简介

这是我个人用于复习的「***Linear Algebra Done Right 3E/4E, by Sheldon Axler***」笔记，一本习题选答与课文补注。范围覆盖所有第三版和第四版的课文和习题（除了第一章 A 节、极少数结合上下文太过显而易见的习题。没有任何日后反复推敲价值的当堂习题和方法套路过于雷同的习题)。这份笔记尚处于缓慢的编撰进度中。

习题答案中,有我完全独立思考得出的,有抄 https://linearalgebras.com/ 的,有抄 https://math.stackexchange.com/ 的,有抄 LADR2eSolutions（By Axler）.pdf ，有抄最新的 LADR4eSolutions 经典最全（By Axler?）.pdf，还有请教别人，乃至请教 AI 得出来的。

这些文档的许可证件，除 LADR4eSolutions 经典最全（By Axler?）.pdf 找不到/没有指明外，都允许复制/引用。

课文补注中，除了我独立思考总结出的易错误区和技巧、难点之外，还（因为我想要兼容那些使用 LADR 第三版纸质书的读者，包括我在内）把 LADR4e 中对课文定理等等的修改也（作了简化和提炼）摘录上去。

题目标为正常数字 **N** 的，为第三版某章某节第 **N** 题（有个别题是第四版又删去的，这里，或直接摘录，或合并简化，仍然作保留；还有个别题是第四版增添条件、设问的，也一并写在第 **N** 题下)。题目标为 '•' 的，为第四版。因为要面向以第三版为主要教材的学习者，所以为了避免混淆，故而将题号（部分题目的实心黑点后有标注具体第四版的数字标号)、甚至章节略去（一些变动过大的章节除外)。题目顺序会有调换，在每章大标题处会交代清楚。除了原书第四版新加入的章节外，均使用原书第三版的索引。这也许对第四版的使用者很不友好，我在此欢迎有心人士将我的作品修改后在同样的 **CC BY NC SA** 条款下作为**衍生作品**发布。

因为使用中文会给我编撰这份笔记带来额外的中英文输入法切换的工作成本，况且对于专业学习者，直接使用英文不会造成任何困扰。但英文词句的冗长性拖慢我编撰/复习的效率，所以我对许多常用术语作了简写。

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作者序

我目前还没有能力和资格评论原书好坏以及线性代数课程教材选用的问题。但作为原书的学习者，我可以说：

相较于（其他课程的）其他教材，以 LADR 作为**自学读本**的**精学**计划，往往在执行中出现一次又一次的时间误判/超时，比如我最开始计划 $40 \times 8\mathrm{h}$ 完成 LADR 的精学，差不多是一天（8h）完成一节，还有额外的复习时间。但在实际学习中，（刨去笔记的功夫）完成到一半时，发现已经耗费了约 $35 \times 8\mathrm{h}$，于是我不得不重新估计 LADR 精学所需的总时间为 $70 \times 8\mathrm{h}$。这一点对于有学时/学期限制/应试要求的线性代数初学者来说很不安全。更主观地讲，这是因为 LADR 更像是一本参考手册，而不是一本细致入微的自学读本；如果把 LADR 作为初学线性代数第一教材和自学读本来学习，会面临不小的困难。

以上或许能劝退相当一部分打算入门的线性代数初学者。S.Axler 说这本书作为第二遍学习线性代数的教材更合适。我认为理由就是，在校的科班生第二遍学习线性代数时，也已经学习过了离散数学、抽象代数、数论、数学分析等课程，这些知识储备统统会化作一个叫 “mathematical maturity” 的东西，让他们面对 LADR 的课文和习题不再少见多怪、茫然无措。据此，我进一步认为，对于完全的初学者，想要完成 LADR 的精学，要么有很好的天赋，要么有与之相匹配的 “mathematical maturity”，再要么，拿出足够的耐心和毅力。幸运的是，在坚持学习 LADR 的过程中，这三样会一同增益。就我个人来说：课文一次看不懂，就多看几遍，一天看不懂，就分三天看；习题一个小时做不出来，就隔六个小时再尝试，一天做不出来，就隔天再尝试。这确实让我收获了独特的学习体验和能力，我迄今也无法在别处得到，因此我很珍视 LADR，我愿意为此编撰一份电子辅助书并免费公开于网络中。这本身并不花费什么，因为实际的时间开销包括了很多不相干的额外项目：初学 LaTeX、调整代码架构、了解许可证选用，诸如此类的各种波折，也不乏戏剧性。

我在学习过程中碰到了很多重大误区：**第一章中**，我一开始误认为 $W = \complement_V U \cup \{0\}$ 是唯一使得 $W \oplus U = V$ 的子空间，但这压根就不是子空间，而且 C 节习题中也提示这样的子空间 W 不唯一。**第二章中**，我随意地将 “线性无关的序列” 等同于有/无限维向量空间的基，没有任何理论依据，我也并不懂什么选择公理。**第三章 B 到 D 节中**，我总觉得子空间是超脱有限维的存在；因为放不下第二章无限维向量空间的基的情结，我刻意寻找那些避开涉及基的解法，一些臆测的结论和容易就找到反例。**第三章 E 节中**，我似乎对商空间有什么误解，觉得 $v + U = v' + U$ 如同变戏法一样，把 v 中一切带有 U 的部分抹除掉，让 v 变得纯粹独立于 U，为此我还单门发明了 Pure V/U 并试着证明一些命题，甚至用它发现了 F 节 23 题无限维情况下不依赖基的解法。后来我猛然发现我最开始的想法多么荒诞，却仍然放不下 Pure V/U 的情结。

Goto

1		B	C				6	A	B	C	D	
2	A	B	C				7	A	B	C	D	F*
3	A	B	C	D	E	F	8	A	B	C	D	
4							9	A	B			
5	A	B^{I}	B^{II}	C	E*		10	A	B			

Abbreviation Table

A B

add	addi(tion)(tive)
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because
bss	basis
bses	bases
B_V	basis of V

C

ch	characteristic
closd	closed under
coeff	coefficient
col	column
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
ctradic	contradict(s)(ion)
ctrapos	constrapositive

D

def	definition
deg	degree
dep	dependen(t)(ce)
deri	derivative(s)
diag	diagonal(iza-ble/ility/tion)
diff	differentia(l)(ting)(tion)
diffce	difference
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

E

-ec	-ec(t)(tor)(tion)(tive)
eig-	eigen-
elem	element(s)
ent	entr(y)(ies)
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expr	expression

F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
G disk	Gershgorin disk
homo	homogeneity
hypo	hypothesis

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
invar	invariant
invard	invariant under
invarsp	invariant subspace
iso	isomorph(ism)(ic)

L

liney	linear(ly)
linity	linearity
len	length

M N O P Q

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
notat	notation(al)
optor	operator
othws	otherwise
poly	polynomial
quotient	quot

R

recurly	recursively
rev	revers(e(s))(ed)(ing)
restr	restrict(ion)(ive)(ing)
req	require(s)(d)/requiring
respectly	respectively

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

T U V W X Y Z

trig	triangular
trslate	translate
trspose	transpose
uniq	unique
uniqnes	uniqueness
val	value
wrto	with respect to

1·B

1 *Prove* $\forall v \in V, -(-v) = v$.

Solus: $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$.

Or. Becs $-(-v) + (-v) = 0$ 又 $v + (-v) = 0$. Now by the uniqnes of add inv. ☐

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2 *Supp* $a \in \mathbf{F}, v \in V$, *and* $av = 0$. *Prove* $a = 0$ *or* $v = 0$.

Solus: Supp $a \neq 0, \exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. ☐

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3 *Supp* $v, w \in V$. *Explain why* $\exists\, !\, x \in V, v + 3x = w$.

Solus: $v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \frac{1}{3}(w - v)$. ☐

Or. [*Existns*] Let $x = \dfrac{1}{3}(w - v)$.

[*Uniqnes*] If $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) − (II) : $3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. ☐

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5 *Show in the def of a vecsp, the add inv cond can be replaced by* [1.29].

Hint: *Supp V satisfies all conds in the def, except we've replaced the add inv cond with* [1.29].

Prove the add inv is true.

Using [1.31]. $0v = 0$ for all $v \in V \Longleftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. ☐

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6 *Let* ∞ *and* $-\infty$ *denote two disti objects, neither of which is in* $\mathbf{R}$.
Define an add and scalar multi on $\mathbf{R} \cup \{\infty, -\infty\}$ *as you could guess.*
The operations of real numbers is as usual. While for $t \in \mathbf{R}$ *define*

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

(I) $t + \infty = \infty + t = \infty + \infty = \infty$,
(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,
(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of add and scalar multi, is $\mathbf{R} \cup \{\infty, -\infty\}$ *a vecsp over* $\mathbf{R}$? *Explain.*

Solus: Not a vecpsp, since the add and scalar mult is not assoc and distr.

By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

Or. By Distr: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. ☐

• **Tips:** *About the Field* $\mathbf{F}$: *Many choices.* [*Req Multi Inv Uniq*]

Exa: $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \ldots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$ *suth* $(m-1)$ *is a prime.*

Ended

1·C

7 8 9 11 12 13 15 16 17 18 21 23 24

• **Note For [1.45]:** If $\mathbf{F} = \{0, 1\}$. Prove if $U + W$ is a direct sum, then $U \cap W = \{0\}$.

Becs $\forall v \in U \cap W, \exists\, !\, (u, w) \in U \times W, v = u + w$.

If $U \cap W \neq \{0\}$, then (u, w) can be $(v, 0)$ or $(0, v)$, ctradic the uniqnes. ☐

• **TIPS 1:** Supp $U, W \subseteq V$. And U, W, V are vecsps $\Rightarrow U, W$ are subsps of V.

Then $U + W$ is also a subsp of V. Becs $\forall u \in U, w \in U, u + w \in V$ since $u, w \in V$.

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7 *Give a nonempty $U \subseteq \mathbf{R}^2$, U is closd taking add invs and add, but is not a subsp of $\mathbf{R}^2$.*

SOLUS: $\big(0 \in U;\ v \in U \Rightarrow -v \in U$. And operations on U are the same as $\mathbf{R}^2$. $\big)$ Let $\mathbf{Z}^2, \mathbf{Q}^2$.

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8 *Give a nonempty $U \subseteq \mathbf{R}^2$, U is closd scalar multi, but is not a subsp of $\mathbf{R}^2$.*

SOLUS: Let $U = \big\{(x, y) \in \mathbf{R}^2 : x = 0 \vee y = 0\big\}$.

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9 *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if $\exists\, p \in \mathbf{N}^+$, $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions $\mathbf{R} \to \mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$? Explain.*

SOLUS: Denote the set by S.

Supp $h(x) = \cos x + \sin\sqrt{2}x \in S$, since $\cos x, \sin\sqrt{2}x \in S$.

Asum $\exists\, p \in \mathbf{N}^+$ suth $h(x) = h(x+p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$

$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Ctradic! $\square$

OR. Becs $\cos x + \sin\sqrt{2}x = \cos(x+p) + \sin\big(\sqrt{2}x + \sqrt{2}p\big)$. By diff twice,

$$\cos x + 2\sin\sqrt{2}x = \cos(x+p) + 2\sin\big(\sqrt{2}x + \sqrt{2}p\big).$$

$$\left.\begin{aligned} \sin\sqrt{2}x &= \sin\big(\sqrt{2}x + \sqrt{2}p\big) \\ \cos x &= \cos(x+p) \end{aligned}\right\} \Rightarrow \text{Let } x = 0,\ p = \frac{m\pi}{\sqrt{2}} = 2k\pi. \text{ Ctradic.}$$ $\square$

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24 *Let $V_E = \big\{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is even}\big\}$, $V_O = \big\{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is odd}\big\}$. Show $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$.*

SOLUS: (a) $V_E \cap V_O = \big\{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x)\big\} = \{0\}$.

(b) $$\left|\begin{aligned} &\text{Let } f_e(x) = \frac{1}{2}\Big[g(x) + g(-x)\Big] \Longrightarrow f_e \in V_E \\ &\text{Let } f_o(x) = \frac{1}{2}\Big[g(x) - g(-x)\Big] \Longrightarrow f_o \in V_O \end{aligned}\right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}},\ g(x) = f_e(x) + f_o(x).$$ $\square$

• *Supp U, W, V_1, V_2, V_3 are subsps of V.*

15 $U + U \ni u + w \in U$. **16** $U + W \ni u + w = w + u \in W + U$. $\square$

17 $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3)$. $\square$

• $(U + W)_{\mathbf{C}} \ni (u_1 + w_1) + \mathrm{i}(u_2 + w_2) = (u_1 + \mathrm{i}u_2) + (w_1 + \mathrm{i}w_2) \in U_{\mathbf{C}} + W_{\mathbf{C}}$. $\square$

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18 *Does the add on the subsps of V have an add id? Which subsps have add invs?*

SOLUS: Supp Ω is the uniq add id.

(a) For any subsp U of V. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

(b) Now supp W is an add inv of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. $\square$

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11 *Prove the intersec of every collec of subsps of V is a subsp of V.*

Solus: Supp $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collec of subsps of V; here Γ is an index set.

We show $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vecs that are in U_α for each $\alpha \in \Gamma$, is a subsp of V.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closd add.

(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closd scalar multi.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of V that is closd add and scalar multi. □

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12 *Supp U, W are subsps of V. Prove $U \cup W$ is a subsp of $V \iff U \subseteq W$ or $W \subseteq U$.*

Solus: (a) Supp $U \subseteq W$. Then $U \cup W = W$ is a subsp of V.

(b) Supp $U \cup W$ is a subsp of V. Asum $U \not\subseteq W,\ U \not\supseteq W$ ($U \cup W \neq U$ and W).

Then $\forall a \in U \wedge a \notin W, \forall b \in W \wedge b \notin U$, we have $a + b \in U \cup W$.

$$\left.\begin{aligned} a + b \in U &\Rightarrow b = (a+b) + (-a) \in U, \text{ctradic} \Rightarrow W \subseteq U. \\ a + b \in W &\Rightarrow a = (a+b) + (-b) \in W, \text{ctradic} \Rightarrow U \subseteq W. \end{aligned}\right| \text{ Ctradic asum.}$$

□

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13 *Prove the union of three subsps of V is a subsp of V if and only if one of the subsps contains the other two.*

This exe is not true if we replace $\mathbf{F}$ with a field containing only two elems.

Solus:

Supp U_1, U_2, U_3 are subsps of V. Denote $U_1 \cup U_2 \cup U_3$ by $\mathcal{U}$.

(a) Supp that one of the subsps contains the other two.

Then $\mathcal{U} = U_1, U_2$ or U_3 is a subsp of V.

(b) Supp that $U_1 \cup U_2 \cup U_3$ is a subsp of V.

Distinctively notice that $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.

Also note that, if $U \cup W = V$ is a vecsp, then in general U and W are not subsps of V.

Hence this literal trick is invalid.

(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.

By applying Exe (12) we conclude that one U_j contains the other two. Thus done.

(II) Asum no U_j is contained in the union of the other two,

and no U_j contains the union of the other two. Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3;\ v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if any $v + \lambda u \in W \cap U_1$ then $v + \lambda u - \lambda u = v \in U_1$.

Now $W \subseteq U_1 \cup U_2 \cup U_3 \Rightarrow W \subseteq U_2 \cup U_3$. $\forall v + \lambda u \in W, v + \lambda u \in U_i, i = 2, 3$.

If $U_2 \subseteq U_3$ or $U_2 \supseteq U_3$, then $\mathcal{U} = U_1 \cup U_i, i = 2, 3$. By Exe (12) done.

Othws, both $U_2, U_3 \neq \{0\}$. Becs $W \subseteq U_2 \cup U_3$ *has at least three elems.*

There must be some U_i that contains at least two elems of W.

$\exists$ disti $\lambda_1, \lambda_2 \in \mathbf{F}, v + \lambda_1 u, v + \lambda_2 u \in U_i, i \in \{2, 3\}$.

Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Ctradic. □

Exa: Let $\mathbf{F} = \mathbf{Z}_2$. $U_1 = \{u, 0\}, U_2 = \{v, 0\}, U_3 = \{v + u, 0\}$. While $\mathcal{U} = \{0, u, v, v + u\}$ is a subsp.

• *Supp* $U = \{(x,x,y,y)\}, W = \{(x,x,x,y)\} \subseteq \mathbf{F}^4$. *Prove* $U + W = \{(x,x,y,z)\}$.

Solus: Let T denote $\{(x,x,y,z)\}$. By def, $U + W \subseteq T$.

And $T \ni (x,x,y,z) \Rightarrow (0,0,y-x,y-x) + (x,x,x,-y+x+z) \in U + W$. Hence $T \subseteq U + W$. $\square$

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21 *Supp* $U = \{(x,y,x+y,x-y,2x)\}$. *Find a* W *suth* $\mathbf{F}^5 = U \oplus W$.

Solus: Let $W = \{(0,0,z,w,u)\}$. Then $U \cap W = \{0\}$.

And $\mathbf{F}^5 \ni (x,y,z,w,u) \Rightarrow (x,y,x+y,x-y,2x) + (0,0,z-x-y,w-x-y,u-2x) \in U + W$.

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23 *Give an exa of vecsps* V_1, V_2, U *suth* $V_1 \oplus U = V_2 \oplus U$, *but* $V_1 \neq V_2$.

Solus: $V = \mathbf{F}^2, U = \{(x,x)\}, V_1 = \{(x,0)\}, V_2 = \{(0,x)\}$.

• **Note For** "$\complement_V U \cup \{0\}$": "$\complement_V U \cup \{0\}$" is supposed to be a subsp W suth $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cup \{0\} \\ u \pm w \in \complement_V U \cup \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cup \{0\}$. Ctradic.

To fix this, denote the set $\{W_1, W_2, \cdots\}$ by $\mathcal{S}_V U$, where for each W_i, $V = U \oplus W_i$. See also in (1.C.23).

• **Tips 2:** *Supp* $V_1 \subseteq V_2$ *in Exe* (23). *Prove* $V_1 = V_2$.

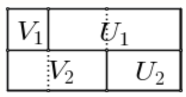
Solus:

Becs the subset V_1 of vecsp V_2 is closd add and scalar multi, V_1 is a subspace of V_2.

Supp W is suth $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$.

If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, ctradic. Hence $W = \{0\}, V_1 = V_2$. $\square$

• *Supp* V_1, V_2, U_1, U_2 *are vecsps,* $V_1 \oplus U_1 = V_2 \oplus U_2$, $V_1 \subseteq V_2$, $U_2 \subseteq U_1$.
Prove or give a counterexa: $V_1 = V_2$, $U_1 = U_2$.



Solus: Let $U_2 = \{0\}$. Give an exa that each of V_1, V_2, U_1 is nonzero. $\square$

• **Tips 3:** *Supp the intersec of any two of the vecsps* U, W, X, Y *is* $\{0\}$.
Give an exa that $(X \oplus U) \cap (Y \oplus W) \neq \{0\}$.

Solus: [Using notas in Chapter 2.] Let $B_X = (e_1), B_U = (e_2 - e_1), B_Y = (\ ), B_W = (e_2)$.

• **Tips 4:** *Let* $V = U + W$, $I = U \cap W$, $U = I \oplus X$, $W = I \oplus Y$. *Prove* $V = I \oplus (X \oplus Y)$.

Solus: We show $X \cap Y = U \cap Y = W \cap X = \{0\}$ by ctradic.

$X \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap X \neq \{0\}, I \cap Y \neq \{0\}$.

$U \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap Y \neq \{0\}$. Simlr for $W \cap X$.

Thus $I + (X + Y) = (I \oplus X) \oplus Y = I \oplus (X \oplus Y)$.

Now we show $V = I + (X + Y)$. $\forall v \in V, v = u + w, \exists (u,w) \in U \times W$

$\Rightarrow \exists (i_u, x_u) \in I \times X$, $(i_w, y_w) \in I \times Y$, $v = (i_u + i_w) + x_u + y_w \in I + (X + Y)$. $\square$

Ended

2·A   1 2 10 11 14 16 17 | 4E: 3,14

1 *Prove [P] (v_1, v_2, v_3, v_4) spans $V \iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V [Q].*

SOLUS: Note that $V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists a_1, \dots, a_n \in \mathbf{F}, v = a_1v_1 + \cdots + a_nv_n$.

Asum $\forall v \in V, \exists a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbf{F}$, (that is, if $\exists a_i$, then we are to find b_i, vice versa)

$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4$
$\quad = b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4$
$\quad = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \cdots + a_4)v_4.$ □

• *Supp $(v_1, \dots, v_m)$ is a list in V. For each k, let $w_k = v_1 + \cdots + v_k$.*
(a) *Show* $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.
(b) *Show [P] $(v_1, \dots, v_m)$ is liney indep $\iff (w_1, \dots, w_m)$ is liney indep [Q].*

SOLUS:

(a) Asum $a_1v_1 + \cdots + a_mv_m = b_1w_1 + \cdots + b_mw_m = b_1v_1 + \cdots + b_k(v_1 + \cdots + v_k) + \cdots + b_m(v_1 + \cdots + v_m)$.
Then $a_k = b_k + \cdots + b_m;\ \ a_{k+1} = b_{k+1} + \cdots + b_m \Rightarrow b_k = a_k - a_{k+1};\ b_m = a_m$. Simlr to Exe (1).

(b) $P \Rightarrow Q:\ \ b_1w_1 + \cdots + b_mw_m = 0 = a_1v_1 + \cdots + a_mv_m$, where $0 = a_k = b_k + \cdots + b_m$.
$Q \Rightarrow P:\ \ a_1v_1 + \cdots + a_mv_m = 0 = b_1w_1 + \cdots + b_mw_m = 0$, where $0 = b_m = a_m,\ 0 = b_k = a_k - a_{k+1}$.

OR. By (a), let $W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$. Supp $(v_1, \dots, v_m)$ is liney dep.
By [2.21](b), a list of len $(m - 1)$ spans W. 又 By [2.23], $(w_1, \dots, w_m)$ liney indep $\Rightarrow m \leqslant m - 1$.
Thus $(w_1, \dots, w_m)$ is liney dep. Now rev the roles of v and w. □

---

2 (a) [P] *A list (v) of len 1 in V is liney indep $\iff v \neq 0$.* [Q]
(b) [P] *A list (v, w) of len 2 in V is liney indep $\iff \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$.* [Q]

SOLUS: (a) $Q \Rightarrow P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ liney indep.
$P \Rightarrow Q : (v)$ liney indep $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.
$\neg Q \Rightarrow \neg P : v = 0 \Rightarrow av = 0$ while we can let $a \neq 0 \Rightarrow (v)$ is liney dep.
$\neg P \Rightarrow \neg Q : (v)$ liney dep $\Rightarrow av = 0$ while $a \neq 0 \Rightarrow v = 0$.

(b) $P \Rightarrow Q : (v, w)$ liney indep $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.
$Q \Rightarrow P :$ no scalar multi $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v, w)$ liney indep.
$\neg P \Rightarrow \neg Q : (v, w)$ liney dep $\Rightarrow$ if $av + bw = 0$, then a or $b \neq 0 \Rightarrow$ scalar multi.
$\neg Q \Rightarrow \neg P :$ scalar multi $\Rightarrow$ if $av + bw = 0$, then a or $b \neq 0 \Rightarrow$ liney dep. □

---

10 *Supp $(v_1, \dots, v_m)$ is liney indep in V and $w \in V$.*
Prove if $(v_1 + w, \dots, v_m + w)$ is linely depe, then $w \in \operatorname{span}(v_1, \dots, v_m)$.

SOLUS:

Note that $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0 \Rightarrow a_1v_1 + \cdots + a_mv_m = -(a_1 + \cdots + a_m)w$.
Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1v_1 + \cdots + a_mv_m = 0$ while $a_i \neq 0$ for some i, ctradic.
OR. We prove the ctrapos: Supp $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $a_1 + \cdots + a_m = 0$.
Thus $a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$. Hence $(v_1 + w, \dots, v_m + w)$ is liney indep. □
OR. $\exists j \in \{1, \dots, m\}, v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and done.
If $j \geqslant 2$, then $\exists a_i \in \mathbf{F}, v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1v_1 + \cdots + a_{j-1}v_{j-1}$.
Where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, ctradic.
Now $w = \lambda^{-1}(a_1v_1 + \cdots + a_{j-1}v_{j-1} - v_j) \Rightarrow w \in \operatorname{span}(v_1, \dots, v_m)$. □

11 *Supp $(v_1, \ldots, v_m)$ is liney indep in V and $w \in V$.*
Show [P] $(v_1, \ldots, v_m, w)$ is liney indep $\iff w \notin \operatorname{span}(v_1, \ldots, v_m)$ [Q].

Solus: Equiv to $(v_1, \ldots, v_m, w)$ liney dep $\iff w \in \operatorname{span}(v_1, \ldots, v_m)$. Using [2.21]. Obviously. $\square$

Note: (a) Supp $(v_1, \ldots, v_m, w)$ is liney indep. Then $(v_1, \ldots, v_m)$ liney indep $\iff w \notin \operatorname{span}(v_1, \ldots, v_m)$.
(b) Supp $(v_1, \ldots, v_m, w)$ is liney dep. Then $(v_1, \ldots, v_m)$ liney indep $\iff w \in \operatorname{span}(v_1, \ldots, v_m)$.

---

14 *Prove [P] V is infinide $\iff \exists$ seq $(v_1, v_2, \ldots)$ in V suth each $(v_1, \ldots, v_m)$ liney indep. [Q]*

Solus:

$P \Rightarrow Q$: Supp V is infinide, so that no list spans V.
Step 1 Pick a $v_1 \neq 0$, (v_1) liney indep.
Step m Pick a $v_m \notin \operatorname{span}(v_1, \ldots, v_{m-1})$, by Exe (11), $(v_1, \ldots, v_m)$ is liney indep.
This process recurly defines the desired seq $(v_1, v_2, \ldots)$.

$\neg P \Rightarrow \neg Q$: Supp V is finide and $V = \operatorname{span}(w_1, ..., w_m)$.
Let $(v_1, v_2, \ldots)$ be a seq in V, then $(v_1, v_2, \ldots, v_{m+1})$ must be liney dep.

Or. $Q \Rightarrow P$: Supp there is such a seq.
Choose an m. Supp a liney indep list $(v_1, \ldots, v_m)$ spans V.
Simlr to [2.16]. $\exists\, v_{m+1} \in V \backslash \operatorname{span}(v_1, \ldots, v_m)$. Hence no list spans V. $\square$

---

16 *Prove the vecsp of all continuous functions in $\mathbf{R}^{[0,1]}$ is infinide.*

Solus: Denote the vecsp by U.

Choose one $m \in \mathbf{N}^+$. Supp $a_0, \ldots, a_m \in \mathbf{R}$ are suth $p(x) = a_0 + a_1 x + \cdots + a_m x^m = 0$, $\forall x \in [0, 1]$.
Then p has infily many roots and hence each $a_k = 0$, othws $\deg p \geqslant 0$, ctradic [4.12].
Thus $(1, x, \ldots, x^m)$ is liney indep in $\mathbf{R}^{[0,1]}$. Simlr to [2.16], U is infinide. $\square$

Or. Note that $\dfrac{1}{1} > \dfrac{1}{2} > \cdots > \dfrac{1}{m}$, $\forall m \in \mathbf{N}^+$. Supp $f_m = \begin{cases} x - \frac{1}{m}, & x \in \left(\frac{1}{m}, 1\right] \\ 0, & x \in \left[0, \frac{1}{m}\right] \end{cases}$

Then $f_1\left(\frac{1}{m}\right) = \cdots = f_m\left(\frac{1}{m}\right) = 0 \neq f_{m+1}\left(\frac{1}{m}\right)$. Hence $f_{m+1} \notin \operatorname{span}(f_1, \ldots, f_m)$. By Exe (14). $\square$

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17 *Supp $p_0, p_1, \ldots, p_m \in \mathcal{P}_m(\mathbf{F})$ suth $p_k(2) = 0$ for each $k \in \{0, \ldots, m\}$.*
Prove $(p_0, p_1, \ldots, p_m)$ is not liney indep in $\mathcal{P}_m(\mathbf{F})$.

Solus:

Supp $(p_0, p_1, \ldots, p_m)$ is liney indep. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z$.
Notice that $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \cdots + a_m p_m(z)$, for if not, let $z = 2$. Thus $z \notin \operatorname{span}(p_0, p_1, \ldots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \ldots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \ldots, p_m)$ has len $(m + 1)$.
Hence $(p_0, p_1, \ldots, p_m)$ is linely depe. For if not, then becs $(1, z, \ldots, z^m)$ of len $(m + 1)$ spans $\mathcal{P}_m(\mathbf{F})$, by the steps in [2.23] trivially, $(p_0, p_1, \ldots, p_m)$ of len $(m + 1)$ spans $\mathcal{P}_m(\mathbf{F})$. Ctradic. $\square$

Or. Note that $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(\underbrace{1, z, \ldots, z^m}_{\text{of len } (m+1)})$. Then $(p_0, p_1, \ldots, p_m, z)$ of len $(m + 2)$ is liney dep.
As shown above, $z \notin \operatorname{span}(p_0, p_1, \ldots, p_m)$. And hence by [2.21](a), $(p_0, p_1, \ldots, p_m)$ is liney dep. $\square$

Ended

2·B 1 8 | 4E: 5 9 11

- **Tips:** *Supp* $\dim V = n$, *and* U *is a subsp of* V *with* $U \neq V$.
  Prove $\exists\, B_V = (v_1, \dots, v_n)$ *suth each* $v_k \notin U$.

Note that $U \neq V \Rightarrow n \geqslant 1$. We will construct B_V via the following process.

Step 1. $\exists\, v_1 \in V \backslash U \Rightarrow v_1 \neq 0$. If $\operatorname{span}(v_1) = V$ then we stop.

Step k. Supp $(v_1, \dots, v_{k-1})$ is liney indep in V, each of which belongs to $V \backslash U$.

Note that $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$. And if $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U = V$, then by (1.C.12),

$\big[$ becs $\operatorname{span}(v_1, \dots, v_{k-1}) \not\subseteq U$, $\big]$ $U \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) \Rightarrow \operatorname{span}(v_1, \dots, v_{k-1}) = V$.

Hence becs $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$, it must be case that $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U \neq V$.

Thus $\exists\, v_k \in V \backslash U$ suth $v_k \notin \operatorname{span}(v_1, \dots, v_{k-1})$.

By (2.A.11), $(v_1, \dots, v_k)$ is liney indep in V. If $\operatorname{span}(v_1, \dots, v_k) = V$, then we stop.

Becs V is finide, this process will stop after n steps. ☐

Or. Supp $U \neq \{0\}$. Let $B_U = (u_1, \dots, u_m)$. Extend to a bss $(u_1, \dots, u_n)$ of V.

Then let $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$. ☐

---

1 *Find all vecsps on whatever* $\mathbf{F}$ *that have exactly one bss.*

Solus: The trivial vecsp $\{0\}$ will do. Indeed, the only bss of $\{0\}$ is the empty list $(\ )$.

Now consider the field $\{0, 1\}$ containing only the add id and multi id,

with $1 + 1 = 0$. Then the list (1) is the uniq bss. Now the vecsp $\{0, 1\}$ will do.

Comment: All vecsp on such $\mathbf{F}$ of dim 1 will do.

Consider other $\mathbf{F}$. Note that this $\mathbf{F}$ contains at least and strictly more than 0 and 1. Failed. ☐

- (4E 9) *Supp* $(v_1, \dots, v_m)$ *is a list in* V. *For* $k \in \{1, \dots, m\}$, *let* $w_k = v_1 + \cdots + v_k$.
  Show $[P]$ $B_V = (v_1, \dots, v_m) \Longleftrightarrow B_V = (w_1, \dots, w_m)$. $[Q]$

Solus: Notice that $B_U = (u_1, \dots, u_n) \Longleftrightarrow \forall u \in U, \exists!\, a_i \in \mathbf{F}, u = a_1 u_1 + \cdots + a_n u_n$.

$P \Rightarrow Q : \forall v \in V, \exists!\, a_i \in \mathbf{F},\ v = a_1 v_1 + \cdots + a_m v_m \Rightarrow v = b_1 w_1 + \cdots + b_m w_m, \exists!\, b_k = a_k - a_{k+1}, b_m = a_m$.

$Q \Rightarrow P : \forall v \in V, \exists!\, b_i \in \mathbf{F},\ v = b_1 w_1 + \cdots + b_m w_m \Rightarrow v = a_1 v_1 + \cdots + a_m v_m, \exists!\, a_k = \sum_{j=k}^{m} b_j$. ☐

Comment: Or. Using [3.C Note For [3.30, 32](a)].

- (4E 5) *Supp* U, W *are finide*, $V = U + W$, $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$.
  Prove $\exists\, B_V$ *consisting of vecs in* $U \cup W$.

Solus: $V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(\overbrace{u_1, \dots, u_m, w_1, \dots, w_n}^{\text{Reduce}})$. By [2.31]. ☐

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8 *Supp* $V = U \oplus W$, $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$.
Prove $B_V = \big(u_1, \dots, u_m, w_1, \dots, w_n\big)$.

Solus: $\forall v \in V, \exists!\, u \in U, w \in W \Rightarrow \exists!\, a_i, b_i \in \mathbf{F}, v = u + w = \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i$.

Or. $V = \operatorname{span}(u_1, \dots, u_m) \oplus \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.

Note that $\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0 \Rightarrow \sum_{i=1}^{m} a_i u_i = -\sum_{i=1}^{n} b_i w_i \in U \cap W = \{0\}$. ☐

• **Note For** ***liney indep seq and*** **[2.34]:** "$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expr.
If we allow using "infini list", then we must assure that $(v_1, \dots, v_n, \dots)$ is a spanning "list"
suth $\forall v \in V, \exists$ smallest $n \in \mathbf{N}^+$, $v = a_1v_1 + \cdots + a_nv_n$. Moreover, given a list $(w_1, \cdots, w_n, \cdots)$ in W,
we can prove $\exists ! T \in \mathcal{L}(V, W)$ with each $Tv_k = w_k$, which has less restr than [3.5].
But the key point is, how can we assure that such a "list" exis. TODO: More details.

• (9.A.3,4 Or 4E 11) *Supp V is on $\mathbf{R}$, and $v_1, \dots, v_n \in V$. Let $B = (v_1, \dots, v_n)$.*
(a) *Show [P] B is liney indep in V $\iff$ B is liney indep in $V_{\mathbf{C}}$. [Q]*
(b) *Show [P] B spans V $\iff$ B spans $V_{\mathbf{C}}$. [Q]*

(a) $P \Rightarrow Q:$ Note that each $v_k \in V_{\mathbf{C}}$. $\quad Q \Rightarrow P:$ If $\lambda_k \in \mathbf{R}$ with $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then each $\operatorname{Re}\lambda_k = \lambda_k = 0$.
$\neg P \Rightarrow \neg Q:$ $\exists\, v_j = a_{j-1}v_{j-1} + \cdots + a_1 v_1 \in V_{\mathbf{C}}$.
$\neg Q \Rightarrow \neg P:$ $\exists\, v_j = \lambda_{j-1}v_{j-1} + \cdots + \lambda_1 v_1 \Rightarrow v_j = (\operatorname{Re}\lambda_{j-1})v_{j-1} + \cdots + (\operatorname{Re}\lambda_1)v_1 \in V$.
(b) $P \Rightarrow Q:$ $\forall u + \mathrm{i}v \in V_{\mathbf{C}},\ u, v \in V \Rightarrow \exists\, a_i, b_i \in \mathbf{R}, u + \mathrm{i}v = \sum_{i=1}^{n} (a_i + \mathrm{i}b_i)v_i$.
$Q \Rightarrow P:$ $\forall v \in V, \exists\, a_i + \mathrm{i}b_i \in \mathbf{C},\ v + \mathrm{i}0 = \left(\sum_{i=1}^{n} a_i v_i\right) + \mathrm{i}\left(\sum_{i=1}^{n} b_i v_i\right) \Rightarrow v \in \operatorname{span}(v_1, \dots, v_m)$.
$\neg Q \Rightarrow \neg P:$ $\exists\, v \in V, v \notin \operatorname{span}(B) \Rightarrow v + \mathrm{i}0 \notin \operatorname{span}(B)$ while $v + \mathrm{i}0 \in V_{\mathbf{C}}$.
$\neg Q \Rightarrow \neg P:$ $\exists\, u + \mathrm{i}v \in V_{\mathbf{C}}, u + \mathrm{i}v \notin \operatorname{span}(B) \Rightarrow u$ or $v \notin \operatorname{span}(B)$. Note that $u, v \in V$. $\square$

Ended

2·C

1 7 9 10 14,16 15 17 | 4E: 10 14,15 16

15 *Supp* $\dim V = n \geqslant 1$. *Prove $\exists$ 1-dim subsps $V_1, \dots, V_n$ suth $V = V_1 \oplus \cdots \oplus V_n$.*

Solus: Supp $B_V = (v_1, \dots, v_n)$. Define V_i by $V_i = \operatorname{span}(v_i)$ for each $i \in \{1, \dots, n\}$.
Then $\forall v \in V, \exists ! a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n \Rightarrow \exists ! u_i \in V_i, v = u_1 + \cdots + u_n$ $\square$

• **Note For Exe (15):** *Supp $v \in V \backslash \{0\}$. Prove $\exists\, B_V = (v_1, \dots, v_n), v = v_1 + \cdots + v_n$.*

Solus: If $n = 1$ then let $v_1 = v$ and done. Supp $n > 1$.
Extend (v) to a bss $(v, v_1, \dots, v_{n-1})$ of V. Let $v_n = v - v_1 - \cdots - v_{n-1}$.
又 $\operatorname{span}(v, v_1, \dots, v_{n-1}) = \operatorname{span}(v_1, \dots, v_n)$. Hence $(v_1, \dots, v_n)$ is also a bss of V. $\square$

Comment: Let $B_V = (v_1, \dots, v_n)$ and supp $v = u_1 + \cdots + u_n$, where each $u_i = a_i v_i \in V_i$.
But $(u_1, \dots, u_n)$ might not be a bss, becs there might be some $u_i = 0$.

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1 [CORO for [2.38,39]] *Supp U is a subsp of V suth* $\dim V = \dim U$. *Then $V = U$.*

Let $B_U = (u_1, \dots, u_m)$. Then $m = \dim V$. 又 $u_i \in V$. By [2.39], $B_V = (u_1, \dots, u_m)$. $\square$

- Let $v_1, \dots, v_n \in V$ and $\dim \operatorname{span}(v_1, \dots, v_n) = n$. Then $(v_1, \dots, v_n)$ is a bss of $\operatorname{span}(v_1, \dots, v_n)$.

Notice that $(v_1, \dots, v_n)$ is a spanning list of $\operatorname{span}(v_1, \dots, v_n)$ *of len* $n = \dim \operatorname{span}(v_1, \dots, v_n)$.

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7 (a) *Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a bss of U.*

(b) *Extend the bss in* (b) *to a bss of $\mathcal{P}_4(\mathbf{F})$.*

(c) *Find a subsp W of $\mathcal{P}_4(\mathbf{F})$ suth $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.*

SOLUS: Using Exe (10).

NOTICE that $\nexists\, p \in \mathcal{P}(\mathbf{F})$ of deg 1 and 2, while $p \in U$. Thus $\dim U \leqslant \dim \mathcal{P}_4(\mathbf{F}) - 2 = 3$.

(a) Consider $B = \big(1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

Let $a_0 + a_3(z-2)(z-5)(z-6) + a_4 z(z-2)(z-5)(z-6) = 0 \Rightarrow a_0 = a_3 = a_4 = 0$.

Thus the list B is liney indep in U. Now $\dim U \geqslant 3 \Rightarrow \dim U = 3$. Thus $B_U = B$.

(b) Extend to a bss of $\mathcal{P}_4(\mathbf{F})$ as $\big(1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. $\square$

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9 *Supp $(v_1, \dots, v_m)$ is liney indep in V, $w \in V$. Prove* $\dim \operatorname{span}\big(v_1 + w, \dots, v_m + w\big) \geq m - 1$.

SOLUS: Using the result of (2.A.10, 11).

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$, for each $i = 1, \dots, m$.

$(v_1, \dots, v_m)$ liney indep $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ liney indep $\Rightarrow \underbrace{(v_2 - v_1, \dots, v_m - v_1)}_{\text{of len } (m-1)}$ liney indep.

又 If $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1 + w, \dots, v_m + w)$ is liney indep.

Hence $m \geqslant \dim \operatorname{span}\big(v_1 + w, \dots, v_m + w\big) \geq m - 1$. $\square$

- (4E 16) *Supp V is finide, U is a subsp of V with $U \neq V$. Let $n = \dim V$, $m = \dim U$.*

Prove $\exists$ $(n-m)$ subsps $U_1, \dots, U_{n-m}$, each of dim $(n-1)$, suth $\bigcap_{i=1}^{n-m} U_i = U$.

SOLUS: Let $B_U = (v_1, \dots, v_m)$, $B_V = (v_1, \dots, v_m, u_1, \dots, u_{n-m})$.

Define $U_i = \operatorname{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m})$ for each i. Then $U \subseteq U_i$ for each i.

And becs $\forall v \in \bigcap_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \cdots + b_{n-m} u_{n-m} \in U_i \Rightarrow$ each $b_i = 0 \Rightarrow v \in U$.

Hence $\bigcap_{i=1}^{n-m} U_i \subseteq U$. $\square$

- **NOTE FOR Exe 10:** For each nonconst $p \in \operatorname{span}(1, z, \dots, z^m)$, $\exists$ smallest $m \in \mathbf{N}^+$, which is $\deg p$.

(a) If $p_0, p_1, \dots, p_m$ are suth all $a_{k,k} \neq 0$, and

$p_0 = a_{0,0}$, each $p_k = a_{0,k} + a_{1,k} z + \cdots + a_{k,k} z^k$.

Then the upper-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m} \\ 0 & a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix}$.

(b) If $p_0, p_1, \dots, p_m$ are suth all $a_{k,k} \neq 0$, and

$p_0 = a_{0,0} + \cdots + a_{m,0} x^m$, each $p_k = a_{k,k} x^k + \cdots + a_{m,k} x^m$.

Then the lower-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,0} & a_{m,1} & \cdots & a_{m,m} \end{pmatrix}$.

COMMENT: Define $\xi_k(p)$ by the coeff of z^k in $p \in \mathcal{P}_m(\mathbf{F})$.

Then $\mathcal{M}\big(\xi_k, (1, z, \dots, z^m), (1)\big) = \mathcal{E}^{(1,k)} \in \mathbf{F}^{1, m+1}$.

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10 *Supp* $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ *are suth each* $\deg p_k = k$.
Prove $(p_0, p_1, \dots, p_m)$ *is a bss of* $\mathcal{P}_m(\mathbf{F})$.

Solus: Using induc on m.

(i) $k = 1$. $\deg p_0 = 0$; $\deg p_1 = 1 \Rightarrow \operatorname{span}(p_0, p_1) = \operatorname{span}(1, x)$.

(ii) $1 \leqslant k \leqslant m-1$. Asum $\operatorname{span}(p_0, p_1, \dots, p_k) = \operatorname{span}(1, x, \dots, x^k)$.

Then $\operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \subseteq \operatorname{span}(1, x, \dots, x^k, x^{k+1})$.

又 $\deg p_{k+1} = k+1$, $p_{k+1}(x) = a_{k+1}x^{k+1} + r_{k+1}(x)$; $a_{k+1} \neq 0$, $\deg r_{k+1} \leqslant k$.

$\Rightarrow x^{k+1} = \dfrac{1}{a_{k+1}}\Big(p_{k+1}(x) - r_{k+1}(x)\Big) \in \operatorname{span}(1, x, \dots, x^k, p_{k+1}) = \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

$\therefore x^{k+1} \in \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \Rightarrow \operatorname{span}(1, x, \dots, x^k, x^{k+1}) \subseteq \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m)$. □

Or. 用比较系数法. Denote the coeff of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Supp $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is liney indep.

Step 1. For $k = m$, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m$, $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.
Now $L = a_{m-1}p_{m-1}(x) + \cdots + a_0 p_0(x)$.

Step k. For $0 \leqslant k \leqslant m$, we have $a_m = \cdots = a_{k+1} = 0$.
Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.
Now if $k = 0$, then done. Othws, we have $L = a_{k-1}p_{k-1}(x) + \cdots + a_0 p_0(x)$. □

• **Tips:** *Supp* $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ *are suth the lowest term of each* p_k *is of deg* k.
Prove $(p_0, p_1, \dots, p_m)$ *is a bss of* $\mathcal{P}_m(\mathbf{F})$.

Solus: Using induc on m.

Let each p_k be defined by $p_k(x) = a_{k,k}x^k + \cdots + a_{m,k}x^m$, where $a_{k,k} \neq 0$.

(i) $k = 1$. $p_m(x) = a_{m,m}x^m$; $p_{m-1}(x) = a_{m-1,m-1}x^{m-1} + a_{m,m-1}x^m \Longrightarrow \operatorname{span}(x^m, x^{m-1}) = \operatorname{span}(p_m, p_{m-1})$.

(ii) $1 \leqslant k \leqslant m-1$. Asum $\operatorname{span}(x^m, \dots, x^{m-k}) = \operatorname{span}(p_m, \dots, p_{m-k})$.

Then $\operatorname{span}(p_m, \dots, p_{m-(k+1)}) \subseteq \operatorname{span}(x^m, \dots, x^{m-(k+1)})$.

又 $p_{m-(k+1)}$ has the form $a_{m-(k+1),m-(k+1)}x^{m-(k+1)} + r_{m-(k+1)}(x)$;
where the lowest term of $r_{m-(k+1)} \in \mathcal{P}_m(\mathbf{F})$ is of deg $(m-k)$.

$$\Rightarrow x^{m-(k+1)} = \frac{1}{a_{m-(k+1),m-(k+1)}}\Big(p_{m-(k+1)}(x) - r_{m-(k+1)}(x)\Big) \in \operatorname{span}(x^m, \dots, x^{m-k}, p_{m-(k+1)}) = \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$$

$\therefore x^{m-(k+1)} \in \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$

$\Rightarrow \operatorname{span}(x^m, \dots, x^{m-k}, x^{m-(k+1)}) \subseteq \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}(p_m, \dots, p_1, p_0)$. □

Or. 用比较系数法. Denote the coeff of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Supp $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is liney indep.

Step 1. For $k = 0$, $\xi_0(L) = a_0 \xi_0(p_0) = \xi_0(R) = 0$ 又 $\deg p_0 = 0$, $\xi_0(p_0) \neq 0 \Rightarrow a_0 = 0$.
Now $L = a_1 p_1(x) + \cdots + a_m p_m(x)$.

Step k. For $0 \leqslant k \leqslant m$, we have $a_{k-1} = \cdots = a_0 = 0$.
Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.
Now if $k = m$, then done. Othws, we have $L = a_{k+1}p_{k+1}(x) + \cdots + a_m p_m(x)$. □

- **NOTE FOR [2.11]:** ***Good definition for a general term always aviods undefined behaviours.***
  If $\deg p = 0$, then $p(z) = a_0 \neq 0$, but not literally $a_0 z^0$, by which if p is defined, then it comes to 0^0.
  To make it clear, we specify that *in* $\mathcal{P}(\mathbf{F})$, $a_0 z^0 = a_0$, where z^0 appears just for nota conveni.
  Becs by def, the term $a_0 z^0$ in a poly only represents the const term of the poly, which is a_0.
  For conveni, we asum $z^0 = 1$ in formula deduction and poly def. Absolutely without 0^0.

- (4E 10) *Supp m is a positive integer. For $0 \leqslant k \leqslant m$, let $p_k(x) = x^k(1-x)^{m-k}$.*
  Show $(p_0, \dots, p_m)$ is a bss of $\mathcal{P}_m(\mathbf{F})$.

SOLUS: *We may see $p_0 = 1$ and $p_m(x) = x^m$, from the expansion below, by the* NOTE FOR [2.11] *above.*

Note that each $p_k(x) = \sum_{j=0}^{m-k} C_{m-k}^j (-1)^j \cdot x^{j+k} \cdot 1^j = \underbrace{(-1)^0 \cdot x^k \cdot 1^0}_{\text{of deg k}} + \underbrace{\sum_{j=1}^{m-k} C_{m-k}^j (-1)^j \cdot x^{j+k} \cdot 1^j}_{\text{of deg m; denote it by } q_k(x)}$.
And, each $q_k \in \operatorname{span}(x^{k+1}, \dots, x^m)$. Using TIPS above. ☐

OR. Simlr to the TIPS above. We will recurly prove each $x^{m-k} \in \operatorname{span}(p_m, \dots, p_{m-k})$.
(i) $k = 1$. $p_m(x) = x^m \in \operatorname{span}(p_m)$; $p_{m-1}(x) = x^{m-1} - x^m \Rightarrow x^{m-1} \in \operatorname{span}(p_{m-1}, p_m)$.
(ii) $k \in \{1, \dots, m-1\}$. Supp for each $k \in \{0, \dots, k\}$, we have $x^{m-k} \in \operatorname{span}(p_{m-k}, \dots, p_m), \exists!\, a_m \in \mathbf{F}$.
Note that $x^{m-(k+1)} = p_{m-(k+1)}(x) + \sum_{j=1}^{k+1} C_{k+1}^j (-1)^{j+1} x^{m-(k+1)+j} \in \operatorname{span}(p_{m-(k+1)}, x^{m-k}, \dots, x^m)$.
Thus $x^{m-(k+1)} \in \operatorname{span}(p_{m-(k+1)}, p_{m-k}, \dots, p_m)$. ☐

COMMENT: The base step and the induc step can be indep.

