简介

这是我个人用于复习的「***Linear Algebra Done Right 3E/4E, by Sheldon Axler***」笔记，一本习题选答与课文补注。因为使用中文会给我编撰这份笔记带来额外的中英文输入法切换的工作成本，况且对于专业学习者，直接使用英文不会造成任何困扰。但英文词句的冗长性拖慢我复习的效率，所以我对许多常用术语作了简写。这份笔记的内容范围和标识说明，我已经在自述中写得很清楚，不再赘述。这份笔记尚处于缓慢的编撰进度中。

Goto

Abbreviation Table

def	definition	vec	vector
vecsp	vector space	subsp	subspace
add	addition/additive	multi	multiplication/multiplicative/multiple
assoc	associative/associativity	distr	distributive properties/property
inv	inverse	existns	existence
uniqnes	uniqueness	linely inde	linearly independent/independence
linely dep	linearly dependent/dependence	dim	dimension(al)
req	require(d)	B_V	basis of V
inje	injective	surj	surjective
col	column	with resp	with respect
standard basis	std basis	iso	isomorphism/isomorphic
correspd	correspond(ing)	poly	polynomial
eigval	eigenvalue	eigvec	eigenvector
mini poly	minimal polynomial	char poly	characteristic polynomial

1·B

1 *Prove that* $\forall v \in V, -(-v) = v$.

SOLUTION:

$$\left.\begin{aligned} -(-v) + (-v) &= 0 \\ v + (-v) &= 0 \end{aligned}\right\} \Rightarrow \text{By the uniqnes of add inv, we are done.}$$

OR. $-(-v) = (-1)\big((-1)v\big) = \big((-1)(-1)\big)v = 1 \cdot v = v$. $\square$

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2 *Suppose* $a \in \mathbf{F}, v \in V$, *and* $av = 0$. *Prove that* $a = 0$ *or* $v = 0$.

SOLUTION:

Suppose $a \neq 0, \exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. $\square$

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3 *Suppose* $v, w \in V$. *Explain why* $\exists\,!\, x \in V, v + 3x = w$.

SOLUTION:

[*Existns*] Let $x = \dfrac{1}{3}(w - v)$.

[*Uniqnes*] Suppose $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) $-$ (II) : $3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. $\square$

OR. $v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \frac{1}{3}(w - v)$. $\square$

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5 *Show that in the def of a vecsp, the add inv condition can be replaced by* [1.29].

Hint: *Suppose V satisfies all conds in the def, except we've replaced the add inv cond with* [1.29]. *Prove that the add inv is true.*

Using [1.31]. $0v = 0$ for all $v \in V \iff (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. $\square$

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6 *Let* ∞ *and* $-\infty$ *denote two distinct objects, neither of which is in* $\mathbf{R}$.
Define an add and scalar multi on $\mathbf{R} \cup \{\infty, -\infty\}$ *as you could guess.*
The operations of real numbers is as usual. While for $t \in \mathbf{R}$ *define*

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

(I) $t + \infty = \infty + t = \infty + \infty = \infty$,
(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,
(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of add and scalar multi, is $\mathbf{R} \cup \{\infty, -\infty\}$ *a vecsp over* $\mathbf{R}$? *Explain.*

SOLUTION:

Not a vecpsp, since the add and scalar mult is not assoc and distr.
By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.
OR. By Distr: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. $\square$

• **TIPS:** *About the Field* $\mathbf{F}$: *Many choices.*
EXAMPLE: $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \dots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$. (See Euler's Theorem.)

ENDED

1·C 7 8 9 11 12 13 15 16 17 18 21 22 23 24

7 *Give a nonempty $U \subseteq \mathbf{R}^2$,*
U is closed under taking add invs and under add, but is not a subsp of $\mathbf{R}^2$.

Solution: $\big(\, 0 \in U;\ v \in U \Rightarrow -v \in U$. And operations on U are the same as $\mathbf{R}^2$. $\big)$ Let $\mathbf{Z}^2, \mathbf{Q}^2$.

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8 *Give a nonempty $U \subseteq \mathbf{R}^2$, U is closed under scalar multi, but is not a subsp of $\mathbf{R}^2$.*

Solution: Let $U = \{(x,y) \in \mathbf{R}^2 : x = 0 \vee y = 0\}$.

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9 *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if $\exists\, p \in \mathbf{N}^+$, $f(x) = f(x+p)$ for all $x \in \mathbf{R}$.*
Is the set of periodic functions $\mathbf{R} \to \mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution: Denote the set by S.

Suppose $h(x) = \cos x + \sin\sqrt{2}x \in S$, since $\cos x, \sin\sqrt{2}x \in S$.

Assume $\exists\, p \in \mathbf{N}^+$ such that $h(x) = h(x+p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$

$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Contradiction! ☐

Or. Because $[\mathrm{I}] : \cos x + \sin\sqrt{2}x = \cos(x+p) + \sin\left(\sqrt{2}x + \sqrt{2}p\right)$. By differentiating twice,

$$[\mathrm{II}] : \cos x + 2\sin\sqrt{2}x = \cos(x+p) + 2\sin\left(\sqrt{2}x + \sqrt{2}p\right).$$

$$\left.\begin{array}{ll} [\mathrm{II}]-[\mathrm{I}] : & \sin\sqrt{2}x = \sin\left(\sqrt{2}x+\sqrt{2}p\right) \\ 2[\mathrm{I}]-[\mathrm{II}] : & \cos x = \cos(x+p) \end{array}\right\} \Rightarrow \text{Let } x = 0,\ p = \frac{m\pi}{\sqrt{2}} = 2k\pi.\ \text{Contradicts.}$$ ☐

• *Suppose U, W, V_1, V_2, V_3 are subsps of V.*

15 $$U + U \ni u + w \in U.$$ ☐

16 $$U + W \ni u + w = w + u \in W + U.$$ ☐

17 $$(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3).$$ ☐

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18 *Does the add on the subsps of V have an add identity? Which subsps have add invs?*

Solution: Suppose Ω is the unique add identity.

(a) For any subsp U of V. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

(b) Now suppose W is an add inv of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. ☐

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11 *Prove that the intersection of every collection of subsps of V is a subsp of V.*

Solution: Suppose $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collection of subsps of V; here Γ is an arbitrary index set.

We show that $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vecs that are in U_α for each $\alpha \in \Gamma$, is a subsp of V.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under add.

(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under scalar multi.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of V that is closed under add and scalar multi. ☐

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12 *Suppose U, W are subsps of V. Prove that $U \cup W$ is a subsp of $V \iff U \subseteq W$ or $W \subseteq U$.*

SOLUTION:

(a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subsp of V.

(b) Suppose $U \cup W$ is a subsp of V. Suppose $U \not\subseteq W$ and $U \not\supseteq W$ ($U \cup W \neq U$ and W).

Then $\forall a \in U \wedge a \notin W, b \in W \wedge b \notin U,\ a + b \in U \cup W$.

$$\left.\begin{array}{l}\text{If } a+b \in U \Rightarrow b = (a+b) + (-a) \in U, \text{ contradicts!} \\ \text{If } a+b \in W \Rightarrow a = (a+b) + (-b) \in W, \text{ contradicts!}\end{array}\right\} \Rightarrow U \cup W = U \text{ or } W. \text{ Contradicts!}$$

Thus $U \subseteq W$ and $U \supseteq W$. □

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13 *Prove that the union of three subsps of V is a subsp of V if and only if one of the subsps contains the other two.*

This exercise is not true if we replace **F** *with a field containing only two elements.*

SOLUTION:

Suppose U_1, U_2, U_3 are subsps of V. Denote $U_1 \cup U_2 \cup U_3$ by $\mathcal{U}$.

(a) Suppose that one of the subsps contains the other two.

Then $\mathcal{U} = U_1, U_2$ or U_3 is a subsp of V.

(b) Suppose that $U_1 \cup U_2 \cup U_3$ is a subsp of V.

Distinctively notice that $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.

Also note that, if $U \cup W = V$ is a vecsp, then in general U and W are not subsps of V.

Hence this literal trick is invalid.

(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.

By applying Problem (12) we conclude that one U_j contains the other two. Thus we are done.

(II) Assume that no U_j is contained in the union of the other two, and no U_j contains the union of the other two.

Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3;\ v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if any $v + \lambda u \in W \cap U_1$ then $v + \lambda u - \lambda u = v \in U_1$.

Now $W \subseteq U_1 \cup U_2 \cup U_3 \Rightarrow W \subseteq U_2 \cup U_3$. $\forall v + \lambda u \in W, v + \lambda u \in U_i, i = 2, 3$.

If $U_2 \subseteq U_3$ or $U_2 \supseteq U_3$, then $\mathcal{U} = U_1 \cup U_i, i = 2, 3$. By Problem (12) we are done.

Otherwise, both $U_2, U_3 \neq \{0\}$. Because $W \subseteq U_2 \cup U_3$ *has at least three elements.*

There must be some U_i that contains at least two elements of W.

$\exists$ distinct $\lambda_1, \lambda_2 \in \mathbf{F}, v + \lambda_1 u, v + \lambda_2 u \in U_i, i \in \{2, 3\}$.

Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Contradicts. □

EXAMPLE: Let $\mathbf{F} = \mathbf{Z}_2$. $U_1 = \{u, 0\}, U_2 = \{v, 0\}, U_3 = \{v + u, 0\}$. While $\mathcal{U} = \{0, u, v, v + u\}$ is a subsp.

• **EXAMPLE:** *Suppose $U = \{(x, x, y, y) \in \mathbf{F}^4\}, W = \{(x, x, x, y) \in \mathbf{F}^4\}$. Prove that $U + W = \{(x, x, y, z) \in \mathbf{F}^4\}$.*

Let T denote $\{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$. By def, $U + W \subseteq T$.

And $T \ni (x, x, y, z) \Rightarrow (0, 0, y - x, y - x) + (x, x, x, -y + x + z) \in U + W$. Hence $T \subseteq U + W$. □

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21 *Suppose* $U = \{(x, y, x+y, x-y, 2x) \in \mathbf{F}^5\}$. *Find a* W *such that* $\mathbf{F}^5 = U \oplus W$.

Solution: Let $W = \{(0,0,z,w,u) \in \mathbf{F}^5\}$. Then $U \cap W = \{0\}$.

And $\mathbf{F}^5 \ni (x,y,z,w,u) \Rightarrow (x,y,x+y,x-y,2x) + (0,0,z-x-y,w-x-y,u-2x) \in U+W$.

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23 *Give an example of vecsps* V_1, V_2, U *such that* $V_1 \oplus U = V_2 \oplus U$, *but* $V_1 \neq V_2$.

Solution: $V = \mathbf{F}^2$, $U = \{(x,x) \in \mathbf{F}^2\}$, $V_1 = \{(x,0) \in \mathbf{F}^2\}$, $V_2 = \{(0,x) \in \mathbf{F}^2\}$.

• **Tips:** *Suppose* $V_1 \subseteq V_2$ *in Exercise* (23). *Prove or give a counterexample*: $V_1 = V_2$.

Solution:

Because the subset V_1 of vecsp V_2 is closed under add and scalar multi, V_1 is a subspace of V_2.
Suppose W is such that $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$.
If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, contradicts. Hence $W = \{0\}$, $V_1 = V_2$. □

• *Suppose* V_1, V_2, U_1, U_2 *are vecsps*, $V_1 \oplus U_1 = V_2 \oplus U_2$, $V_1 \subseteq V_2$, $U_2 \subseteq U_1$.
Prove or give a counterexample: $V_1 = V_2$, $U_1 = U_2$.

V_1	U_1	
	V_2	U_2

Solution: A counterexample: [Using notations in Chapter 2.]

Let $V = \mathbf{F}^3, B_V = (e_1, e_2, e_3)$, $V_1 = \operatorname{span}(e_1)$, $U_1 = \operatorname{span}(e_2, e_3)$, $V_2 = \operatorname{span}(e_1, e_2)$, $U_2 = \operatorname{span}(e_3)$.
Now $V_1 \subseteq V_2$, $U_2 \subseteq U_1$ and $V_1 \oplus U_1 = V_2 \oplus U_2$. But $V_1 \neq V_2$, $U_1 \neq U_2$. □

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24 *Let* $V_E = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is even}\}$, $V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is odd}\}$. *Show that* $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$.

Solution: (a) $V_E \cap V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x)\} = \{0\}$.

$$\text{(b)} \left| \begin{array}{l} \text{Let } f_e(x) = \frac{1}{2}[g(x) + g(-x)] \Longrightarrow f_e \in V_E \\ \text{Let } f_o(x) = \frac{1}{2}[g(x) - g(-x)] \Longrightarrow f_o \in V_O \end{array} \right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}}, g(x) = f_e(x) + f_o(x).$$ □

Ended

2·A

1 2 6 10 11 14 16 17 | 4E: 3,14

2 (a) $[P]$ *A list* (v) *of length* 1 *in* V *is linely inde* $\Longleftrightarrow v \neq 0$. $[Q]$
(b) $[P]$ *A list* (v, w) *of length* 2 *in* V *is linely inde* $\Longleftrightarrow \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$. $[Q]$

Solution:

(a) $Q \overset{1}{\Rightarrow} P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ linely inde.
$P \overset{2}{\Rightarrow} Q : (v)$ linely inde $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.

Or. $\left| \begin{array}{l} \neg Q \overset{3}{\Rightarrow} \neg P : v = 0 \Rightarrow av = 0 \text{ while we can let } a \neq 0 \Rightarrow (v) \text{ is linely dep.} \\ \neg P \overset{4}{\Rightarrow} \neg Q : (v) \text{ linely dep} \Rightarrow av = 0 \text{ while } a \neq 0 \Rightarrow v = 0. \end{array} \right.$

Comment: (1) with (3) and (2) with (4) will do as well. □

(b) $P \overset{1}{\Rightarrow} Q : (v, w)$ linely inde $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.
$Q \overset{2}{\Rightarrow} P$: no scalar multi $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v, w)$ linely inde.

Or. $\left| \begin{array}{l} \neg P \overset{3}{\Rightarrow} \neg Q : (v, w) \text{ linely dep} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{scalar multi} \\ \neg Q \overset{4}{\Rightarrow} \neg P : \text{scalar multi} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{linely dep.} \end{array} \right.$

Comment: (1) with (3) and (2) with (4) will do as well. □

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1 *Prove that* $[P]$ (v_1, v_2, v_3, v_4) *spans* $V \iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ *also spans* V $[Q]$.

SOLUTION:

Notice that $V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists\, a_1, \dots, a_n \in \mathbf{F}, v = a_1v_1 + \cdots + a_nv_n$.

Assume that $\forall v \in V, \exists\, a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbf{F}$, (that is, if $\exists\, a_i$, then we are to find b_i, vice versa)

$$\begin{aligned} v &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 \\ &= b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4. \end{aligned}$$

Now we can let $b_i = \sum_{r=1}^{i} a_r$ if we are to prove Q with P already assumed;

or let $a_i = b_i - b_{i-1}$ with $b_0 = 0$, if we are to prove P with Q already assumed. $\square$

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6 *Prove that* $[P]$ (v_1, v_2, v_3, v_4) *is linely inde*

$\iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ *is linely inde.* $[Q]$

SOLUTION:

$P \Rightarrow Q : a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4v_4 = 0$

$\Rightarrow a_1v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0 \Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0$

$Q \Rightarrow P : a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$

$\Rightarrow a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \cdots + a_4)v_4 = 0$

$\Rightarrow a_1 = a_1 + a_2 = a_1 + a_2 + a_3 = a_1 + \cdots + a_4 = 0.$ $\square$

• *Suppose* $(v_1, \dots, v_m)$ *is a list of vecs in* V. *For each* k, *let* $w_k = v_1 + \cdots + v_k$.

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

(b) *Show that* $[P]$ $(v_1, \dots, v_m)$ *is linely inde* $\iff (w_1, \dots, w_m)$ *is linely inde* $[Q]$.

SOLUTION:

(a) Assume $a_1v_1 + \cdots + a_mv_m = b_1w_1 + \cdots + b_mw_m = b_1v_1 + \cdots + b_k(v_1 + \cdots + v_k) + \cdots + b_m(v_1 + \cdots + v_m)$.

Then $a_k = b_k + \cdots + b_m;\ \ a_{k+1} = b_{k+1} + \cdots + b_m \Rightarrow b_k = a_k - a_{k+1};\ b_m = a_m$. Similar to Problem (1).

(b) $P \Rightarrow Q :\ \ b_1w_1 + \cdots + b_mw_m = 0 = a_1v_1 + \cdots + a_mv_m$, where $0 = a_k = b_k + \cdots + b_m$.

$Q \Rightarrow P :\ \ a_1v_1 + \cdots + a_mv_m = 0 = b_1w_1 + \cdots + b_mw_m = 0$, where $0 = b_m = a_m,\ 0 = b_k = a_k - a_{k+1}$.

OR. Because $W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

By [2.21](b), a list of length $(m-1)$ spans W, then by [2.23],

$(w_1, \dots, w_m)$ linely dep $\Longrightarrow (v_1, \dots, v_m)$ linely dep. Conversely it is true as well. $\square$

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10 *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* V *and* $w \in V$.

Prove that if $(v_1 + w, \dots, v_m + w)$ *is linely depe, then* $w \in \operatorname{span}(v_1, \dots, v_m)$.

SOLUTION:

Suppose $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0, \exists\, a_i \neq 0 \Rightarrow a_1v_1 + \cdots + a_mv_m = -(a_1 + \cdots + a_m)w$.

Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1v_1 + \cdots + a_mv_m = 0$ while $a_i \neq 0$ for some i, contradicts. $\square$

OR. By contrapositive: Prove that $w \notin \operatorname{span}(v_1, \dots, v_m) \Longrightarrow (v_1 + w, \dots, v_m + w)$ is linely inde.

Suppose $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0 \Rightarrow a_1v_1 + \cdots + a_mv_m = -(a_1 + \cdots + a_m)w$.

Now by assumption, $a_1 + \cdots + a_m = 0$. Then $a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$. $\square$

OR. $\exists\, j \in \{1, \dots, m\}, v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and we are done.

If $j \geqslant 2$, then $\exists\, a_i \in \mathbf{F}, v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1v_1 + \cdots + a_{j-1}v_{j-1}$.

Where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, contradicts.

Now $w = \lambda^{-1}(a_1v_1 + \cdots + a_{j-1}v_{j-1} - v_j) \Rightarrow w \in \operatorname{span}(v_1, \dots, v_m)$. $\square$

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11 *Suppose $(v_1, \dots, v_m)$ is linely inde in V and $w \in V$.*
Show that $[P]$ $(v_1, \dots, v_m, w)$ is linely inde $\iff w \notin \operatorname{span}(v_1, \dots, v_m)$ $[Q]$.

Solution: $\neg Q \Rightarrow \neg P$: Suppose $w \in \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1, \dots, v_m, w)$ is linely depe.
$\neg P \Rightarrow \neg Q$: Suppose $(v_1, \dots, v_m, w)$ is linely dep. Then by [2.21](a), $w \in \operatorname{span}(v_1, \dots, v_m)$. □

---

14 *Prove that $[P]$ V is infinite-dim $\iff [Q]$ there is a sequence $(v_1, v_2, \dots)$ in V such that $(v_1, \dots, v_m)$ is linely inde for each $m \in \mathbf{N}^+$.*

Solution:

$P \Rightarrow Q$: Suppose V is infinite-dim, so that no list spans V.
Step 1 Pick a $v_1 \neq 0$, (v_1) linely inde.
Step m Pick a $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$, by Problem (11), $(v_1, \dots, v_m)$ is linely inde.
This process recursively defines the desired sequence $(v_1, v_2, \dots)$.

$\neg P \Rightarrow \neg Q$: Suppose V is finite-dim and $V = \operatorname{span}(w_1, ..., w_m)$.
Let $(v_1, v_2, \dots)$ be a sequence in V, then $(v_1, v_2, \dots, v_{m+1})$ must be linely dep.

Or. $Q \Rightarrow P$: Suppose there is such a sequence.
Choose an m. Suppose a linely inde list $(v_1, \dots, v_m)$ spans V.
Similar to [2.16]. $\exists\, v_{m+1} \in V \backslash \operatorname{span}(v_1, \dots, v_m)$. Hence no list spans V. □

---

16 *Prove that the vecsp of all continuous functions in $\mathbf{R}^{[0,1]}$ is infinite-dim.*

Solution: Denote the vecsp by U.

Choose one $m \in \mathbf{N}^+$. Suppose $a_0, \dots, a_m \in \mathbf{R}$ are such that $p(x) = a_0 + a_1 x + \dots + a_m x^m = 0$, $\forall x \in [0, 1]$.
Then p has infinitely many roots and hence each $a_k = 0$, otherwise $\deg p \geqslant 0$, contradicts [4.12].
Thus $(1, x, \dots, x^m)$ is linely inde in $\mathbf{R}^{[0,1]}$. Similar to [2.16], U is infinite-dim. □

Or. Note that $\frac{1}{1} > \frac{1}{2} > \dots > \frac{1}{m}$, $\forall m \in \mathbf{N}^+$. Suppose $f_m = \begin{cases} x - \frac{1}{m}, & x \in \left(\frac{1}{m}, 1\right] \\ 0, & x \in \left[0, \frac{1}{m}\right] \end{cases}$

Then $f_1\left(\frac{1}{m}\right) = \dots = f_m\left(\frac{1}{m}\right) = 0 \neq f_{m+1}\left(\frac{1}{m}\right)$. Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. By Problem (14). □

---

17 *Suppose $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$.*
Prove that $(p_0, p_1, \dots, p_m)$ is not linely inde in $\mathcal{P}_m(\mathbf{F})$.

Solution:

Suppose $(p_0, p_1, \dots, p_m)$ is linely inde. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z$.
Notice that $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \dots + a_m p_m(z)$, for if not, let $z = 2$. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has length $(m+1)$.
Hence $(p_0, p_1, \dots, p_m)$ is linely depe in $\mathcal{P}_m(\mathbf{F})$.
For if not, then because $(1, z, \dots, z^m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$,
by the steps in [2.23] trivially, $(p_0, p_1, \dots, p_m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. □

Or. Note that $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(\underbrace{1, z, \dots, z^m}_{\text{of length } (m+1)})$. Then $(p_0, p_1, \dots, p_m, z)$ of length $(m+2)$ is linely dep.

As shown above, $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$. And hence by [2.21](a), $(p_0, p_1, \dots, p_m)$ is linely dep. □

Ended

2·B 1 7 8 | 4E: 5 9

7 *Prove or give a counterexample: If (v_1, v_2, v_3, v_4) is a basis of V and U is a subsp of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then (v_1, v_2) is a basis of U.*

Solution: A counterexample:

Let $V = \mathbf{R}^4$ and e_j be the j^{th} standard basis.

Let $v_1 = e_1, v_2 = e_2, v_3 = e_3 + e_4, v_4 = e_4$. Then $(v_1, \dots, v_4)$ is a basis of $\mathbf{R}^4$.

Let $U = \operatorname{span}(e_1, e_2, e_3) = \operatorname{span}(v_1, v_2, v_3 - v_4)$. Then $v_3 \notin U$ and (v_1, v_2) is not a basis of U. $\square$

• **Note For "$\complement_V U \cup \{0\}$":** "$\complement_V U \cup \{0\}$" is supposed to be a subsp W such that $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cup \{0\} \\ u \pm w \in \complement_V U \cup \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cup \{0\}$. Contradicts.

To fix this, denote the set $\{W_1, W_2 \dots\}$ by $\mathcal{S}_V U$, where for each W_i, $V = U \oplus W_i$. See also in (1.C.23).

---

1 *Find all vecsps on whatever $\mathbf{F}$ that have exactly one basis.*

Solution:

The trivial vecsp $\{0\}$ will do. Indeed, the only basis of $\{0\}$ is the empty list ( ).

Now consider a field containing only the add identity 0 and the multi identity 1,

and let $1 + 1 = 0$. Hence the vecsp $\{0, 1\}$ will do, the list (1) is the unique basis. So is $\mathbf{Z}_2$.

And more generally, consider $\mathbf{F} = \mathbf{Z}_m$, $\forall m - 1 \in \mathbf{N}^+$. For each $s, t \in \{1, \dots, m\}$,

$\mathbf{F} = \operatorname{span}(K_s) = \operatorname{span}(K_t)$. We get more than one basis. So are $\mathbf{Q}, \mathbf{R}, \mathbf{C}$ and all vecsps on such $\mathbf{F}$.

Consider other $\mathbf{F}$. Note that this $\mathbf{F}$ contains at least and strictly more than 0 and 1. We fail. $\square$

• *Suppose $(v_1, \dots, v_m)$ is a list of vecs in V. For $k \in \{1, \dots, m\}$, let $w_k = v_1 + \cdots + v_k$. Show that $[P]$ $B_V = (v_1, \dots, v_m) \iff B_W = (w_1, \dots, w_m)$. $[Q]$*

Solution:

Notice that $B_U = (u_1, \dots, u_n) \iff \forall u \in U, \exists!\, a_i \in \mathbf{F}, u = a_1 u_1 + \cdots + a_n u_n$.

$P \Rightarrow Q: \forall v \in V, \exists!\, a_i \in \mathbf{F},\ v = a_1 v_1 + \cdots + a_m v_m \Rightarrow v = b_1 w_1 + \cdots + b_m v_m, \exists!\, b_k = a_k - a_{k+1}, b_m = a_m$.

$Q \Rightarrow P: \forall v \in V, \exists!\, b_i \in \mathbf{F},\ v = b_1 w_1 + \cdots + b_m w_m \Rightarrow v = a_1 v_1 + \cdots + a_m v_m, \exists!\, a_k = \sum_{j=k}^{m} b_j$. $\square$

• *Suppose U, W are finite-dim and $V = U + W$. Let $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$. Prove that $\exists\, B_V$ consisting of vecs in $U \cup W$.*

Solution:

Because $V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.

By [2.31], B_V can be reduced from $(u_1, \dots, u_m, w_1, \dots, w_n)$. $\square$

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8 *Suppose $V = U \oplus W$. Let $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$. Prove that $B_V = (u_1, \dots, u_m, w_1, \dots, w_n)$.*

Solution:

$\forall v \in V, \exists!\, u \in U, w \in W \Rightarrow \exists!\, a_i, b_i \in \mathbf{F}, v = u + w = (a_1 u_1 + \cdots + a_m u_m) + (b_1 w_1 + \cdots + b_n w_n)$. $\square$

Or. $V = \operatorname{span}(u_1, \dots, u_m) \oplus \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.

Note that $\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0 \Rightarrow \sum_{i=1}^{m} a_i u_i = -\sum_{i=1}^{n} b_i w_i \in U \cap W = \{0\}$. $\square$

• **Note For *linely inde sequence and* [2.34]:** "$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expression.
If we allow using "infinite list", then we must guarantee that $(v_1, \dots, v_n, \dots)$ is a spanning "list" such that for all $v \in V$, there exists a smallest positive integer n such that $v = a_1 v_1 + \cdots + a_n v_n$,
The key point is, how can we guarantee that such a "list" exists?
TODO: More details.

Ended

2·C

1 7 9 10 14,16 15 17 | 4E: 10 14,15 16

1 [Corollary for [2.38,39]] *Suppose U is a subsp of V such that* $\dim V = \dim U$. *Then* $V = U$.
Let $B_U = (u_1, \dots, u_m)$. Then $m = \dim V$. 又 $u_i \in V$. By [2.39], $B_V = (u_1, \dots, u_m)$. □

• Let $v_1, \dots, v_n \in V$ and $\dim \operatorname{span}(v_1, \dots, v_n) = n$. Then $(v_1, \dots, v_n)$ is a basis of $\operatorname{span}(v_1, \dots, v_n)$.
Notice that $(v_1, \dots, v_n)$ *is a spanning list of* $\operatorname{span}(v_1, \dots, v_n)$ *of length* $n = \dim \operatorname{span}(v_1, \dots, v_n)$.

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15 *Suppose V is finite-dim and* $\dim V = n \geqslant 1$.
Prove that $\exists$ one-dim subsps $V_1, \dots, V_n$ of V such that $V = V_1 \oplus \cdots \oplus V_n$.

