1·B

- (OR [9.2,9.3]. OR Problem (1) in 9.A)

Suppose V is a real vector space. The complexification of V, denoted by $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.

- *Addition on $V_{\mathbb{C}}$ is defined by*
$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
for all $u_1, v_1, u_2, v_2 \in V$.
- *Complex scalar multiplication on $V_{\mathbb{C}}$ is defined by*
$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$
for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions above, $V_{\mathbb{C}}$ is a complex vector space.

Think of V as a subset of $V_{\mathbb{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbb{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.

SOLUTION:

- Commutativity: $(u_1 + \mathrm{i}v_1) + (u_2 + \mathrm{i}v_2) = (u_2 + \mathrm{i}v_2) + (u_1 + \mathrm{i}v_1)$.
- Associativity:

(I) $[(u_1 + \mathrm{i}v_1) + (u_2 + \mathrm{i}v_2)] + (u_3 + \mathrm{i}v_3) = (u_1 + \mathrm{i}v_1) + [(u_2 + \mathrm{i}v_2) + (u_3 + \mathrm{i}v_3)]$.

(II) $\begin{cases} [(a+b\mathrm{i})(c+d\mathrm{i})](u+\mathrm{i}v) = [(ac-bd)+(ad+bc)\mathrm{i}](u+\mathrm{i}v) = [(ac-bd)u-(ad+bc)v]+\mathrm{i}[(ac-bd)v+(ad+bc)u] \\ (a+b\mathrm{i})[(c+d\mathrm{i})(u+\mathrm{i}v)] = (a+b\mathrm{i})[(cu-dv)+\mathrm{i}(cv+du)] = [a(cu-dv)-b(cv+du)]+\mathrm{i}[a(cv+du)+b(cu-dv)] \end{cases}$

- Additive identity.
- Additive inverse: $(u_1 + \mathrm{i}v_1) + (-u_1 + \mathrm{i}(-v_1)) = 0$.
- Multiplication identity.
- Distributive properties:

(I) $\begin{cases} (a+b\mathrm{i})[(u_1+\mathrm{i}v_1)+(u_2+\mathrm{i}v_2)] = (a+b\mathrm{i})[(u_1+u_2)+\mathrm{i}(v_1+v_2)] \\ \qquad = [a(u_1+u_2)-b(v_1+v_2)]+\mathrm{i}[a(v_1+v_2)+b(u_1+u_2)] \\ (a+b\mathrm{i})(u_1+\mathrm{i}v_1)+(a+b\mathrm{i})(u_2+\mathrm{i}v_2) = [(au_1-bv_1)+\mathrm{i}(av_1+bu_1)]+[(au_2-bv_2)+\mathrm{i}(av_2+bu_2)] \end{cases}$

(II) $\begin{cases} [(a+b\mathrm{i})+(c+d\mathrm{i})](u+\mathrm{i}v) = [(a+c)+(b+d)\mathrm{i}](u+\mathrm{i}v) = [(a+c)u-(b+d)v]+\mathrm{i}[(a+c)v+(b+d)u] \\ (a+b\mathrm{i})(u+\mathrm{i}v)+(c+d\mathrm{i})(u+\mathrm{i}v) = [(au-bv)+\mathrm{i}(av+bu)]+[(cu-dv)+\mathrm{i}(cv+du)] \end{cases}$

$\square$

- *Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.*

SOLUTION:

- Addition on V^S is defined by $(f + g)(x) = f(x) + g(x)$ for any $x \in S$ and $f, g \in V^S$.
- Scalar Multiplication on V^S is defined by $(\lambda f)(x) = \lambda f(x)$ for any $x \in S, \lambda \in \mathbf{F}, f \in V^S$.

Commutativity. Associativity.

Additive identity: $0\,(x) = 0$.

Additive inverse: $f(x) + (-f)(x) = 0$.

Multiplication identity: $I(x) = x$.

Distributive properties: $(\lambda(f + g))(x) = \lambda(f(x) + g(x)) = (\lambda f)(x) + (\lambda g)(x)$;
$((\lambda + \mu)f)(x) = (\lambda + \mu)f(x) = \lambda f(x) + \mu f(x)$. $\square$

---

1 *Prove that $-(-v) = v$ for every $v \in V$.*

SOLUTION: $\left.\begin{array}{l} (-(-v)) + (-v) = 0 \\ v + (-v) = 0 \end{array}\right\} \Rightarrow$ By the uniqueness of additive inverse. $\square$

OR. $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$ $\square$

---

2 *Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.*

Solution: If $a = 0$, then we are done.

Otherwise, $\exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. $\square$

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3 *Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.*

Solution:

[Existence] Let $x = \frac{1}{3}(w - v)$.

[Uniqueness] Suppose $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II).

Then (I) $-$ (II) : $3(x_1 - x_2) = 0 \Rightarrow$ By Problem (2), $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$. $\square$

---

5 *Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that $0v = 0$ for all $v \in V$. Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.*

The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.

Solution: Using [1.31]. $0v = 0$ for all $v \in V \Leftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. $\square$

---

6 *Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$.*
Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

and (I) $t + \infty = \infty + t = \infty + \infty = \infty$,

(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,

(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution: Not a vector space. By Associativity: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

Or By Distributive properties: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. $\square$

Ended

1·C

2 (1.35)

(b) *The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$*

Denote the set by U. $\forall x \in [0, 1]$ we have $\left.\begin{array}{l} (一)\ 0 \in U;\ f(x) = 0 \Leftrightarrow f = 0 \\ (二)\ \forall f, g \in U,\ (f+g)(x) = f(x) + g(x) \\ (三)\ \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R},\ (\lambda f)(x) = \lambda f(x) \end{array}\right\} \Rightarrow \square$

(c) *The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$*

Denote the set by U. $\left.\begin{array}{l} (一)\ 0 \in U \\ (二)\ \forall f, g \in U,\ (f' + g') = f' + g' \\ (三)\ \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R}, (\lambda f)' = \lambda (f)' \end{array}\right\} \Rightarrow \square$

(d) *The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.*

Denote the set by U. Suppose $b = 0$. Then

$$\left.\begin{array}{l}(一)\ 0 \in U \\ (二)\ \forall f, g \in U,\ (f+g)'(2) = f'(2) + g'(2) = 0 \\ (三)\ \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R},\ (\lambda f)'(2) = \lambda f'(2)\end{array}\right\} \Rightarrow U \text{ is a subspace of } \mathbf{R}^{(0,3)}.$$

Suppose U is a subspace of $\mathbf{R}^{(0,3)}$. Suppose $f = 0 \Rightarrow f \in U$. Then $f'(2) = 0 = b$. $\square$

(e) *The set of all sequences with limit 0, of complex numbers, is a subspace of* $\mathbf{C}^\infty$.

Denote the set by A.

$$\left.\begin{array}{l}(一)\ (0,0,\dots) \in A \\ (二)\ \forall\ (a_1,a_2,\dots), (b_1,b_2,\dots) \in A \Longleftrightarrow \lim\limits_{n\to\infty} a_n = 0,\ \lim\limits_{n\to\infty} b_n = 0. \\ \quad \text{Thus } \lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = 0 \Rightarrow (a_1+b_1, a_2+b_2, \dots) \in A \\ (三)\ \forall\ (a_1,a_2,\dots) \in A, \forall \lambda \in \mathbf{F} = \mathbf{C},\ \lim\limits_{n\to\infty}(\lambda a_n) = \lambda \lim\limits_{n\to\infty} a_n = 0 \Rightarrow \lambda(a_1,a_2,\dots) \in A\end{array}\right\} \Rightarrow \square$$

---

4 *Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$*

SOLUTION: Denote the set by V_b.

(a) Suppose V_b is a subspace of $\mathbf{R}^{[0,1]}$, then $\forall f \in V_b$, we have $\int_0^1 f = b$.

Because $kf \in V_b$ for any $k \in \mathbf{R}$. Hence $\int_0^1 (kf) = k\int_0^1 f = kb = b \Leftrightarrow b = 0$.

OR. Because $g = 0 \in V_b \Rightarrow \int_0^1 g = 0 = b$.

(b) Suppose $b = 0$. $\forall f, g \in V_b = V_0, \lambda \in \mathbf{R}$, $\int_0^1 (f + \lambda g) = \int_0^1 f + \int_0^1 g = 0$. $\square$

---

5 *Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?*

ANSWER: No. Because $\mathbf{R}^2$ is not closed on scalar multiplication on $\mathbf{C}$.

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6 (a) *Is $\{(a,b,c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?*

ANSWER: True. As can be easily checked (note that $a^3 = b^3 \Leftrightarrow a = b$).

(b) *Is $\{(a,b,c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of $\mathbf{C}^3$?*

ANSWER: No. Note that $(\dfrac{-1 \pm \sqrt{3}\mathrm{i}}{2})^3 = 1$.

---

7 *Prove or give a counterexample: If U is a nonempty subset of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbf{R}^2$.*

SOLUTION: Let $U = \mathbf{Z}^2$, $(\mathbf{Z}^*)^2$, $(\mathbf{Q}^*)^2$, $\mathbf{Q}^2 \backslash \{0\}$, or $\mathbf{R}^2 \backslash \{0\}$.

---

8 *Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.*

SOLUTION: $U = \{(x,y) \in \mathbf{R}^2 : x = 0 \vee y = 0\}$.

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9 *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.*

SOLUTION: Denote the set by S.

Suppose $h(x) = \sin\sqrt{2}x + \cos x \in S$, since both $f(x) = \sin\sqrt{2}$, $g(x) = \cos x$ are periodic functions $\mathbf{R} \to \mathbf{R}$.

Assume $\exists\ p \in \mathbf{N}^+$ such that $h(x) = h(x+p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$

$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Contradiction! $\square$

---

11 *Prove that the intersection of every collection of subspaces of V is a subspace of V.*

Solution:

Suppose $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collection of subspaces of V; here Γ is an arbitrary index set.

We need to prove that $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vectors that are in U_α for each $\alpha\in\Gamma$, is a subspace of V.

(一) $0\in\bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u,v\in\bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u+v\in U_\alpha,\ \forall\alpha\in\Gamma \Rightarrow u+v\in\bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under addition.

(三) $u\in\bigcap_{\alpha\in\Gamma} U_\alpha, \lambda\in\mathbf{F} \Rightarrow \lambda u\in U_\alpha,\ \forall\alpha\in\Gamma \Rightarrow \lambda u\in\bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under scalar multiplication.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of V that is closed under addition and scalar multiplication.

Hence $\bigcap_{\alpha\in\Gamma} U_\alpha$ is a subspace of V. $\square$

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12 *Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.*

Solution: Suppose U and W are subspaces of V.

(a) Suppose $U\subseteq W$. Then $U\cup W=W$ is a subspace of V.

(b) Suppose $U\cup W$ is a subspace of V. Suppose $U\not\subseteq W$ and $U\not\supseteq W$ ($U\cup W\neq U$ and W).

Then $\forall a\in U$ but $a\notin W$; $b\in W$ but $b\notin U$. $a+b\in U\cup W$.

$$\left.\begin{array}{l}\text{(1) Consider } a+b\in U\Rightarrow b=(a+b)+(-a)\in U, \text{ contradicts!}\\ \text{(2) Consider } a+b\in W\Rightarrow a=(a+b)+(-b)\in W, \text{ contradicts!}\end{array}\right\}\Rightarrow U\cup W=U \text{ or } W. \text{ Contradicts!}$$

Thus $U\subseteq W$ and $U\supseteq W$. $\square$

---

13 *Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.*

This exercise is surprisingly harder than Exercise 12, possibly because this exercise is not true if we replace **F** *with a field containing only two elements.*

Solution: Suppose A,B,C are subspaces of V.

(a) If any two of them are subsets of the third one, then $A\cup B\cup C=A, B$ or C, which is a subspace of V.

$$\text{(b)* If } A\cup B\cup C \text{ is a subspace of } V, \text{ suppose } \left\{\begin{array}{c} A\not\supseteq B \text{ and } C\\ B\not\supseteq A \text{ and } C\\ C\not\supseteq A \text{ and } B\end{array}\right\}\Longleftrightarrow A\cap B\cap C\neq A, B \text{ and } C.$$

$\forall a\in A$ but $a\notin B,C$; $\forall b\in B$ but $b\notin A,C$; $\forall c\in C$ but $c\notin A,B$; by assumption, $a+b+c\in A\cup B\cup C$.

(I) $A\cup B$ is a subspace $\Rightarrow$ By Problem (12), $A\subseteq B$ or $A\supseteq B$.

(II) $A\cup C$ is a subspace $\Rightarrow$ By Problem (12), $A\subseteq C$ or $A\supseteq C$.

(III) $B\cup C$ is a subspace $\Rightarrow$ By Problem (12), $B\subseteq C$ or $B\supseteq C$.

Any two of (I), (II) and (III) must be true.

$$\text{(一). (I) and (II) are true. Then } \left.\begin{array}{r} C\subseteq B\subseteq A\\ \text{or } C\supseteq B\supseteq A\\ \text{or } B\supseteq A,C\\ \text{or } \underbrace{B\subseteq A,C}_{(1)}\end{array}\right\}\Rightarrow\left\{\begin{array}{r} A\supseteq B,C\\ \text{or } B\supseteq A,C\\ \text{or } C\supseteq A,B\end{array}\right.$$

$$\text{(二). (II) and (III) are true. Then } \left.\begin{array}{r} A\subseteq C\subseteq B\\ \text{or } A\supseteq C\supseteq B\\ \text{or } C\supseteq A,B\\ \text{or } \underbrace{C\subseteq A,B}_{(2)}\end{array}\right\}\Rightarrow\left\{\begin{array}{r} A\supseteq B,C\\ \text{or } B\supseteq A,C\\ \text{or } C\supseteq A,B\end{array}\right.$$

(三). (I) and (III) are true. Then $\left.\begin{array}{r} B \subseteq A \subseteq C \\ \text{or } B \supseteq A \supseteq C \\ \text{or } A \supseteq B, C \\ \text{or } \underbrace{A \subseteq B, C}_{(3)} \end{array}\right\} \Rightarrow \left\{\begin{array}{r} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array}\right.$

Among these, any two of (1), (2) and (3) must be true. $\left.\begin{array}{l} \left.\begin{array}{c}(1)\\(2)\end{array}\right\} \Rightarrow B \subseteq C \subseteq A \\ \left.\begin{array}{c}(2)\\(3)\end{array}\right\} \Rightarrow C \subseteq A \subseteq B \\ \left.\begin{array}{c}(1)\\(3)\end{array}\right\} \Rightarrow B \subseteq A \subseteq C \end{array}\right\} \Rightarrow \square$

• *Suppose $U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$ and $W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$. Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.*

SOLUTION:

(a) $U + W = \{(x+y, x-y, 2(x+y)) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} = \{(x', y', 2x')) \in \mathbf{F}^3 : x', y' \in \mathbf{F}\}$.

(b) $U + W$ is a plane of which $(1, 0, 2), (0, 1, 0)$ is a basis. $\square$

---

15 *Suppose U is a subspace of V. What is $U + U$?*

SOLUTION: $\left.\begin{array}{r} \forall x, y \in U, x + y \in U \Rightarrow U + U \subseteq U \\ \forall x \in U, 0 \in U, x + 0 \in U + U \Rightarrow U \subseteq U + U \end{array}\right\} \Rightarrow U + U = U. \square$

---

16 *Suppose U and W are subspaces of V. Prove that $U + W = W + U$?*

SOLUTION: $\forall x \in U, y \in W,$ $\left.\begin{array}{r} x + y = y + x \in W + U \Rightarrow U + W \subseteq W + U \\ y + x = x + y \in U + W \Rightarrow W + U \subseteq U + W \end{array}\right\} \Rightarrow U + W = W + U. \ \square$

---

17 *Suppose V_1, V_2, V_3 are subspaces of V. Prove that $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$.*

SOLUTION:

Let $x \in V_1, y \in V_2, z \in V_3$. Denote $(V_1 + V_2) + V_3$ by L, $V_1 + (V_2 + V_3)$ by R.

$\left.\begin{array}{l} \forall u \in L,\ \exists\ x, y, z,\ u = (x + y) + z = x + (y + z) \in R \Rightarrow L \subseteq R \\ \forall u \in R,\ \exists\ x, y, z,\ u = x + (y + z) = (x + y) + z \in L \Rightarrow R \subseteq L \end{array}\right\} \Rightarrow (V_1 + V_2) + V_3 = V_1 + (V_2 + V_3).\ \square$

---

18 *Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?*

SOLUTION:

Suppose Ω is the additive identity.

For any subspace U of V. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

Now suppose W is an additive inverse of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. $\square$

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19 *Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that $V_1 + U = V_2 + U$, then $V_1 = V_2$.*

SOLUTION: A counterexample:

$V = \mathbf{F}^3, U = \{(x, 0, 0) \in \mathbf{F}^3 : x \in \mathbf{F}\}$,

$V_1 = \{(x, x, y)) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}, V_2 = \{(x, y, z)) \in \mathbf{F}^3 : x, y, z \in \mathbf{F}\}$.

EXAMPLE: *Suppose* $U = \{(x,x,y,y) \in \mathbf{F}^4 : x,y \in \mathbf{F}\}, W = \{(x,x,x,y) \in \mathbf{F}^4 : x,y \in \mathbf{F}\}$.
Prove that $U + W = \{(x,x,y,z) \in \mathbf{F}^4 : x,y,z \in \mathbf{F}\}$.

SOLUTION: Let T denote $\{(x,x,y,z) \in \mathbf{F}^4 : x,y,z \in \mathbf{F}\}$.

(a) By definition, $U+W = \{(x_1+x_2, x_1+x_2, y_1+x_2, y_1+y_2) \in \mathbf{F}^4 : (x_1,x_1,y_1,y_1) \in U, (x_2,x_2,x_2,y_2) \in W\}$.
$\Rightarrow \forall v \in U+W,\ \exists\, t \in T,\ \ v = t \Rightarrow U + W \subseteq T$.

(b) $\forall x,y,z \in \mathbf{F}$, let $u = (0,0,y-x,y-x) \in U,\ \ w = (x,x,x,-y+x+z) \in W$
$\Rightarrow (x,x,y,z) = u + w \in U + W$. Hence $\forall t \in T,\ \exists\, u \in U, w \in W,\ \ t = u + w \Rightarrow T \subseteq U + W$. $\square$

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21 *Suppose* $U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5 : x,y \in \mathbf{F}\}$.
Find a subspace W *of* $\mathbf{F}^5$ *such that* $\mathbf{F}^5 = U \oplus W$.

SOLUTION:

(a) Let $W = \{(0,0,z,w,u) \in \mathbf{F}^5 : z,w,u \in \mathbf{F}\}$. Then $W \cap U = \{0\}$.

(b) $\forall x,y,z,w,u \in \mathbf{F}$, let $u = (x,y,x+y,x-y,2x) \in U$,
$$w = (0,0,z-x-y,w-x-y,u-2x) \in W$$
$\Rightarrow (x,y,z,w,u) = u + w \Rightarrow \mathbf{F}^5 \subseteq U + W$. $\square$

---

22 *Suppose* $U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5 : x,y \in \mathbf{F}\}$.
Find three subspaces W_1, W_2, W_3 *of* $\mathbf{F}^5$, *none of which equals* $\{0\}$,
such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

SOLUTION:

(1) Let $W_1 = \{(0,0,z,0,0) \in \mathbf{F}^5 : z \in \mathbf{F}\}$. Then $W_1 \cap U = \{0\}$.
Let $U_1 = U \oplus W_1$. Then $U_1 = \{(x,y,z,x-y,2x) \in \mathbf{F}^5 : x,y,z \in \mathbf{F}\}$. (Check it!)

(2) Let $W_2 = \{(0,0,0,w,0) \in \mathbf{F}^5 : w \in \mathbf{F}\}$. Then $W_2 \cap U_1 = \{0\}$.
Let $U_2 = U_1 \oplus W_2$. Then $U_2 = \{(x,y,z,w,2x) \in \mathbf{F}^5 : x,y,z,w \in \mathbf{F}\}$.

(3) Let $W_3 = \{(0,0,0,0,u) \in \mathbf{F}^5 : u \in \mathbf{F}\}$. Then $W_3 \cap U_2 = \{0\}$.
Let $U_3 = U_2 \oplus W_3$. Then $U_3 = \{(x,y,z,w,u) \in \mathbf{F}^5 : x,y,z,w,u \in \mathbf{F}\}$.

Thus $\mathbf{F}^5 = ((U \oplus W_1) \oplus W_2) \oplus W_3$. $\square$

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23 *Prove or give a counterexample: If* V_1, V_2, U *are subspaces of* V *such that*
$V = V_1 \oplus U$ *and* $V = V_2 \oplus U$, *then* $V_1 = V_2$.

HINT: *When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in* $\mathbf{F}^2$.

SOLUTION: A counterexample:
$V = \mathbf{F}^2, U = \{(x,x) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_1 = \{(x,0) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_2 = \{(0,x) \in \mathbf{F}^2 : x \in \mathbf{F}\}$.

---

24 *Let* V_e *denote the set of real-valued even functions on* $\mathbf{R}$
and let V_o *denote the set of real-valued odd functions on* $\mathbf{R}$. *Show that* $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

SOLUTION:

(a) $V_e \cap V_o = \{f : f(x) = f(-x) = -f(-x)\} = \{0\}$.

(b) $$\left.\begin{array}{l} f_e \in V_e \Leftrightarrow f_e(x) = f_e(-x) \Leftarrow \text{let } f_e(x) = \dfrac{g(x)+g(-x)}{2} \\ f_o \in V_o \Leftrightarrow f_o(x) = -f_o(-x) \Leftarrow \text{let } f_o(x) = \dfrac{g(x)-g(-x)}{2} \end{array}\right\} \Rightarrow \forall g \in \mathbb{R}^{\mathbb{R}}, g(x) = f_e(x) + f_o(x). \ \square$$

ENDED

2·A

2 (a) *Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.*

(b) *Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.*

SOLUTION:

(a) Suppose $v \neq 0$. Then let $av = 0, a \in \mathbf{F}$. Getting $a = 0$. Thus (v) is linearly independent.
Suppose (v) is linearly independent. $av = 0 \Rightarrow a = 0$. Then $v \neq 0$, for if not, $a \neq 0 \Rightarrow av = 0$. Contradicts.

(b) Denote the list by (v, w), where $v, w \in V$. If (v, w) is linearly independent, suppose $av + bw = 0 \Rightarrow a = b = 0$.
Without loss of generality, suppose $v \neq cw\ \forall c \in \mathbf{F}$. Then let $av + bw = 0$, getting $a = b = 0 \Rightarrow (v, w)$ is linearly independent.

---

1 *Prove that if (v_1, v_2, v_3, v_4) spans V, then the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V.*

SOLUTION: Assume that $\forall v \in V,\ \exists\ a_1, \ldots, a_4 \in \mathbf{F}$,

$$\begin{aligned} v &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 \\ &= b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4, \text{ letting } b_i = \sum_{r=1}^{i} a_r. \end{aligned}$$

Thus $\forall v \in V,\ \exists\ b_i \in \mathbf{F}$, $v = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4$.

Hence the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V. □

---

6 *Suppose (v_1, v_2, v_3, v_4) is linearly independent in V.*
Prove that the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ is also linearly independent.

SOLUTION: $a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4v_4 = 0$

$\Rightarrow a_1v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0$

$\Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \Rightarrow a_1 = \cdots = a_4 = 0 \Rightarrow$ □

---

7 *Prove that if $(v_1, v_2, \ldots, v_m)$ is a linearly independent list of vectors in V, then $(5v_1 - 4v_2, v_2, v_3, \ldots, v_m)$ is linearly independent.*

SOLUTION: $a_1(5v_1 - 4v_2) + a_2v_2 + a_3v_3 + a_4v_4 = 0$

$\Rightarrow 5a_1v_1 + (a_2 - 4a_1)v_2 + a_3v_3 + a_4v_4 = 0$

$\Rightarrow 5a_1 = a_2 - 4a_1 = a_3 = a_4 = 0 \Rightarrow a_1 = \cdots = a_4 = 0$ □

• *Suppose $(v_1, \ldots, v_m)$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let $w_k = v_1 + \cdots + v_k$.*

(a) *Show that span$(v_1, \ldots, v_m)$ = span$(w_1, \ldots, w_m)$.*

(b) *Show that $(v_1, \ldots, v_m)$ is linearly independent if and only if $(w_1, \ldots, w_m)$ is linearly independent.*

SOLUTION:

(a) Let span$(v_1, \ldots, v_m) = U$. Assume that $\forall v \in U,\ \exists\ a_i \in \mathbf{F}$,

$$v = a_1v_1 + \cdots + a_mv_m = b_1w_1 + \cdots + b_mw_m = \sum_{j=1}^{m}(\sum_{i=j}^{m} b_i)v_j$$

$\Rightarrow b_1 = a_1,\ \ b_i = a_i - \sum_{r=1}^{i-1} b_r$. Thus $\exists\ b_i \in \mathbf{F}$ such that $v = b_1w_1 + \cdots + b_mw_m$.

又 Each $w_i \in U \Rightarrow$ span$(v_1, \ldots, v_m)$ = span$(w_1, \ldots, w_m)$.

(b) $a_1w_1 + \cdots + a_mw_m = 0$

$\Rightarrow (a_1 + \cdots + a_m)v_1 + \cdots + (a_i + \cdots + a_m)v_i + \cdots + a_mv_m = 0$

$\Rightarrow a_m = \cdots = (a_m + \cdots + a_i) = \cdots = (a_m + \cdots + a_1) = 0.$ □

---

10 *Suppose $(v_1, \dots, v_m)$ is linearly independent in V and $w \in V$.*

(a) *Prove that if $(v_1 + w, \dots, v_m + w)$ is linearly dependent, then $w \in span(v_1, \dots, v_m)$.*

(b) *Show that $(v_1, \dots, v_m, w)$ is linearly independent $\Longleftrightarrow w \notin span(v_1, \dots, v_m)$.*

SOLUTION:

(a) Suppose $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0,\ \exists\ a_i \neq= 0 \Rightarrow a_1v_1 + \cdots + a_mv_m = 0 = -(a_1 + \cdots + a_m)w$.

Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1v_1 + \cdots + a_mv_m = 0$ while $a_i \neq 0$ for some i, contradicts.

Hence $w \in \text{span}(v_1, \dots, v_m)$.

(b) Suppose $w \in \text{span}(v_1, \dots, v_m)$. Then $(v_1, \dots, v_m, w)$ is linearly dependent.

Thus have we proven the "$\Rightarrow$" by its contrapositive.

Suppose $w \notin \text{span}(v_1, \dots, v_m)$. Then by [2.23], $(v_1, \dots, v_m, w)$ is linearly independent. $\square$

---

14 *Prove that V is infinite-dim if and only if there is a sequence $(v_1, v_2, \dots)$ in V such that $(v_1, \dots, v_m)$ is linearly independent for every $m \in \mathbf{N}^+$.*

SOLUTION: Similar to [2.16].

Suppose there is a sequence $(v_1, v_2, \dots)$ in V such that $(v_1, \dots, v_m)$ is linearly independent for any $m \in \mathbf{N}^+$.

Choose an m. Suppose a linearly independent list $(v_1, \dots, v_m)$ spans V.

Then there exists $v_{m+1} \in V$ but $v_{m+1} \notin \text{span}(v_1, \dots, v_m)$. Hence no list spans V. Thus V is infinite-dim.

Conversely it is true as well. For if not, V must be finite-dim, contradicting the assumption. $\square$

---

15 *Prove that $\mathbf{F}^\infty$ is infinite-dim.*

SOLUTION: Let $e_i = (0, \dots, 0, 1, 0, \dots) \in \mathbf{F}^\infty$ for every $m \in \mathbf{N}^+$, where '1' is on the ith entry of e_i.

Suppose $\mathbf{F}^\infty$ is finite-dim. Then let $\text{span}(e_1, \dots, e_m) = V$. But $e_{m+1} \notin \text{span}(e_1, \dots, e_m)$. Contradicts. $\square$

---

16 *Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dim.*

SOLUTION: Denote the vec-sp by U. Note that for each $m \in \mathbf{N}^+$, $(1, x, \dots, x^m)$ is linearly independent.

Because if $a_0, \dots, a_m \in \mathbf{R}$ are such that $a_0 + a_1x + \cdots + a_mx^m = 0,\ \forall x \in [0, 1]$, then the polynomial has infinitely many roots and hence $a_0 = \cdots = a_m = 0$. $\Big\}$ Similar to [2.16], U is infinite-dim.

OR. Note that for $a_n = \dfrac{1}{n}$, $\quad a_1 < a_2 < \cdots < a_m,\ \ \forall m \in \mathbf{N}^+$.

Suppose $f_n = \begin{cases} x - \dfrac{1}{n}, & x \in [\frac{1}{n}, 1) \\ 0, & x \in [0, \frac{1}{n}) \end{cases}$. Then for any m, $f_1(\frac{1}{m}) = \cdots = f_m(\frac{1}{m})$, while $f_{m+1}(\frac{1}{m}) \neq 0$.

Hence $f_{m+1} \notin \text{span}(f_1, \dots, f_m)$. Thus by Problem (14), U is infinite-dim.

---

17 *Suppose $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$. Prove that $(p_0, p_1, \dots, p_m)$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.*

SOLUTION:

Suppose $(p_0, p_1, \dots, p_m)$ is linearly independent. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z\ \forall z \in \mathbf{F}$.

But $\forall a_i \in \mathbf{F}$, $z \neq a_0p_0(z) + \cdots + a_mp_m(z)$, for if not, let $z = 2$, contradicts. Thus $z \notin \text{span}(p_0, p_1, \dots, p_m)$.

Then $\text{span}(p_0, p_1, \dots, p_m) \subsetneqq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has length $m + 1$.

Hence $(p_0, p_1, \dots, p_m)$ is linearly dependent in $\mathcal{P}_m(\mathbf{F})$.

For if not, notice that the list $(1, z, \dots, z^m)$ spans $\mathcal{P}_m(\mathbf{F})$,

thus by [2.23], $(p_0, p_1, \dots, p_m)$ spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. $\square$

ENDED

2·B

NOTE FOR *linearly independent sequence and [2.34].*

"$V = \operatorname{span}(v_1, \ldots, v_n, \ldots)$" is an invalid expression.

If we allow using "infinite list", then we must guarantee that $(v_1, \ldots, v_n, \ldots)$ is a spanning "list" such that for all $v \in V$, there exists a certain positive integer such that $v = a_1 v_{\alpha_1} + \cdots + a_n v_{\alpha_n}$, where $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ is an finite index set. The key point is, how do we find such a "list"?

NOTE FOR "$\complement_V U \cap \{0\}$": "$\complement_V U \cap \{0\}$" is supposed to be "$W$", where $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cap \{0\} \\ u \pm w \in \complement_V U \cap \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cap \{0\}$. Contradicts.

NEW NOTATION: Denote the set $\{W_1, W_2 \ldots\}$ by $\mathcal{S}_V U$, where for each $W_i, V = U \oplus W_i$. See also in (1.C.23).

---

1 *Find all vector spaces that have exactly one basis.* **SOLUTION:** $\mathbf{F} = \mathbf{C}, \mathbf{R}, \mathbf{Q}, \{0,1\}, \mathcal{P}_0(\mathbf{F})$.

---

6 *Suppose (v_1, v_2, v_3, v_4) is a basis of V. Prove that $(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4)$ is also a basis.*

SOLUTION: $\forall v \in V,\ \exists!\ a_1, \ldots, a_4 \in \mathbf{F},\ v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$.

Assune that $v = b_1(v_1 + v_2) + b_2(v_2 + v_3) + b_3(v_3 + v_4) + b_4 v_4$. Then $v = b_1 v_1 + (b_1 + b_2) v_2 + (b_2 + b_3) v_3 + (b_3 + b_4) v_4$.

$\Rightarrow \exists!\ b_1 = a_1,\ \ b_2 = a_2 - b_1,\ \ b_3 = a_3 - b_2,\ \ b_4 = a_4 - b_3 \in \mathbf{F}$. $\square$

---

7 *Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \in U$, then v_1, v_2 is a basis of U.*

SOLUTION: Let $V = \mathbf{F}^4, v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_3 = (0,0,1,1), v_4 = (0,0,0,1)$.

And $U = \{(x, y, z, 0) \in \mathbf{R}^4 : x, y, z \in \mathbf{F}\}$. We have a counterexample.

• *Suppose V is finite-dim and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.*

SOLUTION: Let $(u_1, \ldots, u_m)$ and $(w_1, \ldots, w_n)$ be bases of U and W respectively.

Then $V = \operatorname{span}(u_1, \ldots, u_m) + \operatorname{span}(w_1, \ldots, w_n) = \operatorname{span}(u_1, \ldots, u_m, w_1, \ldots, w_n)$.

Hence, by [2.31], we get a basis of V consisting of vectors in U or W. $\square$

---

8 *Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $(u_1, \ldots, u_m)$ is a basis of U and $(w_1, \ldots, w_n)$ is a basis of W. Prove that $(u_1, \ldots, u_m, w_1, \ldots, w_n)$ is a basis of V.*

SOLUTION:

$\forall v \in V,\ \exists!\, a_i, b_i \in \mathbf{F},\ v = (a_1 u_1 + \cdots + a_m u_m) + (b_1 w_1 + \cdots + b_n w_n)$

$\Rightarrow (a_1 u_1 + \cdots + a_m u_m) = -(b_1 w_1 + \cdots + b_n w_n) \in U \cap W = \{0\}$. Thus $a_1 = \cdots = a_m = b_1 = \cdots = b_n$. $\square$

• (OR 9.4) *Suppose V is a real vector space. Show that if $(v_1, \ldots, v_n)$ is a basis of V (as a real vector space), then $(v_1, \ldots, v_n)$ is also a basis of the complexification $V_{\mathbb{C}}$ (as a complex vector space).*

See Section 1B (4e) for the definition of the complexification $V_{\mathbb{C}}$.

SOLUTION:

$\forall u + \mathrm{i}v \in V_{\mathbb{C}},\ \exists!\ u, v \in V, a_i, b_i \in \mathbf{R},$

$u + \mathrm{i}v = (a_1 v_1 + \cdots + a_n v_n) + \mathrm{i}(b_1 v_1 + \cdots + b_n v_n) = (a_1 + b_1 \mathrm{i}) v_1 + \cdots + (a_n + b_n \mathrm{i}) v_n$

$\Rightarrow\ u + \mathrm{i}v = c_1 v_1 + \cdots + c_n v_n,\ \exists!\, c_i = a_i + b_i \mathrm{i} \in \mathbf{C}$

$\Rightarrow$ By the uniqueness of c_i and [2.29], $(v_1, \ldots, v_n)$ is a basis of $V_{\mathbb{C}}$. $\square$

ENDED

2·C

1 *Suppose V is finite-dim and U is a subspace of V such that* $\dim V = \dim U$.

Let $(u_1, \ldots, u_m)$ be a basis of U. Then $n = \dim U = \dim V$. 又 $u_i \in V$.

Then by [2.39], $(u_1, \ldots, u_m)$ is a basis of V. Thus $V = U$.

---

2 *Show that the subspaces of* $\mathbf{R}^2$ *are precisely* $\{0\}$*, all lines in* $\mathbf{R}^2$ *containing the origin, and* $\mathbf{R}^2$.

SOLUTION:

Suppose U is a subspace of $\mathbf{R}^2$. Let $\dim U = n$.

If $n = 0$, then $U = \{0\}$.

If $n = 1$, then $U = \text{span}(v)$, where v is a vector in $\mathbf{R}^2$. Thus U can be any line in $\mathbf{R}^2$ containing the origin.

If $n = 2$, then $U = \text{span}(v, w)$, where v, w are vectors in $\mathbf{R}^2$ and (v, w) is linearly independent $\Rightarrow U = \mathbf{R}^2$. $\square$

---

3 *Show that the subspaces of* $\mathbf{R}^3$ *are precisely* $\{0\}$*, all lines in* $\mathbf{R}^3$ *containing the origin, all planes in* $\mathbf{R}^3$ *containing the origin, and* $\mathbf{R}^3$.

SOLUTION:

Suppose U is a subspace of $\mathbf{R}^3$. Let $\dim U = n$.

If $n = 0$, then $U = \{0\}$.

If $n = 1$, then $U = \text{span}(v)$, where v is a vector in $\mathbf{R}^3$. Thus U can be any line in $\mathbf{R}^3$ containing the origin.