OR. For any $m, k \in \mathbf{N}^+$ suth $k \leqslant m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k(1-x)^{m-k}$.
Define the stmt $S(m)$ by $S(m) : (p_{0,m}, \dots, p_{m,m})$ is liney indep (and therefore is a bss).
We use induc on to show $S(m)$ holds for all $m \in \mathbf{N}^+$.
(i) $m = 0$. $p_{0,0} = 1$, and $ap_{0,0} = 0 \Rightarrow a = 0$.
$m = 1$. Let $a_0(1-x) + a_1 x = 0, \forall x \in \mathbf{F}$. Then take $x = 1, x = 0 \Rightarrow a_1 = a_0 = 0$.
(ii) $1 \leqslant m$. Asum $S(m)$ and $S(m-1)$ holds. Now we show $S(m+1)$ holds.
Supp $\sum_{k=0}^{m+1} a_k p_{k,m+1}(x) = \sum_{k=0}^{m+1} a_k \left[x^k(1-x)^{m+1-k}\right] = 0, \forall x \in \mathbf{F}$.
Now $a_0(1-x)^{m+1} + \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k} + a_{m+1}x^{m+1} = 0, \forall x \in \mathbf{F}$.
While $x = 0 \Rightarrow a_0 = 0$; and $x = 1 \Rightarrow a_{m+1} = 0$.
Then $0 = \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k}$
$= x(1-x) \sum_{k=1}^{m} a_k x^{k-1} (1-x)^{m-k}$, note that $m - k = (m-1) - (k-1)$
$= x(1-x) \sum_{k=0}^{m-1} a_{k+1} x^k (1-x)^{m-1-k} = x(1-x) \sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x)$.
Hence $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0, \forall x \in \mathbf{F} \setminus \{0, 1\}$. Which has infily many zeros.
Moreover, $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0$. By asum, $a_1 = \cdots = a_{m-1} = a_m = 0$.
Thus $(p_{0,m+1}, \dots, p_{m+1,m+1})$ is liney indep and $S(m+1)$ holds. ☐

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14 *Supp $V_1, \dots, V_m$ are finide. Prove* $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$.

SOLUS: For each V_i, let $B_{V_i} = \mathcal{E}_i$. Then $V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim V_i = \operatorname{card} \mathcal{E}_i$.
Now $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.

CORO: $V_1 + \cdots + V_m$ is direct
$\iff$ For each $k \in \{1, \dots, m-1\}$, $(V_1 \oplus \cdots \oplus V_k) \cap V_{k+1} = \{0\}$, $(\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$
$\iff \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$
$\iff \dim(V_1 \oplus \cdots \oplus V_m) = \dim V_1 + \cdots + \dim V_m$. ☐

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17 *Supp V_1, V_2, V_3 are subsps of a finide vecsp, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexa.

SOLUS:

$\left[\textit{Simlr to}\right]$ Given three sets A, B and C.

Becs $|X \cup Y| = |X| + |Y| - |X \cap Y|$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

Now $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$.

And $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$.

Hence $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$.

Note that $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned}\dim(V_1 + V_2 + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim\big((V_1 + V_2) \cap V_3\big) \quad (1)\\ &= \dim(V_2 + V_3) + \dim(V_1) - \dim\big((V_2 + V_3) \cap V_1\big) \quad (2)\\ &= \dim(V_1 + V_3) + \dim(V_2) - \dim\big((V_1 + V_3) \cap V_2\big) \quad (3).\end{aligned}$$

Notice that in general, $(X + Y) \cap Z \neq (X \cap Z) + (Y \cap Z)$.

For exa, $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

COMMENT: If $X \subseteq Y$, then $(X + Y) \cap Z = Y \cap Z$; $\dim(X + Y + Z) = \dim(Y) + \dim(Z) - \dim(Y \cap Z)$, and the wrong formual holds. Simlr for $Y \subseteq Z, X \subseteq Z$, and $X, Y \subseteq Z$.

However, it's true that $(X + Y) \cap Z \supseteq (X \cap Z) + (Y \cap Z) = \big(X + (Y \cap Z)\big) \cap Z$.

Becs $(X \cap Z) + (Y \cap Z) \ni v = x + y = z_1 + z_2 \in \big(X + (Y \cap Z)\big) \cap Z \Rightarrow v \in (X + Y) \cap Z$.

Where $\exists\, x = z_1 \in X \cap Z, y = z_2 \in Y \cap Z$.

COMMENT: $\dim\big((X + Y) \cap Z\big) \geqslant \dim(X \cap Z) + \dim(Y \cap Z) - \dim(X \cap Y \cap Z)$.

- **CORO:** *Supp V_1, V_2, V_3 are finide, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} - \frac{\dim\big((V_1 + V_2) \cap V_3\big) + \dim\big((V_1 + V_3) \cap V_2\big) + \dim\big((V_2 + V_3) \cap V_1\big)}{3}.$$

- **TIPS:** Becs $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big)$.

  And $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$. We have (1), and (2), (3) simlr.

  (1) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 + V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big)$.

  (2) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_3) - \dim\big(V_2 + (V_1 \cap V_3)\big)$.

  (3) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim\big(V_3 + (V_1 \cap V_2)\big)$.

- *Supp V_1, V_2, V_3 are subsps of V with*

  (a) $\dim V = 10$, $\dim V_1 = \dim V_2 = \dim V_3 = 7$. *Prove $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*

  By TIPS, $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 \underline{- 2\dim V} > 0$.

  (b) $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. *Prove $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*

  By TIPS, $\dim(V_1 \cap V_2 \cap V_3) \underline{\underline{\geq 2\dim V}} - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)) \geq 0$. $\square$

ENDED

3·A   3 4 5 7 8 10 11 12 13 | 4E: 10 11 17

• **TIPS 1:** $T : V \to W$ is liney $\iff \left| \begin{array}{l} (一)\ \forall v,u \in V, T(v+u) = Tv + Tu; \\ (二)\ \forall v,u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv). \end{array} \right| \iff T(v+\lambda u) = Tv + \lambda Tu.$

• (9.A.2,6 OR 4E 3.B.33) *Supp that V, W are on $\mathbf{R}$, and $T \in \mathcal{L}(V,W)$. Show*
(a) $T_{\mathbf{C}} \in \mathcal{L}(V_{\mathbf{C}}, W_{\mathbf{C}})$. (b) null $(T_{\mathbf{C}}) = (\text{null}\, T)_{\mathbf{C}}$, range $(T_{\mathbf{C}}) = (\text{range}\, T)_{\mathbf{C}}$. (c) $T_{\mathbf{C}}$ is inv $\iff T$ is inv.

SOLUS: (a) $T_{\mathbf{C}}\big((u_1 + \mathrm{i}v_1) + (x+\mathrm{i}y)(u_2 + \mathrm{i}v_2)\big) = T(u_1 + xu_2 - yv_2) + \mathrm{i}T(v_1 + xv_2 + yu_2)$
$= T_{\mathbf{C}}(u_1 + \mathrm{i}v_1) + (x+\mathrm{i}y)T_{\mathbf{C}}(u_2 + \mathrm{i}v_2)$.
(b) $u + \mathrm{i}v \in \text{null}\,(T_{\mathbf{C}}) \iff u, v \in \text{null}\, T \iff u + \mathrm{i}v \in (\text{null}\, T)_{\mathbf{C}}$.
$w + \mathrm{i}x \in \text{range}\,(T_{\mathbf{C}}) \iff w, x \in \text{range}\, T \iff w + \mathrm{i}x \in (\text{range}\, T)_{\mathbf{C}}$.
(c) $\forall w, x \in W, \exists!\, u, v \in V, T_{\mathbf{C}}(u+\mathrm{i}v) = w + \mathrm{i}x \iff Tu = w, Tv = x$. OR. By (b). □

• (9.A.5) *Supp V is on $\mathbf{R}$, and $S, T \in \mathcal{L}(V,W)$. Prove $(S+\lambda T)_{\mathbf{C}} = S_{\mathbf{C}} + \lambda T_{\mathbf{C}}$.*

SOLUS: $(S+\lambda T)_{\mathbf{C}}(u+\mathrm{i}v) = (S+\lambda T)(u) + \mathrm{i}(S+\lambda T)(v)$
$= Su + \mathrm{i}Sv + \lambda(Tu + \mathrm{i}Tv) = (S_{\mathbf{C}} + \lambda T_{\mathbf{C}})(u+\mathrm{i}v)$. □

• *Supp U, V, W are on $\mathbf{R}$, $S \in \mathcal{L}(V,W), T \in \mathcal{L}(U,V)$. Prove $(ST)_{\mathbf{C}} = S_{\mathbf{C}} T_{\mathbf{C}}$.*

SOLUS: $\forall u + \mathrm{i}x \in U_{\mathbf{C}}, (ST)_{\mathbf{C}}(u+\mathrm{i}x) = STu + \mathrm{i}STx = S_{\mathbf{C}}(Tu + \mathrm{i}Tx) = (S_{\mathbf{C}}T_{\mathbf{C}})(u+\mathrm{i}x)$. □

• **NOTE FOR Restr:** *U is a subsp of V.*
(a) $\forall S, T \in \mathcal{L}(V,W), \lambda \in \mathbf{F}$, $(T+\lambda S)\big|_U = T|_U + \lambda S|_U$.
(b) $\forall S \in \mathcal{L}(W,X), T \in \mathcal{L}(V,W)$, $(ST)\big|_U = ST|_U$.

• (4E 1.B.7) *Supp $V \neq \emptyset$ and W is a vecsp. Let $W^V = \{f : V \to W\}$.*
(a) *Define a natural add and scalar multi on W^V.*
(b) *Prove W^V is a vecsp with these defs.*

SOLUS:
(a) $W^V \ni f + g : x \to f(x) + g(x)$; where $f(x) + g(x)$ is the vec add on W.
$W^V \ni \lambda f : x \to \lambda f(x)$; where $\lambda f(x)$ is the scalar multi on W.
(b) Commu: $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$.
Assoc: $\big((f+g)+h\big)(x) = \big(f(x)+g(x)\big) + h(x)$
$= f(x) + \big(g(x) + h(x)\big) = \big(f + (g+h)\big)(x)$.
Add Id: $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$.
Add Inv: $(f+g)(x) = f(x) + g(x) = f(x) + \big(-f(x)\big) = 0 = 0(x)$.
Multi Id: $(1f)(x) = 1f(x) = f(x)$. (NOTICE that the smallest $\mathbf{F}$ is $\{0,1\}$.)
Distr: $\big(a(f+g)\big)(x) = a(f+g)(x) = a\big(f(x)+g(x)\big)$
$= af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x)$.
Simlr, $\big((a+b)f\big)(x) = (af + bf)(x)$.
So far, we have used the same properties in W.
Which means that ***if** W^V **is a vecsp, then** W **must be a vecsp.*** □

• **TIPS 2:** $T \in \mathcal{L}(V, W) \iff T \in \mathcal{L}(V, \operatorname{range} T) \iff T \in \mathcal{L}(V, U)$, if range T is a subsp of U.
CORO: $\{T \in \mathcal{L}(V, W) : \operatorname{range} T \subseteq U\} = \{T \in \mathcal{L}(V, U)\} = \mathcal{L}(V, U)$.

---

5 *Becs* $\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is liney}\}$ *is a subsp of* W^V, $\mathcal{L}(V, W)$ *is a vecsp.*

---

3 *Supp* $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. *Prove* $\exists A_{j,k} \in \mathbf{F}$ *suth for any* $(x_1, \dots, x_n) \in \mathbf{F}^n$,

$$T(x_1, \dots, x_n) = \begin{pmatrix} A_{1,1}x_1 + \cdots + A_{1,n}x_n, \\ \vdots \quad\quad \ddots \quad\quad \vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n \end{pmatrix}$$

SOLUS:

Let $T(1, 0, 0, \dots, 0, 0) = (A_{1,1}, \dots, A_{m,1})$, Note that $(1, 0, \dots, 0, 0), \cdots, (0, 0, \dots, 0, 1)$ is a bss of $\mathbf{F}^n$.
$T(0, 1, 0, \dots, 0, 0) = (A_{1,2}, \dots, A_{m,2})$, Then by [3.5], done. $\square$
$\vdots$
$T(0, 0, 0, \dots, 0, 1) = (A_{1,n}, \dots, A_{m,n})$.

---

4 *Supp* $T \in \mathcal{L}(V, W)$, *and* $v_1, \dots, v_m \in V$ *suth* $(Tv_1, \dots, Tv_m)$ *is liney indep in* W.
Prove $(v_1, \dots, v_m)$ *is liney indep.*

SOLUS: Supp $a_1 v_1 + \cdots + a_m v_m = 0$. Then $a_1 T v_1 + \cdots + a_m T v_m = 0$. Thus $a_1 = \cdots = a_m = 0$. $\square$

---

7 *Show every liney map from a* 1*-dim vecsp to itself is a multi by some scalar.*
More precisely, prove if $\dim V = 1$ *and* $T \in \mathcal{L}(V)$, *then* $\exists \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$.

SOLUS: Let u be a nonzero vec in $V \Rightarrow V = \operatorname{span}(u)$. Becs $Tu \in V \Rightarrow Tu = \lambda u$ for some λ.
Supp $v \in V \Rightarrow v = au$, $\exists ! \, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda a u = \lambda v$. $\square$

---

8 *Give a map* $\varphi : \mathbf{R}^2 \to \mathbf{R}$ *suth* $\forall a \in \mathbf{R}, v \in \mathbf{R}^2, \varphi(av) = a\varphi(v)$ *but* φ *is not liney.*

SOLUS: Define $T(x, y) = \begin{cases} x + y, & \text{if } (x, y) \in \operatorname{span}(3, 1), \\ 0, & \text{othws.} \end{cases}$ OR. Define $T(x, y) = \sqrt[3]{(x^3 + y^3)}$. $\square$

---

9 *Give a map* $\varphi : \mathbf{C} \to \mathbf{C}$ *suth* $\forall w, z \in \mathbf{C}, \varphi(w + z) = \varphi(w) + \varphi(z)$ *but* φ *is not liney.*

SOLUS: Define $\varphi(u + \mathrm{i}v) = u = \operatorname{Re}(u + \mathrm{i}v)$ OR. Define $\varphi(u + \mathrm{i}v) = v = \operatorname{Im}(u + \mathrm{i}v)$. $\square$

• *Prove if* $q \in \mathcal{P}(\mathbf{R})$ *and* $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ *is defined by* $\underbrace{Tp = q \circ p}_{\textit{composition}}$, *then* T *is not liney.*

SOLUS: ***Composition and product are not the same in*** $\mathcal{P}(\mathbf{F})$.

NOTICE that $(p \circ q)(x) = p(q(x))$, while $(pq)(x) = p(x)q(x) = q(x)p(x)$.
Becs in general, $[q \circ (p_1 + \lambda p_2)](x) = q(p_1(x) + \lambda p_2(x)) \neq (qp_1)(x) + \lambda(qp_2)(x)$.
EXA: Let q be defined by $q(x) = x^2$, then $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$. $\square$

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10 *Supp* U *is a subsp of* V *with* $U \neq V$.
Supp $S \in \mathcal{L}(U, W)$ *with* $S \neq 0$. *Define* $T : V \to W$ *by* $Tv = \begin{cases} Sv, & \textit{if } v \in U, \\ 0, & \textit{if } v \in V \backslash U. \end{cases}$
Prove T *is not a liney map on* V.

SOLUS: Asum T is a liney map. Supp $v \in V \backslash U$, $u \in U$ suth $Su \neq 0$.
Then $v + u \in V \backslash U$, for if not, $v = (v + u) - u \in U$;
while $T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$. Ctradic. $\square$

---

11 *Supp U is a subsp of V and $S \in \mathcal{L}(U, W)$.*
Prove $\exists\, T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U$. (OR. $\exists\, T \in \mathcal{L}(V, W), T|_U = S$.)
*In other words, every liney map on a subsp of V can be **extended** to a liney map on the entire V.*

SOLUS: Supp W is suth $V = U \oplus W$. Then $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.
Define $T \in \mathcal{L}(V, W)$ by $T(u_v + w_v) = Su_v$. □
OR. [*Finide Req*] Define by $T\Big(\sum_{i=1}^{m} a_i u_i\Big) = \sum_{i=1}^{n} a_i S u_i$. Let $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$. □

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12 *Supp nonzero V is finide and W is infinide. Prove $\mathcal{L}(V, W)$ is infinide.*

SOLUS: Using (2.A.14).
Let $B_V = (v_1, \dots, v_n)$ be a bss of V. Let $(w_1, \dots, w_m)$ be liney indep in W for any $m \in \mathbf{N}^+$.
Define $T_{x,y} : V \to W$ by $T_{x,y}(v_z) = \delta_{z,x} w_y$, $\forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$
$\forall v = \sum_{i=1}^{n} a_i v_i,\ u = \sum_{i=1}^{n} b_i v_i,\ \lambda \in \mathbf{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) w_y = T_{x,y}(v) + \lambda T_{x,y}(u)$.
Linity checked. Now supp $a_1 T_{x,1} + \dots + a_m T_{x,m} = 0$.
Then $\big(a_1 T_{x,1} + \dots + a_m T_{x,m}\big)(v_x) = 0 = a_1 w_1 + \dots + a_m w_m \Rightarrow a_1 = \dots = a_m = 0$. 又 m arb.
Thus $(T_{x,1}, \dots, T_{x,m})$ is a liney indep list in $\mathcal{L}(V, W)$ for any x and len m. Hence by (2.A.14). □

---

13 *Supp $(v_1, \dots, v_m)$ is linely depe in V and $W \neq \{0\}$.*
Prove $\exists\, w_1, \dots, w_m \in W, \nexists\, T \in \mathcal{L}(V, W)$ suth $Tv_k = w_k, \forall k = 1, \dots, m$.

SOLUS:
We prove by ctradic. By liney dependence lemma, $\exists\, j \in \{1, \dots, m\}, v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$.
Supp $a_1 v_1 + \dots + a_m v_m = 0$, where $a_j \neq 0$. Now let $w_j \neq 0$, while $w_1 = \dots = w_{j-1} = w_{j+1} = w_m = 0$.
Define $T \in \mathcal{L}(V, W)$ by $Tv_k = w_k$ for each k. Then $T\big(a_1 v_1 + \dots + a_m v_m\big) = 0 = a_1 w_1 + \dots + a_m w_m$.
And $0 = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Ctradic. □

OR. We prove the ctrapos: Supp $\forall w_1, \dots, w_m \in W, \exists\, T \in \mathcal{L}(V, W), Tv_k = w_k$ for each w_k.
Now we show $(v_1, \dots, v_n)$ is liney indep. Supp $\exists\, a_i \in \mathbf{F}, a_1 v_1 + \dots + a_n v_n = 0$.
Choose one $w \in W \backslash \{0\}$. By asum, for $(\overline{a_1} w, \dots, \overline{a_m} w), \exists\, T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$ for each v_k.
Now we have $0 = T\Big(\sum_{k=1}^{m} a_k v_k\Big) = \sum_{k=1}^{m} a_k T v_k = \sum_{k=1}^{m} a_k \overline{a_k} w = \Big(\sum_{k=1}^{m} |a_k|^2\Big) w$.
Then $\sum_{k=1}^{m} |a_k|^2 = 0$. Thus $a_1 = \dots = a_m = 0$. Hence $(v_1, \dots, v_n)$ is liney indep. □

• (4E 17) *Supp V is finide. Show all two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUS: Let $B_V = (v_1, \dots, v_n)$. If $\mathcal{E} = 0$, then done.
Supp $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \backslash \{0\}$.
Supp $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \dots + a_n v_n$, where $a_k \neq 0$.
Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y} :\ v_x \mapsto v_y,\ v_z \mapsto 0\ (z \neq x)$. OR. $R_{x,y} v_z = \delta_{z,x} v_y$.
Then $\big(R_{1,1} + \dots + R_{n,n}\big) v_j = v_j \Rightarrow \sum_{r=1}^{n} R_{r,r} = I$. Asum each $R_{x,y} \in \mathcal{E}$.
Hence $\forall T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. Now we prove the asum.
Notice that $\forall x, y \in \mathbf{N}^+,\ (R_{k,y} S)(v_i) = a_k v_y \Rightarrow \big((R_{k,y} S) \circ R_{x,i}\big)(v_z) = \delta_{z,x}(a_k v_y)$.
Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Now $S \in \mathcal{E} \Rightarrow R_{k,y} S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}$. □

• (4E 3.B.32) *Supp* $\dim V = n$. *Supp* $\varphi : \mathcal{L}(V) \to \mathbf{F}$ *is liney.*
Show if $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$, *then* $\varphi = 0$.

Solus: Using notas in (4E 3.A.17). Using the result in Note For [3.60].

Supp $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \dots, n\}, \varphi(R_{i,j}) \neq 0$. Becs $R_{i,j} = R_{x,j} \circ R_{i,x}, \ \forall x = 1, \dots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, becs $R_{i,x} = R_{y,x} \circ R_{i,y}, \ \forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0, \ \forall x, y = 1, \dots, n$.

Let $k \neq i, j \neq l$ and then $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Ctradic. □

Or. Note that by (4E 3.A.17), $\exists\, S, T \in \mathcal{L}(V), ST - TS \neq 0$.

Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \operatorname{null} \varphi \neq \{0\}$.

Note that $\forall\, E \in \operatorname{null} \varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \operatorname{null} \varphi$.

Hence $\operatorname{null} \varphi$ is a nonzero two-sided ideal of $\mathcal{L}(V)$. □

• *Supp* V *is finide,* $T \in \mathcal{L}(V)$ *is suth* $\forall S \in \mathcal{L}(V), ST = TS$. *Prove* $\exists \lambda \in \mathbf{F}, T = \lambda I$.

Solus: If $V = \{0\}$, then done. Now supp $V \neq \{0\}$.

Asum $\forall v \in V, (v, Tv)$ is linely depe, then by (2.A.2.(b)), $\exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

To prove λ_v is indep of v, we discuss in two cases:

$$\left.\begin{array}{l} (-)\text{ If } (v, w) \text{ is liney indep, } \lambda_{v+w}(v + w) = T(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \\ \qquad\qquad\qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=)\text{ Othws, supp } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w \end{array}\right\} \Rightarrow \lambda_w = \lambda_v.$$

Now we prove the asum. Asum $\exists\, v \in V, (v, Tv)$ is liney indep. Let $B_V = (v, Tv, u_1, \dots, u_n)$.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Ctradic. □

Or. Let $B_V = (v_1, \dots, v_m)$. Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$.

Supp $v \in V$. Define $S_v \in \mathcal{L}(V)$ by $S_v(u) = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. □

Or. For each $k \in \{1, \dots, n\}$, define $S_k \in \mathcal{L}(V)$ by $S_k v_j = \begin{cases} v_k, \ j = k, \\ 0, \ \ j \neq k. \end{cases}$ Or. $S_k v_j = \delta_{j,k} v_k$

Note that $S_k\Big(\sum_{i=1}^n a_i v_i\Big) = a_k v_k$. Then $S_k v = v \iff \exists\,! a_k \in \mathbf{F}, v = a_k v_k$.

Hence $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$.

Define $A^{(j,k)} \in \mathcal{L}(V)$ by $A^{(j,k)}v_j = v_k, \ A^{(j,k)}v_k = v_j, \ A^{(j,k)}v_x = 0, x \neq j, k$.

$$\text{Then } \left|\begin{array}{r} A^{(j,k)}Tv_j = TA^{(j,k)}v_j = Tv_k = a_k v_k \\ A^{(j,k)}Tv_j = A^{(j,k)}a_j v_j = a_j A^{(j,k)}v_j = a_j v_k \end{array}\right\} \Rightarrow a_k = a_j. \text{ Hence } a_k \text{ is indep of } v_k.$$ □

• **Tips 3:** *Supp* $T \in \mathcal{L}(V, W)$. *Prove* $Tv \neq 0 \Rightarrow v \neq 0$.

Solus: Asum $v = 0$. Then $Tv = T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$.

Or. $T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$. Ctradic. □

• *Given the fact that $\mathcal{L}(V,W)$ is a vecsp. Prove or give a counterexa: V, W are vecsps.*
We can assure that $\{0\} \subseteq \mathcal{L}(V,W), \{0\} \subseteq V, \{0\} \subseteq W$.
And by [3.2], the add and homo imply that V is closd add and scalar multi.
(W^V might not be a vecsp.)

Solus:
(I) If $W^V = \{0\}$. Then $\mathcal{L}(V,W) = \{0\}$.
And $W = \{0\}$, for if not, $\exists\, w \in W \setminus \{0\}$, define a map f by $f(x) = w, \forall x \in V$.
And V might not be a vecsp. **Exa:** Let $V = \mathbf{R}$, but with the scalar multi defined by $a \odot v = 0$.
(II) If W^V is a nonzero vecsp $\iff$ W is a nonzero vecsp.
(a) If $\mathcal{L}(V,W) = \{0\}$, then by Exa (I), V might not be vecsp.
(b) If not, then $\exists\, T \in \mathcal{L}(V,W)$, $T \neq 0$. Which means $\exists\, v \in V, Tv \neq 0 \Rightarrow v \neq 0$. TODO
Then both W and V have a nonzero elem.
(i) If $\exists$ inje $T \in \mathcal{L}(V,W)$, then $T(u+v) = T(v+u) \Rightarrow u+v = v+u$. etc. Hence V is a vecsp.
(ii) If not, then we cannot guarantee that V is a vecsp. Exa: ???
(III) If W^V is not a vecsp $\iff$ W is not a vecsp.
(a) If $\mathcal{L}(V,W) = \{0\}$, then by Exa (I), V might not be vecsp.
(b) If not. □

Ended

3·B

3 7 8 9 10 11 12 16 17 18 19 20 21 22 23 24 25 26 28 29 30 | 4E: 21 24 27 31 32 33

3 *Supp $(v_1, \dots, v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1 v_1 + \cdots + z_m v_m$.*
(a) *The surj of T corres to $(v_1, \dots, v_m)$ spanning V.* range $T = \operatorname{span}(v_1, \dots, v_m) = V$.
(b) *The inje of T corres to $(v_1, \dots, v_m)$ being liney indep.* *$(v_1, \dots, v_m)$ liney indep $\iff$ T inje.*

Comment: Let $(e_1, \dots, e_m)$ be std bss of $\mathbf{F}^m$. Then $Te_k = v_k$.

---

7 *Supp $2 \leqslant \dim V = n \leqslant m = \dim W$, if W is finide.*
Show $U = \{ T \in \mathcal{L}(V,W) : \operatorname{null} T \neq \{0\} \}$ is not a subsp of $\mathcal{L}(V,W)$.

Solus: *The set of all inje $T \in \mathcal{L}(V,W)$ is a not subsp either.*
Let $(v_1, \dots, v_n)$ be a bss of V, $(w_1, \dots, w_m)$ be liney indep in W. $[\, 2 \leqslant n \leqslant m.\,]$
Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 : \quad v_1 \mapsto 0, \quad v_2 \mapsto w_2, \quad v_i \mapsto w_i.$
Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 : \quad v_1 \mapsto w_1, \quad v_2 \mapsto 0, \quad v_i \mapsto w_i, \quad i = 3, \dots, n.$
Thus $T_1 + T_2 \notin U$. □

Comment: If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V,W), T$ is inje. Hence $U = \emptyset$.
If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $\forall v \in V, T_0 v = 0 \Rightarrow T_0 = 0$.

---

8 *Supp $2 \leqslant \dim W = m \leqslant \dim V$, if V is finide.*
Show $U = \{ T \in \mathcal{L}(V,W) : \operatorname{range} T \neq W \}$ is not a subsp of $\mathcal{L}(V,W)$.

Solus: *The set of all surj $T \in \mathcal{L}(V,W)$ is not a subsp either.* ***Using the generalized version of*** **[3.5].**
Let $(v_1, \dots, v_n)$ be liney indep in V, $(w_1, \dots, w_m)$ be a bss of W. $[\, n \in \{m, m+1, \dots\}; 2 \leqslant m \leqslant n.\,]$
Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 : \quad v_1 \mapsto 0, \quad v_2 \mapsto w_2, \quad v_j \mapsto w_j, \quad v_{m+i} \mapsto 0.$
Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 : \quad v_1 \mapsto w_1, \quad v_2 \mapsto 0, \quad v_j \mapsto w_j, \quad v_{m+i} \mapsto 0.$
(For each $j = 2, \dots, m;\ i = 1, \dots, n-m$, if V is finide, othws let $i \in \mathbf{N}^+$.) Thus $T_1 + T_2 \notin U$. □

Comment: If $\dim W = 0$, then $W = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V,W), T$ is surj. Hence $U = \emptyset$.
If $\dim W = 1$, then $W = \operatorname{span}(w_0)$. Thus $U = \operatorname{span}(T_0)$, where each $T_0 v_i = 0 \Rightarrow T_0 = 0$.

9 *Supp* $(v_1,\dots,v_n)$ *is liney indep. Prove* $\forall$ *inje* T, $(Tv_1,\dots,Tv_n)$ *is liney indep.*

Solus: $a_1Tv_1+\dots+a_nTv_n=0=T\Big(\sum_{i=1}^n a_iv_i\Big)\iff\sum_{i=1}^n a_iv_i=0\iff a_1=\dots=a_n=0.$ □

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10 *Supp* $\operatorname{span}(v_1,\dots,v_n)=V$. *Show* $\operatorname{span}(Tv_1,\dots,Tv_n)=\operatorname{range}T$.

Solus: (a) $\operatorname{range}T=\{Tv:v\in\operatorname{span}(v_1,\dots,v_n)\}\Rightarrow Tv_1,\dots,Tv_n\in\operatorname{range}T$. By [2.7].

Or. $\operatorname{span}(Tv_1,\dots,Tv_n)\ni a_1Tv_1+\dots+a_nTv_n=T(a_1v_1+\dots+a_nv_n)\in\operatorname{range}T$.

(b) $\forall w\in\operatorname{range}T, w=Tv,\exists v\in V\Rightarrow\exists a_i\in\mathbf{F}, v=\sum_{i=1}^n a_iv_i, w=a_1Tv_1+\dots+a_nTv_n$. □

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11 *Supp* $S_1,\dots,S_n\in\mathcal{L}(V)$ *and* $S=S_1S_2\cdots S_n$ *makes sense.* Then using induc:

(a) $\operatorname{range}S_1\supseteq\operatorname{range}(S_1S_2)\supseteq\dots\supseteq\operatorname{range}(S)$; (b) $\operatorname{null}S_n\subseteq\operatorname{null}(S_{n-1}S_n)\subseteq\dots\subseteq\operatorname{null}(S)$.

- Define $X_p=\{T\in\mathcal{L}(V):p(T)\text{ holds}\}$; $P_p:X_p$ is closd vec multi; $Q_p:X_p$ is a group.

(1) S surj $\iff$ each S_k surj. P_{surj} holds. (2) S inje $\iff$ each S_k inje. P_{inje} holds.

(3) P_{inv} and Q_{inv} hold. (4) Q_p in (1) and (2) holds $\iff$ V is finide.

(5) $P_{inje\ or\ surj}$ holds $\iff$ V is finide $\iff$ $Q_{inje\ or\ surj}$ holds.

- *Supp* $S,T\in\mathcal{L}(V)$. *Prove or give a counterexa*:

(a) $\operatorname{null}S\subseteq\operatorname{null}T\Rightarrow\operatorname{range}T\subseteq\operatorname{range}S$; (b) $\operatorname{range}T\subseteq\operatorname{range}S\Rightarrow\operatorname{null}S\subseteq\operatorname{null}T$.

Solus: Let $B_V=(v_1,v_2,v_3)$. Counterexas:

(a) Let $S:v_1\mapsto0;\ v_2\mapsto0;\ v_3\mapsto v_2.$ $T:v_1\mapsto0;\ v_2\mapsto0;\ v_3\mapsto v_3.$ | Then $\operatorname{null}S=\operatorname{null}T$, but $\operatorname{range}T=\operatorname{span}(v_3)\not\subseteq\operatorname{span}(v_2)=\operatorname{null}T$.

(b) Let $S:v_1\mapsto v_2;\ v_2\mapsto v_2;\ v_3\mapsto v_2.$ $T:v_1\mapsto0;\ v_2\mapsto0;\ v_3\mapsto v_2.$ | Then $\operatorname{range}T=\operatorname{range}S$, but $\operatorname{null}S=\operatorname{span}(v_1-v_2,v_2-v_3,v_3-v_1)\not\subseteq\operatorname{span}(v_1,v_2)=\operatorname{null}T$.

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16 *Supp* $T\in\mathcal{L}(V)$ *suth* $\operatorname{null}T,\operatorname{range}T$ *are finide. Prove* V *is finide.*

Solus: Let $B_{\operatorname{range}T}=(Tv_1,\dots,Tv_n), B_{\operatorname{null}T}=(u_1,\dots,u_m)$.

$\forall v\in V,\exists!a_i\in\mathbf{F}, T(v-a_1v_1-\dots-a_nv_n)=0\Rightarrow\exists!b_i\in\mathbf{F}, v-\sum_{i=1}^n a_iv_i=\sum_{i=1}^m b_iu_i$. □

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17 *Supp* V,W *are finide. Prove* $\exists$ *inje* $T\in\mathcal{L}(V,W)\iff\dim V\leqslant\dim W$.

Solus: (a) Supp $\exists$ inje T. Then $\dim V=\dim\operatorname{range}T\leqslant\dim W$.

(b) Supp $\dim V\leqslant\dim W$. Let $B_V=(v_1,\dots,v_n), B_W=(w_1,\dots,w_m)$.

Define $T\in\mathcal{L}(V,W)$ by $Tv_i=w_i,\ i=1,\dots,n\ (=\dim V)$. □

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18 *Supp* V,W *are finide. Prove* $\exists$ *surj* $T\in\mathcal{L}(V,W)\iff\dim V\geqslant\dim W$.

Solus: (a) Supp $\exists$ surj T. Then $\dim V=\dim W+\dim\operatorname{null}T\Rightarrow\dim W\leqslant\dim V$.

(b) Supp $\dim V\geqslant\dim W$. Let $B_V=(v_1,\dots,v_n), B_W=(w_1,\dots,w_m)$.

Define $T\in\mathcal{L}(V,W)$ by $T\big(a_1v_1+\dots+a_mv_m+\dots+a_nv_n\big)=a_1w_1+\dots+a_mw_m$. □

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19 *Supp* V,W *are finide,* U *is a subsp of* V.

Prove $\exists T\in\mathcal{L}(V,W)$, $\operatorname{null}T=U\iff\underbrace{\dim U}_{m}\geqslant\underbrace{\dim V}_{m+n}-\underbrace{\dim W}_{p}$.

Solus:

(a) Supp $\exists T\in\mathcal{L}(V,W),\operatorname{null}T=U$. Then $\dim U+\dim\operatorname{range}T=\dim V\leqslant\dim U+\dim W$.

(b) Let $B_U=(u_1,\dots,u_m), B_V=(u_1,\dots,u_m,v_1,\dots,v_n), B_W=(w_1,\dots,w_p)$. Supp that $p\geqslant n$.

Define $T\in\mathcal{L}(V,W)$ by $T(a_1v_1+\dots+a_nv_n+b_1u_1+\dots+b_mu_m)=a_1w_1+\dots+a_nw_n$. □

- **Tips 1:** *Supp U is a subsp of V. Then $\forall T \in \mathcal{L}(V,W), U \cap \operatorname{null} T = \operatorname{null} T|_U$.*
- **Tips 2:** *Supp $T \in \mathcal{L}(V,W)$ and $T|_U$ is inje. Let $V = M + N,\ U = X + Y$.*
  Then $\operatorname{range} T = \operatorname{range} T|_M + \operatorname{range} T|_N = \operatorname{range} T|_X + \operatorname{range} T|_Y$.
  (a) *Show* $U = X \oplus Y \iff \operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_Y$.
  (b) *Give an exa suth* $V = M \oplus N,\ \operatorname{range} T \neq \operatorname{range} T|_M \oplus \operatorname{range} T|_N$.

Solus: Supp $U = X \oplus Y$. Asum for some $v \in V$, there exis two disti pairs $(x_1, y_1), (x_2, y_2)$ in $X \times Y$ suth $Tv = Tx_1 + Ty_1 = Tx_2 + Ty_2$. Becs $\forall v \in X \oplus Y, \exists!\,(x,y) \in X \times Y, v = x + y$.
Now $T(x_1 + y_1) = T(x_2 + y_2) \Longrightarrow x_1 + y_1 = x_2 + y_2 \Longrightarrow x_1 = x_2, y_1 = y_2$. Ctradic.
Thus $\forall Tv \in \operatorname{range} T, \exists!\, Tx \in \operatorname{range} T|_X, Ty \in \operatorname{range} T|_Y, Tv = Tx + Ty$. Convly, becs T is inje $\square$

Exa: Let $B_V = (v_1, v_2, v_3), B_W = (w_1, w_2),\ T : v_1 \mapsto 0,\ v_2 \mapsto w_1,\ v_3 \mapsto w_2$.
Let $B_M = (v_1 - v_2, v_3), B_N = (v_2)$. Then $\operatorname{range} T|_M = \operatorname{span}(w_1, w_2), \operatorname{range} T|_N = \operatorname{span}(w_1)$

Comment: Also $\operatorname{null} T|_M = \operatorname{null} T|_N = \{0\}$. Hence $\operatorname{null} T \neq \operatorname{null} T|_M \oplus \operatorname{null} T|_N$.

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12 *Prove $\forall T \in \mathcal{L}(V,W), \exists$ subsp U of V suth*
$U \cap \operatorname{null} T = \operatorname{null} T|_U = \{0\},\ \operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$.
Which is equiv to $T|_U : U \to \operatorname{range} T$ being iso.

Solus: By [2.34] (note that V can be infinide), $\exists$ subsp U of V suth $V = U \oplus \operatorname{null} T$.
$\forall v \in V, \exists!\, w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w + u) = Tu \in \{Tu : u \in U\}$. $\square$

Coro: $[P]$ $\quad T|_U : U \to \operatorname{range} T$ *is iso* $\iff U \oplus \operatorname{null} T = V.$ $\quad [Q]$
We have shown $Q \Rightarrow P$. Now we show $P \Rightarrow Q$ to complete the proof.
$\forall v \in V, Tv \in \operatorname{range} T = \operatorname{range} T|_U \Rightarrow \exists!\, u \in U, Tv = Tu \Rightarrow v - u \in \operatorname{null} T$.
Thus $v = (v - u) + u \in U + \operatorname{null} T$. 又 $u \in U \cap \operatorname{null} T \iff T|_U(u) = 0 \iff u = 0$. $\square$

Or. $\neg Q \Rightarrow \neg P$: Becs $U \oplus \operatorname{null} T \subsetneq V$. We show $\operatorname{range} T \neq \operatorname{range} T|_U$ by ctradic.
Let $X \oplus (U \oplus \operatorname{null} T) = V$. Now $\operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_U$. And X is nonzero.
Asum $\operatorname{range} T = \operatorname{range} T|_U$. Then $\operatorname{range} T|_X = \{0\}$. While $T|_X$ is inje. Ctradic.
Or. $\operatorname{range} T|_X \subseteq \operatorname{range} T|_U \Rightarrow \forall x \in X, Tx \in \operatorname{range} T|_U, \exists\, u \in U, Tu = Tx \Rightarrow x = 0$.

Also, $\neg P \Rightarrow \neg Q$: (a) $\operatorname{range} T|_U \subsetneq \operatorname{range} T$; Or (b) $U \cap \operatorname{null} T \neq \{0\}$.
For (a), $\exists\, x \in V \backslash U, Tx \neq 0 \iff x \notin \operatorname{null} T$. Thus $U + \operatorname{null} T \subsetneq V$. For (b), immed. $\square$

Comment: If $T|_U : U \to \operatorname{range} T$ is iso. Let $R \oplus U = V$. Then R might not be $\operatorname{null} T$.
Or. Extend B_U to $B_V = (u_1, \dots, u_n, r_1, \dots, r_m)$, then $(r_1, \dots, r_m)$ might not be a $B_{\operatorname{null} T}$.

- **Tips 3:** Supp $T \in \mathcal{L}(V,W)$ and U is a subsp suth $V = U \oplus \operatorname{null} T$. Let $\operatorname{null} T = X \oplus Y$.
  Now $\forall v \in V, \exists!\, u_v \in U, (x_v, y_v) \in X \times Y, v = u_v + x_v + y_v$. Define $i \in \mathcal{L}(V, U \oplus X)$ by $i(v) = u_v + x_v$.
  Then $T = T \circ i$. Becs $\forall v \in V, T(v) = T(u_v + x_v + y_v) = T(u_v) = T(u_v + x_v) = T\big(i(v)\big) = (T \circ i)(v)$.

- **Tips 4:** Supp $T \in \mathcal{L}(V,W), T \neq 0$. Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
  By (3.A.4), $R = (v_1, \dots, v_n)$ is liney indep in V. Let $\operatorname{span} R = U$. We will prove $U \oplus \operatorname{null} T = V$.
  (a) $T\big(\sum_{i=1}^n a_i v_i\big) = 0 \iff \sum_{i=1}^n a_i T v_i = 0 \iff a_1 = \cdots = a_n = 0$. Thus $U \cap \operatorname{null} T = \{0\}$.
  (b) $Tv = \sum_{i=1}^n a_i T v_i \iff v - \sum_{i=1}^n a_i v_i \in \operatorname{null} T \iff v = \big(v - \sum_{i=1}^n a_i v_i\big) + \big(\sum_{i=1}^n a_i v_i\big)$.
  Thus $U + \operatorname{null} T = V$. Or. $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$. Using Exe (12). $\square$

Coro: Convly, if $U \oplus \operatorname{null} T = V$ and $B_U = (v_1, \dots, v_n)$, then $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
Becs $\operatorname{range} T = \operatorname{range} T|_U = \operatorname{span}(Tv_1, \dots, Tv_n)$, 又 T is inje.

• [4E 27, OR 5.B.4] *Supp* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove* $V = \operatorname{null} P \oplus \operatorname{range} P$.

SOLUS: (a) If $v \in \operatorname{null} P \cap \operatorname{range} P \Rightarrow Pv = 0$, and $\exists\, u \in V, v = Pu$. Then $v = Pu = P^2u = Pv = 0$.

(b) Note that $\forall v \in V, v = Pv + (v - Pv)$ and $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$.

OR. Becs $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P = \dim\big(\operatorname{null} P \oplus \operatorname{range} P\big)$. $\square$

OR. [*Only in Finide*] Let $B_{\operatorname{range} P^2} = \big(P^2v_1, \dots, P^2v_n\big)$. Then $\big(Pv_1, \dots, Pv_n\big)$ is liney indep.
Let $U = \operatorname{span}(Pv_1, \dots, Pv_n) \Rightarrow V = U \oplus \operatorname{null} P^2$. While $U = \operatorname{range} P = \operatorname{range} P^2$; $\operatorname{null} P = \operatorname{null} P^2$. $\square$

• *Supp* $T \in \mathcal{L}(V), v \in V$, *and* $n \in \mathbf{N}^+$ *suth* $T^{n-1}v \neq 0,\ T^n v = 0$. [See [5.16]]
Prove $\big(v, Tv, \dots, T^{n-1}v\big)$ *is liney indep.*

SOLUS: $a_0 v + a_1 Tv + \cdots + a_{n-1}T^{n-1}v = 0 \Rightarrow a_0 T^{n-1}v = 0 \Rightarrow a_0 = 0$. Simlr for $a_1, \dots, a_{n-1}$. $\square$

• (4E 21) *Supp* V *is finide,* $T \in \mathcal{L}(V, W)$, Y *is a subsp of* W. *Let* $\big\{v \in V : Tv \in Y\big\}$.
(a) *Prove* $\big\{v \in V : Tv \in Y\big\}$ *is a subsp of* V.
(b) *Prove* $\dim\big\{v \in V : Tv \in Y\big\} = \dim \operatorname{null} T + \dim\big(Y \cap \operatorname{range} T\big)$.

SOLUS: Let $\mathcal{K}_Y = \big\{v \in V : Tv \in Y\big\}$.

(a) $\forall u, w \in \mathcal{K}_Y, [\, Tu, Tw \in Y\,], \lambda \in \mathbf{F}, T(u + \lambda w) = Tu + \lambda Tw \in Y \Longrightarrow \mathcal{K}_Y$ is a subsp of V.

(b) Define the range-restr map R of T by $R = T|_{\mathcal{K}_Y} \in \mathcal{L}(\mathcal{K}_Y, Y)$. Now $\operatorname{range} R = Y \cap \operatorname{range} T$.
And $v \in \operatorname{null} T \iff Tv = 0 \in Y \iff Rv = 0 \in \operatorname{range} T \iff v \in \operatorname{null} R$. By [3.22]. $\square$

COMMENT: Now $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = \mathcal{K}_Y$. Where $B_{Y \cap \operatorname{range} T} = (Tv_1, \dots, Tv_m)$.
In particular, $\dim \mathcal{K}_{\operatorname{range} T} = \dim \operatorname{null} T + \dim \operatorname{range} T \Longrightarrow \mathcal{K}_{\operatorname{range} T} = V$.

• (4E 31) *Supp* V *is finide,* X *is a subsp of* V, *and* Y *is a finide subsp of* W.
Prove if $\dim X + \dim Y = \dim V$, *then* $\exists\, T \in \mathcal{L}(V, W), \operatorname{null} T = X, \operatorname{range} T = Y$.

SOLUS: Let $V = U \oplus X, B_U = (v_1, \dots, v_m)$. Then $\forall v \in V, \exists!\, a_i \in \mathbf{F}, x \in X,\ v = \sum_{i=1}^{m} a_i v_i + x$.
Let $B_Y = (w_1, \dots, w_m)$. Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i, Tx = 0$ for each v_i and all $x \in X$.
Now $v \in \operatorname{null} T \iff Tv = a_1 w_1 + \cdots + a_m w_m = 0 \iff v = x \in X$. Hence $\operatorname{null} T = X$.
And $Y \ni w = a_1 w_1 + \cdots + a_m w_m = a_1 Tv_1 + \cdots + a_m Tv_m \in \operatorname{range} T$. Hence $\operatorname{range} T = Y$.
OR. NOTICE that $V = U \oplus \operatorname{null} T$. By Exe (12), $\operatorname{range} T = \operatorname{range} T|_U$.
又 $\dim \operatorname{range} T|_U = \dim U = \dim Y$; $\operatorname{range} T \subseteq Y$.

OR. Let $B_X = (x_1, \dots, x_n)$. Now $\operatorname{range} T = \operatorname{span}\big(Tv_1, \dots, Tv_m, Tx_1, \dots, Tx_n\big) = \operatorname{span}(w_1, \dots, w_m) = Y$. $\square$

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22 *Supp* U, V *are finide,* $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.
Prove $\dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$.

SOLUS: We show $\dim \operatorname{null} ST = \dim \operatorname{null} S|_{\operatorname{range} T} + \dim \operatorname{null} T$.
Becs (a) $\underline{\operatorname{range} T|_{\operatorname{null} ST} = \operatorname{range} T \cap \operatorname{null} S = \operatorname{null} S|_{\operatorname{range} T}}$,
(b) $\underline{\operatorname{null} T|_{\operatorname{null} ST} = \operatorname{null} T \cap \operatorname{null} ST = \operatorname{null} T}$. By [3.22] $\square$

OR. NOTICE that $\underline{u \in \operatorname{null} ST \iff S(Tu) = 0 \iff Tu \in \operatorname{null} S}$.
Thus $\underline{\big\{u \in U : Tu \in \operatorname{null} S\big\} = \mathcal{K}_{\operatorname{null} S \cap \operatorname{range} T} = \operatorname{null} ST}$.
By Exe (4E 21), $\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim\big(\operatorname{null} S \cap \operatorname{range} T\big)$. $\square$

CORO: (1) T surj $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.
(2) T inv $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S,\ \operatorname{null} ST = \operatorname{null} T$.
(3) S inje $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} T$.

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23 *Supp V is finide, $S \in \mathcal{L}(V,W), T \in \mathcal{L}(U,V)$.*
Prove $\dim \operatorname{range} ST \leqslant \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.
COMMENT: If $\dim V = \dim U$. Then $\dim \operatorname{null} ST \geqslant \max\{\dim \operatorname{null} S, \dim \operatorname{null} T\}$.

SOLUS: NOTICE that $\underline{\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{range} S|_{\operatorname{range} T}}$.
Let $\operatorname{range} ST = \operatorname{span}(Su_1, \dots, Su_{\dim \operatorname{range} T})$, where $B_{\operatorname{range} T} = (u_1, \dots, u_{\dim \operatorname{range} T})$.
$\dim \operatorname{range} ST \leqslant \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leqslant \dim \operatorname{range} S$. □

OR. $\underline{\dim \operatorname{range} ST = \dim \operatorname{range} S|_{\operatorname{range} T} = \dim \operatorname{range} T - \dim \operatorname{null} S|_{\operatorname{range} T}} \leqslant \operatorname{range} T$. □
COMMENT: $\dim \operatorname{range} ST = \dim U - \dim \operatorname{null} ST = \dim \operatorname{range} T|_U - \dim \operatorname{range} T|_{\operatorname{null} ST}$.

CORO: (1) $S|_{\operatorname{range} T}$ inje $\iff \dim \operatorname{range} ST = \dim \operatorname{range} T$.
(2) Let $X \oplus \operatorname{null} S = V$. Then $X \subseteq \operatorname{range} T \iff \operatorname{range} ST = \operatorname{range} S$.
And T is surj $\Rightarrow \operatorname{range} ST = \operatorname{range} S$.

• (a) *Supp* $\dim V = n$, $ST = 0$ *where* $S, T \in \mathcal{L}(V)$. *Prove* $\dim \operatorname{range} TS \leqslant \lfloor \frac{n}{2} \rfloor$.
(b) *Give an exa of such S, T with $n = 5$ and* $\dim \operatorname{range} TS = 2$.