Solution: Suppose $B_V = (v_1, \dots, v_n)$. Define V_i by $V_i = \operatorname{span}(v_i)$ for each $i \in \{1, \dots, n\}$.
Then $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n \Rightarrow \exists!\, u_i \in V_i, v = u_1 + \cdots + u_n$ □

• **Corollary:** *Suppose W is finite-dim,* $\dim W = m$ *and $w \in W \backslash \{0\}$.*
Prove that $\exists\, B_W = (w_1, \dots, w_m)$ such that $w = w_1 + \cdots + w_m$.

By Problem (15), $\exists$ one-dim subsps $W_1, \dots, W_m$ of W such that $W = W_1 \oplus \cdots \oplus W_m$.
Note that $\dim W_i = \dim \operatorname{span}(w_i) = 1 \Rightarrow \forall x_i \in W_i, \exists!\, c_i \in \mathbf{F}, x_i = c_i w_i$.
Suppose $w = x_1 + \cdots + x_m$, where each $x_i = c_i w_i \in W_i$. Then $(x_1, \dots, x_m)$ is also a basis of W. □

Or. Note that $w \neq 0 \Rightarrow m \geqslant 1$. If $m = 1$ then let $w_1 = w$ and we are done. Suppose $m > 1$.
Extend (w) to a basis $(w, w_1, \dots, w_{m-1})$ of W. Let $w_m = w - w_1 - \cdots - w_{m-1}$.
又 $\operatorname{span}(w, w_1, \dots, w_{m-1}) = \operatorname{span}(w_1, \dots, w_m)$. Hence $(w_1, \dots, w_m)$ is also a basis of W. □

• **New Theorem:** *Suppose V is finite-dim with* $\dim V = n$ *and U is a subsp of V with $U \neq V$.*
Prove that $\exists\, B_V = (v_1, \dots, v_n)$ such that each $v_k \notin U$.

Note that $U \neq V \Rightarrow n \geqslant 1$. We will construct B_V via the following process.
Step 1. $\exists\, v_1 \in V \backslash U \Rightarrow v_1 \neq 0$. If $\operatorname{span}(v_1) = V$ then we stop.
Step k. Suppose $(v_1, \dots, v_{k-1})$ is linely inde in V, each of which belongs to $V \backslash U$.
Note that $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$. And if $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U = V$, then by (1.C.12),
(because $\operatorname{span}(v_1, \dots, v_{k-1}) \not\subseteq U$,) $U \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) \Rightarrow \operatorname{span}(v_1, \dots, v_{k-1}) = V$.
Hence because $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$, it must be case that $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U \neq V$.
Thus $\exists\, v_k \in V \backslash U$ such that $v_k \notin \operatorname{span}(v_1, \dots, v_{k-1})$.
By (2.A.11), $(v_1, \dots, v_k)$ is linely inde in V. If $\operatorname{span}(v_1, \dots, v_k) = V$, then we stop.
Because V is finite-dim, this process will stop after n steps. □

Or. If $U = \{0\}$ then we are done. Let $\dim U \geqslant 1$.
Let $(u_1, \dots, u_m)$ be a basis of U. Extend to a basis $(u_1, \dots, u_n)$ of V.
Then let $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$. □

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7 (a) *Let* $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. *Find a basis of* U.
(b) *Extend the basis in* (b) *to a basis of* $\mathcal{P}_4(\mathbf{F})$.
(c) *Find a subsp* W *of* $\mathcal{P}_4(\mathbf{F})$ *such that* $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

SOLUTION: Using Problem (10).

NOTICE that $\nexists\, p \in \mathcal{P}(\mathbf{F})$ of deg 1 and 2, while $p \in U$. Thus $\dim U \leqslant \dim \mathcal{P}_4(\mathbf{F}) - 2 = 3$.

(a) Consider $B = \Big(1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\Big)$.
Let $a_0 + a_3(z-2)(z-5)(z-6) + a_4 z(z-2)(z-5)(z-6) = 0 \Rightarrow a_0 = a_3 = a_4 = 0$.
Thus the list B is linely inde in U. Now $\dim U \geqslant 3 \Rightarrow \dim U = 3$. Thus $B_U = B$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $(1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6))$.

(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. □

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9 *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* V *and* $w \in V$.
Prove that $\dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$.

SOLUTION: Using the result of (2.A.10, 11).

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$, for each $i = 1, \dots, m$.
$(v_1, \dots, v_m)$ linely inde $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ linely inde $\Rightarrow \underbrace{(v_2 - v_1, \dots, v_m - v_1)}_{\text{of length } (m-1)}$ linely inde.
又 If $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1 + w, \dots, v_m + w)$ is linely inde.
Hence $m \geqslant \dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$. □

• (4E 2.C.16)
Suppose V *is finite-dim and* U *is a subsp of* V *with* $U \neq V$. *Let* $n = \dim V, m = \dim U$.
Prove that $\exists\, (n-m)$ *subsps* $U_1, \dots, U_{n-m}$, *each of dim* $(n-1)$, *such that* $\bigcap_{i=1}^{n-m} U_i = U$.

SOLUTION:

Let $B_U = (v_1, \dots, v_m)$, $B_V = (v_1, \dots, v_m, u_1, \dots, v_{n-m})$.
Define $U_i = \operatorname{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m})$ for each i. Then $U \subseteq U_i$ for each i.
And because $\forall v \in \bigcap_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \cdots + b_{n-m} u_{n-m} \in U_i \Rightarrow b_i = 0$ for each $i \Rightarrow v \in U$.
Hence $\bigcap_{i=1}^{n-m} U_i \subseteq U$. □

EXAMPLE: Suppose $\dim V = 6, \dim U = 3$.

$$\Big(\overbrace{\underbrace{v_1, v_2, v_3}_{\text{Basis of U}}, v_4, v_5, v_6}^{\text{Basis of V}}\Big), \text{ define } \left.\begin{array}{|l} U_1 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_5, v_6) \\ U_2 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_4, v_6) \\ U_3 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_4, v_5) \end{array}\right\} \Rightarrow \dim U_i = 6 - 1,\ i = \underbrace{1, 2, 3}_{6-3=3}.$$
□

• **NOTE FOR Problem 10:** Each nonconst $p \in \operatorname{span}(1, z, \dots, z^m)$, $\exists$ smallest $m \in \mathbf{N}^+$, which is $\deg p$.

(a) If $p_0, p_1, \dots, p_m$ are such that each
$p_k = a_{0,k} + a_{1,k} z + \cdots + a_{k,k} z^k$, with $a_k \neq 0$.
Then $\mathcal{M}\Big(\xi, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\Big) = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m} \\ 0 & a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix}$, which is upper-trig.

(b) If $p_0, p_1, \dots, p_m$ are such that each
$p_k = a_{k,k} x^k + \cdots + a_{m,k} x^m$, with $a_{k,k} \neq 0$.
Then $\mathcal{M}\Big(\xi, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\Big) = \begin{pmatrix} a_{0,0} & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,0} & a_{m,1} & \cdots & a_{m,m} \end{pmatrix}$, which is lower-trig.

---

10 *Suppose $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

Solution: Using mathematical induction on m.

(i) $k = 0, 1$. $\deg p_0 = 0$; $\deg p_1 = 1 \Rightarrow \operatorname{span}(p_0, p_1) = \operatorname{span}(1, x)$.

(ii) $k \in \{1, \dots, m-1\}$. Assume that $\operatorname{span}(p_0, p_1, \dots, p_k) = \operatorname{span}(1, x, \dots, x^k)$.

Then $\operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \subseteq \operatorname{span}(1, x, \dots, x^k, x^{k+1})$.

又 $\deg p_{k+1} = k+1$, $p_{k+1}(x) = a_{k+1}x^{k+1} + r_{k+1}(x)$; $a_{k+1} \neq 0$, $\deg r_{k+1} \leqslant k$.

$\Rightarrow x^{k+1} = \dfrac{1}{a_{k+1}}\big(p_{k+1}(x) - r_{k+1}(x)\big) \in \operatorname{span}(1, x, \dots, x^k, p_{k+1}) = \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

$\therefore x^{k+1} \in \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \Rightarrow \operatorname{span}(1, x, \dots, x^k, x^{k+1}) \subseteq \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m)$. □

Or. 用比较系数法. Denote the coefficient of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Suppose $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show that $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is linely inde.

Step 1. For $k = m$, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m$, $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.

Now $L = a_{m-1}p_{m-1}(x) + \cdots + a_0 p_0(x)$.

Step k. For $0 \leqslant k \leqslant m$, we have $a_m = \cdots = a_{k+1} = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = 0$, then we are done. Otherwise, we have $L = a_{k-1}p_{k-1}(x) + \cdots + a_0 p_0(x)$. □

• **Tips:** *Suppose $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ are such that the lowest term of each p_k is of deg k. Prove that $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

Solution: Using mathematical induction on m.

Let each p_k be defined by $p_k(x) = a_{k,k}x^k + \cdots + a_{m,k}x^m$, where $a_{k,k} \neq 0$.

(i) $k = 0, 1$. $p_m(x) = a_{m,m}x^m$; $p_{m-1}(x) = a_{m-1,m-1}x^{m-1} + a_{m,m-1}x^m \Longrightarrow \operatorname{span}(x^m, x^{m-1}) = \operatorname{span}(p_m, p_{m-1})$.

(ii) $k \in \{1, \dots, m-1\}$. Assume that $\operatorname{span}(x^m, \dots, x^{m-k}) = \operatorname{span}(p_m, \dots, p_{m-k})$.

Then $\operatorname{span}(p_m, \dots, p_{m-(k+1)}) \subseteq \operatorname{span}(x^m, \dots, x^{m-(k+1)})$.

又 $p_{m-(k+1)}$ has the form $a_{m-(k+1),m-(k+1)}x^{m-(k+1)} + r_{m-(k+1)}(x)$;

where the lowest term of $r_{m-(k+1)} \in \mathcal{P}_m(\mathbf{F})$ is of deg $(m-k)$.

$$\Rightarrow x^{m-(k+1)} = \frac{1}{a_{m-(k+1),m-(k+1)}}\big(p_{m-(k+1)}(x) - r_{m-(k+1)}(x)\big) \in \operatorname{span}(x^m, \dots, x^{m-k}, p_{m-(k+1)}) = \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$$

$\therefore x^{m-(k+1)} \in \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$

$\Rightarrow \operatorname{span}(x^m, \dots, x^{m-k}, x^{m-(k+1)}) \subseteq \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}(p_m, \dots, p_1, p_0)$. □

Or. 用比较系数法. Denote the coefficient of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Suppose $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show that $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is linely inde.

Step 1. For $k = 0$, $\xi_0(L) = a_0 \xi_0(p_0) = \xi_0(R) = 0$ 又 $\deg p_0 = 0$, $\xi_0(p_0) \neq 0 \Rightarrow a_0 = 0$.

Now $L = a_1 p_1(x) + \cdots + a_m p_m(x)$.

Step k. For $0 \leqslant k \leqslant m$, we have $a_{k-1} = \cdots = a_0 = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = m$, then we are done. Otherwise, we have $L = a_{k+1}p_{k+1}(x) + \cdots + a_m p_m(x)$. □

- **NOTE FOR [2.11]:** ***Good definition for a general term always aviods undefined behaviours.***
  If $\deg p = 0$, then $p(z) = a_0 \neq 0$, but $\underline{\text{not literally } a_0 z^0}$, by which if p is defined, then it comes to 0^0.
  To make it clear, we $\underline{\text{specify that } \textit{in } \mathcal{P}(\mathbf{F}), a_0 z^0 = a_0, \text{ where } z^0 \text{ appears just for notational convenience.}}$
  Because by definition, the term $a_0 z^0$ in a poly only represents the const term of the poly, which is a_0.
  So z^0 doesn't make sense at all.

- (4E 2.C.10) *Suppose m is a positive integer. For $0 \leqslant k \leqslant m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show that $(p_0, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

SOLUTION: (*We may see that 0 is not a zero of p_0, and that $p_m(x) = x^m$, by the expansion below, and by the* NOTE FOR [2.11] *above.*)

Note that each $p_k(x) = \sum_{j=0}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j = \underbrace{(-1)^0 \cdot x^k \cdot 1^0}_{\text{of deg k}} + \underbrace{\sum_{j=1}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j}_{\text{of deg m; denote it by } q_k(x)}$.
And, each $q_k \in \operatorname{span}(x^{k+1}, \dots, x^m)$. Using TIPS above. ☐

OR. Similar to the TIPS above. We will recursively prove that each $x^{m-k} \in \operatorname{span}(p_m, \dots, p_{m-k})$.
(i) $k = 0, 1$. $p_m(x) = x^m$; $p_{m-1}(x) = x^{m-1} - x^m \Longrightarrow x^{m-1}$. Now $x^m \in \operatorname{span}(p_m), x^{m-1} \in \operatorname{span}(p_{m-1}, p_m)$.
(ii) $k \in \{1, \dots, m-1\}$. Suppose for each $k \in \{0, \dots, k\}$, we have $x^{m-k} \in \operatorname{span}(p_{m-k}, \dots, p_m), \exists !\, a_m \in \mathbf{F}$.
Note that $x^{m-(k+1)} = p_{m-(k+1)}(x) + \sum_{j=1}^{k+1} C_{k+1}^{j}(-1)^{j+1} x^{m-(k+1)+j} \in \operatorname{span}(p_{m-(k+1)}, x^{m-k}, \dots, x^m)$.
Thus $x^{m-(k+1)} \in \operatorname{span}(p_{m-(k+1)}, p_{m-k}, \dots, p_m)$. ☐

COMMENT: The base step and the inductive step can be independent.

OR. For any $m, k \in \mathbf{N}^+$ such that $k \leqslant m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k(1-x)^{m-k}$.
Define the statement $S(m)$ by $S(m) : (p_{0,m}, \dots, p_{m,m})$ is linely inde (and therefore is a basis).
We use induction on to show that $S(m)$ holds for all $m \in \mathbf{N}^+$.
(i) $m = 1$. Let $a_0(1-x) + a_1 x = 0, \forall x \in \mathbf{F}$. Then take $x = 1, x = 0 \Rightarrow a_1 = a_0 = 0$.

$m = 2$. Let $a_0(1-x)^2 + a_1(1-x)x + a_2 x^2, \forall x \in \mathbf{F}$. Then $\begin{cases} x = 0 \Rightarrow a_0 + a_1 = 0; \\ x = 1 \Rightarrow a_2 = 0; \\ x = 2 \Rightarrow a_0 + 2a_1 = 0. \end{cases}$

(ii) $2 \leqslant m$. Assume that $S(m)$ holds.
Suppose $\sum_{k=0}^{m+2} a_k p_{k,m+2}(x) = \sum_{k=0}^{m+2} a_k \left[x^k(1-x)^{m+2-k}\right] = 0, \forall x \in \mathbf{F}$.
Now $a_0(1-x)^{m+2} + \sum_{k=1}^{m+1} a_k x^k (1-x)^{m+2-k} + a_{m+2} x^{m+2} = 0, \forall x \in \mathbf{F}$.
While $\underline{x = 0 \Rightarrow a_0 = 0;\ x = 1 \Rightarrow a_{m+2} = 0.}$ Then $\sum_{k=1}^{m+1} a_k x^k (1-x)^{m+2-k} = 0$;
And note that $\sum_{k=1}^{m+1} a_k x^k (1-x)^{m+2-k}$

$$= x(1-x) \sum_{k=1}^{m+1} a_k x^{k-1} (1-x)^{m+1-k}$$
$$= x(1-x) \sum_{k=0}^{m} a_{k+1} x^k (1-x)^{m-k} = x(1-x) \sum_{k=0}^{m} a_{k+1} p_{k,m}(x).$$

Hence $x(1-x) \sum_{k=0}^{m} a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbf{F} \Rightarrow \sum_{k=0}^{m} a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbf{F} \backslash \{0, 1\}$.
Because $\sum_{k=0}^{m} a_{k+1} p_{k,m}(x)$ has infinitely many zeros. We have $\sum_{k=0}^{m} a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbf{F}$.
By assumption, $a_1 = \cdots = a_m = a_{m+1} = 0$, while $a_0 = a_{m+2} = 0$,
Thus $(p_{0,m+2}, \dots, p_{m+2,m+2})$ is linely inde and $S(m+2)$ holds.

Since $\forall m \in \mathbf{N}^+, S(m) \Rightarrow S(m+2)$. We have $\left| \begin{array}{r} \forall k \in \mathbf{N}, S(2k+1) \text{ holds} \\ \forall k \in \mathbf{N}^+, S(2k) \text{ holds} \end{array} \right\} \Rightarrow S(m)$ holds. ☐

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14 *Suppose that $V_1, \dots, V_m$ are finite-dim subsps of V.*
Prove that $V_1 + \cdots + V_m$ is finite-dim and $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$.

SOLUTION:

Choose a basis $\mathcal{E}_i$ of $V_i \Rightarrow V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim V_i = \operatorname{card} \mathcal{E}_i$.
Then $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$.
又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.
Thus $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$. □

COMMENT: $\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m \iff$ *$V_1 + \cdots + V_m$ is a direct sum.*
For each k, $(V_1 + \cdots + V_k) \cap V_{k+1} = \{0\} \iff V_1 + \cdots + V_m$ is a direct sum
$\iff (\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$ for each k 又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$
$\iff \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$
$\iff \dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m$. □

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17 *Suppose V_1, V_2, V_3 are subsps of a finite-dim vecsp, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

SOLUTION:

[*Similar to*] Given three sets A, B and C.
Because $|X + Y| = |X| + |Y| - |X \cap Y|$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.
Now $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$.
And $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$.
Hence $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$.

Because $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned}
\dim(V_1 + V_2 + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \quad (1) \\
&= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) \quad (2) \\
&= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) \quad (3)
\end{aligned}$$

Notice that in general, $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$.
For example, $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

• **COROLLARY:** *Suppose V_1, V_2 and V_3 are finite-dim vecsps, then* $\dfrac{(1) + (2) + (3)}{3}$:

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.$$

The formula above may seem strange because the right side does not look like an integer. □

• *Suppose V is a 10-dim vecsp and V_1, V_2, V_3 are subsps of V with*

(a) $\dim V_1 = \dim V_2 = \dim V_3 = 7$. *Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*
By TIPS, $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 \underline{- 2 \dim V} > 0$.

(b) $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. *Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*
By TIPS, $\dim(V_1 \cap V_2 \cap V_3) \underline{\underline{\geq 2 \dim V}} - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)) \geq 0$. □

- **TIPS:**

  Because $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big)$.
  And $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$. We have (1), and (2), (3) similarly.
  (1) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 + V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big)$.
  (2) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_3) - \dim\big(V_2 + (V_1 \cap V_3)\big)$.
  (3) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim\big(V_3 + (V_1 \cap V_2)\big)$.

ENDED

3·A   3 4 5 7 8 10 11 12 13 | 4E: 10 11 17

• **TIPS 1:** $T : V \to W$ is linear $\iff \left| \begin{array}{l} (\text{一})\ \forall v, u \in V, T(v+u) = Tv + Tu; \\ (\text{二})\ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv). \end{array} \right| \iff T(v+\lambda u) = Tv + \lambda Tu.$

• **TIPS 2:** $T \in \mathcal{L}(V, W) \iff T \in \mathcal{L}(V, \operatorname{range} T)$.

• **TIPS 3:** If U is a subsp of W, then $\{T \in \mathcal{L}(V, W) : \operatorname{range} T \subseteq U\} = \mathcal{L}(V, U)$.

• (4E 1.B.7) *Suppose $V \neq \emptyset$ and W is a vecsp. Let $W^V = \{f : V \to W\}$.*
(a) *Define a natural add and scalar multi on W^V.*
(b) *Prove that W^V is a vecsp with these definitions.*

SOLUTION:
(a) $W^V \ni f + g : x \to f(x) + g(x)$; where $f(x) + g(x)$ is the vec add on W.
$W^V \ni \lambda f : x \to \lambda f(x)$; where $\lambda f(x)$ is the scalar multi on W.
(b) Commutativity: $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$.
Associativity: $\big((f+g)+h\big)(x) = \big(f(x)+g(x)\big) + h(x)$
$= f(x) + \big(g(x)+h(x)\big) = \big(f + (g+h)\big)(x)$.
Additive Identity: $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$.
Additive Inverse: $(f+g)(x) = f(x) + g(x) = f(x) + \big(-f(x)\big) = 0 = 0(x)$.
Distributive Properties:
$\big(a(f+g)\big)(x) = a(f+g)(x) = a\big(f(x)+g(x)\big)$
$= af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x)$.
Similarly, $\big((a+b)f\big)(x) = (af + bf)(x)$.
So far, we have used the same properties in W.
Which means that ***if** W^V **is a vecsp, then** W **must be a vecsp.***
Multiplication Identity: $(1f)(x) = 1f(x) = f(x)$. (NOTICE that the smallest $\mathbf{F}$ is $\{0, 1\}$.) ☐

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5 *Because $\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is linear}\}$ is a subsp of W^V, $\mathcal{L}(V, W)$ is a vecsp.*

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3 *Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists A_{j,k} \in \mathbf{F}$ such that for any $(x_1, \dots, x_n) \in \mathbf{F}^n$,*

$$T(x_1, \dots, x_n) = \begin{pmatrix} A_{1,1}x_1 + \cdots + A_{1,n}x_n, \\ \vdots \qquad \ddots \qquad \vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n \end{pmatrix}$$

SOLUTION:
Let $T(1,0,0,\dots,0,0) = (A_{1,1}, \dots, A_{m,1})$, Note that $(1,0,\dots,0,0), \cdots, (0,0,\dots,0,1)$ is a basis of $\mathbf{F}^n$.
$T(0,1,0,\dots,0,0) = (A_{1,2}, \dots, A_{m,2})$, Then by [3.5], we are done. ☐
$\vdots$
$T(0,0,0,\dots,0,1) = (A_{1,n}, \dots, A_{m,n})$.

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4 *Suppose $T \in \mathcal{L}(V, W)$, and $v_1, \dots, v_m \in V$ such that $(Tv_1, \dots, Tv_m)$ is linely inde in W. Prove that $(v_1, \dots, v_m)$ is linely inde.*

SOLUTION: Suppose $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$. ☐

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7 *Show that every linear map from a one-dim vecsp to itself is a multi by some scalar. More precisely, prove that if* $\dim V = 1$ *and* $T \in \mathcal{L}(V)$, *then* $\exists \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$.

SOLUTION: Let u be a nonzero vec in $V \Rightarrow V = \operatorname{span}(u)$. Because $Tu \in V \Rightarrow Tu = \lambda u$ for some λ.
Suppose $v \in V \Rightarrow v = au$, $\exists ! a \in \mathbf{F}$. Then $Tv = T(au) = \lambda a u = \lambda v$. ☐

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8 *Give a function* $\varphi : \mathbf{R}^2 \to \mathbf{R}$ *such that* $\forall a \in \mathbf{R}, v \in \mathbf{R}^2, \varphi(av) = a\varphi(v)$ *but* φ *is not linear.*

SOLUTION: Define $T(x,y) = \begin{cases} x+y, & \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, & \text{otherwise.} \end{cases}$ OR. Define $T(x,y) = \sqrt[3]{(x^3+y^3)}$. ☐

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9 *Give a function* $\varphi : \mathbf{C} \to \mathbf{C}$ *such that* $\forall w, z \in \mathbf{C}, \varphi(w+z) = \varphi(w) + \varphi(z)$ *but* φ *is not linear.* (*Here* $\mathbf{C}$ *is thought of as a complex vecsp.*)

SOLUTION: Suppose $V_{\mathbf{C}}$ is the complexification of a vecsp V. Suppose $\varphi : V_{\mathbf{C}} \to V_{\mathbf{C}}$.
Define $\varphi(u + iv) = u = \operatorname{Re}(u + iv)$ OR. Define $\varphi(u + iv) = v = \operatorname{Im}(u + iv)$. ☐

• *Prove that if* $q \in \mathcal{P}(\mathbf{R})$ *and* $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ *is defined by* $\underbrace{Tp = q \circ p}_{\text{composition}}$, *then* T *is not linear.*

SOLUTION: ***Composition and product are not the same in*** $\mathcal{P}(\mathbf{F})$.
Because in general, $[q \circ (p_1 + \lambda p_2)](x) = q(p_1(x) + \lambda p_2(x)) \neq (q \circ p_1)(x) + \lambda (q \circ p_2)(x)$.
EXAMPLE: Let q be defined by $q(x) = x^2$, then $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$. ☐

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10 *Suppose* U *is a subsp of* V *with* $U \neq V$. *Suppose* $S \in \mathcal{L}(U, W)$ *with* $S \neq 0$ (*which means that* $\exists\, u \in U, Su \neq 0$). *Define* $T : V \to W$ *by* $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \backslash U. \end{cases}$
Prove that T *is not a linear map on* V.

SOLUTION: Suppose T is a linear map. And $v \in V \backslash U$, $u \in U$ such that $Su \neq 0$.
Then $v + u \in V \backslash U$, for if not, $v = (v + u) - u \in U$;
while $T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$. Contradicts. ☐

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11 *Suppose* U *is a subsp of* V *and* $S \in \mathcal{L}(U, W)$.
Prove that $\exists\, T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U$. (OR. $\exists T \in \mathcal{L}(V, W), T|_U = S$.)
In other words, every linear map on a subsp of V *can be extended to a linear map on the entire* V.

SOLUTION: Suppose W is such that $V = U \oplus W$. Then $\forall v \in V, \exists ! u_v \in U, w_v \in W, v = u_v + w_v$.
Define $T \in \mathcal{L}(V, W)$ by $T(u_v + w_v) = Su_v$. ☐
OR. [*Finite-dim Req*] Define by $T\Big(\sum_{i=1}^{m} a_i u_i\Big) = \sum_{i=1}^{n} a_i S u_i$. Let $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$. ☐

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12 *Suppose nonzero* V *is finite-dim and* W *is infinite-dim. Prove that* $\mathcal{L}(V, W)$ *is infinite-dim.*

SOLUTION: Using (2.A.14).
Let $B_V = (v_1, \dots, v_n)$ be a basis of V. Let $(w_1, \dots, w_m)$ be linely inde in W for any $m \in \mathbf{N}^+$.
Define $T_{x,y} : V \to W$ by $T_{x,y}(v_z) = \delta_{z,x} w_y$, $\forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$
$\forall v = \sum_{i=1}^{n} a_i v_i$, $u = \sum_{i=1}^{n} b_i v_i$, $\lambda \in \mathbf{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) w_y = T_{x,y}(v) + \lambda T_{x,y}(u)$.
Linearity checked. Now suppose $a_1 T_{x,1} + \dots + a_m T_{x,m} = 0$.
Then $(a_1 T_{x,1} + \dots + a_m T_{x,m})(v_x) = 0 = a_1 w_1 + \dots + a_m w_m \Rightarrow a_1 = \dots = a_m = 0$. 又 m arbitrary.
Thus $(T_{x,1}, \dots, T_{x,m})$ is a linely inde list in $\mathcal{L}(V, W)$ for any x and length m. Hence by (2.A.14). ☐

---

13 *Suppose $(v_1, \dots, v_m)$ is linely depe in V and $W \neq \{0\}$.*
Prove that $\exists\, w_1, \dots, w_m \in W, \nexists\, T \in \mathcal{L}(V, W)$ such that $Tv_k = w_k$, $\forall k = 1, \dots, m$.