If $n = 2$, then $U = \text{span}(v, w)$, where v, w are vectors in $\mathbf{R}^3$ and (v, w) is linearly independent.

Thus U can be any plane in $\mathbf{R}^3$ containing the origin.

If $n = 3$, then $U = \text{span}(u, v, w)$, where u, v, w are vectors in $\mathbf{R}^3$ and (u, v, w) is linearly independent

$\Rightarrow U = \mathbf{R}^3$. $\square$

---

7 (a) *Let* $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. *Find a basis of* U.

(b) *Extend the basis in* (a) *to a basis of* $\mathcal{P}_4(\mathbf{F})$.

(c) *Find a subspace* W *of* $\mathcal{P}_4(\mathbf{F})$ *such that* $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

SOLUTION:

Suppose $p(z) = az^4 + bz^3 + cz^2 + dz + e$ and $p(2) = p(5) = p(6)$.

$$\text{Then} \begin{cases} p(2) = 16a + 8b + 4c + 2d + e \text{ (I)} \\ p(5) = 625a + 125b + 25c + 5d + e \text{ (II)} \\ p(6) = 1296a + 216b + 36c + 6d + e \text{ (III)} \end{cases}$$

You don't have to compute to know that the dimension of the set of soultions is 3.

(a) A basis: $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(c) Let $W = \text{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. $\square$

---

9 *Suppose* $(v_1, \ldots, v_m)$ *is linearly independent in* V *and* $w \in V$.

Prove that $\dim$ *span*$(v_1 + w, \ldots, v_m + w) \geq m - 1$.

SOLUTION:

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \text{span}(v_1 + w, \ldots, v_n + w)$, for each $i = 1, \ldots, m$.

$(v_1, \ldots, v_m)$ is linearly independent $\Rightarrow (v_1, v_2 - v_1, \ldots, v_m - v_1)$ is linearly independent

$\Rightarrow (v_2 - v_1, \ldots, v_m - v_1)$ is linearly independent of length $m - 1$.

又 By the contrapositive of (2.A.10), $w \notin \text{span}(v_1, \ldots, v_m) \Rightarrow (v_1 + w, \ldots, v_m + w)$ is linearly independent.

$\therefore\ m \geq \dim \text{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$. $\square$

---

10 *Suppose m is a positive integer and $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $(p_0, p_1, \ldots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

SOLUTION: Using mathematical induction on m.

(i) For $p_0, \deg p_0 = 0 \Rightarrow \text{span}\,(p_0) = \text{span}\,(1)$.

(ii) Suppose for $i \geq 1, \text{span}\,(p_0, p_1, \ldots, p_i) = \text{span}(1, x, \ldots, x^i)$.

Then $\text{span}(p_0, p_1, \ldots, p_i, p_{i+1}) \subseteq \text{span}\,(1, x, \ldots, x^i, x^{i+1})$.

又 $\deg p_{i+1} = i+1, \quad p_{i+1}(x) = a_{i+1}x^{i+1} + r_{i+1}(x); \quad a_{i+1} \neq 0, \quad \deg r_{i+1} \leq i$.

$\Rightarrow x^{i+1} = \frac{1}{a_{i+1}}(p_{i+1}(x) - r_{i+1}(x)) \in \text{span}\,(1, x, \ldots, x^i, p_{i+1}) = \text{span}\,(p_0, p_1, \ldots, p_i, p_{i+1})$.

$\therefore\ x^{i+1} \in \text{span}\,(p_0, p_1, \ldots, p_i, p_{i+1}). \Rightarrow \text{span}\,(1, x, \ldots, x^i, x^{i+1}) \subseteq \text{span}\,(p_0, p_1, \ldots, p_i, p_{i+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \text{span}\,(1, x, \ldots, x^m) = \text{span}(p_0, p_1, \ldots, p_m)$. $\square$

• *Suppose m is a positive integer. For $0 \leq k \leq m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show that $(p_0, \ldots, p_m)$ is a basis of $\mathcal{P}(\mathbf{F})$.*

The basis in this exercise leads to what are called Bernstein polynomials. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on $[0,1]$.

SOLUTION: Using mathematical induction.

(i) $k = 0, 1, 2,\ p_m(x) = x^m, \quad p_{m-1}(x) = x^{m-1} - x^m, \quad p_{m-2}(x) = x^{m-2} + x^m - 2x^{m-1}$.

(ii) $k \geq 2$. Suppose for $p_{m-k}(x),\ \exists\,!\ a_i \in \mathbf{F}, x^{m-k} = p_{m-k}(x) + a_m x^m + \cdots + a_{m-k+1}x^{m-k+1}$.

Then for $p_{m-k-1}(x),\ \exists\,!\ c_i \in \mathbf{F}$,

$$x^{m-k-1} = p_{m-k-1}(x) + \mathcal{C}_{k+1}^{1}(-1)^2 x^{m-k} + \cdots + \mathcal{C}_{k+1}^{k}(-1)^{k+1}x^{m-1} + (-1)^{k-2}x^m$$

$$\Rightarrow c_{m-i} = \mathcal{C}_{k+1}^{k+1-i}(-1)^{k-i}.$$

Thus for each $x^i,\ \exists\,!\ b_i \in \mathbf{F}, x^i = b_m p_m(x) + \cdots + b_{m-i}p_{m-i}(x)$.

$\Rightarrow \text{span}(x^m, \ldots, x, 1) = \text{span}\underbrace{(p_m, \ldots, p_1, p_0)}_{\text{Basis}}$. $\square$

• *Suppose V is finite-dim and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*

SOLUTION:

$$\left.\begin{array}{l} \dim V_1 + \dim V_2 > 2\dim V - \dim V_3 \geq \dim V \Rightarrow V_1 \cap V_2 \neq \{0\} \\ \dim V_2 + \dim V_3 > 2\dim V - \dim V_1 \geq \dim V \Rightarrow V_2 \cap V_3 \neq \{0\} \\ \dim V_1 + \dim V_3 > 2\dim V - \dim V_2 \geq \dim V \Rightarrow V_1 \cap V_3 \neq \{0\} \end{array}\right\} \Rightarrow V_1 \cap V_2 \cap V_3 \neq \{0\}. \ \square$$

• *Suppose V is finite-dim and U is a subspace of V with $U \neq V$. Let $n = \dim V, m = \dim U$. Prove that there exist $(n-m)$ subspaces of V, say $U_1, \ldots, U_{n-m}$, each of dimension $(n-1)$, such that $\bigcap\limits_{i=1}^{n-m} U_i = U$.*

SOLUTION: Let $(v_1, \ldots, v_m)$ be a basis of U, extend to a basis of V as $(v_1, \ldots, v_m, \ldots, v_n)$.

Define $U_i = \text{span}\,(v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+i-1}, v_{m+i+1}, \ldots, v_n)$ for each i. Thus we are done.

EXAMPLE: Suppose $\dim V = 6, \dim U = 3$.

$$\underbrace{(\underbrace{v_1, v_2, v_3}_{\text{Basis of U}}, v_4, v_5, v_6)}_{\text{Basis of V}}, \text{ define } \left.\begin{array}{l} U_1 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_5, v_6) \\ U_2 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_6) \\ U_3 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_5) \end{array}\right\} \Rightarrow \dim U_i = 6 - 1,\ i = \underbrace{1, 2, 3}_{6-3=3}.$$

$\square$

---

14 *Suppose that $V_1, \dots, V_m$ are finite-dim subspaces of V.*
Prove that $V_1 + \cdots + V_m$ is finite-dim and $\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m$.

SOLUTION:

Choose a basis $\mathcal{E}_i$ of $V_i \Rightarrow V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim U_i = \operatorname{card} \mathcal{E}_i$.

Then $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$.

又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup E_m) \leq \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leq \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.

Thus $\dim(V_1 + \cdots + V_m) \leq \dim U_1 + \cdots + \dim U_m$.

•*The inequality above is an equality if and only if $V_1 + \cdots + V_m$ is a direct sum.*

For each i, $(V_1 + \cdots + V_i) \cap V_{i+1} = \{0\} \Longleftrightarrow V_1 + \cdots + V_m$ is a direct sum $\Longleftrightarrow$ □

---

17 *Suppose V_1, V_2, V_3 are subspaces of a finite-dim vector space, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

SOLUTION:

Looks like: given three sets A, B and C.

Note that: $\operatorname{card}(X \cup Y) = \operatorname{card}(X) + \operatorname{card}(Y) - \operatorname{card}(X \cap Y)$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

Then: $\operatorname{card}((A \cup B) \cup C) = \operatorname{card}(A \cup B) + \operatorname{card} C - \operatorname{card}((A \cup B) \cap C)$.

And: $\operatorname{card}((A \cup B) \cap C) = \operatorname{card}((A \cap C) \cup (B \cap C)) = \operatorname{card}(A \cap C) + \operatorname{card}(B \cap C) - \operatorname{card}(A \cap B \cap C)$.

Thus: $\operatorname{card}((A \cup B) \cup C) = \operatorname{card} A + \operatorname{card} B + \operatorname{card} C + \operatorname{card}(A \cap B \cap C) - \operatorname{card}(A \cap B) - \operatorname{card}(A \cap C) - \operatorname{card}(B \cap C)$.

Because $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned}\dim(V_1 + V_2 + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \quad (1)\\ &= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) \quad (2)\\ &= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) \quad (3)\end{aligned}$$

Notice that $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$.

For example, $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

• **COROLLARY:** *If V_1, V_2 and V_3 are finite-dim vector spaces, then* $\dfrac{(1) + (2) + (3)}{3}$:

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.$$

The formula above may seem strange because the right side does not look like an integer. □

ENDED

3·A

2 *Suppose $b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by*

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x) dx + c \sin p(0)\right).$$

Show that T is linear if and only if $b = c = 0$.

SOLUTION:

(a) Suppose $b = c = 0$, then $\forall p, q \in \mathcal{P}(\mathbf{R}), T(p + q) = (3(p + q)(4) + 5(p + q)'(6), \int_{-1}^{2} x^3 (p + q)(x) \mathrm{d}x)$.

Because $(p + q)(x) = p(x) + q(x), (p + q)'(x) = p'(x) + q'(x)$,

$\int_{-1}^{2} x^3 (p + q)(x) \mathrm{d}x = \int_{-1}^{2} x^3 p(x) \mathrm{d}x + \int_{-1}^{2} x^3 q(x) \mathrm{d}x$.

$\Rightarrow T(p + q) = Tp + Tq$. Similarly, $\forall \lambda \in \mathbf{F}$, $\lambda Tp = T(\lambda p)$. Thus T is linear.

(b) Suppose T is linear, denote the linear map in (a) by $S \Rightarrow (T-S)$ is linear.

$\Rightarrow (T-S)(p) = (bp(1)p(2), c\sin p(0))$ is linear.

Consider $p(x) = q(x) = \frac{\pi}{2},\ \forall x \in \mathbf{R}$.

$\Rightarrow ((T-S)(p+q) = (T-S)(\pi) = (b\pi^2, 0) = (T-S)(\frac{\pi}{2}) + (T-S)(\frac{\pi}{2}) = (b\frac{\pi^2}{2}, 2c) \Rightarrow b = c = 0.$ $\square$

• **TIPS:** *$T : V \to W$ is linear* $\Longleftrightarrow \begin{cases} \forall v, u \in V, T(v+u) = Tv + Tu \\ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv) \end{cases} \Longleftrightarrow T(v + \lambda u) = Tv + \lambda Tu.$

---

3 *Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists\ A_{j,k} \in \mathbf{F}$ such that*

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for any $(x_1, \dots, x_n) \in \mathbf{F}^n$.

SOLUTION:

Let $T(1,0,0,\dots,0,0) = (A_{1,1}, \dots, A_{m,1})$,

$T(0,1,0,\dots,0,0) = (A_{1,2}, \dots, A_{m,2})$,

$\vdots$

$T(0,0,0,\dots,0,1) = (A_{1,n}, \dots, A_{m,n})$.

Note that $(1,0,\dots,0,0), \cdots, (0,0,\dots,0,1)$ is a basis of $\mathbf{F}^n$.

Then by [3.5], we are done. $\square$

---

4 *Suppose $T \in \mathcal{L}(V, W)$ and $(v_1, \dots, v_m)$ is a list of vectors in V such that $(Tv_1, \dots, Tv_m)$ is linearly independent in W. Prove that $(v_1, \dots, v_m)$ is linearly independent.*

SOLUTION:

Suppose $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$.$\square$

---

5 *Prove that $\mathcal{L}(V, W)$ is a vector space,*

SOLUTION: Note that $\mathcal{L}(V, W)$ is a subspace of W^V. $\square$

---

7 *Show that every linear map from a one-dim vector space to itself is multiplication by some scalar. More precisely, prove that if* $\dim V = 1$ *and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.*

SOLUTION:

Let u be a nonzero vector in $V \Rightarrow V = \operatorname{span}(u)$.

Because $Tu \in V \Rightarrow Tu = \lambda u$ for some λ.

Suppose $v \in V \Rightarrow v = au,\ \exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. $\square$

---

8 *Give an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that $\varphi(av) = a\varphi(v)$ for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.*

SOLUTION:

Define $T(x,y) = \begin{cases} x + y, \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, \qquad \text{otherwise}. \end{cases}$

OR. Define $T(x,y) = \sqrt[3]{(x^3 + y^3)}$. $\square$

---

9 *Give an example of a function $\varphi : \mathbf{C} \to \mathbf{C}$ such that $\varphi(w+z) = \varphi(w) + \varphi(z)$ for all $w, z \in \mathbf{C}$ but φ is not linear.*

(Here $\mathbf{C}$ is thought of as a complex vector space.)

SOLUTION:

Suppose $V_{\mathbb{C}}$ is the complexification of a vector space V. Suppose $\varphi : V_{\mathbb{C}} \to V_{\mathbb{C}}$.

Define $\varphi(u + \mathrm{i}v) = u = \Re(u + \mathrm{i}v)$

OR. Define $\varphi(u + \mathrm{i}v) = v = \Im(u + \mathrm{i}v)$. $\square$

• OR (3.D.16) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

SOLUTION:

Assume that (v, Tv) is linearly dependent for every $v \in V$, then by (2.A.2.(b)), $Tv = \lambda_v v$ for some $\lambda_v \in \mathbf{F}$.

To prove that λ_v is independent of v

(in other words, for any two distinct nonzero vectors v and w in V, we have $\lambda_v \neq \lambda_w$), we discuss in two cases:

$$\left.\begin{array}{l} (-)\text{ If } (v,w) \text{ is linearly independent, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = a_v v + a_w w \\ \qquad\qquad\qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=)\text{ Otherwise, suppose } w = cv,\ a_w w = Tw = cTv = ca_v v = a_v w \Rightarrow (a_w - a_v)w \end{array}\right\} \Rightarrow a_w = a_v.$$

Now we prove the assumption by contradiction. Suppose (v, Tv) is linearly independent for every nonzero vector $v \in V$.

Fix one v. Extend to $(v, Tv, u_1, \dots, u_n)$ a basis of V.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$.

Hence a contradiction arises. □

---

10 *Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$*
(which means that $Su \neq 0$ for some $u \in U$).

Define $T : V \to W$ by $Tv = \begin{cases} Sv, \text{ if } v \in U, \\ 0, \quad \text{if } v \in V \backslash U. \end{cases}$ Prove that T is not a linear map on V.

SOLUTION:

Suppose T is a linear map. And $v \in V\backslash U,\ u \in U$ such that $Su \neq 0$.

Then $v + u \in V\backslash U$, (for if not, $v = (v + u) - u \in U$) while $T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$.

Hence we get a contradiction. □

---

11 *Suppose V is finite-dim. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.*

SOLUTION: Define $T \in \mathcal{L}(V, W)$ by $T(a_1u_1 + \cdots + a_nu_n + a_{n+1}u_{n+1} + \cdots + a_mu_m) = a_1Su_1 + \cdots + a_nSu_n$. Where:
Let $(u_1, \dots, u_n)$ be a basis of U, extend to a basis of V as $(u_1, \dots, u_n, \dots, u_m)$.

---

12 *Suppose V is finite-dim with $\dim V > 0$, and W is infinite-dim.*
Prove that $\mathcal{L}(V, W)$ is infinite-dim.

SOLUTION:

Let $(v_1, \dots, v_n)$ be a basis of V. Let $(w_1, \dots, w_m)$ be linearly independent in W for any $m \in \mathbf{N}^+$.

Define $T_{x,y} \in \mathcal{L}(V, W)$ by $T_{x,y}(v_x) = w_y, \forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$.

Suppose $a_1T_{x,1} + \cdots + a_mT_{x,m} = 0$. Then $(a_1T_{x,1} + \cdots + a_mT_{x,m})(v_x) = 0 = a_1w_1 + \cdots + a_mw_m$.

$\Rightarrow a_1 = \cdots = a_m = 0$. 又 m is arbitrarily chosen.

Thus $(T_{x,1}, \dots, T_{x,m})$ is a linearly independent list in $\mathcal{L}(V, W)$ for any x and length m. Hence by (2.A.14). □

---

13 *Suppose $(v_1, \dots, v_m)$ is a linearly dependent list of vectors in V.*
Suppose also that $W \neq \{0\}$. Prove that there exist $(w_1, \dots, w_m) \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \dots, m$.

SOLUTION: We show it by contradiction.

By linear independence lemma, $\exists\ j \in \{1, \dots, m\}$ such that $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$.

Fix j. Let $w_j \neq 0$, while $w_1 = \cdots = w_{j-1} = w_{j+1} = w_m = 0$.

Define T by $Tv_k = w_k$ for all k. Suppose $a_1v_1 + \cdots + a_mv_m = 0$ (where $a_j \neq 0$).

Then $T(a_1v_1 + \cdots + a_mv_m) = 0 = a_1w_1 + \cdots + a_mw_m = a_jw_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. □

• *Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION:

Let $(v_1, \dots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \backslash \{0\}$.

Suppose $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \cdots + a_n v_n$, where $a_k \neq 0$.

Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y}(v_x) = v_y, R_{x,y}(v_z) = 0\,(\, z \neq x\,)$. Then for any $x, y \in \mathbf{N}^+$,

$(R_{k,y}S)(v_i) = a_k v_y \Rightarrow ((R_{k,y}S) \circ R_{x,i})(v_x) = a_k v_y$, and $((R_{k,y}S) \circ R_{x,i})(v_z) = 0$ for $z \neq x$.

Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Denote by $T_{x,y}$.

Getting $(\frac{1}{a_k} T_{1,1} + \cdots + \frac{1}{a_k} T_{n,n}) v_j = v_j \Rightarrow \sum\limits_{r=1}^{n} \frac{1}{a_k} T_{r,r} = I$.

又 By assumption, $T_{x,y} \in \mathcal{E} \Rightarrow I \in \mathcal{E}$.

Hence for any $T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. □

ENDED

3·B

2 *Suppose $S, T \in \mathcal{L}(V)$ are such that range $S \subseteq$ null T. Prove that $(ST)^2 = 0$.*

SOLUTION: $TS = 0 \Rightarrow STST = (ST)^2 = 0$. □

---

3 *Suppose $(v_1, \dots, v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1 v_1 + \cdots + z_m v_m$.*

(a) *What property of T corresponds to $(v_1, \dots, v_m)$ spanning V?*

(b) *What property of T corresponds to $(v_1, \dots, v_m)$ being linearly independent?*

ANSWER: (a) Surjectivity; (b) Injectivity. □

---

4 *Show that $U = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \text{null}\, T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.*

SOLUTION: Let $(v_1, v_2, v_3, v_4, v_5)$ be a basis of $\mathbf{R}^5$, (w_1, w_2, w_3, w_4) be a basis of $\mathbf{R}^4$.

Define $T_1, T_2 \in U$ as $T_1 v_1 = 0,\ T_1 v_2 = 0,\ T_1 v_3 = 0,\ T_1 v_4 = w_4,\ T_1 v_5 = w_1;$

$T_2 v_1 = 0,\ T_2 v_2 = 0,\ T_2 v_3 = w_3,\ T_2 v_4 = 0,\ T_2 v_5 = w_4.$ Thus $T_1 + T_2 \notin U$.

For $U' = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \text{null}\, T > 0\}$,

define $T_1, T_2 \in U'$ as $T_1 v_1 = 0,\ T_1 v_2 = w_2,\ T_1 v_3 = w_3,\ T_1 v_4 = w_4,\ T_1 v_5 = w_1;$

$T_2 v_1 = w_1,\ T_2 v_2 = w_2,\ T_2 v_3 = 0,\ T_2 v_4 = w_3,\ T_2 v_5 = w_4.$ Thus $T_1 + T_2 \notin U'$. □

---

7 *Suppose V is finite-dim with $2 \leq \dim V \leq \dim W$, if W is finite-dim.*

Show that $U = \{\, T \in \mathcal{L}(V, W) : T$ is not injective $\}$ is not a subspace of $\mathcal{L}(V, W)$.

SOLUTION:

Let $(v_1, \dots, v_n)$ be a basis of V, $(w_1, \dots, w_m)$ be linearly independent in W.

(Let $\dim W = m$, if W is finite, otherwise, we choose $m \in \{n, n+1, \dots\}$ arbitrarily; $2 \leq n \leq m$).

Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 :\quad v_1 \mapsto 0, \qquad v_2 \mapsto w_2, \qquad v_i \mapsto w_i.$

Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 :\quad v_1 \mapsto w_1, \qquad v_2 \mapsto 0, \qquad v_i \mapsto w_i, \qquad i = 3, \dots, n.$

Thus $T_1 + T_2 \notin U$. □

COMMENT: If $\dim V = 0$, then $V = \{0\} =$ span ( ). $\forall\, T \in \mathcal{L}(V, W), T$ is injective. Hence $U = \varnothing$.

If $\dim V = 1$, then $V =$ span (v_0). Thus $U =$ span (T_0), where $T_0 v_0 = 0$.

If V is infinite-dim, the result is true as well.

---

8 *Suppose W is finite-dim with $\dim V \geq \dim W \geq 2$, if V is finite-dim.*
Show that $U = \{\, T \in \mathcal{L}(V,W) : T$ is not surjective $\}$ is not a subspace of $\mathcal{L}(V,W)$.

SOLUTION:

Let $(v_1, \ldots, v_n)$ be linearly independent in V, $(w_1, \ldots, w_m)$ be a basis of W.

(Let $n = \dim V$, if V is finite, otherwise we choose $n \in \{m, m+1, \ldots\}$; $2 \leq m \leq n$).

Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 :\quad v_1 \mapsto 0, \qquad v_2 \mapsto w_2, \qquad v_j \mapsto w_j, \qquad v_{m+i} \mapsto 0.$

Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 :\quad v_1 \mapsto w_1, \qquad v_2 \mapsto 0, \qquad v_j \mapsto w_j, \qquad v_{m+i} \mapsto 0.$

For each $j = 2, \ldots, m;\ i = 1, \ldots, n-m$, if V is finite, otherwise let $i \in \mathbf{N}^+$.

Thus $T_1 + T_2 \notin U$. $\square$

COMMENT: If $\dim W = 0$, then $W = \{0\} =$ span ( ). $\forall\, T \in \mathcal{L}(V,W), T$ is surjective. Hence $U = \varnothing$.

If $\dim W = 1$, then $W =$ span (v_0). Thus $U =$ span (T_0), where $T_0 v_0 = 0$.

If W is infinite-dim, the result is true as well.

---

9 *Suppose $T \in \mathcal{L}(V,W)$ is injective and $(v_1, \ldots, v_n)$ is linearly independent in V.*
Prove that $(Tv_1, \ldots, Tv_n)$ is linearly independent in W.

SOLUTION:

$a_1 T v_1 + \cdots + a_n T v_n = 0 = T(\sum\limits_{i=1}^{n} a_i v_i) \iff \sum\limits_{i=1}^{n} a_i v_i = 0 \iff a_1 = \cdots = a_n = 0.$ $\square$

---

10 *Suppose $(v_1, \ldots, v_n)$ spans V and $T \in \mathcal{L}(V,W)$. Show that $(Tv_1, \ldots, Tv_n)$ spans range T.*

SOLUTION:

(a) range $T = \{\, Tv : v \in V \,\} = \{\, Tv : v \in$ span $(v_1, \ldots, v_n)\}$

$\Rightarrow Tv_1, \ldots, Tv_n \in$ range $T \Rightarrow$ By [2.7] and [3.19], span $(Tv_1, \ldots, Tv_n) \subseteq$ range T.

(b) $\forall w \in$ range T, $\exists v \in V, Tv = w$. 又 $\forall v \in V$, $\exists\ a_i \in \mathbf{F}$, $v = a_1 v_1 + \cdots + a_n v_n$

$\Rightarrow w = Tv = a_1 T v_1 + \cdots + a_n T v_n \Rightarrow$ range $T \subseteq$ span $(Tv_1, \ldots, Tv_n)$. $\square$

---

11 *Suppose $S_1, \ldots, S_n$ are injective linear maps and $S_1 S_2 \ldots S_n$ makes sence.*
Prove that $S_1 S_2 \ldots S_n$ is injective.

SOLUTION: $S_1 S_2 \ldots S_n(v) = 0 \iff S_2 S_3 \ldots S_n(v) = 0 \iff \cdots \iff S_n(v) = 0 \iff v = 0.$ $\square$

---

12 *Suppose that V is finite-dim and that $T \in \mathcal{L}(V,W)$. Prove that*
there exists a subspace U of V such that $U \cap$ null $T = \{0\}$ and range $T = \{\, Tu : u \in U \,\}$.

SOLUTION:

By [2.34], there exists a subspace U of V such that $V = U \oplus$ null T.

$\forall v \in V$, $\exists!\ w \in$ null $T, u \in U, v = w + u$. Then $Tv = T(w+u) = Tu \in \{\, Tu : u \in U \,\} \Rightarrow \square$

COMMENT: V can be infinite-dim. See the above of [2.34].

---

16 *Suppose there exists a linear map on V*
whose null space and range are both finite-dim. Prove that V is finite-dim.

SOLUTION:

Denote the linear map by T. Let $(Tv_1, \ldots, Tv_n)$ be a basis of range T, $(u_1, \ldots, u_m)$ be a basis of null T.

Then for all $v \in V, T(\underbrace{v - a_1 v_1 - \cdots - a_n v_n}_{u \,\in\, \text{null}\, T}) = 0$, where $Tv = a_1 T v_1 + \cdots + a_n T v_n$.

$\Rightarrow u = b_1 u_1 + \cdots + b_m u_m \Rightarrow v = a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m$.

Getting $V \subseteq$ span $(v_1, \ldots, v_n, u_1, \ldots, u_m)$. Thus V is finite-dim. $\square$

---

17 *Suppose V and W are both finite-dim. Prove that there exists an injective $T \in \mathcal{L}(V, W)$ if and only if* $\dim V \le \dim W$.

SOLUTION:

(a) Suppose there exists an injective T. Then $\dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim \operatorname{range} T \le \dim W$.

(b) Suppose $\dim V \le \dim W$, letting $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$ be bases of V and W respectively.
Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i, \;\; i = 1, \dots, n \, (= \dim V\,)$. $\square$

---

18 *Suppose V and W are both finite-dim. Prove that there exists a surjective $T \in \mathcal{L}(V, W)$ if and only if* $\dim V \ge \dim W$.

SOLUTION:

(a) Suppose there exists a surjective T. Then $\dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim W + \dim \operatorname{null} T$
$\Rightarrow \dim W = \dim V - \dim \operatorname{null} T \le \dim V$.

(b) Suppose $\dim V \ge \dim W$, letting $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$ be bases of V and W respectively.
Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_m v_m + a_{m+1} v_{m+1} + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m$. $\square$

---

19 *Suppose V and W are finite-dim and that U is a subspace of V.
Prove that $\exists\, T \in \mathcal{L}(V, W)$, null$\,T = U \iff$* $\dim U \ge \dim V - \dim W$.

SOLUTION:

(a) Suppose $\exists\, T \in \mathcal{L}(V, W)$, null $T = U$. Then $\dim \operatorname{null} T = \dim U \ge \dim V - \dim W$.

(b) Suppose $\underbrace{\dim U}_{m} \ge \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_{p}$ $(\Rightarrow \dim W = p \ge n = \dim V - \dim U\,)$.
Let $(u_1, \dots, u_m)$ be a basis of U, extend to a basis of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$.
Let $(w_1, \dots, w_p)$ be a basis of W.
Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m) = a_1 w_1 + \cdots + a_n w_n$. $\square$

• **TIPS:** *Suppose $T \in \mathcal{L}(V, W)$ and $R = (Tv_1, \dots, Tv_n)$ is linearly independent in range T.*
(Let $\dim$ *range $T = n$, if range T is finite, otherwise choose n arbitrarily.).*
By (3.A.4), $L = (v_1, \dots, v_n)$ is linearly independent in V.

NEW NOTATION: *Denote $\mathcal{K}_R$ by span L, if range T is finite-dim,
otherwise, denote it by an vector space in the set $\mathcal{S}_V$null T.*

NEW THEOREM:

$$\mathcal{K}_R \oplus \operatorname{null} T = V \Leftarrow \begin{cases} \text{(a) } T(\sum_{i=1}^{n} a_i v_i) = 0 \Rightarrow \sum_{i=1}^{n} a_i T v_i = 0 \Rightarrow a_1 = \cdots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \operatorname{null} T = \{0\}. \\ \text{(b) } \forall v \in V, Tv = \sum_{i=1}^{n} a_i T v_i \Rightarrow Tv - \sum_{i=1}^{n} a_i T v_i = T(v - \sum_{i=1}^{n} a_i v_i) = 0 \\ \quad \Rightarrow v - \sum_{i=1}^{n} a_i v_i \in \operatorname{null} T \Rightarrow v = (v - \sum_{i=1}^{n} a_i v_i) + (\sum_{i=1}^{n} a_i v_i) \Rightarrow \mathcal{K}_R + \operatorname{null} T = V. \end{cases}$$

COMMENT: null $T \in \mathcal{S}_V \mathcal{K}_R$.

• *Suppose V is finite-dim, $T \in \mathcal{L}(V, W)$, and U is a subspace of W.
Prove that $\mathcal{K}_U = \{\, v \in V : Tv \in U \,\}$ is a subspace of V
and* $\dim \mathcal{K}_U = \dim$ *null* $T + \dim(U \cap$ *range* $T)$.

SOLUTION: For any $u, w \in \mathcal{K}_U$ and $\lambda \in \mathbf{F}$, $T(u + \lambda w) = Tu + \lambda Tw \in U \Rightarrow T$ is linear
Define $S \in \mathcal{L}(\mathcal{K}_U, U)$ as $Rv = Tv$ for all $v \in \mathcal{K}_U$. Hence range $R = U \cap$ range T.
Suppose $Tv = 0$ for some $v \in V$. 又 $0 \in U \Rightarrow Rv = 0$. Thus null $T \subseteq$ null R. $\square$

---

20 *Suppose* $T \in \mathcal{L}(V, W)$. *Prove that* T *is injective* $\iff \exists\ S \in \mathcal{L}(W, V),\ ST = I \in \mathcal{L}(V)$.

SOLUTION:

(a) Suppose $\exists\ S \in \mathcal{L}(W, V),\ ST = I$. Then if $Tv = 0 \Rightarrow ST(v) = 0 = v$. Hence T is injective.

(b) Suppose T is injective. $\forall w \in$ range $T,\ \exists!\, v \in V, Tv = w$. (if $w = 0$, then $v = 0$)

Define $S : W \to V$ by $Sw = v$ and $Su = 0,\ u \in U$. Where $W = U \oplus$ range T.

$\Rightarrow S(Tv + \lambda Tu) = S(T(v + \lambda u)) = v + \lambda u$ and $S(x + \nu y) = 0,\ x, y \in U$.

Thus $S|_{\text{range}\, T+U} = S|_W \in \mathcal{L}(W, V)$ and $ST = I$. □

OR. Let $R = (Tv_1, \dots, Tv_n)$ be linearly independent in range $T \subseteq W$, $(\dots)$ and then $\mathcal{K}_R \oplus$ null $T = V$.

Supose $W = U \oplus$ range T. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$ and $Su = 0,\ u \in U$. Thus $ST = I$. □

---

21 *Suppose* $T \in \mathcal{L}(V, W)$. *Prove that* T *is surjective* $\iff \exists\ S \in \mathcal{L}(W, V),\ TS = I \in \mathcal{L}(W)$.

SOLUTION:

(a) Suppose $\exists\ S \in \mathcal{L}(W, V),\ TS = I$. Then for any $w \in W, TS(w) = w \in$ range $T \Rightarrow$ range $T = W$. □

(b) Suppose T is surjective. $\forall w \in W,\ \exists v \in V, Tv = w$. Define $S : W \to V$ by $Sw = v$.

But $T(Sv + \lambda Su) = T(Sv) + \lambda T(Su) = v + \lambda u = T(S(v + \lambda u)) \not\Rightarrow Sv + \lambda Su = S(v + \lambda u)$.

So we let $R = (Tv_1, \dots, Tv_n)$ be linearly independent in range $T = W$, $(\dots)$ and then $\mathcal{K}_R \oplus$ null $T = V$.

Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$. Then $TS = I$. □

---

22 *Suppose U and V are finite-dim vec-sps and* $S \in \mathcal{L}(V, W)$ *and* $T \in \mathcal{L}(U, V)$.
Prove that $\dim$ *null* $ST \leq \dim$ *null* $S + \dim$ *null* T.

SOLUTION: Define $R \in \mathcal{L}(\text{null}\, ST, V)$ by $Ru = Tu$ for all $u \in$ null $ST \subseteq U$.

$$\left.\begin{aligned} S(Tu) = 0 = S(Ru) &\Rightarrow \text{range}\, R \subseteq \text{null}\, S \Rightarrow \dim \text{range}\, R \leq \dim \text{null}\, S \\ Tu = 0 = Ru &\Rightarrow \text{null}\, R \supseteq \text{null}\, T \Rightarrow \dim \text{null}\, R = \dim \text{null}\, T \end{aligned}\right\} \Rightarrow \square$$

• **COROLLARY:**

(1) If T is injective, then dim null $T = 0 \Rightarrow$ dim null $ST \leq$ dim null S.

(2) If T is surjective, then range $R =$ null $S \Rightarrow$ dim null $ST =$ dim null $S +$ dim null T.

(3) If S is injective, then range $R = \{0\} \Rightarrow$ dim null $ST =$ dim null $R =$ dim null T.

---

23 *Suppose U and V are finite-dim vec-sps and* $S \in \mathcal{L}(V, W)$ *and* $T \in \mathcal{L}(U, V)$.
Prove that $\dim$ *range* $ST \leq \min\{\dim$ *range* $S, \dim$ *range* $T\}$.

SOLUTION:

range $ST = \{Sv : v \in \text{range}\, T\} = \text{span}\,(Su_1, \dots, Su_{\dim \text{range}\, T})$, letting span $(u_1, \dots, u_{\dim \text{range}\, T}) =$ range T.

dim range $ST \leq$ dim range T 又 dim range $ST \leq$ dim range $S \Rightarrow$ □

• **COROLLARY:**

(1) If S is injective, then dim range $ST =$ dim range T.

(2) If T is surjective, then range $ST =$ range S.

• (a) *Suppose* $\dim V = 5$ *and* $S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$. *Prove that* $\dim$ *range* $TS \leq 2$.
(b) *Give an example of* $S, T \in \mathcal{L}(\mathbf{F}^5)$ *with* $ST = 0$ *and* $\dim$ *range* $TS = 2$.

SOLUTION:

By Problem (23), dim range $TS \leq \min\{\underbrace{\text{dim range}\, S}_{5-\text{dim null}\, T}, \underbrace{\text{dim range}\, T}_{5-\text{dim null}\, S}\}$.

Suppose dim range $TS \geq 3$. Then $\min\{5 - \text{dim null}\, T, 5 - \text{dim null}\, S\} \geq 3$

$\Rightarrow \max\{\text{dim null}\, T, \text{dim null}\, S\} \leq 2$.

又 dim null $ST = 5 \leq$ dim null $S +$ dim null $T \leq 4$. Contradicts. Thus dim range $TS \leq 2$. □

EXAMPLE: $V = \text{span}\,(v_1, \dots, v_5)$

$$T: \quad v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i \ ;$$
$$S: \quad v_1 \mapsto v_4, \quad v_2 \mapsto v_5, \quad v_i \mapsto 0 \ ; \qquad i = 3, 4, 5$$

• *Suppose* $\dim V = n$ *and* $S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$.