SOLUS: Note that $\dim \operatorname{range} TS \leqslant \min\{\dim \operatorname{range} T, \dim \operatorname{range} S\}$. We prove by ctradic.
Asum $\dim \operatorname{range} TS \geqslant \lfloor \frac{n}{2} \rfloor + 1$. Then $\min\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\} \geqslant \lfloor \frac{n}{2} \rfloor + 1$
又 $\dim \operatorname{null} ST = n \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \;\Big|\Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant \lceil \frac{n}{2} \rceil - 1$.
Thus $n \leqslant 2(\lceil \frac{n}{2} \rceil - 1) \Rightarrow \frac{n}{2} \leqslant \lceil \frac{n}{2} \rceil - 1$. Ctradic. □

OR. $\dim \operatorname{null} S = n - \dim \operatorname{range} S \leqslant n - \dim \operatorname{range} TS$. 又 $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S$.
$\dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S \leqslant n - \dim \operatorname{range} TS$. Thus $2 \dim \operatorname{range} TS \leqslant n$. □

OR. Becs $\dim \operatorname{range} TS \leqslant \lfloor \frac{n}{2} \rfloor$, and $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$.
We show $\dim \operatorname{null} TS \geqslant \lceil \frac{n}{2} \rceil$. Note that $\dim \operatorname{null} S + \dim \operatorname{null} T \geqslant n$.
$\dim \operatorname{null} S + \dim \operatorname{null} T|_{\operatorname{range} S} = \dim \operatorname{null} TS$. If $\dim \operatorname{null} S \geqslant \lceil \frac{n}{2} \rceil$. Then done.
Othws, $\dim \operatorname{null} S \leqslant \lceil \frac{n}{2} \rceil - 1 \Rightarrow \dim \operatorname{null} T \geqslant n - \dim \operatorname{null} S \geqslant n - \lceil \frac{n}{2} \rceil + 1 = \lfloor \frac{n}{2} \rfloor + 1 \geqslant \lceil \frac{n}{2} \rceil$.
Thus $\dim \operatorname{null} TS \geqslant \max\{\dim \operatorname{null} S, \dim \operatorname{null} T\} = \lceil \frac{n}{2} \rceil$. □

EXA: Define $T: v_1 \mapsto 0,\ v_2 \mapsto 0,\ v_i \mapsto v_i;\ S: v_1 \mapsto v_4,\ v_2 \mapsto v_5,\ v_i \mapsto 0;\ i = 3, 4, 5$.

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26 *Supp $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ and $\forall p, \deg(Dp) = (\deg p) - 1$. Prove $D \in \mathcal{P}(\mathbf{R})$ is surj.*

SOLUS: [D might not be $D : p \mapsto p'$.] NOTICE that the following proof is wrong:
Becs $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) \subseteq \operatorname{range} D$, and $\deg Dx^n = n - 1$.
又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) = \operatorname{span}(1, x, x^2, \cdots) = \mathcal{P}(\mathbf{R})$.

$\underline{\text{Let } D(C) = 0, Dx^k = p_k}$ of deg $(k-1)$, for all $C \in \mathbf{R} = \mathcal{P}_0(\mathbf{R})$ and for each $k \in \mathbf{N}^+$.
Becs $B_{\mathcal{P}_m(\mathbf{R})} = (p_1, \dots, p_m, p_{m+1})$. And for all $p \in \mathcal{P}(\mathbf{R}), \exists! m = \deg p \in \mathbf{N}^+$.
So that $\exists! a_i \in \mathbf{R}, p = \sum_{i=1}^{m+1} a_i p_i \Rightarrow \exists q = \sum_{i=1}^{m+1} a_i x^i, Dq = p$. □

OR. We will $\underline{\text{recurly define a seq of polys } (p_k)_{k=0}^{\infty} \text{ where } Dp_0 = 1, Dp_k = x^k}$ for each $k \in \mathbf{N}^+$.
So that $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), Dq = p, \exists q = \sum_{k=0}^{\deg p} a_k p_k$.
(i) Becs $\deg Dx = (\deg x) - 1 = 0, Dx = C \in \mathbf{F} \backslash \{0\}$. Let $p_0 = C^{-1}x \Rightarrow Dp_0 = C^{-1}Dx = 1$.
(ii) Supp we have defined $Dp_0 = 1, Dp_k = x^k$ for each $k \in \{1, \dots, n\}$. Becs $\deg D(x^{n+2}) = n + 1$.
Let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$, with $a_{n+1} \neq 0$.
Then $a_{n+1}^{-1} D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_n Dp_n + \cdots + a_1 Dp_1 + a_0 Dp_0)$
$\Rightarrow x^{n+1} = D\left[\underline{a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)}\right]$. Thus defining p_{n+1}, so that $Dp_{n+1} = x^{n+1}$. □

20, 21 (a) *Prove if $ST = I \in \mathcal{L}(V)$, then T is inje and S is surj.*
(b) *Supp $T \in \mathcal{L}(V, W)$. Prove if T is inje, then $\exists\, S \in \mathcal{L}(W, V),\ ST = I$.*
(c) *Supp $S \in \mathcal{L}(W, V)$. Prove if S is surj, then $\exists\, T \in \mathcal{L}(V, W),\ ST = I$.*

SOLUS:

(a) $Tv = 0 \Rightarrow S(Tv) = 0 = v$. OR. $\operatorname{null} T \subseteq \operatorname{null} ST = \{0\}$.
$\forall v \in V, ST(v) = v \in \operatorname{range} S$. OR. $V = \operatorname{range} ST \subseteq \operatorname{range} S$.

(b) Define $S \in \mathcal{L}(\operatorname{range} T, V)$ by $Sw = T^{-1}w$, where T^{-1} is the inv of $T \in \mathcal{L}(V, \operatorname{range} T)$.
Then extend to $S \in \mathcal{L}(W, V)$ by (3.A.11). Now $\forall v \in V, STv = T^{-1}Tv = v$.

OR. $\big[$ *Req V Finide* $\big]$ Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n) \Rightarrow B_V = (v_1, \dots, v_n)$. Let $U \oplus \operatorname{range} T = W$.
Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i, Su = 0$ for each v_i and all $u \in U$. Thus $ST = I$.

(c) By Exe (12), $\exists$ subsp U of $W, W = U \oplus \operatorname{null} S,\ \operatorname{range} S = \operatorname{range} S|_U = V$.
Note that $S|_U : U \to V$ is iso. Define $T = (S|_U)^{-1}$, where $(S|_U)^{-1} : V \to U$.
Then $ST = S \circ (S|_U)^{-1} = S|_U \circ (S|_U)^{-1} = I_V$.

OR. $\big[$ *Req V Finide* $\big]$ Let $B_{\operatorname{range} S} = B_V = (Sw_1, \dots, Sw_n) \Rightarrow \operatorname{span}(w_1, \dots, w_n) \oplus \operatorname{null} S = W$.
Define $T \in \mathcal{L}(V, W)$ by $T(Sw_i) = w_i$. Now $ST\big(a_1 Sw_1 + \cdots + a_n Sw_n\big) = \big(a_1 Sw_1 + \cdots + a_n Sw_n\big)$. $\square$

CORO: For (b), if T is inje and $\exists\, S,\ ST = I$, then by (a), this S is surj. Simlr for (c).

• **TIPS 5:** Supp $S \in \mathcal{L}(U, V)$ is surj. Define $\mathcal{B} \in \mathcal{L}\big(\mathcal{L}(V, W), \mathcal{L}(U, W)\big)$ by $\mathcal{B}(T) = TS$.
Then $\mathcal{B}$ is inje. Becs $\mathcal{B}(T) = TS = 0 \iff T|_{\operatorname{range} S} = 0$. OR. $\operatorname{range} TS = \operatorname{range} T = \{0\}$.

---

24 *Supp $S \in \mathcal{L}(V, M), T \in \mathcal{L}(V, W)$, and $\operatorname{null} S \subseteq \operatorname{null} T$. Prove $\exists\, E \in \mathcal{L}(M, W), T = ES$.*

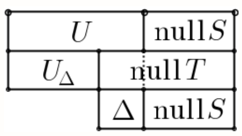
SOLUS:

Let $\underline{V = U \oplus \operatorname{null} S}$
$\underline{\Rightarrow S|_U : U \to \operatorname{range} S \text{ is iso.}}$
Extend $T(S|_U)^{-1}$ to $E \in \mathcal{L}(M, W)$.

$$\begin{tikzcd} \operatorname{range} T & U \arrow[l, "\text{surj } T"'] \arrow[d, "\text{inv } S"] \\ & \operatorname{range} S \arrow[ul, "\text{surj } E"] \end{tikzcd}$$

OR. Define $E : \operatorname{range} S \to W$ by $\underline{E : Sv \mapsto Tv.}$
Extend $E \in \mathcal{L}(\operatorname{range} S, W)$ to $E \in \mathcal{L}(M, W)$. $\square$

COMMENT: Let $\Delta \oplus \operatorname{null} S = \operatorname{null} T,\ U_\Delta \oplus (\Delta \oplus \operatorname{null} S) = V = U_\Delta \oplus \operatorname{null} T$. Redefine $U = U_\Delta \oplus \Delta$.



$$\operatorname{range} S \xleftarrow{S} \begin{array}{l} U_\Delta \xrightarrow{T} \operatorname{range} T \\ \oplus \\ \Delta \xrightarrow{T} \{0\} \end{array}$$

Becs $\Delta = \operatorname{null} T|_U = \operatorname{null} T \cap \operatorname{range} (S|_U)^{-1}$.
Thus $E = T(S|_U)^{-1}$ is not inje $\iff \Delta \neq \{0\}$.
In other words, $\operatorname{range} S|_\Delta = \operatorname{null} E$,
while $E|_{\dots} : \operatorname{range} S|_{U_\Delta} \to \operatorname{range} T$ is iso.

COMMENT: Let $E_1 \in \mathcal{L}(U_\Delta \oplus \operatorname{null} T, U_\Delta)$, and E_2 be an iso of $\operatorname{range} S|_{U_\Delta}$ onto $\operatorname{range} T$.
Define $E_1|_{U_\Delta} = I|_{U_\Delta}$, and $E_2 = T(S|_{U_\Delta})^{-1}$. Then $T = E_2 S E_1$.

CORO: If $\operatorname{null} S = \operatorname{null} T$. Then $\Delta = \{0\}, U_\Delta = U$. $\big[$ *Req W Finide* $\big]$ By (3.D.3),
we can extend inje $T(S|_U)^{-1} \in \mathcal{L}(\operatorname{range} S, W)$ to inv $E \in \mathcal{L}(M, W)$.

OR. $\big[$ *Req* range *S Finide* $\big]$ Let $B_{\operatorname{range} S} = (Sv_1, \dots, Sv_n)$. Then $\underline{V = \operatorname{span}(v_1, \dots, v_n) \oplus \operatorname{null} S.}$
Define $E \in \mathcal{L}(\operatorname{range} S, W)$ by $E(Sv_i) = Tv_i$. Extend to $E \in \mathcal{L}(M, W)$.
Hence $\forall v = \sum_{i=1}^n a_i v_i + u \in V$, $\underline{(\exists!\, u \in \operatorname{null} S \subseteq \operatorname{null} T),\ Tv = \sum_{i=1}^n a_i Tv_i + 0} = E\big(\sum_{i=1}^n a_i Sv_i + 0\big)$. $\square$

CORO: $\big[$ *Req W Finide* $\big]$ Supp $\operatorname{null} S = \operatorname{null} T$. We show $\exists$ inv $E \in \mathcal{L}(M, W), T = ES$.
Redefine $E \in \mathcal{L}(M, W)$ by $E(Tv_i) = Sv_i,\ E(w_j) = x_j$, for each Tv_i and w_j. Where:
Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m), B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n), B_U = (v_1, \dots, v_m)$.
Now $V = U \oplus \operatorname{null} T = U \oplus \operatorname{null} S \Rightarrow B_{\operatorname{range} S} = \big(Sv_1, \dots, Sv_m\big)$. Let $B_M = (Sv_1, \dots, Sv_m, x_1, \dots, x_n)$. $\square$

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25 *Supp* $S \in \mathcal{L}(Y, W), T \in \mathcal{L}(V, W)$, *and* $\operatorname{range} T \subseteq \operatorname{range} S$. *Prove* $\exists\, E \in \mathcal{L}(V, Y), T = SE$.

SOLUS:

Let $Y = U \oplus \operatorname{null} S$

$\Rightarrow S|_U : U \to \operatorname{range} S$ is iso. Becs $(S|_U)^{-1} : \overset{\operatorname{range} T \subseteq}{\operatorname{range} S} \to U$.

<u>Define $E = (S|_U)^{-1} T = (S|_U)^{-1}|_{\operatorname{range} T} T \in \mathcal{L}(V, U) \subseteq \mathcal{L}(V, Y)$.</u> ☐

$$\begin{tikzcd}
U_1 \arrow[r, "inv", "S"'] \arrow[d, phantom, "||"] & \operatorname{range} S \arrow[d, phantom, "||"] & \\
\Delta \arrow[r, "inv", "S"'] \arrow[d, phantom, "\oplus"] & \operatorname{range} S|_\Delta \arrow[d, phantom, "\oplus"] & \\
U_{1\Delta} \arrow[r, "inv", "S"'] & \operatorname{range} T & U_2 \arrow[l, "inv"', "T"] \arrow[ll, bend left, "inv\ E|_{U_2}"]
\end{tikzcd}$$

COMMENT: Let $U_1 = U$. Let $U_2 \oplus \operatorname{null} T = V$.

Let $U_{1\Delta} = \operatorname{range} \left(S|_{U_1}\right)^{-1}|_{\operatorname{range} T} \subseteq U_1 = \Delta \oplus U_{1\Delta}$.

OR. Let $U_{1\Delta} = \operatorname{range} E|_{U_2}$. Let $\Delta \oplus \operatorname{range} E|_{U_2} = U_1$.

[*Req* range T *Finide*] Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$. Now $B_{U_2} = (v_1, \dots, v_n)$.

<u>Let $S(u_i) = Tv_i$ for each Tv_i.</u> Define E by $Ev_i = u_i, Ex = 0$ for all $x \in \operatorname{null} T$ and each v_i. ☐

COMMENT: [*Req* V *Finide*] Note that $\dim U_2 \leqslant \dim U_1 \Longrightarrow \dim \operatorname{null} T = p \geqslant q = \dim \operatorname{null} S$.

Let $B_{\operatorname{null} T} = (x_1, \dots, x_p), B_{\operatorname{null} S} = (y_1, \dots, y_q)$. Redefine $E : v_i \mapsto u_i,\ x_k \mapsto y_k,\ x_j \mapsto 0$,

for each $i \in \{1, \dots, \dim U_2\}, k \in \{1, \dots, \dim \operatorname{null} S\} = K, j \in \{1, \dots, \dim \operatorname{null} T\} \setminus K$.

Note that $(u_1, \dots, u_n)$ is liney indep. Let $X = \operatorname{span}(x_1, \dots, x_q) \oplus \operatorname{span}(v_1, \dots, v_n)$.

Now $E|_X$ is inje, but cannot be re-extend to inv $E \in \mathcal{L}(V, Y)$ suth $T = SE$.

CORO: [*Req* V *Finide*] If $\operatorname{range} T = \operatorname{range} S$, then $\dim \operatorname{null} T = \dim \operatorname{null} S = p$.

Redefine E by $Ev_i = u_i,\ Ex_j = y_j$ for each v_i and x_j. Then $E \in \mathcal{L}(V, Y)$ is inv. ☐

- **COMMENT:** Supp $S, T \in \mathcal{L}(V, W)$. Then $\operatorname{range} S = \operatorname{range} T \nRightarrow \operatorname{null} S, \operatorname{null} T$ iso.

EXA: Forward shift optor on $\mathbf{F}^\infty$ and backward shift optor on $\{(0, x_1, x_2, \cdots) \in \mathbf{F}^\infty\}$.

While $\operatorname{null} S = \operatorname{null} T \Longleftrightarrow E : Sv \mapsto Tv$ and $E^{-1} : Tv \mapsto Sv$ well-defined $\Rightarrow \operatorname{range} S, \operatorname{range} T$ iso.

---

28 *Supp* $T \in \mathcal{L}(V, W)$. *Let* $(Tv_1, \dots, Tv_m)$ *be a bss of* range T *and each* $w_i = Tv_i$.

(a) *Prove* $\exists\, \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ *suth* $\forall v \in V, Tv = \varphi_1(v) w_1 + \cdots + \varphi_m(v) w_m$.

(b) [4E 3.F.5] $\forall v \in V, \exists!\, \varphi_i(v) \in \mathbf{F}, Tv = \varphi_1(v) w_1 + \cdots + \varphi_m(v) w_m$.

Thus defining each $\varphi_i : V \to \mathbf{F}$. *Show each* $\varphi_i \in \mathcal{L}(V, \mathbf{F})$.

SOLUS: *The answer for* (*b*) *with* (*b*) *itself is the answer for* (*a*).

(b) $\sum_{i=1}^m \varphi_i(u + \lambda v) w_i = T(u + \lambda v) = Tu + \lambda Tv = \left(\sum_{i=1}^m \varphi_i(u) w_i\right) + \lambda \left(\sum_{i=1}^m \varphi_i(v) w_i\right)$. ☐

OR. $\forall v \in V, \exists!\, a_i \in \mathbf{F}, Tv = a_1 Tv_1 + \cdots + a_m Tv_m$. Let $B_{(\operatorname{range} T)'} = (\psi_1, \dots, \psi_m)$.

Then $[T'(\psi_i)](v) = (\psi_i \circ T)(v) = a_i$. Thus each $\varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$. ☐

(a) $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = V \Rightarrow \forall v \in V, \exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} T,\ v = \sum_{i=1}^m a_i v_i + u$.

Define $\varphi_i \in \mathcal{L}(V, \mathbf{F})$ by $\varphi_i(v_j) = \delta_{i,j},\ \varphi_i(u) = 0$ for all $u \in \operatorname{null} T$.

Linity: $\forall v, w \in V\ [\exists!\, a_i, b_i \in \mathbf{F}], \lambda \in \mathbf{F}, \varphi_i(v + \lambda w) = a_i + \lambda b_i = \varphi(v) + \lambda \varphi(w)$. ☐

---

29 *Supp* $\varphi \in \mathcal{L}(V, \mathbf{F})$. *Supp* $\varphi(u) \neq 0$. *Prove* $V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}$.

SOLUS: Let $B_{\operatorname{range} \varphi} = (\varphi(u))$. Then by TIPS (4), $\operatorname{span}(u) \oplus \operatorname{null} \varphi = V$. ☐

OR. (a) $v = cu \in \operatorname{null} \varphi \cap \operatorname{span}(u) \Rightarrow c\varphi(u) = 0 \Rightarrow v = 0$. Now $\operatorname{null} \varphi \cap \operatorname{span}(u) = \{0\}$.

(b) $\forall\, v \in V, v = \left(v - \dfrac{\varphi(v)}{\varphi(u)} u\right) + \dfrac{\varphi(v)}{\varphi(u)} u$. Now $V = \operatorname{null} \varphi + \operatorname{span}(u)$. ☐

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30 *Supp* $\varphi, \beta \in \mathcal{L}(V, \mathbf{F})$ *and* $\operatorname{null}\varphi = \operatorname{null}\beta = \eta$. *Prove* $\exists\, c \in \mathbf{F}, \varphi = c\beta$.

SOLUS: If $\eta = V$, then $\varphi = \beta = 0$, done. Now by Exe (29),

$\varphi(u) \neq 0 \iff V = \operatorname{null}\varphi \oplus \operatorname{span}(u) \iff V = \operatorname{null}\beta \oplus \operatorname{span}(u) \iff \beta(u) \neq 0.$

Note that $\forall v \in V, \exists!\, u_0 \in \eta,\ a_v \in \mathbf{F}, v = u_0 + a_v u$
$\Rightarrow \varphi(u_0 + a_v u) = a_v\varphi(u),\ \beta(u_0 + a_v u) = a_v\beta(u).$ | Let $c = \dfrac{\varphi(u)}{\beta(u)} \in \mathbf{F}\backslash\{0\}$. ☐

- (4E 3.F.6) *Supp* $\varphi, \beta \in \mathcal{L}(V, \mathbf{F})$. *Prove* $\operatorname{null}\beta \subseteq \operatorname{null}\varphi \iff \varphi = c\beta, \exists\, c \in \mathbf{F}$.

  CORO: $\operatorname{null}\varphi = \operatorname{null}\beta \iff \varphi = c\beta,\ \exists\, c \in \mathbf{F}\backslash\{0\}$.

SOLUS: Using Exe (29) and (30).

(a) If $\varphi = 0$, then done. Othws, supp $u \notin \operatorname{null}\varphi \supseteq \operatorname{null}\beta$.

Now $V = \operatorname{null}\varphi \oplus \operatorname{span}(u) = \operatorname{null}\beta \oplus \operatorname{span}(u)$. By $\left[1.\text{C TIPS }(2)\right]$, $\operatorname{null}\varphi = \operatorname{null}\beta$. Let $c = \dfrac{\varphi(u)}{\beta(u)}$.

OR. We discuss in two cases. If $\operatorname{null}\beta = \operatorname{null}\varphi$, or if $\varphi = 0$, then done. Othws,
$\exists\, u' \in \operatorname{null}\varphi \backslash \operatorname{null}\beta$, $\exists\, u \notin \operatorname{null}\varphi \supsetneq \operatorname{null}\beta \Rightarrow V = \operatorname{null}\beta \oplus \operatorname{span}(u') = \operatorname{null}\beta \oplus \operatorname{span}(u)$.

$\forall v \in V, v = w + au = w' + bu', \exists!\, w, w' \in \operatorname{null}\beta$
Thus $\varphi(w + au) = a\varphi(u),\ \beta(w' + bu) = b\beta(u')$. | Let $c = \dfrac{a\varphi(u)}{b\beta(u')} \in \mathbf{F}\backslash\{0\}$. Done.

NOTICE that by (b) below, we have $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$, ctradic the asum.

(b) If $c = 0$, then $\operatorname{null}\varphi = V \supseteq \operatorname{null}\beta$, done. Othws, becs $v \in \operatorname{null}\beta \iff v \in \operatorname{null}\varphi$. ☐

OR. By Exe (24), $\operatorname{null}\beta \subseteq \operatorname{null}\varphi \iff \exists\, E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta$. $\left[\text{ If } E \text{ is inv. Then } \operatorname{null}\beta = \operatorname{null}\varphi. \right]$
Now $\exists\, E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta \iff \exists\, c = E(1) \in \mathbf{F}, \varphi = c\beta$. $\left[\ E \text{ is inv} \iff E(1) \neq 0 \iff c \neq 0.\ \right]$ ☐

ENDED

3·C

1 3 4 5 6 9 10 11 13 | 4E: 16 17

- **NOTE FOR [3.30, 32]:** *matrix of span*

  Supp $L_\alpha = (\alpha_1, \dots, \alpha_n)$ and $L_\beta = (\beta_1, \dots, \beta_m)$ are in a vecsp V.
  Let each $\alpha_k = A_{1,k}\beta_1 + \cdots + A_{m,k}\beta_m$, forming $A = \mathcal{M}(\operatorname{span} L_\beta \supseteq L_\alpha) \in \mathbf{F}^{m,n}$.

  Which is *the matrix of span*. Then $\begin{pmatrix}\beta_1 & \cdots & \beta_m\end{pmatrix}\begin{pmatrix}A_{1,1} & \cdots & A_{1,n}\\ \vdots & \ddots & \vdots\\ A_{m,1} & \cdots & A_{m,n}\end{pmatrix} = \begin{pmatrix}\alpha_1 & \cdots & \alpha_n\end{pmatrix}$.

  (a) Supp $m = n$. If $(A_{\cdot,1}, \dots, A_{\cdot,n})$ is a bss of $\mathbf{F}^{n,1}$. We show L_α liney indep $\iff$ L_β liney indep.
  ($\Leftarrow$) Immed. ($\Rightarrow$) Asum L_β is liney dep and $\beta_j = c_1\beta_1 + \cdots + c_{j-1}\beta_{j-1}$. By ctradic. ☐

  (b) Supp $m \geqslant n$. If L_β liney indep. We show $(A_{\cdot,1}, \dots, A_{\cdot,n})$ liney indep $\iff$ L_α liney indep.
  ($\Rightarrow$) Immed. ($\Leftarrow$) By ctradic. ☐

  COMMENT: $\mathcal{M}(\operatorname{span} L_\beta \supseteq L_\alpha) = \mathcal{M}(I, L_\alpha, L_\beta) \iff L_\alpha, L_\beta$ liney indep $\Rightarrow (A_{\cdot,1}, \dots, A_{\cdot,n})$ liney indep.
  Where I is the id optor retr to $\operatorname{span} L_\alpha \subseteq \operatorname{span} L_\beta$.

  (c) Supp $m < n$. Then $(A_{\cdot,1}, \dots, A_{\cdot,n})$ is liney dep, so is L_α.

  Supp $T \in \mathcal{L}(V, W)$ and $B_V = (v_1, \dots, v_m), B_W = (w_1, \dots, w_n)$.
  Then $\mathcal{M}(T, B_V, B_W) = \mathcal{M}\big(\operatorname{span} B_W \supseteq (Tv_1, \dots, Tv_m)\big)$. **COMMENT:** See also (4E 3.D.23).

- **NOTE FOR Trspose:** [3.F.33] Define $\mathcal{T}: A \to A^t$. By [3.111], $\mathcal{T}$ is liney. Becs $(A^t)^t = A$. $\mathcal{T}^2 = I$, $\mathcal{T} = \mathcal{T}^{-1} \Rightarrow \mathcal{T}$ is iso of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$. Define $\mathcal{C}_k : A \to A_{\cdot,k}$, $\mathcal{R}_j : A \to A_{j,\cdot}$, $\mathcal{E}_{j,k} : A \to A_{j,k}$.
  Now we show (a) $\underline{\mathcal{T}\mathcal{R}_j = \mathcal{C}_j\mathcal{T}}$, (b) $\underline{\mathcal{T}\mathcal{C}_k = \mathcal{R}_k\mathcal{T}}$, and (c) $\underline{\mathcal{T}\mathcal{E}_{j,k} = \mathcal{E}_{k,j}\mathcal{T}}$.
  So that $\mathcal{T}\mathcal{C}_k\mathcal{T} = \mathcal{R}_k$, $\mathcal{T}\mathcal{R}_j\mathcal{T} = \mathcal{C}_j$, and $\mathcal{T}\mathcal{E}_{j,k}\mathcal{T} = \mathcal{E}_{k,j}$.

  Let $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{pmatrix}$.

  Note that $(A_{j,k})^t = A_{j,k} = (A^t)_{k,j}$. Thus (c) holds.
  And $(A_{\cdot,k})^t = (A_{1,k} \cdots A_{m,k}) = (A^t_{k,1} \cdots A^t_{k,m}) = (A^t)_{k,\cdot}$
  $\Longrightarrow$ (b) holds. Simlr for (a).

- **NOTE FOR [3.48]:**

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}}_{B} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

- **NOTE FOR [3.47]:** $(AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k}$ ☐
- **NOTE FOR [3.49]:** $\left[(AC)_{\cdot,k}\right]_{j,1} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$ ☐
- **EXE 10:** $\left[(AC)_{j,\cdot}\right]_{1,k} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$ ☐
- **COMMENT:** For [3.49], let $B_U = (u_1, \dots, u_p)$, $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.
  And $C = \mathcal{M}(T, B_U, B_V) \in \mathbf{F}^{n,p}$, $A = \mathcal{M}(S, B_V, B_W) \in \mathbf{F}^{m,n}$.
  Then $\mathcal{M}(Tu_k, B_V) = C_{\cdot,k} \Rightarrow \mathcal{M}(S(Tu_k), B_W) = AC_{\cdot,k}$, 又 $\mathcal{M}\big((ST)(u_k), B_W\big) = (AC)_{\cdot,k}$ ☐
  By NOTE FOR Transpose, $(AC)_{j,\cdot} = \left[\big((AC)^t\big)_{\cdot,j}\right]^t = \big(C^t(A^t)_{\cdot,j}\big)^t = \big((A^t)_{\cdot,j}\big)^tC = A_{j,\cdot}C$ ☐

- **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$. By [4E 3.51(a)], $(Ac)_{\cdot,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ ☐
  OR. $\because (Ac)_{j,1} = \sum_{r=1}^n A_{j,r}c_{r,1} = \left[\sum_{r=1}^n (A_{\cdot,r}c_{r,1})\right]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$
  $\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^n A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ OR. $(Ac)_{j,1} = (Ac)_{j,\cdot} = A_{j,\cdot}c \in \mathbf{F}$. ☐
  OR. Let $B_V = (v_1, \dots, v_n)$. Now $Ac = \mathcal{M}(Tv, B_W) = \mathcal{M}(T(c_1v_1 + \cdots + c_nv_n)) = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$. ☐

- **EXE 11:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$. By [4E 3.51(b)], $(aC)_{1,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ ☐
  OR. $\because (aC)_{1,k} = \sum_{r=1}^n a_{1,r}C_{r,k} = \left[\sum_{r=1}^n a_{1,r}(C_{r,\cdot})\right]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$
  $\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^n a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ OR. $(aC)_{1,k} = (aC)_{\cdot,k} = aC_{\cdot,k} \in \mathbf{F}$. ☐
  OR. $aC = \big((aC)^t\big)^t = \big(C^ta^t\big)^t = \left[a_1^t(C^t)_{\cdot,1} + \cdots + a_n^t(C^t)_{\cdot,n}\right]^t = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$. ☐

- [4E 3.51] Supp $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}$. [See also NOTE FOR [3.49] and Exe (10).]
  (a) For $k = 1, \dots, p$, $(CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^c C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$
  (b) For $j = 1, \dots, m$, $(CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^c C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
- **EXA:** $m = 2, c = 2, p = 3$.

$$(AB)_{\cdot,2} = AB_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} = A_{\cdot,1}B_{1,2} + A_{\cdot,2}B_{2,2} = 6\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix};$$

$$(AB)_{1,\cdot} = A_{1,\cdot}B = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = A_{1,1}B_{1,\cdot} + A_{1,2}B_{2,\cdot} = 1\begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 2\begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \end{pmatrix};$$

• **Column-Row Factoriz** (CR Factoriz) *Supp* $A \in \mathbf{F}^{m,n}, A \neq 0$.
Prove, with p *specified below, that* $\exists\, C \in \mathbf{F}^{m,p}, R \in \mathbf{F}^{p,n}, A = CR$.
(a) *Supp* $S_c = \operatorname{span}\big(A_{\cdot,1}, \cdots, A_{\cdot,n}\big) \subseteq \mathbf{F}^{m,1}, \dim S_c = c,$ *the col rank. Let* $p = c$.
(b) *Supp* $S_r = \operatorname{span}\big(A_{1,\cdot}, \cdots, A_{m,\cdot}\big) \subseteq \mathbf{F}^{1,n}, \dim S_r = r,$ *the row rank. Let* $p = r$.

Solus: Using [4E 3.51]. Notice that $A \neq 0 \Rightarrow c, r \geqslant 1$.

(a) Reduce to bss $B_C = \big(C_{\cdot,1}, \cdots, C_{\cdot,c}\big)$, forming $C \in \mathbf{F}^{m,c}$. Then $\forall k \in \{1, \dots, n\}$,
$A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = \big(CR\big)_{\cdot,k}, \exists!\, R_{1,k}, \cdots, R_{c,k} \in \mathbf{F}$, forming $R \in \mathbf{F}^{c,n}$. Thus $A = CR$.

(b) Reduce to bss $B_R = \big(R_{1,\cdot}, \cdots, R_{r,\cdot}\big)$, forming $R \in \mathbf{F}^{r,n}$. Then $\forall j \in \{1, \dots, m\}$,
$A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,r}R_{r,\cdot} = \big(CR\big)_{j,\cdot}, \exists!\, C_{j,1}, \dots, C_{j,r} \in \mathbf{F}$, forming $C \in \mathbf{F}^{m,r}$. Thus $A = CR$. ☐

Exa: $A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \overset{\text{(I)}}{=\!=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \overset{\text{(II)}}{=\!=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$

(I) $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2\begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix}$, using [4E 3.51(b)].

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} \in \operatorname{span}(A_{1,\cdot}, A_{2,\cdot})$, and $(A_{1,\cdot}, A_{2,\cdot})$ is liney indep. Thus $B_R = \big(A_{1,\cdot}, A_{2,\cdot}\big)$.

(II) $\begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = -\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$. Thus $B_C = \big(A_{\cdot,2}, A_{\cdot,3}\big)$.

• **Column Rank Equals Row Rank** Using nota and result above.
For each $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = \big(CR\big)_{j,\cdot} = C_{j,\cdot}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
For each $A_{\cdot,k} \in S_c$, $A_{\cdot,k} = \big(CR\big)_{\cdot,k} = CR_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$
$\Rightarrow \operatorname{span}\big(A_{1,\cdot}, \cdots, A_{n,\cdot}\big) = S_r = \operatorname{span}\big(R_{1,\cdot}, \cdots, R_{c,\cdot}\big) \Rightarrow \dim S_r = r \leqslant c = \dim S_c$.
$\Rightarrow \operatorname{span}\big(A_{\cdot,1}, \cdots, A_{\cdot,m}\big) = S_c = \operatorname{span}\big(C_{\cdot,1}, \cdots, C_{\cdot,r}\big) \Rightarrow \dim S_c = c \leqslant r = \dim S_r$.
Or. Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leqslant r = \dim S_r = \dim S_c^t$. ☐

• *Supp* $A \in \mathbf{F}^{m,n} \backslash \{0\}$. *Prove* $[P]$ $\operatorname{rank} A = 1 \iff \exists\, c_j, d_k \in \mathbf{F}$, *each* $A_{j,k} = c_j \cdot d_k$. $[Q]$

Solus:

[*Using CR Factoriz*]

$P \Rightarrow Q$: Immed.

$Q \Rightarrow P$: Becs $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1d_1 & \cdots & c_1d_n \\ \vdots & \ddots & \vdots \\ c_md_1 & \cdots & c_md_n \end{pmatrix} \Longrightarrow S_r = \operatorname{span}\left\{ \begin{matrix} \big(\underline{c_1}d_1 \ \cdots \ \underline{c_1}d_n\big), \\ \vdots \\ \big(\underline{c_m}d_1 \ \cdots \ \underline{c_m}d_n\big) \end{matrix} \right\}$.

Or. $S_c = \operatorname{span}\left\{ \begin{pmatrix} c_1\underline{d_1} \\ \vdots \\ c_m\underline{d_1} \end{pmatrix}, \dots, \begin{pmatrix} c_1\underline{d_n} \\ \vdots \\ c_m\underline{d_n} \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$. ☐

[*Not Using CR Factoriz*]

$Q \Rightarrow P$: Using [4E 3.51(a)]. Each $A_{\cdot,k} \in \operatorname{span}\left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$. Then $\operatorname{rank} A = \dim S_c \leqslant 1$
又 $A \neq 0 \Rightarrow \dim S_c \geqslant 1$.

$P \Rightarrow Q$: Becs $\dim S_c = \dim S_r = 1$.

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $\quad d_k' = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$.

$\Rightarrow A_{j,k} = d_k' A_{j,1} = c_j A_{1,k} = c_j d_k' A_{1,1} = c_j d_k$, where $d_k = d_k' A_{1,1}$. ☐

• **TIPS 1:** *Supp* $T \in \mathcal{L}(V,W)$, $B_V = (v_1,\dots,v_n), B_W = (w_1,\dots,w_m)$.

Let $L = (Tv_{\alpha_1},\dots,Tv_{\alpha_k})$, $L_{\mathcal{M}} = (A_{\cdot,\alpha_1},\cdots,A_{\cdot,\alpha_k})$, *where each* $\alpha_i \in \{1,\dots,n\}$.

(a) *Show* $[P]$ L *is liney indep* $\Longleftrightarrow$ $L_{\mathcal{M}}$ *is liney indep.* $[Q]$

(b) *Show* $[P]$ $\operatorname{span} L = W \Longleftrightarrow \operatorname{span} L_{\mathcal{M}} = \mathbf{F}^{m,1}$. $[Q]$ $\qquad$ $\big[$ *Let* $A = \mathcal{M}(T, B_V, B_W)$. $\big]$

SOLUS: (a) Note that $\mathcal{M}: Tv_k \to A_{\cdot,k}$ is iso. of $\operatorname{span} L$ onto $\operatorname{span} L_{\mathcal{M}}$. By (3.B.9).

(b) Reduce to liney indep lists. By (a) and (2.39). $\square$

OR. $c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} = c_1(A_{1,\alpha_1}w_1 + \cdots + A_{m,\alpha_1}w_m) + \cdots + c_k(A_{1,\alpha_k}w_1 + \cdots + A_{m,\alpha_k}w_m)$

$= (c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k})w_1 + \cdots + (c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k})w_m$.

And $c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} = c_1\begin{pmatrix} A_{1,\alpha_1} \\ \vdots \\ A_{m,\alpha_1}\end{pmatrix} + \cdots + c_k\begin{pmatrix} A_{1,\alpha_k} \\ \vdots \\ A_{m,\alpha_k}\end{pmatrix} = \begin{pmatrix} c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k} \\ \vdots \\ c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k}\end{pmatrix}$.

(a) $P \Rightarrow Q$: Supp $c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} = 0$. Let $v = c_1v_{\alpha_1} + \cdots + c_kv_{\alpha_k}$.

Then $Tv = (c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k})w_1 + \cdots + (c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k})w_m = 0w_1 + \cdots + 0w_m$.

Now $c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k} = 0$. Then each $c_i = 0 \Rightarrow L_{\mathcal{M}}$ liney indep.

$Q \Rightarrow P$: Becs $c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k} = 0$. For each $i \in \{1,\dots,m\}$, $c_1A_{i,\alpha_1} + \cdots + c_kA_{i,\alpha_k} = 0$.

Which is equiv to $c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} = 0$. Thus each $c_i = 0 \Rightarrow L$ liney indep.

OR. $\exists\, A_{\cdot,\alpha_j} = c_1A_{\cdot,\alpha_1} + \cdots + c_{j-1}A_{\cdot,\alpha_{j-1}}$

$\Longleftrightarrow$ For each $i \in \{1,\dots,m\}$, $A_{i,\alpha_j} = c_1A_{i,\alpha_1} + \cdots + c_{j-1}A_{i,\alpha_{j-1}}$

$\Longleftrightarrow Tv_{\alpha_j} = A_{1,\alpha_j}w_1 + \cdots + A_{m,\alpha_j}w_m$

$= (c_1A_{1,\alpha_1} + \cdots + c_{j-1}A_{1,\alpha_{j-1}})w_1 + \cdots + (c_1A_{m,\alpha_1} + \cdots + c_{j-1}A_{m,\alpha_{j-1}})w_m$

$\Longleftrightarrow \exists\, Tv_{\alpha_j} = c_1Tv_{\alpha_1} + \cdots + c_{j-1}Tv_{\alpha_{j-1}}$.

(b) Note that each $\mathcal{M}(Tv_{\alpha_i}) = A_{\cdot,\alpha_i}$

$P \Rightarrow Q$: Supp each $w_i = Iw_i = J_{1,i}Tv_{\alpha_1} + \cdots + J_{k,i}Tv_{\alpha_k}$.

$\forall a \in \mathbf{F}^{m,1}, \exists\, w = a_1w_1 + \cdots + a_mw_m \in W,\ a = \mathcal{M}(w, B_W)$.

Becs $w = a_1(J_{1,1}Tv_{\alpha_1} + \cdots + J_{k,1}Tv_{\alpha_k}) + \cdots + a_m(J_{1,m}Tv_{\alpha_1} + \cdots + J_{k,m}Tv_{\alpha_k})$

$= (a_1J_{1,1} + \cdots + a_mJ_{1,m})Tv_{\alpha_1} + \cdots + (a_1J_{k,1} + \cdots + a_mJ_{k,m})Tv_{\alpha_k}$.

Apply $\mathcal{M}$ to both sides, $a = c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k}$, where each $c_i = a_1J_{i,1} + \cdots + a_mJ_{i,m}$.

$Q \Rightarrow P$: $\forall w \in W, \exists\, a = c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} \in \mathbf{F}^{m,1}$, $\mathcal{M}(w, B_W) = a$

$\Rightarrow w = (c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k})w_1 + \cdots + (c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k})w_m = c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k}$.

$\neg Q \Rightarrow \neg P$: $\exists\, w \in W, \exists\, a \in \mathbf{F}^{m,1}, \mathcal{M}(w, B_W) = a$, but $\nexists\, (c_1,\dots,c_k) \in \mathbf{F}^k, a = c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k}$

$\Rightarrow \nexists\, (c_1,\dots,c_k) \in \mathbf{F}^k,\ w = c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k}$. For if not, ctradic. $\square$

NOTE: Let $L = (Tv_1,\dots,Tv_n)$, $L_{\mathcal{M}} = (A_{\cdot,1},\cdots,A_{\cdot,n})$.

Then (a*) By $[$3.B.9, TIPS (4)$]$, T is inje $\Longleftrightarrow$ L is liney indep, so is $L_{\mathcal{M}}$.

And (b*) T is surj $\Longleftrightarrow \operatorname{span} L = W \Longleftrightarrow \operatorname{span} L_{\mathcal{M}} = \mathbf{F}^{m,1}$.

CORO: $B_{\mathbf{F}^{n,1}} = (A_{\cdot,1},\cdots,A_{\cdot,n}) \Longleftrightarrow T$ is inje and surj $\Longleftrightarrow B_{\mathbf{F}^{1,n}} = (A_{\cdot,1},\cdots,A_{\cdot,n})$.

COMMENT: If T is inv. Then by (a*, c) or (b*, d), we have another proof of CORO.

OR. If $m = n$. Then by [3.118] and one of (a*, b*, c, d). Yet another proof.

(c) T surj $\Longleftrightarrow T'$ inje $\Longleftrightarrow (T'(\psi_1),\dots,T'(\psi_m))$ liney indep

$\overset{(a)}{\Longleftrightarrow} \big((A^t)_{\cdot,1},\cdots,(A^t)_{\cdot,m}\big)$ liney indep in $\mathbf{F}^{n,1}$, so is $(A_{1,\cdot},\cdots,A_{m,\cdot})$ in $\mathbf{F}^{1,n}$.

(d) T inje $\Longleftrightarrow T'$ surj $\Longleftrightarrow V' = \operatorname{span}(T'(\psi_1),\dots,T'(\psi_m))$

$\overset{(b)}{\Longleftrightarrow} \mathbf{F}^{n,1} = \operatorname{span}\big((A^t)_{\cdot,1},\cdots,(A^t)_{\cdot,m}\big) \Longleftrightarrow \mathbf{F}^{1,n} = \operatorname{span}(A_{1,\cdot},\cdots,A_{m,\cdot})$.

• **Tips 2**: *Supp p is a poly of n variables in $\mathbf{F}$. Prove $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big)$.*
Where the liney maps $T_1,\dots,T_n$ are suth $p(T_1,\dots,T_n)$ makes sense. See [5.16,17,20].

Solus: Supp the poly p is defined by $p(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^{n} x_i^{k_i}$.
Note that $\mathcal{M}(T^x S^y) = \mathcal{M}(T)^x\mathcal{M}(S)^y$; $\mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$.
Then $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = \mathcal{M}\Big(\sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^{n} T_i^{k_i}\Big)$
$= \sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^{n}\mathcal{M}\big(T_i^{k_i}\big) = p\Big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\Big)$. □

• **Coro:** Supp τ is an algebraic property. Then τ holds for liney maps $\iff$ τ holds for matrices.
Supp $\alpha_1,\dots,\alpha_n$ are dist with each $\alpha_k \in \{1,\dots,n\}$.
Now $p(T_1,\dots,T_n) = p(T_{\alpha_1},\dots,T_{\alpha_n}) \iff p\Big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\Big) = p\Big(\mathcal{M}(T_{\alpha_1}),\dots,\mathcal{M}(T_{\alpha_n})\Big)$.

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13 *Prove the distr holds for matrix add and matrix multi.*
Supp A, B, C are matrices suth $A(B+C)$ make sense, we prove the left distr.

Solus: Supp $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.
Note that $\big[A(B+C)\big]_{j,k} = \sum_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum_{r=1}^{n}\big(A_{j,r}B_{r,k} + A_{j,r}C_{r,k}\big) = \big(AB+AC\big)_{j,k}$.

Or. Define T, S, R suth $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.
$A(B+C) = \mathcal{M}(T(S+R)) \xlongequal{[3.9]} \mathcal{M}(TS+TR) = AB + AC$.
Or. $T(S+R) = TS + TR \Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR) \Rightarrow A(B+C) = AB + AC$. □

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1 *Supp $T \in \mathcal{L}(V,W)$. Show for each pair of B_V and B_W,*
$A = \mathcal{M}(T, B_V, B_W)$ has at least $n = \operatorname{dim range} T$ nonzero ent.

Solus:
Let $U \oplus \operatorname{null} T = V$; $B_U = (v_1,\dots,v_n), B_V = (v_1,\dots,v_m)$.
For each $k \in \{1,\dots,n\}, Tv_k \neq 0 \iff A_{\cdot,k} \neq 0$. Hence every such $A_{\cdot,k}$ has at least one nonzero ent. □

Or. We prove by ctradic. Supp A has at most $(n-1)$ nonzero ent.
Then by Pigeon Hole Principle, at least one of $A_{\cdot,1},\dots,A_{\cdot,n}$ equals 0.
Thus there are at most $(n-1)$ nonzero vecs in $Tv_1,\dots,Tv_n$.
又 $\operatorname{range} T = \operatorname{span}(Tv_1,\dots,Tv_n) \Rightarrow \operatorname{dim range} T = \dim\operatorname{span}(Tv_1,\dots,Tv_n) \leqslant n-1$. Ctradic. □

---

6 *Supp V and W are finide and $T \in \mathcal{L}(V,W)$.*
Prove $\operatorname{dim range} T = 1 \iff \exists\, B_V, B_W$, all ent of $A = \mathcal{M}\big(T, B_V, B_W\big)$ equal 1.

Solus:
(a) Supp $B_V = (v_1,\dots,v_n), B_W = (w_1,\dots,w_m)$ are the bses suth all ent of A equal 1.
Then $Tv_i = w_1 + \cdots + w_m$ for all $i = 1,\dots,n$. Becs $w_1,\dots,w_n$ is liney indep, $w_1 + \cdots + w_n \neq 0$.

(b) Supp $\operatorname{dim range} T = 1$. Then $\dim\operatorname{null} T = \dim V - 1$.
Let $B_{\operatorname{null} T} = (u_2,\dots,u_n)$. Extend to a bss $(u_1,u_2,\dots,u_n)$ of V.
Becs $Tv_1 \neq 0$. Extend to $(Tv_1, w_2,\dots,w_m)$ a bss of W. Let $w_1 = Tv_1 - w_2 - \cdots - w_m$.
Now $B_W = (w_1,\dots,w_m)$. Let $v_1 = u_1$, $v_i = u_1 + u_i$. Now $B_V = (v_1,\dots,v_n)$. □

Or. Supp $B_{\operatorname{range} T} = (w)$. By [2.C Note For (15)], $\exists\, B_W = (w_1,\dots,w_m)$, $w = w_1 + \cdots + w_m$.
By [2.C Tips], $\exists$ a bss $(u_1,\dots,u_n)$ of V suth each $u_k \notin \operatorname{null} T$.
Now each $Tu_k \in \operatorname{range} T = \operatorname{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists\,\lambda_k \in \mathbf{F}\backslash\{0\}$.
Let $v_k = \lambda_k^{-1}u_k \neq 0$, so that each $Tv_k = w = w_1 + \cdots + w_m$. Thus $B_V = (v_1,\dots,v_n)$ will do. □

---

3 *Supp V and W are finide and $T \in \mathcal{L}(V,W)$. Prove $\exists\, B_V, B_W$ suth*
[letting $A = \mathcal{M}(T, B_V, B_W)$] $A_{k,k} = 1, A_{i,j} = 0$, where $1 \leqslant k \leqslant \dim \operatorname{range} T, i \neq j$.

Solus: Let $B_{\operatorname{null} T} = (u_1, \dots, u_m), B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n) \Rightarrow B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$. ☐

Comment: Let each $Tv_k = w_k$. Extend $B_{\operatorname{range} T}$ to $B_W = (w_1, \dots, w_n, \dots, w_p)$. See [3.D Note For [3.60]].

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4 *Supp $B_V = (v_1, \dots, v_m)$ and W is finide. Supp $T \in \mathcal{L}(V,W)$.*
Prove $\exists\, B_W = (w_1, \dots, w_n)$, $\mathcal{M}(T, B_V, B_W)_{\cdot,1} = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}^t$ or $\begin{pmatrix}0 & \cdots & 0\end{pmatrix}^t$.

Solus: If $Tv_1 = 0$, then done. If not then extend (Tv_1) to B_W. ☐

---

5 *Supp $B_W = (w_1, \dots, w_n)$ and V is finide. Supp $T \in \mathcal{L}(V,W)$.*
Prove $\exists\, B_V = (v_1, \dots, v_m)$, $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix}0 & \cdots & 0\end{pmatrix}$ or $\begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}$.

Solus:

Let $(u_1, \dots, u_n)$ be a bss of V. Denote $\mathcal{M}(T, (u_1, \dots, u_n), B_W)$ by A.
If $A_{1,\cdot} = 0$, then $B_V = (u_1, \dots, u_n)$ and done. Othws, supp $A_{1,k} \neq 0$.

Let $v_1 = \dfrac{u_k}{A_{1,k}} \Rightarrow Tv_1 = 1w_1 + \dfrac{A_{2,k}}{A_{1,k}} w_2 + \cdots + \dfrac{A_{n,k}}{A_{1,k}} w_n$. | Let $v_{j+1} = u_j - A_{1,j} v_1$ for each $j \in \{1, \dots, k-1\}$. Let $v_i = u_i - A_{1,i} v_1$ for $i \in \{k+1, \dots, n\}$.

Notice that $Tu_i = A_{1,i} w_1 + \cdots + A_{n,i} w_n$. 又 Each $u_i \in \operatorname{span}(v_1, \dots, v_n) = V$. Let $B_V = (v_1, \dots, v_n)$. ☐

Or. Using Exe (4). Let $B_{W'}$ be the B_V. Now $\exists\, B_{V'}$ suth $\mathcal{M}(T', B_{W'}, B_{V'})_{\cdot,1} = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}^t$ or $\begin{pmatrix}0 & \cdots & 0\end{pmatrix}^t$.
Which is equiv to $\exists\, B_V$ [Using (3.F.31)] suth $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}$ or $\begin{pmatrix}0 & \cdots & 0\end{pmatrix}$. ☐

Ended

3·D

1 2 3 4 5 6 8 9 10 11 12 13 15 16 17 18 19 | 4E: 3 10 15 17 19 20 22 23 24

2 *Supp $\dim V > 1$. Prove the set U of non-inv optors on V is not a subsp of $\mathcal{L}(V)$.*
The set of inv optors is not either. Although multi id/inv, and commu for vec multi hold.