SOLUTION:

We prove by contradiction. By linear dependence lemma, $\exists\, j \in \{1, \dots, m\}, v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$.
Fix j. Let $w_j \neq 0$, while $w_1 = \cdots = w_{j-1} = w_{j+1} = w_m = 0$. Define $T \in \mathcal{L}(V, W)$ by $Tv_k = w_k$ for each k.
Suppose $a_1 v_1 + \cdots + a_m v_m = 0$, where $a_j \neq 0$.
Then $T\big(a_1 v_1 + \cdots + a_m v_m\big) = 0 = a_1 w_1 + \cdots + a_m w_m = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. $\square$

OR. We prove the contrapositive: Suppose $\forall w_1, \dots, w_m \in W, \exists\, T \in \mathcal{L}(V, W), Tv_k = w_k$ for each w_k.
Now we show that $(v_1, \dots, v_n)$ is linely inde. Suppose $\exists\, a_i \in \mathbf{F}, a_1 v_1 + \cdots + a_n v_n = 0$.
Choose one $w \in W \backslash \{0\}$. By assumption, for $(\overline{a_1} w, \dots, \overline{a_m} w), \exists\, T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$ for each v_k.
Now we have $0 = T\Big(\sum_{k=1}^m a_k v_k\Big) = \sum_{k=1}^m a_k T v_k = \sum_{k=1}^m a_k \overline{a_k} w = \Big(\sum_{k=1}^m |a_k|^2\Big) w$.
Then $\sum_{k=1}^m |a_k|^2 = 0 \Longrightarrow$ each $a_k = 0$. Hence $(v_1, \dots, v_n)$ is linely inde. $\square$

• (4E 3.A.17)
Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Let $(v_1, \dots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.
Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \backslash \{0\}$.
Suppose $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \cdots + a_n v_n$, where $a_k \neq 0$.
Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y} : \ v_x \mapsto v_y, \ v_z \mapsto 0 \,(\, z \neq x \,)$. OR. $R_{x,y} v_z = \delta_{z,x} v_y$.
Then $\big(R_{1,1} + \cdots + R_{n,n}\big) v_j = v_j \Rightarrow \sum_{r=1}^n R_{r,r} = I$. Assume that each $R_{x,y} \in \mathcal{E}$.
Hence $\forall T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. Now we prove the assumption.
Notice that $\forall x, y \in \mathbf{N}^+, \ (R_{k,y} S)(v_i) = a_k v_y \Rightarrow \big((R_{k,y} S) \circ R_{x,i}\big)(v_z) = \delta_{z,x}(a_k v_y)$.
Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Now $S \in \mathcal{E} \Rightarrow R_{k,y} S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}$. $\square$

• (4E 3.B.32)
Suppose V is finite-dim with $n = \dim V > 1$.
Show that if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is linear and $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$, then $\varphi = 0$.

SOLUTION:

Using notations in (4E 3.A.17). Using the result in NOTE FOR [3.60].
Suppose $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \dots, n\}, \ \varphi(R_{i,j}) \neq 0$. Because $R_{i,j} = R_{x,j} \circ R_{i,x}, \ \forall x = 1, \dots, n$
$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.
Again, because $R_{i,x} = R_{y,x} \circ R_{i,y}, \ \forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0, \ \forall x, y = 1, \dots, n$.
Let $k \neq i, j \neq l$ and then $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$
$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. $\square$

OR. Note that by (4E 3.A.17), $\exists\, S, T \in \mathcal{L}(V), ST - TS \neq 0$.
Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \operatorname{null} \varphi \neq \{0\}$.
Note that $\forall\, E \in \operatorname{null} \varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \operatorname{null} \varphi$.
Hence $\operatorname{null} \varphi$ is a nonzero two-sided ideal of $\mathcal{L}(V)$. $\square$

- *Suppose V is finite-dim. $T \in \mathcal{L}(V)$ is such that $\forall S \in \mathcal{L}(V), ST = TS$. Prove that $\exists \lambda \in \mathbf{F}, T = \lambda I$.*

Solution:

If $V = \{0\}$, then we are done. Now suppose $V \neq \{0\}$.

Assume that $\forall v \in V, (v, Tv)$ is linely depe, then by (2.A.2.(b)), $\exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

To prove that λ_v is independent of v, we discuss in two cases:

$$\left.\begin{array}{l} (-) \text{ If } (v,w) \text{ is linely inde, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \\ \qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=) \text{ Otherwise, suppose } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w \end{array}\right\} \Rightarrow \lambda_w = \lambda_v.$$

Now we prove the assumption. Assume that $\exists\, v \in V, (v, Tv)$ is linely inde. Let $B_V = (v, Tv, u_1, \dots, u_n)$.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Contradicts. □

Or. Let $(v_1, \dots, v_m)$ be a basis of V.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$. Let $\lambda = \varphi(Tv_1) \in \mathbf{F}$.

For any $v \in V$, define $S_v \in \mathcal{L}(V)$ by $S_v u = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. □

Or. For each $k \in \{1, \dots, n\}$, define $S_k \in \mathcal{L}(V)$ by $S_k v_j = \begin{cases} v_k, & j = k, \\ 0, & j \neq k. \end{cases}$ Or. $S_k v_j = \delta_{j,k} v_k$

Note that $S_k\left(\sum_{i=1}^{n} a_i v_i\right) = a_k v_k$. Then $S_k v = v \iff \exists\,! a_k \in \mathbf{F}, v = a_k v_k$.

Hence $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$.

Define $A^{(j,k)} \in \mathcal{L}(V)$ by $A^{(j,k)} v_j = v_k, A^{(j,k)} v_k = v_j, A^{(j,k)} v_x = 0, x \neq j,k$.

Then $A^{(j,k)} T v_j = T A^{(j,k)} v_j = T v_k = a_k v_k;\ A^{(j,k)} T v_j = A^{(j,k)} a_j v_j = a_j A^{(j,k)} v_j = a_j v_k$.

Hence $a_k = a_j$. Thus a_k is independent of v_k. □

- *Given the fact that $\mathcal{L}(V, W)$ is a vecsp. Prove or give a counterexample: V, W are vecsps.*
  We can guarantee that $\{0\} \subseteq \mathcal{L}(V, W), \{0\} \subseteq V, \{0\} \subseteq W$.
  And by [3.2], the additivity and homogeneity imply that V is closed under add and scalar multi.
  (*We cannot even guarantee that W^V is a vecsp.*)

Solution: TODO: Too tricky to be answered by AI.

(I) If $W^V = \{0\}$. Then $\mathcal{L}(V, W) = \{0\}$.

And $W = \{0\}$, for if not, $\exists\, w \in W \backslash \{0\}$, define a map f by $f(x) = w, \forall x \in V$.

And V might not be a vecsp. Example: ???

(II) If W^V is a nonzero vecsp. Then W is a vecsp.

(a) If $\mathcal{L}(V, W) = \{0\}$, then we cannot guarantee that V is a vecsp. Example: ???

(b) If not, then $\exists\, T \in \mathcal{L}(V, W)$, $T \neq 0$. Which means $\exists\, v \in V, Tv \neq 0 \Rightarrow v \neq 0$.

Then both W and V have a nonzero element.

(i) If $\exists$ inje $T \in \mathcal{L}(V, W)$, then $T(u + v) = T(v + u) \Rightarrow u + v = v + u$. etc. Hence V is a vecsp.

(ii) If not, then we cannot guarantee that V is a vecsp. Example: ???

(III) If W^V is not a vecsp, then W is not a vecsp. Example: ??? □

Ended

3·B

3 7 8 9 10 11 12 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31
4E: 24 27 31 32 33

• *Suppose that V and W are real vecsps and $T \in \mathcal{L}(V, W)$.
Define $T_{\mathbf{C}} : V_{\mathbf{C}} \to W_{\mathbf{C}}$ by $T_{\mathbf{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv$ for all $u, v \in V$.
Show that* (a) *$T_{\mathbf{C}}$ is linear,* (b) *$T_{\mathbf{C}}$ is inje $\iff$ T is inje,* (c) *$T_{\mathbf{C}}$ is surj $\iff$ T is surj.*

SOLUTION:

(a) $\forall u_1 + \mathrm{i}v_1, u_2 + \mathrm{i}v_2 \in V_{\mathbf{C}}, \lambda \in \mathbf{F}$,

$$T\big((u_1 + \mathrm{i}v_1) + \lambda(u_2 + \mathrm{i}v_2)\big) = T\big((u_1 + \lambda u_2) + \mathrm{i}(v_1 + \lambda v_2)\big) = T(u_1 + \lambda u_2) + \mathrm{i}T(v_1 + \lambda v_2)$$
$$= Tu_1 + \mathrm{i}Tv_1 + \lambda Tu_2 + \mathrm{i}\lambda Tv_2 = T(u_1 + \mathrm{i}v_1) + \lambda T(u_2 + \mathrm{i}v_2).$$

(b) Suppose $T_{\mathbf{C}}$ is inje. Let $T(u) = 0 \Rightarrow T_{\mathbf{C}}(u + \mathrm{i}0) = Tu = 0 \Rightarrow u = 0$.
Suppose T is inje. Let $T_{\mathbf{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + \mathrm{i}v = 0$.

(c) Suppose $T_{\mathbf{C}}$ is surj. $\forall w \in W, \exists\, u \in V, T(u + \mathrm{i}0) = Tu = w + \mathrm{i}0 = w \Rightarrow$ T is surj.
Suppose T is surj. $\forall w, x \in W, \exists\, u, v \in V, Tu = w, Tv = x$
$\Rightarrow \forall w + \mathrm{i}x \in W_{\mathbf{C}}, \exists\, u + \mathrm{i}v \in V, T(u + \mathrm{i}v) = w + \mathrm{i}x \Rightarrow T_{\mathbf{C}}$ is surj.

---

3 *Suppose $(v_1, \dots, v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1v_1 + \cdots + z_mv_m$.*
(a) *The surj of T correspds to $(v_1, \dots, v_m)$ spanning V.*
(b) *The inje of T correspds to $(v_1, \dots, v_m)$ being linely inde.*

COMMENT: Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $Te_k = v_k$.
(a) $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V$; (b) $(v_1, \dots, v_m)$ is linely inde $\iff$ T is inje.

---

7 *Suppose V is finite-dim with $2 \leqslant \dim V$. And $\dim V \leqslant \dim W = m$, if W is finite-dim.
Show that $U = \big\{T \in \mathcal{L}(V, W) : \operatorname{null} T \neq \{0\}\big\}$ is not a subsp of $\mathcal{L}(V, W)$.*

SOLUTION: *The set of all inje $T \in \mathcal{L}(V, W)$ is a not subsp either.*

Let $(v_1, \dots, v_n)$ be a basis of V, $(w_1, \dots, w_m)$ be linely inde in W. $(\,2 \leqslant n \leqslant m.\,)$
Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 :\ v_1 \mapsto 0,\quad v_2 \mapsto w_2,\quad v_i \mapsto w_i.$
Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 :\ v_1 \mapsto w_1,\quad v_2 \mapsto 0,\quad v_i \mapsto w_i,\quad i = 3, \dots, n.$
Thus $T_1 + T_2 \notin U$. □

COMMENT: If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\ T \in \mathcal{L}(V, W), T$ is inje. Hence $U = \emptyset$.
If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0v_0 = 0$.

---

8 *Suppose W is finite-dim with $\dim W \geqslant 2$. And $n = \dim V \geqslant \dim W$, if V is finite-dim.
Show that $U = \big\{T \in \mathcal{L}(V, W) : \operatorname{range} T \neq W\big\}$ is not a subsp of $\mathcal{L}(V, W)$.*

SOLUTION: *The set of all surj $T \in \mathcal{L}(V, W)$ is not a subspace either.*

Let $(v_1, \dots, v_n)$ be linely inde in V, $(w_1, \dots, w_m)$ be a basis of W. $\big(\, n \in \{m, m+1, \dots\}; 2 \leqslant m \leqslant n.\,\big)$
Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 :\ v_1 \mapsto 0,\quad v_2 \mapsto w_2,\quad v_j \mapsto w_j,\quad v_{m+i} \mapsto 0.$
Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 :\ v_1 \mapsto w_1,\quad v_2 \mapsto 0,\quad v_j \mapsto w_j,\quad v_{m+i} \mapsto 0.$
$\big($ For each $j = 2, \dots, m;\ i = 1, \dots, n - m$, if V is finite, otherwise let $i \in \mathbf{N}^+$. $\big)$ Thus $T_1 + T_2 \notin U$. □

COMMENT: If $\dim W = 0$, then $W = \{0\} = \operatorname{span}(\,)$. $\forall\ T \in \mathcal{L}(V, W), T$ is surj. Hence $U = \emptyset$.
If $\dim W = 1$, then $W = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0v_0 = 0$.

---

11 *Suppose $S_1, \dots, S_n$ are linear and inje. $S_1S_2 \dots S_n$ makes sence. Prove that $S_1S_2 \dots S_n$ is inje.*

SOLUTION: $S_1S_2 \dots S_n(v) = 0 \iff S_2S_3 \dots S_n(v) = 0 \iff \cdots \iff S_n(v) = 0 \iff v = 0$. □

---

9 *Suppose $(v_1, \dots, v_n)$ is linely inde. Prove that $\forall$ inje T, $(Tv_1, \dots, Tv_n)$ is linely inde.*

SOLUTION: $a_1 T v_1 + \cdots + a_n T v_n = 0 = T\big(\sum_{i=1}^{n} a_i v_i\big) \iff \sum_{i=1}^{n} a_i v_i = 0 \iff a_1 = \cdots = a_n = 0.$ □

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10 *Suppose* $\operatorname{span}(v_1, \dots, v_n) = V$. *Show that* $\operatorname{span}(Tv_1, \dots, Tv_n) = \operatorname{range} T$.

SOLUTION:

(a) $\operatorname{range} T = \{Tv : v \in V\} = \{Tv : v \in \operatorname{span}(v_1, \dots, v_n)\} \Rightarrow Tv_1, \dots, Tv_n \in \operatorname{range} T \Rightarrow$ By [2.7].

OR. $\operatorname{span}(Tv_1, \dots, Tv_n) \ni a_1 T v_1 + \cdots + a_n T v_n = T(a_1 v_1 + \cdots + a_n v_n) \in \operatorname{range} T$.

(b) $\forall\, w \in \operatorname{range} T, \exists\, v \in V, w = Tv.\ (\exists\, a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n) \Rightarrow w = a_1 T v_1 + \cdots + a_n T v_n.$ □

---

16 *Suppose $\exists\, T \in \mathcal{L}(V)$ such that* $\operatorname{null} T, \operatorname{range} T$ *are finite-dim. Prove that V is finite-dim.*

SOLUTION: Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n), B_{\operatorname{null} T} = (u_1, \dots, u_m)$.

$\forall v \in V, T(v - a_1 v_1 - \cdots - a_n v_n) = 0$, letting $Tv = a_1 T v_1 + \cdots + a_n T v_n$.

$\Rightarrow v - a_1 v_1 - \cdots - a_n v_n = b_1 u_1 + \cdots + b_m u_m$. Hence $V \subseteq \operatorname{span}(v_1, \dots, v_n, u_1, \dots, u_m)$. □

---

17 *Suppose V, W are finite-dim. Prove that $\exists$ inje $T \in \mathcal{L}(V, W) \iff$* $\dim V \leqslant \dim W$.

SOLUTION:

(a) Suppose $\exists$ inje T. Then $\dim V = \dim \operatorname{range} T \leqslant \dim W$.

(b) Suppose $\dim V \leqslant \dim W$. Let $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.

Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i,\ \ i = 1, \dots, n\ (\ = \dim V\ )$. □

---

18 *Suppose V, W are finite-dim. Prove that $\exists$ surj $T \in \mathcal{L}(V, W) \iff$* $\dim V \geqslant \dim W$.

SOLUTION:

(a) Suppose $\exists$ surj T. Then $\dim V = \dim W + \dim \operatorname{null} T \Rightarrow \dim W \leqslant \dim V$.

(b) Suppose $\dim V \geqslant \dim W$. Let $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_m v_m + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m$. □

---

19 *Suppose V, W are finite-dim, U is a subsp of V.*

Prove that if $\underbrace{\dim U}_{m} \geqslant \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_{p}$, then $\exists\, T \in \mathcal{L}(V, W)$, $\operatorname{null} T = U$.

SOLUTION:

Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_W = (w_1, \dots, w_p)$.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m) = a_1 w_1 + \cdots + a_n w_n$. □

• (4E 3.B.21)

Suppose V is finite-dim, $T \in \mathcal{L}(V, W)$, U is a subsp of W. Let $\mathcal{K}_U = \{v \in V : Tv \in U\}$.
Prove that $\mathcal{K}_U$ is a subsp of V and $\dim \mathcal{K}_U = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$.

SOLUTION:

$\forall u, w \in \mathcal{K}_U, \lambda \in \mathbf{F}, T(u + \lambda w) = Tu + \lambda T w \in U \Rightarrow \mathcal{K}_U$ is a subsp of V.

Define $S \in \mathcal{L}(\mathcal{K}_U, U)$ as $Rv = Tv$ for all $v \in \mathcal{K}_U$. Hence $\operatorname{range} R = U \cap \operatorname{range} T$.

Suppose $\exists\, v, Tv = 0$. 又 $0 \in U \Rightarrow Rv = 0$. Thus $\operatorname{null} T \subseteq \operatorname{null} R$. □

• **TIPS:** *Suppose U is a subsp of V. Prove that $\forall T \in \mathcal{L}(V, W), U \cap$* $\operatorname{null} T = \operatorname{null} T|_U$.

SOLUTION: Note that $U \cap \operatorname{null} T \subseteq \operatorname{null} T|_U$. On the other hand, suppose $u \in \operatorname{null} T|_U$.

Then $T|_U(u)$ makes sense $\Rightarrow u \in U$. And $T|_U(u) = Tu = 0 \Rightarrow u \in \operatorname{null} T$. □

---

12 *Prove that* $\forall T \in \mathcal{L}(V,W), \exists$ *subsp* U *of* V *such that*

$U \cap \operatorname{null} T = \operatorname{null} T|_U = \{0\}$, $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$.

Which is equivalent to $T|_U : U \to \operatorname{range} T$ *being an iso.*

SOLUTION:

By [2.34] (note that V can be infinite-dim), $\exists$ subsp U of V such that $V = U \oplus \operatorname{null} T$.

$\forall v \in V, \exists!\, w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w+u) = Tu \in \{Tu : u \in U\}$. $\square$

- **NEW NOTATION:**

Suppose $T \in \mathcal{L}(V,W)$ and $R = (Tv_1, \dots, Tv_n)$ is linely inde in range T.

Where $n = \dim \operatorname{range} T$ if finite-dim, otherwise $n \in \mathbf{N}^+$.

By (3.A.4), $L = (v_1, \dots, v_n)$ is linely inde in V.

Denote $\mathcal{K}_R$ by span L, if range T is finite-dim, otherwise, denote it by a vecsp in $\mathcal{S}_V \operatorname{null} T$.

Note that if range T is finite-dim, then $\mathcal{K}_R = \operatorname{range} T$ for any basis R of range T.

- **COMMENT:**

If range T is infinite-dim, we cannot write $\mathcal{K}_R = \operatorname{range} T$. For if we do so, we must guarantee that $\forall Tv \in \operatorname{range} T, \exists!\, n \in \mathbf{N}^+, Tv \in \operatorname{span}(Tv_1, \dots, Tv_n)$, where $(Tv_k)_{k=1}^{\infty}$ is linely inde.

So that range $T \subseteq$ span $(Tv_1, \cdots, Tv_n, \cdots)$. This would be invalid, as we have shown before.

- **NEW THEOREM:** $\mathcal{K}_R \in \mathcal{S}_V \operatorname{null} T$. **COMMENT:** $\operatorname{null} T \in \mathcal{S}_V \mathcal{K}_R$.

Suppose range T is finite-dim. Otherwise, we are done immediately.

(a) $T\left(\sum_{i=1}^n a_i v_i\right) = 0 \Rightarrow \sum_{i=1}^n a_i T v_i = 0 \Rightarrow a_1 = \cdots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \operatorname{null} T = \{0\}$.

(b) $\forall v \in V, Tv = \sum_{i=1}^n a_i T v_i \Rightarrow Tv - \sum_{i=1}^n a_i T v_i = T\left(v - \sum_{i=1}^n a_i v_i\right) = 0$

$\Rightarrow v - \sum_{i=1}^n a_i v_i \in \operatorname{null} T \Rightarrow v = \left(v - \sum_{i=1}^n a_i v_i\right) + \left(\sum_{i=1}^n a_i v_i\right) \Rightarrow \mathcal{K}_R + \operatorname{null} T = V$. $\square$

- *Suppose* V *is finite-dim,* $T \in \mathcal{L}(V,W), B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n), B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$.

Prove or give a counterexample: $(u_1, \dots, u_m)$ *is a basis of* $\operatorname{null} T$.

SOLUTION: A counterexample:

Suppose $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$. Then $\operatorname{span}(Tv_1, Tv_2, Tv_3) = \operatorname{span}(w_1)$.

Extend (v_i) to (v_1, v_2, v_3) for each i. But none of $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is a basis of null T. $\square$

COMMENT: $(v_2 - v_1, v_3 - v_1), (v_1 - v_2, v_3 - v_2)$ or $(v_1 - v_3, v_2 - v_3)$ are all bases of null T.

Always notice that $\mathcal{S}_V \operatorname{span}(v_1, \dots, v_n) = \{U_1, \cdots, \operatorname{null} T, \cdots, U_n, \cdots\}$.

- *Suppose* V *is finite-dim,* X *is a subsp of* V*, and* Y *is a finite-dim subsp of* W*.*

Prove that if $\dim X + \dim Y = \dim V$*, then* $\exists\, T \in \mathcal{L}(V,W), \operatorname{null} T = X, \operatorname{range} T = Y$.

SOLUTION:

Suppose $\dim X + \dim Y = \dim V$. Let $B_X = (u_1, \dots, u_n), B_Y = (w_1, \dots, w_m), B_V = (u_1, \dots, u_n, v_1, \dots, v_m)$.

Define $T \in \mathcal{L}(V,W)$ by $Tv_i = w_i, Tu_j = 0$. Notice that $\forall v \in V, \exists!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i$.

$v \in \operatorname{null} T \iff Tv = 0 \iff a_1 = \cdots = a_m = 0 \iff v \in X$.

$Y \ni w = a_1 w_1 + \cdots + a_m w_m = a_1 T v_1 + \cdots + a_m T v_m \in \operatorname{range} T$.

OR. range $T = \operatorname{span}(Tv_1, \dots, Tv_m, Tu_1, \dots, Tu_n) = \operatorname{span}(Tv_1, \dots, Tv_m) = \operatorname{span}(w_1, \dots, w_m) = Y$. $\square$

• OR (5.B.4) *Suppose* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove that* $V = \operatorname{null} P \oplus \operatorname{range} P$.

SOLUTION:

(a) If $v \in \operatorname{null} P \cap \operatorname{range} P \Rightarrow Pv = 0$ and $\exists\, u \in V, v = Pu$. Then $v = Pu = P^2u = Pv = 0$.

(b) Note that $\forall v \in V, v = Pv + (v - Pv)$ and $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$. □

OR. [*Only in Finite-dim*] Let $(P^2v_1, \dots, P^2v_n)$ be a basis of range P^2. Then $(Pv_1, \dots, Pv_n)$ is linely inde. Let $\mathcal{K} = \operatorname{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \operatorname{null} P^2$. While $\mathcal{K} = \operatorname{range} P = \operatorname{range} P^2$; $\operatorname{null} P = \operatorname{null} P^2$. □

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20 *Suppose W is finite-dim. Prove that* $T \in \mathcal{L}(V, W)$ *is inje* $\iff \exists\, S \in \mathcal{L}(W, V),\ ST = I_V$.

SOLUTION:

(a) Suppose $\exists\, S \in \mathcal{L}(W, V),\ ST = I$. Then if $Tv = 0 \Rightarrow ST(v) = 0 = v$. OR. $\operatorname{null} T \subseteq \operatorname{null} ST = \{0\}$.

(b) Suppose T is inje. Let $R = B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$. Then $\mathcal{K}_R \oplus \operatorname{null} T = V$. Let $U \oplus \operatorname{range} T = W$. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$ and $Su = 0$, where $i \in \{1, \dots, n\}, u \in U$. Thus $ST = I$.

OR. Define $S \in \mathcal{L}(\operatorname{range} T, V)$ by $Sw = T^{-1}w$, *where* T^{-1} *is the inv of* $T \in \mathcal{L}(V, \operatorname{range} T)$. Then extend it to $S \in \mathcal{L}(W, V)$ by (3.A.11). Now $\forall v \in V, STv = T^{-1}Tv = v$. □

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21 *Suppose W is finite-dim. Prove that* $T \in \mathcal{L}(V, W)$ *is surj* $\iff \exists\, S \in \mathcal{L}(W, V),\ TS = I_W$.

SOLUTION:

(a) Suppose $\exists\, S \in \mathcal{L}(W, V),\ TS = I$. Then $\forall\, w \in W, TS(w) = w \in \operatorname{range} T \Rightarrow \operatorname{range} T = W$.

(b) Suppose T is surj. Let $R = B_{\operatorname{range} T} = B_W = (Tv_1, \dots, Tv_n)$. Then $\mathcal{K}_R \oplus \operatorname{null} T = V$. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$. Then $TS = I$.

OR. By Problem (12), $\exists$ subsp U of $V, V = U \oplus \operatorname{null} T, \operatorname{range} T = \{Tu : u \in U\}$. Note that $T|_U : U \to W$ is an iso. Define $S = (T|_U)^{-1}$, where $(T|_U)^{-1} : W \to U$. Then $TS = T \circ (T|_U)^{-1} = T|_U \circ (T|_U)^{-1}$. □

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24 *Suppose* $S, T \in \mathcal{L}(V, W)$. *Prove that* $\operatorname{null} S \subseteq \operatorname{null} T \iff \exists\, E \in \mathcal{L}(W)$ *such that* $T = ES$.

SOLUTION:

Suppose $\exists\, E \in \mathcal{L}(W)$ such that $T = ES$. Then $\operatorname{null} T = \operatorname{null} ES \supseteq \operatorname{null} S$.
Suppose $\operatorname{null} S \subseteq \operatorname{null} T$. Let $W = \operatorname{range} S \oplus U$.
Define $E \in \mathcal{L}(W)$ by $E(Sv + w) = Tv$ for each Sv and each $w \in U$. Now we check that E is linear.
Because $\forall w_1, w_2 \in W, \exists!\, Sv_1, Sv_2 \in \operatorname{range} S, u_1, u_2 \in U, w_1 = Sv_1 + u_1, w_2 = Sv_2 + u_2$.
Now $E(w_1 + \lambda w_2) = E\big((Sv_1 + \lambda Sv_2) + (u_1 + \lambda u_2)\big) = T(v_1 + \lambda v_2) = Tv_1 + \lambda Tv_2 = Ew_1 + \lambda Ew_2$.

OR. Let $V = \mathcal{K} \oplus U$. Then $S|_{\mathcal{K}} : \mathcal{K} \to \operatorname{range} S$ is an iso.
Now extend $T(S|_{\mathcal{K}})^{-1} \in \mathcal{L}(\operatorname{range} S, W)$ to $E \in \mathcal{L}(W, W)$.

OR. [*Requires that* range S *is Finite-dim*] Let $R = B_{\operatorname{range} S} = (Sv_1, \dots, Sv_n)$. Then $V = \mathcal{K}_R \oplus \operatorname{null} S$.
Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i,\ Eu = 0$; for each $i = 1 \dots, n$ and each $u \in \operatorname{null} S$.
Hence $\forall v \in V,\ (\exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} S\ ), Tv = a_1Tv_1 + \cdots + a_nTv_n = E(a_1Sv_1 + \cdots + a_nSv_n) \Rightarrow T = ES$.

OR. [*Requires that W is Finite-dim*] Extend R to a basis $(Sv_1, \dots, Sv_n, w_1, \dots, w_m)$ of W.
Define $E \in \mathcal{L}(W)$ by $E(Sv_k) = Tv_k,\ Ew_j = 0$. Because $\forall\, v \in V, \exists\, a_i \in \mathbf{F}, Sv = a_1Sv_1 + \cdots + a_nSv_n$.
Now $v - (a_1v_1 + \cdots + a_nv_n) \in \operatorname{null} S \Rightarrow v - (a_1v_1 + \cdots + a_nv_n) \in \operatorname{null} T \Rightarrow T\big(v - (a_1v_1 + \cdots + a_nv_n)\big) = 0$.
Thus $Tv = a_1v_1 + \cdots + a_nv_n$. Hence $E(Sv) = a_1E(Sv_1) + \cdots + a_nE(Sv_n) = a_1Tv_1 + \cdots + a_nTv_n = Tv$. □

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25 *Suppose* V *is finite-dim and* $S, T \in \mathcal{L}(V, W)$.
Prove that $\operatorname{range} S \subseteq \operatorname{range} T \iff \exists\, E \in \mathcal{L}(V)$ *such that* $S = TE$.

