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简介

这是我个人用于复习的「Linear Algebra Done Right 3E/4E, by Sheldon Axler 」笔记,一本习题选答与课文补注。范围覆盖所有第三版和 第四版的课文和习题(除了第一章 A 节、极少数结合上下文太过显而易见的习题。没有任何日后反复推敲价值的当堂习题和方法套路过于 雷同的习题)。这份笔记尚处于缓慢的编撰进度中。

习题答案中,有我完全独立思考得出的,有抄 https://linearalgebras.com/的,有抄 https://math.stackexchange.com/的,有抄 LADR2eSolutions (By Axler).pdf ,有抄最新的 LADR4eSolutions 经典最全(By Axler?).pdf ,还有请教别人,乃至请教 AI 得出来的。

这些文档的许可证件,除 LADR4eSolutions 经典最全(By Axler?).pdf 找不到/没有指明外,都允许复制/引用。

课文补注中,除了我独立思考总结出的易错误区和技巧、难点之外,还(因为我想要兼容那些使用 LADR 第三版纸质书的读者,包括我在 内)把 LADR4e中对课文定理等等的修改也(作了简化和提炼)摘录上去。部分课文内容因为比较简单、比如 3E 节的积空间、所以我做 了概念前置,这相当于更改了原书的内容顺序。

题目标为正常数字 N 的,为第三版某章某节第 N 题(有个别题是第四版又删去的,这里,或直接摘录,或合并简化,仍然作保留;还有个 别题是第四版增添条件、设问的,也一并写在第 $\mathbb N$ 题下)。题目标为'ullet'的,为第四版。因为要面向以第三版为主要教材的学习者,所以为 了避免混淆,故而将题号(部分题目的实心黑点后有标注具体第四版的数字标号)、甚至章节略去(一些变动过大的章节除外)。题目顺序 会有调换、在每章大标题处会交代清楚。除了原书第四版新加入的章节外、均使用原书第三版的索引。这也许对第四版的使用者很不友好、 我在此欢迎有心人士将我的作品修改后在同样的 CC BY NC SA 条款下作为衍生作品发布。

因为使用中文会给我编撰这份笔记带来额外的中英文输入法切换的工作成本、况且对于专业学习者、直接使用英文不会造成任何困扰。但 英文词句的冗长性拖慢我编撰/复习的效率,所以我对许多常用术语作了简写。

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作者序

我目前还没有能力和资格评论原书好坏以及线性代数课程教材选用的问题。但作为原书的学习者,我可以说:

相较于(其他课程的)其他教材,以LADR作为自学读本的精学计划,往往在执行中出现一次又一次的时间误判/超时,比 如我最开始计划 40×8h 完成 LADR 的精学,差不多是一天(8h)完成一节,还有额外的复习时间。但在实际学习中,(刨去 笔记的功夫) 完成到一半时,发现已经耗费了约35×8h,于是我不得不重新估计LADR 精学所需的总时间为70×8h。这一 点对于有学时/学期限制/应试要求的线性代数初学者来说很不安全。更主观地讲,这是因为 LADR 更像是一本参考手册,而 不是一本细致入微的自学读本;如果把 LADR 作为初学线性代数第一教材和自学读本来学习,会面临不小的困难。

以上或许能劝退相当一部分打算入门的线性代数初学者。S.Axler 说这本书作为第二遍学习线性代数的教材更合适。我认为理 由就是,在校的科班生第二遍学习线性代数时,也已经学习过了离散数学、抽象代数、数论、数学分析等课程,这些知识储备 统统会化作一个叫 "mathematical maturity" 的东西, 让他们面对 LADR 的课文和习题不再少见多怪、茫然无措。据此, 我进 一步认为,对于完全的初学者,想要完成 LADR 的精学,要么有很好的天赋,要么有与之相匹配的 "mathematical maturity", 再要么,拿出足够的耐心和毅力。幸运的是,在坚持学习 LADR 的过程中,这三样会一同增益。就我个人来说:课文一次看 不懂,就多看几遍,一天看不懂,就分三天看;习题一个小时做不出来,就隔六个小时再尝试,一天做不出来,就隔天再尝 试。这确实让我收获了独特的学习体验和能力,我迄今也无法在别处得到,因此我很珍视 LADR,我愿意为此编撰一份电子 辅助书并免费公开于网络中。这本身并不花费什么,因为实际的时间开销包括了很多不相干的额外项目:初学 LATEX、调整代 码架构、了解许可证选用,诸如此类的各种波折,也不乏戏剧性。

我在学习过程中碰到了很多重大误区: 第一章中,我一开始误认为 $W = C_V U \cup \{0\}$ 是唯一使得 $W \oplus U = V$ 的子空间,但这压根就不是子 空间,而且C节习题中也提示这样的子空间W不唯一。第二章中,我随意地将"线性无关的序列"等同于有/无限维向量空间的基,没有 任何理论依据,我也并不懂什么选择公理。**第三章 B 到 D 节中**,我总觉得子空间是超脱有限维的存在;因为放不下第二章无限维向量空间 的基的情结,我刻意寻找那些避开涉及基的解法,一些臆测的结论和容易就找到反例。第三章 E 节中,我似乎对商空间有什么误解,觉得 v+U=v'+U 如同变戏法一样, 把 v 中一切带有 U 的部分抹除掉, 让 v 变得纯粹独立于 U, 为此我还单门发明了 Pure V/U 并试着证明 一些命题,甚至用它发现了F节23题无限维情况下不依赖基的解法。后来我猛然发现我最开始的想法多么荒诞,却仍然放不下Pure V/U 的情结。这些挫折让我思维变得更加缜密,于是在学习抽象的**第三章 F 节**时比想象中的要顺利。

ABBREVIATION TABLE

AΒ

add	addi(tion)(tive)
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because
bss	basis
bses	bases
B_V	basis of V

\mathbf{C}

ch	characteristic
closd	closed under
coeff	coefficient
col	column
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	constrapositive

D

def	definition	
deg	degree	
dep	dependen(t)(ce)	
deri	derivative(s)	
diag	diagonal(iza-ble/ility/tion)	
diff	differentia(l)(ting)(tion)	
diffce	difference	
dim	dimension(al)	
disti	distinct	
distr	distributive propert(ies)(ty)	
div	div(ide)(ision)	

E

-		
-ec	-ec(t)(tor)(tion)(tive)	
eig-	eigen-	
elem	element(s)	
ent	entr(y)(ies)	
equiv	equivalen(t)(ce)	
exa	example	
exe	exercise	
exis	exist(s)(ing)	
existns	existence	
expr	expression	

FGH

2 0, 11		
factoriz	factorizaion	
fini	finite	
finide	finite-dimensional	
generalized eig-	gig-	
G disk	Gershgorin disk	
homo	homogeneity	
hypo	hypothesis	

Ι

. 1
identity
immediately
induct(ion)(ive)
infinitely
injectiv(e)(ity)
inver(se)(tib-le/ility)
invariant
invariant under
invariant subspace
invariant subspace under
isomorph(ism)(ic)

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
multy	multiplicity
nilp	nilpotent
non0	nonzero
nonC	nonconst
notat	notation(al)
	•

O P Q

optor	operator
othws	otherwise
prod	product
poly	polynomial
quotient	quot

R

recurly	recursively
,	
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)
rotat	rotation

9

<u> </u>		
seq	sequence	
simlr	similar(ly)	
solus	solution	
sp	space	
stmt	statement	
std	standard	
supp	suppose	
surj	surjectiv(e)(ity)	
suth	such that	

$T\;U\;V\;W\;X\;Y\;Z$

trig	triangular
trslate	translate
trspose	transpose
uniq	unique
uniqnes	uniqueness
up-	upper-
val	value
vec	vector
-wd	-ward
-ws	-wise
wrto	with respect to

1.B

1 Prove $\forall v \in V, -(-v) = v$.

Solus: $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$.

2 Supp $a \in \mathbb{F}$, $v \in V$, and av = 0. Prove a = 0 or v = 0.

Solus: Supp $a \neq 0$, $\exists a^{-1} \in \mathbb{F}$, $a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$.

3 Supp $v, w \in V$. Explain why $\exists ! x \in V, v + 3x = w$.

Solus: $v + 3x = w \Leftrightarrow 3x = w - v \Leftrightarrow x = \frac{1}{3}(w - v)$.

Or. [Existns] Let $x = \frac{1}{3}(w - v)$.

[*Uniques*] If $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) - (II) $: 3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$.

5 *Show in the def of a vecsp, the add inv cond can be replaced by* [1.29].

Hint: Supp V satisfies all conds in the def, except we've replaced the add inv cond with [1.29]. Prove the add inv is true.

Solus: Using [1.31]. $0v = 0 \iff [1 + (-1)]v = 1 \cdot v + (-1)v = v + (-v) = 0.$

6 Let ∞ and $-\infty$ denote two disti objects that are not in R.

Define the natural add and scalar multi on $R \cup \{\infty, -\infty\}$ *, that is, for each* $t \in R$ *,*

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t\left(-\infty\right) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad \left(\begin{array}{c} (a) & t + \infty = \infty + t = \infty + \infty = \infty, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{array}\right) \quad \left(\begin{array}{c} (b) & t + \left(-\infty\right) = \left(-\infty\right) + t = \left(-\infty\right) + \left(-\infty\right) = -\infty, \\ \infty & \text{if } t < 0, \end{array}\right)$$

Is $R \cup \{\infty, -\infty\}$ *a vecsp over* R? *Explain.*

Solus: No. Becs the add and scalar multi is not assoc and distr.

By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

Or. By Distr: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$.

• **N**OTE **FOR Fields:** *Many choices.* [Req Multi Inv Uniq]

Exa: $\mathbf{Z}_m = \{K_0, K_1, \dots, K_{m-1}\}$ is a field $\iff m \in \mathbf{N}^+$ is a prime.

ENDED

1 ・ C 注意: 这里我将 3.E 积空间的定义前置;仅涉及概念。 • NOTE FOR Exe (5): $C = R \oplus \{ci : c \in R\} = \{a + bi : a, b \in R\}$ if we let $F = R$ and $i^2 = -1$.	
 NOTE FOR Exe (6): Supp V is a vecsp over R. Then V is not a vecsp over C. See also (9.A.16,17). COMMENT: Supp V is a vecsp over C of dim n. Then V is also a vecsp over R of dim 2n. 	
7,8 Give a non-trivial $U \subseteq \mathbb{R}^2$, U is (a) closd taking add invs and add, but is not a subsp of \mathbb{R}^2 . Let $U = \mathbb{Z}^2$ or \mathbb{Q}^2 , with $0 \in \mathbb{R}^2$ (b) scalar multi, but is not a subsp of \mathbb{R}^2 . Let $U = \{(x,y) \in \mathbb{R}^2 : x = 0 \lor y = 0 \}$	
• Supp U, W, V_1, V_2, V_3 are subsps of V . 15 $U + U \ni u + w \in U$. 16 $U + W \ni u + w = w + u \in W + U$. 17 $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3)$. • $(U + W)_C \ni (u_1 + w_1) + i(u_2 + w_2) = (u_1 + iu_2) + (w_1 + iw_2) \in U_C + W_C$. • $(U \cap W)_C \ni u_1 + iu_2 = w_1 + iw_2 \in U_C \cap W_C$. • $U_C = W_C \iff U = W$. Supp $U_C \ni u + iv \in W_C$. Then $U \ni u, v \in W$.	
18 Does the add on the subsps of V have an add id? Which subsps have add invs? Solus : Supp Ω is the uniq add id. (a) For any subsp U of V , $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$. (b) Supp $U + W = \Omega$. Becs $U + W \supseteq U$, $W \Rightarrow \Omega \supseteq U$, $W \Rightarrow U = W = \Omega = \{0\}$.	
11 Prove the intersec of every collec of subsps of V is a subsp of V . Solus: Supp $\{U_{\alpha}\}_{\alpha\in\Gamma}$ is a collec of subsps of V ; here Γ is an index set. We show $\bigcap_{\alpha\in\Gamma}U_{\alpha}$, which equals the set of vecs in each U_{α} , is a subsp of V . (a) $0\in\bigcap_{\alpha\in\Gamma}U_{\alpha}$. Nonempty. (b) $u,v\in\bigcap_{\alpha\in\Gamma}U_{\alpha}$ \Rightarrow $u+v\in U_{\alpha}$, $\forall \alpha\in\Gamma\Rightarrow u+v\in\bigcap_{\alpha\in\Gamma}U_{\alpha}$. Closd add. (c) $u\in\bigcap_{\alpha\in\Gamma}U_{\alpha}$, $\lambda\in\Gamma\Rightarrow\lambda u\in U_{\alpha}$, $\forall \alpha\in\Gamma\Rightarrow\lambda u\in\bigcap_{\alpha\in\Gamma}U_{\alpha}$. Closd scalar multi. Thus $\bigcap_{\alpha\in\Gamma}U_{\alpha}$ is nonempty subset of V that is closd add and scalar multi.	
• Note For [1.45]: Another proof: Supp $\forall v \in V, \exists ! (u, w) \in U \times W, v = u + w$. Asum non0 $v \in U \cap W$. Then the $(u, w) = (v, 0)$ or $(0, v)$, ctradic the uniques.	
• Tips 1: Supp $U, W \subseteq V$. And U, W, V are vecsps ⇒ U, W are subsps of V . Then $U + W$ is also a subsp of V . Becs $\forall u \in U, w \in U, u + w \in V$ since $u, w \in V$.	
• Supp $U = \{(x, x, y, y)\}, W = \{(x, x, x, y)\} \subseteq \mathbb{F}^4$. Prove $U + W = \{(x, x, y, z)\}$. Solus: Let T denote $\{(x, x, y, z)\}$. By def, $U + W \subseteq T$. And $T \ni (x, x, y, z) \Rightarrow (0, 0, y - x, y - x) + (x, x, x, -y + x + z) \in U + W$. Hence $T \subseteq U + W$.	
21 Supp $U = \{(x, y, x + y, x - y, 2x)\}$. Find a W suth $\mathbf{F}^5 = U \oplus W$. Solus: Let $W = \{(0, 0, z, w, u)\}$. Then $U \cap W = \{0\}$.	

And $\mathbf{F}^5 \ni (x, y, z, w, u) \Rightarrow (x, y, x + y, x - y, 2x) + (0, 0, z - x - y, w - x - y, u - 2x) \in U + W.$

22 Supp $U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5\}.$ Find non0 subsps W_1 , W_2 , W_3 of \mathbf{F}^5 suth $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$. Solus: Let $W_1 = \{(0,0,z,0,0) \in \mathbb{F}^5\} \Rightarrow W_1 \cap U = \{0\}.$ Now $U \oplus W_1 = \{(x, y, z, x - y, 2x) \in \mathbb{F}^5\} = U_1$. Now $U_1 \oplus W_2 = \left\{ \left(x, y, z, w, 2x \right) \in \mathbb{F}^5 \right\} = U_2.$ Let $W_2 = \{(0,0,0,w,0) \in \mathbb{F}^5\} \Rightarrow W_2 \cap U_1 = \{0\}.$ Let $W_3 = \{(0,0,0,0,u) \in \mathbb{F}^5\} \Rightarrow W_3 \cap U_2 = \{0\}.$ Now $U_2 \oplus W_3 = \{(x, y, z, w, u) \in \mathbb{F}^5\} = U_3$. Thus $\mathbf{F}^5 = [(U \oplus W_1) \oplus W_2] \oplus W_3$. **23** Give an exa of vecsps V_1 , V_2 , U suth $V_1 \oplus U = V_2 \oplus U$, but $V_1 \neq V_2$. **Solus:** $V = \mathbb{F}^2$, $U = \{(x, x)\}$, $V_1 = \{(x, 0)\}$, $V_2 = \{(0, x)\}$. • Note For " $\mathbf{C}_V U \cup \{0\}$ ": " $\mathbf{C}_V U \cup \{0\}$ " is supposed to be a subsp W suth $V = U \oplus W$. But if we let $u \in U \setminus \{0\}$ and $w \in W \setminus \{0\}$, then $\begin{cases} w \in C_V U \cup \{0\} \\ u \pm w \in C_V U \cup \{0\} \end{cases} \Rightarrow u \in C_V U \cup \{0\}$. Ctradic. To fix this, denote the set $\{W_1, W_2, \dots\}$ by $S_V U$, where each $W_i \oplus U = V$. • Tips 2: Supp $V_1 \subseteq V_2$ in Exe (23). Prove $V_1 = V_2$. **Solus**: Becs the subset V_1 of vecsp V_2 is closd add and scalar multi, V_1 is a subspace of V_2 . Supp W is suth $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$. If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, ctradic. Hence $W = \{0\}$, $V_1 = V_2$. • Supp V_1, V_2, U_1, U_2 are vecsps, $V_1 \oplus U_1 = V_2 \oplus U_2, V_1 \subseteq V_2, U_2 \subseteq U_1$. $|V_1|$ \dot{U}_1 U_2 Prove or give a countexa: $V_1 = V_2$, $U_1 = U_2$. **Solus:** Let $U_2 = \{0\}$. Give an exa that each of V_1, V_2, U_1 is non0. • Supp the intersec of any two of the vecsps U, W, X, Y is $\{0\}$. Give an exa that $(X \oplus U) \cap (Y \oplus W) \neq \{0\}$. **Solus:** Using notas in Chapter 2. Let $B_X = (e_1), B_U = (e_2 - e_1), B_Y = (), B_W = (e_2).$ • Tips 3: Supp $V = X \oplus Y$, and Z is a subsp of V. Show $X \subseteq Z \Rightarrow Z = X \oplus (Y \cap Z)$. **Solus**: $\forall z \in Z, \exists ! (x,y) \in X \times Y, z = x + y$. Becs $x \in Z \Rightarrow z - x = y \in Z \Rightarrow z \in X + (Y \cap Z)$. $X \cap (Y \cap Z) \subseteq X \cap Y$. • Tips 4: Let V = U + W, $I = U \cap W$, $U = I \oplus X$, $W = I \oplus Y$. Prove $V = I \oplus (X \oplus Y)$. **Solus:** We show $X \cap Y = U \cap Y = W \cap X = \{0\}$ by ctradic. $X \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap X \neq \{0\}, I \cap Y \neq \{0\}.$ $U \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap Y \neq \{0\}$. Simler for $W \cap X$. Thus $I + (X + Y) = (I \oplus X) \oplus Y = I \oplus (X \oplus Y)$. Now we show V = I + (X + Y). $\forall v \in V, v = u + w, \exists (u, w) \in U \times W$ $\Rightarrow \exists \left(i_u, x_u\right) \in I \times X, \left(i_w, y_w\right) \in I \times Y, \ v = \left(i_u + i_w\right) + x_u + y_w \in I + \left(X + Y\right).$

12 Supp U , vv are subsps of v . Prove $U \cup vv$ is a subsp of $v \iff U \subseteq vv$ or $vv \subseteq U$.	
SOLUS: (a) Supp $U \subseteq W$. Then $U \cup W = W$ is a subsp of V .	
(b) Supp $U \cup W$ is a subsp of V . Asum $U \nsubseteq W$, $U \not\supseteq W (U \cup W \neq U \text{ and } W)$.	
Then $\forall a \in U \land a \notin W, \forall b \in W \land b \notin U$, we have $a + b \in U \cup W$.	
$a + b \in U \Rightarrow b = (a + b) + (-a) \in U$, ctradic $\Rightarrow W \subseteq U$. Ctradic asum.	
$a+b \in W \Rightarrow a = (a+b) + (-b) \in W$, ctradic $\Rightarrow U \subseteq W$.	
13 Supp U_1 , U_2 , U_3 are subsps of V , and the union $U_1 \cup U_2 \cup U_3 = \mathcal{U}$ is a subsp of V . Prove one of the subsps contains the other two.	
This exe is not true if we replace \mathbf{F} with a field containing only two elems.	
Solus: Exa: Let $\mathbf{F} = \mathbf{Z}_2$. $U_1 = \{u, 0\}$, $U_2 = \{v, 0\}$, $U_3 = \{v + u, 0\}$. While $\mathcal{U} = \{0, u, v, v + u\}$ is a sub-	sp.
NOTICE that, $U \cup W = V$ is vecsp $\Rightarrow U$, W are subsps of V .	
This trick is invalid: $(A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.	
(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.	
By applying Exe (12) we conclude that one U_j contains the other two. Thus done.	
(II) Asum no one is contained in the union of other two, and no one contains the other two. Say $U_1 \nsubseteq U_2 \cup U_3$ and $U_1 \nsupseteq U_2 \cup U_3$.	
$\exists u \in U_1 \land u \notin U_2 \cup U_3; \ v \in U_2 \cup U_3 \land v \notin U_1. \text{ Let } W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}.$	
Note that $W \cap U_1 = \emptyset$, for if any $v + \lambda u \in W \cap U_1$ then $v + \lambda u - \lambda u = v \in U_1$.	
Now $W \subseteq U_1 \cup U_2 \cup U_3 \Rightarrow W \subseteq U_2 \cup U_3$. $\forall v + \lambda u \in W, v + \lambda u \in U_i, i = 2, 3$.	
If $U_2 \subseteq U_3$ or $U_2 \supseteq U_3$, then $\mathcal{U} = U_1 \cup U_i, i = 2, 3$. By Exe (12) done.	
Othws, both $U_2, U_3 \neq \{0\}$. Becs $W \subseteq U_2 \cup U_3$ has at least three disti elems.	
There must be some U_i that contains at least two disti elems of W .	
$\exists \lambda_1 \neq \lambda_2, \ v + \lambda_1 u \text{ and } v + \lambda_2 u \text{ both in } U_2 \text{ or } U_3 \Rightarrow u \in U_2 \cap U_3, \text{ ctradic.}$	