Solus: Let $B_V = (v_1, \dots, v_n)$. [*If $\dim V = 1$, then $U = \{0\}$ is a subsp of $\mathcal{L}(V)$.*]
Define $S, T \in \mathcal{L}(V)$ by $S(a_1v_1 + \cdots + a_nv_n) = a_1v_1$, $T(a_1v_1 + \cdots + a_nv_n) = a_2v_1 + \cdots + a_nv_n$. ☐

• *Supp $T \in \mathcal{L}(V)$. Prove $\exists$ inv $R, S \in \mathcal{L}(V)$ suth $T = T_1 + T_2$.*

Solus: Let $U \oplus \operatorname{null} T = V, W \oplus \operatorname{range} T = V$. Let $S : \operatorname{null} T \to W$ be an iso.

$$\left.\begin{aligned} &\text{Define } T_1 \in \mathcal{L}(V) \text{ by } T_1(u) = \tfrac{1}{2}Tu, T_1(w) = Sw \\ &\text{Define } T_2 \in \mathcal{L}(V) \text{ by } T_2(u) = \tfrac{1}{2}Tu, T_2(w) = -Sw \end{aligned}\right\} \Rightarrow T = T_1 + T_2 \text{ and } T_1, T_2 \text{ inv.}$$ ☐

• **Tips:** *Supp $V = U \oplus X = W \oplus X$. Prove U, W are iso.*

Solus: $\forall u \in U, \exists!\,(w, x_1) \in W \times X, u = w + x_1$. While $\exists!\,(u', x_2) \in U \times X, w = u' + x_2$.
Now $x_1 = -x_2$, $u = u'$. Thus $\pi : U \to W$ defined by $\pi(u) = w$, is inje.
$\forall w \in W, \exists!\,(u, x_1) \in U \times X, w = u + x_1$. While $\exists!\,(w', x_2) \in W \times X, u = w' + x_2$.
Now $x_1 = -x_2$, $w = w'$. Thus $\pi : U \to W$ defined by $\pi(u) = w$, is surj. ☐

• *Supp X, Y are iso subsp of V.*
Give a counterexa: $\exists$ iso subsps M, N of V, suth $V = M \oplus X = N \oplus Y$.

Exa: Let $V = \mathbf{F}^\infty$. Let $X = \mathbf{F}^\infty, Y = \{(0, x_1, x_2, \cdots) \in \mathbf{F}^\infty\}$. Now X, Y are iso.

3 *Supp V and W are iso and finide, U is a subsp of V, and $S \in \mathcal{L}(U, W)$.*
Prove $\exists$ inv $T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U \iff S$ is inje. [*See also (3.A.11).*]

SOLUS: (a) $\forall u \in U, u = T^{-1}Su \Rightarrow T^{-1}S = I \in \mathcal{L}(U) \Longrightarrow S$ is inje, by (3.B.20).
OR. $\operatorname{null} S = \operatorname{null} T|_U = \operatorname{null} T \cap U = \{0\}$.

(b) Let $B_U = (u_1, \dots, u_m)$. Then S inje $\Rightarrow (Su_1, \dots, Su_m)$ liney indep.
Extend to $B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_W = (Su_1, \dots, Su_m, w_1, \dots, w_n)$.
Define $T \in \mathcal{L}(V, W)$ by $T(u_i) = Su_i$; $Tv_j = w_j$, for each u_i and v_j. □

EXA: Supp V, W are infinide. Then this exe is not true.
Let $V = W = \mathbf{F}^\infty$. Define $S(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$. Now S is inje.
Supp $\exists$ inv $T \in \mathcal{L}(V, W)$ suth $T|_V = S$. Then $T = S$ while S is not surj.

---

8 *Supp $T \in \mathcal{L}(V, W)$ is **surj**. Prove $\exists$ subsp U of V, $T|_U : U \to W$ is iso.*

SOLUS: By (3.B.12). Note that $\operatorname{range} T = W$. OR. [*Req* range T *Finide*] By [3.B TIPS (4)]. □

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18 *Show V and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.*

SOLUS:
Define $\Psi \in \mathcal{L}\big(V, \mathcal{L}(\mathbf{F}, V)\big)$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.
(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is inje.
(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi\big(T(1)\big)$. Hence Ψ is surj. □

OR. Define $\Phi \in \mathcal{L}\big(\mathcal{L}(\mathbf{F}, V), V\big)$ by $\Phi(T) = T(1)$.
(a) Supp $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is inje.
(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surj. □

COMMENT: $\Phi = \Psi^{-1}$. This is a counterexample of the stmt that $\mathcal{L}(V, W)$ and $\mathcal{L}(W, V)$ are iso. See (3.F).

• *Supp $S, T \in \mathcal{L}(V, W)$.* [*For Exe (4) and (5), see the* CORO *in (3.B.24, 25).*]

6 *Supp V and W are finide.* $\dim \operatorname{null} S = \dim \operatorname{null} T = n$.
Prove $S = E_2 T E_1, \exists$ inv $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$.

SOLUS: Define $E_1 :\ v_i \mapsto r_i$; $u_j \mapsto s_j$; for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.
Define $E_2 : Tv_i \mapsto Sr_i$; $x_j \mapsto y_j$; for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m)$; $B_{\operatorname{range} S} = (Sr_1, \dots, Sr_m)$. Let $B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p)$; $B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p)$. Let $B_{\operatorname{null} T} = (u_1, \dots, u_n)$; $B_{\operatorname{null} S} = (s_1, \dots, s_n)$. Thus $B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$; $B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)$.	$\therefore E_1, E_2$ are inv and $S = E_2 T E_1$.

□

• (a) *Supp $T = ES$ and $E \in \mathcal{L}(W)$ is inv. Prove* $\operatorname{null} S = \operatorname{null} T$.
(b) *Supp $T = SE$ and $E \in \mathcal{L}(V)$ is inv. Prove* $\operatorname{range} S = \operatorname{range} T$.
(c) *Supp $T = E_2 S E_1$ and $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$ are inv.*
Prove $\dim \operatorname{null} S = \dim \operatorname{null} T$.

SOLUS: (a) $v \in \operatorname{null} T \iff Tv = 0 = E(Sv) \iff Sv = 0 \iff v \in \operatorname{null} S$.
(b) $w \in \operatorname{range} T \iff \exists v \in V, Tv = S(Ev) \iff \exists u \in V, w = Su \iff w \in \operatorname{range} S$.
(c) Using (3.B.22). $\dim \operatorname{null} E_2 S E_1 \overset{E_2}{\underset{\text{inv}}{=\!=}} \dim \operatorname{null} S E_1 \overset{E_1}{\underset{\text{inv}}{=\!=}} \dim \operatorname{null} S = \dim \operatorname{null} T$. □

• **Note For [3.69]:** Supp V, W are finide and iso, $T \in \mathcal{L}(V, W)$. Then T inv $\Longleftrightarrow$ inje $\Longleftrightarrow$ surj.

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9 [Or 1] *Supp U, V, W are iso and finide, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.*
Prove ST is inv $\Longleftrightarrow$ S, T are inv.

Comment: If any two of U, V, W are not iso or finide, then S, T are inv $\Longrightarrow$ ST is inv.

Solus: Supp S, T are inv. Then $(ST)(T^{-1}S^{-1}) = I_W, (T^{-1}S^{-1})(ST) = I_U$. Hence ST is inv.
Supp ST is inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = I_U, (ST)R = I_W$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0. \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S. \end{array}\right| \begin{array}{l} T \text{ is inje, } S \text{ is surj.} \\ \text{又 } \dim U = \dim V = \dim W. \end{array}$$

Or. By (3.B.23), $\dim W = \dim \operatorname{range} ST \leqslant \min\{\operatorname{range} S, \operatorname{range} T\} \Rightarrow S, T$ are surj. □

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13 *Supp U, V, W, X are iso and finide, $R \in \mathcal{L}(W, X), S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.*
Supp RST is surj. Prove S is inje.

Solus: Using Exe (9). Notice that U, X are finide, so that RST is inv.

$$\text{Let } X = (RST)^{-1} \left| \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ is surj.} \end{array} \right\} \Rightarrow S = R^{-1}(RST)T^{-1}.$$ □

Or. $(RST)^{-1} = \big((RS)T\big)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. □

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10 *Supp V is finide and $S, T \in \mathcal{L}(V)$. Prove $ST = I \Longleftrightarrow TS = I$.*

Solus: (a) Supp $ST = I$.

By (3.B 20, 21)(a), $ST = I \Rightarrow T$ is inje and S is surj. 又 V is finide. S, T are inv.
Or. By Exe (9), V is finide and $ST = I$ is inv $\Rightarrow S, T$ are inv.

Then $\forall v \in V, S\big((TS)v\big) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow TS = I$.
Or. $S^{-1} = T$ 又 $S = S \Rightarrow TS = S^{-1}S = I$.

(b) Reversing the roles of S and T, we conclude that $TS = I \Rightarrow ST = I$. □

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11 *Supp V is finide, $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show T is inv and $T^{-1} = US$.*

Solus: Using Exe (9) and (10). *This result can fail without the hypo that V is finide.*
$(ST)U = U(ST) = (US)T = I \Rightarrow T^{-1} = US$.
Or. $(ST)U = S(TU) = I \Rightarrow U, S$ are inv $\Rightarrow TU = S^{-1}$. 又 $U^{-1} = U^{-1} \Rightarrow T = S^{-1}U^{-1}$. □

Exa: $V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots)$; $T(a_1, \dots) = (0, a_1, \dots)$; $U = I \Rightarrow STU = I$ but T is not inv.

$$\bullet\ (4\text{E } 3)\ \begin{array}{r} T \in \mathcal{L}(V) \\ V \text{ is finide} \end{array} \left| \begin{array}{l} (Tv_1, \dots, Tv_n) \text{ is a bss of } V \text{ for some bss } (v_1, \dots, v_n) \text{ of } V \Longleftrightarrow T \text{ is surj} \\ (Tv_1, \dots, Tv_n) \text{ is a bss of } V \text{ for every bss } (v_1, \dots, v_n) \text{ of } V \Longleftrightarrow T \text{ is inje} \end{array} \right\} \Longleftrightarrow T \text{ is inv.}$$

• (4E 15) *Supp $T \in \mathcal{L}(V)$ and $V = \operatorname{span}(Tv_1, \dots, Tv_m)$. Prove $V = \operatorname{span}(v_1, \dots, v_m)$.*

Solus: Becs $V = \operatorname{span}(Tv_1, \dots, Tv_m) \Rightarrow T$ is surj, and therefore is inv $\Rightarrow T^{-1}$ is inv.
$\forall v \in V, \exists a_i \in \mathbf{F}, v = \sum_{i=1}^m a_i Tv_i \Rightarrow T^{-1}v = \sum_{i=1}^m a_i v_i \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \dots, v_m)$.

Or. Reduce the spanning list $(Tv_1, \dots, Tv_m)$ of V to a bss $(Tv_{\alpha_1}, \dots, Tv_{\alpha_k})$ of V.
Where $k = \dim V$ and each $\alpha_i \in \{1, \dots, k\}$. Then by Exe (4E 3),
$(v_{\alpha_1}, \dots, v_{\alpha_k})$ is also a bss of V, contained in the list $(v_1, \dots, v_m)$. □

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15 *Prove every liney map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multi.*
In other words, prove if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists A \in \mathbf{F}^{m,n}, Tx = Ax,\ \forall x \in \mathbf{F}^{n,1}$.

SOLUS: Let $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$ be std bses of $\mathbf{F}^{n,1}, \mathbf{F}^{m,1}$.
$\forall k = 1, \dots, n,\ T(E_k) = A_{1,k}R_1 + \cdots + A_{m,k}R_m, \exists A_{j,k} \in \mathbf{F}$, forming A.
OR. Let $A = \mathcal{M}(T, B_1, B_2)$. Note that $\mathcal{M}(x, B_1) = x, \mathcal{M}(Tx, B_2) = Tx$.
Hence $Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax$, by [3.65]. □

- **NOTE FOR [3.62]:** $\mathcal{M}(v) = \mathcal{M}\big(I, (v), B_V\big)$. Where I is the id optor restr to span(v).
- **NOTE FOR [3.65]:** $\mathcal{M}(Tv) = \mathcal{M}\big(I, (Tv), B_W\big) = \mathcal{M}(T, B_V, B_W)\mathcal{M}\big(I, (v), B_V\big) = \mathcal{M}\big(T, (v), B_W\big)$.
  If $v = 0$, then span$(v) =$ span$(\ )$, we replace (v) by $B = (\ )$; simlr for $Tv = 0$.

- (4E 23, OR 10.A.4) *Supp that $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bses of V.*
Let $T \in \mathcal{L}(V)$ be suth each $T\alpha_k = \beta_k$. Prove $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)$.
For ease of nota, let $\mathcal{M}\big(T, \alpha \to \beta\big) = \mathcal{M}\big(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)\big)$.

SOLUS:
Denote $\mathcal{M}(T, \alpha \to \alpha)$ by A and $\mathcal{M}(I, \beta \to \alpha)$ by B.
$\forall k \in \{1, \dots, n\}, I\beta_k = \beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = T\alpha_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. □
OR. Note that $\mathcal{M}(T, \alpha \to \beta) = I$. Hence $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)\underbrace{\mathcal{M}(T, \alpha \to \beta)}_{=\mathcal{M}(I, \beta \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. □
OR. Note that $\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I$.
$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}\Big(\underbrace{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}_{=\mathcal{M}(T, \alpha \to \beta)}\Big) = \mathcal{M}(I, \beta \to \alpha)$. □
COMMENT: Let $A' = \mathcal{M}(T, \beta \to \beta)$.
$\beta_k = I\beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n,\ \forall\ k \in \{1, \dots, n\}$.
又 $T\beta_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.
OR. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$.

- **TIPS:** When using $\mathcal{M}^{-1}$, you must first declare bses and the purpose for using $\mathcal{M}^{-1}$.
  That is, to declare $B_U, B_V, B_W,\ \mathcal{M}: \mathcal{L}(V, W) \mapsto \mathbf{F}^{m,n}$, or $\mathcal{M}: v \mapsto \mathbf{F}^{n,1}$.
  So that $\mathcal{M}^{-1}(AC, B_U, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(C, B_U, B_V)$;
  Or $\mathcal{M}^{-1}(Ax, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(x, B_V)$. Where everything is well-defined.

- (4E 22, OR 10.A.1) *Supp $T \in \mathcal{L}(V)$. Prove $\mathcal{M}(T, \alpha \to \beta)$ is inv $\iff$ T itself is inv.*

SOLUS: Notice that $\mathcal{M}: T \mapsto \mathcal{M}(T, \alpha \to \beta)$ is iso. And that $\mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(TS)$.
(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(I) = \mathcal{M}(T)\mathcal{M}(T^{-1}) \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.
(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I,\ \exists!\, S \in \mathcal{L}(V)$ suth $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$
$\Rightarrow \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = I = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$
$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. □

CORO: Supp $A \in \mathbf{F}^{n,n}$. Then A is inv $\iff \exists$ inv $T \in \mathcal{L}(\mathbf{F}^n)$ suth $\mathcal{M}\big(T, (e_1, \dots, e_n), (f_1, \dots, f_n)\big) = A$.

- (4E 24, OR 10.A.2) *Supp $A, B \in \mathbf{F}^{n,n}$. Prove $AB = I \iff BA = I$.* [*Using Exe (10, 15).*]

SOLUS: Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Now $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.
$AB = I \Leftrightarrow A(Bx) = x \iff T(Sx) = x \Leftrightarrow TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I$.
OR. Becs $\mathcal{M}: \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1}) \to \mathbf{F}^{n,n}$ is iso. $\mathcal{M}^{-1}(AB) = TS = ST = \mathcal{M}^{-1}(BA) = I$. □

• **NOTE FOR [3.60]:** Supp $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.
Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{i,x} w_j$. **CORO:** $E_{l,k} E_{i,j} = \delta_{j,l} E_{i,k}$.
Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. And $\left(\mathcal{E}^{(j,i)}\right)_{l,k} = \begin{cases} 1, & \text{if } (i,j) = (l,k); \\ 0, & \text{othws.} \end{cases}$

NOTICE that $\mathcal{M}\colon \mathcal{L}(V, W) \to \mathbf{F}^{m,n}$ is iso. And $E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)} & + & \cdots & + & A_{1,n}\mathcal{E}^{(1,n)} \\ + & & \cdots & & + \\ \vdots & & \ddots & & \vdots \\ + & & \cdots & & + \\ A_{m,1}\mathcal{E}^{(m,1)} & + & \cdots & + & A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff \begin{pmatrix} A_{1,1}E_{1,1} & + & \cdots & + & A_{1,n}E_{n,1} \\ + & & \cdots & & + \\ \vdots & & \ddots & & \vdots \\ + & & \cdots & & + \\ A_{m,1}E_{1,m} & + & \cdots & + & A_{m,n}E_{n,m} \end{pmatrix} = T.$$

$$\text{By [2.42] and [3.61], } B_{\mathcal{L}(V,W)} = \begin{pmatrix} E_{1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \cdots & , E_{n,m} \end{pmatrix}; \quad B_{\mathbf{F}^{m,n}} = \begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & , \mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots & , \mathcal{E}^{(m,n)} \end{pmatrix}.$$

• **TIPS:** Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_p)$, $B_V = (v_1, \dots, v_p, \dots, v_n)$. Let each $w_k = Tv_k$; $B_W = (w_1, \dots, w_p, \dots, w_m)$.
Then $T = E_{1,1} + \cdots + E_{p,p}$, $\mathcal{M}(T, B_V, B_W) = \mathcal{E}^{(1,1)} + \cdots + \mathcal{E}^{(p,p)}$.

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17 *Supp V is finide. Show the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUS: [See also in (3.A).] Using NOTE FOR [3.60].
Let $B_V = (v_1, \dots, v_n)$. If $\mathcal{E} = 0$, then done. Supp $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.
Then $\forall E_{i,j} \in \mathcal{E}$, by asum, $\forall x, y \in \{1, \dots, n\}$, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}$, $E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.
Again, $\forall x, x', y, y' \in \{1, \dots, n\}$, $E_{y,x'}, E_{y',x} \in \mathcal{E}$. Thus $\mathcal{E} = \mathcal{L}(V)$. ☐

• (4E 10) *Supp V, W are finide, U is a subsp of V.*
Let $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V, W) : T|_U = 0\}$.
(a) *Show $\mathcal{E}$ is a subsp of $\mathcal{L}(V, W)$.*
(b) *Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$ and $\dim U$.*
Hint: *Define $\Phi : \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. What is $\operatorname{null} \Phi$? What is $\operatorname{range} \Phi$?*

SOLUS:
(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.
(b) Define Φ as in the hint. Φ is liney, by [3.A NOTE FOR Restriction].
$\Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$. Thus $\operatorname{null} \Phi = \mathcal{E}$.
Extend $S \in \mathcal{L}(U, W)$ to $T \in \mathcal{L}(V, W) \Rightarrow \Phi(T) = S \in \operatorname{range} \Phi$. Thus $\operatorname{range} \Phi = \mathcal{L}(U, W)$.
Thus $\dim \operatorname{null} \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \operatorname{range} \Phi = (\dim V - \dim U) \dim W$. ☐

OR. Let $B_U = (u_1, \dots, u_m)$, $B_V = (u_1, \dots, u_m, v_1, \dots, v_n)$. Let $p = \dim W$. [See NOTE FOR [3.60].]

$$\forall\, T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \operatorname{span} \underbrace{\begin{Bmatrix} E_{1,1}, & \cdots & , E_{m,1}, \\ \vdots & \ddots & \vdots \\ E_{1,p}, & \cdots & , E_{m,p} \end{Bmatrix}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

$$又\ W = \operatorname{span} \begin{Bmatrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{Bmatrix} \subseteq \mathcal{E}.$$

Where $\mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}$.
Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. ☐

- (4E 17) *Supp V is finide and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(T) = ST$.*
  (a) *Show* $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.
  (b) *Show* $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

SOLUS: (a) $\forall T \in \mathcal{L}(V), ST = 0 \iff \operatorname{range} T \subseteq \operatorname{null} S$.
Thus $\operatorname{null} \mathcal{A} = \{T \in \mathcal{L}(V) : \operatorname{range} T \subseteq \operatorname{null} S\} = \mathcal{L}(V, \operatorname{null} S)$.
(b) $\forall R \in \mathcal{L}(V), \operatorname{range} R \subseteq \operatorname{range} S \iff \exists\, T \in \mathcal{L}(V), R = ST$, by (3.B 25).
Thus $\operatorname{range} \mathcal{A} = \{R \in \mathcal{L}(V) : \operatorname{range} R \subseteq \operatorname{range} S\} = \mathcal{L}(V, \operatorname{range} S)$. □

OR. Using NOTE FOR [3.60]. Let $B_{\operatorname{range} S} = (\overbrace{w_1, \dots, w_m}^{Sv_i = w_i}), B_U = (v_1, \dots, v_m)$.
Let $(w_1, \dots, w_n), (v_1, \dots, v_n)$ be bses of V. Now $S = E_{1,1} + \dots + E_{m,m}$. $\mathcal{M}(S, v \to w) = \begin{pmatrix} 1 & \ddots & 0 & \ddots & 0 \\ 0 & \ddots & 1 & \ddots & 0 \\ 0 & \ddots & 0 & \ddots & 0 \end{pmatrix}$.
Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j} : w_x \mapsto \delta_{i,x} v_i$. Let $E_{j,k} R_{i,j} = Q_{i,k},\ R_{j,k} E_{i,j} = G_{i,k}$.
Where $E_{i,k} : v_x \mapsto \delta_{i,x} w_k,\ Q_{i,k} : w_x \mapsto \delta_{i,x} w_k$, and $G_{i,k} : v_x \mapsto \delta_{i,x} v_k$.
For any $T \in \mathcal{L}(V), \exists!\, A_{i,j} \in \mathbf{F}, T = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i} \Longrightarrow \mathcal{M}(T, w \to v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} & \cdots & A_{n,m} \end{pmatrix}$.
$\Longrightarrow \mathcal{A}(T) = ST = \Big(\sum_{r=1}^{m} E_{r,r} \Big)\Big(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i} \Big) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j} Q_{j,i}$.

$$\begin{matrix} \mathcal{M}(S, v \to w)\mathcal{M}(T, w \to v) = \mathcal{M}(ST, w) = \\ \mathcal{M}(\mathcal{A}, R \to Q)\mathcal{M}(T, R) = \mathcal{M}\big(\mathcal{A}(T), Q\big) = \end{matrix} \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

又 $\mathcal{M}(T, R) = \mathcal{M}(T, w \to v)$.
Let $T = I$, we have
$\mathcal{M}(\mathcal{A}, R \to Q) = \mathcal{M}(S, v \to w)$.

$\operatorname{range} \mathcal{A} = \operatorname{span} \begin{Bmatrix} Q_{1,1}, & \cdots & , Q_{n,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,m}, & \cdots & , Q_{n,m} \end{Bmatrix}$, $\operatorname{null} \mathcal{A} = \operatorname{span} \begin{Bmatrix} R_{1,m+1}, & \cdots & , R_{n,m+1}, \\ \vdots & \ddots & \vdots \\ R_{1,n}, & \cdots & , R_{n,n} \end{Bmatrix}$.
(a) $\dim \operatorname{null} \mathcal{A} = n \times (n - m)$;
(b) $\dim \operatorname{range} \mathcal{A} = n \times m$. □

- **NOTE FOR Exe (4E 17):** *Define $\mathcal{B} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{B}(T) = TS$.*
  (a) *Show* $\dim \operatorname{null} \mathcal{B} = (\dim V)(\dim \operatorname{null} S)$.
  (b) *Show* $\dim \operatorname{range} \mathcal{B} = (\dim V)(\dim \operatorname{range} S)$.

SOLUS: (a) $\forall T \in \mathcal{L}(V), TS = 0 \iff \operatorname{range} S \subseteq \operatorname{null} T$.
Thus $\operatorname{null} \mathcal{B} = \{T \in \mathcal{L}(V) : \operatorname{range} S \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V) : T|_{\operatorname{range} S} = 0\}$.
(b) $\forall R \in \mathcal{L}(V), \operatorname{null} S \subseteq \operatorname{null} R \iff \exists\, T \in \mathcal{L}(V), R = TS$, by (3.B.24).
Thus $\operatorname{range} \mathcal{B} = \{R \in \mathcal{L}(V) : \operatorname{null} S \subseteq \operatorname{null} R\} = \{R \in \mathcal{L}(V) : R|_{\operatorname{null} S} = 0\}$.
Using [3.22] and Exe (4E 10). □

OR. Using NOTE FOR [3.60] and nota in Exe (4E 17).
$\mathcal{B}(T) = TS = \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i} \Big)\Big(\sum_{r=1}^{m} E_{r,r} \Big) = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{i,j} G_{j,i} \Longrightarrow \mathcal{M}(TS, v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} & \cdots & 0 \end{pmatrix}$.

$\operatorname{range} \mathcal{B} = \operatorname{span} \begin{Bmatrix} G_{1,1}, & \cdots & , G_{m,1}, \\ \vdots & \ddots & \vdots \\ G_{1,n}, & \cdots & , G_{m,n} \end{Bmatrix}$, $\operatorname{null} \mathcal{B} = \operatorname{span} \begin{Bmatrix} R_{m+1,1}, & \cdots & , R_{n,1}, \\ \vdots & \ddots & \vdots \\ R_{m+1,n}, & \cdots & , R_{n,n} \end{Bmatrix}$.
(a) $\dim \operatorname{null} \mathcal{B} = n \times (n - m)$;
(b) $\dim \operatorname{range} \mathcal{B} = n \times m$. □

- (4E 20) *Supp $q \in \mathcal{P}(\mathbf{R})$. Prove $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.*

SOLUS: Note that $\deg\big[(x^2 + x)p''(x) + 2xp'(x) + p(3)\big] = \deg p$.
Define $T_n \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.
And note that $T_n(p) = 0 \Rightarrow \deg T_n(p) = -\infty = \deg p \Rightarrow p = 0$. Thus T_n is inv.
$\forall q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $n = 0$; if $q \neq 0$, let $n = \deg q$, we have $q \in \mathcal{P}_n(\mathbf{R})$.
Now $\exists\, p \in \mathcal{P}_n(\mathbf{R}), q(x) = T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$. □

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19 *Supp* $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ *is inje. And* $\deg Tp \leqslant \deg p$ *for every nonzero* $p \in \mathcal{P}(\mathbf{R})$.
(a) *Prove* T *is surj.* (b) *Prove for every nonzero* p, $\deg Tp = \deg p$.

Solus: (a) T is inje $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ is inje, so is inv $\iff T$ is surj.

(b) Using induc.

(i) $\deg p = -\infty \geqslant \deg Tp \iff p = 0 = Tp$. And $\deg p = 0 \geqslant \deg Tp \iff p = C \neq 0$.

(ii) Asum $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$. We show $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$ by ctradic.
Supp $\exists r \in \mathcal{P}_{n+1}(\mathbf{R}),\ \deg Tr \leqslant n < n+1 = \deg r$. By (a), $\exists s \in \mathcal{P}_n(\mathbf{R}),\ T(s) = (Tr)$.
又 T is inje $\Rightarrow s = r$. While $\deg s = \deg Ts = \deg Tr < \deg r$. Ctradic. □

---

16 *Supp* V *is finide and* $S \in \mathcal{L}(V)$ *suth* $\forall T \in \mathcal{L}(V), ST = TS$. *Prove* $\exists \lambda \in \mathbf{F}, S = \lambda I$.

Solus: If $S = 0$, done. Now supp $S \neq 0$. [*Using nota in Exe (4E 17). See also in (3.A).*]

Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, B_U) = \mathcal{M}(I, B_{\operatorname{range} S}, B_U)$. Note that $R_{k,1} : w_x \mapsto \delta_{k,x} v_1$.

Then $\forall k \in \{1, \ldots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $\dim \operatorname{null} S = 0,\ \dim \operatorname{range} S = m = n$.

Notice that $G_{i,j} = R_{i,j}S = SR_{i,j} = Q_{i,j}$. Where $G_{i,j} : v_x \mapsto \delta_{i,x} v_j$; $Q_{i,j} : w_x \mapsto \delta_{i,x} w_j$.

For each $w_i, \exists!\, a_{k,i} \in \mathbf{F}, w_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n$. Where $a_{k,i} = \mathcal{M}(I, (w_1, \ldots, w_n), (v_1, \ldots, v_n))_{k,i}$.

Then fix one i. Now for each $j \in \{1, \ldots, n\}, Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}\left(\sum_{k=1}^{n} a_{k,i}v_k\right)$.

Let $\lambda = a_{i,i}$. Hence each $w_j = \lambda v_j$. Now fix one j, we have $a_{1,1}v_j = \cdots = a_{n,n}v_j$, then all $a_{i,i}$ are equal.

Thus each $w_j = \lambda v_j \Longrightarrow \mathcal{M}(S, B_U) = \mathcal{M}(\lambda I)$. □

• (10.A.3, Or 4E 19) *Supp* V *is finide and* $T \in \mathcal{L}(V)$. [*See also in (3.A).*]
Prove $\forall B_V \neq B_V',\ \mathcal{M}(T, B_V) = \mathcal{M}(T, B_V') \Longrightarrow T = \lambda I, \exists \lambda \in \mathbf{F}$.

Solus: Supp $\forall B_V \neq B_V',\ \mathcal{M}(T, B_V) = \mathcal{M}(T, B_V')$. If $T = 0$, then done.

Supp $T \neq 0$, and $v \in V \setminus \{0\}$. Asum (v, Tv) is liney indep.

Extend (v, Tv) to $B_V = (v, Tv, u_3, \ldots, u_n)$. Let $B = \mathcal{M}(T, B_V)$.

$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0,\ \forall i \neq 2$.

By asum, $A = \mathcal{M}(T, B_V') = B, \forall B_V' = (v, w_2, \ldots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$.

$\Rightarrow Tv = w_2$, which is not true if $w_2 = u_3,\ w_3 = Tv,\ w_j = u_j, \forall j \in \{4, \ldots, n\}$. Ctradic.

Hence (v, Tv) is linely depe $\Rightarrow \forall v \in V, \exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show λ_v is indep of v, that is, for all disti $v, w \in V \setminus \{0\}, \lambda_v = \lambda_w$.

$$\left.\begin{array}{l} (v,w) \text{ liney indep} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v,w) \text{ linely depe, } w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \end{array}\right\} \Rightarrow T = \lambda I. \quad \square$$

Or. Let $A = \mathcal{M}(T, B_V)$, where $B_V = (u_1, \ldots, u_m)$ is arb.

Fix one $B_V = (v_1, \ldots, v_m)$ and then $(v_1, \ldots, \frac{1}{2}v_k, \ldots, v_m)$ is also a bss for any given $k \in \{1, \ldots, m\}$.

Fix one k. Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \cdots + A_{k,k}(\frac{1}{2}v_k) + \cdots + A_{m,k}v_m$

$\Rightarrow Tv_k = 2A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + 2A_{m,k}v_m = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{m,k}v_m$.

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1, \ldots, m\}$.

Now we show $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j, k suth $j \neq k$.

Consider $B_V' = (v_1', \ldots, v_j', \ldots, v_k', \ldots, v_m')$, where $v_j' = v_k,\ v_k' = v_j$ and $v_i' = v_i$ for all $i \in \{1, \ldots, m\} \setminus \{j, k\}$.

Now $T(v_k') = A_{1,k}v_1' + \cdots + A_{k,k}v_k' + \cdots + A_{m,k}v_m' = A_{k,k}v_k' = A_{k,k}v_j$, while $T(v_k') = T(v_j) = A_{j,j}v_j$. □

Ended

3·E   1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 20 | 4E: 8 14

1 *A function $T : V \to W$ is liney $\iff$ The graph of T is a subspace of $V \times W$.*

---

2 *Supp $V_1 \times \cdots \times V_m$ is finide. Prove each V_j is finide.*

Solus: For any $k \in \{1, \dots, m\}$, define $S_k \in \mathcal{L}(V_1 \times \cdots \times V_m, V_k)$ by $S_k(v_1, \dots, v_m) = v_k$.
Then S_k is liney map. By [3.22], range $S_k = V_k$ is finide. $\square$

Or. Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \times \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.
We show each U_i is iso to V_i. Then U is finide $\Longrightarrow$ its subsp U_i is finide, so is V_i.

$$\left.\begin{array}{l}\text{Define } R_i \in \mathcal{L}(V_i, U_i) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0)\\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i\end{array}\right\} \Rightarrow \left\{\begin{array}{l} R_i S_j|_{U_j} = \delta_{i,j} I_{U_j}, \\ S_i R_j = \delta_{i,j} I_{V_j}.\end{array}\right.$$ $\square$

---

4 *Prove $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.*

Solus: Using nota in Exe (2): $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0)$; $S_i : (u_1, \dots, u_m) \mapsto u_i$.
Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$.

$$\left.\begin{array}{l}\text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (TR_1, \dots, TR_m).\\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m.\end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ $\square$

---

5 *Prove $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.*

Solus: Using nota in Exe (2): $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0)$; $S_i : (u_1, \dots, u_m) \mapsto u_i$.

Note that $T_i : v \mapsto w_i$, $T : v \mapsto (w_1, \dots, w_m)$. | $$\left.\begin{array}{l}\text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1 T, \dots, S_m T).\\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = R_1 T_1 + \cdots + R_m T_m.\end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ $\square$

---

6 *For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.*

Solus:

Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m$.

(a) Supp $T(v_1, \dots, v_m) = 0$. Then $\forall (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$
For each k, let $a_k = 1, a_j = 0$ for all $j \neq k$. Then each $v_k = 0 \Rightarrow (v_1, \dots, v_m) = 0$. Thus T is inje.

(b) Supp $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be std bss of $\mathbf{F}^m$. Then $\forall (b_1, \dots, b_n) \in \mathbf{F}^m$,
$$\Big[T\big(\psi(e_1), \dots, \psi(e_m)\big)\Big](b_1, \dots, b_m) = b_1\psi(e_1) + \cdots + b_m\psi(e_m) = \psi(b_1 e_1 + \cdots + b_m e_m) = \psi(b_1, \dots, b_m).$$
Thus $T\big(\psi(e_1), \dots, \psi(e_m)\big) = \psi$. Hence T is surj. $\square$

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3 *Give an exa of a vecsp V and its two subsps U_1, U_2 suth
$U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum.* [*V must be infinide.*]

Solus: Note *that at least one of U_1, U_2 must be infinide. Both can be infinide.* [*Req Other Courses.*]
Let $V = \mathbf{F}^\infty = U_1$, $U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$. Then $V = U_1 + U_2$ is not a direct sum.

$$\left.\begin{array}{l}\text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T\big((x_1, x_2, \cdots), (x, 0, \cdots)\big) = (x, x_1, x_2, \cdots)\\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = \big((x_1, x_2, \cdots), (x, 0, \cdots)\big)\end{array}\right\} \Rightarrow S = T^{-1}.$$ $\square$

• **Note For [3.79], def of $v + U$:** Given $v + U$, v is already uniqly determined, as a sort of precond.
Even though $v + U = v' + U$, where v' is *purer* than v.

• **Note For [3.85]:** $v + U = w + U \iff v \in w + U,\ w \in v + U$
$\iff v - w \in U \iff (v + U) \cap (w + U) \neq \emptyset.$

• **NOTE FOR [3.79, 3.83]:**

If U is merely a subset of V, then [3.85, 86] do not hold $\Rightarrow V/U$ not a vecsp.

If V is merely a subset of a vecsp of which U is a subsp, then [3,79, 86] do not hold $\Rightarrow V/U$ not a vecsp.

If U is a vecsp but not a subsp of V, while U, V are subsps of some vecsp, then everything's alright.

Hence if V/U is a vecsp, then V, U are subsps of some vecsp.

COMMENT: Supp U, V are subsps and U is not a subsp of V. Note that $V/U = (V+U)/U$.
Supp $v + U \in V/U$. Then $v \in V$, or possibly $v \in V + U$ as well. To avoid this ambiguity, you have to specify the precond, what subsp that v belongs to.

EXA: Supp $U + W = V$. Then $V/U = (U+W)/U = W/U$. Let $W \cap U = I, U_I \oplus I = U, W_I \oplus I = W$.
Now $U_I \oplus W_I \oplus I = V$. Thus $W/U = (W_I \oplus I)/U = W_I/U$.
$\forall w_1', w_2' \in W_I$ suth $w_1' + U = w_2' + U \in W_I/U$, $w_1' - w_2' \in U \cap W_I = \{0\} \Rightarrow w_1' = w_2'$.

• ***Trivial Cases***: If $v \in U$, then $v + U = 0 + U = \{u : u \in U\} = U$. Now $U = 0 \in V/U$.
If $U = \{0\}$, then $v + U = v + \{0\} = \{v\}$, $V/U = V/\{0\} = \{\{v\} : v \in V\}$.
If $U = \emptyset$, then $v + U = v + \emptyset = \emptyset$, $V/U = V/\emptyset = \{\emptyset\}$.

• **TIPS 1:** V is a subsp of $U \iff \forall v \in V, v + U = 0 + U = U \iff V/U = \{0\} = \{U\}$.

• **NOTE FOR [3.88]:** If U, V are subsp of some vecsp $\mathcal{V}$. Define the quot map $\pi \in \mathcal{L}(V, V/U)$.
Then π is surj by def, and null $\pi = V \cap U$. Thus if $\mathcal{V}$ is finide, then $\dim V = \dim V/U + \dim(V \cap U)$.
OR. Let $I = V \cap U, V_I \oplus I = V$. Becs $V/U = V_I/U$, iso to V_I. Now $\dim V = \dim V_I + \dim I$.

• (4E 8) *Supp* $T \in \mathcal{L}(V, W), w \in \operatorname{range} T$. *Prove* $\{v \in V : Tv = w\} = u + \operatorname{null} T$.

SOLUS: Let $\mathcal{K}_w = \{v \in V : Tv = w\}$. [Not a vecsp.] Supp $u \in \mathcal{K}_w$. Then $u + \operatorname{null} T \subseteq \mathcal{K}_w$.
And $\forall u' \in \mathcal{K}_w$, $u' - u \in \operatorname{null} T \Rightarrow u' \in u + \operatorname{null} T$. Now $\mathcal{K}_w \subseteq u + \operatorname{null} T$. □

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7 *Supp* $\alpha, \beta \in V$, *and* U, W *are subsps of* V. *Prove* $\alpha + U = \beta + W \Rightarrow U = W$.

SOLUS: (a) $\alpha \in \alpha + U = \beta + W \Rightarrow \exists w \in W, \alpha = \beta + w \Rightarrow \alpha - \beta \in W$.
(b) $\beta \in \beta + W = \alpha + U \Rightarrow \exists u \in U, \beta = \alpha + u \Rightarrow \beta - \alpha \in U$.
Now $\beta + U = \alpha + U = \beta + W = \alpha + W$. Thus $\{\alpha + u : u \in U\} = \{\alpha + w : w \in W\} \Rightarrow U = W$.

OR. $\pm(\alpha - \beta) \in U \cap W \Rightarrow \left\{ \begin{array}{l} U \ni u = (\beta - \alpha) + w \in W \Rightarrow U \subseteq W \\ W \ni w = (\alpha - \beta) + u \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W$. □

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8 *Supp* A *is a nonempty subset of* V.
Prove A *is a trslate of some subsp of* $V \iff \lambda v + (1-\lambda)w \in A$, $\forall v, w \in A, \lambda \in \mathbf{F}$.

SOLUS: (a) Supp $A = a + U$. Then $\lambda(a + u_1) + (1-\lambda)(a + u_2) = a + \big(\lambda(u_1 - u_2) + u_2\big) \in A$.
(b) Supp $\lambda v + (1-\lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Supp $\underline{a \in A}$ and let $A' = \{x - a : x \in A\}$.
Then $0 \in A'$ and $\forall (v-a), (w-a) \in A', \lambda \in \mathbf{F}$,
(I) $\lambda(v - a) = [\lambda v + (1-\lambda)a] - a \in A'$.
(II) Becs $\lambda(v-a) + (1-\lambda)(w-a) = [\lambda v + (1-\lambda)w] - a \in A'$.
Let $\lambda = \frac{1}{2}$ here and use (I) above by $\lambda = 2$, we have $(v-a) + (w-a) \in A'$.
OR. Note that $v, a \in A \Rightarrow \lambda v + (1-\lambda)a = 2v - a \in A$. Simlr $2w - a \in A$.
Now $(v - \frac{1}{2}a) + (w - \frac{1}{2}a) = v + w - a \in A \Rightarrow v + w - 2a = (v-a) + (w-a) \in A'$.
Thus $A' = -a + A$ is a subsp of V. Hence $a + A' = a + \{x - a : x \in A\} = A$ is a trslate. □

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9 *Supp $A = \alpha + U$ and $B = \beta + W$ for some $\alpha, \beta \in V$ and some subsps U, W of V. Prove $A \cap B$ is either a trslate of some subsp of V or is $\emptyset$.*

SOLUS: $\forall \alpha + u, \beta + w \in A \cap B \neq \emptyset, \lambda \in \mathbf{F}, \lambda(\alpha + u) + (1-\lambda)(\beta + w) \in A \cap B$. By Exe (8). □

OR. Let $A = \alpha + U,\ B = \beta + W$. Supp $v \in (\alpha + U) \cap (\beta + W) \neq \emptyset$.

Then $v - \alpha \in U \Rightarrow v + U = \alpha + U = A$, and simlr $v + W = \beta + W = B$.

We show $A \cap B = v + (U \cap W)$. Note that $v + (U \cap W) \subseteq A \cap B$.

And $\forall \gamma = v + u = v + w \in A \cap B \Rightarrow u = w \in U \cap W \Rightarrow \gamma \in v + (U \cap W)$. □

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10 *Prove the intersec of any collec of trslates of subsps is either a trslate of some subsps or $\emptyset$.*

SOLUS: Supp $\{A_\alpha\}_{\alpha\in\Gamma}$ is a collec of trslates of subsps of V, where Γ is an index set.

$\forall x, y \in \bigcap_{\alpha\in\Gamma} A_\alpha \neq \emptyset, \lambda \in \mathbf{F}, \lambda x + (1-\lambda) y \in A_\alpha$ for each α. By Exe (8). □

OR. Let each $A_\alpha = w_\alpha + V_\alpha$. Supp $x \in \bigcap_{\alpha\in\Gamma} (w_\alpha + V_\alpha) \neq \emptyset$.

Then $x - w_\alpha \in V_\alpha \Longrightarrow x + V_\alpha = w_\alpha + V_\alpha = A_\alpha$, for each α.

We show $\bigcap_{\alpha\in\Gamma} A_\alpha = \bigcap_{\alpha\in\Gamma} (x + V_\alpha) = x + \bigcap_{\alpha\in\Gamma} V_\alpha$.

$y \in \bigcap_{\alpha\in\Gamma} A_\alpha \Longleftrightarrow$ for each $\alpha,\ y = x + v_\alpha \in A_\alpha$

$\Longleftrightarrow$ each $v_\alpha = y - x \in \bigcap_{\alpha\in\Gamma} V_\alpha \Longleftrightarrow y \in x + \bigcap_{\alpha\in\Gamma} V_\alpha$. □

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11 *Supp $A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.*

(a) *Prove A is a trslate of some subsp of V*

(b) *Prove if B is a trslate of some subsp of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.*

(c) *Prove A is a trslate of some subsp of V of dim $< m$.*

SOLUS: (a) By Exe (8), $\forall u, w \in A, \lambda \in \mathbf{F}, \lambda u + (1-\lambda) w = \left(\lambda \sum_{i=1}^m a_i + (1-\lambda) \sum_{i=1}^m b_i\right) v_i \in A$.

(b) Supp $B = v + U$, where $v \in V$ and U is a subsp of V. Let each $v_k = v + u_k \in B, \exists!\, u_k \in U$.

$\forall w \in A,\ w = \sum_{i=1}^m \lambda_i v_i = \sum_{i=1}^m \lambda_i (v + u_i) = \underline{\sum_{i=1}^m \lambda_i v} + \sum_{i=1}^m \lambda_i u_i = v + \sum_{i=1}^m \lambda_i u_i \in v + U = B$. □

OR. Let $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$. To show $v \in B$, use induc on m by k.

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By Exe (8), $v \in B$.

(ii) $2 \leqslant k < m$. Asum $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. $\left[\forall \lambda_i \text{ suth } \sum_{i=1}^k \lambda_i = 1\right]$

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. Fix one $\mu_\iota \neq 1$.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Longrightarrow \left(\sum_{i=1}^{k+1} \dfrac{\mu_i}{1-\mu_\iota}\right) - \dfrac{\mu_\iota}{1-\mu_\iota} = 1$.

$$\text{Let } w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let $\lambda_i = \dfrac{\mu_i}{1-\mu_\iota}$ for $i \in \{1, \dots, \iota - 1\}$; $\lambda_j = \dfrac{\mu_{j+1}}{1-\mu_\iota}$ for $j \in \{\iota, \dots, k\}$. Then,

$$\left.\begin{array}{r} \sum_{i=1}^k \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B.$$ □

(c) If $m = 1$, then let $A = v_1 + \{0\}$ and done. Now supp $m \geqslant 2$. Fix one $k \in \{1, \dots, m\}$.

$A \ni \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m) v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_m v_m$

$= v_k + \lambda_1 (v_1 - v_k) + \cdots + \lambda_{k-1}(v_{k-1} - v_k) + \lambda_{k+1}(v_{k+1} - v_k) + \cdots + \lambda_m (v_m - v_k)$

$\in v_k + \operatorname{span}(v_1 - v_k, \dots, v_m - v_k)$. □

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14 *Supp* $U = \{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0$ *for only finily many* $k\}$.

(a) *Show* U *is a subsp of* $\mathbf{F}^\infty$. [*Do it in your mind*] (b) *Prove* $\mathbf{F}^\infty / U$ *is infinide.*

Solus: For ease of nota, denote the p^{th} term of $u = (x_1, \cdots, x_p, \cdots) \in \mathbf{F}^\infty$ by $u[p]$.

For each $r \in \mathbf{N}^+$, let $e_r[k] = \begin{cases} 1, & (k-1) \equiv 0 \pmod r \\ 0, & \text{othws} \end{cases}$ simply $e_r = (1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \cdots)$.

For $m \in \mathbf{N}^+$. Let $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 + U \Rightarrow \exists\, u \in U, a_1 e_1 + \cdots + a_m e_m = u$.

Supp $u = (x_1, \cdots, x_L, 0, \cdots)$, where L is the largest suth $u[L] \neq 0$.

Let $s \in \mathbf{N}^+$ be suth $h = s \cdot m! + 1 > L$, and $e_1[h] = \cdots = e_m[h] = 1$.

Notice that for any $p, r \in \{1, \ldots, m\}$, $e_r[s \cdot m! + 1 + p] = e_r[p+1] = 1 \Longleftrightarrow p \equiv 0 \pmod r \Longleftrightarrow r \mid p$.

Let $1 = p_1 \leqslant \cdots \leqslant p_{\tau(p)} = p$ be the disti factors of p. Moreover, $r \mid p \Longleftrightarrow r = p_k$ for some k.

Now $u[h+p] = 0 = \left(\sum_{r=1}^m a_r e_r\right)[p+1] = \sum_{k=1}^{\tau(p)} a_{p_k}$.

Let $q = p_{\tau(p)-1}$. Then $\tau(q) = \tau(p) - 1$, and each $q_k = p_k$. Again, $\left(\sum_{r=1}^m a_r e_r\right)[h+q] = 0 = \sum_{k=1}^{\tau(p)-1} a_{p_k}$.

Thus $a_{p_{\tau(p)}} = a_p = 0$ for all $p \in \{1, \ldots, m\} \Rightarrow (e_1, \ldots, e_m)$ is liney indep in $\mathbf{F}^\infty$.

So is $(e_1 + U, \ldots, e_m + U)$ in $\mathbf{F}^\infty / U$. Becs m is arb. By (2.A.14). $\square$

Or. For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, & \text{if } 2^r \mid p \\ 0, & \text{othws} \end{cases}$ Simlr, let $m \in \mathbf{N}^+$ and $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0$ $\Rightarrow a_1 e_1 + \cdots + a_m e_m = u \in U$.

Supp L is the largest suth $u[L] \neq 0$. And l is suth $2^{ml} > L$. Then for each $k \in \{1, \ldots, m\}$,

$u[2^{ml} + 2^k] = 0 = \left(\sum_{r=1}^m a_r e_r\right)[2^k] = a_1 + \cdots + a_k$. Thus each $a_k = 0$. Simlr. $\square$

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18 *Supp* $T \in \mathcal{L}(V, W)$ *and* U, V *are subsps of* $\mathcal{V}$. *Let* $\pi : V \to V/U$ *be the quot map.*
Prove $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi \Longleftrightarrow U \cap V = \operatorname{null} \pi \subseteq \operatorname{null} T$.

Solus: Supp $\operatorname{null} \pi \subseteq \operatorname{null} T$. By (3.B.24), done. Or. Define $S : (v + U) \mapsto Tv$.

$\forall v_1, v_2 \in V$ suth $v_1 + U = v_2 + U \Longleftrightarrow v_1 - v_2 \in U \cap V \subseteq \operatorname{null} T \Longleftrightarrow Tv_1 = Tv_2$.

Thus S is well-defined. Convly true as well. $\square$

Coro: $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ with $S \mapsto S \circ \pi$ is inje, range $\Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}$.

Comment: If $T = I_V$. Then $S : v + U \mapsto v$ is not well-defined, unless $U \cap V = \{0\} \subseteq \operatorname{null} I_V$.

- **Note For [3.88, 3.90, 3.91]:** Supp $W \oplus U = V$. Then $V/U = W/U$ is iso to W. [Convly not true.]

  Becs $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$. Define $T \in \mathcal{L}(V)$ by $T(v) = w_v$.

  Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$, $\operatorname{range} T \oplus \operatorname{null} T = V$.

  Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, V)$ is defined by $\tilde{T}(v + U) = \tilde{T}(w_v' + U) = Tw_v' = w_v$. [*See Exa below*]

  Now $\pi \circ \tilde{T} = I_{V/U}$, $\tilde{T} \circ \pi|_W = I_W = T|_W$. Hence $\tilde{T}$ is iso of V/U onto W.