SOLUTION:

Suppose $\exists\, E \in \mathcal{L}(V)$ such that $S = TE$. Then $\operatorname{range} S = \operatorname{range} TE \subseteq \operatorname{range} T$.
Suppose $\operatorname{range} S \subseteq \operatorname{range} T$. Let $(v_1, \dots, v_m)$ be a basis of V.
Note that each $Sv_i \in \operatorname{range} T$. Suppose $u_i \in V$ such that $Tu_i = Sv_i$.
Thus defining $E \in \mathcal{L}(V)$ by $Ev_i = u_i$ for each $i \Rightarrow S = TE$. □

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22 *Suppose* U *and* V *are finite-dim vecsps and* $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.
Prove that $\dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$.

SOLUTION:

Define $R \in \mathcal{L}(\operatorname{null} ST, V)$ by $Ru = Tu$ for all $u \in \operatorname{null} ST \subseteq U$.

$$\left.\begin{aligned} S(Tu) = 0 = S(Ru) \Rightarrow \operatorname{range} R \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} R \leqslant \dim \operatorname{null} S \\ Tu = 0 = Ru \Rightarrow \operatorname{null} R \supseteq \operatorname{null} T \Rightarrow \dim \operatorname{null} R = \dim \operatorname{null} T \end{aligned}\right\} \Rightarrow \text{By [3.22], we are done.} \ \square$$

OR. For any $u \in U$, note that $u \in \operatorname{null} ST \iff S(Tu) = 0 \iff Tu \in \operatorname{null} S$.
Thus $\operatorname{null} ST = \mathcal{K}_{\operatorname{null} S \,\cap\, \operatorname{range} T} = \{u \in U : Tu \in \operatorname{null} S\}$. By Problem (4E 3B.21),
$\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T) \leqslant \dim \operatorname{null} T + \dim \operatorname{null} S$. □

COROLLARY: (1) If T is inje, then $\dim \operatorname{null} T = 0 \Rightarrow \dim \operatorname{null} ST \leqslant \dim \operatorname{null} S$.
(2) If T is surj, then $\operatorname{range} R = \operatorname{null} S \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.
(3) If S is inje, then $\operatorname{range} R = \{0\} \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} R = \dim \operatorname{null} T$.

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23 *Suppose* U *and* V *are finite-dim vecsps and* $S \in \mathcal{L}(V, W)$ *and* $T \in \mathcal{L}(U, V)$.
Prove that $\dim \operatorname{range} ST \leqslant \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.

SOLUTION:

$\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{span}(Su_1, \dots, Su_{\dim \operatorname{range} T})$, where $B_{\operatorname{range} T} = (u_1, \dots, u_{\dim \operatorname{range} T})$.
$\dim \operatorname{range} ST \leqslant \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leqslant \dim \operatorname{range} S$. □

OR. Note that $\operatorname{range} S|_{\operatorname{range} T} = \operatorname{range} ST$.
Thus $\dim \operatorname{range} ST = \dim \operatorname{range} S|_{\operatorname{range} T} = \dim \operatorname{range} T - \dim \operatorname{null} S|_{\operatorname{range} T} \leqslant \operatorname{range} T$. □

COROLLARY: (1) If S is inje, then $\dim \operatorname{range} ST = \dim \operatorname{range} T$.
(2) If T is surj, then $\dim \operatorname{range} ST = \dim \operatorname{range} S$.

• (a) *Suppose* $\dim V = 5, S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$. *Prove that* $\dim \operatorname{range} TS \leqslant 2$.
(b) *Let* $\dim V = n$ *in* (a). *Prove that* $\dim \operatorname{range} TS \leqslant \left\lfloor \dfrac{n}{2} \right\rfloor$.
(c) *Give an example of* $S, T \in \mathcal{L}(\mathbf{F}^5)$ *with* $ST = 0$ *and* $\dim \operatorname{range} TS = 2$.

SOLUTION:

(a) By Problem (23), $\dim \operatorname{range} TS \leqslant \min\{\overbrace{\dim \operatorname{range} S}^{5-\dim \operatorname{null} T}, \overbrace{\dim \operatorname{range} T}^{5-\dim \operatorname{null} S}\}$.
We show that $\dim \operatorname{range} TS \leqslant 2$ by contradiction. Assume that $\dim \operatorname{range} TS \geqslant 3$.
Then $\min\{5 - \dim \operatorname{null} T, 5 - \dim \operatorname{null} S\} \geq 3 \Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant 2$.
又 $\dim \operatorname{null} ST = 5 \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \leqslant 4$. Contradicts.

OR. $$\left.\begin{aligned} \dim \operatorname{null} S = 5 - \dim \operatorname{range} S \\ \dim \operatorname{range} TS \leqslant \dim \operatorname{range} S \end{aligned}\right\} \Rightarrow \dim \operatorname{null} S \leqslant 5 - \dim \operatorname{range} TS.$$

And $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S$. □

(b) By Problem (23), $\dim \operatorname{range} TS \leqslant \min\{\overbrace{\dim \operatorname{range} S}^{n-\dim \operatorname{null} T}, \overbrace{\dim \operatorname{range} T}^{n-\dim \operatorname{null} S}\}$. We prove by contradiction.

Assume that $\dim \operatorname{range} TS \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Then $\min\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\} \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$

$\Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant n - \left\lfloor \frac{n}{2} \right\rfloor - 1$.

又 $\dim \operatorname{null} ST = n \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \leqslant 2(n - \left\lfloor \frac{n}{2} \right\rfloor - 1)$

$\Rightarrow \left\lfloor \frac{n}{2} \right\rfloor + 1 \leqslant \frac{n}{2}$. Contradicts. Thus $\dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$. □

OR. $\dim \operatorname{null} S = n - \dim \operatorname{range} S \leqslant n - \dim \operatorname{range} TS$.

And $ST = 0 \Rightarrow \dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S \leqslant n - \dim \operatorname{range} TS$

$\Rightarrow 2 \dim \operatorname{range} TS \leqslant n \Rightarrow \dim \operatorname{range} TS \leqslant \frac{n}{2}$

$\Rightarrow \dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$ (because $\dim \operatorname{range} TS$ is an integer). □

(c) Let $(v_1, \ldots, v_5)$ be a basis of $\mathbf{F}^5$. Define $S, T \in \mathcal{L}(\mathbf{F}^5)$ by:

$$\begin{aligned} T: &\quad v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i \ ; \\ S: &\quad v_1 \mapsto v_4, \quad v_2 \mapsto v_5, \quad v_i \mapsto 0 \ ; \qquad i = 3, 4, 5. \end{aligned}$$ □

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26 *Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ such that $\forall p \in \mathcal{P}(\mathbf{R}), \deg(Dp) = (\deg p) - 1$. Prove that $D \in \mathcal{P}(\mathbf{R})$ is surj.*

SOLUTION:

[*Informal Proof*] | Note that $\deg Dx^n = n - 1$. Because $\operatorname{span}(Dx, Dx^2, \ldots) \subseteq \operatorname{range} D$.
又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, \ldots) = \operatorname{span}(1, x, \ldots) = \mathcal{P}(\mathbf{R})$.

[*Proper Proof*]

We will recursively define a sequence of polys $(p_k)_{k=0}^{\infty}$ where $Dp_k = x^k$.

(i) Because $\dim Dx = (\deg x) - 1 = 0$, we have $Dx = C \in \mathbf{F}$.
Define $p_0 = C^{-1}x$. Then $Dp_0 = C^{-1}Dx = 1$.

(ii) Suppose we have defined $p_0, \ldots, p_n$ such that $Dp_k = x^k$ for each $k \in \{0, \ldots, n\}$.
Because $\deg D(x^{n+2}) = n + 1$. Let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$, where $a_{n+1} \neq 0$.
Then $a_{n+1}^{-1} D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_n Dp_n + \cdots + a_1 Dp_1 + a_0 Dp_0)$
$\Rightarrow x^{n+1} = D\big(a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)\big)$.
Thus defining $p_{n+1} = a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)$, we have $Dp_{n+1} = x^{n+1}$.

Now we get $(p_k)_{k=0}^{\infty}$ by recursion. Hence $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), \exists\, q = \big(\sum_{k=0}^{\deg p} a_k p_k\big), Dq = p$. □

OR. Let $Dx^0 = 0, Dx^k = p_k$ for all $k \in \mathbf{N}^+$. For any $m \in \mathbf{N}^+$, $(p_1, \ldots, p_m)$ is a basis of $\mathcal{P}_{m-1}(\mathbf{R})$.
Because $\forall p' \in \operatorname{range} D, \exists!\, m \in \mathbf{N}, \deg p = m - 1 \Rightarrow \exists!\, a_k \in \mathbf{R}, p' = a_m p_m + \cdots + a_1 p_1$.
Now $Dp = p' = a_m p_m + \cdots + a_1 p_1 = D(a_m x^m + \cdots + a_1 x)$. Thus $\exists\, q \in \mathcal{P}_m(\mathbf{R}), Dq = p$. □

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27 *Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.*

SOLUTION:

Define $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$.
Note that $\deg Bx^n = n - 1$. Similar to Problem (26), we conclude that B is surj. □

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28 *Suppose* $T \in \mathcal{L}(V, W), B_{\operatorname{range} T} = (w_1, \dots, w_m)$.
Prove that $\exists\, \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ *such that* $\forall v \in V, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.

Solution:

Suppose $v_1, \dots, v_m \in V$ such that $Tv_i = w_i$ for each v_i. Then $(v_1, \dots, v_m)$ is linely inde.
Let $B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$. Note that $\forall v \in V, v = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i, \exists\,!\, a_i, b_i \in \mathbf{F}$.
Define $\varphi_i : V \to \mathbf{F}$ by $\varphi_i(v) = a_i v_i$ for each i. We now check the linearity.
$\forall v, u \in V\ (\exists\,!\, a_i, b_i, c_i, d_i \in \mathbf{F}\,), \lambda \in \mathbf{F}, \varphi_i(v + \lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda\varphi(u)$. □

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29 *Suppose* $\varphi \in \mathcal{L}(V, \mathbf{F})$. *Suppose* $u \in V \backslash \operatorname{null} \varphi$. *Prove that* $V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}$.

Solution: If $\varphi = 0$ then we are done. Suppose $\varphi \neq 0$.

(a) $\forall\, v = cu \in \operatorname{null} \varphi \cap \{au : a \in \mathbf{F}\}, \varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$. Hence $\operatorname{null} \varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$.

(b) $$\forall\, v \in V, v = \left(v - \frac{\varphi(v)}{\varphi(u)}u\right) + \frac{\varphi(v)}{\varphi(u)}u. \left| \begin{array}{l} v - \dfrac{\varphi(v)}{\varphi(u)}u \in \operatorname{null} \varphi \\ \dfrac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array} \right\} \Rightarrow V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}.$$ □

Comment: $\varphi \neq 0 \Rightarrow \varphi(v_i) = a_i \neq 0$ for each v_i, for some linely inde list $(v_1, \dots, v_k)$.
Fix one v_k. Then $\forall j \in \{1, \dots, k-1, k+1, \dots, n\}, \operatorname{span}\{a_j v_k - a_k v_j\} \subseteq \operatorname{null} \varphi$.
Hence every vecsp in $\mathcal{S}_V \operatorname{null} \varphi$ is one-dim.

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30 *Suppose* $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$ *and* $\operatorname{null} \varphi_1 = \operatorname{null} \varphi_2 = \operatorname{null} \varphi$. *Prove that* $\exists\, c \in \mathbf{F}, \varphi_1 = c\varphi_2$

Solution:

If $\operatorname{null} \varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $u \in V \backslash \operatorname{null} \varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$.
By Problem (29), $V = \operatorname{null} \varphi \oplus \operatorname{span}(u)$. Hence for any $v \in V, v = w + a_v u, \exists\,!\, w \in \operatorname{null} \varphi, a_v \in \mathbf{F}$.

$$\varphi_1(v) = a_v \varphi_1(u),\ \ \varphi_2(v) = a_v \varphi_2(u) \Rightarrow a_v = \frac{\varphi_1(v)}{\varphi_1(u)} = \frac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \frac{\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}.$$ □

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31 *Prove that* $\exists\, T_1, T_2 \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^2), \operatorname{null} T_1 = \operatorname{null} T_2$ *and* $T_1 \neq cT_2, \forall c \in \mathbf{F}$.

Solution:

Let $(v_1, \dots, v_5)$ be a basis of $\mathbf{R}^5$, (w_1, w_2) be a basis of $\mathbf{R}^2$. Define $T, S \in \mathcal{L}(V, W)$ by

$$\left.\begin{array}{lll} Tv_1 = w_1, & Tv_2 = w_2, & Tv_3 = Tv_4 = Tv_5 = 0 \\ Sv_1 = w_1, & Sv_2 = 2w_2, & Sv_3 = Sv_4 = Sv_5 = 0 \end{array}\right\} \Rightarrow \operatorname{null} T = \operatorname{null} S.$$

Suppose $T = \lambda S$. Then $w_1 = Tv_1 = \lambda Sv_1 = \lambda w_1 \Rightarrow \lambda = 1$.
While $w_2 = Tv_2 = \lambda Sv_2 = 2\lambda w_2 \Rightarrow \lambda = \frac{1}{2}$. Contradicts. □

- **Tips:** *Suppose* $T \in \mathcal{L}(V, W)$ *and* U *is a subsp such that* $V = U \oplus \operatorname{null} T$.
Now $\forall v \in V, \exists\,!\, u_v \in U, w_v \in \operatorname{null} T, v = u_v + w_v$.
Then $T = T \circ i$, *where* $i : V \to U$ *is defined by* $i(v) = u_v$.
Because $\forall v \in V, T(v) = T(u_v + w_v) = T(u_v) = T(i(v)) = (T \circ i)(v)$. □

Ended

• **Note For [3.47]:** $LHS = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k} = RHS.$

• **Note For [3.48]:**

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6&7\\8&9&10\end{pmatrix} = \begin{pmatrix}\begin{pmatrix}1&2\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} & \begin{pmatrix}1&2\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & \begin{pmatrix}1&2\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix}\\ \begin{pmatrix}3&4\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} & \begin{pmatrix}3&4\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & \begin{pmatrix}3&4\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}1\times5+2\times8 & 1\times6+2\times9 & 1\times7+2\times10\\ 3\times5+4\times8 & 3\times6+4\times9 & 3\times7+4\times10\end{pmatrix} = \begin{pmatrix}21&24&27\\47&54&61\end{pmatrix}$$

• [4E 3.51] *Suppose* $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}$.

(a) For $k = 1, \dots, p$, $\quad (CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^{c} C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$

Which means that each cols CR is a linear combination of the cols of C.

(b) For $j = 1, \dots, m$, $\quad (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

Which means that each rows CR is a linear combination of the rows of R.

• **Column-Row Factorization** (CR Factorization) *Suppose* $A \in \mathbf{F}^{m,n}, A \neq 0$.

(a) *Let* $S_c = \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}, \dim S_c = c$, *the col rank.*

Prove that $\exists\, C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,n}, A = CR$.

(b) *Let* $S_r = \operatorname{span}(A_{1,\cdot}, \dots, A_{m,\cdot}) \subseteq \mathbf{F}^{1,n}, \dim S_r = r$, *the row rank.*

Prove that $\exists\, C \in \mathbf{F}^{m,r}, R \in \mathbf{F}^{r,n}, A = CR$.

Solution: Notice that $A \neq 0 \Rightarrow c, r \geqslant 1$.

(a) Let $(C_{\cdot,1}, \dots, C_{\cdot,c})$ be a basis of S_c, forming $C \in \mathbf{F}^{m,c}$. Then $\forall k \in \{1, \dots, n\}$,

$A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}, \exists!\, R_{1,k}, \dots, R_{c,k} \in \mathbf{F}$, forming $R \in \mathbf{F}^{c,n}$. Thus $A = CR$.

(b) Let $(R_{1,\cdot}, \dots, R_{r,\cdot})$ be a basis of S_r, forming $R \in \mathbf{F}^{r,n}$. Then $\forall j \in \{1, \dots, m\}$,

$A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,r}R_{r,\cdot} = (CR)_{j,\cdot}, \exists!\, C_{j,1}, \dots, C_{j,r} \in \mathbf{F}$, forming $C \in \mathbf{F}^{m,r}$. Thus $A = CR$. □

Example:

$$A = \begin{pmatrix}10&7&4&1\\26&19&12&5\\46&33&20&7\end{pmatrix} \overset{(1)}{=} \begin{pmatrix}1&0\\0&1\\2&1\end{pmatrix}\begin{pmatrix}10&7&4&1\\26&19&12&5\end{pmatrix} \overset{(2)}{=} \begin{pmatrix}7&4\\19&12\\33&20\end{pmatrix}\begin{pmatrix}2&-1\\1&0\\0&1\\-1&2\end{pmatrix}$$

(I) $\begin{pmatrix}46&33&20&7\end{pmatrix} = 2\begin{pmatrix}10&7&4&1\end{pmatrix} + \begin{pmatrix}26&19&12&5\end{pmatrix}$.

$\begin{pmatrix}46&33&20&7\end{pmatrix}$ can be uniquely written as a linear combination of $(A_{1,\cdot}, A_{2,\cdot})$.

Hence $\dim S_r = 2$. $(A_{1,\cdot}, A_{2,\cdot})$ is a basis.

(II) $\begin{pmatrix}10\\26\\46\end{pmatrix} = 2\begin{pmatrix}7\\19\\33\end{pmatrix} - \begin{pmatrix}4\\12\\20\end{pmatrix}$; $\begin{pmatrix}1\\5\\7\end{pmatrix} = 2\begin{pmatrix}4\\12\\20\end{pmatrix} - \begin{pmatrix}7\\19\\33\end{pmatrix}$. Hence $\dim S_c = 2$. $(A_{\cdot,2}, A_{\cdot,3})$ is a basis.

- COLUMN RANK EQUALS ROW RANK (Using the notation and result above)

  For each $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$.

  For each $A_{\cdot,k} \in S_c$, $A_{\cdot,k} = (CR)_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$.

  $\Rightarrow \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = S_r = \operatorname{span}(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leqslant c = \dim S_c$.

  $\Rightarrow \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,m}) = S_r = \operatorname{span}(C_{\cdot,1}, \dots, C_{\cdot,r}) \Rightarrow \dim S_c = c \leqslant r = \dim S_r$.

  OR. Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leqslant r = \dim S_r = \dim S_c^t$. $\square$

- [4E 3.C.17, OR 3.F.32] *Suppose $T \in \mathcal{L}(V)$ and $(u_1, \dots, u_n), (v_1, \dots, v_n)$ are bases of V. Prove that the following are equi.* *Here $A = \mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.*

  (a) *T is inje.*

  (b) *The cols of $\mathcal{M}(T)$ are linely inde in $\mathbf{F}^{n,1}$.*

  (c) *The cols of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.*

  (d) *The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.*

  (e) *The rows of $\mathcal{M}(T)$ are linely inde in $\mathbf{F}^{1,n}$.*

SOLUTION: Using TIPS in 2.C.

T is inje $\iff \dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim \operatorname{range} T$

$$\Delta \begin{cases} \iff (Tu_1, \dots, Tu_n) \text{ is a basis of } V; \ \dim \operatorname{range} T = \dim \operatorname{span}(\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n)) = n \\ \iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n)) \text{ is a basis of } \mathbf{F}^{n,1}, \text{ as well as } (A_{\cdot,1}, \dots, A_{\cdot,n}) \end{cases}$$

$\left[\text{又} \dim S_c = \dim \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) = \dim \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = \dim S_r = n \right]$

$\iff (A_{1,\cdot}, \dots, A_{n,\cdot})$ is a basis of $\mathbf{F}^{1,n}$. $\square$

Now we show (Δ) properly, that is T is inje $\iff$ The cols of $\mathcal{M}(T)$ are linely inde.

$(a) \Rightarrow (b)$:

Suppose $b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n} = \begin{pmatrix} b_1 A_{1,1} + \cdots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \cdots + b_n A_{n,n} \end{pmatrix} = 0$. Let $u = b_1 u_1 + \cdots + b_n u_n$.

$$\begin{aligned} \text{Then } Tu &= b_1 Tu_1 + \cdots + b_n Tu_n \\ &= b_1(A_{1,1}v_1 + \cdots + A_{n,1}v_n) + \cdots + b_n(A_{1,n}v_1 + \cdots + A_{n,n}v_n) \\ &= (b_1 A_{1,1} + \cdots + b_n A_{1,n})v_1 + \cdots + (b_1 A_{n,1} + \cdots + b_n A_{n,n})v_n \\ &= 0v_1 + \cdots + 0v_n = 0 \\ &\Rightarrow b_1 = \cdots = b_n = 0. \end{aligned}$$

Thus by (2.39), (b) holds.

$(b) \Rightarrow (a)$:

Suppose $u = b_1 u_1 + \cdots + b_n u_n \in \operatorname{null} T$.

Then $Tu = 0 = (b_1 A_{1,1} + \cdots + b_n A_{1,n})v_1 + \cdots + (b_1 A_{n,1} + \cdots + b_n A_{n,n})v_n$.

Thus $b_1 A_{1,1} + \cdots + b_n A_{1,n} = \cdots = b_1 A_{n,1} + \cdots + b_n A_{n,n} = 0$.

Which is equi to $\begin{pmatrix} b_1 A_{1,1} + \cdots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \cdots + b_n A_{n,n} \end{pmatrix} = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n} = 0 \Rightarrow b_1 = \cdots = b_n = 0$.

Thus by (2.39), (a) holds. $\square$

• $[$4E 3.C.16, OR 3.E.11$]$ *Suppose A is an m-by-n matrix with $A \neq 0$.*
Prove that $\operatorname{rank} A = 1 \iff \exists\, (c_1, \dots, c_m) \in \mathbf{F}^m, (d_1, \dots, d_n) \in \mathbf{F}^n$
such that $A_{j,k} = c_j \cdot d_k$ *for every* $j = 1, \dots, m$ *and* $k = 1, \dots, n$.

SOLUTION:

Using the notation in CR Factorization.

(a) Suppose $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix}$. $(\ \exists\, c_j, d_k \in \mathbf{F}, \forall j, k\ )$

Then $S_c = \left\{ \begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \dots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right\} = \operatorname{span} \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$.

OR. $S_r = \operatorname{span} \left\{ \begin{matrix} \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \end{pmatrix}, \\ \begin{pmatrix} c_2 d_1 & \cdots & c_2 d_n \end{pmatrix}, \\ \vdots \\ \begin{pmatrix} c_m d_1 & \cdots & c_m d_n \end{pmatrix} \end{matrix} \right\} = \operatorname{span} \left\{ \begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} \right\}$. Hence $\operatorname{rank} A = 1$.

OR. Using also the result in $[$4E 3.51(a)$]$.

Every col of A is a scalar multi of C. Then $\operatorname{rank} A \leqslant 1$ 又 $\operatorname{rank} A \geqslant 1$ ($A \neq 0$).

(b) By CR Factorization, $\exists\, C = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbf{F}^{m,1}, R = \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} \in \mathbf{F}^{1,n}$ such that $A = CR$.

OR. Not using CR Factorization. Suppose $\operatorname{rank} A = \dim S_c = \dim S_r = 1$.

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}, \qquad d'_k = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}. \quad (\ \forall j, k\ )$

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$. Letting $d_k = d'_k A_{1,1}$. $\square$

---

1 *Suppose $T \in \mathcal{L}(V, W)$. Show that with resp to each choice of bases of V and W, the matrix of T has at least* $\dim \operatorname{range} T$ *nonzero entries.*

SOLUTION:

Let $B_{\operatorname{null} T} = (v_1, \dots, v_p), B_V = (v_1, \dots, v_n)$. Let $B_W = (w_1, \dots, w_m)$. Denote $\mathcal{M}(T, B_V, B_W)$ by A.

Because at most p of the v_k's can belong to $\operatorname{null} T \iff$ at least $n - p = q$ of the v_k's do not.

For $v_k \notin \operatorname{null} T, Tv_k = A_{1,k} w_1 + \cdots + A_{m,k} w_m \neq 0$. Thus col k has at least one nonzero entry.

Since there are $(n - p) = q$ choices of such k, A has at least $q = \dim \operatorname{range} T$ nonzero entries. $\square$

OR. We prove by contradiction.

Suppose A has at most ($\dim \operatorname{range} T - 1$) nonzero entries.

Then by Pigeon Hole Principle, at least one of $A_{\cdot,p+1}, \dots, A_{\cdot,n}$ equals 0.

Thus there are at most ($\dim \operatorname{range} T - 1$) nonzero vecs in $Tv_{p+1}, \dots, Tv_n$.

While $\operatorname{range} T = \operatorname{span}(Tv_{p+1}, \dots, Tv_n) \Rightarrow \dim \operatorname{range} T = \dim \operatorname{span}(Tv_{p+1}, \dots, Tv_n)$. Contradicts. $\square$

---

3 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$. Prove that $\exists\, B_V, B_W$ such that* $[$ *letting* $A = \mathcal{M}(T, B_V, B_W)$ $]$ $A_{k,k} = 1, A_{i,j} = 0$, *where* $1 \leqslant k \leqslant \dim \operatorname{range} T, i \neq j$.

SOLUTION:

Let $R = (Tv_1, \dots, Tv_n)$ be a basis of range T, extend to $B_W = (Tv_1, \dots, Tv_n, w_1, \dots, w_p)$.
Let $\mathcal{K}_R = \operatorname{span}(v_1, \dots, v_n)$. Let $(u_1, \dots, u_m)$ be a basis of null T. Then $B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$. $\square$

---

4 *Suppose $B_V = (v_1, \dots, v_m)$ and W is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that $\exists\, B_W = (w_1, \dots, w_n)$, $\mathcal{M}(T, B_V, B_W)^t_{\cdot,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$.

SOLUTION: If $Tv_1 = 0$, then we are done. If not then extend (Tv_1). $\square$

---

5 *Suppose $B_W = (w_1, \dots, w_n)$ and V is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that $\exists\, B_V = (v_1, \dots, v_m)$, $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$.

SOLUTION:

Let $(u_1, \dots, u_n)$ be a basis of V. Denote $\mathcal{M}\big(T, (u_1, \dots, u_n), B_W\big)$ by A.
If $A_{1,\cdot} = 0$, then let $B_V = (u_1, \dots, u_n)$, we are done.
Otherwise, $\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \end{pmatrix} \neq 0$, choose one $A_{1,k} \neq 0$.

$$\text{Let } v_1 = \frac{u_k}{A_{1,k}}; \quad \begin{aligned} v_j &= u_{j-1} - A_{1,j-1} v_1 \quad &&\text{for } j = 2, \dots, k; \\ v_i &= u_i - A_{1,i} v_1 &&\text{for } i = k+1, \dots, n. \end{aligned}$$

Now because each $u_k \in \operatorname{span}(v_1, \dots, v_n) \Rightarrow V = \operatorname{span}(v_1, \dots, v_n), B_V = (v_1, \dots, v_n)$.

$$\text{And } Tv_1 = T\Big(\frac{u_k}{A_{1,k}}\Big) = \frac{1}{A_{1,k}}(A_{1,k} w_1 + \cdots + A_{n,k} w_n) = 1 w_1 + \cdots + \frac{A_{n,k}}{A_{1,k}} w_n.$$

$$\forall j \in \{2, \dots, k, \underbrace{k+2, \dots, n+1}_{i \in \{k+1, \dots, n\}}\},\ Tv_j = T(u_{j-1} - A_{1,j-1} v_1) = Tu_{j-1} - T\Big(\frac{A_{1,j-1} u_k}{A_{1,k}}\Big)$$

$$= A_{1,j-1} w_1 + \cdots + A_{n,j-1} w_n - A_{1,j-1}\Big(1 w_1 + \cdots + \frac{A_{n,k}}{A_{1,k}} w_n\Big) = 0 w_1 + \cdots + \Big(A_{n,j-1} - \frac{A_{1,j-1} A_{n,k}}{A_{1,k}}\Big) w_n. \quad \square$$

---

6 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$.*
Prove that $\dim \operatorname{range} T = 1 \iff \exists\, B_V, B_W$, *all entries of $A = \mathcal{M}(T, B_V, B_W)$ equal* 1.

