

# 1.B

- (OR [9.2,9.3]. OR Problem (1) in 9.A)

Suppose  $V$  is a real vector space. The complexification of  $V$ , denoted by  $V_{\mathbb{C}}$ , equals  $V \times V$ . An element of  $V_{\mathbb{C}}$  is an ordered pair  $(u, v)$ , where  $u, v \in V$ , but we write this as  $u + iv$ .

- Addition on  $V_{\mathbb{C}}$  is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for all  $u_1, v_1, u_2, v_2 \in V$ .

- Complex scalar multiplication on  $V_{\mathbb{C}}$  is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all  $a, b \in \mathbf{R}$  and all  $u, v \in V$ .

Prove that with the definitions above,  $V_{\mathbb{C}}$  is a complex vector space.

Think of  $V$  as a subset of  $V_{\mathbb{C}}$  by identifying  $u \in V$  with  $u + i0$ . The construction of  $V_{\mathbb{C}}$  from  $V$  can then be thought of as generalizing the construction of  $\mathbf{C}^n$  from  $\mathbf{R}^n$ .

**SOLUTION:**

- Commutativity:  $(u_1 + iv_1) + (u_2 + iv_2) = (u_2 + iv_2) + (u_1 + iv_1)$ .

- Associativity:

$$(I) [(u_1 + iv_1) + (u_2 + iv_2)] + (u_3 + iv_3) = (u_1 + iv_1) + [(u_2 + iv_2) + (u_3 + iv_3)].$$

$$(II) \begin{cases} [(a + bi)(c + di)](u + iv) = [(ac - bd) + (ad + bc)i](u + iv) = [(ac - bd)u - (ad + bc)v] + i[(ac - bd)v + (ad + bc)u] \\ (a + bi)[(c + di)(u + iv)] = (a + bi)[(cu - dv) + i(cv + du)] = [a(cu - dv) - b(cv + du)] + i[a(cv + du) + b(cu - dv)] \end{cases}$$

- Additive identity.

- Additive inverse:  $(u_1 + iv_1) + (-u_1 + i(-v_1)) = 0$ .

- Multiplication identity.

- Distributive properties:

$$(I) \begin{cases} (a + bi)[(u_1 + iv_1) + (u_2 + iv_2)] = (a + bi)[(u_1 + u_2) + i(v_1 + v_2)] \\ \quad \quad \quad = [a(u_1 + u_2) - b(v_1 + v_2)] + i[a(v_1 + v_2) + b(u_1 + u_2)] \\ (a + bi)(u_1 + iv_1) + (a + bi)(u_2 + iv_2) = [(au_1 - bv_1) + i(av_1 + bu_1)] + [(au_2 - bv_2) + i(av_2 + bu_2)] \end{cases}$$

$$(II) \begin{cases} [(a + bi) + (c + di)](u + iv) = [(a + c) + (b + d)i](u + iv) = [(a + c)u - (b + d)v] + i[(a + c)v + (b + d)u] \\ (a + bi)(u + iv) + (c + di)(u + iv) = [(au - bv) + i(av + bu)] + [(cu - dv) + i(cv + du)] \end{cases}$$

□

- Suppose  $S$  is a nonempty set. Let  $V^S$  denote the set of functions from  $S$  to  $V$ .

Define a natural addition and scalar multiplication on  $V^S$ ,

and show that  $V^S$  is a vector space with these definitions.

**SOLUTION:**

- Addition on  $V^S$  is defined by  $(f + g)(x) = f(x) + g(x)$  for any  $x \in S$  and  $f, g \in V^S$ .

- Scalar Multiplication on  $V^S$  is defined by  $(\lambda f)(x) = \lambda f(x)$  for any  $x \in S, \lambda \in \mathbf{F}, f \in V^S$ .

Commutativity. Associativity.

Additive identity:  $0(x) = 0$ .

Additive inverse:  $f(x) + (-f)(x) = 0$ .

Multiplication identity:  $I(x) = x$ .

Distributive properties:  $(\lambda(f + g))(x) = \lambda(f(x) + g(x)) = (\lambda f)(x) + (\lambda g)(x);$

$$((\lambda + \mu)f)(x) = (\lambda + \mu)f(x) = \lambda f(x) + \mu f(x).$$

□

**1 Prove that  $-(-v) = v$  for every  $v \in V$ .**

**SOLUTION:**  $\left. \begin{array}{l} (-(-v)) + (-v) = 0 \\ v + (-v) = 0 \end{array} \right\} \Rightarrow \text{By the uniqueness of additive inverse. } \square$

OR.  $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v \quad \square$

**2** Suppose  $a \in \mathbf{F}$ ,  $v \in V$ , and  $av = 0$ . Prove that  $a = 0$  or  $v = 0$ .

**SOLUTION:** If  $a = 0$ , then we are done.

Otherwise,  $\exists a^{-1} \in \mathbf{F}$ ,  $a^{-1}a = 1$ , hence  $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$ .  $\square$

**3** Suppose  $v, w \in V$ . Explain why there exists a unique  $x \in V$  such that  $v + 3x = w$ .

**SOLUTION:**

[Existence] Let  $x = \frac{1}{3}(w - v)$ .

[Uniqueness] Suppose  $v + 3x_1 = w$ , (I)  $v + 3x_2 = w$  (II).

Then (I) - (II) :  $3(x_1 - x_2) = 0 \Rightarrow$  By Problem (2),  $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$ .  $\square$

**5** Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that  $0v = 0$  for all  $v \in V$ . Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of  $V$ .

The phrase a “condition can be replaced” in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.

**SOLUTION:** Using [1.31].  $0v = 0$  for all  $v \in V \Leftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$ .  $\square$

**6** Let  $\infty$  and  $-\infty$  denote two distinct objects, neither of which is in  $\mathbf{R}$ .

Define an addition and scalar multiplication on  $\mathbf{R} \cup \{\infty, -\infty\}$  as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for  $t \in \mathbf{R}$  define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

and (I)  $t + \infty = \infty + t = \infty + \infty = \infty$ ,

(II)  $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$ ,

(III)  $\infty + (-\infty) = (-\infty) + \infty = 0$ .

With these operations of addition and scalar multiplication, is  $\mathbf{R} \cup \{\infty, -\infty\}$  a vector space over  $\mathbf{R}$ ? Explain.

**SOLUTION:** Not a vector space. By Associativity:  $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$ .

OR By Distributive properties:  $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$ .  $\square$

**ENDED**

## 1.C

**2** (1.35)

(b) The set of continuous real-valued functions on the interval  $[0, 1]$  is a subspace of  $\mathbf{R}^{[0,1]}$

Denote the set by  $U$ .  $\forall x \in [0, 1]$  we have  $\left. \begin{array}{l} \text{(一)} 0 \in U; f(x) = 0 \Leftrightarrow f = 0 \\ \text{(二)} \forall f, g \in U, (f + g)(x) = f(x) + g(x) \\ \text{(三)} \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R}, (\lambda f)(x) = \lambda f(x) \end{array} \right\} \Rightarrow \square$

(c) The set of differentiable real-valued functions on  $\mathbf{R}$  is a subspace of  $\mathbb{R}^{\mathbb{R}}$

Denote the set by  $U$ .  $\left. \begin{array}{l} \text{(一)} 0 \in U \\ \text{(二)} \forall f, g \in U, (f' + g') = f' + g' \\ \text{(三)} \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R}, (\lambda f)' = \lambda(f)'\end{array} \right\} \Rightarrow \square$

(d) The set of differentiable real-valued functions  $f$  on the interval  $(0, 3)$  such that  $f'(2) = b$  is a subspace of  $\mathbf{R}^{(0,3)}$  if and only if  $b = 0$ .

Denote the set by  $U$ . Suppose  $b = 0$ . Then

$(\neg) 0 \in U$   
 $(\neg) \forall f, g \in U, (f + g)'(2) = f'(2) + g'(2) = 0$   
 $(\neg) \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R}, (\lambda f)'(2) = \lambda f'(2)$

Suppose  $U$  is a subspace of  $\mathbf{R}^{(0,3)}$ . Suppose  $f = 0 \Rightarrow f \in U$ . Then  $f'(2) = 0 = b$ .  $\square$

(e) *The set of all sequences with limit 0, of complex numbers, is a subspace of  $\mathbf{C}^\infty$ .*

Denote the set by  $A$ .

$(\neg) (0, 0, \dots) \in A$   
 $(\neg) \forall (a_1, a_2, \dots), (b_1, b_2, \dots) \in A \iff \lim_{n \rightarrow \infty} a_n = 0, \lim_{n \rightarrow \infty} b_n = 0$   
 Thus  $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = 0 \Rightarrow (a_1 + b_1, a_2 + b_2, \dots) \in A$   
 $(\neg) \forall (a_1, a_2, \dots) \in A, \forall \lambda \in \mathbf{F} = \mathbf{C}, \lim_{n \rightarrow \infty} (\lambda a_n) = \lambda \lim_{n \rightarrow \infty} a_n = 0 \Rightarrow \lambda(a_1, a_2, \dots) \in A$

$\Rightarrow \square$

**4** Suppose  $b \in \mathbf{R}$ . Show that the set of continuous real-valued functions  $f$  on the interval  $[0, 1]$  such that  $\int_0^1 f = b$  is a subspace of  $\mathbf{R}^{[0,1]}$  if and only if  $b = 0$

**SOLUTION:** Denote the set by  $V_b$ .

(a) Suppose  $V_b$  is a subspace of  $\mathbf{R}^{[0,1]}$ , then  $\forall f \in V_b$ , we have  $\int_0^1 f = b$ .

Because  $kf \in V_b$  for any  $k \in \mathbf{R}$ . Hence  $\int_0^1 (kf) = k \int_0^1 f = kb = b \Leftrightarrow b = 0$ .

OR. Because  $g = 0 \in V_b \Rightarrow \int_0^1 g = 0 = b$ .

(b) Suppose  $b = 0$ .  $\forall f, g \in V_b = V_0, \lambda \in \mathbf{R}, \int_0^1 (f + \lambda g) = \int_0^1 f + \int_0^1 g = 0$ .  $\square$

**5** Is  $\mathbf{R}^2$  a subspace of the complex vector space  $\mathbf{C}^2$ ?

**ANSWER:** No. Because  $\mathbf{R}^2$  is not closed on scalar multiplication on  $\mathbf{C}$ .

**6** (a) Is  $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$  a subspace of  $\mathbf{R}^3$ ?

**ANSWER:** True. As can be easily checked (note that  $a^3 = b^3 \Leftrightarrow a = b$ ).

(b) Is  $\{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$  a subspace of  $\mathbf{C}^3$ ?

**ANSWER:** No. Note that  $(\frac{-1 \pm \sqrt{3}i}{2})^3 = 1$ .

**7** Prove or give a counterexample: If  $U$  is a nonempty subset of  $\mathbf{R}^2$  such that  $U$  is closed under addition and under taking additive inverses (meaning  $-u \in U$  whenever  $u \in U$ ), then  $U$  is a subspace of  $\mathbf{R}^2$ .

**SOLUTION:** Let  $U = \mathbf{Z}^2, (\mathbf{Z}^*)^2, (\mathbf{Q}^*)^2, \mathbf{Q}^2 \setminus \{0\}$ , or  $\mathbf{R}^2 \setminus \{0\}$ .

**8** Give an example of a nonempty subset  $U$  of  $\mathbf{R}^2$  such that

$U$  is closed under scalar multiplication, but  $U$  is not a subspace of  $\mathbf{R}^2$ .

**SOLUTION:**  $U = \{(x, y) \in \mathbf{R}^2 : x = 0 \vee y = 0\}$ .

**9** A function  $f : \mathbf{R} \rightarrow \mathbf{R}$  is called periodic if there exists a positive number  $p$  such that  $f(x) = f(x + p)$  for all  $x \in \mathbf{R}$ .

Is the set of periodic functions from  $\mathbf{R}$  to  $\mathbf{R}$  a subspace of  $\mathbb{R}^\mathbb{R}$ ? Explain.

**SOLUTION:** Denote the set by  $S$ .

Suppose  $h(x) = \sin \sqrt{2}x + \cos x \in S$ , since both  $f(x) = \sin \sqrt{2}x, g(x) = \cos x$  are periodic functions  $\mathbf{R} \rightarrow \mathbf{R}$ .

Assume  $\exists p \in \mathbf{N}^+$  such that  $h(x) = h(x + p), \forall x \in \mathbf{R}$ . Let  $x = 0 \Rightarrow h(0) = h(\pm p) = 1$ .

Thus  $1 = \cos p + \sin \sqrt{2}p = \cos p - \sin \sqrt{2}p$

$\Rightarrow \sin \sqrt{2}p = 0, \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$ , while  $p = \frac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$ .

Hence  $2k = \frac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \frac{m}{2k} \in \mathbf{Q}$ . Contradiction!  $\square$

**11 Prove that the intersection of every collection of subspaces of  $V$  is a subspace of  $V$ .**

**SOLUTION:**

Suppose  $\{U_\alpha\}_{\alpha \in \Gamma}$  is a collection of subspaces of  $V$ ; here  $\Gamma$  is an arbitrary index set.

We need to prove that  $\bigcap_{\alpha \in \Gamma} U_\alpha$ , which equals the set of vectors

that are in  $U_\alpha$  for each  $\alpha \in \Gamma$ , is a subspace of  $V$ .

(一)  $0 \in \bigcap_{\alpha \in \Gamma} U_\alpha$ . Nonempty.

(二)  $u, v \in \bigcap_{\alpha \in \Gamma} U_\alpha \Rightarrow u + v \in U_\alpha, \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha \in \Gamma} U_\alpha$ . Closed under addition.

(三)  $u \in \bigcap_{\alpha \in \Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha, \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha \in \Gamma} U_\alpha$ . Closed under scalar multiplication.

Thus  $\bigcap_{\alpha \in \Gamma} U_\alpha$  is nonempty subset of  $V$  that is closed under addition and scalar multiplication.