- **Exa:** Let $V = \mathbf{F}^2, B_U = (e_1), B_W = (e_2 - e_1) \Rightarrow U \oplus W = V$.

  Although $(e_2 - e_1) + U = e_2 + U, \tilde{T}(e_2 + U) = T(e_2) = e_2 - e_1$. Becs $e_2 = e_1 + (e_2 - e_1) \in U \oplus W$.

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17 *Supp* V/U *is finide. Supp* W *is finide and* $V = U + W$. *Show* $\dim W \geqslant \dim V/U$.

Solus: Let $Y \oplus (U \cap W) = W$. Then by [1.C Tips (4)], $V = U \oplus Y$. Note that V/U and Y are iso. $\square$

Or. Let $B_W = (w_1, \ldots, w_n)$. Then $V = U + \operatorname{span}(w_1, \ldots, w_n)$.

$\forall v \in V, \exists\, u \in U$, $v = u + (a_1 w_1 + \cdots + a_n w_n) \Rightarrow v + U = (a_1 w_1 + \cdots + a_n w_n) + U$. $\square$

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12 *Supp U is a subsp of V. Prove is V is iso to $U \times (V/U)$.*

Solus:

[*Req V/U Finide*] Let $B_{V/U} = (v_1 + U, \dots, v_n + U)$.

Note that $\forall v \in V, \exists! a_i \in \mathbf{F}, v + U = \sum_{i=1}^{n} a_i(v_i + U) = \left(\sum_{i=1}^{n} a_i v_i\right) + U$

$\Rightarrow (v - a_1v_1 - \cdots - a_nv_n) \in U \Rightarrow \exists! u \in U, v = \sum_{i=1}^{n} a_i v_i + u.$

Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ and $\psi \in \mathcal{L}(U \times (V/U), V)$

by $\varphi(v) = (u, \sum_{i=1}^{n} a_i v_i + U)$, and $\psi(u, v + U) = \sum_{i=1}^{n} a_i v_i + u$. Then $\psi = \varphi^{-1}$. □

Or. Let $W \oplus U = V$. Define $Tv = u_v, Sv = w_v \Rightarrow \tilde{T} \in \mathcal{L}(V/W, U), \tilde{S} \in \mathcal{L}(V/U, W)$ are iso.

Define $\psi(u, v + U) = u + \tilde{S}(v + U) = u + w_v$. Define $\varphi(v) = (\tilde{T}(v), v + U)$.

$$\left.\begin{array}{l}(\psi \circ \varphi)(u_v + w_v) = \psi(u_v, w_v + U) = u_v + w_v \\ (\varphi \circ \psi)(u, v + U) = \varphi(u + w_v) = (u, w_v + U)\end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ Or Becs ψ or φ is inje and surj. □

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13 *Prove* $B_{V/U} = (v_1 + U, \dots, v_m + U), B_U = (u_1, \dots, u_n) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$.

Solus: $\forall v \in V, \exists! a_i \in \mathbf{F}, v + U = \sum_{i=1}^{m} a_i v_i + U \Rightarrow \exists! b_i \in \mathbf{F}, v - \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{n} b_i u_i \in U$

$\Rightarrow \forall v \in V, \exists! a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^{m} a_i v_i + \sum_{j=1}^{n} b_j u_j$. □

Or. $\sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i = 0 \Rightarrow \left(\sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i\right) + U = 0 \Rightarrow \sum_{i=1}^{m} a_i(v_i + U) = 0$

$\Rightarrow a_1 = \cdots = a_m = 0 \Rightarrow \sum_{i=1}^{n} b_i u_i \Rightarrow b_1 = \cdots = b_n = 0$. 又 $\dim V = m + n$. □

Or. Note that $B = (v_1, \dots, v_m)$ is liney indep, and $[\operatorname{span}(v_1, \dots, v_m) + U] \subseteq V$.

$v \in \operatorname{span} B \cap U \iff v + U = \sum_{i=1}^{m} a_i(v_i + U) = 0 + U \iff v = 0$. Hence $\operatorname{span} B \cap U = \{0\}$.

Becs $\dim[\operatorname{span}(v_1, \dots, v_m) \oplus U] = m + n = \dim V$. Now by (2.B.8). □

• (4E 14) *Supp $V = U \oplus W$, $B_W = (w_1, \dots, w_m)$. Prove $B_{V/U} = (w_1 + U, \dots, w_m + U)$.*

Solus: $\forall v \in V, \exists! u \in U, w \in W, v = u + w$. 又 $\exists! c_i \in \mathbf{F}, w = \sum_{i=1}^{m} c_i w_i \Rightarrow v = \sum_{i=1}^{m} c_i w_i + u$.

Hence $\forall v + U \in V/U, \exists! c_i \in \mathbf{F}, v + U = \sum_{i=1}^{m} c_i w_i + U$. □

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15 *Supp* $\varphi \in \mathcal{L}(V, \mathbf{F}) \backslash \{0\}$. *Prove* $\dim V/(\operatorname{null} \varphi) = 1$.

Solus: By [3.91] (d), $\dim \operatorname{range} \varphi = 1 = \dim V/(\operatorname{null} \varphi)$.

Or. By (3.B.29), $\exists u$, $\operatorname{span}(u) \oplus \operatorname{null} \varphi = V$. Then $B_{V/\operatorname{null} \varphi} = (u + \operatorname{null} \varphi)$. □

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16 *Supp* $\dim V/U = 1$. *Prove* $\exists \varphi \in \mathcal{L}(V, \mathbf{F}), \operatorname{null} \varphi = U$.

Solus: Supp $V_0 \oplus U = V$. Then V_0 is iso to V/U, $\dim V_0 = 1$.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. □

Or. Let $B_{V/U} = (w + U)$. Then $\forall v \in V, \exists! a \in \mathbf{F}, v + U = aw + U$.

Define $\varphi : V \to \mathbf{F}$ by $\varphi(v) = a$. Then $\varphi(v_1 + \lambda v_2) = a_1 + \lambda a_2 = \varphi(v_1) + \lambda\varphi(v_2)$.

Now $u \in U \iff u + U = 0w + U \iff \varphi(u) = 0$. □

• *Supp U, W are subsps of $\mathcal{V}$, and X, Y are subsps of $\mathcal{W}$.*
Supp U, X are iso, W, Y are iso. Prove or give a counterexa: U/W and X/Y are iso.

Solus: A counterexa: Let $\mathcal{V} = \mathcal{W} = \mathbf{F}^2$. Let $U = X = Y = \operatorname{span}(e_1), W = \operatorname{span}(e_2)$.

Then $\dim U/W = \dim U - \dim(U \cap W) = 1 \neq 0 \dim X - \dim(X \cap Y) = \dim X/Y$. □

- **TIPS 2:** *Supp U, W are subsps of V. Let $I = U \cap W$.*
  Prove $V = U + W \iff V/I = U/I \oplus W/I$.

SOLUS: (a) Supp $V = U + W$. Then $\forall v + I \in V/I, \exists\, (u_v, w_v) \in U \times W,\ v + I = (u_v + w_v) + I$.
Note that $U/I, W/I \subseteq V/I$. Thus $V/I = U/I + W/I$.
$\forall u + I = w + I \in (U/I) \cap (W/I)$, $\underline{u - w \in I = U \cap W}$
$\underline{\Rightarrow \exists\, w' \in I, u = w + w' \in U \cap W} \Rightarrow u + I = 0 + I = w + I$. Thus $(U/I) \cap (W/I) = \{0\}$.

(b) Supp $V/I = U/I \oplus W/I$. Then $\forall v \in V, v + I = (u + I) + (w + I)$
$\Rightarrow v - u - w \in I = U \cap W \Rightarrow \exists\, x \in U \cap W, v = u + w + x \in U + W$. ☐

- **TIPS 3:** *Supp U, W are subsps of V and X is a subsp of $U \cap W$.*
  Prove U/W and $\big(U/X\big)/\big(W/X\big)$ are iso.

SOLUS: Let $U_X \oplus X = U, W_X \oplus X = W$. Becs $U/W = U_X/W$, and $U/X = U_X/X$.
Define $T \in \mathcal{L}\Big(\big(U_X/X\big)/\big(W/X\big), U_X/W\Big)$ by $T\big((u_x + X) + W/X\big) = u_x + W$.
$\forall u_1, u_2 \in U_X$ suth $(u_1 + X) + W/X = (u_2 + X) + W/X \Rightarrow u_1 - u_2 + X \in W/X$
$\Rightarrow u_1 - u_2 \in X + W$ 又 $u_1, u_2 \in U_X \Rightarrow u_1 - u_2 \in W \Rightarrow u_1 + W = u_2 + W$. Now T is well-defined.
Inje: $\forall u_x \in U_X$ suth $u_x + W = 0 \Rightarrow u_x \in W_X \Rightarrow (u_x + X) \in W_X/X$.
Surj: $\forall u_x \in U_X, u_x + W = T\big((u_x + X) + W/X\big)$. Hence T is iso. ☐

OR. Define $S \in \mathcal{L}(U_X/X, U_X)$ by $S(u_x + X) = u_x$.
Then $\forall u_1 + X = u_2 + X \in U_X/X,\ u_1 - u_2 \in X$ 又 $u_1, u_2 \in U_X \Rightarrow u_1 = u_2$.
Now S is well-defined. Then $S\Big/^{(W/X)}_{W} = T$ defined above.
Becs range $S|_{W/X \cap U_X/X} \subseteq W$, and $U_X = \operatorname{range} S \Rightarrow U_X \subseteq \operatorname{range} S + W$. Well-defined. Surj.
For $u_x \in U_X,\ u_x + W = 0 \iff u_x \in U_X \cap W \iff u_x + X \in (U_X \cap W)/X = \operatorname{null} S/_W$. Inje. ☐

- Supp $T \in \mathcal{L}(V, W)$, and U, V are subsps of some vecsp, and X, W are subsps of some vecsp.
  Define $T/^U_X : V/U \to W/X$ by $T/^U_X(v + U) = Tv + X$.
  (a) *Prove $T/^U_X$ is well-defined $\iff$ $(\operatorname{range} T|_{U \cap V})/X = \{0\} \iff \operatorname{range} T|_{U \cap V}$ is a subsp of X.*
  Supp $T/^U_X$ is well-defined, and thus is liney. Define $\pi_U \in \mathcal{L}(V, V/U), \pi_X \in \mathcal{L}(W, W/X)$.
  Then $T/^U_X \circ \pi_U = \pi_X \circ T$. Define $T/_X \in \mathcal{L}(V, W/X)$ by $T/_X(v) = Tv + X$.
  (b) $\operatorname{range} T/^U_X = \operatorname{range}\big(T/^U_X \circ \pi_U\big) = \operatorname{range}(\pi_X \circ T) = (\operatorname{range} T)/X$.
  (c) *Prove $T/^U_X$ is surj $\iff W \subseteq \operatorname{range} T + X$.*
  (d) *Show* $\operatorname{null} T/^U_X = \big(\operatorname{null} T/_X\big)/U$. (e) *$T/^U_X$ is inje $\iff$* $\operatorname{null} T/_X \subseteq U$.

SOLUS: (a) For $v, w \in V$. If $v + U = w + U \iff v - w \in U \Rightarrow Tv - Tw \in X \iff Tv + X = Tw + X$.
Then $\forall u \in V \cap U, Tu \in X \Rightarrow \operatorname{range} T|_{U \cap V} \subseteq X$. Convly true as well.

(c) Supp $T/^U_X$ is surj. $\forall w \in W, w + X \in W/X$
$\Rightarrow \exists\, Tv + X = w + X \Rightarrow w - Tv \in X \Rightarrow w \in \operatorname{range} T + X$. Hence $W \subseteq \operatorname{range} T + X$.
Convly, $W \subseteq \operatorname{range} T + X \Rightarrow (\operatorname{range} T)/X = (\operatorname{range} T + X)/X \supseteq W/X$.

(d) $v + U \in \operatorname{null} T/^U_X \iff Tv \in X \iff v \in \operatorname{null} T/_X \iff v + U \in \big(\operatorname{null} T/_X\big)/U$. ☐

- **COMMENT:** Supp $T \in \mathcal{L}(V)$. Define $T/U \in \mathcal{L}(V/U)$ by $T/U = T/^U_U$. Then
  (a) T/U is well-defined $\iff \operatorname{range} T|_{U \cap V}$ is a subsp of $U \iff U \cap V$ is invard T.
  (b) $\operatorname{range} T/U = \operatorname{range}\big(T/U \circ \pi\big) = \operatorname{range}(\pi \circ T) = (\operatorname{range} T)/U$. (c) T/U surj $\iff V \subseteq \operatorname{range} T + U$.
  (d) $\operatorname{null} T/U = \big(\operatorname{null} T/_U\big)/U$. (e) T/U inje $\iff \operatorname{null} T/_U \subseteq U$.

• *Supp $V = \mathbf{R}^{\mathbf{R}}$ and $U = \{f \in \mathbf{R}^{\mathbf{R}} : f(x_1) = \cdots = f(x_m) = 0\}$ is a subsp of V, with each $x_k \in \mathbf{R}$. Prove $\forall W \in \mathcal{S}_V U, \dim W = m$.* ***Hint***: *Find an iso from V/U onto $\mathbf{R}^m$.*

Solus: Define $T \in \mathcal{L}(V/U, \mathbf{R}^m)$ by $T(f+U) = (f(x_1), \dots, f(x_m))$.
$\forall f_1 + U = f_2 + U \in V/U, f_1 - f_2 \in U \Rightarrow f_1(x_k) = f_2(x_k)$. Now T is well-defined.
Inje: Each $f(x_k) = 0 \Rightarrow f + U = 0$. Let $S = T \circ \pi \Rightarrow \tilde{S} = T$. Then S is surj, so is T. □

Ended

3·F

4 5 6 7 8 9 12 13 15 16 17 18 19 20 21 22 23 24 25 26
28 29 30 31 32 33 34 35 36 37 | 4E: 5 6 8 17 23 24 25

• **Note For Exe (1):** Every liney functional is either surj or is a zero map.
Which means, for $\varphi \in V'$, $\varphi = 0 \iff \dim \operatorname{span}(\varphi) = 0 \iff \dim \operatorname{range} \varphi = 0$.
And $\varphi \neq 0 \iff \dim \operatorname{span}(\varphi) = 1 \iff \dim \operatorname{range} \varphi = 1$. Thus $\dim \operatorname{span}(\varphi) = \dim \operatorname{range} \varphi$.

---

4 *Supp U is a subsp of $V \neq U$. Prove $U^0 \neq \{0\}$.*

Solus: Let $X \oplus U = V \Rightarrow X \neq \{0\}$. Supp $s \in X \backslash \{0\}$. Let $Y \oplus \operatorname{span}(s) = X$.
Define $\varphi \in V'$ by $\varphi(u + \lambda s + y) = \lambda$. Hence $\varphi \neq 0$ and $\varphi(u) = 0$ for all $u \in U$. □

Or. [*Req V Finide*] By [3.106], $\dim U^0 = \dim V - \dim U > 0$.
Or. Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1 \dots, v_n)$ with $n \geqslant 1$.
Let $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$. Then each $\varphi \in \operatorname{span}(\varphi_1, \dots, \varphi_n)$ will do. □

Coro: **18** $\{0\}_V^0 = V'$. [*Which means $U^0 = V' \iff U = \{0\}$.*]
19 $U^0 = \{0\} = V^0 \iff U = V$. By the inv and ctrapos of Exe (4).

Comment: ***Another proof of*** **[3.108]**: T is surj $\iff T'$ is inje.
(a) Supp T' is inje. Notice that $\psi \neq 0 \iff T'(\psi) \neq 0 \iff \psi \notin (\operatorname{range} T)^0$.
(b) T is surj $\Rightarrow (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T'$. □

• **Note For [3.102]:** For $U = \emptyset$, U^0 is undefined. If U^0 is in the context, then certainly U is nonempty.

---

25 *Supp U is a subsp of V. Explain why $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.*

Solus: We show $\forall \varphi \in U^0, \varphi(v) = 0 \Rightarrow v \in U$ by ctradic. Asum $v \in V \backslash U$.
Then let $\operatorname{span}(v) \oplus U \oplus X = V$. $\exists \psi \in V', \operatorname{null} \psi = U \oplus X$. 又 $\psi \in U^0 \Rightarrow \psi(v) = 0$. □

Comment: $W \subseteq X = \{v \in V : \varphi(v) = 0, \forall \varphi \in W^0\}$, the ***promotion*** of the subset W of V.
The promotion of every nonempty subset of V is a subsp of V.
Now we show $\operatorname{span} W = X$. $\forall w \in \operatorname{span} W$,

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• *Supp U, W are subsps of V. Prove the promotion of $U \cup W$ is $U + W$.*

Solus: $(U \cup W)^0 = \{\varphi \in V' : \varphi(u) = \varphi(w) = \varphi(u+w) = 0, \forall u \in U, w \in W\} = (U+W)^0$. $\square$

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20 *Supp U, W are nonempty subsets of V. Prove $U \subseteq W \Rightarrow W^0 \subseteq U^0$.*

Solus: $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(u) = 0 \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$. $\square$

---

21 *Supp U, W are subsps of V. Prove $W^0 \subseteq U^0 \Rightarrow U \subseteq W$.*

Solus: Using Exe (25). Now $v \in U \Rightarrow \forall \varphi \in W^0 \subseteq U^0, \varphi(v) = 0 \Rightarrow v \in W$. $\square$

Note: $\varphi \in W^0 \iff \underline{\operatorname{null}\varphi \supseteq W \Rightarrow \operatorname{null}\varphi \supseteq U} \iff \varphi \in U^0$. But cannot conclude $W \supseteq U$.

Comment: (1) If U is merely a subset and W is a subsp. Promote U as X, let $W = Y$.
Then $Y^0 = W^0 \subseteq U^0 = X^0 \Rightarrow Y = W \supseteq X \supseteq U$. Still true.
(2) If W is merely a subset and U is a subsp. Promote W as Y, let $U = X$. For exa,
Let $W = \{(1,0),(0,1)\} \not\supseteq U = \{(x,0) \in \mathbf{R}^2\}$. Then $Y = \mathbf{R}^2 \supseteq X = U,\ Y^0 = \{0\} \subseteq X^0$.

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22 *Supp U and W are subsps of V. Prove $(U+W)^0 = U^0 \cap W^0$.*

Solus: (a) $\varphi \in (U+W)^0 \Rightarrow \forall u \in U, w \in W,$
$\varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$.
| $U \subseteq U + W \Rightarrow (U+W)^0 \subseteq U^0$
| $W \subseteq U + W \Rightarrow (U+W)^0 \subseteq W^0$

(b) $\varphi \in U^0 \cap W^0 \subseteq V' \Rightarrow \forall u \in U, w \in W, \varphi(u+w) = 0 \Rightarrow \varphi \in (U+W)^0$. $\square$

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23 *Supp U and W are subsps of V. Prove $(U \cap W)^0 = U^0 + W^0$.*

Solus:

(a) $\varphi = \psi + \beta \in U^0 + W^0 \Rightarrow \forall v \in U \cap W,$
$\varphi(v) = (\psi+\beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0$.
| Or. $U \cap W \subseteq U \Rightarrow (U \cap W)^0 \supseteq U^0$
| $U \cap W \subseteq W \Rightarrow (U \cap W)^0 \supseteq W^0$

(b) [*Only in Finide; Req U, W Subsps*] By Exe (22), $\dim(U^0 + W^0) = \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0)$
$= 2\dim V - \dim U - \dim W - (\dim V - \dim(U+W)) = \dim V - \dim(U \cap W)$. $\square$

Or. [*Req U, W Subsps*] Let $I = U \cap W$. We show $(U \cap W)^0 \subseteq U^0 + W^0$.
Define $\chi \in \mathcal{L}(V/I, V/U \times V/W)$ by $\chi : v + I \mapsto (v+U, v+W)$.
Well-defined: $v_1 + I = v_2 + I \in V/I \iff v_1 - v_2 \in I$
$\iff v_1 - v_2 \in U$ and $v_1 - v_2 \in W \Rightarrow (v_1 + U, v_1 + W) = (v_2 + U, v_2 + W)$.
Inje: $(v+U, v+W) = 0 \iff v \in U \cap W = I \iff v + I = 0$.
Surj: $\forall v \in V$ suth $(v+U, v+W) \in V/U \times V/W$, becs $\emptyset \neq (v+U) \cap (v+W) = v + I \in V/I$.
Hence $\chi' \in \mathcal{L}((V/U \times V/W)', (V/I)')$ is iso. Now we try finding an iso of $U^0 \times W^0$ onto $(U \cap W)^0$.
By Exe (4E 8), supp $\xi : (V/U)' \times (V/W)' \to (V/U \times V/W)'$ is iso.
By (c) in Exe (37), supp $\Lambda_1 : U^0 \times W^0 \to (V/U)' \times (V/W)'$ and $\Lambda_2 : (V/I)' \to (U \cap W)^0$ are isos.
Hence $(\Lambda_2 \circ \chi' \circ \xi \circ \Lambda_1) : U^0 \times W^0 \to (U \cap W)^0$ is iso. Now we see how it works:
$\forall (\varphi_U, \varphi_W) \in U^0 \times W^0$, $\operatorname{null}\pi_U \subseteq \operatorname{null}\varphi_U \Rightarrow \exists\, \psi_U \in (V/U)',\ \psi_U \circ \pi_U = \varphi_U$, simlr for φ_W,
thus $\Lambda_1 : (\varphi_U, \varphi_W) \mapsto (\psi_U, \psi_W)$. Then $\xi : (\psi_U, \psi_W) \mapsto (\psi_U S_U + \psi_W S_W)$, [See notas in (3.E.2).]
Now $(\psi_U S_U + \psi_W S_W) \xmapsto{\chi'} (\psi_U S_U + \psi_W S_W) \circ \chi \xmapsto{\Lambda_2} (\psi_U S_U + \psi_W S_W) \circ \chi \circ \pi_I$,
which sends v to $\psi_U(v+U) + \psi_W(v+W) = (\varphi_U + \varphi_W)(v)$, which is $\varphi_U + \varphi_W$.
Thus $(\Lambda_2 \circ \chi' \circ \xi \circ \Lambda_1)$ is the surj $\Lambda : U^0 \times W^0 \to U^0 + W^0$ defined in [3.77]. $\square$

Comment: Not true if U or W is merely a subset. Promote $U \cap W$ as I, U as X, and W as Y.

Exa: Let $U = \{(x, x+1) \in \mathbf{R}^2\}, W = \mathbf{R}^2$. Then $U \cap W = I = U \neq \mathbf{R}^2 = X \cap Y$.

• **TIPS 1:** *Prove* $V = U \oplus W \iff V' = U^0 \oplus W^0$.

SOLUS: $U \cap W = \{0\} \iff (U \cap W)^0 = \{0\}_V^0 = V' = U^0 + W^0$.

$V = U + W \iff (U + W)^0 = V_V^0 = \{0\} = U^0 \cap W^0$. □

• *Supp* $V = U \oplus W$. *Prove* $U^0 = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$, *where* $\iota \in \mathcal{L}(V, W) : u_v + w_v \to w_v$.

SOLUS: $\varphi \in U^0 \iff U \subseteq \operatorname{null} \varphi \iff \varphi = \varphi \circ \iota$, by $[$3.B TIPS (3)$]$. □

NOTE: The nota $W_V' = \{\varphi \in V' : \varphi = \varphi \circ \iota\} = U^0$ is not well-defined $[$without a bss$]$.
Simply becs W_V' have no info about the given U. Here is an informal explanation:
Each liney map $T \in \mathcal{L}(V, W)$ that vanishes on a given nontrivial U has its 'P'
(though not uniq) suth '$U \oplus P = V$' with $T : P \mapsto \operatorname{range} T$ being surj.
Hence $\forall W \in \mathcal{S}_V U$, $U^0 = W_V'$. But given nontrivial 'P', the corres 'U' is not uniq.
Fix one W_V', then U^0 is not uniq, with each U_k not equal to each other while each $U_k^0 = W_V'$.

EXA: Let $B_V = (e_1, e_2)$. Let $B_U = (e_1), B_X = (e_2 - e_1), B_Y = (e_2)$.
Then $\iota_X : ae_1 + b(e_2 - e_1) \mapsto b(e_2 - e_1)$, $\iota_Y : ae_1 + be_2 \mapsto be_2$. Now $X_V' = Y_V' = U^0$.
(1) For $V = U \oplus X$, let $B_{U_V'} = (\varphi)$ with $\varphi : e_1 \mapsto 1,\ e_2 - e_1 \mapsto 0 \Rightarrow e_2 \mapsto 1$.
(2) For $V = U \oplus Y$, let $B_{U_V'} = (\psi)$ with $\psi : e_1 \mapsto 1, e_2 \mapsto 0$.
Thus $X^0 = U_V'$ while $Y^0 = U_V' \Rightarrow X^0 = Y^0 \Rightarrow X = Y$, ctradic.
To fix this, we must have a bss of V' as precond, which we'll see in the NOTE for Exa (31).

NOTE: *Supp U is a subsp of V. Then finding the corres subsp in V' firstly req another 'half' $W \in \mathcal{S}_V U$, while finding the corres subsp of V for a subsp of V' must have the another 'half' asumed as precond.*

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31 *Supp V is finide and $B_{V'} = (\varphi_1, \dots, \varphi_n)$. Show $\exists\,!\,B_V$ whose dual bss is the $B_{V'}$.*

SOLUS: For each $k \in \{1, \dots, n\}$, let $\Gamma_k = \{1, \dots, n\} \setminus \{k\}$. Let each $U_k = \bigcap_{j \in \Gamma} \operatorname{null} \varphi_j$.
By Exe (4E 23), $V' = \operatorname{span}(\varphi_1, \dots, \varphi_n) = (\operatorname{null} \varphi_1 \cap \dots \cap \operatorname{null} \varphi_n)^0 \Rightarrow U_k \cap \varphi_k = \{0\}$.
Thus $\forall x_k \in U_k \setminus \{0\}$, $x_k \notin \operatorname{null} \varphi_k$ while $x_k \in \operatorname{null} \varphi_j$ for all $j \in \Gamma$.
Fix one x_k and let $v_k = [\varphi_k(x_k)]^{-1} x_k \Rightarrow \varphi_k(v_k) = 1$, $\varphi_j(v_k) = 0$ for all $j \neq k$.
Simply for each v_k, $\varphi_j(v_k) = \delta_{j,k}$ for all $j \iff$ for each φ_j, $\varphi_j(v_k) = \delta_{j,k}$ for all k.
又 $a_1 v_1 + \dots + a_n v_n = 0 \Rightarrow$ each $\varphi_k(0) = a_k$.
Now we prove the uniqnes part. Supp the dual bss of $B_V' = (u_1, \dots, u_n)$ is the $B_{V'}$.
For each k, we have $\varphi_j(v_k) = \varphi_j(u_k)$ for all $k \Rightarrow v_k - u_k \in \bigcap \operatorname{null} \varphi_j = \{0\}$. □

• **NOTE FOR Exe (31):** Supp V is finide, and Ω is a subsp of V' with $B_\Omega = (\varphi_1, \dots, \varphi_m)$.
The 'W' is not clear when we are to find suth $W_V{}' = \Omega$, becs the another 'half' is undefined.
Extend to $B_{V'} = (\varphi_1, \dots, \varphi_n)$. By Exe (31), $\exists\,!$ corres $B_V = (v_1, \dots, v_n)$.
Let $B_U = (v_{m+1}, \dots, v_n), B_W = (v_1, \dots, v_m)$. Thus we found the W suth $\Omega = W_V'$,
which is well-defined with B_V as precond.

• **TIPS 2:** Supp $\varphi_1, \dots, \varphi_m \in V'$. Let $\operatorname{null}_I = (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m)$.
Supp Ω is a subsp of V'. Let $\operatorname{null}_C = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}$.

(1) If Ω is infinide. Then by def, $\bigcap_{\varphi \in \Omega} \operatorname{null} \varphi = \operatorname{null}_C$.

(2) If $\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$. Then $v \in \operatorname{null}_I \iff$ each $\varphi_k(v) = 0$

$$\iff \forall \varphi = \textstyle\sum_{i=1}^n a_i \varphi_i \in \Omega, \varphi(v) = 0 \iff v \in \operatorname{null}_C.$$

• (4E 23) *Supp V is finide, $\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m) \subseteq V'$. Prove* $\Omega = \big(\operatorname{null}\varphi_1 \cap \cdots \cap \operatorname{null}\varphi_m\big)^0$.

SOLUS: Becs each $\operatorname{span}(\varphi_k) \subseteq (\operatorname{null}\varphi_k)^0$. By NOTE FOR Exe (4E 23) and Exe (23), Immed. $\square$

OR. Reduce to $B_\Omega = (\beta_1, \dots, \beta_p)$. We show $\Omega = (\operatorname{null}\beta_1 \cap \cdots \cap \operatorname{null}\beta_p)^0$. Then by (L1), done.
Let $B_{V'} = (\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_q)$. By Exe (31), let $B_V = (v_1, \dots, v_p, u_1, \dots, u_q)$.
Define each $\Gamma_k = \{1, \dots, p\} \setminus \{k\}$. Then $\operatorname{null}\beta_k = \operatorname{span}\{v_j\}_{j \in \Gamma_k} \oplus \operatorname{span}(u_1, \dots, u_q)$.
Now $\operatorname{null}\beta_1 \cap \cdots \cap \operatorname{null}\beta_p = \operatorname{span}(u_1, \dots, u_q)$. Simlr to (4E 2.C.16).

L1 *Supp each $\varphi_i, \beta_j \in \mathcal{L}(V, W)$. Supp* $\operatorname{span}(\varphi_1, \dots, \varphi_m) = \operatorname{span}(\beta_1, \dots, \beta_n)$.
Prove $\operatorname{null}\varphi_1 \cap \cdots \cap \operatorname{null}\varphi_m = \operatorname{null}\beta_1 \cap \cdots \cap \operatorname{null}\beta_n$.

SOLUS: Denote $\operatorname{null}\psi_a \cap \cdots \cap \operatorname{null}\psi_b$ by $\bigcap_a^b \operatorname{null}\psi_I$. Supp $n < m$, othws rev the roles.
Supp $\varphi_j = c_1\varphi_1 + \cdots + c_{j-1}\varphi_{j-1}$.
Let $N_j \oplus \bigcap_1^{j-1} \operatorname{null}\varphi_I = \operatorname{null}\varphi_j$. Now $\bigcap_1^j \operatorname{null}\varphi_I = \bigcap_1^{j-1} \operatorname{null}\varphi_I \cap \big(\operatorname{null}\varphi_j\big) = \bigcap_1^{j-1} \operatorname{null}\varphi_I$.
Thus $\bigcap_1^m \operatorname{null}\varphi_I = \big[\bigcap_1^{j-1} \operatorname{null}\varphi_I\big] \cap \big[\bigcap_{j+1}^m \operatorname{null}\varphi_I\big]$. Hence $\bigcap_1^n \operatorname{null}\beta_I = \bigcap_1^m \operatorname{null}\varphi_I$.

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26 *Supp V is finide, Ω is a subsp of V'. Prove $\Omega = \big\{v \in V : \varphi(v) = 0$ for every $\varphi \in \Omega\big\}^0$.*

SOLUS: Let $B_\Omega = (\varphi_1, \dots, \varphi_m)$. By TIPS (2) and Exe (4E 23). $\square$

COMMENT: This Exe may not be true if V is infinide.

CORO: Supp V is finide. For every subsp Ω of V', $\exists!$ subsp U of V suth $\Omega = U^0$.
This form of Ω does not depend on a bss and thus is considered more general.

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24 *Supp V is finide and U is a subsp of V.*
Prove, using the pattern of [3.104], that $\dim U + \dim U^0 = \dim V$.

SOLUS: Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \dots \varphi_n)$.
Supp $\psi = \sum_{i=1}^m a_i\psi_i + \sum_{j=1}^n b_j\varphi_j \in U^0 \Rightarrow$ each $\psi(u_k) = a_k = 0$. Thus $U^0 \subseteq \operatorname{span}(\varphi_1, \dots, \varphi_n)$. $\square$

Let $B_{U^0} = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_n) \Rightarrow B_V = (v_1, \dots, v_n)$.
We show $B_U = (v_{m+1}, \dots, v_n)$. Let $B_{W^0} = (\varphi_{m+1}, \dots, \varphi_n)$.
And let corres (I) $B_U = (v_{m+1}, \dots, v_n)$, (II) $B_W = (v_1, \dots, v_m)$.

(I) NOTICE that each $\operatorname{null}\varphi_k = \operatorname{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n) = U_k$; $\dim U_k = \dim V - 1$.
By (4E 2.C.16), $U = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = \bigcap_{k=1}^m U_k = \operatorname{span}(v_{m+1}, \dots, v_n)$.
Hence $\operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0 = \Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$.

(II) NOTICE that $V' = \Omega \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.
And that $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq \operatorname{span}(v_1, \dots, v_m)^0$.
By [1.C TIPS (2)] OR (2.C.1), $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = \operatorname{span}(v_1, \dots, v_m)^0$.
OR. Simlr to (II), let $\Omega = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, immed. $\square$

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9 *Let $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \cdots, \varphi_n)$. Then $\forall \psi \in V', \psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.*

CORO: *For other $B'_V = (u_1, \dots, u_n), B'_{V'} = (\rho_1, \dots, \rho_n), \forall \psi \in V', \psi = \psi(u_1)\rho_1 + \cdots + \psi(u_n)\rho_n$.*

SOLUS:

$\psi(v) = \psi\Big(\sum_{i=1}^n a_i v_i \Big) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i)\varphi_i(v) = \big[\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n\big](v)$.

OR. $\big[\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n\big]\Big(\sum_{i=1}^n a_i v_i \Big) = \psi(v_1)\varphi_1\Big(\sum_{i=1}^n a_i v_i \Big) + \cdots + \psi(v_n)\varphi_n\Big(\sum_{i=1}^n a_i v_i \Big)$. □

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13 *Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x,y,z) = (4x+5y+6z, 7x+8y+9z)$.*

Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ denote the dual bss of std bss of $\mathbf{R}^2$ and $\mathbf{R}^3$.

(a) *Describe the liney functionals $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$*

For any $(x,y,z) \in \mathbf{R}^3$, $\big(T'(\varphi_1)\big)(x,y,z) = 4x+5y+6z$, $\big(T'(\varphi_2)\big)(x,y,z) = 7x+8y+9z$.

(b) *Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as liney combinations of ψ_1, ψ_2, ψ_3.*

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$.

(c) *What is* null T'? *What is* range T'?

$$T(x,y,z) = 0 \iff \begin{cases} 4x+5y+6z = 0 \\ 7x+8y+9z = 0 \end{cases} \iff \begin{cases} x+y+z = 0 \\ y = 2z = 0 \end{cases} \iff (x,y,z) \in \operatorname{span}(e_1 - 2e_2 + e_3).$$

Where (e_1, e_2, e_3) is std bss of $\mathbf{R}^3$.

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a bss, with corres dual bss $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Thus $\operatorname{span}(e_1 - 2e_2 + e_3) = \operatorname{null} T \Rightarrow \operatorname{span}(e_1 - 2e_2 + e_3)^0 = \operatorname{span}(\varepsilon_2, \varepsilon_3) = \operatorname{range} T'$.

Note that $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$.

And $\left| \begin{array}{l} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{array} \right.$

Hence $\varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2$, $\varepsilon_3 = -\psi_1 + \psi_3$. Now $\operatorname{range} T' = \operatorname{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3)$.

OR. $\operatorname{range} T' = \operatorname{span}(T'(\varphi_1), T'(\varphi_2)) = \operatorname{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3)$.

Supp $T'(x\varphi_1 + y\varphi_2) = (4x+7y)\varphi_1 + (5x+8y)\varphi_2 + (6x+9y)\varphi_3 = 0$.

Then $x + y = 4x + 7y = x = y = 0$. Hence $\operatorname{null} T' = \{0\}$.

OR. $\operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \operatorname{span}(-2e_2, e_3) \oplus \operatorname{null} T$.

$\Rightarrow \operatorname{range} T = \{Tx : x \in \operatorname{span}(-2e_2, e_3)\} = \operatorname{span}(T(-2e_2), T(e_3))$

$= \operatorname{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \operatorname{span}(f_1, f_2) = \mathbf{R}^2$. Now $\operatorname{null} T' = (\operatorname{range} T)^0 = \{0\}$. □

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37 *Supp U is a subsp of V and π is the quot map. Thus $\pi' \in \mathcal{L}\big((V/U)', V'\big)$.*

(a) *Show π' is inje*: Becs π is surj. Use [3.108].

(b) *Show* range $\pi' = U^0$: By [3.109](b), $\operatorname{range} \pi' = (\operatorname{null} \pi)^0 = U^0$.

(c) *Conclude that π' is iso from $(V/U)'$ onto U^0*: Immed.

SOLUS: OR. Using (3.E.18), also see (3.E.20).

(a) $\pi'(\varphi) = 0 \iff \forall v \in V\ (\ \forall v + U \in V\ ), \varphi(\pi(v)) = \varphi(v + U) = 0 \iff \varphi = 0$.

(b) $\psi \in \operatorname{range} \pi' \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null} \psi \supseteq U \iff \psi \in U^0$. Hence $\operatorname{range} \pi' = U^0$. □

• *Supp U is a subsp of V. Prove $(V/U)'$ is iso to U^0.* $\Big[$ ***Another proof of*** **[3.106]** $\Big]$

Solus:

Define $\xi : U^0 \to (V/U)'$ by $\xi(\varphi) = \tilde{\varphi}$, where $\tilde{\varphi} \in (V/U)'$ is defined by $\tilde{\varphi}(v + U) = \varphi(v)$.

We show ξ is inje and surj.

Inje: $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V \big(\forall v + U \in V/U \big), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$.

Surj: $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null}(\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$. □

Or. Define $\nu : (V/U)' \to U^0$ by $\nu(\Phi) = \Phi \circ \pi$. Now $\nu \circ \xi = I_{U^0}$, $\xi \circ \nu = I_{(V/U)'} \Rightarrow \xi = \nu^{-1}$. □

• *Supp $V = U \oplus W$. Define $\iota : V \to U$ by $\iota(u + w) = u$. Thus $\iota' \in \mathcal{L}(U', V')$.*

(a) *Show* $\operatorname{null} \iota' = U_U^0 = \{0\}$: $\operatorname{null} \iota' = (\operatorname{range} \iota)_U^0 = U_U^0 = \{0\}$.

(b) *Prove* $\operatorname{range} \iota' = W_V^0$: $\operatorname{range} \iota' = (\operatorname{null} \iota)_V^0 = W_V^0$.

(c) *Prove $\widetilde{\iota'}$ is iso from $U'/\{0\}$ onto W^0:* By (a), (b) and [3.91](d).

Solus:

(a) $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \operatorname{null} \psi$.

(b) Note that $W = \operatorname{null}(\iota) \subseteq \operatorname{null}(\psi \circ \iota)$. Then $\psi \circ \iota \in W^0 \Rightarrow \operatorname{range} \iota' \in W^0$.

Supp $\varphi \in W^0$. Becs $\operatorname{null} \iota = W \subseteq \operatorname{null} \varphi$. By $\big[$3.B Tips (3)$\big]$, $\varphi = \varphi \circ \iota = \iota'(\varphi)$. □

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36 *Supp U is a subsp of V. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*

(a) *Show* $\operatorname{null} i' = U^0$: $\operatorname{null} i' = (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$.

(b) *Prove* $\operatorname{range} i' = U'$: $\operatorname{range} i' = (\operatorname{null} i)_U^0 = \{0\}_U^0 = U'$.

(c) *Prove $\widetilde{i'}$ is iso from V'/U^0 onto U':* By (a), (b) and [3.91](d).

Solus:

(a) $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$. Thus $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$.

(b) Supp $\psi \in U'$. By (3.A.11), $\exists\, \varphi \in V', \varphi|_U = \psi$. Then $i'(\varphi) = \psi$. □

• *Supp $T \in \mathcal{L}(V, W)$. Prove* $\operatorname{range} T' = \big(\operatorname{null} T\big)^0$. $\Big[$ ***Another proof of*** **[3.109](b)** $\Big]$

Solus:

Supp $\Phi \in \big(\operatorname{null} T\big)^0$. Becs by (3.B.12), $T|_U : U \to \operatorname{range} T$ is iso; $V = U \oplus \operatorname{null} T$.

And $\forall v \in V, \exists!\, u_v \in U, w_v \in \operatorname{null} T, v = u_v + w_v$. Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$.

Let $\psi = \Phi \circ \big(T^{-1}|_{\operatorname{range} T}\big)$. Then $T'(\psi) = \psi \circ T = \Phi \circ \big(T^{-1}|_{\operatorname{range} T} \circ T|_V\big)$.

Where $T^{-1}|_{\operatorname{range} T} : \operatorname{range} T \to U$; $\;T : V \to \operatorname{range} T$. Note that $T^{-1}|_{\operatorname{range} T} \circ T|_V = \iota$.

By $\big[$3.B Tips (3)$\big]$, $\Phi = \Phi \circ \iota$. Thus $T'(\psi) = \psi \circ T = \Phi \circ \iota = \Phi$. □

• *Supp $T \in \mathcal{L}(V, W)$. Using [3.108], [3.110].*

$$\textit{Now } T \textit{ is inv} \iff \left| \begin{array}{c} \operatorname{null} T = \{0\} \iff (\operatorname{null} T)^0 = V' = \operatorname{range} T' \\ \operatorname{range} T = W \iff (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T' \end{array} \right| \iff T' \textit{ is inv.}$$

---

15 *Supp $T \in \mathcal{L}(V,W)$. Prove $T' = 0 \iff T = 0$.*

SOLUS:

Supp $T = 0$. Then $\forall \varphi \in W', T'(\varphi) = \varphi \circ T = 0$. Hence $T' = 0$.

Supp $T' = 0$. Then $\operatorname{null} T' = W' = (\operatorname{range} T)^0$, by [3.107](a).

$\big[$ *W can be infinide* $\big]$ By Exe (25),

$\operatorname{range} T = \{w \in W : \varphi(w) = 0, \forall \varphi \in (\operatorname{range} T)^0\} = \{w \in W : \varphi(w) = 0, \forall \varphi \in W'\}$.

Now we prove if $\forall \varphi \in W', \varphi(w) = 0$, then $w = 0$. So that $\operatorname{range} T = \{0\}$ and done.

Asum $w \neq 0$. Then let U be suth $W = U \oplus \operatorname{span}(w)$.

Define $\psi \in W'$ by $\psi(u + \lambda w) = \lambda$. So that $\psi(w) = 1 \neq 0$. □

OR. $\big[$ *Only if W is finide* $\big]$ By [3.106], $\dim \operatorname{range} T = \dim W - \dim(\operatorname{range} T)^0 = 0$. □

---

12 NOTICE that $I_{V'} : V' \to V'$. *Now* $\forall \varphi \in V', I_{V'}(\varphi) = \varphi = \varphi \circ I_V = {I_V}'(\varphi)$. *Thus* $I_{V'} = {I_V}'$.

---

16 *Supp V, W are finide. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(V,W)$.*
Prove Γ is iso of $\mathcal{L}(V,W)$ onto $\mathcal{L}(W',V')$.

SOLUS: By [3.101], Γ is liney.

Supp $\Gamma(T) = T' = 0$. By Exe (15), $T = 0$. Thus Γ is inje.

Becs V, W are finide. $\dim \mathcal{L}(V,W) = \dim \mathcal{L}(W',V')$. Now Γ inje $\Rightarrow$ inv. □

COMMENT: Let $X = \{T \in \mathcal{L}(V,W) : \operatorname{range} T \text{ is finide}\}$.
Let $Y = \{\mathcal{T} \in \mathcal{L}(W',V') : \operatorname{range} \mathcal{T} \text{ is finide}\}$.
Then $\Gamma|_X$ is iso of X onto Y, even if V and W are infinide.

The inje of $\Gamma|_X$ is equiv to the inje of Γ, as shown before.

Now we show $\Gamma|_X$ is surj without the cond that V or W is finide.

Supp $\mathcal{T} \in Y$. Let $B_{\operatorname{range} \mathcal{T}} = (\varphi_1, \dots, \varphi_m)$, with corres $(v_1, \dots, v_m)$. Let $\varphi_k = \mathcal{T}(\psi_k)$.

Let $\mathcal{K}$ be suth $W' = \mathcal{K} \oplus \operatorname{null} \mathcal{T}$. Let $B_{\mathcal{K}} = (\psi_1, \dots, \psi_m)$, with corres $(w_1, \dots, w_m)$.

Define $T \in \mathcal{L}(V,W)$ by $Tv_k = w_k, Tu = 0;\ k \in \{1, \dots, m\}, u \in U$.

$\forall \psi \in \operatorname{null} \mathcal{T}, [T'(\psi)](v) = \psi(Tv) = \psi(a_1 w_1 + \dots + a_p w_p) = 0 = [\mathcal{T}(\psi)](v)$.

$\forall k \in \{1, \dots, m\}, [T'(\psi_k)](v) = \psi_k(Tv) = \psi_k(a_1 w_1 + \dots + a_m w_m) = a_k = \varphi_k(v) = [\mathcal{T}(\psi)](v)$. □

COMMENT: This is another proof of [3.109(a)]: $\dim \operatorname{range} T = \dim \operatorname{range} T'$.

---

5 *Prove $(V_1 \times \dots \times V_m)'$ and $V_1' \times \dots \times V_m'$ are iso.* $\big[$ Using notas in (3.E.2). $\big]$

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \dots \times V_m)' \to V_1' \times \dots \times V_m' \\ \qquad \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = ({R'}_1(T), \dots, {R'}_m(T)). \\ \text{Define } \psi : V_1' \times \dots \times V_m' \to (V_1 \times \dots \times V_m)' \\ \qquad \text{by } \psi(T_1, \dots, T_m) = T_1 S_1 + \dots + T_m S_m = {S'}_1(T_1) + \dots + {S'}_m(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$

□

• (4E 8) *Supp $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \dots, \varphi_n)$.*

$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \dots, a_n) = a_1 v_1 + \dots + a_n v_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

---

6 *Define* $\Gamma : V' \to \mathbf{F}^m$ *by* $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$, *where* $v_1, \dots, v_m \in V$.

(a) *Show* $\operatorname{span}(v_1, \dots, v_m) = V \iff \Gamma$ *is inje.*

(b) *Show* $(v_1, \dots, v_m)$ *is liney indep* $\iff \Gamma$ *is surj.*

Solus:

(a) Notice that $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \dots = \varphi(v_m) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.

If Γ is inje, then $\Gamma(\varphi) = 0 \iff V = \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.

If $V = \operatorname{span}(v_1, \dots, v_m)$, then $\Gamma(\varphi) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$, thus Γ is inje.

(b) Supp Γ is surj. Then let $\Gamma(\varphi_i) = e_i$ for each i, where $(e_1, \dots, e_m)$ is std bss of $\mathbf{F}^m$.

Then by (3.A.4), $(\varphi_1, \dots, \varphi_m)$ is liney indep.

Now $a_1 v_1 + \dots + a_m v_m = 0 \Rightarrow 0 = \varphi_i(a_1 v_1 + \dots + a_m v_m) = a_i$ for each i.

Supp $(v_1, \dots, v_m)$ is liney indep. Let $U = \operatorname{span}(\varphi_1, \dots, \varphi_m), B_{U'} = (\varphi_1, \dots, \varphi_m)$.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists!\, \varphi = a_1\varphi_1 + \dots + a_m\varphi_m$.

Let W be suth $V = U \oplus W$. Now $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.

Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$. So that $\Gamma(\varphi \circ i -) = (a_1, \dots, a_m)$. □

Or. Let $(e_1, \dots, e_m)$ be std bss of $\mathbf{F}^m$ and let $(\psi_1, \dots, \psi_m)$ be corres dual bss.

Define $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$. Then Ψ is iso.

Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $Te_k = v_k$. Now $T(x_1, \dots, x_m) = T(x_1 e_1 + \dots + x_m e_m) = x_1 v_1 + \dots + x_m v_m$.

$\forall \varphi \in V', k \in \{1, \dots, m\}, [T'(\varphi)](e_k) = \varphi(Te_k) = \varphi(v_k) = [\varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m](e_k)$

Now $T'(\varphi) = \varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m = \Psi(\varphi(v_1), \dots, \varphi(v_m)) = \Psi(\Gamma(\varphi))$. Hence $T' = \Psi \circ \Gamma$.

By (3.B.3), (a) $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V \iff T' = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.

(b) $(v_1, \dots, v_m)$ is liney indep $\iff T$ is inje $\iff T' = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

• (4E 25) *Define* $\Gamma : V \to \mathbf{F}^m$ *by* $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$, *where* $\varphi_1, \dots, \varphi_m \in V'$.

(c) *Show* $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ *is inje.*

(d) *Show* $(\varphi_1, \dots, \varphi_m)$ *is liney indep* $\iff \Gamma$ *is surj.*

Solus:

(c) Notice that $\Gamma(v) = 0 \iff \varphi_1(v) = \dots = \varphi_m(v) = 0 \iff v \in (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)$.

By Exe (4E 23) and (18), $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) = \{0\}$.

And $\operatorname{null}\Gamma = (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)$. Hence Γ inje $\iff \operatorname{null}\Gamma = \{0\} \iff \operatorname{span}(\varphi_1, \dots, \varphi_m) = V'$.

(d) Supp $(\varphi_1, \dots, \varphi_m)$ is liney indep. Then by Exe (31), $(v_1, \dots, v_m)$ is liney indep.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}, \exists!\, v = \sum_{i=1}^{m} a_i v_i \in V \Rightarrow \varphi_i(v) = a_i, \Gamma(v) = (a_1, \dots, a_m)$. Hence Γ is surj.

Supp Γ is surj. Let $(e_1, \dots, e_m)$ be std bss of $\mathbf{F}^m$.

Supp $v_i \in V$ suth $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i$, for each i.

Then $(v_1, \dots, v_m)$ is liney indep. And $\varphi_j(v_k) = \delta_{j,k}$.

