简介

这是我个人用于复习的「***Linear Algebra Done Right 3E/4E, by Sheldon Axler***」笔记，一本习题选答与课文补注。因为使用中文会给我编撰这份笔记带来额外的中英文输入法切换的工作成本，况且对于专业学习者，直接使用英文不会造成任何困扰。但英文词句的冗长性拖慢我复习的效率，所以我对许多常用术语作了简写。这份笔记的内容范围和标识说明，我已经在自述中写得很清楚，不再赘述。这份笔记尚处于缓慢的编撰进度中。

我目前还没有能力和资格评论原书好坏以及线性代数课程教材选用的问题。但作为原书的学习者，我可以说，这本书作为初学线性代数的第一教材，虽然不需要其他辅助教材，但要求学习者有足够的耐心和毅力：课文一次看不懂就多看几遍，一天看不懂就分三天看；习题一个小时做不出来，隔六个小时再尝试，一天做不出来，就隔天再尝试。我虽然没有学过除此以外的其他任何线性代数教材，但我认为这样钻研原书是值得的。

Goto

1	2	3	4	5	6	7	8	9	10
A	A	A		A	A	A	A	A	A
B	B	B		B$^{\mathrm{I}}$	B	B	B	B	B
				B$^{\mathrm{II}}$					
C	C	C		C	C	C	C		
		D			D	D	D		
		E		E*					
		F				F*			

Abbreviation Table

def	definition	vec	vector
vecsp	vector space	subsp	subspace
add	addition/additive	multi	multiplication/multiplicative/multiple
assoc	associative/associativity	distr	distributive properties/property
inv	inverse	existns	existence
uniqnes	uniqueness	linely inde	linearly independent/independence
linely dep	linearly dependent/dependence	dim	dimension(al)
coeff	coefficient	degree	deg
req	require(d)/requiring	B_V	basis of V
inje	injective	surj	surjective
col	column	with resp	with respect
standard basis	std basis	iso	isomorphism/isomorphic
correspd	correspond(ing)	poly	polynomial
eigval	eigenvalue	eigvec	eigenvector
mini poly	minimal polynomial	char poly	characteristic polynomial

1·B

1 *Prove that $\forall v \in V, -(-v) = v$.*

Solution: $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$.

Or. Because $-(-v) + (-v) = 0$ 又 $v + (-v) = 0$. Now by the uniqnes of add inv. $\square$

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2 *Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.*

Solution: Suppose $a \neq 0, \exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. $\square$

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3 *Suppose $v, w \in V$. Explain why $\exists\,!\, x \in V, v + 3x = w$.*

Solution: $v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \frac{1}{3}(w - v)$. $\square$

Or. $\left[\,\textit{Existns}\,\right]$ Let $x = \dfrac{1}{3}(w - v)$.

$\left[\,\textit{Uniqnes}\,\right]$ If $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) $-$ (II) : $3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. $\square$

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5 *Show that in the def of a vecsp, the add inv condition can be replaced by [1.29].*

Hint: *Suppose V satisfies all conds in the def, except we've replaced the add inv cond with [1.29]. Prove that the add inv is true.*

Using [1.31]. $0v = 0$ for all $v \in V \Longleftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. $\square$

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6 *Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$.*
Define an add and scalar multi on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess.
The operations of real numbers is as usual. While for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

(I) $t + \infty = \infty + t = \infty + \infty = \infty$,
(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,
(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of add and scalar multi, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vecsp over $\mathbf{R}$? Explain.

Solution: Not a vecpsp, since the add and scalar mult is not assoc and distr.

By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

Or. By Distr: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. $\square$

- **Tips:** *About the Field $\mathbf{F}$: Many choices.*
  Example: $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \ldots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$. [Using Euler's Theorem.]

Ended

1·C

7 8 9 11 12 13 15 16 17 18 21 23 24

- **Note For [1.45]:** If $\mathbf{F} = \{0, 1\}$. Prove that if $U + W$ is a direct sum, then $U \cap W = \{0\}$.
  Because $\forall v \in U \cap W, \exists\,!\,(u, w) \in U \times W, v = u + w$.
  If $U \cap W \neq \{0\}$, then (u, w) can be $(v, 0)$ or $(0, v)$, contradicts the uniqnes. $\square$

• **Tips 1:** Suppose $U, W \subseteq V$. And U, W, V are vecsps $\Rightarrow U, W$ are subsps of V.
Then $U + W$ is also a subsp of V. Because $\forall u \in U, w \in U, u + w \in V$ since $u, w \in V$.

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7 *Give a nonempty $U \subseteq \mathbf{R}^2$,*
U is closed under taking add invs and under add, but is not a subsp of $\mathbf{R}^2$.

Solution: $\big($ $0 \in U$; $v \in U \Rightarrow -v \in U$. And operations on U are the same as $\mathbf{R}^2$. $\big)$ Let $\mathbf{Z}^2, \mathbf{Q}^2$.

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8 *Give a nonempty $U \subseteq \mathbf{R}^2$, U is closed under scalar multi, but is not a subsp of $\mathbf{R}^2$.*

Solution: Let $U = \big\{(x, y) \in \mathbf{R}^2 : x = 0 \vee y = 0\big\}$.

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9 *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if $\exists\, p \in \mathbf{N}^+$, $f(x) = f(x + p)$ for all $x \in \mathbf{R}$.*
Is the set of periodic functions $\mathbf{R} \to \mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution: Denote the set by S.
Suppose $h(x) = \cos x + \sin\sqrt{2}x \in S$, since $\cos x, \sin\sqrt{2}x \in S$.
Assume $\exists\, p \in \mathbf{N}^+$ such that $h(x) = h(x + p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.
Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$
$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.
Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Contradiction! $\square$

Or. Because $[\mathrm{I}] : \cos x + \sin\sqrt{2}x = \cos(x + p) + \sin\big(\sqrt{2}x + \sqrt{2}p\big)$. By differentiating twice,
$[\mathrm{II}] : \cos x + 2\sin\sqrt{2}x = \cos(x + p) + 2\sin\big(\sqrt{2}x + \sqrt{2}p\big)$.

$$\left.\begin{array}{ll} [\mathrm{II}] - [\mathrm{I}] : & \sin\sqrt{2}x = \sin\big(\sqrt{2}x + \sqrt{2}p\big) \\ 2[\mathrm{I}] - [\mathrm{II}] : & \cos x = \cos(x + p) \end{array}\right\} \Rightarrow \text{Let } x = 0,\ p = \frac{m\pi}{\sqrt{2}} = 2k\pi.\ \text{Contradicts.}$$ $\square$

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24 *Let $V_E = \big\{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is even}\big\}, V_O = \big\{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is odd}\big\}$. Show that $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$.*

Solution: (a) $V_E \cap V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x)\} = \{0\}$.

$$\text{(b)} \left|\begin{array}{l} \text{Let } f_e(x) = \dfrac{1}{2}\big[g(x) + g(-x)\big] \Longrightarrow f_e \in V_E \\ \text{Let } f_o(x) = \dfrac{1}{2}\big[g(x) - g(-x)\big] \Longrightarrow f_o \in V_O \end{array}\right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}},\ g(x) = f_e(x) + f_o(x).$$ $\square$

• *Suppose U, W, V_1, V_2, V_3 are subsps of V.*
15 $U + U \ni u + w \in U$. **16** $U + W \ni u + w = w + u \in W + U$. $\square$
17 $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3)$. $\square$
• $(U + W)_{\mathrm{C}} \ni (u_1 + w_1) + \mathrm{i}(u_2 + w_2) = (u_1 + \mathrm{i}u_2) + (w_1 + \mathrm{i}w_2) \in U_{\mathrm{C}} + W_{\mathrm{C}}$. $\square$

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18 *Does the add on the subsps of V have an add identity? Which subsps have add invs?*

Solution: Suppose Ω is the unique add identity.
(a) For any subsp U of V. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.
(b) Now suppose W is an add inv of $U \Rightarrow U + W = \Omega$.
Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. $\square$

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11 *Prove that the intersection of every collection of subsps of V is a subsp of V.*

SOLUTION: Suppose $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collection of subsps of V; here Γ is an index set.

We show that $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vecs that are in U_α for each $\alpha \in \Gamma$, is a subsp of V.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u+v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u+v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under add.

(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under scalar multi.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of V that is closed under add and scalar multi. $\square$

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12 *Suppose U, W are subsps of V. Prove that $U \cup W$ is a subsp of $V \iff U \subseteq W$ or $W \subseteq U$.*

SOLUTION: (a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subsp of V.

(b) Suppose $U \cup W$ is a subsp of V. Assume that $U \not\subseteq W,\ U \not\supseteq W$ ($U \cup W \neq U$ and W).

Then $\forall a \in U \wedge a \notin W, \forall b \in W \wedge b \notin U$, we have $a + b \in U \cup W$.

$a+b \in U \Rightarrow b = (a+b) + (-a) \in U$, contradicts $\Rightarrow W \subseteq U$. | Contradicts the

$a+b \in W \Rightarrow a = (a+b) + (-b) \in W$, contradicts $\Rightarrow U \subseteq W$. | assumption. $\square$

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13 *Prove that the union of three subsps of V is a subsp of V if and only if one of the subsps contains the other two.*

This exercise is not true if we replace $\mathbf{F}$ with a field containing only two elements.

SOLUTION:

Suppose U_1, U_2, U_3 are subsps of V. Denote $U_1 \cup U_2 \cup U_3$ by $\mathcal{U}$.

(a) Suppose that one of the subsps contains the other two.

Then $\mathcal{U} = U_1, U_2$ or U_3 is a subsp of V.

(b) Suppose that $U_1 \cup U_2 \cup U_3$ is a subsp of V.

Distinctively notice that $A \cup B \cup C = (A\cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.

Also note that, if $U \cup W = V$ is a vecsp, then in general U and W are not subsps of V.

Hence this literal trick is invalid.

(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.

By applying Problem (12) we conclude that one U_j contains the other two. Thus we are done.

(II) Assume that no U_j is contained in the union of the other two, and no U_j contains the union of the other two. Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3;\ v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if any $v + \lambda u \in W \cap U_1$ then $v + \lambda u - \lambda u = v \in U_1$.

Now $W \subseteq U_1 \cup U_2 \cup U_3 \Rightarrow W \subseteq U_2 \cup U_3$. $\forall v + \lambda u \in W, v + \lambda u \in U_i, i = 2, 3$.

If $U_2 \subseteq U_3$ or $U_2 \supseteq U_3$, then $\mathcal{U} = U_1 \cup U_i, i = 2, 3$. By Problem (12) we are done.

Otherwise, both $U_2, U_3 \neq \{0\}$. Because $W \subseteq U_2 \cup U_3$ *has at least three elements.*

There must be some U_i that contains at least two elements of W.

$\exists$ distinct $\lambda_1, \lambda_2 \in \mathbf{F}, v + \lambda_1 u, v + \lambda_2 u \in U_i, i \in \{2, 3\}$.

Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Contradicts. $\square$

EXAMPLE: Let $\mathbf{F} = \mathbf{Z}_2$. $U_1 = \{u, 0\}, U_2 = \{v, 0\}, U_3 = \{v + u, 0\}$. While $\mathcal{U} = \{0, u, v, v + u\}$ is a subsp.

• **Example:** *Suppose* $U = \{(x,x,y,y) \in \mathbf{F}^4\}, W = \{(x,x,x,y) \in \mathbf{F}^4\}$.
Prove that $U + W = \{(x,x,y,z) \in \mathbf{F}^4\}$.

Let T denote $\{(x,x,y,z) \in \mathbf{F}^4 : x,y,z \in \mathbf{F}\}$. By def, $U + W \subseteq T$.
And $T \ni (x,x,y,z) \Rightarrow (0,0,y-x,y-x) + (x,x,x,-y+x+z) \in U + W$. Hence $T \subseteq U + W$. $\square$

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21 *Suppose* $U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5\}$. *Find a* W *such that* $\mathbf{F}^5 = U \oplus W$.

Solution: Let $W = \{(0,0,z,w,u) \in \mathbf{F}^5\}$. Then $U \cap W = \{0\}$.
And $\mathbf{F}^5 \ni (x,y,z,w,u) \Rightarrow (x,y,x+y,x-y,2x) + (0,0,z-x-y,w-x-y,u-2x) \in U + W$.

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23 *Give an example of vecsps* V_1, V_2, U *such that* $V_1 \oplus U = V_2 \oplus U$, *but* $V_1 \neq V_2$.

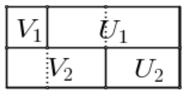
Solution: $V = \mathbf{F}^2, U = \{(x,x) \in \mathbf{F}^2\}, V_1 = \{(x,0) \in \mathbf{F}^2\}, V_2 = \{(0,x) \in \mathbf{F}^2\}$.

• **Tips 2:** *Suppose* $V_1 \subseteq V_2$ *in Exercise* (23). *Prove that* $V_1 = V_2$.

Solution:

Because the subset V_1 of vecsp V_2 is closed under add and scalar multi, V_1 is a subspace of V_2.
Suppose W is such that $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$.
If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, contradicts. Hence $W = \{0\}, V_1 = V_2$. $\square$

• *Suppose* V_1, V_2, U_1, U_2 *are vecsps,* $V_1 \oplus U_1 = V_2 \oplus U_2,\ V_1 \subseteq V_2,\ U_2 \subseteq U_1$.
Prove or give a counterexample: $V_1 = V_2,\ U_1 = U_2$.



Solution: Let $U_2 = \{0\}$. Give an example that each of V_1, V_2, U_1 is nonzero. $\square$

• **Tips 3:** *Suppose the intersection of any two of the vecsps* U, W, X, Y *is* $\{0\}$.
Give an example that $(X \oplus U) \cap (Y \oplus W) \neq \{0\}$.

Solution: [Using notations in Chapter 2.] Let $B_X = (e_1), B_U = (e_2 - e_1), B_Y = (\ ), B_W = (e_2)$.

• **Tips 4:** *Let* $V = U + W,\ I = U \cap W,\ U = I \oplus X,\ W = I \oplus Y$. *Prove that* $V = I \oplus (X \oplus Y)$.

Solution: We show that $X \cap Y = U \cap Y = W \cap X = \{0\}$ by contradiction.
$X \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap X \neq \{0\}, I \cap Y \neq \{0\}$.
$U \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap Y \neq \{0\}$. Similar for $W \cap X$.
Thus $I + (X + Y) = (I \oplus X) \oplus Y = I \oplus (X \oplus Y)$.
Now we show that $V = I + (X + Y)$. $\forall v \in V, v = u + w, \exists\, (u,w) \in U \times W$
$\Rightarrow \exists\, (i_u, x_u) \in I \times X,\ (i_w, y_w) \in I \times Y,\ v = (i_u + i_w) + x_u + y_w \in I + (X + Y)$. $\square$

Ended

2·A

1 2 10 11 14 16 17 | 4E: 3,14

1 *Prove that* $[P]$ (v_1, v_2, v_3, v_4) *spans* $V \iff \big(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\big)$ *also spans* V $[Q]$.

Solution: Note that $V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists\, a_1, \dots, a_n \in \mathbf{F}, v = a_1 v_1 + \dots + a_n v_n$.

Assume that $\forall v \in V, \exists\, a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbf{F}$, (that is, if $\exists\, a_i$, then we are to find b_i, vice versa)

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4 \\ &= a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \dots + a_4) v_4. \end{aligned}$$ □

• *Suppose* $(v_1, \dots, v_m)$ *is a list of vecs in* V. *For each* k, *let* $w_k = v_1 + \dots + v_k$.

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

(b) *Show that* $[P]$ $(v_1, \dots, v_m)$ *is linely inde* $\iff (w_1, \dots, w_m)$ *is linely inde* $[Q]$.

Solution:

(a) Assume $a_1 v_1 + \dots + a_m v_m = b_1 w_1 + \dots + b_m w_m = b_1 v_1 + \dots + b_k(v_1 + \dots + v_k) + \dots + b_m(v_1 + \dots + v_m)$.
Then $a_k = b_k + \dots + b_m$; $a_{k+1} = b_{k+1} + \dots + b_m \Rightarrow b_k = a_k - a_{k+1}$; $b_m = a_m$. Similar to Problem (1).

(b) $P \Rightarrow Q$: $b_1 w_1 + \dots + b_m w_m = 0 = a_1 v_1 + \dots + a_m v_m$, where $0 = a_k = b_k + \dots + b_m$.
$Q \Rightarrow P$: $a_1 v_1 + \dots + a_m v_m = 0 = b_1 w_1 + \dots + b_m w_m = 0$, where $0 = b_m = a_m$, $0 = b_k = a_k - a_{k+1}$.

Or. By (a), let $W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$. Suppose $(w_1, \dots, w_m)$ is linely dep.
By [2.21](b), a list of length $(m-1)$ spans W. 又 By [2.23], $(w_1, \dots, w_m)$ linely inde $\Rightarrow m \leqslant m - 1$.
Thus $(w_1, \dots, w_m)$ is linely dep. Now reversing the roles of v and w. □

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2 (a) $[P]$ *A list* (v) *of length* 1 *in* V *is linely inde* $\iff v \neq 0$. $[Q]$

(b) $[P]$ *A list* (v, w) *of length* 2 *in* V *is linely inde* $\iff \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$. $[Q]$

Solution: (a) $Q \Rightarrow P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ linely inde.
$P \Rightarrow Q : (v)$ linely inde $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.
$\neg Q \Rightarrow \neg P : v = 0 \Rightarrow av = 0$ while we can let $a \neq 0 \Rightarrow (v)$ is linely dep.
$\neg P \Rightarrow \neg Q : (v)$ linely dep $\Rightarrow av = 0$ while $a \neq 0 \Rightarrow v = 0$.

(b) $P \Rightarrow Q : (v, w)$ linely inde $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.
$Q \Rightarrow P$: no scalar multi $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v, w)$ linely inde.
$\neg P \Rightarrow \neg Q : (v, w)$ linely dep $\Rightarrow$ if $av + bw = 0$, then a or $b \neq 0 \Rightarrow$ scalar multi.
$\neg Q \Rightarrow \neg P$: scalar multi $\Rightarrow$ if $av + bw = 0$, then a or $b \neq 0 \Rightarrow$ linely dep. □

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10 *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* V *and* $w \in V$.
Prove that if $\big(v_1 + w, \dots, v_m + w\big)$ *is linely depe, then* $w \in \operatorname{span}(v_1, \dots, v_m)$.

Solution:

Note that $a_1(v_1 + w) + \dots + a_m(v_m + w) = 0 \Rightarrow a_1 v_1 + \dots + a_m v_m = -(a_1 + \dots + a_m) w$.
Then $a_1 + \dots + a_m \neq 0$, for if not, $a_1 v_1 + \dots + a_m v_m = 0$ while $a_i \neq 0$ for some i, contradicts.
Or. We prove the contrapositive: Suppose $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $a_1 + \dots + a_m = 0$.
Thus $a_1 v_1 + \dots + a_m v_m = 0 \Rightarrow a_1 = \dots = a_m = 0$. Hence $(v_1 + w, \dots, v_m + w)$ is linely inde. □

Or. $\exists\, j \in \{1, \dots, m\}, v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and we are done.
If $j \geqslant 2$, then $\exists\, a_i \in \mathbf{F}, v_j + w = a_1(v_1 + w) + \dots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1 v_1 + \dots + a_{j-1} v_{j-1}$.
Where $\lambda = 1 - (a_1 + \dots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, contradicts.
Now $w = \lambda^{-1}(a_1 v_1 + \dots + a_{j-1} v_{j-1} - v_j) \Rightarrow w \in \operatorname{span}(v_1, \dots, v_m)$. □

---

11 *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* V *and* $w \in V$.
Show that $[P]$ $(v_1, \dots, v_m, w)$ *is linely inde* $\iff w \notin \operatorname{span}(v_1, \dots, v_m)$ $[Q]$.

Solution: $\neg Q \Rightarrow \neg P$: Suppose $w \in \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1, \dots, v_m, w)$ is linely depe.
$\neg P \Rightarrow \neg Q$: Suppose $(v_1, \dots, v_m, w)$ is linely dep. Then by $[2.21](\mathrm{a})$, $w \in \operatorname{span}(v_1, \dots, v_m)$. □

---

14 *Prove that* $[P]$ V *is infinite-dim* $\iff [Q]$ *there is a sequence* $(v_1, v_2, \dots)$ *in* V *such that* $(v_1, \dots, v_m)$ *is linely inde for each* $m \in \mathbf{N}^+$.

Solution:

$P \Rightarrow Q$: Suppose V is infinite-dim, so that no list spans V.
Step 1 Pick a $v_1 \neq 0$, (v_1) linely inde.
Step m Pick a $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$, by Problem (11), $(v_1, \dots, v_m)$ is linely inde.
This process recursively defines the desired sequence $(v_1, v_2, \dots)$.

$\neg P \Rightarrow \neg Q$: Suppose V is finite-dim and $V = \operatorname{span}(w_1, \dots, w_m)$.
Let $(v_1, v_2, \dots)$ be a sequence in V, then $(v_1, v_2, \dots, v_{m+1})$ must be linely dep.

Or. $Q \Rightarrow P$: Suppose there is such a sequence.
Choose an m. Suppose a linely inde list $(v_1, \dots, v_m)$ spans V.
Similar to [2.16]. $\exists\, v_{m+1} \in V \backslash \operatorname{span}(v_1, \dots, v_m)$. Hence no list spans V. □

---

16 *Prove that the vecsp of all continuous functions in* $\mathbf{R}^{[0,1]}$ *is infinite-dim.*

Solution: Denote the vecsp by U.
Choose one $m \in \mathbf{N}^+$. Suppose $a_0, \dots, a_m \in \mathbf{R}$ are such that $p(x) = a_0 + a_1 x + \dots + a_m x^m = 0$, $\forall x \in [0,1]$.
Then p has infinitely many roots and hence each $a_k = 0$, otherwise $\deg p \geqslant 0$, contradicts [4.12].
Thus $(1, x, \dots, x^m)$ is linely inde in $\mathbf{R}^{[0,1]}$. Similar to [2.16], U is infinite-dim. □

Or. Note that $\frac{1}{1} > \frac{1}{2} > \dots > \frac{1}{m}$, $\forall m \in \mathbf{N}^+$. Suppose $f_m = \begin{cases} x - \frac{1}{m}, & x \in \left(\frac{1}{m}, 1\right] \\ 0, & x \in \left[0, \frac{1}{m}\right] \end{cases}$

Then $f_1\left(\frac{1}{m}\right) = \dots = f_m\left(\frac{1}{m}\right) = 0 \neq f_{m+1}\left(\frac{1}{m}\right)$. Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. By Problem (14). □

---

17 *Suppose* $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ *such that* $p_k(2) = 0$ *for each* $k \in \{0, \dots, m\}$.
Prove that $(p_0, p_1, \dots, p_m)$ *is not linely inde in* $\mathcal{P}_m(\mathbf{F})$.

Solution:

Suppose $(p_0, p_1, \dots, p_m)$ is linely inde. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z$.
Notice that $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \dots + a_m p_m(z)$, for if not, let $z = 2$. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has length $(m+1)$.
Hence $(p_0, p_1, \dots, p_m)$ is linely depe in $\mathcal{P}_m(\mathbf{F})$.
For if not, then because $(1, z, \dots, z^m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$,
by the steps in [2.23] trivially, $(p_0, p_1, \dots, p_m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. □

Or. Note that $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(\underbrace{1, z, \dots, z^m}_{\text{of length } (m+1)})$. Then $(p_0, p_1, \dots, p_m, z)$ of length $(m+2)$ is linely dep.
As shown above, $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$. And hence by $[2.21](\mathrm{a})$, $(p_0, p_1, \dots, p_m)$ is linely dep. □

Ended

2·B   1 7 8 | 4E: 5 9 11

7 *Prove or give a counterexample: If (v_1, v_2, v_3, v_4) is a basis of V and U is a subsp of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then (v_1, v_2) is a basis of U.*

Solution: A counterexample: Let $V = \mathbf{R}^4$ and $B_V = (e_1, e_2, e_3, e_4)$ be std basis.
Let $v_1 = e_1, v_2 = e_2, v_3 = e_3 + e_4, v_4 = e_4$. Then $(v_1, \dots, v_4)$ is a basis of $\mathbf{R}^4$.
Let $U = \operatorname{span}(e_1, e_2, e_3) = \operatorname{span}(v_1, v_2, v_3 - v_4)$. Then $v_3 \notin U$ and (v_1, v_2) is not a basis of U. $\square$

• **Note For "$\complement_V U \cup \{0\}$":** "$\complement_V U \cup \{0\}$" is supposed to be a subsp W such that $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cup \{0\} \\ u \pm w \in \complement_V U \cup \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cup \{0\}$. Contradicts.

To fix this, denote the set $\{W_1, W_2, \cdots\}$ by $\mathcal{S}_V U$, where for each W_i, $V = U \oplus W_i$. See also in (1.C.23).

• **Tips:** *Suppose V is finite-dim with $\dim V = n$ and U is a subsp of V with $U \neq V$. Prove that $\exists\, B_V = (v_1, \dots, v_n)$ such that each $v_k \notin U$.*

Note that $U \neq V \Rightarrow n \geqslant 1$. We will construct B_V via the following process.

Step 1. $\exists\, v_1 \in V \backslash U \Rightarrow v_1 \neq 0$. If $\operatorname{span}(v_1) = V$ then we stop.

Step k. Suppose $(v_1, \dots, v_{k-1})$ is linely inde in V, each of which belongs to $V \backslash U$.
Note that $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$. And if $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U = V$, then by (1.C.12),
$\big[$ because $\operatorname{span}(v_1, \dots, v_{k-1}) \not\subseteq U$, $\big]$ $U \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) \Rightarrow \operatorname{span}(v_1, \dots, v_{k-1}) = V$.
Hence because $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$, it must be case that $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U \neq V$.
Thus $\exists\, v_k \in V \backslash U$ such that $v_k \notin \operatorname{span}(v_1, \dots, v_{k-1})$.
By (2.A.11), $(v_1, \dots, v_k)$ is linely inde in V. If $\operatorname{span}(v_1, \dots, v_k) = V$, then we stop.

Because V is finite-dim, this process will stop after n steps. $\square$

Or. Suppose $U \neq \{0\}$. Let $B_U = (u_1, \dots, u_m)$. Extend to a basis $(u_1, \dots, u_n)$ of V.
Then let $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$. $\square$

---

1 *Find all vecsps on whatever $\mathbf{F}$ that have exactly one basis.*

Solution: The trivial vecsp $\{0\}$ will do. Indeed, the only basis of $\{0\}$ is the empty list ( ).
Now consider the field $\{0, 1\}$ containing only the add identity and multi identity,
with $1 + 1 = 0$. Then the list (1) is the unique basis. Now the vecsp $\{0, 1\}$ will do.
Comment: All vecsp on such $\mathbf{F}$ of dim 1 will do.
And more generally, consider $\mathbf{F} = \mathbf{Z}_m$, $\forall m - 1 \in \mathbf{N}^+$. For each $s, t \in \{1, \dots, m\}$,
$\mathbf{F} = \operatorname{span}(K_s) = \operatorname{span}(K_t)$. More than one basis. So are $\mathbf{Q}, \mathbf{R}, \mathbf{C}$ and all vecsps on such $\mathbf{F}$.
Consider other $\mathbf{F}$. Note that this $\mathbf{F}$ contains at least and strictly more than 0 and 1. Failed. $\square$

• (4E 9) *Suppose $(v_1, \dots, v_m)$ is a list of vecs in V. For $k \in \{1, \dots, m\}$, let $w_k = v_1 + \cdots + v_k$. Show that $[P]$ $B_V = (v_1, \dots, v_m) \Longleftrightarrow B_W = (w_1, \dots, w_m)$. $[Q]$*

Solution: Notice that $B_U = (u_1, \dots, u_n) \Longleftrightarrow \forall u \in U, \exists!\, a_i \in \mathbf{F}, u = a_1 u_1 + \cdots + a_n u_n$.
$P \Rightarrow Q: \forall v \in V, \exists!\, a_i \in \mathbf{F},\ v = a_1 v_1 + \cdots + a_m v_m \Rightarrow v = b_1 w_1 + \cdots + b_m w_m, \exists!\, b_k = a_k - a_{k+1}, b_m = a_m$.
$Q \Rightarrow P: \forall v \in V, \exists!\, b_i \in \mathbf{F},\ v = b_1 w_1 + \cdots + b_m w_m \Rightarrow v = a_1 v_1 + \cdots + a_m v_m, \exists!\, a_k = \sum_{j=k}^{m} b_j$. $\square$
Comment: See also ??? in (3.F).

• (4E 5) *Suppose U, W are finite-dim, $V = U + W$, $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$. Prove that $\exists\, B_V$ consisting of vecs in $U \cup W$.*

Solution: $V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(\overbrace{u_1, \dots, u_m, w_1, \dots, w_n}^{\text{Reduce}})$. By [2.31]. $\square$

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8 *Suppose $V = U \oplus W$, $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$. Prove that $B_V = \big(u_1, \dots, u_m, w_1, \dots, w_n\big)$.*

Solution: $\forall v \in V, \exists!\, u \in U, w \in W \Rightarrow \exists!\, a_i, b_i \in \mathbf{F}, v = u + w = \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i$.

Or. $V = \operatorname{span}(u_1, \dots, u_m) \oplus \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}\big(u_1, \dots, u_m, w_1, \dots, w_n\big)$.

Note that $\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0 \Rightarrow \sum_{i=1}^{m} a_i u_i = -\sum_{i=1}^{n} b_i w_i \in U \cap W = \{0\}$. $\square$

• (9.A.3,4 Or 4E 11) *Suppose V is on $\mathbf{R}$, and $v_1, \dots, v_n \in V$. Let $B = (v_1, \dots, v_n)$.*
(a) *Show that $[P]$ B is linely inde in $V \iff B$ is linely inde in $V_{\mathbf{C}}$. $[Q]$*
(b) *Show that $[P]$ B spans $V \iff B$ spans $V_{\mathbf{C}}$. $[Q]$*

(a) $P \Rightarrow Q:$ Note that each $v_k \in V_{\mathbf{C}}$. $\quad Q \Rightarrow P:$ If $\lambda_k \in \mathbf{R}$ with $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then each $\operatorname{Re} \lambda_k = \lambda_k = 0$.
$\neg P \Rightarrow \neg Q:$ $\exists\, v_j = a_{j-1} v_{j-1} + \cdots + a_1 v_1 \in V_{\mathbf{C}}$.
$\neg Q \Rightarrow \neg P:$ $\exists\, v_j = \lambda_{j-1} v_{j-1} + \cdots + \lambda_1 v_1 \Rightarrow v_j = (\operatorname{Re} \lambda_{j-1}) v_{j-1} + \cdots + (\operatorname{Re} \lambda_1) v_1 \in V$.