Prove that $\dim TS \le m = \begin{cases} \frac{n}{2}, & \textit{if } 2 \mid n. \\ \frac{n-1}{2}, & \textit{otherwise.} \end{cases}$

SOLUTION:

By Problem (23), $\dim \text{range}\, TS \le \min\{\underbrace{\dim \text{range}\, S}_{n - \dim \text{null}\, T}, \underbrace{\dim \text{range}\, T}_{n - \dim \text{null}\, S}\}$. Suppose $\dim \text{range}\, TS \ge m + 1$.

Then $\min\{n - \dim \text{null}\, T, n - \dim \text{null}\, S\} \ge m + 1$

$\Rightarrow \max\{\dim \text{null}\, T, \dim \text{null}\, S\} \le n - m - 1$.

又 $\dim \text{null}\, ST = n \le \dim \text{null}\, S + \dim \text{null}\, T \le n - m - 1$. Contradicts. Thus $\dim \text{range}\, TS \le m$. $\square$

---

24 *Suppose that* W *is finite-dim and* $S, T \in \mathcal{L}(V, W)$.
Prove that null $S \subseteq$ *null* $T \iff \exists\ E \in \mathcal{L}(W)$ *such that* $T = ES$.

SOLUTION:

Suppose $\text{null}\, S \subseteq \text{null}\, T$. Let $R = (Sv_1, \dots, Sv_n)$ be a basis of range $S \Rightarrow (v_1, \dots, v_n)$ is linearly independent.

Let $\mathcal{K}_R = \text{span}\,(v_1, \dots, v_n) \Rightarrow V = \mathcal{K}_R \oplus \text{null}\, S$.

Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i, \ \ Eu = 0;$ for each $i = 1 \dots, n$ and $u \in \text{null}\, S$.

Hence $\forall v \in V, \ (\exists!\, a_i \in \mathbf{F}, u \in \text{null}\, S\,),\ Tv = a_1 Tv_1 + \cdots + a_n Tv_n = E(a_1 Sv_1 + \cdots + a_n Sv_n) \Rightarrow T = ES$.

Suppose $\exists\ E \in \mathcal{L}(W)$ such that $T = ES$. Then $\text{null}\, T = \text{null}\, ES \supseteq \text{null}\, S$. $\square$

---

25 *Suppose that* V *is finite-dim and* $S, T \in \mathcal{L}(V, W)$.
Prove that range $S \subseteq$ *range* $T \iff \exists\ E \in \mathcal{L}(V)$ *such that* $S = TE$.

SOLUTION:

Suppose range $S \subseteq$ range T. Let $(v_1, \dots, v_m)$ be a basis of V.

Because range $S \subseteq$ range $T \Rightarrow Sv_i \in$ range T for each i. Suppose $u_i \in V$ for each i such that $Tu_i = Tv_i$.

Thus defining $E \in \mathcal{L}(V)$ by $Ev_i = u_i$ for each $i \Rightarrow S = TE$.

Suppose $\exists\ E \in \mathcal{L}(V)$ such that $S = TE$. Then range S = range $TE \subseteq$ range T. $\square$

---

26 *Prove that the differentiation map* $D \in \mathcal{P}(\mathbf{R})$ *is surjective.*

SOLUTION: Note that $\deg Dx^n = n - 1$.

Because $\text{span}\,(Dx, Dx^2, \dots) \subseteq \text{range}\, D$. 又 By (2.A.10), $\text{span}\,(Dx, Dx^2, \dots) = \text{span}\,(1, x, \dots) = \mathcal{P}(\mathbf{R})$. $\square$

---

27 *Suppose* $p \in \mathcal{P}(\mathbf{R})$. *Prove that there exists a polynomial* $q \in \mathcal{P}(\mathbf{R})$ *such that* $5q'' + 3q' = p$.

SOLUTION:

Define $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$.

Note that $\deg Bx^n = n - 1$. Similar to Problem (26), we conclude that B is surjective.

Hence for any $p \in \mathcal{P}(\mathbf{R})$, there exists $q \in \mathcal{P}(\mathbf{R})$ such that $Bq = p$. $\square$

---

28 *Suppose* $T \in \mathcal{L}(V, W)$ *and* $(w_1, \dots, w_m)$ *is a basis of range* T. *Prove that*
$\exists\ \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ *such that for all* $v \in V, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.

SOLUTION:

Suppose $(v_1, \dots, v_m)$ in V such that $Tv_i = w_i$ for each i.

Then $(v_1, \dots, v_m)$ is linearly independent, extend it to a basis of V as $(v_1, \dots, v_m, u_1, \dots, u_n)$.

Note that $\forall v \in V, v = a_1v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_n u_n,\ \exists!\, a_i, b_i \in \mathbf{F} \Rightarrow Tv = a_1 w_1 + \cdots + a_m w_m$.
Define $\varphi_i : V \to \mathbf{F}$ by $\varphi_i(v) = a_i v_i$ for each i. We now check the linearity.
$\forall v, u \in V\,(\exists!\, a_i, b_i, c_i, d_i \in \mathbf{F}), \lambda \in \mathbf{F}, \varphi_i(v + \lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda\varphi(u)$. $\square$

---

29 *Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Suppose $u \in V$ is not in null φ.*
Prove that $V = \text{null}\,\varphi \oplus \{au : a \in \mathbf{F}\}$.

SOLUTION:

(a) Suppose $v = cu \in \text{null}\,\varphi \cap \{au : a \in \mathbf{F}\}$, where $c \in \mathbf{F}$.
Then $\varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$. Hence null $\varphi \cap \{au : a \in \mathbf{F}\}$.

(b) Suppose $v \in V$. Then $v = (v - \dfrac{\varphi(v)}{\varphi(u)}u) + \dfrac{\varphi(v)}{\varphi(u)}u \Rightarrow \varphi(v) = 0$.

$$\left.\begin{array}{l} v - \frac{\varphi(v)}{\varphi(u)}u \in \text{null}\,\varphi \\ \frac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array}\right\} \Rightarrow V = \text{null}\,\varphi \oplus \{au : a \in \mathbf{F}\}. \ \square$$

This may seems strange. Here we explain why.
$\varphi \neq 0 \Rightarrow \exists$ a linearly independent list $(v_1, \ldots, v_n \in V)$ such that $\varphi(v_i) = a_i \neq 0$.
Choose a v_k arbitrarily. Then $\varphi(v_k - \dfrac{\varphi(v_k)}{\varphi(v_j)}v_j) = 0$ for each $j = 1, \ldots, k-1, k+1, \ldots, n$.
Thus span $\{v_k - \dfrac{\varphi(v_k)}{\varphi(v_j)}v_j\}_{j \neq k} \subseteq$ null φ. Hence there is only one nonzero vector in every vec-sp in $\mathcal{S}_V$null φ.

---

30 *Suppose $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$ and null $\varphi_1 =$ null $\varphi_2 =$ null φ. Prove that $\exists\, c \in \mathbf{F}, \varphi_1 = c\varphi_2$*

SOLUTION:

If null $\varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $u \in V/\text{null}\,\varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$.
By Problem (29), $V = \text{null}\,\varphi \oplus \text{span}\,(u)$.
Hence for any $v \in V, v = w + a_v u,\ \exists!\, w \in \text{null}\,\varphi, a_v \in \mathbf{F}$.
$\varphi_1(v) = a_v\varphi_1(u),\ \ \varphi_2(v) = a_v\varphi_2(u) \Rightarrow a_v = \dfrac{\varphi_1(v)}{\varphi_1(u)} = \dfrac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \dfrac{\varphi_1(u)}{\varphi_2(u)} = \dfrac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}$.
Thus $\varphi_1 = c\varphi_2$. $\square$

---

31 *Give an example of $T_1, T_2 \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^2)$ such that null $T_1 =$ null T_2*
and that T_1 is not a scalar multiple of T_2.

SOLUTION:

Let $(v_1, \ldots, v_5)$ be a basis of $\mathbf{R}^5$, (w_1, w_2) be a basis of $\mathbf{R}^2$. Define $T, S \in \mathcal{L}(V, W)$ by

$$\left.\begin{array}{lll} Tv_1 = w_1, & Tv_2 = w_2, & Tv_3 = Tv_4 = Tv_5 = 0 \\ Sv_1 = w_1, & Sv_2 = 2w_2, & Sv_3 = Sv_4 = Sv_5 = 0 \end{array}\right\} \Rightarrow \text{null}\,T = \text{null}\,S.$$

Suppose $T = \lambda S$. Then $w_1 = Tv_1 = \lambda Sv_1 = \lambda w_1 \Rightarrow \lambda = 1$.
While $w_2 = Tv_2 = \lambda Sv_2 = 2\lambda w_2 \Rightarrow \lambda = \frac{1}{2}$. Contradicts. $\square$

• *Suppose V is finite-dim, X is a subspace of V, and Y is a finite-dim subspace of W.*
Prove that there exists $T \in \mathcal{L}(V, W)$ such that null $T = X$ and range $T = Y$
if and only if $\dim X + \dim Y = \dim V$.

SOLUTION:

(a) Suppose $\dim X + \dim Y = \dim V$. Let $(u_1, \ldots, u_n)$ be a basis of X, $R = (w_1, \ldots, w_m)$ be a basis of Y.

Extend $(u_1, \ldots, u_n)$ to a basis of V as $(u_1, \ldots, u_n, v_1, \ldots, v_m)$.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_m v_m + b_1 v_1 + \cdots + b_n v_n) = a_1 w_1 + \cdots + a_m w_m$.

Now we show that null $T = X$ and range $T = Y$

Suppose $v \in V$. Then $\exists !\, a_i, b_j \in \mathbf{F}, v = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_n u_n$.

$$\left.\begin{aligned} v \in \text{null}\, T &\Rightarrow Tv = 0 \\ &\Rightarrow a_1 = \cdots = a_m = 0 \\ &\Rightarrow v \in X \Rightarrow \text{null}\, T \subseteq X. \\ v \in X &\Rightarrow v \in \text{null}\, T \Rightarrow \text{null}\, T \supseteq X. \end{aligned}\right\} \Rightarrow \text{null}\, T = X.$$

$$\left.\begin{aligned} w \in \text{range}\, T &\Rightarrow \exists\, v \in V, Tv = w \Rightarrow \text{let } v = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_n u_n \\ &\Rightarrow Tv = w = a_1 w_1 + \cdots + a_m w_m \Rightarrow w \in Y \Rightarrow \text{range}\, T \subseteq Y. \\ w \in Y &\Rightarrow w = a_1 T v_1 + \cdots + a_m T v_m = T(a_1 v_1 + \cdots + a_m v_m) \\ &\Rightarrow w \in \text{range}\, T \Rightarrow \text{range}\, T \supseteq Y. \end{aligned}\right\} \Rightarrow \text{range}\, T = Y.$$

(b) Conversely it is true as well.

$\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V, W)$. Let $(Tv_1, \ldots, Tv_n)$ be a basis of range T. Extend $(v_1, \ldots, v_n)$ to a basis of V as $(v_1, \ldots, v_n, u_1, \ldots, u_m)$. Prove or give a counterexample: $(u_1, \ldots, u_m)$ is a basis of null T.*

SOLUTION: A counterexample:

Suppose $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$. Then span $(Tv_1, Tv_2, Tv_3) =$ span (w_1).

Extend (v_i) to (v_1, v_2, v_3) for each i. But none of $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is a basis of null T. $\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V, W)$. Let $(u_1, \ldots, u_m)$ be a basis of null T. Extend $(u_1, \ldots, u_m)$ to a basis of V as $(u_1, \ldots, u_m, v_1, \ldots, v_n)$. Prove or give a counterexample: $(Tv_1, \ldots, Tv_n)$ spans range T.*

SOLUTION:

$\forall w \in \text{range}\, T,\ \exists\, v \in V,\ (\exists !\, a_i, b_i \in \mathbf{F}), Tv = T(a_1 v_1 + \cdots + a_n v_n) = w$

$\Rightarrow w \in \text{span}\,(Tv_1, \ldots, Tv_n) \Rightarrow \text{range}\, T \subseteq \text{span}\,(Tv_1, \ldots, Tv_n)$. $\square$

COMMENT: If T is injective, then $(Tv_1, \ldots, Tv_n)$ is a basis of range T.

• (OR (5.B.4)) *Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V =$ null $P \oplus$ range P.*

SOLUTION:

Let $P^2 v_1, \ldots, P^2 v_n$ be a basis of range P^2. Then $(Pv_1, \ldots, Pv_n)$ is linearly independent in V.

$$\left.\begin{aligned} &\text{Let } \mathcal{K} = \text{span}\,(Pv_1, \ldots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \text{null}\, P^2 \\ &\text{又 } \mathcal{K} = \text{range}\, P = \text{range}\, P^2;\ \ \text{null}\, P = \text{null}\, P^2 \end{aligned}\right\} \Rightarrow \square$$

OR.

(a) Suppose $v \in$ null $P \cap$ range P.

Then $\exists\, u \in V, v = Pu, Pv = 0 \Rightarrow v = Pu = P^2 u = Pv = 0$. Hence null $P \cap$ range $P = \{0\}$.

(b) Note that $v = Pv + (v - Pv), Pv^2 = Pv$ for all $v \in V$.

Then $P(v - Pv) = 0 \Rightarrow v - Pv \in$ null P. Hence $V =$ range $P +$ null P. $\square$

• *Suppose V is finite-dim with* $\dim V > 1$. *Show that if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is a linear map such that $\varphi(ST) = \varphi(S) \cdot \varphi(T)$ for all $S, T \in \mathcal{L}(V)$, then $\varphi = 0$.*

HINT: *The description of the two-sided ideals of $\mathcal{L}(V)$ in Section 3A might be useful.*

SOLUTION: Using notations in (3.A.• the last).

Suppose $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \ldots, n\}$, $\varphi(R_{i,j}) \neq 0$.

Because $R_{i,j} = R_{x,j} \circ R_{i,x}$, $\forall x = 1, \ldots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, because $R_{i,x} = R_{y,x} \circ R_{i,y}$, $\forall y = 1, \ldots, n$. Thus $\varphi(R_{y,x}) \neq 0$ for any $x, y = 1, \ldots, n$.

Let $l \neq i, k \neq j$ and then $\varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. $\square$

• *Suppose that V and W are real vector spaces and $T \in \mathcal{L}(V, W)$. Define $T_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ by $T_{\mathbb{C}}(u + iv) = Tu + iTv$ for all $u, v \in V$.*

(a) *Show that $T_{\mathbb{C}}$ is a (complex) linear map from $V_{\mathbb{C}}$ to $W_{\mathbb{C}}$.*

(b) *Show that $T_{\mathbb{C}}$ is injective $\Longleftrightarrow$ T is injective.*

(c) *Show that range $T_{\mathbb{C}} = W_{\mathbb{C}} \Longleftrightarrow$ range $T = W$.*

See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbb{C}}$.

The linear map $T_{\mathbb{C}}$ is called the complexification of the linear map T.

SOLUTION:

(a) $\forall u_1 + \mathrm{i}v_1, u_2 + \mathrm{i}v_2 \in V_{\mathbb{C}}, \lambda \in \mathbf{F}$,

$T((u_1 + \mathrm{i}v_1) + \lambda(u_2 + \mathrm{i}v_2)) = T((u_1 + \lambda u_2) + \mathrm{i}(v_1 + \lambda v_2)) = T(u_1 + \lambda u_2) + \mathrm{i}T(v_1 + \lambda v_2)$

$= Tu_1 + \mathrm{i}Tv_1 + \lambda Tu_2 + \mathrm{i}\lambda Tv_2 = T(u_1 + \mathrm{i}v_1) + \lambda T(u_2 + \mathrm{i}v_2)$. $\square$

(b) $\left.\begin{array}{l}\text{Suppose } T_{\mathbb{C}} \text{ is injective. Let } T(u) = 0 \Rightarrow T_{\mathbb{C}}(u + \mathrm{i}0) = Tu = 0 \Rightarrow u = 0. \\ \text{Suppose } T \text{ is injective. Let } T_{\mathbb{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + \mathrm{i}v = 0.\end{array}\right\} \Rightarrow \square$

(c) $\left.\begin{array}{l}\text{Suppose } T_{\mathbb{C}} \text{ is surjective. } \forall w, x \in W,\ \exists\, u, v \in V, T(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = w + \mathrm{i}x \\ \qquad \Rightarrow Tu = w, Tv = x \Rightarrow \text{T is surjective.} \\ \text{Suppose } T \text{ is surjective. } \forall w, x \in W, \exists\, u, v \in V, Tu = w, Tv = x \\ \qquad \Rightarrow \forall w + \mathrm{i}x \in W_{\mathbb{C}}, \exists\, u + \mathrm{i}v \in V, T(u + \mathrm{i}v) = w + \mathrm{i}x \Rightarrow T_{\mathbb{C}} \text{ is surjective.}\end{array}\right\} \Rightarrow \square$

ENDED

3·C

• **NOTE FOR [3.47]:** $LHS = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n}(A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k} = RHS.$

• **NOTE FOR [3.48]:**

$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\ 8 & 9 & 10\end{pmatrix} = \begin{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} & \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix}\\ \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} & \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}1\times5+2\times8 & 1\times6+2\times9 & 1\times7+2\times10\\ 3\times5+4\times8 & 3\times6+4\times9 & 3\times7+4\times10\end{pmatrix} = \begin{pmatrix}21 & 24 & 27\\ 47 & 54 & 61\end{pmatrix}$$

• **NOTE FOR [3.49]:** $\because [(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$

$\therefore (AC)_{\cdot,k} = A_{\cdot,\cdot}C_{\cdot,k} = AC_{\cdot,k}$

• **EXERCISE 10:** $\because [(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n}(A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$

$\therefore (AC)_{j,\cdot} = A_{j,\cdot}C_{\cdot,\cdot} = A_{j,\cdot}C.$

• *Suppose* $C \in \mathbf{F}^{m,c}$, $R \in \mathbf{F}^{c,p}$.

(a) For $k = 1,\dots,p$, $\quad (CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^{c} C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$

(b) For $j = 1,\dots,m$, $\quad (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

EXAMPLE:

$$P = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\ 8 & 9 & 10\end{pmatrix} \in \mathbf{F}^{2,3}.$$

$$P_{\cdot,1} = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} = 5\begin{pmatrix}1\\3\end{pmatrix} + 8\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}21\\47\end{pmatrix};$$

$$P_{\cdot,2} = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} = 6\begin{pmatrix}1\\3\end{pmatrix} + 9\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}24\\54\end{pmatrix};$$

$$P_{\cdot,3} = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix} = 7\begin{pmatrix}1\\3\end{pmatrix} + 10\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}27\\61\end{pmatrix};$$

$$P_{1,\cdot} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\ 8 & 9 & 10\end{pmatrix} = 1\begin{pmatrix}5 & 6 & 7\end{pmatrix} + 2\begin{pmatrix}8 & 9 & 10\end{pmatrix} = \begin{pmatrix}21 & 24 & 27\end{pmatrix};$$

$$P_{2,\cdot} = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\ 8 & 9 & 10\end{pmatrix} = 3\begin{pmatrix}5 & 6 & 7\end{pmatrix} + 4\begin{pmatrix}8 & 9 & 10\end{pmatrix} = \begin{pmatrix}47 & 54 & 61\end{pmatrix};$$

• **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$

$\because (Ac)_{j,1} = \sum_{r=1}^{n} A_{j,r}c_{r,1} = [\sum_{r=1}^{n}(A_{\cdot,r}c_{r,1})]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$

$\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^{n} A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ OR. By $(Ac)_{\cdot,1} = Ac_{\cdot,1}$ Using (a) above.

• **EXERCISE 10:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$

$\because (aC)_{1,k} = \sum_{r=1}^{n} a_{1,r}C_{r,k} = [\sum_{r=1}^{n} a_{1,r}(C_{r,\cdot})]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$

$\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^{n} a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ OR. By $(aC)_{1,\cdot} = a_{1,\cdot}C$. Using (b) above.

• COLUMN-ROW FACTORIZATION (CR Factorization)

Suppose $A \in \mathbf{F}^{m,n}$, $A \neq 0$. *Let* $S_c = span\,(A_{\cdot,1}, \ldots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}$, $\dim S_c = c$.

And $S_r = span\,(A_{1,\cdot}, \ldots, A_{n,\cdot}) \subseteq \mathbf{F}^{1,n}$, $\dim S_r = r$.

Prove that $A = CR$. $\exists\; C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,n}$.

SOLUTION: Notice that $A \neq 0 \Rightarrow c, r \geq 1$.

Let $(C_{\cdot,1}, \ldots, C_{\cdot,c})$ be a basis of S_c, forming $C \in \mathbf{F}^{m,c}$.

Then for any $A_{\cdot,k}$, $\quad A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}, \quad \exists!\, R_{1,k}, \ldots, R_{c,k} \in \mathbf{F}$.

Hence, by letting $R = \begin{pmatrix} R_{1,1} & \cdots & R_{1,n} \\ \vdots & \ddots & \vdots \\ R_{c,1} & \cdots & R_{c,n} \end{pmatrix}$, we have $A = CR$.

OR. Let $(R_{1,\cdot}, \ldots, R_{c,\cdot})$ be a basis of S_r, forming $R \in \mathbf{F}^{c,n}$.

For any $A_{j,\cdot}$, $\quad A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot} = (CR)_{j,\cdot}, \quad \exists!\, C_{j,1}, \ldots, C_{j,c} \in \mathbf{F}$. Similarly. $\square$

EXAMPLE:

$$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

(1) Because $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2\begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix}$.

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix}$ can be uniquely written as a linear combination of $A_{1,\cdot}, A_{2,\cdot}$.

Hence $\dim S_r = 2$. We choose $(A_{1,\cdot}, A_{2,\cdot})$ as the basis.

(2) Because $\begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix} - \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix}$.

Hence $\dim S_c = 2$. We choose $(A_{\cdot,2}, A_{\cdot,3})$ as the basis.

• COLUMN RANK EQUALS ROW RANK (Using the notation above)

For any $A_{j,\cdot} \in S_r$, $\quad A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

$\Rightarrow \operatorname{span}(A_{1,\cdot}, \ldots, A_{m,\cdot}) = S_r = \operatorname{span}(R_{1,\cdot}, \ldots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leq c = \dim S_c$.

Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leq r = \dim S_r = \dim S_c^t$. $\square$

• *Suppose* $T \in \mathcal{L}(V)$, *and* $u_1, \ldots, u_n$ *and* $v_1, \ldots, v_n$ *are bases of* V.

Prove that the following are equivalent.

(a) T *is injective.*

(b) *The columns of* $\mathcal{M}(T)$ *are linearly independent in* $\mathbf{F}^{n,1}$.

(c) *The columns of* $\mathcal{M}(T)$ *span* $\mathbf{F}^{n,1}$.

(d) *The rows of* $\mathcal{M}(T)$ *span* $\mathbf{F}^{1,n}$.

(e) *The rows of* $\mathcal{M}(T)$ *are linearly independent in* $\mathbf{F}^{1,n}$.

Here $A = \mathcal{M}(T)$ *means* $\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$.

SOLUTION:

T is injective $\iff \dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim \operatorname{range} T$

$\iff (Tu_1, \ldots, Tu_n)$ is linearly independent in V, and therefore is a basis of V

$\iff (\mathcal{M}(Tu_1), \ldots, \mathcal{M}(Tu_n))$ is linearly independent, as well as $(A_{\cdot,1}, \ldots, A_{\cdot,n})$

$\iff (A_{\cdot,1}, \ldots, A_{\cdot,n})$ is a basis of $\mathbf{F}^{n,1}$.

(又 $\dim \operatorname{span}(A_{\cdot,1}, \ldots, A_{\cdot,n}) = \dim \operatorname{span}(A_{1,\cdot}, \ldots, A_{n,\cdot}) = n$)

$\iff (A_{1,\cdot}, \ldots, A_{n,\cdot})$ is a basis of $\mathbf{F}^{1,n}$. $\square$

• *Suppose A is an m-by-n matrix with $A \neq 0$.*
Prove that the rank of A is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbf{F}^m$ and $(d_1, \ldots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j \cdot d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$.

SOLUTION: Using the notation in CR Factorization.

(a) Suppose $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix}$. $(\ \exists\, c_j, d_k \in \mathbf{F}, \forall j, k\ )$

Then $S_c = \operatorname{span}\left(\begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \ldots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right) = \operatorname{span}\left(\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right)$.

OR. $S_r = \operatorname{span}\begin{pmatrix} \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \end{pmatrix}, \\ \begin{pmatrix} c_2 d_1 & \cdots & c_2 d_n \end{pmatrix}, \\ \vdots \\ \begin{pmatrix} c_m d_1 & \cdots & c_m d_n \end{pmatrix} \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} d_1 & \ldots & d_n \end{pmatrix} \right)$. Hence the rank of A is 1.

(b) Suppose the rank of A is $\dim S_c = \dim S_r = 1$

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $\quad d'_k = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$. $(\forall j, k\ )$

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$. Letting $d_k = d'_k A_{1,1}$. ☐

---

1 *Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim$ range T nonzero entries.*

SOLUTION: Let $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ be bases of V and W respectively. We prove by contradiction.

Suppose $A = \mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$ has at most $(\dim \operatorname{range} T - 1)$ nonzero entries.

Then by Pigeon Hole Principle, at least one of $A_{\cdot,k} = 0$.

Thus there are at most $(\dim \operatorname{range} T - 1)$ nonzero vectors in $Tv_1, \ldots, Tv_n$.

While $\operatorname{range} T = \operatorname{span}(Tv_1, \ldots, Tv_n) \Rightarrow \dim \operatorname{range} T \leq \dim \operatorname{range} T - 1$. Hence we get a contradiction. ☐

---

3 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$.*
Prove that there exist a basis of V and a basis of W such that
[letting $A = \mathcal{M}(T)$ with respect to these bases],
$A_{k,k} = 1, A_{i,j} = 0$, *where* $1 \leq k \leq \dim$ *range* $T, i \neq j$.

SOLUTION:

Let $R = (Tv_1, \ldots, Tv_n)$ be a basis of range T, extend it to the basis of W as $(Tv_1, \ldots, Tv_n, w_1, \ldots, w_p)$.

Let $\mathcal{K}_R = \operatorname{span}(v_1, \ldots, v_n)$. Let $(u_1, \ldots, u_m)$ be a basis of null T.

Then $(v_1, \ldots, v_n, u_1, \ldots, u_m)$ is the basis of V.

Thus $T(v_k) = Tv_k, T(u_j) = 0 \Rightarrow A_{k,k} = 1, A_{i,j}$ for each $k \in \{1, \ldots, \dim \operatorname{range} T\}$ and $j \in \{1, \ldots, m\}$. ☐

---

4 *Suppose $(v_1, \ldots, v_m)$ is a basis of V and W is finite-dim. Suppose $T \in \mathcal{L}(V, W)$.*
Prove that there exists a basis $(w_1, \ldots, w_n)$ of W such that all entries in the first column of $A = \mathcal{M}(T, (v_1, \ldots, v_m), (w_1, \ldots, w_n))$ are 0 except for possibly a 1 in the first row, first column.

SOLUTION: If $Tv_1 = 0$, then we are done. Otherwise, extend (Tv_1) to a basis of W, as desired. ☐

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5 *Suppose $(w_1, \ldots, w_n)$ is a basis of W and V is finite-dim. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $(v_1, \ldots, v_m)$ of V such that all entries in the first row of $\mathcal{M}(T, (v_1, \ldots, v_m), (w_1, \ldots, w_n))$ are 0 except for possibly a 1 in the first row, first column.*

SOLUTION:

Let $(u_1, \ldots, u_m)$ be a basis of V. If $A_{1,\cdot} = 0$, then let $v_i = u_i$ for each $i = 1, \ldots, n$, we are done.

Otherwise, $\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \end{pmatrix} \neq 0$, choose one $A_{1,k} \neq 0$.

Let $v_1 = \dfrac{u_k}{A_{1,k}}$; $\quad v_j = u_{j-1} - A_{1,j-1} v_1 \quad$ for $j = 2, \ldots, k$;

$v_i = u_i - A_{1,i} v_1 \quad$ for $i = k+1, \ldots, n$. $\square$

---

6 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $A = \mathcal{M}(T)$ equal 1.*

SOLUTION:

Denote the bases of V and W by $B_V = (v_1, \ldots, v_n)$ and $B_W = (w_1, \ldots, w_m)$ respectively.

(a) Suppose B_V, B_W are the bases such that all entries of A equal 1.

Then $Tv_i = w_1 + \cdots + w_m$ for all $i = 1, \ldots, n$. Hence $\dim \operatorname{range} T = 1$.

(b) Suppose $\dim \operatorname{range} T = 1$. Then $\dim \operatorname{null} T = \dim V - 1$.

Let $(u_2, \ldots, u_n)$ be a basis of $\operatorname{null} T$. Extend it to a basis of V as $(u_1, u_2, \ldots, u_n)$.

Let $w_1 = Tv_1 - w_2 - \cdots - w_m$. Extend it to B_W the basis of W.

Let $v_1 = u_1, v_i = u_1 + u_i$. Extend it to B_V the basis of V. $\square$

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12 *Give an example of 2-by-2 matrices A and B such that $AB \neq BA$.*

SOLUTION: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

---

13 *Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E and F are matrices whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense. Explain why $AB + AC$ and $DF + EF$ both make sense and prove that.*

SOLUTION: Using [3.36], [3.43].

(a) Left distributive: Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Because $[A(B+C)]_{j,k} = \sum\limits_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum\limits_{r=1}^{n} (A_{j,r}B_{r,k} + A_{j,r}C_{r,k})$.

Hence we conclude that $A(B+C) = AB + AC$.

OR. Let $(e_1, \ldots, e_M)$ be the standard basis of $\mathbf{F}^M$, where $M = \max\{m, n, p\}$.

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum\limits_{j=1}^{m} A_{j,k} e_j$ for each $k = 1, \ldots, n$. Then $\mathcal{M}(T) = A$.

Similarly, define S, R such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

Thus $T(S+R) = TS + TR$

$\Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS + TR)$

$\Rightarrow \mathcal{M}(T)[\mathcal{M}(S) + \mathcal{M}(R)] = \mathcal{M}(T)\mathcal{M}(S) + \mathcal{M}(T)\mathcal{M}(R)$

$\Rightarrow A(B+C) = AB + AC.$

(b) Right distributive: Similarly.

Suppose $\mathcal{M}(T) = D, \mathcal{M}(S) = E, \mathcal{M}(R) = F$.

Then $(T+S)R = TR + SR$

$\Rightarrow \mathcal{M}((T+S)R) = \mathcal{M}(TR) + \mathcal{M}(SR)$

$\Rightarrow [\mathcal{M}(T) + \mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)\mathcal{M}(R) + \mathcal{M}(S)\mathcal{M}(R)$

$\Rightarrow (D+E)F = DF + EF.$ $\square$

---

14 *Prove that matrix multiplication is associative. In other words, suppose A, B and C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that $(AB)C = A(BC)$.*

Try to find a clean proof that illustrates the following quote from Emil Artin:

"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

SOLUTION:

Because $[(AB)C]_{j,k} = (AB)_{j,\cdot}C_{\cdot,k} = \sum_{s=1}^{n}(A_{j,s}B_{s,\cdot})C_{\cdot,k} = \sum_{s=1}^{n} A_{j,s}(B_{s,\cdot}C_{\cdot,k}) = \sum_{s=1}^{n} A_{j,s}(BC)_{s,k} = A(BC)_{j,k}$

Hence we conclude that $(AB)C = A(BC)$.

OR. Suppose $A \in \mathbf{F}^{m,n}, B \in \mathbf{F}^{n,p}, C \in \mathbf{F}^{p,s}$.

Let $(e_1, \dots, e_M)$ be the standard basis of $\mathbf{F}^M$, where $M = \max\{m, n, p, s\}$.

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum_{j=1}^{m} A_{j,k}e_j$ for each $k = 1, \dots, n$. Then $\mathcal{M}(T) = A$.

Similarly, define S, R such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

Hence $(TS)R = T(SR) \Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR))$

$$\Rightarrow [\mathcal{M}(T)\mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)[\mathcal{M}(S)\mathcal{M}(R)]$$

$$\Rightarrow (AB)C = A(BC). \ \square$$

---

15 *Suppose A is an n-by-n matrix and $1 \leq j, k \leq n$. Show that the entry in row j, column k, of A^3 (which is defined to mean AAA) is $\sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.*

SOLUTION: $(AAA)_{j,k} = (AA)_{j,\cdot}A_{\cdot,k} = \sum_{p=1}^{n}(A_{j,p}A_{p,\cdot})A_{\cdot,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.

$$OR. \ \ (AAA)_{j,k} = \sum_{r=1}^{n}(AA)_{j,r}A_{r,k} = \sum_{r=1}^{n}(\sum_{p=1}^{n} A_{j,p}A_{p,r})A_{r,k}$$

$$= \sum_{r=1}^{n}(A_{j,1}A_{1,r}A_{r,k} + \cdots + A_{j,n}A_{n,r}A_{r,k})$$

$$= A_{j,1}\sum_{r=1}^{n} A_{1,r}A_{r,k} + \cdots + A_{j,n}\sum_{r=1}^{n} A_{n,r}A_{r,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}. \ \square$$

ENDED

3·D

• *Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and $(T^{-1})^{-1} = T$.*

SOLUTION:

$$\left.\begin{array}{l} TT^{-1} = (T^{-1})^{-1}T^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = T^{-1}(T^{-1})^{-1} = I \in \mathcal{L}(W) \end{array}\right\} \Rightarrow T = (T^{-1})^{-1}, \text{ by the uniqueness of inverse.} \quad \Box$$

---

1 *Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps.*
Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

SOLUTION:

$$\left.\begin{array}{l} (ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(V) \end{array}\right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}, \text{ by the uniqueness of inverse.} \quad \Box$$

---

9 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$.*
Prove that ST is invertible $\iff$ S and T are invertible.

SOLUTION:

Suppose S, T are invertible. Then $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$. Hence ST is invertible.

Suppose ST is invertible. Let $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T \text{ is injective, } S \text{ is surjective.}$$

Notice that V is finite-dim. Hence S, T are invertible. $\Box$

---

10 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that $ST = I \iff TS = I$.*

SOLUTION:

Suppose $ST = I$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T \text{ is injective, } S \text{ is surjective.}$$

Notice that V is finite-dim. Thus T, S are invertible.

OR. By Problem (9), V is finite-dim and $ST = I$ is invertible $\Rightarrow S, T$ are invertible.

$S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v$ (S is invertible).

OR. $ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T$. 又 $S = S \Rightarrow TS = S^{-1}S = I$.

Reversing the roles of S and T, we conclude that $TS = I \Rightarrow ST = I$. $\Box$

---

11 *Suppose V is finite-dim and $S, T, U \in \mathcal{L}(V)$ and $STU = I$.*
Show that T is invertible and that $T^{-1} = US$.

SOLUTION: Using Problem (9) and (10).

$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I$.

$\Rightarrow U^{-1} = ST, \quad T^{-1} = US, \quad S^{-1} = TU.$ $\Box$

---

12 *Show that the result in Exercise 11 can fail without the hypothesis that V is finite-dim.*

SOLUTION:

Let $V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots), T(a_1, \dots) = (0, a_1, \dots), U = I$.

Then $STU = I$ but T^{-1} is not invertible.

---

13 *Suppose V is finite-dim and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective. Prove that S is injective.*

SOLUTION:

By Problem (1) and (9), Notice that V is finite-dim. Then RST is invertible.

$(RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. $\square$

OR. Let $X = (RST)^{-1}$, $\left| \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is injective, and therefore is invertible.} \\ \forall v \in V, v = (RST)Xv \in \text{range } R \Rightarrow R \text{ is surjective, and therefore is invertible.} \end{array} \right.$

Thus $S = R^{-1}(RST)T^{-1}$ is invertible.

---

15 *Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists\, A \in \mathbf{F}^{m,n}, Tx = Ax, \forall x \in \mathbf{F}^{n,1}$.*

SOLUTION:

Let $E_i \in \mathbf{F}^{n,1}$ for each $i = 1, \dots, n$ (where $M = \max\{m, n\}$) be such that $(E_i)_{j,1} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Then $(E_1, \dots, E_n)$ is linearly independent and thus is a basis of $\mathbf{F}^{n,1}$.