Now $a_1\varphi_1 + \dots + a_m\varphi_m = 0 \Rightarrow 0(v_i) = a_i$ for each i. Hence $(\varphi_1, \dots, \varphi_m)$ is liney indep.

Or. Let $\operatorname{span}(v_1, \dots, v_m) = U$. Then $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U)$. Hence $(\varphi_1, \dots, \varphi_m)$ is liney indep. □

Or. Simlr to Exe (6), we get $(e_1, \dots, e_m), (\psi_1, \dots, \psi_m)$ and the iso Ψ.

$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'(\Psi(x_1, \dots, x_m)) = \Gamma'(\Psi(x_1 e_1 + \dots + x_m e_m)) = (x_1\psi_1 + \dots + x_m\psi_m) \circ \Gamma$.

$\forall v \in V, [\Gamma'(\Psi(x_1, \dots, x_m))](v) = [x_1\psi_1 + \dots + x_m\psi_m](\Gamma(v)) = [x_1\varphi_1 + \dots + x_m\varphi_m](v)$.

Now $\Gamma'(\Psi(x_1, \dots, x_m)) = x_1\varphi_1 + \dots + x_m\varphi_m$.

Define $\Phi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Phi = \Psi \circ \Gamma$. $\Phi(x_1, \dots, x_m) = x_1\varphi_1 + \dots + x_m\varphi_m$. Thus by (4E 3.B.3),

(c) the inje of Φ corres to $(\varphi_1, \dots, \varphi_m)$ spanning V'; 又 $\Phi = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.

(d) the surj of Φ corres to $(\varphi_1, \dots, \varphi_m)$ being liney indep; 又 $\Phi = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

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35 *Prove $\big(\mathcal{P}(\mathbf{F})\big)'$ is iso to $\mathbf{F}^\infty$.*

SOLUS:

Define $\theta \in \mathcal{L}\big(\big(\mathcal{P}(\mathbf{F})\big)', \mathbf{F}^\infty\big)$ by $\theta(\varphi) = \big(\varphi(1), \varphi(z), \cdots, \varphi(z^n), \cdots\big)$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall z^k$ in the bss $\big(1, z, \ldots, z^n\big)$ of $\mathcal{P}_n(\mathbf{F})$ $\big(\ \forall n\ \big)$, $\varphi(z^k) = 0 \Rightarrow \varphi = 0$.

$\big[$ NOTICE that $\forall p \in \mathcal{P}(\mathbf{R}), \exists\,! a_i \in \mathbf{F}, m = \deg p,\ p = a_0 z + a_1 z + \cdots + a_m z^m \in \mathcal{P}_m(\mathbf{F})$. $\big]$

Surj: $\forall \big(a_k\big)_{k=1}^\infty \in \mathbf{F}^\infty$, let ψ be suth $\forall k, \psi\big(z^k\big) = a_k$ $\big[$ by [3.5] $\big]$ and thus $\theta(\psi) = \big(a_k\big)_{k=1}^\infty$. ☐

COMMENT: NOTICE that $\mathcal{P}(\mathbf{F})$ is not iso to $\mathbf{F}^\infty$, so is $\mathcal{P}(\mathbf{F})$ to $\big(\mathcal{P}(\mathbf{F})\big)'$

But if we let $\mathbf{F}^\infty = \big\{\big(a_1, \cdots, a_n, \underbrace{0, \cdots, 0, \cdots}_{\text{all zero}}\big) \in \mathbf{F}^\infty \,\big|\, \exists\,! n \in \mathbf{N}^+\big\}$. Then $\mathcal{P}(\mathbf{F})$ is iso to $\mathbf{F}^\infty$.

---

7 *Show the dual bss of $(1, x, \ldots, x^m)$ of $\mathcal{P}_m(\mathbf{R})$ is $(\varphi_0, \varphi_1, \ldots, \varphi_m)$, where $\varphi_k(p) = \dfrac{p^{(k)}(0)}{k!}$.*

Here $p^{(k)}$ denotes the k^{th} deri of p, with the understanding that the 0^{th} deri of p is p.

SOLUS:

$$\forall j, k \in \mathbf{N},\ (x^j)^{(k)} = \begin{cases} j(j-1)\ldots(j-k+1)\cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\ldots(j-j+1) = j! & j = k. \\ 0, & j \leqslant k. \end{cases} \Bigg|\ \text{Then}\ \ (x^j)^{(k)}(0) = \begin{cases} 0\ ,\ j \neq k. \\ k!\ ,\ j = k. \end{cases}$$ ☐

OR. Becs $\forall j, k \in \{1, \ldots, m\}$ suth $j \neq k$, $\varphi_k\big(x^j\big) = \frac{(x^j)^{(k)}(0)}{k!} = \frac{0}{k!} = 0$; $\varphi_k\big(x^k\big) = \frac{(x^k)^{(k)}(0)}{k!} = 1$.

Thus $\frac{p^{(k)}(0)}{k!}$ act exactly the same as φ_k on the same bss $\big(1, \ldots, x^m\big)$, hence is just another def of φ_k. ☐

EXA: Supp $m \in \mathbf{N}^+$. By [2.C.10], $B = \big(1, x-5, \ldots, (x-5)^m\big)$ is a bss of $\mathcal{P}_m(\mathbf{R})$.

Let $\varphi_k = \dfrac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \ldots, m$. Then $\big(\varphi_0, \varphi_1, \ldots, \varphi_m\big)$ is the dual bss of B.

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34 *The double dual space of V, denoted by V'', is defined to be the dual space of V'.*
In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \to V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.

(a) *Show Λ is a liney map from V to V''.*

(b) *Show if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*

(c) *Show if V is finide, then Λ **is iso from** V **onto** V''.*

Supp V is finide. Then V and V' are iso, and finding iso from V onto V' generally requires choosing a bss of V. In contrast, the iso Λ from V onto V'' does not require a choice of bss and thus is considered more natural.

SOLUS:

(a) $\forall \varphi \in V', v, w \in V, a \in \mathbf{F}$, $\big(\Lambda(v+aw)\big)(\varphi) = \varphi(v+aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.
Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence Λ is liney.

(b)
$$\begin{aligned}\big(T''(\Lambda v)\big)(\varphi) &= \big((\Lambda v)\circ T'\big)(\varphi) = (\Lambda v)\big(T'(\varphi)\big) \\ &= \big(T'(\varphi)\big)(v) = (\varphi \circ T)(v) = \varphi(Tv) = \big(\Lambda(Tv)\big)(\varphi).\end{aligned}$$

Hence $T''(\Lambda v) = \big(\Lambda(Tv)\big) \Rightarrow T'' \circ \Lambda = \Lambda \circ T$.

(c) Supp $\Lambda v = 0$. Then $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is inje.
又 Becs V is finide. $\dim V = \dim V' = \dim V''$. Hence Λ is iso. ☐

COMMENT: Supp $\Phi \in V''$ and $\Phi \neq 0$. Then $\exists\, \varphi \in V'$, $\Phi(\varphi) = 1 \Rightarrow \operatorname{null}\Phi \oplus \operatorname{span}(\varphi) = V'$.
And $\varphi \neq 0 \Rightarrow \exists\, v \in V$, $\varphi(v) = 1, \operatorname{null}\varphi \oplus \operatorname{span}(v) = V$. Becs Λ is surj.
Now $\exists\, x \in V, \forall \psi = c\varphi + \rho \in V', \psi(x) = (\Lambda x)(\psi) = \Phi(\psi) = c$.

ENDED

- **Tips:** *Supp $p \in \mathcal{P}(\mathbf{F})$, $\deg p \leqslant m$ and p has at least $(m+1)$ disti zeros.*
  Then by the ctrapos of [4.12], 又 $\deg p = m$, we conclude that $m < 0$. Hence $p = 0$.

  Or. We show if p has at least m disti zeros, then either $p = 0$ or $\deg p \geqslant m$.
  If $p = 0$ then done. If not, then supp p has exactly n disti zeros $\lambda_1, \dots, \lambda_n$.
  Becs $\exists\,!\,\alpha_i \geqslant 1, q \in \mathcal{P}(\mathbf{F})$, and $q \neq 0$, suth $p(z) = \big[(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_n)^{\alpha_n}\big] q(z)$. ☐
- **Comment:** Notice *that by [4.17], some term of the poly factoriz might not be in the form $(x-\lambda_k)^{\alpha_k}$.*

- **Note For [4.7]:** *the uniqnes of coeffs of polys* [***Another proof***]

  If a poly had two different sets of coeffs, then subtracting the two exprs
  would give a poly with some nonzero coeffs but infily many zeros. By Tips. ☐

- **Note For [4.8]:** *div algo for polys* [***Another proof***]

  Supp $\deg p \geqslant \deg s$. Then $\Big(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of len } \deg s}, \underbrace{s, zs, \cdots, z^{\deg p - \deg s} s}_{\text{of len } (\deg p - \deg s + 1)}\Big)$ is a bss of $\mathcal{P}_{\deg p}(\mathbf{F})$.

  Becs $q \in \mathcal{P}(\mathbf{F}), \exists\,!\,a_i, b_j \in \mathbf{F}$,
  $q = a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s$
  $= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1}}_{r} + s \underbrace{\Big(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s}\Big)}_{q}$. Note that r, q are uniq. ☐

- **Note For [4.11]:** *each zero of a poly corres to a deg-one factor;* [***Another proof***]

  First supp $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m, \exists\,!\,a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.
  Then $p(z) = p(z) - p(\lambda) = a_1(z-\lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.
  Hence $\forall k \in \{1, \dots, m\}, z^k - \lambda^k = (z-\lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.
  Thus $p(z) = \sum_{j=1}^{m} a_j (z-\lambda) \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z-\lambda) \sum_{j=1}^{m} a_j \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z-\lambda) q(z)$. ☐

- **Note For [4.13]:** *Every nonconst poly with complex coeffs has a zero in* $\mathbf{C}$. [***Another proof***]

  For any $w \in \mathbf{C}, k \in \mathbf{N}^+$, by polar coordinates, $\exists\, r \geqslant 0, \theta \in \mathbf{R}, r(\cos\theta + \mathrm{i}\sin\theta) = w$.
  By De Moivre' theorem, $w^k = [r(\cos\theta + \mathrm{i}\sin\theta)]^k = r^k(\cos k\theta + \mathrm{i}\sin k\theta)$.
  Hence $\big(r^{1/k}(\cos\frac{\theta}{k} + \mathrm{i}\sin\frac{\theta}{k})\big)^k = w$. Thus every complex number has a k^{th} *root*.

  Supp a nonconst $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.
  Then $|p(z)| \to \infty$ as $|z| \to \infty$ $\left(\text{becs } \dfrac{|p(z)|}{|z_m|} \to |c_m| \text{ as } |z| \to \infty\right)$.
  Thus the continuous function $z \to |p(z)|$ has a global min at some point $\zeta \in \mathbf{C}$.
  To show $p(\zeta) = 0$, asum $p(\zeta) \neq 0$. Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \dfrac{p(z+\zeta)}{p(\zeta)}$.
  The function $z \to |q(z)|$ has a global min value of 1 at $z = 0$.
  Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where $k \in \mathbf{N}^+$ is the smallest suth $a_k \neq 0$.
  Let $\beta \in \mathbf{C}$ be suth $\beta^k = -\dfrac{1}{a_k}$.
  There is a const $c > 1$ so that if $t \in (0,1)$, then $|q(t\beta)| \leqslant |1 + a_k t^k \beta^k| + t^{k+1} c = 1 - t^k(1 - tc)$.
  Now letting $t = 1/(2c)$, we get $|q(t\beta)| < 1$. Ctradic. Hence $p(\zeta) = 0$, as desired. ☐

• (4E 4.2) *Prove if $w, z \in \mathbf{C}$, then $\big||w|-|z|\big| \leqslant |w-z|$.*

SOLUS:

$$\begin{aligned} |w-z|^2 &= (w-z)(\overline{w}-\overline{z}) \\ &= |w|^2+|z|^2-(w\overline{z}+\overline{w}z) \\ &= |w|^2+|z|^2-(\overline{\overline{w}z}+\overline{w}z) \\ &= |w|^2+|z|^2-2Re(\overline{w}z) \\ &\geqslant |w|^2+|z|^2-2|\overline{w}z| \\ &= |w|^2+|z|^2-2|w||z| = \big||w|-|z|\big|^2. \end{aligned}$$

OR. $\left.\begin{array}{l} |w| = |w-z+z| \leqslant |w-z|+|z| \Rightarrow |w|-|z| \leqslant |w-z| \\ |z| = |z-w+w| \leqslant |z-w|+|w| \Rightarrow |z|-|w| \leqslant |w-z| \end{array}\right\}$

Geometric interpretation: The len of each side of a triangle is greater than or equal to the diffce of the lens of the two other sides.

☐

• (4E 4.3) *Supp $\mathbf{F}=\mathbf{C}, \varphi \in V'$. Define $\sigma : V \to \mathbf{R}$ by $\sigma(v) = \operatorname{Re}\varphi(v)$ for each $v \in V$. Show $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$ for all $v \in V$.*

SOLUS: Notice that $\varphi(v) = \operatorname{Re}\varphi(v) + \mathrm{i}\operatorname{Im}\varphi(v) = \sigma(v) + \mathrm{i}\operatorname{Im}\varphi(v)$.

又 $\operatorname{Re}\varphi(\mathrm{i}v) = \operatorname{Re}\big(\mathrm{i}\varphi(v)\big) = -\operatorname{Im}\varphi(v) = \sigma(\mathrm{i}v)$. Hence $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$. ☐

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4 *Supp $m, n \in \mathbf{N}^+$ with $m \leqslant n$, $\lambda_1, \dots, \lambda_m \in \mathbf{F}$. Prove $\exists\, p \in \mathcal{P}(\mathbf{F})$, $\deg p = n$, the zeros of p are $\lambda_1, \dots, \lambda_m$.*

SOLUS: Let $p(z) = (z-\lambda_1)^{n-(m-1)}(z-\lambda_2)\cdots(z-\lambda_m)$. ☐

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5 *Supp $m \in \mathbf{N}$, and $z_1, \dots, z_{m+1}$ are disti in $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$. Prove $\exists!\, p \in \mathcal{P}_m(\mathbf{F}), p(z_k) = w_k$ for each $k \in \{1, \dots, m+1\}$.*

SOLUS:

Define $T \in \mathcal{L}\big(\mathcal{P}_m(\mathbf{F}), \mathbf{F}^{m+1}\big)$ by $Tq = \big(q(z_1), \dots, q(z_m), q(z_{m+1})\big)$.

We now show T is surj, so that such p exis; and that T is inje, so that such p is uniq.

Inje: $Tq = 0 \iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0 \iff q = 0$, by TIPS.

Surj: $\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m+1 = \dim \mathbf{F}^{m+1}$ 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1} \Rightarrow T$ is surj. ☐

OR. Let $p_1 = 1$, $p_k(z) = \prod_{i=1}^{k-1}(z-z_i) = (z-z_1)\cdots(z-z_{k-1})$ for each $k \in \{2, \dots, m+1\}$.

By (2.C.10), $B_p = (p_1, \dots, p_{m+1})$ is a bss of $\mathcal{P}_m(\mathbf{F})$. Let $B_e = (e_1, \dots, e_{m+1})$ be the std bss of $\mathbf{F}^{m+1}$.

NOTICE that $Tp_1 = (1, \dots, 1)$, $Tp_k = \Big(\prod_{i=1}^{k-1}(z_1-z_i), \dots, \underbrace{\prod_{i=1}^{k-1}(z_j-z_i)}_{j^{th}\text{ ent}}, \dots, \prod_{i=1}^{k-1}(z_{m+1}-z_i)\Big)$.

And that $\prod_{i=1}^{k-1}(z_j-z_i) = 0 \iff j \leqslant k-1$, becs $z_1, \dots, z_{m+1}$ are disti.

$$\text{Thus } \mathcal{M}(T, B_p, B_e) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & A_{2,2} & 0 & \cdots & 0 \\ 1 & A_{3,2} & A_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{m+1,2} & A_{m+1,3} & \cdots & A_{m+1,m+1} \end{pmatrix}.$$

Where $A_{j,k} = \prod_{i=1}^{k-1}(z_j-z_i) \neq 0$ for all $j > k-1 \geqslant 1$. The rows of $\mathcal{M}(T)$ is liney indep.

By (4E 3.C.17) 又 $\dim \mathcal{P}_m(\mathbf{F}) = \dim \mathbf{F}^{m+1}$; OR By (3.F.32); T is inv. ☐

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2 *Supp $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

SOLUS: $x^m, x^m + x^{m-1} \in U$ but $\deg\big[(x^m+x^{m-1})-(x^m)\big] \neq m \Rightarrow (x^m+x^{m-1})-(x^m) \notin U$. ☐

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3 *Supp $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : 2 \mid \deg p\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

Solus: $x^2, x^2 + x \in U$ but $\deg\big[(x^2+x) - (x^2)\big]$ is odd and hence $(x^2+x) - (x^2) \notin U$. $\square$

---

6 *Supp nonzero $p \in \mathcal{P}_m(\mathbf{F})$ has deg m. Prove*
$[P]$ p has m disti zeros $\Longleftrightarrow$ p and its deri p' have no common zeros $[Q]$.

Solus:

(a) Supp p has m disti zeros. And $\deg p = m$. By [4.14], $\exists!\, c, \lambda_i \in \mathbf{R}, p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$.
If $m = 0$, then $p = c \neq 0 \Rightarrow p$ has no zeros, and $p' = 0$, done.
If $m = 1$, then $p(z) = c(z - \lambda_1)$, and $p' = c$ has no zeros, done.
For each $j \in \{1, \dots, m\}$, let $q_j \in \mathcal{P}_{m-1}(\mathbf{F})$ be suth $p(z) = (z-\lambda_j)q_j \Rightarrow q_j(\lambda_j) \neq 0$.
Now $p'(z) = (z-\lambda_j){q_j}'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

Or. We show $\neg Q \Rightarrow \neg P$:
Supp $p(z) = (z-\lambda)q(z)$, $p'(z) = (z-\lambda)r(z)$. 又 $p'(z) = (z-\lambda)q'(z) + q(z)$.
Now $p'(\lambda) = q(\lambda) = 0 \Rightarrow q(z) = (z-\lambda)s(z), p(z) = (z-\lambda)^2 s(z)$.
Hence p has strictly less than m disti zeros.

(b) We prove $\neg P \Rightarrow \neg Q$:
Becs nonzero $p \in \mathcal{P}_m(\mathbf{F})$, we supp $\lambda_1, \dots, \lambda_M$ are all the disti zeros of p, where $M < m$.
By Pigeon Hole Principle, $\exists\, \lambda_k$ suth $p(z) = (z-\lambda_k)^2 q(z)$ for some $q \in \mathcal{P}(\mathbf{F})$.
Hence $p'(z) = 2(z-\lambda_k)q(z) + (z-\lambda_k)^2 q'(z) \Rightarrow p'(\lambda_k) = 0 = p(\lambda_k)$. $\square$

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7 *Prove every $p \in \mathcal{P}(\mathbf{R})$ of odd deg has a zero.*

Solus:

Using the nota and proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence λ_1 exis. $\square$

Or. Using calculus only. Supp $p \in \mathcal{P}_m(\mathbf{F}), \deg p = m, m$ is odd.
Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1} a_m$ by δ.
Write $p(x) = x^m \left(\dfrac{a_0}{x^m} + \dfrac{a_1}{x^{m-1}} + \cdots + \dfrac{a_{m-1}}{x} + a_m \right)$.
Thus $p(x)$ is continuous, and $\lim\limits_{x \to -\infty} p(x) = -\delta\infty$; $\lim\limits_{x\to\infty} p(x) = \delta\infty$.
Hence we conclude that p has at least one real zero. $\square$

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9 *Supp $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\bar{z})}$. Prove $q \in \mathcal{P}(\mathbf{R})$.*

Solus:

Notice that by [4.5], $\bar{z}^n = \overline{z^n}$.
Supp $q(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow q(\bar{z}) = a_n \bar{z}^n + \cdots + a_1 \bar{z} + a_0 \Rightarrow \overline{q(\bar{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\bar{z})} = \overline{p(\bar{z})}p(z) = \overline{p(\bar{z})\,\overline{\overline{p(\bar{z})}}} = \overline{q(\bar{z})}$. Hence for each $a_k, \overline{a_k} = a_k \Rightarrow a_k \in \mathbf{R}$. $\square$

Or. Supp $p(z) = a_m z^m + \cdots + a_1 z + a_0$. Now $\overline{p(\bar{z})} = \overline{a_m} z^m + \cdots + \overline{a_1} z + \overline{a_0}$.
Notice that $q(z) = p(z)\overline{p(\bar{z})} = \sum_{k=0}^{2} m \Big(\sum_{i+j=k} a_i \overline{a_j} \Big) z^k$.
Notice that by [4.5], $z - \bar{z} = 2(\operatorname{Im} z) \Rightarrow z = \bar{z} + 2(\operatorname{Im} z)$. So that $z = \bar{z} \Longleftrightarrow \operatorname{Im} z = 0 \Longleftrightarrow z \in \mathbf{R}$.
Now for each $k \in \{0, \dots, 2m\}$, $\overline{\sum_{i+j=k} a_i \overline{a_j}} = \sum_{i+j=k} \overline{a_i \overline{a_j}} = \sum_{i+j=k} a_j \overline{a_i} = \sum_{i+j=k} a_i \overline{a_j} \in \mathbf{R}$. $\square$

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8 *For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \to \mathbf{R}$ by* $(Tp)(x) = \begin{cases} \dfrac{p(x)-p(3)}{x-3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}$

Show (a) *$Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$;* (b) *$T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is liney.*

SOLUS:

(a) For $x \neq 3, T(x^n) = \dfrac{x^n - 3^n}{x-3} = \sum_{i=1}^{n} 3^{i-1}x^{n-i}$. For $x = 3, T(x^n) = n3^{n-1}$.

Note that if $x = 3$, then $\sum_{i=1}^{n} 3^{i-1}x^{n-i} = \sum_{i=1}^{n} 3^{n-1} = n3^{n-1}$.

Hence $T(x^n) = \sum_{i=1}^{n} 3^{i-1}x^{n-i} \Rightarrow T(x^n) \in \mathcal{P}(\mathbf{R})$.

(b) Now we show T is liney: $\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}$,

$$T(p+\lambda q)(x) = \left.\begin{cases} \dfrac{(p+\lambda q)(x)-(p+\lambda q)(3)}{x-3}, & \text{if } x \neq 3, \\ (p+\lambda q)'(3), & \text{if } x = 3 \end{cases}\right| = \big[T(p) + \lambda T(q)\big](x) \text{ for all } x \in \mathbf{R}.$$

□

OR. (a) Note that $\exists!\, q \in \mathcal{P}(\mathbf{R}), p(x) - p(3) = (x-3)q(x) \Rightarrow q(x) = \dfrac{p(x)-p(3)}{x-3}$.

$p'(x) = \big(p(x)-p(3)\big)' = \big((x-3)q(x)\big)' = q(x) + (x-3)q'(x)$.

Hence $p'(3) = q(3)$. Now $Tp = q \in \mathcal{P}(\mathbf{R})$.

(b) $\forall p_1, p_2 \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, \exists!\, q_1, q_2 \in \mathcal{P}(\mathbf{R})$,

$p_1(x) - p_1(3) = (x-3)q_1(x)$ and $p_2(x) - p_2(3) = (x-3)q_2(x)$.

By (a), $Tp_1 = q_1, Tp_2 = q_2$. Note that $(p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x-3)(q_1 + \lambda q_2)(x)$.

Hence by the uniqnes of $q_1 + \lambda q_2$ for $p_1 + \lambda p_2$, we must have $T(p_1 + \lambda p_2) = q_1 + \lambda q_2$. □

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11 *Supp $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*

(a) *Show* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) *Find a bss of $\mathcal{P}(\mathbf{F})/U$.*

SOLUS: NOTICE that $pq \neq p \circ q$, see (4E 3.A.10).

U is a subsp of $\mathcal{P}(\mathbf{F})$ becs $\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, ps_1 + \lambda ps_2 = p(s_1 + \lambda s_2) \in U$.

If $\deg p = 0$, then $U = \mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F})/U = \{0\}$, with the uniq bss $()$. Supp $\deg p \geqslant 1$.

(a) By [4.8], $\forall s \in \mathcal{P}(\mathbf{F}), \exists!\, r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}) \,[\, \exists!\, pq \in U \,], s = (p)q + (r)$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F})$. By the NOTE FOR [3.91] in (3.E), $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p-1}(\mathbf{F})$ are iso.

OR. Define $R : \mathcal{P}(\mathbf{F}) \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $R(s) = r$ for all $s \in \mathcal{P}(\mathbf{F})$ We show R is liney.

$\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, \exists!\, r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q_1, q_2 \in \mathcal{P}(\mathbf{F}), s_1 = (p)q_1 + (r_1);\ s_2 = (p)q_2 + (r_2)$.

又 $\exists!\, r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}), (s_1 + \lambda s_2) = (p)q + (r) = (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2)$.

Note that $r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}) \Rightarrow r_1 + \lambda r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F})$.

OR Note that $\deg(r_1 + \lambda r_2) \leqslant \max\{\deg r_1, \deg(\lambda r_2)\} \leqslant \max\{\deg r_1, \deg r_2\} < \deg p$.

By the uniqnes part of [4.8], $s = s_1 + \lambda s_2;\ r = r_1 + \lambda r_2$. Thus $R(s_1 + \lambda s_2) = R(s_1) + \lambda R(s_2)$.

Becs $Rs = 0 \iff s = pq, \exists!\, q \in \mathcal{P}(\mathbf{F}) \iff s \in U$. And $\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), Rr = r$.

Now $\operatorname{null} R = U,\ \operatorname{range} R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Hence $\tilde{R} : \mathcal{P}(\mathbf{F})/U \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ is defined by $\tilde{R}(s+U) = Rs$. By [3.91(d)], $\tilde{R}$ is iso.

(b) For each $k \in \{0, 1, \ldots, \deg p - 1\}, \tilde{R}(z^k + U) = R(z^k) = z^k \Rightarrow \tilde{R}^{-1}(z^k) = z^k + U$.

Thus $(1 + U, z + U, \ldots, z^{\deg p-1} + U)$ can be a bss of $\mathcal{P}(\mathbf{F})/U$. □

---

10 *Supp $m \in \mathbf{N}, p \in \mathcal{P}_m(\mathbf{C})$ is suth $p(x_k) \in \mathbf{R}$ for each of disti $x_0, x_1, \dots, x_m \in \mathbf{R}$. Prove $p \in \mathcal{P}(\mathbf{R})$.*

Solus:

By Tips and Exe (5), $\exists\,!\, q \in \mathcal{P}_m(\mathbf{R})$ suth $q(x_k) = p(x_k)$. Hence $p = q$. $\square$

Or. Using the Lagrange Interpolating Polynomial.

Define $q(x) = \displaystyle\sum_{j=0}^{m} \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j)$.

又 Each $x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R})$. Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$ for each x_k.
Then $(q-p)$ has $(m+1)$ zeros, while $(q-p) \in \mathcal{P}_m(\mathbf{C})$. By Tips, $q - p = 0 \Rightarrow p = q \in \mathcal{P}(\mathbf{R})$. $\square$

- (4E 4 13) *Supp nonconst $p, q \in \mathcal{P}(\mathbf{C})$ have no common zeros. Let $m = \deg p$, $n = \deg q$. Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r,s) = rp + sq$. Prove T is iso.*
  Coro: $\exists\,!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ *suth $rp + sq = 1$.*

Solus:

T is liney becs $\forall\, r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,
$T\big((r_1,s_1) + \lambda(r_2,s_2)\big) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1,s_1) + \lambda T(r_2,s_2)$.

Let $\lambda_1, \dots, \lambda_M$ and $\mu_1, \dots, \mu_N$ be the disti zeros of p and q respectly. Notice that $M \leqslant m, N \leqslant n$.

Note that the ctrapos of [4.13], $M = 0 \iff m = 0 \Rightarrow s = 0 \iff r = 0 \Leftarrow n = 0 \iff N = 0$.

Now supp $M, N \geqslant 1$. We show $s = 0$. Showing $r = 0$ is almost the same.

Write $p(z) = a(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}$. $\big(\exists\,!\,\alpha_j \geqslant 1, a \in \mathbf{F}.\big)$ Let $\max\{\alpha_1, \dots, \alpha_M\} = A$.

For each $D \in \{0, 1, \dots, A-1\}$, let $I_{D,\alpha} = \{\gamma_{D,1}, \dots, \gamma_{D,J}\}$ be suth each $\alpha_{\gamma_{D,j}} \geqslant D + 1$.

Note that $I_{A-1,\alpha} \subseteq \cdots \subseteq I_{0,\alpha} = \{1, \dots, M\}$. Becs $rp + sq = 0 \Rightarrow (rp+sq)^{(k)} = 0$ for all $k \in \mathbf{N}^+$.

We use induc by D to show $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$ for each $D \in \{0, \dots, A-1\}$.

Notice that $p^{(D)}(\lambda_\gamma) = 0$ for each $D \in \{0, \dots, A-1\}$ and each $\lambda_\gamma \in I_{D,\alpha}$. (Δ)

(i) $D = 0$. $(rp+sq)(\lambda_{\gamma_{0,j}}) = (sq)(\lambda_{\gamma_{0,j}}) = s(\lambda_{\gamma_{0,j}}) = 0$.
   $D = 1$. $(rp+sq)'(\lambda_{\gamma_{1,j}}) = (r'p + rp')(\lambda_{\gamma_{1,j}}) + (s'q + sq')(\lambda_{\gamma_{1,j}}) = (s'q)(\lambda_{\gamma_{1,j}}) = s'(\lambda_{\gamma_{1,j}}) = 0$.

(ii) $2 \leqslant D \leqslant A - 1$. Asum $s^{(d)}(\lambda_{\gamma_{d,j}}) = 0$ for each $d \in \{1, \dots, D-1\}$ and each $\lambda_{\gamma_{d,j}} \in I_{d,\alpha}$.
   $\Big($ Becs $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}. \Big)$ (Δ)

$$\begin{aligned} \text{Now } [rp+sq]^{(D)}(\lambda_{\gamma_{D,j}}) &= \big[C_D^D r^{(D)} p^{(0)} + \cdots + C_D^d r^{(d)} p^{(D-d)} + \cdots + C_D^0 r^{(0)} p^{(D)}\big](\lambda_{\gamma_{D,j}}) \\ &\quad + \big[C_D^D s^{(D)} q^{(0)} + \cdots + C_D^d s^{(d)} q^{(D-d)} + \cdots + C_D^0 s^{(0)} q^{(D)}\big](\lambda_{\gamma_{D,j}}) \\ &= \big[C_D^D s^{(D)} q^{(0)}\big](\lambda_{\gamma_{D,j}}). \text{ Where each } \lambda_{\gamma_{D,j}} \in I_{D,\alpha} \subseteq I_{D-1,\alpha}. \end{aligned}$$

   Hence $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$. The asum holds for all $D \in \{0, \dots, A-1\}$.

Notice that $\forall k = \{0, \dots, A-2\}, s^{(k)}$ and $s^{(k+1)}$ have zeros $\{\lambda_{\gamma_{k+1,1}}, \dots, \lambda_{\gamma_{k+1,J}}\}$ in common.

Now $\forall D \in \{1, A-1\}, s = s^{(0)}, \dots, s^{(D)}$ have zeros $\{\lambda_{\gamma_{D,1}}, \dots, \lambda_{\gamma_{D,J}}\}$ in common.

Thus $\forall D \in \{0, A-1\}$, $s(z)$ is divisible by $(z - \lambda_{\gamma_{D,1}})^{\alpha_{\gamma_{D,1}}} \cdots (z - \lambda_{\gamma_{D,J}})^{\alpha_{\gamma_{D,J}}}$.

Hence we write $s(z) = \Big((z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}\Big) s_0(z)$, while $\deg s \leqslant m - 1 < m = \alpha_1 + \cdots + \alpha_M$.

Thus by Tips, $s = 0$. Following the same pattern, we conclude that $r = 0$.

Hence T is inje. And $\dim\big(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})\big) = \dim \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$ is surj. Thus T is iso. $\square$

Comment: We now prove the stmt that marked by (Δ) above.

L1 *Prove* $\forall p,q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}$.

Solus:

We use induc by $k \in \mathbf{N}^+$.

(i) $k = 1$. $(pq)^{(1)} = (pq)' = C_1^1 p^{(1)} q^{(0)} + C_1^0 p^{(0)} q^{(1)}$.

(ii) $k \geqslant 2$. Asum for $(pq)^{(k-1)} = C_{k-1}^{k-1} p^{(k-1)} q^{(0)} + \cdots + C_{k-1}^j p^{(j)} q^{(k-1-j)} + \cdots + C_{k-1}^0 p^{(0)} q^{(k-1)}$.

Now $(pq)^{(k)} = \left((pq)^{(k-1)}\right)' = \left(\sum_{j=0}^{k-1} C_{k-1}^j p^{(j)} q^{(k-j-1)}\right)' = \sum_{j=0}^{k-1} \left[C_{k-1}^j \left(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)}\right)\right]$.

$$\begin{aligned}
&= \left[C_{k-1}^0 \left(p^{(1)} q^{(k-1)} + \boxed{p^{(0)} q^{(k)}}\right)\right] + \left[C_{k-1}^1 \left(p^{(2)} q^{(k-2)} + p^{(1)} q^{(k-1)}\right)\right] \\
&\quad + \cdots + \left[C_{k-1}^{j-2} \left(p^{(j-1)} q^{(k-j+1)} + p^{(j-2)} q^{(k-j+2)}\right)\right] + \left[C_{k-1}^{j-1} \left(p^{(j)} q^{(k-j)} + p^{(j-1)} q^{(k-j+1)}\right)\right] \\
&\quad + \left[C_{k-1}^{j} \left(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)}\right)\right] + \left[C_{k-1}^{j+1} \left(p^{(j+2)} q^{(k-j-2)} + p^{(j+1)} q^{(k-j-1)}\right)\right] \\
&\quad + \cdots + \left[C_{k-1}^{k-2} \left(p^{(k-1)} q^{(1)} + p^{(k-2)} q^{(2)}\right)\right] + \left[C_{k-1}^{k-1} \left(\boxed{p^{(k)} q^{(0)}} + p^{(k-1)} q^{(1)}\right)\right].
\end{aligned}$$

Hence $(pq)^{(k)} = C_k^0 p^{(0)} q^{(k)} + \cdots + \left[C_{k-1}^j + C_{k-1}^{j-1}\right] \left(p^{(j)} q^{(k-j)}\right) + \cdots + C_k^k p^{(k)} q^{(0)}$. ☐

L2 *Supp* $p(z) = (z-\lambda)^\alpha q(z)$ *and* $\alpha \in \mathbf{N}^+$. *Prove* $p^{(\alpha-1)}(\lambda) = 0$.

Solus:

Supp $p \in \mathcal{P}(\mathbf{F})$. Write $p(z) = (z-\lambda)^A q(z)$, where $A \in \mathbf{N}^+, q(\lambda) \neq 0$.

We use induc to show for all $\alpha \in \{1, \dots, A\}, p^{(\alpha-1)}(\lambda) = 0$.

(i) $\alpha = 1$. $p^{(0)}(\lambda) = 0$.

(ii) $2 \leqslant \alpha \leqslant A$. Asum $p^{(a-2)}(\lambda) = 0$ for all $a \in \{2, \dots, \alpha\}$.

Notice that $p(z) = (z-\lambda)^{\alpha-1} q_{\alpha-1}(z) = (z-\lambda)^\alpha q_\alpha(z)$, where $q_{\alpha-1}(z) = (z-\lambda) q_\alpha(z)$.

Becs $p^{(\alpha-1)}(z) = \Big[C_{\alpha-1}^{\alpha-1}(z-\lambda)^0 q_{\alpha-1}(z) + \cdots + C_{\alpha-1}^k (z-\lambda)^{\alpha-1-k} q_{\alpha-1-k}(z)$

$+ \cdots + C_{\alpha-1}^0 (z-\lambda)^{\alpha-1} {q_{\alpha-1}}^{(\alpha-1)}(z)\Big]$. Now $p^{(\alpha-1)}(\lambda) = C_{\alpha-1}^{\alpha-1} q_{\alpha-1}(\lambda) = 0$. ☐

Ended

5.A

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28
29 30 31 32 33 34 35 36 | 2E: Ch5.20 | 4E: 8 11 15 16 17 36 37 38 39

- **Note For [5.6]:**

More generally, supp we do not know whether V is finide. We show $(a) \iff (b)$.

Supp (a) λ is an eigval of T with an eigvec v. Then $(T - \lambda I)v = 0$.

Hence we get (b), $(T - \lambda I)$ is not inje. And then (d), $(T - \lambda I)$ is not inv.

But (d) $\Rightarrow$ (b) fails, becs S is not inv $\iff$ S is not inje Or S is not surj.

- **Tips:** For $T_1, \dots, T_m \in \mathcal{L}(V)$:

(a) Supp $T_1, \dots, T_m$ are all inje. Then $(T_1 \circ \cdots \circ T_m)$ is inje.

(b) Supp $(T_1 \circ \cdots \circ T_m)$ is not inje. Then at least one of $T_1, \dots, T_m$ is not inje.

(c) At least one of $T_1, \dots, T_m$ is not inje $\nRightarrow$ $(T_1 \circ \cdots \circ T_m)$ is not inje.

Exa: In infinide only. Let $V = \mathbf{F}^\infty$.

$\left.\begin{array}{l}\text{Let } S \text{ be the backward shift (surj but not inje)} \\ \text{Let } T \text{ be the forward shift (inje but not surj)}\end{array}\right\} \Rightarrow$ Then $ST = I$. ☐

• **NOTE FOR [5.2]:** Supp $T \in \mathcal{L}(V)$. Then U is invarsp of V under $T \iff \operatorname{range} T|_U \subseteq U$.

• *Supp V is finide, $T \in \mathcal{L}(V)$, and U is invarsp of V under T.*
Prove there exis invarsp W of dimension $\dim V - dim U$.

SOLUS:

Using the NOTE FOR [3.88,90,91]. Define the eraser S. Now $V = \operatorname{range} S \oplus U$.

Define E_1 by $E_1(u+w) = u$. Define E_2 by $E_2(u+w) = w$. $\left(E_2 = S \circ \pi. \right)$

Note that $T - TE_1 = T(I - E_1) = TE_2$. And $\operatorname{null} TE_2 = \operatorname{null} T \oplus U,\ \operatorname{range} T = \operatorname{range} TE_2 \oplus U$.

Becs $\dim \operatorname{null} TE_2 \geqslant \dim U \iff \dim \operatorname{range} TE_2 \leqslant \dim V - \dim U$.

Let $B_U = (u_1, \dots, u_n),\ B_{\operatorname{range} TE_2} = (v_1, \dots, v_m) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n, \dots, u_p)$.

Let $X = \operatorname{span}(v_1, \dots, v_m, u_{\alpha_1}, \dots, u_{\alpha_{p-\dim U}})$. Where $\alpha_1, \dots, \alpha_{p-\dim U} \in \{1, \dots, p\}$ are disti.

Then $\dim X = \dim V - \dim U$. $\left[\operatorname{range} TE_2 \subseteq \right] X$ is invard TE_2, by Exe (1)(b).

We have $x \in X \Rightarrow TE_2(x) \in X \Rightarrow Tx - TE_1(x) \in X \Rightarrow Tx \in X$. Hence X is invard T. $\square$

$\big($ Note that $E_1(x) \in \operatorname{span}(u_{\beta_1}, \dots, u_{\beta_t})$, where $\beta_1, \dots, \beta_t \in \{\alpha_1, \dots, \alpha_{p-\dim U}\}$ and each $u_{\beta_i} \in U$. $\big)$

COMMENT: Convly, by rev the roles of U and W, we conclude that it is true as well.

• *Supp $T \in \mathcal{L}(V)$ and U is invarsp of V under T.*
Supp $\lambda_1, \dots, \lambda_m$ are the disti eigvals of T corres eigvecs $v_1, \dots, v_m$.

• **TIPS 1:** *Prove $v_1 + \cdots + v_m \in U \iff$ each $v_k \in U$.*

SOLUS:

Supp each $v_k \in U$. Then becs U is a subsp, $v_1 + \cdots + v_m \in U$.

Define the stmt $P(k)$: if $v_1 + \cdots + v_k \in U$, then each $v_j \in U$. We use induc on m.

(i) For $k = 1$, $v_1 \in U$.

(ii) For $2 \leqslant k \leqslant m$. Asum $P(k-1)$ holds. Supp $v = v_1 + \cdots + v_k \in U$.

Then $Tv = \lambda_1 v_1 + \cdots + \lambda_k v_k \in U \Longrightarrow Tv - \lambda_k v = (\lambda_1 - \lambda_k)v_1 + \cdots + (\lambda_{k-1} - \lambda_k)v_{k-1} \in U$.

For each $j \in \{1, \dots, k-1\}, \lambda_j - \lambda_k \neq 0 \Rightarrow (\lambda_j - \lambda_k)v_j = v_j'$ is an eigvec of T corres λ_j.

By asum, each $v_j' \in U$. Thus $v_1, \dots, v_{k-1} \in U$. So that $v_k = v - v_1 - \cdots - v_{k-1} \in U$. $\square$

• **TIPS 2:** *If* $\dim V = m$. *Prove $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m)$, where* $E_k = \operatorname{span}(v_k)$.

SOLUS:

Becs $V = E_1 \oplus \cdots \oplus E_m$. $\forall u \in U, \exists!\, e_j \in E_j, u = e_1 + \cdots + e_m$.

If $e_j \neq 0$, then e_j is an eigvec corres λ_j. Othws $e_j = 0 \in U$. By TIPS (1), each nonzero $e_j \in U$.

Thus $u \in (U \cap E_1) + \cdots + (U \cap E_m) = U$. Becs each $(U \cap E_j) \subseteq E_j$.

For each $k \in \{2, \dots, n\}$, $\big((U \cap E_1) + \cdots + (U \cap E_{k-1})\big) \bigcap (U \cap E_k) \subseteq (E_1 + \cdots + E_{k-1}) \cap E_k = \{0\}$. $\square$

• **TIPS 3:** *Supp W is a nonzero invarsp of V under T. If* $\dim V = m \geqslant 1$.
Prove $W = \operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ for some disti $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$.

SOLUS:

Each $\operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ is invard T.

By TIPS (2), $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m)$. Becs each $\dim E_k = 1$, $U \cap E_k = \{0\}$ or E_k.

There must be at least one k suth $E_k = U \cap E_k$, for if not, $U = \{0\}$ since $V = E_1 \oplus \cdots \oplus E_m$.

Let $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$ be all the disti indices for which $E_k = U \cap E_k$.

Thus $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m) = E_{\alpha_1} \oplus \cdots E_{\alpha_A} = \operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$. $\square$

---

1 *Supp $T \in \mathcal{L}(V)$ and U is a subsp of V.*

(a) *If $U \subseteq \operatorname{null} T$, then U is invard T.* $\forall u \in U \subseteq \operatorname{null} T, Tu = 0 \in U$. $\square$

(b) *If $\operatorname{range} T \subseteq U$, then U is invard T.* $\forall u \in U, Tu \in \operatorname{range} T \subseteq U$. $\square$

• *Supp $S, T \in \mathcal{L}(V)$ are suth $ST = TS$.*

(a) *Prove* $\operatorname{null}(T - \lambda I)$ *is invard S for any $\lambda \in \mathbf{F}$.*

(b) *Prove* $\operatorname{range}(T - \lambda I)$ *is invard S for any $\lambda \in \mathbf{F}$.*

SOLUS:

Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.

(a) $(T - \lambda I)(v) = 0 \Rightarrow (T - \lambda I)(Sv) = \big(S(T - \lambda I)\big)(v) = 0$.

(b) $(T - \lambda I)(u) = v \in \operatorname{range}(T - \lambda I) \Rightarrow Sv = \big(S(T - \lambda I)\big)(u) = (T - \lambda I)(Su) \in \operatorname{range}(T - \lambda I)$. $\square$

• *Supp $S, T \in \mathcal{L}(V)$ are suth $ST = TS$.*

2 *Show $W = \operatorname{null} T$ is invard S.* $\forall u \in W, Tu = 0 \Rightarrow STu = 0 = TSu \Rightarrow Su \in W$. $\square$

3 *Show $U = \operatorname{range} T$ is invard S.* $\forall w \in U, \exists\, v \in V, Tv = w, TSv = STv = Sw \in U$. $\square$

• *Supp $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are invarsps of V under T.*

4 $\forall v_i \in V_i, Tv_i \in V_i \Rightarrow \forall v = v_1 + \cdots + v_m \in V_1 + \cdots + V_m, Tv = Tv_1 + \cdots + Tv_m \in V_1 + \cdots + V_m$. $\square$

5 $\forall v \in \bigcap\limits_{i=1}^{m} V_i, Tv \in V_i, \forall i \in \{1, \ldots, m\} \Rightarrow Tv \in \bigcap\limits_{i=1}^{m} V_i$. Thus $\bigcap\limits_{i=1}^{m} V_i$ is invard T. $\square$

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6 *Supp U is invarsp of V under each $T \in \mathcal{L}(V)$. Show $U = \{0\}$ or $U = V$.*

SOLUS: If $V = \{0\}$. Then done. Supp $V \neq \{0\}$. We show the ctrapos:

Supp $U \neq \{0\}$ and $U \neq V$. Prove $\exists\, T \in \mathcal{L}(V)$ suth U is not invard T.

Let W be suth $V = U \oplus W$. Define $T \in \mathcal{L}(V)$ by $T(u + w) = w$. $\square$

• **TIPS:** Supp $T \in \mathcal{L}(\mathbf{R}^2)$ is the counterclockwise rotation by the angle $\theta \in \mathbf{R}$.

Define $\mathcal{C} \in \mathcal{L}(\mathbf{R}^2, \mathbf{C})$ by $\mathcal{C}(a, b) = a + ib = r(\cos\alpha + \mathrm{i}\sin\alpha) \Rightarrow a = r\cos\alpha, b = r\sin\alpha$, where $r = a^2 + b^2$.

Then $(\cos\theta + \mathrm{i}\sin\theta)(a + ib) = r\big(\cos(\alpha + \theta) + \mathrm{i}\sin(\alpha + \theta)\big) = \mathcal{C}^{-1}T(a, b)$.

Hence $T(a, b) = \big(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta\big)$. Now $\mathcal{M}(T) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

EXA: OR **7** Supp $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find all eigvals of T.

NOTICE that $\mathcal{M}(T) = \begin{pmatrix} \cos 90^\circ & -3\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$. By [5.8](a), we conclude that T has no eigvals.

OR. Supp λ is an eigval with an eigvec (x, y). Then $(\lambda x, \lambda y) = (-3y, x) \Rightarrow -3y = \lambda^2 y \Rightarrow \lambda^2 = -3$.

[Ignoring the possibility of $y = 0$, becs $x = 0 \iff y = 0$.] $\square$

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8 *Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigvals and eigvecs.*

SOLUS: Supp λ is an eigval with an eigvec (w, z). Then $z = \lambda w$ and $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, ignoring the possibility of $z = 0$ ($z = 0 \iff w = 0$).

Hence $\lambda_1 = -1$ and $\lambda_2 = 1$ are all the eigvals of T. And $T(z, z) = (z, z), T(z, -z) = (-z, z)$.

又 $\dim \mathbf{F}^2 = 2$. Thus the set of all eigvecs is $\{(z, z), (z, -z) : z \neq 0\}$. $\square$

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9 *Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigvals and eigvecs.*

Solus: Supp λ is an eigval with an eigvec (z_1, z_2, z_3).
Then $(2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$. We discuss in two cases:
For $\lambda = 0$, $z_2 = z_3 = 0$ and z_1 can be arb ($z_1 \neq 0$).
For $\lambda \neq 0$, $z_2 = 0 = z_1$, and z_3 can be arb ($z_3 \neq 0$), then $\lambda = 5$.
The set of all eigvecs is $\{(0, 0, w), (w, 0, 0) : w \neq 0\}$. □

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10 *Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$*
(a) *Find all eigvals and eigvecs;* (b) *Find all invarsps of V under T.*

Solus:
(a) Supp $x = (x_1, x_2, x_3, \dots, x_n)$ is an eigvec with an eigval λ.
Then $Tx = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$.
Hence $1, \dots, n$ of len $\dim \mathbf{F}^n$ are all the eigvals.
And $\{(0, \dots, 0, x_k, 0, \dots, 0) \in \mathbf{F}^n : x_k \neq 0, k = 1, \dots, n\}$ is the set of all eigvecs.
(b) Let $(e_1, \dots, e_n)$ be the std bss of $\mathbf{F}^n$. Let $V_k = \operatorname{span}(e_k)$. Then $V_1, \dots, V_n$ are invard T.
Hence by Tips (3), every sum of $V_1, \dots, V_n$ is a invarsp of V under T. □

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18 *Define $T \in \mathcal{L}(\mathbf{F}^\infty)$ by $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$. Show T has no eigvals.*

Solus: Supp λ is an eigval of T with an eigvec $(z_1, z_2, \dots)$.
Then $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$. Thus $\lambda z_1 = 0, \lambda z_k = z_{k-1}$.
If $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \dots = z_k \Rightarrow (z_1, z_2, \dots) = 0 \Longrightarrow 0$ is not an eigval.
If $\lambda \neq 0$, then $\lambda z_1 = 0 \Rightarrow z_1 = \dots = z_k = 0 \Longrightarrow \lambda$ is not an eigval. Now no $\lambda \in \mathbf{F}$ is an eigval. □

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19 *Supp $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$.*
In other words, the ent of $\mathcal{M}(T)$ wrto the std bss are all 1's.
Find all eigvals and eigvecs of T.