(b) $P \Rightarrow Q:$ $\forall u + \mathrm{i}v \in V_{\mathbf{C}},\ u, v \in V \Rightarrow \exists\, a_i, b_i \in \mathbf{R}, u + \mathrm{i}v = \sum_{i=1}^{n} (a_i + \mathrm{i} b_i) v_i$.
$Q \Rightarrow P:$ $\forall v \in V, \exists\, a_i + \mathrm{i}b_i \in \mathbf{C},\ v + \mathrm{i}0 = \big(\sum_{i=1}^{n} a_i v_i\big) + \mathrm{i}\,\big(\sum_{i=1}^{n} b_i v_i\big) \Rightarrow v \in \operatorname{span}(v_1, \dots, v_m)$.
$\neg Q \Rightarrow \neg P:$ $\exists\, v \in V, v \notin \operatorname{span}(B) \Rightarrow v + \mathrm{i}0 \notin \operatorname{span}(B)$ while $v + \mathrm{i}0 \in V_{\mathbf{C}}$.
$\neg Q \Rightarrow \neg P:$ $\exists\, u + \mathrm{i}v \in V_{\mathbf{C}}, u + \mathrm{i}v \notin \operatorname{span}(B) \Rightarrow u$ or $v \notin \operatorname{span}(B)$. Note that $u, v \in V$. $\square$

• **Note For *linely inde sequence and* [2.34]:** "$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expression. If we allow using "infinite list", then we must guarantee that $(v_1, \dots, v_n, \dots)$ is a spanning "list" such that $\forall v \in V, \exists$ smallest $n \in \mathbf{N}^+$, $v = a_1 v_1 + \cdots + a_n v_n$. Moreover, given a list $(w_1, \cdots, w_n, \cdots)$ in W, we can prove that $\exists!\, T \in \mathcal{L}(V, W)$ with each $T v_k = w_k$, which has less restrictions than [3.5]. But the key point is, how can we guarantee that such a "list" exists. TODO: More details.

Ended

2·C

1 7 9 10 14,16 15 17 | 4E: 10 14,15 16

15 *Suppose V is finite-dim and $\dim V = n \geqslant 1$. Prove that $\exists$ one-dim subsps $V_1, \dots, V_n$ of V such that $V = V_1 \oplus \cdots \oplus V_n$.*

Solution: Suppose $B_V = (v_1, \dots, v_n)$. Define V_i by $V_i = \operatorname{span}(v_i)$ for each $i \in \{1, \dots, n\}$.
Then $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n \Rightarrow \exists!\, u_i \in V_i, v = u_1 + \cdots + u_n$ $\square$

• **Note For Problem (15):**
Suppose $v \in V \backslash \{0\}$, and $\dim V = n \geqslant 1$. Prove that $\exists\, B_V = (v_1, \dots, v_n), v = v_1 + \cdots + v_n$.

Solution: If $n = 1$ then let $v_1 = v$ and we are done. Suppose $n > 1$.
Extend (v) to a basis $(v, v_1, \dots, v_{n-1})$ of V. Let $v_n = v - v_1 - \cdots - v_{n-1}$.
又 $\operatorname{span}(v, v_1, \dots, v_{n-1}) = \operatorname{span}(v_1, \dots, v_n)$. Hence $(v_1, \dots, v_n)$ is also a basis of V. $\square$

Comment: Let $B_V = (v_1, \dots, v_n)$ and suppose $v = u_1 + \cdots + u_n$, where each $u_i = a_i v_i \in V_i$.
But $(u_1, \dots, u_n)$ might not be a basis, because there might be some $u_i = 0$.

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1 [COROLLARY for [2.38,39]] *Suppose U is a subsp of V such that $\dim V = \dim U$. Then $V = U$.*

Let $B_U = (u_1, \dots, u_m)$. Then $m = \dim V$. 又 $u_i \in V$. By [2.39], $B_V = (u_1, \dots, u_m)$. ☐

- Let $v_1, \dots, v_n \in V$ and $\dim \operatorname{span}(v_1, \dots, v_n) = n$. Then $(v_1, \dots, v_n)$ is a basis of $\operatorname{span}(v_1, \dots, v_n)$.
  Notice that $(v_1, \dots, v_n)$ is a spanning list of $\operatorname{span}(v_1, \dots, v_n)$ *of length* $n = \dim \operatorname{span}(v_1, \dots, v_n)$.

---

7 (a) *Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.*
(b) *Extend the basis in (b) to a basis of $\mathcal{P}_4(\mathbf{F})$.*
(c) *Find a subsp W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.*

SOLUTION: Using Problem (10).

NOTICE that $\nexists\, p \in \mathcal{P}(\mathbf{F})$ of deg 1 and 2, while $p \in U$. Thus $\dim U \leqslant \dim \mathcal{P}_4(\mathbf{F}) - 2 = 3$.

(a) Consider $B = \big(1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.
Let $a_0 + a_3(z-2)(z-5)(z-6) + a_4 z(z-2)(z-5)(z-6) = 0 \Rightarrow a_0 = a_3 = a_4 = 0$.
Thus the list B is linely inde in U. Now $\dim U \geqslant 3 \Rightarrow \dim U = 3$. Thus $B_U = B$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $\big(1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. ☐

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9 *Suppose $(v_1, \dots, v_m)$ is linely inde in V and $w \in V$.*
Prove that $\dim \operatorname{span}\big(v_1 + w, \dots, v_m + w\big) \geq m - 1$.

SOLUTION: Using the result of (2.A.10, 11).

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$, for each $i = 1, \dots, m$.
$(v_1, \dots, v_m)$ linely inde $\Rightarrow \big(v_1, v_2 - v_1, \dots, v_m - v_1\big)$ linely inde $\Rightarrow \underbrace{\big(v_2 - v_1, \dots, v_m - v_1\big)}_{\text{of length } (m-1)}$ linely inde.
又 If $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $\big(v_1 + w, \dots, v_m + w\big)$ is linely inde.
Hence $m \geqslant \dim \operatorname{span}\big(v_1 + w, \dots, v_m + w\big) \geq m - 1$. ☐

- (4E 16) *Suppose V is finite-dim, U is a subsp of V with $U \neq V$. Let $n = \dim V, m = \dim U$.*
  Prove that $\exists\ (n-m)$ subsps $U_1, \dots, U_{n-m}$, each of dim $(n-1)$, such that $\bigcap\limits_{i=1}^{n-m} U_i = U$.

SOLUTION: Let $B_U = (v_1, \dots, v_m)$, $B_V = \big(v_1, \dots, v_m, u_1, \dots, v_{n-m}\big)$.
Define $U_i = \operatorname{span}\big(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m}\big)$ for each i. Then $U \subseteq U_i$ for each i.
And because $\forall v \in \bigcap\limits_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \dots + b_{n-m} u_{n-m} \in U_i \Rightarrow$ each $b_i = 0 \Rightarrow v \in U$.
Hence $\bigcap\limits_{i=1}^{n-m} U_i \subseteq U$. ☐

- **NOTE FOR Problem 10:** For each nonconst $p \in \operatorname{span}(1, z, \dots, z^m)$, $\exists$ smallest $m \in \mathbf{N}^+$, which is $\deg p$.

(a) If $p_0, p_1, \dots, p_m$ are such that all $a_{k,k} \neq 0$, and
$p_0 = a_{0,0}$, each $p_k = a_{0,k} + a_{1,k} z + \dots + a_{k,k} z^k$.
Then the upper-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m} \\ 0 & a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix}$.

(b) If $p_0, p_1, \dots, p_m$ are such that all $a_{k,k} \neq 0$, and
$p_0 = a_{0,0} + \dots + a_{m,0} x^m$, each $p_k = a_{k,k} x^k + \dots + a_{m,k} x^m$.
Then the lower-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,0} & a_{m,1} & \cdots & a_{m,m} \end{pmatrix}$.

COMMENT: Define $\xi_k(p)$ by the coeff of z^k in $p \in \mathcal{P}_m(\mathbf{F})$.
Then $\mathcal{M}\big(\xi_k, (1, z, \dots, z^m), (1)\big) = \mathcal{E}^{(1,k)} \in \mathbf{F}^{1,m+1}$.

---

10 *Suppose $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

Solution: Using mathematical induction on m.

(i) $k = 1$. $\deg p_0 = 0$; $\deg p_1 = 1 \Rightarrow \operatorname{span}(p_0, p_1) = \operatorname{span}(1, x)$.

(ii) $1 \leqslant k \leqslant m-1$. Assume that $\operatorname{span}(p_0, p_1, \dots, p_k) = \operatorname{span}(1, x, \dots, x^k)$.

Then $\operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \subseteq \operatorname{span}(1, x, \dots, x^k, x^{k+1})$.

又 $\deg p_{k+1} = k+1$, $p_{k+1}(x) = a_{k+1}x^{k+1} + r_{k+1}(x)$; $a_{k+1} \neq 0$, $\deg r_{k+1} \leqslant k$.

$\Rightarrow x^{k+1} = \dfrac{1}{a_{k+1}}\Big(p_{k+1}(x) - r_{k+1}(x)\Big) \in \operatorname{span}(1, x, \dots, x^k, p_{k+1}) = \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

$\therefore x^{k+1} \in \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \Rightarrow \operatorname{span}(1, x, \dots, x^k, x^{k+1}) \subseteq \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m)$. □

Or. 用比较系数法. Denote the coeff of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Suppose $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show that $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is linely inde.

Step 1. For $k = m$, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m$, $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.

Now $L = a_{m-1}p_{m-1}(x) + \cdots + a_0 p_0(x)$.

Step k. For $0 \leqslant k \leqslant m$, we have $a_m = \cdots = a_{k+1} = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = 0$, then we are done. Otherwise, we have $L = a_{k-1}p_{k-1}(x) + \cdots + a_0 p_0(x)$. □

• **Tips:** *Suppose $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ are such that the lowest term of each p_k is of deg k. Prove that $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

Solution: Using mathematical induction on m.

Let each p_k be defined by $p_k(x) = a_{k,k}x^k + \cdots + a_{m,k}x^m$, where $a_{k,k} \neq 0$.

(i) $k = 1$. $p_m(x) = a_{m,m}x^m$; $p_{m-1}(x) = a_{m-1,m-1}x^{m-1} + a_{m,m-1}x^m \Longrightarrow \operatorname{span}(x^m, x^{m-1}) = \operatorname{span}(p_m, p_{m-1})$.

(ii) $1 \leqslant k \leqslant m-1$. Assume that $\operatorname{span}(x^m, \dots, x^{m-k}) = \operatorname{span}(p_m, \dots, p_{m-k})$.

Then $\operatorname{span}(p_m, \dots, p_{m-(k+1)}) \subseteq \operatorname{span}(x^m, \dots, x^{m-(k+1)})$.

又 $p_{m-(k+1)}$ has the form $a_{m-(k+1),m-(k+1)}x^{m-(k+1)} + r_{m-(k+1)}(x)$;

where the lowest term of $r_{m-(k+1)} \in \mathcal{P}_m(\mathbf{F})$ is of deg $(m-k)$.

$$\Rightarrow x^{m-(k+1)} = \frac{1}{a_{m-(k+1),m-(k+1)}}\Big(p_{m-(k+1)}(x) - r_{m-(k+1)}(x)\Big) \in \operatorname{span}(x^m, \dots, x^{m-k}, p_{m-(k+1)}) = \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$$

$\therefore x^{m-(k+1)} \in \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$

$\Rightarrow \operatorname{span}(x^m, \dots, x^{m-k}, x^{m-(k+1)}) \subseteq \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}(p_m, \dots, p_1, p_0)$. □

Or. 用比较系数法. Denote the coeff of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Suppose $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show that $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is linely inde.

Step 1. For $k = 0$, $\xi_0(L) = a_0 \xi_0(p_0) = \xi_0(R) = 0$ 又 $\deg p_0 = 0$, $\xi_0(p_0) \neq 0 \Rightarrow a_0 = 0$.

Now $L = a_1 p_1(x) + \cdots + a_m p_m(x)$.

Step k. For $0 \leqslant k \leqslant m$, we have $a_{k-1} = \cdots = a_0 = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = m$, then we are done. Otherwise, we have $L = a_{k+1}p_{k+1}(x) + \cdots + a_m p_m(x)$. □

• **NOTE FOR [2.11]:** ***Good definition for a general term always aviods undefined behaviours.***
If $\deg p = 0$, then $p(z) = a_0 \neq 0$, but not literally $a_0 z^0$, by which if p is defined, then it comes to 0^0.
To make it clear, we specify that *in* $\mathcal{P}(\mathbf{F})$, $a_0 z^0 = a_0$, where z^0 appears just for notational convenience.
Because by definition, the term $a_0 z^0$ in a poly only represents the const term of the poly, which is a_0.
For convenicence, we assume that $z^0 = 1$ in formula deduction and poly def. Absolutely without 0^0.

• (4E 10) *Suppose m is a positive integer. For $0 \leqslant k \leqslant m$, let $p_k(x) = x^k(1-x)^{m-k}$.
Show that $(p_0, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

SOLUTION: *We may see $p_0 = 1$ and $p_m(x) = x^m$, from the expansion below, by the* NOTE FOR *[2.11] above.*

Note that each $p_k(x) = \sum_{j=0}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j = \underbrace{(-1)^0 \cdot x^k \cdot 1^0}_{\text{of deg k}} + \underbrace{\sum_{j=1}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j}_{\text{of deg m; denote it by } q_k(x)}$.
And, each $q_k \in \operatorname{span}(x^{k+1}, \dots, x^m)$. Using TIPS above. □

OR. Similar to the TIPS above. We will recursively prove that each $x^{m-k} \in \operatorname{span}(p_m, \dots, p_{m-k})$.
(i) $k = 1$. $p_m(x) = x^m \in \operatorname{span}(p_m)$; $\quad p_{m-1}(x) = x^{m-1} - x^m \Rightarrow x^{m-1} \in \operatorname{span}(p_{m-1}, p_m)$.
(ii) $k \in \{1, \dots, m-1\}$. Suppose for each $k \in \{0, \dots, k\}$, we have $x^{m-k} \in \operatorname{span}(p_{m-k}, \dots, p_m), \exists!\, a_m \in \mathbf{F}$.
Note that $x^{m-(k+1)} = p_{m-(k+1)}(x) + \sum_{j=1}^{k+1} C_{k+1}^{j}(-1)^{j+1} x^{m-(k+1)+j} \in \operatorname{span}(p_{m-(k+1)}, x^{m-k}, \dots, x^m)$.
Thus $x^{m-(k+1)} \in \operatorname{span}(p_{m-(k+1)}, p_{m-k}, \dots, p_m)$. □

COMMENT: The base step and the inductive step can be independent.

OR. For any $m, k \in \mathbf{N}^+$ such that $k \leqslant m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k(1-x)^{m-k}$.
Define the statement $S(m)$ by $S(m) : (p_{0,m}, \dots, p_{m,m})$ is linely inde (and therefore is a basis).
We use induction on to show that $S(m)$ holds for all $m \in \mathbf{N}^+$.
(i) $m = 0$. $p_{0,0} = 1$, and $ap_{0,0} = 0 \Rightarrow a = 0$.
$m = 1$. Let $a_0(1-x) + a_1 x = 0, \forall x \in \mathbf{F}$. Then take $x = 1, x = 0 \Rightarrow a_1 = a_0 = 0$.
(ii) $1 \leqslant m$. Assume that $S(m)$ and $S(m-1)$ holds. Now we show that $S(m+1)$ holds.
Suppose $\sum_{k=0}^{m+1} a_k p_{k,m+1}(x) = \sum_{k=0}^{m+1} a_k\left[x^k(1-x)^{m+1-k}\right] = 0, \forall x \in \mathbf{F}$.
Now $a_0(1-x)^{m+1} + \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k} + a_{m+1}x^{m+1} = 0, \forall x \in \mathbf{F}$.
While $x = 0 \Rightarrow a_0 = 0$; and $x = 1 \Rightarrow a_{m+1} = 0$.
Then $0 = \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k}$
$= x(1-x)\sum_{k=1}^{m} a_k x^{k-1}(1-x)^{m-k}$, note that $m-k = (m-1)-(k-1)$
$= x(1-x)\sum_{k=0}^{m-1} a_{k+1} x^k (1-x)^{m-1-k} = x(1-x)\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x)$.
Hence $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0, \forall x \in \mathbf{F}\backslash\{0,1\}$. Which has infinitely many zeros.
Moreover, $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0$. By assumption, $a_1 = \cdots = a_{m-1} = a_m = 0$.
Thus $(p_{0,m+1}, \dots, p_{m+1,m+1})$ is linely inde and $S(m+1)$ holds. □

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14 *Suppose $V_1, \dots, V_m$ are finite-dim. Prove that* $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$.

SOLUTION: For each V_i, let $B_{V_i} = \mathcal{E}_i$. Then $V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim V_i = \operatorname{card} \mathcal{E}_i$.
Now $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.

COROLLARY: $V_1 + \cdots + V_m$ is direct
$\iff$ For each $k \in \{1, \dots, m-1\}$, $(V_1 \oplus \cdots \oplus V_k) \cap V_{k+1} = \{0\}$, $(\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$
$\iff \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$
$\iff \dim(V_1 \oplus \cdots \oplus V_m) = \dim V_1 + \cdots + \dim V_m$. □

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17 *Suppose V_1, V_2, V_3 are subsps of a finite-dim vecsp, then*

$$\dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1\cap V_2) - \dim(V_1\cap V_3) - \dim(V_2\cap V_3) + \dim(V_1\cap V_2\cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

SOLUTION:

[*Similar to*] Given three sets A, B and C.

Because $|X\cup Y| = |X| + |Y| - |X\cap Y|$; $(X\cup Y)\cap Z = (X\cap Z)\cup(Y\cap Z)$.

Now $|(A\cup B)\cup C| = |A\cup B| + |C| - |(A\cup B)\cap C|$.

And $|(A\cup B)\cap C| = |(A\cap C)\cup(B\cap C)| = |A\cap C| + |B\cap C| - |A\cap B\cap C|$.

Hence $|(A\cup B)\cup C| = |A| + |B| + |C| + |A\cap B\cap C| - |A\cap B| - |A\cap C| - |B\cap C|$.

Note that $(V_1+V_2)+V_3 = V_1 + (V_2+V_3) = (V_1+V_3)+V_2$.

$$\begin{aligned}\dim(V_1+V_2+V_3) &= \dim(V_1+V_2) + \dim(V_3) - \dim\big((V_1+V_2)\cap V_3\big) \quad (1)\\ &= \dim(V_2+V_3) + \dim(V_1) - \dim\big((V_2+V_3)\cap V_1\big) \quad (2)\\ &= \dim(V_1+V_3) + \dim(V_2) - \dim\big((V_1+V_3)\cap V_2\big) \quad (3).\end{aligned}$$

Notice that in general, $(X+Y)\cap Z \neq (X\cap Z) + (Y\cap Z)$.

For example, $X = \{(x,0)\in\mathbf{R}^2 : x\in\mathbf{R}\}, Y = \{(0,y)\in\mathbf{R}^2 : y\in\mathbf{R}\}, Z = \{(z,z)\in\mathbf{R}^2 : z\in\mathbf{R}\}$.

COMMENT: If $X\subseteq Y$, then $(X+Y)\cap Z = Y\cap Z$; $\dim(X+Y+Z) = \dim(Y) + \dim(Z) - \dim(Y\cap Z)$, and the wrong formual holds. Similar for $Y\subseteq Z, X\subseteq Z$, and $X, Y\subseteq Z$.

However, $(X+Y)\cap Z \supseteq (X\cap Z) + (Y\cap Z)$ holds. Because $\forall v\in(X\cap Z)+(Y\cap Z)$, $\exists\, u = x_1 = z_1 \in X\cap Z, w = y_2 = z_2 \in Y\cap Z$, $v = u + w = x_1 + y_2 = z_1 + z_2 \in (X+Y)\cap Z$.

COMMENT: $\dim\big((X+Y)\cap Z\big) \geqslant \dim(X\cap Z) + \dim(Y\cap Z) - \dim(X\cap Y\cap Z)$.

- **COROLLARY:** *Suppose V_1, V_2, V_3 are finite-dim, then* $\dfrac{(1)+(2)+(3)}{3}$:

$$\begin{aligned}\dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2 + \dim V_3 &- \frac{\dim(V_1\cap V_2) + \dim(V_1\cap V_3) + \dim(V_2\cap V_3)}{3}\\ &- \frac{\dim\big((V_1+V_2)\cap V_3\big) + \dim\big((V_1+V_3)\cap V_2\big) + \dim\big((V_2+V_3)\cap V_1\big)}{3}.\end{aligned}$$

- **TIPS:** Because $\dim(V_1\cap V_2\cap V_3) = \dim V_1 + \dim(V_2\cap V_3) - \dim\big(V_1 + (V_2\cap V_3)\big)$.
  And $\dim(V_2\cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2+V_3)$. We have (1), and (2), (3) similarly.
  (1) $\dim(V_1\cap V_2\cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2+V_3) - \dim\big(V_1 + (V_2\cap V_3)\big)$.
  (2) $\dim(V_1\cap V_2\cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1+V_3) - \dim\big(V_2 + (V_1\cap V_3)\big)$.
  (3) $\dim(V_1\cap V_2\cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1+V_2) - \dim\big(V_3 + (V_1\cap V_2)\big)$.

- *Suppose V_1, V_2, V_3 are subsps of V with*
  (a) $\dim V = 10,\ \dim V_1 = \dim V_2 = \dim V_3 = 7$. *Prove that* $V_1\cap V_2\cap V_3 \neq \{0\}$.
  By TIPS, $\dim(V_1\cap V_2\cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 \underline{- 2\dim V} > 0$.
  (b) $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. *Prove that* $V_1\cap V_2\cap V_3 \neq \{0\}$.
  By TIPS, $\dim(V_1\cap V_2\cap V_3) \underline{\underline{\geq 2\dim V}} - \dim(V_2+V_3) - \dim\big(V_1 + (V_2\cap V_3)\big) \geq 0$. □

ENDED

3·A   3 4 5 7 8 10 11 12 13 | 4E: 10 11 17

• **Tips 1:** $T : V \to W$ is linear $\iff$ $\left| \begin{array}{l} (\text{一})\ \forall v, u \in V, T(v+u) = Tv + Tu; \\ (\text{二})\ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv). \end{array} \right|$ $\iff T(v+\lambda u) = Tv + \lambda Tu.$

• (9.A.2,6 Or 4E 3.B.33) *Suppose that V, W are on $\mathbf{R}$, and $T \in \mathcal{L}(V, W)$. Show that*
(a) $T_{\mathrm{C}} \in \mathcal{L}(V_{\mathrm{C}}, W_{\mathrm{C}})$. (b) $\operatorname{null}(T_{\mathrm{C}}) = (\operatorname{null} T)_{\mathrm{C}}, \operatorname{range}(T_{\mathrm{C}}) = (\operatorname{range} T)_{\mathrm{C}}$. (c) T_{C} is inv $\iff$ T is inv.

Solution: (a) $T_{\mathrm{C}}\big((u_1 + \mathrm{i}v_1) + (x + \mathrm{i}y)(u_2 + \mathrm{i}v_2)\big) = T(u_1 + xu_2 - yv_2) + \mathrm{i}T(v_1 + xv_2 + yu_2)$
$= T_{\mathrm{C}}(u_1 + \mathrm{i}v_1) + (x + \mathrm{i}y)T_{\mathrm{C}}(u_2 + \mathrm{i}v_2)$.
(b) $u + \mathrm{i}v \in \operatorname{null}(T_{\mathrm{C}}) \iff u, v \in \operatorname{null} T \iff u + \mathrm{i}v \in (\operatorname{null} T)_{\mathrm{C}}$.
$w + \mathrm{i}x \in \operatorname{range}(T_{\mathrm{C}}) \iff w, x \in \operatorname{range} T \iff w + \mathrm{i}x \in (\operatorname{range} T)_{\mathrm{C}}$.
(c) $\forall w, x \in W, \exists!\, u, v \in V, T_{\mathrm{C}}(u + \mathrm{i}v) = w + \mathrm{i}x \iff Tu = w, Tv = x$. Or. By (b). □

• (9.A.5) *Suppose V is on $\mathbf{R}$, and $S, T \in \mathcal{L}(V, W)$. Prove that $(S + \lambda T)_{\mathrm{C}} = S_{\mathrm{C}} + \lambda T_{\mathrm{C}}$.*

Solution: $(S + \lambda T)_{\mathrm{C}}(u + \mathrm{i}v) = (S + \lambda T)(u) + \mathrm{i}(S + \lambda T)(v)$
$= Su + \mathrm{i}Sv + \lambda(Tu + \mathrm{i}Tv) = (S_{\mathrm{C}} + \lambda T_{\mathrm{C}})(u + \mathrm{i}v)$. □

• *Suppose U, V, W are on $\mathbf{R}$, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Prove that $(ST)_{\mathrm{C}} = S_{\mathrm{C}}T_{\mathrm{C}}$.*

Solution: $\forall u + \mathrm{i}x \in U_{\mathrm{C}}, (ST)_{\mathrm{C}}(u + \mathrm{i}x) = STu + \mathrm{i}STx = S_{\mathrm{C}}(Tu + \mathrm{i}Tx) = (S_{\mathrm{C}}T_{\mathrm{C}})(u + \mathrm{i}x)$. □

• **Note For Restriction:** *U is a subsp of V.*
(a) $\forall S, T \in \mathcal{L}(V, W), \lambda \in \mathbf{F}$, $(T + \lambda S)|_U = T|_U + \lambda S|_U$.
(b) $\forall S \in \mathcal{L}(W, X), T \in \mathcal{L}(V, W)$, $(ST)|_U = ST|_U$.

• (4E 1.B.7) *Suppose $V \neq \emptyset$ and W is a vecsp. Let $W^V = \{f : V \to W\}$.*
(a) *Define a natural add and scalar multi on W^V.*
(b) *Prove that W^V is a vecsp with these definitions.*

Solution:
(a) $W^V \ni f + g : x \to f(x) + g(x)$; where $f(x) + g(x)$ is the vec add on W.
$W^V \ni \lambda f : x \to \lambda f(x)$; where $\lambda f(x)$ is the scalar multi on W.
(b) Commutativity: $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$.
Associativity: $\big((f + g) + h\big)(x) = \big(f(x) + g(x)\big) + h(x)$
$= f(x) + \big(g(x) + h(x)\big) = \big(f + (g + h)\big)(x)$.
Additive Identity: $(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$.
Additive Inverse: $(f + g)(x) = f(x) + g(x) = f(x) + \big(-f(x)\big) = 0 = 0(x)$.
Distributive Properties:
$\big(a(f + g)\big)(x) = a(f + g)(x) = a\big(f(x) + g(x)\big)$
$= af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x)$.
Similarly, $\big((a + b)f\big)(x) = (af + bf)(x)$.
So far, we have used the same properties in W.
Which means that ***if W^V is a vecsp, then W must be a vecsp.***
Multiplication Identity: $(1f)(x) = 1f(x) = f(x)$. (Notice that the smallest $\mathbf{F}$ is $\{0, 1\}$.) □

• **Tips 2:** $T \in \mathcal{L}(V,W) \iff T \in \mathcal{L}(V, \operatorname{range} T) \iff T \in \mathcal{L}(V,U)$, if range T is a subsp of U.

Corollary: $\{T \in \mathcal{L}(V,W) : \operatorname{range} T \subseteq U\} = \{T \in \mathcal{L}(V,U)\} = \mathcal{L}(V,U)$.

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5 *Because $\mathcal{L}(V,W) = \{T : V \to W \mid T \text{ is linear}\}$ is a subsp of W^V, $\mathcal{L}(V,W)$ is a vecsp.*

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3 *Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists A_{j,k} \in \mathbf{F}$ such that for any $(x_1, \dots, x_n) \in \mathbf{F}^n$,*

$$T(x_1, \dots, x_n) = \begin{pmatrix} A_{1,1}x_1 + \cdots + A_{1,n}x_n, \\ \vdots \quad\quad \ddots \quad\quad \vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n \end{pmatrix}$$

Solution:

Let $T(1,0,0,\dots,0,0) = (A_{1,1}, \dots, A_{m,1})$, Note that $(1,0,\dots,0,0), \cdots, (0,0,\dots,0,1)$ is a basis of $\mathbf{F}^n$.

$T(0,1,0,\dots,0,0) = (A_{1,2}, \dots, A_{m,2})$, Then by [3.5], we are done. □

$\vdots$

$T(0,0,0,\dots,0,1) = (A_{1,n}, \dots, A_{m,n})$.

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4 *Suppose $T \in \mathcal{L}(V,W)$, and $v_1, \dots, v_m \in V$ such that $(Tv_1, \dots, Tv_m)$ is linely inde in W. Prove that $(v_1, \dots, v_m)$ is linely inde.*

Solution: Suppose $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$. □

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7 *Show that every linear map from a one-dim vecsp to itself is a multi by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then $\exists \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$.*

Solution: Let u be a nonzero vec in $V \Rightarrow V = \operatorname{span}(u)$. Because $Tu \in V \Rightarrow Tu = \lambda u$ for some λ.

Suppose $v \in V \Rightarrow v = au$, $\exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. □

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8 *Give a map $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that $\forall a \in \mathbf{R}, v \in \mathbf{R}^2, \varphi(av) = a\varphi(v)$ but φ is not linear.*

Solution: Define $T(x,y) = \begin{cases} x+y, & \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, & \text{otherwise.} \end{cases}$ Or. Define $T(x,y) = \sqrt[3]{(x^3+y^3)}$. □

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9 *Give a map $\varphi : \mathbf{C} \to \mathbf{C}$ such that $\forall w, z \in \mathbf{C}, \varphi(w+z) = \varphi(w) + \varphi(z)$ but φ is not linear.*

Solution: Define $\varphi(u + \mathrm{i}v) = u = \operatorname{Re}(u + \mathrm{i}v)$ Or. Define $\varphi(u + \mathrm{i}v) = v = \operatorname{Im}(u + \mathrm{i}v)$. □

• *Prove that if $q \in \mathcal{P}(\mathbf{R})$ and $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by $\underbrace{Tp = q \circ p}_{\text{composition}}$, then T is not linear.*

Solution: ***Composition and product are not the same in $\mathcal{P}(\mathbf{F})$.***

Notice that $(p \circ q)(x) = p(q(x))$, while $(pq)(x) = p(x)q(x) = q(x)p(x)$.

Because in general, $[q \circ (p_1 + \lambda p_2)](x) = q(p_1(x) + \lambda p_2(x)) \neq (qp_1)(x) + \lambda(qp_2)(x)$.