Similarly, let $(R_1, \dots, R_m)$ be a basis of $\mathbf{F}^{m,1}$.

Suppose $T(E_i) = A_{1,i}R_1 + \dots + A_{m,i}R_m$ for each $i = 1, \dots, n$. Hence by letting $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$. $\square$

COMMENT: $\mathcal{M}(T) = A$. Conversely it is true as well.

- OR (10.A.2) *Suppose $A, B \in \mathbf{F}^{n,n}$. Prove that $AB = I \iff BA = I$.*

SOLUTION: Using Problem (10) and (15).

Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Then $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.

Thus $AB = I \Leftrightarrow A(Bx) = x \iff T(Sx) = x \Leftrightarrow TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I$.$\square$

- **NOTE FOR [3.60]:** Suppose $(v_1, \dots, v_n)$ is a basis of V and $(w_1, \dots, w_m)$ is a basis of W.

Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{ix}w_j$; $\quad \delta_{ix} = \begin{cases} 0, & i \neq x \\ 1, & i = x \end{cases}$ COROLLARY: $E_{l,k}E_{i,j} = \delta_{jl}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. $(\mathcal{E}^{(j,i)})_{l,k} = \begin{cases} 0, & i \neq k \text{ or } j \neq l \\ 1, & i = k \text{ and } j = l \end{cases}$

Because $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$ are isomorphic. And $T = \mathcal{M}^{-1}\mathcal{M}(T), E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

Hence $\forall T \in \mathcal{L}(V, W)$, $\exists!\, A_{i,j} \in \mathbf{F}$ ($\forall i = \{1, \dots, m\}, j = \{1, \dots, n\}$), $\mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)}+ & \cdots & +A_{1,n}\mathcal{E}^{(1,n)} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}\mathcal{E}^{(m,1)}+ & \cdots & +A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff T = \begin{pmatrix} A_{1,1}E_{1,1}+ & \cdots & +A_{1,n}E_{n,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}E_{1,m}+ & \cdots & +A_{m,n}E_{n,m} \end{pmatrix}.$$

$$\therefore \mathcal{L}(V, W) = \operatorname{span} \underbrace{\begin{pmatrix} E_{1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \cdots & , E_{n,m} \end{pmatrix}}_{B}; \quad \mathbf{F}^{m,n} = \operatorname{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & , \mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots & , \mathcal{E}^{(m,n)} \end{pmatrix}}_{B_M}.$$

Hence by [2.42] and [3.61], we conclude that B is a basis of $\mathcal{L}(V, W)$ and that B_M is a basis of $\mathbf{F}^{m,n}$.

∘ *Suppose V is finite-dim and $S \in \mathcal{L}(V)$. Define $A \in \mathcal{L}(\mathcal{L}(V))$ by $A(T) = ST$ for $T \in \mathcal{L}(V)$.*
(a) *Show that* $\dim \textit{null}\, A = (\dim V)(\dim \textit{null}\, S)$.
(b) *Show that* $\dim \textit{range}\, A = (\dim V)(\dim \textit{range}\, S)$.

SOLUTION: Using NOTE FOR [3.60].

Let $(w_1, \ldots, w_m)$ be a basis of range S, extend it to a basis of V as $(w_1, \ldots, w_m, \ldots, w_n)$.
Let $v_i \in V$ such that $Sv_i = w_i$ for $m = 1, \ldots, m$. Extend $(v_1, \ldots, v_m)$ to a basis of V as $(v_1, \ldots, v_m, \ldots, v_n)$.
Define $E_{i,j} \in \mathcal{L}(V)$ by $E_{i,j}(v_x) = \delta_{ix} w_i$.

Thus $S = E_{1,1} + \cdots + E_{m,m}$; $\quad \mathcal{M}(S, (v_1, \ldots, v_n), (w_1, \ldots, w_n)) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$.

Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j}(w_x) = \delta_{ix} v_i$.
Let $E_{j,k} R_{i,j} = Q_{i,k}, \quad R_{j,k} E_{i,j} = G_{i,k}$

Because $\forall T \in \mathcal{L}(V), \quad \exists!\, A_{i,j} \in \mathbf{F}\, (\forall i, j = 1, \ldots, n), \quad T = \begin{pmatrix} A_{1,1}R_{1,1}+ & \cdots & +A_{1,m}R_{m,1}+ & \cdots & +A_{1,n}R_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}R_{1,m}+ & \cdots & +A_{m,m}R_{m,m}+ & \cdots & +A_{m,n}R_{n,m} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{n,1}R_{1,n}+ & \cdots & +A_{n,m}R_{m,n}+ & \cdots & +A_{n,n}R_{n,n} \end{pmatrix}$.

$$\Rightarrow A(T) = ST = (\sum_{r=1}^{m} E_{r,r})(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i}) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{i,j}Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1}+ & \cdots & +A_{1,m}Q_{m,1}+ & \cdots & +A_{1,n}Q_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}Q_{1,m}+ & \cdots & +A_{m,m}Q_{m,m}+ & \cdots & +A_{m,n}Q_{n,m} \end{pmatrix}.$$

Thus null A = span $\begin{pmatrix} R_{1,m+1}, & \cdots & , R_{n,m+1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ R_{1,n}, & \cdots & , R_{n,n} \end{pmatrix}$, $\quad$ range A = span $\begin{pmatrix} Q_{1,1}, & \cdots & , Q_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ Q_{1,m}, & \cdots & , Q_{n,m} \end{pmatrix}$.

Hence (a) $\dim \operatorname{null} A = n \times (n - m)$; $\quad$ (b) $\dim \operatorname{range} A = n \times m$. $\quad \square$

• COMMENT: Define $B \in \mathcal{L}(\mathcal{L}(V))$ by $B(T) = TS$ for $T \in \mathcal{L}(V)$.

Similarly, $B(T) = TS = (\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i})(\sum_{r=1}^{m} E_{r,r}) = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{i,j}G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1}+ & \cdots & +A_{1,m}G_{m,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}G_{1,m}+ & \cdots & +A_{m,m}G_{m,m} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{n,1}G_{1,n}+ & \cdots & +A_{n,m}G_{m,n} \end{pmatrix}$.

• OR (10.A.1) *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V.*
Prove that $\mathcal{M}(T, (v_1, \ldots, v_n))$ is invertible $\iff$ T is invertible.

SOLUTION: Notice that $\mathcal{M}$ is an isomorphism of $\mathcal{L}(V)$ onto $\mathbf{F}^{n,n}$.

(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.
(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I$. $\quad \exists!\, S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$
$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$
$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. $\quad \square$

• OR (10.A.4) *Suppose that* $(u_1,\dots,u_n)$ *and* $(v_1,\dots,v_n)$ *are bases of* V.
Let $T\in\mathcal{L}(V)$ *be such that* $Tv_k=u_k$ *for each* $k=1,\dots,n$.
Prove that $A=\mathcal{M}(T,(v_1,\dots,v_n))=\mathcal{M}(I,(u_1,\dots,u_n),(v_1,\dots,v_n))=B$.

SOLUTION:

$\forall\, k\in\{1,\dots,n\}, Iu_k=u_k=B_{1,k}v_1+\cdots+B_{n,k}v_n=Tv_k=A_{1,k}v_1+\cdots+A_{n,k}v_n\Rightarrow A=B.\quad\square$

OR. Note that $\mathcal{M}(T,(v_1,\dots,v_n),(u_1,\dots,u_n))$ is the identity matrix.

$$A=\mathcal{M}(T,(v_1,\dots,v_n))=\mathcal{M}(I,(u_1,\dots,u_n),(v_1,\dots,v_n))\underbrace{\mathcal{M}(T,(v_1,\dots,v_n),(u_1,\dots,u_n))}_{=I}=B.\quad\square$$

• COMMENT: Denote $\mathcal{M}(T,(u_1,\dots,u_n))$ by A'.

$u_k=Iu_k=B_{1,k}v_1+\cdots+B_{n,k}v_n,\ \forall\, k\in\{1,\dots,n\}$.

又 $Tu_k=T(B_{1,k}v_1+\cdots+B_{n,k}v_n)=B_{1,k}u_1+\cdots+B_{n,k}u_n=A'_{1,k}u_1+\cdots+A'_{n,k}u_n\Rightarrow A'=B$.

OR. $A'=\mathcal{M}(T,(u_1,\dots,u_n))=\mathcal{M}(T,(v_1,\dots,v_n),(u_1,\dots,u_n))\mathcal{M}(I,(u_1,\dots,u_n),(v_1,\dots,v_n))=B$.

---

16 *Suppose* V *is finite-dim and* $S\in\mathcal{L}(V)$.
Prove that $\exists\,\lambda\in\mathbf{F}, S=\lambda I\iff ST=TS$ *for every* $T\in\mathcal{L}(V)$.

SOLUTION: Using the notation and result in $(\circ)$.

Suppose $S=\lambda I$. Then $ST=TS=\lambda T$ for every $T\in\mathcal{L}(V)$. Conversely, if $S=0$, then we are done.

Suppose $S\neq 0, ST=TS, \forall T\in\mathcal{L}(V)$. Let $S=E_{1,1}+\cdots+E_{m,m}\Rightarrow\mathcal{M}(S,(v_1,\dots,v_1))=\mathcal{M}(I,(w_1,\dots,w_n),(v_1,$

Then $\forall k\in\{m+1,\dots,n\}, 0\neq SR_{k,1}=R_{k,1}S$. Hence $n=\dim V=\dim\operatorname{range}S=m$.

Note that $R_{i,j}S=SR_{i,j}\iff Q_{i,j}=G_{i,j}$. Thus $Q_{i,j}(w_i)=w_j=a_{i,i}v_j=G_{i,j}(a_{1,i}v_1+\cdots+a_{n,i}v_n)$. Where:

$a_{i,j}=\mathcal{M}(I,(w_1,\dots,w_n),(v_1,\dots,v_n))_{i,j}\iff w_i=Iw_i=a_{1,i}v_1+\cdots+a_{n,i}v_n$;

For each j, for all i. Thus $a_{i,i}=a_{k,k}=\lambda,\forall\, k\neq i$.

Hence $w_i=\lambda v_i\Rightarrow\mathcal{M}(S)=\begin{pmatrix}\lambda & & 0\\ & \ddots & \\ 0 & & \lambda\end{pmatrix}=\mathcal{M}(\lambda I,(v_1,\dots,v_n))\Rightarrow S=\mathcal{M}^{-1}(\mathcal{M}(\lambda I))=\lambda I.\ \square$

• OR (10.A.3) *Suppose* V *is finite-dim and* $T\in\mathcal{L}(V)$.
Prove that T *has the same matrix with respect to every basis of* V
if and only if T *is a scalar multiple of the identity operator.*

SOLUTION: [Compare with the first solution of Problem (16) in (3.A)]

Suppose $T=\lambda I$ for some $\lambda\in\mathbf{F}$. Then T has the same matrix with respect to every basis of V.

Conversely, if $T=0$, then we are done; Suppose $T\neq 0$. And v is a nonzero vector in V.

Assume that (v,Tv) is linearly independent.

Extend (v,Tv) to a basis of V as $(v,Tv,u_3,\dots,u_n)$. Let $B=\mathcal{M}(T,(v,Tv,u_3,\dots,u_n))$.

$\Rightarrow Tv=B_{1,1}v+B_{2,1}(Tv)+B_{3,1}u_3+\cdots+B_{n,1}u_n\Rightarrow B_{2,1}=1, B_{i,1}=0,\ \forall i\neq 2$.

By assumption, $A=\mathcal{M}(T,(v,w_2,\dots,w_n)=B$ for any basis $(v,w_2,\dots,w_n)$.

Then $A_{2,1}=1, A_{i,1}=0\,(\,i\neq 2\,)\Rightarrow Tv=w_2$,

which is not true if we let $w_2=u_3, w_3=Tv, w_j=u_j\,(\,j=4,\dots,n\,)$. Contradicts.

Hence (v,Tv) is linearly dependent $\Rightarrow\forall v\in V,\exists\,\lambda_v\in\mathbf{F}, Tv=\lambda_v v$.

Now we show that λ_v is independent of v, that is,

to show that for any two nonzero distinct vectors $v,w\in V,\lambda_v=\lambda_w$. Thus $T=\lambda I$ for some $\lambda\in\mathbf{F}$.

$$\left.\begin{aligned}(v,w)\text{ is linearly independent}\Rightarrow T(v+w)&=\lambda_{v+w}(v+w)\\&=\lambda_{v+w}v+\lambda_{v+w}w\\&=\lambda_v v+\lambda_w w=Tv+Tw\Rightarrow\lambda_{v+w}=\lambda_v=\lambda_w\\(v,w)\text{ is linearly dependent}, w=cv\Rightarrow Tw&=\lambda_w w=\lambda_w cv=c\lambda_v v=T(cv)\Rightarrow\lambda_v=\lambda_w\end{aligned}\right\}\Rightarrow\square$$

---

17 *Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Using NOTE FOR [3.60]. Let $(v_1, \ldots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.

Then for any $E_{i,j} \in \mathcal{E}$, $(\forall x, y = 1, \ldots, n)$, by assumption, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.

Again, $E_{y,x'}, E_{y',x} \in \mathcal{E}$ for all $x', y', x, y = 1, \ldots, n$. Thus $\mathcal{E} = \mathcal{L}(V)$. $\square$

---

18 *Show that V and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces.*

SOLUTION:

Define $\varphi \in \mathcal{L}(V, \mathcal{L}(\mathbf{F}, V))$ by $\varphi(v) = \varphi_v$; where $\varphi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\varphi_v(\lambda) = \lambda v$.

$$\left.\begin{array}{l}\text{(a) } \varphi(v) = \varphi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \varphi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0. \text{ Hence } \varphi \text{ is injective.} \\ \text{(b) } \forall \psi \in \mathcal{L}(\mathbf{F}, V), \text{ let } v = \psi(1) \Rightarrow \psi(\lambda) = \lambda v = \varphi_v(\lambda), \forall \lambda \in \mathbf{F} \\ \qquad \Rightarrow \psi = \varphi_{\psi(1)} = \varphi(\psi(1)). \text{ Hence } \varphi \text{ is surjective.}\end{array}\right\} \Rightarrow \varphi \text{ is an isomorphism.} \quad \square$$

• *Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that $q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$.*

SOLUTION:

Note that $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$.

Define $T_n : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.

As can be easily checked, T_n is an operator.

Now how can we prove that $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3) = 0 \iff p = 0$?

Hence T_n is injective and therefore is surjective.

Thus $\forall q \in \mathcal{P}(\mathbf{R}), \deg q = m, \exists\, p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$.

---

19 *Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is injective. $\deg Tp \leq \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.*

(a) *Prove that T is surjective.*

(b) *Prove that for every nonzero p, $\deg Tp = \deg p$.*

SOLUTION:

(a) T is injective $\iff T|_{\mathcal{P}_n(\mathbb{R})} : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ is injective for any $n \in \mathbf{N}^+$

$\iff T|_{\mathcal{P}_n(\mathbb{R})}$ is surjective for any $n \in \mathbf{N}^+ \iff T$ is surjective.

(b) Using mathematical induction.

(i) $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$.

$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$.

(ii) Suppose $\deg f = \deg Tf$ for all $f \in \mathcal{P}_n(\mathbf{R})$. Then suppose $\deg g = n + 1, g \in \mathcal{P}_{n+1}(\mathbf{R})$.

Assume that $\deg Tg < \deg g$ ($\Rightarrow \deg Tg \leq n, Tg \in \mathcal{P}_n(\mathbf{R})$).

Then by (a), $\exists\, f \in \mathcal{P}_n(\mathbf{R}), T(f) = (Tg)$. 又 T is injective $\Rightarrow f = g$.

While $\deg f = \deg Tf = \deg Tg < \deg g$. Contradicts the assumption.

Hence $\deg Tp = \deg p$ for all $p \in \mathcal{P}_{n+1}(\mathbf{R})$.

Thus $\deg Tp = \deg p$ for all $p \in \mathcal{P}(\mathbf{R})$. $\square$

• *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \dots, v_m)$ is a list in V such that $(Tv_1, \dots, Tv_m)$ spans V. Prove that $(v_1, \dots, v_m)$ spans V.*

SOLUTION:

$V = \operatorname{span}(Tv_1, \dots, Tv_m) \Rightarrow T$ is surjective, 又 V is finite-dim $\Rightarrow T$ is invertible $\Rightarrow T^{-1}$ is invertible.

$\forall v \in V,\ \exists\, a_i \in \mathbf{F}, v = a_1 Tv_1 + \cdots + a_n Tv_n$

$\Rightarrow T^{-1}v = a_1 v_1 + \cdots + a_n v_n \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \dots, v_n)$ 又 $\operatorname{range} T^{-1} = V$. $\square$

OR. Reduce $(Tv_1, \dots, Tv_n)$ to a basis of V as $(Tv_{\alpha_1}, \dots, Tv_{\alpha_m})$, where $m = \dim V$ and $\alpha_i \in \{1, \dots, m\}$.
Then $(v_{\alpha_1}, \dots, v_{\alpha_m})$ is linearly independet of length m, therefore is a basis of V, contained in the list $(v_1, \dots, v_m)$. $\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*

$$\left.\begin{array}{l} (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1, \dots, v_n) \text{ of } V \iff T \text{ is surjective} \\ (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1, \dots, v_n) \text{ of } V \iff T \text{ is injective} \end{array}\right\} \iff T \text{ is invertible.}$$

---

2 *Suppose V is finite-dim and $\dim V > 1$.*
Prove that the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.

SOLUTION:

Suppose $\dim V = n > 1$. Let $(v_1, \dots, v_n)$ be a basis of V.
Define $S, T \in \mathcal{L}(V)$ by $S(a_1v_1 + \cdots + a_nv_n) = a_1v_1$ and $T(a_1v_1 + \cdots + a_nv_n) = a_2v_1 + \cdots + a_nv_n$.
Hence $S + T = I$ is invertible.
Thus the set of noninvertible linear maps in $\mathcal{L}(V)$ is not closed under addition and therefore is not a subspace. $\square$
COMMENT: If $\dim V = 1$, then the set of noninvertible operators on V equals $\{0\}$, which is a subspace of $\mathcal{L}(V)$.

---

3 *Suppose V is finite-dim, U is a subspace of V, and $S \in \mathcal{L}(U, V)$.*
Prove that there exists an invertible $T \in \mathcal{L}(V, V)$ such that
$Tu = Su$ for every $u \in U$ if and only if S is injective.

SOLUTION: [*Compare this with (3.A.11).*]

(a) $Tu = Su$ for every $u \in U \Rightarrow u = T^{-1}Su \Rightarrow S$ is injective.

(b) Suppose $(u_1, \dots, u_m)$ be a basis of U and S is injective $\Rightarrow (Su_1, \dots, Su_m)$ is linearly independent in V.
Extend these to bases of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$ and $(Su_1, \dots, Su_m, w_1, \dots, w_n)$.
Define $T \in \mathcal{L}(V)$ by $T(u_i) = Su_i;\ \ Tv_j = w_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.

---

4 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that null S = null T $(= U) \iff S = ET,\ \exists$ invertible $E \in \mathcal{L}(W)$.

SOLUTION:

Define $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i,\ \ E(w_j) = x_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $(Tv_1, \dots, Tv_m)$ be a basis of range T, extend it to a basis of W as $(Tv_1, \dots, Tv_m, w_1, \dots, w_n)$. Let $(u_1, \dots, u_n)$ be a basis of U. Then by (3.B.TIPS), $(v_1, \dots, v_m, u_1, \dots, u_n)$ is a basis of V. 又 $\operatorname{null} S = \operatorname{null} T \Rightarrow V = \operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} S \Rightarrow \operatorname{span}(Sv_1, \dots, Sv_m) = \operatorname{range} S$. And $\dim \operatorname{range} T = \dim \operatorname{range} S = \dim V - \operatorname{null} U = m$. Hence $(Sv_1, \dots, Sv_m)$ is a basis of range S. Thus we let $(Sv_1, \dots, Sv_m, x_1, \dots, x_n)$ be a basis of W.	Hence E is invertible and $S = ET$.

Conversely, $S = ET \Rightarrow \operatorname{null} S = \operatorname{null} ET$.
Then $v \in \operatorname{null} ET \iff ET(v) = 0 \iff Tv = 0 \iff v \in \operatorname{null} T$. Hence $\operatorname{null} ET = \operatorname{null} T = \operatorname{null} S$. $\square$

---

5 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that range S = range T $(= R) \iff S = TE,\ \exists$ invertible $E \in \mathcal{L}(V)$.

SOLUTION:

Define $E \in \mathcal{L}(V)$ as $E:\ v_i \mapsto r_i\ ;\quad u_j \mapsto s_j;$ for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. Where:

Let $(Tv_1, \ldots, Tv_m)$ and $(Sr_1, \ldots, Sr_m)$ be bases of R such that $\forall\, i, Tv_i = Sr_i$. Let $(u_1, \ldots, u_n)$ and $(s_1, \ldots, s_n)$ be bases of null T and null S respectively. Thus $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ and $(r_1, \ldots, r_m, s_1, \ldots, s_n)$ are bases of V.	Hence E is invertible and $S = TE$.

Conversely, $S = TE \Rightarrow$ range S = range TE.
Then $w \in$ range $S \iff \exists\, v \in V, Sv = TE(v) = T(E(v)) = w \in$ range T. Hence range S = range T. $\square$

---

6 *Suppose V and W are finite-dim and $S, T \in \mathcal{L}(V, W)$.* [dim null S = dim null $T = n$]
Prove that $S = E_2 T E_1,\ \exists$ invertible $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W) \iff$ dim null S = dim null T.

SOLUTION:

Define $E_1:\quad v_i \mapsto r_i\ ;\quad u_j \mapsto s_j;$ for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$.
Define $E_2: Tv_i \mapsto Sr_i\ ;\ x_j \mapsto y_j;$ for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. Where:

Let $(Tv_1, \ldots, Tv_m)$ and $(Sr_1, \ldots, Sr_m)$ be bases of range T and range S. Let $(u_1, \ldots, u_n)$ and $(s_1, \ldots, s_n)$ be bases of null T and null S respectively. Thus $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ and $(r_1, \ldots, r_m, s_1, \ldots, s_n)$ are bases of V. Extend $(Tv_1, \ldots, Tv_m)$ and $(Sr_1, \ldots, Sr_m)$ to bases of W as $(Tv_1, \ldots, Tv_m, x_1, \ldots, x_p)$ and $(Sr_1, \ldots, Sr_m, y_1, \ldots, y_p)$.	Thus E_1, E_2 are invertible and $S = E_2 T E_1$.

Conversely, $S = E_2 T E_1 \Rightarrow$ dim null S = dim null $E_2 T E_1$.
$v \in$ null $E_2 T E_1 \iff E_2 T E_1(v) = 0 \iff TE_1(v) = 0$. Hence null $E_2 T E_1$ = null TE_1 = null S.
又 By (3.B.22.COROLLARY), E is invertible $\Rightarrow$ dim null TE_1 = dim null T = dim null S. $\square$

---

8 *Suppose V is finite-dim and $T : V \to W$ is a surjective linear map of V onto W.*
Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.
$T|_U$ is the function whose domain is U, with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.

SOLUTION:

T is surjective $\Rightarrow$ range $T = W \Rightarrow$ dim range T = dim W = dim V − dim null T.
Let $(w_1, \ldots, w_m)$ be a basis of range $T = W \Rightarrow \forall w_i, \exists!\, v_i \in V, Tv_i = w_i$.
$\Rightarrow (v_1, \ldots, v_m)$ is a basis of $\mathcal{K}$. Thus dim $\mathcal{K}$ = dim W.
Thus $T|_{\mathcal{K}}$ maps a basis of $\mathcal{K}$ to a basis of range $T = W$. Denote $\mathcal{K}$ by U. $\square$

• *Suppose V and W are finite-dim and U is a subspace of V.*
Let $\mathcal{E} = \{\, T \in \mathcal{L}(V, W) : U \subseteq \text{null}\, T \,\}$.
(a) *Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.*
(b) *Find a formula for dim $\mathcal{E}$ in terms of dim V, dim W and dim U.*
Hint: Define $\Phi : \mathcal{L}(V, W) \to L(U, W)$ by $\Phi(T) = T|_U$. What is null Φ? What is range Φ?

SOLUTION:

(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, Su = Tu = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.
(b) Define Φ as in the hint.
$T \in$ null $\Phi \iff \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$. Hence null $\Phi = \mathcal{E}$.
$S \in \mathcal{L}(U, W) \Rightarrow \exists T \in \mathcal{L}(V, W), \Phi(T) = S$, by (3.B.11) $\Rightarrow S \in$ range T. Hence range $\Phi = \mathcal{L}(U, W)$.
Thus dim null Φ = dim $\mathcal{E}$ = dim $\mathcal{L}(V, W)$ − dim range Φ = (dim V − dim U) dim W. $\square$

OR. Extend $(u_1, \ldots, u_m)$ a basis of U to $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ a basis of V. Let $p = \dim W$.
(See NOTE FOR [3.60])

$$\forall T \in \mathcal{E}, k \in \{1, \ldots, m\}, TE_{k,k} = 0 \Rightarrow \text{span} \underbrace{\left\{ \begin{matrix} E_{1,1}, & \cdots & , E_{m,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,p}, & \cdots & , E_{m,p} \end{matrix} \right\}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

$$\text{又 } W = \text{span} \left\{ \begin{matrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{matrix} \right\} \subseteq \mathcal{E}. \text{ Where } \mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}.$$

Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. $\square$

ENDED

3·E

2 *Suppose $V_1, \ldots, V_m$ are vec-sps such that $V_1 \times \cdots \times V_m$ is finite-dim.*
Prove that every V_j is finite-dim.

SOLUTION: Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.

Let $(v_1, \ldots, v_M)$ be a basis of U. Note that $\forall u_i \in V_i, \in U_i \subseteq U$, for each i.

$$\left. \begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U) \text{ by } R_i(u_i) = (0, \ldots, 0, u_i, 0, \ldots, 0). \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \ldots, u_i, \ldots, u_m) = u_i \end{array} \right\} \Rightarrow S_i|_{U_i} = R_i^{-1}|_{U_i}.$$

Thus U_i and V_i are isomorphic. 又 U_i is a subspace of a finite-dim vec-sp U. $\square$

---

3 *Give an example of a vec-sp V and its two subspaces U_1, U_2 such that $U_1 \times U_2$ and $U_1 + U_2$ are isomorphic but $U_1 + U_2$ is not a direct sum.*

SOLUTION:

NOTE *that at least one of U_1, U_2 must be infinite-dim.*

For if not, $U_1 \times U_2$ is finite-dim and $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$.

And V must be infinite-dim. For if not, both U_1 and U_2 are finite-dim subspaces.

Let $V = \mathbf{F}^\infty = U_1, U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$.

$$\left. \begin{array}{l} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T((x_1, x_2, \cdots), (x, 0, \cdots)) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = ((x_1, x_2, \cdots), (x, 0, \cdots)) \end{array} \right\} \Rightarrow S = T^{-1}. \ \square$$

---

4 *Suppose $V_1, \ldots, V_m$ are vec-sps.*
Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic.

SOLUTION: Using the notations in Problem (2). Note that $T(u_1, \ldots, u_m) = T(u_1, 0, \ldots, 0) + \cdots + T(0, \ldots, u_m)$.

$$\left. \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \ldots, T_m) \text{ by } \varphi(T) = (TR_1, \ldots, TR_m). \\ \text{Define } \psi : (T_1, \ldots, T_m) \mapsto T \text{ by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array} \right\} \Rightarrow \psi = \varphi^{-1}. \square$$

---

5 *Suppose $W_1, \ldots, W_m$ are vec-sps.*
Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic.

SOLUTION: Using the notations in Problem (2).

Note that $Tv = (w_1, \ldots, w_m)$. Define $T_i \in \mathcal{L}(V, W_i)$ by $T_i(v) = w_i$.

$$\left. \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \ldots, T_m) \text{ by } \varphi(T) = (S_1 T, \ldots, S_m T). \\ \text{Define } \psi : (T_1, \ldots, T_m) \mapsto T \text{ by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array} \right\} \Rightarrow \psi = \varphi^{-1}. \square$$

---

6 *For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m\ times}$. Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are isomorphic.*

SOLUTION:

Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m$.

Suppose $T(v_1, \dots, v_m) = 0$. Then $\forall\, (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$
$\Rightarrow (v_1, \dots, v_m) = 0 \Rightarrow T$ is injective.

Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $\forall\, (b_1, \dots, b_n) \in \mathbf{F}^m$,
$(\,T(\psi(e_1), \dots, \psi(e_m))\,)(b_1, \dots, b_m) = b_1\psi(e_1) + \cdots + b_m\psi(e_m) = \psi(b_1 e_1 + \cdots + b_m e_m) = \psi(b_1, \dots, b_m)$.
Thus $T(\psi(e_1), \dots, \psi(e_m)) = \psi$. Hence T is surjective. $\square$

---

7 *Suppose $v, x \in V$ (chosen arbitrarily) of which U and W are subspaces.*
Suppose $v + U = x + W$. Prove that $U = W$.

SOLUTION:

(a) $\forall u \in U,\ \exists\, w \in W, v + u = x + w$, let $u = 0$, getting $v = x + w \Rightarrow v - x \in W$.

(b) $\forall w \in W,\ \exists\, u \in U, v + u = x + w$, let $w = 0$, getting $x = v + u \Rightarrow x - v \in U$.

Thus $\pm(v - x) \in U \cap W \Rightarrow \left\{ \begin{array}{l} u = (x - v) + w \in W \Rightarrow U \subseteq W \\ w = (v - x) + u \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W.$ $\square$

• *Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$.*
Prove that A is a translate of $U \iff \exists\, c \in \mathbf{R},\ A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$.
[*Do it in your mind.*]

• *Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $U = \{x \in V : Tx = c\}$ is either $\varnothing$ or is a translate of null T.*

SOLUTION:

If $c \in W$ but $c \notin$ range T, then $U = \varnothing$ and we are done.

Suppose $c \in$ range T, then $\exists\, u \in V, Tu = c \Rightarrow u \in U$.

Suppose $y \in$ null $T \Rightarrow y + u \in U \iff T(y + u) = Ty + c = c$. Thus $u+$ null $T \subseteq U$. Hence $u+$ null $T = U$,
for if not, suppose $z \notin u+$ null$T$ but $Tz = c (\Leftrightarrow z \in U)$, then $\forall w \in$ null $T, z \neq u + w \Leftrightarrow z - u \notin$ null$T$.
又 $\tilde{T}(z+$ null $T) = \tilde{T}(u+$ null $T) \Rightarrow z+$ null $T = u+$ null $T \Rightarrow z - u \in$ null T, contradicts. $\square$

• **COROLLARY:** *The set of solutions to a system of linear equations such as [3.28] is either $\varnothing$ or a translate of the null subspace.*

---

8 *Prove that a nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1 - \lambda) w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$.*

SOLUTION:

Suppose $A = a + U$, where U is a subspace of V. $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$,
$\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + [\lambda(u_1 - u_2) + u_2] \in A$.

Suppose $\lambda v + (1 - \lambda) w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $a \in A$ and let $A' = \{x - a : x \in A\}$.

Then $\forall x - a, y - a \in A', \lambda \in \mathbf{F}$,

(I) $\lambda(x - a) = [\lambda x + (1 - \lambda) a] - a \in A'$. Then let $\lambda = 2$.

(II) $\lambda(x - a) + (1 - \lambda)(y - a) = \frac{1}{2}(x - a) + \frac{1}{2}(y - a) = \frac{1}{2}x + (1 - \frac{1}{2})(y) - a \in A'$.
By (I), $2 \times [\frac{1}{2}(x - a) + \frac{1}{2}(y - a)] = (x - a) + (y - a) \in A'$.

Thus A' is a subspace of V. Hence $a + A' = \{(x - a) + a : x \in A\} = A$ is a translate. $\square$

---

9 *Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V.*
Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or is $\varnothing$.

SOLUTION: Suppose $v+u_1, w+u_2 \in A_1 \cap A_2 \neq \varnothing$. By Problem (8),
$\forall \lambda \in \mathbf{F}$, $\lambda(v+u_1)+(1-\lambda)(w+u_2) \in A_1$ and A_2. Thus $A_1 \cap A_2$ is a translate of some subspace of V. $\square$

---

10 *Prove that the intersection of any collection of translates of subspaces of V is either a translate of some subspace or $\varnothing$.*

SOLUTION: Suppose $\{A_\alpha\}_{\alpha\in\Gamma}$ is a collection of translates of subspaces of V, where Γ is an arbitrary index set.
Suppose $x, y \in \bigcap_{\alpha\in\Gamma} A_\alpha \neq \varnothing$, then by Problem (18), $\forall \lambda \in \mathbf{F}$, $\lambda x + (1-\lambda) y \in A_\alpha$ for every $\alpha \in \Gamma$.
Thus $\bigcap_{\alpha\in\Gamma} A_\alpha$ is a translate of some subspace of V. $\square$

---

11 *Suppose* $A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum\limits_{i=1}^{m} \lambda_i = 1\}$, *where each* $v_i \in V, \lambda_i \in \mathbf{F}$.

(a) *Prove that A is a translate of some subspace of V: By Problem (8),*
$$\forall \sum_{i=1}^{m} a_i v_i, \sum_{i=1}^{m} b_i v_i \in A, \lambda \in \mathbf{F}, \quad \lambda \sum_{i=1}^{m} a_i v_i + (1-\lambda) \sum_{i=1}^{m} b_i v_i = (\lambda \sum_{i=1}^{m} a_i + (1-\lambda) \sum_{i=1}^{m} b_i) v_i \in A. \ \square$$

(b) *Prove that if B is a translate of some subspace of V and $\{v_1, \ldots, v_m\} \subseteq B$, then $A \subseteq B$.*

(c) *Prove that A is a translate of some subspace of V and* $\dim V < m$.

SOLUTION:

(b) Let $v = \lambda_1 v_+ \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on m by k.

(i) $k=1, v=\lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.
$k=2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1-\lambda_1$. 又 $v_1, v_2 \in B$. By problem (8), $v \in B$.

(ii) $2 \leq k \leq m$, we assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. ($\forall \lambda_i$ such that $\sum\limits_{i=1}^{k} \lambda_i = 1$)
For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. $\forall i = 1, \ldots, k$, $\exists\, \mu_i \neq 1$, fix one such i by ι.
Then $\sum\limits_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow (\sum\limits_{i=1}^{k+1} \frac{\mu_i}{1-\mu_\iota}) - \frac{\mu_\iota}{1-\mu_\iota} = 1$.
Let $w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k\,terms}$.
Let $\lambda_i = \frac{\mu_i}{1-\mu_\iota}$ for $i = 1, \ldots, \iota-1$; $\lambda_j = \frac{\mu_{j+1}}{1-\mu_\iota}$ for $j = \iota, \ldots, k$. Then,
$$\left.\begin{array}{l} \sum\limits_{i=1}^{k} \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1-\mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \ \square$$

(c) $\forall k = 1, \ldots, m$, $\forall \lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_m$, let $\lambda_k = 1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m$
$\Rightarrow \lambda_1 v_1 + \cdots + \lambda_m v_m$
$= \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m) v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_m v_m$
$= v_k + \lambda_1 (v_1 - v_k) + \cdots + \lambda_{k-1}(v_{k-1} - v_k) + \lambda_{k+1}(v_{k+1} - v_k) + \cdots + \lambda_m (v_m - v_k)$.
Thus $A = v_k +$ span$(v_1 - v_k, \ldots, v_{k-1} - v_k, v_{k+1} - v_k, \ldots, v_m - v_k)$. $\square$

---

12 *Suppose U is a subspace of V such that V/U is finite-dim. Prove that is V is isomorphic to $U \times (V/U)$.*

SOLUTION: Let $(v_1 + U, \ldots, v_n + U)$ be a basis of V/U. Note that
$\forall v \in V$, $\exists\,!\, a_1, \ldots, a_n \in \mathbf{F}$, $v + U = \sum\limits_{i=1}^{n} a_i (v_i + U) = (\sum\limits_{i=1}^{n} a_i v_i) + U$
$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) = u \in U$ for some u; $v = \sum\limits_{i=1}^{n} a_i v_i + u$.
Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ by $\varphi(v) = (u, \sum\limits_{i=1}^{n} a_i v_i + U)$
and $\psi \in \mathcal{L}(U \times (V/U), V)$ by $\psi(u, w + U) = u + w$; $w = \sum\limits_{i=1}^{n} b_i v_i + U$.
So that $\psi = \varphi^{-1}$. $\square$

• *Suppose* $V = U \oplus W$, $(w_1, \ldots, w_m)$ *is a basis of* W.
Prove that $(w_1 + U, \ldots, w_m + U)$ *is a basis of* V/U.