Hence  $\bigcap_{\alpha \in \Gamma} U_\alpha$  is a subspace of  $V$ .  $\square$

**12 Prove that the union of two subspaces of  $V$  is a subspace of  $V$  if and only if one of the subspaces is contained in the other.**

**SOLUTION:** Suppose  $U$  and  $W$  are subspaces of  $V$ .

(a) Suppose  $U \subseteq W$ . Then  $U \cup W = W$  is a subspace of  $V$ .

(b) Suppose  $U \cup W$  is a subspace of  $V$ . Suppose  $U \not\subseteq W$  and  $U \not\supseteq W$  ( $U \cup W \neq U$  and  $W$ ).

Then  $\forall a \in U$  but  $a \notin W$ ;  $b \in W$  but  $b \notin U$ .  $a + b \in U \cup W$ .

(1) Consider  $a + b \in U \Rightarrow b = (a + b) + (-a) \in U$ , contradicts!  
 (2) Consider  $a + b \in W \Rightarrow a = (a + b) + (-b) \in W$ , contradicts!  
 Thus  $U \subseteq W$  and  $U \supseteq W$ .  $\square$

**13 Prove that the union of three subspaces of  $V$  is a subspace of  $V$  if and only if one of the subspaces contains the other two.**

*This exercise is surprisingly harder than Exercise 12, possibly because this exercise is not true if we replace  $\mathbf{F}$  with a field containing only two elements.*

**SOLUTION:** Suppose  $A, B, C$  are subspaces of  $V$ .

(a) If any two of them are subsets of the third one, then  $A \cup B \cup C = A, B$  or  $C$ , which is a subspace of  $V$ .

(b)\* If  $A \cup B \cup C$  is a subspace of  $V$ , suppose  $\left\{ \begin{array}{l} A \not\subseteq B \text{ and } C \\ B \not\subseteq A \text{ and } C \\ C \not\subseteq A \text{ and } B \end{array} \right\} \iff A \cap B \cap C \neq A, B \text{ and } C$ .

$\forall a \in A$  but  $a \notin B, C$ ;  $\forall b \in B$  but  $b \notin A, C$ ;  $\forall c \in C$  but  $c \notin A, B$ ; by assumption,  $a + b + c \in A \cup B \cup C$ .

(I)  $A \cup B$  is a subspace  $\Rightarrow$  By Problem (12),  $A \subseteq B$  or  $A \supseteq B$ .

(II)  $A \cup C$  is a subspace  $\Rightarrow$  By Problem (12),  $A \subseteq C$  or  $A \supseteq C$ .

(III)  $B \cup C$  is a subspace  $\Rightarrow$  By Problem (12),  $B \subseteq C$  or  $B \supseteq C$ .

Any two of (I), (II) and (III) must be true.

(一). (I) and (II) are true. Then  $\left\{ \begin{array}{l} C \subseteq B \subseteq A \\ \text{or } C \supseteq B \supseteq A \\ \text{or } B \supseteq A, C \\ \text{or } B \subseteq A, C \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array} \right.$

(二). (II) and (III) are true. Then  $\left\{ \begin{array}{l} A \subseteq C \subseteq B \\ \text{or } A \supseteq C \supseteq B \\ \text{or } C \supseteq A, B \\ \text{or } C \subseteq A, B \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array} \right.$

$$(三). (I) \text{ and } (III) \text{ are true. Then } \left. \begin{array}{l} B \subseteq A \subseteq C \\ \text{or } B \supseteq A \supseteq C \\ \text{or } A \supseteq B, C \\ \text{or } \underbrace{A \subseteq B, C}_{(3)} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array} \right.$$

Among these, any two of (1), (2) and (3) must be true.

$$\left. \begin{array}{l} (1) \\ (2) \\ (2) \\ (3) \\ (1) \\ (3) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} B \subseteq C \subseteq A \\ C \subseteq A \subseteq B \\ B \subseteq A \subseteq C \end{array} \right\} \Rightarrow \square$$

• Suppose  $U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$  and  $W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$ .

Describe  $U + W$  using symbols, and also give a description of  $U + W$  that uses no symbols.

**SOLUTION:**

$$(a) U + W = \{(x + y, x - y, 2(x + y)) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} = \{(x', y', 2x') \in \mathbf{F}^3 : x', y' \in \mathbf{F}\}.$$

$$(b) U + W \text{ is a plane of which } (1, 0, 2), (0, 1, 0) \text{ is a basis. } \square$$

**15** Suppose  $U$  is a subspace of  $V$ . What is  $U + U$ ?

$$\text{SOLUTION: } \left. \begin{array}{l} \forall x, y \in U, x + y \in U \Rightarrow U + U \subseteq U \\ \forall x \in U, 0 \in U, x + 0 \in U + U \Rightarrow U \subseteq U + U \end{array} \right\} \Rightarrow U + U = U. \square$$

**16** Suppose  $U$  and  $W$  are subspaces of  $V$ . Prove that  $U + W = W + U$ ?

$$\text{SOLUTION: } \left. \begin{array}{l} \forall x \in U, y \in W, x + y = y + x \in W + U \Rightarrow U + W \subseteq W + U \\ y + x = x + y \in U + W \Rightarrow W + U \subseteq U + W \end{array} \right\} \Rightarrow U + W = W + U. \square$$

**17** Suppose  $V_1, V_2, V_3$  are subspaces of  $V$ . Prove that  $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$ .

**SOLUTION:**

Let  $x \in V_1, y \in V_2, z \in V_3$ . Denote  $(V_1 + V_2) + V_3$  by  $L$ ,  $V_1 + (V_2 + V_3)$  by  $R$ .

$$\left. \begin{array}{l} \forall u \in L, \exists x, y, z, u = (x + y) + z = x + (y + z) \in R \Rightarrow L \subseteq R \\ \forall u \in R, \exists x, y, z, u = x + (y + z) = (x + y) + z \in L \Rightarrow R \subseteq L \end{array} \right\} \Rightarrow (V_1 + V_2) + V_3 = V_1 + (V_2 + V_3). \square$$

**18** Does the operation of addition on the subspaces of  $V$  have an additive identity?

Which subspaces have additive inverses?

**SOLUTION:**

Suppose  $\Omega$  is the additive identity.

For any subspace  $U$  of  $V$ .  $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$ . Let  $U = \{0\}$ , then  $\Omega = \{0\}$ .

Now suppose  $W$  is an additive inverse of  $U \Rightarrow U + W = \Omega$ .

Note that  $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$ . Thus  $U = W = \Omega = \{0\}$ .  $\square$

**19** Prove or give a counterexample: If  $V_1, V_2, U$  are subspaces of  $V$  such that  $V_1 + U = V_2 + U$ , then  $V_1 = V_2$ .

**SOLUTION:** An counterexample:

$$V = \mathbf{F}^3, U = \{(x, 0, 0) \in \mathbf{F}^3 : x \in \mathbf{F}\},$$

$$V_1 = \{(x, x, y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}, V_2 = \{(x, y, z) \in \mathbf{F}^3 : x, y, z \in \mathbf{F}\}.$$

**EXAMPLE:** Suppose  $U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$ ,  $W = \{(x, x, x, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$ .

*Prove that  $U + W = \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$ .*

**SOLUTION:** Let  $T$  denote  $\{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$ .

(a) By definition,  $U + W = \{(x_1 + x_2, x_1 + x_2, y_1 + x_2, y_1 + y_2) \in \mathbf{F}^4 : (x_1, x_1, y_1, y_1) \in U, (x_2, x_2, x_2, y_2) \in W\}$ .  
 $\Rightarrow \forall v \in U + W, \exists t \in T, v = t \Rightarrow U + W \subseteq T$ .

(b)  $\forall x, y, z \in \mathbf{F}$ , let  $u = (0, 0, y - x, y - x) \in U$ ,  $w = (x, x, x, -y + x + z) \in W$   
 $\Rightarrow (x, x, y, z) = u + w \in U + W$ . Hence  $\forall t \in T, \exists u \in U, w \in W, t = u + w \Rightarrow T \subseteq U + W$ .  $\square$

**21** Suppose  $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}$ .

*Find a subspace  $W$  of  $\mathbf{F}^5$  such that  $\mathbf{F}^5 = U \oplus W$ .*

**SOLUTION:**

(a) Let  $W = \{(0, 0, z, w, u) \in \mathbf{F}^5 : z, w, u \in \mathbf{F}\}$ . Then  $W \cap U = \{0\}$ .

(b)  $\forall x, y, z, w, u \in \mathbf{F}$ , let  $u = (x, y, x + y, x - y, 2x) \in U$ ,  
 $w = (0, 0, z - x - y, w - x - y, u - 2x) \in W$   
 $\Rightarrow (x, y, z, w, u) = u + w \Rightarrow \mathbf{F}^5 \subseteq U + W$ .  $\square$

**22** Suppose  $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}$ .

*Find three subspaces  $W_1, W_2, W_3$  of  $\mathbf{F}^5$ , none of which equals  $\{0\}$ , such that  $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$ .*

**SOLUTION:**

(1) Let  $W_1 = \{(0, 0, z, 0, 0) \in \mathbf{F}^5 : z \in \mathbf{F}\}$ . Then  $W_1 \cap U = \{0\}$ .

Let  $U_1 = U \oplus W_1$ . Then  $U_1 = \{(x, y, z, x - y, 2x) \in \mathbf{F}^5 : x, y, z \in \mathbf{F}\}$ . ( Check it! )

(2) Let  $W_2 = \{(0, 0, 0, w, 0) \in \mathbf{F}^5 : w \in \mathbf{F}\}$ . Then  $W_2 \cap U_1 = \{0\}$ .

Let  $U_2 = U_1 \oplus W_2$ . Then  $U_2 = \{(x, y, z, w, 2x) \in \mathbf{F}^5 : x, y, z, w \in \mathbf{F}\}$ .

(3) Let  $W_3 = \{(0, 0, 0, 0, u) \in \mathbf{F}^5 : u \in \mathbf{F}\}$ . Then  $W_3 \cap U_2 = \{0\}$ .

Let  $U_3 = U_2 \oplus W_3$ . Then  $U_3 = \{(x, y, z, w, u) \in \mathbf{F}^5 : x, y, z, w, u \in \mathbf{F}\}$ .

Thus  $\mathbf{F}^5 = (U \oplus W_1) \oplus W_2 \oplus W_3$ .  $\square$

**23** Prove or give a counterexample: If  $V_1, V_2, U$  are subspaces of  $V$  such that

$V = V_1 \oplus U$  and  $V = V_2 \oplus U$ , then  $V_1 = V_2$ .

**HINT:** When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in  $\mathbf{F}^2$ .

**SOLUTION:** An counterexample:

$V = \mathbf{F}^2, U = \{(x, x) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_1 = \{(x, 0) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_2 = \{(0, x) \in \mathbf{F}^2 : x \in \mathbf{F}\}$ .

**24** Let  $V_e$  denote the set of real-valued even functions on  $\mathbf{R}$

and let  $V_o$  denote the set of real-valued odd functions on  $\mathbf{R}$ . Show that  $\mathbb{R}^{\mathbf{R}} = V_e \oplus V_o$ .

**SOLUTION:**

(a)  $V_e \cap V_o = \{f : f(x) = f(-x) = -f(-x)\} = \{0\}$ .

(b) 
$$\left. \begin{aligned} f_e \in V_e &\Leftrightarrow f_e(x) = f_e(-x) \Leftarrow \text{let } f_e(x) = \frac{g(x) + g(-x)}{2} \\ f_o \in V_o &\Leftrightarrow f_o(x) = -f_o(-x) \Leftarrow \text{let } f_o(x) = \frac{g(x) - g(-x)}{2} \end{aligned} \right\} \Rightarrow \forall g \in \mathbb{R}^{\mathbf{R}}, g(x) = f_e(x) + f_o(x). \quad \square$$

**ENDED**

## 2.A

- 2 (a) Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.
- (b) Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

**SOLUTION:**

- (a) Suppose  $v \neq 0$ . Then let  $av = 0, a \in \mathbf{F}$ . Getting  $a = 0$ . Thus  $(v)$  is linearly independent.  
 Suppose  $(v)$  is linearly independent.  $av = 0 \Rightarrow a = 0$ . Then  $v \neq 0$ , for if not,  $a \neq 0 \Rightarrow av = 0$ . Contradicts.
- (b) Denote the list by  $(v, w)$ , where  $v, w \in V$ . If  $(v, w)$  is linearly independent, suppose  $av + bw = 0 \Rightarrow a = b = 0$ .  
 Without loss of generality, suppose  $v \neq cw \forall c \in \mathbf{F}$ . Then let  $av + bw = 0$ , getting  $a = b = 0 \Rightarrow (v, w)$  is linearly independent.

1 Prove that if  $(v_1, v_2, v_3, v_4)$  spans  $V$ , then the list  $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$  also spans  $V$ .

**SOLUTION:** Assume that  $\forall v \in V, \exists a_1, \dots, a_4 \in \mathbf{F}$ ,

$$\begin{aligned} v &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 \\ &= b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4, \text{ letting } b_i = \sum_{r=1}^i a_r. \end{aligned}$$

Thus  $\forall v \in V, \exists b_i \in \mathbf{F}, v = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4$ .

Hence the list  $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$  also spans  $V$ .  $\square$

6 Suppose  $(v_1, v_2, v_3, v_4)$  is linearly independent in  $V$ .

Prove that the list  $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$  is also linearly independent.

**SOLUTION:**  $a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4v_4 = 0$

$$\Rightarrow a_1v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0$$

$$\Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \Rightarrow a_1 = \dots = a_4 = 0 \Rightarrow \square$$

7 Prove that if  $(v_1, v_2, \dots, v_m)$  is a linearly independent list of vectors in  $V$ , then  $(5v_1 - 4v_2, v_2, v_3, \dots, v_m)$  is linearly independent.