Example: Let q be defined by $q(x) = x^2$, then $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$. □

---

10 *Suppose U is a subsp of V with $U \neq V$. Suppose $S \in \mathcal{L}(U,W)$ with $S \neq 0$. Define $T : V \to W$ by $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \backslash U. \end{cases}$ Prove that T is not a linear map on V.*

Solution: Assume that T is a linear map. Suppose $v \in V \backslash U$, $u \in U$ such that $Su \neq 0$.

Then $v + u \in V \backslash U$, for if not, $v = (v+u) - u \in U$;

while $T(v+u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$. Contradicts. □

---

11 *Suppose U is a subsp of V and $S \in \mathcal{L}(U, W)$.*
Prove that $\exists\, T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U$. (OR. $\exists\, T \in \mathcal{L}(V, W), T|_U = S$.)
*In other words, every linear map on a subsp of V can be **extended** to a linear map on the entire V.*

SOLUTION: Suppose W is such that $V = U \oplus W$. Then $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.
Define $T \in \mathcal{L}(V, W)$ by $T(u_v + w_v) = Su_v$. □

OR. [*Finite-dim Req*] Define by $T\left(\sum_{i=1}^{m} a_i u_i\right) = \sum_{i=1}^{n} a_i S u_i$. Let $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$. □

---

12 *Suppose nonzero V is finite-dim and W is infinite-dim. Prove that $\mathcal{L}(V, W)$ is infinite-dim.*

SOLUTION: Using (2.A.14).

Let $B_V = (v_1, \dots, v_n)$ be a basis of V. Let $(w_1, \dots, w_m)$ be linely inde in W for any $m \in \mathbf{N}^+$.
Define $T_{x,y} : V \to W$ by $T_{x,y}(v_z) = \delta_{z,x} w_y$, $\forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$
$\forall v = \sum_{i=1}^{n} a_i v_i,\ u = \sum_{i=1}^{n} b_i v_i,\ \lambda \in \mathbf{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) w_y = T_{x,y}(v) + \lambda T_{x,y}(u)$.
Linearity checked. Now suppose $a_1 T_{x,1} + \dots + a_m T_{x,m} = 0$.
Then $(a_1 T_{x,1} + \dots + a_m T_{x,m})(v_x) = 0 = a_1 w_1 + \dots + a_m w_m \Rightarrow a_1 = \dots = a_m = 0$. 又 m arbitrary.
Thus $(T_{x,1}, \dots, T_{x,m})$ is a linely inde list in $\mathcal{L}(V, W)$ for any x and length m. Hence by (2.A.14). □

---

13 *Suppose $(v_1, \dots, v_m)$ is linely depe in V and $W \neq \{0\}$.*
Prove that $\exists\, w_1, \dots, w_m \in W, \nexists\, T \in \mathcal{L}(V, W)$ such that $Tv_k = w_k, \forall k = 1, \dots, m$.

SOLUTION:

We prove by contradiction. By linear dependence lemma, $\exists\, j \in \{1, \dots, m\}, v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$.
Suppose $a_1 v_1 + \dots + a_m v_m = 0$, where $a_j \neq 0$. Now let $w_j \neq 0$, while $w_1 = \dots = w_{j-1} = w_{j+1} = w_m = 0$.
Define $T \in \mathcal{L}(V, W)$ by $Tv_k = w_k$ for each k. Then $T(a_1 v_1 + \dots + a_m v_m) = 0 = a_1 w_1 + \dots + a_m w_m$.
And $0 = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. □

OR. We prove the contrapositive: Suppose $\forall w_1, \dots, w_m \in W, \exists\, T \in \mathcal{L}(V, W), Tv_k = w_k$ for each w_k.
Now we show that $(v_1, \dots, v_n)$ is linely inde. Suppose $\exists\, a_i \in \mathbf{F}, a_1 v_1 + \dots + a_n v_n = 0$.
Choose one $w \in W \backslash \{0\}$. By assumption, for $(\overline{a_1} w, \dots, \overline{a_m} w), \exists\, T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$ for each v_k.
Now we have $0 = T\left(\sum_{k=1}^{m} a_k v_k\right) = \sum_{k=1}^{m} a_k T v_k = \sum_{k=1}^{m} a_k \overline{a_k} w = \left(\sum_{k=1}^{m} |a_k|^2\right) w$.
Then $\sum_{k=1}^{m} |a_k|^2 = 0$. Thus $a_1 = \dots = a_m = 0$. Hence $(v_1, \dots, v_n)$ is linely inde. □

• (4E 17) *Suppose V is finite-dim. Show that all two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Let $B_V = (v_1, \dots, v_n)$. If $\mathcal{E} = 0$, then we are done.
Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \backslash \{0\}$.
Suppose $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \dots + a_n v_n$, where $a_k \neq 0$.
Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y} :\ v_x \mapsto v_y,\ v_z \mapsto 0\ (z \neq x)$. OR. $R_{x,y} v_z = \delta_{z,x} v_y$.
Then $(R_{1,1} + \dots + R_{n,n}) v_j = v_j \Rightarrow \sum_{r=1}^{n} R_{r,r} = I$. Assume that each $R_{x,y} \in \mathcal{E}$.
Hence $\forall T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. Now we prove the assumption.
Notice that $\forall x, y \in \mathbf{N}^+,\ (R_{k,y} S)(v_i) = a_k v_y \Rightarrow ((R_{k,y} S) \circ R_{x,i})(v_z) = \delta_{z,x}(a_k v_y)$.
Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Now $S \in \mathcal{E} \Rightarrow R_{k,y} S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}$. □

- (4E 3.B.32) *Suppose V is finite-dim with $n = \dim V > 1$.*
  Show that if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is linear and $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$, then $\varphi = 0$.

SOLUTION: Using notations in (4E 3.A.17). Using the result in NOTE FOR [3.60].

Suppose $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \ldots, n\},\ \varphi(R_{i,j}) \neq 0$. Because $R_{i,j} = R_{x,j} \circ R_{i,x},\ \ \forall x = 1, \ldots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, because $R_{i,x} = R_{y,x} \circ R_{i,y},\ \ \forall y = 1, \ldots, n$. Thus $\varphi(R_{y,x}) \neq 0,\ \forall x, y = 1, \ldots, n$.

Let $k \neq i, j \neq l$ and then $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. $\square$

OR. Note that by (4E 3.A.17), $\exists\, S, T \in \mathcal{L}(V), ST - TS \neq 0$.

Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \operatorname{null}\varphi \neq \{0\}$.

Note that $\forall\, E \in \operatorname{null}\varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \operatorname{null}\varphi$.

Hence $\operatorname{null}\varphi$ is a nonzero two-sided ideal of $\mathcal{L}(V)$. $\square$

- *Suppose V is finite-dim. $T \in \mathcal{L}(V)$ is such that $\forall S \in \mathcal{L}(V), ST = TS$.*
  Prove that $\exists\, \lambda \in \mathbf{F}, T = \lambda I$.

SOLUTION: If $V = \{0\}$, then we are done. Now suppose $V \neq \{0\}$.

Assume that $\forall v \in V, (v, Tv)$ is linely depe, then by (2.A.2.(b)), $\exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

To prove that λ_v is independent of v, we discuss in two cases:

$$\left.\begin{array}{l} (-)\text{ If } (v, w) \text{ is linely inde, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \\ \qquad\qquad\qquad\qquad\qquad\Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=)\text{ Otherwise, suppose } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w \end{array}\right\} \Rightarrow \lambda_w = \lambda_v.$$

Now we prove the assumption. Assume that $\exists\, v \in V, (v, Tv)$ is linely inde. Let $B_V = (v, Tv, u_1, \ldots, u_n)$.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1 u_1 + \cdots + c_n u_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Contradicts. $\square$

OR. Let $B_V = (v_1, \ldots, v_m)$. Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$.

Suppose $v \in V$. Define $S_v \in \mathcal{L}(V)$ by $S_v(u) = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. $\square$

OR. For each $k \in \{1, \ldots, n\}$, define $S_k \in \mathcal{L}(V)$ by $S_k v_j = \begin{cases} v_k, & j = k, \\ 0, & j \neq k. \end{cases}$ OR. $S_k v_j = \delta_{j,k} v_k$

Note that $S_k\Big(\sum_{i=1}^{n} a_i v_i\Big) = a_k v_k$. Then $S_k v = v \iff \exists\,!\, a_k \in \mathbf{F}, v = a_k v_k$.

Hence $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$.

Define $A^{(j,k)} \in \mathcal{L}(V)$ by $A^{(j,k)} v_j = v_k,\ A^{(j,k)} v_k = v_j,\ A^{(j,k)} v_x = 0, x \neq j, k$.

$$\text{Then}\ \left|\ \begin{array}{r} A^{(j,k)} T v_j = T A^{(j,k)} v_j = T v_k = a_k v_k \\ A^{(j,k)} T v_j = A^{(j,k)} a_j v_j = a_j A^{(j,k)} v_j = a_j v_k \end{array}\right\} \Rightarrow a_k = a_j. \text{ Hence } a_k \text{ is inde of } v_k.$$

$\square$

- **TIPS 3:** *Suppose $T \in \mathcal{L}(V, W)$. Prove that $Tv \neq 0 \Rightarrow v \neq 0$.*

SOLUTION: Assume that $v = 0$. Then $Tv = T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$.

OR. $T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$. Contradicts. $\square$

• *Given the fact that $\mathcal{L}(V,W)$ is a vecsp. Prove or give a counterexample: V,W are vecsps.*

We can guarantee that $\{0\} \subseteq \mathcal{L}(V,W), \{0\} \subseteq V, \{0\} \subseteq W$.

And by [3.2], the additivity and homogeneity imply that V is closed under add and scalar multi.

(We cannot even guarantee that W^V is a vecsp.)

SOLUTION: TODO: Too tricky to be answered by AI.

(I) If $W^V = \{0\}$. Then $\mathcal{L}(V,W) = \{0\}$.

And $W = \{0\}$, for if not, $\exists\, w \in W \backslash \{0\}$, define a map f by $f(x) = w, \forall x \in V$.

And V might not be a vecsp. Example: ???

(II) If W^V is a nonzero vecsp. Then W is a vecsp.

(a) If $\mathcal{L}(V,W) = \{0\}$, then we cannot guarantee that V is a vecsp. Example: ???

(b) If not, then $\exists\, T \in \mathcal{L}(V,W)$, $T \neq 0$. Which means $\exists\, v \in V, Tv \neq 0 \Rightarrow v \neq 0$.

Then both W and V have a nonzero element.

(i) If $\exists$ inje $T \in \mathcal{L}(V,W)$, then $T(u+v) = T(v+u) \Rightarrow u+v = v+u$. etc. Hence V is a vecsp.

(ii) If not, then we cannot guarantee that V is a vecsp. Example: ???

(III) If W^V is not a vecsp, then W is not a vecsp. Example: ??? □

ENDED

3·B

3 7 8 9 10 11 12 16 17 18 19 20 21 22 23 24 25 26 28 29 30 | 4E: 21 24 27 31 32 33

3 *Suppose $(v_1,\dots,v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1,\dots,z_m) = z_1v_1 + \cdots + z_mv_m$.*

(a) *The surj of T correspds to $(v_1,\dots,v_m)$ spanning V.* range $T = \operatorname{span}(v_1,\dots,v_m) = V$.

(b) *The inje of T correspds to $(v_1,\dots,v_m)$ being linely inde.* *$(v_1,\dots,v_m)$ linely inde $\iff$ T inje.*

COMMENT: Let $(e_1,\dots,e_m)$ be the std basis of $\mathbf{F}^m$. Then $Te_k = v_k$.

---

7 *Suppose V is finite-dim with $2 \leqslant \dim V$. And $\dim V \leqslant \dim W = m$, if W is finite-dim. Show that $U = \{T \in \mathcal{L}(V,W) : \operatorname{null} T \neq \{0\}\}$ is not a subsp of $\mathcal{L}(V,W)$.*

SOLUTION: *The set of all inje $T \in \mathcal{L}(V,W)$ is a not subsp either.*

Let $(v_1,\dots,v_n)$ be a basis of V, $(w_1,\dots,w_m)$ be linely inde in W. $[\ 2 \leqslant n \leqslant m.\ ]$

Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 :\ v_1 \mapsto 0,\quad v_2 \mapsto w_2,\quad v_i \mapsto w_i.$

Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 :\ v_1 \mapsto w_1,\quad v_2 \mapsto 0,\quad v_i \mapsto w_i,\quad i = 3,\dots,n.$

Thus $T_1 + T_2 \notin U$. □

COMMENT: If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V,W), T$ is inje. Hence $U = \emptyset$.

If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $\forall v \in V, T_0v = 0 \Rightarrow T_0 = 0$.

---

8 *Suppose W is finite-dim with $\dim W \geqslant 2$. And $n = \dim V \geqslant \dim W$, if V is finite-dim. Show that $U = \{T \in \mathcal{L}(V,W) : \operatorname{range} T \neq W\}$ is not a subsp of $\mathcal{L}(V,W)$.*

SOLUTION: *The set of all surj $T \in \mathcal{L}(V,W)$ is not a subsp either.* ***Using the generalized version of*** **[3.5]**.

Let $(v_1,\dots,v_n)$ be linely inde in V, $(w_1,\dots,w_m)$ be a basis of W. $[\ n \in \{m, m+1, \dots\}; 2 \leqslant m \leqslant n.\ ]$

Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 :\ v_1 \mapsto 0,\quad v_2 \mapsto w_2,\quad v_j \mapsto w_j,\quad v_{m+i} \mapsto 0.$

Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 :\ v_1 \mapsto w_1,\quad v_2 \mapsto 0,\quad v_j \mapsto w_j,\quad v_{m+i} \mapsto 0.$

(For each $j = 2,\dots,m;\ i = 1,\dots,n-m$, if V is finite, otherwise let $i \in \mathbf{N}^+$.) Thus $T_1 + T_2 \notin U$. □

COMMENT: If $\dim W = 0$, then $W = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V,W), T$ is surj. Hence $U = \emptyset$.

If $\dim W = 1$, then $W = \operatorname{span}(w_0)$. Thus $U = \operatorname{span}(T_0)$, where each $T_0v_i = 0 \Rightarrow T_0 = 0$.

---

9 *Suppose* $(v_1,\dots,v_n)$ *is linely inde. Prove that* $\forall$ *inje* T, $(Tv_1,\dots,Tv_n)$ *is linely inde.*

SOLUTION: $a_1Tv_1+\cdots+a_nTv_n=0=T\Big(\sum_{i=1}^n a_iv_i\Big)\iff\sum_{i=1}^n a_iv_i=0\iff a_1=\cdots=a_n=0.$ $\square$

---

10 *Suppose* $\operatorname{span}(v_1,\dots,v_n)=V$. *Show that* $\operatorname{span}(Tv_1,\dots,Tv_n)=\operatorname{range}T$.

SOLUTION: (a) $\operatorname{range}T=\{Tv:v\in\operatorname{span}(v_1,\dots,v_n)\}\Rightarrow Tv_1,\dots,Tv_n\in\operatorname{range}T$. By [2.7].

OR. $\operatorname{span}(Tv_1,\dots,Tv_n)\ni a_1Tv_1+\cdots+a_nTv_n=T(a_1v_1+\cdots+a_nv_n)\in\operatorname{range}T$.

(b) $\forall w\in\operatorname{range}T, w=Tv,\exists\, v\in V\Rightarrow\exists\, a_i\in\mathbf{F}, v=\sum_{i=1}^n a_iv_i, w=a_1Tv_1+\cdots+a_nTv_n.$ $\square$

---

11 *Suppose* $S_1,\dots,S_n\in\mathcal{L}(V)$ *and* $S=S_1S_2\cdots S_n$ *makes sense.* Then using induction:

(a) $\operatorname{range}S_1\supseteq\operatorname{range}(S_1S_2)\supseteq\cdots\supseteq\operatorname{range}(S)$; (b) $\operatorname{null}S_n\subseteq\operatorname{null}(S_{n-1}S_n)\subseteq\cdots\subseteq\operatorname{null}(S)$.

- Define $X_p=\{T\in\mathcal{L}(V):p(T)\text{ holds}\}$; $P_p:X_p$ is closed under vec multi; $Q_p:X_p$ is a group.

(1) S surj $\iff$ each S_k surj. P_{surj} holds. (2) S inje $\iff$ each S_k inje. P_{inje} holds.

(3) P_{inv} and Q_{inv} hold. Q_p in (1) and (2) holds $\iff$ V is finite-dim.

(4) $P_{inje\ or\ surj}$ holds $\iff$ V is finite-dim $\iff$ $Q_{inje\ or\ surj}$ holds.

- *Suppose* $S,T\in\mathcal{L}(V)$. *Prove or give a counterexample*:

(a) $\operatorname{null}S\subseteq\operatorname{null}T\Rightarrow\operatorname{range}T\subseteq\operatorname{range}S$; (b) $\operatorname{range}T\subseteq\operatorname{range}S\Rightarrow\operatorname{null}S\subseteq\operatorname{null}T$.

SOLUTION: Let $B_V=(v_1,v_2,v_3)$. Counterexamples:

(a) Let $S:v_1\mapsto0;\ v_2\mapsto0;\ v_3\mapsto v_2.$ $T:v_1\mapsto0;\ v_2\mapsto0;\ v_3\mapsto v_3.$ Then $\operatorname{null}S=\operatorname{null}T$, but $\operatorname{range}T=\operatorname{span}(v_3)\not\subseteq\operatorname{span}(v_2)=\operatorname{null}T$.

(b) Let $S:v_1\mapsto v_2;\ v_2\mapsto v_2;\ v_3\mapsto v_2.$ $T:v_1\mapsto0;\ v_2\mapsto0;\ v_3\mapsto v_2.$ Then $\operatorname{range}T=\operatorname{range}S$, but $\operatorname{null}S=\operatorname{span}(v_1-v_2,v_2-v_3,v_3-v_1)\not\subseteq\operatorname{span}(v_1,v_2)=\operatorname{null}T$.

---

16 *Suppose* $T\in\mathcal{L}(V)$ *such that* $\operatorname{null}T,\operatorname{range}T$ *are finite-dim. Prove that* V *is finite-dim.*

SOLUTION: Let $B_{\operatorname{range}T}=(Tv_1,\dots,Tv_n), B_{\operatorname{null}T}=(u_1,\dots,u_m)$.

$\forall v\in V,\exists!\,a_i\in\mathbf{F},T(v-a_1v_1-\cdots-a_nv_n)=0\Rightarrow\exists!\,b_i\in\mathbf{F},v-\sum_{i=1}^n a_iv_i=\sum_{i=1}^m b_iu_i.$ $\square$

---

17 *Suppose* V,W *are finite-dim. Prove that* $\exists$ *inje* $T\in\mathcal{L}(V,W)\iff\dim V\leqslant\dim W$.

SOLUTION: (a) Suppose $\exists$ inje T. Then $\dim V=\dim\operatorname{range}T\leqslant\dim W$.

(b) Suppose $\dim V\leqslant\dim W$. Let $B_V=(v_1,\dots,v_n),B_W=(w_1,\dots,w_m)$.

Define $T\in\mathcal{L}(V,W)$ by $Tv_i=w_i,\ i=1,\dots,n\ (=\dim V)$. $\square$

---

18 *Suppose* V,W *are finite-dim. Prove that* $\exists$ *surj* $T\in\mathcal{L}(V,W)\iff\dim V\geqslant\dim W$.

SOLUTION: (a) Suppose $\exists$ surj T. Then $\dim V=\dim W+\dim\operatorname{null}T\Rightarrow\dim W\leqslant\dim V$.

(b) Suppose $\dim V\geqslant\dim W$. Let $B_V=(v_1,\dots,v_n),B_W=(w_1,\dots,w_m)$.

Define $T\in\mathcal{L}(V,W)$ by $T\big(a_1v_1+\cdots+a_mv_m+\cdots+a_nv_n\big)=a_1w_1+\cdots+a_mw_m$. $\square$

---

19 *Suppose* V,W *are finite-dim,* U *is a subsp of* V.

Prove that $\exists\,T\in\mathcal{L}(V,W)$, $\operatorname{null}T=U\iff\underbrace{\dim U}_{m}\geqslant\underbrace{\dim V}_{m+n}-\underbrace{\dim W}_{p}$.

SOLUTION:

(a) Suppose $\exists\,T\in\mathcal{L}(V,W),\operatorname{null}T=U$. Then $\dim U+\dim\operatorname{range}T=\dim V\leqslant\dim U+\dim W$.

(b) Let $B_U=(u_1,\dots,u_m),B_V=(u_1,\dots,u_m,v_1,\dots,v_n),B_W=(w_1,\dots,w_p)$. Suppose that $p\geqslant n$.

Define $T\in\mathcal{L}(V,W)$ by $T(a_1v_1+\cdots+a_nv_n+b_1u_1+\cdots+b_mu_m)=a_1w_1+\cdots+a_nw_n$. $\square$

• **TIPS 1:** *Suppose U is a subsp of V. Then $\forall T \in \mathcal{L}(V,W), U \cap \operatorname{null} T = \operatorname{null} T|_U$.*

• **TIPS 2:** *Suppose $T \in \mathcal{L}(V,W)$ and $T|_U$ is inje. Let $V = M + N,\ U = X + Y$.*
Then $\operatorname{range} T = \operatorname{range} T|_M + \operatorname{range} T|_N = \operatorname{range} T|_X + \operatorname{range} T|_Y$.
(a) *Show that if $U = X \oplus Y$, then $\operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_Y$.*
(b) *Give an example such that $V = M \oplus N,\ \operatorname{range} T \neq \operatorname{range} T|_M \oplus \operatorname{range} T|_N$.*

SOLUTION: Assume that for some $v \in V$, there exist two distinct pairs $(x_1, y_1), (x_2, y_2)$ in $X \times Y$ such that $Tv = Tx_1 + Ty_1 = Tx_2 + Ty_2$. Because $\forall v \in X \oplus Y, \exists!\,(x,y) \in X \times Y, v = x + y$.
Now $T(x_1 + y_1) = T(x_2 + y_2) \Longrightarrow x_1 + y_1 = x_2 + y_2 \Longrightarrow x_1 = x_2, y_1 = y_2$. Contradicts.
Thus $\forall Tv \in \operatorname{range} T, \exists!\, Tx \in \operatorname{range} T|_X, Ty \in \operatorname{range} T|_Y, Tv = Tx + Ty$. □

EXAMPLE: Let $B_V = (v_1, v_2, v_3), B_W = (w_1, w_2),\ T : v_1 \mapsto 0,\ v_2 \mapsto w_1,\ v_3 \mapsto w_2$.
Let $B_M = (v_1 - v_2, v_3), B_N = (v_2)$. Then $\operatorname{range} T|_M = \operatorname{span}(w_1, w_2), \operatorname{range} T|_N = \operatorname{span}(w_1)$

COMMENT: Also $\operatorname{null} T|_M = \operatorname{null} T|_N = \{0\}$. Hence $\operatorname{null} T \neq \operatorname{null} T|_M \oplus \operatorname{null} T|_N$.

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12 *Prove that $\forall T \in \mathcal{L}(V,W), \exists$ subsp U of V such that*
$U \cap \operatorname{null} T = \operatorname{null} T|_U = \{0\},\ \operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$.
Which is equivalent to $T|_U : U \to \operatorname{range} T$ being an iso.

SOLUTION: By [2.34] (note that V can be infinite-dim), $\exists$ subsp U of V such that $V = U \oplus \operatorname{null} T$.
$\forall v \in V, \exists!\, w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w + u) = Tu \in \{Tu : u \in U\}$. □

COROLLARY: $[P]\quad T|_U : U \to \operatorname{range} T$ *is an iso* $\Longleftrightarrow U \oplus \operatorname{null} T = V. \quad [Q]$
We have shown $Q \Rightarrow P$. Now we show that $P \Rightarrow Q$ to complete the proof.
$\forall v \in V, Tv \in \operatorname{range} T = \operatorname{range} T|_U \Rightarrow \exists!\, u \in U, Tu = Tu \Rightarrow v - u \in \operatorname{null} T$.
Thus $v = (v - u) + u \in U + \operatorname{null} T$. 又 $u \in U \cap \operatorname{null} T \Longleftrightarrow T|_U(u) = 0 \Longleftrightarrow u = 0$. □

OR. $\neg Q \Rightarrow \neg P$: Because $U \oplus \operatorname{null} T \subsetneq V$. We show $\operatorname{range} T \neq \operatorname{range} T|_U$ by contradiction.
Let $X \oplus (U \oplus \operatorname{null} T) = V$. Now $\operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_U$. And X is nonzero.
Assume that $\operatorname{range} T = \operatorname{range} T|_U$. Then $\operatorname{range} T|_X = \{0\}$. While $T|_X$ is inje. Contradicts.
OR. $\operatorname{range} T|_X \subseteq \operatorname{range} T|_U \Rightarrow \forall x \in X, Tx \in \operatorname{range} T|_U, \exists\, u \in U, Tu = Tx \Rightarrow x = 0$.

Also, $\neg P \Rightarrow \neg Q$: (a) $\operatorname{range} T|_U \subsetneq \operatorname{range} T$; OR (b) $U \cap \operatorname{null} T \neq \{0\}$.
For (a), $\exists\, x \in V \backslash U, Tx \neq 0 \Longleftrightarrow x \notin \operatorname{null} T$. Thus $U + \operatorname{null} T \subsetneq V$. For (b), immediately. □

COMMENT: If $T|_U : U \to \operatorname{range} T$ is an iso. Let $R \oplus U = V$. Then R might not be $\operatorname{null} T$.
OR. Extend B_U to $B_V = (u_1, \dots, u_n, r_1, \dots, r_m)$, then $(r_1, \dots, r_m)$ might not be a $B_{\operatorname{null} T}$.

• **TIPS 3:** Suppose $T \in \mathcal{L}(V,W)$ and U is a subsp such that $V = U \oplus \operatorname{null} T$. Let $\operatorname{null} T = X \oplus Y$.
Now $\forall v \in V, \exists!\, u_v \in U, (x_v, y_v) \in X \times Y, v = u_v + x_v + y_v$. Define $i \in \mathcal{L}(V, U \oplus X)$ by $i(v) = u_v + x_v$.
Then $T = T \circ i$. Because $\forall v \in V, T(v) = T(u_v + x_v + y_v) = T(u_v) = T(u_v + x_v) = T\big(i(v)\big) = (T \circ i)(v)$.

• **TIPS 4:** Suppose $T \in \mathcal{L}(V,W), T \neq 0$. Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
By (3.A.4), $R = (v_1, \dots, v_n)$ is linely inde in V. Let $\operatorname{span} R = U$. We will prove that $U \oplus \operatorname{null} T = V$.
(a) $T\big(\sum_{i=1}^{n} a_i v_i\big) = 0 \Longleftrightarrow \sum_{i=1}^{n} a_i T v_i = 0 \Longleftrightarrow a_1 = \cdots = a_n = 0$. Thus $U \cap \operatorname{null} T = \{0\}$.
(b) $Tv = \sum_{i=1}^{n} a_i T v_i \Longleftrightarrow v - \sum_{i=1}^{n} a_i v_i \in \operatorname{null} T \Longleftrightarrow v = \big(v - \sum_{i=1}^{n} a_i v_i\big) + \big(\sum_{i=1}^{n} a_i v_i\big)$.
Thus $U + \operatorname{null} T = V$. OR. $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$. Using Problem (12). □

COROLLARY: Conversely, if $U \oplus \operatorname{null} T = V$ and $B_U = (v_1, \dots, v_n)$, then $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
Because $\operatorname{range} T = \operatorname{range} T|_U = \operatorname{span}(Tv_1, \dots, Tv_n)$, 又 T is inje.

• [4E 27, OR 5.B.4] *Suppose* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove that* $V = \operatorname{null} P \oplus \operatorname{range} P$.

SOLUTION: (a) If $v \in \operatorname{null} P \cap \operatorname{range} P \Rightarrow Pv = 0$, and $\exists\, u \in V, v = Pu$. Then $v = Pu = P^2u = Pv = 0$.

(b) Note that $\forall v \in V, v = Pv + (v - Pv)$ and $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$.

OR. Because $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P = \dim\big(\operatorname{null} P \oplus \operatorname{range} P\big)$. $\square$

OR. [*Only in Finite-dim*] Let $B_{\operatorname{range} P^2} = (P^2v_1, \dots, P^2v_n)$. Then $(Pv_1, \dots, Pv_n)$ is linely inde.
Let $U = \operatorname{span}(Pv_1, \dots, Pv_n) \Rightarrow V = U \oplus \operatorname{null} P^2$. While $U = \operatorname{range} P = \operatorname{range} P^2$; $\operatorname{null} P = \operatorname{null} P^2$. $\square$

• *Suppose* $T \in \mathcal{L}(V), v \in V$, *and* $n \in \mathbf{N}^+$ *such that* $T^{n-1}v \neq 0,\ T^n v = 0$. [See [5.16]]
Prove that $\big(v, Tv, \dots, T^{n-1}v\big)$ *is linely inde.*

SOLUTION: $a_0 v + a_1 Tv + \cdots + a_{n-1}T^{n-1}v = 0 \Rightarrow a_0 T^{n-1}v = 0 \Rightarrow a_0 = 0$. Similar for $a_1, \dots, a_{n-1}$. $\square$

• (4E 21) *Suppose* V *is finite-dim*, $T \in \mathcal{L}(V, W)$, Y *is a subsp of* W. *Let* $\{v \in V : Tv \in Y\}$.
(a) *Prove that* $\{v \in V : Tv \in Y\}$ *is a subsp of* V.
(b) *Prove that* $\dim\{v \in V : Tv \in Y\} = \dim \operatorname{null} T + \dim\big(Y \cap \operatorname{range} T\big)$.

SOLUTION: Let $\mathcal{K}_Y = \{v \in V : Tv \in Y\}$.

(a) $\forall u, w \in \mathcal{K}_Y, [\, Tu, Tw \in Y\,], \lambda \in \mathbf{F}, T(u + \lambda w) = Tu + \lambda Tw \in Y \Longrightarrow \mathcal{K}_Y$ is a subsp of V.

(b) Define the range-restricted map R of T by $R = T|_{\mathcal{K}_Y} \in \mathcal{L}(\mathcal{K}_Y, Y)$. Now $\operatorname{range} R = Y \cap \operatorname{range} T$.
And $v \in \operatorname{null} T \iff Tv = 0 \in Y \iff Rv = 0 \in \operatorname{range} T \iff v \in \operatorname{null} R$. By [3.22]. $\square$

COMMENT: Now $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = \mathcal{K}_Y$. Where $B_{Y \cap \operatorname{range} T} = (Tv_1, \dots, Tv_m)$.
In particular, $\dim \mathcal{K}_{\operatorname{range} T} = \dim \operatorname{null} T + \dim \operatorname{range} T \Longrightarrow \mathcal{K}_{\operatorname{range} T} = V$.