SOLUTION:

(a) Suppose $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$ are the bases such that all entries of A equal 1.

Then $Tv_i = w_1 + \cdots + w_m$ for all $i = 1, \dots, n$. Because $w_1, \dots, w_n$ is linely inde, $w_1 + \cdots + w_n \neq 0$.

(b) Suppose $\dim \operatorname{range} T = 1$. Then $\dim \operatorname{null} T = \dim V - 1$.
Let $(u_2, \dots, u_n)$ be a basis of null T. Extend it to a basis of V as $(u_1, u_2, \dots, u_n)$.
Let $w_1 = Tv_1 - w_2 - \cdots - w_m$. Extend to a basis of W and we have B_W.
Let $v_1 = u_1, v_i = u_1 + u_i$. Extend to a basis of V and we have B_V. $\square$

OR. Suppose range T has a basis (w).
By 2.C.15 [COROLLARY], $\exists\, B_W = (w_1, \dots, w_m)$ such that $w = w_1 + \cdots + w_m$.
By 2.C [NEW THEOREM], $\exists$ a basis $(u_1, \dots, u_n)$ of V such that each $u_k \notin \operatorname{null} T$.
$\forall k \in \{1, \dots, n\}, Tu_k \in \operatorname{range} T = \operatorname{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists\, \lambda_k \in \mathbf{F} \backslash \{0\}$.
Let $v_k = \lambda_k^{-1} u_k \neq 0 \Rightarrow B_V = (v_1, \dots, v_n)$. Hence for each v_k, $Tv_k = w = w_1 + \cdots + w_m$. $\square$

• **NOTE FOR [3.49]:** $\because \left[(AC)_{\cdot,k} \right]_{j,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$

$\therefore (AC)_{\cdot,k} = A_{\cdot,\cdot}C_{\cdot,k} = AC_{\cdot,k}$ □

• **EXERCISE 10:** $\because \left[(AC)_{j,\cdot} \right]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$

$\therefore (AC)_{j,\cdot} = A_{j,\cdot}C_{\cdot,\cdot} = A_{j,\cdot}C.$ □

• **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$

$\because (Ac)_{j,1} = \sum_{r=1}^{n} A_{j,r}c_{r,1} = \left[\sum_{r=1}^{n} (A_{\cdot,r}c_{r,1}) \right]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$

$\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^{n} A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ OR. By $(Ac)_{\cdot,1} = Ac_{\cdot,1}$ Using (a) above. □

• **EXERCISE 11:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$

$\because (aC)_{1,k} = \sum_{r=1}^{n} a_{1,r}C_{r,k} = \left[\sum_{r=1}^{n} a_{1,r}(C_{r,\cdot}) \right]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$

$\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^{n} a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ OR. By $(aC)_{1,\cdot} = a_{1,\cdot}C$. Using (b) above. □

• *Suppose p is a poly of n variables in* $\mathbf{F}$. *Prove that* $\mathcal{M}\big(p(T_1, \dots, T_n)\big) = p\big(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n)\big)$.

Where the linear maps $T_1, \dots, T_n$ *are such that* $p(T_1, \dots, T_n)$ *makes sense. See [5.B.16,17,20].*

SOLUTION:

Suppose the poly p is defined by $p(x_1, \dots, x_n) = \sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} \prod_{i=1}^{n} x_i^{k_i}$.

Note that $\mathcal{M}(T^xS^y) = \mathcal{M}(T)^x\mathcal{M}(S)^y$; $\mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$.

Then $\mathcal{M}\big(p(T_1, \dots, T_n)\big) = \mathcal{M}\Big(\sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} \prod_{i=1}^{n} T_i^{k_i} \Big)$

$= \sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} \prod_{i=1}^{n} \mathcal{M}(T_i^{k_i}) = p\big(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n)\big).$ □

---

13 *Prove that the distr holds for matrix add and matrix multi.*

Suppose A, B, C *are matrices such that* $A(B + C)$ *make sense, we prove the left distr.*

SOLUTION:

Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Note that $\left[A(B + C)\right]_{j,k} = \sum_{r=1}^{n} A_{j,r}(B + C)_{r,k} = \sum_{r=1}^{n} (A_{j,r}B_{r,k} + A_{j,r}C_{r,k}) = (AB + AC)_{j,k}$. □

OR. Define T, S, R such that $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$A(B + C) = \mathcal{M}(T(S + R)) \overset{[3.9]}{=} \mathcal{M}(TS + TR) = AB + AC.$

Or $T(S + R) = TS + TR \Rightarrow \mathcal{M}(T(S + R)) = \mathcal{M}(TS + TR) \Rightarrow A(B + C) = AB + AC.$ □

---

14 *Prove that matrix multi is associ.*

Suppose A, B, C *are matrices such that* $(AB)C$ *makes sense, we prove that* $(AB)C = A(BC)$.

SOLUTION:

Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$. We will show that $LHS = \left[(AB)C\right]_{j,k} = \left[A(BC)\right]_{j,k} = RHS$.

$LHS = (AB)_{j,\cdot}C_{\cdot,k} = \sum_{s=1}^{n} (A_{j,s}B_{s,\cdot})C_{\cdot,k} = \sum_{s=1}^{n} A_{j,s}(B_{s,\cdot}C_{\cdot,k}) = \sum_{s=1}^{n} A_{j,s}(BC)_{s,k} = RHS.$ □

OR. Define T, S, R such that $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$(AB)C = \mathcal{M}(T(SR)) \overset{[3.9]}{=} \mathcal{M}(TSR) \overset{[3.9]}{=} \mathcal{M}((TS)R) = A(BC).$

OR. $(TS)R = T(SR) \Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR)) \Rightarrow (AB)C = A(BC).$ □

---

15 *Suppose $A \in \mathbf{F}^{n,n}, j,k \in \{1,\dots,n\}$. Show that $(A^3)_{j,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.*

SOLUTION: $(AAA)_{j,k} = (AA)_{j,\cdot}A_{\cdot,k} = \sum_{p=1}^{n}(A_{j,p}A_{p,\cdot})A_{\cdot,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.

OR. $(AAA)_{j,k} = \sum_{r=1}^{n}(AA)_{j,r}A_{r,k} = \sum_{r=1}^{n}\left(\sum_{p=1}^{n} A_{j,p}A_{p,r}\right)A_{r,k}$

$= \sum_{r=1}^{n}\left[A_{j,1}(A_{1,r}A_{r,k}) + \cdots + A_{j,n}(A_{n,r}A_{r,k})\right]$

$= A_{j,1}\sum_{r=1}^{n} A_{1,r}A_{r,k} + \cdots + A_{j,n}\sum_{r=1}^{n} A_{n,r}A_{r,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$. □

• *Prove that the commutativity does not hold in $\mathbf{F}^{m,n}$.*

SOLUTION:

Suppose $\dim V = n, \dim W = m$ and the commutativity holds in $\mathbf{F}^{n,m}$.

$\forall T \in \mathcal{L}(V,W), S \in \mathcal{L}(W,V), \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$.

Hence $ST = TS$. Which in general is not true. (See 3.D) □

• [10.A.3, OR 4E 3.D.19] *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that $\forall B_V \neq B_V', \ \mathcal{M}(T,B_V) = \mathcal{M}(T,B_V') \iff T = \lambda\mathcal{M}(I), \exists \lambda \in \mathbf{F}$.

SOLUTION: [Compare with the first solution of (3.D.16) in 3.A]

Suppose $T = \lambda I$ for some $\lambda \in \mathbf{F}$. Then $T = \lambda\mathcal{M}(I)$.

Suppose $\forall B_V \neq B_V', \ \mathcal{M}(T,B_V) = \mathcal{M}(T,B_V')$. If $T = 0$, then we are done.

Suppose $T \neq 0$, and $v \in V\backslash\{0\}$. Assume that (v,Tv) is linely inde.

Extend (v,Tv) to $B_V = (v,Tv,u_3,\dots,u_n)$. Let $B = \mathcal{M}()(T,B_V)$.

$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \ \forall i \neq 2$.

By assumption, $A = \mathcal{M}(T,B_V') = B, \forall B_V' = (v,w_2,\dots,w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$.

$\Rightarrow Tv = w_2$, which is not true if we let $w_2 = u_3, \ w_3 = Tv, \ w_j = u_j, \forall j \in \{4,\dots,n\}$. Contradicts.

Hence (v,Tv) is linely depe $\Rightarrow \forall v \in V, \exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show that λ_v is independent of v, that is, to show that for all $v \neq w \in V\backslash 0, \lambda_v = \lambda_w$.

$$\left.\begin{array}{l}(v,w) \text{ is linely inde} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v,w) \text{ is linely depe, } w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv)\end{array}\right\} \Rightarrow T = \lambda I, \exists \lambda \in \mathbf{F}.$$ □

OR. Conversely, denote $\mathcal{M}(T,B_V)$ by A, where $B_V = (u_1,\dots,u_m)$ is arbitrary.

Fix one $B_V = (v_1,\dots,v_m)$ and then $(v_1,\dots,\frac{1}{2}v_k,\dots,v_m)$ is also a basis for any given $k \in \{1,\dots,m\}$.

Fix one k. Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \cdots + A_{k,k}(\frac{1}{2}v_k) + \cdots + A_{m,k}v_m$

$\Rightarrow Tv_k = 2A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + 2A_{m,k}v_m = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{m,k}v_m$.

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1,\dots,m\}$.

Now we show that $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j,k such that $j \neq k$.

Consider the basis $B_V' = (v'_1,\dots,v'_j,\dots,v'_k,\dots,v'_m)$,

where $v'_j = v_k, \ v_k{}' = v_j$ and $v'_i = v_i$ for all $i \in \{1,\dots,m\} \setminus \{j,k\}$.

Remember that $\mathcal{M}(T,B_V') = \mathcal{M}(T,B_V) = A$.

Hence $T(v'_k) = A_{1,k}v'_1 + \cdots + A_{k,k}v'_k + \cdots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_j$, while $T(v'_k) = T(v_j) = A_{j,j}v_j$.

Thus $A_{k,k} = A_{j,j}$. □

ENDED

3·D 1 2 3 4 5 6 8 9 10 11 12 13 15 16 17 18 19 | 4E: 1 3 10 15 17 19 20 22 23 24

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*

$$\left.\begin{array}{l}(Tv_1,\dots,Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1,\dots,v_n) \text{ of } V \iff T \text{ is surj} \\ (Tv_1,\dots,Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1,\dots,v_n) \text{ of } V \iff T \text{ is inje}\end{array}\right\} \iff T \text{ is inv.}$$

• OR (10.A.1) *Suppose $T \in \mathcal{L}(V), B_V = (v_1,\dots,v_n)$. Prove that $\mathcal{M}(T, B_V)$ is inv $\iff$ T is inv.*

SOLUTION: Notice that $\mathcal{M} \in \mathcal{L}(\mathcal{L}(V), \mathbf{F}^{n,n})$ is an iso.

(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I$. $\exists! S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$
$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$
$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. ☐

• *Suppose $T \in \mathcal{L}(V, W)$ is inv. Show that T^{-1} is inv and $(T^{-1})^{-1} = T$.*

SOLUTION: $\left.\begin{array}{l}TT^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = I \in \mathcal{L}(W)\end{array}\right\} \Rightarrow T = (T^{-1})^{-1}$, by the uniqnes of inverse. ☐

---

1 *Suppose $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$ are inv. Prove that ST is inv and $(ST)^{-1} = T^{-1}S^{-1}$.*

SOLUTION: $\left.\begin{array}{l}(ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(V)\end{array}\right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}$, by the uniqnes of inv. ☐

• *Suppose $T \in \mathcal{L}(V)$ and $V = \operatorname{span}(Tv_1,\dots,Tv_m)$. Prove that $V = \operatorname{span}(v_1,\dots,v_m)$.*

SOLUTION:

Because $V = \operatorname{span}(Tv_1,\dots,Tv_m) \Rightarrow T$ is surj, 又 V is finite-dim $\Rightarrow T$ is inv $\Rightarrow T^{-1}$ is inv.
$\forall v \in V, \exists a_i \in \mathbf{F}, v = a_1Tv_1 + \cdots + a_mTv_m \Rightarrow T^{-1}v = a_1v_1 + \cdots + a_mv_m \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1,\dots,v_m)$.

OR. Reduce the spanning list $(Tv_1,\dots,Tv_m)$ of V to a basis $(Tv_{\alpha_1},\dots,Tv_{\alpha_k})$ of V.
Where $k = \dim V$ and each $\alpha_i \in \{1,\dots,k\}$. Then by Problem (4E 3),
$(v_{\alpha_1},\dots,v_{\alpha_k})$ is also a basis of V, contained in the list $(v_1,\dots,v_m)$. ☐

---

2 *Suppose V is finite-dim and* $\dim V > 1$.
Prove that the set U of non-inv operators on V is not a subsp of $\mathcal{L}(V)$.
The set of inv operators is not either. Although multi identity/inv, and commutativity for vec multi hold.

SOLUTION: Let $B_V = (v_1,\dots,v_n)$. [*If* $\dim V = 1$, *then $U = \{0\}$ is a subsp of $\mathcal{L}(V)$.*]
Define $S, T \in \mathcal{L}(V)$ by $S(a_1v_1 + \cdots + a_nv_n) = a_1v_1$, $T(a_1v_1 + \cdots + a_nv_n) = a_2v_1 + \cdots + a_nv_n$.
Hence $S, T \in U$ while $S + T \notin U$. ☐

---

3 *Suppose V is finite-dim, U is a subsp of V, and $S \in \mathcal{L}(U, V)$.*
Prove that $\exists$ inv $T \in \mathcal{L}(V), Tu = Su, \forall u \in U \iff S$ is inje. [*Compare this with (3.A.11).*]

SOLUTION:

(a) $\forall u \in U, u = T^{-1}Su \Longrightarrow S$ is inje. OR. $\operatorname{null} S = \operatorname{null} T \cap U = \{0\} \cap U = \{0\}$.

(b) Let $(u_1,\dots,u_m)$ be a basis of U. Then S inje $\Longrightarrow (Su_1,\dots,Su_m)$ linely inde.
Extend these to bases of V as $(u_1,\dots,u_m,v_1,\dots,v_n)$ and $(Su_1,\dots,Su_m,w_1,\dots,w_n)$.
Define $T \in \mathcal{L}(V)$ by $T(u_i) = Su_i$; $Tv_j = w_j$, for each $i \in \{1,\dots,m\}, j \in \{1,\dots,n\}$. ☐

---

4 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{null} S = \operatorname{null} T (= U) \Longleftrightarrow S = ET$, $\exists$ *inv* $E \in \mathcal{L}(W)$.

SOLUTION:

Define $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i$, $E(w_j) = x_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m)$, extend to $B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n)$.
Let $\mathcal{K} = \operatorname{span}(v_1, \dots, v_m)$. 又 $\operatorname{null} S = \operatorname{null} T \Longrightarrow V = \mathcal{K} \oplus \operatorname{null} S \Leftrightarrow \mathcal{K} \in \mathcal{S}_V \operatorname{null} S$.
$\Rightarrow \operatorname{span}(Sv_1, \dots, Sv_m) = \operatorname{range} S$ 又 $\dim \operatorname{range} T = \dim \operatorname{range} S = m$.
Hence $B_{\operatorname{range} S} = (Sv_1, \dots, Sv_m)$. Thus we let $B'_W = (Sv_1, \dots, Sv_m, x_1, \dots, x_n)$.

$\therefore E$ is inv and $S = ET$.

Conversely, $S = ET \Rightarrow \operatorname{null} S = \operatorname{null} ET$.
Then $v \in \operatorname{null} ET \Longleftrightarrow ET(v) = 0 \Longleftrightarrow Tv = 0 \Longleftrightarrow v \in \operatorname{null} T$. Hence $\operatorname{null} ET = \operatorname{null} T = \operatorname{null} S$. □

---

5 *Suppose that V is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{range} S = \operatorname{range} T (= R) \Longleftrightarrow S = TE$, $\exists$ *inv* $E \in \mathcal{L}(V)$.

SOLUTION:

Define $E \in \mathcal{L}(V)$ as $E: \ v_i \mapsto r_i \ ; \quad u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $B_R = (Tv_1, \dots, Tv_m);\ B'_R = (Sr_1, \dots, Sr_m)$ such that $\forall\, i, Tv_i = Sr_i$.
Let $B_{\operatorname{null} T} = (u_1, \dots, u_n);\ B_{\operatorname{null} S} = (s_1, \dots, s_n)$.
Thus $B_V = (v_1, \dots, v_m, u_1, \dots, u_n);\ B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)$.

$\therefore E$ is inv and $S = TE$.

Conversely, $S = TE \Rightarrow \operatorname{range} S = \operatorname{range} TE$.
Then $w \in \operatorname{range} S \Longleftrightarrow \exists\, v \in V, Sv = TE(v) = T\big(E(v)\big) = w \in \operatorname{range} T$. Hence $\operatorname{range} S = \operatorname{range} T$. □

---

6 *Suppose V and W are finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $S = E_2 T E_1, \exists$ inv $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$ $\Longleftrightarrow \dim \operatorname{null} S = \dim \operatorname{null} T = n$.

SOLUTION:

Define $E_1: \quad v_i \mapsto r_i \ ; \quad u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.
Define $E_2: Tv_i \mapsto Sr_i \ ; \ x_j \mapsto y_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m);\ B_{\operatorname{range} S} = (Sr_1, \dots, Sr_m)$.
Extend to $B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p);\ B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p)$.
Let $B_{\operatorname{null} T} = (u_1, \dots, u_n);\ B_{\operatorname{null} S} = (s_1, \dots, s_n)$.
Thus $B_V = (v_1, \dots, v_m, u_1, \dots, u_n);\ B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)$.

$\therefore E_1, E_2$ are inv and $S = E_2 T E_1$.

Conversely, $S = E_2 T E_1 \Rightarrow \dim \operatorname{null} S = \dim \operatorname{null} E_2 T E_1$.
$v \in \operatorname{null} E_2 T E_1 \Longleftrightarrow E_2 T E_1(v) = 0 \Longleftrightarrow TE_1(v) = 0$. Hence $\operatorname{null} E_2 T E_1 = \operatorname{null} TE_1 = \operatorname{null} S$.
又 By (3.B.22.COROLLARY), E is inv $\Rightarrow \dim \operatorname{null} TE_1 = \dim \operatorname{null} T = \dim \operatorname{null} S$. □

---

8 *Suppose V is finite-dim and $T: V \to W$ is a **surj** linear map of V onto W.*
Prove that there is a subsp U of V such that $T|_U$ is an iso of U onto W.

SOLUTION:

Let $B_{\operatorname{range} T} = B_W = (w_1, \dots, w_m) \Rightarrow \forall w_i, \exists\,!\, v_i \in V, Tv_i = w_i$. Let $B_{\mathcal{K}} = (v_1, \dots, v_m)$.
Then $\dim \mathcal{K} = \dim W$. Thus $T|_{\mathcal{K}}$ is an iso of $\mathcal{K}$ onto W. □

OR. By (3.B.12), there is a subsp U of V such that
$U \cap \operatorname{null} T = \{0\} = \operatorname{null} T|_U, \operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$. □

---

9 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that ST is inv $\iff$ S and T are inv.*

SOLUTION:

Suppose S, T are inv. Then $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$. Hence ST is inv.

Suppose ST is inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T \text{ is inje, } S \text{ is surj. While } V \text{ is finite-dim.}$$ □

OR. Because by (3.B.23), $\dim V = \dim \operatorname{range} ST \leqslant \min\{\operatorname{range} T, \operatorname{range} S\}$. □

---

10 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that $ST = I \iff TS = I$.*

SOLUTION:

Suppose $ST = I$. $\left.\begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T$ is inje, S is surj. While V is finite-dim.

OR. By Problem (9), V is finite-dim and $ST = I$ is inv $\Rightarrow S, T$ are inv.

$S\big((TS)v\big) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow S$ is inv.

OR. $ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T$. 又 $S = S \Rightarrow TS = S^{-1}S = I$.

Reversing the roles of S and T, we conclude that $TS = I \Rightarrow ST = I$. □

---

11 *Suppose V is finite-dim, $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is inv and $T^{-1} = US$.*

SOLUTION: Using Problem (9) and (10). *This result can fail without the hypothesis that V is finite-dim.*

$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I$.

$\Rightarrow U^{-1} = ST, \qquad T^{-1} = US, \qquad S^{-1} = TU.$ □

EXAMPLE: $V = \mathbf{R}^{\infty}, S(a_1, a_2, \dots) = (a_2, \dots);\ T(a_1, \dots) = (0, a_1, \dots);\ U = I \Rightarrow STU = I$ but T is not inv.

---

13 *Suppose V is finite-dim, $R, S, T \in \mathcal{L}(V)$ are such that RST is surj. Prove that S is inje.*

SOLUTION: By Problem (1) and (9), Notice that V is finite-dim. Then RST is inv.

Let $X = (RST)^{-1}$ $\left|\begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ is surj.} \end{array}\right\} \Rightarrow S = R^{-1}(RST)T^{-1}$ is inv. □

OR. $(RST)^{-1} = \big((RS)T\big)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. □

---

15 *Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multi.*
In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists\, A \in \mathbf{F}^{m,n}, Tx = Ax,\ \forall x \in \mathbf{F}^{n,1}$.

SOLUTION:

Let $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$ be the standard bases of $\mathbf{F}^{n,1}, \mathbf{F}^{m,1}$.

$\forall k = 1, \dots, n$, suppose $T(E_k) = A_{1,k}R_1 + \dots + A_{m,k}R_m, \exists\, A_{j,k} \in \mathbf{F}$, forming $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$. □

OR. Let $A = \mathcal{M}(T, B_1, B_2)$. Note that $\mathcal{M}(x, B_1) = x, \mathcal{M}(y, B_2) = y$.

Hence $Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax$, by [3.65]. □

• OR (10.A.2) *Suppose $A, B \in \mathbf{F}^{n,n}$. Prove that $AB = I \iff BA = I$.*

SOLUTION: Using Problem (10) and (15).

Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Then $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.

Thus $AB = I \Leftrightarrow A(Bx) = x \iff T(Sx) = x \Leftrightarrow TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I$. □

• **NOTE FOR [3.60]:** Suppose $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.

Define $E_{i,j} \in \mathcal{L}(V,W)$ by $E_{i,j}(v_x) = \delta_{i,x} w_j$; See (3.A.12). **COROLLARY:** $E_{l,k}E_{i,j} = \delta_{j,l}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. And $\left(\mathcal{E}^{(j,i)}\right)_{l,k} = \begin{cases} 0, & i \neq k \vee j \neq l \\ 1, & i = k \wedge j = l \end{cases}$

Because $\mathcal{L}(V,W)$ and $\mathbf{F}^{m,n}$ are iso. And $T = \mathcal{M}^{-1}\mathcal{M}(T)$; $E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

Hence $\forall T \in \mathcal{L}(V,W), \exists !\, A_{i,j} \in \mathbf{F} \left(\forall i \in \{1, \dots, m\}, j \in \{1, \dots, n\} \right), \mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)}+ & \cdots & +A_{1,n}\mathcal{E}^{(1,n)} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}\mathcal{E}^{(m,1)}+ & \cdots & +A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff \begin{pmatrix} A_{1,1}E_{1,1}+ & \cdots & +A_{1,n}E_{n,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}E_{1,m}+ & \cdots & +A_{m,n}E_{n,m} \end{pmatrix} = T.$$

$$\therefore \mathcal{L}(V,W) = \operatorname{span} \underbrace{\begin{pmatrix} E_{1,1}, & \cdots & ,E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,m}, & \cdots & ,E_{n,m} \end{pmatrix}}_{B}; \quad \mathbf{F}^{m,n} = \operatorname{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & ,\mathcal{E}^{(1,n)}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ \mathcal{E}^{(m,1)}, & \cdots & ,\mathcal{E}^{(m,n)} \end{pmatrix}}_{B_{\mathcal{M}}}.$$

Hence by [2.42] and [3.61], we conclude that B is a basis of $\mathcal{L}(V,W)$ and that $B_{\mathcal{M}}$ is a basis of $\mathbf{F}^{m,n}$.

• *Suppose V, W are finite-dim, U is a subsp of V.*
Let $\mathcal{E} = \{T \in \mathcal{L}(V,W) : U \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V,W) : T|_U = 0\}$.
(a) *Show that $\mathcal{E}$ is a subsp of $\mathcal{L}(V,W)$.*
(b) *Find a formula for* $\dim \mathcal{E}$ *in terms of* $\dim V$, $\dim W$ *and* $\dim U$.
Hint: *Define $\Phi : \mathcal{L}(V,W) \to \mathcal{L}(U,W)$ by $\Phi(T) = T|_U$. What is* null Φ*? What is* range Φ*?*

SOLUTION:

(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.

(b) Define Φ as in the hint.

Because $T \in \operatorname{null} \Phi \iff \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$.

Hence $\operatorname{null} \Phi = \mathcal{E}$.

Because $S \in \mathcal{L}(U,W) \Rightarrow \exists\, T \in \mathcal{L}(V,W), \Phi(T) = S$, by (3.A.11) $\Rightarrow S \in \operatorname{range} T$.

Hence $\operatorname{range} \Phi = \mathcal{L}(U,W)$.

Thus $\dim \operatorname{null} \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V,W) - \dim \operatorname{range} \Phi = (\dim V - \dim U) \dim W$. □

OR. Extend $(u_1, \dots, u_m)$ a basis of U to $(u_1, \dots, u_m, v_1, \dots, v_n)$ a basis of V. Let $p = \dim W$.
(See NOTE FOR [3.60])

$$\forall\, T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \operatorname{span} \underbrace{\begin{Bmatrix} E_{1,1}, & \cdots & ,E_{m,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,p}, & \cdots & ,E_{m,p} \end{Bmatrix}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

$$\text{又 } W = \operatorname{span} \begin{Bmatrix} E_{m+1,1}, & \cdots & ,E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{m+1,p}, & \cdots & ,E_{n,p} \end{Bmatrix} \subseteq \mathcal{E}. \text{ Where } \mathcal{L}(V,W) = R \oplus W \Rightarrow \mathcal{L}(V,W) = R + \mathcal{E}.$$

Then $\dim \mathcal{E} = \dim \mathcal{L}(V,W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. □

∘ *Suppose V is finite-dim and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(T) = ST$.*

(a) *Show that* $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.

(b) *Show that* $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

SOLUTION:

(a) $\forall T \in \mathcal{L}(V), ST = 0 \Longleftrightarrow \operatorname{range} T \subseteq \operatorname{null} S$.

Thus $\operatorname{null} \mathcal{A} = \big\{ T \in \mathcal{L}(V) : \operatorname{range} T \subseteq \operatorname{null} S \big\} = \mathcal{L}(V, \operatorname{null} S)$.

(b) $\forall R \in \mathcal{L}(V), \operatorname{range} R \subseteq \operatorname{range} S \Longleftrightarrow \exists\, T \in \mathcal{L}(V), R = ST$, by (3.B 25).

Thus $\operatorname{range} \mathcal{A} = \big\{ R \in \mathcal{L}(V) : \operatorname{range} R \subseteq \operatorname{range} S \big\} = \mathcal{L}(V, \operatorname{range} S)$.

□

OR. Using NOTE FOR [3.60].

Let $B_{\operatorname{range} S} = (\underbrace{w_1, \dots, w_m}_{Sv_i = w_i}), B_{\mathcal{K}} = (v_1, \dots, v_m); (w_1, \dots, w_n), (v_1, \dots, v_n)$ are bases of V.