**SOLUTION:**  $a_1(5v_1 - 4v_2) + a_2v_2 + a_3v_3 + \dots + a_mv_m = 0$

$$\Rightarrow 5a_1v_1 + (a_2 - 4a_1)v_2 + a_3v_3 + \dots + a_mv_m = 0$$

$$\Rightarrow 5a_1 = a_2 - 4a_1 = a_3 = \dots = a_m = 0 \Rightarrow a_1 = \dots = a_m = 0 \square$$

• Suppose  $(v_1, \dots, v_m)$  is a list of vectors in  $V$ . For  $k \in \{1, \dots, m\}$ , let  $w_k = v_1 + \dots + v_k$ .

(a) Show that  $\text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$ .

(b) Show that  $(v_1, \dots, v_m)$  is linearly independent if and only if  $(w_1, \dots, w_m)$  is linearly independent.

**SOLUTION:**

(a) Let  $\text{span}(v_1, \dots, v_m) = U$ . Assume that  $\forall v \in U, \exists a_i \in \mathbf{F}$ ,

$$v = a_1v_1 + \dots + a_mv_m = b_1w_1 + \dots + b_mw_m = \sum_{j=1}^m \left( \sum_{i=j}^m b_i \right) v_j$$

$$\Rightarrow b_1 = a_1, \quad b_i = a_i - \sum_{r=1}^{i-1} b_r. \text{ Thus } \exists b_i \in \mathbf{F} \text{ such that } v = b_1w_1 + \dots + b_mw_m.$$

又 Each  $w_i \in U \Rightarrow \text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$ .

(b)  $a_1w_1 + \dots + a_mw_m = 0$

$$\Rightarrow (a_1 + \dots + a_m)v_1 + \dots + (a_i + \dots + a_m)v_i + \dots + a_mv_m = 0$$

$$\Rightarrow a_m = \dots = (a_m + \dots + a_i) = \dots = (a_m + \dots + a_1) = 0. \square$$

**10** Suppose  $(v_1, \dots, v_m)$  is linearly independent in  $V$  and  $w \in V$ .

(a) Prove that if  $(v_1 + w, \dots, v_m + w)$  is linearly dependent, then  $w \in \text{span}(v_1, \dots, v_m)$ .

(b) Show that  $(v_1, \dots, v_m, w)$  is linearly independent  $\iff w \notin \text{span}(v_1, \dots, v_m)$ .

**SOLUTION:**

(a) Suppose  $a_1(v_1 + w) + \dots + a_m(v_m + w) = 0$ ,  $\exists a_i \neq 0 \Rightarrow a_1v_1 + \dots + a_mv_m = 0 = -(a_1 + \dots + a_m)w$ .

Then  $a_1 + \dots + a_m \neq 0$ , for if not,  $a_1v_1 + \dots + a_mv_m = 0$  while  $a_i \neq 0$  for some  $i$ , contradicts.

Hence  $w \in \text{span}(v_1, \dots, v_m)$ .

(b) Suppose  $w \in \text{span}(v_1, \dots, v_m)$ . Then  $(v_1, \dots, v_m, w)$  is linearly dependent.

Thus have we proven the “ $\Rightarrow$ ” by its contrapositive.

Suppose  $w \notin \text{span}(v_1, \dots, v_m)$ . Then by [2.23],  $(v_1, \dots, v_m, w)$  is linearly independent.  $\square$

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**14** Prove that  $V$  is infinite-dim if and only if there is a sequence  $(v_1, v_2, \dots)$  in  $V$  such that  $(v_1, \dots, v_m)$  is linearly independent for every  $m \in \mathbf{N}^+$ .

**SOLUTION:** Similar to [2.16].

Suppose there is a sequence  $(v_1, v_2, \dots)$  in  $V$  such that  $(v_1, \dots, v_m)$  is linearly independent for any  $m \in \mathbf{N}^+$ .

Choose an  $m$ . Suppose a linearly independent list  $(v_1, \dots, v_m)$  spans  $V$ .

Then there exists  $v_{m+1} \in V$  but  $v_{m+1} \notin \text{span}(v_1, \dots, v_m)$ . Hence no list spans  $V$ . Thus  $V$  is infinite-dim.

Conversely it is true as well. For if not,  $V$  must be finite-dim, contradicting the assumption.  $\square$

---

**15** Prove that  $\mathbf{F}^\infty$  is infinite-dim.

**SOLUTION:** Let  $e_i = (0, \dots, 0, 1, 0, \dots) \in \mathbf{F}^\infty$  for every  $m \in \mathbf{N}^+$ , where ‘1’ is on the  $i^{\text{th}}$  entry of  $e_i$ .

Suppose  $\mathbf{F}^\infty$  is finite-dim. Then let  $\text{span}(e_1, \dots, e_m) = V$ . But  $e_{m+1} \notin \text{span}(e_1, \dots, e_m)$ . Contradicts.  $\square$

---

**16** Prove that the real vector space of all continuous real-valued functions on the interval  $[0, 1]$  is infinite-dimensional.

**SOLUTION:** Denote the vec-sp by  $U$ . Note that for each  $m \in \mathbf{N}^+$ ,  $(1, x, \dots, x^m)$  is linearly independent.

Because if  $a_0, \dots, a_m \in \mathbf{R}$  are such that  $a_0 + a_1x + \dots + a_mx^m = 0$ ,  $\forall x \in [0, 1]$ , then the polynomial has infinitely many roots and hence  $a_0 = \dots = a_m = 0$ .  $\left. \begin{array}{l} \text{Because if } a_0, \dots, a_m \in \mathbf{R} \text{ are such that } a_0 + a_1x + \dots + a_mx^m = 0, \forall x \in [0, 1], \\ \text{then the polynomial has infinitely many roots and hence } a_0 = \dots = a_m = 0. \end{array} \right\} \text{Similar to [2.16], } U \text{ is infinite-dim.}$

OR. Note that for  $a_n = \frac{1}{n}$ ,  $a_1 < a_2 < \dots < a_m$ ,  $\forall m \in \mathbf{N}^+$ .

Suppose  $f_n = \begin{cases} x - \frac{1}{n}, & x \in [\frac{1}{n}, 1) \\ 0, & x \in [0, \frac{1}{n}) \end{cases}$ . Then for any  $m$ ,  $f_1(\frac{1}{m}) = \dots = f_m(\frac{1}{m})$ , while  $f_{m+1}(\frac{1}{m}) \neq 0$ .

Hence  $f_{m+1} \notin \text{span}(f_1, \dots, f_m)$ . Thus by Problem (14),  $U$  is infinite-dim.

---

**17** Suppose  $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$  such that  $p_k(2) = 0$  for each  $k \in \{0, \dots, m\}$ .

Prove that  $(p_0, p_1, \dots, p_m)$  is not linearly independent in  $\mathcal{P}_m(\mathbf{F})$ .

**SOLUTION:**

Suppose  $(p_0, p_1, \dots, p_m)$  is linearly independent. Define  $p \in \mathcal{P}_m(\mathbf{F})$  by  $p(z) = z \forall z \in \mathbf{F}$ .

But  $\forall a_i \in \mathbf{F}$ ,  $z \neq a_0p_0(z) + \dots + a_mp_m(z)$ , for if not, let  $z = 2$ , contradicts. Thus  $z \notin \text{span}(p_0, p_1, \dots, p_m)$ .

Then  $\text{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$  while the list  $(p_0, p_1, \dots, p_m)$  has length  $m + 1$ .

Hence  $(p_0, p_1, \dots, p_m)$  is linearly dependent in  $\mathcal{P}_m(\mathbf{F})$ .

For if not, notice that the list  $(1, z, \dots, z^m)$  spans  $\mathcal{P}_m(\mathbf{F})$ ,

thus by [2.23],  $(p_0, p_1, \dots, p_m)$  spans  $\mathcal{P}_m(\mathbf{F})$ . Contradicts.  $\square$



## 2.B

**NOTE FOR linearly independent sequence and [2.34].**

“ $V = \text{span}(v_1, \dots, v_n, \dots)$ ” is an invalid expression.

If we allow using “infinite list”, then we must guarantee that  $(v_1, \dots, v_n, \dots)$  is a spanning “list” such that for all  $v \in V$ , there exists a certain positive integer such that  $v = a_1 v_{\alpha_1} + \dots + a_n v_{\alpha_n}$ , where  $\Gamma = \{\alpha_1, \dots, \alpha_n\}$  is an finite index set. The key point is, how do we find such a “list”?

**NOTE FOR “ $\mathbb{C}_V U \cap \{0\}$ ”:** “ $\mathbb{C}_V U \cap \{0\}$ ” is supposed to be “ $W$ ”, where  $V = U \oplus W$ .

But if we let  $u \in U \setminus \{0\}$  and  $w \in W \setminus \{0\}$ , then 
$$\left. \begin{array}{l} w \in \mathbb{C}_V U \cap \{0\} \\ u \pm w \in \mathbb{C}_V U \cap \{0\} \end{array} \right\} \Rightarrow u \in \mathbb{C}_V U \cap \{0\}. \text{ Contradicts.}$$

**NEW NOTATION:** Denote the set  $\{W_1, W_2, \dots\}$  by  $\mathcal{S}_V U$ , where for each  $W_i, V = U \oplus W_i$ . See also in (1.C.23).

**1 Find all vector spaces that have exactly one basis. SOLUTION:**  $\mathbf{F} = \mathbf{C}, \mathbf{R}, \mathbf{Q}, \{0,1\}, \mathcal{P}_0(\mathbf{F})$ .

**6 Suppose  $(v_1, v_2, v_3, v_4)$  is a basis of  $V$ . Prove that  $(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4)$  is also a basis.**

**SOLUTION:**  $\forall v \in V, \exists! a_1, \dots, a_4 \in \mathbf{F}, v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$ .

Assume that  $v = b_1(v_1 + v_2) + b_2(v_2 + v_3) + b_3(v_3 + v_4) + b_4 v_4$ . Then  $v = b_1 v_1 + (b_1 + b_2)v_2 + (b_2 + b_3)v_3 + (b_3 + b_4)v_4$ .  
 $\Rightarrow \exists! b_1 = a_1, b_2 = a_2 - b_1, b_3 = a_3 - b_2, b_4 = a_4 - b_3 \in \mathbf{F}. \square$

**7 Prove or give a counterexample: If  $v_1, v_2, v_3, v_4$  is a basis of  $V$  and  $U$  is a subspace of  $V$  such that  $v_1, v_2 \in U$  and  $v_3 \notin U$  and  $v_4 \in U$ , then  $v_1, v_2$  is a basis of  $U$ .**

**SOLUTION:** Let  $V = \mathbf{F}^4, v_1 = (1, 0, 0, 0), v_2 = (0, 1, 0, 0), v_3 = (0, 0, 1, 1), v_4 = (0, 0, 0, 1)$ .

And  $U = \{(x, y, z, 0) \in \mathbf{R}^4 : x, y, z \in \mathbf{F}\}$ . We have an counterexample.

• **Suppose  $V$  is finite-dim and  $U, W$  are subspaces of  $V$  such that  $V = U + W$ . Prove that there exists a basis of  $V$  consisting of vectors in  $U \cup W$ .**

**SOLUTION:** Let  $(u_1, \dots, u_m)$  and  $(w_1, \dots, w_n)$  be bases of  $U$  and  $W$  respectively.

Then  $V = \text{span}(u_1, \dots, u_m) + \text{span}(w_1, \dots, w_n) = \text{span}(u_1, \dots, u_m, w_1, \dots, w_n)$ .

Hence, by [2.31], we get a basis of  $V$  consisting of vectors in  $U$  or  $W$ .  $\square$

**8 Suppose  $U$  and  $W$  are subspaces of  $V$  such that  $V = U \oplus W$ . Suppose also that  $(u_1, \dots, u_m)$  is a basis of  $U$  and  $(w_1, \dots, w_n)$  is a basis of  $W$ .**

**Prove that  $(u_1, \dots, u_m, w_1, \dots, w_n)$  is a basis of  $V$ .**

**SOLUTION:**

$\forall v \in V, \exists! a_i, b_i \in \mathbf{F}, v = (a_1 u_1 + \dots + a_m u_m) + (b_1 w_1 + \dots + b_n w_n)$

$\Rightarrow (a_1 u_1 + \dots + a_m u_m) = -(b_1 w_1 + \dots + b_n w_n) \in U \cap W = \{0\}$ . Thus  $a_1 = \dots = a_m = b_1 = \dots = b_n = 0$ .  $\square$

• (OR 9.4) **Suppose  $V$  is a real vector space.**

**Show that if  $(v_1, \dots, v_n)$  is a basis of  $V$  (as a real vector space),**

**then  $(v_1, \dots, v_n)$  is also a basis of the complexification  $V_{\mathbb{C}}$  (as a complex vector space).**

See Section 1B (4e) for the definition of the complexification  $V_{\mathbb{C}}$ .

**SOLUTION:**

$\forall u + iv \in V_{\mathbb{C}}, \exists! u, v \in V, a_i, b_i \in \mathbf{R},$

$u + iv = (a_1 v_1 + \dots + a_n v_n) + i(b_1 v_1 + \dots + b_n v_n) = (a_1 + b_1 i)v_1 + \dots + (a_n + b_n i)v_n$

$\Rightarrow u + iv = c_1 v_1 + \dots + c_n v_n, \exists! c_i = a_i + b_i i \in \mathbf{C}$

$\Rightarrow$  By the uniqueness of  $c_i$  and [2.29],  $(v_1, \dots, v_n)$  is a basis of  $V_{\mathbb{C}}$ .  $\square$

## 2.C

**1** Suppose  $V$  is finite-dim and  $U$  is a subspace of  $V$  such that  $\dim V = \dim U$ .

Let  $(u_1, \dots, u_m)$  be a basis of  $U$ . Then  $n = \dim U = \dim V$ .  $\forall u_i \in V$ .

Then by [2.39],  $(u_1, \dots, u_m)$  is a basis of  $V$ . Thus  $V = U$ .

**2** Show that the subspaces of  $\mathbf{R}^2$  are precisely  $\{0\}$ , all lines in  $\mathbf{R}^2$  containing the origin, and  $\mathbf{R}^2$ .

**SOLUTION:**

Suppose  $U$  is a subspace of  $\mathbf{R}^2$ . Let  $\dim U = n$ .