• (4E 31) *Suppose* V *is finite-dim*, X *is a subsp of* V, *and* Y *is a finite-dim subsp of* W.
Prove that if $\dim X + \dim Y = \dim V$, *then* $\exists\, T \in \mathcal{L}(V, W), \operatorname{null} T = X, \operatorname{range} T = Y$.

SOLUTION: Let $V = U \oplus X, B_U = (v_1, \dots, v_m)$. Then $\forall v \in V, \exists!\, a_i \in \mathbf{F}, x \in X,\ v = \sum_{i=1}^m a_i v_i + x$.
Let $B_Y = (w_1, \dots, w_m)$. Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i, Tx = 0$ for each v_i and all $x \in X$.
Now $v \in \operatorname{null} T \iff Tv = a_1w_1 + \cdots + a_mw_m = 0 \iff v = x \in X$. Hence $\operatorname{null} T = X$.
And $Y \ni w = a_1w_1 + \cdots + a_mw_m = a_1Tv_1 + \cdots + a_mTv_m \in \operatorname{range} T$. Hence $\operatorname{range} T = Y$.
OR. NOTICE that $V = U \oplus \operatorname{null} T$. By Problem (12), $\operatorname{range} T = \operatorname{range} T|_U$.
又 $\dim \operatorname{range} T|_U = \dim U = \dim Y$; $\operatorname{range} T \subseteq Y$.

OR. Let $B_X = (x_1, \dots, x_n)$. Now $\operatorname{range} T = \operatorname{span}\big(Tv_1, \dots, Tv_m, Tx_1, \dots, Tx_n\big) = \operatorname{span}(w_1, \dots, w_m) = Y$. $\square$

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22 *Suppose* U, V *are finite-dim*, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.
Prove that $\dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$.

SOLUTION: We show that $\dim \operatorname{null} ST = \dim \operatorname{null} S|_{\operatorname{range} T} + \dim \operatorname{null} T$.
Because (a) $\underline{\operatorname{range} T|_{\operatorname{null} ST} = \operatorname{range} T \cap \operatorname{null} S = \operatorname{null} S|_{\operatorname{range} T}}$,
(b) $\underline{\operatorname{null} T|_{\operatorname{null} ST} = \operatorname{null} T \cap \operatorname{null} ST = \operatorname{null} T}$. By [3.22] $\square$

OR. NOTICE that $\underline{u \in \operatorname{null} ST \iff S(Tu) = 0 \iff Tu \in \operatorname{null} S}$.
Thus $\underline{\{u \in U : Tu \in \operatorname{null} S\} = \mathcal{K}_{\operatorname{null} S \cap \operatorname{range} T} = \operatorname{null} ST}$.
By Problem (4E 21), $\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim\big(\operatorname{null} S \cap \operatorname{range} T\big)$. $\square$

COROLLARY: (1) T surj $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.
(2) T inv $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S,\ \operatorname{null} ST = \operatorname{null} T$.
(3) S inje $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} T$.

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23 *Suppose V is finite-dim, $S \in \mathcal{L}(V,W), T \in \mathcal{L}(U,V)$.*
Prove that $\dim \operatorname{range} ST \leqslant \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.

Comment: If $\dim V = \dim U$. Then $\dim \operatorname{null} ST \geqslant \max\{\dim \operatorname{null} S, \dim \operatorname{null} T\}$.

Solution: Notice that $\underline{\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{range} S|_{\operatorname{range} T}}$.
Let $\operatorname{range} ST = \operatorname{span}(Su_1, \dots, Su_{\dim \operatorname{range} T})$, where $B_{\operatorname{range} T} = (u_1, \dots, u_{\dim \operatorname{range} T})$.
$\dim \operatorname{range} ST \leqslant \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leqslant \dim \operatorname{range} S$. $\square$

Or. $\underline{\dim \operatorname{range} ST = \dim \operatorname{range} S|_{\operatorname{range} T} = \dim \operatorname{range} T - \dim \operatorname{null} S|_{\operatorname{range} T}} \leqslant \operatorname{range} T$. $\square$

Comment: $\dim \operatorname{range} ST = \dim U - \dim \operatorname{null} ST = \dim \operatorname{range} T|_U - \dim \operatorname{range} T|_{\operatorname{null} ST}$.

Corollary: (1) $S|_{\operatorname{range} T}$ inje $\iff \dim \operatorname{range} ST = \dim \operatorname{range} T$.
(2) Let $X \oplus \operatorname{null} S = V$. Then $X \subseteq \operatorname{range} T \iff \operatorname{range} ST = \operatorname{range} S$.
And T is surj $\Rightarrow \operatorname{range} ST = \operatorname{range} S$.

• (a) *Suppose* $\dim V = n$, $ST = 0$ *where* $S, T \in \mathcal{L}(V)$. *Prove that* $\dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$.
(b) *Give an example of such S, T with $n = 5$ and* $\dim \operatorname{range} TS = 2$.

Solution: Note that $\dim \operatorname{range} TS \leqslant \min\{\dim \operatorname{range} T, \dim \operatorname{range} S\}$. We prove by contradiction.
Assume that $\dim \operatorname{range} TS \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$. Then $\min\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\} \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$
又 $\dim \operatorname{null} ST = n \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \Big| \Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant \left\lceil \frac{n}{2} \right\rceil - 1$.
Thus $n \leqslant 2\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \Rightarrow \frac{n}{2} \leqslant \left\lceil \frac{n}{2} \right\rceil - 1$. Contradicts. $\square$

Or. $\dim \operatorname{null} S = n - \dim \operatorname{range} S \leqslant n - \dim \operatorname{range} TS$. 又 $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S$.
$\dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S \leqslant n - \dim \operatorname{range} TS$. Thus $2 \dim \operatorname{range} TS \leqslant n$. $\square$

Or. Because $\dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$, and $\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n$.
We show that $\dim \operatorname{null} TS \geqslant \left\lceil \frac{n}{2} \right\rceil$. Note that $\dim \operatorname{null} S + \dim \operatorname{null} T \geqslant n$.
$\dim \operatorname{null} S + \dim \operatorname{null} T|_{\operatorname{range} S} = \dim \operatorname{null} TS$. If $\dim \operatorname{null} S \geqslant \left\lceil \frac{n}{2} \right\rceil$. Then we are done.
Otherwise, $\dim \operatorname{null} S \leqslant \left\lceil \frac{n}{2} \right\rceil - 1 \Rightarrow \dim \operatorname{null} T \geqslant n - \dim \operatorname{null} S \geqslant n - \left\lceil \frac{n}{2} \right\rceil + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1 \geqslant \left\lceil \frac{n}{2} \right\rceil$.
Thus $\dim \operatorname{null} TS \geqslant \max\{\dim \operatorname{null} S, \dim \operatorname{null} T\} = \left\lceil \frac{n}{2} \right\rceil$. $\square$

Example: Define $T: v_1 \mapsto 0,\ v_2 \mapsto 0,\ v_i \mapsto v_i;\quad S: v_1 \mapsto v_4,\ v_2 \mapsto v_5,\ v_i \mapsto 0;\ i = 3,4,5$.

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26 *Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ and $\forall p, \deg(Dp) = (\deg p) - 1$. Prove that $D \in \mathcal{P}(\mathbf{R})$ is surj.*

Solution: $[\,D$ might not be $D : p \mapsto p'.\,]$ Notice that the following proof is wrong:
Because $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) \subseteq \operatorname{range} D$, and $\deg Dx^n = n - 1$.
又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) = \operatorname{span}(1, x, x^2, \cdots) = \mathcal{P}(\mathbf{R})$.

$\underline{\text{Let } D(C) = 0, Dx^k = p_k}$ of deg $(k-1)$, for all $C \in \mathbf{R} = \mathcal{P}_0(\mathbf{R})$ and for each $k \in \mathbf{N}^+$.
Because $B_{\mathcal{P}_m(\mathbf{R})} = (p_1, \dots, p_m, p_{m+1})$. And for all $p \in \mathcal{P}(\mathbf{R}), \exists!\, m = \deg p \in \mathbf{N}^+$.
So that $\exists!\, a_i \in \mathbf{R}, p = \sum_{i=1}^{m+1} a_i p_i \Rightarrow \exists\, q = \sum_{i=1}^{m+1} a_i x^i, Dq = p$. $\square$

Or. We will $\underline{\text{recursively define a sequence of polys } (p_k)_{k=0}^{\infty} \text{ where } Dp_0 = 1, Dp_k = x^k}$ for each $k \in \mathbf{N}^+$.
So that $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), Dq = p, \exists\, q = \sum_{k=0}^{\deg p} a_k p_k$.

(i) Because $\deg Dx = (\deg x) - 1 = 0, Dx = C \in \mathbf{F} \backslash \{0\}$. Let $p_0 = C^{-1}x \Rightarrow Dp_0 = C^{-1}Dx = 1$.

(ii) Suppose we have defined $Dp_0 = 1, Dp_k = x^k$ for each $k \in \{1, \dots, n\}$. Because $\deg D(x^{n+2}) = n + 1$.
Let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$, with $a_{n+1} \neq 0$.
Then $a_{n+1}^{-1} D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_n Dp_n + \cdots + a_1 Dp_1 + a_0 Dp_0)$
$\Rightarrow x^{n+1} = D\left[\underline{a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)}\right]$. Thus defining p_{n+1}, so that $Dp_{n+1} = x^{n+1}$. $\square$

20, 21 (a) *Prove that if $ST = I \in \mathcal{L}(V)$, then T is inje and S is surj.*
(b) *Suppose $T \in \mathcal{L}(V, W)$. Prove that if T is inje, then $\exists\, S \in \mathcal{L}(W, V),\ ST = I$.*
(c) *Suppose $S \in \mathcal{L}(W, V)$. Prove that if S is surj, then $\exists\, T \in \mathcal{L}(V, W),\ ST = I$.*

SOLUTION:

(a) $Tv = 0 \Rightarrow S(Tv) = 0 = v$. OR. $\operatorname{null} T \subseteq \operatorname{null} ST = \{0\}$.
$\forall v \in V, ST(v) = v \in \operatorname{range} S$. OR. $V = \operatorname{range} ST \subseteq \operatorname{range} S$.

(b) Define $S \in \mathcal{L}(\operatorname{range} T, V)$ by $Sw = T^{-1}w$, where T^{-1} is the inv of $T \in \mathcal{L}(V, \operatorname{range} T)$.
Then extend to $S \in \mathcal{L}(W, V)$ by (3.A.11). Now $\forall v \in V, STv = T^{-1}Tv = v$.

OR. $\big[$ *Req V Finite-dim* $\big]$ Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n) \Rightarrow B_V = (v_1, \dots, v_n)$. Let $U \oplus \operatorname{range} T = W$.
Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i, Su = 0$ for each v_i and all $u \in U$. Thus $ST = I$.

(c) By Problem (12), $\exists$ subsp U of $W, W = U \oplus \operatorname{null} S,\ \operatorname{range} S = \operatorname{range} S|_U = V$.
Note that $S|_U : U \to V$ is an iso. Define $T = (S|_U)^{-1}$, where $(S|_U)^{-1} : V \to U$.
Then $ST = S \circ (S|_U)^{-1} = S|_U \circ (S|_U)^{-1} = I_V$.

OR. $\big[$ *Req V Finite-dim* $\big]$ Let $B_{\operatorname{range} S} = B_V = (Sw_1, \dots, Sw_n) \Rightarrow \operatorname{span}(w_1, \dots, w_n) \oplus \operatorname{null} S = W$.
Define $T \in \mathcal{L}(V, W)$ by $T(Sw_i) = w_i$. Now $ST(a_1Sw_1 + \cdots + a_nSw_n) = (a_1Sw_1 + \cdots + a_nSw_n)$. □

COROLLARY: For (b), if T is inje and $\exists\, S,\ ST = I$, then by (a), this S is surj. Similar for (c).

• **TIPS 5:** Suppose $S \in \mathcal{L}(U, V)$ is surj. Define $\mathcal{B} \in \mathcal{L}\big(\mathcal{L}(V, W), \mathcal{L}(U, W)\big)$ by $\mathcal{B}(T) = TS$.
Then $\mathcal{B}$ is inje. Because $\mathcal{B}(T) = TS = 0 \iff T|_{\operatorname{range} S} = 0$. OR. $\operatorname{range} TS = \operatorname{range} T = \{0\}$.

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24 *Suppose $S, T \in \mathcal{L}(V, W)$, and $\operatorname{null} S \subseteq \operatorname{null} T$. Prove that $\exists\, E \in \mathcal{L}(W), T = ES$.*

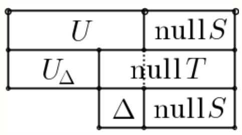
SOLUTION:

Let $\underline{V = U \oplus \operatorname{null} S}$
$\underline{\Rightarrow S|_U : U \to \operatorname{range} S \text{ is an iso.}}$
Extend $T(S|_U)^{-1}$ to $E \in \mathcal{L}(W)$.

$$\begin{tikzcd} \operatorname{range} T & U \arrow[l, "surj\ T"'] \arrow[d, "inv\ S"] \\ & \operatorname{range} S \arrow[ul, "surj\ E"] \end{tikzcd}$$

OR. Define $E : \operatorname{range} S \to W$ by $\underline{E : Sv \mapsto Tv}$.
Extend $E \in \mathcal{L}(\operatorname{range} S, W)$ to $E \in \mathcal{L}(W)$. □

COMMENT: Let $\Delta \oplus \operatorname{null} S = \operatorname{null} T,\ U_\Delta \oplus (\Delta \oplus \operatorname{null} S) = V = U_\Delta \oplus \operatorname{null} T$. Redefine $U = U_\Delta \oplus \Delta$.



$$\operatorname{range} S \xleftarrow{S} \begin{array}{l} U_\Delta \xrightarrow{T} \operatorname{range} T \\ \oplus \\ \Delta \xrightarrow{T} \{0\} \end{array}$$

Because $\Delta = \operatorname{null} T|_U = \operatorname{null} T \cap \operatorname{range} (S|_U)^{-1}$.
Thus $E = T(S|_U)^{-1}$ is not inje $\iff \Delta \neq \{0\}$.
In other words, $\operatorname{range} S|_\Delta = \operatorname{null} E$,
while $E|_{\dots} : \operatorname{range} S|_{U_\Delta} \to \operatorname{range} T$ is an iso.

COMMENT: Let $E_1 \in \mathcal{L}(U_\Delta \oplus \operatorname{null} T, U_\Delta)$, and E_2 be an iso of $\operatorname{range} S|_{U_\Delta}$ onto $\operatorname{range} T$.
Define $E_1|_{U_\Delta} = I|_{U_\Delta}$, and $E_2 = T(S|_{U_\Delta})^{-1}$. Then $T = E_2SE_1$.

COROLLARY: If $\operatorname{null} S = \operatorname{null} T$. Then $\Delta = \{0\}, U_\Delta = U$.
By (3.D.3), we can extend inje $T(S|_U)^{-1} \in \mathcal{L}(\operatorname{range} S, W)$ to inv $E \in \mathcal{L}(W)$.

OR. $\big[$ *Req* range S *Finite-dim* $\big]$ Let $B_{\operatorname{range} S} = (Sv_1, \dots, Sv_n)$. Then $\underline{V = \operatorname{span}(v_1, \dots, v_n) \oplus \operatorname{null} S}$.
Let $U \oplus \operatorname{range} S = W$. Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i,\ Eu = 0$ for all $u \in U$ and each v_i.
Hence $\forall v \in V$, $\underline{(\exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} S \subseteq \operatorname{null} T\,), Tv = a_1Tv_1 + \cdots + a_nTv_n} = E(a_1Sv_1 + \cdots + a_nSv_n)$ □

COROLLARY: $\big[$ *Req W Finite-dim* $\big]$ Suppose $\operatorname{null} S = \operatorname{null} T$. We show that $\exists$ inv $E \in \mathcal{L}(W), T = ES$.
Redefine $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i,\ E(w_j) = x_j$, for each Tv_i and w_j. Where:
Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m), B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n), B_U = (v_1, \dots, v_m)$.
Now $V = U \oplus \operatorname{null} T = U \oplus \operatorname{null} S \Rightarrow B_{\operatorname{range} S} = (Sv_1, \dots, Sv_m)$. Let $B'_W = (Sv_1, \dots, Sv_m, x_1, \dots, x_n)$. □

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25 *Suppose $S, T \in \mathcal{L}(V, W)$, and* $\operatorname{range} T \subseteq \operatorname{range} S$. *Prove that* $\exists\, E \in \mathcal{L}(V), T = SE$.

SOLUTION:

Let $V = U \oplus \operatorname{null} S \Rightarrow S|_U : U \to \operatorname{range} S$ is an iso. Because $(S|_U)^{-1} : \underset{\supseteq \operatorname{range} T}{\overline{\operatorname{range} S}} \to U$.

$\underline{\text{Define } E = (S|_U)^{-1} T = (S|_U)^{-1}|_{\operatorname{range} T} T \in \mathcal{L}(V, U) \subseteq \mathcal{L}(V).}$ $\square$

COMMENT: Let $U_1 = U$. Let $U_2 \oplus \operatorname{null} T = V = U_1 \oplus \operatorname{null} S$.

Let $U_{1\Delta} = \operatorname{range}\big(S|_{U_1}\big)|_{\operatorname{range} T} \subseteq U_1 = \Delta \oplus U_{1\Delta}$.

OR. Let $U_{1\Delta} = \operatorname{range} E|_{U_2}$. Let $\Delta \oplus \operatorname{range} E|_{U_2} = U_1$.

Thus $U_1 \oplus \operatorname{null} S = U_{1\Delta} \oplus \underbrace{\underline{(\Delta \oplus \operatorname{null} S)} = U_2 \oplus \underline{\operatorname{null} T}}_{\text{iso, by (3.D.TIPS).}}$.

$$\begin{tikzcd}
U_1 \arrow[r, "inv", "S"'] \arrow[d, phantom, "||"] & \operatorname{range} S \arrow[d, phantom, "||"] & \\
\Delta \arrow[r, "inv", "S"'] \arrow[d, phantom, "\oplus"] & \operatorname{range} S|_\Delta \arrow[d, phantom, "\oplus"] & \\
U_{1\Delta} \arrow[r, "inv", "S"'] & \operatorname{range} T & U_2 \arrow[l, "inv"', "T"] \arrow[ll, bend left, "inv\ E|_{U_2}"]
\end{tikzcd}$$

If $\Delta \neq \{0\}$, assume $\exists$ inv $E \in \mathcal{L}(V)$ re-extended from $E|_{U_2}$ still satisfying $T = SE$,

then let $\Delta \xrightarrow{E^{-1}} \Theta$; $\operatorname{null} S \xrightarrow{E^{-1}} \operatorname{null} T_\Theta$. Now $\Theta \oplus \operatorname{null} T_\Theta = \operatorname{null} T$.

Then $\Theta \xrightarrow{E} \Delta \neq \{0\}$, while $\operatorname{null} S \cap \Delta = \{0\}$. Thus $T|_\Theta = SE|_\Theta \neq 0$, contradicts.

COROLLARY: If $\Delta = \{0\}$, then $U_1 = U_{1\Delta} \Rightarrow \operatorname{range} S = \operatorname{range} T$. 又 $\operatorname{null} S, \operatorname{null} T$ are iso.

By (3.D.3), we can re-extend inje $E|_{U_2} \in \mathcal{L}\big(U_2, U_1 \oplus \operatorname{null} S\big)$ to inv $E \in \mathcal{L}\big(U_2 \oplus \operatorname{null} T, U_1 \oplus \operatorname{null} S\big)$.

Thus we have $\Delta \neq \{0\} \iff E|_{U_2} \in \mathcal{L}(U_2, V)$ cannot be re-extended to inv $E \in \mathcal{L}(V)$ freely.

OR. [*Req* $\operatorname{range} T$ *Finite-dim*] Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$. Then $\underline{V = \operatorname{span}(v_1, \dots, v_n) \oplus \operatorname{null} T.}$
$\underline{\text{Let } S(u_i) = Tv_i \text{ for each } Tv_i.}$ Define E by $Ev_i = u_i, Ex = 0$ for all $x \in \operatorname{null} T$ and each v_i. $\square$

COMMENT: [*Req V Finite-dim*] Note that $\dim U_2 \leqslant \dim U_1 \Longrightarrow \dim \operatorname{null} T = p \geqslant q = \dim \operatorname{null} S$.
Let $B_{\operatorname{null} T} = (x_1, \dots, x_p), B_{\operatorname{null} S} = (y_1, \dots, y_q)$. Redefine $E : v_i \mapsto u_i,\ x_k \mapsto y_k,\ x_j \mapsto 0$,
for each $i \in \{1, \dots, \dim U_2\}, k \in \{1, \dots, \dim \operatorname{null} S\}, j \in \{\dim \operatorname{null} S + 1, \dots, \dim \operatorname{null} T\}$.
Note that $(u_1, \dots, u_n)$ is linely inde. Let $X = \operatorname{span}(x_1, \dots, x_q) \oplus \operatorname{span}(v_1, \dots, v_n)$.
Now $E|_X$ is inje, but cannot be re-extend to inv $E \in \mathcal{L}(V)$ without loss of functionality.

COROLLARY: [*Req V Finite-dim*] If $\operatorname{range} T = \operatorname{range} S$, then $\dim \operatorname{null} T = \dim \operatorname{null} S = p$.
Redefine E by $Ev_i = u_i,\ Ex_j = y_j$ for each v_i and x_j. Then $E \in \mathcal{L}(V)$ is inv. $\square$

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28 *Suppose $T \in \mathcal{L}(V, W)$. Let $B_{\operatorname{range} T} = (w_1, \dots, w_m)$.*

(a) *Prove that $\exists\, \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ such that $\forall v \in V, Tv = \varphi_1(v) w_1 + \dots + \varphi_m(v) w_m$.*

(b) [4E 3.F.5] $\forall v \in V, \exists!\, \varphi_i(v) \in \mathbf{F}, Tv = \varphi_1(v) w_1 + \dots + \varphi_m(v) w_m$.
Thus defining each $\varphi_i : V \to \mathbf{F}$. Show that each $\varphi_i \in \mathcal{L}(V, \mathbf{F})$.

SOLUTION: (a) Using TIPS (4). Let each $w_i = Tv_i$. Then $(v_1, \dots, v_m)$ is linely inde.
And $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = V$. Now $\forall v \in V, \exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} T,\ v = \sum_{i=1}^m a_i v_i + u$.
Define $\varphi_i \in \mathcal{L}(V, \mathbf{F})$ by $\varphi_i(v_j) = \delta_{i,j},\ \varphi_i(u) = 0$ for all $u \in \operatorname{null} T$.
Linearity: $\forall v, w \in V\ [\exists!\, a_i, b_i \in \mathbf{F}], \lambda \in \mathbf{F}, \varphi_i(v + \lambda w) = a_i + \lambda b_i = \varphi(v) + \lambda \varphi(w)$. $\square$

(b) $\sum_{i=1}^m \varphi_i(u + \lambda v) w_i = T(u + \lambda v) = Tu + \lambda Tv = \Big(\sum_{i=1}^m \varphi_i(u) w_i\Big) + \lambda \Big(\sum_{i=1}^m \varphi_i(v) w_i\Big)$. $\square$

OR. Using (3.F). Let each $w_i = Tv_i \Rightarrow (v_1, \dots, v_m)$ is linely inde.
Now $\forall v \in V, \exists!\, a_i \in \mathbf{F}, Tv = a_1 Tv_1 + \dots + a_m Tv_m$. Let $B_{(\operatorname{range} T)'} = (\psi_1, \dots, \psi_m)$.
Then $[T'(\psi_i)](v) = (\psi_i \circ T)(v) = a_i$. Where $T : V \to \operatorname{range} T$; $T' : (\operatorname{range} T)' \to V'$.
Thus each $\varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$. $\square$

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29 *Suppose* $\varphi \in \mathcal{L}(V, \mathbf{F})$. *Suppose* $\varphi(u) \neq 0$. *Prove that* $V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}$.

SOLUTION: Let $B_{\operatorname{range} \varphi} = (\varphi(u))$. Then by TIPS (4), $\operatorname{span}(u) \oplus \operatorname{null} \varphi = V$. $\square$

OR. (a) $v = cu \in \operatorname{null} \varphi \cap \operatorname{span}(u) \Rightarrow c\varphi(u) = 0 \Rightarrow v = 0$. Now $\operatorname{null} \varphi \cap \operatorname{span}(u) = \{0\}$.

(b) $\forall\, v \in V, v = \left(v - \dfrac{\varphi(v)}{\varphi(u)} u\right) + \dfrac{\varphi(v)}{\varphi(u)} u$. Now $V = \operatorname{null} \varphi + \operatorname{span}(u)$. $\square$

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30 *Suppose* $\varphi, \beta \in \mathcal{L}(V, \mathbf{F})$ *and* $\operatorname{null} \varphi = \operatorname{null} \beta = \eta$. *Prove that* $\exists\, c \in \mathbf{F}, \varphi = c\beta$.

SOLUTION: If $\eta = V$, then $\varphi = \beta = 0$, we are done. Now by Problem (29),

$\varphi(u) \neq 0 \Longleftrightarrow V = \operatorname{null} \varphi \oplus \operatorname{span}(u) \Longleftrightarrow V = \operatorname{null} \beta \oplus \operatorname{span}(u) \Longleftrightarrow \beta(u) \neq 0$.

Note that $\forall v \in V, \exists!\, u_0 \in \eta,\ a_v \in \mathbf{F}, v = u_0 + a_v u$
$\Rightarrow \varphi(u_0 + a_v u) = a_v \varphi(u),\ \beta(u_0 + a_v u) = a_v \beta(u)$.
| Let $c = \dfrac{\varphi(u)}{\beta(u)} \in \mathbf{F} \backslash \{0\}$. $\square$

- (4E 3.F.6) *Suppose* $\varphi, \beta \in \mathcal{L}(V, \mathbf{F})$. *Prove that* $\operatorname{null} \beta \subseteq \operatorname{null} \varphi \Longleftrightarrow \varphi = c\beta, \exists\, c \in \mathbf{F}$.

  COROLLARY: $\operatorname{null} \varphi = \operatorname{null} \beta \Longleftrightarrow \varphi = c\beta,\ \exists\, c \in \mathbf{F} \backslash \{0\}$.

SOLUTION: Using Problem (29) and (30).

(a) If $\varphi = 0$, then we are done. Otherwise, suppose $u \notin \operatorname{null} \varphi \supseteq \operatorname{null} \beta$.

Now $V = \operatorname{null} \varphi \oplus \operatorname{span}(u) = \operatorname{null} \beta \oplus \operatorname{span}(u)$. By [1.C TIPS (2)], $\operatorname{null} \varphi = \operatorname{null} \beta$. Let $c = \dfrac{\varphi(u)}{\beta(u)}$.

OR. We discuss in two cases. If $\operatorname{null} \beta = \operatorname{null} \varphi$, or if $\varphi = 0$, then we are done. Otherwise, $\exists\, u' \in \operatorname{null} \varphi \backslash \operatorname{null} \beta$, $\exists\, u \notin \operatorname{null} \varphi \supsetneq \operatorname{null} \beta \Rightarrow V = \operatorname{null} \beta \oplus \operatorname{span}(u') = \operatorname{null} \beta \oplus \operatorname{span}(u)$.

$\forall v \in V, v = w + au = w' + bu', \exists!\, w, w' \in \operatorname{null} \beta$
Thus $\varphi(w + au) = a\varphi(u),\ \beta(w' + bu) = b\beta(u')$.
| Let $c = \dfrac{a\varphi(u)}{b\beta(u')} \in \mathbf{F} \backslash \{0\}$. We are done.

NOTICE that by (b) below, we have $\operatorname{null} \varphi \subseteq \operatorname{null} \beta$, contradicts the assumption.

(b) If $c = 0$, then $\operatorname{null} \varphi = V \supseteq \operatorname{null} \beta$, we are done. Otherwise, because $v \in \operatorname{null} \beta \Longleftrightarrow v \in \operatorname{null} \varphi$. $\square$

OR. By Problem (24), $\operatorname{null} \beta \subseteq \operatorname{null} \varphi \Longleftrightarrow \exists\, E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta$. [If E is inv. Then $\operatorname{null} \beta = \operatorname{null} \varphi$.]
Now $\exists\, E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta \Longleftrightarrow \exists\, c = E(1) \in \mathbf{F}, \varphi = c\beta$. [E is inv $\Longleftrightarrow E(1) \neq 0 \Longleftrightarrow c \neq 0$.] $\square$

ENDED

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1 3 4 5 6 9 10 11 13 | 4E: 16 17

- **NOTE FOR Transpose:** [3.F.33] Define $\mathcal{T} : A \to A^t$. By [3.111], $\mathcal{T}$ is linear. Because $(A^t)^t = A$.
  $\mathcal{T}^2 = I$, $\mathcal{T} = \mathcal{T}^{-1} \Rightarrow \mathcal{T}$ is an iso of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$. Define $\mathcal{C}_k : A \to A_{\cdot,k}$, $\mathcal{R}_j : A \to A_{j,\cdot}$, $\mathcal{E}_{j,k} : A \to A_{j,k}$.
  Now we show that (a) $\underline{\mathcal{T}\mathcal{R}_j = \mathcal{C}_j\mathcal{T}}$, (b) $\underline{\mathcal{T}\mathcal{C}_k = \mathcal{R}_k\mathcal{T}}$, and (c) $\underline{\mathcal{T}\mathcal{E}_{j,k} = \mathcal{E}_{k,j}\mathcal{T}}$.
  So that furthermore, $\mathcal{T}\mathcal{C}_k\mathcal{T} = \mathcal{R}_k$, $\mathcal{T}\mathcal{R}_j\mathcal{T} = \mathcal{C}_j$, and $\mathcal{T}\mathcal{E}_{j,k}\mathcal{T} = \mathcal{E}_{k,j}$.

  Let $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{pmatrix}$.

  Note that $(A_{j,k})^t = A_{j,k} = (A^t)_{k,j}$. Thus (c) holds.
  And $(A_{\cdot,k})^t = (A_{1,k} \cdots A_{m,k}) = (A^t_{k,1} \cdots A^t_{k,m}) = (A^t)_{k,\cdot}$
  $\Longrightarrow$ (b) holds. Similar for (a).