ENDED

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1 Prove [P] (v_1, v_2, v_3, v_4) spans V \iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4) also spans V [Q].
Solus: Note that V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists a_1, \dots, a_n \in F, v = a_1v_1 + \dots + a_nv_n.
   Asum \forall v \in V, \exists a_1, \dots, a_4, b_1, \dots, b_4 \in F, (that is, if \exists a_i, then we are to find b_i, vice versa)
   v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = b_1 (v_1 - v_2) + b_2 (v_2 - v_3) + b_3 (v_3 - v_4) + b_4 v_4
     = b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4
     = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \dots + a_4)v_4.
                                                                                                                                       • (4E 3, 14) Supp (v_1, \dots, v_m) is a list in V. For each k, let w_k = v_1 + \dots + v_k.
  (a) Show span(v_1, \ldots, v_m) = \text{span}(w_1, \ldots, w_m).
  (b) Show [P](v_1, ..., v_m) is liney indep \iff (w_1, ..., w_m) is liney indep [Q].
Solus:
   (a) Asum a_1v_1 + \dots + a_mv_m = b_1w_1 + \dots + b_mw_m = b_1v_1 + \dots + b_k(v_1 + \dots + v_k) + \dots + b_m(v_1 + \dots + v_m).
        Then a_k = b_k + \dots + b_m; a_{k+1} = b_{k+1} + \dots + b_m \Rightarrow b_k = a_k - a_{k+1}; b_m = a_m. Simly to Exe (1).
   (b) P \Rightarrow Q: b_1w_1 + \dots + b_mw_m = 0 = a_1v_1 + \dots + a_mv_m, where 0 = a_k = b_k + \dots + b_m.
        Q \Rightarrow P: a_1v_1 + \dots + a_mv_m = 0 = b_1w_1 + \dots + b_mw_m = 0, where 0 = b_m = a_m, 0 = b_k = a_k - a_{k+1}.
        Or. By (a), let W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m). Supp (v_1, \dots, v_m) is liney dep.
        By [2.21](b), a list of len (m-1) spans W. X By [2.23], (w_1, ..., w_m) liney indep \Rightarrow m \leq m-1.
        Thus (w_1, ..., w_m) is liney dep. Now rev the roles of v and w.
                                                                                                                                       [Q]
2 (a) [P]
                   A list (v) of len 1 in V is liney indep \iff v \neq 0.
   (b) [P] A list (v, w) of len 2 in V is liney indep \iff \forall \lambda, \mu \in F, v \neq \lambda w, w \neq \mu v.
                                                                                                                                    [Q]
Solus: (a) Q \Rightarrow P : v \neq 0 \Rightarrow \text{ if } av = 0 \text{ then } a = 0 \Rightarrow (v) \text{ liney indep.}
                P \Rightarrow Q : (v) liney indep \Rightarrow v \neq 0, for if v = 0, then av = 0 \Rightarrow a = 0.
                \neg Q \Rightarrow \neg P : v = 0 \Rightarrow av = 0 while we can let a \neq 0 \Rightarrow (v) is liney dep.
                \neg P \Rightarrow \neg Q : (v) \text{ liney dep} \Rightarrow av = 0 \text{ while } a \neq 0 \Rightarrow v = 0.
           (b) P \Rightarrow Q : (v, w) liney indep \Rightarrow if av + bw = 0, then a = b = 0 \Rightarrow no scalar multi.
                Q \Rightarrow P: no scalar multi \Rightarrow if av + bw = 0, then a = b = 0 \Rightarrow (v, w) liney indep.
                \neg P \Rightarrow \neg Q : (v, w) liney dep \Rightarrow if av + bw = 0, then a or b \neq 0 \Rightarrow scalar multi.
                \neg Q \Rightarrow \neg P: scalar multi \Rightarrow if av + bw = 0, then a or b \neq 0 \Rightarrow liney dep.
                                                                                                                                       10 Supp (v_1, ..., v_m) is liney indep in V and w \in V.
    Prove if (v_1 + w, ..., v_m + w) is linely dep, then w \in \text{span}(v_1, ..., v_m).
Solus:
   Note that a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0 \Rightarrow a_1v_1 + \cdots + a_mv_m = -(a_1 + \cdots + a_m)w.
   Then a_1 + \cdots + a_m \neq 0, for if not, a_1v_1 + \cdots + a_mv_m = 0 while a_i \neq 0 for some i, ctradic.
   OR. We prove the ctrapos: Supp w \notin \text{span}(v_1, \dots, v_m). Then a_1 + \dots + a_m = 0.
   Thus a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0. Hence (v_1 + w, \dots, v_m + w) is liney indep.
                                                                                                                                       Or. \exists j \in \{1, ..., m\}, v_j + w \in \text{span}(v_1 + w, ..., v_{j-1} + w). If j = 1 then v_1 + w = 0 and done.
   If j \ge 2, then \exists a_i \in \mathbf{F}, v_i + w = a_1(v_1 + w) + \dots + a_{j-1}(v_{j-1} + w) \iff v_i + \lambda w = a_1v_1 + \dots + a_{j-1}v_{j-1}.
   Where \lambda = 1 - (a_1 + \dots + a_{j-1}). Note that \lambda \neq 0, for if not, v_j + \lambda w = v_j \in \text{span}(v_1, \dots, v_{j-1}), ctradic.
   Now w = \lambda^{-1}(a_1v_1 + \dots + a_{j-1}v_{j-1} - v_j) \Rightarrow w \in \text{span}(v_1, \dots, v_m).
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11 Supp (v_1, ..., v_m) is liney indep in V and w \in V.
    Show [P](v_1, ..., v_m, w) is liney indep \iff w \notin \text{span}(v_1, ..., v_m)[Q].
Solus: Equiv to (v_1, ..., v_m, w) liney dep \iff w \in \text{span}(v_1, ..., v_m). Using [2.21]. Obviously.
                                                                                                                          Note: (a) Supp (v_1, ..., v_m, w) is liney indep. Then (v_1, ..., v_m) liney indep \iff w \notin \text{span}(v_1, ..., v_m).
         (b) Supp (v_1, ..., v_m, w) is liney dep. Then (v_1, ..., v_m) liney indep \iff w \in \text{span}(v_1, ..., v_m).
14 Prove [P] V is infinide \iff \exists seq(v_1, v_2, ...) in V suth each (v_1, ..., v_m) liney indep. [Q]
Solus: P \Rightarrow Q: Supp V is infinide, so that no list spans V. Define the desired seq recurly via:
                     Step 1 Pick a v_1 \neq 0, (v_1) liney indep.
                     Step m Pick a v_m \notin \text{span}(v_1, \dots, v_{m-1}), by Exe (11), (v_1, \dots, v_m) is liney indep.
          \neg P \Rightarrow \neg Q: Supp V is finide and V = \text{span}(w_1, ..., w_m).
                        Let (v_1, v_2, \dots) be a seq in V, then (v_1, v_2, \dots, v_{m+1}) must be liney dep.
          OR. Q \Rightarrow P: Supp there is such a seq.
                           Choose an m. Supp a liney indep list (v_1, ..., v_m) spans V.
                           Simlr to [2.16]. \exists v_{m+1} \in V \setminus \text{span}(v_1, \dots, v_m). Hence no list spans V.
                                                                                                                          17 Prove (p_0, p_1, ..., p_m) cannot be liney indep in \mathcal{P}_m(\mathbf{F}) with each p_k(2) = 0.
Solus:
  Supp (p_0, p_1, ..., p_m) is liney indep. Define p \in \mathcal{P}_m(\mathbf{F}) by p(z) = z.
  NOTICE that \forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \dots + a_m p_m(z), for if not, let z = 2. Thus z \notin \text{span}(p_0, p_1, \dots, p_m).
  Then span(p_0, p_1, \dots, p_m) \subseteq \mathcal{P}_m(\mathbf{F}) while the list (p_0, p_1, \dots, p_m) has len (m+1).
  Hence (p_0, p_1, \dots, p_m) is linely dep. For if not, then becs (1, z, \dots, z^m) of len (m + 1) spans \mathcal{P}_m(\mathbf{F}),
  by the steps in [2.23] trivially, (p_0, p_1, ..., p_m) of len (m + 1) spans \mathcal{P}_m(\mathbf{F}). Ctradic.
                                                                                                                          OR. Becs (1, z, ..., z^m) of len (m + 1) spans \mathcal{P}_m(\mathbf{F}). Then (p_0, p_1, ..., p_m, z) of len (m + 2) is liney dep.
  As shown above, z \notin \text{span}(p_0, p_1, \dots, p_m). And hence by [2.21](a), (p_0, p_1, \dots, p_m) is liney dep.
                                                                                                                   ENDED
2.B
• Note For liney indep seq and [2.34]: "V = \text{span}(v_1, ..., v_n, ...)" is an invalid expr.
 If we allow using "infini list", then we must assure that (v_1, \dots, v_n, \dots) is a spanning "list"
 suth \forall v \in V, \exists smallest n \in \mathbb{N}^+, v = a_1v_1 + \cdots + a_nv_n. Moreover, given a list (w_1, \cdots, w_n, \cdots) in W,
 we can prove \exists ! T \in \mathcal{L}(V, W) with each Tv_k = w_k, which has less restr than [3.5].
  But the key point is, how can we assure that such a "list" exis? [See higher courses]
1 Find all vecsps on whatever F that have exactly one bss.
Solus: The trivial vecsp \{0\} will do. Indeed, the only bss of \{0\} is the empty list ( ).
          Now consider the field \{0,1\} containing only the add id and multi id,
          with 1 + 1 = 0. Then the list (1) is the uniq bss. Now the vecsp \{0, 1\} will do.
          COMMENT: All vecsp on such F of dim 1 will do.
```

Consider other F. Note that this F contains at least and strictly more than 0 and 1. Failed.

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• (4E9) Supp (v_1, ..., v_m) is a list in V. For k \in \{1, ..., m\}, let w_k = v_1 + \cdots + v_k.
            Show [P] B_V = (v_1, ..., v_m) \iff B_V = (w_1, ..., w_m). [Q]
Solus: Notice that B_U = (u_1, ..., u_n) \iff \forall u \in U, \exists ! a_i \in F, u = a_1u_1 + \cdots + a_nu_n.
   P \Rightarrow Q: \forall v \in V, \exists ! a_i \in \mathbf{F}, \ v = a_1 v_1 + \dots + a_m v_m \Rightarrow v = b_1 w_1 + \dots + b_m w_m, \exists ! b_k = a_k - a_{k+1}, b_m = a_m.
   Q\Rightarrow P:\forall v\in V, \exists !\, b_i\in \mathbf{F},\ v=b_1w_1+\cdots+b_mw_m\Rightarrow v=a_1v_1+\cdots+a_mv_m, \exists !\, a_k=\textstyle\sum_{j=k}^m b_j.
                                                                                                                                                                COMMENT: OR. Using [3.C \text{ Note For } [3.30, 32](a)].
8 Supp B_{II} = (u_1, ..., u_m), B_W = (w_1, ..., w_n).
   Prove V = U \oplus W \iff B_V = (u_1, \dots, u_m, w_1, \dots, w_n).
Solus: \forall v \in V, \exists ! u \in U, w \in W \Rightarrow \exists ! a_i, b_i \in F, v = u + w = \sum_{i=1}^m a_i u_i + \sum_{i=1}^n b_i w_i.
             Or. V = \text{span}(u_1, ..., u_m) \oplus \text{span}(w_1, ..., w_n) = \text{span}(u_1, ..., u_m, w_1, ..., w_n).
                    Note that \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0 \Rightarrow \sum_{i=1}^{m} a_i u_i = -\sum_{i=1}^{n} b_i w_i \in U \cap W = \{0\}.
                                                                                                                                                               • (9.A.3,4 Or 4E 11) Supp V is on R, and v_1, ..., v_n \in V. Let B = (v_1, ..., v_n).
  (a) Show [P] B is liney indep in V \iff B is liney indep in V_C. [Q]
  (b) Show [P] B spans V \iff B spans V_C. [Q]
Solus:
   (a) P \Rightarrow Q: Note that each v_k \in V_C. Supp \lambda_1 v_1 + \cdots + \lambda_n v_n = 0 with F = C.
                      Then (\text{Re}\lambda_1)v_1 + \cdots + (\text{Re}\lambda_n)v_n = 0 \Rightarrow \text{each Re}\lambda_i = 0, siml for \text{Im}\lambda_i.
         Q \Rightarrow P: If \lambda_k \in \mathbb{R} with \lambda_1 v_1 + \cdots + \lambda_n v_n = 0, then each \operatorname{Re} \lambda_k = \lambda_k = 0.
         \neg P \Rightarrow \neg Q : \exists v_i = a_{i-1}v_{i-1} + \dots + a_1v_1 \in V_C.
         \neg Q \Rightarrow \neg P: \ \exists \ v_i = \lambda_{i-1}v_{i-1} + \dots + \lambda_1v_1 \in V \Rightarrow v_i = (\operatorname{Re}\lambda_{i-1})v_{i-1} + \dots + (\operatorname{Re}\lambda_1)v_1 \in V.
   (b) P \Rightarrow Q: \forall u + iv \in V_C, u, v \in V \Rightarrow \exists a_i, b_i \in \mathbb{R}, u + iv = \sum_{i=1}^n (a_i + ib_i)v_i.
         Q \Rightarrow P: \ \forall v \in V, \exists a_i + ib_i \in C, \ v + i0 = \left(\sum_{i=1}^n a_i v_i\right) + i\left(\sum_{i=1}^n b_i v_i\right) \Rightarrow v \in \operatorname{span}(v_1, \dots, v_m).
         \neg P \Rightarrow \neg Q : \exists v \in V, v \notin \operatorname{span} B \text{ with } \mathbf{F} = \mathbf{R} \Rightarrow v + \mathrm{i} 0 \notin \operatorname{span} B \text{ with } \mathbf{F} = \mathbf{C}.
         \neg Q \Rightarrow \neg P : \exists u + iv \in V_C, u + iv \notin \operatorname{span} B \Rightarrow (\operatorname{Re} 1)u + (\operatorname{Re} i)v = u \text{ or } (\operatorname{Im} 1)u + (\operatorname{Im} i)v = v \notin \operatorname{span} B. \quad \Box
• Tips: Supp dim V = n, and U is a subsp of V with U \neq V.
            Prove \exists B_V = (v_1, \dots, v_n) suth each v_k \notin U.
  Note that U \neq V \Rightarrow n \geqslant 1. We will construct B_V via the following process.
  Step 1. \exists v_1 \in V \setminus U \Rightarrow v_1 \neq 0. If span(v_1) = V then we stop.
  Step k. Supp (v_1, ..., v_{k-1}) is liney indep in V, each of which belongs to V \setminus U.
               Note that span(v_1, \dots, v_{k-1}) \neq V. And if span(v_1, \dots, v_{k-1}) \cup U = V, then by (1.C.12),
               becs \operatorname{span}(v_1, \dots, v_{k-1}) \not\subseteq U, U \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) \Rightarrow \operatorname{span}(v_1, \dots, v_{k-1}) = V.
              Hence becs span(v_1, \dots, v_{k-1}) \neq V, it must be case that span(v_1, \dots, v_{k-1}) \cup U \neq V.
               Thus \exists v_k \in V \setminus U suth v_k \notin \text{span}(v_1, \dots, v_{k-1}).
               By (2.A.11), (v_1, \dots, v_k) is liney indep in V. If span(v_1, \dots, v_k) = V, then we stop.
  Becs V is finide, this process will stop after n steps.
                                                                                                                                                                OR. Supp U \neq \{0\}. Let B_U = (u_1, \dots, u_m). Extend to a bss (u_1, \dots, u_n) of V.
         Then let B_V = (u_1 - u_k, ..., u_m - u_k, u_{m+1}, ..., u_k, ..., u_n).
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15 Supp dim V = n \ge 1. Prove \exists 1-dim subsps V_1, \dots, V_n suth V = V_1 \oplus \dots \oplus V_n.
SOLUS: Supp B_V = (v_1, ..., v_n). Let each V_i = \text{span}(v_i).
            Then \forall v \in V, \exists ! a_i \in F, v = a_1v_1 + \dots + a_nv_n \Rightarrow \exists ! u_i \in V_i, v = u_1 + \dots + u_n
                                                                                                                                                • Note For Exe (15): Supp \ v \in V \setminus \{0\}. Prove \exists B_V = (v_1, \dots, v_n), v = v_1 + \dots + v_n.
Solus: If n = 1 then let v_1 = v and done. Supp n > 1.
            Extend (v) to a bss (v, v_1, ..., v_{n-1}) of V. Let v_n = v - v_1 - \cdots - v_{n-1}.
            \mathbb{X} span(v, v_1, \dots, v_{n-1}) = \operatorname{span}(v_1, \dots, v_n). Hence (v_1, \dots, v_n) is also a bss of V.
                                                                                                                                                COMMENT: Let B_V = (v_1, ..., v_n) and supp v = u_1 + ... + u_n, where each u_i = a_i v_i \in V_i.
                But (u_1, ..., u_n) might not be a bss, becs there might be some u_i = 0.
1 [CORO for [2.38,39]] Supp U is a subsp of V suth dim V = \dim U. Then V = U.
Solus: Let B_U = (u_1, ..., u_m). Then m = \dim V. X, u_i \in V. By [2.39], B_V = (u_1, ..., u_m).
                                                                                                                                                • Let v_1, \ldots, v_n \in V and dim span(v_1, \ldots, v_n) = n. Then (v_1, \ldots, v_n) is a bss of span(v_1, \ldots, v_n).
  Notice that (v_1, \dots, v_n) is a spanning list of span(v_1, \dots, v_n) of len n = \dim \text{span}(v_1, \dots, v_n).
9 Supp (v_1, \ldots, v_m) is liney indep in V, w \in V. Prove \dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \ge m - 1.
Solus: Using (2.A.10, 11).
   Note that each v_i - v_1 = (v_i + w) - (v_1 + w) \in \text{span}(v_1 + w, ..., v_n + w).
   (v_1,\ldots,v_m) liney indep \Rightarrow (v_1,v_2-v_1,\ldots,v_m-v_1) liney indep \Rightarrow (v_2-v_1,\ldots,v_m-v_1) liney indep.
   \mathbb{Z} If w \notin \text{span}(v_1, \dots, v_m). Then (v_1 + w, \dots, v_m + w) is liney indep. of len (m-1)
   Hence m \ge \dim \operatorname{span}(v_1 + w, \dots, v_m + w) \ge m - 1.
                                                                                                                                                • (4E 16) Supp V is finide, U is a subsp of V with U \neq V. Let n = \dim V, m = \dim U.
            Prove \exists (n-m) subsps U_1, \ldots, U_{n-m}, each of dim (n-1), suth \bigcap_{i=1}^{n} U_i = U.
Solus: Let B_U = (v_1, \dots, v_m), B_V = (v_1, \dots, v_m, u_1, \dots, u_{n-m}).
           Define each U_i = \operatorname{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m}) \Rightarrow U \subseteq U_i.
           And becs \forall v \in \bigcap_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \dots + b_{n-m} u_{n-m} \in U_i \Rightarrow \text{each } b_i = 0 \Rightarrow v \in U.
           Hence \bigcap_{i=1}^{n-m} U_i \subseteq U.
                                                                                                                                                14 Supp V_1, \ldots, V_m are finide. Prove \dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.
Solus: For each V_i, let B_{V_i} = \mathcal{E}_i. Then V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m); dim V_i = \operatorname{card} \mathcal{E}_i.
   Now dim(V_1 + \cdots + V_m) = dim span(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leq \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leq \operatorname{card}(\mathcal{E}_1 + \cdots + \operatorname{card}(\mathcal{E}_m))
Coro: V_1 + \cdots + V_m is direct
          \Leftrightarrow For each k \in \{1, ..., m-1\}, (V_1 \oplus \cdots \oplus V_k) \cap V_{k+1} = \{0\}, (\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset
          \iff dim span(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card}\mathcal{E}_1 + \cdots + \operatorname{card}\mathcal{E}_m
          \iff dim(V_1 \oplus \cdots \oplus V_m) = \dim V_1 + \cdots + \dim V_m.
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\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3
                               -\dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).
Solus:
  (1) |(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|.
  (2) |(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|.
  Thus |(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|.
  Becs (V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2.
  \dim(V_1 + V_2 + V_3) = \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3)
                         = \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1)
                                                                                        (2)
                         = \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2)  (3).
  Generally, (X + Y) \cap Z \neq (X \cap Z) + (Y \cap Z). Exa: X = \{(x,0)\}, Y = \{(0,y)\}, Z = \{(z,z)\} \subseteq F^2.
  COMMENT: If X \subseteq Y, then (X + Y) \cap Z = Y \cap Z; \dim(X + Y + Z) = \dim Y + \dim Z - \dim(Y \cap Z),
                and the wrong formula holds. Simlr for Y \subseteq Z, X \subseteq Z, and X, Y \subseteq Z.
  Note: However, it's true that (X + Y) \cap Z \supseteq (X \cap Z) + (Y \cap Z) = (X + (Y \cap Z)) \cap Z.
           Becs (X \cap Z) + (Y \cap Z) \ni v = x + y = z_1 + z_2 \in (X + (Y \cap Z)) \cap Z \Rightarrow v \in (X + Y) \cap Z.
• Tips: Becs dim (V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim(V_1 + (V_2 \cap V_3)).
         And dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3). We have (1), and (2), (3) simlr.
         (1) \dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)).
         (2) \dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_3) - \dim(V_2 + (V_1 \cap V_3)).
         (3) \dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim(V_3 + (V_1 \cap V_2)).
• Supp V_1, V_2, V_3 are subsps of V with
 (a) dim V = 10, dim V_1 = \dim V_2 = \dim V_3 = 7. Prove V_1 \cap V_2 \cap V_3 \neq \{0\}.
      By Tips, \dim(V_1 \cap V_2 \cap V_3) \ge \dim V_1 + \dim V_2 + \dim V_3 - 2\dim V > 0.
 (b) dim V_1 + dim V_2 + dim V_3 > 2 dim V. Prove V_1 \cap V_2 \cap V_3 \neq \{0\}.
      By Tips, \dim(V_1 \cap V_2 \cap V_3) \ge 2\dim V - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)) \ge 0.
                                                                                                                      • Supp \mathcal{C} is a collectof k-dim subsps of V with any two of them have a (k-1)-dim intersec.
 Prove either all contain a (k-1)-dim intersec, or all contained in a (k+1)-dim subsp.
Solus: If V is finide and dim V = k, then \mathcal{C} = \{V\}, done. We use induc on k. (i) k = 1. Immed.
          (ii) k > 1. Asum it holds for k - 1. If \exists common (k - 1)-dim intersec, then done.
              Othws, we show all X \in \mathcal{C} are contained in a (k + 1)-dim subsp.
              Supp U, W \in \mathcal{C} \Rightarrow \dim(U + W) = k + 1. Then for X \in \mathcal{C}, X \cap U, X \cap W are (k - 1)-dim.
              Now by asum, \dim(X \cap U + X \cap W) = k \Rightarrow X = (X \cap U) + (X \cap W) \Rightarrow X \subseteq U + W. \square
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17 Supp V_1 , V_2 , V_3 are subsps of a finide vecsp. Explain and give a countexa:

ENDED

3·A 注意: 这里我将 3.B 的值空间、零空间、单满射、和 3.D 的可逆性定义前置; 仅涉及概念。	
• TIPS 1: $T: V \to W$ is liney $\iff \begin{vmatrix} (-) \ \forall v, u \in V, T(v+u) = Tv + Tu; \\ (\underline{-}) \ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv). \end{vmatrix} \iff T(v+\lambda u) = Tv + \lambda v$.Ти.
Note: Supp V is a vecsp. For $U \subseteq V$, U is a subsp of $V \iff \forall u_1, u_2 \in U, \lambda \in \mathbb{F}, u_1 + \lambda u_2 \in U$.	
• (3.E.1) A function $T: V \to W$ is liney \iff The graph of T is a subspace of $V \times W$.	
• Tips 2: Supp $T \in \mathcal{L}(V, W)$. Prove $Tv \neq 0 \Rightarrow v \neq 0$.	
Solus: Asum $v = 0$. Then $Tv = T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$. Or. $T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$. Ctradic.	
11 Supp U is a subsp of V and $S \in \mathcal{L}(U, W)$. Prove $\exists T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U.$ (Or. $\exists T \in \mathcal{L}(V, W), T _U = S.$) In other words, every liney map on a subsp of V can be extended to a liney map on the entire V . Solus: Supp W is suth $V = U \oplus W$. Then $\forall v \in V, \exists ! u_v \in U, w_v \in W, v = u_v + w_v$.	
Define $T \in \mathcal{L}(V, W)$ by $T(u_v + w_v) = Su_v$. Or. $[Finide\ Req]$ Define by $T(\sum_{i=1}^m a_i u_i) = \sum_{i=1}^n a_i Su_i$. Let $B_V = (\overbrace{u_1, \dots, u_n}, \dots, u_m)$.	
OR. [Finite Req.] Define by $I\left(\sum_{i=1}u_iu_i\right) = \sum_{i=1}u_iSu_i$. Let $B_V = (u_1, \dots, u_n, \dots, u_m)$.	
• Note For Restr: U is a subsp of V . (a) $(T + \lambda S) _{U} = T _{U} + \lambda S _{U}$. (b) $(ST) _{U} = ST _{U}$.	
• TIPS 3: $T \in \mathcal{L}(V, W)$. (a) If U is a subsp of W . Then range $T \subseteq U \iff T \in \mathcal{L}(V, U) \subseteq \mathcal{L}(V, W)$. (b) If U is a subsp of V . Then $U \subseteq \operatorname{null} T \iff T _U = 0$.	
• (4E 4.3) Supp $\mathbf{F} = \mathbf{C}$, $\varphi \in \mathcal{L}(V, \mathbf{F})$, $\sigma = \text{Re} \circ \varphi$. Show all $\varphi(v) = \sigma(v) - i\sigma(iv)$. Solus: $\varphi(v) = \sigma(v) + i \operatorname{Im} \varphi(v)$. $\mathbb{Z} \operatorname{Re} \varphi(iv) = \operatorname{Re}(i\varphi(v)) = -\operatorname{Im} \varphi(v) = \sigma(iv)$.	
• (9.A.5) Supp V is on \mathbb{R} , and $S, T \in \mathcal{L}(V, W)$. Prove $(S + \lambda T)_{\mathbb{C}} = S_{\mathbb{C}} + \lambda T_{\mathbb{C}}$. Solus: $(S + \lambda T)_{\mathbb{C}}(u + iv) = (S + \lambda T)(u) + i(S + \lambda T)(v)(S_{\mathbb{C}} + \lambda T_{\mathbb{C}})(u + iv)$.	
• Supp U, V, W are on $\mathbf{R}, S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Prove $(ST)_{\mathbf{C}} = S_{\mathbf{C}}T_{\mathbf{C}}$. Solus: $\forall u + \mathrm{i}x \in U_{\mathbf{C}}, (ST)_{\mathbf{C}}(u + \mathrm{i}x) = STu + \mathrm{i}STx = S_{\mathbf{C}}(Tu + \mathrm{i}Tx) = (S_{\mathbf{C}}T_{\mathbf{C}})(u + \mathrm{i}x)$.	
• (9.A.2,6 OR 4E 3.B.33) $Supp\ V$, W are on \mathbb{R} , and $T \in \mathcal{L}(V,W)$. $Show$ (a) $T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}}, W_{\mathbb{C}})$. (b) $null(T_{\mathbb{C}}) = (null\ T)_{\mathbb{C}}$, $range(T_{\mathbb{C}}) = (range\ T)_{\mathbb{C}}$. (c) $T_{\mathbb{C}}$ is inv $\Leftrightarrow T$ is inv. Solus: (a) $T_{\mathbb{C}}((u_1 + iv_1) + (x + iy)(u_2 + iv_2)) = T(u_1 + xu_2 - yv_2) + iT(v_1 + xv_2 + yu_2)$ $= T_{\mathbb{C}}(u_1 + iv_1) + (x + iy)T_{\mathbb{C}}(u_2 + iv_2).$ (b) $u + iv \in null(T_{\mathbb{C}}) \Leftrightarrow u, v \in null\ T \Leftrightarrow u + iv \in (null\ T)_{\mathbb{C}}$. $w + ix \in range(T_{\mathbb{C}}) \Leftrightarrow w, x \in range\ T \Leftrightarrow w + ix \in (range\ T)_{\mathbb{C}}.$ (c) $\forall w, x \in W, \exists ! u, v \in V, T_{\mathbb{C}}(u + iv) = w + ix \Leftrightarrow Tu = w, Tv = x$. Or. By (b).	
7 Supp dim $V = 1$ and $T \in \mathcal{L}(V)$. Prove $\exists \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$. Solus: Let u be a non0 vec in $V \Rightarrow B_V = (u)$. Becs $Tu \in V \Rightarrow Tu = \lambda u$ for some λ .	
Supp $v \in V \Rightarrow v = au$, $\exists ! a \in F$. Then $Tv = T(au) = \lambda au = \lambda v$.	

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3 Supp T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m). Prove \exists A_{i,k} \in \mathbf{F}, T(x_1, ..., x_n) = (\sum_{i=1}^n A_{1,i} x_i, ..., \sum_{i=1}^n A_{m,i} x_i).
Solus: Let T(1,0,0,\ldots,0,0) = (A_{1,1},\ldots,A_{m,1}),
                T(0,1,0,\ldots,0,0) = (A_{1,2},\ldots,A_{m,2}),
                T(0,0,0,\dots,0,1) = (A_{1,n},\dots,A_{m,n}).
                                                                                                                                        8 Give a map \varphi: \mathbb{R}^2 \to \mathbb{R} suth \forall a \in \mathbb{R}, v \in \mathbb{R}^2, \varphi(av) = a\varphi(v) but \varphi is not liney.
Solus: Define T(x,y) = \begin{cases} x+y, & \text{if } (x,y) \in \text{span}(3,1), \\ 0, & \text{othws.} \end{cases}
                                                                                  Or. Define T(x,y) = \sqrt[3]{(x^3 + y^3)}.
                                                                                                                                        9 Give a map \varphi: \mathbb{C} \to \mathbb{C} suth \forall w, z \in \mathbb{C}, \varphi(w+z) = \varphi(w) + \varphi(z) but \varphi is not liney.
Solus: Define \varphi(u+iv) = u = \text{Re}(u+iv) OR. Define \varphi(u+iv) = v = \text{Im}(u+iv).
                                                                                                                                        • Prove if q \in \mathcal{P}(\mathbf{R}) and T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R}) is defined by Tp = q \circ p, then T is not liney.
SOLUS: Composition and product are not the same in \mathcal{P}(F).
   NOTICE that (p \circ q)(x) = p(q(x)), while (pq)(x) = p(x)q(x) = q(x)p(x).
   Becs in general, [q \circ (p_1 + \lambda p_2)](x) = q(p_1(x) + \lambda p_2(x)) \neq (qp_1)(x) + \lambda (qp_2)(x).
   Exa: Let q be defined by q(x) = x^2, then q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2.
                                                                                                                                        10 Supp U is a subsp of V with U \neq V, and S \in \mathcal{L}(U, W) is non0.
    Prove T is not liney, where we define T: V \to W by Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \setminus U. \end{cases}
Solus: Asum T is liney. Supp v \in V \setminus U, u \in U suth Su \neq 0.
           Then v + u \in V \setminus U, for if not, v = (v + u) - u \in U;
           while T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0. Ctradic.
                                                                                                                                        • (3.B.7) Supp 2 \le \dim V = n \le m = \dim W, if W is finide.
           Show U = \{T \in \mathcal{L}(V, W) : T \text{ is not inje}\}\ is not a subsp of \mathcal{L}(V, W).
Solus: The set of all inje T \in \mathcal{L}(V, W) is a not subspecither.
   Let (v_1, \ldots, v_n) be a bss of V, (w_1, \ldots, w_m) be liney indep in W. [2 \le n \le m]
  Define T_1 \in \mathcal{L}(V, W) as T_1: v_1 \mapsto 0, v_2 \mapsto w_2, v_i \mapsto w_i.

Define T_2 \in \mathcal{L}(V, W) as T_2: v_1 \mapsto w_1, v_2 \mapsto 0, v_i \mapsto w_i, i = 3, ..., n.

Thus T_1 + T_2 \notin U. \square
COMMENT: If dim V = 0, then V = \{0\} = \text{span}(). \forall T \in \mathcal{L}(V, W), T \text{ is inje. Hence } U = \emptyset.
               If dim V = 1, then V = \text{span}(v_0). Thus U = \text{span}(T_0), where \forall v \in V, T_0 v = 0 \Rightarrow T_0 = 0.
• (3.B.8) Supp 2 \le \dim W = m \le \dim V, if V is finide.
           Show U = \{T \in \mathcal{L}(V, W) : T \text{ is not surj}\}\ is not a subsp of \mathcal{L}(V, W).
Solus: The set of all surj T \in \mathcal{L}(V, W) is not a subspecifier. Using the generalized version of [3.5].
   Let (v_1, \ldots, v_n) be liney indep in V, (w_1, \ldots, w_m) be a bss of W. [n \in \{m, m+1, \ldots\}; 2 \le m \le n]
   Define T_1 \in \mathcal{L}(V, W) as T_1: v_1 \mapsto 0, v_2 \mapsto w_2, v_i \mapsto w_i, v_{m+i} \mapsto 0.
   Define T_2 \in \mathcal{L}(V, W) as T_2: v_1 \mapsto w_1, v_2 \mapsto 0, v_i \mapsto w_i, v_{m+i} \mapsto 0.
   ( For each j=2,\ldots,m;\ i=1,\ldots,n-m, if V is finide, othws let i\in\mathbb{N}^+. ) Thus T_1+T_2\notin U.
                                                                                                                                        COMMENT: If dim W = 0, then W = \{0\} = \text{span}(). \forall T \in \mathcal{L}(V, W), T \text{ is surj. Hence } U = \emptyset.
               If dim W = 1, then W = \text{span}(w_0). Thus U = \text{span}(T_0), where each T_0v_i = 0 \Rightarrow T_0 = 0.
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4 Supp T \in \mathcal{L}(V, W), (Tv_1, ..., Tv_m) liney indep in W. Prove (v_1, ..., v_m) liney indep.
Solus: Let a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1Tv_1 + \cdots + a_mTv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0.
                                                                                                                                                 12 Supp non0 V is finide and W is infinide. Prove \mathcal{L}(V,W) is infinide.
Solus: Let B_V = (v_1, ..., v_n). Let (w_1, ..., w_m) be liney indep in W for any m \in \mathbb{N}^+.
   Define T_{x,y}: V \to W by T_{x,y}(v_z) = \delta_{z,x} w_y, \forall x \in \{1, ..., n\}, y \in \{1, ..., m\}, \text{ where } \delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}
   \forall v = \sum_{i=1}^{n} a_i v_i, \ u = \sum_{i=1}^{n} b_i v_i, \ \lambda \in \mathbf{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) w_y = T_{x,y}(v) + \lambda T_{x,y}(u).
   Linity checked. Now supp a_1T_{x,1} + \cdots + a_mT_{x,m} = 0.
   Then (a_1T_{x,1} + \dots + a_mT_{x,m})(v_x) = 0 = a_1w_1 + \dots + a_mw_m \Rightarrow a_1 = \dots = a_m = 0. \mathbb{X} m arb.
   Thus (T_{x,1}, ..., T_{x,m}) is a liney indep list in \mathcal{L}(V, W) for any x and len m. Hence by (2.A.14).
                                                                                                                                                 13 Supp (v_1, ..., v_m) is linely dep in V and W \neq \{0\}.
     Prove \exists w_1, \dots, w_m \in W, \nexists T \in \mathcal{L}(V, W) suth Tv_k = w_k, \forall k = 1, \dots, m.
SOLUS:
   We prove by ctradic. By liney dep lemma, \exists j \in \{1, ..., m\}, v_i \in \text{span}(v_1, ..., v_{i-1}).
   Supp a_1v_1 + \dots + a_mv_m = 0, where a_j \neq 0. Now let w_j \neq 0, while w_1 = \dots = w_{j-1} = w_{j+1} = w_m = 0.
   Define T \in \mathcal{L}(V, W) with each Tv_k = w_k. Then T(a_1v_1 + \dots + a_mv_m) = 0 = a_1w_1 + \dots + a_mw_m.
   And 0 = a_i w_i while a_i \neq 0 and w_i \neq 0. Ctradic.
                                                                                                                                                 OR. We prove the ctrapos: Supp \forall w_1, \dots, w_m \in W, \exists T \in \mathcal{L}(V, W), each Tv_k = w_k.
   Now we show (v_1, ..., v_n) is liney indep. Supp \exists a_i \in \mathbf{F}, a_1v_1 + \cdots + a_nv_n = 0.
   Choose one w \in W \setminus \{0\}. By asum, for (\overline{a_1}w, ..., \overline{a_m}w), \exists T \in \mathcal{L}(V, W), each Tv_k = \overline{a_k}w.
   Now we have 0 = T(\sum_{k=1}^{m} a_k v_k) = \sum_{k=1}^{m} a_k T v_k = \sum_{k=1}^{m} a_k \overline{a_k} w = (\sum_{k=1}^{m} |a_k|^2) w.
   Then \sum_{k=1}^{m} |a_k|^2 = 0. Thus a_1 = \cdots = a_m = 0. Hence (v_1, \ldots, v_n) is liney indep.
                                                                                                                                                 • (4E 11) Supp V is finide, T \in \mathcal{L}(V) is suth \forall S \in \mathcal{L}(V), ST = TS. Prove \exists \lambda \in \mathbf{F}, T = \lambda I.
Solus: If V = \{0\}, then done. Now supp V \neq \{0\}.
   Asum \forall v \in V, (v, Tv) is linely dep, then \exists \lambda_v \in F, Tv = \lambda_v v.
   To prove \lambda_v is indep of v, we discuss in two cases:
  \begin{array}{l} (-) \text{ If } (v,w) \text{ is liney indep, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \\ (=) \text{ Othws, supp } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v) w \end{array} \right\} \Rightarrow \lambda_w = \lambda_v.
   Now we prove the asum. Asum \exists v \in V, (v, Tv) is liney indep. Let B_V = (v, Tv, u_1, \dots, u_n).
   Define S \in \mathcal{L}(V) by S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0. Ctradic.
                                                                                                                                                 Or. Let B_V = (v_1, \dots, v_m). Define \varphi \in \mathcal{L}(V, \mathbf{F}) by \varphi(v_1) = \dots = \varphi(v_m) = 1.
         Supp v \in V. Define S_v \in \mathcal{L}(V) by S_v(u) = \varphi(u)v.
          Then Tv = T(\varphi(v_1)v) = T(S_vv_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v.
                                                                                                                                                 Or. Define S_k\left(\sum_{i=1}^n a_i v_i\right) = a_k v_k. Then S_k v = v \iff \exists ! a_k \in \mathbf{F}, v = a_k v_k.
         Hence S_k(Tv_k) = T(S_kv_k) = Tv_k \Rightarrow Tv_k = a_kv_k.
         Define A^{(j,k)} \in \mathcal{L}(V) by A^{(j,k)}v_j = v_k, A^{(j,k)}v_k = v_j, A^{(j,k)}v_x = 0, x \neq j, k.
                    A^{(j,k)}Tv_j = TA^{(j,k)}v_j = Tv_k = a_k v_k
A^{(j,k)}Tv_j = A^{(j,k)}a_j v_j = a_j A^{(j,k)}v_j = a_j v_k
\Rightarrow a_k = a_j. \text{ Hence } a_k \text{ is indep of } v_k.
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• (4E 17) Supp V is finide. Show all two-sided ideals of \mathcal{L}(V) are \{0\} and \mathcal{L}(V).
  A subsp \mathcal{E} of \mathcal{L}(V) is called a two-sided ideal of \mathcal{L}(V) if TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V).
Solus: If \mathcal{E} = \{0\}, then done. Supp 0 \neq S \in \mathcal{E}, a two-sided ideal of \mathcal{L}(V). Let B_V = (v_1, \dots, v_n).
   Define R_{x,y} \in \mathcal{L}(V): v_x \mapsto v_y, v_z \mapsto 0 \ (z \neq x). Or. R_{x,y}v_z = \delta_{z,x}v_y. Asum each R_{x,y} \in \mathcal{E}.
   Then (R_{1,1} + \cdots + R_{n,n})v_j = v_j \Rightarrow \sum_{r=1}^n R_{r,r} = I \Rightarrow \mathcal{L}(V) \ni T = I \circ T = T \circ I \in \mathcal{E}.
   Or. Let each Tv_j = w_j = A_{1,j}v_1 + \cdots + A_{n,j}v_n \Rightarrow T = \sum_{x=1}^n \sum_{y=1}^n A_{y,x}R_{x,y} \in \mathcal{E}. Now we prove the asum.
   Supp Sv_i \neq 0 and Sv_i = a_1v_1 + \cdots + a_nv_n, where a_k \neq 0.
   Then (R_{k,y}S)v_i = a_k v_y \Rightarrow ((R_{k,y}S) \circ R_{x,i})v_z = \delta_{z,x}(a_k v_y), for all x, y \in \{1, ..., n\}.
   Thus R_{k,y}SR_{x,i} = a_kR_{x,y}. Now S \in \mathcal{E} \Rightarrow R_{k,y}S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}.
                                                                                                                                          COMMENT: Not true if infinide. Consider the subsp X = \{T \in \mathcal{L}(V) : \text{range } T \text{ is finide} \}.
   For any T \in X, \forall E \in \mathcal{L}(V), range TE \subseteq \text{range } T; range ET = \text{span}(Ew_1, \dots, Ew_n) \Rightarrow TE, ET \in X.
• (4E 3.B.32) Supp dim V = n. Supp \varphi : \mathcal{L}(V) \to \mathbf{F} is liney.
                Show if \forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T), then \varphi = 0.
Solus: Using notas in (4E 17) and Note For [3.60].
           Supp \varphi \neq 0 \Rightarrow \exists i, j \in \{1, ..., n\}, \varphi(R_{i,j}) \neq 0. Becs R_{i,j} = R_{x,j} \circ R_{i,x}, \forall x = 1, ..., n
           \Rightarrow \varphi(R_{i,i}) = \varphi(R_{x,i}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,i}) \neq 0 \text{ and } \varphi(R_{i,x}) \neq 0.
           Again, becs R_{i,x} = R_{y,x} \circ R_{i,y}, \forall y = 1, ..., n. Thus \varphi(R_{y,x}) \neq 0, \forall x, y = 1, ..., n.
           Let k \neq i, j \neq l and then \varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})
           \Rightarrow \varphi(R_{lk}) = 0 \text{ or } \varphi(R_{i,i}) = 0. Ctradic.
                                                                                                                                          OR. Becs \exists S, T \in \mathcal{L}(V), ST - TS \neq 0. While \varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0.
                Note that \forall E \in \text{null } \varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \text{null } \varphi.
                Thus null \varphi is a nonzero two-sided ideal of \mathcal{L}(V). By (4E 17).
                                                                                                                                          • (4E 1.B.7) Supp V \neq \emptyset and W is a vecsp. Let W^V = \{ f : V \rightarrow W \}.
  (a) Define a natural add and scalar multi on W^V. (b) Prove W^V is a vecsp with these defs.
Solus:
   (a) W^V \ni f + g : x \to f(x) + g(x); where f(x) + g(x) is the vec add on W.
        W^V \ni \lambda f: x \to \lambda f(x); where \lambda f(x) is the scalar multi on W.
   (b) Commu: (f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x).
        Assoc: ((f+g)+h)(x) = (f(x)+g(x))+h(x)
                                          = f(x) + (g(x) + h(x)) = (f + (g + h))(x).
        Add Id: (f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x).
        Add Inv: (f+g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0 = 0(x).
        Multi Id: (1f)(x) = 1f(x) = f(x). (NOTICE that the smallest F is \{0,1\}.)
        Distr: (a(f+g))(x) = a(f+g)(x) = a(f(x) + g(x))
                                                         = af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x).
                 Simlr, ((a+b)f)(x) = (af+bf)(x).
        So far, we have used the same properties in W. [If W^V is a vecsp, then W must be a vecsp.]
5 Becs \mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is liney}\}\ is a subsp of W^V, \mathcal{L}(V, W) is a vecsp.
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Solus: We can assure that $\{0\} \subseteq \mathcal{L}(V, W), \{0\} \subseteq V, \{0\} \subseteq W$. (I) If $W^V = \{0\}$. Then $\mathcal{L}(V, W) = \{0\}$. And $W = \{0\}$, for if not, $\exists w \in W \setminus \{0\}$, define a map f by f(x) = w, $\forall x \in V$. And *V* might not be a vecsp. Exa: Let $V = \mathbb{R}$, but with the scalar multi defined by $a \odot v = 0$. (II) If W^V is a non0 vecsp \iff W is a non0 vecsp. (a) If $\mathcal{L}(V, W) = \{0\}$, then by Exa (I), V might not be vecsp. (b) If not, then $\exists T \in \mathcal{L}(V, W), T \neq 0$. Which means $\exists v \in V, Tv \neq 0 \Rightarrow v \neq 0$. TODO Then both *W* and *V* have a non0 elem. (i) If \exists inje $T \in \mathcal{L}(V, W)$, then $T(u+v) = T(v+u) \Rightarrow u+v = v+u$. etc. Hence V is a vecsp. (ii) If not, then we cannot guarantee that *V* is a vecsp. Exa: ??? (III) If W^V is not a vecsp \iff W is not a vecsp. (a) If $\mathcal{L}(V, W) = \{0\}$, then by Exa (I), V might not be vecsp. (b) If not. • Note For F^S : Supp $S \neq \emptyset$, $C_S = \{ f \in \mathbf{F}^S : \exists \text{ finily many } x, \text{ suth } f(x) \neq 0 \}$. Then C_S is a subsp of \mathbf{F}^S . (a) If $S = \{x_1, ..., x_n\}$. Find a bss of \mathbf{F}^S and conclude $\mathbf{F}^S = C_S$. \mathbf{F}^S infinide $\Rightarrow S$ infini. (b) If S has infily many elem. Prove \mathbf{F}^S is infinide. \mathbf{F}^S finide $\Rightarrow S$ fini. (c) Supp V is on F. Prove \exists surj $T \in \mathcal{L}(C_V, V)$. **Solus**: (a) Define each $f_i(x_j) = \delta_{i,j}$. Supp $f \in C_S$, let each $y_k = f(x_k) = (y_1 f_1 + \dots + y_n f_n)(x_k)$. Then $f = y_1 f_1 + \dots + y_n f_n \in \operatorname{span}(f_1, \dots, f_n)$. \mathbb{X} If f = 0, then each $y_k = 0$. (b) Let $S = \{x_1, \dots, x_n, \dots\}$. Define each $f_i(x_i) = \delta_{i,i} \Rightarrow f_i \in C_S$. $\mathbb{X}(f_1, \dots, f_n, \dots)$ liney indep.

(c) Define $T: C_V \to V$ by $T(f) = \sum_{x \in V} f(x)x$. Note that $f(x) \neq 0$ for finily many $x \in V$.

Define each $f(v_k) = a_k$ and f(x) = 0 for $x \notin \{v_1, \dots, v_n\}$. Then T(f) = v.

Becs for any $v \in V$, \exists liney indep (v_1, \dots, v_n) suth $v = a_1v_1 + \dots + a_nv_n$. [See higher courses]

Coro: *S* fini \iff **F**^{*S*} finide.

• Given the fact that $\mathcal{L}(V, W)$ is a vecsp. Prove or give a countexa: V, W are vecsps.

By [3.2], the add and homo imply that V is closd add and scalar multi. While W^V might not be a vecsp.

ENDED

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3 Supp (v_1, \ldots, v_m) in V. Define T \in \mathcal{L}(\mathbf{F}^m, V) by T(z_1, \ldots, z_m) = z_1v_1 + \cdots + z_mv_m.
   (a) The surj of T corres to (v_1, ..., v_m) spanning V.
                                                                                               range T = \text{span}(v_1, \dots, v_m) = V.
  (b) The inje of T corres to (v_1, ..., v_m) being liney indep.
                                                                                              (v_1, \dots, v_m) liney indep \iff T inje.
COMMENT: Let (e_1, ..., e_m) be std bss of \mathbf{F}^m. Then Te_k = v_k.
9 Supp (v_1, ..., v_n) is liney indep. Prove for any inje T, (Tv_1, ..., Tv_n) is liney indep.
Solus: a_1Tv_1 + \cdots + a_nTv_n = 0 = T\left(\sum_{i=1}^n a_iv_i\right) \Longleftrightarrow \sum_{i=1}^n a_iv_i = 0 \Longleftrightarrow a_1 = \cdots = a_n = 0.
                                                                                                                                        10 Supp span(v_1, ..., v_n) = V. Show span(Tv_1, ..., Tv_n) = \operatorname{range} T.
Solus: (a) range T = \{Tv : v \in \text{span}(v_1, \dots, v_n)\} \Rightarrow Tv_1, \dots, Tv_n \in \text{range } T. By [2.7].
                Or. span(Tv_1, \dots, Tv_n) \ni a_1Tv_1 + \dots + a_nTv_n = T(a_1v_1 + \dots + a_nv_n) \in \text{range } T.
           (b) \forall w \in \text{range } T, w = Tv, \exists v \in V \Rightarrow \exists a_i \in F, v = \sum_{i=1}^n a_i v_i, w = a_1 T v_1 + \dots + a_n T v_n.
                                                                                                                                        11 Supp S_1, \ldots, S_n are liney and inje suth S_1 S_2 \cdots S_n makes sense. Prove S_1 S_2 \cdots S_n inje.
\textbf{Solus:} \ \ S_1S_2\cdots S_nv=0 \Rightarrow S_2\cdots S_nv=0 \Rightarrow \cdots \Rightarrow S_nv=0 \Rightarrow v=0.
                                                                                                                                        • (4E 5.A.33) Supp T \in \mathcal{L}(V), m \in \mathbb{N}^+. Prove T inje \iff T^m inje, and T surj \iff T^m surj.
Solus: (a) T^m inje \Rightarrow if Tv = 0, then T^{m-1}Tv = T^mv = 0 \Rightarrow v = 0, thus T inje. Convly immed.
           (b) T^m \operatorname{surj} \Rightarrow \forall u \in V, \exists v \in V \Rightarrow \exists w = T^{m-1}v, T^m v = u = Tw.
                T \operatorname{surj} \Rightarrow \forall u \in V, \exists v_1, \dots, v_m \in V, T(v_1) = T^2 v_2 = \dots = T^m v_m = u.
                                                                                                                                        16 Supp T \in \mathcal{L}(V) suth null T, range T are finide. Prove V is finide.
Solus: Let B_{\text{range }T} = (Tv_1, \dots, Tv_n), B_{\text{null }T} = (u_1, \dots, u_m).
           \forall v \in V, \exists ! a_i \in \mathbf{F}, T(v - a_1v_1 - \dots - a_nv_n) = 0 \Rightarrow \exists ! b_i \in \mathbf{F}, v - \sum_{i=1}^n a_iv_i = \sum_{i=1}^m b_iu_i.
                                                                                                                                        17 Supp V, W are finide. Prove \exists inje T \in \mathcal{L}(V, W) \iff \dim V \leqslant \dim W.
Solus: (a) Supp \exists inje T. Then dim V = \dim \operatorname{range} T \leq \dim W.
           (b) Supp dim V \leq \dim W. Let B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m). Define each Tv_i = w_i.
18 Supp V, W are finide. Prove \exists surj T \in \mathcal{L}(V, W) \iff \dim V \geqslant \dim W.
Solus: (a) Supp \exists surj T. Then dim V = \dim W + \dim \operatorname{null} T \Rightarrow \dim W \leq \dim V.
           (b) Supp dim V \ge \dim W. Let B_V = (v_1, ..., v_n), B_W = (w_1, ..., w_m).
                Define T \in \mathcal{L}(V, W) by T(a_1v_1 + \dots + a_mv_m + \dots + a_nv_n) = a_1w_1 + \dots + a_mw_m.
                                                                                                                                        19 Supp V, W are finide, U is a subsp of V.
     Prove \exists T \in \mathcal{L}(V, W), \text{null } T = U \iff \underline{\dim U} \geqslant \underline{\dim V} - \underline{\dim W}.
SOLUS:
   (a) Supp \exists T \in \mathcal{L}(V, W), null T = U. Then dim U + \dim \operatorname{range} T = \dim V \leq \dim U + \dim W.
   (b) Let B_U = (u_1, ..., u_m), B_V = (u_1, ..., u_m, v_1, ..., v_n), B_W = (w_1, ..., w_p). Supp p \ge n.
        Define T \in \mathcal{L}(V, W) by T(a_1v_1 + \dots + a_nv_n + b_1u_1 + \dots + b_mu_m) = a_1w_1 + \dots + a_nw_n.
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• Tips 1: Supp U is a subsp of V. Then \forall T \in \mathcal{L}(V, W), U \cap \text{null } T = \text{null } T|_{U}.
• Tips 2: Supp T \in \mathcal{L}(V, W) and T|_{U} is inje. Let V = M + N, U = X + Y.
              Then range T = \operatorname{range} T|_{M} + \operatorname{range} T|_{N} = \operatorname{range} T|_{X} + \operatorname{range} T|_{Y}.
              (a) Show U = X \oplus Y \iff \text{range } T = \text{range } T|_X \oplus \text{range } T|_Y.
              (b) Give an exa suth V = M \oplus N, range T \neq \text{range } T|_M \oplus \text{range } T|_N.
Solus: Supp U = X \oplus Y. Asum for some v \in V, there exis two disti pairs (x_1, y_1), (x_2, y_2) in X \times Y
           suth Tv = Tx_1 + Ty_1 = Tx_2 + Ty_2. Becs \forall v \in X \oplus Y, \exists ! (x,y) \in X \times Y, v = x + y.
           Now T(x_1 + y_1) = T(x_2 + y_2) \Longrightarrow x_1 + y_1 = x_2 + y_2 \Longrightarrow x_1 = x_2, y_1 = y_2. Ctradic.
           Thus \forall Tv \in \operatorname{range} T, \exists ! Tx \in \operatorname{range} T|_X, Ty \in \operatorname{range} T|_Y, Tv = Tx + Ty. Convly, becs T is inje. \Box
EXA: Let B_V = (v_1, v_2, v_3), B_W = (w_1, w_2), T : v_1 \mapsto 0, v_2 \mapsto w_1, v_3 \mapsto w_2.
       Let B_M = (v_1 - v_2, v_3), B_N = (v_2). Then range T|_M = \text{span}(w_1, w_2), range T|_N = \text{span}(w_1)
COMMENT: Also null T|_M = \text{null } T|_N = \{0\}. Hence null T \neq \text{null } T|_M \oplus \text{null } T|_N.
12 Prove \forall T \in \mathcal{L}(V, W), \exists subsp U of V suth
     U \cap \text{null } T = \text{null } T|_{U} = \{0\}, \text{ range } T = \{Tu : u \in U\} = \text{range } T|_{U}.
     Which is equiv to T|_U : U \rightarrow \text{range } T \text{ being iso.}
Solus: By [2.34] (note that V can be infinide), \exists subsp U of V suth V = U \oplus \text{null } T.
            \forall v \in V, \exists ! w \in \text{null } T, u \in U, v = w + u. \text{ Then } Tv = T(w + u) = Tu \in \{Tu : u \in U\}.
                                                                                                                                                 T|_{U}: U \to \operatorname{range} T \text{ is iso} \iff U \oplus \operatorname{null} T = V. [Q]
Coro: [P]
          We have shown Q \Rightarrow P. Now we show P \Rightarrow Q to complete the proof.
           \forall v \in V, Tv \in \text{range } T = \text{range } T|_U \Rightarrow \exists ! u \in U, Tv = Tu \Rightarrow v - u \in \text{null } T.
          Thus v = (v - u) + u \in U + \text{null } T. \forall u \in U \cap \text{null } T \iff T|_U(u) = 0 \iff u = 0.
                                                                                                                                                 Or. \neg Q \Rightarrow \neg P: Becs U \oplus \text{null } T \subseteq V. We show range T \neq \text{range } T|_U by ctradic.
          Let X \oplus (U \oplus \text{null } T) = V. Now range T = \text{range } T|_X \oplus \text{range } T|_U. And X is non0.
          Asum range T = \text{range } T|_{U}. Then range T|_{X} = \{0\}. While T|_{X} is inje. Ctradic.
          Or. range T|_X \subseteq \text{range } T|_U \Rightarrow \forall x \in X, Tx \in \text{range } T|_U, \exists u \in U, Tu = Tx \Rightarrow x = 0.
          Also, \neg P \Rightarrow \neg Q: (a) range T|_U \subseteq \text{range } T; OR (b) U \cap \text{null } T \neq \{0\}.
          For (a), \exists x \in V \setminus U, Tx \neq 0 \iff x \notin \text{null } T. Thus U + \text{null } T \subseteq V. For (b), immed.
                                                                                                                                                 COMMENT: If T|_U: U \to \operatorname{range} T is iso. Let R \oplus U = V. Then R might not be null T.
                Or. Extend B_U to B_V = (u_1, \dots, u_n, r_1, \dots, r_m), then (r_1, \dots, r_m) might not be a B_{\text{null }T}.
• Tips 3: Supp T \in \mathcal{L}(V, W) and U is a subsp suth V = U \oplus \text{null } T. Let \text{null } T = X \oplus Y.
  Now \forall v \in V, \exists ! u_v \in U, (x_v, y_v) \in X \times Y, v = u_v + x_v + y_v. Define i \in \mathcal{L}(V, U \oplus X) by i(v) = u_v + x_v.
  Then T = T \circ i. Becs \forall v \in V, T(v) = T(u_v + x_v + y_v) = T(u_v) = T(u_v + x_v) = T(i(v)) = (T \circ i)(v).
• TIPS 4: Supp T \in \mathcal{L}(V, W), T \neq 0. Let B_{\text{range } T} = (Tv_1, \dots, Tv_n).
  By (3.A.4), R = (v_1, ..., v_n) is liney indep in V. Let span R = U. We will prove U \oplus \text{null } T = V.
  (a) T\left(\sum_{i=1}^{n} a_i v_i\right) = 0 \iff \sum_{i=1}^{n} a_i T v_i = 0 \iff a_1 = \dots = a_n = 0. Thus U \cap \text{null } T = \{0\}.
  (b) Tv = \sum_{i=1}^{n} a_i Tv_i \iff v - \sum_{i=1}^{n} a_i v_i \in \text{null } T \iff v = \left(v - \sum_{i=1}^{n} a_i v_i\right) + \left(\sum_{i=1}^{n} a_i v_i\right).
        Thus U + \text{null } T = V. Or. range T = \{Tu : u \in U\} = \text{range } T|_{U}. Using Exe (12).
                                                                                                                                                 Coro: Convly, if U \oplus \text{null } T = V \text{ and } B_U = (v_1, \dots, v_n), then B_{\text{range } T} = (Tv_1, \dots, Tv_n).
          Becs range T = \text{range } T|_U = \text{span}(Tv_1, \dots, Tv_n), \ \ensuremath{\mathbb{X}} T \text{ is inje.}
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• (4E 3.D.15) Supp T \in \mathcal{L}(V) and V = \operatorname{span}(Tv_1, \dots, Tv_m). Prove V = \operatorname{span}(v_1, \dots, v_m).
Solus: Becs V = \text{span}(Tv_1, ..., Tv_m) \Rightarrow T \text{ surj} \Rightarrow T, T^{-1} \text{ inv.}
            \forall v \in V, \exists a_i \in \mathbf{F}, v = \sum_{i=1}^m a_i T v_i \Rightarrow T^{-1} v = \sum_{i=1}^m a_i v_i \Rightarrow \mathrm{range}\, T^{-1} \subseteq \mathrm{span}\big(v_1, \dots, v_m\big).
            OR. Reduce to a bss (Tv_{\alpha_1}, ..., Tv_{\alpha_k}), where k = \dim V, each \alpha_i \in \{1, ..., m\}. By (4E 3.D.3). \square
• (4E 27) Supp P \in \mathcal{L}(V) and P^2 = P. Prove V = \text{null } P \oplus \text{range } P.
Solus: (a) If v \in \text{null } P \cap \text{range } P \Rightarrow Pv = 0, and \exists u \in V, v = Pu. Then v = Pu = P^2u = Pv = 0.
            (b) Note that \forall v \in V, v = Pv + (v - Pv) and P(v - Pv) = 0 \Rightarrow v - Pv \in \text{null } P.
                  OR. Becs dim V = \dim \text{null } P + \dim \text{range } P = \dim (\text{null } P \oplus \text{range } P).
                                                                                                                                                        Or. Becs P|_{\text{range }P}: Pv \mapsto Pv^2 = Pv \Rightarrow P|_{\text{range }P} = I is iso. By Coro in Exe (12).
                                                                                                                                                        • (4E 21) Supp V is finide, T \in \mathcal{L}(V, W), Y is a subsp of W. Let \mathcal{K}_Y = \{v \in V : Tv \in Y\}.
  (a) Prove \{v \in V : Tv \in Y\} is a subsp of V.
  (b) Prove \dim\{v \in V : Tv \in Y\} = \dim \operatorname{null} T + \dim(Y \cap \operatorname{range} T).
Solus: (a) \forall u, w \in \mathcal{K}_Y, [Tu, Tw \in Y], \lambda \in F, T(u + \lambda w) = Tu + \lambda Tw \in Y \Longrightarrow \mathcal{K}_Y is a subspoof V.
            (b) Define the range-restr map R of T by R = T|_{\mathcal{K}_Y} \in \mathcal{L}(\mathcal{K}_Y, Y). Now range R = Y \cap \text{range } T.
                  And v \in \text{null } T \iff Tv = 0 \in Y \iff Rv = 0 \in \text{range } T \iff v \in \text{null } R. By [3.22].
                                                                                                                                                        COMMENT: Now span(v_1, ..., v_m) \oplus \text{null } T = \mathcal{K}_Y. Where B_{Y \cap \text{range } T} = (Tv_1, ..., Tv_m).
                 \text{In particular, } \dim \mathcal{K}_{\operatorname{range} T} = \dim \operatorname{null} T + \dim \operatorname{range} T \Longrightarrow \mathcal{K}_{\operatorname{range} T} = V.
• (4E 31) Supp V is finide, X is a subsp of V, and Y is a finide subsp of W.
  Prove if dim X + dim Y = dim V, then \exists T \in \mathcal{L}(V, W), null T = X, range T = Y.
Solus: Let V = U \oplus X, B_U = (v_1, \dots, v_m). Then \forall v \in V, \exists ! a_i \in F, x \in X, v = \sum_{i=1}^m a_i v_i + x.
            Let B_Y = (w_1, ..., w_m). Define T \in \mathcal{L}(V, W) with each Tv_i = w_i, Tx = 0.
            Now v \in \text{null } T \iff Tv = a_1w_1 + \dots + a_mw_m = 0 \iff v = x \in X. Hence \text{null } T = X.
            And Y \ni w = a_1 w_1 + \dots + a_m w_m = a_1 T v_1 + \dots + a_m T v_m \in \operatorname{range} T. Hence \operatorname{range} T = Y.
            OR. NOTICE that V = U \oplus \text{null } T. By Exe (12), range T = \text{range } T|_{U}.
                  \mathbb{Z} dim range T|_U = \dim U = \dim Y; range T \subseteq Y.
   Or. Let B_X = (x_1, \dots, x_n). Now range T = \operatorname{span}(Tv_1, \dots, Tv_m, Tx_1, \dots, Tx_n) = \operatorname{span}(w_1, \dots, w_m) = Y. \square
22 Supp U, V are finide, S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V).
     Prove dim null ST \leq \dim \text{null } S + \dim \text{null } T.
Solus: We show dim null ST = \dim \text{null } S|_{\text{range } T} + \dim \text{null } T.
            Becs (a) range T|_{\text{null }ST} = \text{range } T \cap \text{null } S = \text{null } S|_{\text{range }T},
                    (b) \underline{\operatorname{null} T}|_{\operatorname{null} ST} = \operatorname{null} T \cap \operatorname{null} ST = \operatorname{null} T. By [3.22]
                                                                                                                                                        OR. NOTICE that u \in \text{null } ST \iff S(Tu) = 0 \iff Tu \in \text{null } S.
                  Thus \{u \in U : Tu \in \text{null } S\} = \mathcal{K}_{\text{null } S \cap \text{range } T} = \text{null } ST.
                  By Exe (4E 21), dim null ST = \dim \text{null } T + \dim (\text{null } S \cap \text{range } T).
                                                                                                                                                        Coro: (1) T \operatorname{surj} \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T.
           (2) T \text{ inv} \Rightarrow \dim \text{null } ST = \dim \text{null } ST = \text{null } T.
           (3) S \text{ inje} \Rightarrow \dim \text{null } ST = \dim \text{null } T.
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23 Supp V is finide, S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V).
      Prove dim range ST \leq \min \{ \dim \operatorname{range} S, \dim \operatorname{range} T \}.
      COMMENT: If dim V = \dim U. Then dim null ST \ge \max\{\dim \text{null } S, \dim \text{null } T\}.
SOLUS: NOTICE that range ST = \{Sv : v \in \text{range } T\} = \text{range } S|_{\text{range } T}.
              Let range ST = \text{span}(Su_1, ..., Su_{\dim \text{range}T}), where B_{\text{range}T} = (u_1, ..., u_{\dim \text{range}T}).
              \dim \operatorname{range} ST \leqslant \dim \operatorname{range} T \setminus \dim \operatorname{range} ST \leqslant \dim \operatorname{range} S.
                                                                                                                                                                             OR. \operatorname{dim}\operatorname{range} ST = \operatorname{dim}\operatorname{range} S|_{\operatorname{range} T} = \operatorname{dim}\operatorname{range} T - \operatorname{dim}\operatorname{null} S|_{\operatorname{range} T} \leqslant \operatorname{range} T.
                                                                                                                                                                             COMMENT: \dim \operatorname{range} ST = \dim U - \dim \operatorname{null} ST = \dim \operatorname{range} T|_{U} - \dim \operatorname{range} T|_{\operatorname{null} ST}.
Coro: (1) S|_{\text{range }T} inje \iff dim range ST = \dim \text{range }T.
             (2) Let X \oplus \text{null } S = V. Then X \subseteq \text{range } T \iff \text{range } ST = \text{range } S.
                   And T is surj \Rightarrow range ST = \text{range } S.
• (a) Supp dim V = n, ST = 0 where S, T \in \mathcal{L}(V). Prove dim range TS \leq \lfloor \frac{n}{2} \rfloor.
  (b) Give an exa of such S, T with n = 5 and dim range TS = 2.
Solus: Note that dim range TS \leq \min \{ \dim \operatorname{range} T, \dim \operatorname{range} S \}. We prove by ctradic.
   Asum dim range TS \ge \left| \frac{n}{2} \right| + 1. Then min \left\{ n - \dim \operatorname{null} T, n - \dim \operatorname{null} S \right\} \ge \left| \frac{n}{2} \right| + 1
    \mathbb{Z} \dim \operatorname{null} ST = n \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \mid \Rightarrow \max \left\{ \dim \operatorname{null} T, \dim \operatorname{null} S \right\} \leqslant \left\lceil \frac{n}{2} \right\rceil - 1.
   Thus n \le 2\left(\left\lceil \frac{n}{2}\right\rceil - 1\right) \Rightarrow \frac{n}{2} \le \left\lceil \frac{n}{2}\right\rceil - 1. Ctradic.
                                                                                                                                                                             OR. dim null S = n - \dim \operatorname{range} S \leq n - \dim \operatorname{range} TS. X ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S.
    \dim \operatorname{range} TS \leq \dim \operatorname{range} T \leq \dim \operatorname{null} S \leq n - \dim \operatorname{range} TS. Thus 2 \dim \operatorname{range} TS \leq n.
                                                                                                                                                                             OR. Becs dim range TS \leq \left\lfloor \frac{n}{2} \right\rfloor, and \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = n.
   We show dim null TS \ge \lceil \frac{n}{2} \rceil. Note that dim null S + \dim \text{null } T \ge n.
   \dim \operatorname{null} S + \dim \operatorname{null} T|_{\operatorname{range} S} = \dim \operatorname{null} TS. If \dim \operatorname{null} S \geqslant \left\lceil \frac{n}{2} \right\rceil. Then done.
   Othws, \dim \operatorname{null} S \leqslant \left\lceil \frac{n}{2} \right\rceil - 1 \Rightarrow \dim \operatorname{null} T \geqslant n - \dim \operatorname{null} S \geqslant n - \left\lceil \frac{n}{2} \right\rceil + 1 = \left\lceil \frac{n}{2} \right\rceil + 1 \geqslant \left\lceil \frac{n}{2} \right\rceil.
   Thus dim null TS \ge \max\{\dim \text{null } S, \dim \text{null } T\} = \left\lceil \frac{n}{2} \right\rceil.
                                                                                                                                                                             Exa: Define T: v_1 \mapsto 0, v_2 \mapsto 0, v_i \mapsto v_i; S: v_1 \mapsto v_4, v_2 \mapsto v_5, v_i \mapsto 0; i = 3,4,5.
20, 21 (a) Prove if ST = I \in \mathcal{L}(V), then T is inje and S is surj.
             (b) Supp T \in \mathcal{L}(V, W). Prove if T is inje, then \exists surj S \in \mathcal{L}(W, V), ST = I.
             (c) Supp S \in \mathcal{L}(W, V). Prove if S is surj, then \exists inje T \in \mathcal{L}(V, W), ST = I.
SOLUS:
    (a) Tv = 0 \Rightarrow S(Tv) = 0 = v. Or. \text{null } T \subseteq \text{null } ST = \{0\}.
          \forall v \in V, ST(v) = v \in \text{range } S. \text{ Or. } V = \text{range } ST \subseteq \text{range } S.
    (b) Define S \in \mathcal{L}(\operatorname{range} T, V) by Sw = T^{-1}w, where T^{-1} is the inv of T \in \mathcal{L}(V, \operatorname{range} T).
          Then extend to S \in \mathcal{L}(W, V) by (3.A.11). Now \forall v \in V, STv = T^{-1}Tv = v.
          Or. [Req \ V \ Finide] Let B_{range \ T} = (Tv_1, ..., Tv_n) \Rightarrow B_V = (v_1, ..., v_n). Let U \oplus range \ T = W.
          Define S \in \mathcal{L}(W, V) with each S(Tv_i) = v_i, Su = 0 for u \in U. Thus ST = I.
    (c) By Exe (12), \exists subsp U of W, W = U \oplus \text{null } S, range S = \text{range } S|_U = V.
          Note that S|_U: U \to V is iso. Define T = (S|_U)^{-1}, where (S|_U)^{-1}: V \to U.
          Then ST = S \circ (S|_{U})^{-1} = S|_{U} \circ (S|_{U})^{-1} = I_{V}.
          Or. [Req V Finide] Let B_{\text{range }S} = B_V = (Sw_1, ..., Sw_n) \Rightarrow \text{span}(w_1, ..., w_n) \oplus \text{null } S = W.
          Define T \in \mathcal{L}(V, W) by T(Sw_i) = w_i. Now ST(a_1Sw_1 + \cdots + a_nSw_n) = (a_1Sw_1 + \cdots + a_nSw_n). \square
```

• Tips 5: Supp $S \in \mathcal{L}(U, V)$ is surj. Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(U, W))$ by $\mathcal{B}(T) = TS$. Then \mathcal{B} is inje. Becs $\mathcal{B}(T) = TS = 0 \iff T|_{\text{range }S} = 0$. Or. range $TS = \text{range }T = \{0\}$. **24** Supp $S \in \mathcal{L}(V, M)$, $T \in \mathcal{L}(V, W)$, and $\text{null } S \subseteq \text{null } T$. Prove $\exists E \in \mathcal{L}(M, W)$, T = ES. Solus: Let $V = U \oplus \text{null } S$ range $T \leftarrow U$ $\Rightarrow S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ $S|_U : U \rightarrow \text{range } S \text{ is iso.}$ **COMMENT:** Let $\Delta \oplus \text{null } S = \text{null } T$, $U_{\Delta} \oplus (\Delta \oplus \text{null } S) = V = U_{\Delta} \oplus \text{null } T$. Redefine $U = U_{\Delta} \oplus \Delta$. Becs $\Delta = \operatorname{null} T|_U = \operatorname{null} T \cap \operatorname{range}(S|_U)^{-1}$. $\begin{array}{c|c}
\hline \text{null } S \\
\hline \text{null } T \\
\hline \Delta \text{ null } S
\end{array}$ $\begin{array}{c|c}
\hline \text{range } S & \stackrel{S}{\leftarrow} & \stackrel{U_{\Delta}}{\rightarrow} & \text{range } T \\
\hline \Delta \text{ null } S
\end{array}$ $\begin{array}{c|c}
\hline \text{Thus } E = T(S|_{U})^{-1} \text{ is not inje} \iff \Delta \neq \{0\}.$ In other words, range $S|_{\Delta} = \text{null } E$, $\begin{array}{c|c}
\hline \text{the proof } S|_{\Delta} = \text{range } T \text{ is is not inje} \iff \Delta \neq \{0\}.$ while $E|_{...}$: range $S|_{U_{\Lambda}} \rightarrow \text{range } T$ is iso. **COMMENT:** Let $E_1 \in \mathcal{L}(U_\Delta \oplus \text{null } T, U_\Delta)$, and E_2 be an iso of range $S|_{U_\Delta}$ onto range T. Define $E_1|_{U_{\Lambda}} = I|_{U_{\Lambda}}$, and $E_2 = T(S|_{U_{\Lambda}})^{-1}$. Then $T = E_2 S E_1$. **CORO:** If null S = null T. Then $\Delta = \{0\}$, $U_{\Delta} = U$. [Reg W Finide] By (3.D.3), we can extend inje $T(S|_{U})^{-1} \in \mathcal{L}(\text{range } S, W)$ to inv $E \in \mathcal{L}(M, W)$. Or. [Req range S Finide] Let $B_{\text{range }S} = (Sv_1, \dots, Sv_n)$. Then $\underline{V} = \text{span}(v_1, \dots, v_n) \oplus \text{null } S$. Define $E \in \mathcal{L}(\text{range } S, W)$ by $E(Sv_i) = Tv_i$. Extend to $E \in \mathcal{L}(M, W)$. Hence $\forall v = \sum_{i=1}^{n} a_i v_i + u \in V$, $(\exists ! u \in \text{null } S \subseteq \text{null } T)$, $Tv = \sum_{i=1}^{n} a_i T v_i + 0 = E(\sum_{i=1}^{n} a_i S v_i + 0)$. **Coro:** [Reg W Finide] Supp null S = null T. We show $\exists \text{ inv } E \in \mathcal{L}(M, W), T = ES$. Redefine $E \in \mathcal{L}(M, W)$ by $E(Tv_i) = Sv_i$, $E(w_i) = x_i$, for each Tv_i and w_i . Where: Let $B_{\text{range }T} = (Tv_1, ..., Tv_m), B_W = (Tv_1, ..., Tv_m, w_1, ..., w_n), B_U = (v_1, ..., v_m).$ Now $V = U \oplus \text{null } T = U \oplus \text{null } S \Rightarrow B_{\text{range } S} = (Sv_1, \dots, Sv_m)$. Let $B_M = (Sv_1, \dots, Sv_m, x_1, \dots, x_n)$. **25** Supp $S \in \mathcal{L}(Y, W), T \in \mathcal{L}(V, W)$, and range $T \subseteq \text{range } S$. Prove $\exists E \in \mathcal{L}(V, Y), T = SE$. **Solus:** Let $Y = U \oplus \text{null } S$ $\Rightarrow S|_U: U \rightarrow \operatorname{range} S \text{ is iso. Becs } (S|_U)^{-1}: \operatorname{range} S \rightarrow U.$ $\begin{array}{c|c} U_1 \xrightarrow{inv} \operatorname{range} S \\ || & \downarrow || \\ \Delta \xrightarrow{inv} \operatorname{range} S|_{\Delta} \\ \oplus & \oplus \\ U_{1\Delta} \xrightarrow{inv} \operatorname{range} T \xleftarrow{inv}_T U_2 \\ \uparrow & \downarrow \\ inv E|_T \end{array}$ Define $E = (S|_U)^{-1}T = (S|_U)^{-1}|_{\text{range }T}T \in \mathcal{L}(V, U) \subseteq \mathcal{L}(V, Y).$ Comment: Let $U_1 = U$. Let $U_2 \oplus \text{null } T = V$. Let $U_{1\Delta} = \operatorname{range}(S|_{U_1})^{-1}|_{\operatorname{range} T} \subseteq U_1 = \Delta \oplus U_{1\Delta}$. Or. Let $U_{1\Delta} = \text{range } E|_{U_2}$. Let $\Delta \oplus \text{range } E|_{U_2} = U_1$. [Req range T Finide] Let $B_{\text{range }T} = (Tv_1, ..., Tv_n)$. Now $B_{U_2} = (v_1, ..., v_n)$. Let $S(u_i) = Tv_i$ for each Tv_i . Define E with each $Ev_i = u_i$, Ex = 0 for $x \in \text{null } T$. **COMMENT:** $\lceil Req \ V \ Finide \rceil$ Note that dim $U_2 \leq \dim U_1 \Longrightarrow \dim \operatorname{null} T = p \geq q = \dim \operatorname{null} S$. Let $B_{\text{null }T} = (x_1, \dots, x_v), B_{\text{null }S} = (y_1, \dots, y_a)$. Redefine $E: v_i \mapsto u_i, x_k \mapsto y_k, x_i \mapsto 0$, for each $i \in \{1, \dots, \dim U_2\}, k \in \{1, \dots, \dim \operatorname{null} S\} = K, j \in \{1, \dots, \dim \operatorname{null} T\} \setminus K$. Note that (u_1, \dots, u_n) is liney indep. Let $X = \text{span}(x_1, \dots, x_n) \oplus \text{span}(v_1, \dots, v_n)$. Now $E|_X$ is inje, but cannot be re-extend to inv $E \in \mathcal{L}(V, Y)$ suth T = SE. **Coro:** $[Reg\ V\ Finide]$ If range $T = \text{range}\ S$, then $\dim \text{null}\ T = \dim \text{null}\ S = p$. Redefine *E* by $Ev_i = u_i$, $Ex_j = y_j$ for each v_i and x_j . Then $E \in \mathcal{L}(V, Y)$ is inv.