If  $n = 0$ , then  $U = \{0\}$ .

If  $n = 1$ , then  $U = \text{span}(v)$ , where  $v$  is a vector in  $\mathbf{R}^2$ . Thus  $U$  can be any line in  $\mathbf{R}^2$  containing the origin.

If  $n = 2$ , then  $U = \text{span}(v, w)$ , where  $v, w$  are vectors in  $\mathbf{R}^2$  and  $(v, w)$  is linearly independent  $\Rightarrow U = \mathbf{R}^2$ .  $\square$

**3** Show that the subspaces of  $\mathbf{R}^3$  are precisely  $\{0\}$ , all lines in  $\mathbf{R}^3$  containing the origin, all planes in  $\mathbf{R}^3$  containing the origin, and  $\mathbf{R}^3$ .

**SOLUTION:**

Suppose  $U$  is a subspace of  $\mathbf{R}^3$ . Let  $\dim U = n$ .

If  $n = 0$ , then  $U = \{0\}$ .

If  $n = 1$ , then  $U = \text{span}(v)$ , where  $v$  is a vector in  $\mathbf{R}^3$ . Thus  $U$  can be any line in  $\mathbf{R}^3$  containing the origin.

If  $n = 2$ , then  $U = \text{span}(v, w)$ , where  $v, w$  are vectors in  $\mathbf{R}^3$  and  $(v, w)$  is linearly independent.

Thus  $U$  can be any plane in  $\mathbf{R}^3$  containing the origin.

If  $n = 3$ , then  $U = \text{span}(u, v, w)$ , where  $u, v, w$  are vectors in  $\mathbf{R}^3$  and  $(u, v, w)$  is linearly independent  $\Rightarrow U = \mathbf{R}^3$ .  $\square$

**7** (a) Let  $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$ . Find a basis of  $U$ .

(b) Extend the basis in (a) to a basis of  $\mathcal{P}_4(\mathbf{F})$ .

(c) Find a subspace  $W$  of  $\mathcal{P}_4(\mathbf{F})$  such that  $\mathcal{P}_4(\mathbf{F}) = U \oplus W$ .

**SOLUTION:**

Suppose  $p(z) = az^4 + bz^3 + cz^2 + dz + e$  and  $p(2) = p(5) = p(6)$ .

$$\text{Then } \begin{cases} p(2) = 16a + 8b + 4c + 2d + e \text{ (I)} \\ p(5) = 625a + 125b + 25c + 5d + e \text{ (II)} \\ p(6) = 1296a + 216b + 36c + 6d + e \text{ (III)} \end{cases}$$

You don't have to compute to know that the dimension of the set of solutions is 3.

(a) A basis:  $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$ .

(b) Extend to a basis of  $\mathcal{P}_4(\mathbf{F})$  as  $1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$ .

(c) Let  $W = \text{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$ , so that  $\mathcal{P}_4(\mathbf{F}) = U \oplus W$ .  $\square$

**9** Suppose  $(v_1, \dots, v_m)$  is linearly independent in  $V$  and  $w \in V$ .

Prove that  $\dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$ .

**SOLUTION:**

Note that  $v_i - v_1 = (v_i + w) - (v_1 + w) \in \text{span}(v_1 + w, \dots, v_m + w)$ , for each  $i = 1, \dots, m$ .

$(v_1, \dots, v_m)$  is linearly independent  $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$  is linearly independent

$\Rightarrow (v_2 - v_1, \dots, v_m - v_1)$  is linearly independent of length  $m - 1$ .

$\forall$  By the contrapositive of (2.A.10),  $w \notin \text{span}(v_1, \dots, v_m) \Rightarrow (v_1 + w, \dots, v_m + w)$  is linearly independent.

$\therefore m \geq \dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$ .  $\square$

**10** Suppose  $m$  is a positive integer and  $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$  are such that each  $p_k$  has degree  $k$ . Prove that  $(p_0, p_1, \dots, p_m)$  is a basis of  $\mathcal{P}_m(\mathbf{F})$ .

**SOLUTION:** Using mathematical induction on  $m$ .

(i) For  $p_0, \deg p_0 = 0 \Rightarrow \text{span}(p_0) = \text{span}(1)$ .

(ii) Suppose for  $i \geq 1, \text{span}(p_0, p_1, \dots, p_i) = \text{span}(1, x, \dots, x^i)$ .

Then  $\text{span}(p_0, p_1, \dots, p_i, p_{i+1}) \subseteq \text{span}(1, x, \dots, x^i, x^{i+1})$ .

又  $\deg p_{i+1} = i + 1, p_{i+1}(x) = a_{i+1}x^{i+1} + r_{i+1}(x); a_{i+1} \neq 0, \deg r_{i+1} \leq i$ .

$\Rightarrow x^{i+1} = \frac{1}{a_{i+1}}(p_{i+1}(x) - r_{i+1}(x)) \in \text{span}(1, x, \dots, x^i, p_{i+1}) = \text{span}(p_0, p_1, \dots, p_i, p_{i+1})$ .

$\therefore x^{i+1} \in \text{span}(p_0, p_1, \dots, p_i, p_{i+1}) \Rightarrow \text{span}(1, x, \dots, x^i, x^{i+1}) \subseteq \text{span}(p_0, p_1, \dots, p_i, p_{i+1})$ .

Thus  $\mathcal{P}_m(\mathbf{F}) = \text{span}(1, x, \dots, x^m) = \text{span}(p_0, p_1, \dots, p_m)$ .  $\square$

• Suppose  $m$  is a positive integer. For  $0 \leq k \leq m$ , let  $p_k(x) = x^k(1-x)^{m-k}$ .

Show that  $(p_0, \dots, p_m)$  is a basis of  $\mathcal{P}(\mathbf{F})$ .

The basis in this exercise leads to what are called Bernstein polynomials. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on  $[0, 1]$ .

**SOLUTION:** Using mathematical induction.

(i)  $k = 0, 1, 2, p_m(x) = x^m, p_{m-1}(x) = x^{m-1} - x^m, p_{m-2}(x) = x^{m-2} + x^m - 2x^{m-1}$ .

(ii)  $k \geq 2$ . Suppose for  $p_{m-k}(x), \exists ! a_i \in \mathbf{F}, x^{m-k} = p_{m-k}(x) + a_m x^m + \dots + a_{m-k+1} x^{m-k+1}$ .

Then for  $p_{m-k-1}(x), \exists ! c_i \in \mathbf{F}$ ,

$$x^{m-k-1} = p_{m-k-1}(x) + \mathcal{C}_{k+1}^1(-1)^2 x^{m-k} + \dots + \mathcal{C}_{k+1}^k(-1)^{k+1} x^{m-1} + (-1)^{k-2} x^m$$

$$\Rightarrow c_{m-i} = \mathcal{C}_{k+1}^{k+1-i}(-1)^{k-i}.$$

Thus for each  $x^i, \exists ! b_i \in \mathbf{F}, x^i = b_m p_m(x) + \dots + b_{m-i} p_{m-i}(x)$ .

$\Rightarrow \text{span}(x^m, \dots, x, 1) = \text{span}(\underbrace{p_m, \dots, p_1, p_0}_{\text{Basis}})$ .  $\square$

• Suppose  $V$  is finite-dim and  $V_1, V_2, V_3$  are subspaces of  $V$  with

$\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$ . Prove that  $V_1 \cap V_2 \cap V_3 \neq \{0\}$ .

**SOLUTION:** 
$$\left. \begin{array}{l} \dim V_1 + \dim V_2 > 2 \dim V - \dim V_3 \geq \dim V \Rightarrow V_1 \cap V_2 \neq \{0\} \\ \dim V_2 + \dim V_3 > 2 \dim V - \dim V_1 \geq \dim V \Rightarrow V_2 \cap V_3 \neq \{0\} \\ \dim V_1 + \dim V_3 > 2 \dim V - \dim V_2 \geq \dim V \Rightarrow V_1 \cap V_3 \neq \{0\} \end{array} \right\} \Rightarrow V_1 \cap V_2 \cap V_3 \neq \{0\}. \quad \square$$

• Suppose  $V$  is finite-dim and  $U$  is a subspace of  $V$  with  $U \neq V$ . Let  $n = \dim V, m = \dim U$ . Prove that there exist  $(n-m)$  subspaces of  $V$ , say  $U_1, \dots, U_{n-m}$ , each of dimension  $(n-1)$ , such that  $\bigcap_{i=1}^{n-m} U_i = U$ .

**SOLUTION:** Let  $(v_1, \dots, v_m)$  be a basis of  $U$ , extend to a basis of  $V$  as  $(v_1, \dots, v_m, \dots, v_n)$ .

Define  $U_i = \text{span}(v_1, \dots, v_m, v_{m+1}, \dots, v_{m+i-1}, v_{m+i+1}, \dots, v_n)$  for each  $i$ . Thus we are done.

**EXAMPLE:** Suppose  $\dim V = 6, \dim U = 3$ .

$$\left. \begin{array}{l} U_1 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_5, v_6) \\ \underbrace{(v_1, v_2, v_3, v_4, v_5, v_6)}_{\text{Basis of } V}, \text{ define } U_2 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_6) \\ U_3 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_5) \end{array} \right\} \Rightarrow \dim U_i = 6 - 1, i = \underbrace{1, 2, 3}_{6-3=3}.$$

$\square$

**14** Suppose that  $V_1, \dots, V_m$  are finite-dim subspaces of  $V$ .

Prove that  $V_1 + \dots + V_m$  is finite-dim and  $\dim(V_1 + \dots + V_m) \leq \dim V_1 + \dots + \dim V_m$ .

**SOLUTION:**

Choose a basis  $\mathcal{E}_i$  of  $V_i \Rightarrow V_1 + \dots + V_m = \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$ ;  $\dim U_i = \text{card } \mathcal{E}_i$ .

Then  $\dim(V_1 + \dots + V_m) = \dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$ .

$\times \dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leq \text{card}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leq \text{card } \mathcal{E}_1 + \dots + \text{card } \mathcal{E}_m$ .

Thus  $\dim(V_1 + \dots + V_m) \leq \dim U_1 + \dots + \dim U_m$ .

• The inequality above is an equality if and only if  $V_1 + \dots + V_m$  is a direct sum.

For each  $i$ ,  $(V_1 + \dots + V_i) \cap V_{i+1} = \{0\} \iff V_1 + \dots + V_m$  is a direct sum  $\iff \square$

**17** Suppose  $V_1, V_2, V_3$  are subspaces of a finite-dim vector space, then

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

**SOLUTION:**

Looks like: given three sets  $A, B$  and  $C$ .

Note that:  $\text{card}(X \cup Y) = \text{card}(X) + \text{card}(Y) - \text{card}(X \cap Y)$ ;  $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$ .

Then:  $\text{card}((A \cup B) \cup C) = \text{card}(A \cup B) + \text{card } C - \text{card}((A \cup B) \cap C)$ .

And:  $\text{card}((A \cup B) \cap C) = \text{card}((A \cap C) \cup (B \cap C)) = \text{card}(A \cap C) + \text{card}(B \cap C) - \text{card}(A \cap B \cap C)$ .

Thus:  $\text{card}((A \cup B) \cup C) = \text{card } A + \text{card } B + \text{card } C + \text{card}(A \cap B \cap C) - \text{card}(A \cap B) - \text{card}(A \cap C) - \text{card}(B \cap C)$ .

Because  $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$ .

$$\dim(V_1 + V_2 + V_3) = \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \quad (1)$$

$$= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) \quad (2)$$

$$= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) \quad (3)$$

Notice that  $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$ .

For example,  $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}$ ,  $Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}$ ,  $Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$ .

• **COROLLARY:** If  $V_1, V_2$  and  $V_3$  are finite-dim vector spaces, then  $\frac{(1) + (2) + (3)}{3}$  :

$$\begin{aligned} \dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\ &\quad - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}. \end{aligned}$$

The formula above may seem strange because the right side does not look like an integer.  $\square$

**ENDED**

## 3.A

**2** Suppose  $b, c \in \mathbf{R}$ . Define  $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathbf{R}^2$  by

$$Tp = (3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^2 x^3 p(x) dx + c \sin p(0)).$$

Show that  $T$  is linear if and only if  $b = c = 0$ .

**SOLUTION:**

(a) Suppose  $b = c = 0$ , then  $\forall p, q \in \mathcal{P}(\mathbf{R})$ ,  $T(p + q) = (3(p + q)(4) + 5(p + q)'(6), \int_{-1}^2 x^3 (p + q)(x) dx)$ .

Because  $(p + q)(x) = p(x) + q(x)$ ,  $(p + q)'(x) = p'(x) + q'(x)$ ,

$$\int_{-1}^2 x^3 (p + q)(x) dx = \int_{-1}^2 x^3 p(x) dx + \int_{-1}^2 x^3 q(x) dx.$$

$\Rightarrow T(p + q) = Tp + Tq$ . Similarly,  $\forall \lambda \in \mathbf{F}$ ,  $\lambda Tp = T(\lambda p)$ . Thus  $T$  is linear.

(b) Suppose  $T$  is linear, denote the linear map in (a) by  $S \Rightarrow (T - S)$  is linear.

$\Rightarrow (T - S)(p) = (bp(1)p(2), c \sin p(0))$  is linear.

Consider  $p(x) = q(x) = \frac{\pi}{2}, \forall x \in \mathbf{R}$ .

$\Rightarrow ((T - S)(p + q) = (T - S)(\pi) = (b\pi^2, 0) = (T - S)(\frac{\pi}{2}) + (T - S)(\frac{\pi}{2}) = (b\frac{\pi^2}{2}, 2c) \Rightarrow b = c = 0. \quad \square$

---

• **TIPS:**  $T : V \rightarrow W$  is linear  $\iff \begin{cases} \forall v, u \in V, T(v + u) = Tv + Tu \\ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv) \end{cases} \iff T(v + \lambda u) = Tv + \lambda Tu.$

---

**3** Suppose  $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ . Prove that  $\exists A_{j,k} \in \mathbf{F}$  such that

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$$

for any  $(x_1, \dots, x_n) \in \mathbf{F}^n$ .