- **NOTE FOR [3.48]:**

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}}_{B} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

- **NOTE FOR [3.47]:** $(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k}$ ☐
- **NOTE FOR [3.49]:** $\left[(AC)_{\cdot,k}\right]_{j,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$ ☐
- **EXERCISE 10:** $\left[(AC)_{j,\cdot}\right]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$ ☐
- **COMMENT:** For [3.49], let $B_U = (u_1, \dots, u_p)$, $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.
  And $C = \mathcal{M}(T, B_U, B_V) \in \mathbf{F}^{n,p}$, $A = \mathcal{M}(S, B_V, B_W) \in \mathbf{F}^{m,n}$.
  Then $\mathcal{M}(Tu_k, B_V) = C_{\cdot,k} \Rightarrow \mathcal{M}(S(Tu_k), B_W) = AC_{\cdot,k}$, 又 $\mathcal{M}((ST)(u_k), B_W) = (AC)_{\cdot,k}$ ☐
  By NOTE FOR Transpose, $(AC)_{j,\cdot} = \left[\left((AC)^t\right)_{\cdot,j}\right]^t = \left(C^t(A^t)_{\cdot,j}\right)^t = \left((A^t)_{\cdot,j}\right)^t C = A_{j,\cdot}C$ ☐

- **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$. By [4E 3.51(a)], $(Ac)_{\cdot,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ ☐
  OR. $\because (Ac)_{j,1} = \sum_{r=1}^{n} A_{j,r}c_{r,1} = \left[\sum_{r=1}^{n} (A_{\cdot,r}c_{r,1})\right]_{j,1} = \left(c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}\right)_{j,1}$
  $\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^{n} A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ OR. $(Ac)_{j,1} = (Ac)_{j,\cdot} = A_{j,\cdot}c \in \mathbf{F}$. ☐
  OR. Let $B_V = (v_1, \dots, v_n)$. Now $Ac = \mathcal{M}(Tv, B_W) = \mathcal{M}(T(c_1v_1 + \cdots + c_nv_n)) = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$. ☐

- **EXERCISE 11:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$. By [4E 3.51(b)], $(aC)_{1,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ ☐
  OR. $\because (aC)_{1,k} = \sum_{r=1}^{n} a_{1,r}C_{r,k} = \left[\sum_{r=1}^{n} a_{1,r}(C_{r,\cdot})\right]_{1,k} = \left(a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}\right)_{1,k}$
  $\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^{n} a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ OR. $(aC)_{1,k} = (aC)_{\cdot,k} = aC_{\cdot,k} \in \mathbf{F}$. ☐
  OR. $aC = \left((aC)^t\right)^t = \left(C^ta^t\right)^t = \left[a^t_1(C^t)_{\cdot,1} + \cdots + a^t_n(C^t)_{\cdot,n}\right]^t = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$. ☐

- [4E 3.51] Suppose $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}$. [See also NOTE FOR [3.49] and Problem (10).]
  (a) For $k = 1, \dots, p$, $(CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^{c} C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$
  (b) For $j = 1, \dots, m$, $(CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
- **EXAMPLE:** $m = 2, c = 2, p = 3$.

$$(AB)_{\cdot,2} = AB_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} = A_{\cdot,1}B_{1,2} + A_{\cdot,2}B_{2,2} = 6\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix};$$

$$(AB)_{1,\cdot} = A_{1,\cdot}B = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = A_{1,1}B_{1,\cdot} + A_{1,2}B_{2,\cdot} = 1\begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 2\begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \end{pmatrix};$$

• Column-Row Factorization (CR Factorization) *Suppose* $A \in \mathbf{F}^{m,n}, A \neq 0$. *Prove, with p specified below, that* $\exists\, C \in \mathbf{F}^{m,p}, R \in \mathbf{F}^{p,n}, A = CR$.

(a) *Suppose* $S_c = \operatorname{span}(A_{\cdot,1}, \cdots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}, \dim S_c = c$, *the col rank. Let* $p = c$.

(b) *Suppose* $S_r = \operatorname{span}(A_{1,\cdot}, \cdots, A_{m,\cdot}) \subseteq \mathbf{F}^{1,n}, \dim S_r = r$, *the row rank. Let* $p = r$.

Solution: Using [4E 3.51]. Notice that $A \neq 0 \Rightarrow c, r \geqslant 1$.

(a) Reduce to basis $B_C = (C_{\cdot,1}, \cdots, C_{\cdot,c})$, forming $C \in \mathbf{F}^{m,c}$. Then $\forall k \in \{1, \dots, n\}$,
$A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$, $\exists!\, R_{1,k}, \cdots, R_{c,k} \in \mathbf{F}$, forming $R \in \mathbf{F}^{c,n}$. Thus $A = CR$.

(b) Reduce to basis $B_R = (R_{1,\cdot}, \cdots, R_{r,\cdot})$, forming $R \in \mathbf{F}^{r,n}$. Then $\forall j \in \{1, \dots, m\}$,
$A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,r}R_{r,\cdot} = (CR)_{j,\cdot}$, $\exists!\, C_{j,1}, \dots, C_{j,r} \in \mathbf{F}$, forming $C \in \mathbf{F}^{m,r}$. Thus $A = CR$. $\square$

Example: $A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \overset{\text{(I)}}{=\!=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \overset{\text{(II)}}{=\!=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$

(I) $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2\begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix}$, using [4E 3.51(b)].

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} \in \operatorname{span}(A_{1,\cdot}, A_{2,\cdot})$, and $(A_{1,\cdot}, A_{2,\cdot})$ is linely inde. Thus $B_R = (A_{1,\cdot}, A_{2,\cdot})$.

(II) $\begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = -\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$. Thus $B_C = (A_{\cdot,2}, A_{\cdot,3})$.

• Column Rank Equals Row Rank Using notation and result above.

For each $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

For each $A_{\cdot,k} \in S_c$, $A_{\cdot,k} = (CR)_{\cdot,k} = CR_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$

$\Rightarrow \operatorname{span}(A_{1,\cdot}, \cdots, A_{n,\cdot}) = S_r = \operatorname{span}(R_{1,\cdot}, \cdots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leqslant c = \dim S_c$.

$\Rightarrow \operatorname{span}(A_{\cdot,1}, \cdots, A_{\cdot,m}) = S_c = \operatorname{span}(C_{\cdot,1}, \cdots, C_{\cdot,r}) \Rightarrow \dim S_c = c \leqslant r = \dim S_r$.

Or. Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leqslant r = \dim S_r = \dim S_c^t$. $\square$

• *Suppose* $A \in \mathbf{F}^{m,n} \backslash \{0\}$. *Prove that* $[P]$ $\operatorname{rank} A = 1 \iff \exists\, c_j, d_k \in \mathbf{F}$, *each* $A_{j,k} = c_j \cdot d_k$. $[Q]$

Solution:

[*Using CR Factorization*]

$P \Rightarrow Q$: Immediately.

$Q \Rightarrow P$: Because $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}\begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1d_1 & \cdots & c_1d_n \\ \vdots & \ddots & \vdots \\ c_md_1 & \cdots & c_md_n \end{pmatrix} \Longrightarrow S_r = \operatorname{span}\left\{ \begin{matrix} \begin{pmatrix} \underline{c_1}d_1 & \cdots & \underline{c_1}d_n \end{pmatrix}, \\ \vdots \\ \begin{pmatrix} \underline{c_m}d_1 & \cdots & \underline{c_m}d_n \end{pmatrix} \end{matrix} \right\}$.

Or. $S_c = \operatorname{span}\left\{ \begin{pmatrix} c_1\underline{d_1} \\ \vdots \\ c_m\underline{d_1} \end{pmatrix}, \dots, \begin{pmatrix} c_1\underline{d_n} \\ \vdots \\ c_m\underline{d_n} \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$. $\square$

[*Not Using CR Factorization*]

$Q \Rightarrow P$: Using [4E 3.51(a)]. Each $A_{\cdot,k} \in \operatorname{span}\left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$. Then $\operatorname{rank} A = \dim S_c \leqslant 1$
又 $A \neq 0 \Rightarrow \dim S_c \geqslant 1$.

$P \Rightarrow Q$: Because $\dim S_c = \dim S_r = 1$.

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $\quad d'_k = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$.

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$, where $d_k = d'_k A_{1,1}$. $\square$

• **TIPS 1:** *Suppose* $T \in \mathcal{L}(V,W)$, $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.
Let $L = (Tv_{\alpha_1}, \dots, Tv_{\alpha_k})$, $M = (A_{\cdot,\alpha_1}, \cdots, A_{\cdot,\alpha_k})$, *where each* $\alpha_i \in \{1, \dots, n\}$.
(a) *Show that* $[P]$ L *is linely inde* $\iff$ M *is linely inde.* $[Q]$
(b) *Show that* $[P]$ $\operatorname{span} L = W \iff \operatorname{span} M = \mathbf{F}^{m,1}$. $[Q]$ $\quad$ [*Let* $A = \mathcal{M}(T, B_V, B_W)$.]

SOLUTION:

(a) Note that $\mathcal{M}\colon Tv_k \to A_{\cdot,k}$ is an iso of W onto $\mathbf{F}^{m,1}$. $\quad$ (b) Reduce L to B'_W, M to $B_{\mathbf{F}^{m,1}}$. Similarly. $\square$

OR. $$c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} = c_1(A_{1,\alpha_1}w_1 + \cdots + A_{m,\alpha_1}w_m) + \cdots + c_k(A_{1,\alpha_k}w_1 + \cdots + A_{m,\alpha_k}w_m)$$
$$= (c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k})w_1 + \cdots + (c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k})w_m.$$

And $$c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} = c_1\begin{pmatrix} A_{1,\alpha_1} \\ \vdots \\ A_{m,\alpha_1}\end{pmatrix} + \cdots + c_k\begin{pmatrix} A_{1,\alpha_k} \\ \vdots \\ A_{m,\alpha_k}\end{pmatrix} = \begin{pmatrix} c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k} \\ \vdots \\ c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k}\end{pmatrix}.$$

(a) $P \Rightarrow Q$: Suppose $c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} = 0$. Let $v = c_1v_{\alpha_1} + \cdots + c_kv_{\alpha_k}$.
Then $Tv = (c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k})w_1 + \cdots + (c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k})w_m = 0w_1 + \cdots + 0w_m$.
Now $c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k} = 0$. Then each $c_i = 0 \Rightarrow M$ linely inde.

$Q \Rightarrow P$: Because $c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k} = 0$. For each $i \in \{1, \dots, m\}$, $c_1A_{i,\alpha_1} + \cdots + c_kA_{i,\alpha_k} = 0$.
Which is equi to $c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} = 0$. Thus each $c_i = 0 \Rightarrow L$ linely inde.

OR. $\exists A_{\cdot,\alpha_j} = c_1A_{\cdot,\alpha_1} + \cdots + c_{j-1}A_{\cdot,\alpha_{j-1}}$
$\iff$ For each $i \in \{1, \dots, m\}$, $A_{i,\alpha_j} = c_1A_{i,\alpha_1} + \cdots + c_{j-1}A_{i,\alpha_{j-1}}$
$\iff Tv_{\alpha_j} = A_{1,\alpha_j}w_1 + \cdots + A_{m,\alpha_j}w_m$
$= (c_1A_{1,\alpha_1} + \cdots + c_{j-1}A_{1,\alpha_{j-1}})w_1 + \cdots + (c_1A_{m,\alpha_1} + \cdots + c_{j-1}A_{m,\alpha_{j-1}})w_m$
$\iff \exists\, Tv_{\alpha_j} = c_1Tv_{\alpha_1} + \cdots + c_{j-1}Tv_{\alpha_{j-1}}$.

(b) Note that each $\mathcal{M}(Tv_{\alpha_i}) = A_{\cdot,\alpha_i}$
$P \Rightarrow Q$: Suppose each $w_i = Iw_i = J_{1,i}Tv_{\alpha_1} + \cdots + J_{k,i}Tv_{\alpha_k}$.
$\forall a \in \mathbf{F}^{m,1}, \exists\, w = a_1w_1 + \cdots + a_mw_m \in W$, $a = \mathcal{M}(w, B_W)$.
Because $w = a_1(J_{1,1}Tv_{\alpha_1} + \cdots + J_{k,1}Tv_{\alpha_k}) + \cdots + a_m(J_{1,m}Tv_{\alpha_1} + \cdots + J_{k,m}Tv_{\alpha_k})$
$= (a_1J_{1,1} + \cdots + a_mJ_{1,m})Tv_{\alpha_1} + \cdots + (a_1J_{k,1} + \cdots + a_mJ_{k,m})Tv_{\alpha_k}$.
Apply $\mathcal{M}$ to both sides, $a = c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k}$, where each $c_i = a_1J_{i,1} + \cdots + a_mJ_{i,m}$.

$Q \Rightarrow P$: $\forall w \in W, \exists\, a = c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} \in \mathbf{F}^{m,1}$, $\mathcal{M}(w, B_W) = a$
$\Rightarrow w = (c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k})w_1 + \cdots + (c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k})w_m = c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k}$.

$\neg Q \Rightarrow \neg P$: $\exists\, w \in W, \exists\, a \in \mathbf{F}^{m,1}, \mathcal{M}(w, B_W) = a$, but $\nexists\, c_i \in \mathbf{F}, a = c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k}$
$\Rightarrow \nexists\, c_i \in \mathbf{F}$, $w = c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k}$. $\square$

COROLLARY: Let $L = (Tv_1, \dots, Tv_n)$, $M = (A_{\cdot,1}, \cdots, A_{\cdot,n})$.
Then (a*) By [3.B.9, TIPS (4)], T is inje $\iff$ L is linely inde, so is M.
And (b*) T is surj $\iff \operatorname{span} L = W \iff \operatorname{span} M = \mathbf{F}^{m,1}$.
COROLLARY: $B_{\mathbf{F}^{n,1}} = (A_{\cdot,1}, \cdots, A_{\cdot,n}) \iff T$ is inje and surj $\iff B_{\mathbf{F}^{1,n}} = (A_{\cdot,1}, \cdots, A_{\cdot,n})$.
COMMENT: If T is inv. Then by (a*, c) or (b*, d), we have another proof of COROLLARY.
OR. If $m = n$. Then by [3.118] and one of (a*, b*, c, d). Yet another proof.
(c) T surj $\iff T'$ inje $\iff (T'(\psi_1), \dots, T'(\psi_m))$ linely inde
$\overset{(a)}{\iff} ((A^t)_{\cdot,1}, \cdots, (A^t)_{\cdot,m})$ linely inde in $\mathbf{F}^{n,1}$, so is $(A_{1,\cdot}, \cdots, A_{m,\cdot})$ in $\mathbf{F}^{1,n}$.
(d) T inje $\iff T'$ surj $\iff V' = \operatorname{span}(T'(\psi_1), \dots, T'(\psi_m))$
$\overset{(b)}{\iff} \mathbf{F}^{n,1} = \operatorname{span}((A^t)_{\cdot,1}, \cdots, (A^t)_{\cdot,m}) \iff \mathbf{F}^{1,n} = \operatorname{span}(A_{1,\cdot}, \cdots, A_{m,\cdot})$.

• **TIPS 2:** *Suppose p is a poly of n variables in $\mathbf{F}$.*
Prove that $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big)$.
Where the linear maps $T_1,\dots,T_n$ are such that $p(T_1,\dots,T_n)$ makes sense. See [5.16,17,20].

SOLUTION: Suppose the poly p is defined by $p(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^n x_i^{k_i}$.
Note that $\mathcal{M}(T^xS^y) = \mathcal{M}(T)^x\mathcal{M}(S)^y$; $\mathcal{M}(T^x+S^y) = \mathcal{M}(T)^x+\mathcal{M}(S)^y$.
Then $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = \mathcal{M}\big(\sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^n T_i^{k_i}\big)$
$= \sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^n \mathcal{M}(T_i^{k_i}) = p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big)$. □

• **COROLLARY:** Suppose τ is an algebraic property. Then τ holds for linear maps $\Longleftrightarrow$ τ holds for matrices.
Each $\alpha_k \in \{1,\dots,n\}$. Now $p(T_1,\dots,T_n) = p(T_{\alpha_1},\dots,T_{\alpha_n})$
$\Longleftrightarrow p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big) = p\big(\mathcal{M}(T_{\alpha_1}),\dots,\mathcal{M}(T_{\alpha_n})\big)$.

---

13 *Prove that the distr holds for matrix add and matrix multi.*
Suppose A,B,C are matrices such that $A(B+C)$ make sense, we prove the left distr.

SOLUTION: Suppose $A \in \mathbf{F}^{m,n}$ and $B,C \in \mathbf{F}^{n,p}$.
Note that $\big[A(B+C)\big]_{j,k} = \sum_{r=1}^n A_{j,r}(B+C)_{r,k} = \sum_{r=1}^n \big(A_{j,r}B_{r,k}+A_{j,r}C_{r,k}\big) = \big(AB+AC\big)_{j,k}$.
OR. Define T,S,R such that $\mathcal{M}(T)=A, \mathcal{M}(S)=B, \mathcal{M}(R)=C$.
$A(B+C) = \mathcal{M}(T(S+R)) \overset{[3.9]}{=\!=} \mathcal{M}(TS+TR) = AB+AC$.
OR. $T(S+R) = TS+TR \Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR) \Rightarrow A(B+C) = AB+AC$. □

---

1 *Suppose $T \in \mathcal{L}(V,W)$. Show that for each pair of B_V and B_W,*
$A = \mathcal{M}(T,B_V,B_W)$ has at least $n = \operatorname{dim}\operatorname{range} T$ nonzero entries.

SOLUTION:
Using [3.B TIPS (4)]. Let $U \oplus \operatorname{null} T = V$; $B_U = (v_1,\dots,v_n), B_V = (v_1,\dots,v_m)$.
For each $k \in \{1,\dots,n\}, Tv_k \neq 0 \Longleftrightarrow A_{\cdot,k} \neq 0$. Hence every such $A_{\cdot,k}$ has at least one nonzero entry. □

OR. We prove by contradiction. Suppose A has at most $(n-1)$ nonzero entries.
Then by Pigeon Hole Principle, at least one of $A_{\cdot,1},\dots,A_{\cdot,n}$ equals 0.
Thus there are at most $(n-1)$ nonzero vecs in $Tv_1,\dots,Tv_n$.
又 $\operatorname{range} T = \operatorname{span}(Tv_1,\dots,Tv_n) \Rightarrow \operatorname{dim}\operatorname{range} T = \operatorname{dim}\operatorname{span}(Tv_1,\dots,Tv_n) \leqslant n-1$. Contradicts. □

---

6 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$.*
Prove that $\operatorname{dim}\operatorname{range} T = 1 \Longleftrightarrow \exists B_V, B_W$, all entries of $A = \mathcal{M}\big(T,B_V,B_W\big)$ equal 1.

SOLUTION:
(a) Suppose $B_V = (v_1,\dots,v_n), B_W = (w_1,\dots,w_m)$ are the bases such that all entries of A equal 1.
Then $Tv_i = w_1+\dots+w_m$ for all $i = 1,\dots,n$. Because $w_1,\dots,w_n$ is linely inde, $w_1+\dots+w_n \neq 0$.

(b) Suppose $\operatorname{dim}\operatorname{range} T = 1$. Then $\operatorname{dim}\operatorname{null} T = \operatorname{dim} V - 1$.
Let $B_{\operatorname{null} T} = (u_2,\dots,u_n)$. Extend to a basis $(u_1,u_2,\dots,u_n)$ of V.
Let $w_1 = Tv_1 - w_2 - \dots - w_m$. Extend to B_W. Let $v_1 = u_1$, $v_i = u_1+u_i$. Extend to B_V. □

OR. Suppose $B_{\operatorname{range} T} = (w)$. By [2.C NOTE FOR (15)], $\exists B_W = (w_1,\dots,w_m)$, $w = w_1+\dots+w_m$.
By [2.C TIPS], $\exists$ a basis $(u_1,\dots,u_n)$ of V such that each $u_k \notin \operatorname{null} T$.
Now each $Tu_k \in \operatorname{range} T = \operatorname{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists \lambda_k \in \mathbf{F}\backslash\{0\}$.
Let $v_k = \lambda_k^{-1}u_k \neq 0$, so that each $Tv_k = w = w_1+\dots+w_m$. Thus $B_V = (v_1,\dots,v_n)$ will do. □

---

3 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$. Prove that $\exists\, B_V, B_W$ such that* $[$ *letting* $A = \mathcal{M}(T, B_V, B_W)$ $]$ $A_{k,k} = 1, A_{i,j} = 0$, *where* $1 \leqslant k \leqslant \dim \operatorname{range} T, i \neq j$.

SOLUTION: Using $\left[3.\text{B TIPS } (4)\right]$. Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n), B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$. $\square$

COMMENT: Let each $Tv_k = w_k$. Extend $B_{\operatorname{range} T}$ to $B_W = (w_1, \dots, w_n, \dots, w_p)$. See $\left[3.\text{D NOTE FOR } [3.60]\right]$.

---

4 *Suppose $B_V = (v_1, \dots, v_m)$ and W is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that $\exists\, B_W = (w_1, \dots, w_n)$, $\mathcal{M}(T, B_V, B_W)_{\cdot,1} = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}^t$ or $\begin{pmatrix}0 & \cdots & 0\end{pmatrix}^t$.

SOLUTION: If $Tv_1 = 0$, then we are done. If not then extend (Tv_1) to B_W. $\square$

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5 *Suppose $B_W = (w_1, \dots, w_n)$ and V is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that $\exists\, B_V = (v_1, \dots, v_m)$, $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix}0 & \cdots & 0\end{pmatrix}$ or $\begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}$.

SOLUTION:

Let $(u_1, \dots, u_n)$ be a basis of V. Denote $\mathcal{M}(T, (u_1, \dots, u_n), B_W)$ by A.

If $A_{1,\cdot} = 0$, then $B_V = (u_1, \dots, u_n)$ and we are done. Otherwise, suppose $A_{1,k} \neq 0$.

Let $v_1 = \dfrac{u_k}{A_{1,k}} \Rightarrow Tv_1 = 1w_1 + \dfrac{A_{2,k}}{A_{1,k}} w_2 + \cdots + \dfrac{A_{n,k}}{A_{1,k}} w_n.$ | Let $v_j = u_{j-1} - A_{1,j-1} v_1$ for each $j \in \{2, \dots, k\}$. Let $v_i = u_i - A_{1,i} v_1$ for $i \in \{k+1, \dots, n\}$.

NOTICE that $Tu_i = A_{1,i} w_1 + \cdots + A_{n,i} w_n$. 又 Each $u_i \in \operatorname{span}(v_1, \dots, v_n) = V$. Let $B_V = (v_1, \dots, v_n)$. $\square$

OR. Using Problem (4). Let $B_{W'}$ be the B_V.

Now $\exists\, B_{V'}$ such that $\mathcal{M}(T', B_{W'}, B_{V'})_{\cdot,1} = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}^t$ or $\begin{pmatrix}0 & \cdots & 0\end{pmatrix}^t$.

Which is equiv to $\exists\, B_V$ $\left[\text{Using } (3.\text{F}.31)\right]$ such that $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}$ or $\begin{pmatrix}0 & \cdots & 0\end{pmatrix}$. $\square$

ENDED

3·D

1 2 3 4 5 6 8 9 10 11 12 13 15 16 17 18 19 | 4E: 3 10 15 17 19 20 22 23 24

2 *Suppose V is finite-dim and $\dim V > 1$.*
Prove that the set U of non-inv operators on V is not a subsp of $\mathcal{L}(V)$.
The set of inv operators is not either. Although multi identity/inv, and commutativity for vec multi hold.

SOLUTION: Let $B_V = (v_1, \dots, v_n)$. $[$ *If $\dim V = 1$, then $U = \{0\}$ is a subsp of $\mathcal{L}(V)$.* $]$
Define $S, T \in \mathcal{L}(V)$ by $S(a_1 v_1 + \cdots + a_n v_n) = a_1 v_1,\ T(a_1 v_1 + \cdots + a_n v_n) = a_2 v_1 + \cdots + a_n v_n$.
Hence $S, T \in U$ while $S + T \notin U$. $\square$

• **TIPS:** *Suppose $U \oplus X = W \oplus Y$, and X, Y are iso. Prove that U, W are iso.*

SOLUTION: Let ξ be an iso of X onto Y. That is, $\forall y \in Y, \exists!\, x \in X, \xi(x) = y$.
$\forall u \in U, \exists!\, w \in W, y \in Y, u = w + y \Rightarrow \exists!\, x \in X, u = w + \xi(x)$. Define $\pi : u \mapsto w$.
Now suppose $u_1, u_2 \in U$, then each $u_i = w_i + \xi(x_i), \exists!\, w_i \in W, x_i \in X$.
Linearity: $\forall, \lambda \in \mathbf{F}, \pi(u_1 + \lambda u_2) = w_1 + \lambda w_2 = \pi(u_1) + \lambda \pi(u_2)$.
Injectivity: $\pi(u_1) = \pi(u_2) \Rightarrow w_1 = w_2 \Rightarrow \xi(x_1) = \xi(x_2) \Rightarrow x_1 = x_2 \Rightarrow u_1 = u_2$.
Surjectivity: $\forall w \in W, \pi(w) = w \in \operatorname{range} \pi$. Thus π is an iso of U onto W. $\square$

---

3 *Suppose V and W are iso, U is a subsp of V, and $S \in \mathcal{L}(U,W)$.*
Prove that $\exists$ inv $T \in \mathcal{L}(V,W), Tu = Su, \forall u \in U \iff S$ is inje. [*See also* (3.A.11).]

SOLUTION: (a) $\forall u \in U, u = T^{-1}Su \Rightarrow T^{-1}S = I \in \mathcal{L}(U) \Longrightarrow S$ is inje, by (3.B.20).
OR. $\operatorname{null} S = \operatorname{null} T|_U = \operatorname{null} T \cap U = \{0\}$.

(b) Let $X \oplus U = V$. Because $S : U \to V$ is inje. By (3.B.12), $S : U \to \operatorname{range} S$ is an iso.
Let $Y \oplus \operatorname{range} S = V$. Then by TIPS, X and Y are iso. Let $E : X \to Y$ be an iso.
Define $T \in \mathcal{L}(V,W)$ by $Tu = Su, Tw = Ew$ for all $u \in U, w \in X$.

OR. [*Req V Finite-dim*] Let $B_U = (u_1, \dots, u_m)$. Then S inje $\Rightarrow (Su_1, \dots, Su_m)$ linely inde.
Extend to $B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_W = (Su_1, \dots, Su_m, w_1, \dots, w_n)$.
Define $T \in \mathcal{L}(V,W)$ by $T(u_i) = Su_i;\ \ Tv_j = w_j$, for each u_i and v_j. □

---

8 *Suppose $T \in \mathcal{L}(V,W)$ is **surj**. Prove that $\exists$ subsp U of V, $T|_U : U \to W$ is an iso.*

SOLUTION: By (3.B.12). Note that $\operatorname{range} T = W$. OR. [*Req* range T *Finite-dim*] By [3.B TIPS (4)]. □

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18 *Show that V and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.*

SOLUTION:

Define $\Psi \in \mathcal{L}\big(V, \mathcal{L}(\mathbf{F}, V)\big)$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.
(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is inje.
(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi\big(T(1)\big)$. Hence Ψ is surj. □

OR. Define $\Phi \in \mathcal{L}\big(\mathcal{L}(\mathbf{F}, V), V\big)$ by $\Phi(T) = T(1)$.
(a) Suppose $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is inje.
(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surj. □
COMMENT: $\Phi = \Psi^{-1}$.

• *Suppose $S, T \in \mathcal{L}(V,W)$.* [*For Problem (4) and (5), see the* COROLLARY *in (3.B.24, 25).*]

6 *Suppose V and W are finite-dim.* $\dim \operatorname{null} S = \dim \operatorname{null} T = n$.
Prove that $S = E_2TE_1, \exists$ inv $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$.

SOLUTION: Define $E_1 :\ \ v_i \mapsto r_i\ ;\ \ u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.
Define $E_2 : Tv_i \mapsto Sr_i\ ;\ \ x_j \mapsto y_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m)$; $B_{\operatorname{range} S} = (Sr_1, \dots, Sr_m)$. Let $B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p)$; $B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p)$. Let $B_{\operatorname{null} T} = (u_1, \dots, u_n)$; $B_{\operatorname{null} S} = (s_1, \dots, s_n)$. Thus $B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$; $B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)$.	$\therefore E_1, E_2$ are inv and $S = E_2TE_1$.

□

• (a) *Suppose $T = ES$ and $E \in \mathcal{L}(W)$ is inv. Prove that* $\operatorname{null} S = \operatorname{null} T$.
(b) *Suppose $T = SE$ and $E \in \mathcal{L}(V)$ is inv. Prove that* $\operatorname{range} S = \operatorname{range} T$.
(c) *Suppose $T = E_2SE_1$ and $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$ are inv.*
Prove that $\dim \operatorname{null} S = \dim \operatorname{null} T$.

SOLUTION: (a) $v \in \operatorname{null} T \iff Tv = 0 = E(Sv) \iff Sv = 0 \iff v \in \operatorname{null} S$.
(b) $w \in \operatorname{range} T \iff \exists\, v \in V, Tv = S(Ev) \iff \exists\, u \in V, w = Su \iff w \in \operatorname{range} S$.
(c) Using (3.B.22). $\dim \operatorname{null} E_2SE_1 \overset{E_2}{\underset{\text{inv}}{=\!=}} \dim \operatorname{null} SE_1 \overset{E_1}{\underset{\text{inv}}{=\!=}} \dim \operatorname{null} S = \dim \operatorname{null} T$. □

• **NOTE FOR [3.69]:** Suppose V, W are finite-dim and iso, $T \in \mathcal{L}(V, W)$. Then T inv $\iff$ inje $\iff$ surj.

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9 [OR 1] *Suppose U, V, W are iso and finite-dim, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Prove that ST is inv $\iff$ S, T are inv.*

COMMENT: If any two of U, V, W are not iso or finite-dim, then S, T are inv $\Longrightarrow$ ST is inv.

SOLUTION: Suppose S, T are inv. Then $(ST)(T^{-1}S^{-1}) = I_W, (T^{-1}S^{-1})(ST) = I_U$. Hence ST is inv.
Suppose ST is inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = I_U, (ST)R = I_W$.

$Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0.$ | T is inje, S is surj.
$\forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S.$ | 又 $\dim U = \dim V = \dim W$.

