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ENDED

## 5.E\* (4E)    1 2 3 4 5 6 7 8 9 10

- 1 Give an example of two commuting operators  $S, T \in \mathbf{F}^4$  such that  
there is an invar subsp of  $\mathbf{F}^4$  under  $S$  but not under  $T$   
and an invar subsp of  $\mathbf{F}^4$  under  $T$  but not under  $S$ .

SOLUTION:

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- 2** Suppose  $\mathcal{E}$  is a subset of  $\mathcal{L}(V)$  and every element of  $\mathcal{E}$  is diagable.  
 Prove that  $\exists$  a basis of  $V$  with resp to which  
 every element of  $\mathcal{E}$  has a diag matrix  $\iff$  every pair of elements of  $\mathcal{E}$  commutes.  
 This exercise extends [5.76], which considers the case in which  $\mathcal{E}$  contains only two elements.  
 For this exercise,  $\mathcal{E}$  may contain any number of elements, and  $\mathcal{E}$  may even be an infinite set.

**SOLUTION:**

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- 3** Suppose  $S, T \in \mathcal{L}(V)$  are such that  $ST = TS$ . Suppose  $p \in \mathcal{P}(\mathbf{F})$ .  
 (a) Prove that  $\text{null } p(S)$  is invar under  $T$ .  
 (b) Prove that  $\text{range } p(S)$  is invar under  $T$ .  
 See NOTE FOR[5.17] for the special case  $S = T$ .

**SOLUTION:**

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- 4** Prove or give a counterexample:  
 A diag matrix  $A$  and an upper-trig matrix  $B$  of the same size commute.

**SOLUTION:**

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- 5** Prove that a pair of operators on a finite-dim vecsp commute  $\iff$  their dual operators commute.

**SOLUTION:**

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- 6** Suppose  $V$  is a finite-dim complex vecsp and  $S, T \in \mathcal{L}(V)$  commute.  
 Prove that  $\exists \alpha, \lambda \in \mathbf{C}$  such that  $\text{range}(S - \alpha I) + \text{range}(T - \lambda I) \neq V$ .

**SOLUTION:**

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- 7** Suppose  $V$  is a complex vecsp,  $S \in \mathcal{L}(V)$  is diagable, and  $T$  commutes with  $S$ .  
 Prove that  $\exists$  basis  $B$  of  $V$  such that  $S$  has a diag matrix with resp to  $B$   
 and  $T$  has an upper-trig matrix with resp to  $B$ .

**SOLUTION:**

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- 8** Suppose  $m = 3$  in Example [5.72]  
 and  $D_x, D_y$  are the commuting partial differentiation operators on  $\mathcal{P}_3(\mathbf{R}^2)$  from that example.  
 Find a basis of  $\mathcal{P}_3(\mathbf{R}^2)$  with resp to which  $D_x$  and  $D_y$  each have an upper-trig matrix.

**SOLUTION:**

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- 9** Suppose  $V$  is a finite-dim nonzero complex vecsp.  
 Suppose that  $\mathcal{E} \subseteq \mathcal{L}(V)$  is such that  $S$  and  $T$  commute for all  $S, T \in \mathcal{E}$ .  
 (a) Prove that  $\exists v \in V$  is an eigvec for every element of  $\mathcal{E}$ .  
 (b) Prove that  $\exists$  a basis of  $V$  with resp to which every element of  $\mathcal{E}$  has an upper-trig matrix.

**SOLUTION:**

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- 10** Give an example of two commuting operators  $S, T$  on a finite-dim real vecsp such that  
 $S + T$  has a eigval that does not equal an eigval of  $S$  plus an eigval of  $T$   
 and  $ST$  has a eigval that does not equal an eigval of  $S$  times an eigval of  $T$ .

**SOLUTION:**