**SOLUTION:**

Let  $T(1, 0, 0, \dots, 0, 0) = (A_{1,1}, \dots, A_{m,1})$ , Note that  $(1, 0, \dots, 0, 0), \dots, (0, 0, \dots, 0, 1)$  is a basis of  $\mathbf{F}^n$ .

$T(0, 1, 0, \dots, 0, 0) = (A_{1,2}, \dots, A_{m,2})$ , Then by [3.5], we are done.  $\square$

$\vdots$

$T(0, 0, 0, \dots, 0, 1) = (A_{1,n}, \dots, A_{m,n})$ .

---

**4** Suppose  $T \in \mathcal{L}(V, W)$  and  $(v_1, \dots, v_m)$  is a list of vectors in  $V$  such that

$(Tv_1, \dots, Tv_m)$  is linearly independent in  $W$ . Prove that  $(v_1, \dots, v_m)$  is linearly independent.

**SOLUTION:**

Suppose  $a_1v_1 + \dots + a_mv_m = 0$ . Then  $a_1Tv_1 + \dots + a_mTv_m = 0$ . Thus  $a_1 = \dots = a_m = 0. \square$

---

**5** Prove that  $\mathcal{L}(V, W)$  is a vector space,

**SOLUTION:** Note that  $\mathcal{L}(V, W)$  is a subspace of  $W^V$ .  $\square$

---

**7** Show that every linear map from a one-dimensional vector space to itself

is multiplication by some scalar. More precisely, prove that if  $\dim V = 1$  and  $T \in \mathcal{L}(V)$ , then there exists  $\lambda \in \mathbf{F}$  such that  $Tv = \lambda v$  for all  $v \in V$ .

**SOLUTION:**

Let  $u$  be a nonzero vector in  $V \Rightarrow V = \text{span}(u)$ .

Because  $Tu \in V \Rightarrow Tu = \lambda u$  for some  $\lambda$ .

Suppose  $v \in V \Rightarrow v = au, \exists! a \in \mathbf{F}$ . Then  $Tv = T(au) = \lambda au = \lambda v. \quad \square$

---

**8** Give an example of a function  $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}$  such that

$\varphi(av) = a\varphi(v)$  for all  $a \in \mathbf{R}$  and all  $v \in \mathbf{R}^2$  but  $\varphi$  is not linear.

**SOLUTION:**

Define  $T(x, y) = \begin{cases} x + y, & \text{if } (x, y) \in \text{span}(3, 1), \\ 0, & \text{otherwise.} \end{cases}$

OR. Define  $T(x, y) = \sqrt[3]{(x^3 + y^3)}$ .  $\square$

---

**9** Give an example of a function  $\varphi : \mathbf{C} \rightarrow \mathbf{C}$  such that

$\varphi(w + z) = \varphi(w) + \varphi(z)$  for all  $w, z \in \mathbf{C}$  but  $\varphi$  is not linear.

(Here  $\mathbf{C}$  is thought of as a complex vector space.)

**SOLUTION:**

Suppose  $V_{\mathbf{C}}$  is the complexification of a vector space  $V$ . Suppose  $\varphi : V_{\mathbf{C}} \rightarrow V_{\mathbf{C}}$ .

Define  $\varphi(u + iv) = u = \Re(u + iv)$

OR. Define  $\varphi(u + iv) = v = \Im(u + iv)$ .  $\square$

---

• OR (3.D.16) Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

Prove that  $T$  is a scalar multiple of the identity if and only if  $ST = TS$  for every  $S \in \mathcal{L}(V)$ .

**SOLUTION:**

Assume that  $(v, Tv)$  is linearly dependent for every  $v \in V$ , then by (2.A.2.(b)),  $Tv = \lambda_v v$  for some  $\lambda_v \in \mathbf{F}$ .

To prove that  $\lambda_v$  is independent of  $v$

(in other words, for any two distinct nonzero vectors  $v$  and  $w$  in  $V$ , we have  $\lambda_v \neq \lambda_w$ ), we discuss in two cases:

$$\left. \begin{aligned} (-) \text{ If } (v, w) \text{ is linearly independent, } \lambda_{v+w}(v+w) &= T(v+w) = Tv + Tw = a_v v + a_w w \\ &\Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=) \text{ Otherwise, suppose } w &= cv, a_w w = Tw = cTv = ca_v v = a_v w \Rightarrow (a_w - a_v)w = 0 \end{aligned} \right\} \Rightarrow a_w = a_v.$$

Now we prove the assumption by contradiction. Suppose  $(v, Tv)$  is linearly independent for every nonzero vector  $v \in V$ .

Fix one  $v$ . Extend to  $(v, Tv, u_1, \dots, u_n)$  a basis of  $V$ .

Define  $S \in \mathcal{L}(V)$  by  $S(av + bTv + c_1u_1 + \dots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$ .

Hence a contradiction arises.  $\square$

**10** Suppose  $U$  is a subspace of  $V$  with  $U \neq V$ . Suppose  $S \in \mathcal{L}(U, W)$  and  $S \neq 0$  (which means that  $Su \neq 0$  for some  $u \in U$ ).

Define  $T : V \rightarrow W$  by  $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \setminus U. \end{cases}$  Prove that  $T$  is not a linear map on  $V$ .

**SOLUTION:**

Suppose  $T$  is a linear map. And  $v \in V \setminus U$ ,  $u \in U$  such that  $Su \neq 0$ .

Then  $v + u \in V \setminus U$ , (for if not,  $v = (v + u) - u \in U$ ) while  $T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$ .

Hence we get a contradiction.  $\square$

**11** Suppose  $V$  is finite-dim. Prove that every linear map on a subspace of  $V$  can be extended to a linear map on  $V$ . In other words, show that if  $U$  is a subspace of  $V$  and  $S \in \mathcal{L}(U, W)$ , then there exists  $T \in \mathcal{L}(V, W)$  such that  $Tu = Su$  for all  $u \in U$ .

**SOLUTION:** Define  $T \in \mathcal{L}(V, W)$  by  $T(a_1u_1 + \dots + a_nu_n + a_{n+1}u_{n+1} + \dots + a_mu_m) = a_1Su_1 + \dots + a_nSu_n$ . Where:

Let  $(u_1, \dots, u_n)$  be a basis of  $U$ , extend to a basis of  $V$  as  $(u_1, \dots, u_n, \dots, u_m)$ .

**12** Suppose  $V$  is finite-dim with  $\dim V > 0$ , and  $W$  is infinite-dim. Prove that  $\mathcal{L}(V, W)$  is infinite-dim.

**SOLUTION:**

Let  $(v_1, \dots, v_n)$  be a basis of  $V$ . Let  $(w_1, \dots, w_m)$  be linearly independent in  $W$  for any  $m \in \mathbf{N}^+$ .

Define  $T_{x,y} \in \mathcal{L}(V, W)$  by  $T_{x,y}(v_x) = w_y, \forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$ .

Suppose  $a_1T_{x,1} + \dots + a_mT_{x,m} = 0$ . Then  $(a_1T_{x,1} + \dots + a_mT_{x,m})(v_x) = 0 = a_1w_1 + \dots + a_mw_m$ .

$\Rightarrow a_1 = \dots = a_m = 0$ . 又  $m$  is arbitrarily chosen.

Thus  $(T_{x,1}, \dots, T_{x,m})$  is a linearly independent list in  $\mathcal{L}(V, W)$  for any  $x$  and length  $m$ . Hence by (2.A.14).  $\square$

**13** Suppose  $(v_1, \dots, v_m)$  is a linearly dependent list of vectors in  $V$ . Suppose also that  $W \neq \{0\}$ . Prove that there exist  $(w_1, \dots, w_m) \in W$  such that no  $T \in \mathcal{L}(V, W)$  satisfies  $Tv_k = w_k$  for each  $k = 1, \dots, m$ .

**SOLUTION:** We show it by contradiction.

By linear independence lemma,  $\exists j \in \{1, \dots, m\}$  such that  $v_j \in \text{span}(v_1, \dots, v_{j-1})$ .

Fix  $j$ . Let  $w_j \neq 0$ , while  $w_1 = \dots = w_{j-1} = w_{j+1} = \dots = w_m = 0$ .

Define  $T$  by  $Tv_k = w_k$  for all  $k$ . Suppose  $a_1v_1 + \dots + a_mv_m = 0$  (where  $a_j \neq 0$ ).

Then  $T(a_1v_1 + \dots + a_mv_m) = 0 = a_1w_1 + \dots + a_mw_m = a_jw_j$  while  $a_j \neq 0$  and  $w_j \neq 0$ . Contradicts.  $\square$

• Suppose  $V$  is finite-dim. Show that the only two-sided ideals of  $\mathcal{L}(V)$  are  $\{0\}$  and  $\mathcal{L}(V)$ .

A subspace  $\mathcal{E}$  of  $\mathcal{L}(V)$  is called a two-sided ideal of  $\mathcal{L}(V)$  if  $TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$ .

**SOLUTION:**

Let  $(v_1, \dots, v_n)$  be a basis of  $V$ . If  $\mathcal{E} = 0$ , then we are done.

Suppose  $\mathcal{E} \neq 0$  and  $\mathcal{E}$  is a two-sided ideal of  $\mathcal{L}(V)$ . Let  $S \in \mathcal{E} \setminus \{0\}$ .

Suppose  $Sw_i \neq 0$  and  $Sw_i = a_1v_1 + \dots + a_nv_n$ , where  $a_k \neq 0$ .

Define  $R_{x,y} \in \mathcal{L}(V)$  by  $R_{x,y}(v_x) = v_y, R_{x,y}(v_z) = 0 (z \neq x)$ . Then for any  $x, y \in \mathbf{N}^+$ ,

$(R_{k,y}S)(v_i) = a_kv_y \Rightarrow ((R_{k,y}S) \circ R_{x,i})(v_x) = a_kv_y$ , and  $((R_{k,y}S) \circ R_{x,i})(v_z) = 0$  for  $z \neq x$ .

Thus  $R_{k,y}SR_{x,i} = a_kR_{x,y}$ . Denote by  $T_{x,y}$ .

Getting  $(\frac{1}{a_k}T_{1,1} + \dots + \frac{1}{a_k}T_{n,n})v_j = v_j \Rightarrow \sum_{r=1}^n \frac{1}{a_k}T_{r,r} = I$ .

又 By assumption,  $T_{x,y} \in \mathcal{E} \Rightarrow I \in \mathcal{E}$ .

Hence for any  $T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$ .  $\square$

**ENDED**

### 3.B

2 Suppose  $S, T \in \mathcal{L}(V)$  are such that  $\text{range } S \subseteq \text{null } T$ . Prove that  $(ST)^2 = 0$ .

**SOLUTION:**  $TS = 0 \Rightarrow STST = (ST)^2 = 0$ .  $\square$

3 Suppose  $(v_1, \dots, v_m)$  in  $V$ . Define  $T \in \mathcal{L}(\mathbf{F}^m, V)$  by  $T(z_1, \dots, z_m) = z_1v_1 + \dots + z_mv_m$ .

(a) What property of  $T$  corresponds to  $(v_1, \dots, v_m)$  spanning  $V$ ?

(b) What property of  $T$  corresponds to  $(v_1, \dots, v_m)$  being linearly independent?

**ANSWER:** (a) Surjectivity; (b) Injectivity.  $\square$

4 Show that  $U = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \text{null } T > 2\}$  is not a subspace of  $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$ .

**SOLUTION:** Let  $(v_1, v_2, v_3, v_4, v_5)$  be a basis of  $\mathbf{R}^5$ ,  $(w_1, w_2, w_3, w_4)$  be a basis of  $\mathbf{R}^4$ .

Define  $T_1, T_2 \in U$  as  $T_1v_1 = 0, T_1v_2 = 0, T_1v_3 = 0, T_1v_4 = w_4, T_1v_5 = w_1$ ;

$T_2v_1 = 0, T_2v_2 = 0, T_2v_3 = w_3, T_2v_4 = 0, T_2v_5 = w_4$ . Thus  $T_1 + T_2 \notin U$ .

For  $U' = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \text{null } T > 0\}$ ,

define  $T_1, T_2 \in U'$  as  $T_1v_1 = 0, T_1v_2 = w_2, T_1v_3 = w_3, T_1v_4 = w_4, T_1v_5 = w_1$ ;

$T_2v_1 = w_1, T_2v_2 = w_2, T_2v_3 = 0, T_2v_4 = w_3, T_2v_5 = w_4$ . Thus  $T_1 + T_2 \notin U'$ .  $\square$

7 Suppose  $V$  is finite-dim with  $2 \leq \dim V \leq \dim W$ , if  $W$  is finite-dim.

Show that  $U = \{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$  is not a subspace of  $\mathcal{L}(V, W)$ .

**SOLUTION:**

Let  $(v_1, \dots, v_n)$  be a basis of  $V$ ,  $(w_1, \dots, w_m)$  be linearly independent in  $W$ .

(Let  $\dim W = m$ , if  $W$  is finite, otherwise, we choose  $m \in \{n, n+1, \dots\}$  arbitrarily;  $2 \leq n \leq m$ ).

Define  $T_1 \in \mathcal{L}(V, W)$  as  $T_1 : v_1 \mapsto 0, v_2 \mapsto w_2, v_i \mapsto w_i$ .

Define  $T_2 \in \mathcal{L}(V, W)$  as  $T_2 : v_1 \mapsto w_1, v_2 \mapsto 0, v_i \mapsto w_i, i = 3, \dots, n$ .

Thus  $T_1 + T_2 \notin U$ .  $\square$

**COMMENT:** If  $\dim V = 0$ , then  $V = \{0\} = \text{span}(\cdot)$ .  $\forall T \in \mathcal{L}(V, W), T$  is injective. Hence  $U = \emptyset$ .

If  $\dim V = 1$ , then  $V = \text{span}(v_0)$ . Thus  $U = \text{span}(T_0)$ , where  $T_0v_0 = 0$ .

If  $V$  is infinite-dim, the result is true as well.