OR. By (3.B.23), $\dim W = \dim \operatorname{range} ST \leqslant \min\{\operatorname{range} S, \operatorname{range} T\} \Rightarrow S, T$ are surj. □

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13 *Suppose U, V, W, X are iso and finite-dim, $R \in \mathcal{L}(W, X), S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Suppose RST is surj. Prove that S is inje.*

SOLUTION: Using Problem (9). Notice that U, X are finite-dim, so that RST is inv.

$$\text{Let } X = (RST)^{-1} \left| \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ is surj.} \end{array} \right\} \Rightarrow S = R^{-1}(RST)T^{-1}.$$ □

OR. $(RST)^{-1} = \big((RS)T\big)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. □

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10 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that $ST = I \iff TS = I$.*

SOLUTION: (a) Suppose $ST = I$.

By $(3.B\ 20, 21)(a)$, $ST = I \Rightarrow T$ is inje and S is surj. 又 V is finite-dim. S, T are inv.
OR. By Problem (9), V is finite-dim and $ST = I$ is inv $\Rightarrow S, T$ are inv.

Then $\forall v \in V, S\big((TS)v\big) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow TS = I$.
OR. $S^{-1} = T$ 又 $S = S \Rightarrow TS = S^{-1}S = I$.

(b) Reversing the roles of S and T, we conclude that $TS = I \Rightarrow ST = I$. □

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11 *Suppose V is finite-dim, $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is inv and $T^{-1} = US$.*

SOLUTION: Using Problem (9) and (10). *This result can fail without the hypothesis that V is finite-dim.*
$(ST)U = U(ST) = (US)T = I \Rightarrow T^{-1} = US$.
OR. $(ST)U = S(TU) = I \Rightarrow U, S$ are inv $\Rightarrow TU = S^{-1}$. 又 $U^{-1} = U^{-1} \Rightarrow T = S^{-1}U^{-1}$. □

EXAMPLE: $V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots)$; $T(a_1, \dots) = (0, a_1, \dots)$; $U = I \Rightarrow STU = I$ but T is not inv.

• (4E 3) $T \in \mathcal{L}(V)$, V is finite-dim | $(Tv_1, \dots, Tv_n)$ is a basis of V for some basis $(v_1, \dots, v_n)$ of $V \iff T$ is surj; $(Tv_1, \dots, Tv_n)$ is a basis of V for every basis $(v_1, \dots, v_n)$ of $V \iff T$ is inje $\Big\} \iff T$ is inv.

• (4E 15) *Suppose $T \in \mathcal{L}(V)$ and $V = \operatorname{span}(Tv_1, \dots, Tv_m)$. Prove that $V = \operatorname{span}(v_1, \dots, v_m)$.*

SOLUTION: Because $V = \operatorname{span}(Tv_1, \dots, Tv_m) \Rightarrow T$ is surj, and therefore is inv $\Rightarrow T^{-1}$ is inv.
$\forall v \in V, \exists a_i \in \mathbf{F}, v = \sum_{i=1}^m a_i Tv_i \Rightarrow T^{-1}v = \sum_{i=1}^m a_i v_i \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \dots, v_m)$.

OR. Reduce the spanning list $(Tv_1, \dots, Tv_m)$ of V to a basis $(Tv_{\alpha_1}, \dots, Tv_{\alpha_k})$ of V.
Where $k = \dim V$ and each $\alpha_i \in \{1, \dots, k\}$. Then by Problem (4E 3),
$(v_{\alpha_1}, \dots, v_{\alpha_k})$ is also a basis of V, contained in the list $(v_1, \dots, v_m)$. □

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15 *Prove that every linear map from* $\mathbf{F}^{n,1}$ *to* $\mathbf{F}^{m,1}$ *is given by a matrix multi.*
In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, *then* $\exists\, A \in \mathbf{F}^{m,n}, Tx = Ax,\ \forall x \in \mathbf{F}^{n,1}$.

SOLUTION: Let $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$ be the std bases of $\mathbf{F}^{n,1}, \mathbf{F}^{m,1}$.
$\forall k = 1, \dots, n,\ T(E_k) = A_{1,k}R_1 + \cdots + A_{m,k}R_m, \exists\, A_{j,k} \in \mathbf{F}$, forming $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.
OR. Let $A = \mathcal{M}(T, B_1, B_2)$. Note that $\mathcal{M}(x, B_1) = x, \mathcal{M}(Tx, B_2) = Tx$.
Hence $Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax$, by [3.65]. □

- **NOTE FOR [3.62]:** $\mathcal{M}(v) = \mathcal{M}\big(I, (v), B_V\big)$. Where I is the identity operator restricted to $\operatorname{span}(v)$.
- **NOTE FOR [3.65]:** $\mathcal{M}(Tv) = \mathcal{M}\big(I, (Tv), B_W\big) = \mathcal{M}(T, B_V, B_W)\mathcal{M}\big(I, (v), B_V\big) = \mathcal{M}\big(T, (v), B_W\big)$.
  If $v = 0$, then $\operatorname{span}(v) = \operatorname{span}(\,)$, we replace (v) by $B = (\,)$; similar for $Tv = 0$.

- (4E 23, OR 10.A.4) *Suppose that* $(\beta_1, \dots, \beta_n)$ *and* $(\alpha_1, \dots, \alpha_n)$ *are bases of* V.
  Let $T \in \mathcal{L}(V)$ *be such that each* $T\alpha_k = \beta_k$. *Prove that* $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)$.
  For ease of notation, let $\mathcal{M}\big(T, \alpha \to \beta\big) = \mathcal{M}\big(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)\big)$.

SOLUTION:

Denote $\mathcal{M}(T, \alpha \to \alpha)$ by A and $\mathcal{M}(I, \beta \to \alpha)$ by B.
$\forall k \in \{1, \dots, n\}, I\beta_k = \beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. □

OR. Note that $\mathcal{M}(T, \alpha \to \beta) = I$. Hence $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)\underbrace{\mathcal{M}(T, \alpha \to \beta)}_{= \mathcal{M}(I, \beta \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. □

OR. Note that $\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I$.
$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}\Big(\underbrace{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}_{= \mathcal{M}(T, \alpha \to \beta)}\Big) = \mathcal{M}(I, \beta \to \alpha)$. □

COMMENT: Let $A' = \mathcal{M}(T, \beta \to \beta)$.
$\beta_k = I\beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n,\ \forall\, k \in \{1, \dots, n\}$.
又 $T\beta_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.
OR. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$.

- **TIPS:** When using $\mathcal{M}^{-1}$, you must first declare bases and the purpose for using $\mathcal{M}^{-1}$.
  That is, to declare $B_U, B_V, B_W,\ \mathcal{M}: \mathcal{L}(V, W) \mapsto \mathbf{F}^{m,n}$, or $\mathcal{M}: v \mapsto \mathbf{F}^{n,1}$.
  So that $\mathcal{M}^{-1}(AC, B_U, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(C, B_U, B_V)$;
  Or $\mathcal{M}^{-1}(Ax, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(x, B_V)$. Where everything is well-defined.

- (4E 22, OR 10.A.1) *Suppose* $T \in \mathcal{L}(V)$. *Prove that* $\mathcal{M}(T, B_V)$ *is inv* $\Longleftrightarrow$ T *itself is inv.*

SOLUTION: Notice that $\mathcal{M}: T \mapsto \mathcal{M}(T, B_V)$ is an iso. And that $\mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(TS)$.
(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(I) = \mathcal{M}(T)\mathcal{M}(T^{-1}) \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.
(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I,\ \exists!\, S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$
$\Rightarrow \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = I = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$
$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. □

- (4E 24, OR 10.A.2) *Suppose* $A, B \in \mathbf{F}^{n,n}$. *Prove that* $AB = I \Longleftrightarrow BA = I$. [*Using Problem (10, 15).*]

SOLUTION: Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Now $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.
$AB = I \Leftrightarrow A(Bx) = x \Longleftrightarrow T(Sx) = x \Leftrightarrow TS = I \Longleftrightarrow ST = I \Longleftrightarrow \mathcal{M}(S)\mathcal{M}(T) = BA = I$.
OR. Because $\mathcal{M}: \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1}) \to \mathbf{F}^{n,n}$ is an iso. $\mathcal{M}^{-1}(AB) = TS = ST = \mathcal{M}^{-1}(BA) = I$. □

• **NOTE FOR [3.60]:** Suppose $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.
Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{i,x} w_j$. **COROLLARY:** $E_{l,k}E_{i,j} = \delta_{j,l}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. And $\left(\mathcal{E}^{(j,i)}\right)_{l,k} = \begin{cases} 1, & \text{if } (i,j) = (l,k); \\ 0, & \text{otherwise.} \end{cases}$

NOTICE that $\mathcal{M}: \mathcal{L}(V, W) \to \mathbf{F}^{m,n}$ is an iso. And $E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)} & + & \cdots & + & A_{1,n}\mathcal{E}^{(1,n)} \\ + & & \cdots & & + \\ \vdots & & \ddots & & \vdots \\ + & & \cdots & & + \\ A_{m,1}\mathcal{E}^{(m,1)} & + & \cdots & + & A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} A_{1,1}E_{1,1} & + & \cdots & + & A_{1,n}E_{n,1} \\ + & & \cdots & & + \\ \vdots & & \ddots & & \vdots \\ + & & \cdots & & + \\ A_{m,1}E_{1,m} & + & \cdots & + & A_{m,n}E_{n,m} \end{pmatrix} = T.$$

$$\text{By [2.42] and [3.61], } B_{\mathcal{L}(V,W)} = \begin{pmatrix} E_{1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \cdots & , E_{n,m} \end{pmatrix}; \quad B_{\mathbf{F}^{m,n}} = \begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & , \mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots & , \mathcal{E}^{(m,n)} \end{pmatrix}.$$

• **TIPS:** Let $B_{\text{range}\, T} = (Tv_1, \dots, Tv_p)$, $B_V = (v_1, \dots, v_p, \dots, v_n)$. Let each $w_k = Tv_k$; $B_W = (w_1, \dots, w_p, \dots, w_m)$.
Then $T = E_{1,1} + \cdots + E_{p,p}$, $\mathcal{M}(T, B_V, B_W) = \mathcal{E}^{(1,1)} + \cdots + \mathcal{E}^{(p,p)}$.

---

17 *Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: [See also in (3.A).] Using NOTE FOR [3.60].
Let $B_V = (v_1, \dots, v_n)$. If $\mathcal{E} = 0$, then we are done. Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.
Then $\forall E_{i,j} \in \mathcal{E}$, by assumption, $\forall x, y \in \{1, \dots, n\}$, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}$, $E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.
Again, $\forall x, x', y, y' \in \{1, \dots, n\}$, $E_{y,x'}, E_{y',x} \in \mathcal{E}$. Thus $\mathcal{E} = \mathcal{L}(V)$. □

• (4E 10) *Suppose V, W are finite-dim, U is a subsp of V.*
Let $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V, W) : T|_U = 0\}$.
(a) *Show that $\mathcal{E}$ is a subsp of $\mathcal{L}(V, W)$.*
(b) *Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$ and $\dim U$.*
Hint: *Define $\Phi : \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. What is $\operatorname{null}\Phi$? What is $\operatorname{range}\Phi$?*

SOLUTION:
(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.
(b) Define Φ as in the hint. Φ is linear, by [3.A NOTE FOR Restriction].
$\Phi(T) = 0 \Longleftrightarrow \forall u \in U, Tu = 0 \Longleftrightarrow T \in \mathcal{E}$. Thus $\operatorname{null}\Phi = \mathcal{E}$.
Extend $S \in \mathcal{L}(U, W)$ to $T \in \mathcal{L}(V, W) \Rightarrow \Phi(T) = S \in \operatorname{range}\Phi$. Thus $\operatorname{range}\Phi = \mathcal{L}(U, W)$.
Thus $\dim \operatorname{null}\Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \operatorname{range}\Phi = (\dim V - \dim U)\dim W$. □

OR. Let $B_U = (u_1, \dots, u_m)$, $B_V = (u_1, \dots, u_m, v_1, \dots, v_n)$. Let $p = \dim W$. [See NOTE FOR [3.60].]

$$\forall\, T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \underbrace{\operatorname{span}\begin{Bmatrix} E_{1,1}, & \cdots & , E_{m,1}, \\ \vdots & \ddots & \vdots \\ E_{1,p}, & \cdots & , E_{m,p} \end{Bmatrix}}_{\text{Denote it by R}} \cap\ \mathcal{E} = \{0\}.$$

$$又\ W = \operatorname{span}\begin{Bmatrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{Bmatrix} \subseteq \mathcal{E}.$$

Where $\mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}$.
Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U)\dim W$. □

• (4E 17) *Suppose V is finite-dim and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(T) = ST$.*
(a) *Show that* $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Show that* $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

SOLUTION: (a) $\forall T \in \mathcal{L}(V), ST = 0 \iff \operatorname{range} T \subseteq \operatorname{null} S$.
Thus $\operatorname{null} \mathcal{A} = \{T \in \mathcal{L}(V) : \operatorname{range} T \subseteq \operatorname{null} S\} = \mathcal{L}(V, \operatorname{null} S)$.
(b) $\forall R \in \mathcal{L}(V), \operatorname{range} R \subseteq \operatorname{range} S \iff \exists\, T \in \mathcal{L}(V), R = ST$, by (3.B 25).
Thus $\operatorname{range} \mathcal{A} = \{R \in \mathcal{L}(V) : \operatorname{range} R \subseteq \operatorname{range} S\} = \mathcal{L}(V, \operatorname{range} S)$. □

OR. Using NOTE FOR [3.60]. Let $B_{\operatorname{range} S} = (\overbrace{w_1, \dots, w_m}^{Sv_i = w_i}), B_U = (v_1, \dots, v_m)$.
Let $(w_1, \dots, w_n), (v_1, \dots, v_n)$ be bases of V. Now $S = E_{1,1} + \cdots + E_{m,m}$. $\mathcal{M}(S, v \to w) = \begin{pmatrix} 1 & \ddots & 0 & \ddots & 0 \\ 0 & \ddots & 1 & \ddots & 0 \\ 0 & \ddots & 0 & \ddots & 0 \end{pmatrix}$.
Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j} : w_x \mapsto \delta_{i,x} v_i$. Let $E_{j,k} R_{i,j} = Q_{i,k},\ R_{j,k} E_{i,j} = G_{i,k}$.
Where $E_{i,k} : v_x \mapsto \delta_{i,x} w_k,\ Q_{i,k} : w_x \mapsto \delta_{i,x} w_k$, and $G_{i,k} : v_x \mapsto \delta_{i,x} v_k$.
For any $T \in \mathcal{L}(V), \exists!\, A_{i,j} \in \mathbf{F}, T = \sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \Longrightarrow \mathcal{M}(T, w \to v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} & \cdots & A_{n,m} \end{pmatrix}$.
$\Longrightarrow \mathcal{A}(T) = ST = \Big(\sum_{r=1}^m E_{r,r}\Big)\Big(\sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i}\Big) = \sum_{i=1}^m \sum_{j=1}^n A_{i,j} Q_{j,i}$.

$$\begin{aligned} \mathcal{M}(S, v \to w)\mathcal{M}(T, w \to v) = \mathcal{M}(ST, w) = \\ \mathcal{M}(\mathcal{A}, R \to Q)\mathcal{M}(T, R) = \mathcal{M}\big(\mathcal{A}(T), Q\big) = \end{aligned} \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

又 $\mathcal{M}(T, R) = \mathcal{M}(T, w \to v)$.
Let $T = I$, we have
$\mathcal{M}(\mathcal{A}, R \to Q) = \mathcal{M}(S, v \to w)$.

$$\operatorname{range} \mathcal{A} = \operatorname{span}\left\{\begin{matrix} Q_{1,1}, & \cdots & , Q_{n,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,m}, & \cdots & , Q_{n,m} \end{matrix}\right\},\ \operatorname{null} \mathcal{A} = \operatorname{span}\left\{\begin{matrix} R_{1,m+1}, & \cdots & , R_{n,m+1}, \\ \vdots & \ddots & \vdots \\ R_{1,n}, & \cdots & , R_{n,n} \end{matrix}\right\}.$$

(a) $\dim \operatorname{null} \mathcal{A} = n \times (n - m)$;
(b) $\dim \operatorname{range} \mathcal{A} = n \times m$. □

• **NOTE FOR Problem (4E 17):** *Define $\mathcal{B} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{B}(T) = TS$.*
(a) *Show that* $\dim \operatorname{null} \mathcal{B} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Show that* $\dim \operatorname{range} \mathcal{B} = (\dim V)(\dim \operatorname{range} S)$.

SOLUTION: (a) $\forall T \in \mathcal{L}(V), TS = 0 \iff \operatorname{range} S \subseteq \operatorname{null} T$.
Thus $\operatorname{null} \mathcal{B} = \{T \in \mathcal{L}(V) : \operatorname{range} S \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V) : T|_{\operatorname{range} S} = 0\}$.
(b) $\forall R \in \mathcal{L}(V), \operatorname{null} S \subseteq \operatorname{null} R \iff \exists\, T \in \mathcal{L}(V), R = TS$, by (3.B.24).
Thus $\operatorname{range} \mathcal{B} = \{R \in \mathcal{L}(V) : \operatorname{null} S \subseteq \operatorname{null} R\} = \{R \in \mathcal{L}(V) : R|_{\operatorname{null} S} = 0\}$.
Using [3.22] and Problem (4E 10). □

OR. Using NOTE FOR [3.60] and notation in Problem (4E 17).
$\mathcal{B}(T) = TS = \Big(\sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i}\Big)\Big(\sum_{r=1}^m E_{r,r}\Big) = \sum_{i=1}^n \sum_{j=1}^m A_{i,j} G_{j,i} \Longrightarrow \mathcal{M}(TS, v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} & \cdots & 0 \end{pmatrix}$.

$$\operatorname{range} \mathcal{B} = \operatorname{span}\left\{\begin{matrix} G_{1,1}, & \cdots & , G_{m,1}, \\ \vdots & \ddots & \vdots \\ G_{1,n}, & \cdots & , G_{m,n} \end{matrix}\right\},\ \operatorname{null} \mathcal{B} = \operatorname{span}\left\{\begin{matrix} R_{m+1,1}, & \cdots & , R_{n,1}, \\ \vdots & \ddots & \vdots \\ R_{m+1,n}, & \cdots & , R_{n,n} \end{matrix}\right\}.$$

(a) $\dim \operatorname{null} \mathcal{B} = n \times (n - m)$;
(b) $\dim \operatorname{range} \mathcal{B} = n \times m$. □

• (4E 20) *Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.*

SOLUTION: Note that $\deg\big[(x^2 + x)p''(x) + 2xp'(x) + p(3)\big] = \deg p$.
Define $T_n \in \mathcal{L}\big(\mathcal{P}_n(\mathbf{R})\big)$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.
And note that $T_n(p) = 0 \Rightarrow \deg T_n(p) = -\infty = \deg p \Rightarrow p = 0$. Thus T_n is inv.
$\forall q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $n = 0$; if $q \neq 0$, let $n = \deg q$, we have $q \in \mathcal{P}_n(\mathbf{R})$.
Now $\exists\, p \in \mathcal{P}_n(\mathbf{R}), q(x) = T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$. □

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19 *Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is inje. And $\deg Tp \leqslant \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.*
(a) *Prove that T is surj;* (b) *Prove that for every nonzero p, $\deg Tp = \deg p$.*

Solution: (a) T is inje $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ is inje, so is inv $\iff T$ is surj.

(b) Using mathematical induction.

(i) $\deg p = -\infty \geqslant \deg Tp \iff p = 0 = Tp$. And $\deg p = 0 \geqslant \deg Tp \iff p = C \neq 0$.

(ii) Assume $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$. We show $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$ by contradiction. Suppose $\exists\, r \in \mathcal{P}_{n+1}(\mathbf{R}),\ \deg Tr \leqslant n < n+1 = \deg r$. Then by (a), $\exists\, s \in \mathcal{P}_n(\mathbf{R}),\ T(s) = (Tr)$. 又 T is inje $\Rightarrow s = r$. While $\deg s = \deg Ts = \deg Tr < \deg r$. Contradicts. □

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16 *Suppose V is finite-dim and $S \in \mathcal{L}(V)$ such that $\forall T \in \mathcal{L}(V), ST = TS$.*
Prove that $\exists\, \lambda \in \mathbf{F}, S = \lambda I$. [*Using notation in Problem (4E 17). See also in (3.A).*]

Solution: If $S = 0$, we are done. Now suppose $S \neq 0$.

Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, B_U) = \mathcal{M}(I, B_{\operatorname{range} S}, B_U)$. Note that $R_{k,1} : w_x \mapsto \delta_{k,x} v_1$.

Then $\forall k \in \{1, \ldots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $\dim \operatorname{null} S = 0,\ \dim \operatorname{range} S = m = n$.

Notice that $G_{i,j} = R_{i,j}S = SR_{i,j} = Q_{i,j}$. Where $G_{i,j} : v_x \mapsto \delta_{i,x} v_j;\ Q_{i,j} : w_x \mapsto \delta_{i,x} w_j$.

For each $w_i, \exists!\, a_{k,i} \in \mathbf{F}, w_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n$. Where $a_{k,i} = \mathcal{M}(I, (w_1, \ldots, w_n), (v_1, \ldots, v_n))_{k,i}$.

Then fix one i. Now for each $j \in \{1, \ldots, n\}, Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}\big(\sum_{k=1}^{n} a_{k,i}v_k\big)$.

Let $\lambda = a_{i,i}$. Hence each $w_j = \lambda v_j$. Now fix one j, we have $a_{1,1}v_j = \cdots = a_{n,n}v_j$, then all $a_{i,i}$ are equal.

Thus each $w_j = \lambda v_j \Longrightarrow \mathcal{M}(S, B_U) = \mathcal{M}(\lambda I)$. □

• (10.A.3, Or 4E 19) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.* [*See also in (3.A).*]
Prove that $\forall B_V \neq B_V',\ \mathcal{M}(T, B_V) = \mathcal{M}(T, B_V') \Longrightarrow T = \lambda I, \exists\, \lambda \in \mathbf{F}$.

Solution: Suppose $\forall B_V \neq B_V',\ \mathcal{M}(T, B_V) = \mathcal{M}(T, B_V')$. If $T = 0$, then we are done.

Suppose $T \neq 0$, and $v \in V \backslash \{0\}$. Assume that (v, Tv) is linely inde.

Extend (v, Tv) to $B_V = (v, Tv, u_3, \ldots, u_n)$. Let $B = \mathcal{M}(T, B_V)$.

$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0,\ \forall i \neq 2$.

By assumption, $A = \mathcal{M}(T, B_V') = B, \forall B_V' = (v, w_2, \ldots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$.

$\Rightarrow Tv = w_2$, which is not true if $w_2 = u_3,\ w_3 = Tv,\ w_j = u_j, \forall j \in \{4, \ldots, n\}$. Contradicts.

Hence (v, Tv) is linely depe $\Rightarrow \forall v \in V, \exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show that λ_v is independent of v, that is, for all distinct $v, w \in V \backslash \{0\}, \lambda_v = \lambda_w$.

$$\left.\begin{array}{l} (v,w) \text{ linely inde} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v,w) \text{ linely depe, } w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \end{array}\right\} \Rightarrow T = \lambda I.$$ □

Or. Let $A = \mathcal{M}(T, B_V)$, where $B_V = (u_1, \ldots, u_m)$ is arbitrary.

Fix one $B_V = (v_1, \ldots, v_m)$ and then $(v_1, \ldots, \frac{1}{2}v_k, \ldots, v_m)$ is also a basis for any given $k \in \{1, \ldots, m\}$.

Fix one k. Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \cdots + A_{k,k}(\frac{1}{2}v_k) + \cdots + A_{m,k}v_m$

$\Rightarrow Tv_k = 2A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + 2A_{m,k}v_m = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{m,k}v_m$.

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1, \ldots, m\}$.

Now we show that $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j, k such that $j \neq k$.

Consider $B_V' = (v_1', \ldots, v_j', \ldots, v_k', \ldots, v_m')$, where $v_j' = v_k,\ v_k' = v_j$ and $v_i' = v_i$ for all $i \in \{1, \ldots, m\} \backslash \{j, k\}$.

Now $T(v_k') = A_{1,k}v_1' + \cdots + A_{k,k}v_k' + \cdots + A_{m,k}v_m' = A_{k,k}v_k' = A_{k,k}v_j$, while $T(v_k') = T(v_j) = A_{j,j}v_j$. □

Ended

1 *A function $T : V \to W$ is linear $\Longleftrightarrow$ The graph of T is a subspace of $V \times W$.*

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2 *Suppose $V_1 \times \cdots \times V_m$ is finite-dim. Prove that each V_j is finite-dim.*

Solution:

For any $k \in \{1, \dots, m\}$, define $S_k \in \mathcal{L}(V_1 \times \cdots \times V_m, V_k)$ by $S_k(v_1, \dots, v_m) = v_k$.
Then S_k is linear map. By [3.22], range $S_k = V_k$ is finite-dim. $\square$

Or. Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.
We show that each U_i is iso to V_i. Then U is finite-dim $\Longrightarrow$ its subsp U_i is finite-dim, so is V_i.

Let $B_U = (v_1, \dots, v_M)$ $\left| \begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U_i) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} R_i S_j|_{U_j} = \delta_{i,j} I_{U_j}, \\ S_i R_j = \delta_{i,j} I_{V_j}. \end{array} \right.$ $\square$

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4 *Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.*

Solution: Using notation in Problem (2): $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0)$; $S_i : (u_1, \dots, u_m) \mapsto u_i$.

Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$.

$\left. \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (TR_1, \dots, TR_m). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array} \right\} \Rightarrow \psi = \varphi^{-1}.$ $\square$

---

5 *Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.*

Solution: Using notation in Problem (2): $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0)$; $S_i : (u_1, \dots, u_m) \mapsto u_i$.

$\begin{array}{l} \text{Note that } T_i : v \mapsto w_i, \\ T : v \mapsto (w_1, \dots, w_m). \end{array} \left| \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1 T, \dots, S_m T). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = R_1 T_1 + \cdots + R_m T_m. \end{array} \right\} \Rightarrow \psi = \varphi^{-1}.$ $\square$

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6 *For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.*

Solution:

Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m$.

(a) Suppose $T(v_1, \dots, v_m) = 0$. Then $\forall (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$
For each k, let $a_k = 1, a_j = 0$ for all $j \neq k$. Then each $v_k = 0 \Rightarrow (v_1, \dots, v_m) = 0$. Thus T is inje.

(b) Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be the std basis of $\mathbf{F}^m$. Then $\forall (b_1, \dots, b_n) \in \mathbf{F}^m$,
$\Big[T\big(\psi(e_1), \dots, \psi(e_m)\big)\Big](b_1, \dots, b_m) = b_1 \psi(e_1) + \cdots + b_m \psi(e_m) = \psi(b_1 e_1 + \cdots + b_m e_m) = \psi(b_1, \dots, b_m)$.
Thus $T\big(\psi(e_1), \dots, \psi(e_m)\big) = \psi$. Hence T is surj. $\square$

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3 *Give an example of a vecsp V and its two subsps U_1, U_2 such that $U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum.* [*V must be infinite-dim.*]

Solution: Note *that at least one of U_1, U_2 must be infinite-dim. And at least one must be finite-dim??*

Let $V = \mathbf{F}^\infty = U_1$, $U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$. Then $V = U_1 + U_2$ is not a direct sum.

$\left. \begin{array}{l} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T\big((x_1, x_2, \cdots), (x, 0, \cdots)\big) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = \big((x_1, x_2, \cdots), (x, 0, \cdots)\big) \end{array} \right\} \Rightarrow S = T^{-1}.$ $\square$

• **NOTE FOR [3.79, 3.83]:** If $U = \{0\}$, then $v + U = v + \{0\} = \{v\}$, $V/U = V/\{0\} = \{\{v\} : v \in V\}$.
If $U = V$, then $v + V = 0 + V$, $V/V = \{v + V : v \in V\} = \{0\}$.
If $U = \emptyset$, then $v + U = v + \emptyset = \emptyset$, $V/U = V/\emptyset = \{\emptyset\}$.

• **COMMENT:** If U is merely a subset of V, then [3.85, 3.86] do not hold, and V/U is not a vecsp.
Because $\big((v - w) + u\big) \in U$ or $u - u' \in U$ needs that U is closed under add.
And because $(v - \hat{v}) + (w - \hat{w}) \in U$ and $\lambda(v - \hat{v}) \in U$ assume that U is a subsp.

If U is a vecsp but not a subsp of V, then everything will be all right.
If U is a vecsp and $U \cap V = \{0\}$, then $v + U = w + U \Rightarrow v = w$.

• **NOTE FOR [3.85]:** $v + U = w + U \iff v \in w + U,\ w \in v + U$
$\iff v - w \in U \iff (v + U) \cap (w + U) \neq \emptyset$.

• (4E 8) *Suppose $T \in \mathcal{L}(V, W), w \in \operatorname{range} T$. Prove that $\{v \in V : Tv = w\} = u + \operatorname{null} T$.*

SOLUTION: Let $\mathcal{K}_u = \{v \in V : Tv = w\}$. [Not a vecsp.] Suppose $u \in \mathcal{K}_u$. Then $u + \operatorname{null} T \subseteq \mathcal{K}_u$.
And $\forall u' \in \mathcal{K}_u$, $u' - u \in \operatorname{null} T \Rightarrow u' \in u + \operatorname{null} T$. Now $\mathcal{K}_u \subseteq u + \operatorname{null} T$. $\square$

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7 *Suppose $v, x \in V$, and U, W are subsps of V. Prove that $v + U = x + W \Rightarrow U = W$.*

SOLUTION: (a) $v \in v + U = x + W \Rightarrow \exists\, w_v \in W, v = x + w_v \Rightarrow v - x \in W$.
(b) $x \in x + W = v + U \Rightarrow \exists\, u_x \in U, x = v + u_x \Rightarrow x - v \in U$.
Now $x + U = v + U = x + W = v + W$. Thus $\{v + u : u \in U\} = \{v + w : w \in W\} \Rightarrow U = W$.

OR. $\pm(v - x) \in U \cap W \Rightarrow \left\{ \begin{array}{l} u_1 = (x - v) + w_1 \in W \Rightarrow U \subseteq W \\ w_2 = (v - x) + u_2 \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W.$ $\square$

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8 *Suppose A is a nonempty subset of V.*
Prove that A is a translate of some subsp of $V \iff \lambda v + (1 - \lambda) w \in A$, $\forall v, w \in A, \lambda \in \mathbf{F}$.

SOLUTION:

(a) Suppose $A = a + U$. Then $\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + \big(\lambda(u_1 - u_2) + u_2\big) \in A$.

(b) Suppose $\lambda v + (1 - \lambda) w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $\underline{a \in A}$ and let $A' = \{x - a : x \in A\}$.
Then $0 \in A'$ and $\forall (v - a), (w - a) \in A', \lambda \in \mathbf{F}$,

(I) $\lambda(v - a) = [\lambda v + (1 - \lambda) a] - a \in A'$.