Solus:
Supp λ is an eigval of T with an eigvec $(x_1, \dots, x_n)$.
Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$.
Thus $\lambda x_1 = \dots = \lambda x_n = x_1 + \dots + x_n$.

$$\left.\begin{array}{r} \text{For } \lambda = 0, x_1 + \dots + x_n = 0 \\ \text{For } \lambda \neq 0, x_1 = \dots = x_n \Longrightarrow \lambda x_k = n x_k \end{array}\right\} \Rightarrow 0, n \text{ are the eigvals of } T.$$

And the set of all eigvecs of T is $\{(x_1, \dots, x_n) \in \mathbf{F}^n \backslash \{0\} : x_1 + \dots + x_n = 0 \vee x_1 = \dots = x_n\}$. □

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20 *Define $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.*
(a) *Show every elem of $\mathbf{F}$ is an eigval of S;* (b) *Find all eigvecs of S.*

Solus:
Supp λ is an eigval of S with an eigvec $(z_1, z_2, \dots)$.
Then $S(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$. Thus for each $k \in \mathbf{N}^+, \lambda z_k = z_{k+1}$.
If $\lambda = 0$, then $\lambda z_1 = z_2 = \dots = z_k = 0$ for all k, while z_1 can be nonzero. Thus 0 is an eigval.
If $\lambda \neq 0$, then $\lambda^k z_1 = \lambda^{k-1} z_2 = \dots = \lambda z_k = z_{k+1}$, let $z_1 \neq 0 \Longrightarrow (1, \lambda, \lambda^2, \dots, \lambda^k, \dots)$ is an eigvec.
Now each $\lambda \in \mathbf{F}$ is an eigval of T, with corres eigvecs in $\operatorname{span}((1, \lambda, \lambda^2, \dots, \lambda^k, \dots))$. □

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11 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigvals and eigvecs.*

SOLUS:

Note that $\forall p \in \mathcal{P}(\mathbf{R})\backslash\{0\}, \deg p' < \deg p$. And $\deg 0 = -\infty$. Supp λ is an eigval with an eigvec p.
Asum $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p = p'$. Ctradic. Thus $\lambda = 0$.
Therefore $\deg \lambda p = -\infty = \deg p' \Rightarrow p \in \mathcal{P}_0(\mathbf{R})$. Hence the eigvecs are all the nonzero consts. $\square$

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12 *Define $T \in \mathcal{L}\big(\mathcal{P}_n(\mathbf{R})\big)$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigvals and eigvecs.*

SOLUS:

Supp λ is an eigval of T with an eigvec p, then $(Tp)(x) = xp'(x) = \lambda p(x)$.
Let $p = a_0 + a_1 x + \cdots + a_n x^n$. Then $xp'(x) = a_1 x + 2a_2 x^2 + \cdots + na_n x^n = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n$.
Define $S \in \mathcal{L}(\mathbf{F}^{n+1}, \mathcal{P}_n(\mathbf{R}))$ by $S(a_0, a_1, \dots, a_n) = a_0 + a_1 x + \cdots + a_n x^n$.
Then $(S^{-1}TS)(a_0, a_1, \dots, a_n) = (0 \cdot a_0, 1 \cdot a_1, 2 \cdot a_2, \dots, n \cdot a_n)$. Thus $0, 1, \dots, n$ are the eigvals of $S^{-1}TS$.
By Exe (15), $0, 1, \dots, n$ are the eigvals of T. The set of all eigvecs is $\{cx^\lambda : c \neq 0, \lambda = 0, 1, \dots, n\}$. $\square$

• *Supp V is finide, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$.*

13 *Prove $\forall \lambda \in \mathbf{F}, \exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}, (T - \alpha I)$ is inv.*

SOLUS:

Let $\alpha_k \in \mathbf{F}$ be suth $|\alpha_k - \lambda| = \frac{1}{1000+k}$ for each $k = 1, \dots, \dim V + 1$.
Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ disti eigvals.
Hence $\exists\, k = 1, \dots, \dim V + 1$ suth α_k is not an eigval of T and therefore $(T - \alpha_k I)$ is inv. $\square$

• (4E 5.A.11) *Prove $\exists\, \delta > 0$ suth $(T - \alpha I)$ is inv for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$.*

SOLUS:

If T has no eigvals, then $(T - \alpha I)$ is inje for all $\alpha \in \mathbf{F}$ and done.
Supp $\lambda_1, \dots, \lambda_m$ are all the disti eigvals of T.
Let $\delta > 0$ be suth, for each eigval λ_k, $\lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.
So that for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$, $(T - \alpha I)$ is not inje. $\square$

OR. Let $\delta = \min\{|\lambda - \lambda_k| : k \in \{1, \dots, m\}, \lambda_k \neq \lambda\}$.
Then $\delta > 0$ and each $\lambda_k \neq \alpha$ $\big[\iff (T - \alpha I)$ is inv $\big]$ for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$. $\square$

• (5.B.4 OR 4E 3.B.27) *Supp λ is an eigval of $P \in \mathcal{L}(V), P^2 = P$. Prove $\lambda = 0$ or $\lambda = 1$.*

SOLUS: Supp λ is an eigval with an eigvec v. Then $P(Pv) = Pv \Rightarrow \lambda^2 v = \lambda v$. Thus $\lambda = 1$ or 0. $\square$

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14 *Supp $V = U \oplus W$, where U and W are nonzero subsps of V.*
Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.
Find all eigvals and eigvecs of P.

SOLUS:

Supp λ is an eigval of P with an eigvec $(u + w)$.
Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$.
OR. Note that $P|_{\operatorname{range} P} = I|_{\operatorname{range} P} \iff P^2 = P$. By (4E 5.A.8), 1 and 0 are the eigvals.
By [1.44], $(\lambda - 1)u = \lambda w = 0$, hence $\lambda = 0 \iff u = 0$, and $\lambda = 1 \iff w = 0$.
Thus $Pu = u, Pw = 0$. Hence the eigvals are 0 and 1, the set of all eigvecs of P is $U \cup W$. $\square$

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15 *Supp $T \in \mathcal{L}(V)$. Supp $S \in \mathcal{L}(V)$ is inv.*

(a) *Prove T and $S^{-1}TS$ have the same eigvals.*

(b) *What is the relationship between the eigvecs of T and the eigvecs of $S^{-1}TS$?*

SOLUS:

(a) λ is an eigval of T with an eigvec $v \Rightarrow S^{-1}TS(\underline{S^{-1}v}) = S^{-1}Tv = S^{-1}(\lambda v) = \underline{\lambda S^{-1}v}$.

λ is an eigval of $S^{-1}TS$ with an eigvec $v \Rightarrow S(S^{-1}TS)v = T\underline{Sv} = \underline{\lambda Sv}$.

OR. Note that $S(S^{-1}TS)S^{-1} = T$. Hence every eigval of $S^{-1}TS$ is an eigval of $S(S^{-1}TS)S^{-1} = T$.

OR. $Tv = \lambda v \iff (TS)(u) = \lambda Su \iff (S^{-1}TS)(u) = \lambda u$. Where $v = Su$.

$(S^{-1}TS)(u) = \lambda u \iff (S^{-1}T)(v) = \lambda S^{-1}v \iff Tv = \lambda v$. Where $u = S^{-1}v$.

(b) Becs λ is an eigval of $T \iff \lambda$ is an eigval of $S^{-1}TS$.

(See [5.36].) Now $E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\}$; $E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}$. □

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17 *Give an exa of an optor on $\mathbf{R}^4$ that has no real eigvals.*

SOLUS:

Let (e_1, e_2, e_3, e_4) be the std bss of $\mathbf{R}^4$.

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$.

Supp λ is an eigval of T with an eigvec (x, y, z, w). Then we get $\begin{cases} (1-\lambda)x + y + z + w = 0, \\ -x + (1-\lambda)y - z - w = 0, \\ 3x + 8y + (11-\lambda)z + 5w = 0, \\ 3x - 8y - 11z + (5-\lambda)w = 0. \end{cases}$

This set of liney equations has no solutions.

[You can type it on https://zh.numberempire.com/equationsolver.php to check.]

OR. Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Supp λ is an eigval of T with an eigvec (x, y, z, w).

Then $T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \implies \begin{cases} -y = \lambda x, x = \lambda y \implies -xy = \lambda^2 xy \\ -w = \lambda z, z = \lambda w \implies -zw = \lambda^2 zw \end{cases}$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Othws, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, ctradic.

Simlr, $y = z = w = 0$. Then we fail. Thus T has no eigvals. □

• (4E 5.A.16) *Supp $B_V = (v_1, \dots, v_n), T \in \mathcal{L}(V), \mathcal{M}(T, (v_1, \dots, v_n)) = A$.*
Prove if λ is an eigval of T, then $|\lambda| \leqslant n \max\{|A_{j,k}| : 1 \leqslant j, k \leqslant n\}$.

SOLUS:

Supp v is an eigval of T corres to λ. Let $v = c_1 v_1 + \cdots + c_n v_n$.

Becs $\lambda c_1 v_1 + \cdots + \lambda c_n v_n = c_1 T v_1 + \cdots + c_n T v_n = \sum_k^n c_k \left(\sum_j^n A_{j,k} v_j \right)$.

We have $\lambda c_j = \sum_{k=1}^n c_k A_{j,k} \implies |\lambda||c_j| = \sum_{k=1}^n |c_k||A_{j,k}|$ for each $j \in \{1, \dots, n\}$

Let $|c_j| = \max\{|c_1|, \dots, |c_n|\}$. Note that $|c_j| \neq 0$, for if not, $c_1 = \cdots = c_n = 0 \Rightarrow v = 0$, ctradic.

Let $M = \max\{|A_{j,k}| : 1 \leqslant j, k \leqslant n\}$. Note that for each j, $\sum_{k=1}^n |A_{j,k}| \leqslant \sum_{k=1}^n M = nM$.

Thus $|\lambda||c_j| = \sum_{k=1}^n |c_k||A_{j,k}| \implies |\lambda| \leqslant \sum_{k=1}^n |A_{j,k}| \dfrac{|c_k|}{|c_j|} \leqslant \sum_{k=1}^n |A_{j,k}| \leqslant nM$. □

• (4E 5.A.15) *Supp* $T \in \mathcal{L}(V)$, *and* $\lambda \in \mathbf{F}$.
Show λ *is an eigval of* $T \iff \lambda$ *is an eigval of the dual optor* $T' \in \mathcal{L}(V')$.

SOLUS:

(a) Supp λ is an eigval of T with an eigvec v.
Let U be invar suth $V = \operatorname{span}(v) \oplus U$ $\big[$ by (4E 5.A.39) $\big]$.
Define $\psi \in V'$ by $\psi(cv+u) = c$.
Now $\big[T'(\psi)\big](cv+u) = \psi\big(cv + Tu\big) = \lambda cv = \lambda\psi(cv+u)$. Hence $T'(\psi) = \lambda\psi$.

(b) Supp λ is an eigval T' with an eigvec ψ. Then $T'(\psi) = \psi \circ T = \lambda\psi$.
Note that $\psi \neq 0, \psi(Tv) = \lambda\psi(v)$ Thus $\exists\, v \in V\backslash\{0\}, Tv = \frac{\psi(Tv)}{\psi(v)}v = \lambda v$. $\square$

OR. [*Only in Finide*] Using [5.6], (4E 3.F.17), [3.101] and (3.F.12).
λ is an eigval of $T \iff (T - \lambda I_V)$ is not inv
$\iff (T - \lambda I_V)' = T' - \lambda I_{V'}$ is not inv $\iff \lambda$ is an eigval of T'. $\square$

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24 *Supp* $A \in \mathbf{F}^{n,n}$. *Define* $T \in \mathcal{L}(\mathbf{F}^{n,1})$ *by* $Tx = Ax$.
(a) *Supp the sum of the ent in each row of* A *equals* 1. *Prove* 1 *is an eigval of* T.
(b) *Supp the sum of the ent in each col of* A *equals* 1. *Prove* 1 *is an eigval of* T.

SOLUS:

Supp λ is an eigval of T with an eigvec x. Then $Tx = Ax = \begin{pmatrix} \sum_{k=1}^{n} A_{1,k}x_k \\ \vdots \\ \sum_{k=1}^{n} A_{n,k}x_k \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

(a) Supp $\sum_{r=1}^{n} A_{R,c} = 1$ for each $R \in \{1, \dots, n\}$.
Then if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, and hence is an eigval of T.

(b) Supp $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C \in \{1, \dots, n\}$.
Then $\sum_{r=1}^{n} (Ax)_{r,\cdot} = \sum_{r=1}^{n} (Ax)_{r,1} = \sum_{c=1}^{n} (A_{1,c} + \cdots + A_{n,c})x_c = \sum_{c=1}^{n} x_c = \lambda(x_1 + \cdots + x_n)$.
Hence $\lambda = 1$ for all $x \in \mathbf{F}^{n,1}$ suth $\sum_{c=1}^{n} x_{c,1} \neq 0$. $\square$

OR. We show $(T - I)$ is not inv, so that $\lambda = 1$ is an eigval.
Becs $(T-I)x = (A-I)x = \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^{n} A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.
Then $y_1 + \cdots + y_n = \sum_{r=1}^{n}\sum_{c=1}^{n} (A_{r,c}x_c - x_r) = \sum_{c=1}^{n} x_c \sum_{r=1}^{n} A_{r,c} - \sum_{r=1}^{n} x_r = 0$.
Thus range $(T-I) \subseteq \Big\{ \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}^t \in \mathbf{F}^{n,1} : y_1 + \cdots + y_n = 0 \Big\}$. Hence $(T-I)$ is not surj. $\square$

OR. Let $(e_1, \dots, e_n)$ be the std bss of $\mathbf{F}^{n,1}$. Define $\psi \in (\mathbf{F}^{n,1})'$ by $\psi(e_k) = 1$.
Thus $\big(\psi \circ (T-I)\big)(e_k) = \psi\Big(\big(\sum_{j=1}^{n} A_{j,k}e_j \big) - e_k \Big) = \big(\sum_{j=1}^{n} A_{j,k} \big) - 1 = 0$.
Which means that $\psi \circ (T-I) = 0$. 又 $\psi \neq 0$. Hence $(T-I)$ is not inje. $\square$

OR. Define $S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Sx = A^t x$. Becs the rows of A^t are the cols of A.
Now by (a), 1 is an eigval of S. Let $(\varphi_1, \dots, \varphi_n)$ be the dual bss of $(e_1, \dots, e_n)$.
Define $\Phi \in \mathcal{L}\Big(\mathbf{F}^{n,1}, (\mathbf{F}^{1,n})'\Big)$ by $\Phi(e_k) = \varphi_k$. Note that $\mathcal{M}(T') = A^t$.
Now $(\Phi^{-1}T'\Phi)(e_k) = (\Phi^{-1}T')(\varphi_k) = \Phi^{-1}\big(\sum_{j=1}^{n} A_{k,j}\varphi_j \big) = \sum_{j=1}^{n} A_{k,j}e_j = A^t e_k = Se_k$.
Thus 1 is an eigval of $S = \Phi^{-1}T'\Phi$, so of T', $\big[$ by Exe (15) $\big]$, so of T, $\big[$ by (4E 5.A.15) $\big]$. $\square$

• *Supp* $A \in \mathbf{F}^{n,n}$. *Define* $T \in \mathcal{L}(\mathbf{F}^{1,n})$ *by* $Tx = xA$.
(a) *Supp the sum of the ent in each col of* A *equals* 1. *Prove* 1 *is an eigval of* T.
(b) *Supp the sum of the ent in each row of* A *equals* 1. *Prove* 1 *is an eigval of* T.

SOLUS:

Supp λ is an eigval with an eigvec x. Then $\left(\sum_{r=1}^{n} x_r A_{r,1} \quad \cdots \quad \sum_{r=1}^{n} x_r A_{r,n}\right) = \lambda\left(x_1 \quad \cdots \quad x_n\right)$.

(a) Supp $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C \in \{1, \dots, n\}$.
Thus if $x_1 = \cdots = x_n$, then $\lambda = 1$, hence is an eigval of T.

(b) Supp $\sum_{c=1}^{n} A_{R,c} = 1$ for each $R \in \{1, \dots, n\}$.
Thus $\sum_{c=1}^{n} (xA)_{\cdot,c} = \sum_{c=1}^{n} (A_{c,1} + \cdots + A_{c,n}) x_c = \sum_{c=1}^{n} x_c = \lambda(x_1 + \cdots + x_n)$.
Hence $\lambda = 1$, for all x suth $\sum_{r=1}^{n} x_{1,r} \neq 0$. □

OR. We show $(T - I)$ is not inv, so that $\lambda = 1$ is an eigval.
Becs $(T - I)x = x(A - \mathcal{M}(I)) = \left(\sum_{c=1}^{n} x_c A_{c,1} - x_1 \quad \cdots \quad \sum_{c=1}^{n} x_c A_{c,n} - x_n\right) = \left(y_1 \quad \cdots \quad y_n\right)$.
Then $y_1 + \cdots + y_n = \sum_{c=1}^{n} \sum_{r=1}^{n} (x_r A_{r,c} - x_c) = \sum_{r=1}^{n} x_r \sum_{c=1}^{n} A_{r,c} - \sum_{c=1}^{n} x_c = 0$.
Thus range $(T - I) \subseteq \left\{ \left(y_1 \quad \cdots \quad y_n\right) \in \mathbf{F}^{1,n} : y_1 + \cdots + y_n = 0 \right\}$. Hence $(T - I)$ is not surj. □

OR. Let $(e_1, \dots, e_n)$ be the std bss of $\mathbf{F}^{1,n}$. Define $\psi \in (\mathbf{F}^{n,1})'$ by $\psi(e_k) = 1$.
Becs $Te_k = e_k A = \left(A_{k,1} \quad \cdots \quad A_{k,n}\right) = \sum_{i=1}^{n} A_{k,i} e_i$. **CORO:** $\mathcal{M}(T) = A^t$.
$\left(\psi \circ (T - I)\right)(e_k) = \left(\sum_{i=1}^{n} A_{k,i}\right) - 1 = 0$. Then $\psi \circ (T - I) = 0$. 又 $\psi \neq 0$. $(T - I)$ is not inje. □

OR. Define $S \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Sx = xA^t$. Becs the rows of A are the cols of A^t.
Now by (a), 1 is an eigval of S. Let $(\varphi_1, \dots, \varphi_n)$ be the dual bss of $(e_1, \dots, e_n)$.
Define $\Phi \in \mathcal{L}\left(\mathbf{F}^{1,n}, (\mathbf{F}^{1,n})'\right)$ by $\Phi(e_k) = \varphi_k$. Becs $\left[T'(\varphi_k)\right](e_j) = \varphi_k\left(\sum_{i=1}^{n} A_{j,i} e_i\right) = A_{j,k}$.
By (3.F.9), $T'(\varphi_k) = \sum_{j=1}^{n} A_{j,k} \varphi_j$. **CORO:** $\mathcal{M}(T') = A = \mathcal{M}(T)^t$. FIXME: $\mathcal{M}(T) e_k = A^t e_k = e_k A$
Now $(\Phi^{-1} T' \Phi)(e_k) = (\Phi^{-1} T')(\varphi_k) = \Phi^{-1}\left(\sum_{j=1}^{n} A_{j,k} \varphi_j\right) = \sum_{j=1}^{n} A_{j,k} e_j = e_k A^t = Se_k$.
Thus 1 is an eigval of $S = \Phi^{-1} T' \Phi$, so of T', [by Exe (15)], so of T, [by (4E 5.A.15)]. □

• *Supp* $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.
(a) [OR (9.11)] $\lambda \in \mathbf{R}$. *Prove* λ *is an eigval of* $T \iff \lambda$ *is an eigval of* T_{C}.
(b) [OR **16** OR [9.16]] $\lambda \in \mathbf{C}$. *Prove* λ *is an eigval of* $T_{\mathrm{C}} \iff \overline{\lambda}$ *is an eigval of* T_{C}.

SOLUS:

(a) Supp λ is an eigval of T with an eigvec v.
Then $Tv = \lambda v \Longrightarrow T_{\mathrm{C}}(v + \mathrm{i}0) = Tv + \mathrm{i}T0 = \lambda v$. Thus λ is an eigval of T_{C}.
Supp λ is an eigval of T_{C} with an eigvec $v + \mathrm{i}u$.
Then $T_{\mathrm{C}}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \Longrightarrow Tv = \lambda v, Tu = \lambda u$. Thus λ is an eigval of T.
(Note that $v + \mathrm{i}u$ is nonzero $\iff$ at least one of v, u is nonzero).

(b) Supp λ is an eigval of T_{C} with an eigvec $v + \mathrm{i}u$. Then $T_{\mathrm{C}}(v + \mathrm{i}u) = Tv + \mathrm{i}Tu = \lambda(v + \mathrm{i}u)$.
Note that $\overline{T_{\mathrm{C}}(v + \mathrm{i}u)} = \overline{Tv + \mathrm{i}Tu} = Tv - \mathrm{i}Tu = T_{\mathrm{C}}(v - \mathrm{i}u) = T_{\mathrm{C}}(\overline{v + \mathrm{i}u})$.
And that $\overline{\lambda(v + \mathrm{i}u)} = \overline{\lambda}v - \mathrm{i}\overline{\lambda}u = \overline{\lambda}(v - \mathrm{i}u) = \overline{\lambda}(\overline{v + \mathrm{i}u})$.
Hence $\overline{\lambda}$ is an eigval of T_{C}. To prove the other direction, notice that $\overline{\overline{\lambda}} = \lambda$. □

OR. Supp $\lambda = a + \mathrm{i}b$ is an eigval of T_{C} with an eigvec $v + \mathrm{i}u$.
Becs $T_{\mathrm{C}}(v + \mathrm{i}u) = \lambda(v + \mathrm{i}u) = (av - bu) + \mathrm{i}(au + bv) = Tv + \mathrm{i}Tu \Longrightarrow Tv = av - bu, Tu = au + bv$.
Now $T_{\mathrm{C}}(\overline{v + \mathrm{i}u}) = Tv - \mathrm{i}Tu = (av - bu) - \mathrm{i}(au + bv) = (a - \mathrm{i}b)(v - \mathrm{i}u) = \overline{\lambda}(\overline{v - \mathrm{i}u})$. Simlr. □

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21 *Supp $T \in \mathcal{L}(V)$ is inv.*

(a) *Supp $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove λ is an eigval of $T \iff \lambda^{-1}$ is an eigval of T^{-1}.*

(b) *Prove T and T^{-1} have the same eigvecs.*

Solus: (a) $Tv = \lambda v \iff v = \lambda T^{-1}v \iff \lambda^{-1}v = T^{-1}v$. Where $v \neq 0$.

(b) Notice that T is inv $\Longrightarrow 0$ is not an eigval of T or T^{-1}. By (a), immed. □

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22 *Supp $T \in \mathcal{L}(V)$ and $\exists$ nonzero vecs u, w in V suth $Tu = 3w,\ Tw = 3u$.*
Prove 3 or -3 is an eigval of T.

Solus: $T(u+w) = 3(u+w),\ T(u-w) = 3(w-u) = -3(u-w)$. Note that $u - w \neq 0$ or $u + w \neq 0$.

Or. $T(Tu) = 9u \Rightarrow T^2 - 9 = (T - 3I)(T + 3I)$ is not injective $\Rightarrow 3$ or -3 is an eigval. □

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23 *Supp $S, T \in \mathcal{L}(V)$. Prove ST and TS have the same eigvals.*

Solus: Supp λ is an eigval of ST with an eigvec v. Then $T(STv) = \lambda Tv = TS(Tv)$.

If $Tv = 0$ (while $v \neq 0$), then T is not inje $\Rightarrow (TS - 0I)$ and $(ST - 0I)$ are not inje.

Thus $\lambda = 0$ is an eigval of ST and TS with the same eigvec v.

Othws, $Tv \neq 0$, then λ is an eigval of TS. Reversing the roles of T and S. □

• (2E 20) *Supp $T \in \mathcal{L}(V)$ has $\dim V$ disti eigvals and $S \in \mathcal{L}(V)$ has the same eigvecs (but might not with the same eigvals). Prove $ST = TS$.*

Solus: Let $n = \dim V$. For each $j \in \{1, \dots, n\}$, let v_j be an eigvec with eigval λ_j of T and α_j of S.

Then $B_V = (v_1, \dots, v_n)$. Becs $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$ for each j. Hence $ST = TS$. □

• (4E 5.A.37) *Supp V is finide and $T \in \mathcal{L}(V)$.*
Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(S) = TS$ for each $S \in \mathcal{L}(V)$.
Prove the set of eigvals of T equals the set of eigvals of $\mathcal{A}$.

Solus:

(a) Supp λ is an eigval of T with an eigvec $v = v_1$. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.

Note that $\operatorname{span}(v) \subseteq \operatorname{null}(T - \lambda I)$. Define $S \in \mathcal{L}(V)$ by $S(v_j) = v$ for each $j \in \{1, \dots, n\}$.

Or. Define $S \in \mathcal{L}(V)$ by $Sv_1 = v_1,\ Sv_j = 0$ for $j \geqslant 2$. Then $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$.

Then $(T - \lambda I)S = 0$. Thus $\mathcal{A}(S) = TS = \lambda S$ while $S \neq 0$. Hence λ is an eigval of $\mathcal{A}$.

(b) Supp λ is an eigval of $\mathcal{A}$ with an eigvec S.

Then $\exists\, v \in V, 0 \neq u = S(v) \in V \Rightarrow Tu = (TS)v = (\lambda S)v = \lambda u$. Thus λ is an eigval T.

Or. Becs $TS - \lambda S = (T - \lambda I)S = 0 \Rightarrow \{0\} \subsetneq \operatorname{range} S \subseteq \operatorname{null}(T - \lambda I)$. $(T - \lambda I)$ is not inje. □

Comment: If $\mathcal{A}(S) = ST, \forall S \in \mathcal{L}(V)$. Then the eigvals of $\mathcal{A}$ are not the eigvals of T.

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25 *Supp $T \in \mathcal{L}(V)$ and u, w are eigvecs of T suth $u + w$ is also an eigvec of T.*
Prove u and w corres to the same eigval.

Solus: Supp $\lambda_1, \lambda_2, \lambda_0$ are eigvals of T with eigvecs to $u, w, u + w$ respectly.

Then $T(u + w) = \lambda_0(u + w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$.

If (u, w) is linely depe, then let $w = cu$, therefore $\lambda_2 cu = Tw = cTu = \lambda_1 cu \Rightarrow \lambda_2 = \lambda_1$.

Othws, (u, w) is liney indep. Then $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. □

Or. Asum $\lambda_1 \neq \lambda_2$. Then (u, w) is liney indep. Thus $\lambda_0 - \lambda_1 = \lambda_0 - \lambda_2$. Ctradic. □

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26 *Supp $T \in \mathcal{L}(V)$ is suth every nonzero vec in V is an eigvec of T.*
Prove T is a scalar multi of the id optor.

SOLUS: If $\dim V = 0, 1$ then done. Supp $\dim V \geqslant 2$.

Becs $\forall v \in V, \exists ! \lambda_v \in \mathbf{F}, Tv = \lambda_v v$. For any two disti nonzero vecs $v, w \in V$,
$T(v+w) = \lambda_{v+w}(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$. □

OR. For any two nonzero vecs $u, v \in V, u, v$ are eigvecs.
If $u + v \neq 0$, then $u + v$ is also an eigvec. Othws, $u + v = 0$, then $Tu = -Tv = \lambda u = -\lambda v$.
Thus by Exe (25), $\forall u, v \in V, Tu = \lambda u, Tv = \lambda v \Rightarrow \forall v \in V, Tv = \lambda v$. □

27, 28 *Supp V is finide and $k \in \{1, \dots, \dim V - 1\}$.*
Supp $T \in \mathcal{L}(V)$ is suth every subsp of V of dim k is invard T.
Prove T is a scalar multi of the id optor.

SOLUS: If $\dim V \leqslant 1$ then done. Supp $\dim V \geqslant 2$.

We prove the ctrapos: *If T is not a scalar multi of I. Then $\exists$ subsp U of dim k not invard T.*
By Exe (26), $\exists\, v \in V$ and $v \neq 0$ suth v is not an eigvec of T.
Thus (v, Tv) is liney indep. Extend to $B_V = (v, Tv, u_1, \dots, u_n)$.
Let $U = \operatorname{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$ is not invarsp of V under T. □

OR. Supp $0 \neq v = v_1 \in V$. Extend to $B_V = (v_1, \dots, v_n)$. Supp $Tv_1 = c_1 v_1 + \cdots + c_n v_n, \exists ! c_i \in \mathbf{F}$.
Consider a k-dim subsp $U = \operatorname{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$. Where $\alpha_1, \dots, \alpha_{k-1} \in \{2, \dots, n\}$ are disti.
Becs every subsp such U is invar. $Tv_1 = c_1 v_1 + \cdots + c_n v_n \in U \Longrightarrow c_2 = \cdots = c_n = 0$.
For if not, $\exists\, c_i \neq 0$, let $W = \operatorname{span}(v_1, v_{\beta_1}, \dots, v_{\beta_{k-1}})$, where each $\beta_j \in \{2, \dots, i-1, i+1, \dots, n\}$.
Hence $Tv_1 = c_1 v_1$. Becs $v_1 = v \in V$ is arb. We conclude that $T = \lambda I$ for some $\lambda \in \mathbf{F}$. □

OR. For each $k \in \{1, \dots, \dim V - 1\}$, define $P(k)$: if every subsp of dim k is invar, then $T = \lambda I$.
(i) If every subsp of dim 1 is invar, then by Exe (26), $T = \lambda I$. Thus $P(1)$ holds.
(ii) Asum $P(k)$ holds for $k \in \{1, \dots, \dim V - 1\}$. And every subsp of dim $k+1$ is invar.
Let U be a subsp of dim k. If $\dim U = \dim V - 1$ then extend B_U to B_V and done.
Supp $\dim U \in \{1, \dots, \dim V - 2\}$. Choose two liney indep vecs $v, w \notin U$.
Becs $U \oplus \operatorname{span}(v)$ and $U \oplus \operatorname{span}(w)$ of dim $k+1$ are invar.
Supp $u \in U$. Let $Tu = a_1 u_1 + bv = a_2 u_2 + cw, \exists ! u_1, u_2 \in U, a_1, a_2, b, c \in \mathbf{F}$.
Now $a_1 u_1 - a_2 u_2 = cw - bv \in U \cap \operatorname{span}(v) = \{0\} \Rightarrow b = c = 0$. Thus $Tu \in U$.
Becs $P(k)$ holds, we conclude that $T = \lambda I$. Thus $P(k+1)$ holds. □

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29 *Supp $T \in \mathcal{L}(V)$ and* range T *is finide.*
Prove T has at most $1 + \dim \operatorname{range} T$ *disti eigvals.*

SOLUS:

Let $\lambda_1, \dots, \lambda_m$ be the disti eigvals of T with corres eigvecs $v_1, \dots, v_m$.
(Becs range T is finide. The corres eigvals are fini.)
Then $(v_1, \dots, v_m)$ liney indep $\Longrightarrow (\lambda_1 v_1, \dots, \lambda_m v_m)$ liney indep, if each $\lambda_k \neq 0$.
Othws, $\exists ! \lambda_k = 0$. Now $(\lambda_1 v_1, \dots, \lambda_{k-1} v_{k-1}, \lambda_{k+1} v_{k+1}, \dots, \lambda_m v_m)$ is liney indep.
Hence, by [2.23], $m - 1 \leqslant \dim \operatorname{range} T$. □

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30 *Supp $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigvals. Prove $\exists\, x, Tx - 9x = (-4, 5, \sqrt{7})$.*

SOLUS: T has $\dim \mathbf{R}^3$ eigvals not including $9 \Rightarrow (T - 9I)$ is inv. $x = (T - 9I)^{-1}(-4, 5, \sqrt{7})$. □

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31 *Supp V is finide, and $v_1, \dots, v_m \in V$. Prove*
$(v_1, \dots, v_m)$ is liney indep $\iff v_1, \dots, v_m$ are eigvecs of some T corres to disti eigvals.

SOLUS: Supp $(v_1, \dots, v_m)$ is liney indep. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.
Define $T \in \mathcal{L}(V)$ by $Tv_k = k \cdot v_k$ for each $k \in \{1, \dots, m, \dots, n\}$. Convly by [5.10]. $\square$

• *Supp $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ are disti.*
(a) **32** *Prove $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is liney indep in $\mathbf{R}^{\mathbf{R}}$.*
HINT: Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$. Define $D \in \mathcal{L}(V)$ by $Df = f'$. Find eigvals and eigvecs of D.
(b) [4E 36] *Show $\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$ is liney indep in $\mathbf{R}^{\mathbf{R}}$.*

SOLUS:
(a) Define V and $D \in \mathcal{L}(V)$ as in HINT. Then becs for each k, $D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.
Thus $\lambda_1, \dots, \lambda_n$ are disti eigvals of D. By [5.10], $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is liney indep in $\mathbf{R}^{\mathbf{R}}$. $\square$
(b) Let $V = \operatorname{span}\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.
Then becs $D\big(\cos(\lambda_k x)\big) = -\lambda_k \sin(\lambda_k x)$. 又 $D\big(\sin(\lambda_k x)\big) = \lambda_k \cos(\lambda_k x)$.
Thus $D^2\big(\cos(\lambda_k x)\big) = -\lambda_k^2 \cos(\lambda_k x)$.
Notice that $\lambda_1, \dots, \lambda_n$ are disti $\Longrightarrow -\lambda_1^2, \dots, -\lambda_n^2$ are disti. And $\dim V = n$.
Hence $-\lambda_1^2, \dots, -\lambda_n^2$ are all the eigvals of D^2 with corres eigvecs $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$.
And then $\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$ is liney indep in $\mathbf{R}^{\mathbf{R}}$. $\square$

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33 *Supp $T \in \mathcal{L}(V)$. Prove $T/(\operatorname{range} T) = 0$.*

SOLUS: $v + \operatorname{range} T \in V/\operatorname{range} T \Longrightarrow v + \operatorname{range} T \in \operatorname{null}\big(T/(\operatorname{range} T)\big)$. Hence $T/(\operatorname{range} T) = 0$. $\square$

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34 *Supp $T \in \mathcal{L}(V)$. Prove $T/(\operatorname{null} T)$ is inje $\iff (\operatorname{null} T) \bigcap (\operatorname{range} T) = \{0\}$.*

SOLUS: NOTICE that $\big(T/(\operatorname{null} T)\big)(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0 \iff Tu \in (\operatorname{null} T) \bigcap (\operatorname{range} T)$.
Now $T/(\operatorname{null} T)$ is inje $\iff u + \operatorname{null} T = 0 \iff Tu = 0 \iff (\operatorname{null} T) \bigcap (\operatorname{range} T) = \{0\}$. $\square$

• *Supp V is finide, $T \in \mathcal{L}(V)$, and U is invarsp of V under T.*
Define $T/U : V/U \to V/U$ by $(T/U)(v + U) = Tv + U$ for each $v \in V$.
(a) *Show T/U is well-defined and is liney. Req U invard T.*
(b) [OR **35**] *Show each eigval of T/U is an eigval of T.*

SOLUS:
(a) $v + U = w + U \iff v - w \in U \Longrightarrow T(v - w) \in U \iff Tv + U = Tw + U$.
Hence T/U is well-defined. Now we show T/U is liney.
$(T/U)\big((v + U) + \lambda(w + U)\big) = T(v + \lambda w) + U = (T/U)(v + U) + \lambda(T/U)(w)$. Checked.
(b) Supp λ is an eigval of T/U with an eigvec $v + U$. Then $Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v = u \in U$.
If $u = 0 \Rightarrow Tv = \lambda v$, then done. Othws, we discuss in two cases.
If $(T - \lambda I)|_U$ is inv. Then $\exists!\, w \in U, (T - \lambda I)(w) = u = (T - \lambda I)v \Rightarrow T(v + w) = \lambda(v + w)$.
Note that $v + w \neq 0$, for if not, $v \in U \Rightarrow v + U = 0$, ctradic. Thus λ is an eigval of T.
If $(T - \lambda I)|_U$ is not inv. Then becs V is finide, $(T - \lambda I)|_U$ is not inje,
so that $\exists\, w \in \operatorname{null}(T - \lambda I)|_U, w \neq 0, (T - \lambda I)w = 0 \Rightarrow Tw = \lambda w$. $\square$

OR. Let $B_U = (u_1, \dots, u_m)$. Then $\big((T - \lambda I)v, (T - \lambda I)u_1, \dots, (T - \lambda I)u_m\big)$ is liney indep in U.
So that $a_0(T - \lambda I)v + a_1(T - \lambda I)u_1 + \cdots + a_m(T - \lambda I)u_m = 0, \exists\, a_0, a_1, \dots, a_m \in \mathbf{F}$ with some $a_i \neq 0$.
Let $w = a_0 v + a_1 u_1 + \cdots + a_m u_m \Longrightarrow Tw = \lambda w$. Note that $w \neq 0$, for if not, $a_0 v \in U$, each $a_i = 0$. $\square$

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36 *Prove or give a counterexa: The result in Exercise 35 is still true if V is infinide.*

Solus: A counterexa:

Consider $V = \{ f \in \mathbf{R}^{\mathbf{R}} : \exists !\, m \in \mathbf{N}, f \in \operatorname{span}(1, e^x, \dots, e^{mx}) \}$. Note that V is infinide.

And a subsp $U = \{ f \in \mathbf{R}^{\mathbf{R}} : \exists !\, m \in \mathbf{N}^+, f \in \operatorname{span}(e^x, \dots, e^{mx}) \}$.

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then range $T = U$ is invard T.

Consider $(T/U)(1+U) = e^x + U = 0 \Longrightarrow 0$ is an eigval of T/U but is not an eigval of T.

$\big[$ null $T = \{0\}$, for if not, $\exists f \in V \backslash \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, ctradic. $\big]$ ☐

• (4E 5.A.39) *Supp V is finide and $T \in \mathcal{L}(V)$.*
Prove T has an eigval $\Longleftrightarrow \exists$ invarsp U under T of dimension $\dim V - 1$.

Solus:

(a) Supp λ is an eigval of T with an eigvec v. (If $\dim V = 1$, then $U = \{0\}$ and done.)

Extend $v_1 = v$ to $B_V = (v_1, v_2 \dots, v_n)$.

Step 1. If $\exists\, w_1 \in \operatorname{span}(v_2, \dots, v_n)$ suth $0 \neq Tw_1 \in \operatorname{span}(v_1)$.
Then extend $w_1 = \alpha_{1,2}$ to a bss of $\operatorname{span}(v_2, \dots, v_n)$ as $(\alpha_{1,2}, \dots, \alpha_{1,n})$.
Othws, we stop at step 1.

Step 2. If $\exists\, w_2 \in \operatorname{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$ suth $0 \neq Tw_2 \in \operatorname{span}(v_1, w_1)$.
Then extend $w_2 = \alpha_{2,3}$ to a bss of $\operatorname{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$ as $(\alpha_{2,3}, \dots, \alpha_{2,n})$.
Othws, we stop at step 2.

Step k. If $\exists\, w_k \in \operatorname{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$ suth $0 \neq Tw_k \in \operatorname{span}(v_1, w_1, \dots, w_{k-1})$,
Then extend $w_k = \alpha_{k,k+1}$ to a bss of $\operatorname{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$ as $(\alpha_{k,k+1}, \dots, \alpha_{k,n})$.
Othws, we stop at step k.

Finally, we stop at step m, thus we get $(v_1, w_1, \dots, w_{m-1})$ and $(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})$,
range $T|_{\operatorname{span}(w_1,\dots,w_{m-1})} = \operatorname{span}(v_1, w_1, \dots, w_{m-2}) \Rightarrow \dim \operatorname{null} T|_{\operatorname{span}(w_1,\dots,w_{m-1})} = 0$,
$\underbrace{\operatorname{span}(v_1, w_1, \dots, w_{m-1})}_{\dim m}$ and $\underbrace{\operatorname{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})}_{\dim (n-m)}$ are invard T.

Let $U = \operatorname{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n}) \oplus \operatorname{span}(v_1, w_1, \dots, w_{m-2})$ and done. ☐

Comment: Both $\operatorname{span}(v_2, \dots, v_n)$ and $U \oplus \operatorname{span}(w_{m-1})$ are in $\mathcal{S}_V \operatorname{span}(v_1)$.
If $T|_U$ is inv, then by the simlr algo, we can extend U to invarsp.

Or. Note that $\dim \operatorname{null}(T - \lambda I) \geqslant 1$. And $\dim \operatorname{range}(T - \lambda I) \leqslant \dim V - 1$.
Let $B_{\operatorname{range}(T-\lambda I)} = (w_1, \dots, w_m)$, $B_V = (w_1, \dots, w_m, u_1, \dots, u_n)$.
If $m = \dim V - 1$. $\big[\Longleftrightarrow n = 0. \big]$ Then range $(T - \lambda I)$ is invarsp of dim $\dim V - 1$.
Othws, choose $k \in \{1, \dots, n\}$ and then let $U = \operatorname{span}(w_1, \dots, w_m, u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n)$.
By Exe (1)(b), U is invard $(T - \lambda I)$. Now $u \in U \Rightarrow (T - \lambda I)(u) \in U \Rightarrow Tu \in U$.

(b) Supp U is invarsp under T of dim $m = \dim V - 1$. (If $m = 0$, then done.)

Let $B_U = (u_1, \dots, u_m), B_V = (u_0, u_1, \dots, u_m)$. We discuss in cases:
(I) If $Tu_0 \in U$, then range $T = U$ so that T is not surj $\Longleftrightarrow$ null $T \neq \{0\} \Longleftrightarrow 0$ is an eigval of T.
(II) If $Tu_0 \notin U$, then $Tu_0 = a_0 u_0 + a_1 u_1 + \dots + a_m u_m$.
If range $T|_U = U \Longleftrightarrow a_1 = \dots = a_m = 0 \Longleftrightarrow Tu_0 \in \operatorname{span}(u_0)$ then done.
Othws, $T|_U : U \to U$ is not surj, so is not inje. Thus 0 is an eigval of $T|_U$, so of T. ☐

Or. Consider $T/U \in \mathcal{L}(V/U)$. Becs $\dim V/U = 1$. $\exists \lambda \in \mathbf{F}, T/U = \lambda I$. By Exe (35). ☐

Ended

5.B: I [*See 5.B: II below.*]

Comment: 下面，为了照顾原书 5.B 节两版过大的差距，特别将此节补注分成 I 和 II 两部分。又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本征值」(相当一部分是从原第 3 版 8.C 节挪过来的) 是对原第 3 版「多项式作用于算子」与「本征值的存在性」(也即第 3 版 5.B 前半部分) 的极大扩充，这一扩充也大大改变了原第 3 版后半部分的「上三角矩阵」这一小节，故而将第 4 版 5.B 节放在第 3 版前面。

I 部分除了覆盖第 4 版 5.B 节全部和第 3 版 5.B 节前半部分与之相关的所有习题，还会覆盖第 4 版 5.A 节末。
II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节，还会覆盖第 4 版 5.C 节；并且，下面 5.C 还会覆盖第 4 版 5.D 节。

「注：[8.40] Or (4E 5.22) —— min poly;
[8.44,8.45] Or (4E 5.25,5.26) —— how to find the min poly;
[8.49] Or (4E 5.27) —— eigvals are the zeros of the min poly;
[8.46] Or (4E 5.29) —— $q(T) = 0 \Leftrightarrow q$ is a poly multi of the min poly.」

1 2 3 5 6 7 8 10 11 12 13 18 19 | 2E: Ch5.24
4E: 5.A.32 5.A.33 3 7 8 9 10 11 12 13 14 15
16 17 18 19 20 21 22 23 24 25 26 27 28 29

• (4E 5.A.33) *Supp $T \in \mathcal{L}(V)$ and m is a positive integer.*
(a) *Prove T is inje $\Longleftrightarrow T^m$ is inje.*
(b) *Prove T is surj $\Longleftrightarrow T^m$ is surj.*

Solus:
(a) Supp T^m is inje. Then $Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$.
Supp T is inje. Then $T^m v = T^{m-1}v = \cdots = T^2 v = Tv = v = 0$.
(b) Supp T^m is surj. $\forall u \in V, \exists v \in V, T^m v = u = Tw$, let $w = T^{m-1}v$.
Supp T is surj. Then $\forall u \in V, \exists v_1, \dots, v_m \in V, T(v_1) = T^2 v_2 = \cdots = T^m v_m = u$. ☐

• **Note For [5.17]:**
Supp $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{F})$. Prove null $p(T)$ *and* range $p(T)$ *are invard T.*
Solus: Using the commu in [5.10].
(a) Supp $u \in \operatorname{null} p(T)$. Then $p(T)u = 0$.
Thus $p(T)(Tu) = \big(p(T)T\big)u\big(Tp(T)\big)u = T\big(p(T)u\big) = 0$. Hence $Tu \in \operatorname{null} p(T)$. ☐
(b) Supp $u \in \operatorname{range} p(T)$. Then $\exists v \in V$ suth $u = p(T)v$.
Thus $Tu = T\big(p(T)v\big) = p(T)(Tv) \in \operatorname{range} p(T)$. ☐

• **Note For [5.21]:** *Every optor on a finide nonzero complex vecsp has an eigval.*
Supp V is a finide complex vecsp of dim $n > 0$ and $T \in \mathcal{L}(V)$.
Choose a nonzero $v \in V$. $(v, Tv, T^2 v, \dots, T^n v)$ of len $n+1$ is linely depe.
Supp $a_0 I + a_1 T + \cdots + a_n T^n = 0$. Then $\exists a_j \neq 0$.
Thus $\exists$ nonconst p of smallest deg ($\deg p > 0$) *suth* $p(T)v = 0$.
Becs $\exists \lambda \in \mathbf{C}$ suth $p(\lambda) = 0 \Rightarrow \exists q \in \mathcal{P}(\mathbf{C}), p(z) = (z-\lambda)q(z), \forall z \in \mathbf{C}$.
Thus $0 = p(T)v = (T - \lambda I)\big(q(T)v\big)$. By the min of $\deg p$ and $\deg q < \deg p$, $q(T)v \neq 0$.
Then $(T - \lambda I)$ is not inje. Thus λ is an eigval of T with eigvec $q(T)v$.

• **Exa:** *an optor on a complex vecsp with no eigvals*
Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by $(Tp)(z) = zp(z)$.

Supp $p \in \mathcal{P}(\mathbf{C})$ is a nonzero poly. Then $\deg Tp = \deg p + 1$, and thus $Tp \neq \lambda p$, $\forall \lambda \in \mathbf{C}$.
Hence T has no eigvals.

---

13 *Supp V is a complex vecsp and $T \in \mathcal{L}(V)$ has no eigvals.*
Prove every subsp of V invard T is either $\{0\}$ or infinide.

Solus: Supp U is a finide nonzero invarsp on $\mathbf{C}$. Then by [5.21], $T|_U$ has an eigval. □

---

16 *Supp $0 \neq v \in V$. Define $S \in \mathcal{L}\big(\mathcal{P}_{\dim V}(\mathbf{C}), V\big)$ by $S(p) = p(T)v$. Prove [5.21].*

Solus:

Becs $\dim \mathcal{P}_{\dim V}(\mathbf{C}) = \dim V + 1$. Then S is not inje. Hence $\exists\, 0 \neq p \in \mathcal{P}_{\dim V}(\mathbf{C}), p(T)v = 0$.
Using [4.14], write $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$. Apply T to both sides: $p(T) = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$.
Thus at least one of $(T-\lambda_j I)$ is not inje (becs $p(T)$ is not inje). □

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17 *Supp $0 \neq v \in V$. Define $S \in \mathcal{L}\big(\mathcal{P}_{(\dim V)^2}(\mathbf{C}), \mathcal{L}(V)\big)$ by $S(p) = p(T)$. Prove [5.21].*

Solus:

Becs $\dim \mathcal{P}_{(\dim V)^2}(\mathbf{C}) = (\dim V)^2 + 1$. Then S is not inje.
Hence $\exists\, p \in \mathcal{P}_{(\dim V)^2}(\mathbf{C}) \backslash \{0\}, 0 = S(p) = p(T) = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$, where $c \neq 0$.
Thus $(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0 \Longrightarrow \exists\, j, (T-\lambda_j)$ is not inje. □

Comment: $\exists$ monic $q \in \operatorname{null} S \neq \{0\}$ of smallest deg, $S(q) = q(T) = 0$, then q is the *min poly.*

- **Note For [8.40]:** *def for min poly*

Supp V is finide and $T \in \mathcal{L}(V)$.
Supp $M_T^0 = \{p_j\}_{j\in\Gamma}$ is the set of all monic poly that give 0 whenever T is applied.
Prove $\exists\,!\, p_k \in M_T^0, \deg p_k = \min\{\deg p_j\}_{j\in\Gamma} \leqslant \dim V$.

Solus: Or. Another Proof :

[*Existns Part*] We use induc on $\dim V$.

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, done.

(ii) Supp $\dim V \geqslant 1$.

Asum $\dim V > 0$ *and that the desired result is true for all optors on all vecsps of smaller dim.*
Let $u \in V, u \neq 0$. The list $(u, Tu, \dots, T^{\dim V}u)$ of len $(1 + \dim V)$ is linely depe.
Then $\exists\,!\, T^m$ of smallest deg suth $T^m u \in \operatorname{span}(u, Tu, \dots, T^{m-1}u)$.
Thus $\exists\, c_j \in \mathbf{F}, c_0 u + c_1 Tu + \cdots + c_{m-1}T^{m-1}u + T^m u = 0$.
Define q by $q(z) = c_0 + c_1 z + \cdots + c_{m-1}z^{m-1} + z^m$.
Then $0 = T^k\big(q(T)u\big) = q(T)(T^k u), \forall k \in \{1, \dots, m-1\} \subseteq \mathbf{N}$.
Becs $(u, Tu, \dots, T^{m-1}u)$ is liney indep.
Thus $\dim \operatorname{null} q(T) \geq m \Rightarrow \dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leqslant \dim V - m$.

Let $W = \operatorname{range} q(T)$.
By *asum*, $\exists\, s \in M_T^0$ of smallest deg (and $\deg s \leqslant \dim W$,) so that $s(T|_W) = 0$.
Hence $\forall v \in V, \big((sq)(T)\big)(v) = s(T)\big(q(T)v\big) = 0$.
Thus $sq \in M_T^0$ and $\deg sq \leqslant \dim V$.