**8** Suppose  $W$  is finite-dim with  $\dim V \geq \dim W \geq 2$ , if  $V$  is finite-dim.

Show that  $U = \{ T \in \mathcal{L}(V, W) : T \text{ is not surjective} \}$  is not a subspace of  $\mathcal{L}(V, W)$ .

**SOLUTION:**

Let  $(v_1, \dots, v_n)$  be linearly independent in  $V$ ,  $(w_1, \dots, w_m)$  be a basis of  $W$ .

( Let  $n = \dim V$ , if  $V$  is finite, otherwise we choose  $n \in \{m, m+1, \dots\}$ ;  $2 \leq m \leq n$  ).

Define  $T_1 \in \mathcal{L}(V, W)$  as  $T_1 : v_1 \mapsto 0, v_2 \mapsto w_2, v_j \mapsto w_j, v_{m+i} \mapsto 0$ .

Define  $T_2 \in \mathcal{L}(V, W)$  as  $T_2 : v_1 \mapsto w_1, v_2 \mapsto 0, v_j \mapsto w_j, v_{m+i} \mapsto 0$ .

For each  $j = 2, \dots, m$ ;  $i = 1, \dots, n-m$ , if  $V$  is finite, otherwise let  $i \in \mathbb{N}^+$ .

Thus  $T_1 + T_2 \notin U$ .  $\square$

**COMMENT:** If  $\dim W = 0$ , then  $W = \{0\} = \text{span}(\cdot)$ .  $\forall T \in \mathcal{L}(V, W)$ ,  $T$  is surjective. Hence  $U = \emptyset$ .

If  $\dim W = 1$ , then  $W = \text{span}(v_0)$ . Thus  $U = \text{span}(T_0)$ , where  $T_0 v_0 = 0$ .

If  $W$  is infinite-dim, the result is true as well.

**9** Suppose  $T \in \mathcal{L}(V, W)$  is injective and  $(v_1, \dots, v_n)$  is linearly independent in  $V$ .

Prove that  $(Tv_1, \dots, Tv_n)$  is linearly independent in  $W$ .

**SOLUTION:**

$$a_1 Tv_1 + \dots + a_n Tv_n = 0 = T\left(\sum_{i=1}^n a_i v_i\right) \iff \sum_{i=1}^n a_i v_i = 0 \iff a_1 = \dots = a_n = 0. \quad \square$$

**10** Suppose  $(v_1, \dots, v_n)$  spans  $V$  and  $T \in \mathcal{L}(V, W)$ . Show that  $(Tv_1, \dots, Tv_n)$  spans range  $T$ .

**SOLUTION:**

$$(a) \text{ range } T = \{Tv : v \in V\} = \{Tv : v \in \text{span}(v_1, \dots, v_n)\}$$

$$\Rightarrow Tv_1, \dots, Tv_n \in \text{range } T \Rightarrow \text{By [2.7] and [3.19], } \text{span}(Tv_1, \dots, Tv_n) \subseteq \text{range } T.$$

$$(b) \forall w \in \text{range } T, \exists v \in V, Tv = w. \text{ } \forall v \in V, \exists a_i \in \mathbf{F}, v = a_1 v_1 + \dots + a_n v_n$$

$$\Rightarrow w = Tv = a_1 Tv_1 + \dots + a_n Tv_n \Rightarrow \text{range } T \subseteq \text{span}(Tv_1, \dots, Tv_n). \quad \square$$

**11** Suppose  $S_1, \dots, S_n$  are injective linear maps and  $S_1 S_2 \dots S_n$  makes sense.

Prove that  $S_1 S_2 \dots S_n$  is injective.

**SOLUTION:**  $S_1 S_2 \dots S_n(v) = 0 \iff S_2 S_3 \dots S_n(v) = 0 \iff \dots \iff S_n(v) = 0 \iff v = 0. \quad \square$

**12** Suppose that  $V$  is finite-dim and that  $T \in \mathcal{L}(V, W)$ . Prove that

there exists a subspace  $U$  of  $V$  such that  $U \cap \text{null } T = \{0\}$  and  $\text{range } T = \{Tu : u \in U\}$ .

**SOLUTION:**

By [2.34], there exists a subspace  $U$  of  $V$  such that  $V = U \oplus \text{null } T$ .

$\forall v \in V, \exists! w \in \text{null } T, u \in U, v = w + u$ . Then  $Tv = T(w + u) = Tu \in \{Tu : u \in U\} \Rightarrow \square$

**COMMENT:**  $V$  can be infinite-dim. See the above of [2.34].

**16** Suppose there exists a linear map on  $V$

whose null space and range are both finite-dim. Prove that  $V$  is finite-dim.

**SOLUTION:**

Denote the linear map by  $T$ . Let  $(Tv_1, \dots, Tv_n)$  be a basis of range  $T$ ,  $(u_1, \dots, u_m)$  be a basis of null  $T$ .

Then for all  $v \in V, T(\underbrace{v - a_1 v_1 - \dots - a_n v_n}_{u \in \text{null } T}) = 0$ , where  $Tv = a_1 Tv_1 + \dots + a_n Tv_n$ .

$$\Rightarrow u = b_1 u_1 + \dots + b_m u_m \Rightarrow v = a_1 v_1 + \dots + a_n v_n + b_1 u_1 + \dots + b_m u_m.$$

Getting  $V \subseteq \text{span}(v_1, \dots, v_n, u_1, \dots, u_m)$ . Thus  $V$  is finite-dim.  $\square$



**17** Suppose  $V$  and  $W$  are both finite-dim. Prove that there exists an injective  $T \in \mathcal{L}(V, W)$  if and only if  $\dim V \leq \dim W$ .

**SOLUTION:**

- (a) Suppose there exists an injective  $T$ . Then  $\dim V = \dim \text{range } T + \dim \text{null } T = \dim \text{range } T \leq \dim W$ .  
 (b) Suppose  $\dim V \leq \dim W$ , letting  $(v_1, \dots, v_n)$  and  $(w_1, \dots, w_m)$  be bases of  $V$  and  $W$  respectively.  
 Define  $T \in \mathcal{L}(V, W)$  by  $Tv_i = w_i, i = 1, \dots, n (= \dim V)$ .  $\square$

**18** Suppose  $V$  and  $W$  are both finite-dim. Prove that there exists a surjective  $T \in \mathcal{L}(V, W)$  if and only if  $\dim V \geq \dim W$ .

**SOLUTION:**

- (a) Suppose there exists a surjective  $T$ . Then  $\dim V = \dim \text{range } T + \dim \text{null } T = \dim W + \dim \text{null } T \Rightarrow \dim W = \dim V - \dim \text{null } T \leq \dim V$ .  
 (b) Suppose  $\dim V \geq \dim W$ , letting  $(v_1, \dots, v_n)$  and  $(w_1, \dots, w_m)$  be bases of  $V$  and  $W$  respectively.  
 Define  $T \in \mathcal{L}(V, W)$  by  $T(a_1v_1 + \dots + a_mv_m + a_{m+1}v_{m+1} + \dots + a_nv_n) = a_1w_1 + \dots + a_mw_m$ .  $\square$

**19** Suppose  $V$  and  $W$  are finite-dim and that  $U$  is a subspace of  $V$ .

Prove that  $\exists T \in \mathcal{L}(V, W), \text{null } T = U \iff \dim U \geq \dim V - \dim W$ .

**SOLUTION:**

- (a) Suppose  $\exists T \in \mathcal{L}(V, W), \text{null } T = U$ . Then  $\dim \text{null } T = \dim U \geq \dim V - \dim W$ .  
 (b) Suppose  $\underbrace{\dim U}_m \geq \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_p (\Rightarrow \dim W = p \geq n = \dim V - \dim U)$ .

Let  $(u_1, \dots, u_m)$  be a basis of  $U$ , extend to a basis of  $V$  as  $(u_1, \dots, u_m, v_1, \dots, v_n)$ .

Let  $(w_1, \dots, w_p)$  be a basis of  $W$ .

Define  $T \in \mathcal{L}(V, W)$  by  $T(a_1v_1 + \dots + a_nv_n + b_1u_1 + \dots + b_mu_m) = a_1w_1 + \dots + a_nw_n$ .  $\square$

• **TIPS:** Suppose  $T \in \mathcal{L}(V, W)$  and  $R = (Tv_1, \dots, Tv_n)$  is linearly independent in  $\text{range } T$ .

(Let  $\dim \text{range } T = n$ , if  $\text{range } T$  is finite, otherwise choose  $n$  arbitrarily. ).

By (3.A.4),  $L = (v_1, \dots, v_n)$  is linearly independent in  $V$ .

**NEW NOTATION:** Denote  $\mathcal{K}_R$  by  $\text{span } L$ , if  $\text{range } T$  is finite-dim,

otherwise, denote it by an vector space in the set  $\mathcal{S}_V \text{null } T$ .

**NEW THEOREM:**

$$\mathcal{K}_R \oplus \text{null } T = V \iff \begin{cases} \text{(a) } T(\sum_{i=1}^n a_i v_i) = 0 \Rightarrow \sum_{i=1}^n a_i T v_i = 0 \Rightarrow a_1 = \dots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \text{null } T = \{0\}. \\ \text{(b) } \forall v \in V, T v = \sum_{i=1}^n a_i T v_i \Rightarrow T v - \sum_{i=1}^n a_i T v_i = T(v - \sum_{i=1}^n a_i v_i) = 0 \\ \Rightarrow v - \sum_{i=1}^n a_i v_i \in \text{null } T \Rightarrow v = (v - \sum_{i=1}^n a_i v_i) + (\sum_{i=1}^n a_i v_i) \Rightarrow \mathcal{K}_R + \text{null } T = V. \end{cases}$$

**COMMENT:**  $\text{null } T \in \mathcal{S}_V \mathcal{K}_R$ .

• Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V, W)$ , and  $U$  is a subspace of  $W$ .

Prove that  $\mathcal{K}_U = \{v \in V : Tv \in U\}$  is a subspace of  $V$

and  $\dim \mathcal{K}_U = \dim \text{null } T + \dim(U \cap \text{range } T)$ .

**SOLUTION:** For any  $u, w \in \mathcal{K}_U$  and  $\lambda \in \mathbf{F}$ ,  $T(u + \lambda w) = Tu + \lambda Tw \in U \Rightarrow T$  is linear

Define  $S \in \mathcal{L}(\mathcal{K}_U, U)$  as  $Rv = Tv$  for all  $v \in \mathcal{K}_U$ . Hence  $\text{range } R = U \cap \text{range } T$ .

Suppose  $Tv = 0$  for some  $v \in V$ .  $\nexists 0 \in U \Rightarrow Rv = 0$ . Thus  $\text{null } T \subseteq \text{null } R$ .  $\square$

**20** Suppose  $T \in \mathcal{L}(V, W)$ . Prove that  $T$  is injective  $\iff \exists S \in \mathcal{L}(W, V)$ ,  $ST = I \in \mathcal{L}(V)$ .

**SOLUTION:**

(a) Suppose  $\exists S \in \mathcal{L}(W, V)$ ,  $ST = I$ . Then if  $Tv = 0 \Rightarrow ST(v) = 0 = v$ . Hence  $T$  is injective.

(b) Suppose  $T$  is injective.  $\forall w \in \text{range } T$ ,  $\exists! v \in V, Tv = w$ . (if  $w = 0$ , then  $v = 0$ )

Define  $S : W \rightarrow V$  by  $Sw = v$  and  $Su = 0$ ,  $u \in U$ . Where  $W = U \oplus \text{range } T$ .

$\Rightarrow S(Tv + \lambda Tu) = S(T(v + \lambda u)) = v + \lambda u$  and  $S(x + \nu y) = 0$ ,  $x, y \in U$ .

Thus  $S|_{\text{range } T+U} = S|_W \in \mathcal{L}(W, V)$  and  $ST = I$ .  $\square$

OR. Let  $R = (Tv_1, \dots, Tv_n)$  be linearly independent in  $\text{range } T \subseteq W$ ,  $(\dots)$  and then  $\mathcal{K}_R \oplus \text{null } T = V$ .

Suppose  $W = U \oplus \text{range } T$ . Define  $S \in \mathcal{L}(W, V)$  by  $S(Tv_i) = v_i$  and  $Su = 0$ ,  $u \in U$ . Thus  $ST = I$ .  $\square$

**21** Suppose  $T \in \mathcal{L}(V, W)$ . Prove that  $T$  is surjective  $\iff \exists S \in \mathcal{L}(W, V)$ ,  $TS = I \in \mathcal{L}(W)$ .

**SOLUTION:**

(a) Suppose  $\exists S \in \mathcal{L}(W, V)$ ,  $TS = I$ . Then for any  $w \in W$ ,  $TS(w) = w \in \text{range } T \Rightarrow \text{range } T = W$ .  $\square$

(b) Suppose  $T$  is surjective.  $\forall w \in W$ ,  $\exists v \in V, Tv = w$ . Define  $S : W \rightarrow V$  by  $Sw = v$ .

But  $T(Sv + \lambda Su) = T(Sv) + \lambda T(Su) = v + \lambda u = T(S(v + \lambda u)) \neq Sv + \lambda Su = S(v + \lambda u)$ .

So we let  $R = (Tv_1, \dots, Tv_n)$  be linearly independent in  $\text{range } T = W$ ,  $(\dots)$  and then  $\mathcal{K}_R \oplus \text{null } T = V$ .

Define  $S \in \mathcal{L}(W, V)$  by  $S(Tv_i) = v_i$ . Then  $TS = I$ .  $\square$

**22** Suppose  $U$  and  $V$  are finite-dim vec-sps and  $S \in \mathcal{L}(V, W)$  and  $T \in \mathcal{L}(U, V)$ .

Prove that  $\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T$ .