SOLUTION: Note that for any $v \in V$,

$\exists\, !\; u \in U, w \in W, v = u + w$ 又 $\exists\, !\; c_i \in \mathbf{F}$ such that $w = \sum_{i=1}^{m} c_i w_i \Rightarrow v = u + \sum_{i=1}^{m} c_i w_i$.

Thus $v + U = \sum_{i=1}^{m} c_i w_i + U \Rightarrow v + U \in \operatorname{span}(w_1 + U, \ldots, w_m + U) \Rightarrow V/U \subseteq \operatorname{span}(w_1 + U, \ldots, w_m + U)$.

Now suppose $a_1(w_1 + U) + \cdots + a_m(w_m + U) = 0 + U \Rightarrow \sum_{i=1}^{m} a_i w_i \in U$ while $U \cap W = \{0\}$.

Then $\sum_{i=1}^{m} a_i w_i = 0 \Rightarrow a_1 = \cdots = a_m = 0$. $\square$

---

13 *Suppose* $(v_1 + U, \ldots, v_m + U)$ *is a basis of* V/U *and* $(u_1, \ldots, u_n)$ *is a basis of* U.
Prove that $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ *is a basis of* V.

SOLUTION: By Problem (12), U and V/U are finite-dim$\Rightarrow U \times (V/U)$ is finite-dim, so is V.

$\dim V = \dim(U \times (V/U)) = \dim U + \dim V/U = m + n$.

OR. Note that for any $v \in V, v + U = \sum_{i=1}^{m} a_i v_i + U,\ \exists\, !\; a_i \in \mathbf{F} \Rightarrow v = \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} b_i v_i,\ \exists\, !\; b_i \in \mathbf{F}$.

$\Rightarrow v \in \operatorname{span}(v_1, \ldots, v_m, u_1, \ldots, u_n) \supseteq V$.

又 Notice that $(\sum_{i=1}^{m} a_i v_i) + U = 0 + U (\Rightarrow \sum_{i=1}^{m} a_i v_i \in U) \iff a_1 = \cdots = a_m = 0$.

Hence $\operatorname{span}(v_1, \ldots, v_m) \cap U = \{0\} \Rightarrow \operatorname{span}(v_1, \ldots, v_m) \oplus U = V$

Thus $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ is linearly independent, so is a basis of V. $\square$

---

14 *Suppose* $U = \{(x_1, x_2, \ldots) \in \mathbf{F}^\infty : x_k \neq 0$ *for only finitely many* $k\}$.

(a) *Show that* U *is a subspace of* $\mathbf{F}^\infty$. *[Do it in your mind]*

(b) *Prove that* $\mathbf{F}^\infty/U$ *is infinite-dim.*

SOLUTION:

For $u = (x_1, \ldots, x_p, \ldots) \in \mathbf{F}^\infty$, denote x_p by $u[p]$. For each $r \in \mathbf{N}^+$.

Define $e_r[p] = \begin{cases} 1\,, (p-1) \equiv 0 \,(\operatorname{mod} r) \\ 0\,, \text{otherwise} \end{cases}$, simply $e_r = (1, \underbrace{0, \ldots, 0}_{(p-1)\ times}, 1, \underbrace{0, \ldots, 0}_{(p-1)\ times}, 1, \ldots) \in \mathbf{F}^\infty$.

Choose $m \in \mathbf{N}^+$ arbitrarily.

Suppose $a_1(e_1 + U) + \cdots + a_m(e_m + U) = (a_1 e_1 + \cdots + a_m e_m) + U = 0 + U = 0$.

$\Rightarrow a_1 e_1 + \cdots + a_m e_m = u$ for some $u \in U$.

Then suppose $u = (x_1, \ldots, x_t, 0, \ldots) \Rightarrow u[t + i] = 0, \forall i \in \mathbf{N}^+$,

then let $j = s \cdot m! + 1 \geq t$ $(\exists\ s \in \mathbf{N}^+)$ so that $e_1[j] = \cdots = e_m[j] = 1,\ u[j + i] = 0$.

Now we have: $u[j+i] = (\Sigma_{r=1}^{m} a_r e_r)[j+i] = \Sigma_{r=1}^{m} a_r e_r[s \cdot m! + 1 + i] = a_{i_1} + \cdots + a_{i_{\tau(i)}} = 0$,

$\Rightarrow (\Sigma_{r=1}^{m} a_r e_r)[j+i] = a_{i_1} + \cdots + a_{i_{\tau(i)}} = 0$. (Δ)

where $i_1, \ldots, i_{\tau(i)}$ are distinct ordered factors of i ($1 = i_1 \leq \cdots \leq i_{\tau(i)} = i$).

$\Big($ Note that by definition, $e_r[s \cdot m! + 1 + i] = 1 \iff s \cdot m! + i \equiv i \equiv 0\, (\operatorname{mod} r\,) \iff r \mid i$. $\Big)$

Let $i' = i_{\tau(i)-1}$. Notice that $i'_l = i_l, \forall l \in \{1, \ldots, \tau(i')\}$; and $\tau(i') = \tau(i) - 1$.

Again by (Δ), $(\Sigma_{r=1}^{m} a_r e_r)[j + i'] = a_{i'_1} + \cdots + a_{i'_{\tau(i')}} = a_{i_1} + \cdots + a_{i_{\tau(i)-1}} = 0$.

Thus $a_{i_{\tau(i)}} = a_i = 0$ for any $i \in \{1, \ldots, m\}$.

Hence $(e_1, \ldots, e_m)$ is linearly independent in $\mathbf{F}^\infty$, so is $(e_1, \ldots, e_m, \ldots)$, since $m \in \mathbf{N}^+$.

又 $e_i \notin U \Rightarrow (e_1 + U, e_2 + U, \ldots)$ is linearly independent in $\mathbf{F}^\infty/U$. By [2.B.14]. $\square$

---

15 *Suppose* $\varphi \in \mathcal{L}(V, \mathbf{F}) \setminus \{0\}$. *Prove that* $\dim V/(\textit{null}\ \varphi) = 1$.

Solution: By [3.91] (d), dim range $\varphi = 1 = \dim V/(\text{null}\ \varphi)$. $\square$

Note For [3.88, 3.90, 3.91]

For any $W \in \mathcal{S}_V U$, because $V = U \oplus W$. $\forall v \in V$, $\exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.

Define $T \in \mathcal{L}(V, W)$ by $T(v) = w_v$. Hence null $T = U$, range $T = W$.

Then $\tilde{T} \in \mathcal{L}(V/\text{null}\, T, W)$ is defined as $\tilde{T}(v + U) = Tv = w_v$.

Thus $\tilde{T}$ is injective (by [3.91(b)]) and surjective (range $\tilde{T}$ = range $T = W$),

and therefore is an isomorphism. We conclude that V/U and W, namely any vec-sp in $\mathcal{S}_V$, are isomorphic.

---

16 *Suppose* $\dim V/U = 1$. *Prove that* $\exists\ \varphi \in \mathcal{L}(V, \mathbf{F})$ *such that null* $\varphi = U$.

Solution:

Suppose V_0 is a subspace of V such that $V = U \oplus V_0$. Then V_0 and V/U are isomorphic. $\dim V_0 = 1$.

Define a linear map $\varphi : v \mapsto \lambda$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. $\square$

---

17 *Suppose* V/U *is finite-dim.* W *is a subspace of* V.

(a) *Show that if* $V = U + W$, *then* $\dim W \geq \dim V/U$.

(b) *Suppose* $\dim W = \dim V/U$ *and* $V = U \oplus W$. *Find such* W.

Solution:

Let $(w_1, \ldots, w_n)$ be a basis of W

(a) $\forall v \in V,\ \exists\ u \in U, w \in W$ such that $v = u + w \Rightarrow v + U = w + U$

Then $V/U \subseteq \text{span}(w_1 + U, \ldots, w_n + U) \Rightarrow V/U = \text{span}(w_1 + U, \ldots, w_n + U)$.

Hence $\dim V/U = \dim \text{span}(w_1 + U, \ldots, w_n + U) \leq \dim W$.

(b) Let $W \in \mathcal{S}_V U$. In other words,

reduce $(w_1{+}U, \ldots, w_n{+}U)$ to a basis of V/U as $(w_{\alpha_1}{+}U, \ldots, w_{\alpha_m}{+}U)$ and let $W = \text{span}\,(w_{\alpha_1}, \ldots, w_{\alpha_m})$. $\square$

---

18 *Suppose* $T \in \mathcal{L}(V, W)$ *and* U *is a subspace of* V. *Let* π *denote the quotient map.*

Prove that $\exists\ S \in \mathcal{L}(V/U, W)$ *such that* $T = S \circ \pi$ *if and only if* $U \subseteq$ *null* T.

Solution:

(a) Define $S \in \mathcal{L}(V/U, W)$ by $S(v + U) = Tv$. We have to check it is *well-defined*.

Suppose $v_1 + U = v_2 + U$, while $v_1 \neq v_2$.

Then $(v_1 - v_2) \in U \Rightarrow S((v_1 - v_2) + U) = T(v_1 - v_2) = 0 \Rightarrow Tv_1 = Tv_2$. Checked. $\square$

(b) Suppose $\exists\ S \in \mathcal{L}(V/U, W),\ T = S \circ \pi$. Then $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq$ null T.$\square$

---

20 *Define* $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ *by* $\Gamma(S) = S \circ \pi (= \pi'(S))$.

(a) *Prove that* Γ *is linear:* By [3.9] distributive properties and [3.6]. $\square$

(b) *Prove that* Γ *is injective:*

$$\begin{aligned}
&\Gamma(S) = 0 \\
\iff &\forall v \in V, S(\pi(v)) = 0 \\
\iff &\forall v + U \in V/U, S(v + U) = 0 \\
\iff &S = 0. \ \square
\end{aligned}$$

(c) *Prove that range* Γ $(=$ *range* π' $) = \{T \in \mathcal{L}(V, W) : U \subseteq \textit{null}\ T\}$*:*

By Problem (18). $\square$

Ended

3·F

• By (18) in (3.D) *we know that* $\varphi : V \to \mathcal{L}(\mathbf{F}, V)$ *is an isomorphism. Now we prove that* $(v_1, \dots, v_m)$ *is linearly independent* $\iff$ $(\varphi(v_1), \dots, \varphi(v_m))$ *is linearly independent.*

SOLUTION:

(a) Suppose $(v_1, \dots, v_m)$ is linearly independent and $\vartheta \in \operatorname{span}(\varphi(v_1), \dots, \varphi(v_m))$.

Let $\vartheta = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m)$. Then $\vartheta(1) = 0 = a_1v_1 + \cdots + a_mv_m \Rightarrow a_1 = \cdots = a_m = 0$.

OR Because φ is injective. Suppose $a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) = 0 = \varphi(a_1v_1 + \cdots + a_mv_m)$.

Then $a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

Thus $(\varphi(v_1), \dots, \varphi(v_m))$ is linearly independent.

(b) Suppose $(\varphi(v_1), \dots, \varphi(v_m))$ is linearly independent and $v \in \operatorname{span}(v_1, \dots, v_m)$.

Let $v = 0 = a_1v_1 + \cdots + a_mv_m$. Then $\varphi(v) = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

Thus $v_1, \dots, v_m$ is linearly independent. $\square$

---

1 *Explain why each linear functional is surjective or is the zero map.*

SOLUTION: $\left.\begin{array}{r}\text{For any } \varphi \in V' \text{ and } \varphi \neq 0,\ \exists\, v \in V,\ \text{such that } \varphi(v) \neq 0. \text{ (a)} \\ \dim \operatorname{range} \varphi = \dim \mathbf{F} = 1. \text{ (b)}\end{array}\right\} \Rightarrow \square$

---

4 *Suppose V is finite-dim and U is a subspace of V such that $U \neq V$. Prove that $\exists\, \varphi \in V'$ and $\varphi \neq 0$ such that $\varphi(u) = 0$ for every $u \in U$.*

SOLUTION:

Let $(u_1, \dots, u_m)$ be a basis of U, extend to $(u_1, \dots, u_m, u_{m+1}, \dots, u_{m+n})$ a basis of V.

Choose $k \in \{1, \dots, n\}$ arbitrarily. Define $\varphi \in V'$ by $\varphi(u_i) = \begin{cases} 1, & \text{if } i = m + k. \\ 0, & \text{otherwise.} \end{cases}$

OR: Equivalent to proving that $U^0 \neq \{0\}$. By [3.106], $\dim U^0 = \dim V - \dim U > 0$. $\square$

• *Suppose $T \in \mathcal{L}(V, W)$ and $(w_1, \dots, w_m)$ is a basis of range$T$. Hence $\forall v \in V,\ Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,\ \ \exists!\, \varphi_1(v), \dots, \varphi_m(v)$, thus defining functions $\varphi_1, \dots, \varphi_m$ from V to $\mathbf{F}$. Show that each $\varphi_i \in V'$.*

SOLUTION:

For each $w_i, \exists\, v_i \in V,\ Tv_i = w_i$, getting a linearly independent list $(v_1, \dots, v_m)$.

Now we have $Tv = a_1Tv_1 + \cdots + a_mTv_m,\ \forall v \in V, \exists!\, a_i \in \mathbf{F}$.

Let $(\psi_1, \dots, \psi_m)$ be the dual basis of range$T$. Then $(T'(\psi_i))(v) = \psi_i \circ T(v) = a_i$.

Thus letting $\varphi_i = \psi_i \circ T$. $\square$

• *Suppose $\varphi,\ \beta \in V'$. Prove that null$\varphi \subseteq$ nullβ if and only if $\beta = c\varphi.\ \exists\, c \in \mathbf{F}$.*

SOLUTION: Using (3.B.29, 30)

(a) Suppose null$\varphi \subseteq$ nullβ. Choose a $u \notin$ nullβ. $V = \operatorname{null}\beta \oplus \{au : a \in \mathbf{F}\}$.

If nullφ = nullβ, then let $c = \frac{\beta(u)}{\varphi(u)}$, we are done.

Otherwise, suppose $u' \in$ nullβ, but $u' \notin$ nullφ, then $V = \operatorname{null}\varphi \oplus \{bu' : b \in \mathbf{F}\}$.

$\forall v \in V, v = w + au = w' + bu',\ \exists!\, w, w' \in \operatorname{null}\varphi,\ a, b \in \mathbf{F}$.

Thus $\beta(v) = a\beta(u),\ \varphi(v) = b\varphi(u')$. Let $c = \frac{a\beta(u)}{b\varphi(u')}$. We are done

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$.

If $c = 0$, then null$\beta = V \supseteq$ nullφ, we are done.

Otherwise, $\left.\begin{array}{l}\forall v \in \operatorname{null}\varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \operatorname{null}\varphi \subseteq \operatorname{null}\beta. \\ \forall v \in \operatorname{null}\beta, \beta(v) = 0 = \varphi(v) \Rightarrow \operatorname{null}\beta \subseteq \operatorname{null}\varphi.\end{array}\right\} \Rightarrow \operatorname{null}\varphi = \operatorname{null}\beta$

$\Rightarrow$ null$\varphi \subseteq$ nullβ. $\square$

---

5 *Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic.*

SOLUTION: Using notations in (3.E.2).

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to V_1' \times \cdots \times V_m' \\ \quad \text{by } \varphi(T) = (T \circ R_1, \ldots, T \circ R_m) = (R_1'(T), \ldots, R_m'(T)). \\ \text{Define } \psi : V_1' \times \cdots \times V_m' \to (V_1 \times \cdots \times V_m)' \\ \quad \text{by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m = S_1'(T_1) + \cdots + S_m'(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}. \quad \square$$

• *Suppose $(v_1, \ldots, v_n)$ is a basis of V and $(\varphi_1, \ldots, \varphi_n)$ is the dual basis of V'.*

$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = (\varphi_1(v), \ldots, \varphi_n(v)). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

---

35 *Prove that $(\mathcal{P}(\mathbf{R}))'$ and $\mathbf{R}^\infty$ are isomorphic.*

SOLUTION:

Define $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{R}))', \mathbf{R}^\infty)$ by $\theta(\varphi) = (\varphi(1), \varphi(x), \ldots, \varphi(x^n), \ldots)$.

Injectivity: $\theta(\varphi) = 0 \Rightarrow \forall x^k$ in the basis $(1, x, \ldots, x^n, \ldots)$ of $\mathcal{P}_n(\mathbf{R})$ for any n, $\varphi(x^k) = 0 \Rightarrow \varphi = 0$.

Surjectivity: $\forall\, (a_0, a_1, \ldots, a_n, \ldots) \in \mathbf{F}^\infty$, let ψ be such that $\psi(x^k) = a_k$ and thus $\theta(\psi) = (a_0, a_1, \ldots, a_n, \ldots)$.

Hence θ is an isomorphism from $(\mathcal{P}(\mathbf{R}))'$ onto $\mathbf{R}^\infty$. $\square$

---

7 *Suppose m is a positive integer. Show that the dual basis of the basis $(1, x, \ldots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where $\varphi_k = \dfrac{p^{(k)}(0)}{k!}$. Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.*

SOLUTION:

For each j and k, $(x^j)^{(k)} = \begin{cases} j(j-1)\ldots(j-k+1) \cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\ldots(j-j+1) = j!\,, & j = k. \\ 0\,, & j \leq k. \end{cases}$ Then $(x^j)^{(k)}(0) = \begin{cases} 0\,, & j \neq k. \\ k!\,, & j = k. \end{cases}$

Thus $\varphi_k = \psi_k$, where $\psi_1, \ldots, \psi_m$ is the dual basis of $(1, x, \ldots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$. $\square$

---

8 *Suppose m is a positive integer.*

(a) By [2.C.10], $B = (1, x-5, \ldots, (x-5)^m)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) Let $\varphi_k = \frac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \ldots, m$. Then $(\varphi_0, \varphi_1, \ldots, \varphi_m)$ is the dual basis of B.

---

9 *Suppose $(v_1, \ldots, v_n)$ is a basis of V and $(\varphi_1, \cdots, \varphi_n)$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.*

SOLUTION: $\psi(v) = \psi(\sum\limits_{i=1}^{n} a_i v_i) = \sum\limits_{i=1}^{n} a_i \psi(v_i) = \sum\limits_{i=1}^{n} \psi(v_i)\varphi_i(v) = [\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n](v) \Rightarrow \square$

COMMENT: For any other basis $(u_1, \ldots, u_n)$ of V and the corresponding dual basis of $(\rho_1, \ldots, \rho_n)$,
$\psi = \rho(u_1)\rho_1 + \cdots + \rho(u_n)\rho_n$.

---

12 *Show that the dual map of the identity operator on V is the identity operator on V'.*

SOLUTION: $I'(\varphi) = \varphi \circ I = \varphi, \ \forall \varphi \in V'$. $\square$

• *Suppose W is finite-dim and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \Longleftrightarrow T = 0$.*

SOLUTION: $T = 0 \Leftrightarrow T'(\varphi) = \varphi \circ T = 0$ for all $\varphi \in V' \Leftrightarrow T' = 0$. $\square$

---

13 *Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ denote the dual basis of the standard basis of $\mathbf{R}^2$ and $\mathbf{R}^3$.*

(a) *Describe the linear functionals $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$*

For any $(x, y, z) \in \mathbf{R}^3$, $(T'(\varphi_1))(x, y, z) = 4x + 5y + 6z$, $(T'(\varphi_2))(x, y, z) = 7x + 8y + 9z$.

(b) *Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.*

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$. $\square$

14 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for each $x \in \mathbf{R}$.*

(a) *Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe $T'(\varphi) \in \mathcal{P}(\mathbf{R})'$.*

$(T'(\varphi))(p) = [x^2p(x) + p''(x)]'(4) = [2xp(x) + x^2p'(x) + p'''(x)](4) = 8p(4) + 16p'(4) + p'''(4)$.

(b) *Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\varphi))(x^3)$.*

$(T'(\varphi))(x^3) = \int_0^1 (x^5 + 6x)\mathrm{d}x = \int_0^1 (\frac{1}{6}x^6 + 3x^2)'\mathrm{d}x = \frac{6}{19}$. □

• *Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$.*
Prove that T is invertible if and only if $T' \in \mathcal{L}(W', V')$ is invertible.

Solution: By [3.108] and [3.110]. □

---

16 *Suppose V and W are finite-dim. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(L, W)$.*
Prove that Γ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

Solution:

V, W are finite-dim $\Rightarrow$ dim $\mathcal{L}(V, W) =$ dim $\mathcal{L}(W', V')$. And by [3.101], Γ is linear.

又 Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus T is injective $\Rightarrow$ T is invertible. □

---

17 *Suppose $U \subseteq V$. Explain why $U^0 = \{\varphi \in V' : U \subseteq null\varphi\}$.*

Solution: Because for $\varphi \in V'$, $U \subseteq$ null$\varphi \iff \forall u \in U, \varphi(u) = 0$. By definition in [3.102]. □

---

18 *$U \subseteq V$. We have $U = \{0\} \iff \forall\varphi \in V', U \subseteq null\varphi \iff U^0 = V'$.*

19 *U is a subspace of V. Prove that $U = V \iff U_V^0 = \{0\} = V_V^0$.*

Solution:

Suppose $U_V^0 = \{0\}$. Then $U = V$.

Conversely, suppose $U = V$, then $U_V^0 = \{\varphi \in V' : V \subseteq \text{null}\,\varphi\}$, therefore $U_V^0 = \{0\}$.

20, 21 *Suppose U and W are subsets of V. Prove that $U \subseteq W \iff W^0 \subseteq U^0$.*

Solution:

(a) $U \subseteq W \Rightarrow \forall w \in W, u \in U \cap W = U,\ \ \forall\varphi \in W^0, \varphi(w) = 0 = \varphi(u) \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.

(b) $W^0 \subseteq U^0 \Rightarrow \forall w \in W, u \in U, \varphi(w) = 0 \Rightarrow \varphi(u) = 0$. Then null$\varphi \supseteq W \Rightarrow$ null$\varphi \supseteq U$. Thus $W \supseteq U$. □.

• **Corollary:** $W^0 = U^0 \iff U = W$.

---

22 *Prove that $(U + W)^0 = U^0 \cap W^0$.*

Solution:

(a) $\left.\begin{array}{c} U \subseteq U + W \\ W \subseteq U + W \end{array}\right\} \Rightarrow \left.\begin{array}{c} (U + W)^0 \subseteq U^0 \\ (U + W)^0 \subseteq W^0 \end{array}\right\} \Rightarrow (U + W)^0 \subseteq U^0 \cap W^0.$

(b) $\forall\varphi \in U^0 \cap W^0, \varphi(u + w) = 0$, where $u \in U, w \in W \Rightarrow \varphi \in (U + W)^0$. Thus $(U + W)^0 \supseteq U^0 \cap W^0$. □

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23 *Prove that $(U \cap W)^0 = U^0 + W^0$.*

Solution:

(a) $\left.\begin{array}{c} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array}\right\} \Rightarrow \left.\begin{array}{c} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array}\right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \supseteq U^0 \cap W^0.$

(b) $\forall\varphi \in U^0, \psi \in W^0$ and $\forall v \in U \cap W$, $(\varphi + \psi)(v) = \varphi(v) + \psi(v) = 0$. Thus $U^0 + W^0 \subseteq (U \cap W)^0$. □

• **Corollary:**

Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subspaces of V.
Then $(\sum_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \bigcap_{\alpha_i \in \Gamma}(V_{\alpha_i}^0)$;
And $(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \sum_{\alpha_i \in \Gamma}(V_{\alpha_i}^0)$.

---

24 *Suppose V is finite-dim and U is a subspace of V.*
Prove, using the pattern of [3.104], that dim$U$+ dimU^0 = dim$V$.

SOLUTION:

Let $(u_1, \ldots, u_m)$ be a basis of U, extend to a basis of V as $(u_1, \ldots, u_m, \ldots, u_n)$, and let $(\varphi_1, \ldots, \varphi_m, \ldots, \varphi_n)$ be the dual basis.

(a) Suppose $\varphi \in \text{span}(\varphi_{m+1}, \ldots, \varphi_n)$, then $\exists\, a_i \in \mathbf{F}$, $\varphi = a_{m+1}\varphi_{m+1} + \cdots + a_n\varphi_n$.
For all $u \in U$, $\varphi(u) = 0$. Thus $\varphi \in U^0$, getting $\text{span}(\varphi_{m+1}, \ldots, \varphi_n) \subseteq U^0$.

(b) Suppose $\varphi \in U^0$, then $\exists\, a_i \in \mathbf{F}$, $\varphi = a_1\varphi_1 + \cdots + a_m\varphi_m + \cdots + a_n\varphi_n$.
For all $u_i \in U$, $0 = \varphi(u_i) = \sum\limits_{i=1}^{n} \varphi(u_i) = a_i$. Then $\varphi = a_{m+1}\varphi_{m+1} + \cdots + a_n\varphi_n$.
Thus $\varphi \in \text{span}(\varphi_{m+1}, \ldots, \varphi_n)$, getting $\text{span}(\varphi_{m+1}, \ldots, \varphi_n) \supseteq U^0$.

Hence $\text{span}(\varphi_{m+1}, \ldots, \varphi_n) = U^0$, $\quad \dim U^0 = n - m = \dim V - \dim U$. $\square$

---

25 *Suppose U is a subspace of V. Explain why $U = \{v \in V : \varphi(v) = 0$ for every $\varphi \in U^0\}$.*

SOLUTION: Note that $U = \{v \in V : v \in U\}$ is a subspace of V and $\varphi(v) = 0$ for every $\varphi \in U^0 \iff v \in U$. $\square$

---

26 *Suppose V is finite-dim and Ω is a subspace of V'.*
Prove that $\Omega = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}^0$.

SOLUTION: Using the corollary in Problem (20, 21).

Suppose $U = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}$.

Getting $U^0 = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}^0$. We need to show that $\Omega = U^0$.

$$\left.\begin{array}{l} \text{(a) } \forall \varphi \in \Omega, v \in U, \varphi(v) = 0 \Rightarrow \varphi \in U^0 \Rightarrow \Omega \subseteq U^0. \\ \text{(b) } v \in U \Leftrightarrow \left\{ \begin{array}{l} \forall \varphi \in \Omega, \varphi(v) = 0 \\ \forall \psi \in U^0, \psi(v) = 0 \end{array} \right. \quad \text{Thus } \Omega \supseteq U^0. \end{array}\right\} \Rightarrow \square$$

---

27 *Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ and nullT' = span(φ), where φ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that range$T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$.*

SOLUTION: By Problem (26), $\text{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \psi(p) = 0, \forall \psi \in \text{span}(\varphi)\}^0$,
Hence $\text{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \varphi(p) = p(8) = 0\}^0$, 又 $\text{span}(\varphi) = \text{null}T' = (\text{range}T)^0$.

By the corollary in Problem (20, 21), range$T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$. $\square$

28, 29 *Suppose V, W are finite-dim, $T \in \mathcal{L}(V, W)$.*
(a) *Suppose $\exists\, \varphi \in W'$ such that nullT' = span(φ). Prove that range$T$ = nullφ.*
(b) *Suppose $\exists\, \varphi \in V'$ such that rangeT' = span(φ). Prove that null$T$ = nullφ.*

SOLUTION: Using Problem (26), [3.107] and [3.109].

Because $\text{span}(\varphi) = \{v \in V : \forall \psi \in \text{span}(\varphi), \psi(v) = 0\}^0 = \{v \in V : \varphi(v) = 0\}^0 = (\text{null}\varphi)^0$.

$$\left.\begin{array}{l} \text{(a) } (\text{range}T)^0 = \text{null}T' = \text{span}(\varphi) = (\text{null}\varphi)^0 \iff \text{range}T = \text{null}\varphi. \\ \text{(b) } (\text{null}T)^0 = \text{range}T' = \text{span}(\varphi) = (\text{null}\varphi)^0 \iff \text{null}T = \text{null}\varphi. \end{array}\right\} \Rightarrow \square$$

---

31 *Suppose V is finite-dim and $(\varphi_1, \ldots, \varphi_n)$ is a basis of V'.*
Show that there exists a basis of V whose dual basis is $(\varphi_1, \ldots, \varphi_n)$.

SOLUTION: Using (3.B.29,30).

For each φ_i, $\text{null}\varphi_i \oplus \{au_i : a \in \mathbf{F}\} = V$.

Because $\varphi_1, \ldots, \varphi_m$ is linearly independent. $\text{null}\varphi_i \neq \text{null}\varphi_j$ for all $i, j \in \mathbf{N}^+$ such that $i \neq j$.

Thus $(u_1, \ldots, u_m)$ is linearly independent, for if not, then $\exists\, i, j$ such that $\text{null}\varphi_i = \text{null}\varphi_j$, contradicts.

又 $\dim V' = m = \dim V$. Then $(u_1, \ldots, u_m)$ is a basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$. $\square$.

• *Suppose dim and $\varphi_1, \ldots, \varphi_m \in V'$. Prove that the following three sets are equal to each other.*
(a) $span(\varphi_1, \ldots, \varphi_m)$
(b) $((null\varphi_1) \cap \cdots \cap (null\varphi_m))^0$
(c) $\{\varphi \in V' : (\ null\varphi_1) \cap \cdots \cap (null\varphi_m) \subseteq null\varphi\}$

SOLUTION: By Problem (17), (b) and (c) are equivalent. By Problem (26) and the corollary in Problem (23),

$$\left.\begin{aligned} ((\text{null}\varphi_1) \cap \cdots \cap (\text{null}\varphi_m))^0 = (\text{null}\varphi_1)^0 + \cdots + (\text{null}\varphi_m)^0. \\ \text{又 span}(\varphi_i) = \{v \in V : \forall \psi \in \text{span}(\varphi_i), \psi(v) = 0\}^0 = (\text{null}\varphi_i)^0. \end{aligned}\right\} \Rightarrow (a) = (b). \quad \square$$

30 OR COROLLARY:

Suppose V is finite-dim and $\varphi_1, \ldots, \varphi_m$ is a linearly independent list in V'.
Then $dim((null\varphi_1) \cap \cdots \cap (null\varphi_m)) = (dimV) - m$.

---

6 *Define $\Gamma : V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m))$, where $v_1, \ldots, v_m \in V$.*
(a) *Show that $span(v_1, \ldots, v_m) = V \iff \Gamma$ is injective.*
(b) *Show that $(v_1, \ldots, v_m)$ is linearly independent $\iff \Gamma$ is surjective.*

SOLUTION:

(a)
- Suppose Γ is injective. Then let $\Gamma(\varphi) = 0$, getting $\varphi = 0 \Leftrightarrow \text{null}\varphi = V = \text{span}(v_1, \ldots, v_m)$.
- Suppose $\text{span}(v_1, \ldots, v_m) = V$. Then let $\Gamma(\varphi) = 0$, getting $\varphi(v_i) = 0$ for each i, $\text{null}\varphi = \text{span}(v_1, \ldots, v_m) = V$, thus $\varphi = 0$, Γ is injective.

(b)
- Suppose Γ is surjective. Then let $\Gamma(\varphi_i) = e_i$ for each i, where $(e_1, \ldots, e_m)$ is the standard basis of $\mathbf{F}^m$.
  Then $(\varphi_1, \ldots, \varphi_m)$ is linearly independent, suppose $a_1 v_1 + \cdots + a_m v_m = 0$,
  then for each i, we have $\varphi_i(a_1 v_1 + \cdots + a_m v_m) = a_i = 0$. Thus $v_1, \ldots, v_n$ is linearly independent.
- Suppose $(v_1, \ldots, v_m)$ is linearly independent. Let $(\varphi_1, \ldots, \varphi_m)$ be the dual basis of $\text{span}(v_1, \ldots, v_m)$.
  Thus for each $(a_1, \ldots, a_m) \in \mathbf{F}^m$, we have $\varphi = a_1\varphi_1 + \cdots + a_m\varphi_m$ so that $\Gamma(\varphi) = (a_1, \ldots, a_m)$. $\square$

• *Define $\Gamma : V \to \mathbf{F}^m$ by $\Gamma(v) = (\varphi_1(v), \ldots, \varphi_m(v))$, where $\varphi_1, \ldots, \varphi_m \in V'$.*
(c) *Show that $span(\varphi_1, \ldots, \varphi_m) = V' \iff \Gamma$ is injective.*
(d) *Show that $(\varphi_1, \ldots, \varphi_m)$ is linearly independent $\iff \Gamma$ is surjective.*

SOLUTION:

(c)
- Suppose Γ is injective. Then $\Gamma(v) = 0 \Leftrightarrow \forall i, \varphi_i(v) = 0 \Leftrightarrow v \in (\text{null}\varphi_1) \cap \cdots \cap (\text{null}\,\varphi_m) \Leftrightarrow v = 0$.
  Getting $(\text{null}\,\varphi_1) \cap \cdots \cap (\text{null}\varphi_m) = \{0\}$. By Problem (•) above, $\text{span}\,(\varphi_1, \ldots, \varphi_m) = V'$
- Suppose $\text{span}\,(\varphi_1, \ldots, \varphi_m) = V'$. Again by Problem (•), $(\text{null}\,\varphi_1) \cap \cdots \cap (\text{null}\,\varphi_m) = \{0\}$.
  Thus $\Gamma(v) = 0 \Rightarrow \forall i, \varphi_i(v) = 0 \Rightarrow v = 0$.

(d)
- Suppose $(\varphi_1, \ldots, \varphi_m)$ is linearly independent. Then by Problem (31), $(v_1, \ldots, v_m)$ is linearly independent.
  Thus for any $(a_1, \ldots, a_m) \in \mathbf{F}$, by letting $v = \sum_{i=1}^m a_i v_i$, then $\varphi_i(v) = a_i \Rightarrow \Gamma(v) = (a_1, \ldots, a_m)$.
- Suppose Γ is surjective. Let $e_1, \ldots, e_m$ be a basis of $\mathbf{F}^m$.
  For every e_i, $\exists\, v_i \in V$ such that $\Gamma(v_i) = (\varphi_1(v_i), \ldots, \varphi_m(v)) = e_i$,
  fix v_i ($\Rightarrow (v_1, \ldots, v_m)$ is linearly independent). Thus $\varphi_i(v_i) = 1, \varphi_i(v_j) = 0$.
  Hence $(\varphi_1, \ldots, \varphi_m)$ is the dual basis of the basis $v_1, \ldots, \varphi_m$ of $\text{span}\,(v_1, \ldots, v_m)$. $\square$

---

33 *Suppose $A \in \mathbf{F}^{m,n}$. Define $T : A \to A^t$. Prove that T is an isomorphism of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$*

SOLUTION: By [3.111], T is linear. Note that $(A^t)^t = A$.

$$\left.\begin{aligned} &\text{(a) For any } B \in \mathbf{F}^{n,m}, \text{ let } A = B^t \text{ so that } T(A) = B. \text{ Thus } T \text{ is surjective.} \\ &\text{(b) If } T(A) = 0 \text{ for some } A \in \mathbf{F}^{n,m}, \text{ then } A = 0. \text{ Thus } T \text{ is injective.} \\ &\quad \text{for if not, } \exists\, j, k \in \mathbf{N}^+ \text{ such that } A_{j,k} \neq 0, \text{ then } T(A)_{k,j} \neq 0, \text{ contradicts.} \end{aligned}\right\} \Rightarrow \square$$

---

32 *Suppose $T \in \mathcal{L}(V)$, and $(u_1, \dots, u_m)$ and $(v_1, \dots, v_m)$ are bases of V. Prove that T is invertible $\iff$ The rows of $\mathcal{M}(T, (u_1, \dots, u_m), (v_1, \dots, v_m))$ form a basis of $\mathbf{F}^{1,n}$.*

SOLUTION: Note that T is inveritlbe $\Rightarrow T'$ is invertible. And $\mathcal{M}(T') = \mathcal{M}(T)^t = A^t$, denote it by B.

Let $(\varphi_1, \dots, \varphi_m)$ be the dual basis of $(v_1, \dots, v_m)$, $(\psi_1, \dots, \psi_m)$ be the dual basis of $(u_1, \dots, u_m)$.

(a) Suppose T is invertible, so is T'. Because $T'(\varphi_1), \dots, T'(\varphi_m)$ is linearly independent.