[*Uniqnes Part*]

Supp $p, q \in M_T^0$ are of the smallest deg. Then $(p-q)(T) = 0$. 又 $\deg(p-q) = m < \min\{\deg p_j\}_{j\in\Gamma}$.
Hence $p - q = 0$, for if not, $\exists\,!\, c \in \mathbf{F}, c(p-q) \in M_T^0$. Ctradic. □

• (4E 5.31, 4E 5.B.25 and 26) *min poly of restr optor and min poly of quot optor*
Supp V is finide, $T \in \mathcal{L}(V)$, and U is invarsp of V under T.
Let p be the min poly of T.
(a) *Prove p is a poly multi of the min poly of $T|_U$.*
(b) *Prove p is a poly multi of the min poly of T/U.*
(c) *Prove (min poly of $T|_U$) $\times$ (min poly of T/U) is a poly multi of p.*
(d) *Prove the set of eigvals of T equals*
the union of the set of eigvals of $T|_U$ and the set of eigvals of T/U.

SOLUS:
(a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46]. ☐
(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v+U) = p(T)v + U = 0.$ ☐
(c) Supp r is the min poly of $T|_U$, s is the min poly of T/U.
Becs $\forall v \in V, s(T/U)(v+U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.
又 $\forall u \in U, r(T|_U)u = r(T)u = 0$.
Thus $\forall v \in V$ but $v \notin U$, $(rs)(T)v = r\big(s(T)v\big) = 0$.
And $\forall u \in U$, $(rs)(T)u = r\big(s(T)u\big) = 0$ (becs $s(T)u = s(T|_U)u \in U$).
Hence $\forall v \in V$, $(rs)(T)v = 0 \Rightarrow (rs)(T) = 0$. ☐
(d) By [8.49], immed. ☐

• (4E 5.B.27) *Supp $\mathbf{F} = \mathbf{R}$, V is finide, and $T \in \mathcal{L}(V)$.*
Prove the min poly p of T_{C} equals the min poly q of T.

SOLUS:
(a) $\forall u + \mathrm{i}0 \in V_{\mathrm{C}}, p(T_{\mathrm{C}})(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p$ is a poly multi of q.
(b) $q(T) = 0 \Rightarrow \forall u + \mathrm{i}v \in V_{\mathrm{C}}, q(T_{\mathrm{C}})(u + \mathrm{i}v) = q(T)u + \mathrm{i}q(T)v = 0 \Rightarrow q$ is a poly multi of p. ☐

• (4E 5.B.28) *Supp V is finide and $T \in \mathcal{L}(V)$.*
Prove the min poly p of $T' \in \mathcal{L}(V')$ equals the min poly q of T.

SOLUS:
(a) $\forall \varphi \in V', p(T')\varphi = \varphi \circ \big(p(T)\big) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \operatorname{null}\varphi \Rightarrow p(T) = 0, p$ is a poly multi of q.
(b) $q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ \big(q(T)\big) = q(T')\varphi = 0 \Rightarrow q(T) = 0, q$ is a poly multi of p. ☐

• (4E 5.32) *Supp $T \in \mathcal{L}(V)$ and p is the min poly.*
Prove T is not inje $\Longleftrightarrow$ the const term of p is 0.

SOLUS:
T is not inje $\Longleftrightarrow$ 0 is an eigval of $T \Longleftrightarrow$ 0 is a zero of $p \Longleftrightarrow$ the const term of p is 0. ☐

OR. Becs $p(0) = (z-0)(z-\lambda_1)\cdots(z-\lambda_m) = 0 \Rightarrow T(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0$
又 p is the min poly $\Rightarrow q$ define by $q(z) = (z-\lambda_1)\cdots(z-\lambda_m)$ is suth $q(T) \neq 0$.
Hence $0 = p(T) = Tq(T) \Rightarrow T$ is not inje.
Convly, supp $(T - 0I)$ is not inje, then 0 is a zero of p, so that the const term is 0. ☐

• (4E 5.B.22)
Supp V is finide, $T \in \mathcal{L}(V)$. Prove T is inv $\Longleftrightarrow I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$.

SOLUS: Denote the min poly by p, where for all $z \in \mathbf{F}, p(z) = a_0 + a_1 z + \cdots + z^m$.
Notice that V is finide. T is inv $\Longleftrightarrow T$ is inje $\Longleftrightarrow p(0) \neq 0$.

Hence $p(T) = 0 = a_0 I + a_1 T + \cdots + T^m$, where $a_0 \neq 0$ and $m \leqslant \dim V$. $\square$

---

6 *Supp* $T \in \mathcal{L}(V)$ *and* U *is a subsp of* V *invard* T.
Prove U *is invard* $p(T)$ *for every poly* $p \in \mathcal{P}(\mathbf{F})$.

SOLUS:

$\forall u \in U, Tu \in U \Rightarrow Iu, Tu, T(Tu), \ldots, T^m u \in U \Longrightarrow \forall a_k \in \mathbf{F}, (a_0 I + a_1 T + \cdots + a_m T^m)u \in U$. $\square$

• (4E 5.B.10, 23) *Supp* V *is finide,* $T \in \mathcal{L}(V)$ *and* p *is the min poly with deg* m. *Supp* $v \in V$.
(a) *Prove* $\operatorname{span}(v, Tv, \ldots, T^{m-1}v) = \operatorname{span}(v, Tv, \ldots, T^{j-1}v)$ *for some* $j \leqslant m$.
(b) *Prove* $\operatorname{span}(v, Tv, \ldots, T^{m-1}v) = \operatorname{span}(v, Tv, \ldots, T^{m-1}v, \ldots, T^n v)$.

SOLUS:

COMMENT: By NOTE FOR[8.40], j has an upper bound $m-1$, m has an upper bound $\dim V$.

Write $p(z) = a_0 + a_1 z + \cdots + z^m$ ($m \leqslant \dim V$). If $v = 0$, then done. Supp $v \neq 0$.

(a) Supp $j \in \mathbf{N}^+$ is the smallest suth $T^j v \in \operatorname{span}(v, Tv, \ldots, T^{j-1}v) = U_0$. Then $j \leqslant m$.
Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1} T^{j-1} v$. And becs $T(T^k v) = T^{k+1} \in U_0$. U_0 is invard T.
By Exe (6), $\forall k \in \mathbf{N}$, $T^{j+k} v = T^k (T^j v) \in U_0$.
Thus $U_0 = \operatorname{span}(v, Tv, \ldots, T^{j-1}v, \ldots, T^n v)$ for all $n \geqslant j-1$. Let $n = m-1$ and done.

(b) Let $U = \operatorname{span}(v, Tv, \ldots, T^{m-1}v)$.
By (a), $U = U_0 = \operatorname{span}(v, Tv, \ldots, T^{j-1}, \ldots, T^{m-1}, \ldots, T^n)$ for all $n \geqslant m-1$. $\square$

• (4E 5.B.21) *Supp* V *is finide and* $T \in \mathcal{L}(V)$.
Prove the min poly p *has deg at most* $1 + \dim \operatorname{range} T$.
If $\dim \operatorname{range} T < \dim V - 1$, *then this result gives a better upper bound for the deg of min poly.*

SOLUS:

If T is inje, then range $T = V$ and done. Now choose $0 \neq v \in \operatorname{null} T$, then $Tv + 0 \cdot v = 0$.
1 is the smallest positive integer suth $T^1 v \in \operatorname{span}(v, \ldots, T^0 v)$. Define q by $q(z) = z \Rightarrow q(T)v = 0$.
Let $W = \operatorname{range} q(T) = \operatorname{range} T$. $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ of smallest deg ($\deg s \leqslant \dim W$), $s(T|_W) = 0$.
Hence sq is the min poly (see NOTE FOR[8.40]) and $\deg(sq) = \deg s + \deg q \leqslant \dim \operatorname{range} T + 1$. $\square$

---

19 *Supp* V *is finide,* $\dim V > 1, T \in \mathcal{L}(V)$. *Prove* $\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V)$.

SOLUS: If $\forall S \in \mathcal{L}(V), \exists p \in \mathcal{P}(\mathbf{F}), S = p(T)$. Then by [5.20], $\forall S_1, S_2 \in \mathcal{L}(V), S_1 S_2 = S_2 S_1$.
Note that $\dim \geqslant 2$. By (3.A.14), $\exists S_1, S_2 \in \mathcal{L}(V), S_1 S_2 \neq S_2 S_1$. Ctradic. $\square$

• *Supp* V *is finide and* $T \in \mathcal{L}(V)$. *Let* $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.
Prove $\dim \mathcal{E}$ *equals the deg of the min poly of* T.

SOLUS:

Becs the list $(I, T, \ldots, T^{(\dim V)^2})$ of len $\dim \mathcal{L}(V) + 1$ is linely depe in $\dim \mathcal{L}(V)$.
Supp $m \in \mathbf{N}^+$ is the smallest suth $T^m = a_0 I + \cdots + a_{m-1} T^{m-1}$.
Then q defined by $q(z) = z^m - a_{m-1} z^{m-1} - \cdots - a_0$ is the min poly (see [8.40]).
For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k (T^m) \in \operatorname{span}(I, T, \ldots, T^{m-1}) = U$.
Hence $\operatorname{span}(I, T, \ldots, T^{(\dim V)^2}) = \operatorname{span}(I, T, \ldots, T^{(\dim V)^2 - 1}) = U$.
Note that by the min of m, $(I, T, \ldots, T^{m-1})$ is liney indep.
Thus $\dim U = m = \dim \operatorname{span}(I, T, \ldots, T^{(\dim V)^2 - 1}) = \dim \operatorname{span}(I, T, \ldots, T^n)$ for all $m < n \in \mathbf{N}^+$.

Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.

(a) Supp $p(T) = 0$. 又 $\deg p \leqslant m-1 \Rightarrow p = 0$. Then φ is inje.

(b) $\forall S = a_0 I + a_1 T + \cdots + a_{m-1} T^{m-1} \in \mathcal{E}$, define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by

$p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} \Rightarrow \varphi(p) = S$. Then φ is surj.

Hence $\mathcal{E}$ and $\mathcal{P}_{m-1}(\mathbf{F})$ are iso. 又 $\dim \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. □

• (4E 5.B.13) *Supp $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$ is defined by*

$$q(z) = a_0 + a_1 z + \cdots + a_n z^n, \text{ where } a_n \neq 0, \text{ for all } z \in \mathbf{F}.$$

Denote the min poly of T by p defined by

$$p(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m \text{ for all } z \in \mathbf{F}.$$

Prove $\exists!\, r \in \mathcal{P}(\mathbf{F})$ suth $q(T) = r(T), \deg r < \deg p$.

Solus:

If $\deg q < \deg p$, then done.

If $\deg q = \deg p$, notice that $p(T) = 0 = c_0 I + c_1 T + \cdots + c_{m-1} T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0 I - c_1 T - \cdots - c_{m-1} T^{m-1},$$

define r by $r(z) = q(z) + \left[-a_m z^m + a_m\left(-c_0 - c_1 z - \cdots - c_{m-1} z^{m-1} \right) \right]$

$= (a_0 - a_m c_0) + (a_1 - a_m c_1) z + \cdots + (a_{m-1} - a_m c_{m-1}) z^{m-1}$,

hence $r(T) = 0, \deg r < m$ and done.

Now supp $\deg q \geqslant \deg p$. We use induc on $\deg q$.

(i) $\deg q = \deg p$, then the desired result is true, as shown above.

(ii) $\deg q > \deg p$, asum the desired result is true for $\deg q = n$.

Supp $f \in \mathcal{P}(\mathbf{F})$ suth $f(z) = b_0 + b_1 z + \cdots + b_n z^n + b_{n+1} z^{n+1}$.

Apply the asum to g defined by $g(z) = b_0 + b_1 z + \cdots + b_n z^n$,

getting s defined by $s(z) = d_0 + d_1 z + \cdots + d_{m-1} z^{m-1}$.

Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1} T^{n+1} = s(T) + b_{n+1} T^{n+1}$.

Apply the asum to t defined by $t(z) = z^n$,

getting δ defined by $\delta(z) = c_0{}' + c_1{}' z + \cdots + c_{m-1}{}' z^{m-1}$.

Thus $t(T) = T^n = c_0{}' + c_1{}' z + \cdots + c_{m-1}{}' z^{m-1} = \delta(T)$.

又 $\operatorname{span}(v, Tv, \dots, T^{m-1} v)$ is invard T.

Hence $\exists!\, k_j \in \mathbf{F}, T^{n+1} = T(T^n) = k_0 + k_1 z + \cdots + k_{m-1} z^{m-1}$.

And $f(T) = s(T) + b_{n+1}(k_0 + k_1 T + \cdots + k_{m-1} T^{m-1})$

$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1) z + \cdots + (d_{m-1} + k_{m-1}) z^{m-1} = h(T)$, thus defining h. □

• (4E 5.B.14) *Supp V is finide, $T \in \mathcal{L}(V)$ has min poly p*

$$\textit{defined by } p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m, a_0 \neq 0.$$

Find the min poly of T^{-1}.

Solus:

Notice that V is finide. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is inv.

Then $p(T) = a_0 I + a_1 T + \cdots + T^m = 0$. Apply T^{-m} to both sides,

$a_0 (T^{-1})^m + a_1 (T^{-1})^{m-1} + \cdots + a_{m-1} T^{-1} + I = 0$.

Define q by $q(z) = z^m + \dfrac{a_1}{a_0} z^{m-1} + \cdots + \dfrac{a_{m-1}}{a_0} z + \dfrac{1}{a_0}$ for all $z \in \mathbf{F}$.

We now show $(T^{-1})^k \notin \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$

for every $k \in \{1, \dots, m-1\}$ by ctradic, so that q is exactly the min poly of T^{-1}.

Supp $(T^{-1})^k \in \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$.

Then let $\left(T^{-1}\right)^k = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. Apply T^k to both sides,

getting $I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T$, hence $T^k \in \operatorname{span}(I, T, \ldots, T^{k-1})$.

Thus f defined by $f(z) = z^k + \dfrac{b_1}{b_0} z^{k-1} + \cdots + \dfrac{b_{k-1}}{b_0} z - \dfrac{1}{b_0}$ is a poly multi of p.

While $\deg f < \deg p$. Ctradic. $\square$

- **Note For [8.49]:**

Supp V is a finide complex vecsp and $T \in \mathcal{L}(V)$.
By [4.14], the min poly has the form $(z-\lambda_1)\cdots(z-\lambda_m)$,
where $\lambda_1, \ldots, \lambda_m$ are all the eigvals of T, ***possibly with repetitions****.*

- **Comment:**

A nonzero poly has at most as many disti zeros as its deg (see [4.12]).
Thus by the upper bound for the deg of min poly given in Note For*[8.40], and by [8.49,]*
we can ***give an alternative proof of*** [**5.13**].

- Notice (See also 4E 5.B.20,24)

Supp $\alpha_1, \ldots, \alpha_n$ are all the disti eigvals of T,
and therefore are all the disti zeros of the min poly.
Also, the min poly of T is a poly multi of, but not equal to, $(z-\alpha_1)\cdots(z-\alpha_n)$.
If we define q by $q(z) = (z-\alpha_1)^{\dim V - (n-1)} \cdots (z-\alpha_n)^{\dim V - (n-1)}$,
then q is a poly multi of the ch poly (see [8.34] and [8.26])
(Becs $\dim V > n$ and $n-1>0$, $n\big[\dim V - (n-1)\big] > \dim V$.)
The ch poly has the form $(z-\alpha_1)^{\gamma_1}\cdots(z-\alpha_n)^{\gamma_n}$, where $\gamma_1 + \cdots + \gamma_n = \dim V$.
The min poly has the form $(z-\alpha_1)^{\delta_1}\cdots(z-\alpha_n)^{\delta_n}$, where $0 \leqslant \delta_1 + \cdots + \delta_n \leqslant \dim V$.

---

10 *Supp $T \in \mathcal{L}(V)$, λ is an eigval of T with an eigvec v.*
Prove for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.

Solus:

Supp p is defined by $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbf{F}$. Becs for any $n \in \mathbf{N}^+$, $T^n v = \lambda^n v$.
Thus $p(T)v = a_0 v + a_1 T v + \cdots + a_m T^m v = a_0 v + a_1 \lambda v + \cdots + a_m \lambda^m v = p(\lambda)v$. $\square$

Comment: For any $p \in \mathcal{P}(\mathbf{F})$ suth $p(z) = (z-\lambda_1)^{\alpha_1}\cdots(z-\lambda_m)^{\alpha_m}$, the result is true as well.

Now we prove that $(T-\lambda_1 I)^{\alpha_1}\cdots(T-\lambda_m I)^{\alpha_m} v = (\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_m)^{\alpha_m} v$.
Define q_i by $q_i(z) = (z-\lambda_i)^{\alpha_i}$ for all $z \in \mathbf{F}$.
Becs $(a+b)^n = a^n + C_n^1 a^{n-1} b + \cdots + C_n^k a^{n-k} b^k + \cdots + C_n^n b^n$.
Let $a = z, b = \lambda_i, n = \alpha_i$, so we can write $q_i(z)$ in the form $a_0 + a_1 z + \cdots + a_m z^m$.
Hence $q_i(T)v = q_i(\lambda)v \Rightarrow (T-\lambda_i I)^{\alpha_i} v = (\lambda-\lambda_i)^{\alpha_i} v$.
Then for each $k \in \{2, \ldots, m\}$, $(T-\lambda_{k-1} I)^{\alpha_{k-1}}(T-\lambda_k I)^{\alpha_k} v$

$$\begin{aligned}
&= q_{k-1}(T)(q_k(T)v) \\
&= q_{k-1}(T)(q_k(\lambda)v) \\
&= q_{k-1}(\lambda)(q_k(\lambda)v) \\
&= (\lambda-\lambda_{k-1})^{\alpha_{k-1}}(\lambda-\lambda_k)^{\alpha_k} v.
\end{aligned}$$

So that $(T-\lambda_1 I)^{\alpha_1}\cdots(T-\lambda_m I)^{\alpha_m} v$

$$\begin{aligned}
&= q_1(T)\Big(q_2(T)\Big(\ldots(q_m(T)v)\ldots\Big)\Big) \\
&= q_1(\lambda)(q_2(\lambda)(\ldots(q_m(\lambda)v)\ldots))
\end{aligned}$$

$$= (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v. \qquad \square$$

1 *Supp $T \in \mathcal{L}(V)$ and $\exists\, n \in \mathbf{N}^+$ suth $T^n = 0$.*
Prove $(I - T)$ is inv and $(I - T)^{-1} = I + T + \cdots + T^{n-1}$.

Solus: Note that $1 - x^n = (1 - x)(1 + x + \cdots + x^{n-1})$.

$$\left.\begin{aligned}(I-T)(1+T+\cdots+T^{n-1}) &= I - T^n = I\\ (1+T+\cdots+T^{n-1})(I-T) &= I - T^n = I\end{aligned}\right\} \Rightarrow (I-T)^{-1} = 1 + T + \cdots + T^{n-1}. \qquad \square$$

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2 *Supp $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$.*
Supp λ is an eigval of T. Prove $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Solus:

Supp v is an eigvec corres to λ. Then for any $p \in \mathcal{P}(\mathbf{F}), p(T)v = p(\lambda)v$.
Hence $0 = (T - 2I)(T - 3I)(T - 4I)v = (\lambda - 2)(\lambda - 3)(\lambda - 4)v$ while $v \neq 0 \Rightarrow \lambda = 2, 3$ or 4. $\square$

Comment: Note that $(T - 2I)(T - 3I)(T - 4I) = 0$ is not inje, so that $2, 3, 4$ are eigvals of T.
But it doesn't mean that all the eigvals of T are exactly $2, 3, 4$.

---

7 [See 5.A.22] *Supp $T \in \mathcal{L}(V)$. Prove 9 is an eigval of $T^2 \Longleftrightarrow 3$ or -3 is an eigval of T.*

Solus:

(a) Supp λ is an eigval of T with an eigvec v.
Then $(T - 3I)(T + 3I)v = (\lambda - 3)(\lambda + 3)v = 0 \Rightarrow \lambda = \pm 3$.

(b) Supp 3 or -3 is an eigval of T with an eigvec v. Then $Tv = \pm 3v \Rightarrow T^2 v = T(Tv) = 9v$ $\square$

Or. 9 is an eigval of $T^2 \Longleftrightarrow (T^2 - 9I) = (T - 3I)(T + 3I)$ is not inje $\Longleftrightarrow \pm 3$ is an eigval. $\square$

---

3 *Supp $T \in \mathcal{L}(V), T^2 = I$ and -1 is not an eigval of T. Prove $T = I$.*

Solus:

$T^2 - I = (T + I)(T - I)$ is not inje, 又 -1 is not an eigval of $T \Longrightarrow$ By Tips. $\square$

Or. Note that $\forall v \in V, v = [\frac{1}{2}(I - T)v] + [\frac{1}{2}(I + T)v]$.

$$\left.\begin{aligned}(I+T)\big((I-T)v\big) = 0 &\Longrightarrow (I-T)v \in \operatorname{null}(I+T)\\ (I-T)\big((I+T)v\big) = 0 &\Longrightarrow (I+T)v \in \operatorname{null}(I-T)\end{aligned}\right\} \Rightarrow V = \operatorname{null}(I+T) + \operatorname{null}(I-T).$$

又 -1 is not an eigval of $T \Longleftrightarrow (I + T)$ is inje $\Longleftrightarrow \operatorname{null}(I + T) = \{0\}$.
Hence $V = \operatorname{null}(I - T) \Rightarrow \operatorname{range}(I - T) = \{0\}$. Thus $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$. $\square$

• (4E 5.A.32) *Supp $T \in \mathcal{L}(V)$ has no eigvals and $T^4 = I$. Prove $T^2 = -I$.*

Solus:

Becs $T^4 - I = (T^2 - I)(T^2 + I) = 0$ is not inje $\Rightarrow (T^2 - I)$ or $(T^2 + I)$ is not inje.
又 T has no eigvals $\Rightarrow (T^2 - I) = (T - I)(T + I)$ is inje. Hence $T^2 + I = 0 \in \mathcal{L}(V)$, for if not, $\exists\, v \in V, (T^2 + I)v \neq 0$ while $(T^2 - I)\big((T^2 + I)v\big) = 0$ but $(T^2 - I)$ is inje. Ctradic.
Or. $\forall v \in V, 0 = (T^2 - I)(T^2 + I)v \Longleftrightarrow 0 = (T^2 + I)v$. Hence $T^2 + I = 0$. $\square$

Or. Note that $\forall v \in V, v = \left[\frac{1}{2}(I - T^2)v\right] + \left[\frac{1}{2}(I + T^2)v\right]$.

$$\left.\begin{aligned}(I+T^2)\big((I-T^2)v\big) = 0 &\Longrightarrow (I-T^2)v \in \operatorname{null}(I+T^2)\\ (I-T^2)\big((I+T^2)v\big) = 0 &\Longrightarrow (I+T^2)v \in \operatorname{null}(I-T^2)\end{aligned}\right\} \Rightarrow V = \operatorname{null}(I+T^2) + \operatorname{null}(I-T^2).$$

又 T has no eigvals $\Longleftrightarrow (I - T^2)$ is inje $\Longleftrightarrow \operatorname{null}(I - T^2) = \{0\}$.
Hence $V = \operatorname{null}(I + T^2) \Rightarrow \operatorname{range}(I + T^2) = \{0\}$. Thus $I + T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$. $\square$

---

8 [OR (4E 5.A.31)] *Give an exa of $T \in \mathcal{L}(\mathbf{R}^2)$ suth $T^4 = -I$.*

SOLUS:

Define $\mathrm{i} \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y) = (-y,x)$. Just like $\mathrm{i} : \mathbf{C} \to \mathbf{C}$ defined by $\mathrm{i}(x+\mathrm{i}y) = -y + \mathrm{i}x$.

Define $\mathrm{i}^n \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y) = \Big(\mathrm{Re}\big(\mathrm{i}^n x + \mathrm{i}^{n+1} y\big), \mathrm{Im}\big(\mathrm{i}^n x + \mathrm{i}^{n+1} y\big)\Big)$.

$T^4 + I = (T^2 + \mathrm{i}I)(T^2 - \mathrm{i}I) = (T + \mathrm{i}^{1/2}I)(T - \mathrm{i}^{1/2}I)(T - (-\mathrm{i})^{1/2}I)(T + (-\mathrm{i})^{1/2}I)$.

Note that $\mathrm{i}^{1/2} = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}$, $(-\mathrm{i})^{1/2} = \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}$. Hence $T = \pm(\pm\mathrm{i})^{1/2}I$.

Let $T = \mathrm{i}^{1/2}I$ defined by $\mathrm{i}^{1/2}(x,y) = \Big(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\Big)$. $\square$

OR. Becs $\mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix}$. Using $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}$.

We define $T \in \mathcal{L}(\mathbf{R}^2)$ suth $\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix}$. $\square$

• (4E 5.B.12) *Find the min poly of T defined in (5.A.10).*

SOLUS: By (5.A.9) and [8.40, 8.49], $1, 2, \ldots, n$ are all the zeros of the min poly of T. $\square$

• (4E 5.B.3) *Find the min poly of T defined in (5.A.19).*

SOLUS:

If $n = 1$ then 1 is the only eigval of T, and $(z-1)$ is the min poly.

Becs n and 0 are all the eigvals of T, 又 $\forall k \in \{1, \ldots, n\}, Te_k = e_1 + \cdots + e_n$; $T^2 e_k = n(e_1 + \cdots + e_n)$.

Hence $T^2 e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T-n) = 0$. Thus $\big(z(z-n)\big)$ is the min poly. $\square$

• (4E 5.B.8) *Find the min poly of T. Where $T \in \mathcal{L}(\mathbf{R}^2)$ is the optor of counterclockwise rotation by θ, where $\theta \in \mathbf{R}^+$.*

SOLUS:

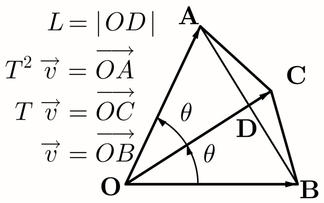
If $\theta = \pi + 2k\pi$, then $T(w,z) = (-w,-z), T^2 = I$ and the min poly is $z+1$.

If $\theta = 2k\pi$, then $T = I$ and the min poly is $z - 1$.

Othws (v, Tv) is liney indep. Then $\mathrm{span}(v, Tv) = \mathbf{R}^2$. Note that $\nexists b \in \mathbf{F}, T - bI = 0$.

Thus supp the min poly p is defined by $p(z) = z^2 + bz + c$ for all $z \in \mathbf{R}$.

Becs



$$Tv = \frac{|\vec{v}|}{2L}(T^2 v + v) \Rightarrow T = \frac{|\vec{v}|}{2L}(T^2 + I)$$

$$L = |\vec{v}|\cos\theta \Rightarrow \frac{|\vec{v}|}{2L} = \frac{1}{2\cos\theta}$$

Hence $p(T) = T^2 - 2\cos\theta\, T + I = 0$ and $z^2 - 2\cos\theta\, z + 1$ is the min poly of T. $\square$

OR. Let (e_1, e_2) be the std bss of $\mathbf{R}^2$. We use the pattern shown in [8.44].

Becs $Te_1 = \cos\theta\, e_1 + \sin\theta\, e_2$, $T^2 e_1 = \cos 2\theta\, e_1 + \sin 2\theta\, e_2$.

Thus $ce_1 + bTe_1 = -T^2 e_1 \iff \begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix}\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$. Now $\det = \sin\theta \neq 0, c = 1, b = 2\cos\theta$. $\square$

OR. $\mathcal{M}(T, (e_1, e_2)) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. By (4E 5.B.11), the min poly is $(z \pm 1)$ or $(z^2 - 2\cos\theta\, z + 1)$. $\square$

• (4E 5.B.11) *Supp V is a two-dim vecsp, $T \in \mathcal{L}(V)$, and the matrix of T wrto some B_V is* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.
(a) *Show* $T^2 - (a+d)T + (ad-bc)I = 0$.
(b) *Show the min poly of T equals*

$$\begin{cases} z-a & \text{if } b=c=0 \text{ and } a=d, \\ z^2-(a+d)z+(ad-bc) & \text{othws.} \end{cases}$$

SOLUS:

(a) Supp the bss is (v,w). Becs $\begin{cases} Tv = av+bw \Rightarrow (T-aI)v = bw, \text{ then apply } (T-dI) \text{ to both sides.} \\ Tw = cv+dw \Rightarrow (T-dI)w = cv, \text{ then apply } (T-aI) \text{ to both sides.} \end{cases}$

Hence $(T-aI)(T-dI) = bcI \Rightarrow T^2-(a+d)T+(ad-bc)I = 0$.

(b) If $b=c=0$ and $a=d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the min poly is $z-a$.

Othws, by (a), $z^2-(a+d)z+(ad-bc)$ is a poly multi of the min poly.

Now we prove that $T \notin \operatorname{span}(I)$, so that then the min poly of T has exactly deg 2.

(At least one of the asum of (I),(II) below is true.)

(I) Supp $a=d$, then $Tv = av+bw \notin \operatorname{span}(v), Tw = cv+aw \notin \operatorname{span}(w)$.

(II) Supp at most one of b,c is not 0. If $b=0$, then $Tw \notin \operatorname{span}(w)$; If $c=0$, then $Tv \notin \operatorname{span}(v)$. □

• *Supp $S,T \in \mathcal{L}(V)$, S is inv, and $p \in \mathcal{P}(\mathbf{F})$. Prove $Sp(TS) = p(ST)S$.*

SOLUS:

We prove $S(TS)^m = (ST)^m S$ for each $m \in \mathbf{N}$ by induc.

(i) If $m = 0,1$. Then $S(TS)^0 = I = (ST)^0 S$; $S(TS)^1 = (ST)S$.

(ii) If $m > 1$. Asum $S(TS)^m = (ST)^m S$.

Then $S(TS)^{m+1} = S(TS)^m(TS) = (ST)^m STS = (ST)^{m+1}S$.

Hence $\forall p \in \mathcal{P}(\mathbf{F}), Sp(TS) = \sum_{k=1}^m a_k S(TS)^k = \sum_{k=1}^m a_k p(ST)^k S = \big[\sum_{k=1}^m a_k (TS)^k\big] S$. □

COMMENT: $p(TS) = S^{-1}p(ST)S,\ p(ST) = Sp(TS)S^{-1}$.

CORO: **5** Becs S is inv, $T \in \mathcal{L}(V)$ is arb $\Longleftrightarrow R = ST$ is arb.

Hence $\forall R \in \mathcal{L}(V)$, inv $S \in \mathcal{L}(V), p(S^{-1}RS) = S^{-1}p(R)S$.

• (4E 5.B.7) *Supp $S,T \in \mathcal{L}(V)$. Let p,q be the min polys of ST, TS respectly.*
(a) *If $V = \mathbf{F}^2$. Give an exa suth $p \neq q$;* (b) *If S or T is inv. Prove $p = q$.*

SOLUS:

(a) Define S by $S(x,y) = (x,x)$. Define T by $T(x,y) = (0,y)$.

Then $ST(x,y) = 0,\ TS(x,y) = (0,x)$ for all $(x,y) \in \mathbf{F}^2$. Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.

Hence the min poly of ST does not equal to the min poly of TS.

(b) Supp S is inv. Becs p,q are monic.

$$\left.\begin{array}{l} p(ST) = 0 = Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a poly multi of } q \\ q(TS) = 0 = S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a poly multi of } p \end{array}\right\} \Rightarrow p = q.$$

Reversing the roles of S and T, we conclude that if T is inv, then $p = q$ as well. □

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11 *Supp $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{C})$, and $\alpha \in \mathbf{C}$.*
Prove α is an eigval of $p(T) \Longleftrightarrow \alpha = p(\lambda)$ for some eigval λ of T.

SOLUS:

(a) Supp α is an eigval of $p(T) \Leftrightarrow \big(p(T) - \alpha I\big)$ is not inje.

Write $p(z) - \alpha = c(z - \lambda_1)\cdots(z - \lambda_m) \Rightarrow p(T) - \alpha I = c(T - \lambda_1 I)\cdots(T - \lambda_m I)$.
By TIPS, $\exists\,(T - \lambda_j I)$ not inje. Thus $p(\lambda_j) - \alpha = 0$.

(b) Supp $\alpha = p(\lambda)$ and λ is an eigval of T with an eigvec v. Then $p(T)v = p(\lambda)v = \alpha v$. $\square$

OR. Define q by $q(z) = p(z) - \alpha$. λ is a zero of q.
Becs $q(T)v = \big(p(T) - \alpha I\big)v = q(\lambda)v = \big(p(\lambda) - \alpha\big)v = 0$.
Hence $q(T)$ is not inje $\Rightarrow \big(p(T) - \alpha I\big)$ is not inje. $\square$

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12 [OR (4E.5.B.6)] *Give an exa of an optor on $\mathbf{R}^2$ that shows the result above does not hold if $\mathbf{C}$ is replaced with $\mathbf{R}$.*

SOLUS:
Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(w,z) = (-z, w)$.
By Exe (4E 5.B.11), $\mathcal{M}\big(T, ((1,0),(0,1))\big) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ the min poly of T is $z^2 + 1$.
Define p by $p(z) = z^2$. Then $p(T) = T^2 = -I$. Thus $p(T)$ has eigval -1.
While $\nexists\,\lambda \in \mathbf{R}$ suth $-1 = p(\lambda) = \lambda^2$. $\square$

• (4E 5.B.17) *Supp V is finide, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$, and p is the min poly of T. Show the min poly of $(T - \lambda I)$ is the poly q defined by $q(z) = p(z + \lambda)$.*

SOLUS:
$q(T - \lambda I) = 0 \Rightarrow q$ is poly multi of the min poly of $(T - \lambda I)$.
Supp the deg of the min poly of $(T - \lambda I)$ is n, and the deg of the min poly of T is m.
By definition of min poly,
n is the smallest suth $(T - \lambda I)^n \in \operatorname{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{n-1})$;
m is the smallest suth $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.
又 $T^k \in \operatorname{span}(I, T, \dots, T^{k-1}) \iff (T - \lambda)^k \in \operatorname{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{k-1})$.
Thus $n = m$. 又 q is monic. By the uniqnes of min poly. $\square$

• (4E 5.B.18) *Supp V is finide, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}\backslash\{0\}$, and p is the min poly of T. Show the min poly of λT is the poly q defined by $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.*

SOLUS:
$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow q$ is a poly multi of the min poly of λT.
Supp the deg of the min poly of λT is n, and the deg of the min poly of T is m.
By definition of min poly,
n is the smallest suth $(\lambda T)^n \in \operatorname{span}(\lambda I, \lambda T, \dots, (\lambda T)^{n-1})$;
m is the smallest suth $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.
又 $(\lambda T)^k \in \operatorname{span}(\lambda I, \lambda T, \dots, (\lambda T)^{k-1}) \iff T^k \in \operatorname{span}(I, T \dots, T^{k-1})$.
Thus $n = m$. 又 q is monic. By the uniqnes of min poly. $\square$

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18 [OR (4E 5.B.15)] *Supp V is a finide complex vecsp with $\dim V > 0$ and $T \in \mathcal{L}(V)$. Define $f : \mathbf{C} \to \mathbf{R}$ by $f(\lambda) = \dim \operatorname{range}(T - \lambda I)$. Prove f is not a continuous function.*

SOLUS: Note that V is finide.
Let λ_0 be an eigval of T. Then $(T - \lambda_0 I)$ is not surj. Hence $\dim \operatorname{range}(T - \lambda_0 I) < \dim V$.
Becs T has finily many eigvals. There exis a seq of number $\{\lambda_n\}$ suth $\lim\limits_{n\to\infty} \lambda_n = \lambda_0$.

And λ_n is not an eigval of T for each $n \Rightarrow \dim \operatorname{range}(T-\lambda_n I) = \dim V \neq \dim \operatorname{range}(T-\lambda_0 I)$.
Thus $f(\lambda_0) \neq \lim_{n\to\infty} f(\lambda_n)$. $\square$

- (4E 5.B.9) *Supp $T \in \mathcal{L}(V)$ is suth wrto some bss of V,*
  all ent of the matrix of T are rational numbers.
  Explain why all coeffs of the min poly of T are rational numbers.

Solus:

Let $(v_1,\dots,v_n)$ denote the bss suth $\mathcal{M}(T,(v_1,\dots,v_n))_{j,k} = A_{j,k} \in \mathbf{Q}$ for all $j,k = 1,\dots,n$.
Denote $\mathcal{M}(v_j,(v_1,\dots,v_n))$ by x_j for each v_j.
Supp p is the min poly of T and $p(z) = z^m + \cdots + c_1 z + c_0$. Now we show each $c_j \in \mathbf{Q}$.
Note that $\forall s \in \mathbf{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbf{Q}^{n,n}$ and $T^s v_k = A^s_{1,k}v_1 + \cdots + A^s_{n,k}v_n$ for all $k \in \{1,\dots,n\}$.

$$\text{Thus}\begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \cdots + c_1A + c_0I)x_1 = \sum\limits_{j=1}^{n} (A^m + \cdots + c_1A + c_0I)_{j,1}x_j = 0; \\ \qquad\vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \cdots + c_1A + c_0I)x_n = \sum\limits_{j=1}^{n} (A^m + \cdots + c_1A + c_0I)_{j,n}x_j = 0; \end{cases}$$

$$\text{More clearly,}\begin{cases} (A^m + \cdots + c_1A + c_0I)_{1,1} = \cdots = (A^m + \cdots + c_1A + c_0I)_{n,1} = 0; \\ \qquad\qquad\qquad\vdots \quad \ddots \quad \vdots \\ (A^m + \cdots + c_1A + c_0I)_{1,n} = \cdots = (A^m + \cdots + c_1A + c_0I)_{n,n} = 0; \end{cases}$$

Hence we get a system of n^2 liney equations in m unknowns $c_0, c_1, \dots, c_{m-1}$.
We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbf{Q}$. $\square$

- [Or (4E 5.B.16), Or (8.C.18)] *Supp $a_0,\dots,a_{n-1} \in \mathbf{F}$. Let T be the optor on $\mathbf{F}^n$ suth*

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ wrto the std bss } (e_1,\dots,e_n).$$

Show the min poly of T is p defined by $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$.

*$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the min poly of some optor. Hence a formula or an algo that could produce exact eigvals for each optor on each $\mathbf{F}^n$ could then produce exact zeros for each poly [by 8.36(b)]. Thus there is no such formula or algo. However, efficient numeric methods exis for obtaining very good approximations for the eigvals of an optor.*

Solus: Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ is liney indep. 又 The deg of min poly is at most n.

$T^n e_1 = \cdots = T^{n-k}e_{1+k} = \cdots = Te_n = -a_0e_1 - a_1e_2 - a_2e_3 - \cdots - a_{n-1}e_n$
$= (-a_0I - a_1T - a_2T^2 - \cdots - a_{n-1}T^{n-1})e_1$. Thus $p(T)e_1 = 0 = p(T)e_j$ for each $e_j = T^{j-1}e_1$. $\square$

- **Eigenvalues On Odd-Dimensional Real Vector Spaces**
- **Even-Dimensional Null Space**

Supp $\mathbf{F} = \mathbf{R}$, V is finide, $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$.
Prove $\dim \operatorname{null}(T^2 + bT + cI)$ *is an even number.*

Solus:

Denote $\operatorname{null}(T^2 + bT + cI)$ by R. Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.
Supp λ is an eigval of T_R with an eigvec $v \in R$.
Then $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \left((\lambda + b)^2 + c - \frac{b^2}{4}\right)v$.

Becs $c - \frac{b^2}{4} > 0$ and we have $v = 0$. Thus T_R has no eigvals.

Let U be invarsp of R that has the largest, even dim among all invarsps.

Asum $U \neq R$. Then $\exists\, w \in R$ but $w \notin U$. Let W be suth $(w, T|_R w)$ is a bss of W.

Becs $T|_R^2 w = -bT|_R w - cw \in W$. Hence W is invarsp of dim 2.

Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,

for if not, becs $w \notin U, T|_R w \in U$,

$U \cap W$ is invard $T|_R$ of one dim (impossible becs $T|_R$ has no eigvecs).

Hence $U + W$ is even-dim invarsp under $T|_R$, ctradic the max of dim U.

Thus the asum was incorrect. Hence $R = \operatorname{null}(T^2 + bT + cI) = U$ has even dim. □

• Operators On Odd-Dimensional Vector Spaces Have Eigenvalues

(a) *Supp* $\mathbf{F} = \mathbf{C}$. Then by [5.21], done.

(b) *Supp* $\mathbf{F} = \mathbf{R}$, *V is finide, and* $\dim V = n$ *is an odd number.*

Let $T \in \mathcal{L}(V)$ and the min poly is p. Prove T has an eigval.

Solus:

(i) If $n = 1$, then done.

(ii) Supp $n \geqslant 3$. Asum every optor, on odd-dim vecsps of dim less than n, has an eigval.

If p is a poly multi of $(x - \lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] λ is an eigval of T and done.

Now supp $b, c \in \mathbf{R}$ suth $b^2 < 4c$ and p is a poly multi of $x^2 + bx + c$ (see [4.17]).

Then $\exists\, q \in \mathcal{P}(\mathbf{R})$ suth $p(x) = q(x)(x^2 + bx + c)$ for all $x \in \mathbf{R}$.

Now $0 = p(T) = \big(q(T)\big)(T^2 + bT + cI)$, which means that $q(T)|_{\operatorname{range}(T^2+bT+cI)} = 0$.

Becs $\deg q < \deg p$ and p is the min poly of T, hence $\operatorname{range}(T^2 + bT + cI) \neq V$.

又 $\dim V$ is odd and $\dim \operatorname{null}(T^2 + bT + cI)$ is even (by our previous result).

Thus $\dim V - \dim \operatorname{null}(T^2 + bT + cI) = \dim \operatorname{range}(T^2 + bT + cI)$ is odd.

By [5.18], $\operatorname{range}(T^2 + bT + cI)$ is invarsp of V under T that has odd dim less than n.

Our induc hypo now implies that $T|_{\operatorname{range}(T^2+bT+cI)}$ has an eigval.

By induc. □

• (2E Ch5.24) *Supp* $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$ *has no eigvals.*

Prove every invarsp of V under T is even-dim.

Solus:

Supp U is such a subsp. Then $T|_U \in \mathcal{L}(U)$. We prove by ctradic.

If $\dim U$ is odd, then $T|_U$ has an eigval and so is T, so that $\exists$ invarsp of 1 dim, ctradic. □

• (4E 5.B.29) *Show every optor on a finide vecsp of dim* ≥ 2 *has a* 2*-dim invarsp.*

Solus:

Using induc on $\dim V$.

(i) $\dim V = 2$, done.

(ii) $\dim V > 2$. Asum the desired result is true for vecsp of smaller dim.

Supp p is the min poly of deg m and $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.

If $T = \lambda I$ ($\Leftrightarrow m = 1 \vee m = -\infty$), then done. ($m \neq 0$ becs $\dim V \neq 0$.)

Now define a q by $q(z) = (z - \lambda_1)(z - \lambda_2)$.

By asum, $T|_{\operatorname{null} q(T)}$ has invarsp of dim 2. □

5.B: II

9 14 15 20 | 4E: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 11, 12, 13, 14

- (4E 5.C.1) *Prove or give a counterexa*:
  If $T \in \mathcal{L}(V)$ and T^2 has an upper-trig matrix, then T has an upper-trig matrix.

Solus:

- (4E 5.C.2) *Supp A and B are upper-trig matrices of the same size, with $\alpha_1, \dots, \alpha_n$ on the diag of A and $\beta_1, \dots, \beta_n$ on the diag of B.*
  (a) *Show $A + B$ is an upper-trig matrix with $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ on the diag.*
  (b) *Show AB is an upper-trig matrix with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diag.*

Solus:

- (4E 5.C.3)
  Supp $T \in \mathcal{L}(V)$ is inv and $B = (v_1, \dots, v_n)$ is a bss of V suth $\mathcal{M}(T, B) = A$ is upper trig, with $\lambda_1, \dots, \lambda_n$ on the diag.
  Show the matrix of $\mathcal{M}(T^{-1}, B) = A^{-1}$ is also upper trig, with $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ on the diag.

Solus:

---

9 [4E 5.C.7] *Supp V is finide, $T \in \mathcal{L}(V)$, and $v \in V$.*
(a) *Prove $\exists\,!$ monic poly p_v of smallest deg suth $p_v(T)v = 0$.*
(b) *Prove the min poly of T is a poly multi of p_v.*

Solus:

---

14 [Or (4E 5.C.4)] *Give an optor T suth wrto some bss, $\mathcal{M}(T)_{k,k} = 0$ for each k, while T is inv.*

Solus:

---

15 [Or (4E 5.C.5)] *Give an optor T suth wrto some bss, $\mathcal{M}(T)_{k,k} \neq 0$ for each k, while T is not inv.*

Solus:

---

20 [Or (Or 4E 5.C.6)]
Supp $\mathbf{F} = \mathbf{C}$, V is finide, and $T \in \mathcal{L}(V)$.
Prove if $k \in \{1, \dots, \dim V\}$, then V has a k dim subsp invard T.

Solus:

- (4E 5.C.8) *Supp V is finide, $T \in \mathcal{L}(V)$, and $\exists\, v \in V \backslash \{0\}$ suth $T^2 v + 2Tv = -2v$.*
  (a) *Prove if $\mathbf{F} = \mathbf{R}$, then $\nexists$ a bss of V wrto which T has an upper-trig matrix.*
  (b) *Prove if $\mathbf{F} = \mathbf{C}$ and A is an upper-trig matrix that equals the matrix of T wrto some bss of V, then $-1 + \mathrm{i}$ or $-1 - \mathrm{i}$ appears on the diag of A.*

Solus:

- (4E 5.C.9) *Supp $B \in \mathbf{F}^{n,n}$ with complex ent.*

Prove $\exists$ inv $A \in \mathbf{F}^{n,n}$ with complex ent suth $A^{-1}BA$ is an upper-trig matrix.

Solus:

- (4E 5.C.10) *Supp $T \in \mathcal{L}(V)$ and $(v_1, \dots, v_n)$ is a bss of V.*
  Show the following are equi.
  (a) *The matrix of T wrto $(v_1, \dots, v_n)$ is lower trig.*
  (b) $\operatorname{span}(v_k, \dots, v_n)$ *is invard T for each $k = 1, \dots, n$.*
  (c) $Tv_k \in \operatorname{span}(v_k, \dots, v_n)$ *for each $k = 1, \dots, n$.*

Solus:

- (4E 5.C.11) *Supp $\mathbf{F} = \mathbf{C}$ and V is finide.*
  Prove if $T \in \mathcal{L}(V)$, then T has a lower-trig matrix wrto some bss.

Solus:

- (4E 5.C.12)
  Supp V is finide, $T \in \mathcal{L}(V)$ has an upper-trig matrix wrto some bss,
  and U is a subsp of V that is invard T.
  (a) *Prove $T|_U$ has an upper-trig matrix wrto some bss of U.*
  (b) *Prove T/U has an upper-trig matrix wrto some bss of V/U.*

Solus:

- (4E 5.C.13) *Supp V is finide, $T \in \mathcal{L}(V)$. Supp U is invarsp of V under T*
  suth $T|_U, T/U$ have upper-trig matrix.
  Prove T has upper-trig matrix.

Solus:

- (4E 5.C.14) *Supp V is finide and $T \in \mathcal{L}(V)$.*
  Prove T has upper-trig matrix $\Longleftrightarrow$ T' has upper-trig matrix.

Solus:

Ended

5.C

XXXX

Ended

5.E* [4E] 1 2 3 4 5 6 7 8 9 10

1 *Give commu optors $S, T \in \mathbf{F}^4$ suth*
$\exists$ subsp of $\mathbf{F}^4$ invard S but not T and $\exists$ subsp of $\mathbf{F}^4$ invard T but not S.

Solus:

---

2 *Supp $\mathcal{E}$ is a subset of $\mathcal{L}(V)$ and every elem of $\mathcal{E}$ is diag.*
Prove $\exists$ a bss of V wrto which
every elem of $\mathcal{E}$ has a diag matrix $\Longleftrightarrow$ every pair of elems of $\mathcal{E}$ commu.

This exercise extends [5.76], which considers the case in which $\mathcal{E}$ contains only two elems.
For this exercise, $\mathcal{E}$ may contain any number of elems, and $\mathcal{E}$ may even be an infini set.

SOLUS:

---

3 *Supp $S, T \in \mathcal{L}(V)$ are suth $ST = TS$. Supp $p \in \mathcal{P}(\mathbf{F})$.*
(a) *Prove* null $p(S)$ *is invard T.* (b) *Prove* range $p(S)$ *is invard T.*
See NOTE FOR[5.17] *for the special case $S = T$.*

SOLUS:

---

4 *Prove or give a counterexa: A diag matrix A and upper-trig matrix B of the same size commu.*

SOLUS:

---

5 *Prove a pair of optors on a finide vecsp commu $\iff$ their dual optors commu.*

SOLUS:

---

6 *Supp V is a finide complex vecsp and $S, T \in \mathcal{L}(V)$ commu.*
Prove $\exists\, \alpha, \lambda \in \mathbf{C}$ suth $\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V$.

SOLUS:

---

7 *Supp V is a complex vecsp, $S \in \mathcal{L}(V)$ is diag, and T commu with S.*
Prove $\exists$ bss B of V suth S has a diag matrix wrto B
and T has upper-trig matrix wrto B.

SOLUS:

---

8 *Supp $m = 3$ in [5.72] and D_x, D_y are the commu partial diff optors on $\mathcal{P}_3(\mathbf{R}^2)$ from [5.72].*
Find a bss of $\mathcal{P}_3(\mathbf{R}^2)$ wrto which D_x and D_y each have upper-trig matrix.

SOLUS:

---

9 *Supp V is a finide nonzero complex vecsp.*
Supp that $\mathcal{E} \subseteq \mathcal{L}(V)$ is suth S and T commu for all $S, T \in \mathcal{E}$.
(a) *Prove $\exists$ eigvec $v \in V$ for every elem of $\mathcal{E}$.*
(b) *Prove $\exists$ a bss of V wrto which every elem of $\mathcal{E}$ has upper-trig matrix.*

SOLUS:

---

10 *Give commu optors S, T on a finide real vecsp suth*
$S + T$ has a eigval that does not equal an eigval of S plus an eigval of T
and ST has a eigval that does not equal an eigval of S times an eigval of T.

SOLUS:

ENDED