(II) Because $\lambda(v - a) + (1 - \lambda)(w - a) = [\lambda v + (1 - \lambda) w] - a \in A'$.
Let $\lambda = \frac{1}{2}$ here and use (I) above by $\lambda = 2$, we have $(v - a) + (w - a) \in A'$.

OR. Note that $v, a \in A \Rightarrow \lambda v + (1 - \lambda) a = 2v - a \in A$. Similarly $2w - a \in A$.
Now $(v - \frac{1}{2}a) + (w - \frac{1}{2}a) = v + w - a \in A \Rightarrow v + w - 2a = (v - a) + (w - a) \in A'$.

Thus $A' = -a + A$ is a subsp of V. Hence $a + A' = a + \{x - a : x \in A\} = A$ is a translate. $\square$

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9 *Suppose $A = v + U$ and $B = x + W$ for some $v, x \in V$ and some subsps U, W of V. Prove that $A \cap B$ is either a translate of some subsp of V or is $\emptyset$.*

Solution: $\forall v+u, x+w \in A \cap B \neq \emptyset, \lambda \in \mathbf{F}, \lambda(v+u) + (1-\lambda)(x+w) \in A \cap B$. By Problem (8). $\square$

OR. Let $A = v + U,\ B = x + W$. Suppose $\alpha \in (v + U) \cap (x + W) \neq \emptyset$.

Then $\alpha - v \in U \Rightarrow \alpha + U = v + U = A$, and $\alpha - x \in W \Rightarrow \alpha + W = x + W = B$.

We show that $A \cap B = \alpha + (U \cap W)$. Note that $\alpha + (U \cap W) \subseteq A \cap B$.

And $\forall \beta = \alpha + u = \alpha + w \in A \cap B \Rightarrow u = w \in U \cap W \Rightarrow \beta \in \alpha + (U \cap W)$. $\square$

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10 *Prove that the intersection of any collection of translates of subsps is either a translate of some subsps or $\emptyset$.*

Solution: Suppose $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collection of translates of subsps of V, where Γ is an index set.

$\forall x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset, \lambda \in \mathbf{F}, \lambda x + (1-\lambda)y \in A_\alpha$ for each α. By Problem (8). $\square$

OR. Let each $A_\alpha = w_\alpha + V_\alpha$. Suppose $x \in \bigcap_{\alpha \in \Gamma}(w_\alpha + V_\alpha) \neq \emptyset$.

Then $x - w_\alpha \in V_\alpha \Longrightarrow x + V_\alpha = w_\alpha + V_\alpha = A_\alpha$, for each α.

We show that $\bigcap_{\alpha \in \Gamma} A_\alpha = \bigcap_{\alpha \in \Gamma}(x + V_\alpha) = x + \bigcap_{\alpha \in \Gamma} V_\alpha$.

$y \in \bigcap_{\alpha \in \Gamma} A_\alpha \Longleftrightarrow$ for each α, $y = x + v_\alpha \in A_\alpha$

$\Longleftrightarrow$ each $v_\alpha = y - x \in \bigcap_{\alpha \in \Gamma} V_\alpha \Longleftrightarrow y \in x + \bigcap_{\alpha \in \Gamma} V_\alpha$. $\square$

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11 *Suppose $A = \left\{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\right\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.*

(a) *Prove that A is a translate of some subsp of V*

(b) *Prove that if B is a translate of some subsp of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.*

(c) *Prove that A is a translate of some subsp of V of dim less than m.*

Solution: (a) By Problem (8), $\forall u, w \in A, \lambda \in \mathbf{F}, \lambda u + (1-\lambda)w = \left(\lambda \sum_{i=1}^m a_i + (1-\lambda)\sum_{i=1}^m b_i\right)v_i \in A$.

(b) Suppose $B = v + U$, where $v \in V$ and U is a subsp of V. $\underline{\text{Let each } v_k = v + u_k \in B, \exists!\, u_k \in U.}$

$\forall w \in A,\ w = \sum_{i=1}^m \lambda_i v_i = \sum_{i=1}^m \lambda_i (v + u_i) = \underline{\sum_{i=1}^m \lambda_i v} + \sum_{i=1}^m \lambda_i u_i = v + \sum_{i=1}^m \lambda_i u_i \in v + U = B$. $\square$

OR. Let $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on m by k.

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By Problem (8), $v \in B$.

(ii) $2 \leqslant k < m$. Assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. $\left[\forall \lambda_i \text{ such that } \sum_{i=1}^k \lambda_i = 1\right]$

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. Fix one $\mu_\iota \neq 1$.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Longrightarrow \left(\sum_{i=1}^{k+1} \dfrac{\mu_i}{1-\mu_\iota}\right) - \dfrac{\mu_\iota}{1-\mu_\iota} = 1.$

$$\text{Let } w = \underbrace{\frac{\mu_1}{1-\mu_\iota}v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota}v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota}v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota}v_{k+1}}_{k \text{ terms}}.$$

Let $\lambda_i = \dfrac{\mu_i}{1-\mu_\iota}$ for $i \in \{1, \dots, \iota - 1\}$; $\lambda_j = \dfrac{\mu_{j+1}}{1-\mu_\iota}$ for $j \in \{\iota, \dots, k\}$. Then,

$$\left.\begin{array}{r} \sum_{i=1}^k \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda)v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B.$$ $\square$

(c) If $m = 1$, then let $A = v_1 + \{0\}$ and we are done. Now suppose $m \geqslant 2$. Fix one $k \in \{1, \dots, m\}$.

$A \ni \lambda_1 v_1 + \cdots + \lambda_{k-1}v_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m)v_k + \lambda_{k+1}v_{k+1} + \cdots + \lambda_m v_m$

$= v_k + \lambda_1(v_1 - v_k) + \cdots + \lambda_{k-1}(v_{k-1} - v_k) + \lambda_{k+1}(v_{k+1} - v_k) + \cdots + \lambda_m(v_m - v_k)$

$\in v_k + \operatorname{span}(v_1 - v_k, \dots, v_m - v_k)$. $\square$

• **NOTE FOR [3.88, 3.90, 3.91]:** Suppose $W \in \mathcal{S}_V U$. Then V/U is iso to W.
Because $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$. Define $T \in \mathcal{L}(V)$ by $T(v) = w_v$.
Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$, $\operatorname{range} T \oplus \operatorname{null} T = V$.
Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, V)$ is defined by $\tilde{T}(v+U) = \tilde{T}(w'_v + U) = Tw'_v = w_v$. [See TIPS (1) below]
Now $\pi \circ \tilde{T} = I_{V/U}$, $\tilde{T} \circ \pi|_W = I_W = T|_W$. Hence $\tilde{T}$ is an iso of V/U onto W.

• **TIPS 1:** Suppose U is a subsp of V. Define $S \in \mathcal{L}(V/U, V)$ by $S(v+U) = v$.
Then $\operatorname{range} S$ is the *purest* in $\mathcal{S}_V U$. Now $\operatorname{null} S = \{0\}$, $U \oplus \operatorname{range} S = V$.
Let $E = S \circ \pi$. Because S is inje and π is surj, $\operatorname{null} E = \operatorname{null} \pi = U$, $\operatorname{range} E = \operatorname{range} S$.
Then $\operatorname{range} E \oplus \operatorname{null} E = V$. NOTICE that $E : V \to W$ is the *purest eraser*. Now we explain why:

EXAMPLE: Let $V = \mathbf{F}^2, B_U = (e_1), B_W = (e_2 - e_1) \Rightarrow U \oplus W = V$.
Notice that $T(e_2 - e_1) = (e_2 - e_1)$, while $(e_2 - e_1) + U = e_2 + U$, but
because $e_2 = e_1 + (e_2 - e_1)$, now still, $\tilde{T}\big((e_2 - e_1) + U\big) = e_2 - e_1 = Te_2$.
In contrast, $S\big((e_2 - e_1) + U\big) = S(e_2 + U) = e_2$, $E(e_2 - e_1) = e_2$.
And $\operatorname{range} E = \operatorname{range} S = \operatorname{span}(e_2)$ is the *purest* in $\mathcal{S}_V U$.

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12 *Suppose U is a subsp of V. Prove that is V is iso to $U \times (V/U)$.*

SOLUTION:
[*Req V/U Finite-dim*] Let $B_{V/U} = (v_1 + U, \dots, v_n + U)$.
Note that $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^n a_i(v_i + U) = \left(\sum_{i=1}^n a_i v_i\right) + U$
$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) \in U \Rightarrow \exists!\, u \in U, v = \sum_{i=1}^n a_i v_i + u$.
Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ and $\psi \in \mathcal{L}(U \times (V/U), V)$
by $\varphi(v) = (u,\ v + U)$ and $\psi(u,\ v+U) = v + u$. Then $\psi = \varphi^{-1}$. □

OR. Define $S \in \mathcal{L}(V/U, V)$ by $S(v+U) = v$.
By NOTE FOR [3.88, 90, 91], $\operatorname{range} S \oplus U = V$. Thus $\forall v \in V, \exists!\, u \in U, w \in \operatorname{range} S, v = u + w$.
Define $T \in \mathcal{L}(U \times (V/U), V)$ by $T(u, v+U) = u + S(v+U) = u + w = v$. Then T is surj.
And $T(u, v+U) = u + S(v+U) = 0 \Longrightarrow \pi\big(T(u, v+U)\big) = v + U = 0$, and $u = -S(v+U) = 0$.
OR. Define $R \in \mathcal{L}(V, U \times (V/U))$ by $R(v) = \big(u, (w+U)\big)$. Now $R \circ T = I_{U\times(V/U)}$, $T \circ R = I_V$. □

• (4E 14) *Suppose $V = U \oplus W$, $B_W = (w_1, \dots, w_m)$. Prove that $B_{V/U} = (w_1 + U, \dots, w_m + U)$.*

SOLUTION: $\forall v \in V, \exists!\, u \in U, w \in W, v = u + w$. 又 $\exists!\, c_i \in \mathbf{F}, w = \sum_{i=1}^m c_i w_i \Rightarrow v = \sum_{i=1}^m c_i w_i + u$.
Hence $\forall v + U \in V/U, \exists!\, c_i \in \mathbf{F}, v + U = \sum_{i=1}^m c_i w_i + U$. □

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13 *Prove that $B_{V/U} = (v_1 + U, \dots, v_m + U), B_U = (u_1, \dots, u_n) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$.*

SOLUTION: $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^m a_i v_i + U \Rightarrow \exists!\, b_i \in \mathbf{F}, v - \sum_{i=1}^m a_i v_i = \sum_{i=1}^n b_i u_i \in U$
$\Rightarrow \forall v \in V, \exists!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^m a_i v_i + \sum_{j=1}^n b_j u_j$. □

OR. $\sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i = 0 \Rightarrow \left(\sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i\right) + U = 0 \Rightarrow \sum_{i=1}^m a_i(v_i + U) = 0$
$\Rightarrow a_1 = \cdots = a_m = 0 \Rightarrow \sum_{i=1}^n b_i u_i \Rightarrow b_1 = \cdots = b_n = 0$. 又 $\dim V = m + n$. □

OR. Note that $B = (v_1, \dots, v_m)$ is linely inde, and $[\operatorname{span}(v_1, \dots, v_m) + U] \subseteq V$.
$v \in \operatorname{span} B \cap U \Longleftrightarrow v + U = \sum_{i=1}^m a_i(v_i + U) = 0 + U \Longleftrightarrow v = 0$. Hence $\operatorname{span} B \cap U = \{0\}$.
Because $\dim[\operatorname{span}(v_1, \dots, v_m) \oplus U] = m + n = \dim V$. Now by (2.B.8). □

• **NOTE FOR Problem (13) and (4E 14):** Let $U \oplus W = V$. Define $S(w+U) = w$. [See also TIPS (1).]
(a) Let $B_W = (w_1, \dots, w_m) \Rightarrow B_{V/U} = (w_1 + U, \dots, w_m + U)$. Then $S(w_k + U)$ might not equal w_k.
(b) Let $B_{V/U} = (w_1 + U, \dots, w_m + U)$, then let $B_W = (w_1, \dots, w_m)$. Now each $S(w_k + U) = w_k$.

• **NEW NOTATION:** Pure $V/U = W \Longleftrightarrow V = U \oplus W,\ W = \operatorname{range} S$.

• **NEW THEOREM:** The uniqnes of Pure V/U follows from range S.

• **TIPS 2:** *Suppose U, W are subsps of V. Let $I = U \cap W$.*
Prove that $V = U + W \Longleftrightarrow V/I = U/I \oplus W/I$.

SOLUTION: (a) Suppose $U + W$. Then $\forall x \in V/I, \exists\, v \in V, (u_v, w_v) \in U \times W, x = v + I = (u_v + w_v) + I$.
Note that $U/I, W/I \subseteq V/I$. Thus $V/I = U/I + W/I$.
$\forall x \in (U/I) \cap (W/I), \exists\, u + I \in U/I, w + I \in W/I, x = u + I = w + I \Rightarrow u - w \in I = U \cap W$
$\Rightarrow \exists\, w' \in I, u = w + w' \in U \cap W \Rightarrow x = u + I = 0 + I$. Thus $(U/I) \cap (W/I) = \{0\}$.
(b) Suppose $V/I = U/I \oplus W/I$. Then $\forall v \in V, v + I = (u + I) + (w + I)$
$\Rightarrow v - u - w \in I = U \cap W \Rightarrow \exists\, x \in U \cap W, v = u + w + x \in U + W$. □

• **TIPS 3:** *Suppose I is a subsp of U. Suppose U is a subsp of V.*
Let $V = S_V I \oplus I = S_V U \oplus U$. Let $U = S_U I \oplus I$. Then $V = S_V U \oplus S_U I \oplus I$.
Suppose $S_V I =$ Pure V/I, similar for $S_V U, S_U I$. Prove that $S_V I = S_V U \oplus S_U I$.

SOLUTION: $\forall v_i \in S_V I, v_i = v_u + u, \exists!\, v_u \in S_V U, u \in U \Rightarrow \exists!\, u_i \in S_U I, i \in I, v_i = v_u + u_i + i$.
又 $v_i \in$ Pure V/I. Hence $i = 0$, and $v_i \in S_V U \oplus S_U I$. Now because $S_V U, U \subseteq S_V I$. □

---

15 *Suppose $\varphi \in \mathcal{L}(V, \mathbf{F}) \backslash \{0\}$. Prove that* $\dim V/(\operatorname{null} \varphi) = 1$.

SOLUTION: By [3.91] (d), $\dim \operatorname{range} \varphi = 1 = \dim V/(\operatorname{null} \varphi)$.
OR. By (3.B.29), $\exists\, u,\ \operatorname{span}(u) \oplus \operatorname{null} \varphi = V$. Then $B_{V/\operatorname{null} \varphi} = (u + \operatorname{null} \varphi)$. □

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16 *Suppose* $\dim V/U = 1$. *Prove that $\exists\, \varphi \in \mathcal{L}(V, \mathbf{F}), \operatorname{null} \varphi = U$.*

SOLUTION: Suppose $V_0 \oplus U = V$. Then V_0 is iso to V/U. $\dim V_0 = 1$.
Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. □

OR. Let $B_{V/U} = (w + U)$. Then $\forall v \in V, \exists!\, a \in \mathbf{F}, v + U = aw + U$.
Define $\varphi : V \to \mathbf{F}$ by $\varphi(v) = a$. Then $\varphi(v_1 + \lambda v_2) = a_1 + \lambda a_2 = \varphi(v_1) + \lambda \varphi(v_2)$.
Now $u \in U \Longleftrightarrow u + U = 0w + U \Longleftrightarrow \varphi(u) = 0$. □

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17 *Suppose V/U is finite-dim, W is a subsp of V.*
(a) *Show that if $V = U + W$, then* $\dim W \geqslant \dim V/U$.
(b) *Show that $\exists\, W \in \mathcal{S}_V U$,* $\dim W = \dim V/U$.

SOLUTION: Let $B_W = (w_1, \dots, w_n)$.
(a) $\forall v \in V, \exists\, u \in U, w \in W,\ v = u + w \Longrightarrow v + U = w + U = (a_1 w_1 + \cdots + a_n w_n) + U, \exists!\, a_i \in \mathbf{F}$.
Then $V/U \subseteq \operatorname{span}(w_1 + U, \dots, w_n + U)$. Hence $\dim V/U \leqslant \dim \operatorname{span}(w_1 + U, \dots, w_n + U)$.
(b) Reduce $(w_1 + U, \dots, w_n + U)$ to $B_{V/U} = (w_1 + U, \dots, w_m + U)$, and let $W = \operatorname{span}(w_1, \dots, w_m)$. □
OR. Let $B_{V/U} = (v_1 + U, \dots, v_m + U)$ and define $\tilde{T} \in \mathcal{L}(V/U, V)$ by $\tilde{T}(v_k + U) = v_k$.
Note that $\pi \circ \tilde{T} = I$. By (3.B.20), $\tilde{T}$ is inje. And $(v_1, \dots, v_m)$ is linely inde.
Let $W = \operatorname{range} \tilde{T} = \operatorname{span}(v_1, \dots, v_m)$. Then $\tilde{T} \in \mathcal{L}(V/U, W)$ is an iso. Thus $\dim W = \dim V/U$.
And $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = a_1 v_1 + \cdots + a_m v_m + U \Rightarrow \exists!\, w \in W, u \in U, v = w + u$. □

---

18 *Suppose $T \in \mathcal{L}(V, W)$ and U is a subsp of V. Let $\pi : V \to V/U$ be the quotient map. Prove that $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \subseteq \operatorname{null} T$.*

SOLUTION:

(a) Suppose $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi$. Then $U = \operatorname{null} \pi \subseteq \operatorname{null}(S \circ \pi) = \operatorname{null} T$.

(b) Suppose $U = \operatorname{null} \pi \subseteq \operatorname{null} T$. By (3.B.24), we are done. OR. Define $S : (v + U) \mapsto Tv$.
$v_1 + U = v_2 + U \iff v_1 - v_2 \in \operatorname{null} T \iff Tv_1 = Tv_2$. Thus S is well-defined. Hence $S \circ \pi = T$. $\square$

COROLLARY: Define $\Gamma : S \mapsto S \circ \pi$. Then Γ is inje, range $\Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}$.

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14 *Suppose $U = \{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0$ for only finitely many $k\}$.*

(a) *Show that U is a subsp of $\mathbf{F}^\infty$.* [*Do it in your mind*]

(b) *Prove that $\mathbf{F}^\infty/U$ is infinite-dim.*

SOLUTION: For ease of notation, denote the p^{th} term of $u = (x_1, \cdots, x_p, \cdots) \in \mathbf{F}^\infty$ by $u[p]$.

For each $r \in \mathbf{N}^+$, let $e_r[k] = \begin{cases} 1, & (k-1) \equiv 0 \pmod r \\ 0, & \text{otherwise} \end{cases}$ $\Bigg|$ simply $e_r = (1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \cdots)$.

For $m \in \mathbf{N}^+$. Let $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 + U \Rightarrow \exists\, u \in U, a_1 e_1 + \cdots + a_m e_m = u$.

Suppose $u = (x_1, \cdots, x_L, 0, \cdots)$, where L is the largest such that $u[L] \neq 0$.

Let $s \in \mathbf{N}^+$ be such that $h = s \cdot m! + 1 > L$, and $e_1[h] = \cdots = e_m[h] = 1$.

NOTICE that for any $p, r \in \{1, \ldots, m\}$, $e_r[s \cdot m! + 1 + p] = e_r[p+1] = 1 \iff p \equiv 0 \pmod r \iff r \,|\, p$.

Let $1 = p_1 \leqslant \cdots \leqslant p_{\tau(p)} = p$ be the distinct factors of p. Moreover, $r \,|\, p \iff r = p_k$ for some k.

Now $u[h+p] = 0 = \left(\sum_{r=1}^{m} a_r e_r\right)[p+1] = \sum_{k=1}^{\tau(p)} a_{p_k}$.

Let $q = p_{\tau(p)-1}$. Then $\tau(q) = \tau(p) - 1$, and each $q_k = p_k$. Again, $\left(\sum_{r=1}^{m} a_r e_r\right)[h+q] = 0 = \sum_{k=1}^{\tau(p)-1} a_{p_k}$.

Thus $a_{p_{\tau(p)}} = a_p = 0$ for all $p \in \{1, \ldots, m\} \Rightarrow (e_1, \ldots, e_m)$ is linely inde in $\mathbf{F}^\infty$.

So is $(e_1 + U, \ldots, e_m + U)$ in $\mathbf{F}^\infty/U$. Because m is arbitrary. By (2.A.14). $\square$

OR. For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, \text{if } 2^r \,|\, p \\ 0, \text{otherwise} \end{cases}$.

Similarly, let $m \in \mathbf{N}^+$ and $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 \Rightarrow a_1 e_1 + \cdots + a_m e_m = u \in U$.

Suppose L is the largest such that $u[L] \neq 0$. And l is such that $2^{ml} > L$.

Then for each $k \in \{1, \ldots, m\}$, $u[2^{ml} + 2^k] = 0 = \left(\sum_{r=1}^{m} a_r e_r\right)[2^k] = a_1 + \cdots + a_k$.

Thus $a_1 = \cdots = a_m = 0$ and $(e_1, \ldots, e_m)$ is linely inde. Similarly. $\square$

ENDED

3·F

4 5 6 7 8 9 12 13 15 16 17 18 19 20 21 22 23 24 25 26
28 29 30 31 32 33 34 35 36 37 | 4E: 5 6 8 17 23 24 25

4 *Suppose U is a subsp of V and $U \neq V$. Prove that $\exists\, \varphi \in V' \backslash \{0\}, \varphi(u) = 0$ for all $u \in U$.*

SOLUTION: Let $X \oplus U = V \Rightarrow X \neq \{0\}$. Suppose $s \in X \backslash \{0\}$. Let $Y \oplus \operatorname{span}(s) = X$.
Define $\varphi \in V'$ by $\varphi(u + \lambda s + y) = \lambda$. Hence $\varphi \neq 0$ and $\varphi(u) = 0$ for all $u \in U$. □

OR. [*Req V Finite-dim*] By [3.106], $\dim U^0 = \dim V - \dim U > 0$.
OR. Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1 \dots, v_n)$ with $n \geqslant 1$.
Let $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$. Then each $\varphi \in \operatorname{span}(\varphi_1, \dots, \varphi_n)$ will do. □

COMMENT: ***Another proof of* [3.108]**: T is surj $\Longleftrightarrow$ T' is inje.
(a) Suppose T' is inje. Note that $T'(\psi) = 0 \Rightarrow \psi = 0$.
Then $\nexists\, \psi \in W' \backslash \{0\}, (T'(\psi))(v) = \psi(Tv) = 0$ for all $v \in V$.
Thus if we assume that range $T \neq W$ then contradicts. Hence range $T = W$.
(b) Suppose T is surj. Then $(\operatorname{range} T)^0 = \{0\} = \operatorname{null} T'$. □

• *Suppose V is a vecsp and U is a subsp of V.*

17 $U^0 = \{\varphi \in V' : U \subseteq null\varphi\}$. Noticing $\varphi \in V', U \subseteq \operatorname{null} \varphi \Longleftrightarrow \forall u \in U, \varphi(u) = 0$.

18 $U^0 = V' \Longleftrightarrow \forall \varphi \in V', U \subseteq \operatorname{null} \varphi \Longleftrightarrow U = \{0\}$. [*Which means* $\{0\}_V^0 = V'$.]
OR. $U^0 = V' \Longleftrightarrow \dim U^0 = \dim V' = \dim V \Longleftrightarrow \dim U = 0 \Longleftrightarrow U = \{0\}$.

19 $U_V^0 = \{0\} = V_V^0 \Longleftrightarrow U = V$. By the inverse and contrapositive of Problem (4). OR. By [3.106].

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25 *Suppose U is a subsp of V. Explain why $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.*

SOLUTION: Note that $U = \{v \in V : v \in U\}$ is a subsp. Now we show $\forall \varphi \in U^0, \varphi(v) = 0 \Rightarrow v \in U$.
Assume that $v \in V \backslash U$. Then let $\operatorname{span}(v) \oplus U \oplus X = V$. $\exists\, \psi \in V', \operatorname{null} \psi = U \oplus X$.
又 $\psi \in U^0 \Rightarrow \psi(v) = 0$. Contradicts. Hence $v \in U \Longleftrightarrow \forall \varphi \in U^0, \varphi(v) = 0$. □

COMMENT: $W \subseteq X = \{v \in V : \varphi(v) = 0, \forall \varphi \in W^0\}$, the ***promotion*** of the subset W of V.

---

20 *Suppose U and W are subsets of V. Prove that $U \subseteq W \Rightarrow W^0 \subseteq U^0$.*

SOLUTION: $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(u) = 0 \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$. □

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21 *Suppose U and W are subsps of V. Prove that $U \subseteq W \Leftarrow W^0 \subseteq U^0$.*

SOLUTION: [*Req U, W Subsp*] Using Problem (25). $v \in U \Rightarrow \forall \varphi \in W^0 \subseteq U^0, \varphi(v) = 0 \Rightarrow v \in W$. □

COMMENT: For (b), $\varphi \in W^0 \Longleftrightarrow \underline{\operatorname{null} \varphi \supseteq W \Rightarrow \operatorname{null} \varphi \supseteq U} \Longleftrightarrow \varphi \in U^0$. But cannot conclude $W \supseteq U$.

COMMENT: This result is not true if U ***and*** W are merely subsets.

EXAMPLE: Let $U = \{(x, x+1) \in \mathbf{R}^2\}, W = \{(x, \sqrt{x}) \in \mathbf{R}^2\}$.
Then the promotion $X, Y = \mathbf{R}^2$, and $X^0 = Y^0$, but $U \neq W$.

COMMENT: True if W is a subsp. Because the promotion of every subset of W is a subsp of W.
(1) If U is merely a subset and W is a subsp. Promote U to X, let $W = Y$.
Then $Y^0 = W^0 \subseteq U^0 = X^0 \Rightarrow Y = W \supseteq X \supseteq U$.
(2) If W is merely a subset and U is a subsp. Promote W to Y, let $U = X$.
EXAMPLE: Let $W = \{(1,0)\} \cup \{(0,1)\}, U = \{(x,0) \in \mathbf{R}^2\}$.
Then $Y = \mathbf{R}^2$. And $Y^0 = \{0\} \subseteq U^0$. But $U \not\subseteq W$.

• **Corollary:** If U is a subset and W is a subsp. Then $U \subseteq W \iff W^0 \subseteq U^0$.
If U, W are subsps. Then $U = W \iff W^0 = U^0$.

---

22 *Suppose U and W are subsps of V. Prove that $(U+W)^0 = U^0 \cap W^0$.*

Solution: (a) $\varphi \in (U+W)^0 \Rightarrow \forall u \in U, w \in W,$ $\varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$. | $U \subseteq U+W \Rightarrow (U+W)^0 \subseteq U^0$; $W \subseteq U+W \Rightarrow (U+W)^0 \subseteq W^0$

(b) $\varphi \in U^0 \cap W^0 \subseteq V' \Rightarrow \forall u \in U, w \in W, \varphi(u+w) = 0 \Rightarrow \varphi \in (U+W)^0$. □

---

23 *Suppose U and W are subsps of V. Prove that $(U \cap W)^0 = U^0 + W^0$.*

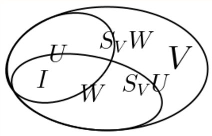
Solution:

(a) $\varphi = \psi + \beta \in U^0 + W^0 \Rightarrow \forall v \in U \cap W,$
$\varphi(v) = (\psi+\beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0$. | Or. $U \cap W \subseteq U \Rightarrow (U \cap W)^0 \supseteq U^0$; $U \cap W \subseteq W \Rightarrow (U \cap W)^0 \supseteq W^0$

(b) [*Only in Finite-dim; Req U, W Subsps*] Using Problem (22).

$$\begin{aligned}\dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0)\\ &= 2\dim V - \dim U - \dim W - \big(\dim V - \dim(U+W)\big) = \dim V - \dim(U \cap W).\end{aligned}$$

Or. [*Req U, W Subsps*] Let $I = U \cap W$. Using [3E Tips (3)].
Now $S_V I = S_V U \oplus S_U I = S_V W \oplus S_W I$. For $\varphi \in (U \cap W)^0 = I^0$.
Let $\operatorname{span}(x) = \text{Pure } V/\operatorname{null}\varphi$. If $x = 0$ then we are done.



Now $0 \neq x \in S_V I \Rightarrow \exists!\,(u_v, i_u, w_v, i_w) \in S_V U \times S_U I \times S_V W \times S_W I$,
$x = u_v + i_u = w_v + i_w$. Define $\varphi \in U^0, \beta \in W^0$ by $\varphi : u_v \mapsto 1, u \mapsto 0$, and $\beta : i_u \mapsto 1, i \mapsto 0$,
for all $u \in \text{Pure } V/\operatorname{span}(u_v)$ and $i \in \text{Pure } V/\operatorname{span}(i_u)$. Or Define $\psi \in W^0, \gamma \in U^0$, similarly.
Then $\varphi = \varphi + \beta = \psi + \gamma \in U^0 + W^0$. □

Comment: If U or W is merely a subset. By Problem (25), promote $U \cap W$ as I, U as X, and W as Y.
Now we show that $X \cap Y = I$. So that $(U \cap W)^0 = I^0 = (X \cap Y)^0 = X^0 + Y^0 = U^0 + W^0$.

• **Corollary:** (1) $(U \cap W)^0 = U^0 + W^0 \supseteq U^0 \cap W^0 = (U+W)^0$.
(2) $\left(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \sum_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$, $\left(\sum_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \bigcap_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$.

• **Tips 1:** (a) *Prove that $V = U \oplus W \iff V' = U^0 \oplus W^0$.*
(b) *Suppose $U \oplus W = V$. Prove that $U^0 = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$,*
where $\iota \in \mathcal{L}(V, W) : u_v + w_v \to u_v$. **New Notation:** Denote W^0 by U'_V, and U^0 by W'_V.

Solution: (a) $U \cap W = \{0\} \iff (U \cap W)^0 = \{0\}_V^0 = V' = U^0 + W^0$.
$V = U + W \iff (U+W)^0 = V_V^0 = \{0\} = U^0 \cap W^0$.
(b) Notice that by [3.B Tips (3)], $\varphi \in W^0 \iff W \subseteq \operatorname{null}\varphi \iff \varphi = \varphi \circ \iota$. □

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31 *Suppose U is a subsp of V. Let $B_{U'_V} = (\varphi_1, \ldots, \varphi_n)$. Show that the correspd B_U exists.*

Solution: Let each $\underline{\operatorname{null}\varphi_i \oplus \operatorname{span}(u_i) = V}$ with $\varphi_i(u_i) = 1$.
Now $a_1u_1 + \cdots + a_nu_n = 0 \Rightarrow$ <u>Each $a_i = \varphi_i(a_1u_1 + \cdots + a_nu_n) = 0$, by def of dual basis.</u> □

Comment: Cannot extend B_U freely.