**SOLUTION:** Define  $R \in \mathcal{L}(\text{null } ST, V)$  by  $Ru = Tu$  for all  $u \in \text{null } ST \subseteq U$ .

$$\left. \begin{array}{l} S(Tu) = 0 = S(Ru) \Rightarrow \text{range } R \subseteq \text{null } S \Rightarrow \dim \text{range } R \leq \dim \text{null } S \\ Tu = 0 = Ru \Rightarrow \text{null } R \supseteq \text{null } T \Rightarrow \dim \text{null } R = \dim \text{null } T \end{array} \right\} \Rightarrow \square$$

• **COROLLARY:**

(1) If  $T$  is injective, then  $\dim \text{null } T = 0 \Rightarrow \dim \text{null } ST \leq \dim \text{null } S$ .

(2) If  $T$  is surjective, then  $\text{range } R = \text{null } S \Rightarrow \dim \text{null } ST = \dim \text{null } S + \dim \text{null } T$ .

(3) If  $S$  is injective, then  $\text{range } R = \{0\} \Rightarrow \dim \text{null } ST = \dim \text{null } R = \dim \text{null } T$ .

**23** Suppose  $U$  and  $V$  are finite-dim vec-sps and  $S \in \mathcal{L}(V, W)$  and  $T \in \mathcal{L}(U, V)$ .

Prove that  $\dim \text{range } ST \leq \min\{\dim \text{range } S, \dim \text{range } T\}$ .

**SOLUTION:**

$\text{range } ST = \{Sv : v \in \text{range } T\} = \text{span}(Su_1, \dots, Su_{\dim \text{range } T})$ , letting  $\text{span}(u_1, \dots, u_{\dim \text{range } T}) = \text{range } T$ .

$\dim \text{range } ST \leq \dim \text{range } T \wedge \dim \text{range } ST \leq \dim \text{range } S \Rightarrow \square$

• **COROLLARY:**

(1) If  $S$  is injective, then  $\dim \text{range } ST = \dim \text{range } T$ .

(2) If  $T$  is surjective, then  $\text{range } ST = \text{range } S$ .

• (a) Suppose  $\dim V = 5$  and  $S, T \in \mathcal{L}(V)$  are such that  $ST = 0$ . Prove that  $\dim \text{range } TS \leq 2$ .

(b) Give an example of  $S, T \in \mathcal{L}(\mathbf{F}^5)$  with  $ST = 0$  and  $\dim \text{range } TS = 2$ .

**SOLUTION:**

By Problem (23),  $\dim \text{range } TS \leq \min\{\underbrace{\dim \text{range } S}_{5 - \dim \text{null } T}, \underbrace{\dim \text{range } T}_{5 - \dim \text{null } S}\}$ .

Suppose  $\dim \text{range } TS \geq 3$ . Then  $\min\{5 - \dim \text{null } T, 5 - \dim \text{null } S\} \geq 3$

$\Rightarrow \max\{\dim \text{null } T, \dim \text{null } S\} \leq 2$ .

$\wedge \dim \text{null } ST = 5 \leq \dim \text{null } S + \dim \text{null } T \leq 4$ . Contradicts. Thus  $\dim \text{range } TS \leq 2$ .  $\square$

EXAMPLE:  $V = \text{span}(v_1, \dots, v_5)$

$$T : \quad v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i ;$$

$$S : \quad v_1 \mapsto v_4, \quad v_2 \mapsto v_5, \quad v_i \mapsto 0 ; \quad i = 3, 4, 5$$

• Suppose  $\dim V = n$  and  $S, T \in \mathcal{L}(V)$  are such that  $ST = 0$ .

$$\text{Prove that } \dim TS \leq m = \begin{cases} \frac{n}{2}, & \text{if } 2 \mid n. \\ \frac{n-1}{2}, & \text{otherwise.} \end{cases}$$

**SOLUTION:**

By Problem (23),  $\dim \text{range } TS \leq \min\{\underbrace{\dim \text{range } S}_{n - \dim \text{null } T}, \underbrace{\dim \text{range } T}_{n - \dim \text{null } S}\}$ . Suppose  $\dim \text{range } TS \geq m + 1$ .

$$\text{Then } \min\{n - \dim \text{null } T, n - \dim \text{null } S\} \geq m + 1$$

$$\Rightarrow \max\{\dim \text{null } T, \dim \text{null } S\} \leq n - m - 1.$$

又  $\dim \text{null } ST = n \leq \dim \text{null } S + \dim \text{null } T \leq n - m - 1$ . Contradicts. Thus  $\dim \text{range } TS \leq m$ .  $\square$

**24** Suppose that  $W$  is finite-dim and  $S, T \in \mathcal{L}(V, W)$ .

Prove that  $\text{null } S \subseteq \text{null } T \iff \exists E \in \mathcal{L}(W)$  such that  $T = ES$ .

**SOLUTION:**

Suppose  $\text{null } S \subseteq \text{null } T$ . Let  $R = (Sv_1, \dots, Sv_n)$  be a basis of  $\text{range } S \Rightarrow (v_1, \dots, v_n)$  is linearly independent.

Let  $\mathcal{K}_R = \text{span}(v_1, \dots, v_n) \Rightarrow V = \mathcal{K}_R \oplus \text{null } S$ .

Define  $E \in \mathcal{L}(W)$  by  $E(Sv_i) = Tv_i$ ,  $Eu = 0$ ; for each  $i = 1 \dots, n$  and  $u \in \text{null } S$ .

Hence  $\forall v \in V$ ,  $(\exists! a_i \in \mathbf{F}, u \in \text{null } S)$ ,  $Tv = a_1Tv_1 + \dots + a_nTv_n = E(a_1Sv_1 + \dots + a_nSv_n) \Rightarrow T = ES$ .

Suppose  $\exists E \in \mathcal{L}(W)$  such that  $T = ES$ . Then  $\text{null } T = \text{null } ES \supseteq \text{null } S$ .  $\square$

**25** Suppose that  $V$  is finite-dim and  $S, T \in \mathcal{L}(V, W)$ .

Prove that  $\text{range } S \subseteq \text{range } T \iff \exists E \in \mathcal{L}(V)$  such that  $S = TE$ .

**SOLUTION:**

Suppose  $\text{range } S \subseteq \text{range } T$ . Let  $(v_1, \dots, v_m)$  be a basis of  $V$ .

Because  $\text{range } S \subseteq \text{range } T \Rightarrow Sv_i \in \text{range } T$  for each  $i$ . Suppose  $u_i \in V$  for each  $i$  such that  $Tu_i = Sv_i$ .

Thus defining  $E \in \mathcal{L}(V)$  by  $Ev_i = u_i$  for each  $i \Rightarrow S = TE$ .

Suppose  $\exists E \in \mathcal{L}(V)$  such that  $S = TE$ . Then  $\text{range } S = \text{range } TE \subseteq \text{range } T$ .  $\square$

• Suppose  $P \in \mathcal{L}(V)$  and  $P^2 = P$ . Prove that  $V = \text{null } P \oplus \text{range } P$ .

**SOLUTION:**

Let  $P^2v_1, \dots, P^2v_n$  be a basis of  $\text{range } P^2$ . Then  $(Pv_1, \dots, Pv_n)$  is linearly independent in  $V$ .

$$\left. \begin{array}{l} \text{Let } \mathcal{K} = \text{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \text{null } P^2 \\ \text{又 } \mathcal{K} = \text{range } P = \text{range } P^2; \text{ null } P = \text{null } P^2 \end{array} \right\} \Rightarrow \square$$

**26** Prove that the differentiation map  $D \in \mathcal{P}(\mathbf{R})$  is surjective.

**SOLUTION:** Note that  $\deg Dx^n = n - 1$ .

Because  $\text{span}(Dx, Dx^2, \dots) \subseteq \text{range } D$ . 又 By (2.A.10),  $\text{span}(Dx, Dx^2, \dots) = \text{span}(1, x, \dots) = \mathcal{P}(\mathbf{R})$ .  $\square$

**27** Suppose  $p \in \mathcal{P}(\mathbf{R})$ . Prove that there exists a polynomial  $q \in \mathcal{P}(\mathbf{R})$  such that  $5q'' + 3q' = p$ .

**SOLUTION:**

Define  $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$  by  $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$ .

Note that  $\deg Bx^n = n - 1$ . Similar to Problem (26), we conclude that  $B$  is surjective.

Hence for any  $p \in \mathcal{P}(\mathbf{R})$ , there exists  $q \in \mathcal{P}(\mathbf{R})$  such that  $Bq = p$ .  $\square$

**28** Suppose  $T \in \mathcal{L}(V, W)$  and  $(w_1, \dots, w_m)$  is a basis of range  $T$ . Prove that

$\exists \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$  such that for all  $v \in V$ ,  $Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m$ .

**SOLUTION:**

Suppose  $(v_1, \dots, v_m)$  in  $V$  such that  $Tv_i = w_i$  for each  $i$ .

Then  $(v_1, \dots, v_m)$  is linearly independent, extend it to a basis of  $V$  as  $(v_1, \dots, v_m, u_1, \dots, u_n)$ .

Note that  $\forall v \in V$ ,  $v = a_1v_1 + \dots + a_mv_m + b_1u_1 + \dots + b_nu_n$ ,  $\exists! a_i, b_i \in \mathbf{F} \Rightarrow Tv = a_1w_1 + \dots + a_mw_m$ .

Define  $\varphi_i : V \rightarrow \mathbf{F}$  by  $\varphi_i(v) = a_i$  for each  $i$ . We now check the linearity.

$\forall v, u \in V$  ( $\exists! a_i, b_i, c_i, d_i \in \mathbf{F}$ ),  $\lambda \in \mathbf{F}$ ,  $\varphi_i(v + \lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda \varphi(u)$ .  $\square$

**29** Suppose  $\varphi \in \mathcal{L}(V, \mathbf{F})$  and  $\varphi \neq 0$ . Suppose  $u \in V$  is not in null  $\varphi$ .

Prove that  $V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$ .

**SOLUTION:**

(a) Suppose  $v = cu \in \text{null } \varphi \cap \{au : a \in \mathbf{F}\}$ , where  $c \in \mathbf{F}$ .

Then  $\varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$ . Hence  $\text{null } \varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$ .

(b) Suppose  $v \in V$ . Then  $v = (v - \frac{\varphi(v)}{\varphi(u)}u) + \frac{\varphi(v)}{\varphi(u)}u \Rightarrow \varphi(v) = 0$ .

$$\left. \begin{array}{l} v - \frac{\varphi(v)}{\varphi(u)}u \in \text{null } \varphi \\ \frac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array} \right\} \Rightarrow V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}. \quad \square$$

*This may seem strange. Here we explain why.*

$\varphi \neq 0 \Rightarrow \exists$  a linearly independent list  $(v_1, \dots, v_n \in V)$  such that  $\varphi(v_i) = a_i \neq 0$ .

Choose a  $v_k$  arbitrarily. Then  $\varphi(v_k - \frac{\varphi(v_k)}{\varphi(v_j)}v_j) = 0$  for each  $j = 1, \dots, k-1, k+1, \dots, n$ .

Thus  $\text{span}\{v_k - \frac{\varphi(v_k)}{\varphi(v_j)}v_j\}_{j \neq k} \subseteq \text{null } \varphi$ . Hence there is only one nonzero vector in every vec-sp in  $\mathcal{S}_V \text{null } \varphi$ .

**30** Suppose  $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$  and  $\text{null } \varphi_1 = \text{null } \varphi_2 = \text{null } \varphi$ . Prove that  $\exists c \in \mathbf{F}$ ,  $\varphi_1 = c\varphi_2$

**SOLUTION:**

If  $\text{null } \varphi = V$ , then  $\varphi_1 = \varphi_2 = 0$ , we are done. Suppose  $u \in V/\text{null } \varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$ .

By Problem (29),  $V = \text{null } \varphi \oplus \text{span}(u)$ .

Hence for any  $v \in V$ ,  $v = w + a_vu$ ,  $\exists! w \in \text{null } \varphi, a_v \in \mathbf{F}$ .

$$\varphi_1(v) = a_v\varphi_1(u), \quad \varphi_2(v) = a_v\varphi_2(u) \Rightarrow a_v = \frac{\varphi_1(v)}{\varphi_1(u)} = \frac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \frac{\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}.$$

Thus  $\varphi_1 = c\varphi_2$ .  $\square$

**31** Give an example of  $T_1, T_2 \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^2)$  such that  $\text{null } T_1 = \text{null } T_2$  and that  $T_1$  is not a scalar multiple of  $T_2$ .

**SOLUTION:**

Let  $(v_1, \dots, v_5)$  be a basis of  $\mathbf{R}^5$ ,  $(w_1, w_2)$  be a basis of  $\mathbf{R}^2$ . Define  $T, S \in \mathcal{L}(V, W)$  by

$$\left. \begin{array}{lll} Tv_1 = w_1, & Tv_2 = w_2, & Tv_3 = Tv_4 = Tv_5 = 0 \\ Sv_1 = w_1, & Sv_2 = 2w_2, & Sv_3 = Sv_4 = Sv_5 = 0 \end{array} \right\} \Rightarrow \text{null } T = \text{null } S.$$

Suppose  $T = \lambda S$ . Then  $w_1 = Tv_1 = \lambda Sv_1 = \lambda w_1 \Rightarrow \lambda = 1$ .

While  $w_2 = Tv_2 = \lambda Sv_2 = 2\lambda w_2 \Rightarrow \lambda = \frac{1}{2}$ . Contradicts.  $\square$

- Suppose  $V$  is finite-dim,  $X$  is a subspace of  $V$ , and  $Y$  is a finite-dim subspace of  $W$ . Prove that there exists  $T \in \mathcal{L}(V, W)$  such that  $\text{null } T = X$  and  $\text{range } T = Y$  if and only if  $\dim X + \dim Y = \dim V$ .

**SOLUTION:**

(a) Suppose  $\dim X + \dim Y = \dim V$ . Let  $(u_1, \dots, u_n)$  be a basis of  $X$ ,  $R = (w_1, \dots, w_m)$  be a basis of  $Y$ .

Extend  $(u_1, \dots, u_n)$  to a basis of  $V$  as  $(u_1, \dots, u_n, v_1, \dots, v_m)$ .