Noticing that $T'(\varphi_i) = B_{1,i}\psi_1 + \cdots + B_{m,i}\psi_m$.

Thus the columns of B, namely the rows of A, are linearly independent (check it by contradiction).

(b) Suppose the rows of A are linearly independent, so are the columns of B.

Then $(T'(\varphi_1), \dots, T'(\varphi_m))$ is a basis of range T', namely V'. Thus T' is surjective.

Hence T' is invertible, so is T. $\square$

---

34 *The double dual space of V, denoted by V'', is defined to be the dual space of V'. In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \to V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.*

(a) *Show that Λ is a linear map from V to V''.*

(b) *Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*

(c) *Show that if V is finite-dim, then Λ is an isomorphism from V onto V''.*

Suppose V is finite-dim. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

SOLUTION:

(a) $\forall \varphi \in V'$, $\forall v, w \in V, a \in \mathbf{F}$, $(\Lambda(v + aw))(\varphi) = \varphi(v + aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.

Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence Λ is linear.

(b) $(T''(\Lambda v))(\varphi) = ((\Lambda v) \circ (T'))(\varphi) = (\Lambda v)(T'(\varphi)) = (T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = (\Lambda(Tv))(\varphi)$.

Hence $T''(\Lambda v) = (\Lambda(Tv))$, getting $T'' \circ \Lambda = \Lambda \circ T$.

(c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is injective.

又 Because V is finite-dim. $\dim V = \dim V' = \dim V''$. Hence Λ is an isomorphism. $\square$

---

36 *Suppose U is a subspace of V. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*

(a) *Show that null $i' = U^0$: null $i' = (\text{range } i)^0 = U^0 \Leftarrow \text{range } i = U$.* $\square$

(b) *Prove that if V is finite-dim, then range $i' = U'$: range $i' = (\text{null } i)^0_U = (\{0\})^0_U = U'$.* $\square$

(c) *Prove that if V is finite-dim, then $\tilde{i'}$ is an isomorphism from V'/U^0 onto U':*

Note that $\tilde{i'} : V'/\text{null } i' \to \text{range } i' \Rightarrow \tilde{i'} : V'/U^0 \to U'$. By (a), (b) and [3.91(d)]. $\square$

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

---

37 *Suppose U is a subspace of V and π is the quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.*

(a) *Show that π' is injective: Because π is surjective. Use [3.108].* $\square$

(b) *Show that range $\pi' = U^0$.*

(c) *Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.*

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

In fact, there is no assumption here that any of these vector spaces are finite-dim.

SOLUTION: [3.109] is not available. Using (3.E.18), also see (3.E.20).

(b) $\psi \in \text{range } \pi' \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \text{null } \psi \supseteq U \iff \psi \in U^0$. Hence range $\pi' = U^0$.

(c) $\psi \in U^0 \iff \text{null } \psi \supseteq U \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi = \pi(\varphi)$. Thus π' is surjective. And by (a). $\square$

ENDED

- **NOTE FOR [4.8]:** *division algorithm for polynomials*

  Suppose $p, s \in \mathcal{P}(\mathbf{F})$, with $s \neq 0$. Then $\exists!\, q, r \in \mathcal{P}(\mathbf{F})$ such that $p = sq + r$ and $\deg r < \deg s$. *Another Proof:*

  Suppose $\deg p \geq \deg s$. Then $(\underbrace{1, z, \dots, z^{\deg s-1}}_{\text{of length } \deg s}, \underbrace{s, zs, \cdots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)})$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.

  Because $q \in \mathcal{P}(\mathbf{F})$, $\exists!\, a_i, b_j \in \mathbf{F}$,

  $$\begin{aligned} q &= a_0 + a_1 z + \cdots + a_{\deg s-1} z^{\deg s-1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s \\ &= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s-1} z^{\deg s-1}}_{r} + s \underbrace{\left(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s}\right)}_{q}. \end{aligned}$$

  With r, q as defined uniquely above, we are done. $\square$

- **NOTE FOR [4.11]:** *each zero of a polynomial corresponds to a degree-one factor;* *Another Proof:*

  First suppose $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, $\exists!\, a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.

  Then $p(z) = p(z) - p(\lambda) = a_1(z - \lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.

  Hence $\forall k \in \{1, \dots, m\}$, $z^k - \lambda^k = (z - \lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.

  Thus $p(z) = \sum\limits_{j=1}^{m} a_j (z - \lambda) \sum\limits_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) \sum\limits_{j=1}^{m} a_j \sum\limits_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) q(z)$.

- **NOTE FOR [4.13]:** *fundamental theorem of algebra, first version*

  Every nonconstant polynomial with complex coefficients has a zero in $\mathbf{C}$. *Another Proof:*

  De Moivre's theorem (which you can prove using induction on k and the addition formulas for cosine and sine), states that if $k \in \mathbf{N}^+, \theta \in \mathbf{R}$, then $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$.

  Suppose $w \in \mathbf{C}, k \in \mathbf{N}^+$ and using polar coordinates. $\exists\, r \geq 0, \theta \in \mathbf{R}$ such that $r(\cos\theta + \mathrm{i}\sin\theta) = w$.

  Hence $(r^{1/k}(\cos\frac{\theta}{k} + \mathrm{i}\sin\frac{\theta}{k}))^k = w$. Thus every complex number has a k^{th} *root*, a fact that we will soon use.

  Suppose a nonconstant $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.

  Then $|p(z)| \to \infty$ as $|z| \to \infty$ (because $\dfrac{|p(z)|}{|z_m|} \to |c_m|$ as $|z| \to \infty$).

  Thus the continuous function $z \to |p(z)|$ has a global minimum at some point $\zeta \in \mathbf{C}$.

  To show that $p(\zeta) = 0$, suppose that $p(\zeta) \neq 0$.

  Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \dfrac{p(z + \zeta)}{p(\zeta)}$.

  The function $z \to |q(z)|$ has a global minimum value of 1 at $z = 0$.

  Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where k is the smallest positive integer such that $a_k \neq 0$.

  Let $\beta \in \mathbf{C}$ be such that $\beta^k = -\dfrac{1}{a_k}$.

  There is a constant $c > 1$ such that if $t \in (0, 1)$,

  then $|q(t\beta)| \leq |1 + a_k t^k \beta^k| + t^{k+1} c = 1 - t^k(1 - tc)$.

  Thus taking t to be $1/(2c)$ in the inequality above, we have $|q(t\beta)| < 1$,

  which contradicts the assumption that the global minimum of $z \to |q(z)|$ is 1.

  Hence $p(\zeta) = 0$, as desired. $\square$

• *Prove that if $w, z \in \mathbf{C}$, then $||w| - |z|| \leq |w - z|$.* *The inequality here is called the* **reverse triangle inequality**.

SOLUTION:

$$
\begin{aligned}
|w-z|^2 &= (w-z)(\overline{w}-\overline{z}) \\
&= |w|^2 + |z|^2 - (w\overline{z} + \overline{w}z) \\
&= |w|^2 + |z|^2 - (\overline{\overline{w}z} + \overline{w}z) \\
&= |w|^2 + |z|^2 - 2Re(\overline{w}z) \\
&\geq |w|^2 + |z|^2 - 2|\overline{w}z| \\
&= |w|^2 + |z|^2 - 2|w||z| = ||w| - |z||^2. \qquad \square
\end{aligned}
$$

Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides.

• *Suppose V is a complex vector space and $\varphi \in V'$.*
Define $: V \to \mathbf{R}$ by $\sigma(v) = \Re\varphi(v)$ for each $v \in V$.
Show that $\varphi(v) = \sigma(v) - i\sigma(iv)$ for all $v \in V$.

SOLUTION:

Notice that $\varphi(v) = \Re\varphi(v) + \mathrm{i}\Im\varphi(v) = \sigma(v) + \mathrm{i}\Im\varphi(v)$. 又 $\Re\varphi(\mathrm{i}v) = \Re[\mathrm{i}\varphi(v)] = -\Im\varphi(v) = \sigma(\mathrm{i}v)$.

Hence $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$. $\square$

---

2 *Suppose m is a positive integer. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subspace of $\mathcal{P}(\mathbf{F})$?*

SOLUTION:

$x^m, x^m + x^{m-1} \in U$ but $\deg[(x^m + x^{m-1}) - (x^m)] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$.

Hence U is not closed under addition, and therefore is not a subspace. $\square$

---

3 *Suppose m is a positive integer. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p$ is even $\}$ a subspace of $\mathcal{P}(\mathbf{F})$?*

SOLUTION:

$x^2, x^2 + x \in U$ but $\deg[(x^2 + x) - (x^2)]$ is odd and hence $(x^2 + x) - (x^2) \notin U$.

Thus U is not closed under addition, and therefore is not a subspace. $\square$

---

4 *Suppose that m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \dots, \lambda_m \in \mathbf{F}$.*
Prove that $\exists\, p \in \mathcal{P}(\mathbf{F})$ such that $\deg p = n$, the zeros of p are $\lambda_1, \dots, \lambda_m$.

SOLUTION: Let $p(z) = (z - \lambda_1)^{n-(m-1)}(z - \lambda_2)\cdots(z - \lambda_m)$. $\square$

---

5 *Suppose that $m \in \mathbf{N}$, $z_1, \dots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$.*
Prove that $\exists!\, p \in \mathcal{P}_m(\mathbf{F})$ such that $p(z_k) = w_k$ for each $k = 1, \dots, m+1$.

This result can be proved without using linear algebra. However, try to find the clearer, shorter proof that uses some linear algebra.

SOLUTION:

Define $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by $Tq = (q(z_1), \dots, q(z_m), q(z_{m+1}))$. As can be easily checked, T is linear.

We need to show that T is surjective, so that such p exists; and that T is injective, so that such p is unique.

$Tq = 0 \iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0$

$\iff q \in \mathcal{P}_m(\mathbf{F})$ is the zero polynomial, for if not,

q has at least $m+1$ distinct roots, while $\deg q = m$. Contradicts (by [4.12]). Hence T is injective.

$\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m + 1 = \dim \mathbf{F}^{m+1}$. 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1}$. Hence T is surjective. $\square$

---

6 *Suppose $p \in \mathcal{P}_m(\mathbf{C})$ has degree m. Prove that*
p has m distinct zeros $\iff$ p and its derivative p' have no zeros in common.

SOLUTION:

(a) Suppose p has m distinct zeros. By [4.14] and $\deg p = m$, let $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$, $\exists!\, c, \lambda_i \in \mathbf{C}$.

For each $j \in \{1,\dots,m\}$, let $\dfrac{p(z)}{(z-\lambda_j)} = q_j \in \mathcal{P}_{m-1}(\mathbf{C})$, then $p(z) = (z-\lambda_j)q_j(z)$ and $q_j(\lambda_j) \neq 0$.

$p'(z) = (z-\lambda_j)q_j'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

(b) To prove the implication on the other direction, we prove the contrapositive:
Suppose p has less than m distinct roots.
We must show that p and its derivative p' have at least one zero in common.
Let λ be a zero of p, then write $p(z) = (z-\lambda)^n q(z)$, $\exists!\, n \in \mathbf{N}^+, q \in \mathcal{P}_{m-n}(\mathbf{C})$.
$p'(z) = (z-\lambda)^n q'(z) + n(z-\lambda)^{n-1}q(z) \Rightarrow p'(\lambda) = 0$, λ is a common root of p' and p. $\square$

---

7 *Prove that every polynomial of odd degree with real coefficients has a real zero.*

SOLUTION:

Using the notation proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow$ m is odd. Hence λ_1 exists. $\square$

OR. Using calculus but not using [4.17].
Suppose $p \in \mathcal{P}_m(\mathbf{F}), \deg p = m, m$ is odd.

Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1}a_m$ by δ

Write $p(x) = x^m(\dfrac{a_0}{x^m} + \dfrac{a_1}{x^{m-1}} + \cdots + \dfrac{a_{m-1}}{x} + a_m)$.

Thus $p(x)$ is continuous, and $\lim\limits_{x\to-\infty} p(x) = -\delta\infty\,$; $\quad \lim\limits_{x\to\infty} p(x) = \delta\infty$.

Hence we conclude that p has at least one real zero. $\square$

---

8 *For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \to \mathbf{R}$ by* $(Tp)(x) = \begin{cases} \dfrac{p(x)-p(3)}{x-3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3 \end{cases}$ *for all $x \in \mathbf{R}$.*

Show that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$ and that $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is a linear map.

SOLUTION:

For $x \neq 3, T(x^n) = \dfrac{x^n - 3^n}{x-3} = \sum\limits_{i=1}^{n} 3^{i-1}x^{n-i}$.

For $x = 3, T(x^n) = 3^{n-1}\cdot n$. Note that if $x = 3$, then $\sum\limits_{i=1}^{n} 3^{i-1}x^{n-i} = \sum\limits_{i=1}^{n} 3^{n-1} = 3^{n-1}\cdot n$.

Hence for all $x \in \mathbf{R}$ and for all $n \in \mathbf{N}$, $T(x^n) = \sum\limits_{i=1}^{n} 3^{i-1}x^{n-i} \in \mathcal{P}(\mathbf{R})$.

Because T is linear, we conclude that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$.

Now we show that T is linear:

$\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, T(p+\lambda q)(x) = \begin{cases} \dfrac{(p+\lambda q)(x) - (p+\lambda q)(3)}{x-3} & \text{if } x \neq 3, \\ (p+\lambda q)'(3) & \text{if } x = 3 \end{cases}$ for all $x \in \mathbf{R}$.

Notice that $(p+\lambda q)(x) - (p+\lambda q)(3) = (p(x)-p(3)) + (\lambda q(x) - \lambda q(3))$;
$\quad (p+\lambda q)'(3) = p'(3) + \lambda q'(3)$.

Thus $T(p+\lambda q)(x) = (T(p) + \lambda T(q))(x)$ for all $x \in \mathbf{R}$. $\square$

---

9 *Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\overline{z})}$.*
Prove that q is a polynomial with real coefficients.

SOLUTION:

$p(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow p(\overline{z}) = \underline{a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0} \Rightarrow \overline{p(\overline{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\overline{z})} = \overline{p(\overline{z})}p(z) = \overline{p(\overline{z})\overline{p(\overline{\overline{z}})}} = \overline{q(\overline{z})}$.

Hence letting $q(z) = c_m x^m + \cdots + c_1 x + c_0 \;\Rightarrow\; \overline{c_k} = c_k$, $c_k \in \mathbf{R}$ for each k. ☐

---

10 *Suppose $m \in \mathbf{N}$ and $p \in \mathcal{P}_m(\mathbf{C})$ is such that there are $(m+1)$ distinct real numbers $x_0, x_1, \ldots, x_m$ with $p(x_k) \in \mathbf{R}$ for each x_k.*
Prove that all coefficients of p are real.

SOLUTION: Let $p(x_k) = y_k$ for each k. By Problem (5), $\exists!\, q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_k) = y_k$. Hence $p = q$. ☐

OR. Using the Lagrange Interpolating Polynomial.

Define $q(x) = \displaystyle\sum_{j=0}^{m} \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j)$.

又 For each j, $x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R}) \subseteq \mathcal{P}_m(\mathbf{C})$.

Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$ for each $k \in \{0, 1, \ldots, m\}$.

Then $(q-p)$ has $(m+1)$ distinct zeros, while $(q-p) \in \mathcal{P}_m(\mathbf{C})$. Hence by [4.12], $q - p = 0 \Rightarrow p = q$. ☐

---

11 *Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*
(a) *Show that* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.
(b) *Find a basis of* $\mathcal{P}(\mathbf{F})/U$.

SOLUTION:

U is a subspace of $\mathcal{P}(\mathbf{F})$ because $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, pf + \lambda pg = p(f + \lambda g) \in U$.

NOTE: Define $P :\to \mathcal{P}(\mathbf{F})$ by $(Pq)(x) = p(q(x)) = (p \circ q)(x)\ (\neq p(x)q(x)\,)$. P is not linear.

(a) By [4.8], $\forall f \in \mathcal{P}(\mathbf{F})$, $\exists!\, q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r)$; $\deg r < \deg p$.

Hence $\forall f \in \mathcal{P}(\mathbf{F})$, $\exists!\, pq \in U, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), f = (pq) + (r); r \notin U$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F})$. Therefore $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p-1}(\mathbf{F})$ are isomorphic.

OR. $\forall f \in \mathcal{P}(\mathbf{F})$, $\exists!\, q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r)$; $\deg r < \deg p$.

Define $R : \mathcal{P}(\mathbf{F}) \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $(Rf)(z) = r(z)$ for each $z \in \mathbf{F}$.

$\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, R(f + \lambda g)(z) = R(f) + \lambda R(g)$.

BECAUSE: $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}$,

$\exists!\, q_1, r_1 \in \mathcal{P}(\mathbf{F}), f = (p)q_1 + (r_1),\ \deg r_1 < \deg p$;

$\exists!\, q_2, r_2 \in \mathcal{P}(\mathbf{F}), g = (p)q_2 + (r_2),\ \deg r_2 < \deg p$;

$\exists!\, q_3, r_3 \in \mathcal{P}(\mathbf{F}), \lambda g = (p)q_3 + (r_3) = (p)(\lambda q_2) + (\lambda r_2),\ \deg r_3 < \deg p$ and $\deg \lambda r_2 < \deg p$.

$\Rightarrow q_3 = \lambda q_2, r_3 = \lambda r_2$.

$\exists!\, q_0, r_0 \in \mathcal{P}(\mathbf{F}), (f + \lambda g) = (p)q_0 + (r_0)$

$= (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2),\ \deg r_0 < \deg p$ and $\deg(r_1 + \lambda r_2) < \deg p$.

$\Rightarrow q_1 + \lambda q_2 = q_0;\ \ r_1 + \lambda r_2 = r_0$.

Hence R is linear.

$R(f) = 0 \iff f = pq$, $\exists!\, q \in \mathcal{P}(\mathbf{F})$. Thus $\operatorname{null} R = U$.

$\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F})$, let $f = p + r$, then $R(f) = r$. Thus $\operatorname{range} R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Finally, by [3.91(d)], $\mathcal{P}(\mathbf{F})/\operatorname{null} R$, namely $\mathcal{P}(\mathbf{F})/U$, and $\operatorname{range} R$, namely $\mathcal{P}_{\deg p-1}(\mathbf{F})$, are isomorphic.

(b) $(1 + U, x + U, \ldots, x^{\deg p - 1}) + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. ☐

• *Suppose nonconstant $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p$, $n = \deg q$. Use* (a)—(c) *below to prove that $\exists\,!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*

(a) *Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r, s) = rp + sq$. Show that the linear map T is injective.*

(b) *Show that the linear map T in* (a) *is surjective.*

(c) *Use* (b) *to conclude that $\exists\,!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*

SOLUTION:

(a) T is linear because $\forall\, r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,

$$T((r_1, s_1) + \lambda(r_2, s_2)) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1, s_1) + \lambda T(r_2, s_2).$$

Suppose $T(r, s) = rp + sq = 0$. Notice that p, q have no zeros in common.

Then $r = s = 0$, for if not, write $\dfrac{q(z)}{r(z)} = \dfrac{p(z)}{s(z)}$, while for any zero λ of q, $\dfrac{q(\lambda)}{r(z)} = 0 \neq \dfrac{p(\lambda)}{s(z)}$. Hence $\square$

(b) $\dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) = \dim \mathcal{P}_{n-1}(\mathbf{C}) + \dim \mathcal{P}_{m-1}(\mathbf{C}) = n + m = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.

又 T is injective. Hence $\dim \operatorname{range} T = \dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) - \dim \operatorname{null} T = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.

Thus $\operatorname{range} T = \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$ is surjective, and therefore is an isomorphism. $\square$

(c) Immediately. $\square$

ENDED

5.A

• **NOTE FOR [5.10]:** *linearly independent eigenvectors*

Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

Another Proof:

Suppose the desired result is false.

Then there exists a smallest positive integer $m > 1$

(because an eigenvector is, by definition, nonzero) such that

there exists a linearly dependent list $(v_1, \dots, v_m)$ of eigenvectors of T

corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_m$ of T.

Thus there exist $a_1, \dots, a_m \in \mathbf{F}$, none of which are 0 (because of the minimality of m),

such that $a_1 v_1 + \dots + a_{m-1} v_{m-1} + a_m v_m = 0$.

Apply $T - \lambda_m I$ to both sides of the equation above, getting $a_1(\lambda_1 - \lambda_m)v_1 + \dots + a_{m-1}(\lambda_{m-1} - \lambda_m)v_{m-1} = 0$.

Because the eigenvalues $\lambda_1, \dots, \lambda_m$ are distinct, none of the coefficients above equal 0.

Thus $v_1, \dots, v_{m-1}$is a linearly dependent list of $m - 1$ eigenvectors of T corresponding to distinct eigenvalues.

Contradicts the minimality of m. $\square$

---

1 *Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.*

(a) *Prove that if $U \subseteq \operatorname{null} T$, then U is invariant under T.*

(b) *Prove that if $\operatorname{range} T \subseteq U$, then U is invariant under T.*

SOLUTION:

(a) $\forall u \in U \subseteq \operatorname{range} T, Tu = 0 \in U$. $\square$

(b) $\forall u \in U \subseteq V, Tu \in \operatorname{range} T \subseteq U$. $\square$

- *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*
  (a) *Prove that null $(T - \lambda I)$ is invariant under S, where λ is chosen arbitrarily.*
  (b) *Prove that range $(T - \lambda I)$ is invariant under S, where λ is chosen arbitrarily.*

SOLUTION:

Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.

(a) Suppose $v \in$ null $(T - \lambda I)$, then $(T - \lambda I)(Sv) = S(T - \lambda I)v = S(0) = 0$.
Hence $Sv \in$ null $(T - \lambda I)$ and therefore null $(T - \lambda I)$ is invariant under S.

(b) Suppose $v \in$ range $(T - \lambda I)$, therefore $\exists\, u \in V, (T - \lambda I)u = v$.
Then $Sv = S(T - \lambda I)u = (T - \lambda I)(Su) \in$ range $(T - \lambda I)$.
Hence $Sv \in$ range $(T - \lambda I)$ and therefore range $(T - \lambda I)$ is invariant under S. $\square$

COMMENT: Reversing the roles of S and T, letting $\lambda = 0$, we can conclude that
null S and range S is invariant under T, which is what we will prove in Problem (2) and (3) below.

---

2 *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that null S is invariant under T.*

SOLUTION: $\forall u \in$ null $S, Su = 0 \Rightarrow TSu = 0 = STu \Rightarrow Tu \in$ null S. $\square$

---

3 *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that range S is invariant under T.*

SOLUTION: $\forall w \in$ range S, $\exists\, v \in V, Sv = w, STv = TSv = Tw \in$ range S. $\square$

---

4 *Suppose $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are subspaces of V invariant under T.
Prove that $V_1 + \cdots + V_m$ is invariant under T.*

SOLUTION:

For each $i = 1, \ldots, m$, $\forall v_i \in V_i, Tv_i \in V_i$

Hence $\forall v = v_1 + \cdots + v_m \in V_1 + \cdots + V_m, Tv = Tv_1 + \cdots + Tv_m \in V_1 + \cdots + V_m$. $\square$

---

5 *Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection
of subspaces of V invariant under T is invariant under T.*

SOLUTION:

Suppose $\{V_\alpha\}_{\alpha \in \Gamma}$ is a collection of subspaces of V invariant under T; here Γ is an arbitrary index set.

We need to prove that $\bigcap_{\alpha \in \Gamma} V_\alpha$, which equals the set of vectors
that are in V_α for each $\alpha \in \Gamma$, is invariant under T.

For each $\alpha \in \Gamma$, $\forall v_\alpha \in V_\alpha, Tv_\alpha \in V_i$.

Hence $\forall v \in \bigcap_{\alpha \in \Gamma} V_\alpha, Tv \in V_\alpha, \forall \alpha \in \Gamma \Rightarrow Tv \in \bigcap_{\alpha \in \Gamma} V_\alpha$. Thus $\bigcap_{\alpha \in \Gamma} V_\alpha$ is invariant under T. $\square$

---

6 *Prove or give a counterexample:
If V is finite-dim and U is a subspace of V that is invariant under every operator on V,
then $U = \{0\}$ or $U = V$.*

SOLUTION:

Notice that V might be $\{0\}$. In this case we are done. Suppose dim $V \geq 1$. We prove by contrapositive:
Suppose $U \neq \{0\}$ and $U \neq V$, then $\exists\, T \in \mathcal{L}(V)$ such that U is not invariant under T.

Let W be such that $V = U \oplus W$.

Let $(u_1, \ldots, u_m)$ be a basis of U and $(w_1, \ldots, w_n)$ be a basis of W.

Hence $(u_1, \ldots, u_m, w_1, \ldots, w_n)$ is a basis of V.

Define $T \in \mathcal{L}(V)$ by $T(a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n) = b_1w_1 + \cdots + b_nw_n$. $\square$

---

7 *Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.*

SOLUTION:

Suppose $\lambda \in \mathbf{R}$ and $(x, y) \in \mathbf{R}^2 \backslash \{0\}$ such that $T(x, y) = (-3y, x) = \lambda(x, y)$. Then $-3y = \lambda x$ and $x = \lambda y$.

Thus $-3y = \lambda^2 y \Rightarrow \lambda^2 = -3$, ignoring the possibility of $y = 0$ (because if $y = 0$, then $x = 0$).

Hence the set of solution for this equation is $\varnothing$, and therefore T has no eigenvalues in $\mathbf{R}$. $\square$

---

8 *Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.*

SOLUTION:

Suppose $\lambda \in \mathbf{F}$ and $(w, z) \in \mathbf{F}^2$ such that $T(w, z) = (z, w) = \lambda(w, z)$. Then $z = \lambda w$ and $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, ignoring the possibility of $z = 0$ ($z = 0 \Rightarrow w = 0$).

Hence $\lambda_1 = -1$ and $\lambda_2 = 1$ are all eigenvalues of T.

For $\lambda_1 = -1, z = -w, w = -z$; For $\lambda_2 = 1, z = w$.

Thus the set of all eigenvectors is $\{(z, -z), (z, z) : z \in \mathbf{F} \wedge z \neq 0\}$. $\square$

---

9 *Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$.
Find all eigenvalues and eigenvectors of T.*

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector $(z_1, z_2, z_3) \in \mathbf{F}^3$.

Then $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$.

Thus $2z_2 = \lambda z_1, \quad 0 = \lambda z_2, \quad 5z_3 = \lambda z_3$.

We discuss in two cases:

For $\lambda = 0,\ z_2 = z_3 = 0$ and z_1 can be arbitrary ($z_1 \neq 0$).

For $\lambda \neq 0,\ z_2 = 0 = z_1$, and z_3 can be arbitrary ($z_3 \neq 0$), then $\lambda = 5$.

The set of all eigenvectors is $\{(0, 0, z), (z, 0, 0) : z \in \mathbf{F} \wedge z \neq 0\}$. $\square$

• *Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$.
Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.*

SOLUTION: (See also at (3.B), just below Problem (25), where (5.B.4) is answered.)

Suppose λ is an eigenvalue, $v \in V \backslash \{0\}$ such that $Pv = \lambda v$.

Then because $P(Pv) = Pv$, hence $\lambda^2 v = \lambda v$.

Thus $\lambda = 1$ or 0. $\square$

---

10 *Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$*

(a) *Find all eigenvalues and eigenvectors of T.*

(b) *Find all invariant subspaces of V under T.*

SOLUTION:

(a) Suppose $v = (x_1, x_2, x_3, \ldots, x_n)$ is an eigenvector of T with an eigenvalue λ.

Then $Tv = \lambda v = (x_1, 2x_2, 3x_3, \ldots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots, \lambda x_n)$.

Hence $1, \ldots, n$ are eigenvalues of T.

And $\{(0, \ldots, 0, x_\lambda, 0, \ldots, 0) \in \mathbf{F}^n : \lambda = 1, \ldots, n,\ x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$ is the set of all eigenvectors of T.

(b) Let $V_\lambda = \{(0, \ldots, 0, x_\lambda, 0, \ldots, 0) \in \mathbf{F}^n : x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$. Then $V_1, \ldots, V_n$ are invariant under T.

Hence by Problem (4), every sum of $V_1, \ldots, V_n$ is a invariant subspace of V under T. $\square$

---

11 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.*

SOLUTION:

Note that in general, $\deg p' < \deg p$ ($\deg 0 = -\infty$).

Suppose λ is an eigenvalue of T with an eigenvector p.

Suppose $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p \neq p'$. Contradicts. Thus $\lambda = 0$.

Therefore $\deg \lambda p = -\infty = \deg p \Rightarrow p$ is a nonzero constant polynomial.

Hence the set of all eigenvectors is $\{C : C \in \mathbf{R} \wedge C \neq 0\} = \mathcal{P}_0(\mathbf{R}) \backslash \{0\}$. $\square$

---

12 *Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$.
Find all eigenvalues and eigenvectors of T.*

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector p, then $(Tp)(x) = xp'(x) = \lambda p(x)$.

Let $p = a_0 + a_1 x + \cdots + a_n x^n$.

Then $xp'(x) = a_1 x + 2a_2 x^2 + \cdots + n a_n x^n = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n$.

Similar to Problem (10), $0, 1, \ldots, n$ are eigenvalues of T.

The set of all eigenvectors of T is $\{cx^\lambda : \lambda = 0, 1, \ldots, n,\ c \in \mathbf{F} \wedge c \neq 0\}$. $\square$

---

13 *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.
Prove that $\exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}$ and $(T - \alpha I)$ is invertible.*

SOLUTION:

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \dfrac{1}{1000 + k}$ for each $k = 1, \ldots, \dim V + 1$.

Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigenvalues.

Hence $\exists\, k = 1, \ldots, \dim V + 1$ such that α_k is not an eigenvalue of T and therefore $(T - \alpha_k I)$ is invertible. $\square$

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\alpha \in \mathbf{F}$.
Prove that $\exists\, \delta > 0$ such that $(T - \lambda I)$ is invertible for all $\lambda \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.*

SOLUTION:

Choose $\delta > 0$ arbitrarily.

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \dfrac{\delta}{k}$ for each $k = 1, \ldots, \dim V + 1$.

Similar to Problem (13), $\exists\, k$ such that α_k is not an eigenvalue. $\square$

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14 *Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V.
Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.
Find all eigenvalues and eigenvectors of P.*

SOLUTION:

Suppose λ is an eigenvalue of P with an eigenvector $(u + w)$.

Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$. By [1.44] and $V = U \oplus W$, $(\lambda - 1)u = \lambda w = 0$.

Thus if $\lambda = 1$, then $w = 0$; if $\lambda = 0$, then $u = 0$.

Hence the eigenvalues of P are 0 and 1, the set of all eigenvectors in P is $U \cup W$. $\square$

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15 *Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.*

(a) *Prove that T and $S^{-1}TS$ have the same eigenvalues.*

(b) *What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?*

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector v.

Then $S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v$.

Thus λ is also an eigenvalue of $S^{-1}TS$ with an eigenvector $S^{-1}v$.

Suppose λ is an eigenvalue of $S^{-1}TS$ with an eigenvector v.

Then $S(S^{-1}TS)v = TSv = \lambda Sv$.

Thus λ is also an eigenvalue of T with an eigenvector Sv. $\square$

OR. Note that $S(S^{-1}TS)S^{-1} = T$.

Hence every eigenvalue of $S^{-1}TS$ is an eigenvalue of $S(S^{-1}TS)S^{-1} = T$.

And every eigenvector v of $S^{-1}TS$ is $S^{-1}v$, every eigenvector u of T is Su. $\square$

---

17 *Give an example of an operator on $\mathbf{R}^4$ that has no (real) eigenvalues.*

SOLUTION:

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$. Where (e_1, e_2, e_3, e_4) is the standard basis of $\mathbf{R}^4$.

Suppose λ is an eigenvalue of T with an eigenvector (x, y, z, w).

Then $T(x,y,z,w) = \lambda(x,y,z,w) \Rightarrow \begin{cases} (1-\lambda)x + y + z + w = 0 \\ -x + (1-\lambda)y - z - w = 0 \\ 3x + 8y + (11-\lambda)z + 5w = 0 \\ 3x - 8y - 11z + (5-\lambda)w = 0 \end{cases}$

This linear equation has no solutions.

(You can type it on https://zh.numberempire.com/equationsolver.php to check.)

OR. Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Suppose λ is an eigenvalue of T with an eigenvector (x, y, z, w).

Then $T(x,y,z,w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \Rightarrow \begin{cases} -y = \lambda x \\ x = \lambda y \\ -w = \lambda z \\ z = \lambda w \end{cases} \Rightarrow \begin{cases} -xy = \lambda^2 xy \\ -zw = \lambda^2 zw \end{cases}$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Otherwise, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, contradicts.

Similarly, $y = z = w = 0$. Then we fail.

Thus T has no eigenvalues. $\square$

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*

Show that λ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector v.

Then $(T - \lambda I_V)$ is not invertible. 又 V is finite-dim.

Thus by [3.108, 110], [3.101] and Problem (12) in (3.F), $(T - \lambda I_V)' = T' - \lambda I_{V'}$ is not invertible.

Hence λ is an eigenvalue of T'.

(b) Suppose λ is an eigenvalue T' with an eigenvector ψ. Then $T'(\psi) = \psi \circ T = \lambda \psi$.

又 $\psi \neq 0 \Rightarrow \exists v \in V$ such that $\psi(v) \neq 0$. Note that $\psi(Tv) = \lambda \psi(v)$.

Thus $\lambda = \dfrac{\psi(Tv)}{\psi(v)} \Rightarrow Tv = \dfrac{\psi(Tv)}{\psi(v)} v = \lambda v$. Hence λ is an eigenvalue of T. $\square$

- TODO *Suppose* $(v_1, \ldots, v_n)$ *is a basis of* V *and* $T \in \mathcal{L}(V)$.
  Prove that if λ *is an eigenvalue of* T*, then*
  $|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\}$*, where* $\mathcal{M}(T, (v_1, \ldots, v_n))$.

SOLUTION:

Suppose λ is an eigenvalue of T, and therefore is an eigenvalue of $\mathcal{M}(T)$, with an eigenvector v.
We discuss in two cases:
If $\mathcal{M}(T)$ is invertible ($\Longleftrightarrow T$ is invertible), then $\mathcal{M}(Tv) = \mathcal{M}(\lambda v) \Rightarrow \frac{1}{\lambda}\mathcal{M}(v) = \mathcal{M}(T^{-1}v)$.
Otherwise, $(T - 0I)$ is not invertible and therefore $\lambda = 0$ is an eigenvalue. And other λs ?

- *Suppose* $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.
  (a) (OR (9.11)) $\lambda \in \mathbf{R}$. *Prove that* λ *is an eigenvalue of* $T \Longleftrightarrow \lambda$ *is an eigenvalue of* $T_{\mathbb{C}}$.
  (b) (OR Problem (16)) $\lambda \in \mathbf{C}$. *Prove that* λ *is an eigenvalue of* $T_{\mathbb{C}} \Longleftrightarrow \overline{\lambda}$ *is an eigenvalue of* $T_{\mathbb{C}}$.

SOLUTION:

(a) Suppose $v \in V$ is an eigenvector corresponding to the eigenvalue λ.
Then $Tv = \lambda v \Rightarrow T_{\mathbb{C}}(v + \mathrm{i}0) = Tv + \mathrm{i}T0 = \lambda v$.
Thus λ is an eigenvalue of T.
Suppose $v + \mathrm{i}u \in V_{\mathbb{C}}$ is an eigenvector corresponding to the eigenvalue λ.
Then $T_{\mathbb{C}}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \Rightarrow Tv = \lambda v, Tu = \lambda u$. (Note that v or u might be zero).
Thus λ is an eigenvalue of $T_{\mathbb{C}}$.