Define $E_{i,j} \in \mathcal{L}(V)$ by $E_{i,j}(v_x) = \delta_{i,x} w_i$.

$$\text{Thus } S = E_{1,1} + \cdots + E_{m,m}; \quad \mathcal{M}\big(S, (v_1, \dots, v_n), (w_1, \dots, w_n)\big) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j}(w_x) = \delta_{i,x} v_i$.

Let $E_{j,k} R_{i,j} = Q_{i,k}, \quad R_{j,k} E_{i,j} = G_{i,k}$.

$$\text{Because } \forall T \in \mathcal{L}(V),\ \exists\,!\, A_{i,j} \in \mathbf{F}, \quad T = \begin{pmatrix} A_{1,1}R_{1,1}+ & \cdots & +A_{1,m}R_{m,1}+ & \cdots & +A_{1,n}R_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}R_{1,m}+ & \cdots & +A_{m,m}R_{m,m}+ & \cdots & +A_{m,n}R_{n,m} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{n,1}R_{1,n}+ & \cdots & +A_{n,m}R_{m,n}+ & \cdots & +A_{n,n}R_{n,n} \end{pmatrix}.$$

$$\Rightarrow \mathcal{A}(T) = ST = \Big(\sum_{r=1}^{m} E_{r,r}\Big)\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j} R_{j,i}\Big)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} A_{i,j} Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1}+ & \cdots & +A_{1,m}Q_{m,1}+ & \cdots & +A_{1,n}Q_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}Q_{1,m}+ & \cdots & +A_{m,m}Q_{m,m}+ & \cdots & +A_{m,n}Q_{n,m} \end{pmatrix}.$$

$$\text{Thus } \operatorname{null} \mathcal{A} = \operatorname{span} \begin{pmatrix} R_{1,m+1}, & \cdots & ,R_{n,m+1}, \\ \vdots & \ddots & \vdots \\ R_{1,n}, & \cdots & ,R_{n,n} \end{pmatrix}, \quad \operatorname{range} \mathcal{A} = \operatorname{span} \begin{pmatrix} Q_{1,1}, & \cdots & ,Q_{n,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,m}, & \cdots & ,Q_{n,m} \end{pmatrix}.$$

Hence (a) $\dim \operatorname{null} \mathcal{A} = n \times (n - m)$; (b) $\dim \operatorname{range} \mathcal{A} = n \times m$. □

• **COMMENT:** Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(T) = TS$. Similarly to Problem ($\circ$),

(a) $\forall T \in \mathcal{L}(V), TS = 0 \iff \operatorname{range} S \subseteq \operatorname{null} T$.

Thus $\operatorname{null} \mathcal{B} = \big\{T \in \mathcal{L}(V) : \operatorname{range} S \subseteq \operatorname{null} T\big\} = \big\{T \in \mathcal{L}(V) : T|_{\operatorname{range} S} = 0\big\}$.

(b) $\forall R \in \mathcal{L}(V), \operatorname{null} S \subseteq \operatorname{null} R \iff \exists\, T \in \mathcal{L}(V), R = TS$, by (3.B.24).

Thus $\operatorname{range} \mathcal{B} = \big\{R \in \mathcal{L}(V) : \operatorname{null} S \subseteq \operatorname{null} R\big\} = \big\{R \in \mathcal{L}(V) : R|_{\operatorname{null} S} = 0\big\}$.

Hence $\dim \operatorname{null} \mathcal{B} = (\dim V - \dim \operatorname{range} S)(\dim V)$;

$\dim \operatorname{range} \mathcal{B} = (\dim V - \dim \operatorname{null} S)(\dim V)$. □

OR. Using NOTE FOR [3.60] and the notation in Problem ($\circ$).

$$\mathcal{B}(T) = TS = \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i}\Big)\Big(\sum_{r=1}^{m} E_{r,r}\Big) = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{i,j}G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1}+ & \cdots & +A_{1,m}G_{m,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}G_{1,m}+ & \cdots & +A_{m,m}G_{m,m} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{n,1}G_{1,n}+ & \cdots & +A_{n,m}G_{m,n} \end{pmatrix}.$$

Thus $\operatorname{null} \mathcal{B} = \operatorname{span}\begin{pmatrix} R_{m+1,1}, & \cdots & ,R_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ R_{m+1,n}, & \cdots & ,R_{n,n} \end{pmatrix}$,

$\operatorname{range} \mathcal{B} = \operatorname{span}\begin{pmatrix} G_{1,1}, & \cdots & ,G_{m,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ G_{1,n}, & \cdots & ,G_{m,n} \end{pmatrix}$. Hence (a) $\dim \operatorname{null} \mathcal{B} = n \times (n-m)$;

(b) $\dim \operatorname{range} \mathcal{B} = n \times m$. □

---

17 *Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Using NOTE FOR [3.60]. Let $(v_1, \dots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.

Then $\forall E_{i,j} \in \mathcal{E}$, ($\forall x, y = 1, \dots, n$), by assumption, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.

Again, $E_{y,x'}, E_{y',x} \in \mathcal{E}$ for all $x', y', x, y = 1, \dots, n$. Thus $\mathcal{E} = \mathcal{L}(V)$. □

• OR (10.A.4) *Suppose that $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bases of V.*
Let $T \in \mathcal{L}(V)$ be such that $T\alpha_k = \beta_k, \forall k$. Prove that $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)$
For ease of notation, let $\mathcal{M}(T, \alpha \to \beta) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n))$, $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n))$.

SOLUTION:

Denote $\mathcal{M}(T, \alpha \to \alpha)$ by A and $\mathcal{M}(I, \beta \to \alpha)$ by B.

$\forall k \in \{1, \dots, n\}, Iu_k = u_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. □

OR. Note that $\mathcal{M}(T, \alpha \to \beta) = I$. Hence $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)\underbrace{\mathcal{M}(T, \alpha \to \beta)}_{=\mathcal{M}(I, \beta \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. □

OR. Note that $\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I$.

$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}\Big(\underbrace{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}_{=\mathcal{M}(T, \alpha \to \beta)}\Big) = \mathcal{M}(I, \beta \to \alpha)$. □

COMMENT: Denote $\mathcal{M}(T, \beta \to \beta)$ by A'.

$u_k = Iu_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n,\ \forall\, k \in \{1, \dots, n\}$.

又 $Tu_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = B_{1,k}\beta_1 + \cdots + B_{n,k}\beta_n = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.

OR. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$.

---

16 *Suppose V is finite-dim and $S \in \mathcal{L}(V)$ such that $\forall T \in \mathcal{L}(V), ST = TS$. Prove that $\exists \lambda \in \mathbf{F}, S = \lambda I$.*

Solution: Using the notation and result in ($\circ$).

Suppose $ST = TS$ for every $T \in \mathcal{L}(V)$. If $S = 0$, we are done. Now suppose $S \neq 0$.

Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, B_{\mathcal{K}}) = \mathcal{M}(I, B_{\text{range } S}, B_{\mathcal{K}})$.

Then $\forall k \in \{m+1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $n = \dim V = \dim \operatorname{range} S = m$.

Notice that $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$. Thus $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \cdots + a_{n,i}v_n)$.

Where $a_{i,j} = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n$;

And For each j, for all i. Thus $a_{i,i} = a_{k,k} = \lambda, \forall\, k \neq i$.

Hence $w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \mathcal{M}\big(\lambda I, (v_1, \dots, v_n)\big) \Rightarrow S = \mathcal{M}^{-1}(\mathcal{M}(\lambda I))\lambda I$. ☐

---

18 *Show that V and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.*

Solution:

Define $\Psi \in \mathcal{L}\big(V, \mathcal{L}(\mathbf{F}, V)\big)$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.

(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is inje.

(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi\big(T(1)\big)$. Hence Ψ is surj. ☐

Or. Define $\Phi \in \mathcal{L}\big(\mathcal{L}(\mathbf{F}, V), V\big)$ by $\Phi(T) = T(1)$.

(a) Suppose $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is inje.

(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surj. ☐

Comment: $\Phi = \Psi^{-1}$.

• *Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.*

Solution:

Note that $\deg\big[(x^2 + x)p''(x) + 2xp'(x) + p(3)\big] = \deg p$.

Define $T_n : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$. Then $T_n \in \mathcal{L}\big(\mathcal{P}_n(\mathbf{R})\big)$.

And note that $T_n(p) = 0 \Rightarrow \deg(T_n p) = -\infty = \deg p \Rightarrow p = 0$. Thus T_n is inv.

$\forall q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $m = 0$; if $q \neq 0$, let $m = \deg q$, we have $q \in \mathcal{P}_m(\mathbf{R})$.

Hence $\exists\, p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$. ☐

---

19 *Suppose $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$ is inje. $\deg Tp \leqslant \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.*
(a) *Prove that T is surj;* (b) *Prove that for every nonzero p, $\deg Tp = \deg p$.*

Solution:

(a) T is inje $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ is inje and therefore is inv $\iff T$ is surj.

(b) Using mathematical induction.

(i) $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$;
$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$.

(ii) Assume that $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$.
Suppose $\exists\, r \in \mathcal{P}_{n+1}(\mathbf{R}),\ \deg Tr \leqslant n < \deg r = n + 1$.
Then by (a), $\exists\, s \in \mathcal{P}_n(\mathbf{R}),\ T(s) = (Tr)$.
又 T is inje $\Rightarrow s = r$. While $\deg s = \deg Ts = \deg Tr < \deg r$.
Contradicts. Thus $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$. ☐

Ended

3·E

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 20 | 4E: 8 14

1 *A function $T : V \to W$ is linear $\iff$ T is a subspace of $V \times W$.*

---

2 *Suppose $V_1 \times \cdots \times V_m$ is finite-dim. Prove that each V_j is finite-dim.*

Solution:

For any $k \in \{1, \dots, m\}$, define $p_k : V_1 \times \cdots \times V_m \to V_k$ by $p_k(v_1, \dots, v_m) = v_k$.
Then p_k is a surj linear map. By [3.22], range $p_k = V_k$ is finite-dim. □

Or. Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.
Let $(v_1, \dots, v_M)$ be a basis of U. Note that $\forall u_i \in V_i, \in U_i \subseteq U$, for each i.

$$\left.\begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{array}\right\} \Rightarrow S_i|_{U_i} = R_i^{-1}|_{U_i}.$$

Thus U_i and V_i are iso. 又 U_i is a subsp of a finite-dim vecsp U. □

---

3 *Give an example of a vecsp V and its two subsps U_1, U_2 such that $U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum.*

Solution: V must be infinite-dim. For if not, both U_1 and U_2 are finite-dim subsps. By [3.76, 3.78].

Note *that at least one of U_1, U_2 must be infinite-dim. And at least one must be infinite-dim???* TODO

For if not, $U_1 \times U_2$ is finite-dim and $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$.

Let $V = \mathbf{F}^\infty = U_1,\ U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$.

$$\left.\begin{array}{l} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T\big((x_1, x_2, \cdots), (x, 0, \cdots)\big) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = \big((x_1, x_2, \cdots), (x, 0, \cdots)\big) \end{array}\right\} \Rightarrow S = T^{-1}.$$ □

---

4 *Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.*

Solution: Using the notation in Problem (2).

Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$.

$$\left.\begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (TR_1, \dots, TR_m). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m. \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ □

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5 *Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.*

Solution: Using the notation in Problem (2).

Note that $Tv = (w_1, \dots, w_m)$. Define $T_i \in \mathcal{L}(V, W_i)$ by $T_i(v) = w_i$.

$$\left.\begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1T, \dots, S_mT). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m. \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ □

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6 *For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.*

Solution:

Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1v_1 + \cdots + a_mv_m$.

(a) Suppose $T(v_1, \dots, v_m) = 0$. Then $\forall\ (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1v_1 + \cdots + a_mv_m = 0$
$\Rightarrow (v_1, \dots, v_m) = 0 \Rightarrow T$ is inje.

(b) Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $\forall\ (b_1, \dots, b_n) \in \mathbf{F}^m$,
$\Big[T\big(\psi(e_1), \dots, \psi(e_m)\big)\Big](b_1, \dots, b_m) = b_1\psi(e_1) + \cdots + b_m\psi(e_m) = \psi(b_1e_1 + \cdots + b_me_m) = \psi(b_1, \dots, b_m)$.
Thus $T\big(\psi(e_1), \dots, \psi(e_m)\big) = \psi$. Hence T is surj. □

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14 *Suppose* $U = \{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0$ *for only finitely many* $k\}$.
(a) *Show that* U *is a subsp of* $\mathbf{F}^\infty$. [*Do it in your mind*]
(b) *Prove that* $\mathbf{F}^\infty / U$ *is infinite-dim.*

SOLUTION: For ease of notation, denote the p^{th} term of $u = (x_1, \cdots, x_p, \cdots) \in \mathbf{F}^\infty$ by $u[p]$.

For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, (p-1) \equiv 0 \,(\bmod r) \\ 0, \text{otherwise} \end{cases}$ | simply $e_r = (1, \underbrace{0, \cdots, 0}_{(p-1)\ times}, 1, \underbrace{0, \cdots, 0}_{(p-1)\ times}, 1, \cdots)$.

Choose one $m \in \mathbf{N}^+$. Let $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 + U \Rightarrow \exists\, u \in U, a_1 e_1 + \cdots + a_m e_m = u$.

Suppose $u = (x_1, \cdots, x_L, 0, \cdots)$, where L is the largest such that $u[L] \neq 0$.

Let $s \in \mathbf{N}^+$ be such that $h = s \cdot m! + 1 > L$ and $e_1[h] = \cdots = e_m[h] = 1$.

Note that by definition, $e_r[s \cot m! + 1 + p] = e_r[p+1] = 1 \iff p \equiv 0\,(\bmod r) \iff r \,|\, p$.

Now for any $p \in \{1, \dots, m\}$, $u[h+p] = \left(\sum_{r=1}^m a_r e_r\right)[p+1] = \sum_{k=1}^{\tau(p)} a_{p_k} = 0$ $\quad (\Delta)$

where $1 = p_1 \leqslant \cdots \leqslant p_{\tau(p)} = p$ are all the distinct factors of p.

Let $q = p_{\tau(p)-1}$. Notice that $\tau(q) = \tau(p) - 1$ and $q_k = p_k, \forall k \in \{1, \dots, \tau(q)\}$.

Again by (Δ), $\left(\sum_{r=1}^m a_r e_r\right)[h+q] = \sum_{k=1}^{\tau(p)-1} a_{p_k} = 0$. Thus $a_{p_{\tau(p)}} = a_p = 0$ for any $p \in \{1, \dots, m\}$.

Hence $\forall m \in \mathbf{N}^+$, $(e_1, \dots, e_m)$ is linely inde in $\mathbf{F}^\infty$, so is $(e_1 + U, \dots, e_m + U)$ in $\mathbf{F}^\infty / U$. By (2.A.14). □

OR. For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, \text{if } 2^r \,|\, p \\ 0, \text{otherwise} \end{cases}$.

Similarly, let $m \in \mathbf{N}^+$ and $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 \Rightarrow a_1 e_1 + \cdots + a_m e_m = u \in U$.

Suppose L is the largest such that $u[L] \neq 0$. And l is such that $2^{ml} > L$.

Then $\forall k \in \{1, \dots, m\}, u[2^{ml} + 2^k] = \left(\sum_{r=1}^m a_r e_r\right)[2^k] = a_1 + \cdots + a_k = 0$.

Thus $a_1 = \cdots = a_m = 0$ and $(e_1, \dots, e_m)$ is linely inde. Similarly. □

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7 *Suppose* $v, x \in V$ *and* U *and* W *are subsps of* V. *Prove that* $v + U = x + W \Rightarrow U = W$.

SOLUTION:

(a) $\forall u_1 \in U, \exists\, w_1 \in W, v + u_1 = x + w_1$, let $u_1 = 0$, now $v = x + w_1' \Rightarrow v - x \in W$.
(b) $\forall w_2 \in W, \exists\, u_2 \in U, v + u_2 = x + w_2$, let $w_2 = 0$, now $x = v + u_2' \Rightarrow x - v \in U$.

Thus $\pm(v - x) \in U \cap W \Rightarrow \left.\begin{cases} u_1 = (x - v) + w_1 \in W \Rightarrow U \subseteq W \\ w_2 = (v - x) + u_2 \in U \Rightarrow W \subseteq U \end{cases}\right\} \Rightarrow U = W.$ □

- Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$.
  Then A is a translate of $U \iff \exists\, c \in \mathbf{R}, A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$.

- *Suppose* $T \in \mathcal{L}(V, W)$ *and* $c \in W$. *Prove that* $U = \{x \in V : Tx = c\}$ *is either* $\emptyset$ *or is a translate of* null T.

SOLUTION:

If $c \in W$ but $c \notin \operatorname{range} T$, then $U = \emptyset$, we are done. Now suppose $c \in \operatorname{range} T$ and $x \in U$.
$\forall x + y \in x + \operatorname{null} T$ ($\forall y \in \operatorname{null} T$), $x + y \in U$. Hence $x + \operatorname{null} T \subseteq U$.
$\forall u \in U, u - x \in \operatorname{null} T \Rightarrow u = x + (u - x)x + \operatorname{null} T$. Hence $U \subseteq x + \operatorname{null} T$. □

COROLLARY: *The set of solutions to a system of linear equations such as* [3.28] *is either* $\emptyset$ *or a translate.*

---

8 *Suppose A is a nonempty subset of V.*
Prove that A is a translate of some subsp of $V \iff \lambda v + (1-\lambda)w \in A\ , \forall v, w \in A, \lambda \in \mathbf{F}$.

SOLUTION:

Suppose $A = a + U$. Then $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$,
$\lambda(a+u_1) + (1-\lambda)(a+u_2) = a + \big(\lambda(u_1 - u_2) + u_2\big) \in A$.

Suppose $\lambda v + (1-\lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $a \in A$ and let $A' = \{x - a : x \in A\}$.

Then $0 \in A'$ and $\forall x - a, y - a \in A'$, $(\ \forall x, y \in A\ )$, $\lambda \in \mathbf{F}$,
(I) $\lambda(x-a) = [\lambda x + (1-\lambda)a] - a \in A'$.
(II) $\lambda(x-a) + (1-\lambda)(y-a) = \frac{1}{2}(x-a) + \frac{1}{2}(y-a) = \frac{1}{2}x + (1-\frac{1}{2})y - a \in A'$.
OR. By (I), $2 \times \left[\frac{1}{2}(x-a) + \frac{1}{2}(y-a)\right] = (x-a) + (y-a) \in A'$.
Thus A' is a subsp of V. Hence $a + A' = \{(x-a) + a : x \in A\} = A$ is a translate. □

OR. Suppose $x - a, y - a \in A', \lambda \in \mathbf{F}$.
Note that $x, a \in A \Rightarrow \lambda x + (1-\lambda)a = 2x - a \in A$. Similarly $2y - a \in A$.
(I) $(x - \frac{1}{2}a) + (y - \frac{1}{2}a) = x + y - a \in A \Rightarrow x + y - 2a = (x-a) + (y-a) \in A'$.
(II) $\lambda(x-a) = (\lambda x + (1-\lambda)a) - a \in A'$.
Thus $-x + A$ is a subsp of V. Hence $A = x + (-x + A)$ is a translate of the subsp $(-x + A)$. □

---

9 *Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subsps U_1, U_2 of V.*
Prove that the intersection $A_1 \cap A_2$ is either a translate of some subsp of V or is $\emptyset$.

SOLUTION:

Suppose $v + u_1, w + u_2 \in A_1 \cap A_2 \neq \emptyset$. By Problem (8),
$\forall \lambda \in \mathbf{F}, \lambda(v+u_1) + (1-\lambda)(w+u_2) \in A_1 \cap A_2$. Thus $A_1 \cap A_2$ is a translate of some subsp of V. □

OR. Let $A_1 = v + U_1, A_2 = w + U_2$. Suppose $x \in (v + U_1) \cap (w + U_2) \neq \emptyset$.
Then $\exists u_1 \in U_1, x = v + u_1 \Rightarrow x - v \in U_1$, $\exists u_2 \in U_2, x = w + u_2 \Rightarrow x - w \in U_2$.
Note that by [3.85], $A_1 = v + U_1 = x + U_1, A_2 = w + U_2 = x + U_2$. We show that $A_1 \cap A_2 = x + (U_1 \cap U_2)$.
(a) $y \in A_1 \cap A_2 \Rightarrow \exists u_1 \in U_1, u_2 \in U_2, y = x + u_1 = x + u_2 \Rightarrow u_1 = u_2 \in U_1 \cap U_2 \Rightarrow y \in x + (U_1 \cap U_2)$.
(b) $y = x + u \in x + (U_1 \cap U_2) = (x + U_1) \cap (x + U_2) \Rightarrow y \in A_1 \cap A_2$. □

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10 *Prove that the intersection of any collection of translates of subsps of V is either a translate of some subsp or $\emptyset$.*

SOLUTION:

Suppose $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collection of translates of subsps of V, where Γ is an arbitrary index set.

Suppose $x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset$, then by Problem (8), $\forall \lambda \in \mathbf{F}, \lambda x + (1-\lambda)y \in A_\alpha$ for every $\alpha \in \Gamma$.
Thus $\bigcap_{\alpha \in \Gamma} A_\alpha$ is a translate of some subsp of V. □

OR. Let $A_\alpha = w_\alpha + V_\alpha$ for each $\alpha \in \Gamma$. Suppose $x \in \bigcap_{\alpha \in \Gamma} (w_\alpha + V_\alpha) \neq \emptyset$.
Then for each $A_\alpha, \exists v_\alpha \in V_\alpha, x = w_\alpha + v_\alpha \Rightarrow x - w_\alpha \in V_\alpha \Rightarrow A_\alpha = w_\alpha + V_\alpha = x + V_\alpha$.
(a) $y \in \bigcap_{\alpha \in \Gamma} A_\alpha \Rightarrow \forall \alpha \in \Gamma, \exists v_\alpha, y = x + v_\alpha \Rightarrow \forall \alpha, \beta \in \Gamma, v_\alpha = v_\beta \Rightarrow y \in x + \bigcap_{\alpha \in \Gamma} V_\alpha$.
(b) $y = x + v \in x + \bigcap_{\alpha \in \Gamma} V_\alpha = \bigcap_{\alpha \in \Gamma} (x + V_\alpha) \Rightarrow y \in \bigcap_{\alpha \in \Gamma} A_\alpha$. Hence $\bigcap_{\alpha \in \Gamma} A_\alpha = x + \bigcap_{\alpha \in \Gamma} V_\alpha$. □

• **NOTE FOR [3.79, 3.83]:** If $U = \{0\}$, then $v + U = v + \{0\} = \{v\}$, $V/U = V/\{0\} = \{\{v\} : v \in V\}$.

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11 *Suppose* $A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^{m} \lambda_i = 1\}$, *where each* $v_i \in V, \lambda_i \in \mathbf{F}$.

(a) *Prove that* A *is a translate of some subsp of* V

(b) *Prove that if* B *is a translate of some subsp of* V *and* $\{v_1, \dots, v_m\} \subseteq B$, *then* $A \subseteq B$.

(c) *Prove that* A *is a translate of some subsp of* V *of dim less than* m.

Solution:

(a) By Problem (8), $\forall u, w \in A, \lambda \in \mathbf{F}, \exists a_i, b_i \mathbf{F}$,
$\lambda u + (1-\lambda) w = \left(\lambda \sum_{i=1}^{m} a_i + (1-\lambda) \sum_{i=1}^{m} b_i\right) v_i \in A.$ □

(b) Suppose $B = v + U$, where $v \in V$ and U is a subsp of V. Suppose $\exists !\, u_k \in U, v_k = v + u_k \in B$.
Then for all $v = \sum_{i=1}^{m} \lambda_i v_i \in A$, $v = \sum_{i=1}^{m} \lambda_i (v + u_i) = v + \sum_{i=1}^{m} \lambda_i u_i \in v + U = B$. □

Or. Let $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on m by k.

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By Problem (8), $v \in B$.

(ii) $2 \leqslant k \leqslant m$, we assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. $\left(\forall \lambda_i \text{ such that } \sum_{i=1}^{k} \lambda_i = 1\right)$

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. $\forall i = 1, \dots, k$, $\exists\, \mu_i \neq 1$, fix one such i by ι.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow \left(\sum_{i=1}^{k+1} \frac{\mu_i}{1-\mu_\iota}\right) - \frac{\mu_\iota}{1-\mu_\iota} = 1.$

$$\text{Let } w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let $\lambda_i = \frac{\mu_i}{1-\mu_\iota}$ for $i = 1, \dots, \iota - 1$; $\lambda_j = \frac{\mu_{j+1}}{1-\mu_\iota}$ for $j = \iota, \dots, k$. Then,

$$\left.\begin{array}{r} \sum_{i=1}^{k} \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B.$$ □

(c) If $m = 1$, then let $A = v_1 + \{0\}$ and we are done.
Choose one $k \in \{1, \dots, m\}$. Given $\lambda_i \in \mathbf{F}$, where $i \in \{1, \dots, k-1, k+1, \dots, m\}$.
Let $\lambda_k = 1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m$
Then $\lambda_1 v_1 + \cdots + \lambda_k v_k + \cdots + \lambda_m v_m = v_k + \sum_{i=1}^{m} \lambda_i (v_i - v_k)$.
Thus $A = v_k + \operatorname{span}(v_1 - v_k, \dots, v_{k-1} - v_k, v_{k+1} - v_k, \dots, v_m - v_k)$. □

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18 *Suppose* $T \in \mathcal{L}(V, W)$ *and* U *is a subsp of* V. *Let* π *denote the quotient map.*
Prove that $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \subseteq \operatorname{null} T$.

Solution:

(a) Suppose $U \subseteq \operatorname{null} T$. Define $S \in \mathcal{L}(V/U, W)$ by $S(v + U) = Tv$. Then $S \circ \pi = T$.
Now we show that this map is *well-defined*.
$v_1 + U = v_2 + U \iff (v_1 - v_2) \in U \iff S\big((v_1 - v_2) + U\big) = T(v_1 - v_2) = 0 \iff Tv_1 = Tv_2$.

(b) Suppose $\exists\, S, T = S \circ \pi$. Then $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null} T$. □

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20 *Define* $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ *by* $\Gamma(S) = S \circ \pi$. *Prove that*:

(a) Γ *is linear*: By [3.9] distr and [3.6].

(b) Γ *is inje*: $\Gamma(S) = 0 = S \circ \pi \iff \forall v \in V, S(\pi(v)) = 0 \iff \forall v + U \in V/U, S(v + U) = 0 \iff S = 0$.

(c) $\operatorname{range} \Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}$: By Problem (18). □

• **NOTE FOR [3.88, 3.90, 3.91]:** Suppose $W \in \mathcal{S}_V U$. Then V/U and W are iso.

For any $W \in \mathcal{S}_V U$, because $V = U \oplus W$, $\forall v \in V, \exists!\, u_v \in U, w_v \in W$ such that $v = u_v + w_v$.

Define $T \in \mathcal{L}(V, W)$ by $T(v) = w_v$. Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$, $\operatorname{range} T \oplus \operatorname{null} T = V$.

Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, W)$ is defined by $\tilde{T}(v + U) = Tv = w_v$.

Now $\pi \circ \tilde{T} = I_{V/U}$, $\tilde{T} \circ \pi = I_W = T|_W$ Hence $\tilde{T}$ is an iso of V/U onto W.

• **COMMENT:** Note that $v = u_v + w_v = (u_v - u') + (w'_v + u')$, where $w'_v \notin W \iff u' \neq 0$.

Define $S \in \mathcal{L}(V/U, V)$ by $S(v + U) = v$. Hence $\operatorname{null} S = \{0\}$, $\operatorname{range} S \in \mathcal{S}_V U$, $\operatorname{range} S \oplus U = V$.

Let $E = S \circ \pi$. Now $\operatorname{null} E = \operatorname{null} \pi = U$. Because π is surj 又 $\operatorname{range}(S \circ \pi) \subseteq \operatorname{range} S$. $\operatorname{range} E = \operatorname{range} S$.

Then $\operatorname{range} E \oplus \operatorname{null} E = V$. NOTICE that $E : V \to \operatorname{range} S$ is a pure *eraser*. Now we explain why:

EXAMPLE: Suppose $B_V = (v_1, v_2, v_3)$, $U = \operatorname{span}(v_1)$. Then it is uniquely fixed that $\operatorname{range} S = \operatorname{span}(v_2, v_3)$.