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• COMMENT: Supp S, T \in \mathcal{L}(V, W). Then range S = \text{range } T \Rightarrow \text{null } S, null T iso.
  EXA: Forwd shift optor on \mathbf{F}^{\infty} and backwd shift optor on \{(0, x_1, x_2, \dots) \in \mathbf{F}^{\infty}\}.
  While \operatorname{null} S = \operatorname{null} T \iff E : Sv \mapsto Tv and E^{-1} : Tv \mapsto Sv well-defined \Rightarrow range S, range T iso.
• Supp S, T \in \mathcal{L}(V, W).
• (3.D.6) Supp V and W are finide. dim null S = dim null T = n.
             Prove S = E_2TE_1, \exists inv E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W).
Solus: Define E_1: v_i \mapsto r_i; u_i \mapsto s_i; for each i \in \{1, ..., m\}, j \in \{1, ..., n\}.
             Define E_2: Tv_i \mapsto Sr_i; x_i \mapsto y_j; for each i \in \{1, ..., m\}, j \in \{1, ..., n\}. Where:
                Let B_{\text{range }T} = (Tv_1, \dots, Tv_m); B_{\text{range }S} = (Sr_1, \dots, Sr_m).
                Let B_W = (Tv_1, ..., Tv_m, x_1, ..., x_p); B'_W = (Sr_1, ..., Sr_m, y_1, ..., y_p). \mid :: E_1, E_2 are inv
                Let B_{\text{null }T} = (u_1, \dots, u_n); B_{\text{null }S} = (s_1, \dots, s_n).
                                                                                                                        and S = E_2 T E_1.
                                                                                                                                                              Thus B_V = (v_1, ..., v_m, u_1, ..., u_n); B'_V = (r_1, ..., r_m, s_1, ..., s_n).
• (a) Supp T = ES and E \in \mathcal{L}(W) is inv. Prove \text{null } S = \text{null } T.
  (b) Supp T = SE and E \in \mathcal{L}(V) is inv. Prove range S = \text{range } T.
  (c) Supp T = E_2SE_1 and E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W) are inv.
        Prove dim null S = \dim \text{null } T.
Solus: (a) v \in \text{null } T \iff Tv = 0 = E(Sv) \iff Sv = 0 \iff v \in \text{null } S.
             (b) w \in \operatorname{range} T \iff \exists v \in V, w = Tv = S(Ev) \iff w \in \operatorname{range} S.
             (c) By the CORO in Exe (22), dim null E_2SE_1 = \frac{E_2}{\sin y} \dim \text{null } SE_1 = \frac{E_1}{\sin y} \dim \text{null } S = \dim \text{null } T.
                                                                                                                                                              28 Supp T \in \mathcal{L}(V, W). Let (Tv_1, ..., Tv_m) be a bss of range T and each w_i = Tv_i.
     (a) Prove \exists \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F}) suth \forall v \in V, Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m.
     (b) [4E 3.F.5] \forall v \in V, \exists ! \varphi_i(v) \in F, Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m.
                          Thus defining each \varphi_i: V \to \mathbf{F}. Show each \varphi_i \in \mathcal{L}(V, \mathbf{F}).
Solus: The answer for (b) with (b) itself is the answer for (a).
   (b) \sum_{i=1}^{m} \varphi_i(u + \lambda v) w_i = T(u + \lambda v) = Tu + \lambda Tv = \left(\sum_{i=1}^{m} \varphi_i(u) w_i\right) + \lambda \left(\sum_{i=1}^{m} \varphi_i(v) w_i\right).
                                                                                                                                                              Or. \forall v \in V, \exists ! a_i \in F, Tv = a_1 Tv_1 + \dots + a_m Tv_m. Let B_{(\text{range }T)}, = (\psi_1, \dots, \psi_m).
         Then [T'(\psi_i)](v) = (\psi_i \circ T)(v) = a_i. Thus each \varphi_i = \psi_i \circ T = T'(\psi_i) \in V'.
                                                                                                                                                              (a) \operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = V \Rightarrow \forall v \in V, \exists ! a_i \in F, u \in \operatorname{null} T, v = \sum_{i=1}^m a_i v_i + u.
         Define \varphi_i \in \mathcal{L}(V, \mathbf{F}) by \varphi_i(v_i) = \delta_{i,i}, \varphi_i(u) = 0 for all u \in \text{null } T.
         Linity: \forall v, w \in V \ [\exists ! a_i, b_i \in \mathbf{F}], \lambda \in \mathbf{F}, \varphi_i(v + \lambda w) = a_i + \lambda b_i = \varphi(v) + \lambda \varphi(w).
                                                                                                                                                              29 Supp \varphi \in \mathcal{L}(V, \mathbf{F}). Supp \varphi(u) \neq 0. Prove V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}. By Tips (4), immed.
Solus: (a) v = cu \in \text{null } \varphi \cap \text{span}(u) \Rightarrow c\varphi(u) = 0 \Rightarrow v = 0. Now \text{null } \varphi \cap \text{span}(u) = \{0\}.
             (b) For v \in V, let a_v = \varphi(v). Then v = [v - (a_v/a_u)u] + (a_v/a_u)u \Rightarrow V = \operatorname{null} \varphi + \operatorname{span}(u). \square
30 Supp \varphi, \beta \in \mathcal{L}(V, \mathbf{F}) and \text{null } \varphi = \text{null } \beta = \eta. Prove \exists c \in \mathbf{F}, \varphi = c\beta.
Solus: If \eta = V, then \varphi = \beta = 0, done. Now by Exe (29),
             \varphi(u) \neq 0 \iff V = \text{null } \varphi \oplus \text{span}(u) \iff V = \text{null } \beta \oplus \text{span}(u) \iff \beta(u) \neq 0.
             Note that \forall v \in V, \exists ! u_0 \in \eta, \ a_v \in F, v = u_0 + a_v u \Rightarrow \varphi(u_0 + a_v u) = a_v \varphi(u), \ \beta(u_0 + a_v u) = a_v \beta(u). Let c = \frac{\varphi(u)}{\beta(u)} \in F \setminus \{0\}.
```

• (4E 3.F.6) $Supp\ \varphi, \beta \in \mathcal{L}(V, \mathbf{F})$. $Prove\ null\ \beta \subseteq null\ \varphi \Longleftrightarrow \varphi = c\beta, \exists\ c \in \mathbf{F}$. Coro: $null\ \varphi = null\ \beta \Longleftrightarrow \varphi = c\beta, \ \exists\ c \in \mathbf{F} \setminus \{0\}$. Solus: Using Exe (29) and (30).

(a) If $\varphi = 0$, then done. Othws, $\sup u \notin null\ \varphi \supseteq null\ \beta$.

Now $V = null\ \varphi \oplus \operatorname{span}(u) = null\ \beta \oplus \operatorname{span}(u)$. By $[1.C\ Tips\ (2)]$, $null\ \varphi = null\ \beta$. Let $c = \frac{\varphi(u)}{\beta(u)}$.

Or. We discuss in two cases. If $null\ \beta = null\ \varphi$, or if $\varphi = 0$, then done. Othws, $\exists\ u' \in null\ \varphi \setminus null\ \beta, \ \exists\ u \notin null\ \varphi \supseteq null\ \beta \Rightarrow V = null\ \beta \oplus \operatorname{span}(u') = null\ \beta \oplus \operatorname{span}(u)$. $\forall v \in V, v = w + au = w' + bu', \ \exists\ !w, w' \in null\ \beta$ $Thus\ \varphi(w + au) = a\varphi(u), \ \beta(w' + bu) = b\beta(u')$.

Notice that by (b) below, we have $null\ \varphi \subseteq null\ \beta$, ctradic the asum.

(b) If c = 0, then $null\ \varphi = V \supseteq null\ \beta$, done. Othws, $becs\ v \in null\ \beta \Longleftrightarrow v \in null\ \varphi$.

OR. By Exe (24), $\operatorname{null} \varphi \iff \exists E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta$. [If E is inv. Then $\operatorname{null} \varphi = \operatorname{null} \varphi$.]
Now $\exists E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta \iff \exists c = E(1) \in \mathbf{F}, \varphi = c\beta$. [E is inv $\iff E(1) \neq 0 \iff c \neq 0$.]

ENDED

3.C

• **Note For** [3.30, 32]: *matrix of span*

Supp $L_{\alpha} = (\alpha_1, ..., \alpha_n)$ and $L_{\beta} = (\beta_1, ..., \beta_m)$ are in a vecsp V.

Let each $\alpha_k = A_{1,k}\beta_1 + \dots + A_{m,k}\beta_m$, forming $A = \mathcal{M}(\operatorname{span} L_\beta \supseteq L_\alpha) \in \mathbf{F}^{m,n}$.

Which is the matrix of span. Then $(\beta_1 \cdots \beta_m) \begin{pmatrix} A_{1,1} \cdots A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} \cdots A_{m,n} \end{pmatrix} = (\alpha_1 \cdots \alpha_n).$

- (a) Supp m = n. If $(A_{\cdot,1}, \dots, A_{\cdot,n})$ is a bss of $\mathbf{F}^{n,1}$. We show L_{α} liney indep $\iff L_{\beta}$ liney indep. (\Leftarrow) Immed. (\Rightarrow) Asum L_{β} is liney dep and $\beta_j = c_1\beta_1 + \dots + c_{j-1}\beta_{j-1}$. By ctradic.
- (b) Supp $m \ge n$. If L_{β} liney indep. We show $(A_{\cdot,1}, \dots, A_{\cdot,n})$ liney indep $\iff L_{\alpha}$ liney indep. (\Rightarrow) Immed. (\Leftarrow) By ctradic.

Comment: $\mathcal{M}(\operatorname{span} L_{\beta} \supseteq L_{\alpha}) = \mathcal{M}(I, L_{\alpha}, L_{\beta}) \iff L_{\alpha}, L_{\beta} \text{ liney indep} \Rightarrow (A_{\cdot,1}, \dots, A_{\cdot,n}) \text{ liney indep}.$ Where I is the id optor retr to $\operatorname{span} L_{\alpha} \subseteq \operatorname{span} L_{\beta}$.

(c) Supp m < n. Then $(A_{\cdot,1}, \dots, A_{\cdot,n})$ is liney dep, so is L_{α} .

Supp $T \in \mathcal{L}(V, W)$ and $B_V = (v_1, \dots, v_m), B_W = (w_1, \dots, w_n).$

Then $\mathcal{M}(T, B_V, B_W) = \mathcal{M}(\operatorname{span} B_W \supseteq (Tv_1, \dots, Tv_m))$. Comment: See also (4E 3.D.23).

• Note For Trspose: [3.F.33] Define $\mathcal{T}: A \to A^t$. By [3.111], \mathcal{T} is liney. Becs $(A^t)^t = A$. $\mathcal{T}^2 = I$, $\mathcal{T} = \mathcal{T}^{-1} \Rightarrow \mathcal{T}$ is iso of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$. Define $\mathcal{C}_k: A \to A_{.,k}$, $\mathcal{R}_j: A \to A_{j,\cdot}$, $\mathcal{E}_{j,k}: A \to A_{j,k}$. Now we show (a) $\mathcal{T}\mathcal{R}_j = \mathcal{C}_j\mathcal{T}$, (b) $\mathcal{T}\mathcal{C}_k = \mathcal{R}_k\mathcal{T}$, and (c) $\mathcal{T}\mathcal{E}_{j,k} = \mathcal{E}_{k,j}\mathcal{T}$. So that $\mathcal{T}\mathcal{C}_k\mathcal{T} = \mathcal{R}_k$, $\mathcal{T}\mathcal{R}_j\mathcal{T} = \mathcal{C}_j$, and $\mathcal{T}\mathcal{E}_{j,k}\mathcal{T} = \mathcal{E}_{k,j}$.

$$\operatorname{Let} A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{pmatrix}. \quad \begin{array}{c} \operatorname{Note \ that} \ (A_{j,k})^t = A_{j,k} = (A^t)_{k,j}. \ \operatorname{Thus} \ (c) \ \operatorname{holds}. \\ \operatorname{And} \ (A_{\cdot,k})^t = (A_{1,k} & \cdots & A_{m,k}) = (A^t_{k,1} & \cdots & A^t_{k,m}) = (A^t)_{k,k}. \\ \Longrightarrow \ (b) \ \operatorname{holds}. \ \operatorname{Simlr \ for} \ (a). \end{array}$$

• Note For [3.48]:
$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}}_{B} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

• Note For [3.47]:
$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k} = \sum_{r=1}^{n} (A_{j,r})_{1,r} (C_{\cdot,k})_{r,1} = (A_{j,r} C_{\cdot,k})_{1,1} = A_{j,r} C_{\cdot,k}$$

• Note For [3.49]:
$$[(AC)_{\cdot,k}]_{i,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k} = \sum_{r=1}^{n} A_{j,r} (C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$$

• Exe 10:
$$[(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r} C_{r,k} = (A_{j,\cdot}C)_{1,k}$$

• Comment: For [3.49], let $B_U = (u_1, \dots, u_p)$, $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.

And
$$C = \mathcal{M}(T, B_U, B_V) \in \mathbf{F}^{n,p}, A = \mathcal{M}(S, B_V, B_W) \in \mathbf{F}^{m,n}$$
.

Then
$$\mathcal{M}(Tu_k, B_V) = C_{\cdot,k} \Rightarrow \mathcal{M}(S(Tu_k), B_W) = AC_{\cdot,k}, \ \ \ \ \mathcal{M}((ST)(u_k), B_W) = (AC)_{\cdot,k} \ \ \Box$$

By Note For Transpose,
$$(AC)_{j,\cdot} = \left[\left((AC)^t \right)_{\cdot,j} \right]^t = \left(C^t (A^t)_{\cdot,j} \right)^t = \left((A^t)_{\cdot,j} \right)^t C = A_{j,\cdot} C \quad \Box$$

• Note For [3.52]: $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$. By $[4E \ 3.51(a)], (Ac)_{\cdot,1} = c_1 A_{\cdot,1} + \dots + c_n A_{\cdot,n} \square$

Or.
$$: (Ac)_{j,1} = \sum_{r=1}^{n} A_{j,r} c_{r,1} = \left[\sum_{r=1}^{n} (A_{\cdot,r} c_{r,1}) \right]_{j,1} = (c_1 A_{\cdot,1} + \dots + c_n A_{\cdot,n})_{j,1}$$

$$\therefore Ac = A_{.,c_{.,1}} = \sum_{r=1}^{n} A_{.,r} c_{r,1} = c_1 A_{.,1} + \dots + c_n A_{.,n} \text{ Or. } (Ac)_{j,1} = (Ac)_{j,.} = A_{j,.} c \in \mathbf{F}.$$

OR. Let
$$B_V = (v_1, ..., v_n)$$
. Now $Ac = \mathcal{M}(Tv, B_W) = \mathcal{M}(T(c_1v_1 + ... + c_nv_n)) = c_1A_{.,1} + ... + c_nA_{.,n}$. \Box

• EXE 11: $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$. By $[4E \ 3.51(b)], (aC)_{1,..} = a_1C_{1,..} + \dots + a_nC_{n,..} \square$

Or.
$$: (aC)_{1,k} = \sum_{r=1}^{n} a_{1,r} C_{r,k} = \left[\sum_{r=1}^{n} a_{1,r} (C_{r,\cdot}) \right]_{1,k} = (a_1 C_{1,\cdot} + \dots + a_n C_{n,\cdot})_{1,k}$$

$$\therefore aC = a_1 \cdot C_{\cdot \cdot} = \sum_{r=1}^{n} a_{1,r} C_{r,\cdot} = a_1 C_{1,\cdot} + \dots + a_n C_{n,\cdot} \text{ Or. } (aC)_{1,k} = (aC)_{\cdot,k} = aC_{\cdot,k} \in \mathbf{F}.$$

Or.
$$aC = ((aC)^t)^t = (C^t a^t)^t = [a_1^t (C^t)_{\cdot,1} + \dots + a_n^t (C^t)_{\cdot,n}]^t = a_1 C_{1,\cdot} + \dots + a_n C_{n,\cdot}.$$

• [4E 3.51] Supp $C \in \mathbf{F}^{m,c}$, $R \in \mathbf{F}^{c,p}$.

See also Note For [3.49] and Exe (10).

(a) For
$$k = 1, ..., p$$
, $(CR)_{.k} = CR_{.k} = C_{..}R_{.k} = \sum_{r=1}^{c} C_{.r}R_{r,k} = R_{1,k}C_{.,1} + \cdots + R_{c,k}C_{.c}$

(b) For
$$j = 1, ..., m$$
, $(CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

• Exa: m = 2, c = 2, p = 3.

$$(AB)_{\cdot,2} = AB_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} = A_{\cdot,1}B_{1,2} + A_{\cdot,2}B_{2,2} = 6 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix};$$

$$(AB)_{1,\cdot} = A_{1,\cdot}B = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = A_{1,1}B_{1,\cdot} + A_{1,2}B_{2,\cdot} = 1 \begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 2 \begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \end{pmatrix};$$

- CR Factoriz Supp non0 $A \in \mathbb{F}^{m,n}$. Prove, with p below, that $\exists C \in \mathbb{F}^{m,p}$, $R \in \mathbb{F}^{p,n}$, A = CR.
 - (a) $Supp \operatorname{col} A = \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}$, $\dim \operatorname{col} A = c$, the col rank. Let p = c.
- (b) $Supp \text{ row } A = \operatorname{span}(A_{1,\cdot}, \dots, A_{m,\cdot}) \subseteq \mathbf{F}^{1,n}$, $\dim \operatorname{row} A = r$, the row rank. Let p = r.

Solus: Using [4E 3.51]. Notice that $A \neq 0 \Rightarrow c, r \geqslant 1$.

- (a) Reduce to bss $B_C = (C_{\cdot,1}, \cdots, C_{\cdot,c})$, forming $C \in \mathbf{F}^{m,c}$. Then $\forall k \in \{1, \dots, n\}$, $A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$, $\exists ! R_{1,k}, \cdots, R_{c,k} \in \mathbf{F}$, forming $R \in \mathbf{F}^{c,n}$. Thus A = CR.
- (b) Reduce to bss $B_R = (R_{1,\cdot}, \cdots, R_{r,\cdot})$, forming $R \in \mathbf{F}^{r,n}$. Then $\forall j \in \{1, \dots, m\}$, $A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,r}R_{r,\cdot} = (CR)_{j,\cdot}$, $\exists ! C_{j,1}, \dots, C_{j,r} \in \mathbf{F}$, forming $C \in \mathbf{F}^{m,r}$. Thus A = CR. \square

Exa:
$$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \xrightarrow{\text{(I)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \xrightarrow{\text{(II)}} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

- (I) $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2\begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix}$, using [4E 3.51(b)]. $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} \in \operatorname{span}(A_{1,\cdot}, A_{2,\cdot})$, and $(A_{1,\cdot}, A_{2,\cdot})$ is liney indep. Thus $B_R = \begin{pmatrix} A_{1,\cdot}, A_{2,\cdot} \end{pmatrix}$.
- (II) $\begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2 \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}; \quad \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = -\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}. \text{ Thus } B_C = (A_{\cdot,2}, A_{\cdot,3}).$
- Col Rank = Row Rank Using CR Factoriz. Let A = CY by (a) and A = XR by (b).
 - (a) $A_{j,\cdot} = (CY)_{j,\cdot} = C_{j,\cdot}Y = C_{j,1}Y_{1,\cdot} + \dots + C_{j,c}Y_{c,\cdot} \in \text{row } A = \text{span}(A_{1,\cdot},\dots,A_{n,\cdot}) = \text{span}(Y_{1,\cdot},\dots,Y_{c,\cdot}).$
 - (b) $A_{\cdot,k} = (XR)_{\cdot,k} = XR_{\cdot,k} = R_{1,k}X_{\cdot,1} + \dots + R_{r,k}X_{\cdot,r} \in \text{col} A = \text{span}(A_{\cdot,1},\dots,A_{\cdot,m}) = \text{span}(X_{\cdot,1},\dots,X_{\cdot,r}).$ Thus (a) dim row $A = r \le c = \dim \text{col} A$ and (b) dim $\text{col} A = c \le r = \dim \text{row} A$
 - Thus (a) dim row $A = r \le c = \dim \operatorname{col} A$, and (b) dim $\operatorname{col} A = c \le r = \dim \operatorname{row} A$. \square OR. Apply the result of (a) to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim \operatorname{row} A^t = \dim \operatorname{col} A = c \le r = \dim \operatorname{row} A = \dim \operatorname{col} A^{t}$
- (4E 16) Supp $A \in \mathbf{F}^{m,n} \setminus \{0\}$. Prove [P] rank $A = 1 \iff \exists c_j, d_k \in \mathbf{F}$, each $A_{j,k} = c_j \cdot d_k$. [Q] Solus:

[Using CR Factoriz]

$$P \Rightarrow Q : \text{ Immed.}$$

$$Q \Rightarrow P : \text{ Becs } A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 \cdots d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix} \Rightarrow \text{row } A = \text{span} \left\{ \begin{pmatrix} \underline{c_1} d_1 & \cdots & \underline{c_1} d_n \\ \vdots & \vdots \\ \underline{c_m} d_1 & \cdots & \underline{c_m} d_n \end{pmatrix} \right\}.$$

$$OR. \text{ col } A = \text{span} \left\{ \begin{pmatrix} c_1 \underline{d_1} \\ \vdots \\ c_m \underline{d_1} \end{pmatrix}, \dots, \begin{pmatrix} c_1 \underline{d_n} \\ \vdots \\ c_m \underline{d_n} \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}.$$

[Not Using CR Factoriz]

 $Q \Rightarrow P : \text{Using [4E 3.51(a)]}. \text{ Each } A_{\cdot,k} \in \text{span} \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}. \text{ Then } \text{rank } A = \dim \text{col } A \leqslant 1$ $P \Rightarrow Q : \text{ Becs } \dim \text{col } A = \dim \text{row } A = 1.$

Let
$$c_j = \frac{A_{j,1}}{A_{1,1}} = \frac{A_{j,2}}{A_{1,2}} = \dots = \frac{A_{j,n}}{A_{1,n}}, \quad d'_k = \frac{A_{1,k}}{A_{1,1}} = \frac{A_{2,k}}{A_{2,1}} = \dots = \frac{A_{m,k}}{A_{m,1}}.$$

$$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$$
, where $d_k = d'_k A_{1,1}$.

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• Tips 1: Supp T \in \mathcal{L}(V, W), B_V = (v_1, ..., v_n), B_W = (w_1, ..., w_m).
                  Let L = (Tv_{\alpha_1}, \dots, Tv_{\alpha_k}), L_{\mathcal{M}} = (A_{\cdot,\alpha_1}, \dots, A_{\cdot,\alpha_k}), where each \alpha_i \in \{1, \dots, n\}.
                  (a) Show [P] L is liney indep \iff L_{\mathcal{M}} is liney indep. [Q]
                  (b) Show[P] \operatorname{span} L = W \iff \operatorname{span} L_{\mathcal{M}} = \mathbf{F}^{m,1}.[Q]
                                                                                                                                                 [ Let A = \mathcal{M}(T, B_V, B_W).]
Solus: (a) Note that \mathcal{M}: Tv_k \to A_{\cdot,k} is iso. of span L onto span L_{\mathcal{M}}. By (3.B.9).
                (b) Reduce to liney indep lists. By (a) and (2.39).
                                                                                                                                                                                               Or. c_1 T v_{\alpha_1} + \dots + c_k T v_{\alpha_k} = c_1 (A_{1,\alpha_1} w_1 + \dots + A_{m,\alpha_1} w_m) + \dots + c_k (A_{1,\alpha_k} w_1 + \dots + A_{m,\alpha_k} w_m)
                                                    = (c_1 A_{1,\alpha_1} + \dots + c_k A_{1,\alpha_k}) w_1 + \dots + (c_1 A_{m,\alpha_1} + \dots + c_k A_{m,\alpha_k}) w_m.
             \text{And } c_1 A_{\cdot,\alpha_1} + \dots + c_k A_{\cdot,\alpha_k} = c_1 \begin{pmatrix} A_{1,\alpha_1} \\ \vdots \\ A_{m,\alpha_1} \end{pmatrix} + \dots + c_k \begin{pmatrix} A_{1,\alpha_k} \\ \vdots \\ A_{m,\alpha_k} \end{pmatrix} = \begin{pmatrix} c_1 A_{1,\alpha_1} + \dots + c_k A_{1,\alpha_k} \\ \vdots \\ c_1 A_{m,\alpha_1} + \dots + c_k A_{m,\alpha_k} \end{pmatrix}. 
    (a) P \Rightarrow Q: Supp c_1 A_{\cdot,\alpha_1} + \dots + c_k A_{\cdot,\alpha_k} = 0. Let v = c_1 v_{\alpha_1} + \dots + c_k v_{\alpha_k}.
                            Then Tv = (c_1 A_{1,\alpha_1} + \dots + c_k A_{1,\alpha_k}) w_1 + \dots + (c_1 A_{m,\alpha_1} + \dots + c_k A_{m,\alpha_k}) w_m = 0 w_1 + \dots + 0 w_m.
                            Now c_1 T v_{\alpha_1} + \cdots + c_k T v_{\alpha_k} = 0. Then each c_i = 0 \Rightarrow L_{\mathcal{M}} liney indep.
           Q\Rightarrow P: \text{ Becs } c_1Tv_{\alpha_1}+\cdots+c_kTv_{\alpha_k}=0. \text{ For each } i\in \left\{1,\ldots,m\right\},\ c_1A_{i,\alpha_1}+\cdots+c_kA_{i,\alpha_k}=0.
                            Which is equiv to c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} = 0. Thus each c_i = 0 \Rightarrow L liney indep.
           Or. \exists A_{\cdot,\alpha_i} = c_1 A_{\cdot,\alpha_1} + \dots + c_{i-1} A_{\cdot,\alpha_{i-1}}
                    \Leftrightarrow For each i \in \{1, \dots, m\}, A_{i,\alpha_i} = c_1 A_{i,\alpha_1} + \dots + c_{i-1} A_{i,\alpha_{i-1}}
                    \iff Tv_{\alpha_i} = A_{1,\alpha_i}w_1 + \dots + A_{m,\alpha_i}w_m
                                     = (c_1 A_{1,\alpha_1} + \dots + c_{j-1} A_{1,\alpha_{j-1}}) w_1 + \dots + (c_1 A_{m,\alpha_1} + \dots + c_{j-1} A_{m,\alpha_{j-1}}) w_m
                    \iff \exists Tv_{\alpha_i} = c_1 Tv_{\alpha_1} + \dots + c_{i-1} Tv_{\alpha_{i-1}}.
    (b) Note that each \mathcal{M}(Tv_{\alpha_i}) = A_{\cdot,\alpha_i}
           P \Rightarrow Q: Supp each w_i = Iw_i = J_{1,i}Tv_{\alpha_1} + \cdots + J_{k,i}Tv_{\alpha_k}.
                             \forall a \in \mathbf{F}^{m,1}, \exists w = a_1 w_1 + \dots + a_m w_m \in W, \ a = \mathcal{M}(w, B_W).
                             Becs w = a_1(J_{1,1}Tv_{\alpha_1} + \dots + J_{k,1}Tv_{\alpha_k}) + \dots + a_m(J_{1,m}Tv_{\alpha_1} + \dots + J_{k,m}Tv_{\alpha_k})
                                           = (a_1J_{1,1} + \cdots + a_mJ_{1,m})Tv_{\alpha_1} + \cdots + (a_1J_{k,1} + \cdots + a_mJ_{k,m})Tv_{\alpha_k}.
                            Apply \mathcal{M} to both sides, a = c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k}, where each c_i = a_1 J_{i,1} + \cdots + a_m J_{i,m}.
           Q \Rightarrow P: \forall w \in W, \exists a = c_1 A_{\cdot,\alpha_1} + \dots + c_k A_{\cdot,\alpha_k} \in \mathbf{F}^{m,1}, \ \mathcal{M}(w, B_W) = a
                            \Rightarrow w = (c_1 A_{1,\alpha_1} + \dots + c_k A_{1,\alpha_k}) w_1 + \dots + (c_1 A_{m,\alpha_1} + \dots + c_k A_{m,\alpha_k}) w_m = c_1 T v_{\alpha_1} + \dots + c_k T v_{\alpha_k}.
             \neg Q \Rightarrow \neg P : \exists w \in W, \exists a \in \mathbf{F}^{m,1}, \mathcal{M}(w, B_W) = a, \text{ but } \not\exists \left(c_1, \dots, c_k\right) \in \mathbf{F}^k, a = c_1 A_{\cdot, \alpha_1} + \dots + c_k A_{\cdot, \alpha_k} 
                                 \Rightarrow \nexists (c_1,\ldots,c_k)\in \mathbf{F}^k, w=c_1Tv_{\alpha_1}+\cdots+c_kTv_{\alpha_k}. For if not, ctradic.
Note: Let L = (Tv_1, ..., Tv_n), L_{\mathcal{M}} = (A_{.1}, ..., A_{.n}).
              Then (a*) By [3.B.9, \text{Tips}(4)], T is inje \iff L is liney indep, so is L_{\mathcal{M}}.
              And (b*) T is surj \iff span L = W \iff span L_{\mathcal{M}} = \mathbf{F}^{m,1}.
             Coro: B_{\mathbf{F}^{n,1}} = (A_{\cdot,1}, \cdots, A_{\cdot,n}) \iff T is inje and surj \iff B_{\mathbf{F}^{1,n}} = (A_{\cdot,1}, \cdots, A_{\cdot,n}).
              COMMENT: If T is inv. Then by (a^*, c) or (b^*, d), we have another proof of CORO.
                                   Or. If m = n. Then by [3.118] and one of (a^*, b^*, c, d). Yet another proof.
             (c) T \operatorname{surj} \iff T' \operatorname{inje} \iff (T'(\psi_1), \dots, T'(\psi_m)) liney indep
                                 \stackrel{\text{(a)}}{\Longleftrightarrow} ((A^t)_{\cdot,1},\cdots,(A^t)_{\cdot,m}) liney indep in \mathbf{F}^{n,1}, so is (A_{1,\cdot},\cdots,A_{m,\cdot}) in \mathbf{F}^{1,n}.
              (d) T inje \iff T' surj \iff V' = \text{span}(T'(\psi_1), ..., T'(\psi_m))
                                 \stackrel{\text{(b)}}{\Longleftrightarrow} \mathbf{F}^{n,1} = \operatorname{span}\left( (A^t)_{\cdot,1}, \cdots, (A^t)_{\cdot,m} \right) \Longleftrightarrow \mathbf{F}^{1,n} = \operatorname{span}\left( A_{1,\cdot}, \cdots, A_{m,\cdot} \right).
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• Tips 2: Supp p is a poly of n variables in \mathbf{F}. Prove \mathcal{M}(p(T_1, ..., T_n)) = p(\mathcal{M}(T_1), ..., \mathcal{M}(T_n)).
             Where the liney maps T_1, ..., T_n are suth p(T_1, ..., T_n) makes sense. See [5.16,17,20].
Solus: Supp the poly p is defined by p(x_1, ..., x_n) = \sum_{k_1, ..., k_n} \alpha_{k_1, ..., k_n} \prod_{i=1}^n x_i^{k_i}.
           Note that \mathcal{M}(T^x S^y) = \mathcal{M}(T)^x \mathcal{M}(S)^y; \mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y.
           Then \mathcal{M}(p(T_1,\ldots,T_n)) = \mathcal{M}(\sum_{k_1,\ldots,k_n} \alpha_{k_1,\ldots,k_n} \prod_{i=1}^n T_i^{k_i})
                                            = \sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} \prod_{i=1}^n \mathcal{M}(T_i^{k_i}) = p(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)).
                                                                                                                                          • Coro: Supp \tau is an algebraic property. Then \tau holds for liney maps \Longleftrightarrow \tau holds for matrices.
            Supp \alpha_1, ..., \alpha_n are disti with each \alpha_k \in \{1, ..., n\}.
            Now p(T_1, ..., T_n) = p(T_{\alpha_1}, ..., T_{\alpha_n}) \iff p(\mathcal{M}(T_1), ..., \mathcal{M}(T_n)) = p(\mathcal{M}(T_{\alpha_1}), ..., \mathcal{M}(T_{\alpha_n})).
13 Prove the distr holds for matrix add and matrix multi.
     Supp A, B, C are matrices suth A(B+C) make sense, we prove the left distr.
Solus: Supp A \in \mathbf{F}^{m,n} and B, C \in \mathbf{F}^{n,p}.
           Note that [A(B+C)]_{j,k} = \sum_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum_{r=1}^{n} (A_{j,r}B_{r,k} + A_{j,r}C_{r,k}) = (AB+AC)_{j,k}.
           OR. Define T, S, R suth \mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C.
           A(B+C) = \mathcal{M}(T(S+R)) \xrightarrow{[3.9]} \mathcal{M}(TS+TR) = AB + AC.
           Or. T(S+R) = TS + TR \Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR) \Rightarrow A(B+C) = AB + AC.
                                                                                                                                          1 Supp T \in \mathcal{L}(V, W). Show for each pair of B_V and B_W,
  A = \mathcal{M}(T, B_V, B_W) has at least n = \dim \operatorname{range} T non0 ent.
SOLUS:
   Let U \oplus \text{null } T = V; B_U = (v_1, ..., v_n), B_V = (v_1, ..., v_m).
   Each Tv_k \neq 0 \iff A_{\cdot,k} \neq 0. Hence every such A_{\cdot,k} has at least one non0 ent.
                                                                                                                                          OR. We prove by ctradic. Supp A has at most (n-1) non0 ent.
   Then by Pigeon Hole Principle, at least one of A_{.1}, ..., A_{.n} equals 0.
   Thus there are at most (n-1) non0 vecs in Tv_1, ..., Tv_n.
   \mathbb{X} range T = \operatorname{span}(Tv_1, \dots, Tv_n) \Rightarrow \operatorname{dim}\operatorname{range} T = \operatorname{dim}\operatorname{span}(Tv_1, \dots, Tv_n) \leqslant n - 1. Ctradic.
                                                                                                                                          6 Supp V and W are finide and T \in \mathcal{L}(V, W).
   Prove dim range T = 1 \iff \exists B_V, B_W, all ent of A = \mathcal{M}(T, B_V, B_W) equal 1.
SOLUS:
   (a) Supp B_V = (v_1, ..., v_n), B_W = (w_1, ..., w_m) are the bses suth all ent of A equal 1.
        Then Tv_i = w_1 + \dots + w_m for all i = 1, \dots, n. Becs w_1, \dots, w_n is liney indep, w_1 + \dots + w_n \neq 0.
   (b) Supp dim range T = 1. Then dim null T = \dim V - 1.
        Let B_{\text{null }T} = (u_2, \dots, u_n). Extend to a bss (u_1, u_2, \dots, u_n) of V.
        Becs Tv_1 \neq 0. Extend to (Tv_1, w_2, \dots, w_m) a bss of W. Let w_1 = Tv_1 - w_2 - \dots - w_m.
        Now B_W = (w_1, ..., w_m). Let v_1 = u_1, v_i = u_1 + u_i. Now B_V = (v_1, ..., v_n).
                                                                                                                                          OR. Supp B_{\text{range }T} = (w). By [2.C Note For (15)], \exists B_W = (w_1, ..., w_m), w = w_1 + ... + w_m.
        By [2.C Tips], \exists a bss (u_1, ..., u_n) of V suth each u_k \notin \text{null } T.
        Now each Tu_k \in \text{range } T = \text{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists \lambda_k \in \mathbb{F} \setminus \{0\}.
        Let v_k = \lambda_k^{-1} u_k \neq 0, so that each Tv_k = w = w_1 + \dots + w_m. Thus B_V = (v_1, \dots, v_n) will do.
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3 Supp V and W are finide and T \in \mathcal{L}(V, W). Prove \exists B_V, B_W suth
   [ letting A = \mathcal{M}(T, B_V, B_W) ] A_{k,k} = 1, A_{i,j} = 0, where 1 \le k \le \dim \operatorname{range} T, i \ne j.
Solus: Let B_{\text{null }T} = (u_1, \dots, u_m), B_{\text{range }T} = (Tv_1, \dots, Tv_n) \Rightarrow B_V = (v_1, \dots, v_n, u_1, \dots, u_m).
                                                                                                                                                        COMMENT: Let each Tv_k = w_k. Extend B_{\text{range }T} to B_W = (w_1, \dots, w_n, \dots, w_p). See [3.D Note For [3.60]].
4 Supp B_V = (v_1, ..., v_m) and W is finide. Supp T \in \mathcal{L}(V, W).
   Prove \exists B_W = (w_1, ..., w_n), \ \mathcal{M}(T, B_V, B_W)_{1} = (1 \ 0 \ ... \ 0)^t \ or \ (0 \ ... \ 0)^t.
Solus: If Tv_1 = 0, then done. If not then extend (Tv_1) to B_W.
                                                                                                                                                         5 Supp B_W = (w_1, ..., w_n) and V is finide. Supp T \in \mathcal{L}(V, W).
   Prove \exists B_V = (v_1, ..., v_m), \ \mathcal{M}(T, B_V, B_W)_{1.} = (0 \ ... \ 0) \ or \ (1 \ 0 \ ... \ 0).
SOLUS:
   Let (u_1, ..., u_n) be a bss of V. Denote \mathcal{M}(T, (u_1, ..., u_n), B_W) by A.
   If A_{1,.} = 0, then B_V = (u_1, ..., u_n) and done. Othws, supp A_{1,k} \neq 0.
   \text{Let } v_1 = \frac{u_k}{A_{1,k}} \Rightarrow Tv_1 = 1w_1 + \frac{A_{2,k}}{A_{1,k}}w_2 + \dots + \frac{A_{n,k}}{A_{1,k}}w_n. \ \left| \begin{array}{l} \text{Let } v_{j+1} = u_j - A_{1,j}v_1 \text{ for each } j \in \left\{1,\dots,k-1\right\}. \\ \text{Let } v_i = u_i - A_{1,i}v_1 \text{ for } i \in \left\{k+1,\dots,n\right\}. \end{array} \right|
   NOTICE that Tu_i = A_{1,i}w_1 + \cdots + A_{n,i}w_n. \mathbb{X} Each u_i \in \text{span}(v_1, \dots, v_n) = V. Let B_V = (v_1, \dots, v_n).
                                                                                                                                                        Or. Using Exe (4). Let B_W, be the B_V. Now \exists B_V, suth \mathcal{M}(T', B_{W'}, B_{V'})_{:,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^t or \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}^t.
   Which is equiv to \exists B_V \text{ [Using (3.F.31)] suth } \mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}.
• (10.A.3, Or 4E 3.D.19) Supp V is finide and T \in \mathcal{L}(V).
                                                                                                                                  [See also in (3.A).]
  Prove \forall B_V \neq B_V', \mathcal{M}(T, B_V) = \mathcal{M}(T, B_V') \Longrightarrow T = \lambda I, \exists \lambda \in \mathbf{F}.
Solus: Supp \forall B_V \neq B_V', \mathcal{M}(T, B_V) = \mathcal{M}(T, B_V'). If T = 0, then done.
            Supp T \neq 0, and v \in V \setminus \{0\}. Asum (v, Tv) is liney indep.
            Extend (v, Tv) to B_V = (v, Tv, u_3, ..., u_n). Let B = \mathcal{M}(T, B_V).
            \Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \dots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \forall i \neq 2.
            By asum, A = \mathcal{M}(T, B'_V) = B, \forall B'_V = (v, w_2, ..., w_n). Then A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2.
            \Rightarrow Tv = w_2, which is not true if w_2 = u_3, w_3 = Tv, w_i = u_i, \forall j \in \{4, ..., n\}. Ctradic.
            Hence (v, Tv) is linely dep \Rightarrow \forall v \in V, \exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v.
            Now we show \lambda_v is indep of v, that is, for all disti v, w \in V \setminus \{0\}, \lambda_v = \lambda_w.
            (v,w) liney indep \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \} \Rightarrow T = \lambda I.
                                                                                                                                                        (v, w) linely dep, w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv)
   Or. Let A = \mathcal{M}(T, B_V), where B_V = (u_1, ..., u_m) is arb.
   Fix one B_V = (v_1, \dots, v_m) and then (v_1, \dots, \frac{1}{2}v_k, \dots, v_m) is also a bss for any given k \in \{1, \dots, m\}.
   Fix one k. Now we have T(\frac{1}{2}v_k) = A_{1,k}v_1 + \dots + A_{k,k}(\frac{1}{2}v_k) + \dots + A_{m,k}v_m
   \Rightarrow Tv_k = 2A_{1,k}v_1 + \dots + A_{k,k}v_k + \dots + 2A_{m,k}v_m = A_{1,k}v_1 + \dots + A_{k,k}v_k + \dots + A_{m,k}v_m.
   Then A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0 for all j \neq k. Thus Tv_k = A_{k,k}v_k, \forall k \in \{1, ..., m\}.
   Now we show A_{k,k} = A_{j,j} for all j \neq k. Choose j,k suth j \neq k.
   Consider B'_{V} = (v'_{1}, ..., v'_{i}, ..., v'_{m}), where v'_{i} = v_{k}, v'_{k} = v_{i} and v'_{i} = v_{i} for all i \in \{1, ..., m\} \setminus \{j, k\}.
   Now T(v'_k) = A_{1,k}v'_1 + \dots + A_{k,k}v'_k + \dots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_i, while T(v'_k) = T(v_i) = A_{i,i}v_i. \square
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• (3.E.2) Supp V_1 \times \cdots \times V_m is finide. Prove each V_i is finide.
Solus: For any k \in \{1, ..., m\}, define S_k \in \mathcal{L}(V_1 \times \cdots \times V_m, V_k) by S_k(v_1, ..., v_m) = v_k.
             Then S_k is liney map. By [3.22], range S_k = V_k is finide.
                                                                                                                                                                     Or. Denote V_1 \times \cdots \times V_m by U. Denote \{0\} \times \cdots \times \{0\} \times V_i \times \{0\} \cdots \times \{0\} by U_i.
             We show each U_i is iso to V_i. Then U is finide \Longrightarrow its subsp U_i is finide, so is V_i.
               \begin{aligned} & \text{Define } R_i \in \mathcal{L}(V_i, U_i) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \\ & \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{aligned} \right\} \Rightarrow \left\{ \begin{array}{l} R_i S_j |_{U_j} = \delta_{i,j} I_{U_j}, \\ S_i R_j = \delta_{i,j} I_{V_j}. \end{array} \right. 
                                                                                                                                                                     • (3.E.4) Prove \mathcal{L}(V_1 \times \cdots \times V_m, W) and \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) are iso.
Solus: Using nota in (3.E.2): R_i : u_i \mapsto (0, ..., u_i, ..., 0); S_i : (u_1, ..., u_m) \mapsto u_i.
   Note that T(u_1, ..., u_m) = T(u_1, 0, ..., 0) + ... + T(0, ..., u_m).
   Define \psi: (T_1, \dots, T_m) \mapsto T by \psi(T_1, \dots, T_m) = T_1 S_1 + \dots + T_m S_m. \rbrace \Rightarrow \psi = \varphi^{-1}.
                                                                                                                                                                     • (3.E.5) Prove \mathcal{L}(V, W_1 \times \cdots \times W_m) and \mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m) are iso.
Solus: Using nota in (3.E.2): R_i : u_i \mapsto (0, ..., u_i, ..., 0); S_i : (u_1, ..., u_m) \mapsto u_i.
 Note that T_i: v \mapsto w_i, Define \varphi: T \mapsto (T_1, \dots, T_m) by \varphi(T) = (S_1T, \dots, S_mT).
  T: v \mapsto (w_1, \dots, w_m). \mid \text{Define } \psi: (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = R_1 T_1 + \dots + R_m T_m. \right\} \Rightarrow \psi = \varphi^{-1}.
18 Show V and \mathcal{L}(\mathbf{F}, V) are iso vecsps.
Solus: Define \Psi \in \mathcal{L}(V, \mathcal{L}(F, V)) by \Psi(v) = \Psi_v; where \Psi_v \in \mathcal{L}(F, V) and \Psi_v(\lambda) = \lambda v.
              (a) \Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbb{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0. Now \Psi inje.
             (b) \forall T \in \mathcal{L}(\mathbf{F}, V), let v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi(T(1)) \in \operatorname{range} \Psi. \square
             Or. Define \Phi \in \mathcal{L}(\mathcal{L}(\mathbf{F}, V), V) by \Phi(T) = T(1).
              (a) Supp \Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbb{F} \Rightarrow T = 0. Now \Phi inje.
              (b) For any v \in V, define T \in \mathcal{L}(F, V) by T(\lambda) = \lambda v. Then \Phi(T) = T(1) = v \in \text{range }\Phi.
COMMENT: \Phi = \Psi^{-1}. This is a countexa of the stmt that \mathcal{L}(V, W) and \mathcal{L}(W, V) are iso if infinde. See (3.F).
• (3.E.6) Supp m \in \mathbb{N}^+. Prove V^m and \mathcal{L}(\mathbb{F}^m, V) are iso.
Solus: Using (3.D.18) and (3.E.4), immed.
                                                                                                                                                                     Or. Define T:(v_1,\ldots,v_m)\to\varphi, where \varphi:(a_1,\ldots,a_m)\mapsto a_1v_1+\cdots+a_mv_m.
   (a) Supp T(v_1, ..., v_m) = 0. Then \forall (a_1, ..., a_n) \in \mathbb{F}^m, \varphi(a_1, ..., a_m) = a_1 v_1 + ... + a_m v_m = 0
          For each k, let a_k = 1, a_j = 0 for all j \neq k. Then each v_k = 0 \Rightarrow (v_1, \dots, v_m) = 0. Thus T is inje.
   (b) Supp \psi \in \mathcal{L}(\mathbf{F}^m, V). Let (e_1, \dots, e_m) be std bss of \mathbf{F}^m. Then \forall (b_1, \dots, b_n) \in \mathbf{F}^m,
           \left[ T(\psi(e_1), \dots, \psi(e_m)) \right] (b_1, \dots, b_m) = b_1 \psi(e_1) + \dots + b_m \psi(e_m) = \psi(b_1 e_1 + \dots + b_m e_m) = \psi(b_1, \dots, b_m). 
          Thus T(\psi(e_1), \dots, \psi(e_m)) = \psi. Hence T is surj.
• Supp T \in \mathcal{L}(V). Prove \exists inv R, S \in \mathcal{L}(V) suth T = T_1 + T_2.
Solus: Let U \oplus \text{null } T = V, W \oplus \text{range } T = V. Let S : \text{null } T \to W be an iso.
             Define T_1 \in \mathcal{L}(V) by T_1(u) = \frac{1}{2}Tu, T_1(w) = Sw
Define T_2 \in \mathcal{L}(V) by T_2(u) = \frac{1}{2}Tu, T_2(w) = -Sw
                                                                                                     \Rightarrow T = T_1 + T_2 \text{ and } T_1, T_2 \text{ inv.}
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2 Supp dim V \ge 2. The set U of non-inv optors on V is not a subsp of \mathcal{L}(V).
   The set of inv optors is not either. Although multi id/inv, and commu for vec multi hold.
Solus: Similar to (3.B.7 or 8). [ If dim V = 1, then U = \{0\} is a subsp of \mathcal{L}(V).]
                                                                                                                                • Tips: Supp V = U \oplus X = W \oplus X. Prove U, W are iso.
Solus: \forall u \in U, \exists ! (w, x_1) \in W \times X, u = w + x_1. While \exists ! (u', x_2) \in U \times X, w = u' + x_2.
          Now x_1 = -x_2, u = u'. Thus \pi : U \to W defined by \pi(u) = w, is inje.
          \forall w \in W, \exists ! (u, x_1) \in U \times X, w = u + x_1. \text{ While } \exists ! (w', x_2) \in W \times X, u = w' + x_2.
          Now x_1 = -x_2, w = w'. Thus \pi : U \to W defined by \pi(u) = w, is surj.
                                                                                                                                COMMENT: Let V = \mathbb{F}^{\infty}. Let X = \mathbb{F}^{\infty}, Y = \{(0, x_1, x_2, \dots) \in \mathbb{F}^{\infty}\}. Now X, Y are iso subsps of V.
              But \nexists iso subsps M, N of V, suth V = M \oplus X = N \oplus Y.
• (3.E.3) Give an exa of a vecsp V and its two subsps U_1, U_2 suth
           U_1 \times U_2 and U_1 + U_2 are iso but U_1 + U_2 is not a direct sum.
                                                                                                         [V must be infinide.]
Solus: Note that at least one of U_1, U_2 must be infinide. Both can be infinide. [Req Other Courses.]
  Let V = \mathbf{F}^{\infty} = U_1, U_2 = \{(x, 0, \dots) \in \mathbf{F}^{\infty} : x \in \mathbf{F}\}. Then V = U_1 + U_2 is not a direct sum.
  Define T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) by T((x_1, x_2, \dots), (x, 0, \dots)) = (x, x_1, x_2, \dots)
                                                                                                                                Define S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) by S(x, x_1, x_2, \dots) = ((x_1, x_2, \dots), (x, 0, \dots))
3 Supp V and W are iso and finide, U is a subsp of V, and S \in \mathcal{L}(U, W).
  Prove \exists inv T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U \iff S is inje.
                                                                                                         See also (3.A.11).
Solus: (a) \forall u \in U, u = T^{-1}Su \Rightarrow T^{-1}S = I \in \mathcal{L}(U). Thus by (3.B.20), S is inje.
               Or. \operatorname{null} S = \operatorname{null} T|_{U} = \operatorname{null} T \cap U = \{0\}.
          (b) Let B_U = (u_1, ..., u_m). Then S inje \Rightarrow (Su_1, ..., Su_m) liney indep.
               Extend to B_V = (u_1, ..., u_m, v_1, ..., v_n), B_W = (Su_1, ..., Su_m, w_1, ..., w_n).
               Define T \in \mathcal{L}(V, W) by T(u_i) = Su_i; Tv_i = w_i, for each u_i and v_i.
                                                                                                                                EXA: Supp V, W are infinide. Let V = W = \mathbf{F}^{\infty}. Define S(x_1, x_2, \dots) = (0, x_1, x_2, \dots).
      Now S is inje. Supp \exists inv T \in \mathcal{L}(V, W) suth T|_{V} = S. Then T = S while S is not surj.
8 Supp T \in \mathcal{L}(V, W) is surj. Prove \exists subsp U of V, T|_{U} : U \to W is iso.
Solus: By (3.B.12). Note that range T = W. Or. [ Reg range T Finide ] By [3.B TIPS (4)].
• COMMENT: If S \in \mathcal{L}(V) is iso, T \in \mathcal{L}(U, W) is iso, and W \subsetneq V, then ST = S|_W T is merely inje.
9 [OR 1] Supp U, V, W are iso and finide, S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V).
            Prove ST is inv \iff S,T are inv.
            Note: Suppone of U, V, W infinide \Rightarrow all infinde. Then S, T inv \Longrightarrow ST inv.
Solus: Supp S, T inv. Then (ST)(T^{-1}S^{-1}) = I_W, (T^{-1}S^{-1})(ST) = I_U. Hence ST inv.
          Supp ST inv. Let R = (ST)^{-1} \Rightarrow R(ST) = I_U, (ST)R = I_W.
          Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0.
                                                                   T inje, S surj.
          \forall v \in V, v = (ST)Rv = S(TRv) \in \text{range } S. \mid \emptyset \text{ dim } U = \text{dim } V = \text{dim } W.
          OR. By (3.B.23), dim W = \dim \operatorname{range} ST \leq \min \{\operatorname{range} S, \operatorname{range} T\} \Rightarrow S, T \operatorname{surj}.
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• TIPS: Supp each $S_k \in \mathcal{L}(V_k, W_k)$, $W_k \subseteq V_{k+1} \Rightarrow S_m \circ S_{m-1} \circ \cdots \circ S_2 \circ S_1$ makes sense. (a) By the ctrapos of (3.B.11), $S_m \circ \cdots \circ S_1$ not inje $\Rightarrow \exists S_k$ not inje. Convly not true unless k = 1. (b) By Exe (9), if all V_k finide and iso to each other, then $S_m \circ \cdots \circ S_1$ inje \Rightarrow inv, so are all S_k . (c) $\operatorname{null} S_1 \subseteq \operatorname{null}(S_2S_1) \subseteq \cdots \subseteq \operatorname{null}(S_m \cdots S_2S_1); S_m \circ \cdots \circ S_1 \text{ inje} \Rightarrow \operatorname{each} S_k \circ \cdots \circ S_1 \text{ inje}.$ Supp each $W_k = V_{k+1}$, for if $W_k \subsetneq V_{k+1}$, then S_1, S_2 surj $\not\Rightarrow S_2 \circ S_1 \in \mathcal{L}(V_1, W_2)$ surj. (d) Each $S_k \text{ surj} \Rightarrow S_m \circ \cdots \circ S_1 \text{ surj}$. Convly not true unless all V_k finide and iso to each other. (e) range $S_m \supseteq \text{range}(S_m S_{m-1}) \supseteq \cdots \supseteq \text{range}(S_m S_{m-1} \cdots S_1); \ S_m \circ \cdots \circ S_1 \text{ surj} \Rightarrow \text{each } S_m \circ \cdots \circ S_k \text{ surj.}$ **13** Supp U, V, W, X are iso and finide, $R \in \mathcal{L}(W, X), S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Supp RST is surj. Prove S is inje. **Solus**: Using Exe (9). Notice that U, X are finide, so that RST inv. Let $X = (RST)^{-1} \mid Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ inje.}$ $\forall v \in V, v = (RST)Xv \in \text{range } R \Rightarrow R \text{ surj.} \end{cases} \Rightarrow S = R^{-1}(RST)T^{-1}.$ Or. $(RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. **10** Supp V is finide and $S, T \in \mathcal{L}(V)$. Prove $ST = I \iff TS = I$. **Solus**: Supp ST = I. By $(3.B\ 20,\ 21)(a)$, $ST = I \Rightarrow T$ inje and S surj. X V finide. S, T inv. OR. By Exe (9), V finide and ST = I inv $\Rightarrow S, T$ inv. Then $\forall v \in V, S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow TS = I.$ Or. $S^{-1} = T \times S = S \Rightarrow TS = S^{-1}S = I$. Rev the roles and done. **11** Supp V is finide, S, T, $U \in \mathcal{L}(V)$ and STU = I. Show T is inv and $T^{-1} = US$. **Solus**: Using Exe (9) and (10). This result can fail without the hypo that V is finide. $(ST)U = U(ST) = (US)T = I \Rightarrow T^{-1} = US.$ Or. $(ST)U = S(TU) = I \Rightarrow U, S \text{ inv} \Rightarrow TU = S^{-1}. \ \ \ \ \ U^{-1} = U^{-1} \Rightarrow T = S^{-1}U^{-1}.$ Exa: $V = \mathbb{R}^{\infty}$, $S(a_1, a_2, \dots) = (a_2, \dots)$; $T(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$; $U = I \Rightarrow STU = I$ but T is not inv. **15** Prove every liney map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multi. In other words, prove if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists A \in \mathbf{F}^{m,n}, Tx = Ax, \forall x \in \mathbf{F}^{n,1}$. **Solus**: Let $B_1 = (E_1, ..., E_n), B_2 = (R_1, ..., R_m)$ be std bses of $\mathbf{F}^{n,1}, \mathbf{F}^{m,1}$. $\forall k=1,\ldots,n,\ T\big(E_k\big)=A_{1,k}R_1+\cdots+A_{m,k}R_m, \exists\, A_{i,k}\in \mathbb{F}, \text{ forming }A.$ OR. Let $A = \mathcal{M}(T, B_1, B_2)$. Note that $\mathcal{M}(x, B_1) = x$, $\mathcal{M}(Tx, B_2) = Tx$. Hence $Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2) \mathcal{M}(x, B_1) = Ax$, by [3.65]. • Note For [3.62]: $\mathcal{M}(v) = \mathcal{M}(I, (v), B_V)$. Where *I* is the id optor restr to span(*v*). • Note For [3.65]: $\mathcal{M}(Tv) = \mathcal{M}(I, (Tv), B_W) = \mathcal{M}(T, B_V, B_W) \mathcal{M}(I, (v), B_V) = \mathcal{M}(T, (v), B_W).$ If v = 0, then span(v) = span(), we replace (v) by B = (); simlr for Tv = 0. • TIPS: When using \mathcal{M}^{-1} , you must first declare bses and the purpose for using \mathcal{M}^{-1} . That is, to declare $B_U, B_V, B_W, \mathcal{M} : \mathcal{L}(V, W) \mapsto \mathbf{F}^{m,n}$, or $\mathcal{M} : v \mapsto \mathbf{F}^{n,1}$.

So that $\mathcal{M}^{-1}(AC, B_U, B_W) = \mathcal{M}^{-1}(A, B_V, B_W) \mathcal{M}^{-1}(C, B_U, B_V);$

Or. $\mathcal{M}^{-1}(Ax, B_W) = \mathcal{M}^{-1}(A, B_V, B_W) \mathcal{M}^{-1}(x, B_V)$. Where everything is well-defined.

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Solus: Notice that \mathcal{M}: T \mapsto \mathcal{M}(T, \alpha \to \beta) is iso. And that \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(TS).
    (a) T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(I) = \mathcal{M}(T)\mathcal{M}(T^{-1}) \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}.
    (b) \mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I, \exists ! S \in \mathcal{L}(V) \text{ suth } \mathcal{M}(T)^{-1} = \mathcal{M}(S)
            \Rightarrow \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = I = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)
            \Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}.
                                                                                                                                                                                                      Coro: Supp A \in \mathbf{F}^{n,n}. Then A is inv \iff \exists inv T \in \mathcal{L}(\mathbf{F}^n) suth \mathcal{M}(T, (e_1, \dots, e_n), (f_1, \dots, f_n)) = A.
• (4E 24, OR 10.A.2) Supp A, B \in \mathbf{F}^{n,n}. Prove AB = I \iff BA = I.
                                                                                                                                                                       [Using Exe (10, 15).]
Solus: Define T, S \in \mathcal{L}(\mathbf{F}^{n,1}) by Tx = Ax, Sx = Bx for all x \in \mathbf{F}^{n,1}. Now \mathcal{M}(T) = A, \mathcal{M}(S) = B.
                AB = I \Leftrightarrow A(Bx) = x \Leftrightarrow T(Sx) = x \Leftrightarrow TS = I \Leftrightarrow ST = I \Leftrightarrow \mathcal{M}(S)\mathcal{M}(T) = BA = I.
                OR. Becs \mathcal{M}: \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1}) \to \mathbf{F}^{n,n} is iso. \mathcal{M}^{-1}(AB) = TS = ST = \mathcal{M}^{-1}(BA) = I.
                                                                                                                                                                                                      • New Nota: For ease of nota, let \mathcal{M}(T, \alpha \to \beta) = \mathcal{M}(T, (\alpha_1, ..., \alpha_n), (\beta_1, ..., \beta_n)).
• (4E 23, OR 10.A.4) Supp (\beta_1, \ldots, \beta_n) and (\alpha_1, \ldots, \alpha_n) are bses of V.
   Let T \in \mathcal{L}(V) be such each T\alpha_k = \beta_k. Prove \mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha).
Solus: Denote \mathcal{M}(T, \alpha \to \alpha) by A and \mathcal{M}(I, \beta \to \alpha) by B.
                Each I\beta_k = \beta_k = B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n = T\alpha_k = A_{1,k}\alpha_1 + \dots + A_{n,k}\alpha_n \Rightarrow A = B.
                                                                                                                                                                                                      Or. Note that \mathcal{M}(T, \alpha \to \beta) = I. Hence \mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)\mathcal{M}(T, \alpha \to \beta) = \mathcal{M}(I, \beta \to \alpha). \square
    Or. Note that \mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I.
    Hence \mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1} \left[ \mathcal{M}(T, \beta \to \beta) \mathcal{M}(I, \alpha \to \beta) \right] = \mathcal{M}(I, \beta \to \alpha).
                                                                                                                                                                                                      COMMENT: \mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B. Or. Let A' = \mathcal{M}(T, \beta \to \beta).
    Simlr. Now each T\beta_k = T(B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n) = A'_{1,k}\beta_1 + \dots + A'_{n,k}\beta_n \Rightarrow A' = B.
• Note For [3.60]: Supp B_V = (v_1, ..., v_n), B_W = (w_1, ..., w_m).
   Define E_{i,j} \in \mathcal{L}(V, W) by E_{i,j}(v_x) = \delta_{i,x}w_j. Denote \mathcal{M}(E_{i,j}) by \mathcal{E}^{(j,i)}. And (\mathcal{E}^{(j,i)})_{l,k} = \delta_{i,l}\delta_{j,k}.
   Coro: E_{l,k}E_{i,j} = \delta_{j,l}E_{i,k}, \ \mathcal{E}^{(k,l)}\mathcal{E}^{(j,i)} = \delta_{l,j}\mathcal{E}^{(k,i)}.
   \begin{array}{l} \text{Becs } \mathcal{M} \colon \mathcal{L}(V,W) \to \mathbf{F}^{m,n} \text{ is iso.} \\ E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}. \text{ By [2.42] and [3.61]:} \end{array} \qquad B_{\mathcal{L}(V,W)} = \begin{pmatrix} E_{1,1}, \, \cdots, E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, \, \cdots, E_{n,m} \end{pmatrix}; \quad B_{\mathbf{F}^{m,n}} = \begin{pmatrix} \mathcal{E}^{(1,1)}, \, \cdots, \mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, \, \cdots, \, \mathcal{E}^{(m,n)}, \end{pmatrix}. 
• Tips: Let B_{\text{range }T}=\left(Tv_1,\ldots,Tv_p\right), B_V=\left(v_1,\ldots,v_p,\ldots,v_n\right). Let each w_k=Tv_k.
              Extend to B_W = (w_1, ..., w_p, ..., w_m). Then T = E_{1,1} + ... + E_{p,p}, \mathcal{M}(T) = \mathcal{E}^{(1,1)} + ... + \mathcal{E}^{(p,p)}.
• Supp A \in \mathbf{F}^{n,n}, rank A = r. Define T, S \in \mathcal{L}(\mathbf{F}^{n,n}) by T(X) = AX, S(Y) = YA^t.
   Find the dim and a bss of range ST.
Solus: Becs A\mathcal{E}^{(j,k)} = \left[\sum_{x=1}^{n} A_{x,j} \mathcal{E}^{(x,j)}\right] \mathcal{E}^{(j,k)} = \sum_{x=1}^{n} A_{x,j} \mathcal{E}^{(x,k)}. Let B_{\text{col}A} = (C_{\cdot,1}, \dots, C_{\cdot,r}).
                Each A_{\cdot,j} = R_{1,j}C_{\cdot,1} + \dots + R_{r,j}C_{\cdot,r} \Rightarrow \text{range } T = \{C_{j,k} = \sum_{x=1}^{n} C_{x,j}\mathcal{E}^{(x,k)} : 1 \leq j \leq r, 1 \leq k \leq n\}.
                Becs C_{j,k}A^t = C_{j,k} \left[ \sum_{y=1}^n A_{k,y}^t \mathcal{E}^{(k,y)} \right] = \sum_{x=1}^n \sum_{y=1}^n C_{x,j} A_{y,k} \mathcal{E}^{(x,y)}.

Simlr, range ST = \left\{ \mathcal{X}_{j,k} = \sum_{x=1}^n \sum_{y=1}^n C_{x,j} C_{y,k} \mathcal{E}^{(x,y)} : 1 \leq j,k \leq r \right\}.
                (\mathcal{X}_{1,k},\ldots,\mathcal{X}_{r,k}) and (\mathcal{X}_{i,1},\ldots,\mathcal{X}_{i,r}) are liney indep.
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• (4E 22, OR 10.A.1) Supp $T \in \mathcal{L}(V)$. Prove $\mathcal{M}(T, \alpha \to \beta)$ is inv $\iff T$ itself is inv.

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Solus: (a) \forall T \in \mathcal{L}(U, V), ST = 0 \iff \text{range } T \subseteq \text{null } S. Thus \text{null } A = \mathcal{L}(U, \text{null } S).
                                 (b) \forall R \in \mathcal{L}(U, W), range R \subseteq \text{range } S \iff \exists T \in \mathcal{L}(U, V), R = ST, by (3.B 25).
                                                Thus range A = \mathcal{L}(U, \text{range } S).
                                                                                                                                                                                                                                                                                                                                                                                                             OR. Let B_{\text{range }S} = (w_1, \dots, w_s) with each w_i = Sv_i. Let B_W = (w_1, \dots, w_n), B_{\text{null }S} = (v_{s+1}, \dots, v_p).
         Let B_U = (u_1, \dots, u_m). Define E_{i,j} \in \mathcal{L}(V, W) : v_x \mapsto \delta_{i,x} w_j. Now S = E_{1,1} + \dots + E_{s,s}.
         Define R_{i,j} \in \mathcal{L}(U, V) : u_x \mapsto \delta_{i,x} v_j. Let E_{k,j} R_{i,k} = Q_{i,j} : u_x \mapsto \delta_{i,x} w_j.
        For any T \in \mathcal{L}(V), \exists ! A_{i,j} \in \mathbf{F}, T = \sum_{j=1}^{p} \sum_{i=1}^{m} A_{j,i} R_{i,j} \Longrightarrow \mathcal{M}(T, u \to v) = \begin{bmatrix} \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{s,1} & \cdots & A_{s,s} & \cdots & A_{s,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{p,1} & \cdots & A_{p,s} & \cdots & A_{p,m} \end{bmatrix}.
\Longrightarrow \mathcal{A}(T) = ST = \left(\sum_{k=1}^{s} E_{k,k}\right) \left(\sum_{j=1}^{p} \sum_{i=1}^{m} A_{j,i} R_{i,j}\right) = \sum_{j=1}^{s} \sum_{i=1}^{m} A_{i,j} Q_{j,i}.
      \mathcal{M}(S,v\to w)\mathcal{M}(T,u\to v) = \mathcal{M}(ST,u\to w) = \begin{pmatrix} A_{1,1} \cdots A_{1,s} \cdots A_{1,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{s,1} \cdots A_{s,s} \cdots A_{s,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{s,n} \cdots & 0 & \cdots & 0 \end{pmatrix} \quad \begin{matrix} \mathcal{X} \mathcal{M}(T,R) = \mathcal{M}(T,u\to v). \\ \mathcal{M}(T,R) = \mathcal{M}(T,R) = I, \\ \mathcal{M}(T,
        \operatorname{range} \mathcal{A} = \operatorname{span} \begin{cases} Q_{1,1}, \cdots, Q_{m,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,s}, \cdots, Q_{m,s} \end{cases}, \ \operatorname{null} \mathcal{A} = \operatorname{span} \begin{cases} R_{1,s+1}, \cdots, R_{m,s+1}, \\ \vdots & \ddots & \vdots \\ R_{1,p}, & \cdots, R_{m,p} \end{cases} \end{cases}  (a) \operatorname{dim} \operatorname{null} \mathcal{A} = m \times (p-s); (b) \operatorname{dim} \operatorname{range} \mathcal{A} = m \times s.
                                                                                                                                                                                                                                                                                                                                                                                                             • (4E 10) Supp V, W finide, U is a subsp of V, \mathcal{E} = \{T \in \mathcal{L}(V, W) : T|_{U} = 0\}.
      Prove dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \mathcal{L}(U, W).
Solus: Define \Phi : \mathcal{L}(V, W) \to \mathcal{L}(U, W) by \Phi(T) = T|_U. By [3.A Note For Restriction], \Phi is liney.
                                 \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}. \text{ Thus null } \Phi = \mathcal{E}.
                                Extend S \in \mathcal{L}(U, W) to T \in \mathcal{L}(V, W) \Rightarrow \Phi(T) = S \in \text{range } \Phi. Thus range \Phi = \mathcal{L}(U, W). \square
                                Or. Let B_U = (u_1, ..., u_m), B_V = (u_1, ..., u_m, ..., u_n), B_W = (w_1, ..., w_v).
                                Define E_{i,j} \in \mathcal{L}(V, W) : u_x \mapsto \delta_{i,x} w_j.
                              \forall T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \operatorname{span} \left\{ \begin{array}{c} \vdots \\ \vdots \\ E_{1,p}, \dots, E_{m,p} \end{array} \right\} \cap \mathcal{E} = \{0\}.
\forall C = \operatorname{span} \left\{ \begin{array}{c} E_{m+1,1}, \dots, E_{n,1}, \\ \vdots \\ E_{m+1,p}, \dots, E_{n,p} \end{array} \right\} \subseteq \mathcal{E}.
| C = \operatorname{span} \left\{ \begin{array}{c} E_{m+1,1}, \dots, E_{m,p}, \\ \vdots \\ E_{m+1,p}, \dots, E_{m,p} \end{array} \right\} \cap \mathcal{E} = \{0\}.
| C = \operatorname{span} \left\{ \begin{array}{c} E_{m+1,1}, \dots, E_{m,p}, \\ \vdots \\ E_{m+1,p}, \dots, E_{m,p} \end{array} \right\} \subseteq \mathcal{E}.
| C = \operatorname{span} \left\{ \begin{array}{c} E_{m+1,1}, \dots, E_{m,p}, \\ \vdots \\ E_{m+1,p}, \dots, E_{m,p} \end{array} \right\} \subseteq \mathcal{E}.
| C = \operatorname{span} \left\{ \begin{array}{c} E_{m+1,1}, \dots, E_{m,p}, \\ \vdots \\ E_{m+1,p}, \dots, E_{m,p} \end{array} \right\} \subseteq \mathcal{E}.
• Supp U, V, W finide, S \in \mathcal{L}(U, V), \mathcal{B} \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(U, W)) : T \mapsto TS.
      Show dim null \mathcal{B} = (\dim W)(\dim \text{null } S), dim range \mathcal{B} = (\dim W)(\dim \text{range } S).
Solus: (a) \forall T \in \mathcal{L}(V, W), TS = 0 \iff \text{range } S \subseteq \text{null } T. \text{ Thus null } \mathcal{B} = \{T \in \mathcal{L}(V, W) : T|_{\text{range } S} = 0\}.
                                 (b) \forall R \in \mathcal{L}(U, W), \text{null } S \subseteq \text{null } R \iff \exists T \in \mathcal{L}(V, W), R = TS, by (3.B.24).
                                                Thus range \mathcal{B} = \{R \in \mathcal{L}(U, W) : R|_{\text{null }S} = 0\}. Now by Exe (4E 10).
                                                                                                                                                                                                                                                                                                                                                                                                             Or. Let B_{\text{range }S} = (v_1, ..., v_r) with each u_i = Sv_i. Let B_V = (v_1, ..., v_m), B_{\text{null }S} = (u_{r+1}, ..., u_n).
         Let B_W = (w_1, \dots, w_p). Define E_{i,j} \in \mathcal{L}(U, V) : u_x \mapsto \delta_{i,x} v_j \Rightarrow S = E_{1,1} + \dots + E_{r,r}.
         Define R_{i,j} \in \mathcal{L}(V, W) : v_x \mapsto \delta_{i,x} w_j. Let R_{k,j} E_{i,k} = Q_{i,j} : u_x \mapsto \delta_{i,x} w_j.
        Define R_{i,j} \in \mathcal{L}(V, W) : v_x \mapsto \delta_{i,x} w_j. Let R_{k,j} E_{i,k} = Q_{i,j} : u_x \mapsto \delta_{i,x} w_j.

\mathcal{B}(T) = TS = \left(\sum_{j=1}^p \sum_{i=1}^m A_{j,i} R_{i,j}\right) \left(\sum_{k=1}^r E_{k,k}\right) = \sum_{j=1}^p \sum_{i=1}^r A_{j,i} Q_{i,j} \Rightarrow \mathcal{M}(TS, v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,r} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{r,1} & \cdots & A_{r,r} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{p,1} & \cdots & A_{p,r} & \cdots & 0 \end{pmatrix}.
        range \mathcal{B} = \operatorname{span} \begin{Bmatrix} Q_{1,1}, \cdots, Q_{r,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,p}, \cdots, Q_{r,p} \end{Bmatrix}, \operatorname{null} \mathcal{B} = \operatorname{span} \begin{Bmatrix} R_{r+1,1}, \cdots, R_{n,1}, \\ \vdots & \ddots & \vdots \\ R_{r+1,n}, \cdots, R_{n,n} \end{Bmatrix}.
```