(b) Suppose λ is an eigenvalue of $T_{\mathbb{C}}$ with an eigenvector $v + \mathrm{i}u$.
Let $(v_1, \ldots, v_n)$ be a basis of V. Write $v = \sum_{i=1}^{n} a_i v_i, u = \sum_{i=1}^{n} b_i v_i$, where $a_i, b_i \in \mathbf{R}$.
Then $T_{\mathbb{C}}(v + \mathrm{i}u) = Tv + \mathrm{i}Tu = \lambda v + \mathrm{i}\lambda u = \lambda \sum_{i=1}^{n}(a_i + \mathrm{i}b_i)v_i$. Conjugating two sides, we have:
$\overline{T_{\mathbb{C}}(v + \mathrm{i}u)} = \overline{Tv + \mathrm{i}Tu} = \overline{Tv} - \mathrm{i}\overline{Tu} = Tv - \mathrm{i}Tu = T_{\mathbb{C}}(\overline{v + \mathrm{i}u}) = \overline{\lambda \sum_{i=1}^{n}(a_i + \mathrm{i}b_i)v_i} = \overline{\lambda} \sum_{i=1}^{n}(a_i - \mathrm{i}b_i)v_i$.
Hence $\overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$. To prove the other direction, notice that $\overline{\overline{\lambda}} = \lambda$. $\square$

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18 *Show that the forward shift operator* $T \in \mathcal{L}(\mathbf{F}^\infty)$
defined by $T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$ *has no eigenvalues.*

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector $(z_1, z_2, \ldots)$.
Then $T(z_1, z_2, \ldots) = (\lambda z_1, \lambda z_2, \ldots) = (0, z_1, z_2, \ldots)$.
Thus $\lambda z_1 = 0, \lambda z_2 = z_1, \ldots, \lambda z_k = z_{k-1}, \ldots$.
Let $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \lambda z_k = z_{k-1}$, therefore $(z_1, z_2, \ldots) = 0$ is not an eigenvector.
Suppose $\lambda \neq 0$. Then $\lambda z_1 = 0 \Rightarrow z_1 = 0 \Rightarrow z_2 = 0 = z_k$ for all $k \in \mathbf{N}^+$.
And then $(z_1, z_2, \ldots) = 0$ is not an eigenvector. Hence T has no eigenvalues. $\square$

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19 *Suppose $n \in \mathbf{N}^+$.*
Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.
In other words, the entries of $\mathcal{M}(T)$ with respect to the standard basis are all 1's.
Find all eigenvalues and eigenvectors of T.

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector $(x_1, \dots, x_n)$.
Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.
Thus $\lambda x_1 = \cdots = \lambda x_n = x_1 + \cdots + x_n$.
For $\lambda = 0$, $x_1 + \cdots + x_n = 0$.
For $\lambda \neq 0$, $x_1 = \cdots = x_n$ and then $\lambda x_k = n x_k$ for each k.
Hence $0, n$ are eigenvectors of T.
And the set of all eigenvectors of T is $\{(x_1, \dots, x_n) \in \mathbf{F}^n : x_1 + \cdots + x_n = 0 \vee x_1 = \cdots = x_n\}$. $\square$

---

20 *Define the backward shift operator $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.*
(a) *Show that every element of $\mathbf{F}$ is an eigenvalue of S.*
(b) *Find all eigenvectors of S.*

SOLUTION:

Suppose λ is an eigenvalue of S with an eigenvector $(z_1, z_2, \dots)$.
Then $S(z_1, z_2, z_3 \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$.
Thus $\lambda z_1 = z_2, \lambda z_2 = z_3, \dots, \lambda z_k = z_{k+1}, \dots$.
For $\lambda = 0$, $\lambda z_1 = z_2 = 0 = \lambda z_2 = z_3 = \cdots = z_k$ for all k.
While z_1 can be arbitrary, so that $(z_1, 0, \dots)$ is an eigenvector with $z_1 \neq 0$.
For $\lambda \neq 0$, $\lambda^k z_1 = \lambda^{k-1} z_2 = \cdots = \lambda z_k = z_{k+1}$ for all k.
Then $(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots)$ is an eigenvector with $z_1 \neq 0$.
Hence (a) each element of $\lambda \in \mathbf{F}$ is an eigenvalue of T.
And (b) the set of all eigenvectors of T is $\{(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots) \in \mathbf{F}^\infty : \lambda \in \mathbf{F}, z_1 \neq 0\}$ $\square$

---

21 *Suppose $T \in \mathcal{L}(V)$ is invertible.*
(a) *Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$.*
Prove that λ is an eigenvalue of $T \iff \frac{1}{\lambda}$ is an eigenvalue of T^{-1}.
(b) *Prove that T and T^{-1} have the same eigenvectors.*

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector v.
Then $T^{-1}Tv = \lambda T^{-1} v = v \Rightarrow T^{-1}v = \frac{1}{\lambda}v$. Hence $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.
(b) Suppose $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} with an eigenvector v.
Then $TT^{-1}v = v = \frac{1}{\lambda}Tv \Rightarrow Tv = \lambda v$. Hence λ is an eigenvalue of T.
OR. Note that $(T^{-1})^{-1} = T$ and $\frac{1}{\frac{1}{\lambda}} = \lambda$. $\square$

---

22 *Suppose $T \in \mathcal{L}(V)$ and $\exists$ nonzero vectors u, w in V such that $Tu = 3w$ and $Tw = 3u$.*
Prove that 3 or -3 is an eigenvalue of T.

SOLUTION: COMMENT: $Tu = 3w, Tw = 3u \Rightarrow T(Tu) = 9u \Rightarrow T^2$ has an eigenvalue 9.

$Tu = 3w, Tw = 3u \Rightarrow T(u + w) = 3(u + w), T(u - w) = 3(w - u) = -3(u - w)$.
Hence 3 or -3 is an eigenvalue of T. $\square$

---

23 *Suppose V is finite-dim, $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.*

SOLUTION:

Suppose λ is an eigenvalue of ST with an eigenvector v. Then $T(STv) = \lambda Tv = TS(Tv)$.

If $Tv \neq 0$, then λ is an eigenvalue of TS.

Otherwise, $\lambda = 0$, $(\, v \neq 0, \lambda v = 0 = STv\,)$, then T is not invertible

$\Rightarrow TS$ is not invertible $\Rightarrow (TS - 0I)$ is not invertible $\Rightarrow \lambda = 0$ is an eigenvalue of TS.

Reversing the roles of T and S, we conclude that ST and TS have the same eigenvalues. $\square$

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24 *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = Ax$, where elements of $\mathbf{F}^n$ are thought of as n-by-1 column vectors.*

(a) *Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.*

(b) *Suppose the sum of the entries in each column of A equals 1. Prove that 1 is an eigenvalue of T.*

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $Tx = Ax = \begin{pmatrix} \sum\limits_{c=1}^{n} A_{1,c}x_c \\ \vdots \\ \sum\limits_{c=1}^{n} A_{n,c}x_c \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. While $\sum\limits_{r=1}^{n} A_{R,c} = 1$ for each $R = 1, \dots, n$.

Thus if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, and hence is an eigenvalue of T.

(b) Suppose λ is an eigenvalue of T with an eigenvector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $Tx = Ax = \begin{pmatrix} \sum\limits_{r=1}^{n} A_{1,r}x_r \\ \vdots \\ \sum\limits_{r=1}^{n} A_{n,r}x_r \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. While $\sum\limits_{r=1}^{n} A_{r,C} = 1$ for each $C = 1, \dots, n$.

Thus $\sum\limits_{r=1}^{n} (Ax)_{r,\cdot} = \sum\limits_{r=1}^{n} (Ax)_{r,1}$

$$= \sum_{c=1}^{n} (A_{1,c} + \cdots + A_{n,c})x_c = \sum_{c=1}^{n} x_c = \lambda \begin{pmatrix} x_1 \\ + \\ \vdots \\ + \\ x_n \end{pmatrix}.$$

Hence $\lambda = 1$, for all x such that $\sum\limits_{c=1}^{n} x_{c,1} \neq 0$. $\square$

OR. Prove that $(T - I)$ is not invertible, so that we can conclude $\lambda = 1$ is an eigenvalue.

Because $(T - I)x = (A - \mathcal{M}(I))x = \begin{pmatrix} \sum\limits_{r=1}^{n} A_{1,r}x_r - x_1 \\ \vdots \\ \sum\limits_{r=1}^{n} A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

Then $y_1 + \cdots + y_n = \sum\limits_{r=1}^{n}\sum\limits_{c=1}^{n} (A_{r,c}x_c - x_r) = \sum\limits_{c=1}^{n} x_c \sum\limits_{r=1}^{n} A_{r,c} - \sum\limits_{r=1}^{n} x_r = 0$.

Thus range $(T - I) \subseteq \{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbf{F}^n : y_1 + \cdots + y_n = 0 \}$. Hence $(T - I)$ is not surjective. $\square$

• *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = xA$,*
where elements of $\mathbf{F}^n$ are thought of as 1-by-n row vectors.
(a) *Suppose the sum of the entries in each column of A equals 1.*
Prove that 1 is an eigenvalue of T.
(b) *Suppose the sum of the entries in each row of A equals 1.*
Prove that 1 is an eigenvalue of T.

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.
Then $Tx = xA = \begin{pmatrix} \sum_{r=1}^{n} x_r A_{r,1} & \cdots & \sum_{r=1}^{n} x_r A_{r,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. While $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C = 1, \ldots, n$.
Thus if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, hence is an eigenvalue of T.

(b) Suppose λ is an eigenvalue of T with an eigenvector $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.
Then $Tx = xA = \begin{pmatrix} \sum_{c=1}^{n} x_c A_{c,1} & \cdots & \sum_{c=1}^{n} x_c A_{c,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. While $\sum_{c=1}^{n} A_{R,c} = 1$ for each $R = 1, \ldots, n$.
Thus $\sum_{c=1}^{n} (xA)_{\cdot,c} = \sum_{c=1}^{n} (xA)_{1,c} = \sum_{c=1}^{n} (A_{c,1} + \cdots + A_{c,n}) x_c = \sum_{c=1}^{n} x_c = \lambda \begin{pmatrix} x_1 & + & \cdots & + & x_n \end{pmatrix}$.
Hence $\lambda = 1$, for all x such that $\sum_{r=1}^{n} x_{1,r} \neq 0$. $\square$

OR. Prove that $(T - I)$ is not invertible, so that we can conclude $\lambda = 1$ is an eigenvalue.
Because $(T - I)x = x(A - \mathcal{M}(I)) == \begin{pmatrix} \sum_{c=1}^{n} x_c A_{c,1} - x_1 & \cdots & \sum_{c=1}^{n} x_c A_{c,n} - x_n \end{pmatrix} = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}$.
Then $y_1 + \cdots + y_n = \sum_{c=1}^{n} \sum_{r=1}^{n} (x_r A_{r,c} - x_c) = \sum_{r=1}^{n} x_r \sum_{c=1}^{n} A_{r,c} - \sum_{c=1}^{n} x_c = 0$.
Thus range $(T - I) \subseteq \{ \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}. \in \mathbf{F}^n : y_1 + \cdots + y_n = 0 \}$. Hence $(T - I)$ is not surjective. $\square$

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25 *Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T*
such that $u + w$ is also an eigenvector of T.
Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

SOLUTION:

Suppose $\lambda_1, \lambda_2, \lambda_0$ are eigenvalues of T corresponding to $u, w, u + w$ respectively.
Then $T(u + w) = \lambda_0(u + w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$.
Notice that $u, w, u + w$ are nonzero.
If (u, w) is linearly dependent, then let $w = cu$, therefore

$$\lambda_2 cu = Tw = cTu = \lambda_1 cu \qquad \Rightarrow \lambda_2 = \lambda_1.$$
$$\lambda_0(u + w) = T(u + w) = \lambda_1 u + \lambda_1 cu = \lambda_1(u + w) \quad \Rightarrow \lambda_0 = \lambda_1.$$

Otherwise, $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. $\square$

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26 *Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T.*
Prove that T is a scalar multiple of the identity operator.

SOLUTION:

Because $\forall v \in V, \exists!\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.
Then for any two distinct nonzero vectors $v, w \in V$,
$T(v + w) = \lambda_{v+w}(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$.
If (v, w) is linearly independent, then let $w = cv$, therefore

$$\lambda_v cv = cTv = Tw = \lambda_w w \qquad \Rightarrow \lambda_w = \lambda_v.$$
$$\lambda_{v+w}(v + w) = T(v + w) = Tv + Tw = \lambda_v(v + cv) \Rightarrow \lambda_{v+w} = \lambda_v.$$

Otherwise, $\lambda_v = \lambda_{v+w} = \lambda_w$. $\square$

27, 28 *Suppose V is finite-dim and $k \in \{1, \dots, \dim V - 1\}$. Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V of dim k is invariant under T. Prove that T is a scalar multiple of the identity operator.*

SOLUTION:

We prove the contrapositive:

If $T \neq \lambda I, \forall \lambda \in \mathbf{F}$, then $\exists$ a subspace U of V such that $\dim U = k$, and U is invariant under T.

By Problem (26), $\exists\, v \in V$ and $v \neq 0$ such that v is not an eigenvector of T.

Thus (v, Tv) is linearly independent. Extend to a basis of V as $(v, Tv, u_1, \dots, u_n)$.

Let $U = \text{span}\,(v, u_1, \dots, u_{k-1}) \Rightarrow U$ is not an invariant subspace of V under T.

OR. Suppose $0 \neq v = v_1 \in V$ and extend to a basis of V as $(v_1, \dots, v_n)$.

Suppose $Tv_1 = c_1 v_1 + \cdots + c_n v_n$, $\exists!\, c_i \in \mathbf{F}$.

Consider a k - dim subspace $U = \text{span}\,(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$,

where $\alpha_j \in \{2, \dots, n\}$ for each j, and $\alpha_1, \dots, \alpha_{k-1}$ are distinct and are chosen arbitrarily.

Because every subspace such U is invariant.

Thus $Tv_1 = c_1 v_1 + \cdots + c_n v_n \in U$

$\Rightarrow c_2 = \cdots = c_n = 0$,

for if not, for each $c_i \neq 0$, choose U_i such that $\alpha_j \in \{\underbrace{2, \dots, i-1, i+1, \dots, n}_{\text{length }(n-2)}\}$ for each j,

hence for $Tv_1 = c_1 v_1 + \cdots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \cdots + c_n v_n \in U_i$, we conclude that $c_i = 0$.

$\Rightarrow Tv_1 = c_1 v_1$, 又 $v_1 = v \in V$ is arbitrarily chosen $\Rightarrow T = \lambda I$ for some λ. $\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue $\iff \exists$ a subspace U of V such that $\dim U = \dim V - 1$, U is invariant under T.*

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector v.

(If $\dim V = 1$, then $U = \{0\}$ and we are done.)

Extend $v_1 = v$ to a basis of V as $(v_1, v_2 \dots, v_n)$.

Step 1 If $\exists\, w_1 \in \text{span}\,(v_2, \dots, v_n)$ such that $0 \neq Tw_1 \in \text{span}\,(v_1)$,

then extend $w_1 = \alpha_{1,1}$ to a basis of span $(v_2, \dots, v_n)$ as $(\alpha_{1,1}, \dots, \alpha_{1,n-1})$.

Otherwise, we stop at step 1.

Step 2 If $\exists\, w_2 \in \text{span}\,(\alpha_{1,2}, \dots, \alpha_{1,n-1})$ such that $0 \neq Tw_2 \in \text{span}\,(v_1, w_1)$,

then extend $w_2 = \alpha_{2,1}$ to a basis of span $(\alpha_{1,2}, \dots, \alpha_{1,n-1})$ as $(\alpha_{2,1}, \dots, \alpha_{2,n-2})$.

Otherwise, we stop at step 2.

$\vdots$

Step k If $\exists\, w_k \in \text{span}\,(\alpha_{k-1,2}, \dots, \alpha_{k-1,n-k+1})$ such that $0 \neq Tw_k \in \text{span}\,(v_1, w_1, \dots, w_{k-1})$,

then extend $w_k = \alpha_{k,1}$ to a basis of span $(\alpha_{k-1,2}, \dots, \alpha_{k-1,n-k+1})$ as $(\alpha_{k,1}, \dots, \alpha_{k,n-k})$.

Otherwise, we stop at step k.

$\vdots$

Finally, we stop at step m, thus we get $(v_1, w_1, \dots, w_{m-1})$ and $(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1})$,

range $T|_{\text{span}\,(w_1,\dots,w_{m-1})} = \text{span}\,(v_1, w_1, \dots, w_{m-2}) \Rightarrow \dim \text{null}\, T|_{\text{span}\,(w_1,\dots,w_{m-1})} = 0$,

span $(\underbrace{v_1, w_1, \dots, w_{m-1}}_{\text{length dim } m})$ and span $(\underbrace{\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}}_{\text{length dim}(n-m)})$ are invariant under T.

Let $U = \text{span}\,(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}) \oplus \text{span}\,(v_1, w_1, \dots, w_{m-2})$ and we are done. $\square$

COMMENT: Both span $(v_2, \dots, v_n)$ and span $(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}) \oplus \text{span}\,(w_1, \dots, w_{m-1})$ are in $\mathcal{S}_V \text{span}\,(v_1)$.

(b) Suppose U is an invariant subpsace of V under T with $\dim U = m = \dim V - 1$.

(If $m = 0$, then $\dim V = 1$ and we are done).

Let $(u_1, \ldots, u_m)$ be a basis of U, extend to a basis of V as $(u_0, u_1, \ldots, u_m)$.

We discuss in cases:

For $Tu_0 \in U$, then range $T = U$ so that T is not surjective $\iff$ null $T \neq \{0\} \iff 0$ is an eigenvalue of T.

For $Tu_0 \notin U$, then $Tu_0 = a_0u_0 + a_1u_1 + \cdots + a_mu_m$.

(1) If $Tu_0 \in$ span (u_0), then we are done.

(2) Otherwise, if range $T|_U = U$, then $Tu_0 = a_0u_0$ and we are done;

otherwise, $T|_U : U \to U$ is not surjective ($\Rightarrow$ not injective), suppose range $T|_U \neq \{0\}$

(Suppose range $T|_U = \{0\}$. If $\dim U = 0$ then we are done.

Otherwise $\exists\, u \in U \backslash \{0\}, Tu = 0$ and we are done.)

then $\exists\, u \in U \backslash \{0\}, Tu = 0$, we are done. $\square$

---

29 *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*

Prove that T has at most $1 + \dim$ range T distinct eigenvalues.

SOLUTION:

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T and let $v_1, \ldots, v_m$ be the corresponding eigenvectors.

For every $\lambda_k \neq 0, T(\frac{1}{\lambda_k}v_k) = v_k$. And if $T = T - 0I$ is not invertible, then $\exists!\, \lambda_A = 0$ and $Tv_A = \lambda_A v_A = 0$.

Thus for $\lambda_k \neq 0, \forall\, k$, $(Tv_1, \ldots, Tv_m)$ is a linearly independent list of length m in range T.

And for $\lambda_A = 0$, there is a linearly independent list of length at most $(m-1)$ in range T.

Hence, by [2.23], $m \leq \dim$ range $T + 1$. $\square$

---

30 *Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigenvalues of T.*

Prove that $\exists\, x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

SOLUTION: Because 9 is not an eigenvalue. Hence $(T - 9I)$ is surjective. $\square$

---

31 *Suppose V is finite-dim and $v_1, \ldots, v_m \in V$.*

Prove that $(v_1, \ldots, v_m)$ is linearly independent

$\iff \exists\, T \in \mathcal{L}(V)$ such that $v_1, \ldots, v_m$ are eigenvectors of T

corresponding to distinct eigenvalues.

SOLUTION:

Suppose $(v_1, \ldots, v_m)$ is linearly independent, extend it to a basis of V as $(v_1, \ldots, v_m, \ldots, v_n)$.

Then define $T \in \mathcal{L}(V)$ by $Tv_k = kv_k$ for each $k \in \{1, \ldots, m, \ldots, n\}$.

Conversely by [5.10] it is true as well. $\square$

---

32 *Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct real numbers.*

Prove that $(e^{\lambda_1}x, \ldots, e^{\lambda_n}x)$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

HINT: *Let $V =$ span $(e^{\lambda_1}x, \ldots, e^{\lambda_n}x)$, and define an operator $D \in \mathcal{L}(V)$ by $Df = f'$.*

Find eigenvalues and eigenvectors of D.

SOLUTION:

Define V and $D \in \mathcal{L}(V)$ as in HINT. Then because for each k, $D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.

Thus $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of D. By [5.10], $(e^{\lambda_1}x, \ldots, e^{\lambda_n}x)$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$. $\square$

• *Suppose $\lambda_1, \ldots, \lambda_n$ are distinct positive numbers.*
Prove that $(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

SOLUTION:

Let $V = \operatorname{span}(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.
Then because $D(\cos(\lambda_k x)) = -\lambda_k \sin(\lambda_k x)$. 又 $D(\sin(\lambda_k x)) = \lambda_k \cos(\lambda_k x)$.
Thus $D^2(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$.
Notice that $\lambda_1, \ldots, \lambda_n$ are distinct $\Rightarrow -\lambda_1^2, \ldots, -\lambda_n^2$ are distinct.
Hence $-\lambda_1^2, \ldots, -\lambda_n^2$ are distinct eigenvalues of D^2
with the corresponding eigenvectors $\cos(\lambda_1 x), \ldots, \cos(\lambda_n x)$ respectively.
And then $(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$. $\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Define $A \in \mathcal{L}(\mathcal{L}(V))$ by $A(S) = TS$ for each $S \in \mathcal{L}(V)$.
Prove that the set of eigenvalues of T equals the set of eigenvalues of A.

SOLUTION:

(a) Suppose $\lambda_1, \ldots, \lambda_m$ are all eigenvalues of T with eigenvectors $v_1, \ldots, v_m$ respectively.
Extend to a basis of V as $(v_1, \ldots, v_m, \ldots, v_n)$.
Then for each $k \in \{1, \ldots, m\}$, $\operatorname{span}(v_k) \subseteq \operatorname{null}(T - \lambda_k I)$.
Define $S_k \in \mathcal{L}(V)$ by $S_k(v_j) = v_k$ for each $j \in \{1, \ldots, n\}$,
so that range $S_k = \operatorname{span}(v_k)$ for each $k \in \{1, \ldots, m\}$, then $A(S_k) = TS_k = \lambda_k S_k$.
Thus the eigenvalues of T are eigenvalues of A.

(b) Suppose $\lambda_1, \ldots, \lambda_m$ are all eigenvalues of A with eigenvectors $S_1, \ldots, S_m$ respectively.
Then for each $k \in \{1, \ldots, m\}$, because $\forall v \in V, u = S_k(v) \in V \Rightarrow Tu = \lambda_k u$.
Thus the eigenvalues of A are eigenvalues of T. $\square$

• COMMENT: Define $B \in \mathcal{L}(\mathcal{L}(V))$ by $B(S) = ST, \forall S \in \mathcal{L}(V)$. Then the eigenvalues of B are not the eigenvalues of T.

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T.*
The quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U \text{ for each } v \in V.$$

(a) *Show that the definition of T/U makes sense*
(which requires using the condition that U is invariant under T)
and show that T/U is an operator on V/U.

(b) (OR Problem 35) *Show that each eigenvalue of T/U is an eigenvalue of T.*

SOLUTION:

(a) Suppose $v + U = w + U$ ($\iff v - w \in U$).
Then because U is invariant under T, $T(v - w) \in U \iff Tv + U = Tw + U$.
Hence the definition of T/U makes sense.

(b) Suppose λ is an eigenvalue of T/U with an eigenvector $v + U$.
Then $(T/U)(v + U) = \lambda(v + U) = Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v \in U$.
If $(T - \lambda I)v = 0 \Rightarrow Tv = \lambda v$, then we are done.
Otherwise, then $(T|_U - \lambda I) : U \to U$ is invertible,
hence $\exists!\, w \in U, (T|_U - \lambda I)(w) = (T - \lambda I)v \Rightarrow T(v - w) = \lambda(v - w)$.
Note that $v - w \neq 0$ (for if not, $v \in U \Rightarrow v + U = 0 + U$ is not an eigenvector).
Thus λ is an eigenvalue of T. $\square$

---

36 *Prove or give a counterexample:*

The result of (b) *in Exercise 35 is still true if* V *is infinite-dim.*

SOLUTION: A counterexample:

Consider $V = \text{span}\,(1, e^x, e^{2x}, \dots)$ in $\mathbb{R}^{\mathbb{R}}$, and a subspace $U = \text{span}\,(e^x, e^{2x}, \dots)$ of V.

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then range $T = U$ is invariant under T.

Consider $(T/U)(1 + U) = e^x + U = 0$

$\Rightarrow 0$ is an eigenvalue of T/U but is not an eigenvalue of T

(null $T = \{0\}$, for if not, $\exists\, f \in V\backslash\{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, contradicts). $\square$

COMMENT: Using [5.6] requires that V is finite-dim.

---

33 *Suppose* $T \in \mathcal{L}(V)$*. Prove that* $T/(\textit{range}\, T) = 0$.

SOLUTION:

$\forall v + \text{range}\, T \in V/\text{range}\, T, v + \text{range}\, T \in \text{null}\,(T/(\text{range}\, T))$

$\Rightarrow \text{null}\,(T/(\text{range}\, T)) = V/\text{range}\, T \Rightarrow T/(\text{range}\, T)$ is a zero map. $\square$

---

34 *Suppose* $T \in \mathcal{L}(V)$*. Prove that* $T/(\textit{null}\, T)$ *is injective* $\iff (\textit{null}\, T) \cap (\textit{range}\, T) = \{0\}$.

SOLUTION:

(a) Suppose $T/(\text{null}\, T)$ is injective.

Then $(T/(\text{null}\, T))(u + \text{null}\, T) = Tu + \text{null}\, T = 0$

$\iff Tu \in \text{null}\, T$ 又 $Tu \in \text{range}\, T \iff u + \text{null}\, T = 0 \iff u \in \text{null}\, T \iff Tu = 0.$

Thus $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$.

(b) Suppose $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$.

Then $(T/(\text{null}\, T))(u + \text{null}\, T) = Tu + \text{null}\, T = 0$

$\iff Tu \in \text{null}\, T$ 又 $Tu \in \text{range}\, T \iff Tu = 0 \iff u \in \text{null}\, T \iff u + \text{null}\, T = 0.$

Thus $T/(\text{null}\, T)$ is injective. $\square$

• **NOTE FOR [5.6]:** More generally, suppose we do not know whether V is finite-dim. Then $(a) \iff (b)$.

Suppose (a) λ is an eigenvalue of T. Then $(T - \lambda I)v = 0$ for a corresponding eigenvector v. Hence we get (b), $(T - \lambda I)$ is not injective. And then (d), $(T - \lambda I)$ is not invertible. But $(d) \not\Rightarrow (b)$ (because S is not invertible $\iff$ S is not injective *or* S is not surjective).

ENDED

COMMENT: 下面是第 5 章 B 节。为了照顾 5.B 节两版过大的差距，特别将 5.B 补注分成 **I** 和 **II** 两部分。又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本征值」(相当一部分是从原第 3 版 8.C 节挪过来的) 是对原第 3 版「多项式作用于算子」与「本征值的存在性」(也即第 3 版 5.B 前半部分) 的极大扩充，这一扩充也大大改变了原第 3 版后半部分的「上三角矩阵」这一小节，故而将第 4 版 5.B 节放在第 3 版 5.B 节前面。

I 部分除了覆盖第 4 版 5.B 节和第 3 版 5.B 节前半部分与之相关的所有习题，还会覆盖第 4 版 5.A 节末。

II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节，还会覆盖第 4 版 5.C 节；并且，下面 **5.C** 还会覆盖第 4 版 5.D 节。

「注：[8.40] OR (4E 5.22) —— minimal polynomial;

[8.44,8.45] OR (4E 5.25,5.26) —— how to find the minimal polynomial;

[8.49] OR (4E 5.27) —— eigenvalues are the zeros of the minimal polynomial;

[8.46] OR (4E 5.29) —— $q(T) = 0 \Leftrightarrow q$ is a poly multiple of the mini poly.」

5.B: I [*See* **5.B: II** *below.*]

- *Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.*
  (a) *Prove that T is injective $\iff T^m$ is injective.*
  (b) *Prove that T is surjective $\iff T^m$ is surjective.*

SOLUTION:

(a) Suppose T^m is injective. Then $Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$. $\square$

Suppose T is injective.

Then $T^m v = T(T^{m-1}v) = 0$

$$\Rightarrow T^{m-1}v = 0 = T(T^{m-2}v) \Rightarrow \cdots$$

$$\Rightarrow T^2 v = TTv = 0$$

$$\Rightarrow Tv = 0 \Rightarrow v = 0. \quad \square$$

(b) Suppose T^m is surjective. $\forall u \in V$, $\exists\, v \in V, T^m v = u = Tw$, let $w = T^{m-1}v$. $\square$

Suppose T is surjective.

Then $\forall u \in V$, $\exists\, v \in V, T(\underline{v}) = u$

$$\Rightarrow \exists\, v_2 \in V, Tv_2 = \underline{v}, T^2(\underline{v_2}) = u$$

$$\vdots$$

$$\Rightarrow \exists\, v_k \in V, Tv_k = \underline{v_{k-1}}, T^k(\underline{v_k}) = u$$

$$\vdots$$

$$\Rightarrow \exists\, v_{m-1} \in V, Tv_{m-1} = \underline{v_{m-2}}, T^{m-1}(\underline{v_{m-1}}) = u$$

$$\Rightarrow \exists\, v_m \in V, Tv_m = \underline{v_{m-1}}, T^{m-1}(Tv_m) = u. \quad \square$$

- **NOTE FOR [5.17]:** *Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$.*
  Prove that null $p(T)$ and range $p(T)$ are invariant under T.

SOLUTION: Using the commutativity in [5.10].

(a) Suppose $u \in$ null $p(T)$. Then $p(T)u = 0$.

Thus $p(T)(Tu) = (p(T)T)u(Tp(T))u = T(p(T)u) = 0$. Hence $Tu \in$ null $p(T)$. $\square$

(b) Suppose $u \in$ range $p(T)$. Then $\exists\, v \in V$ such that $u = p(T)v$.

Thus $Tu = T(p(T)v) = p(T)(Tv) \in$ range $p(T)$. $\square$

- **NOTE FOR [5.21]:** *Every operator on a finite-dim nonzero complex vector space has an eigval.*

Suppose V is a finite-dim complex vector space of dim $n > 0$ and $T \in \mathcal{L}(V)$.

Choose a nonzero $v \in V$. $(v, Tv, T^2v, \dots, T^n v)$ of length $n+1$ is linearly dependent.

Suppose $a_0 I + a_1 T + \cdots + a_n T^n = 0$. Then $\exists\, a_j \neq 0$.

Thus $\exists$ nonconst p of smallest degree ($\deg p > 0$) *such that* $p(T)v = 0$.

Because $\exists\, \lambda \in \mathbf{C}$ such that $p(\lambda) = 0 \Rightarrow \exists\, q \in \mathcal{P}(\mathbf{C}), p(z) = (z - \lambda)q(z), \forall z \in \mathbf{C}$.

Thus $0 = p(T)v = (T - \lambda I)(q(T)v)$. By the minimality of $\deg p$ and $\deg q < \deg p$, $q(T)v \neq 0$.

Then $(T - \lambda I)$ is not injective. Thus λ is an eigval of T with eigvec $q(T)v$.

- **EXAMPLE:** *an operator on a complex vector space with no eigvals*

Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by $(Tp)(z) = zp(z)$.

Suppose $p \in \mathcal{P}(\mathbf{C})$ is a nonzero poly. Then $\deg Tp = \deg p + 1$, and thus $Tp \neq \lambda p$, $\forall \lambda \in \mathbf{C}$.

Hence T has no eigvals. Because $\mathcal{P}(\mathbf{C})$ is infinite-dim, this example does not contradict the result above.

---

13 *Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ has no eigvals.*
Prove that every subspace of V invariant under T is either $\{0\}$ or infinite-dim.

SOLUTION: Suppose U is a finite-dim nonzero invariant subspace on $\mathbf{C}$. Then by [5.21], $T|_U$ has an eigval. $\square$

16 *Define $S \in \mathcal{L}(\mathcal{P}_n(\mathbf{C}), V)$ by $S(p) = p(T)v$. Prove [5.21] using [3.23].*

SOLUTION:

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17 *Define $S \in \mathcal{L}(\mathcal{P}_{n^2}(\mathbf{C}), V)$ by $S(p) = p(T)v$. Prove [5.21] using [3.23].*

SOLUTION:

• **NOTE FOR [8.40]:** *definition for minimal polynomial*

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that $\exists!$ monic poly $p \in \mathcal{P}(\mathbf{F})$ of smallest degree such that $p(T) = 0$. Furthermore, $\deg p \leq \dim V$.

SOLUTION OR Another Proof **:**

[*Existence Part*] We use induction on $\dim V$.

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, we are done.

(ii) Suppose $\dim V \geq 1$.

Assume that $\dim V > 0$ *and that the desired result is true for all operators on all vecsps of smaller dim.*

Let $u \in V, u \neq 0$. The list $(u, Tu, \dots, T^{\dim V}u)$ of length $(1 + \dim V)$ is linearly dependent.

Then $\exists!\, T^m$ of smallest degree such that $T^m u \in \operatorname{span}(u, Tu, \dots, T^{m-1}u)$.

Thus $\exists\, c_j \in \mathbf{F}, c_0 u + c_1 Tu + \dots + c_{m-1}T^{m-1}u + T^m u = 0$.

Define q by $q(z) = c_0 + c_1 z + \dots + c_{m-1}z^{m-1} + z^m$.

Then $0 = T^k(q(T)u) = q(T)(T^k u), \forall k \in \{1, \dots, m-1\} \subseteq \mathbf{N}$.

Because $(u, Tu, \dots, T^{m-1}u)$ is linearly independent.

Thus $\dim \operatorname{null} q(T) \geq m \Rightarrow \dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leq \dim V - m$.

Let $W = \operatorname{range} q(T)$.

By *assumption*, $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ and $\deg s \leq \dim W$, so that $s(T|_W) = 0$.

Hence $\forall v \in V, ((sq)(T))(v) = s(T)(q(T)v) = 0$.

Thus sq is a monic poly such that $\deg sq \leq \dim V$ and $(sq)(T) = 0$.

[*Uniqueness Part*]

Let $p, q \in \mathcal{P}(\mathbf{F})$ be monic polys of smallest degree such that $p(T) = q(T) = 0$

$\Rightarrow (p - q)(T) = 0$ 又 $\deg(p - q) < \deg p$.

If $p - q = a_m z^m + \dots + a_1 z_1 + a_0 \neq 0$, then $\frac{1}{a_m}(p - q)$ is a monic poly of smaller degree than p.

Hence contradicts the minimality of $\deg p$. Thus $p - q = 0$ and we are done. $\square$

• (4E 5.31, 4E 5.B.25 and 26) *mini poly of restriction operator and mini poly of quotient operator*

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invariant subspace of V under T.

Let p be the mini poly of T.

(a) *Prove that p is a polynomial multiple of the mini poly of $T|_U$.*

(b) *Prove that p is a polynomial multiple of the mini poly of T/U.*

(c) *Prove that (mini poly of $T|_U$) $\times$ (mini poly of T/U) is a polynomial multiple of p.*

(d) *Prove that the set of eigvals of T equals*

the union of the set of eigvals of $T|_U$ and the set of eigvals of T/U.

SOLUTION:

(a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46].$\square$

(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v + U) = p(T)v + U = 0.$ $\square$

(c) Suppose r is the mini poly of $T|_U$, s is the mini poly of T/U.

Because $\forall v \in V, s(T/U)(v+U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.

又 $\forall u \in U, r(T|_U)u = r(T)u = 0$.

Thus $\forall v \in V$ but $v \notin U, (rs)(T)v = r(s(T)v) = 0$.

And $\forall u \in U, (rs)(T)u = r(s(T)u) = 0$ (because $s(T)u = s(T|_U)u \in U$).

Hence $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0$. $\square$

(d) By [8.49], immediately. $\square$

• (4E 5.32) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$ and p is the mini poly.*

T is not invertible $\iff$ 0 is an eigval of T $\iff$ 0 is a zero of p $\iff$ the const term of p is 0. $\square$

• (4E 5.B.22)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$. Prove that T is invertible $\iff$ $I \in$ span $(T, T^2, \dots, T^{\dim V})$.

SOLUTION:

• (4E 5.B.23)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Let $n = \dim V$.

Prove that if $v \in V$, then span $(v, Tv, \dots, T^{n-1}v)$ is invariant under T.

SOLUTION:

• (4E 5.B.27)

Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, and $T \in \mathcal{L}(V)$.

Prove that the mini poly p of $T_{\mathbb{C}}$ equals the mini poly q of T.