While we might have $\operatorname{range} T = \operatorname{span}(v_2 - 2v_1, v_3) = W$, depending on the choice of W.

Now $E : v_2 \mapsto v_2;\ \ v_2 - 2v_1 \mapsto v_2$. While $T : v_2 \mapsto v_2 - 2v_1;\ \ v_2 - 2v_1 \mapsto v_2 - 2v_1$.

---

12 *Suppose U is a subsp of V such that V/U is finite-dim. Prove that is V is iso to $U \times (V/U)$.*

SOLUTION:

Let $(v_1 + U, \dots, v_n + U)$ be a basis of V/U.

Note that $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^{n} a_i(v_i + U) = \left(\sum_{i=1}^{n} a_i v_i\right) + U$

$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) \in U \Rightarrow \exists!\, u \in U, v = \sum_{i=1}^{n} a_i v_i + u$.

Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ by $\varphi(v) = (u, v + U)$,

and $\psi \in \mathcal{L}(U \times (V/U), V)$ by $\psi(u, v + U) = v + u$, where $\exists!\, a_i \in \mathbf{F}, v = \sum_{i=1}^{n} a_i v_i + U$. ☐

OR. [*V/U, U and V can be infinite-dim*] Define $S \in \mathcal{L}(V/U, V)$ by $S(v + U) = v$.

By the NOTE FOR [3.88,3.90,3.91], $\operatorname{range} S \oplus U = V$. Thus $\forall v \in V, \exists!\, u \in U, w \in \operatorname{range} S, v = u + w$.

Define $T \in \mathcal{L}(U \times (V/U), V)$ by $T(u, v + U) = u + S(v + U) = u + w = v$. Then T is surj.

And $T(u, v + U) = u + S(v + U) = 0 \Longrightarrow \pi\big(T(u, v + U)\big) = v + U = 0$, and $u = -S(v + U) = 0$.

OR. Define $R \in \mathcal{L}(V, U \times (V/U))$ by $R(v) = (u, (w + U))$. Now $R \circ T = I_{U \times (V/U)}$, $T \circ R = I_V$. ☐

• (4E 3.E.14) *Suppose $V = U \oplus W$, $(w_1, \dots, w_m)$ is a basis of W.*
Prove that $(w_1 + U, \dots, w_m + U)$ is a basis of V/U.

SOLUTION: $\forall v \in V, \exists!\, u \in U, w \in W, v = u + w$. 又 $\exists!\, c_i \in \mathbf{F}, w = \sum_{i=1}^{m} c_i w_i \Rightarrow v = \sum_{i=1}^{m} c_i w_i + u$.

Hence $\forall v + U \in V/U, \exists!\, c_i \in \mathbf{F}, v + U = \sum_{i=1}^{m} c_i w_i + U$. ☐

---

13 *Suppose $(v_1 + U, \dots, v_m + U)$ is a basis of V/U and $(u_1, \dots, u_n)$ is a basis of U.*
Prove that $(v_1, \dots, v_m, u_1, \dots, u_n)$ is a basis of V.

SOLUTION: Notice that $(v_1, \dots, v_m)$ is linely inde.

By Problem (12), U and V/U are finite-dim $\Longrightarrow U \times (V/U)$ is finite-dim, so is V.

$\dim V = \dim\big(U \times (V/U)\big) = m + n$. 又 Each $v_i = S(v_i + U)$, where we define $S(v + U) = v$.

Note that $\sum_{i=1}^{m} a_i v_i \in U \iff \left(\sum_{i=1}^{m} a_i v_i\right) + U = 0 + U \iff a_1 = \cdots = a_m = 0$.

Hence $\operatorname{span}(v_1, \dots, v_m) \cap U = \{0\} \Rightarrow \operatorname{span}(v_1, \dots, v_m) \oplus U = V$. By (2.B.8), we are done. ☐

OR. Note that $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^{m} a_i v_i + U \Rightarrow \exists!\, b_i \in \mathbf{F}, v - \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} b_i u_i \in U$

$\Rightarrow \forall v \in V, \exists!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^{m} a_i v_i + \sum_{j=1}^{m} b_j u_j$. ☐

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15 *Suppose* $\varphi \in \mathcal{L}(V, \mathbf{F}) \backslash \{0\}$. *Prove that* $\dim V/(\operatorname{null} \varphi) = 1$.

SOLUTION:

By (3.B.29), $\exists\, u \in V, V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}$. By (4E 3.E.14), $(u + \operatorname{null} \varphi)$ is a basis of $V/\operatorname{null} \varphi$.
OR. By [3.91] (d), $\dim \operatorname{range} \varphi = 1 = \dim V/(\operatorname{null} \varphi)$. ☐

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16 *Suppose* $\dim V/U = 1$. *Prove that* $\exists\, \varphi \in \mathcal{L}(V, \mathbf{F})$ *such that null* $\varphi = U$.

SOLUTION:

Suppose V_0 is a subsp of V such that $V = U \oplus V_0$. Then V_0 and V/U are iso. $\dim V_0 = 1$.
Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. ☐

OR. Let $(w + U)$ be a basis of V/U. Then $\forall v \in V, \exists!\, a \in \mathbf{F}, v + U = aw + U$.
Define $\varphi : V \to \mathbf{F}$ by $\varphi(v) = a$. Assume that φ is linear.
Then $u \in U \iff u + U = 0w + U \iff \varphi(u) = 0 \iff u \in \operatorname{null} \varphi$. Thus $U = \operatorname{null} \varphi$. ☐
Now we prove the assumption.
$\forall x, y \in V, \lambda \in \mathbf{F}, \exists!\, a, b \in \mathbf{F}, x + U = aw + U, \lambda y + U = \lambda bw + U \Rightarrow (x + \lambda y) + U = (a + \lambda b)w + U$.
Then $\varphi(x + \lambda y) = a + \lambda b = \varphi(x) + \lambda \varphi(y)$.

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17 *Suppose* V/U *is finite-dim.* W *is a subsp of* V.
(a) *Show that if* $V = U + W$, *then* $\dim W \geqslant \dim V/U$.
(b) *Find a* W *such that* $\dim W = \dim V/U$ *and* $V = U \oplus W$.

SOLUTION: Let $(w_1, \dots, w_n)$ be a basis of W

(a) $\forall v \in V, \exists\, u \in U, w \in W$ such that $v = u + w \Rightarrow v + U = w + U$
And $\exists!\, a_i \in \mathbf{F}, v + U = (a_1 w_1 + \dots + a_n w_n) + U$. Then $V/U \subseteq \operatorname{span}(w_1 + U, \dots, w_n + U)$.
Hence $\dim V/U = \dim \operatorname{span}(w_1 + U, \dots, w_n + U) \leqslant \dim W$.

(b) Let $W \in \mathcal{S}_V U$. In other words, reduce $(w_1 + U, \dots, w_n + U)$
to a basis $(w_1 + U, \dots, w_m + U)$ of V/U and let $W = \operatorname{span}(w_1, \dots, w_m)$. ☐

OR. Let $(v_1 + U, \dots, v_m + U)$ be a basis of V/U and define $\tilde{T} \in \mathcal{L}(V/U, V)$ by $\tilde{T}(v_k + U) = v_k$.
Note that $\pi \circ \tilde{T} = I$. By (3.B.20), $\tilde{T}$ is inje. And $(v_1, \dots, v_m)$ is linely inde.
Let $W = \operatorname{range} \tilde{T} = \operatorname{span}(v_1, \dots, v_m)$. Then $\tilde{T} \in \mathcal{L}(V/U, W)$ is an iso. Thus $\dim W = \dim V/U$.
And $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = a_1 v_1 + \dots + a_m v_m + U$

$$\Rightarrow v - (a_1 v_1 + \dots + a_m v_m) \in U \Rightarrow \exists!\, w \in W, u \in U, v = w + u.$$ ☐

ENDED

3·F

4 5 6 7 8 9 12 13 15 16 17 18 19 20 21 22 23 24 25 26
28 29 30 31 32 33 34 35 36 37 | 4E: 5 6 8 17 23 24 25

20, 21 *Suppose* U *and* W *are subsets of* V. *Prove that* $U \subseteq W \iff W^0 \subseteq U^0$.

SOLUTION:

(a) Suppose $U \subseteq W$. Then $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(w) = 0 \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.
(b) Suppose $W^0 \subseteq U^0$. Then $\varphi \in W^0 \Rightarrow \varphi \in U^0$. Hence $\operatorname{null} \varphi \supseteq W \Rightarrow \operatorname{null} \varphi \supseteq U$. Thus $W \supseteq U$.

OR. For a subsp U of V, let $A_U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = U$, by Problem (25).
Suppose $W^0 \subseteq U^0$. Then $\forall \varphi \in W^0, v \in A_U, \varphi(v) = 0 \Rightarrow v \in A_W$. Thus $A_U \subseteq A_W$. ☐

COROLLARY: $W^0 = U^0 \iff U = W$.

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22 *Suppose U and W are subsps of V. Prove that $(U+W)^0 = U^0 \cap W^0$.*

SOLUTION:

(a) $\left.\begin{matrix} U \subseteq U+W \\ W \subseteq U+W \end{matrix}\right\} \Rightarrow \left.\begin{matrix} (U+W)^0 \subseteq U^0 \\ (U+W)^0 \subseteq W^0 \end{matrix}\right\} \Rightarrow (U+W)^0 \subseteq U^0 \cap W^0.$

OR. Suppose $\varphi \in (U+W)^0$. Then $\forall u \in U, w \in W, \varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$.

(b) Suppose $\varphi \in U^0 \cap W^0 \subseteq V'$. Then $\forall u \in U, w \in W, \varphi(u+w) = 0 \Rightarrow \varphi \in (U+W)^0$. ☐

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23 *Suppose U and W are subsets of V. Prove that $(U \cap W)^0 = U^0 + W^0$.*

SOLUTION:

(a) $\left.\begin{matrix} U \cap W \subseteq U \\ U \cap W \subseteq W \end{matrix}\right\} \Rightarrow \left.\begin{matrix} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{matrix}\right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \ [\supseteq U^0 \cap W^0 = (U+W)^0.]$

OR. Suppose $\varphi = \psi + \beta \in U^0 + W^0$. Then $\forall v \in U \cap W$, $\varphi(v) = (\psi+\beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0$.

(b) [*Only in Finite-dim; Requires that U, W are subsps*] Using Problem (22).

$$\begin{aligned} \dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\ &= 2\dim V - \dim U - \dim W - \big(\dim V - \dim(U+W)\big) = \dim V - \dim(U \cap W). \end{aligned}$$

OR. Suppose $\varphi \in (U \cap W)^0$. Let X, Y be such that $V = U \oplus X = W \oplus Y$.
Define $\psi \in U^0, \beta \in W^0$ by $\psi(u+x) = \frac{1}{2}\varphi(x), \beta(w+y) = \frac{1}{2}\varphi(y)$.
$\forall v = u + x = w + y \in V, \varphi(v) = \varphi(x) = \varphi(y)$. Now $\varphi(v) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y) = \psi(v) + \beta(v)$.
Hence $\varphi \in U^0 + W^0$. Now $(U \cap W)^0 \subseteq U^0 + W^0$. ☐

• **COROLLARY:**

(a) *Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subsets of V. Then $\left(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \sum_{\alpha_i \in \Gamma} \left(V_{\alpha_i}^0\right)$.*

(b) *Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subsps of V. Then $\left(\sum_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \bigcap_{\alpha_i \in \Gamma} \left(V_{\alpha_i}^0\right)$.*

(c) *Suppose $V = U \oplus W$. Then $V' = U^0 \oplus W^0$. And $U_V' = W^0$, $W_V' = U^0$.*

Where $U_V' = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$. And $\iota \in \mathcal{L}(V, U)$ is defined by $\iota(u_v + w_v) = u_v$.

• (4E 3.F.23) *Suppose $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following sets are the same.*

(a) $\operatorname{span}(\varphi_1, \dots, \varphi_m)$

(b) $\big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big)^0 \overset{(c)}{=\!=} \{\varphi \in V' : (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) \subseteq \operatorname{null}\varphi\}$

SOLUTION: By Problem (17), (c) holds.

By Problem (26) [*May require finite-dim*] and the COROLLARY in Problem (23),

$$\left.\begin{matrix} \big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big)^0 = (\operatorname{null}\varphi_1)^0 + \cdots + (\operatorname{null}\varphi_m)^0 \\ \operatorname{span}(\varphi_i) = \{v \in V : \forall \psi \in \operatorname{span}(\varphi_i), \psi(v) = 0\}^0 = (\operatorname{null}\varphi_i)^0 \end{matrix}\right\} \Rightarrow \text{(a)} = \text{(b)}.$$ ☐

OR. Note that by COROLLARY in Problem (4E 6), for each φ_i, we have
$\forall c \in \mathbf{F}\backslash\{0\}, \psi = c\varphi_i \in \operatorname{span}(\varphi_i) \iff \operatorname{null}\psi = \operatorname{null}\varphi_i \iff \psi \in (\operatorname{null}\psi)^0 = (\operatorname{null}\varphi_i)^0$.
And $0 \in \operatorname{span}(\varphi_i), 0 \in (\operatorname{null}\varphi_i)^0$. Hence $\operatorname{span}(\varphi_i) = (\operatorname{null}\varphi_i)^0$. Similarly. ☐

OR. [*Only in Finite-dim*] Suppose $\varphi \in V'$. Note that $\dim(\operatorname{null}\varphi)^0 = \dim\operatorname{range}\varphi = \dim\operatorname{span}(\varphi)$.
And because $\forall c \in \mathbf{F}, v \in \operatorname{null}\varphi, c\varphi(v) = 0 \Rightarrow \operatorname{span}(\varphi) \subseteq (\operatorname{null}\varphi)^0$. Similarly. ☐

COROLLARY: **30** Suppose V is finite-dim and $\varphi_1, \dots, \varphi_m$ is a linely inde list in V'.
Then $\dim\big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big) = (\dim V) - m$.

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31 *Suppose V is finite-dim and $B_{V'} = (\varphi_1, \dots, \varphi_n)$. Show that the correspd B_V exists.*

SOLUTION:

Using (3.B.29). Let $\varphi_i(u_i) = 1$ and then $V = \operatorname{null} \varphi_i \oplus \operatorname{span}(u_i)$ for each φ_i.

Suppose $a_1 u_1 + \dots + a_n u_n = 0$. Then $0 = \varphi_i(a_1 u_1 + \dots + a_n u_n) = a_i$ for each i.

Thus $B_V = (\varphi_1, \dots, \varphi_n)$. And $\varphi_i(u_x) = \delta_{i,x}$. $\square$

OR. For each $k \in \{1, \dots, n\}$, define $\Gamma_k = \{1, \dots, k-1, k+1, \dots, n\}$ and $U_k = \bigcap_{j \in \Gamma_k} \operatorname{null} \varphi_j$.

By Problem (30) OR (4E 2.C.16), $\dim U_k = 1$. Thus $\exists\, u_k \in V, U_k = \operatorname{span}(u_k) \neq 0$.

又 By Problem (30), $(\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_n) = \{0\} = U \cap \operatorname{null} \varphi_k$.

Then if $\varphi_k(u_k) = 0 \Rightarrow u_k \in \operatorname{null} \varphi_k$ while $u_k \in U \Rightarrow u_k \in \{0\}$, contradicts.

Thus $\varphi_k(u_k) \neq 0$. Let $v_k = \big(\varphi_k(u_k)\big)^{-1} u_k \Rightarrow \varphi_k(v_k) = 1$. Now for $j \neq k, u_k \in \operatorname{null} \varphi_j \Rightarrow \varphi_j(v_k) = 0$.

Similarly, suppose $a_1 v_1 + \dots + a_n v_n = 0 \Rightarrow a_1 = \dots = a_n = 0$. $B_V = (v_1, \dots, v_n)$. And $\varphi_j(v_k) = \delta_{j,k}$. $\square$

---

25 *Suppose U is a subsp of V. Explain why $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.*

SOLUTION: Note that $U = \{v \in V : v \in U\}$ is a subsp of V; And $v \in U \iff \varphi(v) = 0, \forall \varphi \in U^0$. $\square$

COROLLARY: $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}^0$.

COMMENT: $\{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = \big((\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m) \cap \cdots\big)$, where $\varphi_k \in U^0$, always remains a subsp, whether the subset U is a subsp or not.

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26 *Suppose Ω is a subsp of V'. Prove that $\Omega = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}^0$.*

SOLUTION:

Suppose $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}$, *which is the set of vecs that each $\varphi \in \Omega$ sends to zero in common.*

Then $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}^0$. 又 $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}^0$.

Immediately by the COROLLARY in Problem (20,21), we may conclude that $\Omega = U^0$. $\square$

OR. [*Requires Ω finite-dim*] Let $(\varphi_1, \dots, \varphi_m)$ be a basis of Ω. Then by def, $U \subseteq (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m)$.

$\forall \varphi \in \Omega, \exists!\, a_i \in \mathbf{F}, \varphi = a_1 \varphi_1 + \dots + a_m \varphi_m \Rightarrow \forall v \in (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m), \varphi(v) = 0 \Rightarrow v \in U$.

Hence $(\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m) = U$. 又 $\operatorname{span}(\varphi_1, \dots, \varphi_m) = \Omega$. By Problem (23), we are done. $\square$

COROLLARY: For every subsp Ω of V', $\exists!$ subsp U of V such that $\Omega = U^0$.

COMMENT: [*Only in Finite-dim*] Using Problem (31) and the COROLLARY(c) in Problem (22, 23).

Let $B_\Omega = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n), B_V = (v_1, \dots, v_m, \dots, v_n)$.

$V' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \overset{\text{(I)}}{=\!=} \operatorname{span}(v_{m+1}, \dots, v_n)^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.

$\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m) \overset{\text{(II)}}{=\!=} \operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0$; $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \overset{\text{(III)}}{=\!=} \operatorname{span}(v_1, \dots, v_m)^0$.

$\iff U = \operatorname{span}(v_{m+1}, \dots, v_n) = (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m)$. [***Another proof of*** [*3.106*] OR. Problem (24)]

(I) Using the COROLLARY(c), immediately.

(II) NOTICE that each $\operatorname{null} \varphi_k = \operatorname{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n) = U_k$; $\dim U_k = \dim V - 1$.

By (4E 2.C.16), $U = (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m) = \bigcap_{k=1}^{m} U_k = \operatorname{span}(v_{m+1}, \dots, v_n)$.

Hence $\operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0 = \Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$.

(III) NOTICE that $V' = \Omega \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.

And that $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq \operatorname{span}(v_1, \dots, v_m)^0$.

By the TIPS in (1.C), $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = \operatorname{span}(v_1, \dots, v_m)$.

OR. Similar to (II), let $\Omega = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, immediately. $\square$

• *Suppose* $T \in \mathcal{L}(V,W), \varphi_k \in V', \psi_k \in W'$.

28 *Prove that* $\operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) \iff \operatorname{range} T = (\operatorname{null}\psi_1) \cap \dots \cap (\operatorname{null}\psi_m)$.

29 *Prove that* $\operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \iff \operatorname{null} T = (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)$.

SOLUTION: Using [3.107], [3.109], Problem (23) and the COROLLARY in Problem (20, 21).

(28) $(\operatorname{range} T)^0 = \operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) = \big((\operatorname{null}\psi_1) \cap \dots \cap (\operatorname{null}\psi_m)\big)^0$.

(29) $(\operatorname{null} T)^0 = \operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) = \big((\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)\big)^0$. □

COROLLARY: Using the COMMENT in Problem (26).

$\operatorname{null} T = \operatorname{span}(v_1, \dots, v_m) \iff \operatorname{null} T = (\operatorname{null}\varphi_{m+1}) \cap \dots \cap (\operatorname{null}\varphi_n) \iff \operatorname{range} T' = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$.

——Where $B_V = (v_1, \dots, v_m, \dots, v_n) \iff B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$.

$\operatorname{range} T = \operatorname{span}(w_1, \dots, w_m) \iff \operatorname{range} T = (\operatorname{null}\psi_{m+1}) \cap \dots \cap (\operatorname{null}\psi_n) \iff \operatorname{null} T' = \operatorname{span}(\psi_{m+1}, \dots, \psi_n)$.

——Where $B_W = (w_1, \dots, w_m, \dots, w_n) \iff B_{W'} = (\psi_1, \dots, \psi_m, \dots, \psi_n)$.

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9 *Let* $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \cdots, \varphi_n)$. *Then* $\forall \psi \in V', \psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n$.

COROLLARY: *For other* $B'_V = (u_1, \dots, u_n), B'_{V'} = (\rho_1, \dots, \rho_n), \forall \psi \in V', \psi = \psi(u_1)\rho_1 + \dots + \psi(u_n)\rho_n$.

SOLUTION:

$\psi(v) = \psi\Big(\sum_{i=1}^n a_i v_i\Big) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i)\varphi_i(v) = \big[\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n\big](v)$.

OR. $\big[\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n\big]\Big(\sum_{i=1}^n a_i v_i\Big) = \psi(v_1)\varphi_1\Big(\sum_{i=1}^n a_i v_i\Big) + \dots + \psi(v_n)\varphi_n\Big(\sum_{i=1}^n a_i v_i\Big)$. □

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13 *Define* $T : \mathbf{R}^3 \to \mathbf{R}^2$ *by* $T(x,y,z) = (4x + 5y + 6z, 7x + 8y + 9z)$.

Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ *denote the dual basis of the standard basis of* $\mathbf{R}^2$ *and* $\mathbf{R}^3$.

(a) *Describe the linear functionals* $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$

For any $(x,y,z) \in \mathbf{R}^3$, $\big(T'(\varphi_1)\big)(x,y,z) = 4x + 5y + 6z$, $\big(T'(\varphi_2)\big)(x,y,z) = 7x + 8y + 9z$.

(b) *Write* $T'(\varphi_1)$ *and* $T'(\varphi_2)$ *as linear combinations of* ψ_1, ψ_2, ψ_3.

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$.

(c) *What is* $\operatorname{null} T'$? *What is* $\operatorname{range} T'$?

$$T(x,y,z) = 0 \iff \begin{cases} 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases} \iff \begin{cases} x + y + z = 0 \\ y = 2z = 0 \end{cases} \iff (x,y,z) \in \operatorname{span}(e_1 - 2e_2 + e_3).$$

Where (e_1, e_2, e_3) is standard basis of $\mathbf{R}^3$.

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a basis, with the correspd dual basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Thus $\operatorname{span}(e_1 - 2e_2 + e_3) = \operatorname{null} T \Rightarrow \operatorname{span}(e_1 - 2e_2 + e_3)^0 = \operatorname{span}(\varepsilon_2, \varepsilon_3) = \operatorname{range} T'$.

Note that $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$.

And $\left| \begin{array}{l} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{array} \right.$

Hence $\varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2$, $\varepsilon_3 = -\psi_1 + \psi_3$. Now $\operatorname{range} T' = \operatorname{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3)$.

OR. $\operatorname{range} T' = \operatorname{span}(T'(\varphi_1), T'(\varphi_2)) = \operatorname{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3)$.

Suppose $T'(x\varphi_1 + y\varphi_2) = (4x + 7y)\varphi_1 + (5x + 8y)\varphi_2 + (6x + 9y)\varphi_3 = 0$.

Then $x + y = 4x + 7y = x = y = 0$. Hence $\operatorname{null} T' = \{0\}$.

OR. $\operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \operatorname{span}(-2e_2, e_3) \oplus \operatorname{null} T$.

$\Rightarrow \operatorname{range} T = \{Tx : x \in \operatorname{span}(-2e_2, e_3)\} = \operatorname{span}(T(-2e_2), T(e_3))$

$= \operatorname{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \operatorname{span}(f_1, f_2) = \mathbf{R}^2$. Now $\operatorname{null} T' = (\operatorname{range} T)^0 = \{0\}$. □

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24 *Suppose V is finite-dim and U is a subsp of V.*
Prove, using the pattern of [3.104], that $\dim U + \dim U^0 = \dim V$.

SOLUTION:

By Problem (31) and the COMMENT in Problem (26), $B_U = (v_1, \dots, v_m) \iff B_{U^0} = (\varphi_{m+1}, \dots, \varphi_n)$. $\square$

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37 *Suppose U is a subsp of V and π is the quotient map. Thus $\pi' \in \mathcal{L}\big((V/U)', V'\big)$.*

(a) *Show that π' is inje*: Because π is surj. Use [3.108].

(b) *Show that* $\operatorname{range} \pi' = U^0$: By [3.109](b), $\operatorname{range} \pi' = (\operatorname{null} \pi)^0 = U^0$.

(c) *Conclude that π' is an iso from $(V/U)'$ onto U^0*: Immediately.

SOLUTION: OR. Using (3.E.18), also see (3.E.20).

(a) $\pi'(\varphi) = 0 \iff \forall v \in V\ (\ \forall v + U \in V\ ), \varphi(\pi(v)) = \varphi(v + U) = 0 \iff \varphi = 0$.

(b) $\psi \in \operatorname{range} \pi' \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null} \psi \supseteq U \iff \psi \in U^0$. Hence $\operatorname{range} \pi' = U^0$. $\square$

• *Suppose U is a subsp of V. Prove that $(V/U)'$ and U^0 are iso.* [*Another proof of* **[3.106]**]

SOLUTION:

Define $\xi : U^0 \to (V/U)'$ by $\xi(\varphi) = \tilde{\varphi}$, where $\tilde{\varphi} \in (V/U)'$ is defined by $\tilde{\varphi}(v + U) = \varphi(v)$.

We show that ξ is inje and surj.

Inje: $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V\ (\ \forall v + U \in V/U\ ), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$.

Surj: $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null}(\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$. $\square$

OR. Define $\nu : (V/U)' \to U^0$ by $\nu(\Phi) = \Phi \circ \pi$. Now $\nu \circ \xi = I_{U^0},\ \xi \circ \nu = I_{(V/U)'} \Rightarrow \xi = \nu^{-1}$. $\square$

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4 *Suppose U is a subsp of V and $U \neq V$. Prove that* $\underbrace{\exists\, \varphi \in V' \backslash \{0\}, \varphi(u) = 0 \text{ for all } u \in U}_{\iff U^0_V \neq \{0\}}$.

SOLUTION:

Let X be such that $V = U \oplus X$. Then $X \neq \{0\}$. Suppose $s \in X$ and $x \neq 0$.

Let Y be such that $X = \operatorname{span}(s) \oplus Y$. Now $V = U \oplus \big(\operatorname{span}(s) \oplus Y\big)$.

Define $\varphi \in V'$ by $\varphi(u + \lambda s + y) = \lambda$. Hence $\varphi \neq 0$ and $\varphi(u) = 0$ for all $u \in U$. $\square$

OR. [*Requires that V is finite-dim*] By [3.106], $\dim U^0 = \dim V - \dim U > 0$. Then $U^0 \neq \{0\}$.

OR. Let $B_V = (\underbrace{u_1, \dots, u_m}_{B_U}, v_1, \dots, v_n)$ with $n \geqslant 1$. Let $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$. Let $\varphi = \varphi_i$.

OR. Define $\varphi \in V'$ by $\varphi(u_1) = \dots = \varphi(u_m) = 0$ and $\varphi(v_1) = \dots = \varphi(v_n) = 1$. $\square$

COMMENT: [*Another proof of* **[3.108]**]: T is surj $\iff T'$ is inje.

(a) Suppose T' is inje. Note that $T'(\psi) = 0 \Rightarrow \psi = 0$.

Then $\nexists\, \psi \in W' \backslash \{0\}, \big(T'(\psi)\big)(v) = \psi(Tv) = 0$ for all $w \in \operatorname{range} T\ (\ \forall v \in V\ )$.

Thus if we assume that $\operatorname{range} T \neq W$ then contradicts. Hence $\operatorname{range} T = W$.