• (4E 17) Supp U, V, W finide, $S \in \mathcal{L}(V, W), A \in \mathcal{L}(\mathcal{L}(U, V), \mathcal{L}(U, W)) : T \mapsto ST$.

Show dim null $\mathcal{A} = (\dim U)(\dim \operatorname{null} S)$, dim range $\mathcal{A} = (\dim U)(\dim \operatorname{range} S)$.

```
16 Supp V is finide and S \in \mathcal{L}(V) suth \forall T \in \mathcal{L}(V), ST = TS. Prove \exists \lambda \in \mathbf{F}, S = \lambda I.

Solus: If S = 0, done. Now supp S \neq 0. [Using nota in Exe (4E 17). See also in (3.A).] Let S = E_{1,1} + \dots + E_{m,m} \Rightarrow \mathcal{M}(S, B_U) = \mathcal{M}(I, B_{\mathrm{range}S}, B_U). Note that R_{k,1} : w_x \mapsto \delta_{k,x} v_1. Then \forall k \in \{1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S. Hence dim null S = 0, dim range S = m = n.

Notice that G_{i,j} = R_{i,j}S = SR_{i,j} = Q_{i,j}. Where G_{i,j} : v_x \mapsto \delta_{i,x} v_j; Q_{i,j} : w_x \mapsto \delta_{i,x} w_j. For each w_i, \exists ! a_{k,i} \in \mathbf{F}, w_i = a_{1,i}v_1 + \dots + a_{n,i}v_n. Where a_{k,i} = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))_{k,i}. Then fix one i. Now for each j \in \{1, \dots, n\}, Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}\left(\sum_{k=1}^n a_{k,i}v_k\right). Let \lambda = a_{i,i}. Hence each w_j = \lambda v_j. Now fix one j, we have a_{1,1}v_j = \dots = a_{n,n}v_j, then all a_{i,i} are equal. Thus each w_j = \lambda v_j \Longrightarrow \mathcal{M}(S, B_U) = \mathcal{M}(\lambda I).
```

17 Supp V is finide. Show the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

SOLUS: If $\mathcal{E} = \{0\}$, then done. Supp $0 \neq T \in \mathcal{E}$, a two-sided ideal of $\mathcal{L}(V)$. Let $w = Tv \neq 0$. Extend $v = v_1$ to $B_V = (v_1, \dots, v_n) \Rightarrow Tv_1 = a_1v_1 + \dots + a_nv_n$. Supp $a_k \neq 0$. Then each $E_{k,y}TE_{x,1} = E_{k,y} \big[a_1E_{x,1} + \dots + a_kE_{x,k} + \dots + a_nE_{k,n} \big] = a_kE_{x,y} \in \mathcal{E}$.

ENDED

3.E

- Note For [3.79], def of v + U: Given v + U, v is already uniqly determined, as a sort of precond. Even though v + U = v' + U, where v' is *purer* than v.
- Note For [3.85]: $v + U = w + U \iff v \in w + U, \ w \in v + U \iff v w \in U \iff (v + U) \cap (w + U) \neq \emptyset.$

• Note For [3.79, 3.83]:

If *U* is merely a subset of *V*, then [3.85, 86] do not hold $\Rightarrow V/U$ not a vecsp.

If *V* is merely a subset of a vecsp of which *U* is a subsp, then [3,79, 86] do not hold $\Rightarrow V/U$ not a vecsp. If *U* is a vecsp but not a subsp of *V*, while *U*, *V* are subsps of some vecsp, then everything's alright.

Hence if V/U is a vecsp, then V, U are subsps of some vecsp.

COMMENT: Supp U, V are subsps and U is not a subsp of V. Note that V/U = (V + U)/U.

Supp $v + U \in V/U$. Then $v \in V$, or possibly $v \in V + U$ as well. To avoid this ambiguity,

you have to specify the precond, what subsp that v belongs to.

Exa: Supp U + W = V. Then V/U = (U + W)/U = W/U. Let $W \cap U = I$, $U_I \oplus I = U$, $W_I \oplus I = W$.

Now $U_I \oplus W_I \oplus I = V$. Thus $W/U = (W_I \oplus I)/U = W_I/U$.

 $\forall w_1', w_2' \in W_I \text{ suth } w_1' + U = w_2' + U \in W_I/U, \ w_1' - w_2' \in U \cap W_I = \{0\} \Rightarrow w_1' = w_2'.$

• *Trivial Cases*: If $v \in U$, then $v + U = 0 + U = \{u : u \in U\} = U$. Now $U = 0 \in V/U$.

If $U = \{0\}$, then $v + U = v + \{0\} = \{v\}$, $V/U = V/\{0\} = \{\{v\} : v \in V\}$.

If $U = \emptyset$, then $v + U = v + \emptyset = \emptyset$, $V/U = V/\emptyset = \{\emptyset\}$.

- TIPS 1: V is a subsp of $U \iff \forall v \in V, v + U = 0 + U = U \iff V/U = \{0\} = \{U\}.$
- Note For [3.88]: If U, V are subspof some vecsp \mathcal{V} . Define the quot map $\pi \in \mathcal{L}(V, V/U)$. Then π is surj by def, and null $\pi = V \cap U$. Thus if \mathcal{V} is finide, then dim $V = \dim V/U + \dim (V \cap U)$. Or. Let $I = V \cap U$, $V_I \oplus I = V$. Becs $V/U = V_I/U$, iso to V_I . Now dim $V = \dim V_I + \dim I$.
- (4E 8) Supp $T \in \mathcal{L}(V, W)$, $w \in \text{range } T$. Prove $\{v \in V : Tv = w\} = u + \text{null } T$.

Solus: Let $\mathcal{K}_w = \{v \in V : Tv = w\}$. [Not a vecsp.] Supp $u \in \mathcal{K}_w$. Then $u + \text{null } T \subseteq \mathcal{K}_w$. And $\forall u' \in \mathcal{K}_w$, $u' - u \in \text{null } T \Rightarrow u' \in u + \text{null } T$. Now $\mathcal{K}_w \subseteq u + \text{null } T$.

7 Supp $\alpha, \beta \in V$, and U, W are subsps of V. Prove $\alpha + U = \beta + W \Rightarrow U = W$.

Solus: (a) $\alpha \in \alpha + U = \beta + W \Rightarrow \exists w \in W, \alpha = \beta + w \Rightarrow \alpha - \beta \in W$.

(b) $\beta \in \beta + W = \alpha + U \Rightarrow \exists u \in U, \beta = \alpha + u \Rightarrow \beta - \alpha \in U.$

Now $\beta + U = \alpha + U = \beta + W = \alpha + W$. Thus $\{\alpha + u : u \in U\} = \{\alpha + w : w \in W\} \Rightarrow U = W$.

Or. $\pm(\alpha - \beta) \in U \cap W \Rightarrow \left\{ \begin{array}{l} U \ni u = (\beta - \alpha) + w \in W \Rightarrow U \subseteq W \\ W \ni w = (\alpha - \beta) + u \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W.$

8 Supp A is a nonempty subset of V.

Prove A is a trislate of some subsp of $V \iff \lambda v + (1 - \lambda)w \in A$, $\forall v, w \in A, \lambda \in F$.

Solus: (a) Supp A = a + U. Then $\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + (\lambda(u_1 - u_2) + u_2) \in A$.

- (b) Supp $\lambda v + (1 \lambda)w \in A$, $\forall v, w \in A, \lambda \in \mathbf{F}$. Supp $\underline{a \in A}$ and let $A' = \{x a : x \in A\}$. Then $0 \in A'$ and $\forall (v a), (w a) \in A', \lambda \in \mathbf{F}$,
 - (I) $\lambda(v-a) = [\lambda v + (1-\lambda)a] a \in A'$.
 - (II) Becs $\lambda(v-a) + (1-\lambda)(w-a) = [\lambda v + (1-\lambda)w] a \in A'$. Let $\lambda = \frac{1}{2}$ here and use (I) above by $\lambda = 2$, we have $(v-a) + (w-a) \in A'$.

Or. Note that $v, a \in A \Rightarrow \lambda v + (1 - \lambda)a = 2v - a \in A$. Simly $2w - a \in A$.

Now $(v - \frac{1}{2}a) + (w - \frac{1}{2}a) = v + w - a \in A \Rightarrow v + w - 2a = (v - a) + (w - a) \in A'$.

Thus A' = -a + A is a subsp of V. Hence $a + A' = a + \{x - a : x \in A\} = A$ is a trslate.

Prove $A \cap B$ *is either a trslate of some subsp of* V *or is* \emptyset . **Solus**: $\forall \alpha + u, \beta + w \in A \cap B \neq \emptyset, \lambda \in F, \lambda(\alpha + u) + (1 - \lambda)(\beta + w) \in A \cap B$. By Exe (8). Or. Let $A = \alpha + U$, $B = \beta + W$. Supp $v \in (\alpha + U) \cap (\beta + W) \neq \emptyset$. Then $v - \alpha \in U \Rightarrow v + U = \alpha + U = A$, and simlr $v + W = \beta + W = B$. We show $A \cap B = v + (U \cap W)$. Note that $v + (U \cap W) \subseteq A \cap B$. And $\forall \gamma = v + u = v + w \in A \cap B \Rightarrow u = w \in U \cap W \Rightarrow \gamma \in v + (U \cap W)$. **10** *Prove the intersec of any collec of trslates of subsps is either a trslate of some subsps or* \emptyset . **Solus**: Supp $\{A_{\alpha}\}_{{\alpha}\in\Gamma}$ is a collectof trslates of subspst of V, where Γ is an index set. $\forall x, y \in \bigcap_{\alpha \in \Gamma} A_{\alpha} \neq \emptyset, \lambda \in \Gamma, \lambda x + (1 - \lambda)y \in A_{\alpha} \text{ for each } \alpha. \text{ By Exe } (8).$ Or. Let each $A_{\alpha} = w_{\alpha} + V_{\alpha}$. Supp $x \in \bigcap_{\alpha \in \Gamma} (w_{\alpha} + V_{\alpha}) \neq \emptyset$. Then $x - w_{\alpha} \in V_{\alpha} \Longrightarrow x + V_{\alpha} = w_{\alpha} + V_{\alpha} = A_{\alpha}$, for each α . We show $\bigcap_{\alpha \in \Gamma} A_{\alpha} = \bigcap_{\alpha \in \Gamma} (x + V_{\alpha}) = x + \bigcap_{\alpha \in \Gamma} V_{\alpha}$. $y \in \bigcap_{\alpha \in \Gamma} A_{\alpha} \iff \text{for each } \alpha, \ y = x + v_{\alpha} \in A_{\alpha}$ \Leftrightarrow each $v_{\alpha} = y - x \in \bigcap_{\alpha \in \Gamma} V_{\alpha} \Leftrightarrow y \in x + \bigcap_{\alpha \in \Gamma} V_{\alpha}$. **11** Supp $A = \{\lambda_1 v_1 + \dots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\}$, where each $v_i \in V, \lambda_i \in F$. (a) *Prove A is a trslate of some subsp of V* (b) Prove if B is a trslate of some subsp of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$. (c) Prove A is a trslate of some subsp of V of dim < m. Solus: (a) By Exe (8), $\forall u, w \in A, \lambda \in \mathbb{F}, \lambda u + (1 - \lambda)w = \left(\lambda \sum_{i=1}^{m} a_i + (1 - \lambda) \sum_{i=1}^{m} b_i\right)v_i \in A.$ (b) Supp B = v + U, where $v \in V$ and U is a subsp of V. Let each $v_k = v + u_k \in B$, $\exists ! u_k \in U$. $\forall w \in A, \ w = \sum_{i=1}^{m} \lambda_i v_i = \sum_{i=1}^{m} \lambda_i (v + u_i) = \sum_{i=1}^{m} \lambda_i v + \sum_{i=1}^{m} \lambda_i u_i = v + \sum_{i=1}^{m} \lambda_i u_i \in v + U = B.$ Or. Let $v = \lambda_1 v_1 + \dots + \lambda_m v_m \in A$. To show $v \in B$, use induc on m by k. (i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. $\forall v_1 \in B$. Hence $v \in B$. (ii) $2 \le k < m$. Asum $v = \lambda_1 v_1 + \dots + \lambda_k v_k \in A \subseteq B$. $\left[\forall \lambda_i \text{ suth } \sum_{i=1}^k \lambda_i = 1 \right]$ For $u = \mu_1 v_1 + \dots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. Fix one $\mu_i \neq 1$. Then $\sum_{i=1}^{k+1} \mu_i - \mu_i = 1 - \mu_i \Longrightarrow \left[\sum_{i=1}^{k+1} \frac{\mu_i}{1 - \mu_i} \right] - \frac{\mu_i}{1 - \mu_i} = 1.$ Let $w = \underbrace{\frac{\mu_1}{1 - \mu_i} v_1 + \dots + \frac{\mu_{i-1}}{1 - \mu_i} v_{i-1} + \frac{\mu_{i+1}}{1 - \mu_i} v_{i+1} + \dots + \frac{\mu_{k+1}}{1 - \mu_i} v_{k+1}}_{l \text{ torus}}.$ Let $\lambda_i = \frac{\mu_i}{1 - \mu_i}$ for $i \in \{1, \dots, \iota - 1\}$; $\lambda_j = \frac{\mu_{j+1}}{1 - \mu_i}$ for $j \in \{\iota, \dots, k\}$. Then, $\sum_{i=1}^{k} \lambda_i = 1 \Rightarrow w \in B$ $v_i \in B \Rightarrow u' = \lambda w + (1 - \lambda)v_i \in B$ \rightarrow Let $\lambda = 1 - \mu_i$. Thus $u' = u \in B \Rightarrow A \subseteq B$. (c) If m = 1, then let $A = v_1 + \{0\}$ and done. Now supp $m \ge 2$. Fix one $k \in \{1, ..., m\}$. $A \ni \lambda_1 v_1 + \dots + \lambda_{k-1} v_{k-1} + \left(1 - \lambda_1 - \dots - \lambda_{k-1} - \lambda_{k+1} - \dots - \lambda_m\right) v_k + \lambda_{k+1} v_{k+1} + \dots + \lambda_m v_m$ $= v_k + \lambda_1(v_1 - v_k) + \dots + \lambda_{k-1}(v_{k-1} - v_k) + \lambda_{k+1}(v_{k+1} - v_k) + \dots + \lambda_m(v_m - v_k)$ $\in v_k + \operatorname{span}(v_1 - v_k, \dots, v_m - v_k).$

9 Supp $A = \alpha + U$ and $B = \beta + W$ for some $\alpha, \beta \in V$ and some subsps U, W of V.

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18 Supp T \in \mathcal{L}(V, W) and U, V are subsps of V. Let \pi : V \to V/U be the quot map.
     Prove \exists S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \cap V = \text{null } \pi \subseteq \text{null } T.
Solus: Supp null \pi \subseteq null T. By (3.B.24), done. Or. Define S: (v + U) \mapsto Tv.
            \forall v_1, v_2 \in V \text{ suth } v_1 + U = v_2 + U \Longleftrightarrow v_1 - v_2 \in U \cap V \subseteq \text{null } T \Longleftrightarrow Tv_1 = Tv_2.
            Thus S is well-defined. Convly true as well.
                                                                                                                                                 Coro: \Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W) with S \mapsto S \circ \pi is inje, range \Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\}.
COMMENT: If T = I_V. Then S : v + U \rightarrow v is not well-defined, unless U \cap V = \{0\} \subseteq \text{null } I_V.
• Note For [3.88, 3.90, 3.91]: Supp W \oplus U = V. Then V/U = W/U is iso to W. [Convly not true.]
  Becs \forall v \in V, \exists ! u_v \in U, w_v \in W, v = u_v + w_v. Define T \in \mathcal{L}(V) by T(v) = w_v.
  Hence \operatorname{null} T = U, \operatorname{range} T = W, \operatorname{range} T \oplus \operatorname{null} T = V.
  Then \tilde{T} \in \mathcal{L}(V/\text{null }T,V) is defined by \tilde{T}(v+U) = \tilde{T}(w_v'+U) = Tw_v' = w_v. [See Exa below]
  Now \pi \circ \tilde{T} = I_{V/U}, \tilde{T} \circ \pi|_W = I_W = T|_W. Hence \tilde{T} = (\pi|_W)^{-1} is iso of V/U onto W.
• Exa: Let V = \mathbf{F}^2, B_U = (e_1), B_W = (e_2 - e_1) \Rightarrow U \oplus W = V.
Solus: Although (e_2 - e_1) + U = e_2 + U, \tilde{T}(e_2 + U) = T(e_2) = e_2 - e_1. Becs e_2 = e_1 + (e_2 - e_1) \in U \oplus W.
17 Supp V/U is finide. Supp W is finide and V = U + W. Show dim W \ge \dim V/U.
Solus: Let Y \oplus (U \cap W) = W. Then by [1.C TIPS (4)], V = U \oplus Y. Note that V/U and Y are iso.
                                                                                                                                                 Or. Let B_W = (w_1, ..., w_n). Then V = U + \text{span}(w_1, ..., w_n).
           \forall v \in V, \exists u \in U, v = u + (a_1 w_1 + \dots + a_n w_n) \Rightarrow v + U = (a_1 w_1 + \dots + a_n w_n) + U.
                                                                                                                                                 Note: If dim W = \dim V/U. Then B_{V/U} = (w_1 + U, ..., w_n + U). Supp v = \sum_{i=1}^n a_i w_i \in U \cap W
          \Rightarrow v + U = 0 = \sum_{i=1}^{n} a_i(w_i + U) \Rightarrow \text{each } a_i = 0. \text{ Thus } V = U \oplus W.
12 Supp U is a subsp of V. Prove is V is iso to U \times (V/U).
Solus:
   [ Req V/U Finide ] Let B_{V/U} = (v_1 + U, ..., v_n + U).
   Now \forall v \in V, \exists ! a_i \in F, v + U = \sum_{i=1}^n a_i v_i + U \Rightarrow v - \sum_{i=1}^n a_i v_i \in U \Rightarrow \exists ! u \in U, v = \sum_{i=1}^n a_i v_i + u.
   Thus define \varphi \in \mathcal{L}(V, U \times (V/U))
                                                         and \psi \in \mathcal{L}(U \times (V/U), V)
                by \varphi(v) = (u, \sum_{i=1}^{n} a_i v_i + U), and \psi(u, v + U) = \sum_{i=1}^{n} a_i v_i + u. Then \psi = \varphi^{-1}.
                                                                                                                                                 Or. Let W \oplus U = V. Define Tv = u_v, Sv = w_v \Rightarrow \tilde{T} \in \mathcal{L}(V/W, U), \tilde{S} \in \mathcal{L}(V/U, W) are iso.
   Define \psi(u, v + U) = u + \tilde{S}(v + U) = u + w_v. Define \varphi(v) = (\tilde{T}(v), v + U).
    \frac{(\psi \circ \varphi)(u_v + w_v) = \psi(u_v, w_v + U) = u_v + w_v}{(\varphi \circ \psi)(u, v + U) = \varphi(u + w_v) = (u, w_v + U)} \right\} \Rightarrow \psi = \varphi^{-1}. \text{ Or Becs } \psi \text{ or } \varphi \text{ is inje and surj.} 
                                                                                                                                                 13 Prove B_{V/U} = (v_1 + U, ..., v_m + U), B_U = (u_1, ..., u_n) \Rightarrow B_V = (v_1, ..., v_m, u_1, ..., u_n).
Solus: \forall v \in V, \exists ! a_i \in F, v + U = \sum_{i=1}^m a_i v_i + U \Rightarrow \exists ! b_i \in F, v - \sum_{i=1}^m a_i v_i = \sum_{i=1}^n b_i u_i \in U
           \Rightarrow \forall v \in V, \exists ! a_i, b_i \in F, v = \sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i.
                                                                                                                                                 Or. \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i = 0 \Rightarrow \sum_{i=1}^{m} a_i (v_i + U) = 0 \Rightarrow \text{each } a_i = 0 \Rightarrow \text{each } b_i = 0.
                                                                                                                                                 OR. Note that B = (v_1, \dots, v_m) is liney indep, and [\operatorname{span}(v_1, \dots, v_m) + U] \subseteq V.
           v \in \operatorname{span} B \cap U \iff v + U = \sum_{i=1}^{m} a_i (v_i + U) = 0 + U \iff v = 0. Hence \operatorname{span} B \cap U = \{0\}.
           Becs dim [\operatorname{span}(v_1, \dots, v_m) \oplus U] = m + n = \dim V. Now by (2.B.8).
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• (4E 14) Supp V = U \oplus W, B_W = (w_1, ..., w_m). Prove B_{V/U} = (w_1 + U, ..., w_m + U).
Solus: \forall v \in V, \exists ! u \in U, w \in W, v = u + w. \not \exists ! c_i \in F, w = \sum_{i=1}^m c_i w_i \Rightarrow v = \sum_{i=1}^m c_i w_i + u.
          Hence \forall v + U \in V/U, \exists ! c_i \in F, v + U = \sum_{i=1}^m c_i w_i + U.
                                                                                                                                  Or. Becs \pi|_W: W \to W/U is inv, and V/U = W/U.
                                                                                                                                  15 Supp \varphi \in \mathcal{L}(V, \mathbf{F}) \setminus \{0\}. Prove dim V/(\text{null }\varphi) = 1.
SOLUS: By [3.91] (d), dim range \varphi = 1 = \dim V / (\operatorname{null} \varphi).
          Or. By (3.B.29), \exists u, span(u) \oplus \text{null } \varphi = V. Then B_{V/\text{null } \varphi} = (u + \text{null } \varphi).
                                                                                                                                  16 Supp dim V/U = 1. Prove \exists \varphi \in \mathcal{L}(V, \mathbf{F}), null \varphi = U.
Solus: Supp V_0 \oplus U = V. Then V_0 is iso to V/U, dim V_0 = 1.
          Define \varphi \in \mathcal{L}(V, \mathbf{F}) by \varphi(v_0) = 1, \varphi(u) = 0, where v_0 \in V_0, u \in U.
                                                                                                                                  Or. Let B_{V/U} = (w + U). Then \forall v \in V, \exists ! a \in F, v + U = aw + U.
          Define \varphi: V \to \mathbf{F} by \varphi(v) = a. Then \varphi(v_1 + \lambda v_2) = a_1 + \lambda a_2 = \varphi(v_1) + \lambda \varphi(v_2).
          Now u \in U \iff u + U = 0w + U \iff \varphi(u) = 0.
                                                                                                                                  • Supp U, W are subsps of V, and X, Y are subsps of W.
  Supp U, X are iso, W, Y are iso. Prove or give a countexa: U/W and X/Y are iso.
Solus: A countexa: Let \mathcal{V} = \mathcal{W} = \mathbf{F}^2. Let U = X = Y = \operatorname{span}(e_1), W = \operatorname{span}(e_2).
          Then dim U/W = \dim U - \dim(U \cap W) = 1 \neq 0 = \dim X - \dim(X \cap Y) = \dim X/Y.
          Or. Let \mathcal{V}=U=W=\mathbf{F}^{\infty}=X, Y=\left\{ \left( 0,x_{1},x_{2},\cdots\right) \right\}. Then U/W=\left\{ 0\right\}, while \dim X/Y=1. \square
• Tips 2: Supp U, W are vecsps, I = U \cap W. Prove V = U + W \iff V/I = U/I \oplus W/I.
Solus: (a) Supp V = U + W. Then \forall v + I \in V/I, \exists (u_v, w_v) \in U \times W, v + I = (u_v + w_v) + I.
               Note that U/I, W/I \subseteq V/I. Thus V/I = U/I + W/I.
               \forall u + I = w + I \in (U/I) \cap (W/I), u - w \in I = U \cap W
               \Rightarrow \exists w' \in I, u = w + w' \in U \cap W \Rightarrow u + I = 0 + I = w + I. \text{ Thus } (U/I) \cap (W/I) = \{0\}.
           (b) Supp V/I = U/I \oplus W/I. Then \forall v \in V, v + I = (u + I) + (w + I)
                \Rightarrow v - u - w \in I = U \cap W \Rightarrow \exists x \in U \cap W, v = u + w + x \in U + W.
                                                                                                                                  • Tips 3: Supp U, W are subsps of V and X is a subsp of U \cap W.
            Prove U/W and (U/X)/(W/X) are iso.
Solus: Let U_X \oplus X = U, W_X \oplus X = W. Becs U/W = U_X/W, and U/X = U_X/X.
  Define T \in \mathcal{L}((U_X/X)/(W/X), U_X/W) by T((u_x + X) + W/X) = u_x + W.
   \forall u_1, u_2 \in U_X \text{ suth } (u_1 + X) + W/X = (u_2 + X) + W/X \Rightarrow u_1 - u_2 + X \in W/X
  \Rightarrow u_1 - u_2 \in X + W \not \subset u_1, u_2 \in U_X \Rightarrow u_1 - u_2 \in W \Rightarrow u_1 + W = u_2 + W. Now T is well-defined.
  Inje: \forall u_x \in U_X \text{ suth } u_x + W = 0 \Rightarrow u_x \in W_X \Rightarrow (u_x + X) \in W_X/X.
   Surj: \forall u_x \in U_X, u_x + W = T((u_x + X) + W/X). Hence T is iso.
                                                                                                                                  Or. Define S \in \mathcal{L}(U_X/X, U_X) by S(u_x + X) = u_x. Becs \forall u_1 + X = u_2 + X \in U_X/X,
  u_1 - u_2 \in X \times u_1, u_2 \in U_X \Rightarrow u_1 = u_2. Now S well-defined, and S/W^{(W/X)} = T defined above.
  Becs range S|_{W/X \cap U_X/X} \subseteq W, and U_X = \operatorname{range} S \Rightarrow U_X \subseteq \operatorname{range} S + W. Well-defined. Surj.
  For u_x \in U_X, u_x + W = 0 \iff u_x \in U_X \cap W \iff u_x + X \in (U_X \cap W)/X = \text{null } S/_W. Inje.
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Define T/X^U : V/U \to W/X by T/X^U(v+U) = Tv + X.
  (a) Prove T/X is well-defined \iff (\operatorname{range} T|_{U \cap V})/(X \cap W) = \{0\} \iff \operatorname{range} T|_{U \cap V} is a subsp of X \cap W.
  Supp T/X^U is well-defined, and thus is liney. Define \pi_U \in \mathcal{L}(V, V/U), \pi_X \in \mathcal{L}(W, W/X).
  Then T/X \circ \pi_U = \pi_X \circ T. Define T/X \in \mathcal{L}(V, W/X) by T/X (v) = Tv + X.
  (b) range T/X^U = \text{range}(T/X^U \circ \pi_U) = \text{range}(\pi_X \circ T) = (\text{range } T)/X.
  (c) Prove T/_X^U is surj \iff W = range T + X \cap W.
  (d) Show \operatorname{null} T/_X^U = \left(\operatorname{null} T/_X\right)/U. (e) T/_X^U is inje \iff \operatorname{null} T/_X \subseteq U.
Solus: (a) For v, w \in V. If v + U = w + U \iff v - w \in U \Rightarrow Tv - Tw \in X \cap W \iff Tv + X = Tw + X.
                 Then \forall u \in V \cap U, Tu \in X \Rightarrow \operatorname{range} T|_{U \cap V} \subseteq X \cap W. Convly true as well.
            (c) Supp T/X^U is surj. \forall w \in W, w + X \in W/X \Rightarrow \exists v + U \in V/U, Tv + X = w + X
                 \Rightarrow w - Tv \in X \cap W \Rightarrow w \in \operatorname{range} T + X \cap W. Hence W \subseteq \operatorname{range} T + X \cap W.
                 Convly, W = \operatorname{range} T + X \cap W \Rightarrow (\operatorname{range} T)/X = (\operatorname{range} T + X \cap W)/X = W/X.
            (\mathrm{d})\ v + U \in \mathrm{null}\ T/_X^U \Longleftrightarrow Tv \in X \Longleftrightarrow v \in \mathrm{null}\ T/_X \Longleftrightarrow v + U \in (\mathrm{null}\ T/_X)/U.
                                                                                                                                                 • COMMENT: Supp T \in \mathcal{L}(V). Define T/U \in \mathcal{L}(V/U) by T/U = T/U. Then
  (a) T/U well-defined \iff U \cap V invard T. (b) range T/U = \text{range}(\pi \circ T) = (\text{range } T)/U.
  (c) T/U \operatorname{surj} \iff V = \operatorname{range} T + U \cap V. (d) \operatorname{null} T/U = \left(\operatorname{null} T/U\right)/U. (e) T/U \operatorname{inje} \iff \operatorname{null} T/U \subseteq U.
• (5.A.33) Supp T \in \mathcal{L}(V). Prove T/\text{range } T = 0.
                                                                                                          By (b) or (d) above, immed.
Solus: v + \text{range } T \in V/\text{range } T \Rightarrow v + \text{range } T \in \text{null}(T/\text{range } T). Thus T/\text{range } T = 0.
• (5.A.34) Supp T \in \mathcal{L}(V). Prove T/\text{null } T is inje \iff null T \cap \text{range } T = \{0\}.
Solus: Notice that (T/\text{null }T)(u+\text{null }T)=Tu+\text{null }T=0 \iff Tu \in \text{null }T\cap \text{range }T.
           Now T/\text{null } T is inje \iff u + \text{null } T = 0 \iff Tu = 0 \iff \text{null } T \cap \text{range } T = \{0\}.
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• Supp $T \in \mathcal{L}(V, W)$, and U, V are subsps of some vecsp, and X, W are subsps of some vecsp.