SOLUTION:

$$\left.\begin{array}{l} \forall u + \mathrm{i}0 \in V_{\mathbb{C}}, p(T_{\mathbb{C}})(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p \text{ is a polynomial multiple of } q. \\ q(T) = 0 \in \mathcal{L}(V) \Rightarrow \forall u + \mathrm{i}v \in V_{\mathbb{C}}, q(T_{\mathbb{C}})(u + \mathrm{i}v) = q(T)u + \mathrm{i}q(T)v = 0 \Rightarrow q \text{ is a polynomial multiple of } p. \end{array}\right\} \Rightarrow \square$$

• (4E 5.B.28)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that the mini poly p of $T' \in \mathcal{L}(V')$ equals the mini poly q of T.

SOLUTION:

$$\left.\begin{array}{r} \forall \varphi \in V', p(T')\varphi = \varphi \circ (p(T)) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \operatorname{null} \varphi \Rightarrow p(T) = 0 \\ \Rightarrow p(T) = 0 \Rightarrow p \text{ is a polynomial multiple of } q. \\ q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ (q(T)) = q(T')\varphi = 0 \Rightarrow q \text{ is a polynomial multiple of } p. \end{array}\right\} \Rightarrow \square$$

• (4E 5.B.10)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $v \in V$.

Prove that $\forall m \in \mathbf{N}, m \geq \dim V - 1$, span $(v, Tv, \dots, T^m v) =$ span $(v, Tv, \dots, T^{\dim V - 1}v)$.

SOLUTION:

• (4E 5.B.19)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.

Prove that $\dim \mathcal{E}$ equals the degree of the minimal polynomial of T.

SOLUTION:

19 *Suppose that V is finite-dim,* $\dim V > 1$, *and* $T \in \mathcal{L}(V)$.
Prove that $\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V)$.

SOLUTION:

• (4E 5.B.21)
Suppose V is finite-dim and $T \in \mathcal{L}(V)$.
Prove that the mini poly of T has degree at most $1 + \dim \mathit{range}\, T$.
If $\dim \mathit{range}\, T < \dim V - 1$, *then this result gives a better upper bound than* NOTE FOR *[8.40] for the degree of mini poly.*

SOLUTION:

• **NOTE FOR [8.49]:**
Suppose V is a finite-dim complex vecsp and $T \in \mathcal{L}(V)$.
Prove that the mini poly of T has the form $(z - \lambda_1) \cdots (z - \lambda_m)$,
where $\lambda_1, \ldots, \lambda_m$ *is a list of all eigvals of T, possibly with repetitions.*

SOLUTION: To get the desired result, use [8.49] and the 2nd version of the fundamental theorem of algebra [4.14].

• **NOTICE**: Suppose $\alpha_1, \ldots, \alpha_n$ are the distinct eigvals of T.
Then $\alpha_1, \ldots, \alpha_n$ are the distinct zeros of the mini poly.
Also, the mini poly of T is a polynomial multiple of $q(z) = (z - \alpha_1) \cdots (z - \alpha_n)$, not equal to q.
If we define q by $q(z) = (z - \alpha_1)^{\dim V - 1} \cdots (z - \alpha_n)^{\dim V - 1}$, then q is a polynomial multiple of the mini poly.
The mini poly has the form $(z - \alpha_1)^{\delta_1} \cdots (z - \alpha_n)^{\delta_n}$, where $0 \leq \delta_1 + \cdots + \delta_n \leq \dim V$ and for each $j, 1 \leq \delta_j \in \mathbf{N}$.

• **COMMENT**: *A nonzero poly has at most as many distinct zeros as its degree (see [4.12]).*
Thus [8.49], along with the result that the mini poly of an operator on V has degree at most $\dim V$,
***gives an alternative proof of* [5.13]**, *which states that an operator on V has at most* $\dim V$ *distinct eigenvalues.*
The mini poly and the **characteristic polynomial** *are definitely not the same (Compare with [8.34]).*

---

10 *Suppose* $T \in \mathcal{L}(V)$, λ *is an eigval of T with an eigvec v.*
Prove that for any $p \in \mathcal{P}(\mathbf{F}), p(T)v = p(\lambda)v$.

SOLUTION:

Suppose p is defined by $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbf{F}$. Because for any $n \in \mathbf{N}^+, T^n v = \lambda^n v$.
Thus $p(T)v = a_0 v + a_1 Tv + \cdots + a_m T^m v = a_0 v + a_1 \lambda v + \cdots + a_m \lambda^m v = p(\lambda)v$. $\square$

• **COMMENT:** For any $p \in \mathcal{P}(\mathbf{F})$ such that $p(z) = (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_m)^{\alpha_m}$, the result is true as well.
Now we prove that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v = (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v$.
Define q_i by $q_i(z) = (z - \lambda_i)^{\alpha_i}$ for all $z \in \mathbf{F}$.
Because $(a+b)^n = a^n + \mathcal{C}_n^1 a^{n-1} b + \cdots + \mathcal{C}_n^k a^{n-k} b^k + \cdots + \mathcal{C}_n^n b^n$.
Let $a = z, b = \lambda_i, n = \alpha_i$, so we can write $q_i(z)$ in the form $a_0 + a_1 z + \cdots + a_m z^m$.
Hence $q_i(T)v = q_i(\lambda)v \Rightarrow (T - \lambda_i I)^{\alpha_i} v = (\lambda - \lambda_i)^{\alpha_i} v$.
Then for each $k \in \{2, \ldots, m\}$, $(T - \lambda_{k-1} I)^{\alpha_{k-1}} (T - \lambda_k I)^{\alpha_k} v$

$$\begin{aligned} &= q_{k-1}(T)(q_k(T)v) \\ &= q_{k-1}(T)(q_k(\lambda)v) \\ &= q_{k-1}(\lambda)(q_k(\lambda)v) \\ &= (\lambda - \lambda_{k-1})^{\alpha_{k-1}} (\lambda - \lambda_k)^{\alpha_k} v. \end{aligned}$$

So that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v$

$$\begin{aligned} &= q_1(T)(q_2(T)(\ldots(q_m(T)v)\ldots)) \\ &= q_1(\lambda)(q_2(\lambda)(\ldots(q_m(\lambda)v)\ldots)) \end{aligned}$$

$= (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v. \quad \square$

• (4E 5.B.14)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has mini poly $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, $a_0 \neq 0$. Find the mini poly of T^{-1}.

SOLUTION: Notice that V is finite-dim. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is invertible.

Then $p(T) = a_0 I + a_1 T + \cdots + a_m T^m = 0$.

Apply T^{-m}, getting $a_0 (T^{-1})^m + a_1 (T^{-1})^{m-1} + \cdots + a_{m-1} T^{-1} + a_m I = (T^{-1})^m (0)$.

Define q by $q(z) = z^m + \frac{a_1}{a_0} z^{m-1} + \cdots + \frac{a_{m-1}}{a_0} T + \frac{a_m}{a_0}$ for all $z \in \mathbf{F}$.

We now show that $(T^{-1})^k \notin \operatorname{span}(I, T^{-1}, \ldots, (T^{-1})^{k-1})$ for every $k \in \{1, \ldots, m-1\}$ by contradiction, so that q is exactly the mini poly of T^{-1}.

Suppose $(T^{-1})^k \in \operatorname{span}(I, T^{-1}, \ldots, (T^{-1})^{k-1})$.

Then let $(T^{-1})^k = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. Apply T^k to both sides, getting $I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T$, hence $T^k \in \operatorname{span}(I, T, \ldots, T^{k-1})$.

Thus f defined by $f(z) = z^k + \frac{b_1}{b_0} z^{k-1} + \cdots + \frac{b_{k-1}}{b_0} z - \frac{1}{b_0}$ is a polynomial multiple of the mini poly p of T.

While $\deg f < \deg p$. Contradicts. $\square$

---

1 *Suppose $T \in \mathcal{L}(V)$ and $\exists\, n \in \mathbf{N}^+$ such that $T^n = 0$. Prove that $(I - T)$ is invertible and $(I - T)^{-1} = I + T + \cdots + T^{n-1}$.*

SOLUTION: Note that $1 - x^n = (1 - x)(1 + x + \cdots + x^{n-1})$.

$$\left. \begin{array}{l} (I - T)(1 + T + \cdots + T^{n-1}) = I - T^n = I \\ (1 + T + \cdots + T^{n-1})(I - T) = I - T^n = I \end{array} \right\} \Rightarrow (I - T)^{-1} = 1 + T + \cdots + T^{n-1}. \quad \square$$

• **TIPS:** *For $T_1, \ldots, T_m \in \mathcal{L}(V)$:*

(a) *Suppose $T_1, \ldots, T_m$ are all injective. Prove that $(T_1 \circ \cdots \circ T_m)$ is injective.*

(b) *Suppose $(T_1 \circ \cdots \circ T_m)$ is not injective. Prove that at least one of $T_1, \ldots, T_m$ is not injective.*

(c) *Prove or give a counterexample:*

At least one of $T_1, \ldots, T_m$ is not injective $\Rightarrow (T_1 \circ \cdots \circ T_m)$ is not injective.

SOLUTION:

(c) A counterexample: On infinite-dim only.

Let $S \in \mathcal{L}(\mathbf{F}^\infty)$ be the backward shift operator (surjective but not injective).

Let $T \in \mathcal{L}(\mathbf{F}^\infty)$ be the forward shift operators (injective but not surjective).

Then $ST = I$ is injective. $\square$

---

2 *Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigval of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.*

SOLUTION:

Suppose v is an eigvec corresponding to λ.

Then for any $p \in \mathcal{P}(\mathbf{F}), p(T)v = p(\lambda)v$.

Hence $0 = (T - 2I)(T - 3I)(T - 4I)v = (\lambda - 2)(\lambda - 3)(\lambda - 4)v$ while $v \neq 0$.

Thus $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$. $\square$

OR. Because $(T - 2I)(T - 3I)(T - 4I) = 0$ is not injective.

Then at least one of $(T - 2I), (T - 3I), (T - 4I)$ is not injective. $\square$

---

3 *Suppose $T \in \mathcal{L}(V)$, $T^2 = I$ and -1 is not an eigval of T. Prove that $T = I$.*

SOLUTION:

$T^2 - I = (T+I)(T-I)$ is not injective $\Rightarrow (T+I)$ or $(T-I)$ is not injective.

又 -1 is not an eigval of $T \Rightarrow (T-I)$ is not injective. $\square$

OR. Note that $v = [\frac{1}{2}(I-T)v] + [\frac{1}{2}(I+T)v]$ for all $v \in V$.

And $(I-T^2)v = (I-T)(I+T)v = 0$ for all $v \in V$,

$$\left.\begin{array}{l} (I+T)(\frac{1}{2}(I-T)v) = \frac{1}{2}(I-T^2)v = 0 \Rightarrow \frac{1}{2}(I-T)v \in \operatorname{null}(I+T) \\ (I-T)(\frac{1}{2}(I+T)v) = \frac{1}{2}(I-T^2)v = 0 \Rightarrow \frac{1}{2}(I+T)v \in \operatorname{null}(I-T) \end{array}\right\} \Rightarrow V = \operatorname{null}(I+T) + \operatorname{null}(I-T).$$

又 -1 is not an eigval of $T \Rightarrow (I+T)$ is injective $\Rightarrow \operatorname{null}(I+T) = \{0\}$.

Hence $V = \operatorname{null}(I-T) \Rightarrow \operatorname{range}(I-T) = \{0\}$. Thus $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$. $\square$

• (4E 5.A.32) *Suppose $T \in \mathcal{L}(V)$ has no eigvals and $T^4 = I$. Prove that $T^2 = -I$.*

SOLUTION:

Because $T^4 - I = (T^2 - I)(T^2 + I) = 0$ is not injective $\Rightarrow (T^2 - I)$ or $(T^2 + I)$ is not injective.

又 T has no eigvals $\Rightarrow (T^2 - I) = (T-I)(T+I)$ is injective, for if not, $(T-I)$ or $(T+I)$ is not injective. Contradicts.

Hence $T^2 + I = 0 \in \mathcal{L}(V)$, for if not, $\exists\, v \in V, (T^2+I)v \neq 0$ while $(T^2-I)((T^2+I)v) = 0$. Contradicts. $\square$

OR. Note that $v = [\frac{1}{2}(I-T^2)v] + [\frac{1}{2}(I+T^2)v]$ for all $v \in V$.

And $(I-T^4)v = (I-T^2)(I+T^2)v = 0$ for all $v \in V$,

$$\left.\begin{array}{l} (I+T^2)(\frac{1}{2}(I-T^2)v) = \frac{1}{2}(I-T^4)v = 0 \Rightarrow \frac{1}{2}(I-T^2)v \in \operatorname{null}(I+T^2) \\ (I-T^2)(\frac{1}{2}(I+T^2)v) = \frac{1}{2}(I-T^4)v = 0 \Rightarrow \frac{1}{2}(I+T^2)v \in \operatorname{null}(I-T^2) \end{array}\right\} \Rightarrow V = \operatorname{null}(I+T^2) + \operatorname{null}(I-T^2).$$

又 T has no eigvals $\Rightarrow (I-T^2)$ is injective $\Rightarrow \operatorname{null}(I-T^2) = \{0\}$.

Hence $V = \operatorname{null}(I+T^2) \Rightarrow \operatorname{range}(I+T^2) = \{0\}$. Thus $I + T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$. $\square$

---

7 *Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigval of $T^2 \iff 3$ or -3 is an eigval of T.*

SOLUTION: COMMENT: Note that V can be infinite-dim. See also in (5.A.22).

(a) Suppose 9 is an eigval of T^2. Then $(T^2 - 9I)v = (T-3I)(T+3I)v = 0$ for some v.

Thus $(T-3I)$ or $(T+3I)$ is not injective.

(b) Suppose 3 or -3 is an eigval of T with an eigenvector v. Then $Tv = \pm 3v \Rightarrow T^2 v = T(Tv) = 9v$ $\square$

---

8 (OR 4E 5.A.31) *Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$.*

SOLUTION:

Simply by computing: $p(z) = z^4 + 1 = (z^2 + \mathrm{i})(z^2 - \mathrm{i}) = (z + \mathrm{i}^{1/2})(z - \mathrm{i}^{1/2})(z - (-\mathrm{i})^{1/2})(z + (-\mathrm{i})^{1/2})$.

Note that $\mathrm{i}^{1/2} = \dfrac{\sqrt{2}}{2} + \mathrm{i}\dfrac{\sqrt{2}}{2}$, $(-\mathrm{i})^{1/2} = \dfrac{\sqrt{2}}{2} - \mathrm{i}\dfrac{\sqrt{2}}{2}$.

Hence $T = \pm(\pm\mathrm{i})^{1/2}$.

Define T by $T(x,y) = (\dfrac{\sqrt{2}}{2}x - \dfrac{\sqrt{2}}{2}y, \dfrac{\sqrt{2}}{2}x + \dfrac{\sqrt{2}}{2}y)$.

$$\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix} \Rightarrow \mathcal{M}(T)^4 = \mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathcal{M}(I). \quad \square$$

$$\left(\text{Using } \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}. \right)$$

• (4E 5.B.20)

Suppose $T \in L(\mathbf{F}^4)$ is such that the eigvals of T are $3, 5, 8$.
Prove that $(T-3I)^2(T-5I)^2(T-8I)^2 = 0$.

SOLUTION:

Because the eigvals of T are the zeros of the mini poly, and also are the zeros of $q(z) = (z-3)^2(z-5)^2(z-8)^2$.

又 $\dim V = 4$. And the degree of mini poly is less than or equal to $\dim V$.

Thus the mini poly of T is p defined by $p(z) = (z-3)^{1+A}(z-5)^{1+B}(z-8)^{1+C}$, where $0 \leq A+B+C \leq 1, A, B, C \in \mathbf{N}$.

Hence q is a polynomial multiple of p. $\square$

• (4E 5.B.24)

Suppose V is a finite-dim complex vecsp, $T \in \mathcal{L}(V)$ is such that 5 and 6 are eigvals of T and that T has no other eigvals. Prove that $(T-5I)^{\dim V-1}(T-6I)^{\dim V-1} = 0$.

SOLUTION:

Because the eigvals of T are the zeros of the mini poly, and also are the zeros of $q(z) = (z-5)^{\dim V-1}(z-6)^{\dim V-1}$.

又 The degree of mini poly is less than or equal to $\dim V$.

Thus the mini poly of T is p defined by $p(z) = (z-5)^{1+A}(z-6)^{1+B}$, where $0 \leq A+B \leq \dim V - 2, A, B \in \mathbf{N}$.

Hence q is a polynomial multiple of p. $\square$

• (4E 5.B.12 See also at 5.A.9)

Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$. Find the mini poly.

SOLUTION:

$T(x_1, \ldots, 0) =$ By (5.A.9) and [8.49], $1, 2, \ldots, n$ are zeros of the mini poly of T.

(又 Each eigvals of T corresponds to exact one-dim subspace of $\mathbf{F}^n$.)

Define a poly q by $q(z) = (z-1)(z-2)\cdots(z-n)$, for all $z \in \mathbf{F}$. (Then q is the char poly of T.)

Because $q(T)e_j = [(T-I)\cdots(T-(j-1)I)(T-(j+1)I)\cdots(T-nI)](T-jI)e_j = 0$ for each j,

where $(e_1, \ldots, e_n)$ is the standard basis. Thus $\forall v \in \mathbf{F}^n, q(T)v = 0$. Hence q is the mini poly of T. $\square$

• *Suppose $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n)$.*
[*See also at* (5.A.19)] *Find the mini poly of T.*

SOLUTION:

Because n and 0 are all eigvals of T,

with the sets of eigvecs $U_1 = \{(x, \ldots, x) \in \mathbf{F}^n : x \in \mathbf{F} \wedge x \neq 0\}$

and $U_2 = \{(x_1, \ldots, x_n) \in \mathbf{F}^n : x_1 + \cdots + x_n = 0 \wedge \exists x_i \neq 0\}$ respectively.

(又 $\dim U_1 = 1, \dim U_2 = n-1 \Rightarrow z^{n-1}(z-n)$ is the char poly of T.)

Because for all $e_k, Te_k = e_1 + \cdots + e_n$; $\quad T^2 e_k = n(e_1 + \cdots + e_n)$.

Hence $T^2 e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T-n)$. Thus $z(z-n)$ is the mini poly of T. $\square$

• (See [5.8]) *Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (-z, w)$. Find the mini poly of T.*

SOLUTION:

Because i and $-\mathrm{i}$ are eigvals of T. 又 The degree of mini poly has at most $\dim V = 2$.

Hence $(z-\mathrm{i})(z+\mathrm{i})$ is the mini poly of T. $\square$

• (4E 5.B.8)

Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is the operator of counterclockwise rotation by the angel θ, where $x \in \mathbf{R}^+$. Find the minimal polynomial of T.

SOLUTION:

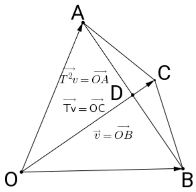
If $\theta = \pi$, then $T(w, z) = (-w, -z), T^2 = I$ and the mini poly is $z + 1$.

If $2\pi | \theta$, then $T = I$ and the mini poly is $z - 1$.

Now suppose (v, Tv) is linearly independent.

Then span $(v, Tv) = \mathbf{R}^2$.

Suppose the mini poly p is defined by $p(z) = z^2 + bz + c$ for all $z \in \mathbf{R}$.

Because 

$$\begin{cases} Tv = \dfrac{|\vec{v}|}{2L}(T^2v + v) \Rightarrow T = \dfrac{|\vec{v}|}{2L}(T^2 + I) \\ \\ L = |\vec{v}|\cos\theta \Rightarrow \dfrac{|\vec{v}|}{2L} = \dfrac{1}{2\cos\theta} \end{cases}$$

Hence $p(T) = T^2 - 2\cos\theta T + I = 0$. $z^2 - 2\cos\theta z + 1$ is the mini poly of T. $\square$

- (4E 5.B.11)

Suppose V is a two-dim vecsp, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) *Show that $T^2 - (a+d)T + (ad - bc)I = 0$.*

(b) *Show that the mini poly of T equals*

$$\begin{cases} z - a & \textit{if } b = c = 0 \textit{ and } a = d, \\ z^2 - (a+d)z + (ad - bc) & \textit{otherwise.} \end{cases}$$

SOLUTION:

(a) Suppose the basis is (v, w). Because $\begin{cases} Tv = av + bw \Rightarrow (T - aI)v = bw, \text{ then apply } (T - dI) \text{ to both sides.} \\ Tw = cv + dw \Rightarrow (T - dI)w = cv, \text{ then apply } (T - aI) \text{ to both sides.} \end{cases}$

Hence $(T - aI)(T - dI) = bcI \Rightarrow T^2 - (a+d)T + (ad - bc)I = 0$.

(b) If $b = c = 0$ and $a = d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the mini poly is $z - a$.

Otherwise, by (a), $z^2 - (a+d)z + (ad - bc)$ is a polynomial multiple of the mini poly.

Now we prove that $T \notin \operatorname{span}(I)$, so that then the mini poly of T has exactly degree 2.

(At least one of the assumption of (I),(II) below is true.)

(I) Suppose $a = d$, then $Tv = av + bw \notin \operatorname{span}(v), Tw = cv + aw \notin \operatorname{span}(w)$.

(II) Suppose at most one of b, c is not 0. If $b = 0$, then $Tw \notin \operatorname{span}(w)$; If $c = 0$, then $Tv \notin \operatorname{span}(v)$. $\square$

---

5 *Suppose $S, T \in \mathcal{L}(V)$, S is invertible, and $p \in \mathcal{P}(\mathbf{F})$. Prove that $p(TS) = S^{-1}p(ST)S$.*

SOLUTION:

- (4E 5.B.7)

(a) *Give an example of $S, T \in \mathcal{L}(\mathbf{F}^2)$ such that the mini poly of ST does not equal the mini poly of TS.*

(b) *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that if S or T is invertible, then the mini poly of ST equals the mini poly of TS.*

SOLUTION:

(a) Let $V = \mathbf{F}^\infty$, $S \in \mathcal{L}(\mathbf{F}^\infty)$ is the forward shift operator, $T \in \mathcal{L}(\mathbf{F}^\infty)$ is the backward shift operator.

Then $ST(x_1, x_2, x_3, \ldots) = (0, x_2, x_3, \ldots) \Rightarrow 0, 1$ are all the eigvals of ST, $(ST)^2 - (ST) = 0$.

$TS(x_1, x_2, \ldots) = (x_1, x_2, \ldots) \Rightarrow 1$ is the only eigval of TS, $TS = I$.

Hence the mini poly of ST does not equal to the mini poly of TS.

(b) Suppose S is invertible.

Suppose T is invertible. Similarly,

$\square$

---

6 *Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T.*
Prove that U is invariant under $p(T)$ for every poly $p \in \mathcal{P}(\mathbf{F})$.

SOLUTION:

$$\forall u \in U, Tu \in U \Rightarrow \forall u \in U, Iu, Tu, T(Tu), \ldots, T^m u \in U \Rightarrow \forall u \in U, (a_0 I + a_1 T + \cdots + a_m T^m)u \in U. \quad \square$$

• (4E 5.B.9)

Suppose $T \in \mathcal{L}(V)$ is such that with respect to some basis of V,
all entries of the matrix of T are rational numbers.
Explain why all coefficients of the mini poly of T are rational numbers.

SOLUTION:

---

11 *Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{C})$, and $\alpha \in \mathbf{C}$.*
Prove that α is an eigval of $p(T) \Longleftrightarrow \alpha = p(\lambda)$ for some eigval λ of T.

SOLUTION:

---

12 *Give an example of an operator on $\mathbf{R}^2$*
that shows the result above does not hold if $\mathbf{C}$ is replaced with $\mathbf{R}$.

SOLUTION:

• (4E 5.B.13)

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$.
Prove that $\exists\,!\, r \in \mathcal{P}(\mathbf{F})$ such that $p(T) = r(T)$
and $\deg r$ is less than the degree of the mini poly of T.

SOLUTION:

---

18 (OR 4E 5.B.15)

Suppose V is a finite-dim complex vector space with $\dim V > 0$ and $T \in \mathcal{L}(V)$.
Define $f : \mathbf{C} \to \mathbf{R}$ by $f(\lambda) = \dim \textit{range}\,(T - \lambda I)$. Prove that f is not a continuous function.

SOLUTION:

• OR (4E 5.B.16), OR (8.C.18)

Suppose $a_0, \ldots, a_{n-1} \in \mathbf{F}$. Let T be the operator on $\mathbf{F}^n$ such that

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \textit{ with respect to the standard basis.}$$

Show that the mini poly of T is $a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$.

*$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the mini poly of some operator. Hence a formula or an algorithm that could produce exact eigvals for each operator on each $\mathbf{F}^n$ could then produce exact zeros for each poly [by 8.36(b)]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an operator.*

SOLUTION:

• (4E 5.B.17)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and p is the mini poly of T. Suppose $\lambda \in \mathbf{F}$.
Show that the mini poly of $(T - \lambda I)$ is the polynomial q defined by $q(z) = p(z + \lambda)$.

SOLUTION:

• (4E 5.B.18)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and p is the mini poly of T. Suppose $\lambda \in \mathbf{F} \backslash \{0\}$.
Show that the mini poly of λT is the polynomial q defined by $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.

SOLUTION:

• EIGENVALUES ON ODD-DIMENSIONAL REAL VECTOR SPACES (Eigvals on Odd-dim Real Vecsps)

• EVEN-DIMENSIONAL NULL SPACE

Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$.
Prove that $\dim$ null $(T^2 + bT + cI)$ is an even number.

SOLUTION:

Denote null $(T^2 + bT + cI)$ by R. Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.

Suppose λ is an eigval of T_R with an eigvec $v \in R$.

Then $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \left((\lambda + b)^2 + c - \dfrac{b^2}{4} \right) v.$

Because $c - \dfrac{b^2}{4} > 0$ and we have $v = 0$. Thus T_R has no eigvals.

Let U be an invariant subspace of R that has the largest, even dim among all invariant subspaces.

Assume that $U \neq R$. Then $\exists\, w \in R$ but $w \notin U$. Let W be such that $(w, T|_R w)$ is a basis of W.

Because $T|_R^2 w = -bT|_R w - cw \in W$. Hence W is an invariant subspace of dim 2.

Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,

for if not, because $w \notin U, T|_R w \in U$,

$U \cap W$ is invariant under $T|_R$ of one dim (impossible because $T|_R$ has no eigvecs).

Hence $U + W$ is even-dim invariant subspace under $T|_R$, contradicting the maximality of $\dim U$.

Thus the assumption was incorrect. Hence $R = \text{null}\,(T^2 + bT + cI) = U$ has even dim. $\square$

• OPERATORS ON ODD-DIMENSIONAL VECTOR SPACES HAVE EIGENVALUES

(a) Suppose $\mathbf{F} = \mathbf{C}$. Then by [5.21], we are done.

(b) *Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, and $\dim V = n \neq 0$ is an odd number.*
Let $T \in \mathcal{L}(V)$ and the mini poly is p. Prove that T has an eigval.

SOLUTION:

(i) If $n = 1$, then we are done.

(ii) Suppose $n \geq 3$. Assume that every operator, on odd-dim vecsps of dim less than n, has an eigval.

If p is a polynomial multiple of $(x - \lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] λ is an eigval of T and we are done.

Now suppose $b, c \in \mathbf{R}$ such that $b^2 < 4c$ and p is a polynomial multiple of $x^2 + bx + c$ (see [4.17]).

Then $\exists\, q \in \mathcal{P}(\mathbf{R})$ such that $p(x) = q(x)(x^2 + bx + c)$ for all $x \in \mathbf{R}$.

Now $0 = p(T) = (q(T))(T^2 + bT + cI)$, which means that $q(T)|_{\text{range}\,(T^2+bT+cI)} = 0$.

Because $\deg q < \deg p$ and p is the mini poly of T, hence range $(T^2 + bT + cI) \neq V$.

又 $\dim V$ is odd and $\dim \operatorname{null}(T^2 + bT + cI)$ is even (by our previous result).

Thus $\dim V - \dim \operatorname{null}(T^2 + bT + cI) = \dim \operatorname{range}(T^2 + bT + cI)$ is odd.

By [5.18], range $(T^2 + bT + cI)$ is an invariant subspace of V under T that has odd dim less than n.

Our induction hypothesis now implies that $T|_{\text{range}\,(T^2+bT+cI)}$ has an eigenvalue.

By mathematical induction. □

• *Suppose* $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$ *has no eigvals.*
Prove that every invariant subspace of V *under* T *is even-dim.*

SOLUTION:

Suppose U is such a subspace. Then $T|_U \in \mathcal{L}(U)$. We prove by contradiction.

If $\dim U$ is odd, then $T|_U$ has an eigval and so is T, so that $\exists$ invariant subspace of 1 dim, contradicts. □

• (4E 5.B.29)
Show that every operator on a finite-dim vecsp of dim ≥ 2
has an invariant subspace of dim 2.

Exercise (4E 5.C.6) will give an improvement of this result when $\mathbf{F} = \mathbf{C}$.

SOLUTION:

For any such vecsp V and $T \in \mathcal{L}(V)$, then $\exists$ invariant U_0 of even-dim.

Again for $T|_{U_0} \in \mathcal{L}(U_0)$, if $\dim U_0 > 2$, then $\exists$ invariant U_1 of even-dim;
otherwise, $\dim U_0 = 2$ and we are done.

Thus after some steps, we get an invariant subspace U of V and $\dim U = 2$. □

ENDED

5.B: II

• (4E 5.C.1)
Prove or give a counterexample:
If $T \in \mathcal{L}(V)$ *and* T^2 *has an upper-trig matrix with respect to some basis of* V,
then T *has an upper-trig matrix with respect to some basis of* V.

SOLUTION:

• (4E 5.C.2)
Suppose A *and* B *are upper-trig matrices of the same size,*
with $\alpha_1, \ldots, \alpha_n$ *on the diagonal of* A *and* $\beta_1, \ldots, \beta_n$ *on the diagonal of* B.
(a) *Show that* $A + B$ *is an upper-trig matrix with* $\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n$ *on the diagonal.*
(b) *Show that* AB *is an upper-trig matrix with* $\alpha_1\beta_1, \ldots, \alpha_n\beta_n$ *on the diagonal.*

SOLUTION:

• (4E 5.C.3)
Suppose $T \in \mathcal{L}(V)$ *is invertible and* $(v_1, \ldots, v_n)$ *is a basis of* V *with respect*
to which the matrix of T *is upper trig, with* $\lambda_1, \ldots, \lambda_n$ *on the diagonal.*

Show that the matrix of T^{-1} is also upper trig with respect to the same basis, with $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$ on the diagonal.

SOLUTION:

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9 (4E 5.C.7)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $v \in V$.

(a) *Prove that $\exists!$ monic poly p_v of smallest degree such that $p_v(T)v = 0$.*

(b) *Prove that the mini poly of T is a polynomial multiple of p_v.*

SOLUTION:

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14 (OR 4E 5.C.4)

Give an operator T such that with respect to some basis, $\mathcal{M}(T)_{k,k} = 0$ for each k, while T is invertible.

SOLUTION:

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15 (OR 4E 5.C.5)

Give an operator T such that with respect to some basis, $\mathcal{M}(T)_{k,k} \neq 0$ for each k, while T is not invertible.

SOLUTION:

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20 (OR 4E 5.C.6)

Suppose $\mathbf{F} = \mathbf{C}$, V is finite-dim, and $T \in \mathcal{L}(V)$.

Prove that if $k \in \{1, \ldots, \dim V\}$, then V has a k dim subspace invariant under T.

SOLUTION:

• (4E 5.C.8)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\exists\, v \in V \backslash \{0\}$ such that $T^2v + 2Tv = -2v$.

(a) *Prove that if $\mathbf{F} = \mathbf{R}$, then $\nexists$ a basis of V with respect to which T has an upper-trig matrix.*

(b) *Prove that if $\mathbf{F} = \mathbf{C}$ and A is an upper-trig matrix that equals the matrix of T with respect to some basis of V, then $-1 + i$ or $-1 - i$ appears on the diagonal of A.*

SOLUTION:

• (4E 5.C.9)

Suppose $B \in \mathbf{F}^{n,n}$ with complex entries.

Prove that $\exists$ invertible $A \in \mathbf{F}^{n,n}$ with complex entries such that $A^{-1}BA$ is an upper-trig matrix.

SOLUTION:

• (4E 5.C.10)

Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V.
Show that the following are equivalent.
(a) *The matrix of T with respect to $(v_1, \ldots, v_n)$ is lower trig.*
(b) *span $(v_k, \ldots, v_n)$ is invariant under T for each $k = 1, \ldots, n$.*
(c) *$Tv_k \in$ span $(v_k, \ldots, v_n)$ for each $k = 1, \ldots, n$.*
A square matrix is called lower trig if all entries above the diagonal are 0.

SOLUTION:

- (4E 5.C.11)

Suppose $\mathbf{F} = \mathbf{C}$ and V is finite-dim.
Prove that if $T \in \mathcal{L}(V)$, then T has a lower-trig matrix with respect to some basis.

SOLUTION:

- (4E 5.C.12)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has an upper-trig matrix with respect to some basis, and U is a subspace of V that is invariant under T.
(a) *Prove that $T|_U$ has an upper-trig matrix with respect to some basis of U.*
(b) *Prove that T/U has an upper-trig matrix with respect to some basis of V/U.*

SOLUTION:

- (4E 5.C.13)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.
Suppose $\exists U$ of V that is invariant under T such that
$T|_U$ has an upper-trig matrix with respect to some basis of U
and also T/U has an upper-trig matrix with respect to some basis of V/U.
Prove that T has an upper-trig matrix with respect to some basis of V.

SOLUTION:

- (4E 5.C.14)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.
Prove that T has an upper-trig matrix with respect to some basis of V
$\iff T'$ has an upper-trig matrix with respect to some basis of V'.

SOLUTION:

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5.C

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5.E* (4E)

1 *Give an example of two commuting operators $S, T \in \mathbf{F}^4$ such that there is a subspace of $\mathbf{F}^4$ that is invariant under S but not under T and there is a subspace of $\mathbf{F}^4$ that is invariant under T but not under S.*

SOLUTION:

---

2 *Suppose $\mathcal{E}$ is a subset of $\mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagonalizable.*
Prove that $\exists$ a basis of V with respect to which
every element of $\mathcal{E}$ has a diagonal matrix $\Longleftrightarrow$ every pair of elements of $\mathcal{E}$ commutes.

This exercise extends [5.76], which considers the case in which $\mathcal{E}$ contains only two elements.

For this exercise, $\mathcal{E}$ may contain any number of elements, and $\mathcal{E}$ may even be an infinite set.

SOLUTION:

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3 *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.*

(a) *Prove that null $p(S)$ is invariant under T.*

(b) *Prove that range $p(S)$ is invariant under T.*

See NOTE FOR [5.17] for the special case $S = T$.

SOLUTION:

4 *Prove or give a counterexample:*
If A is a diagonal matrix and B is an upper-trig matrix of the same size as A, then A and B commute.

SOLUTION:

---

5 *Prove that a pair of operators on a finite-dim vecsp commute $\iff$ their dual operators commute.*

SOLUTION:

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6 *Suppose V is a finite-dim complex vecsp and $S, T \in \mathcal{L}(V)$ commute.*
Prove that $\exists\, \alpha, \lambda \in \mathbf{C}$ such that range $(S - \alpha I) +$ range $(T - \lambda I) \neq V$.

SOLUTION:

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7 *Suppose V is a complex vecsp, $S \in \mathcal{L}(V)$ is diagonalizable, and T commutes with S.*
Prove that $\exists$ basis B of V such that S has a diagonal matrix with respect to B and T has an upper-trig matrix with respect to B.

SOLUTION:

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8 *Suppose $m = 3$ in Example [5.72]*
and D_x, D_y are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbf{R}^2)$ from that example.
Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ with respect to which D_x and D_y each have an upper-trig matrix.

SOLUTION:

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9 *Suppose V is a finite-dim nonzero complex vecsp.*
Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.
(a) *Prove that $\exists\, v \in V$ is an eigvec for every element of $\mathcal{E}$.*
(b) *Prove that $\exists$ a basis of V with respect to which every element of $\mathcal{E}$ has an upper-trig matrix.*

SOLUTION:

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10 *Give an example of two commuting operators S, T on a finite-dim real vecsp such that*
$S + T$ has a eigval that does not equal an eigval of S plus an eigval of T
and ST has a eigval that does not equal an eigval of S times an eigval of T.

SOLUTION:

ENDED

• **N**

SOLUTION:

- **NOTE FOR []:**