(b) Suppose T is surj. Then $(\operatorname{range} T)^0 = W^0_W = \{0\} = \operatorname{null} T'$. $\square$

• *Suppose V is a vecsp and U is a subsp of V.*

17 $U^0 = \{\varphi \in V' : U \subseteq null \varphi\}$. Noticing $\varphi \in V', U \subseteq \operatorname{null} \varphi \iff \forall u \in U, \varphi(u) = 0$.

18 $U^0 = V' \iff \forall \varphi \in V', U \subseteq \operatorname{null} \varphi \iff U = \{0\}$. [*Which means* $\{0\}^0_V = V'$.]

OR. $U^0 = V' \iff \dim U^0 = \dim V' = \dim V \iff \dim U = 0 \iff U = \{0\}$.

19 $U^0_V = \{0\} = V^0_V \iff U = V$. By the inverse and contrapositive of Problem (4). OR. By [3.106].

• *Suppose $V = U \oplus W$. Define $\iota : V \to U$ by $\iota(u+w) = u$. Thus $\iota' \in \mathcal{L}(U', V')$.*

(a) *Show that* $\operatorname{null}\iota' = U_U^0 = \{0\}$: $\operatorname{null}\iota' = (\operatorname{range}\iota)_U^0 = U_U^0 = \{0\}$.

(b) *Prove that* $\operatorname{range}\iota' = W_V^0$: $\operatorname{range}\iota' = (\operatorname{null}\iota)_V^0 = W_V^0$.

(c) *Prove that $\widetilde{\iota'}$ is an iso from $U'/\{0\}$ onto W^0*: By (a), (b) and [3.91](d).

SOLUTION:

(a) $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \operatorname{null}\psi$.

(b) Note that $W = \operatorname{null}(\iota) \subseteq \operatorname{null}(\psi \circ \iota)$. Then $\psi \circ \iota \in W^0 \Rightarrow \operatorname{range}\iota' \in W^0$.

Suppose $\varphi \in W^0$. Because $\operatorname{null}\iota = W \subseteq \operatorname{null}\varphi$. By TIPS in (3.B), $\varphi = \varphi \circ \iota = \iota'(\varphi)$. □

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36 *Suppose U is a subsp of V. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*

(a) *Show that* $\operatorname{null} i' = U^0$: $\operatorname{null} i' = (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$.

(b) *Prove that* $\operatorname{range} i' = U'$: $\operatorname{range} i' = (\operatorname{null} i)_U^0 = \{0\}_U^0 = U'$.

(c) *Prove that $\widetilde{i'}$ is an iso from V'/U^0 onto U'*: By (a), (b) and [3.91](d).

SOLUTION:

(a) $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$. Thus $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$.

(b) Suppose $\psi \in U'$. By (3.A.11), $\exists\, \varphi \in V', \varphi|_U = \psi$. Then $i'(\varphi) = \psi$. □

• *Suppose $T \in \mathcal{L}(V, W)$. Prove that* $\operatorname{range} T' = (\operatorname{null} T)^0$. [***Another proof of*** **[3.109](b)**]

SOLUTION:

Suppose $\Phi \in (\operatorname{null} T)^0$. Because by (3.B.12), $T|_U : U \to \operatorname{range} T$ is an iso; $V = U \oplus \operatorname{null} T$.

And $\forall v \in V, \exists!\, u_v \in U, w_v \in \operatorname{null} T, v = u_v + w_v$. Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$.

Let $\psi = \Phi \circ (T|_{\operatorname{range} T}^{-1})$. Then $T'(\psi) = \psi \circ T = \Phi \circ (T^{-1}|_{\operatorname{range} T} \circ T|_V)$.

Where $T^{-1}|_{\operatorname{range} T} : \operatorname{range} T \to U$; $T : V \to \operatorname{range} T$. Note that $T^{-1}|_{\operatorname{range} T} \circ T|_V = \iota$.

By TIPS in (3.B), $\Phi = \Phi \circ \iota$. Thus $T'(\psi) = \psi \circ T = \Phi \circ \iota = \Phi$. □

• *Suppose $T \in \mathcal{L}(V, W)$.* *Using* [3.108], [3.110].

$$\textit{Now } T \textit{ is inv} \iff \left| \begin{array}{c} \operatorname{null} T = \{0\} \iff (\operatorname{null} T)^0 = V' = \operatorname{range} T' \\ \operatorname{range} T = W \iff (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T' \end{array} \right| \iff T' \textit{ is inv.}$$

---

15 *Suppose $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \iff T = 0$.*

SOLUTION:

Suppose $T = 0$. Then $\forall \varphi \in W', T'(\varphi) = \varphi \circ T = 0$. Hence $T' = 0$.

Suppose $T' = 0$. Then $\operatorname{null} T' = W' = (\operatorname{range} T)^0$, by [3.107](a).

[*W can be infinite-dim*] By Problem (25),

$\operatorname{range} T = \{w \in W : \varphi(w) = 0, \forall \varphi \in (\operatorname{range} T)^0\} = \{w \in W : \varphi(w) = 0, \forall \varphi \in W'\}$.

Now we prove that if $\forall \varphi \in W', \varphi(w) = 0$, then $w = 0$. So that $\operatorname{range} T = \{0\}$ and we are done.

Assume that $w \neq 0$. Then let U be such that $W = U \oplus \operatorname{span}(w)$.

Define $\psi \in W'$ by $\psi(u + \lambda w) = \lambda$. So that $\psi(w) = 1 \neq 0$. □

OR. [*Only if W is finite-dim*] By [3.106], $\dim \operatorname{range} T = \dim W - \dim(\operatorname{range} T)^0 = 0$. □

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12 NOTICE that $I_{V'} : V' \to V'$. *Now $\forall \varphi \in V', I_{V'}(\varphi) = \varphi = \varphi \circ I_V = I_V{}'(\varphi)$. Thus $I_{V'} = I_V{}'$.*

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16 *Suppose V, W are finite-dim. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(V, W)$. Prove that Γ is an iso of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.*

SOLUTION: By [3.101], Γ is linear.

Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus Γ is inje.

Because V, W are finite-dim. $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$. Now Γ inje $\Rightarrow$ inv. □

COMMENT: Let $X = \{T \in \mathcal{L}(V, W) : \operatorname{range} T \text{ is finite-dim}\}$.

Let $Y = \{\mathcal{T} \in \mathcal{L}(W', V') : \operatorname{range} \mathcal{T} \text{ is finite-dim}\}$.

Then $\Gamma|_X$ is an iso of X onto Y, even if V and W are infinite-dim.

The inje of $\Gamma|_X$ is equiv to the inje of Γ, as shown before.

Now we show that $\Gamma|_X$ is surj without the cond that V or W is finite-dim.

Suppose $\mathcal{T} \in Y$. Let $B_{\operatorname{range}\mathcal{T}} = (\varphi_1, \dots, \varphi_m)$, with the correspd $(v_1, \dots, v_m)$. Let $\varphi_k = \mathcal{T}(\psi_k)$.

Let $\mathcal{K}$ be such that $W' = \mathcal{K} \oplus \operatorname{null}\mathcal{T}$. Let $B_{\mathcal{K}} = (\psi_1, \dots, \psi_m)$, with the correspd $(w_1, \dots, w_m)$.

Define $T \in \mathcal{L}(V, W)$ by $Tv_k = w_k, Tu = 0;\ k \in \{1, \dots, m\}, u \in U$.

$\forall \psi \in \operatorname{null}\mathcal{T}, [T'(\psi)](v) = \psi(Tv) = \psi(a_1 w_1 + \cdots + a_p w_p) = 0 = [\mathcal{T}(\psi)](v)$.

$\forall k \in \{1, \dots, m\}, [T'(\psi_k)](v) = \psi_k(Tv) = \psi_k(a_1 w_1 + \cdots + a_m w_m) = a_k = \varphi_k(v) = [\mathcal{T}(\psi)](v)$. □

COMMENT: This is another proof of [3.109(a)]: $\dim \operatorname{range} T = \dim \operatorname{range} T'$.

- (4E 3.F.6) *Suppose $\varphi, \beta \in V'$. Prove that* $\operatorname{null}\varphi \subseteq \operatorname{null}\beta \iff \beta = c\varphi, \exists\, c \in \mathbf{F}$.

COROLLARY: $\operatorname{null}\varphi = \operatorname{null}\beta \iff \beta = c\varphi, \exists\, c \in \mathbf{F}\backslash\{0\}$.

SOLUTION:

Using (3.B.29, 30).

(a) Suppose $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$. Suppose $u \notin \operatorname{null}\beta$, then $u \notin \operatorname{null}\varphi$.

Now $V = \operatorname{null}\beta \oplus \operatorname{span}(u) = \operatorname{null}\varphi \oplus \operatorname{span}(u)$. By TIPS in (1.C), $\operatorname{null}\beta = \operatorname{null}\varphi$. Let $c = \dfrac{\beta(u)}{\varphi(u)}$.

OR. We discuss in two cases. If $\operatorname{null}\varphi = \operatorname{null}\beta$, then we are done.

Otherwise, $\operatorname{null}\beta \neq \operatorname{null}\varphi$. Then $\exists\, u' \in \operatorname{null}\beta \backslash \operatorname{null}\varphi$.

Now $V = \operatorname{null}\varphi \oplus \operatorname{span}(u') = \operatorname{null}\varphi \oplus \operatorname{span}(u)$. $\forall v \in V, v = w + au = w' + bu', \exists!\, w, w' \in \operatorname{null}\varphi$.

Thus $\beta(v) = a\beta(u),\ \varphi(v) = b\varphi(u')$. Let $c = \dfrac{a\beta(u)}{b\varphi(u')}$. We are done.

NOTICE that by (b) below, we have $\operatorname{null}\beta \subseteq \operatorname{null}\varphi,\ u = u'$. Thus contradicts the assumption.

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$. If $c = 0$, then $\operatorname{null}\beta = V \supseteq \operatorname{null}\varphi$, we are done.

$$\text{Otherwise, } \left.\begin{array}{l} \forall v \in \operatorname{null}\varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \operatorname{null}\varphi \subseteq \operatorname{null}\beta \\ \forall v \in \operatorname{null}\beta, \beta(v) = 0 = \varphi(v) \Rightarrow \operatorname{null}\beta \subseteq \operatorname{null}\varphi \end{array}\right\} \Rightarrow \operatorname{null}\varphi = \operatorname{null}\beta.$$ □

OR. By (3.B.24), $\operatorname{null}\varphi \subseteq \operatorname{null}\beta \iff \exists\, E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi$. (if ***E* is *inv***, then $\operatorname{null}\varphi = \operatorname{null}\beta$)

Now we show that [P] $\exists\, E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi \iff \exists\, c \in \mathbf{F}, \beta = c\varphi$. [Q].

$[P] \Rightarrow [Q]:$ Let $c = E(1)$. Then $\forall v \in V, \beta(v) = E(\varphi(v)) = \varphi(v)E(1) = c\varphi(v)$. ($\boldsymbol{E(1) \neq 0}$)

$[Q] \Rightarrow [P]:$ Define $E \in \mathcal{L}(\mathbf{F})$ by $E(x) = cx$. Then $\forall v \in V, \beta(v) = c\varphi(v) = E(\varphi(v))$. ($\boldsymbol{c \neq 0}$) □

---

5 *Prove that $(V_1 \times \cdots \times V_m)'$ and ${V'}_1 \times \cdots \times {V'}_m$ are iso.* [Using notations in (3.E.2).]

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to {V'}_1 \times \cdots \times {V'}_m \\ \qquad \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = ({R'}_1(T), \dots, {R'}_m(T)). \\ \text{Define } \psi : {V'}_1 \times \cdots \times {V'}_m \to (V_1 \times \cdots \times V_m)' \\ \qquad \text{by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m = {S'}_1(T_1) + \cdots + {S'}_m(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ □

- *In* (3.D.18), $\varphi : V \to \mathcal{L}(\mathbf{F}, V)$ *is an iso. Now we prove that*
  $[P]$ $(v_1, \dots, v_m)$ *is linely inde* $\iff \big(\varphi(v_1), \dots, \varphi(v_m)\big)$ *is linely inde.* $[Q]$

Solution:

$[P] \Rightarrow [Q]$: Notice that φ is inje and by (3.B.9).

Or. Suppose $\vartheta \in \operatorname{span}\big(\varphi(v_1), \dots, \varphi(v_m)\big)$. Let $\vartheta = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m)$.

Then $\vartheta(1) = 0 = a_1 v_1 + \cdots + a_m v_m \Rightarrow a_1 = \cdots = a_m = 0$.

$[Q] \Rightarrow [P]$: Suppose $v \in \operatorname{span}(v_1, \dots, v_m)$. Let $v = 0 = a_1 v_1 + \cdots + a_m v_m$.

Then $\varphi(v) = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) \Rightarrow a_1 = \cdots = a_m = 0$. □

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32 *Let* $B_\alpha = (\alpha_1, \dots, \alpha_m), B_\alpha{}' = (\varphi_1, \dots, \varphi_m)$, $B_\beta = (v_1, \dots, v_m), B_\beta{}' = (\psi_1, \dots, \psi_m)$.
Prove that $\forall T \in \mathcal{L}(V), T$ *is inv* $\iff$ *the rows of* $A = \mathcal{M}\big(T, B_\alpha, B_\beta\big)$ *form a basis of* $\mathbf{F}^{1,n}$.

Solution: Note that T is inveritlbe $\iff T'$ is inv. And $A^t = \mathcal{M}(T', B_\beta{}', B_\alpha{}')$.

(a) Suppose T is inv, so is T'. Because $\big(T'(\varphi_1), \dots, T'(\varphi_m)\big)$ is linely inde.
Notice that $T'(\varphi_i) = A^t_{1,i}\psi_1 + \cdots + A^t_{m,i}\psi_m$. By the (Δ) part in (4E 3.C.17),
the cols of A^t, namely the rows of A, are linely inde.

(b) Suppose the rows of A are linely inde, so are the cols of A^t. Notice that A^t has $\dim V'$ cols.
Then $B_{\operatorname{range} T'} = B_{V'} = \big(T'(\varphi_1), \dots, T'(\varphi_m)\big)$. Thus T' is surj. Hence T' is inv, so is T. □

---

33 *Suppose* $A \in \mathbf{F}^{m,n}$. *Define* $T : A \to A^t$. *Prove that* T *is an iso of* $\mathbf{F}^{m,n}$ *onto* $\mathbf{F}^{n,m}$

Solution: By [3.111], T is linear. Note that $(A^t)^t = A$, $T \circ T = I$. □

- Define $T \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Tx = xA$, where $A \in \mathbf{F}^{n,n}$, for all $x \in \mathbf{F}^{1,n}$.
  Let $B_e = (e_1, \dots, e_n)$ be the standard basis of $\mathbf{F}^{1,n}$, with the dual basis $B_\varphi = (\varphi_1, \dots, \varphi_n)$.
  What is $\mathcal{M}(T)$? Because $Te_k = e_k A = \sum_{j=1}^n A_{k,j} e_j = \sum_{j=1}^n A^t_{j,k} e_j$. Now $\mathcal{M}(T) = A^t$.
  Note that $A = \mathcal{M}(A, B_e) \in \mathbf{F}^{n,n}$, $\mathcal{M}(Te_k) = \mathcal{M}(Te_k, B_e) \in \mathbf{F}^{n,1}$,
  $\mathcal{M}(e_k) = \mathcal{M}(e_k, B_e) \in \mathbf{F}^{n,1}$, $\mathcal{M}(e_k A) = \mathcal{M}(e_k A, B_e) \in \mathbf{F}^{n,1}$.
  Now $\mathcal{M}(Te_k) = \mathcal{M}(T)_{\cdot,k} = \mathcal{M}(e_k A) = A^t_{\cdot,k} \Longrightarrow \mathcal{M}(T)\mathcal{M}(e_k) = \mathcal{M}(T)_{\cdot,k} = \mathcal{M}(e_k)\mathcal{M}(A)$.
  Then $\mathcal{M}(e_k)\mathcal{M}(A)$ does not make sense. And now??? FIXME: BASIS NOT AGREED

- (4E 3.F.8) *Suppose* $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \dots, \varphi_n)$.

$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = \big(\varphi_1(v), \dots, \varphi_n(v)\big). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

- (4E 3.F.5) *Suppose* $T \in \mathcal{L}(V, W)$. $B_{\operatorname{range} T} = (w_1, \dots, w_m)$.
  Hence $\forall v \in V$, $Tv = \varphi_1(v) w_1 + \cdots + \varphi_m(v) w_m$, $\exists!\, \varphi_1(v), \dots, \varphi_m(v)$,
  thus defining $\varphi_i : V \to \mathbf{F}$ *for each* $i \in \{1, \dots, m\}$. *Show that each* $\varphi_i \in V'$.

Solution:

$$\begin{aligned} \forall u, v \in V, \lambda \in \mathbf{F}, T(u + \lambda v) &= \textstyle\sum_{i=1}^m \varphi_i(u + \lambda v) w_i \\ = Tu + \lambda Tv &= \Big(\textstyle\sum_{i=1}^m \varphi_i(u) w_i\Big) + \lambda\Big(\textstyle\sum_{i=1}^m \varphi_i(v) w_i\Big) = \textstyle\sum_{i=1}^m \big(\varphi_i(u) + \lambda\varphi_i(v)\big) w_i. \end{aligned}$$ □

Or. For each $w_i, \exists\, v_i \in V$, $Tv_i = w_i$, then $(v_1, \dots, v_m)$ is linely inde.
Now we have $Tv = a_1 Tv_1 + \cdots + a_m Tv_m$, $\forall v \in V, \exists!\, a_i \in \mathbf{F}$. Let $B_{(\operatorname{range} T)'} = (\psi_1, \dots, \psi_m)$.
Then $\big(T'(\psi_i)\big)(v) = \psi_i \circ T(v) = a_i$. Where $T : V \to \operatorname{range} T$; $T' : (\operatorname{range} T)' \to V'$.
Thus for each $i \in \{1, \dots, m\}, \varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$. □

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6 *Define* $\Gamma : V' \to \mathbf{F}^m$ *by* $\Gamma(\varphi) = \big(\varphi(v_1), \dots, \varphi(v_m)\big)$, *where* $v_1, \dots, v_m \in V$.

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = V \iff \Gamma$ *is inje.*

(b) *Show that* $(v_1, \dots, v_m)$ *is linely inde* $\iff \Gamma$ *is surj.*

Solution:

(a) Notice that $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \dots = \varphi(v_m) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.

If Γ is inje, then $\Gamma(\varphi) = 0 \iff V = \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.

If $V = \operatorname{span}(v_1, \dots, v_m)$, then $\Gamma(\varphi) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$, thus Γ is inje.

(b) Suppose Γ is surj. Then let $\Gamma(\varphi_i) = e_i$ for each i, where $(e_1, \dots, e_m)$ is the standard basis of $\mathbf{F}^m$.

Then by (3.A.4), $(\varphi_1, \dots, \varphi_m)$ is linely inde.

Now $a_1 v_1 + \dots + a_m v_m = 0 \Rightarrow 0 = \varphi_i(a_1 v_1 + \dots + a_m v_m) = a_i$ for each i.

Suppose $(v_1, \dots, v_m)$ is linely inde. Let $U = \operatorname{span}(\varphi_1, \dots, \varphi_m), B_{U'} = (\varphi_1, \dots, \varphi_m)$.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists!\, \varphi = a_1\varphi_1 + \dots + a_m\varphi_m$.

Let W be such that $V = U \oplus W$. Now $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.

Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$. So that $\Gamma(\varphi \circ i -) = (a_1, \dots, a_m)$. ☐

Or. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$ and let $(\psi_1, \dots, \psi_m)$ be the correspd dual basis.

Define $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$. Then Ψ is an iso.

Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $Te_k = v_k$. Now $T(x_1, \dots, x_m) = T(x_1 e_1 + \dots + x_m e_m) = x_1 v_1 + \dots + x_m v_m$.

$\forall \varphi \in V', k \in \{1, \dots, m\}, \big[T'(\varphi)\big](e_k) = \varphi(Te_k) = \varphi(v_k) = \big[\varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m\big](e_k)$

Now $T'(\varphi) = \varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m = \Psi\big(\varphi(v_1), \dots, \varphi(v_m)\big) = \Psi\big(\Gamma(\varphi)\big)$. Hence $T' = \Psi \circ \Gamma$.

By (3.B.3), (a) $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V \iff T' = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.

(b) $(v_1, \dots, v_m)$ is linely inde $\iff T$ is inje $\iff T' = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. ☐

• (4E 3.F.25) *Define* $\Gamma : V \to \mathbf{F}^m$ *by* $\Gamma(v) = \big(\varphi_1(v), \dots, \varphi_m(v)\big)$, *where* $\varphi_1, \dots, \varphi_m \in V'$.

(c) *Show that* $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ *is inje.*

(d) *Show that* $(\varphi_1, \dots, \varphi_m)$ *is linely inde* $\iff \Gamma$ *is surj.*

Solution:

(c) Notice that $\Gamma(v) = 0 \iff \varphi_1(v) = \dots = \varphi_m(v) = 0 \iff v \in (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)$.

By Problem (4E 23) and (18), $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) = \{0\}$.

And $\operatorname{null}\Gamma = (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)$. Hence Γ inje $\iff \operatorname{null}\Gamma = \{0\} \iff \operatorname{span}(\varphi_1, \dots, \varphi_m) = V'$.

(d) Suppose $(\varphi_1, \dots, \varphi_m)$ is linely inde. Then by Problem (31), $(v_1, \dots, v_m)$ is linely inde.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}, \exists!\, v = \sum_{i=1}^{m} a_i v_i \in V \Rightarrow \varphi_i(v) = a_i, \Gamma(v) = (a_1, \dots, a_m)$. Hence Γ is surj.

Suppose Γ is surj. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$.

Suppose $v_i \in V$ such that $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i$, for each i.

Then $(v_1, \dots, v_m)$ is linely inde. And $\varphi_j(v_k) = \delta_{j,k}$.

Now $a_1\varphi_1 + \dots + a_m\varphi_m = 0 \Rightarrow 0(v_i) = a_i$ for each i. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde.

Or. Let $\operatorname{span}(v_1, \dots, v_m) = U$. Then $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U)$. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde. ☐

Or. Similar to Problem (6), we get $(e_1, \dots, e_m), (\psi_1, \dots, \psi_m)$ and the iso Ψ.

$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'\big(\Psi(x_1, \dots, x_m)\big) = \Gamma'\big(\Psi(x_1 e_1 + \dots + x_m e_m)\big) = (x_1\psi_1 + \dots + x_m\psi_m) \circ \Gamma$.

$\forall v \in V, \big[\Gamma'\big(\Psi(x_1, \dots, x_m)\big)\big](v) = \big[x_1\psi_1 + \dots + x_m\psi_m\big]\big(\Gamma(v)\big) = \big[x_1\varphi_1 + \dots + x_m\varphi_m\big](v)$.

Now $\Gamma'\big(\Psi(x_1, \dots, x_m)\big) = x_1\varphi_1 + \dots + x_m\varphi_m$.

Define $\Phi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Phi = \Psi \circ \Gamma$. $\Phi(x_1, \dots, x_m) = x_1\varphi_1 + \dots + x_m\varphi_m$. Thus by (4E 3.B.3),

(c) the inje of Φ correspds to $(\varphi_1, \dots, \varphi_m)$ spanning V'; 又 $\Phi = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.

(d) the surj of Φ correspds to $(\varphi_1, \dots, \varphi_m)$ being linely inde; 又 $\Phi = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. ☐

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35 *Prove that $\big(\mathcal{P}(\mathbf{F})\big)'$ and $\mathbf{F}^\infty$ are iso.*

Solution:

Define $\theta \in \mathcal{L}\Big(\big(\mathcal{P}(\mathbf{F})\big)', \mathbf{F}^\infty\Big)$ by $\theta(\varphi) = \big(\varphi(1), \varphi(z), \cdots, \varphi(z^n), \cdots\big)$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall z^k$ in the basis $(1, z, \dots, z^n)$ of $\mathcal{P}_n(\mathbf{F})$ $(\ \forall n\ )$, $\varphi(z^k) = 0 \Rightarrow \varphi = 0$.

$\big[$ Notice that $\forall p \in \mathcal{P}(\mathbf{R}), \exists!\, a_i \in \mathbf{F}, m = \deg p,\ p = a_0 z + a_1 z + \cdots + a_m z^m \in \mathcal{P}_m(\mathbf{F})$. $\big]$

Surj: $\forall \big(a_k\big)_{k=1}^\infty \in \mathbf{F}^\infty$, let ψ be such that $\forall k, \psi\big(z^k\big) = a_k$ $\big[$ by [3.5] $\big]$ and thus $\theta(\psi) = \big(a_k\big)_{k=1}^\infty$. ☐

Comment: Notice that $\mathcal{P}(\mathbf{F})$ and $\mathbf{F}^\infty$ are not iso, so are $\mathcal{P}(\mathbf{F})$ and $\big(\mathcal{P}(\mathbf{F})\big)'$

But if we let $\mathbf{F}^\infty = \big\{(a_1, \cdots, a_n, \underbrace{0, \cdots, 0, \cdots}_{\text{all zero}}) \in \mathbf{F}^\infty \ \big|\ \exists!\, n \in \mathbf{N}^+\big\}$. Then $\mathcal{P}(\mathbf{F})$ and $\mathbf{F}^\infty$ are iso.

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7 *Show that the dual basis of $(1, x, \dots, x^m)$ of $\mathcal{P}_m(\mathbf{R})$ is $(\varphi_0, \varphi_1, \dots, \varphi_m)$, where $\varphi_k(p) = \dfrac{p^{(k)}(0)}{k!}$.*

Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

Solution:

$$\forall j, k \in \mathbf{N},\ (x^j)^{(k)} = \begin{cases} j(j-1)\dots(j-k+1)\cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\dots(j-j+1) = j! & j = k. \\ 0, & j \leqslant k. \end{cases} \Bigg|\ \text{Then}\ \ (x^j)^{(k)}(0) = \begin{cases} 0\ ,\ \ j \neq k. \\ k!\ ,\ \ j = k. \end{cases}$$ ☐

Or. Because $\forall j, k \in \{1, \dots, m\}$ such that $j \neq k$, $\varphi_k\big(x^j\big) = \frac{(x^j)^{(k)}(0)}{k!} = \frac{0}{k!} = 0$; $\varphi_k\big(x^k\big) = \frac{(x^k)^{(k)}(0)}{k!} = 1$.

Thus $\frac{p^{(k)}(0)}{k!}$ act exactly the same as φ_k on the same basis $(1, \dots, x^m)$, hence is just another def of φ_k.☐

Example: Suppose $m \in \mathbf{N}^+$. By [2.C.10], $B = \big(1, x-5, \dots, (x-5)^m\big)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

Let $\varphi_k = \dfrac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \dots, m$. Then $(\varphi_0, \varphi_1, \dots, \varphi_m)$ is the dual basis of B.

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34 *The double dual space of V, denoted by V'', is defined to be the dual space of V'.*
In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \to V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.

(a) *Show that Λ is a linear map from V to V''.*

(b) *Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*

(c) *Show that if V is finite-dim, then Λ* ***is an iso from V onto V''****.*

Suppose V is finite-dim. Then V and V' are iso, and finding an iso from V onto V' generally requires choosing a basis of V. In contrast, the iso Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

Solution:

(a) $\forall \varphi \in V', v, w \in V, a \in \mathbf{F}, \big(\Lambda(v + aw)\big)(\varphi) = \varphi(v + aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.

Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence Λ is linear.

(b) $$\begin{aligned} \big(T''(\Lambda v)\big)(\varphi) &= \big((\Lambda v) \circ T'\big)(\varphi) = (\Lambda v)\big(T'(\varphi)\big) \\ &= \big(T'(\varphi)\big)(v) = (\varphi \circ T)(v) = \varphi(Tv) = \big(\Lambda(Tv)\big)(\varphi). \end{aligned}$$

Hence $T''(\Lambda v) = \big(\Lambda(Tv)\big) \Rightarrow T'' \circ \Lambda = \Lambda \circ T$.

(c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is inje.

又 Because V is finite-dim. $\dim V = \dim V' = \dim V''$. Hence Λ is an iso. ☐

Ended