ENDED

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• Note For Exe (1): Every liney functional is either surj or is a zero map.
  Which means, for \varphi \in V', \varphi = 0 \iff \dim \operatorname{span}(\varphi) = 0 \iff \dim \operatorname{range} \varphi = 0.
  And \varphi \neq 0 \iff \dim \operatorname{span}(\varphi) = 1 \iff \dim \operatorname{range} \varphi = 1. Thus \dim \operatorname{span}(\varphi) = \dim \operatorname{range} \varphi.
4 Supp U is a subsp of V \neq U. Prove U^0 \neq \{0\}.
Solus: Let X \oplus U = V \Rightarrow X \neq \{0\}. Supp s \in X \setminus \{0\}. Let Y \oplus \text{span}(s) = X.
           Define \varphi \in V' by \varphi(u + \lambda s + y) = \lambda. Hence \varphi \neq 0 and \varphi(u) = 0 for all u \in U.
                                                                                                                                          Or. [ Req V Finide ] By [3.106], dim U^0 = \dim V - \dim U > 0.
                Or. Let B_U = (u_1, ..., u_m), B_V = (u_1, ..., u_m, v_1, ..., v_n) with n \ge 1.
                Let B_V = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n). Then each \varphi \in \text{span}(\varphi_1, \dots, \varphi_n) will do.
                                                                                                                                          19 U^0 = \{0\} = V^0 \iff U = V. By the inv and ctrapos of Exe (4).
Coro:
COMMENT: Another proof of [3.108]: T is surj \iff T' is inje.
                (a) Supp T' is inje. Notice that \psi \neq 0 \iff T'(\psi) \neq 0 \iff \psi \notin (\text{range } T)^0.
                (b) T is surj \Rightarrow (range T)<sup>0</sup> = {0} = null T'.
                                                                                                                                          • Note For [3.102] and Exe (18): For U = \emptyset, U^0 = \{ \varphi \in V' : U \subseteq \text{null } \varphi \} = V'. While \{ 0 \}_V^0 = V'.
  Not a ctradic becs \emptyset is not a subsp. Now U^0 = V' can be true with U = \emptyset \neq \{0\}.
25 Supp U is a subsp of V. Explain why U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}.
Solus: Asum \forall \varphi \in U^0, \varphi(v) = 0 while v \in V \setminus U. Then let \text{span}(v) \oplus U \oplus X = V.
           \exists \varphi \in V', \text{null } \varphi = U \oplus X \Rightarrow \varphi \in U^0. \not \subseteq \varphi(v) = 0 \Rightarrow 0 \neq v \in \text{null } \varphi \cap \text{span}(v). Ctradic.
                                                                                                                                          COMMENT: X \subseteq W = \{v \in V : \varphi(v) = 0, \forall \varphi \in X^0\}, the promotion of the subset X of V.
• Supp U, W are subsps of V. Prove the promotion of U \cup W is U + W.
Solus: (U \cup W)^0 = \{ \varphi \in V' : \varphi(u) = \varphi(w) = \varphi(u+w) = 0, \forall u \in U, w \in W \} = (U+W)^0.
                                                                                                                                          • Supp X = \{x_1, ..., x_m\} \subseteq V. Prove the promotion of X is span(x_1, ..., x_m).
Solus: X^0 = \{ \varphi \in V' : \varphi(\lambda x_i + \mu x_k) = 0, \forall j, k \in \{1, ..., m\}, \lambda, \mu \in F \} = \text{span}(x_1, ..., x_m)^0.
                                                                                                                                          COMMENT: The promotion of every finite subset X of V is the smallest subsp of V containing X.
20 Supp U, W are subsets of V. Prove U \subseteq W \Rightarrow W^0 \subseteq U^0.
Solus: \forall \varphi \in W^0, u \in U \subseteq W, \varphi(u) = 0 \Rightarrow \varphi \in U^0. Thus W^0 \subseteq U^0.
                                                                                                                                          21 Supp U, W are subsps of V. Prove W^0 \subseteq U^0 \Rightarrow U \subseteq W.
Solus: Using Exe (25). Now v \in U \Rightarrow \forall \varphi \in W^0 \subseteq U^0, \varphi(v) = 0 \Rightarrow v \in W.
                                                                                                                                          Note: \varphi \in W^0 \iff \text{null } \varphi \supseteq W \Rightarrow \text{null } \varphi \supseteq U \iff \varphi \in U^0. But cannot conclude W \supseteq U.
COMMENT: (1) If U is merely a subset and W is a subsp. Promote U as X, let W = Y.
                     Then Y^0 = W^0 \subseteq U^0 = X^0 \Rightarrow Y = W \supseteq X \supseteq U. Still true.
                (2) If W is merely a subset and U is a subsp. Promote W as Y, let U = X. For exa,
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Let $W = \{(1,0), (0,1)\} \not\supseteq U = \{(x,0) \in \mathbb{R}^2\}$. Then $Y = \mathbb{R}^2 \supseteq X = U$, $Y^0 = \{0\} \subseteq X^0$.

```
22 Supp U and W are subsps of V. Prove (U + W)^0 = U^0 \cap W^0.
Solus: (a) \varphi \in (U+W)^0 \Rightarrow \forall u \in U, w \in W, \mid U \subseteq U+W \Rightarrow (U+W)^0 \subseteq U^0
                 \varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0. W \subseteq U + W \Rightarrow (U + W)^0 \subseteq W^0
            (b) \varphi \in U^0 \cap W^0 \subseteq V' \Rightarrow \forall u \in U, w \in W, \varphi(u+w) = 0 \Rightarrow \varphi \in (U+W)^0.
                                                                                                                                                  37 Supp U is a subsp of V and \pi is the quot map. Thus \pi' \in \mathcal{L}((V/U)', V').
     (a) Show \pi' is inje: Becs \pi is surj. Use [3.108].
    (b) Show range \pi' = U^0: By [3.109](b), range \pi' = (\text{null } \pi)^0 = U^0.
    (c) Conclude that \pi' is iso from (V/U)' onto U^0: Immed.
 \text{Solus:} \ \ (\text{a}) \ \text{Or.} \ \pi'(\varphi) = 0 \Longleftrightarrow \forall v \in V \ \big( \ \forall v + U \in V \ \big), \\ \varphi\big(\pi(v)\big) = \varphi(v + U) = 0 \Longleftrightarrow \varphi = 0. 
            (b) Or. \psi \in \operatorname{range} \pi' \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null} \psi \supseteq U \iff \psi \in U^0.
                                                                                                                                                  • Supp U is a subsp of V. Prove (V/U)' is iso to U^0.
                                                                                                            [ Another proof of [3.106] ]
Solus: Define \xi: U^0 \to (V/U)' by \xi(\varphi) = \widetilde{\varphi}, where \widetilde{\varphi} \in (V/U)' is defined by \widetilde{\varphi}(v+U) = \varphi(v).
            Inje: \xi(\varphi) = 0 = \widetilde{\varphi} \Rightarrow \forall v \in V (\forall v + U \in V/U), \widetilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0.
            Surj: \Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u+U) = \Phi(0+U) = 0 \Rightarrow U \subseteq \text{null}(\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi.
            Or. Define \nu:(V/U)'\to U^0 by \nu(\Phi)=\Phi\circ\pi. Now \nu\circ\xi=I_{U^0},\ \xi\circ\nu=I_{(V/U)},\ \Rightarrow\xi=\nu^{-1}.
23 Supp U and W are subsps of V. Prove (U \cap W)^0 = U^0 + W^0.
Solus:
   (a) \varphi = \psi + \beta \in U^0 + W^0 \Rightarrow \forall v \in U \cap W, OR. U \cap W \subseteq U \Rightarrow (U \cap W)^0 \supseteq U^0
        \varphi(v) = (\psi + \beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0.
                                                                              U \cap W \subseteq W \Rightarrow (U \cap W)^0 \supseteq W^0
   (b) \lceil Only \text{ in Finide} \rceil By Exe (22), \dim(U^0 + W^0) = \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0)
         = 2\dim V - \dim U - \dim W - (\dim V - \dim(U + W)) = \dim V - \dim(U \cap W).
                                                                                                                                                  Or. Let I = U \cap W. We show (U \cap W)^0 \subseteq U^0 + W^0.
        Define \chi \in \mathcal{L}(V/I, V/U \times V/W) by \chi : v + I \mapsto (v + U, v + W).
        Well-defined: v_1 + I = v_2 + I \in V/I \iff v_1 - v_2 \in I
                             \iff v_1 - v_2 \in U \text{ and } v_1 - v_2 \in W \Rightarrow (v_1 + U, v_1 + W) = (v_2 + U, v_2 + W).
         Inje: (v + U, v + W) = 0 \iff v \in U \cap W = I \iff v + I = 0.
        Surj: \forall v \in V \text{ suth } (v + U, v + W) \in V/U \times V/W, \text{ becs } \emptyset \neq (v + U) \cap (v + W) = v + I \in V/I.
        Thus \chi' \in \mathcal{L}((V/U \times V/W)', (V/I)') is iso. Now we find an iso of U^0 \times W^0 onto (U \cap W)^0.
        By (3.E.4), supp \xi: (V/U)' \times (V/W)' \rightarrow (V/U \times V/W)' is iso.
        By (c) in Exe (37), supp \Lambda_1: U^0 \times W^0 \to (V/U)' \times (V/W)' and \Lambda_2: (V/I)' \to (U \cap W)^0 are isos.
        Hence (\Lambda_2 \circ \chi' \circ \xi \circ \Lambda_1) : U^0 \times W^0 \to (U \cap W)^0 is iso. Now we see how it works:
        \forall (\varphi_U, \varphi_W) \in U^0 \times W^0, \text{null } \pi_U \subseteq \text{null } \varphi_U \Rightarrow \exists \psi_U \in (V/U)', \ \psi_U \circ \pi_U = \varphi_U, \text{ simlr for } \varphi_W,
        thus \Lambda_1: (\varphi_U, \varphi_W) \mapsto (\psi_U, \psi_W). Then \xi: (\psi_U, \psi_W) \mapsto (\psi_U S_U + \psi_W S_W), [See notas in (3.E.2). ]
        Now (\psi_U S_U + \psi_W S_W) \stackrel{\chi'}{\longrightarrow} (\psi_U S_U + \psi_W S_W) \circ \chi \stackrel{\Lambda_2}{\longmapsto} (\psi_U S_U + \psi_W S_W) \circ \chi \circ \pi_I,
        which sends v to \psi_U(v+U) + \psi_W(v+W) = (\varphi_U + \varphi_W)(v), which is \varphi_U + \varphi_W.
        Thus (\Lambda_2 \circ \chi' \circ \xi \circ \Lambda_1) is the surj \Lambda : U^0 \times W^0 \to U^0 + W^0 defined in [3.77].
                                                                                                                                                  COMMENT: Not true if U or W is merely a subset. Promote U \cap W as I, U as X, and W as Y.
Exa: Let U = \{(x, x + 1) \in \mathbb{R}^2\}, W = \mathbb{R}^2. Then U \cap W = I = U \neq \mathbb{R}^2 = X \cap Y.
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• Tips 1: Prove V = U \oplus W \iff V' = U^0 \oplus W^0.
Solus: U \cap W = \{0\} \iff (U \cap W)^0 = \{0\}_V^0 = V' = U^0 + W^0.
            V = U + W \iff (U + W)^0 = V_V^0 = \{0\} = U^0 \cap W^0.
                                                                                                                                                     • Supp V = U \oplus W. Define \iota : V \to U by \iota(u + w) = u. Thus \iota' \in \mathcal{L}(U', V').
  (a) Show \operatorname{null} \iota' = \{0\}: \operatorname{null} \iota' = (\operatorname{range} \iota)_U^0 = U_U^0 = \{0\}. Or. \iota'(\psi) = \psi \circ \iota = 0 \Longleftrightarrow U \subseteq \operatorname{null} \psi.
  (b) Prove range \iota' = W_V^0: range \iota' = (\text{null } \iota)_V^0 = W_V^0. Now \tilde{\iota}' is iso from U'/\{0\} onto W^0
Solus: (b) Or. Note that W = \text{null } \iota \subseteq \text{null } (\psi \circ \iota). Then \psi \circ \iota \in W^0 \Rightarrow \text{range } \iota' \in W^0.
                       Supp \varphi \in W^0. Becs null \iota = W \subseteq \text{null } \varphi. By [3.B Tips (3)], \varphi = \varphi \circ \iota = \iota'(\varphi).
                                                                                                                                                     • Supp V = U \oplus W. Prove U^0 = \{ \varphi \in V' : \varphi = \varphi \circ \iota \}, where \iota \in \mathcal{L}(V, W) : u_v + w_v \to w_v.
Solus: \varphi \in U^0 \iff U \subseteq \text{null } \varphi \iff \varphi = \varphi \circ \iota, \text{ by } [3.B \text{ Tips } (3)].
                                                                                                                                                     Note: The nota W_V' = \{ \varphi \in V' : \varphi = \varphi \circ \iota \} = U^0 \text{ is not well-defined [without a bss].}
          Simply becs W'_V have no info about the given U. Here is an informal explanation:
          Each liney map T \in \mathcal{L}(V, W) that vanishes on a given nontrivial U has its P'
          (though not uniq) suth U \oplus P = V' with T : P \mapsto \operatorname{range} T being surj.
          Hence \forall W \in \mathcal{S}_V U, U^0 = W_V'. But given nontrivial 'P', the corres 'U' is not uniq.
          Fix one W'_V, then U^0 is not uniq, with each U_k not equal to each other while each U_k^0 = W'_V.
EXA: Let B_V = (e_1, e_2). Let B_U = (e_1), B_X = (e_2 - e_1), B_Y = (e_2).
        Then \iota_X : ae_1 + b(e_2 - e_1) \mapsto b(e_2 - e_1), \ \iota_Y : ae_1 + be_2 \mapsto be_2. Now X_V' = Y_V' = U^0.
        (1) For V = U \oplus X, let B_{U_V'} = (\varphi) with \varphi : e_1 \mapsto 1, e_2 - e_1 \mapsto 0 \Rightarrow e_2 \mapsto 1.
        (2) For V = U \oplus Y, let B_{U_V'} = (\psi) with \psi : e_1 \mapsto 1, e_2 \mapsto 0.
        Thus X^0 = U_V' while Y^0 = U_V' \Rightarrow X^0 = Y^0 \Rightarrow X = Y, ctradic.
        To fix this, we must have a bss of V' as precond, which we'll see in the NOTE FOR Exa (31).
Note: Supp U is a subsp of V. Then finding the corres subsp in V' firstly reg another 'half' W \in S_V U,
          while finding the corres subsp of V for a subsp of V' must have the another 'half' asumed as precond.
31 Supp V is finide and B_{V'} = (\varphi_1, \dots, \varphi_n). Show \exists ! B_V whose dual bss is the B_{V'}.
Solus: For each k \in \{1, \dots, n\}, let \Gamma_k = \{1, \dots, n\} \setminus \{k\}. Let each U_k = \bigcap_{j \in \Gamma} \operatorname{null} \varphi_j.
            By Exe (4E 23), V' = \operatorname{span}(\varphi_1, \dots, \varphi_n) = (\operatorname{null} \varphi_1 \cap \dots \cap \operatorname{null} \varphi_n)^0 \Rightarrow U_k \cap \varphi_k = \{0\}.
            Thus \forall x_k \in U_k \setminus \{0\}, x_k \notin \text{null } \varphi_k \text{ while } x_k \in \text{null } \varphi_j \text{ for all } j \in \Gamma.
            Fix one x_k and let v_k = [\varphi_k(x_k)]^{-1}x_k \Rightarrow \varphi_k(v_k) = 1, \varphi_i(v_k) = 0 for all i \neq k.
            Simply for each v_k, \varphi_i(v_k) = \delta_{i,k} for all j \iff for each \varphi_i, \varphi_i(v_k) = \delta_{i,k} for all k.
            \not \subset a_1v_1 + \dots + a_nv_n = 0 \Rightarrow \operatorname{each} \varphi_k(0) = a_k.
            Now we prove the uniques part. Supp the dual bss of B'_V = (u_1, \dots, u_n) is the B_{V_V}.
            For each k, we have \varphi_i(v_k) = \varphi_i(u_k) for all k \Rightarrow v_k - u_k \in \bigcap \text{null } \varphi_i = \{0\}.
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• Note For Exe (31): Supp V is finide, and Ω is a subsp of V' with $B_{\Omega} = (\varphi_1, \ldots, \varphi_m)$. The 'W' is not clear when we are to find suth $W_V' = \Omega$, becs the another 'half' is undefined. Extend to $B_V = (\varphi_1, \ldots, \varphi_n)$. By Exe (31), \exists ! corres $B_V = (v_1, \ldots, v_n)$. Let $B_U = (v_{m+1}, \ldots, v_n)$, $B_W = (v_1, \ldots, v_m)$. Thus we found the W suth $\Omega = W_V'$, which is well-defined with B_V as precond.

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• TIPS 2: Supp \varphi_1, \dots, \varphi_m \in V'. Denote [\operatorname{null} \psi_a \cap \dots \cap \operatorname{null} \varphi_b] by \bigcap_a^b \operatorname{null} \varphi_I.
                 Supp \Omega is a subsp of V'. Denote \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\} by C^0 \Omega.
  If \Omega is infinide, then by def, \bigcap_{\in \Omega} \operatorname{null} \varphi = C^0 \Omega. If \Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m),
  then v \in \bigcap_{1}^{m} \operatorname{null} \varphi_{I} \iff \operatorname{each} \varphi_{k}(v) = 0 \iff \forall \varphi = \sum_{i=1}^{n} a_{i} \varphi_{i} \in \Omega, \varphi(v) = 0 \iff v \in C^{0} \Omega.
• (4E 23) Supp V is finide, \Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m) \subseteq V'. Prove \Omega = (\operatorname{null} \varphi_1 \cap \dots \cap \operatorname{null} \varphi_m)^0.
Solus: Becs each span(\varphi_k) \subseteq (null \varphi_k)<sup>0</sup>. By Note For Exe (1) and Exe (23), Immed.
               Or. Reduce to B_{\Omega} = (\beta_1, \dots, \beta_v). We show \Omega = (\text{null } \beta_1 \cap \dots \cap \text{null } \beta_v)^0, then done by Tips (3).
               Let B_V = (\beta_1, ..., \beta_v, \gamma_1, ..., \gamma_a). By Exe (31), let B_V = (v_1, ..., v_v, u_1, ..., u_a).
               Define each \Gamma_k = \{1, \dots, p\} \setminus \{k\}. Then \text{null } \beta_k = \text{span}\{v_i\}_{i \in \Gamma_k} \oplus \text{span}(u_1, \dots, u_q).
               Now (\text{null }\beta_1 \cap \cdots \cap \text{null }\beta_p) = \text{span}(u_1, \dots, u_q). Similr to (4E 2.C.16).
               Supp \varphi = \sum_{i=1}^p a_i \beta_i + \sum_{j=1}^q b_j \gamma_j \in \text{span}(u_1, \dots, u_q)^0. Then each \varphi(u_k) = 0 = b_k
               Thus span(u_1, \dots, u_q)^0 \subseteq \text{span}(\beta_1, \dots, \beta_p) = \Omega.
                                                                                                                                                                                            • Tips 3: Supp each \varphi_i, \beta_i \in \mathcal{L}(V, W). Supp span(\varphi_1, \dots, \varphi_m) = \text{span}(\beta_1, \dots, \beta_n).
                  Prove \operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m = \operatorname{null} \beta_1 \cap \cdots \cap \operatorname{null} \beta_n.
Solus: Becs each \beta_k \in \text{span}(\varphi_1, \dots, \varphi_m).
               \forall v \in \bigcap_{1}^{m} \text{null } \varphi_{I}, \beta_{k}(v) = 0. \text{ Thus } \bigcap_{1}^{m} \text{null } \varphi_{I} \subseteq \bigcap_{1}^{n} \text{null } \beta_{I}. \text{ Rev the roles and done.}
                                                                                                                                                                                            Note: Supp \varphi_i = c_1 \varphi_1 + \dots + c_{i-1} \varphi_{i-1}.
              Let N_i \oplus \bigcap_{1}^{j-1} \operatorname{null} \varphi_i = \operatorname{null} \varphi_i. Now \bigcap_{1}^{j} \operatorname{null} \varphi_i = \bigcap_{1}^{j-1} \operatorname{null} \varphi_i \cap (\operatorname{null} \varphi_i) = \bigcap_{1}^{j-1} \operatorname{null} \varphi_i.
              Thus \bigcap_{1}^{m} \operatorname{null} \varphi_{I} = \left[\bigcap_{1}^{j-1} \operatorname{null} \varphi_{I}\right] \cap \left[\bigcap_{i+1}^{m} \operatorname{null} \varphi_{I}\right]. Hence \bigcap_{1}^{n} \operatorname{null} \beta_{I} = \bigcap_{1}^{m} \operatorname{null} \varphi_{I}.
26 Supp V is finide, \Omega is a subsp of V'. Prove \Omega = (C^0 \Omega)^0.
Solus: Let B_{\Omega} = (\varphi_1, \dots, \varphi_m). By Tips (2) and Exe (4E 23).
                                                                                                                                                                                            Exa: Immed, \Omega \subseteq (C^0 \Omega)^0. Now we give a countexa for \Omega \supseteq (C^0 \Omega)^0.
          Let V = \{(x_1, x_2, \dots) \in \mathbb{F}^{\infty} : x_k \neq 0 \text{ for only finily many } k\}. Then V' = (\mathbb{F}^{\infty})'.
          Let \Omega = \left\{ \varphi \in \operatorname{span}(\varphi_{\alpha_1}, \dots, \varphi_{\alpha_m}) : \exists m, \alpha_k \in \mathbb{N}^+ \right\} \subsetneq V'. Then C^0 \Omega = \left\{ 0 \right\} \Rightarrow (C^0 \Omega)^0 = V'.
Coro: (1) C^0 span(\varphi_1, ..., \varphi_m) = \text{null } \varphi_1 \cap ... \cap \text{null } \varphi_m.
              (2) Supp V is finide. For every subsp \Omega of V', \exists! subsp U of V suth \Omega = U^0.
                     This form of \Omega does not depend on a bss and thus is considered more general.
• Supp span(\varphi_1, ..., \varphi_m) \subseteq V'. Let each U_k \oplus \text{null } \varphi_k = V.
  Prove or give a countexa: (U_1 + \cdots + U_m) \oplus (\operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m) = V.
Solus: Let V = \mathbb{R}^2. Define \varphi_1 = \varphi_2 : (x,y) \mapsto x. Let B_{U_1} = (e_1), B_{U_2} = (e_1 + e_2) \Rightarrow U_1 + U_2 = V.
               Or. Let B_{V'}=\left(\varphi_1,\varphi_2\right) be corres to the std bss. Let B_{U_1}=B_{U_2}=\left(e_1+e_2\right)\Rightarrow U_1+U_2\subsetneq V.
• Tips 4: Let B_{U^0} = (\varphi_1, ..., \varphi_m), B_{V'} = (\varphi_1, ..., \varphi_n) \Rightarrow B_V = (v_1, ..., v_n).
                  We show (a) B_U = (v_{m+1}, \dots, v_n); (b) U = \text{null } \varphi_1 \cap \dots \cap \text{null } \varphi_m.
                  (a) Becs span(v_{m+1},...,v_n)^0 = \text{span}(\varphi_1,...,\varphi_m) = U^0. Now by Exe (20, 21).
                         Or. Becs by (b), U = \bigcap_{1}^{m} \text{null } \varphi_{I} = \text{span}(v_{m+1}, \dots, v_{n}).
                  (b) Each null \varphi_k = \operatorname{span}\{B_V \setminus \{v_k\}\} \Rightarrow \bigcap_{1}^m \operatorname{null} \varphi_I = \operatorname{span}(v_{m+1}, \dots, v_n). Now by (a).
                         Or. Becs span(\varphi_1, \dots, \varphi_m) = U^0 = (\text{null } \varphi_1 \cap \dots \cap \text{null } \varphi_m)^0. Now by Exe (20, 21).
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24 Prove, using the pattern of [3.104], that dim U + \dim U^0 = \dim V.
\textbf{Solus:} \ \ \text{By Tips } \textbf{(4)}. \ \ \text{Or. Let } B_U = \left(u_1, \ldots, u_m\right), B_V = \left(u_1, \ldots, u_m, v_1, \ldots, v_n\right), B_{V'} = \left(\psi_1, \ldots, \psi_m, \varphi_1, \ldots, \varphi_n\right).
             Supp \psi = \sum_{i=1}^{m} a_i \psi_i + \sum_{j=1}^{n} b_j \varphi_j \in U^0 \Rightarrow \text{each } \psi(u_k) = a_k = 0. \text{ Thus } U^0 \subseteq \text{span}(\varphi_1, \dots, \varphi_n).
• Supp T \in \mathcal{L}(V, W), each \varphi_k \in V', and each \psi_k \in W'.
28 Prove null T' = \operatorname{span}(\psi_1, \dots, \psi_m) \iff \operatorname{range} T = (\operatorname{null} \psi_1) \cap \dots \cap (\operatorname{null} \psi_m).
29 Prove range T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \iff \operatorname{null} T = (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m).
SOLUS: (\text{range } T)^0 = \text{null } T' = \text{span}(\psi_1, \dots, \psi_m) = (\text{null } \psi_1 \cap \dots \cap \text{null } \psi_m)^0.
             (\operatorname{null} T)^0 = \operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) = (\operatorname{null} \varphi_1 \cap \dots \cap \operatorname{null} \varphi_m)^0.
                                                                                                                                                                  34 Define \Lambda: V \to \mathbf{F}^{V'} by \Lambda v = \overline{v}, and \overline{v}: V' \to \mathbf{F} by \overline{v}(\varphi) = \varphi(v).
     (a) Show \overline{v} \in V'' and \Lambda \in \mathcal{L}(V, V'').
     (b) Show if T \in \mathcal{L}(V), then T'' \circ \Lambda = \Lambda \circ T, where T'' = (T')'.
     (c) Show if V is finide, then \Lambda is iso from V onto V''.
SOLUS: (a) \overline{v}(\varphi + \lambda \psi) = (\varphi + \lambda \psi)(v) = \varphi(v) + \lambda \psi(v) = \overline{v}(\varphi) + \lambda \overline{v}(\psi).
                   \overline{v + \lambda w}(\varphi) = \varphi(v + \lambda w) = \varphi(v) + \lambda \varphi(w) = \overline{v}(\varphi) + \lambda \overline{w}(\varphi).
             (b) (T''\overline{v})(\varphi) = (\overline{v} \circ T')(\varphi) = \overline{v}(T'(\varphi)) = (T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = \overline{Tv}(\varphi).
             (c) \overline{v} = 0 \Rightarrow \forall \varphi \in V', \overline{v}(\varphi) = \varphi(v) = 0 \Rightarrow v = 0. Inje. Now becs V finide.
                                                                                                                                                                  COMMENT: Supp \Phi \in V'' and \Phi \neq 0. Then \exists \varphi \in V', \Phi(\varphi) = 1 \Rightarrow \text{null } \Phi \oplus \text{span}(\varphi) = V'.
                  And \varphi \neq 0 \Rightarrow \exists v \in V, \varphi(v) = 1, \text{null } \varphi \oplus \text{span}(v) = V. Becs \Lambda is surj.
                  Now \exists x \in V, \forall \psi = c\varphi + \rho \in V', \psi(x) = \overline{x}(\psi) = \Phi(\psi) = c.
36 Supp U is a subsp of V. Define i: U \to V by i(u) = u. Thus i' \in \mathcal{L}(V', U').
     (a) Show null i' = U^0: null i' = (\text{range } i)^0 = U^0 \Leftarrow \text{range } i = U.
     (b) Prove range i' = U': range i' = (\text{null } i)_U^0 = \{0\}_U^0 = U'.
     (c) Prove \tilde{i}' is iso from V'/U^0 onto U': Immed.
Solus: (a) Or. \forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_{U}. Thus i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^{0}.
             (b) Or. Supp \psi \in U'. By (3.A.11), \exists \varphi \in V', \varphi|_U = \psi. Then i'(\varphi) = \psi.
                                                                                                                                                                   • Supp T \in \mathcal{L}(V, W). Prove range T' \supseteq (\text{null } T)^0.
                                                                                                                  | Another proof of [3.109](b) |
Solus: Let V = U \oplus \text{null } T. Let R = (T|_U)^{-1}|_{\text{range } T}. Define \iota \in \mathcal{L}(V, U) by \iota(u + w) = u.
             \forall \Phi \in (\text{null } T)^0, let \psi = \Phi \circ R, then T'(\psi) = \psi \circ T = \Phi \circ (R \circ T|_V) = \Phi \circ \iota = \Phi \in \text{range } T'.
Coro: [3.108] and [3.110] hold without the hypo of finide. Now T inv \iff T' inv.
12 Note that I'_V, I_{V'}: V' \to V'. For \varphi \in V', I_{V'}(\varphi) = \varphi = \varphi \circ I_V = I'_V(\varphi). Thus I_{V'} = I'_V.
15 Supp T \in \mathcal{L}(V, W). Prove T' = 0 \Rightarrow T = 0.
                                                                                               Coro: If V, W finide, then \Gamma : T \mapsto T' is iso.
Solus: Supp T' = 0. Then (range T)^0 = null T' = W'.
             By Exe (25), range T = \{ w \in W : \varphi(w) = 0, \forall \varphi \in (\text{range } T)^0 = W' \}.
             Asum w \neq 0 suth \forall \varphi \in W', \varphi(w) = 0. Let U \oplus \text{span}(w) = W.
             Define \psi \in W' by \psi(u + \lambda w) = \lambda \Rightarrow \psi(w) \neq 0. Ctradic. Now range T = \{0\}.
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• Note For Exe (16):
  Let B_V = (v_1, ..., v_n), B_V, = (\varphi_1, ..., \varphi_n), B_W = (w_1, ..., w_m), B_W, = (\psi_1, ..., \psi_m).
  Define each E_{i,k} \in \mathcal{L}(V,W) : v_x \mapsto \delta_{i,x} w_k, and each \exists_{k,j} \in \mathcal{L}(W',V') : \psi_x \mapsto \delta_{k,x} \varphi_j.
  Note that each E'_{j,k}(\psi_x) = \psi_x \circ E_{j,k} = \delta_{k,x} \varphi_j = \exists_{k,j} (\psi_x) \Rightarrow E'_{j,k} = \exists_{k,j}.
  \mathcal{L}(V,W) \ni T = \sum_{j=1}^{n} \sum_{k=1}^{m} A_{k,j} E_{j,k} \iff \mathcal{T} = \sum_{j=1}^{n} \sum_{k=1}^{m} A_{k,j} \exists_{k,j} \in \mathcal{L}(W',V'). Uniqly by Exe (16).
  Coro: ST = TS \iff S'T' = T'S'. By Exe (16). Or. Becs AC = CA \iff A^tC^t = C^tA^t.
• (4E 8) Describe the relation of B_V = (v_1, ..., v_n) and the corres B_{V'} = (\varphi_1, ..., \varphi_n) using isos.
Solus: Define \Gamma: V \to \mathbf{F}^n by \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)), and \Gamma^{-1}(a_1, \dots, a_n) = a_1v_1 + \dots + a_nv_n.
6 Define \Gamma: V' \to \mathbf{F}^m by \Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m)), where v_1, \dots, v_m \in V.
   (a) Show span(v_1, ..., v_m) = V \iff \Gamma is inje.
   (b) Show (v_1, ..., v_m) is liney indep \iff \Gamma is surj.
Solus: Let (e_1, \dots, e_m) be the std bss of \mathbf{F}^m.
   (a) Becs \Gamma(\varphi) = 0 \iff \varphi(v_1) = \dots = \varphi(v_m) = 0 \iff \text{null } \varphi = \text{span}(v_1, \dots, v_m). Immed.
   (b) Supp \Gamma is surj. Let each e_k = \Gamma(\varphi_k) \Rightarrow \varphi_k(v_j) = \delta_{j,k}. Now a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow \text{each } a_k = \varphi_k(0).
         Supp (v_1, ..., v_m) is liney indep. Let U = \text{span}(v_1, ..., v_m), B_{U'} = (\psi_1, ..., \psi_m). Let W \oplus U = V.
         Define \iota : u_v + w_v \mapsto u_v. Each \psi_k \circ \iota = \varphi_k \in V' \Rightarrow \varphi_k(v_i) = \psi_k(v_i) = \delta_{i,k} \Rightarrow \text{each } e_k = \Gamma(\varphi_k).
   OR. Let (\psi_1, ..., \psi_m) be dual bss of the std bss of \mathbf{F}^m. Define an iso \Psi : \mathbf{F}^m \to (\mathbf{F}^m)' by \Psi(e_k) = \psi_k.
   Define T \in \mathcal{L}(\mathbf{F}^m, V) by Te_k = v_k. Now T(x_1, \dots, x_m) = T(x_1e_1 + \dots + x_me_m) = x_1v_1 + \dots + x_mv_m.
   \forall \varphi \in V', k \in \big\{1, \dots, m\big\}, \big[T'(\varphi)\big](e_k) = \varphi(Te_k) = \varphi(v_k) = \big[\varphi(v_1)\psi_1 + \dots + \varphi(v_m)\psi_m\big](e_k)
   Now T'(\varphi) = \varphi(v_1)\psi_1 + \dots + \varphi(v_m)\psi_m = \Psi(\varphi(v_1), \dots, \varphi(v_m)) = \Psi(\Gamma(\varphi)). Hence T' = \Psi \circ \Gamma.
   By (3.B.3), (a) range T = \operatorname{span}(v_1, \dots, v_m) = V \iff T' inje \iff \Gamma inje.
                     (b) (v_1, ..., v_m) is liney indep \iff T is inje \iff T' surj \iff \Gamma surj.
                                                                                                                                                                 • (4E 25) Define \Gamma: V \to \mathbf{F}^m by \Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v)), where \varphi_1, \dots, \varphi_m \in V'.
  (c) Show span(\varphi_1, ..., \varphi_m) = V' \iff \Gamma is inje.
  (d) Show (\varphi_1, ..., \varphi_m) is liney indep \iff \Gamma is surj.
SOLUS: Let (e_1, \dots, e_m) be the std bss of \mathbf{F}^m.
    (c) Becs \Gamma(v) = 0 \iff \varphi_1(v) = \dots = \varphi_m(v) = 0 \iff v \in (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m).
         By Exe (4E 23), \operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \operatorname{null} \Gamma = (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m) = \{0\}.
   (d) Supp (\varphi_1, ..., \varphi_m) is liney indep. [Req\ Finide] Extend to B_V = (\varphi_1, ..., \varphi_n).
         Then by Exe (31), B_V = (v_1, ..., v_n) and each \varphi_k(v_i) = \delta_{i,k} \Rightarrow \text{each } e_k = \Gamma(\varphi_k).
          Supp \Gamma is surj. Let each e_k = \Gamma(v_k) = (\varphi_1(v_k), \dots, \varphi_m(v_k)).
          Now a_1 \varphi_1 + \dots + a_m \varphi_m = 0 \Rightarrow \text{each } a_k = 0(v_k).
          Or. Let U = \text{span}(v_1, \dots, v_m). Then B_{U_i} = (\varphi_1|_{U_i}, \dots, \varphi_m|_{U_i}) \Rightarrow (\varphi_1, \dots, \varphi_m) liney indep.
                                                                                                                                                                 Or. Let (\psi_1, \dots, \psi_m) be dual bss of the std bss of \mathbf{F}^m. Define an iso \Psi : \mathbf{F}^m \to (\mathbf{F}^m)' by \Psi(e_k) = \psi_k.
   \forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'(\Psi(x_1, \dots, x_m)) = (x_1 \psi_1 + \dots + x_m \psi_m) \circ \Gamma.
   \forall v \in V, \left[\Gamma'(\Psi(x_1, \dots, x_m))\right](v) = \left[x_1\psi_1 + \dots + x_m\psi_m\right](\varphi_1(v), \dots, \varphi_m(v)) = x_1\varphi_1(v) + \dots + x_m\varphi_m(v).
   Now \Gamma'(\Psi(x_1,...,x_m)) = x_1\varphi_1 + \cdots + x_m\varphi_m. Define \Phi: \mathbb{F}^m \to V' by \Phi = \Gamma' \circ \Psi. Thus by (3.B.3),
   (c) \Gamma inje \iff \Gamma' surj \iff \Phi surj \iff (\varphi_1, ..., \varphi_m) spanning V'.
    (d) \Gamma surj \iff \Gamma' inje \iff \Phi inje \iff (\varphi_1, \dots, \varphi_m) being liney indep.
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9 Show \forall \psi \in V', \psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n, where B_V = (v_1, \dots, v_n), B_{V'} = (\varphi_1, \dots, \varphi_n). Solus: \psi(v) = a_1\psi(v_1) + \dots + a_n\psi(v_n) = \psi(v_1)\varphi_1(v) + \dots + \psi(v_n)\varphi_n(v).
```

13 Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ by T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z).

Let (φ_1, φ_2) , (ψ_1, ψ_2, ψ_3) denote the dual bss of std bss of \mathbb{R}^2 and \mathbb{R}^3 .

- (a) Describe the liney functionals $T'(\varphi_1)$, $T'(\varphi_2)$. For any $(x,y,z) \in \mathbb{R}^3$, $(T'(\varphi_1))(x,y,z) = 4x + 5y + 6z$, $(T'(\varphi_2))(x,y,z) = 7x + 8y + 9z$.
- (b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as liney combinas of ψ_1, ψ_2, ψ_3 . $T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3, \ T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3.$
- (c) What is null T'? What is range T'?

$$T(x,y,z) = 0 \iff \begin{cases} 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases} \iff \begin{cases} x = z, & \text{Thus null } T = \text{span}(e_1 - 2e_2 + e_3), \\ y = -2z. & \text{where } (e_1,e_2,e_3) \text{ is std bss of } \mathbb{R}^3. \end{cases}$$

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a bss, with corres dual bss $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Thus span $(e_1 - 2e_2 + e_3) = \text{null } T \Rightarrow \text{span}(e_1 - 2e_2 + e_3)^0 = \text{span}(\varepsilon_2, \varepsilon_3) = \text{range } T'$.

Note that $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$.

And
$$\varepsilon_{2}(e_{2}) = -\frac{1}{2}$$
, $\varepsilon_{2}(e_{1}) = \varepsilon_{2}(e_{1} - 2e_{2} + e_{3}) + \varepsilon_{2}(2e_{2}) - \varepsilon_{2}(e_{3}) = 1$, $\varepsilon_{3}(e_{2}) = 0$, $\varepsilon_{3}(e_{3}) = \varepsilon_{3}(e_{1} - 2e_{2} + e_{3}) + \varepsilon_{3}(2e_{2}) - \varepsilon_{3}(e_{3}) = -1$.

Hence $\varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2$, $\varepsilon_3 = -\psi_1 + \psi_3$. Now range $T' = \text{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3)$.

Or. range $T' = \text{span}(T'(\varphi_1), T'(\varphi_2)) = \text{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3)$.

Supp $T'(x\varphi_1 + y\varphi_2) = (4x + 7y)\psi_1 + (5x + 8y)\psi_2 + (6x + 9y)\psi_3 = 0.$

Then x + y = 4x + 7y = x = y = 0. Hence null $T' = \{0\}$.

OR. $\operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \operatorname{span}(-2e_2, e_3) \oplus \operatorname{null} T$.

$$\Rightarrow$$
 range $T = \{Tx : x \in \text{span}(-2e_2, e_3)\} = \text{span}(T(-2e_2), T(e_3))$

= span
$$(-10f_1 - 16f_2, 6f_1 + 9f_2)$$
 = span (f_1, f_2) = \mathbb{R}^2 . Now null $T' = (\text{range } T)^0 = \{0\}$.

Or. For any $A, B \in \mathbb{R}$, asum (x, y, z) is suth A = 4x + 5y + 6z, B = 7x + 8y + 9z.

By computing x = z + 4/3(b-a), y = -2z + (7a - 4b)/3, z = z. An exa for (4E 3.E.8).

Hence (x, y, z) exis \Rightarrow $(A, B) \in \text{range } T$. Now $T \text{ surj } \Rightarrow T' \text{ inje.}$

ENDED

Exes about Sequences and Number Theory before Chapter 4

• (2.A.16) Prove the vecsp U of all continuous functions in $\mathbb{R}^{[0,1]}$ is infinide.

Solus: By $[3.A \text{ Note For } \mathbf{F}^S]$, immed. Or. Choose $m \in \mathbb{N}^+$. Let $p(x) = a_0 + a_1 x + \dots + a_m x^m = 0 \in \mathbb{R}^{[0,1]}$. Then *p* has infily many roots and hence each $a_k = 0$, othws deg $p \ge 0$, ctradic [4.12]. Thus $(1, x, ..., x^m)$ is liney indep in $\mathbb{R}^{[0,1]}$. Simlr to [2.16], U is infinide. OR. Note that $\frac{1}{1} > \frac{1}{2} > \dots > \frac{1}{m}$, $\forall m \in \mathbb{N}^+$. Supp $f_m = \begin{cases} x - \frac{1}{m}, & x \in (\frac{1}{m}, 1] \\ 0, & x \in [0, \frac{1}{m}] \end{cases}$ Then $f_1\left(\frac{1}{m}\right) = \dots = f_m\left(\frac{1}{m}\right) = 0 \neq f_{m+1}\left(\frac{1}{m}\right)$. Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. By (2.A.14). • (3.F.35) Prove $(\mathcal{P}(\mathbf{F}))'$ is iso to \mathbf{F}^{∞} . **Solus:** Define $\theta \in \mathcal{L}[(\mathcal{P}(\mathbf{F}))', \mathbf{F}^{\infty}]$ by $\theta(\varphi) = (\varphi(1), \varphi(z), \cdots, \varphi(z^m), \cdots)$. Notice that $\forall p \in \mathcal{P}(\mathbf{R}), \exists ! c_i \in \mathbf{F}, m = \deg p, \ p(z) = c_0 + c_1 z + \dots + c_m z^m \in \mathcal{P}_m(\mathbf{F}).$ Inje: $\theta(\varphi) = 0 \Rightarrow \forall p \in \mathcal{P}(\mathbf{F}), \varphi(p) = c_0 \varphi(1) + c_1 \varphi(z) + \dots + c_m \varphi(z^m) = 0.$ Surj: Define $\psi_x(p) = x_0c_0 + \dots + x_mc_m$ for any $x = (x_0, x_1, \dots) \in \mathbb{F}^{\infty}$. Now each $\psi_x(z^k) = x_k$. $\forall p, q \in \mathcal{P}(\mathbf{F})$, supp $\deg p = m \geqslant n = \deg q$, [which is why we do not write $(p + \lambda q)$.] $\psi_{x}(\lambda p + \mu q) = \sum_{j=0}^{n} x_{j}(\lambda a_{j} + \mu b_{j}) + \sum_{k=1}^{m-n} x_{n+k} \lambda a_{n+k} = \lambda \psi_{x}(p) + \mu \psi_{x}(q).$ **COMMENT:** $\mathcal{P}(\mathbf{F})$ is not iso to \mathbf{F}^{∞} , so is $\mathcal{P}(\mathbf{F})$ to $(\mathcal{P}(\mathbf{F}))'$. But $\mathcal{P}(\mathbf{F})$ is iso to $\mathbf{F}^{\mathbf{N}}$, which the 'U' in (3.E.14). • (3.E.14) Supp $U = \{(x_1, x_2, \dots) \in \mathbb{F}^{\infty} : x_k \neq 0 \text{ for only finily many } k\}$. Denote it by \mathbb{F}^N . (a) Show U is a subsp of \mathbf{F}^{∞} . [Do it in your mind] (b) Prove \mathbf{F}^{∞}/U is infinide.

SOLUS: For ease of nota, denote the p^{th} term of $u = (x_1, \cdots, x_p, \cdots) \in \mathbf{F}^{\infty}$ by u[p]. For each $r \in \mathbf{N}^+$, let $e_r[k] = \begin{cases} 1, & (k-1) \equiv 0 \pmod{r} \\ 0, & \text{othws} \end{cases}$ simply $e_r = (1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \cdots)$.

For $m \in \mathbb{N}^+$. Let $a_1(e_1 + U) + \dots + a_m(e_m + U) = 0 + U \Rightarrow \exists u \in U, a_1e_1 + \dots + a_me_m = u$.

Supp $u = (x_1, \dots, x_L, 0, \dots)$, where L is the largest suth $u[L] \neq 0$.

Let $s \in \mathbb{N}^+$ be suth $h = s \cdot m! + 1 > L$, and $e_1[h] = \cdots = e_m[h] = 1$.

Notice that for any $p, r \in \{1, ..., m\}$, $e_r[s \cdot m! + 1 + p] = e_r[p + 1] = 1 \iff p \equiv 0 \pmod{r} \iff r \mid p$.

Let $1 = p_1 \leqslant \cdots \leqslant p_{\tau(p)} = p$ be the disti factors of p. Moreover, $r \mid p \iff r = p_k$ for some k.

Now
$$u[h+p] = 0 = \sum_{r=1}^{m} a_r e_r [p+1] = \sum_{k=1}^{\tau(p)} a_{p_k}$$
.

Let $q = p_{\tau(p)-1}$. Then $\tau(q) = \tau(p) - 1$, and each $q_k = p_k$. Again, $\sum_{r=1}^m a_r e_r [h+q] = 0 = \sum_{k=1}^{\tau(p)-1} a_{p_k}$. Thus $a_{p_{\tau(p)}} = a_p = 0$ for all $p \in \{1, \dots, m\} \Rightarrow (e_1, \dots, e_m)$ is liney indep in \mathbf{F}^{∞} .