Example: Let $B_V = (e_1, e_2 - e_1)$. Let the correspd $B_{V'} = (\varphi_1, \varphi_2)$. Let $U'_V = \operatorname{span}(\varphi_1)$.
Then we get e_1. Extend to $B'_V = (e_1, e_2)$. Then the correspd $B_{V'}$ is not $(\varphi_1, \varphi_2, \varphi_3)$.

• **TIPS 2:** Suppose $\varphi_1, \dots, \varphi_m \in V'$. Let $\operatorname{null}_I = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.
Suppose Ω is a subsp of V'. Let $\operatorname{null}_C = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}$.
If $\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$. Then $\operatorname{null}_I = \operatorname{null}_C$.
Because $v \in \operatorname{null}_I \iff$ each $\varphi_i(v) = 0 \iff \forall \varphi \in \Omega, \varphi_i(v) = 0 \iff v \in \operatorname{null}_C$.

COMMENT: If Ω is infinite-dim. Then $\operatorname{null}_I = \bigcap_{\varphi \in \Omega} \operatorname{null}\varphi = \operatorname{null}_C$.

• **TIPS 3:** *Let* $\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m) \subseteq V'$. *Prove that* (a) $\Omega = (\operatorname{null}_I)^0$; (b) $\Omega = (\operatorname{null}_C)^0$.

SOLUTION: Here (a) is [4E 23], (b) is Problem (26).

(a) For each $\varphi_k = 0$, $\operatorname{span}(\varphi_k) = \{0\} = (\operatorname{null}\varphi_k)^0$.
For each $\varphi_k \neq 0$. Using (3.B.29) and TIPS (1). Let $\varphi(v_k) \neq 0 \Rightarrow \operatorname{null}\varphi_k \oplus \operatorname{span}(v_k) = V$.
Then $(\operatorname{null}\varphi_k)^0 = (\operatorname{span}(v_k))'_V = \{\varphi \in V' : \varphi = \varphi \circ \iota\} = \operatorname{span}(\varphi_k)$, where $\iota : cv_k + u_0 \to cv_k$.
Thus $\Omega = \operatorname{span}(\varphi_1) + \cdots + \operatorname{span}(\varphi_m) = (\operatorname{null}\varphi_1)^0 + \cdots + (\operatorname{null}\varphi_m)^0$
$= \big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big)^0 = (\operatorname{null}_I)^0$. □

OR. $\dim(\operatorname{null}\varphi)^0 = \underline{\dim\operatorname{range}\varphi = \dim\operatorname{span}(\varphi)}$. 又 $\operatorname{span}(\varphi) \subseteq (\operatorname{null}\varphi)^0$. OR. By Problem (26) □

OR. $c \in \mathbf{F}\backslash\{0\} \iff \operatorname{null}(c\varphi_i) = \operatorname{null}\varphi_i \iff c\varphi_i \in (\operatorname{null}(c\varphi_i))^0 = (\operatorname{null}\varphi_i)^0$.
And $0 \in \operatorname{span}(\varphi_i), 0 \in (\operatorname{null}\varphi_i)^0$. Hence $\operatorname{span}(\varphi_i) = (\operatorname{null}\varphi_i)^0$. □

(b) $\forall \varphi \in \Omega, \operatorname{null}_C \subseteq \operatorname{null}\varphi \Rightarrow \varphi \in (\operatorname{null}_C)^0$. Hence $\Omega = (\operatorname{null}_I)^0 \subseteq (\operatorname{null}_C)^0$. OR. By TIPS (2). □

• **NOTE FOR Problem (26):** For every subsp Ω of V', $\exists!$ subsp U of V such that $\Omega = U^0$.

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24 *Suppose V is finite-dim and U is a subsp of V.*
Prove, using the pattern of [3.104], that $\dim U + \dim U^0 = \dim V$.

SOLUTION: Let $B_\Omega = B_{U^0} = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$.
And let the correspd (I) $B_U = (v_{m+1}, \dots, v_n)$, (II) $B_W = (v_1, \dots, v_m)$.
(I) NOTICE that each $\operatorname{null}\varphi_k = \operatorname{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n) = U_k$; $\dim U_k = \dim V - 1$.
By (4E 2.C.16), $U = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = \bigcap_{k=1}^m U_k = \operatorname{span}(v_{m+1}, \dots, v_n)$.
Hence $\operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0 = \Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$.
(II) NOTICE that $V' = \Omega \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.
And that $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq \operatorname{span}(v_1, \dots, v_m)^0$.
By [1.C TIPS (2)] OR (2.C.1), $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = \operatorname{span}(v_1, \dots, v_m)^0$.
OR. Similar to (II), let $\Omega = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, immediately. □

• *Suppose* $T \in \mathcal{L}(V, W), \varphi_k \in V', \psi_k \in W'$.

28 *Prove that* $\operatorname{null}T' = \operatorname{span}(\psi_1, \dots, \psi_m) \iff \operatorname{range}T = (\operatorname{null}\psi_1) \cap \cdots \cap (\operatorname{null}\psi_m)$.

29 *Prove that* $\operatorname{range}T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \iff \operatorname{null}T = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.

SOLUTION: Using [3.107], [3.109], Problem (23) and the COROLLARY in Problem (20, 21).
(28) $(\operatorname{range}T)^0 = \operatorname{null}T' = \operatorname{span}(\psi_1, \dots, \psi_m) = \big((\operatorname{null}\psi_1) \cap \cdots \cap (\operatorname{null}\psi_m)\big)^0$.
(29) $(\operatorname{null}T)^0 = \operatorname{range}T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) = \big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big)^0$. □

COROLLARY: Using the COMMENT in Problem (26).
$\operatorname{null}T = \operatorname{span}(v_1, \dots, v_m) \iff \operatorname{null}T = (\operatorname{null}\varphi_{m+1}) \cap \cdots \cap (\operatorname{null}\varphi_n) \iff \operatorname{range}T' = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$.
——Where $B_V = (v_1, \dots, v_m, \dots, v_n) \iff B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$.

$\operatorname{range}T = \operatorname{span}(w_1, \dots, w_m) \iff \operatorname{range}T = (\operatorname{null}\psi_{m+1}) \cap \cdots \cap (\operatorname{null}\psi_n) \iff \operatorname{null}T' = \operatorname{span}(\psi_{m+1}, \dots, \psi_n)$.
——Where $B_W = (w_1, \dots, w_m, \dots, w_n) \iff B_{W'} = (\psi_1, \dots, \psi_m, \dots, \psi_n)$.

9 *Let $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \cdots, \varphi_n)$. Then $\forall \psi \in V', \psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.*

Corollary: *For other $B'_V = (u_1, \dots, u_n), B'_{V'} = (\rho_1, \dots, \rho_n), \forall \psi \in V', \psi = \psi(u_1)\rho_1 + \cdots + \psi(u_n)\rho_n$.*

Solution:

$\psi(v) = \psi\Big(\sum_{i=1}^n a_i v_i \Big) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i)\varphi_i(v) = \big[\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n\big](v)$.

Or. $\big[\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n\big]\Big(\sum_{i=1}^n a_i v_i \Big) = \psi(v_1)\varphi_1\Big(\sum_{i=1}^n a_i v_i \Big) + \cdots + \psi(v_n)\varphi_n\Big(\sum_{i=1}^n a_i v_i \Big)$. $\square$

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13 *Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x,y,z) = (4x + 5y + 6z, 7x + 8y + 9z)$.*

Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ denote the dual basis of the std basis of $\mathbf{R}^2$ and $\mathbf{R}^3$.

(a) *Describe the linear functionals $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$*

For any $(x,y,z) \in \mathbf{R}^3$, $\big(T'(\varphi_1)\big)(x,y,z) = 4x + 5y + 6z$, $\big(T'(\varphi_2)\big)(x,y,z) = 7x + 8y + 9z$.

(b) *Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.*

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$.

(c) *What is* null T'? *What is* range T'?

$$T(x,y,z) = 0 \iff \begin{cases} 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases} \iff \begin{cases} x + y + z = 0 \\ y = 2z = 0 \end{cases} \iff (x,y,z) \in \operatorname{span}(e_1 - 2e_2 + e_3).$$

Where (e_1, e_2, e_3) is std basis of $\mathbf{R}^3$.

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a basis, with the correspd dual basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Thus $\operatorname{span}(e_1 - 2e_2 + e_3) = \operatorname{null} T \Rightarrow \operatorname{span}(e_1 - 2e_2 + e_3)^0 = \operatorname{span}(\varepsilon_2, \varepsilon_3) = \operatorname{range} T'$.

Note that $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$.

And $\left| \begin{array}{l} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{array} \right.$

Hence $\varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2$, $\varepsilon_3 = -\psi_1 + \psi_3$. Now $\operatorname{range} T' = \operatorname{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3)$.

Or. $\operatorname{range} T' = \operatorname{span}(T'(\varphi_1), T'(\varphi_2)) = \operatorname{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3)$.

Suppose $T'(x\varphi_1 + y\varphi_2) = (4x + 7y)\varphi_1 + (5x + 8y)\varphi_2 + (6x + 9y)\varphi_3 = 0$.

Then $x + y = 4x + 7y = x = y = 0$. Hence $\operatorname{null} T' = \{0\}$.

Or. $\operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \operatorname{span}(-2e_2, e_3) \oplus \operatorname{null} T$.

$\Rightarrow \operatorname{range} T = \{Tx : x \in \operatorname{span}(-2e_2, e_3)\} = \operatorname{span}(T(-2e_2), T(e_3))$

$= \operatorname{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \operatorname{span}(f_1, f_2) = \mathbf{R}^2$. Now $\operatorname{null} T' = (\operatorname{range} T)^0 = \{0\}$. $\square$

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37 *Suppose U is a subsp of V and π is the quotient map. Thus $\pi' \in \mathcal{L}\big((V/U)', V'\big)$.*

(a) *Show that π' is inje*: Because π is surj. Use [3.108].

(b) *Show that* range $\pi' = U^0$: By [3.109](b), $\operatorname{range} \pi' = (\operatorname{null} \pi)^0 = U^0$.

(c) *Conclude that π' is an iso from $(V/U)'$ onto U^0*: Immediately.

Solution: Or. Using (3.E.18), also see (3.E.20).

(a) $\pi'(\varphi) = 0 \iff \forall v \in V\ (\ \forall v + U \in V\ ), \varphi(\pi(v)) = \varphi(v + U) = 0 \iff \varphi = 0$.

(b) $\psi \in \operatorname{range} \pi' \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null} \psi \supseteq U \iff \psi \in U^0$. Hence $\operatorname{range} \pi' = U^0$. $\square$

• *Suppose U is a subsp of V. Prove that $(V/U)'$ is iso to U^0.* $\Big[$ ***Another proof of* [3.106]** $\Big]$

Solution:

Define $\xi : U^0 \to (V/U)'$ by $\xi(\varphi) = \tilde{\varphi}$, where $\tilde{\varphi} \in (V/U)'$ is defined by $\tilde{\varphi}(v + U) = \varphi(v)$.

We show that ξ is inje and surj.

Inje: $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V\ (\ \forall v + U \in V/U\ ), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$.

Surj: $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null}(\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$. $\square$

OR. Define $\nu : (V/U)' \to U^0$ by $\nu(\Phi) = \Phi \circ \pi$. Now $\nu \circ \xi = I_{U^0},\ \xi \circ \nu = I_{(V/U)'} \Rightarrow \xi = \nu^{-1}$. $\square$

• *Suppose $V = U \oplus W$. Define $\iota : V \to U$ by $\iota(u + w) = u$. Thus $\iota' \in \mathcal{L}(U', V')$.*

(a) *Show that* $\operatorname{null} \iota' = U_U^0 = \{0\}$: $\operatorname{null} \iota' = (\operatorname{range} \iota)_U^0 = U_U^0 = \{0\}$.

(b) *Prove that* $\operatorname{range} \iota' = W_V^0$: $\operatorname{range} \iota' = (\operatorname{null} \iota)_V^0 = W_V^0$.

(c) *Prove that $\widetilde{\iota'}$ is an iso from $U'/\{0\}$ onto W^0*: By (a), (b) and [3.91](d).

SOLUTION:

(a) $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \operatorname{null} \psi$.

(b) Note that $W = \operatorname{null}(\iota) \subseteq \operatorname{null}(\psi \circ \iota)$. Then $\psi \circ \iota \in W^0 \Rightarrow \operatorname{range} \iota' \in W^0$.

Suppose $\varphi \in W^0$. Because $\operatorname{null} \iota = W \subseteq \operatorname{null} \varphi$. By [3.B TIPS (3)], $\varphi = \varphi \circ \iota = \iota'(\varphi)$. $\square$

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36 *Suppose U is a subsp of V. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*

(a) *Show that* $\operatorname{null} i' = U^0$: $\operatorname{null} i' = (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$.

(b) *Prove that* $\operatorname{range} i' = U'$: $\operatorname{range} i' = (\operatorname{null} i)_U^0 = \{0\}_U^0 = U'$.

(c) *Prove that $\widetilde{i'}$ is an iso from V'/U^0 onto U'*: By (a), (b) and [3.91](d).

SOLUTION:

(a) $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$. Thus $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$.

(b) Suppose $\psi \in U'$. By (3.A.11), $\exists\, \varphi \in V', \varphi|_U = \psi$. Then $i'(\varphi) = \psi$. $\square$

• *Suppose $T \in \mathcal{L}(V, W)$. Prove that* $\operatorname{range} T' = (\operatorname{null} T)^0$. [***Another proof of* [3.109](b)**]

SOLUTION:

Suppose $\Phi \in (\operatorname{null} T)^0$. Because by (3.B.12), $T|_U : U \to \operatorname{range} T$ is an iso; $V = U \oplus \operatorname{null} T$.

And $\forall v \in V, \exists!\, u_v \in U, w_v \in \operatorname{null} T, v = u_v + w_v$. Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$.

Let $\psi = \Phi \circ (T^{-1}|_{\operatorname{range} T})$. Then $T'(\psi) = \psi \circ T = \Phi \circ (T^{-1}|_{\operatorname{range} T} \circ T|_V)$.

Where $T^{-1}|_{\operatorname{range} T} : \operatorname{range} T \to U;\quad T : V \to \operatorname{range} T$. Note that $T^{-1}|_{\operatorname{range} T} \circ T|_V = \iota$.

By [3.B TIPS (3)], $\Phi = \Phi \circ \iota$. Thus $T'(\psi) = \psi \circ T = \Phi \circ \iota = \Phi$. $\square$

• *Suppose $T \in \mathcal{L}(V, W)$.* *Using* [3.108], [3.110].

$$\textit{Now } T \textit{ is inv} \iff \left| \begin{array}{c} \operatorname{null} T = \{0\} \iff (\operatorname{null} T)^0 = V' = \operatorname{range} T' \\ \operatorname{range} T = W \iff (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T' \end{array} \right| \iff T' \textit{ is inv.}$$

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15 *Suppose $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \iff T = 0$.*

SOLUTION:

Suppose $T = 0$. Then $\forall \varphi \in W', T'(\varphi) = \varphi \circ T = 0$. Hence $T' = 0$.

Suppose $T' = 0$. Then $\operatorname{null} T' = W' = (\operatorname{range} T)^0$, by [3.107](a).

[*W can be infinite-dim*] By Problem (25),

$\operatorname{range} T = \{w \in W : \varphi(w) = 0, \forall \varphi \in (\operatorname{range} T)^0\} = \{w \in W : \varphi(w) = 0, \forall \varphi \in W'\}$.

Now we prove that if $\forall \varphi \in W', \varphi(w) = 0$, then $w = 0$. So that $\operatorname{range} T = \{0\}$ and we are done.

Assume that $w \neq 0$. Then let U be such that $W = U \oplus \operatorname{span}(w)$.

Define $\psi \in W'$ by $\psi(u + \lambda w) = \lambda$. So that $\psi(w) = 1 \neq 0$. $\square$

OR. [*Only if W is finite-dim*] By [3.106], $\dim \operatorname{range} T = \dim W - \dim(\operatorname{range} T)^0 = 0$. $\square$

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12 NOTICE that $I_{V'} : V' \to V'$. *Now* $\forall \varphi \in V', I_{V'}(\varphi) = \varphi = \varphi \circ I_V = I_V{}'(\varphi)$. *Thus* $I_{V'} = I_V{}'$.

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16 *Suppose V, W are finite-dim. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(V, W)$.*
Prove that Γ is an iso of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

SOLUTION: By [3.101], Γ is linear.

Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus Γ is inje.

Because V, W are finite-dim. $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$. Now Γ inje $\Rightarrow$ inv. $\square$

COMMENT: Let $X = \{T \in \mathcal{L}(V, W) : \operatorname{range} T \text{ is finite-dim}\}$.

Let $Y = \{\mathcal{T} \in \mathcal{L}(W', V') : \operatorname{range} \mathcal{T} \text{ is finite-dim}\}$.

Then $\Gamma|_X$ is an iso of X onto Y, even if V and W are infinite-dim.

The inje of $\Gamma|_X$ is equiv to the inje of Γ, as shown before.

Now we show that $\Gamma|_X$ is surj without the cond that V or W is finite-dim.

Suppose $\mathcal{T} \in Y$. Let $B_{\operatorname{range} \mathcal{T}} = (\varphi_1, \dots, \varphi_m)$, with the correspd $(v_1, \dots, v_m)$. Let $\varphi_k = \mathcal{T}(\psi_k)$.

Let $\mathcal{K}$ be such that $W' = \mathcal{K} \oplus \operatorname{null} \mathcal{T}$. Let $B_{\mathcal{K}} = (\psi_1, \dots, \psi_m)$, with the correspd $(w_1, \dots, w_m)$.

Define $T \in \mathcal{L}(V, W)$ by $Tv_k = w_k, Tu = 0$; $k \in \{1, \dots, m\}, u \in U$.

$\forall \psi \in \operatorname{null} \mathcal{T}, [T'(\psi)](v) = \psi(Tv) = \psi(a_1 w_1 + \cdots + a_p w_p) = 0 = [\mathcal{T}(\psi)](v)$.

$\forall k \in \{1, \dots, m\}, [T'(\psi_k)](v) = \psi_k(Tv) = \psi_k(a_1 w_1 + \cdots + a_m w_m) = a_k = \varphi_k(v) = [\mathcal{T}(\psi)](v)$. $\square$

COMMENT: This is another proof of [3.109(a)]: $\dim \operatorname{range} T = \dim \operatorname{range} T'$.

---

5 *Prove that $(V_1 \times \cdots \times V_m)'$ and $V'_1 \times \cdots \times V'_m$ are iso.* [Using notations in (3.E.2).]

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to V'_1 \times \cdots \times V'_m \\ \quad \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = (R'_1(T), \dots, R'_m(T)). \\ \text{Define } \psi : V'_1 \times \cdots \times V'_m \to (V_1 \times \cdots \times V_m)' \\ \quad \text{by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m = S'_1(T_1) + \cdots + S'_m(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ $\square$

• (4E 8) *Suppose $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \dots, \varphi_n)$.*

$$\left.\begin{array}{l} \text{Define } \Gamma : V \to \mathbf{F}^n \text{ by } \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)). \\ \text{Define } \Lambda : \mathbf{F}^n \to V \text{ by } \Lambda(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

---

6 *Define $\Gamma : V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$, where $v_1, \dots, v_m \in V$.*

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = V \iff \Gamma$ *is inje.*

(b) *Show that $(v_1, \dots, v_m)$ is linely inde $\iff \Gamma$ is surj.*

SOLUTION:

(a) NOTICE that $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \cdots = \varphi(v_m) = 0 \iff \operatorname{null} \varphi = \operatorname{span}(v_1, \dots, v_m)$.

If Γ is inje, then $\Gamma(\varphi) = 0 \iff V = \operatorname{null} \varphi = \operatorname{span}(v_1, \dots, v_m)$.

If $V = \operatorname{span}(v_1, \dots, v_m)$, then $\Gamma(\varphi) = 0 \iff \operatorname{null} \varphi = \operatorname{span}(v_1, \dots, v_m)$, thus Γ is inje.

(b) Suppose Γ is surj. Then let $\Gamma(\varphi_i) = e_i$ for each i, where $(e_1, \dots, e_m)$ is the std basis of $\mathbf{F}^m$.

Then by (3.A.4), $(\varphi_1, \dots, \varphi_m)$ is linely inde.

Now $a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow 0 = \varphi_i(a_1 v_1 + \cdots + a_m v_m) = a_i$ for each i.

Suppose $(v_1, \dots, v_m)$ is linely inde. Let $U = \operatorname{span}(\varphi_1, \dots, \varphi_m), B_{U'} = (\varphi_1, \dots, \varphi_m)$.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists!\, \varphi = a_1 \varphi_1 + \cdots + a_m \varphi_m$.

Let W be such that $V = U \oplus W$. Now $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.

Define $\iota \in \mathcal{L}(V,U)$ by $\iota(v) = u_v$. So that $\Gamma(\varphi \circ i -) = (a_1, \dots, a_m)$. $\square$

OR. Let $(e_1, \dots, e_m)$ be the std basis of $\mathbf{F}^m$ and let $(\psi_1, \dots, \psi_m)$ be the correspd dual basis.
Define $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$. Then Ψ is an iso.
Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $Te_k = v_k$. Now $T(x_1, \dots, x_m) = T(x_1e_1 + \dots + x_me_m) = x_1v_1 + \dots + x_mv_m$.
$\forall \varphi \in V', k \in \{1, \dots, m\}, [T'(\varphi)](e_k) = \varphi(Te_k) = \varphi(v_k) = [\varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m](e_k)$
Now $T'(\varphi) = \varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m = \Psi(\varphi(v_1), \dots, \varphi(v_m)) = \Psi(\Gamma(\varphi))$. Hence $T' = \Psi \circ \Gamma$.
By (3.B.3), (a) $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V \iff T' = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.
(b) $(v_1, \dots, v_m)$ is linely inde $\iff T$ is inje $\iff T' = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. $\square$

- (4E 25) *Define* $\Gamma : V \to \mathbf{F}^m$ *by* $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$, *where* $\varphi_1, \dots, \varphi_m \in V'$.
  (c) *Show that* $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ *is inje.*
  (d) *Show that* $(\varphi_1, \dots, \varphi_m)$ *is linely inde* $\iff \Gamma$ *is surj.*

SOLUTION:

(c) NOTICE that $\Gamma(v) = 0 \iff \varphi_1(v) = \dots = \varphi_m(v) = 0 \iff v \in (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m)$.
By Problem (4E 23) and (18), $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m) = \{0\}$.
And $\operatorname{null} \Gamma = (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m)$. Hence Γ inje $\iff \operatorname{null} \Gamma = \{0\} \iff \operatorname{span}(\varphi_1, \dots, \varphi_m) = V'$.
(d) Suppose $(\varphi_1, \dots, \varphi_m)$ is linely inde. Then by Problem (31), $(v_1, \dots, v_m)$ is linely inde.
Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}, \exists ! v = \sum_{i=1}^m a_iv_i \in V \Rightarrow \varphi_i(v) = a_i, \Gamma(v) = (a_1, \dots, a_m)$. Hence Γ is surj.
Suppose Γ is surj. Let $(e_1, \dots, e_m)$ be the std basis of $\mathbf{F}^m$.
Suppose $v_i \in V$ such that $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i$, for each i.
Then $(v_1, \dots, v_m)$ is linely inde. And $\varphi_j(v_k) = \delta_{j,k}$.
Now $a_1\varphi_1 + \dots + a_m\varphi_m = 0 \Rightarrow 0(v_i) = a_i$ for each i. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde.
OR. Let $\operatorname{span}(v_1, \dots, v_m) = U$. Then $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U)$. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde. $\square$

OR. Similar to Problem (6), we get $(e_1, \dots, e_m)$, $(\psi_1, \dots, \psi_m)$ and the iso Ψ.
$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'(\Psi(x_1, \dots, x_m)) = \Gamma'(\Psi(x_1e_1 + \dots + x_me_m)) = (x_1\psi_1 + \dots + x_m\psi_m) \circ \Gamma$.
$\forall v \in V, [\Gamma'(\Psi(x_1, \dots, x_m))](v) = [x_1\psi_1 + \dots + x_m\psi_m](\Gamma(v)) = [x_1\varphi_1 + \dots + x_m\varphi_m](v)$.
Now $\Gamma'(\Psi(x_1, \dots, x_m)) = x_1\varphi_1 + \dots + x_m\varphi_m$.
Define $\Phi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Phi = \Psi \circ \Gamma$. $\Phi(x_1, \dots, x_m) = x_1\varphi_1 + \dots + x_m\varphi_m$. Thus by (4E 3.B.3),
(c) the inje of Φ correspds to $(\varphi_1, \dots, \varphi_m)$ spanning V'; 又 $\Phi = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.
(d) the surj of Φ correspds to $(\varphi_1, \dots, \varphi_m)$ being linely inde; 又 $\Phi = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. $\square$

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35 *Prove that* $(\mathcal{P}(\mathbf{F}))'$ *is iso to* $\mathbf{F}^\infty$.

SOLUTION:

Define $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{F}))', \mathbf{F}^\infty)$ by $\theta(\varphi) = (\varphi(1), \varphi(z), \cdots, \varphi(z^n), \cdots)$.
Inje: $\theta(\varphi) = 0 \Rightarrow \forall z^k$ in the basis $(1, z, \dots, z^n)$ of $\mathcal{P}_n(\mathbf{F})$ ($\forall n$), $\varphi(z^k) = 0 \Rightarrow \varphi = 0$.
[NOTICE that $\forall p \in \mathcal{P}(\mathbf{R}), \exists ! a_i \in \mathbf{F}, m = \deg p,\ p = a_0z + a_1z + \dots + a_mz^m \in \mathcal{P}_m(\mathbf{F})$.]
Surj: $\forall (a_k)_{k=1}^\infty \in \mathbf{F}^\infty$, let ψ be such that $\forall k, \psi(z^k) = a_k$ [by [3.5]] and thus $\theta(\psi) = (a_k)_{k=1}^\infty$. $\square$

COMMENT: NOTICE that $\mathcal{P}(\mathbf{F})$ is not iso to $\mathbf{F}^\infty$, so is $\mathcal{P}(\mathbf{F})$ to $(\mathcal{P}(\mathbf{F}))'$
But if we let $\mathbf{F}^\infty = \{(a_1, \cdots, a_n, \underbrace{0, \cdots, 0, \cdots}_{\text{all zero}}) \in \mathbf{F}^\infty \mid \exists ! n \in \mathbf{N}^+\}$. Then $\mathcal{P}(\mathbf{F})$ is iso to $\mathbf{F}^\infty$.

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7 *Show that the dual basis of* $(1, x, \dots, x^m)$ *of* $\mathcal{P}_m(\mathbf{R})$ *is* $(\varphi_0, \varphi_1, \dots, \varphi_m)$, *where* $\varphi_k(p) = \dfrac{p^{(k)}(0)}{k!}$.
Here $p^{(k)}$ *denotes the* k^{th} *derivative of* p, *with the understanding that the* 0^{th} *derivative of* p *is* p.

SOLUTION:

$$\forall j,k\in\mathbf{N},\ (x^j)^{(k)}=\begin{cases} j(j-1)\dots(j-k+1)\cdot x^{(j-k)}, & j\geq k.\\ j(j-1)\dots(j-j+1)=j! & j=k.\\ 0, & j\leqslant k.\end{cases}\quad \text{Then}\quad (x^j)^{(k)}(0)=\begin{cases}0\ , & j\neq k.\\ k!\ , & j=k.\end{cases}\qquad \square$$

OR. Because $\forall j,k\in\{1,\dots,m\}$ such that $j\neq k$, $\varphi_k(x^j)=\frac{(x^j)^{(k)}(0)}{k!}=\frac{0}{k!}=0$; $\varphi_k(x^k)=\frac{(x^k)^{(k)}(0)}{k!}=1$. Thus $\frac{p^{(k)}(0)}{k!}$ act exactly the same as φ_k on the same basis $(1,\dots,x^m)$, hence is just another def of φ_k.$\square$

EXAMPLE: Suppose $m\in\mathbf{N}^+$. By [2.C.10], $B=\big(1,x-5,\dots,(x-5)^m\big)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

Let $\varphi_k=\dfrac{p^{(k)}(5)}{k!}$ for each $k=0,1,\dots,m$. Then $(\varphi_0,\varphi_1,\dots,\varphi_m)$ is the dual basis of B.

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34 *The double dual space of V, denoted by V'', is defined to be the dual space of V'. In other words, $V''=\mathcal{L}(V',\mathbf{F})$. Define $\Lambda:V\to V''$ by $(\Lambda v)(\varphi)=\varphi(v)$.*

(a) *Show that Λ is a linear map from V to V''.*

(b) *Show that if $T\in\mathcal{L}(V)$, then $T''\circ\Lambda=\Lambda\circ T$, where $T''=(T')'$.*

(c) *Show that if V is finite-dim, then Λ* ***is an iso from V onto V''****.*

Suppose V is finite-dim. Then V and V' are iso, and finding an iso from V onto V' generally requires choosing a basis of V. In contrast, the iso Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

SOLUTION:

(a) $\forall\varphi\in V',v,w\in V,a\in\mathbf{F}$, $\big(\Lambda(v+aw)\big)(\varphi)=\varphi(v+aw)=\varphi(v)+a\varphi(w)=(\Lambda v)(\varphi)+a(\Lambda w)(\varphi)$. Thus $\Lambda(v+aw)=\Lambda v+a\Lambda w$. Hence Λ is linear.

(b) $$\begin{aligned}\big(T''(\Lambda v)\big)(\varphi)&=\big((\Lambda v)\circ T'\big)(\varphi)=(\Lambda v)\big(T'(\varphi)\big)\\ &=\big(T'(\varphi)\big)(v)=(\varphi\circ T)(v)=\varphi(Tv)=\big(\Lambda(Tv)\big)(\varphi).\end{aligned}$$

Hence $T''(\Lambda v)=\big(\Lambda(Tv)\big)\Rightarrow T''\circ\Lambda=\Lambda\circ T$.

(c) Suppose $\Lambda v=0$. Then $\forall\varphi\in V'$, $(\Lambda v)(\varphi)=\varphi(v)=0\Rightarrow v=0$. Thus Λ is inje.
又 Because V is finite-dim. $\dim V=\dim V'=\dim V''$. Hence Λ is an iso. $\square$

ENDED