Or. For each $r \in \mathbb{N}^+$, let $e_r[p] = \begin{cases} 1, \text{ if } 2^r | p| & \text{Simlr, let } m \in \mathbb{N}^+ \text{ and } a_1(e_1 + U) + \dots + a_m(e_m + U) = 0 \\ 0, \text{ othws} & \Rightarrow a_1 e_1 + \dots + a_m e_m = u \in U. \end{cases}$

Supp *L* is the largest suth $u[L] \neq 0$. And *l* is suth $2^{ml} > L$. Then for each $k \in \{1, ..., m\}$,

$$u[2^{ml} + 2^k] = 0 = \sum_{r=1}^m a_r e_r[2^k] = a_1 + \dots + a_k$$
. Thus each $a_k = 0$. Simlr.

Exes about Polys before Chapter 4

• (1.C.9) A function $f: \mathbb{R} \to \mathbb{R}$ is called periodic if $\exists p \in \mathbb{N}^+$, f(x) = f(x+p) for all $x \in \mathbb{R}$. *Is the set of periodic functions* $R \to R$ *a subsp of* R^R ? *Explain.*

SoLUS: Denote the set by *S*.

Supp $h(x) = \cos x + \sin \sqrt{2x} \in S$, since $\cos x$, $\sin \sqrt{2x} \in S$.

Asum $\exists p \in \mathbb{N}^+$ suth $h(x) = h(x+p), \forall x \in \mathbb{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin \sqrt{2p} = \cos p - \sin \sqrt{2p}$

 $\Rightarrow \sin \sqrt{2}p = 0$, $\cos p = 1 \Rightarrow p = 2k\pi$, $k \in \mathbb{Z}$, while $p = \frac{m\pi}{\sqrt{2}}$, $m \in \mathbb{Z}$.

Hence $2k = \frac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \frac{m}{2k} \in \mathbb{Q}$. Ctradic!

OR. Becs $\cos x + \sin \sqrt{2}x = \cos(x+p) + \sin(\sqrt{2}x + \sqrt{2}p)$. By diff twice, $\cos x + 2\sin\sqrt{2}x = \cos(x+p) + 2\sin(\sqrt{2}x + \sqrt{2}p).$

$$\frac{\sin\sqrt{2}x = \sin\left(\sqrt{2}x + \sqrt{2}p\right)}{\cos x = \cos(x + p)} \Rightarrow \text{Let } x = 0, \ p = \frac{m\pi}{\sqrt{2}} = 2k\pi. \text{ Ctradic.}$$

• (1.C.24) Let $V_E = \{ f \in \mathbb{R}^{\mathbb{R}} : f \text{ is even} \}, V_O = \{ f \in \mathbb{R}^{\mathbb{R}} : f \text{ is odd} \}. \text{ Show } V_E \oplus V_O = \mathbb{R}^{\mathbb{R}}.$

Solus: (a) $V_E \cap V_O = \{ f \in \mathbb{R}^R : f(x) = f(-x) = -f(-x) \} = \{ 0 \}.$

(b)
$$\left| \begin{array}{l} \text{Let } f_e(x) = \frac{1}{2} \Big[g(x) + g(-x) \Big] \Longrightarrow f_e \in V_E \\ \text{Let } f_o(x) = \frac{1}{2} \Big[g(x) - g(-x) \Big] \Longrightarrow f_o \in V_O \end{array} \right| \Rightarrow \forall g \in \mathbb{R}^R, \ g(x) = f_e(x) + f_o(x).$$

• (2.C.7) (a) Let $U = \{ p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6) \}$. Find a bss of U. (b) Extend the bss in (a) to a bss of $\mathcal{P}_4(\mathbf{F})$, and find a W suth $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solus: Using (2.C.10).

Notice that $\nexists p \in \mathcal{P}(\mathbf{F})$ of deg 1 and 2, while $p \in U$. Thus dim $U \leq \dim \mathcal{P}_4(\mathbf{F}) - 2 = 3$.

- (a) Consider B = (1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)).Let $a_0 + a_3(z-2)(z-5)(z-6) + a_4z(z-2)(z-5)(z-6) = 0 \Rightarrow a_0 = a_3 = a_4 = 0$. Thus the list *B* is liney indep in *U*. Now dim $U \ge 3 \Rightarrow \dim U = 3$. Thus $B_U = B$.
- (b) Extend to a bss of $\mathcal{P}_4(\mathbf{F})$ as $(1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6))$. Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbb{F}\}$, so that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

• Note For (2.C.10): For each nonC $p \in \text{span}(1, z, ..., z^m)$, $\exists \text{ smallest } m \in \mathbb{N}^+$, which is deg p.

(a) If p_0, p_1, \dots, p_m are suth all $a_{k,k} \neq 0$, and

$$p_{0} = a_{0,0}, \text{ each } p_{k} = a_{0,k} + a_{1,k}z + \dots + a_{k,k}z^{k}.$$
Then the upper-trig $\mathcal{M}\left(I, (p_{0}, p_{1}, \dots, p_{m}), (1, z, \dots, z^{m})\right) = \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,m} \\ 0 & a_{1,1} & \dots & a_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{m,m} \end{pmatrix}.$
If $p_{0}, p_{1}, \dots, p_{m}$ are suth all $a_{k,k} \neq 0$, and

(b) If p_0, p_1, \dots, p_m are suth all $a_{k,k} \neq 0$, and If p_0, p_1, \dots, p_m are sum an $a_{k,k} \neq 0$, and $p_0 = a_{0,0} + \dots + a_{m,0}x^m$, each $p_k = a_{k,k}x^k + \dots + a_{m,k}x^m$.

Then the lower-trig $\mathcal{M}\left(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\right) = \begin{pmatrix} a_{0,0} & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$.

COMMENT: Define $\xi_k(p)$ by the coeff of z^k in $p \in \mathcal{P}_m(\mathbf{F})$. Then $\mathcal{M}(\xi_k, (1, z, ..., z^m), (1)) = \mathcal{E}^{(1,k)} \in \mathbf{F}^{1,m+1}$. • (2.C.10) Supp $m \in \mathbb{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ are suth each $\deg p_k = k$. *Prove* $(p_0, p_1, ..., p_m)$ *is a bss of* $\mathcal{P}_m(\mathbf{F})$. **Solus**: Using induc on *m*. (i) k = 1. $\deg p_0 = 0$; $\deg p_1 = 1 \Rightarrow \operatorname{span}(p_0, p_1) = \operatorname{span}(1, x)$. (ii) $1 \le k \le m - 1$. Asum span $(p_0, p_1, ..., p_k) = \text{span}(1, x, ..., x^k)$. Then span $(p_0, p_1, ..., p_k, p_{k+1}) \subseteq \text{span}(1, x, ..., x^k, x^{k+1}).$ $\mathbb{Z} \operatorname{deg} p_{k+1} = k+1, \ p_{k+1}(x) = a_{k+1}x^{k+1} + r_{k+1}(x); \ a_{k+1} \neq 0, \ \operatorname{deg} r_{k+1} \leqslant k.$ $\Rightarrow x^{k+1} = \frac{1}{a_{k+1}} \Big(p_{k+1}(x) - r_{k+1}(x) \Big) \in \text{span}(1, x, \dots, x^k, p_{k+1}) = \text{span}(p_0, p_1, \dots, p_k, p_{k+1}).$ $x_{k+1} \in \text{span}(p_0, p_1, \dots, p_k, p_{k+1}) \Rightarrow \text{span}(1, x, \dots, x^k, x^{k+1}) \subseteq \text{span}(p_0, p_1, \dots, p_k, p_{k+1}).$ Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m).$ OR. By comparing coeffs. Denote the coeff of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$. Supp $L = a_m p_m(x) + \dots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \dots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}.$ We show $a_m = \cdots = a_0 = 0$ via the following process. So that (p_0, p_1, \dots, p_m) is liney indep. **Step 1.** For k = m, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0 \ \, \text{\mathbb{Z} deg $p_m = m$, $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.}$ Now $L = a_{m-1}p_{m-1}(x) + \dots + a_0p_0(x)$. **Step k.** For $0 \le k \le m$, we have $a_m = \cdots = a_{k+1} = 0$. Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0 \ \mathbb{Z} \deg p_k = k, \ \xi_k(p_k) \neq 0 \Rightarrow a_k = 0.$ Now if k = 0, then done. Othws, we have $L = a_{k-1}p_{k-1}(x) + \cdots + a_0p_0(x)$. • Tips: Supp $m \in \mathbb{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbb{F})$ are suth the lowest term of each p_k is of deg k. *Prove* $(p_0, p_1, ..., p_m)$ *is a bss of* $\mathcal{P}_m(\mathbf{F})$. **Solus**: Using induc on *m*. Let each p_k be defined by $p_k(x) = a_{k,k}x^k + \cdots + a_{m,k}x^m$, where $a_{k,k} \neq 0$. (i) k = 1. $p_m(x) = a_{m,m}x^m$; $p_{m-1}(x) = a_{m-1,m-1}x^{m-1} + a_{m,m-1}x^m \Longrightarrow \operatorname{span}(x^m, x^{m-1}) = \operatorname{span}(p_m, p_{m-1})$. (ii) $1 \le k \le m-1$. Asum span $(x^m, ..., x^{m-k}) = \text{span}(p_m, ..., p_{m-k})$. Then span $(p_m, \dots, p_{m-(k+1)}) \subseteq \operatorname{span}(x^m, \dots, x^{m-(k+1)})$. $\mathbb{Z} p_{m-(k+1)}$ has the form $a_{m-(k+1),m-(k+1)} x^{m-(k+1)} + r_{m-(k+1)}(x)$; where the lowest term of $r_{m-(k+1)} \in \mathcal{P}_m(\mathbf{F})$ is of deg (m-k). $\Rightarrow x^{m-(k+1)} = \frac{1}{a_{m-(k+1),m-(k+1)}} \Big(p_{m-(k+1)}(x) - r_{m-(k+1)}(x) \Big) \in \operatorname{span}(x^m, \dots, x^{m-k}, p_{m-(k+1)}) \\ = \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$ $\therefore x^{m-(k+1)} \in \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$ $\Rightarrow \operatorname{span}(x^m, \dots, x^{m-k}, x^{m-(k+1)}) \subseteq \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$ Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}(p_m, \dots, p_1, p_0).$ OR. By comparing coeffs. Denote the coeff of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$. Supp $L = a_m p_m(x) + \dots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \dots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}.$ We show $a_m = \cdots = a_0 = 0$ via the following process. So that (p_0, p_1, \dots, p_m) is liney indep. **Step 1.** For k = 0, $\xi_0(L) = a_0 \xi_0(p_0) = \xi_0(R) = 0 \ \ \ \deg p_0 = 0$, $\xi_0(p_0) \neq 0 \Rightarrow a_0 = 0$. Now $L = a_1 p_1(x) + \dots + a_m p_m(x)$. **Step k.** For $0 \le k \le m$, we have $a_{k-1} = \cdots = a_0 = 0$. Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0 \ \mathbb{Z} \deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$. Now if k = m, then done. Othws, we have $L = a_{k+1}p_{k+1}(x) + \cdots + a_mp_m(x)$.

• Note For [2.11]: Good definition for a general term always aviods undefined behaviours. If deg p = 0, then $p(z) = a_0 \neq 0$, but not literally $a_0 z^0$, by which if p is defined, then it comes to 0^0 . To make it clear, we specify that $in \mathcal{P}(\mathbf{F})$, $a_0 z^0 = a_0$, where z^0 appears just for nota conveni. Becs by def, the term a_0z^0 in a poly only represents the const term of the poly, which is a_0 . For conveni, we asum $z^0 = 1$ in formula deduction and poly def. Absolutely without 0^0 .

• (4E 2.C.10) Supp m is a positive integer. For $0 \le k \le m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show (p_0, \ldots, p_m) is a bss of $\mathcal{P}_m(\mathbf{F})$.

Solus: We may see $p_0 = 1$ and $p_m(x) = x^m$, from the expansion below, by the Note For [2.11] above.

Note that each
$$p_k(x) = \sum_{j=0}^{m-k} C_{m-k}^j (-1)^j \cdot x^{j+k} \cdot 1^j = \underbrace{(-1)^0 \cdot x^k \cdot 1^0}_{\text{of deg k}} + \underbrace{\sum_{j=1}^{m-k} C_{m-k}^j (-1)^j \cdot x^{j+k} \cdot 1^j}_{\text{of deg m; denote it by } q_k(x)}.$$

OR. Simlr to the TIPS above. We will recurly prove each $x^{m-k} \in \text{span}(p_m, \dots, p_{m-k})$.

(i)
$$k = 1$$
. $p_m(x) = x^m \in \text{span}(p_m)$; $p_{m-1}(x) = x^{m-1} - x^m \Rightarrow x^{m-1} \in \text{span}(p_{m-1}, p_m)$.

(ii)
$$k \in \{1, \dots, m-1\}$$
. Supp for each $j \in \{0, \dots, k\}$, we have $x^{m-j} \in \text{span}(p_{m-j}, \dots, p_m)$, $\exists ! a_m \in \mathbb{F}$. Note that $x^{m-(k+1)} = p_{m-(k+1)}(x) + \sum_{j=1}^{k+1} C_{k+1}^j (-1)^{j+1} x^{m-(k+1)+j} \in \text{span}(p_{m-(k+1)}, x^{m-k}, \dots, x^m)$. Thus $x^{m-(k+1)} \in \text{span}(p_{m-(k+1)}, p_{m-k}, \dots, p_m)$.

COMMENT: The base step and the induc step can be indep.

Or. For any $m, k \in \mathbb{N}^+$ suth $k \leq m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k (1-x)^{m-k}$. Define the stmt $S(m):(p_{0,m},...,p_{m,m})$ is liney indep (and therefore is a bss). We use induc on to show S(m) holds for all $m \in \mathbb{N}^+$.

(i)
$$m = 0$$
. $p_{0,0} = 1$, and $ap_{0,0} = 0 \Rightarrow a = 0$. $m = 1$. Let $a_0(1-x) + a_1x = 0$, $\forall x \in \mathbf{F}$. Then take $x = 1$, $x = 0 \Rightarrow a_1 = a_0 = 0$.

(ii) $1 \le m$. Asum S(m) and S(m-1) holds. Now we show S(m+1) holds. Supp $\sum_{k=0}^{m+1} a_k p_{k,m+1}(x) = \sum_{k=0}^{m+1} a_k [x^k (1-x)^{m+1-k}] = 0, \forall x \in \mathbf{F}.$

$$\sup_{k=0} u_k \rho_{k,m+1}(x) - \sum_{k=0} u_k [x (1-x)] = 0, \forall x \in \mathbf{F}.$$

Now
$$a_0(1-x)^{m+1} + \sum_{k=1}^m a_k x^k (1-x)^{m+1-k} + a_{m+1} x^{m+1} = 0, \forall x \in \mathbf{F}.$$

While
$$x = 0 \Rightarrow a_0 = 0$$
; and $x = 1 \Rightarrow a_{m+1} = 0$.

Then
$$0 = \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k}$$

 $= x(1-x) \sum_{k=1}^{m} a_k x^{k-1} (1-x)^{m-k}$, note that $m-k = (m-1) - (k-1)$
 $= x(1-x) \sum_{k=0}^{m-1} a_{k+1} x^k (1-x)^{m-1-k} = x(1-x) \sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x)$.

Hence $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0, \forall x \in \mathbb{F} \setminus \{0,1\}$. Which has infily many zeros.

Moreover,
$$\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0$$
. By asum, $a_1 = \dots = a_{m-1} = a_m = 0$.

Thus
$$(p_{0,m+1},...,p_{m+1,m+1})$$
 is liney indep and $S(m+1)$ holds.

• (4E 3.D.20) Supp $q \in \mathcal{P}(\mathbf{R})$. Prove $\exists p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.

Solus: Note that $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$.

Define
$$T_n \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$$
 by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.

And note that $T_n(p) = 0 \Rightarrow \deg T_n(p) = -\infty = \deg p \Rightarrow p = 0$. Thus T_n is inv.

$$\forall q \in \mathcal{P}(\mathbf{R})$$
, if $q = 0$, let $n = 0$; if $q \neq 0$, let $n = \deg q$, we have $q \in \mathcal{P}_n(\mathbf{R})$.

Now
$$\exists p \in \mathcal{P}_n(\mathbf{R}), q(x) = T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$
 for all $x \in \mathbf{R}$.

```
• (3.D.19) Supp T \in \mathcal{L}(\mathcal{P}(\mathbf{R})) is inje. And \deg Tp \leqslant \deg p for every non0 p \in \mathcal{P}(\mathbf{R}).
               (a) Prove T is surj. (b) Prove for every non0 p, \deg Tp = \deg p.
Solus: (a) T is inje \iff \forall n \in \mathbb{N}^+, T|_{\mathcal{P}_n(\mathbb{R})} \in \mathcal{L}(\mathcal{P}_n(\mathbb{R})) is inje, so is inv \iff T is surj.
             (b) Using induc.
                   (i) \deg p = -\infty \geqslant \deg Tp \iff p = 0 = Tp. And \deg p = 0 \geqslant \deg Tp \iff p = C \neq 0.
                   (ii) Asum \forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts. We show \forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p by ctradic.
                         Supp \exists r \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tr \leqslant n < n+1 = \deg r. By (a), \exists s \in \mathcal{P}_n(\mathbf{R}), T(s) = (Tr).
                         \not T is inje \Rightarrow s = r. While deg s = \deg Ts = \deg Tr < \deg r. Ctradic.
                                                                                                                                                           • (3.B.26) Supp D \in \mathcal{L}(\mathcal{P}(\mathbf{R})) and \forall p, \deg(Dp) = (\deg p) - 1. Prove D \in \mathcal{P}(\mathbf{R}) is surj.
Solus: [D \text{ might not be } D: p \mapsto p'.] Notice that the following proof is wrong:
            Becs span(Dx, Dx^2, Dx^3, \dots) \subseteq \text{range } D, and deg Dx^n = n - 1.
             \nabla By (2.C.10), span(Dx, Dx^2, Dx^3, ...) = span(1, x, x^2, ...) = \mathcal{P}(\mathbf{R}).
   Let D(C) = 0, Dx^k = p_k of deg (k-1), for all C \in \mathcal{P}_0(\mathbf{R}) and each k \in \mathbf{N}^+. Notice that \mathbf{R} \neq \mathcal{P}_0(\mathbf{R}).
   Becs B_{\mathcal{P}_m(\mathbf{R})} = (p_1, \dots, p_m, p_{m+1}). And for all p \in \mathcal{P}(\mathbf{R}), \exists ! m = \deg p \in \mathbf{N}^+.
   So that \exists ! a_i \in \mathbf{R}, p = \sum_{i=1}^{m+1} a_i p_i \Rightarrow \exists q = \sum_{i=1}^{m+1} a_i x^i, Dq = p.
                                                                                                                                                           OR. We will recurly define a seq of polys (p_k)_{k=0}^{\infty} where Dp_0 = 1, Dp_k = x^k for each k \in \mathbb{N}^+.
   So that \forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), Dq = p, \exists q = \sum_{k=0}^{\deg p} a_k p_k.
   (i) Becs deg Dx = (\deg x) - 1 = 0, Dx = C \in \mathbb{F} \setminus \{0\}. Let p_0 = C^{-1}x \Rightarrow Dp_0 = C^{-1}Dx = 1.
   (ii) Supp we have defined Dp_0 = 1, Dp_k = x^k for each k \in \{1, ..., n\}. Becs deg D(x^{n+2}) = n + 1.
         Let D(x^{n+2}) = a_{n+1}x^{n+1} + a_nx^n + \dots + a_1x + a_0, with a_{n+1} \neq 0.
         Then a_{n+1}^{-1}D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_nDp_n + \dots + a_1Dp_1 + a_0Dp_0)
         \Rightarrow x^{n+1} = D\Big[\underline{a_{n+1}^{-1}(x^{n+2} - a_n p_n - \dots - a_1 p_1 - a_0 p_0)}\Big]. \text{ Thus defining } p_{n+1}, \text{ so that } Dp_{n+1} = x^{n+1}. \quad \Box
• Supp V = \mathbb{R}^{\mathbb{R}} with a subsp U = \{ f \in \mathbb{R}^{\mathbb{R}} : f(x_1) = \dots = f(x_m) = 0 \}, where each x_k \in \mathbb{R}.
  Prove if W \in S_V U, then dim W = m.
                                                                                                        Hint: Find an iso from V/U onto \mathbb{R}^m.
Solus: Define T \in \mathcal{L}(V/U, \mathbb{R}^m) by T(f + U) = (f(x_1), \dots, f(x_m)).
            \forall f + U = g + U \in V/U, f - g \in U \Rightarrow f(x_k) = g(x_k). Well-defined.
            Inje: Each f(x_k) = 0 \Rightarrow f + U = 0. Surj: Let S = T \circ \pi \Rightarrow \tilde{S} = T. Becs S is surj.
                                                                                                                                                           • (3.F.7) Show the dual bss of (1, x, ..., x^m) of \mathcal{P}_m(\mathbf{R}) is (\varphi_0, \varphi_1, ..., \varphi_m), where \varphi_k(p) = \frac{p^{(k)}(0)}{k!}.
SOLUS: The uniques of dual bss is guaranteed by [3.5].
            For j, k \in \mathbb{N}, (x^{j})^{(k)} = \begin{cases} j(j-1) \dots (j-k+1) \cdot x^{(j-k)}, & j \geqslant k. \\ j(j-1) \dots (j-j+1) = j! & j = k. \\ 0 & i < k \end{cases} \Rightarrow (x^{j})^{(k)}(0) = \begin{cases} 0, & j \neq k. \\ k!, & j = k. \\ \cdots & \cdots & \cdots \end{cases}
Exa: By [2.C.10], B_m = (1,7x-5,...,(7x-5)^m) is a bss of \mathcal{P}_m(\mathbf{R}). Let each \varphi_k = \frac{p^{(k)}(5/7)}{7 \cdot k!}.
```

- TIPS 1: Supp $p \in \mathcal{P}_n(\mathbf{F})$ has at least n+1 disti zeros. Then by the ctrapos of [4.12], $\deg p < 0 \Rightarrow p = 0$. OR. We show if $p \in \mathcal{P}(\mathbf{F})$ has at least m disti zeros, then either p=0 or $\deg p \geqslant m$. If p=0 then done. If not, then supp p has exactly m disti zeros $\lambda_1,\ldots,\lambda_m$. Becs $\exists ! \alpha_i \geqslant 1, q \in \mathcal{P}(\mathbf{F})$, and $q \neq 0$, suth $p(z) = \left\lceil (z-\lambda_1)^{\alpha_1}\cdots(z-\lambda_m)^{\alpha_m}\right\rceil q(z)$.
- **COMMENT**: Notice that by [4.17], some term of the poly factoriz might not be in the form $(x \lambda_k)^{\alpha_k}$.
- Note For [4.7]: the uniques of coeffs of polys

[Another proof]

If a poly had two different sets of coeffs, then subtracting the two exprs would give a poly with some non0 coeffs but infily many zeros. By Tips.

- Note For [4.8]: $div\ algo\ for\ polys$ $\sup_{\text{of len } (\deg p \deg s + 1)} [Another\ proof]$ Supp $\deg p \geqslant \deg s$. Then $(\underbrace{1,z,\ldots,z^{\deg s-1}},\underbrace{s,zs,\cdots,z^{\deg p-\deg s}})$ is a bss of $\mathcal{P}_{\deg p}(\mathbf{F})$. Becs $q \in \mathcal{P}(\mathbf{F})$, $\exists !\ a_i,b_j \in \mathbf{F}$, $q = a_0 + a_1z + \cdots + a_{\deg s-1}z^{\deg s-1} + b_0s + b_1zs + \cdots + b_{\deg p-\deg s}z^{\deg p-\deg s}s$ $= \underbrace{a_0 + a_1z + \cdots + a_{\deg s-1}z^{\deg s-1}}_{r} + s\underbrace{\left(b_0 + b_1z + \cdots + b_{\deg p-\deg s}z^{\deg p-\deg s}\right)}_{q}.$ Note that r,q are uniq. \square
- **Note For [4.11]:** each zero of a poly corres to a deg-one factor;

[Another proof]

First supp $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \dots + a_m z^m$, $\exists ! a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.

Then $p(z) = p(z) - p(\lambda) = a_1(z - \lambda) + \dots + a_m(z^m - \lambda^m)$ for all $z \in F$.

Hence $\forall k \in \{1, ..., m\}, z^k - \lambda^k = (z - \lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + ... + z^{k-(j+1)}\lambda^j + ... + z\lambda^{k-2} + z^0\lambda^{k-1}).$

Thus $p(z) = \sum_{j=1}^{m} a_j(z - \lambda) \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) \sum_{j=1}^{m} a_j \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) q(z).$

• (4E2) Prove if $w, z \in \mathbb{C}$, then $||w| - |z|| \leq |w - z|$.

Solus: $|w-z|^2 = (w-z)(\overline{w}-\overline{z}) = |w|^2 + |z|^2 - 2Re(w\overline{z}) \geqslant |w|^2 + |z|^2 - 2|w\overline{z}| = ||w| - |z||^2$. Or. $|w| = |w-z+z| \leqslant |w-z| + |z| \Rightarrow |w| - |z| \leqslant |w-z|$. $|z| = |z-w+w| \leqslant |z-w| + |w| \Rightarrow |z| - |w| \leqslant |w-z|$.

5 Supp $m \in \mathbb{N}$, and z_1, \dots, z_{m+1} are disti in \mathbb{F} , and $w_1, \dots, w_{m+1} \in \mathbb{F}$. Prove $\exists ! p \in \mathcal{P}_m(\mathbb{F}), p(z_k) = w_k$ for each $k \in \{1, \dots, m+1\}$.

Solus:

Define $T \in \mathcal{L}(\mathcal{P}_m(\mathbf{F}), \mathbf{F}^{m+1})$ by $Tq = (q(z_1), \dots, q(z_m), q(z_{m+1}))$.

Becs $Tq = 0 \iff q(z_1) = \dots = q(z_m) = q(z_{m+1}) = 0 \iff q = 0$, by Tips. Now T iso. Immed.

Or. Let $p_1 = 1$, $p_k(z) = \prod_{i=1}^{k-1} (z - z_i) = (z - z_1) \cdots (z - z_{k-1})$ for each $k \in \{2, \dots, m+1\}$.

By (2.C.10), $B_p = (p_1, ..., p_{m+1})$ is a bss of $\mathcal{P}_m(\mathbf{F})$. Let $B_e = (e_1, ..., e_{m+1})$ be the std bss of \mathbf{F}^{m+1} .

Now $Tp_1 = (1, ..., 1)$, $Tp_k = \left(\prod_{i=1}^{k-1} (z_1 - z_i), ..., \prod_{i=1}^{k-1} (z_j - z_i), ..., \prod_{i=1}^{k-1} (z_{m+1} - z_i)\right)$;

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & A_{2,2} & 0 & \cdots & 0 \\ 1 & A_{3,2} & A_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{m+1,2} & A_{m+1,2} & \cdots & A_{m+1,m+1} \end{pmatrix} \text{ And } \prod_{i=1}^{k-1} (z_j - z_i) = 0 \iff j \leqslant k-1, \text{ becs } z_1, \dots, z_{m+1} \text{ are disti.}$$

$$= \mathcal{M}(T, B_p, B_e). \text{ Where } A_{j,k} = \prod_{i=1}^{k-1} (z_j - z_i) \neq 0 \text{ for all } j > k-1 \geqslant 1.$$
Now the rows of $\mathcal{M}(T)$ liney indep. By (4E 3.C.17) OR (3.F.32). \square

```
If m = 0, then p = c \neq 0 \Rightarrow p has no zeros, and p' = 0, done.
                   If m = 1, then p(z) = c(z - \lambda_1), and p' = c has no zeros, done.
                   For each j \in \{1, ..., m\}, let q_i(z - \lambda_i) = p(z) \Rightarrow q_i(\lambda_i) \neq 0.
                   Now p'(z) = (z - \lambda_i)q_i'(z) + q_i(z) \Rightarrow p'(\lambda_i) = q_i(\lambda_i) \neq 0.
                   Or. \neg Q \Rightarrow \neg P: Supp p(z) = (z - \lambda)q(z), p'(z) = (z - \lambda)r(z).
                   Becs p'(z) = (z - \lambda)q'(z) + q(z) \Rightarrow p'(\lambda) = q(\lambda) = 0 \Rightarrow q(z) = (z - \lambda)s(z).
                   Now p(z) = (z - \lambda)^2 s(z). Hence p has strictly less than m disti zeros.
             (b) \neg P \Rightarrow \neg Q: Becs 0 \neq p \in \mathcal{P}_m(\mathbf{F}). Supp all disti zeros are \lambda_1, \dots, \lambda_M, with M < m.
                   By Pigeon Hole Principle, (z - \lambda_k)^2 q(z) = p(z) for some \lambda_k \Rightarrow p'(\lambda_k) = 0 = p(\lambda_k).
                                                                                                                                                                 7 Prove every p \in \mathcal{P}(\mathbf{R}) of odd deg has a zero.
Solus: Using the nota and proof of [4.17]. \deg p = 2M + m is odd \Rightarrow m is odd. Hence \lambda_1 exis.
                                                                                                                                                                 OR. Supp p \in \mathcal{P}(\mathbf{R}) of odd deg m. Let p(x) = a_0 + a_1 x + \cdots + a_m x^m.
             Write p(x) = x^m \left( \frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \dots + \frac{a_{m-1}}{x} + a_m \right) \Rightarrow p(x) continuous. Let \delta = |a_m|^{-1} a_m.
             Then \lim_{x \to \infty} p(x) = -\delta \infty; \lim_{x \to \infty} p(x) = \delta \infty \Rightarrow p has at least one real zero.
                                                                                                                                                                 8 Supp p \in \mathcal{P}(\mathbf{R}). Define Tp : \mathbf{R} \to \mathbf{R} by (Tp)(x) = \begin{cases} \frac{p(x) - p(3)}{x - 3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}
Show (a) Tp \in \mathcal{P}(\mathbf{R}); (b) T \in \mathcal{L}(\mathcal{P}(\mathbf{R})).
Solus:
    (a) For x \neq 3, T(x^n) = \frac{x^n - 3^n}{x - 3} = \sum_{i=1}^n 3^{i-1} x^{n-i}.
          For x = 3, T(x^n) = n3^{n-1} = \sum_{i=1}^n 3^{n-1} = \sum_{i=1}^n 3^{i-1}x^{n-i}. Now each T(x^n) = \sum_{i=1}^n 3^{i-1}x^{n-i} \in \mathcal{P}(\mathbf{R}).
   (b) T(p + \lambda q)(x) = \begin{cases} \frac{(p + \lambda q)(x) - (p + \lambda q)(3)}{x - 3}, & \text{if } x \neq 3, \\ (p + \lambda q)'(3), & \text{if } x = 3 \end{cases} = [T(p) + \lambda T(q)](x) \text{ for all } x \in \mathbb{R}.
                                                                                                                                                                 Or. (a) Becs \exists ! q \in \mathcal{P}(\mathbf{R}), p(x) - p(3) = (x - 3)q(x). For x \neq 3, q(x) = \frac{p(x) - p(3)}{x - 3}
                p'(x) = (p(x) - p(3))' = q(x) + (x - 3)q'(x). For x = 3, p'(3) = q(3). Now Tp = q.
          (b) Let q_k(x)(x-3) = p_k(x) - p_k(3). Now by (a), Tp_k = q_k.
                Then (p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x - 3)(q_1 + \lambda q_2)(x). By the uniques of q_1 + \lambda q_2. \Box
11 Supp p \in \mathcal{P}(\mathbf{F}) with p \neq 0. Let U = \{pq : q \in \mathcal{P}(\mathbf{F})\}.
      (a) Show dim \mathcal{P}(\mathbf{F})/U = \deg p; (b) Find a bss of \mathcal{P}(\mathbf{F})/U.
Solus: Note that pq \neq p \circ q, see (4E 3.A.10). Let deg p = m as precond.
   If deg p = 0, then U = \mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F})/U = \{0 + U\}, with the uniq bss (). Supp deg p \ge 1.
    (a) Becs \forall s \in \mathcal{P}(\mathbf{F}), \exists ! r \in \mathcal{P}_{m-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}) \Rightarrow \exists ! pq \in U, s = (p)q + (r) \Rightarrow \mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{m-1}(\mathbf{F}).
          By [3.E Note For [3.88, 90, 91]] Or Define R(s) = r \Rightarrow \text{null } R = U, and R surj. Immed.
    (b) Let (1, z, ..., z^{m-1}) be a bss of \mathcal{P}_{m-1}(\mathbf{F}). By (4E 3.E.14) Or \widetilde{R}^{-1}: \mathcal{P}_{m-1}(\mathbf{F}) \to \mathcal{P}(\mathbf{F})/U, immed.
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6 Supp non0 $p \in \mathcal{P}_m(\mathbf{F})$ has deg m. Prove

[P] p has m disti zeros \iff p and its deri p' have no common zeros. [Q]

Solus: (a) Supp p of deg m has m disti zeros. By [4.14], $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$.

```
9 Supp p \in \mathcal{P}(C). Define q: C \to C by q(z) = p(z)p(\overline{z}). Prove q \in \mathcal{P}(R).
Solus: By [4.5], \overline{z}^n = \overline{z^n}. For any f(z) = a_n z^n + \dots + a_1 z + a_0, \overline{f(\overline{z})} = \overline{a_n} z^n + \dots + \overline{a_1} z + \overline{a_0}.
             Becs q(z) = p(z)\overline{p(\overline{z})} = \overline{p(\overline{z})}p(z) = \overline{q(\overline{z})}. Each c_k = \overline{c_k} \Rightarrow c_k \in \mathbb{R}.
                                                                                                                                                                   Or. Becs q(z) = p(z)\overline{p(\overline{z})} = \sum_{k=0}^{2n} \left( \sum_{i+j=k} c_i \overline{c_j} \right) z^k. For each k \in \{0, \dots, 2n\},
             \sum_{i+j=k} c_i \overline{c_j} = \sum_{i+j=k} c_i \overline{c_j} = \sum_{i+j=k} c_i \overline{c_i} = \sum_{i+j=k} c_i \overline{c_j} \in \mathbf{R}.
                                                                                                                                                                  10 Supp disti x_0, x_1, ..., x_m \in \mathbb{R}, and p \in \mathcal{P}_m(\mathbb{C}) suth each p(x_k) \in \mathbb{R}. Prove p \in \mathcal{P}(\mathbb{R}).
Solus: By Tips and Exe (5), \exists ! q \in \mathcal{P}_m(\mathbf{R}) suth q(x_k) = p(x_k). Hence p = q.
                                                                                                                                                                   OR. Define q(x) = \sum_{j=0}^{m} \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j).
   \mathbb{X} Each x_i, p(x_i) \in \mathbb{R} \Rightarrow q \in \mathcal{P}_m(\mathbb{R}). Becs each q(x_k) = 1 \cdot p(x_k) \Rightarrow (q - p)(x_k) = 0.
    (q-p) has (m+1) zeros. By Tips, q-p=0 \Rightarrow p=q \in \mathcal{P}(\mathbf{R}).
                                                                                                                                                                   • (4E 13) Supp nonC p, q \in \mathcal{P}(C) have no common zeros. Let m = \deg p, n = \deg q.
              Define T: \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C}) by T(r,s) = rp + sq. Prove T is inje.
  Coro: \exists ! r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C}) \text{ suth } rp + sq = 1.
Solus: Immed, T is liney. Supp T(r,s) = rp + sq = 0.
   Then rp = -sq. Becs p, q are coprime \Rightarrow p \mid s, while \deg s \leqslant m - 1 \Rightarrow s = 0 \Rightarrow r = 0.
                                                                                                                                                                  Or. Let \lambda_1, \dots, \lambda_M and \mu_1, \dots, \mu_N be the disti zeros of p and q respectly. Notice that M \leq m, N \leq n.
   By the ctrapos of [4.13], M = 0 \iff m = 0 \Rightarrow s = 0 \iff r = 0 \iff n = 0 \iff N = 0.
   Now supp M, N \ge 1. We show s = 0. Similar for r = 0. Or. s = 0 \Rightarrow r = 0.
   Write p(z) = a(z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_M)^{\alpha_M}. (\exists! \alpha_i \ge 1, a \in \mathbf{F}.) Let \max\{\alpha_1, \dots, \alpha_M\} = A.
   For each D \in \{0, 1, ..., A - 1\}, let I_{>D} = \{I_{D,1}, ..., I_{D,I_D}\} be suth each \alpha[I_{D,i}] = \alpha_{I_{D,i}} \ge D + 1.
   Now \{M\} = I_{>A-1} \subseteq \cdots \subseteq I_{>0} = \{1, ..., M\}. Becs rp + sq = 0 \Rightarrow (rp + sq)^{(k)} = 0 for all k \in \mathbb{N}^+.
   We use induc by D to show s^{(D)}(\lambda[I_{D,i}]) = 0 for each D \in \{0, ..., A-1\}.
   NOTICE that p^{(D)}(\lambda[I_{D,i}]) = 0 for each D \in \{0, ..., A-1\} and each I_{D,i} \in I_{>D}.
                                                                                                                                                              (L2)
   (i) D = 0. Each (rp + sq)(\lambda[I_{0,i}]) = (sq)(\lambda[I_{0,i}]) = s(\lambda[I_{0,i}]) = 0. Where q(\lambda[I_{0,i}]) \neq 0.
        D = 1. \text{ Each } (r'p + rp') (\lambda [I_{1,i}]) + (s'q + sq') (\lambda [I_{1,i}]) = (s'q) (\lambda [I_{1,i}]) = s' (\lambda [I_{1,i}]) = 0.
                    Where p'(\lambda[I_{1,i}]) = 0, and each I_{1,i} \subseteq I_{0,i} \Rightarrow s(\lambda[I_{1,i}]) = 0.
   (ii) 2 \leqslant D \leqslant A - 1. Asum s^{(d)}(\lambda[I_{d,j}]) = 0 for each d \in \{0,1,\ldots,D-1\} and each \lambda[I_{d,j}] \in I_{>d}.
          Each [rp + sq]^{(D)}(\lambda[I_{D,i}]) = [C_D^D r^{(D)} p^{(0)} + \dots + C_D^d r^{(d)} p^{(D-d)} + \dots + C_D^0 r^{(0)} p^{(D)}](\lambda[I_{D,i}])
                                                                                                                                                              (L1)
                                                         + \left[C_D^D s^{(D)} q^{(0)} + \dots + C_D^d s^{(d)} q^{(D-d)} + \dots + C_D^0 s^{(0)} q^{(D)}\right] (\lambda \left[I_{D,i}\right])
                                                     = [C_D^D s^{(D)} q^{(0)}](\lambda[I_{D,i}]). Where each \lambda[I_{D,i}] \in I_{>D} \subseteq I_{D-1,\alpha}.
          Hence s^{(D)}(\lambda[I_{D,j}]) = 0. The asum holds for all D \in \{0, ..., A-1\}.
   NOTICE that \forall k = \{0, ..., A-2\}, s^{(k)} \text{ and } s^{(k+1)} \text{ have zeros } \{\lambda \lceil I_{k+1,1} \rceil, ..., \lambda \lceil I_{k+1,I_{k+1}} \rceil \} in common.
   Now \forall D \in \{1, ..., A-1\}, s = s^{(0)}, ..., s^{(D)} \text{ have zeros } \{\lambda[I_{D,1}], ..., \lambda[I_{D,I_D}]\} \text{ in common.}
   Thus s(z) is divisible by (z - \lambda [I_{D,1}])^{\alpha [I_{D,1}]} \cdots (z - \lambda [I_{D,I_D}])^{\alpha [I_{D,I_D}]}, for each D \in \{0, ..., A - 1\}.
   Hence s(z) = \left[ (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_M)^{\alpha_M} \right] s_0(z), while deg s < m = \alpha_1 + \cdots + \alpha_M. Now by Tips.
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L1 Prove \forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \dots + C_k^l p^{(j)} q^{(k-j)} + \dots + C_k^0 p^{(0)} q^{(k)}.

Solus: We use induc by k \in \mathbf{N}^+. (i) k = 1. (pq)^{(1)} = (pq)' = C_1^1 p^{(1)} q^{(0)} + C_1^0 p^{(0)} q^{(1)}. (ii) k \geqslant 2.

Asum for (pq)^{(k-1)} = C_{k-1}^{k-1} p^{(k-1)} q^{(0)} + \dots + C_{k-1}^{j} p^{(j)} q^{(k-1-j)} + \dots + C_{k-1}^{0} p^{(0)} q^{(k-1)}.

Now (pq)^{(k)} = ((pq)^{(k-1)})' = \left(\sum_{j=0}^{k-1} C_{k-1}^j p^{(j)} q^{(k-j)}\right)' = \sum_{j=0}^{k-1} \left[C_{k-1}^i \left(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)}\right)\right].

= \left[C_{k-1}^0 \left(p^{(1)} q^{(k-1)} + p^{(0)} q^{(k)}\right)\right] + \left[C_{k-1}^1 \left(p^{(2)} q^{(k-2)} + p^{(1)} q^{(k-j)} + p^{(j-1)} q^{(k-j+1)}\right)\right]

+ \dots + \left[C_{k-1}^{j-2} \left(p^{(j-1)} q^{(k-j+1)} + p^{(j)} q^{(k-j)}\right)\right] + \left[C_{k-1}^{j+1} \left(p^{(j+2)} q^{(k-j-2)} + p^{(j+1)} q^{(k-j-1)}\right)\right]

+ \dots + \left[C_{k-1}^{k-2} \left(p^{(k-1)} q^{(1)} + p^{(k-2)} q^{(2)}\right)\right] + \left[C_{k-1}^{k-1} \left(p^{(j)} q^{(k-j)} + p^{(k-1)} q^{(1)}\right)\right].

Hence (pq)^{(k)} = C_k^0 p^{(0)} q^{(k)} + \dots + \left[C_{k-1}^j + C_{k-1}^{j-1}\right] \left(p^{(j)} q^{(k-j)}\right) + \dots + C_k^k p^{(k)} q^{(0)}.

- \mathbf{L2} \ Supp \ \alpha \in \mathbf{N}^+ \ suth \ p(z) = (z - \lambda)^\alpha q(z). \ Prove \ p^{(\alpha-1)}(\lambda) = 0.

Solus: \left[(z - \lambda)^\alpha q(z)\right]^{(\alpha-1)} = \sum_{j=1}^{\alpha-1} C_{\alpha-1}^j \left[(z - \lambda)^\alpha\right]^j q^{(\alpha-1-j)}. \ Immed.

- \mathbf{Thes 2:} \ Supp \ non0 \ p, q \in \mathcal{P}(\mathbf{F}) \ are \ multi \ of \ each \ other. \ Prove \ p = cq \ for \ a \ c \neq 0.

Solus: Let p = rq, q = sp \Rightarrow p = rsp \Rightarrow r(z)s(z) = 1 \ \text{for all } z \ \text{with } p(z) \neq 0, \text{ while such } z \ \text{is fini.}
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Thus (rs)(z) = 1 for infily many z, so for all z. Now deg $rs = 1 = \deg r + \deg s$.

ENDED

下面第5章中,3e和4e差距过大。我认为是因为4e将原来3e第8章的极小多项式和第2章线性无关最小性和第4章的多项式的原理结合,以极小多项式为工具重写了第5章几个核心定理,并引入一些结论承接读者一些很自然的想法,让第5章的定理和习题更加富有动机和系统性。

我编撰这份笔记,主要面向 3e 纸质书的读者,所以题号和定理索引都采用 3e (除少数 4e 新增章节)。所以 3e 读者会对第 5章这样的 4e 变化感到茫然无措。为了内容的紧密性,我决定将 3e 第 8章提前到第 5章后,对应到 4e 只有第 8章前三节。3e 第 8章个别涉及第 6、7章的习题,会拆散塞人对应章节的笔记中。额外说明,3e 的第 9章已经被拆散塞人前章了。

• Note For [5.6]: If V is infinide. Then (a) \Leftrightarrow (b) \Rightarrow (d), while (b) \Rightarrow (c), and (b) \Rightarrow (d). • **COMMENT**: λ not an eigval of $T \iff T - \lambda I$ inje \iff inv, if finide. • Supp V is finide, $T \in \mathcal{L}(V)$, and U is invarsp of V under T. *Prove or give a countexa: there exis invarsp of dimension* $\dim V - \dim U$. **Solus**: If dim *U* or dim *V* is odd, then \exists eigval of *T* by [4E 5.34], and \exists invarsp of dim dim V - 1. TODO • Supp $T \in \mathcal{L}(V)$, $\lambda_1, \dots, \lambda_m$ are the disti eigenstates corres v_1, \dots, v_m , and U invard T. • Tips 1: Supp $v_1 + \cdots + v_m \in U$. Prove each $v_k \in U$. **Solus**: Consider the stmt P(k): if $v_1 + \cdots + v_k \in U$, then each $v_i \in U$. (i) For k = 1, $v_1 \in U$, P(1) holds. (ii) For $2 \le k \le m$. Asum P(k-1) holds. Supp $v = v_1 + \cdots + v_k \in U$. Then $Tv = \lambda_1 v_1 + \dots + \lambda_k v_k \in U \Longrightarrow Tv - \lambda_k v = (\lambda_1 - \lambda_k)v_1 + \dots + (\lambda_{k-1} - \lambda_k)v_{k-1} \in U$. For each $j \in \{1, ..., k-1\}$, $\lambda_j - \lambda_k \neq 0 \Rightarrow (\lambda_j - \lambda_k)v_j = v_j'$ is an eigvec of T corres λ_j . By asum, each $v_i' \in U$. Thus $v_1, \dots, v_{k-1} \in U$. So that $v_k = v - v_1 - \dots - v_{k-1} \in U$. • Tips 2: Supp dim $V = m \Rightarrow B_V = (v_1, ..., v_m)$. Let each $E_k = \operatorname{span}(v_k)$. *Prove* $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m)$. **Solus:** Becs $V = E_1 \oplus \cdots \oplus E_m \Rightarrow \forall v \in U, v = c_1v_1 + \cdots + c_mv_m$, uniqly. $c_i \neq 0 \Rightarrow c_i v_i$ eigvec corres λ_i . Or $c_k = 0 \Rightarrow c_k v_k = 0 \in U$. By Tips (1), each $c_i v_i \in U$. Thus $v \in (U \cap E_1) \oplus \cdots \oplus (U \cap E_m) \supseteq U$. **Coro:** Becs each dim $E_j = 1 \Rightarrow (U \cap E_j) = E_j$ or $\{0\}$. Let E_{k_1}, \dots, E_{k_M} be all suth each $E_{k_j} = U \cap E_{k_j}$. • Tips 3: Supp U is a non0 invarsp of V under T. Let $\dim V = m$. Then $U = \operatorname{span}(v_{k_1}, \dots, v_{k_M})$. **2, 3** Supp $S, T \in \mathcal{L}(V)$ suth ST = TS. Prove null T, range T invard S. **Solus**: (a) $Tv = 0 \Rightarrow TSv = STv = 0$. (b) $Tu = v \Rightarrow Sv = STu = TSu \in \text{range } T$. **Coro:** Simlr in [5.20], p(S)q(T) = q(T)p(S). Then q(T) invard p(S). **6** Supp U is invarsp of non0 V under any $T \in \mathcal{L}(V)$. Show U = V or $\{0\}$. **Solus**: We show the ctrapos: Supp $U \neq \{0\}$ and $U \neq V$. Prove $\exists T \in \mathcal{L}(V)$, U is not invard T. Let $W \oplus U = V$. Define $T \in \mathcal{L}(V)$ by T(u + w) = w. • (4E 8 OR 5.B.4) Supp λ is eigral of $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove $\lambda = 0$ or 1. **Solus**: $v \neq 0$, $Pv = \lambda v = \lambda^2 v = P(Pv)$. Thus $\lambda = 1$ or 0. **14** Supp $V = U \oplus W$, and U, W non0. Define P(u + w) = u. Find all eigvals and eigvecs. **SOLUS:** Supp $u + w \neq 0$ and $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$.

Becs $(\lambda - 1)u = \lambda w = 0$. Now $\lambda = 0 \iff u = 0$, and $\lambda = 1 \iff w = 0$. Thus Pu = u, Pw = 0. \square

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• TIPS 4: Supp T \in \mathcal{L}(\mathbb{R}^2) is the countclockws rotat by \theta \in \mathbb{R}. Define \mathcal{C}(a,b) = a + ib.
  Becs (\cos \theta + i \sin \theta)(a + ib) = r(\cos(\alpha + \theta) + i \sin(\alpha + \theta)).
  Hence T(a,b) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta). Now \mathcal{M}(T) = \begin{pmatrix} \cos\theta & -\sin\theta\\ \sin\theta & \cos\theta \end{pmatrix}.
10 Define T \in \mathcal{L}(\mathbf{F}^n) by T(x_1, x_2, x_3, ..., x_n) = (x_1, 2x_2, 3x_3, ..., nx_n).
     (a) Find all eigvals and eigvecs; (b) Find all invarsps of V under T.
Solus: Let (e_1, ..., e_n) be the std bss of \mathbf{F}^n. The eigends are \{1, ..., n\} of len dim \mathbf{F}^n.
            Let each E_k = \text{span}(e_k). The set of all eigences is (E_1 \cup \cdots \cup E_n) \setminus \{0\}.
            By TIPS (3), invarsps are precisely span(e_{k_1}, \dots, e_{k_m}) for k_j \in \{1, \dots, n\}.
                                                                                                                                                       18 Define T \in \mathcal{L}(\mathbf{F}^{\infty}) by T(z_1, z_2, \dots) = (0, z_1, z_2, \dots). Show T has no eigenst.
Solus: Supp z_k \neq 0 and T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots). Thus \lambda z_1 = 0, \lambda z_k = z_{k-1}.
            (-)\ \lambda=0\Rightarrow \lambda z_2=z_1=0=\cdots=z_k.\ \ (=)\ \lambda\neq 0\Rightarrow z_1=0\Rightarrow z_2=\cdots=z_k=0.
                                                                                                                                                       19 Supp n \in \mathbb{N}^+. Define T \in \mathcal{L}(\mathbb{F}^n) by T(x_1, ..., x_n) = (x_1 + ... + x_n, ..., x_1 + ... + x_n).
     In other words, the ent of \mathcal{M}(T) wrto the std bss are all 1's. Find all eigvals and eigvecs of T.
Solus: Supp x_k \neq 0 and T(x_1, ..., x_n) = (\lambda x_1, ..., \lambda x_n) = (x_1 + ... + x_n, ..., x_1 + ... + x_n).
            Then (I) \lambda = 0 \Rightarrow x_1 + \dots + x_n = 0. If n > 1, then \lambda = 0 is eigval; othws not, becs T = I.
                     (II) \lambda \neq 0 \Rightarrow x_1 = \dots = x_n \Rightarrow \lambda x_k = nx_k. Now n is eigval.
                                                                                                                                                       OR. Becs range T = \{(x, ..., x) \in \mathbb{F}^n\} of dim 1. By Exe (29). Simlr.
                                                                                                                                                       Or. Supp n > 1. Becs null T = \{(-x_2 - \dots - x_n, x_2, \dots, x_n)\} of dim n - 1 > 0 \Rightarrow 0 is eigval.
            Notice that n is also eigval corres (x, ..., x) \neq 0. We show 0, n are the only eigvals.
            Supp non0 x \in \mathbb{F}^n and \lambda \in \mathbb{F} with Tx = \lambda x. Becs range T = \text{span}((1, ..., 1)), \exists ! \alpha \in \text{range } T,
            \lambda x = \alpha \Rightarrow x corres \lambda and \alpha corres n are liney dep. By the ctrapos of [5.10], \lambda = n.
                                                                                                                                                       20 Define S \in \mathcal{L}(\mathbf{F}^{\infty}) by S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots).
     Show every elem of F is an eigval of S, and find all eigvecs of S.
Solus: Supp z_k \neq 0 and S(z_1, z_2, ...) = (\lambda z_1, \lambda z_2, ...) = (z_2, z_3, ...). Then each \lambda z_k = z_{k+1}.
            (I) \lambda = 0 \Rightarrow \operatorname{each} z_k = \dots = z_2 = \lambda z_1 = 0. Let z_1 \neq 0 \Rightarrow E(0, S) = \operatorname{span}(e_1).
            (II) \lambda \neq 0 \Rightarrow \lambda^k z_1 = \lambda^{k-1} z_2 = \cdots = \lambda z_k = z_{k+1}, let z_1 \neq 0 \Rightarrow E(\lambda, S) = \text{span}[(1, \lambda^1, \dots, \lambda^k, \dots)].\square
• Supp V is finide, T \in \mathcal{L}(V), \lambda \in \mathbf{F}.
13 Prove \exists \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000} suth (T - \alpha I) is inv.
Solus: Let each |\alpha_k - \lambda| = \frac{1}{1000 + k}, where k \in \{1, \dots, \underline{\dim V + 1}\}. Then \exists \alpha_k not an eigval.
                                                                                                                                                       • (4E 11) Prove \exists \delta > 0 suth (T - \alpha I) is inv for all \alpha \in \mathbf{F} suth 0 < |\alpha - \lambda| < \delta.
Solus: If T has no eigvals, then (T - \alpha I) is inje for all \alpha \in \mathbb{F}, done.
            Supp \lambda_1, \dots, \lambda_m are all the disti eigvals of T unequal to \lambda.
            Let \delta > 0 be suth, for each eigval \lambda_k, \lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta).
            So that for all \alpha \in \mathbf{F} suth 0 < |\alpha - \lambda| < \delta, (T - \alpha I) is inv.
                                                                                                                                                       Or. Let \delta = \min\{|\lambda - \lambda_k| : k \in \{1, ..., m\}, \lambda_k \neq \lambda\}.
            Then \delta > 0 and each \lambda_k \neq \alpha \ [\iff (T - \alpha I) \text{ is inv }] for all \alpha \in \mathbf{F} suth 0 < |\alpha - \lambda| < \delta.
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Solus: For 0 \neq p \in \mathcal{P}(\mathbf{R}), \deg p' < \deg p. And \deg 0 = -\infty. Supp p' = \lambda p.
            Asum \lambda \neq 0. Then \deg \lambda p = \deg p' < \deg \lambda p, ctradic. Thus \lambda = 0.
            Therefore \deg \lambda p = -\infty = \deg p' \Rightarrow p \in \mathcal{P}_0(\mathbf{R}).
                                                                                                                                                  15 Supp T \in \mathcal{L}(V). Supp S \in \mathcal{L}(V) is inv.
     (a) Prove T and S^{-1}TS have the same eigvals.
     (b) Describe the relationship between eigvecs of T and eigvecs of S^{-1}TS.
Solus: (a) \lambda is an eigval of T with an eigvec v \Rightarrow S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v.
                 \lambda is an eigval of S^{-1}TS with an eigvec v \Rightarrow S(S^{-1}TS)v = T\underline{Sv} = \underline{\lambda Sv}.
                 OR. Note that S(S^{-1}TS)S^{-1} = T. Every eigval of S^{-1}TS is an eigval of S(S^{-1}TS)S^{-1} = T.
                 Or. Tv = \lambda v \iff TSu = \lambda Su \iff (S^{-1}TS)u = \lambda u. Where v = Su.
                       (S^{-1}TS)u = \lambda u \iff S^{-1}Tv = \lambda S^{-1}v \iff Tv = \lambda v. Where u = S^{-1}v.
            (b) Becs \lambda is eigval of T \iff of S^{-1}TS.
                 Now E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\}; E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}.
                                                                                                                                                  • (4E 15) Show \lambda is eigral of T \iff of T'.
Solus: [Req Finide; For [5.6]] T - \lambda I_V \text{ not inv} \iff (T - \lambda I_V)' = T' - \lambda I_V, \text{ not inv}.
                                                                                                                                                  (a) Supp \lambda is eigval with v. Let U be invar with U \oplus \text{span}(v) = V, by Exe (4E 39).
                 Define \psi \in V' by \psi(cv + u) = c. Then [T'(\psi)](cv + u) = \psi(c\lambda v + Tu) = \lambda c = \lambda \psi(cv + u).
            (b) A countexa: Let T be the forwd shift optor on V = \mathbf{F}^{\infty}. No eigvals for T, by Exe (18).
                 Define \psi \in V' by \psi(x_1, x_2, \dots) = x_1. Then [T'(\psi)](x_1, x_2, \dots) = \psi(0, x_1, x_2, \dots) = 0.
                                                                                                                                                  • Supp \mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V).
  (a) [(4E\ 17)\ OR\ [9.11]]\ \lambda \in \mathbf{R}. Prove \lambda is eigval of T \iff \lambda is eigval of T_{\mathbf{C}}.
  (b) [16 Or [9.16]] \lambda \in \mathbb{C}. Prove \lambda is eigral of T_{\mathbb{C}} \iff \overline{\lambda} is eigral of T_{\mathbb{C}}.
Solus: (a) Tv = \lambda v \Rightarrow T_C(v + i0) = \lambda v. T_C(v + iu) = \lambda v + i\lambda u \Rightarrow Tv = \lambda v, Tu = \lambda u.
            (b) Supp T_{\mathbf{C}}(v + iu) = Tv + iTu = \lambda(v + iu).
                 Becs \overline{T_{\rm C}(v+{\rm i}u)} = \overline{Tv+{\rm i}Tu} = Tv-{\rm i}Tu = T_{\rm C}(v-{\rm i}u) = T_{\rm C}(\overline{v+{\rm i}u}).
                 And \overline{\lambda(v+iu)} = \overline{\lambda}v - i\overline{\lambda}u = \overline{\lambda}(v-iu) = \overline{\lambda}(\overline{v+iu}).
                                                                                                                                                  Or. Supp \lambda = a + ib is eigval of T_C with v + iu.
                 Becs T_{\rm C}(v+\mathrm{i}u)=\lambda(v+\mathrm{i}u)=(\underline{av}-\underline{bu})+\mathrm{i}(\underline{au}+\underline{bv})=\underline{Tv}+\mathrm{i}Tu.
                 Now T_{\mathbf{C}}(\overline{v+\mathrm{i}u}) = Tv - \mathrm{i}Tu = (av - bu) - \mathrm{i}(au + bv) = (a - \mathrm{i}b)(v - \mathrm{i}u) = \overline{\lambda}(\overline{v+\mathrm{i}u}).
                                                                                                                                                  21 Supp T \in \mathcal{L}(V) is inv. Then 0 is not eigral of T or T^{-1}.
    (a) Supp \lambda \in \mathbf{F} with \lambda \neq 0. Prove \lambda is eigral of T \iff \lambda^{-1} is eigral of T^{-1}.
    (b) Prove T, T^{-1} have the same eigvecs.
Solus: Tv = \lambda v \iff v = \lambda T^{-1}v \iff \lambda^{-1}v = T^{-1}v. Where v \neq 0.
                                                                                                                                                  23 Supp V is finide, and S,T \in \mathcal{L}(V). Prove ST and TS have the same eigensts.
Solus: [False if infinide. See Exe (18, 20).] Supp v \neq 0 and STv = \lambda v \Rightarrow T(STv) = \lambda Tv = TS(Tv).
            If Tv = 0, then T not inje, so are TS, ST. Othws, \lambda is eigval of TS. Rev the roles in asum.
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11 Define $T: \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by Tp = p'. Find all eigends and eigences.

•	Supp $T \in \mathcal{L}(V)$ has $n = \dim V$ disti eigvals and $S \in \mathcal{L}(V)$ has the same eigvecs eight not with the same eigvals. Prove $ST = TS$.	
Solus:	Let each $\lambda_i v_i = T v_i$, $\mu_i v_i = S v_i$. Where μ_1, \dots, μ_n might have repeti.	
	Becs $B_V = (v_1,, v_n)$. Each $(ST)v_j = \mu_j \lambda_j v_j = (TS)v_j \Rightarrow ST = TS$.	
• (5.C.1	2) Supp V is finide, $R, T \in \mathcal{L}(V)$ has same dim V eigvals $\lambda_1, \dots, \lambda$. Prove \exists inv $S \in \mathcal{L}(V)$ suth $R = S^{-1}TS$.	
Solus:	Let $(u_1,, u_n)$, $(v_1,, v_n)$ be the corres eigences of R , T respectly, so be the bses of T . Becs each $Ru_k = \lambda_k u_k$, $Tv_k = \lambda_k v_k$. Define each $S(u_k) = v_k$.	
•	Supp V is finide, $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(S) = TS$.	
Prov	the set of eigvals of T equals the set of eigvals of A .	
Solus:	(a) For $v \neq 0$ and $Tv = \lambda v$, let $v_1 = v \Rightarrow B_V = (v_1, \dots, v_n)$.	
	Define $S \in \mathcal{L}(V) : v_j \mapsto v$, Or $v_j \mapsto \delta_{1,j}v_1$. Then each $(T - \lambda I)Sv_j = 0$.	
	Thus $(T - \lambda I)S = 0 \Rightarrow \mathcal{A}(S) = TS = \lambda S$ with $S \neq 0$.	
	(b) Supp $S \neq 0$ and $TS = \lambda S$. Then $\exists v \in V \setminus S$. Let $u = Sv \Rightarrow Tu = TSv = \lambda Sv = \lambda u$.	
	Or. $TS - \lambda S = (T - \lambda I)S = 0 \Rightarrow \{0\} \neq \text{range } S \subseteq \text{null}(T - \lambda I) \Rightarrow (T - \lambda I) \text{ not inje.}$	
• Tips 5	5: Supp $S, T \in \mathcal{L}(V), p \in \mathcal{P}(F)$. Prove $Sp(TS) = p(ST)S$.	
	We prove each $S(TS)^m = (ST)^m S$ by induc. (i) $m = 0, 1$. Immed.	
00200	(ii) $m > 1$. $S(TS)^{m-1} = (ST)^{m-1}S \Rightarrow S(TS)^m = S(TS)^{m-1}(TS) = (ST)^{m-1}(ST)S = (ST)^mS$. 🗆
Соммі	ENT: If <i>S</i> is inv. Then $p(TS) = S^{-1}p(ST)S$, $p(ST) = Sp(TS)S^{-1}$.	
	Becs S is inv, $T \in \mathcal{L}(V)$ is arb $\iff ST = R \in \mathcal{L}(V)$ is arb. Hence $p(S^{-1}RS) = S^{-1}p(R)S$.	
•	$pp \ T \in \mathcal{L}(V) \ is \ suth \ \forall v \in V, \exists ! \lambda_v \in \mathbf{F}, Tv = \lambda_v v. \ Prove \ T = \lambda I.$	
Solus:	Supp V non0. Becs $\forall v \in V, \exists ! \lambda_v \in \mathbf{F}, Tv = \lambda_v v$. For any distinon0 $v, w \in V$,	
	$T(v+w) = \lambda_{v+w}(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w.$	
	Or. For any non0 $u, v \in V$, u, v are eigvecs. If $u + v \neq 0$, then $u + v$ is also eigvec.	
	Othws done. By Exe (25), $\forall u, v \in V, Tu = \lambda u, Tv = \lambda v \Rightarrow \forall v \in V, Tv = \lambda v$.	
27 28	<i>Supp</i> dim $V > 1$, $k ∈ \{1,, dim V - 1\}$.	
21,20	Supp every subsp of dim k is invard a $T \in \mathcal{L}(V)$. Prove $T = \lambda I$.	
Solus:	We prove the ctrapos. Supp $\exists v \in V \setminus \{0\}$ not eigvec.	
	Then (v, Tv) liney indep $\Rightarrow B_V = (v, Tv, u_1, \dots, u_n)$. Let $U = \text{span}(v, u_1, \dots, u_{k-1})$.	
	Or. Supp $v = v_1 \in V \setminus \{0\} \Rightarrow B_V = (v_1,, v_n)$. Let $Tv_1 = c_1v_1 + \cdots + c_nv_n$.	
	Let $B_U = (v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$. Becs every such U invar. Now $Tv_1 \in U \Rightarrow Tv_1 = c_1v_1$.	
	By Exe (26), done. [For $0 \neq c_j \in \{c_2,, c_n\}$, let $B_W = (v_1, v_{\beta_1},, v_{\beta_{k-1}})$ with each $\beta_i \neq j$.]	
29 Su	$pp \ T \in \mathcal{L}(V)$, range T is finide. Prove T has at most $1 + \dim range T$ disti eigvals.	
•	Becs range T finide \Rightarrow not too many. Let $\lambda_1, \dots, \lambda_m$ be the disti eigvals of T with corres v_1, \dots, v_m	
	Then $(v_1,, v_m)$ liney indep $\Rightarrow (\lambda_1 v_1,, \lambda_m v_m)$ liney indep, if each $\lambda_k \neq 0$. Othws,	m
	$\exists ! \lambda_k = 0$. Now $\{\lambda_j v_j : j \neq k\}$ liney indep. By [2.23], $m - 1 \leq \dim \operatorname{range} T$.	

• Supp $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are disti. (a) 32 Prove $(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$ is liney indep in \mathbb{R}^R . (b) $[4E 36]$ Show $(\cos \lambda_1 x, \ldots, \cos \lambda_n x)$ is liney indep in \mathbb{R}^R .	
SOLUS: (a) Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$. Define $D \in \mathcal{L}(V)$ by $Df = f'$. Then becs each $\lambda_k e^{\lambda_k x} = D(e^{\lambda_k x})$. Now $\lambda_1, \dots, \lambda_n$ are disti eigvals of D . By [5.10].	
(b) Define V, D simlr. Becs $D(\cos \lambda_k x) = -\lambda_k \sin \lambda_k x$. $\nabla D(\sin \lambda_k x) = \lambda_k \cos \lambda_k x$. Thus $D^2(\cos \lambda_k x) = -\lambda_k^2 \cos \lambda_k x$. Now $-\lambda_1^2, \dots, -\lambda_n^2$ are disti eigvals of D^2 . Simlr.	
24 Supp $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Tx = Ax$. Prove 1 is eigend of T if: (a) the sum of the ent in each row of A equals 1. (b) each col of A .	
Solus: Supp $x \neq 0$ and $Ax = (A_{j,1}x_1 + \dots + A_{j,n}x_n)_{j=1}^n = \lambda(x_j)_{j=1}^n = \lambda x$.	
(a) Supp $A_{R,1} + \cdots + A_{R,n} = 1$. Let $x_1 = \cdots = x_n$. Immed. (b) Supp $A_{1,C} + \cdots + A_{n,C} = 1$. Then $\left[\sum_{R=1}^n A_{R,\cdot}\right] x = \sum_{k=1}^n \left(A_{1,k} + \cdots + A_{n,k}\right) x_k$. Now each $(Ax)_{R,1} = (x)_{R,1} = (\lambda x)_{R,1}$. Thus for x with $\sum_{k=1}^n x_k \neq 0$, $\lambda = 1$ is the corres eigval.	
OR. Becs $(T-I)x = (A-I)x = ((A_{j,1}x_1 + \dots + A_{j,n}x_n) - x_j)_{j=1}^n = (y_j)_{j=1}^n$. Now $y_1 + \dots + y_n = \sum_{j=1}^n \sum_{k=1}^n (A_{j,k}x_k - x_j) = \sum_{k=1}^n x_k \left[\sum_{j=1}^n A_{j,k} \right] - \sum_{j=1}^n x_j = 0$.	
Thus range $(T - I) \subseteq \{(y_1,, y_n) : y_1 + + y_n = 0\}$. Now $(T - I)$ is not inv.	
OR. Let $(e_1,, e_n)$ be the std bss of $\mathbf{F}^{n,1}$. Define $\psi \in (\mathbf{F}^{n,1})'$ with each $\psi(e_k) = 1$. Becs $Ae_k = A_{\cdot,k} = \sum_{j=1}^n A_{j,k}e_j \Rightarrow \psi(T-I)e_k = \psi\left(\sum_{j=1}^n A_{j,k}e_j - e_k\right) = \sum_{j=1}^n A_{j,k} - 1 = 0$. Thus $\psi(T-I) = 0 \Rightarrow (T-I)$ not inje.	
OR. Define $S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Sx = A^t x$. Becs the rows of $\mathcal{M}(S) = A^t$ are the cols of $\mathcal{M}(T) = A$. Let $(\varphi_1, \dots, \varphi_n)$ be the dual bss of (e_1, \dots, e_n) . Define $\Phi \in \mathcal{L}[\mathbf{F}^{n,1}, (\mathbf{F}^{1,n})']$ by $\Phi(e_k) = \varphi_k$. Now $(\Phi^{-1}T'\Phi)e_k = (\Phi^{-1}T')\varphi_k = \Phi^{-1}(\sum_{j=1}^n A_{j,k}^t \varphi_j) = \sum_{j=1}^n A_{j,k}^t e_j = A^t e_k = Se_k$. Becs by (a), 1 is eigval of $S = \Phi^{-1}T'\Phi$. So of T' , by Exe (15). So of T , by Exe (4E 15).	
Decs by (a), I is eigval of $S = \Psi$ I Ψ . So of I, by Exe (4E 15).	
• Supp $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Tx = xA$. Prove 1 is eigeal of T if: (a) the sum of the ent in each col of A equals 1. (b) each row of A .	
Solus: Supp $x \neq 0$ and $xA = (x_1A_{1,k} + \dots + x_nA_{n,k})_{k=1}^n = \lambda(x_k)_{k=1}^n = \lambda x$. (a) Supp $A_{1,C} + \dots + A_{n,C} = 1$. Let $x_1 = \dots = x_n$. Immed.	
(b) Supp $A_{R,1} + \dots + A_{R,n} = 1$. Then $\sum_{C=1}^{n} x A_{\cdot,C} = \sum_{j=1}^{n} (A_{j,1} + \dots + A_{j,n}) x_{j}$. Now each $(xA)_{1,C} = (x)_{1,C} = (\lambda x)_{1,C}$. Thus for x suth $\sum_{k=1}^{n} x_{k} \neq 0$, $\lambda = 1$ is the corres eigval.	
OR. Becs $(T-I)x = x(A-I) = ((x_1A_{1,k} + \dots + x_nA_{n,k}) - x_k)_{k=1}^n = (y_k)_{k=1}^n$. Now $y_1 + \dots + y_n = \sum_{k=1}^n \sum_{j=1}^n (x_jA_{j,k} - x_k) = \sum_{j=1}^n x_j \left[\sum_{k=1}^n A_{j,k}\right] - \sum_{k=1}^n x_k = 0$.	
Thus range $(T-I) \subseteq \{(y_1, \dots, y_n) : y_1 + \dots + y_n = 0\}$. Now $(T-I)$ is not inv.	
OR. Define $(e_1,, e_n)$ and $\psi(e_k) = 1$ simlr in Exe (24). Becs $e_j A = A_{j,.} = \sum_{k=1}^n A_{j,k} e_k$. Now $\psi(T - I)e_j = \sum_{k=1}^n A_{j,k} - 1 = 0 \Rightarrow \psi \circ (T - I) = 0$. Simlr.	
OR. Define $S \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Sx = xA^t$. NOTICE that $\mathcal{M}(S) \neq A$ and $\mathcal{M}(T) \neq A^t$. [Noted by AI.] Let $(\varphi_1, \dots, \varphi_n)$ be the dual bss. Define Φ by $\Phi(e_k) = \varphi_k$.	
Becs $[T'(\varphi_k)](e_j) = \varphi_k(\sum_{i=1}^n A_{j,i}e_i) = A_{j,k}$. By (3.F.9), $T'(\varphi_k) = \sum_{j=1}^n A_{j,k}\varphi_j$.	
Now $(\Phi^{-1}T'\Phi)e_k = (\Phi^{-1}T')\varphi_k = \Phi^{-1}(\sum_{j=1}^n A_{j,k}\varphi_j) = \sum_{j=1}^n A_{j,k}e_j = e_kA^t = Se_k$. Simlr.	

• (4E 16) Supp $B_V = (v_1, ..., v_n), T \in \mathcal{L}(V)$, and λ is eigval. Let A_M be the max of all ent of $A = \mathcal{M}(T, B_V)$. Prove $|\lambda| \leq A_M \cdot \dim V$. **Solus**: Supp λ is eigval with to v. Let $v = c_1 v_1 + \cdots + c_n v_n$. Becs $\lambda c_1 v_1 + \dots + \lambda c_n v_n = c_1 T v_1 + \dots + c_n T v_n = \sum_{k=1}^n c_k \left[\sum_{j=1}^n A_{j,k} v_j \right] = \sum_{j=1}^n \left[\sum_{k=1}^n c_k A_{j,k} \right] v_j$. Thus $\lambda c_j = \sum_{k=1}^n c_k A_{j,k} \Rightarrow \text{each } |\lambda| |c_j| = \sum_{k=1}^n |c_k| |A_{j,k}|$. Let $|c_M| = \max\{|c_1|, \dots, |c_n|\}$. Becs $v \neq 0 \Rightarrow |c_M| \neq 0$. Now $|\lambda||c_M| = \sum_{k=1}^n |c_k||A_{M,k}| \Rightarrow |\lambda| \leqslant \sum_{k=1}^n |A_{M,k}| \leqslant nM$. **35** Supp V is finide, $T \in \mathcal{L}(V)$, and U is invard T. Show λ is eigeal of $T/U \Rightarrow$ of T. Solus: Supp $v + U \neq 0$ and $Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v = u \in U$. If u = 0, done. Othws, two cases. If $(T - \lambda I)|_{U}$ inje \Rightarrow surj. Then $(T - \lambda I)v = u = (T - \lambda I)|_{U}(w)$, $\exists w \in U \Rightarrow T(v + w) = \lambda(v + w)$. If $(T - \lambda I)|_U = T|_U - \lambda I_U$ not inje. Then λ is eigval of $T|_U \Rightarrow$ of T. Or. Let $B_U = (u_1, ..., u_m) \Rightarrow (Tv - \lambda v, Tu_1 - \lambda u_1, ..., Tu_m - \lambda u_m)$ of len (m+1) liney dep in U. So that $a_0(T - \lambda I)v + a_1(T - \lambda I)u_1 + \dots + a_m(T - \lambda I)u_m = 0, \exists a_k \neq 0.$ Then $Tw = \lambda w$, where $w = a_0 v + a_1 u_1 + \dots + a_m u_m \neq 0 \Leftarrow w \notin U \Leftarrow v \notin U$. **36** Give a countexa: The result in Exe (35) is still true if V is infinide. Solus: Let $V = \{ f \in \mathbb{R}^{\mathbb{R}} : \exists ! m \in \mathbb{N}, f \in \text{span}(1, e^x, \dots, e^{mx}) \}.$ Let $U = \{ f \in \mathbb{R}^{\mathbb{R}} : \exists ! m \in \mathbb{N}^+, f \in \text{span}(e^x, ..., e^{mx}) \}.$ Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then range T = U invard inje T. Note that $(T/U)(1+U) = e^x + U = 0$. While 0 is not an eigval of T. • (4E 39) Supp $T \in \mathcal{L}(V)$, V is finide. Prove \exists eigval of $T \iff \exists$ invarsp of dim dim V-1. **Solus**: (a) Supp λ is eigval with v. Becs dim null $(T - \lambda I) \ge 1 \iff \dim \operatorname{range}(T - \lambda I) \le \dim V - 1 = N$. Let $B_{\text{range}(T-\lambda I)} = (w_1, \dots, w_m)$, $B_V = (w_1, \dots, w_m, u_1, \dots, u_n)$, $B_U = (w_1, \dots, w_m, u_1, \dots, u_{N-m})$. Becs *U* invard $(T - \lambda I)$. Now $u \in U \Rightarrow (T - \lambda I)u \in U \Rightarrow Tu \in U$. **NOTE:** *U* might not be in S_V span(v). (b) Supp *U* is invarspd *T* with dim $U = \dim V - 1 \Rightarrow \dim V/U = 1$. By (3.A.7), Exe (35). • (3.C.16) Let $\{F_n\}$ be the Fibonacci Seq. Define $T \in \mathcal{L}(\mathbb{R}^2) : (x,y) \mapsto (y,x+y)$. (a) Find all eigvals and eigvecs. (b) Show $T^n(0,1) = (F_n, F_{n+1})$ and find the formula. **SOLUS:** (a) Supp $\lambda(x,y) = (y,x+y)$ with x or y non0. Note that $x=0 \iff y=0$, and 0 is not eigval. Then $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $v_1 = (1, \frac{1+\sqrt{5}}{2})$; and $\lambda_2 = \frac{1-\sqrt{5}}{2}$, $v_2 = (1, \frac{1-\sqrt{5}}{2})$. Becs dim $\mathbb{R}^2 = 2$. (b) $T(0,1) = (F_1, F_2)$. Asum $T^k(0,1) = (F_k, F_{k+1})$. Then $T^{k+1}(0,1) = (F_{k+1}, F_k + F_{k+1})$. $T^{n}(0,1) = T^{n} \left[\frac{1}{\sqrt{5}} (v_{1} - v_{2}) \right] = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n} v_{1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n} v_{2} \right]$. Take the first slot. ENDED

5.B: I

- (I) 覆盖本节 4e 全部、上节 4e 末、3e 前半部分与之相关的所有习题。
 - 注意:本节 4e 和 3e 的 8.C 节、9.A 节有交集,许多略去的习题和注解可以在那两节 3e 找到。
- (II) 覆盖本节 3e 后半部分「上三角矩阵」和下节 4e;并且,下节还会覆盖下下节 4e。

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9 Supp V finide, T \in \mathcal{L}(V), and non0 v \in V. Let p \in \mathcal{P}(F) be non0 of smallest deg
  with p(T)v = 0. Show every zero of p is eigval of T.
Solus: Let p(z) = (z - \lambda)q(z) \Rightarrow p(T)v = 0 = (T - \lambda I)q(T)v \Rightarrow T(q(T)v) = \lambda q(T)v.
                                                                                                                                      • Tips 1: Supp V is finide, and v \in V.
            (a) Prove \exists! monic p_v of smallest deg suth p_v(T)v = 0.
             (b) Prove p_v is the min q of T|_{\text{null }p_v(T)}.
                                                                                         So that the min of T is a multi of p_v.
Solus: (a) [Existns] If v=0, then let p_v(z)=1. Supp v\neq 0. Then (v,Tv,\ldots,T^{\dim V}v) liney dep.
                              \exists smallest m suth -T^m v = c_0 v + c_1 T v + \cdots + c_{m-1} T^{m-1} v. Thus define p_v.
                              OR. Let U = \text{span}(v, Tv, ..., T^{m-1}v) of dim m invard T. Let p_v be the min of T|_{U}.
                [Uniques] Supp q_v is monic of smallest deg [= deg p_v] and q_v(T)v = 0.
                               Then (p_v - q_v)(T)v = 0, while \deg p_v = m = \deg q_v \Rightarrow \deg (p_v - q_v) < m.
           (b) Becs p_v(T|_{\text{null }p_v(T)}) = 0 \Rightarrow p_v is multi of q. \mathbb{X} q(T)v = 0 \Rightarrow q = p_v, by the min of \deg p_v. \square
10 Supp T \in \mathcal{L}(V), \lambda is eigral of T with v. Prove if p \in \mathcal{P}(\mathbf{F}), then p(T)v = p(\lambda)v.
Solus: Define p(z) = a_0 + a_1 z + \dots + a_m z^m. Becs for each k \in \mathbb{N}^+, T^k v = \lambda^k v, and T^0 v = v.
           Now p(T)v = a_0v + a_1\lambda v + \dots + a_m\lambda^m v = p(\lambda)v.
                                                                                                                                      CORO: p(T)v = \left[c(T-\lambda_1)^{\alpha_1}\cdots(T-\lambda_m)^{\alpha_m}\right]v = \left[c(\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_m)^{\alpha_m}\right]v.
11 Supp \mathbf{F} = \mathbf{C}, T \in \mathcal{L}(V), nonC p \in \mathcal{P}(\mathbf{F}).
    Prove \alpha is eigval of p(T) \iff \alpha = p(\lambda) for some eigval \lambda of T.
Solus: Supp p(T) - \alpha I not inje. Let p(z) - \alpha = c(z - \lambda_1) \cdots (z - \lambda_m), with c \neq 0, becs p nonC.
           Then \exists (T - \lambda_i I) not inje. Now p(\lambda_i) - \alpha = 0. Convly true immed.
                                                                                                                                      13 Supp F = C, T \in \mathcal{L}(V) has no eigvals. Prove every invarsp either \{0\} or infinide.
Solus: Supp U is a finide non0 invarsp. Then by [5.21], \exists eigval of T|_{U}, so of T.
                                                                                                                                      • Supp non0 v \in V. Prove [5.21] using the given map below.
16 Define S: \mathcal{P}_{\dim V}(\mathbf{C}) \to V by S(p) = p(T)v. Then S not inje \Rightarrow \exists non0 p \in \text{null } S.
17 Define S: \mathcal{P}_{\dim V^2}(\mathbf{C}) \to \mathcal{L}(V) by S(p) = p(T). Then S not inje \Rightarrow \exists non0 p \in \text{null } S.
Solus: Let p(z) = c(z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow (T - \lambda_1 I) \cdots (T - \lambda_m I) not inje.
                                                                                                                                      Note: \exists monic q \in \text{null } S|_W of smallest deg, q(T) = 0, then q is the min poly.
18 [4E 15] Supp \mathbf{F} = \mathbf{C}, V finide and non0, T \in \mathcal{L}(V).
              Define f: \mathbb{C} \to \mathbb{N} by f(\lambda) = \dim \operatorname{range}(T - \lambda I). Prove f is not continuous.
Solus: Let \lambda_0 be eigval of T. Then (T - \lambda_0 I) is not surj. Hence dim range (T - \lambda_0 I) < \dim V.
           Becs T has finily many eigvals. \exists \text{ seq } \{\lambda_n\} with each \lambda_n not eigval of T, suth \lim_{n \to \infty} \lambda_n = \lambda_0
           Becs each f(\lambda_n) = \dim \operatorname{range}(T - \lambda_n I) = \dim V \neq f(\lambda_0) \Rightarrow f(\lambda_0) \neq \lim_{n \to \infty} f(\lambda_n).
                                                                                                                                      19 Supp V is finide, dim V > 1, T \in \mathcal{L}(V). Prove \{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V).
Solus: If \forall S \in \mathcal{L}(V), \exists p \in \mathcal{P}(F), S = p(T). Then by [5.20], \forall S_1, S_2 \in \mathcal{L}(V), S_1S_2 = S_2S_1.
           Note that dim V \ge 2. By (3.A.14) Or (3.D.16 Or 4E 3.A.11).
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1 Supp T \in \mathcal{L}(V) and T^n = 0. Prove (I - T) is inv and (I - T)^{-1} = I + T + \cdots + T^{n-1}.
Solus: Becs p(z) = 1 - x^n = (1 - x)(1 + x + \dots + x^{n-1}). Consider p(T) = I, by [5.20].
                                                                                                                                        • Supp T \in \mathcal{L}(V) has no eigends and T^4 = I. Prove T^2 = -I.
Solus: Becs T^4 - I = (T^2 - I)(T^2 + I) = 0 not inje, so is (T^2 - I) or (T^2 + I), while T has no eigvals.
           (T-I), (I+T) inje, so is (T^2-I) \Rightarrow \forall v \in V, 0 = (T^2-I)(T^2+I)v \iff 0 = (T^2+I)v.
           Or. Note that \forall v \in V, v = (I - T^2)v/2 + (I + T^2)v/2. X = I - T^4 = (I \pm T^2)(I \mp T^2).
           Then range (I \mp T^2) \subseteq \text{null}(I \pm T^2) \Rightarrow V = \text{null}(I - T^2) + \text{null}(I - T^2).
           \not T has no eigvals \iff (I - T^2) inje \iff null(I - T^2) = \{0\} \supseteq \text{range}(I + T^2).
                                                                                                                                        8 Give an exa of T \in \mathcal{L}(\mathbb{R}^2) suth T^4 = -I.
Solus: Define i^n \in \mathcal{L}(\mathbb{R}^2) by i^n(x,y) = (\operatorname{Re}(i^n x + i^{n+1} y), \operatorname{Im}(i^n x + i^{n+1} y)).
   T^{4} + I = (T^{2} + iI)(T^{2} - iI) = (T + i^{1/2}I)(T - i^{1/2}I)(T - (-i)^{1/2}I)(T + (-i)^{1/2}I).
   Note that i^{1/2} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, (-i)^{1/2} = i^{3/2} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}. Hence T = \pm (\pm i)^{1/2}I.
   Let T = i^{1/2}I defined by i^{1/2}(x,y) = \left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right).
                                                                                                                                        Or. Becs \mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix}. Using \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}
                                                                                                                                        • (4E7) Supp S, T \in \mathcal{L}(V) with the min polys p, q respectly. Supp S or T is inv. Prove p = q.
Solus: S \text{ inv} \Rightarrow p(TS) = S^{-1}p(ST)S = 0 and q(ST) = Sq(TS)S^{-1} = 0 \Rightarrow p = q. Rev the roles.
                                                                                                                                        • (4E 21) Supp V finide, T \in \mathcal{L}(V). Prove the min p has deg at most 1 + \dim \operatorname{range} T.
Solus: Let q be the min of T|_{\text{range }T}. Then q(T)Tv = 0 \Rightarrow zq(z) of deg < 1 + dim range T is multi of p.\Box
• Supp T \in \mathcal{L}(V). Then each (T')^k(\varphi) = (T')^{k-1}(\varphi \circ T) = \cdots = \varphi \circ T^k.
  If U invarspd T, then each \pi(T^k v) = T^k v + U = (T/U)^k (v + U) = (T/U)^k \pi(v).
• (4E 5.31, 4E 25, 26) Supp V is finide, U invarspd T \in \mathcal{L}(V), with the min p.
  Supp r the min of T|_U, and s of T/U. For q \in \mathcal{P}(\mathbf{F}), define Z_q as the set of zeros of q.
  Then Z_v is the set of eigvals of T. Simlr for Z_r, Z_s.
  (a) Prove p is a multi of r and of s. (b) Show rs is multi of p. (c) Prove Z_p = Z_r \cup Z_s.
Solus: (a) p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0.
                p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v+U) = p(T)v + U = 0.
           (b) \forall v \in V, s(T/U)(v+U) = s(T)v + U = 0 \Rightarrow (rs)(T)v = r(T)(s(T)v) = 0.
           (c) By (b), Z_p \subseteq Z_r \cup Z_s. Let ar = p and bs = p. Then for r(\lambda) = 0 or s(\lambda) = 0,
                [which is equiv to \lambda \in Z_r \cup Z_{s'}] then p(\lambda) = 0 \iff \lambda \in Z_p.
                                                                                                                                        • (4E 28) Supp V is finide and T \in \mathcal{L}(V). Prove the min p of T' equals the min q of T.
Solus: (a) \forall \varphi \in V', p(T')(\varphi) = \varphi \circ p(T) = 0 \Rightarrow p(T) \in \text{null } \varphi. Thus p(T) = 0.
           (b) q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi(q(T)) = q(T')(\varphi) = 0. Thus q(T') = 0.
                                                                                                                                        Or. By (3.F.15), for any s \in \mathcal{P}(\mathbf{F}), s(T') = s(T)' = 0 \iff s(T) = 0. Simlr.
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• (4E 8) Find the min p of T \in \mathcal{L}(\mathbb{R}^2), the countclockws rotat optor by \theta \in \mathbb{R}^+.
Solus: If \theta = 2k\pi, then p(z) = z - 1. If \theta = \pi + 2k\pi, then p(z) = z + 1.
                                                                                                                L = |OD|
           Othws, let span(v, Tv) = \mathbb{R}^2. Let L = x^2 + y^2, where v = (x, y).
                                                                                                            T^2 \overrightarrow{v} = \overrightarrow{OA}
           Supp p(z) = z^2 + bz + c. Let P = L\cos\theta \Rightarrow L/2P = 1/(2\cos\theta).
                                                                                                             T \overrightarrow{v} = \overrightarrow{OC}
                                                                                                                \overrightarrow{v} = \overrightarrow{OB}
           Then Tv = (L/2P)(T^2v + v) \Rightarrow T = (L/2P)(T^2 + I).
           Hence p(T) = T^2 - 2\cos\theta T + I = 0.
           Or. Let (e_1,e_2) be the std bss of \mathbb{R}^2. Becs Te_1=\cos\theta\;e_1+\sin\theta\;e_2,\;T^2e_1=\cos2\theta\;e_1+\sin2\theta\;e_2.
           ce_1 + bTe_1 = -T^2e_1 \iff \begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix} \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}. Now det =\sin\theta \neq 0, c=1, b=2\cos\theta. \square
• (4E 11) Supp V is 2-dim, T \in \mathcal{L}(V) with the min p, and \mathcal{M}(T,(v,w)) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}. (a) Show q(z) = z^2 - (a+d)z + (ad-bc) is a multi of p.
            (b) Show if b = c = 0 and a = d, then p(z) = z - a; othws p = q.
SOLUS: (a) Tv = av + bw \Rightarrow (T - aI)v = bw \Rightarrow (T - dI)(T - aI)v = bTw - bdw = bcv.
                 Tw = cv + dw \Rightarrow (T - dI)w = cv \Rightarrow (T - aI)(T - dI)w = cTv - acv = bcw.
            (b) If b = c = 0 and a = d. Then \mathcal{M}(T) = a\mathcal{M}(I) \Rightarrow T = aI. Othws, we show T \notin \text{span}(I),
                 so that \deg p = \dim V. Let (1) a = d, (2) b = 0, (3) c = 0. Then (1), (2) and (3) cannot be all true.
                 (I) Asum (1) is true, with (2) or (3) not true. Then Tv = av + bw, or Tw = cv + aw \notin \text{span}(w).
                 (II) Asum (2) or (3) are true, with (1) not true. Then Tv = av + bw, or Tw = cv + dw.
                                                                                                                                              • (8.C.18 OR 4E 16) Define T \in \mathcal{L}(\mathbf{F}^n) : (x_1, \dots, x_n) \mapsto (-a_0 x_n, x_1 - a_1 x_n, \dots, x_{n-1} - a_{n-1} x_n).
                         Show the min p of T is q(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + z^n.
Solus: Becs Te_1 = e_2, T^2e_1 = e_3, ..., T^{n-1}e_1 = e_n, T^ne_1 = T^{n-k}e_{k+1} = Te_n = -(a_0e_1 + \cdots + a_{n-1}e_n).
           Let -T^n = c_0 I + c_1 T + \cdots + c_{n-1} T^{n-1} \Rightarrow \text{each } c_k = a_k. Becs n = \dim V. No greater deg.
                                                                                                                                              • (4E 17) Supp V finide, T \in \mathcal{L}(V) with the min p, and \lambda \in \mathbf{F}.
            Show the min s of (T - \lambda I) is q(z) = p(z + \lambda).
Solus: Becs q(T - \lambda I) = p(T) = 0 \Rightarrow q a multi of s \Rightarrow \deg q = \deg p \geqslant \deg s.
           Define r(z) = s(z - \lambda) \Rightarrow r(T) = s(T - \lambda I) = 0 \Rightarrow \deg r = \deg s \geqslant \deg p.
                                                                                                                                              OR. Becs T^k \in \text{span}(I, T, ..., T^{k-1}) \iff (T - \lambda)^k \in \text{span}(I, (T - \lambda I), ..., (T - \lambda I)^{k-1}).
                                                                                                                                              • (4E 18) Supp V is finide, T \in \mathcal{L}(V) with the min p of deg m, and \lambda \neq 0.
            Show the min s of \lambda T is q(z) = \lambda^m p(z/\lambda).
Solus: Becs q(\lambda T) = \lambda^m p(T) = 0 \Rightarrow q is multi s \Rightarrow \deg q = \deg p \geqslant \deg s.
           Define r(z) = s(\lambda z) \Rightarrow r(T) = s(\lambda T) = 0 \Rightarrow \deg r = \deg s \geqslant \deg p.
                                                                                                                                              OR. Becs (\lambda T)^k \in \text{span}(\lambda I, \lambda T, ..., (\lambda T)^{k-1}) \iff T^k \in \text{span}(I, T, ..., T^{k-1}).
                                                                                                                                              • (4E 10, 23) Supp V is finide, T \in \mathcal{L}(V), with the min p of deg m.
                Supp non0 v \in V. Let each U_k = \text{span}(v, Tv, ..., T^k v).
                Prove \exists j \in \{1, ..., m\}, U_{j-1} = U_n for all n \ge j - 1.
Solus: Supp j is the smallest suth T^jv = a_0v + a_1Tv + \cdots + a_{i-1}T^{j-1}v \in U_{i-1} \Rightarrow j \leq m.
           Then U_{i-1} is invard T, so is each U_n = \operatorname{span}(v, Tv, \dots, T^{j-1}v, \dots, T^nv).
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• (4E 13) Supp V finide, T \in \mathcal{L}(V), with the min p(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m.
            Prove if q(z) = a_0 + a_1 z + \dots + a_n z^n, then \exists ! r \in \mathcal{P}_{\text{deg } p-1}(\mathbf{F}), q(T) = r(T).
Solus: Becs p \neq 0. By the div algo, immed. [r = 0 \text{ if } q = p]
                                                                                                                                      Or. Becs T^m = -c_0I - c_1T - \cdots - c_{m-1}T^{m-1}. For \deg q < m = \deg p, the repres of q(T) is uniq.
           Supp deg q \ge \deg p. For each k \in \mathbb{N}, \exists ! b_{i,k} \in \mathbb{F}, T^{m+k} = b_{0,k}I + b_{1,k}T + \cdots + b_{m-1,k}T^{m-1}.
• (4E 14) Supp V finide, T \in \mathcal{L}(V), with the min p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m,
            and a_0 \neq 0. Give a repres of s, the min of T^{-1}.
Solus: Define q(z) = z^m + \frac{a_1}{a_0} z^{m-1} + \dots + \frac{a_{m-1}}{a_0} z + \frac{1}{a_0} \Rightarrow q(T^{-1}) = T^{-m} p(T) = 0.
           Becs deg s \le \deg q, while (T^{-1})^{-1} = T \Rightarrow \deg q \le \deg s.
                                                                                                                                      Or. Becs each T^{-k} \notin \text{span}(I, T^{-1}, ..., T^{-(k-1)}) for k \in \{1, ..., m-1\}. Done.
           For if not, supp T^{-k} = b_0 I + b_1 T^{-1} + \dots + b_{k-1} T^{k-1}. Note that T inv \Rightarrow \exists b_i \neq 0.
           Now T^k(T^{-k}) = I = b_0 T^k + b_1 T^{k-1} + \dots + b_{k-1} T \Rightarrow T^j \in \text{span}(I, T, \dots, T^{k-1}).
                                                                                                                                      • (8.C.11) Supp V finide and T \in \mathcal{L}(V) inv. Prove \exists q \in \mathcal{P}(\mathbf{F}), T^{-1} = q(T).
Solus: By (4E 22), I = a_1 T + \dots + a_m T^m \Rightarrow T^{-1} = a_1 I + a_2 T + \dots + a_m T^{m-1}.
                                                                                                                                      • (4E 19) Supp V finide, T \in \mathcal{L}(V), with the min p(z) = c_0 + c_1 z + \dots + c_{m-1} z^{m-1} + z^m.
           Let \mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}\, a subsp of \mathcal{L}(V). Prove dim \mathcal{E} = \deg p.
Solus: Becs T^m = c_0 I + c_1 T + \dots + c_{m-1} T^{m-1} \Rightarrow U = \operatorname{span}(I, T, \dots, T^{m-1}) \Rightarrow U invard T
           \Rightarrow each T^{m+k} = T^k(T^m) \in U \Rightarrow \mathcal{E} = \operatorname{span}(I, T, \dots, T^{\dim \mathcal{L}(V)-1}) = \operatorname{span}(I, T, \dots, T^{m-1}) = U.
           Or. Define \Phi \in \mathcal{L}(\mathcal{P}(\mathbf{F}), \mathcal{L}(V)) by \Phi(q) = q(T) \Rightarrow \operatorname{range} \Phi = \mathcal{E}.
           Becs \Phi(q) = q(T) = 0 \iff q \text{ is a multi of the min } p \iff q \in \{ps : s \in \mathcal{P}(\mathbf{F})\} = \text{null } \Phi.
           Now by (4.11), dim \mathcal{P}(\mathbf{F})/\text{null }\Phi = \deg p = m. By [3.91](d).
                                                                                                                                      • (4E 29) Supp V is finide, dim V = n \ge 2, and T \in \mathcal{L}(V). Show T has a 2-dim invarsp.
Solus: See [9.8] for a graceful proof. Or. Let each V_k be an arb vecsp of dim k with an arb T_k \in \mathcal{L}(V_k).
   Define the stmt P(k): every optor on a V_k has invarsp of dim 2. (i) k=2. Immed.
   (ii) k \ge 2. Asum P(k) holds. Let p be the min of T_{k+1} = T. Note that V_{k+1} non0 \Rightarrow p nonC, \deg p \ge 1.
   (a) If p(z) = (z - \lambda)q(z), then by (4E 5.A.39), \exists U invarspd T of dim k.
        By asum, the optor T|_U on a k-dim vecsp has invarsp of dim 2, so has T.
   (b) Othws, T_{k+1} has no eigvals \Rightarrow p of deg \geqslant 1 has no zeros, thus F = R, and deg p is even.
        Let p(z) = (z^2 + b_1 z + c_1) \cdots (z^2 + b_m z + c_m) \Rightarrow \exists (T^2 + b_i T + c_i) not inje
        \Rightarrow \exists v \neq 0, (T^2 + b_i T + c_i)v = 0 \Rightarrow T^2 v \in \text{span}(v, Tv), \text{ invard } T, \text{ while } \dim \text{span}(v, Tv) = 2.
• Note For [4E 5.33]: Supp F = R, V is finide, T \in \mathcal{L}(V), and b^2 < 4c for b, c \in F.
                              Prove dim null(T^2 + bT + cI)^j is even for each j \in \mathbb{N}^+.
Solus: Using induc on j. (i) Immed. (ii) j > 1. Asum it holds for j - 1.
           Replace V with \operatorname{null}(T^2 + bT + cI)^j and T with T restr to \operatorname{null}(T^2 + bT + cI)^j.
           Then (T^2 + bT + cI)^j = 0 \Rightarrow z^2 + bz + c is a multi of the min of T \Rightarrow no eigvecs of T.
           Let U be invarspd T and has the largest even dim of all such invarsp. If V = U, done. Othws,
           for w \in V \setminus U \Rightarrow W = (w, Tw) invard T of dim 2 \Rightarrow U + W of dim (\dim U + 2) invard T.
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5.B: II 注意:这一节的题号使用第四版 5.C 节.

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2 Supp A and B are up-trig (and square) matrices of the same size,
   with \alpha_1, \ldots, \alpha_n on the diag of A and \beta_1, \ldots, \beta_n on the diag of B.
   (a) Show A + B up-trig with \alpha_1 + \beta_1, ..., \alpha_n + \beta_n on the diag.
   (b) Show AB up-trig with \alpha_1\beta_1, \dots, \alpha_n\beta_n on the diag.
Solus: (a) By def, immed. (b) Becs A_{i,k} = B_{i,k} = 0 for j > k. By def, for each p \in \{1, ..., n\},
            (AB)_{p,p} = A_{p,1}B_{1,p} + \dots + A_{p,p-1}B_{p-1,p} + A_{p,p}B_{p,p} + A_{p,p+1}B_{p+1,p} + \dots + A_{p,n}B_{n,p} = A_{p,p}B_{p,p}.
                                                                                                                                                   3 Supp T inv, B_V = (v_1, ..., v_n), \mathcal{M}(T) = A is up-trig,
   with \lambda_1, \ldots, \lambda_n on diag. Show A^{-1} is also up-trig, with \lambda_1^{-1}, \ldots, \lambda_n^{-1} on diag.
Solus: Becs \lambda_k on diag of A \iff \lambda_k eigval of T \iff \lambda_k^{-1} eigval of T^{-1} \iff \lambda_k^{-1} on diag of A^{-1}.
                                                                                                                                                    Or. Let each Tv_k = u_k + \lambda_k v_k, where u_k \in \text{span}(v_1, \dots, v_{k-1}). We use induc on k.
   (i) k=1. Tv_1=\lambda_1v_1\Rightarrow T^{-1}v_1=\lambda_1^{-1}v_1\in \mathrm{span}\big(v_1\big), invard T^{-1}; and \lambda_1^{-1} is the 1st ent on diag.
   (ii) k \ge 2. Asum span(v_1, \dots, v_{k-1}) invard T^{-1}.
         Note that Tv_k = u_k + \lambda_k v_k \Rightarrow v_k = T^{-1}(c_1 v_1 + \dots + c_{k-1} v_{k-1}) + \lambda_k T^{-1} v_k.
         Thus T^{-1}v_k = \lambda_k^{-1}v_k - \lambda_k^{-1}T^{-1}u_k \in \operatorname{span}(v_1,\ldots,v_k), invard T; and \lambda_k^{-1} is the k^{\operatorname{th}} ent on diag.
                                                                                                                                                   8 Supp V is finide, and v \in V is non0 suth q(T)v = 0, where q(z) = z^2 + 2z + 2.
   (a) Supp \mathbf{F} = \mathbf{R}. Prove \not\exists B_V suth \mathcal{M}(T) up-trig.
   (b) Supp \mathbf{F} = \mathbf{C}, and \exists B_V suth A = \mathcal{M}(T) up-trig. Prove -1 + \mathrm{i} or -1 - \mathrm{i} on diag.
Solus: Define p_v as (4E 3.C.7). Note that \deg p_v \geqslant 1 becs v \neq 0. \boxtimes q(T|_{\operatorname{null} p_v(v)}) = 0.
            Now q of deg 2 is a multi of the min of T|_{\text{null }p_n(v)}, which is p_v, of which the min of T is a multi.
            (a) Note that q has no 1-deg factors \Rightarrow deg p_v \ge 2. By [4E 5.44].
            (b) q(z) = (z + 1 + i)(z + 1 - i) \Rightarrow -1 - i or -1 + i zero of p_v \Rightarrow is eigval \Rightarrow on diag.
                                                                                                                                                   9 Supp B \in \mathbb{C}^{n,n}. Prove \exists inv A \in \mathbb{C}^{n,n} suth A^{-1}BA is up-trig.
Solus: Define T \in \mathbb{C}^n with B = \mathcal{M}(T, (e_1, \dots, e_n)). Let C = \mathcal{M}(T, (f_1, \dots, f_n)) be up-trig.
            Let A = \mathcal{M}(I, f \to e). Then C = A^{-1}BA.
                                                                                                                                                   10 Supp B_V = (v_1, ..., v_n), A = \mathcal{M}(T, B_V). Show the following are equiv:
     (a) A is low-trig. (b) Each Tv_k \in \text{span}(v_k, ..., v_n). (c) Each \text{span}(v_k, ..., v_n) invard T.
Solus: By def, (a) and (b) are equiv, and (c) \Rightarrow (b). Now supp (b) holds. For any k \in \{1, ..., n\}.
            Tv_k \in \operatorname{span}(v_k, \dots, v_n), Tv_{k+1} \in \operatorname{span}(v_{k+1}, \dots, v_n), \dots, Tv_n \in \operatorname{span}(v_n). Thus (c) holds.
                                                                                                                                                   • Tips 1: Supp B_V = (v_1, \dots, v_n), B_{V'} = (\varphi_1, \dots, \varphi_n), T \in \mathcal{L}(V), A = \mathcal{M}(T, B_V).
              (a) A up-trig \iff T = \sum_{k=1}^{n} \sum_{j=1}^{k} A_{j,k} E_{k,j} \iff T' = \sum_{k=1}^{n} \sum_{j=1}^{k} A_{k,j}^{t} \exists_{j,k} \iff A^{t} \text{ low-trig.}

(b) A low-trig \iff T = \sum_{k=1}^{n} \sum_{j=1}^{k} A_{k,j} E_{j,k} \iff T' = \sum_{k=1}^{n} \sum_{j=1}^{k} A_{j,k}^{t} \exists_{k,j} \iff A^{t} \text{ up-trig.}
• Tips 2: Supp (\alpha_1, ..., \alpha_n), (\beta_1, ..., \beta_n) are bses of V, with each \alpha_k = \beta_{n-k+1}.
              Prove \mathcal{M}(T, \alpha \to \alpha) up-trig \iff \mathcal{M}(T, \beta \to \beta) low-trig.
Solus: For each k \in \{1, ..., n\}, T\beta_{n-k+1} = T\alpha_k \in \operatorname{span}(\alpha_1, ..., \alpha_k) = \operatorname{span}(\beta_n, ..., \beta_{n-k+1}).
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Coro: (a) Supp $\mathbf{F} = \mathbf{C}$. Then $\exists B_V$ suth $\mathcal{M}(T, B_V)$ low-trig. (b) T up-trig $\iff T'$ up-trig.

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12, 13 Supp V finide, T \in \mathcal{L}(V). Prove T|_{U}, T/U up-trig for some U invarsp \iff T up-trig.
Solus: Supp B_U = (u_1, \dots, u_p), B_{V/U} = (w_1 + U, \dots, w_q + U) suth \mathcal{M}(T|_U), \mathcal{M}(T/U) up-trig.
            Then each Tu_k \in \text{span}(u_1, \dots, u_k) and each Tw_i + U \in \text{span}(w_1 + U, \dots, w_i + U).
            By (3.E.13), B_V = (u_1, ..., u_p, w_1, ..., w_q). Now each Tw_i \in \text{span}(u_1, ..., u_p, w_1, ..., w_i).
                                                                                                                                                         OR. By (4E 5.B.25)(b) and [4E 5.44], immed. Convly, by [4E 5.44], immed.
                                                                                                                                                         ENDED
5.C & [4E] 5.D
                                             注意:这一节的题号主要使用第四版 5.D 节.
15 Supp F = C, V is finide, T \in \mathcal{L}(V) with the min p. Then using Exe (4.6),
     T \operatorname{diag} \iff \exists (z - \lambda)^2 \operatorname{in} p \iff p, p' \operatorname{have} \operatorname{no} \operatorname{common} \operatorname{zeros} \iff \gcd(p, p') = 1.
3 Supp T \in \mathcal{L}(V) is diag. Prove V = \text{null } T \oplus \text{range } T.
Solus: Let U = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T), where each \lambda_k \neq 0 and B_{E(\lambda_k, T)} = (v_{1,k}, \dots, v_{M_k, k}).
            Becs null T = E(0, T) \Rightarrow whether 0 is eigval or not, V = U \oplus \text{null } T. Now we show U = \text{range } T.
            By (3.B.12), range T = \{Tu : u \in U\} = \{\sum_{k=1}^{m} \lambda_k (a_{1,k}v_{1,k} + \dots + a_{M_k,k}v_{M_k,k}) : a_{j,k} \in F\} = U.
Exa: Convly not true. Define the inv T \in \mathcal{L}(\mathbb{R}^2) : (x,y) \mapsto (-y,x). No eigvals.
L1 Supp T \in \mathcal{L}(V), \alpha, \beta \in \mathbf{F} and \alpha \neq \beta. Prove \operatorname{null}(T - \alpha I) \subseteq \operatorname{range}(T - \beta I).
SOLUS: \forall v \in \text{null}(T - \alpha I), Tv = \alpha v \Rightarrow (T - \beta I)[v/(\alpha - \beta)] = v \in \text{range}(T - \beta I).
                                                                                                                                                         5 Supp \mathbf{F} = \mathbf{C}, V is finide, and T \in \mathcal{L}(V).
   Supp V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I) for all \lambda \in \mathbb{C}. Prove T is diag.
Solus: (i) dim V = 1. Immed. (ii) dim V > 1. Asum it holds for vecsps of smaller dim.
             \exists eigval \lambda_0 \Rightarrow U = \text{range}(T - \lambda_0 I) invard T \Rightarrow U = \text{null}(T|_U - \lambda I) \oplus \text{range}(T|_U - \lambda I).
            While V = E(\lambda_0, T) \oplus U \Rightarrow \dim U < \dim V. By asum, T|_U is diag wrto B_U of eigvecs.
                                                                                                                                                         Or. Supp T not diag. We show \exists \lambda \in \mathbb{C}, \text{null}(T - \lambda I) \cap \text{range}(T - \lambda I) \neq \{0\}.
   Let the min of T be p(z) = (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_m)^{\alpha_m}, where each \alpha_k \ge 1 and \exists \alpha_i > 1.
   Let q(z)(z - \lambda_i) = p(z) \Rightarrow 0 = p(T) = (T - \lambda_i)q(T) \Rightarrow \operatorname{range} q(T) \subseteq \operatorname{null}(T - \lambda_i I).
   Let q(z) = (z - \lambda_i)s(z) \Rightarrow \operatorname{range} q(T) \subseteq \operatorname{range}(T - \lambda_i I). Note that q(T) \neq 0.
                                                                                                                                                         Or. Let \lambda_1, \dots, \lambda_m be disti eigvals. Now V = \text{null}(T - \lambda_k I) \oplus \text{range}(T - \lambda_k I) for each \lambda_k.
   Asum V = \left[\bigoplus_{i=1}^{j} \text{null}(T - \lambda_j)\right] \oplus \left[\bigcap_{i=1}^{j} \text{range}(T - \lambda_j)\right] \text{ for } j \in \{1, \dots, m-1\}.
   Becs \bigcap_{i=1}^{J} \operatorname{range}(T - \lambda_i I) \supseteq \operatorname{null}(T - \lambda_{i+1} I). By (L1) and [1.C TIPS (3)],
   \bigcap_{i=1}^{J} \operatorname{range}(T - \lambda_i I) = \operatorname{null}(T - \lambda_{i+1} I) \oplus \left[\bigcap_{i=1}^{J} \operatorname{range}(T - \lambda_i I) \cap \operatorname{range}(T - \lambda_{i+1} I)\right].
   By induc, V = \lceil \text{null}(T - \lambda_1 I) \oplus \cdots \oplus \text{null}(T - \lambda_m I) \rceil \oplus \lceil \text{range}(T - \lambda_1 I) \cap \cdots \cap \text{range}(T - \lambda_m I) \rceil.
   Asum U = \bigcap_{k=1}^m \operatorname{range}(T - \lambda_k I) \neq \{0\}. Becs U invard T. Thus \exists \mu = \lambda_j eigval of T|_U. Ctradic.
                                                                                                                                                         13 Supp A, B \in \mathbb{F}^{n,n} and A is diag with dist ents on diag. Prove AB = BA \iff B is diag.
Solus: Notice that for any diag C, each C_{i,k} = 0 for j \neq k.
            Becs (I) A_{i,i}B_{i,k} = A_{i,1}B_{1,k} + \dots + [A_{i,i}B_{i,k}] + \dots + A_{i,n}B_{n,k} = (AB)_{i,k}.
            And (II) B_{i,k}A_{k,k} = B_{i,1}A_{1,k} + \dots + [B_{i,k}A_{k,k}] + \dots + B_{i,n}A_{n,k} = (BA)_{i,k}.
            Supp B diag. If j = k, then (BA)_{i,k} = (AB)_{i,k}, othws true as well.
            Supp AB = BA \Rightarrow A_{i,j}B_{j,k} = A_{k,k}B_{j,k}. Asum B_{j,k} \neq 0 with j \neq k. Then A_{i,j} = A_{k,k}, ctradic.
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14 Supp \mathbf{F} = \mathbf{C}, k \in \mathbf{N}^+, and T \in \mathcal{L}(V) is inv. Prove T^k diag \Rightarrow T diag.
Solus: Let the min of T^k be p(z) = (z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow \operatorname{each} \lambda_k non0 and disti.
            Becs any non0 \lambda \in \mathbb{C} has k disti k^{\text{th}} roots. Let \{\mu_{1,j}, \dots, \mu_{k,j}\} be the roots of z^k = \lambda_j.
            For x, y \in \{1, ..., n\}, x \neq y \iff \mu_{p,x}^k = \lambda_x \neq \lambda_y = \mu_{q,y}^k for each p, q \in \{1, ..., k\} \Rightarrow \mu_{p,x} \neq \mu_{q,y}.
            Thus all \mu's are dist. Let s(z)=(z^k-\lambda_1)\cdots(z^k-\lambda_m)=\prod_{j=1}^m\prod_{i=1}^k(z-\mu_{i,j})\Rightarrow s(T)=0.
EXA: Not true if F = R. Define T \in \mathcal{L}(R^2) : (x,y) \mapsto (-y,x). No eigvals.
• Supp \mathbf{F} = \mathbf{C}, n \in \mathbf{N}^+, p \in \mathcal{P}(\mathbf{F}). Prove T \in \mathcal{L}(V) is diag \iff null p(T) = \text{null}[p(T)]^n.
Solus: (a) Supp T diag. Let p(z) = (z - \alpha_1) \cdots (z - \alpha_m). We show each \text{null}(T - \alpha_k I)^n = \text{null}(T - \alpha_k I).
                  Done if T - \alpha_k I = S inje. Supp S not inje. Notice that \text{null } S|_{\text{range } S} = \text{null } S \cap \text{range } S = \{0\}.
                  By (3.B.22), dim null S^2 = \dim \text{null } S \Rightarrow \text{null } S^2 = \text{null } S. Asum null S^j = \text{null } S for j \ge 2.
                  Becs dim null(S^{j}S) = dim(null S^{j} \cap \text{range } S) + dim null S. By induc.
            (b) Supp \operatorname{null}(T - \lambda I) = \operatorname{null}(T - \lambda I)^n for all \lambda \in \mathbb{C}. Let \lambda_1, \dots, \lambda_m be disti eigvals of T.
                  Define p(z) = (z - \lambda_1) \cdots (z - \lambda_m). Then [p(T)]^{\dim V} = 0 \Rightarrow p(T) = 0 \Rightarrow p is the min.
16 Supp T \in \mathcal{L}(V) diag. Let \lambda_1, \ldots, \lambda_m be the disti eigens of T.
     Prove U invarspd T \Rightarrow \exists subsp\ U_k \text{ of } E(\lambda_k, T) \text{ suth } U = U_1 \oplus \cdots \oplus U_m.
Solus: Let each U_k = U \cap E(\lambda_k, T). Becs \forall u \in U, \exists ! v_k \in E(\lambda_j, T), u = v_1 + \dots + v_m.
            Now by [5.A \text{ Tips } (1)], each v_k \in U_j \Rightarrow u \in U_1 \oplus \cdots \oplus U_m.
                                                                                                                                                      18 Supp T \in \mathcal{L}(V) is diag. Prove T/U \in \mathcal{L}(V/U) is diag for any U invarspd T.
Solus: By [5.A \text{ Tips } (3)], \exists B_U = (v_1, \dots, v_m) consists of eigences of T.
            Extend to eigvecs B_V = (v_1, ..., v_m, w_1, ..., w_p) \Rightarrow B_{V/U} = (w_1 + U, ..., w_p + U).
            Becs for each w_k, \exists eigval \lambda of T, Tw_k = \lambda w_k \Rightarrow (T/U)(w_k + U) = \lambda w_k + U.
                                                                                                                                                      OR. Becs the min of T is multi of that of T/U. By [4E 5.62].
                                                                                                                                                      EXA: Define T \in \mathcal{L}(\mathbf{F}^2) : (x,y) \mapsto (y,0). Then 0 is the only eigend with E(0,T) = \operatorname{span}(e_1) = U.
        Then T|_U = 0, T/U = 0. Now T|_U, T/U diag while T not diag.
                                                                                                                                                      22 Supp V finide, T \in \mathcal{L}(V), A = \mathcal{M}(T, B_V) \in \mathbf{F}^{n,n}.
     Prove if each |A_{j,j}| > \sum_{k=1}^{n} |A_{j,k}| - A_{j,j}, then T is inv.
Solus: If T inv \Rightarrow 0 is eigval, then 0 is in G disk for some j, now \left|0 - A_{j,j}\right| \leqslant \sum_{k=1}^{n} \left|A_{j,k}\right| - A_{j,j}, ctradic
COMMENT: If each |A_{k,k}| > \sum_{j=1}^{n} |A_{j,k}| - A_{k,k}, then becs [5.67] still holds by Exe (4E 23), T is inv.
23 Redefine G disks suth the radius of the k^{th} disk is the sum of the absolute vals
     of the ents in col k, excluding the diag ent. Show [4E 5.67] still holds.
Solus: Supp T \in \mathcal{L}(V), B_V = (v_1, \dots, v_n), A = \mathcal{M}(T, B_V). Similar to [5.67]. Let B_{V'} = (\varphi_1, \dots, \varphi_n).
            Supp T'(\psi) = \lambda \psi with \psi = c_1 \varphi_1 + \dots + c_n \varphi_n \neq 0 \Rightarrow \lambda \psi = \sum_{j=1}^n \left( \sum_{k=1}^n A_{j,k}^t c_k v_j \right) = \sum_{j=1}^n c_j \lambda v_j.
            Let |c_j| = \max\{|c_1|, \dots, |c_n|\}. Now \lambda c_j = \sum_{k=1}^n A_{j,k}^t c_k \Rightarrow |\lambda - A_{j,j}^t| \leqslant \sum_{j \neq k=1}^n |A_{k,j}|.
                                                                                                                                                      Or. Becs \lambda is eigval of T \iff of T'. For A^t = \mathcal{M}(T', B_{V_t}), by [5.67],
           \lambda \in \left\{ z \in \mathbf{F} : \left| z - A_{j,j} \right| \le \sum_{j \ne k=1}^{n} \left| A_{j,k}^{t} \right| = \sum_{j \ne k=1}^{n} \left| A_{k,j} \right| \right\} \text{ for some } j \in \{1, \dots, n\}.
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5.E [4E]

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8 Find a bss of \mathcal{P}_m(\mathbf{R}^2) suth D_x, D_y up-trig in [5.72].
Solus: Let B = (1, x, y, x^2, xy, y^2, \dots, x^m, x^{m-1}y, \dots, xy^{m-1}, y^m) in \mathcal{P}_m(\mathbb{R}^2).
            Supp a liney combina of B is 0; \sum_{j=0}^{m} \sum_{k=0}^{m-j} a_{j,k} x^{j} y^{k} = 0.
            Let x = 0 \Rightarrow \text{each } a_{0,k} = 0, and y = 0 \Rightarrow \text{each } a_{k,0} = 0. Now \sum_{j=1}^{m-1} \sum_{k=1}^{m-1-j} a_{j,k} x^j y^k = 0. Take ((x_1, y_1), \dots, (x_q, y_q)) [where q = 1 + \dots + m] suth all \sum_{j=1}^{m-1} \sum_{k=1}^{m-1-j} x_s^j y_t^k a_{j,k} = 0
            form a system of q equations having uniq solus (0, ..., 0). Thus B is liney indep.
            Apply D_x to each vec in B \Rightarrow B_x = (0, 1, 0, 2x, y, 0, \dots, mx^{m-1}, (m-1)x^{m-2}y, \dots, y^{m-1}, 0).
            Apply D_y to each vec in B \Rightarrow B_y = (0, 0, 1, 0, x, 2y, \dots, \dots, 0, x^{m-1}, \dots, (m-1)xy^{m-2}, my^{m-1}).
6 Supp \mathbf{F} = \mathbf{C}, V is finide, and S, T \in \mathcal{L}(V) commu.
   Prove \exists \alpha, \lambda \in \mathbb{C} suth range(S - \alpha I) + \text{range}(T - \lambda I) \neq V.
Solus: Supp A, C \in \mathbb{F}^{n,n} are up-trig matrices of S, T wrto a B_V = (v_1, \dots, v_n) suth A, C commu.
            Let \alpha = A_{n,n}, \lambda = C_{n,n}. Then range (S - \alpha I), range (T - \lambda I) \subseteq \text{span}(v_1, \dots, v_{n-1}).
                                                                                                                                                          7 Supp \mathbf{F} = \mathbf{C}, and S, T \in \mathcal{L}(V) commu, S diag. Prove \exists B_V suth S diag and T up-trig.
Solus: Let \lambda_1, \dots, \lambda_m be disti eigends of S \Rightarrow V = E(\lambda_1, S) \oplus \dots \oplus E(\lambda_m, S).
            Becs each E_k = E(\lambda_k, S) invard T. Let each T|_{E_k} be up-trig with B_{E_k} = (v_{1,k}, \dots, v_{M_k,k}).
            Then S diag while T up-trig with the same B_V = (v_{1,1}, \dots, v_{M_n,n}).
                                                                                                                                                          OR. Using induc on n = \dim V. (i) n = 1. Immed. (ii) n > 1. Asum it holds for smaller V.
             \exists eigval \lambda of S, U = \text{null}(S - \lambda I), W = \text{range}(S - \lambda I) \Rightarrow V = \text{null}(S - \lambda I) \oplus \text{range}(S - \lambda I).
             Apply the asum to T|_{U}, S|_{U} and T|_{W}, S|_{W}, then put B_{U}, B_{W} together.
                                                                                                                                                          2 Supp \mathcal{E} \subseteq \mathcal{L}(V) and every elem of \mathcal{E} diag.
   Prove each pair of elems of \mathcal{E} commu \Rightarrow \exists B_V suth all elem of \mathcal{E} diag.
Solus: Let dim V = n \Rightarrow \dim \mathcal{L}(V) = n^2.
   \exists \{T_1, \dots, T_m\} \subseteq \mathcal{E} with each elem of \mathcal{E} in span(T_1, \dots, T_m) and m \leqslant n^2
   For each T_k, becs V = \bigoplus_{\lambda_k} E(\lambda_k, T_k) and E(\lambda_k, T_k) non0 for finily many \lambda_k \in \mathbf{F}.
   Becs U_k = E(\lambda_1, T_1) \cap \cdots \cap E(\lambda_k, T_k) = E(\lambda_k, T_k|_{U_{k-1}}) = \bigoplus_{\lambda_{k+1}} E(\lambda_{k+1}, T_{k+1}|_{U_k}) = \bigoplus_{\lambda_{k+1}} U_{k+1}.
   Hence V = \bigoplus_{\lambda_1} E(\lambda_1, T_1) = \bigoplus_{\lambda_1, \dots, \lambda_m} [E(\lambda_1, T_1) \cap \dots \cap E(\lambda_m, T_m)]. Take bss of each summand.
   Then we form B_V. For any T \in \mathcal{E}, \mathcal{M}(T, B_V) = c_1 \mathcal{M}(T_1, B_V) + \cdots + c_m \mathcal{M}(T_m, B_V).
                                                                                                                                                          9 Supp \mathbf{F} = \mathbf{C}, V finide and non0. Supp \mathcal{E} \subseteq \mathcal{L}(V) is suth all S, T \in \mathcal{E} commu.
   (a) Prove \exists eigrec v \in V of all elem of \mathcal{E}. (b) \exists B_V suth all elem of \mathcal{E} has up-trig matrix.
Solus: Simil to Exe (2). \exists \{T_1, \dots, T_m\} \subseteq \mathcal{E}. Let U_0 = V, U_k = E(\lambda_1, T_1) \cap \dots \cap E(\lambda_k, T_k).
            (a) Let \lambda_1, \dots, \lambda_m be eigvals of T_1, \dots, T_m respectly with each \lambda_k eigval of T_k|_{U_k} \Rightarrow U_k \neq 0
                  Now for non0 v \in U_m, \forall T = c_1 T_1 + \dots + c_m T_m \in \mathcal{E}, Tv = (c_1 \lambda_1 + \dots + c_m \lambda_m)v.
             (b) Using induc on dim V. (i) Immed. (ii) dim V > 1. Asum it holds for smaller V.
                  Let v_1 be a common eigvec of all T_k. Let W \oplus \text{span}(v_1) = V, P : av_1 + w \mapsto w.
                  Simlr in [4E 5.80], each pair of \{\hat{T}_1, \dots, \hat{T}_m\} commu. By asum, \exists B_W \Rightarrow \exists B_V.
                  Now each \mathcal{M}(T_k, B_V) up-trig \Rightarrow \forall T \in \mathcal{E}, \mathcal{M}(T) = c_1 \mathcal{M}(T_1) + \cdots + c_m \mathcal{M}(T_m), wrto B_V. \square
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8 Note: V denotes a finide non0 vecsp over \mathbf{F}. An Exe marked by \blacksquare is true if infinide or partially finide.
A.3 Supp T \in \mathcal{L}(V) inv. Prove G(\lambda, T) = G(\lambda^{-1}, T^{-1}) for any non0 \lambda \in \mathbf{F}.
Solus: (T - \lambda I)^j v = 0 = \sum_{i=0}^j C_i^i (-\lambda)^{j-i} T^i v. Apply (-\lambda)^{-j} T^{-j} to both sides. (T^{-1} - \lambda^{-1})^j v = 0.
                                                                                                                                                         OR. We use induc on j to show each \operatorname{null}(T - \lambda I)^j = \operatorname{null}(T^{-1} - \lambda^{-1})^j. (i) Immed. (ii) j > 1.
            Asum true for (j-1). \forall v \in \text{null}(T-\lambda I)^j, (T-\lambda I)v \in \text{null}(T-\lambda I)^{j-1} = \text{null}(T^{-1}-\lambda^{-1}I)^{j-1}.
            Thus 0 = (T^{-1} - \lambda^{-1}I)^{j-1}(T - \lambda I)v = (T - \lambda I)(T^{-1} - \lambda^{-1}I)^{j-1}v. By (i) and rev the roles.
A.5 Supp T \in \mathcal{L}(V), T^{n-1}v \neq 0, T^nv = 0. Prove (v, Tv, ..., T^{n-1}v) is liney indep.
SOLUS: a_0v + a_1Tv + \dots + a_{n-1}T^{n-1}v = 0 \Rightarrow a_0T^{n-1}v = 0 \Rightarrow a_0 = 0. Similar for a_1, \dots, a_{n-1}.
• Note For [8.19] Or [4E 8.18]: If the min of T is z^m. Then \exists v suth T^{m-1}v \neq 0. If m = \dim V.
  Now B_V = (T^{m-1}v, \dots, Tv, v). Let each w_k = T^{m-k}v. Then Tw_1 = 0 and each T(w_k) = w_{k-1}.
A.6 Supp T \in \mathcal{L}(V) nilp, n = \dim V, T^{n-1} \neq 0. Prove \nexists S \in \mathcal{L}(V), S^k = T for all k > 1.
Solus: Asum \exists suth S. Then \text{null } S^{kn} = \text{null } T^n = V = \text{null } S^{kn-1} = \cdots = \text{null } S^n.
            Note that \exists j suth null S^{kn-j} = \text{null } T^m for some m \in \{1, ..., n-1\}.
                                                                                                                                                         • (4E A.4) Supp T \in \mathcal{L}(V), \lambda \in \mathbf{F}, the min of T is a multi of (z - \lambda)^m for m \in \mathbf{N}^+.
               Prove dim null(T - \lambda I)^m \ge m.
                                                                                                                  Coro: dim G(\lambda, T) \ge m.
Solus: Becs \lambda is eigval of T. We show z^m is the min of (T - \lambda I)|_{\text{null}(T - \lambda I)^m}.
            Using induc on m. (i) m = 1. Becs dim E(\lambda, T) \ge 1. (ii) m > 1. Asum it holds for (m - 1).
            \dim \operatorname{null}(T - \lambda I)(T - \lambda I)^{m-1} = \dim \operatorname{null}(T - \lambda I)|_{\operatorname{null}(T - \lambda I)^{m-1}} + \underline{\dim \operatorname{null}(T - \lambda I)^{m-1}}.
                                                                                                                                                         Or. Let p(z) = (z - \lambda)^m q(z) the min of T.
            We show each inclusion of \{0\} \subseteq \operatorname{null}(T - \lambda I) \subseteq \cdots \subseteq \operatorname{null}(T - \lambda)^m is strict by ctradic.
            Asum \operatorname{null}(T - \lambda I)^k = \operatorname{null}(T - \lambda I)^{k+1} for k \in \{1, ..., m-1\}.
            Then \operatorname{null}(T - \lambda I)^k = \operatorname{null}(T - \lambda I)^m \Rightarrow (T - \lambda I)^m q(T)v = 0 = (T - \lambda I)^k q(T)v.
                                                                                                                                                         • (4E A.3) Supp T \in \mathcal{L}(V). Prove V = \text{null } T \oplus \text{range } T \iff \text{null } T^2 = \text{null } T.
Solus: (a) \operatorname{null} T^2 = \operatorname{null} T = \operatorname{null} T^{\dim V} \Rightarrow \dim \operatorname{range} T^{\dim V} = \dim \operatorname{range} T.
             (b) V = \text{null } T \oplus U, U = \text{range } T, \mathbb{X} \text{ dim null } T^2 = \text{dim null } T + \text{dim null } T|_{\text{range } T}.
                                                                                                                                                         OR. (a) Supp null T^2 = \text{null } T. Then Tu \in \text{null } T \cap \text{range } T \iff T^2u = 0 \iff Tu = 0.
                   (b) Supp null T \cap \text{range } T = \{0\}. Then T^2u = 0 \iff Tu \in \text{null } T \iff Tu = 0.
A.17 Supp T \in \mathcal{L}(V), range T^m = \operatorname{range} T^{m+1}. Show range T^m = \operatorname{range} T^{m+1} = \cdots.
Solus: By Exe (A.19), \operatorname{null} T^m = \operatorname{null} T^{m+1} = \cdots \Rightarrow \operatorname{dim} \operatorname{range} T^m = \operatorname{dim} \operatorname{range} T^{m+1} = \cdots.
                                                                                                                                                         OR. Supp w = T^{m+k}v. Then becs T^mv \in \operatorname{range} T^{m+1}, \exists T^{m+1}u = T^mv. Thus w = T^{m+k+1}u.
A.18 Supp T \in \mathcal{L}(V), dim V = n. Show range T^n = \text{range } T^{n+1} = \cdots.
Solus: By Exe (A.19), becs \operatorname{null} T^{\dim V} = \operatorname{null} T^{\dim V+1} = \cdots. Simlr.
                                                                                                                                                         OR. Asum range T^n \supseteq \operatorname{range} T^{n+1}. By Exe (A.17), V = \operatorname{range} T^0 \supseteq \operatorname{range} T \supseteq \cdots \supseteq \operatorname{range} T^{n+1}.
            Now each dim range T^{k+1} \leq \dim \operatorname{range} T^k - 1 \Rightarrow \dim \operatorname{range} T^{n+1} \leq \dim \operatorname{range} T^0 - (n+1). \square
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A.10 Supp T \in \mathcal{L}(V) not nilp, n = \dim V. Show V = \operatorname{null} T^{n-1} \oplus \operatorname{range} T^{n-1}.
Solus: Notice that \operatorname{null} T^{n-1} \neq \operatorname{null} T^n \Rightarrow \operatorname{dim} \operatorname{null} T^n = n \iff T^n = 0. Thus \operatorname{null} T^{n-1} = \operatorname{null} T^n.
            \not \subseteq V = \text{null } T^n \oplus \text{range } T^n, \text{range } T^n \subseteq \text{range } T^{n-1} \Rightarrow V = \text{null } T^{n-1} + \text{range } T^{n-1}.
            OR. Then dim range T^{n-1} = \dim \operatorname{range} T^n \Rightarrow \operatorname{range} T^{n-1} = \operatorname{range} T^n.
                                                                                                                                                     Or. By Exe (4E A.3), \operatorname{null} T^{2(n-1)} = \operatorname{null} T^{n-1} \iff V = \operatorname{null} T^{n-1} \oplus \operatorname{range} T^{n-1}.
                                                                                                                                                     • (4E A.18) Supp T \in \mathcal{L}(V) nilp. Prove T^{1+\dim \operatorname{range} T} = 0.
\textbf{Solus:} \ \ \text{Let} \ U \oplus \text{null} \ T = V. \ \text{Then} \ \text{range} \ T^m|_U = \text{range} \ (T|_U)^m = \text{range} \ T^m. \ \text{While} \ U = \dim \text{range} \ T.
                                                                                                                                                     OR. Let dim range T = k. Asum T^{k+1} \neq 0. Let m be suth T^m = 0 \neq T^{m-1}. Then k + 2 \leq m.
            Let v be suth T^{m-1}v \neq 0 = T^mv \Rightarrow (v, Tv, ..., T^{m-1}v) liney indep \Rightarrow k \geqslant m-1 \geqslant k+1.
                                                                                                                                                     • (4E A.12) Supp T \in \mathcal{L}(V) and all v \in V is a gigvec of T. Prove V = G(\lambda, T).
Solus: Becs for any liney indep (v, w), (v, w, v + w) of gigves is liney dep; say corres \alpha, \beta, \gamma repectly.
            If \alpha = \beta then done. If \alpha = \gamma, v, v + w \in G(\alpha, T) \Rightarrow w \in G(\alpha, T). If \beta = \gamma, then simlr.
            Thus \alpha = \beta = \gamma. Any two liney indep v, w corres one eigval.
B.5 [4E A.15] Supp \mathbf{F} = \mathbf{C}, T \in \mathcal{L}(V). Prove non0 T diag \Rightarrow each G(\lambda, T) = E(\lambda, T).
Solus: Supp V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T); \lambda_1 = 0 if possible, in this case m > 1.
            Supp w \in G(\lambda_i, T). Then w = v_1 + \dots + v_m, where each v_i \in E(\lambda_i, T).
            Becs (T - \lambda_i I)^k w = 0 = \sum_{i=1}^m \lambda_i (\lambda_i - \lambda_i)^k v_i \Rightarrow w = v_i \in E(\lambda_i, T), othws ctradic.
                                                                                                                                                     Or. Supp G(\lambda_i, T) \supseteq E(\lambda_i, T). Let w \in G(\lambda_i, T) \setminus E(\lambda_i, T)
            Then Iw \neq 0 \neq (T - \lambda_i I)w. Let (T - \lambda_i I)^k w = 0 \neq (T - \lambda_i I)^{k-1}w.
            By [5.B(I) \text{ TIPS } (1)], the min of T is a multi of (z - \lambda_i)^k. \mathbb{X} k \ge 2.
                                                                                                                                                     • (4E A.16) Supp S, T \in \mathcal{L}(V) nilp and commu. Prove S + T, ST are nilp
Solus: By [4E 5.80], \exists B_V suth S, T up-trig (with only 0's on diags). By (4E 5.C.2).
                                                                                                                                                     Or. Let S^p = T^q = 0. Becs S, T commu, (ST)^{\max\{p,q\}} = 0 = (S+T)^{p+q} = \sum_{i=0}^{p+q} C_{p+q}^i S^i T^{p+q-i}.
B.10 Supp \mathbf{F} = \mathbf{C}, T \in \mathcal{L}(V). Prove \exists commu D, N \in \mathcal{L}(V), T = D + N, D diag, N nilp.
Solus: Note: D \text{ diag}, N \text{ nilp} \not\Rightarrow D, N \text{ commu. Exa: } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
            We use induc on dim V = n. (i) Immed. (ii) n > 1. Asum it holds for smaller V.
            Becs V = G_1 \oplus U, where U = G_2 \oplus \cdots \oplus G_m, and each G_k = G(\lambda_k, T).
            \exists B_{G_1} \text{ suth } T|_{G_1} = (T - \lambda_1 I)|_{G_1} + \lambda_1 I|_{G_1} = N_1 + D_1 \text{ up-trig and } N_1, D_1 \text{ commu.}
            \exists commu D_2, N_2 \in \mathcal{L}(U), T|_U = D_2 + N_2, D_2 diag, N_2 nilp; wrto some B_U, by (4E 5.E.7).
            Let B_V = B_{G_1} \cup B_U. Define P_1, P_2 \in \mathcal{L}(V) by P_1(v_1 + u) = v_1, P_2(v_1 + u) = u.
            Let D = D_1 P_1 + D_2 P_2, N = N_1 P_1 + N_2 P_2. Becs P_i D_k = \delta_{i,k} D_i, P_i N_k = \delta_{i,k} N_i.
            Thus D + N = (D_1 + N_1)P_1 + (D_2 + N_2)P_2 = T, and DN = D_1N_1P_1 + D_2N_2P_2 = NP.
                                                                                                                                                     Or. Let V = G_1 \oplus \cdots G_m \Rightarrow \forall v \in V, \exists ! v_k \in G_k, v = v_1 + \cdots + v_m.
            Define D \in \mathcal{L}(V) : v \mapsto (\lambda_1 v_1 + \dots + \lambda_m v_m) \Rightarrow D|_{G_k} = \lambda_k I.
            Let N = T - D \Rightarrow N|_{G_k} = (T - D)|_{G_k} = (T - \lambda_k I)|_{G_k} is nilp.
            Then N^M v = N^M v_1 + \cdots + N^M v_m = 0, where M = \max\{d_1, \dots, d_m\}. Now N is nilp.
            Becs DN = DT - D^2, ND = TD - D^2, \mathcal{Z} each TDv_k = \lambda_k Tv_k = DTv_k \Rightarrow TD = DT.
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• (4E B.7) Supp T \in \mathcal{L}(V) and \lambda is an eigral with multy d. Prove G(\lambda, T) = \text{null}(T - \lambda I)^d.
Solus: Let N = T - \lambda I, and \text{null } N \subseteq \cdots \subseteq \text{null } N^m = \text{null } N^{m+1}. Choose B_{\text{null } N}.
           Extend to B_{\text{null }N^2} \Rightarrow \cdots \Rightarrow B_{\text{null }N^m}, with each time adding at least one bss vec. Thus m \leqslant d.
           Or. Let the min of T be p(z) = (z - \lambda)^m q(z) with q(\lambda) \neq 0.
            By (4E B.6), G(\lambda, T) = \text{null}(T - \lambda I)^m. By (4E A.4), d \ge m.
                                                                                                                                              Or. Let the min of N = (T - \lambda I)^m |_{G(\lambda, T)} be z^m.
           By (4E 5.B.17), the min of N + \lambda I = T|_{G(\lambda, T)} is s(z) = (z - \lambda)^m.
           Becs the char of T [See [9.21] for F = R] is a multi of min of T, which is a multi of s.
                                                                                                                                              • (4E B.6) Supp T \in \mathcal{L}(V) and \lambda is an eigval. Explain why
             the exponent of z - \lambda in the factoriz of the min of T
             is the smallest m \in \mathbb{N}^+ suth (T - \lambda I)^m|_{G(\lambda, T)} = 0.
Solus: Let G = G(\lambda, T), N = (T - \lambda I)|_{G}, and N^m = 0 \neq N^{m-1} \Rightarrow the min of T|_{G} is s(z) = (z - \lambda)^m.
            Thus the min p of T is a multi of s. Now we show the deg of (z - \lambda) in p is no more than m.
           Let \lambda_1 = \lambda and V_C = G_C \oplus U, where G = G(\lambda_1, T), U = G(\lambda_2, T_C) \oplus \cdots \oplus G(\lambda_n, T_C).
           Asum the min p(z) = (z - \lambda_1)^{m+k} (z - \lambda_2)^{\alpha_2} \cdots (z - \lambda_n)^{\alpha_n}, where k \in \mathbb{N}^+.
           Let r(z) = (z - \lambda)^m (z - \lambda_2)^{\alpha_2} \cdots (z - \lambda_n)^{\alpha_n} \Rightarrow r(T) = 0. Ctradic the min of p.
                                                                                                                                              OR. Let the min of T be p(z) = (z - \lambda)^m q(z), with q(\lambda) \neq 0.
           We show \operatorname{null}(T - \lambda I)^m \supseteq \operatorname{null}(T - \lambda I)^{m+1}.
           Supp v \in \text{null}(T - \lambda I)^{m+1} \iff (T - \lambda I)^m v \in \text{null}(T - \lambda I) = E(\lambda, T).
           Then 0 = p(T)v = q(T) \lceil (T - \lambda I)^m v \rceil = q(\lambda) \lceil (T - \lambda I)^m v \rceil \Rightarrow v \in \text{null}(T - \lambda I)^m.
           We show m is the smallest. Let k be suth null(T - \lambda I)^k = G(\lambda, T).
           Let s(z) = (z - \lambda)^k q(z). We show s(T) = 0 and done. Consider 0 = p(T)v = (T - \lambda I)^m q(T)v.
           If q(T)v = 0 \Rightarrow s(T)v = 0. Othws, q(T)v \in \text{null}(T - \lambda I)^m = \text{null}(T - \lambda I)^k \Rightarrow s(T)v = 0.
B.9 Supp A, C are block diag matrices, and A_k, C_k are of the same size n_k for k \in \{1, ..., m\}.
       Show AC is block diag and the k^{th} block on the diag of AC is A_kC_k.
Solus: Let A = \mathcal{M}(S), C = \mathcal{M}(T), AC = \mathcal{M}(ST) \in \mathbb{F}^{n,n}, where n = n_1 + \dots + n_m.
           Let B_1 = (e_1, \dots, e_{n_1}), and B_k = (e_{n_1 + \dots + n_{k-1} + 1}, \dots, e_{n_1 + \dots + n_k}) for k \in \{2, \dots, m\}.
           Let each U_k = \operatorname{span} B_k invard S, T. Becs \mathcal{M}(S|_{U_k}, B_k) = A_k, \mathcal{M}(T|_{U_k}, B_k) = C_k.
           Now \mathcal{M}[(ST)|_{U_k}] = \mathcal{M}(S|_{U_k}T|_{U_k}) = A_kC_k.
• Supp T \in \mathcal{L}(V), \lambda_1, ..., \lambda_m are the disti eigvals corres v_1, ..., v_m, and U invard T.
• B.Tips 1: Supp v_1 + \cdots + v_m \in U. Prove each v_k \in U.
Solus: Let each S_k = (T - \lambda_k I)^{m_k}, where (T - \lambda_k I)^{m_k} v_k \neq 0 = (T - \lambda_k I)^{m_k + 1} v_k.
           Let each R_k = (T - \lambda_1 I)^n \cdots (T - \lambda_{k-1})^n (T - \lambda_{k+1} I)^n \cdots (T - \lambda_m I)^n, where n = \dim V.
           Then each R_k S_k (v_1 + \dots + v_m) = R_k w_k, where w_k = S_k v_k \in E(\lambda_k, T).
           Becs U invard T \Rightarrow invard R_k S_k = p(T). Thus w_k \in U \Rightarrow v_k \in U.
                                                                                                                                              • B.Tips 2: Supp \mathbf{F} = \mathbf{C}. Prove \exists subsp U_k of G(\lambda_k, T) suth U = U_1 \oplus \cdots \oplus U_m.
Solus: Let each U_k = U \cap G(\lambda_k, T). Becs \forall u \in U, \exists ! v_k \in G(\lambda_k, T), u = v_1 + \dots + v_m.
           Now by [B Tips (1)], each v_k \in U_k \Rightarrow u \in U_1 \oplus \cdots \oplus U_m.
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C.20 [4E B.20] Supp $\mathbf{F} = \mathbf{C}$, and each V_k non0 invarspd $T \in \mathcal{L}(V)$ of $V = V_1 \oplus \cdots \oplus V_m$.				
Let p_k be the char of $T _{V_k}$. Prove the char of T is $p_1 \cdots p_m$.				
Solus:				
D.7 [4E B.21] Supp monic $p, q \in \mathcal{P}(\mathbf{C})$ have the same zeros, and q is a multi of p .				

Solus:

ENDED

9.A	Note:	<i>V</i> denotes a finide non0 vecsp over F .
/ U =	T TO I L.	v deficies a finale fiction veesp over 1.

- Note For [9.12]: Another proof: $\overline{T_{\rm C}(u+{\rm i}v)}=\overline{Tu+{\rm i}Tv}=Tu-{\rm i}Tv=T_{\rm C}(u-{\rm i}v)=T_{\rm C}(\overline{u+{\rm i}v}).$ $\overline{(T_{\rm C}-\lambda I)(u+{\rm i}v)}=\overline{T_{\rm C}(u+{\rm i}v)-\lambda(u+{\rm i}v)}=T_{\rm C}(u-{\rm i}v)-\overline{\lambda}(u-{\rm i}v)=(T_{\rm C}-\overline{\lambda}I)(u-{\rm i}v).$ We use induc on m to show $\overline{(T_{\rm C}-\lambda I)^m(u+{\rm i}v)}=(T_{\rm C}-\overline{\lambda}I)^m(u-{\rm i}v).$ (i) Immed. (ii) m>1. Asum it holds for $k\leqslant m$. Let $(T_{\rm C}-\lambda I)^{m-1}(u+{\rm i}v)=x+{\rm i}y.$ Becs $\overline{(T_{\rm C}-\lambda I)^{m-1}(u+{\rm i}v)}=x-{\rm i}y.$
- Note For [9.17]: Detailed proof:

Let $B = (u_1 + iv_1, \dots, u_m + iv_m)$ be a bss of $G(\lambda, T_C)$. By [9.12], $\overline{B} = (u_1 - iv_1, \dots, u_m - iv_m)$ in $G(\overline{\lambda}, T_C)$.

- (a) If $a_1(u_1 iv_1) + \cdots + a_m(u_m iv_m) = 0$. Conjugating, now each $\overline{a_k} = 0$. Liney indep.
- (b) $\forall u iv \in G(\overline{\lambda}, T_{\mathbf{C}}), u + iv \in G(\lambda, T_{\mathbf{C}}) \Rightarrow u + iv \in \operatorname{span} B \Rightarrow u iv \in \operatorname{span} \overline{B}.$
- **13** Supp $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$, and $b^2 < 4c$ for $b, c \in \mathbf{F}$. Prove dim null $(T^2 + bT + cI)^j$ is even for each $j \in \mathbf{N}^+$.
- Solus: Let $z^2 + bz + c = (z \lambda)(z \overline{\lambda})$. Supp $(T_C \lambda I)^j (T_C \overline{\lambda} I)^j v = 0$ Note that $v = u + w \in G(\lambda, T_C) \oplus G(\overline{\lambda}, T_C) \Rightarrow u \in \text{null}(T_C - \lambda I)^j, w \in \text{null}(T_C - \overline{\lambda} I)^j$. Thus $\text{null}(T_C^2 + bT_C + cI)^j = \text{null}(T_C - \lambda I)^j \oplus \text{null}(T_C - \overline{\lambda} I)^j$. By [9.4] and [9.12].
- **17** Supp $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$ suth $T^2 = -I$. Define complex scalar multi on V as $(a + b\mathbf{i})v = av + bTv$. Then V itself is already a complex vecsp with these defs. Show the dim of V as a complex vecsp is half of the dim of V as the usual real vecsp.
- **Solus**: Supp $V \neq \{0\}$. Let $N = \dim V$ as real vecsp. We construct a real B_V via a (N-1)-step process. Let (v_1, Tv_1) be liney indep in V as real vecsp. Let $v_2 \notin \operatorname{span}(v_1, Tv_1) \Rightarrow (v_1, Tv_1, v_2)$ liney indep.
 - Step 1. We show (v_1, Tv_1, v_2, Tv_2) liney indep in V as real vecsp. Asum $Tv_2 = a_1v_1 + b_1Tv_1 + a_2v_2$. Then $-v_2 = a_1Tv_1 b_1v_1 + a_2Tv_2$. Note that $a_2 \neq 0$ and $a_2^2 = -1$ while $a_2 \in \mathbb{R}$, ctradic.
 - Step k. $[k \le N-1]$ We show $(v_1, Tv_1, \dots, v_k, Tv_k, v_{k+1}, Tv_{k+1})$ liney indep in V as real vecsp. Simlr. Asum $Tv_{k+1} = a_1v_1 + b_1Tv_1 + \dots + a_{k+1}v_{k+1}$. Then $-v_{k+1} = a_1Tv_1 b_1v_1 + \dots + a_{k+1}Tv_{k+1}$. \square
- **18** Supp $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and all eigends of $T_{\mathbf{C}}$ are real. Show (a) $\exists B_V$ suth T up-trig; (b) $\exists B_V$ of gigvecs of T.
- **Solus**: (a) By [9.10] and [4E 5.44], immed. Or. Using induc on dim V. (i) Immed. (ii) dim V>1. Asum it holds for smaller V. Supp all eigvals of $T_{\rm C}$ are real. Let $U={\rm range}(T-\lambda I)$. Then all eigvals of $T_{\rm C}|_{U_{\rm C}}=(T|_U)_{\rm C}$ are real. By asum, simil to [5.27].
 - (b) By (a), (4E 8.A.11) and [4E 5.44], immed. Or. $V_{\rm C} = G(\lambda_1, T_{\rm C}) \oplus \cdots \oplus G(\lambda_m, T_{\rm C})$. Becs each $G(\lambda_k, T_{\rm C}) = G(\lambda_k, T)_{\rm C}$ and $U_{\rm C} + W_{\rm C} = (U + W)_{\rm C}$. By [9.4](b).

ENDED

6.A	
6.B	Ended
	Ended
6.C	Ended
7.A	T
7. B	Ended
	Ended