Define  $T \in \mathcal{L}(V, W)$  by  $T(a_1v_1 + \dots + a_mv_m + b_1u_1 + \dots + b_nu_n) = a_1w_1 + \dots + a_mw_m$ .

Now we show that  $\text{null } T = X$  and  $\text{range } T = Y$

Suppose  $v \in V$ . Then  $\exists! a_i, b_j \in \mathbf{F}, v = a_1v_1 + \dots + a_mv_m + b_1u_1 + \dots + b_nu_n$ .

$$\left. \begin{array}{l} v \in \text{null } T \Rightarrow Tv = 0 \\ \Rightarrow a_1 = \dots = a_m = 0 \\ \Rightarrow v \in X \Rightarrow \text{null } T \subseteq X. \\ v \in X \Rightarrow v \in \text{null } T \Rightarrow \text{null } T \supseteq X. \end{array} \right\} \Rightarrow \text{null } T = X.$$

$$\left. \begin{array}{l} w \in \text{range } T \Rightarrow \exists v \in V, Tv = w \Rightarrow \text{let } v = a_1v_1 + \dots + a_mv_m + b_1u_1 + \dots + b_nu_n \\ \Rightarrow Tv = w = a_1w_1 + \dots + a_mw_m \Rightarrow w \in Y \Rightarrow \text{range } T \subseteq Y. \\ w \in Y \Rightarrow w = a_1Tv_1 + \dots + a_mTv_m = T(a_1v_1 + \dots + a_mv_m) \\ \Rightarrow w \in \text{range } T \Rightarrow \text{range } T \supseteq Y. \end{array} \right\} \Rightarrow \text{range } T = Y.$$

(b) Conversely it is true as well.

□

- Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V, W)$ . Let  $(Tv_1, \dots, Tv_n)$  be a basis of  $\text{range } T$ . Extend  $(v_1, \dots, v_n)$  to a basis of  $V$  as  $(v_1, \dots, v_n, u_1, \dots, u_m)$ . Prove or give a counterexample:  $(u_1, \dots, u_m)$  is a basis of  $\text{null } T$ .

**SOLUTION:** An counterexample:

Suppose  $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$ . Then  $\text{span}(Tv_1, Tv_2, Tv_3) = \text{span}(w_1)$ .

Extend  $(v_i)$  to  $(v_1, v_2, v_3)$  for each  $i$ . But none of  $(v_1, v_2), (v_1, v_3), (v_2, v_3)$  is a basis of  $\text{null } T$ .

- Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V, W)$ . Let  $(u_1, \dots, u_m)$  be a basis of  $\text{null } T$ . Extend  $(u_1, \dots, u_m)$  to a basis of  $V$  as  $(u_1, \dots, u_m, v_1, \dots, v_n)$ . Prove or give a counterexample:  $(Tv_1, \dots, Tv_n)$  spans  $\text{range } T$ .

**SOLUTION:**

$\forall w \in \text{range } T, \exists v \in V, (\exists! a_i, b_i \in \mathbf{F}), Tv = T(a_1v_1 + \dots + a_nv_n) = w$

$\Rightarrow w \in \text{span}(Tv_1, \dots, Tv_n) \Rightarrow \text{range } T \subseteq \text{span}(Tv_1, \dots, Tv_n). \quad \square$

COMMENT: If  $T$  is injective, then  $(Tv_1, \dots, Tv_n)$  is a basis of  $\text{range } T$ .

- Suppose  $V$  is finite-dim with  $\dim V > 1$ . Show that if  $\varphi : \mathcal{L}(V) \rightarrow \mathbf{F}$  is a linear map such that  $\varphi(ST) = \varphi(S) \cdot \varphi(T)$  for all  $S, T \in \mathcal{L}(V)$ , then  $\varphi = 0$ .  
HINT: The description of the two-sided ideals of  $\mathcal{L}(V)$  in Section 3A might be useful.

**SOLUTION:** Using notations in (3.A.● the last).

Suppose  $\varphi \neq 0 \Rightarrow \exists i, j \in \{1, \dots, n\}, \varphi(R_{i,j}) \neq 0$ .

Because  $R_{i,j} = R_{x,j} \circ R_{i,x}, \forall x = 1, \dots, n$

$$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0 \text{ and } \varphi(R_{i,x}) \neq 0.$$

Again, because  $R_{i,x} = R_{y,x} \circ R_{i,y}, \forall y = 1, \dots, n$ . Thus  $\varphi(R_{y,x}) \neq 0$  for any  $x, y = 1, \dots, n$ .

Let  $l \neq i, k \neq j$  and then  $\varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$$\Rightarrow \varphi(R_{l,k}) = 0 \text{ or } \varphi(R_{i,j}) = 0. \text{ Contradicts. } \square$$

- Suppose that  $V$  and  $W$  are real vector spaces and  $T \in \mathcal{L}(V, W)$ .

Define  $T_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$  by  $T_{\mathbb{C}}(u + iv) = Tu + iTv$  for all  $u, v \in V$ .

(a) Show that  $T_{\mathbb{C}}$  is a (complex) linear map from  $V_{\mathbb{C}}$  to  $W_{\mathbb{C}}$ .

(b) Show that  $T_{\mathbb{C}}$  is injective  $\iff T$  is injective.

(c) Show that  $\text{range } T_{\mathbb{C}} = W_{\mathbb{C}} \iff \text{range } T = W$ .

See Exercise 8 in Section 1B for the definition of the complexification  $V_{\mathbb{C}}$ .

The linear map  $T_{\mathbb{C}}$  is called the complexification of the linear map  $T$ .

**SOLUTION:**

(a)  $\forall u_1 + iv_1, u_2 + iv_2 \in V_{\mathbb{C}}, \lambda \in \mathbf{F},$

$$\begin{aligned} T((u_1 + iv_1) + \lambda(u_2 + iv_2)) &= T((u_1 + \lambda u_2) + i(v_1 + \lambda v_2)) = T(u_1 + \lambda u_2) + iT(v_1 + \lambda v_2) \\ &= Tu_1 + iTv_1 + \lambda Tu_2 + i\lambda Tv_2 = T(u_1 + iv_1) + \lambda T(u_2 + iv_2). \quad \square \end{aligned}$$

(b)  $\left. \begin{array}{l} \text{Suppose } T_{\mathbb{C}} \text{ is injective. Let } T(u) = 0 \Rightarrow T_{\mathbb{C}}(u + i0) = Tu = 0 \Rightarrow u = 0. \\ \text{Suppose } T \text{ is injective. Let } T_{\mathbb{C}}(u + iv) = Tu + iTv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + iv = 0. \end{array} \right\} \Rightarrow \square$

$\left. \begin{array}{l} \text{Suppose } T_{\mathbb{C}} \text{ is surjective. } \forall w, x \in W, \exists u, v \in V, T(u + iv) = Tu + iTv = w + ix \\ \Rightarrow Tu = w, Tv = x \Rightarrow T \text{ is surjective.} \end{array} \right\}$

(c)  $\left. \begin{array}{l} \text{Suppose } T \text{ is surjective. } \forall w, x \in W, \exists u, v \in V, Tu = w, Tv = x \\ \Rightarrow \forall w + ix \in W_{\mathbb{C}}, \exists u + iv \in V, T(u + iv) = w + ix \Rightarrow T_{\mathbb{C}} \text{ is surjective.} \end{array} \right\} \Rightarrow \square$

**ENDED**

### 3.C

• **NOTE FOR [3.47]:**  $LHS = (AC)_{j,k} = \sum_{r=1}^n A_{j,r} C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r} (C_{\cdot,k})_{r,1} = (A_{j,\cdot} C_{\cdot,k})_{1,1} = A_{j,\cdot} C_{\cdot,k} = RHS.$

• **NOTE FOR [3.48]:**

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix} \\ = \begin{pmatrix} 1 \times 5 + 2 \times 8 & 1 \times 6 + 2 \times 9 & 1 \times 7 + 2 \times 10 \\ 3 \times 5 + 4 \times 8 & 3 \times 6 + 4 \times 9 & 3 \times 7 + 4 \times 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

• **NOTE FOR [3.49]:**  $\because [(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r} C_{r,k} = \sum_{r=1}^n A_{j,r} (C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$   
 $\therefore (AC)_{\cdot,k} = A_{\cdot,\cdot} C_{\cdot,k} = AC_{\cdot,k}$

• **EXERCISE 10:**  $\because [(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r} C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r} C_{r,k} = (A_{j,\cdot} C)_{1,k}$   
 $\therefore (AC)_{j,\cdot} = A_{j,\cdot} C_{\cdot,\cdot} = A_{j,\cdot} C.$

• **Suppose**  $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}.$

(a) For  $k = 1, \dots, p,$   $(CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot} R_{\cdot,k} = \sum_{r=1}^c C_{\cdot,r} R_{r,k} = R_{1,k} C_{\cdot,1} + \dots + R_{c,k} C_{\cdot,c}$

(b) For  $j = 1, \dots, m,$   $(CR)_{j,\cdot} = C_{j,\cdot} R = C_{j,\cdot} R_{\cdot,\cdot} = \sum_{r=1}^c C_{j,r} R_{r,\cdot} = C_{j,1} R_{1,\cdot} + \dots + C_{j,c} R_{c,\cdot}$

EXAMPLE:

$$P = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} \in \mathbf{F}^{2,3}.$$

$$P_{\cdot,1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 8 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 21 \\ 47 \end{pmatrix};$$

$$P_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix};$$

$$P_{\cdot,3} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 10 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 27 \\ 61 \end{pmatrix};$$

$$P_{1,\cdot} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = 1 \begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 2 \begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \end{pmatrix};$$

$$P_{2,\cdot} = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = 3 \begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 4 \begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 47 & 54 & 61 \end{pmatrix};$$

• **NOTE FOR [3.52]:**  $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow AC \in \mathbf{F}^{m,1}$

$$\because (Ac)_{j,1} = \sum_{r=1}^n A_{j,r} c_{r,1} = [\sum_{r=1}^n (A_{\cdot,r} c_{r,1})]_{j,1} = (c_1 A_{\cdot,1} + \dots + c_n A_{\cdot,n})_{j,1}$$

$$\therefore Ac = A_{\cdot,\cdot} c_{\cdot,1} = \sum_{r=1}^n A_{\cdot,r} c_{r,1} = c_1 A_{\cdot,1} + \dots + c_n A_{\cdot,n} \quad \text{OR. By } (Ac)_{\cdot,1} = Ac_{\cdot,1} \text{ Using (a) above.}$$

• **EXERCISE 10:**  $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$

$$\because (aC)_{1,k} = \sum_{r=1}^n a_{1,r} C_{r,k} = [\sum_{r=1}^n a_{1,r} (C_{r,\cdot})]_{1,k} = (a_1 C_{1,\cdot} + \dots + a_n C_{n,\cdot})_{1,k}$$

$$\therefore aC = a_{1,\cdot} C_{\cdot,\cdot} = \sum_{r=1}^n a_{1,r} C_{r,\cdot} = a_1 C_{1,\cdot} + \dots + a_n C_{n,\cdot} \quad \text{OR. By } (aC)_{1,\cdot} = a_{1,\cdot} C. \text{ Using (b) above.}$$

• COLUMN-ROW FACTORIZATION (CR Factorization)

Suppose  $A \in \mathbf{F}^{m,n}$ ,  $A \neq 0$ . Let  $S_c = \text{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}$ ,  $\dim S_c = c$ .

And  $S_r = \text{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) \subseteq \mathbf{F}^{1,n}$ ,  $\dim S_r = r$ .

Prove that  $A = CR$ .  $\exists C \in \mathbf{F}^{m,c}$ ,  $R \in \mathbf{F}^{c,n}$ .

**SOLUTION:** Notice that  $A \neq 0 \Rightarrow c, r \geq 1$ .

Let  $(C_{\cdot,1}, \dots, C_{\cdot,c})$  be a basis of  $S_c$ , forming  $C \in \mathbf{F}^{m,c}$ .

Then for any  $A_{\cdot,k}$ ,  $A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \dots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$ ,  $\exists! R_{1,k}, \dots, R_{c,k} \in \mathbf{F}$ .

Hence, by letting  $R = \begin{pmatrix} R_{1,1} & \dots & R_{1,n} \\ \vdots & \ddots & \vdots \\ R_{c,1} & \dots & R_{c,n} \end{pmatrix}$ , we have  $A = CR$ .

OR. Let  $(R_{1,\cdot}, \dots, R_{c,\cdot})$  be a basis of  $S_r$ , forming  $R \in \mathbf{F}^{c,n}$ .

For any  $A_{j,\cdot}$ ,  $A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \dots + C_{j,c}R_{c,\cdot} = (CR)_{j,\cdot}$ ,  $\exists! C_{j,1}, \dots, C_{j,c} \in \mathbf{F}$ . Similarly.  $\square$

**EXAMPLE:**

$$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \stackrel{(1)}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \stackrel{(2)}{=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

(1) Because  $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2 \begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix}$ .

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix}$  can be uniquely written as a linear combination of  $A_{1,\cdot}, A_{2,\cdot}$ .

Hence  $\dim S_r = 2$ . We choose  $(A_{1,\cdot}, A_{2,\cdot})$  as the basis.

$$(2) \text{ Because } \begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2 \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}; \quad \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix} - \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix}.$$

Hence  $\dim S_c = 2$ . We choose  $(A_{\cdot,2}, A_{\cdot,3})$  as the basis.

• COLUMN RANK EQUALS ROW RANK (Using the notation above)

For any  $A_{j,\cdot} \in S_r$ ,  $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R_{1,\cdot} + \dots + C_{j,c}R_{c,\cdot}$ .

$\Rightarrow \text{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = S_r = \text{span}(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leq c = \dim S_c$ .

Apply the result to  $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leq r = \dim S_r = \dim S_c^t$ .  $\square$

• Suppose  $T \in \mathcal{L}(V)$ , and  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  are bases of  $V$ .

Prove that the following are equivalent.

(a)  $T$  is injective.

(b) The columns of  $\mathcal{M}(T)$  are linearly independent in  $\mathbf{F}^{n,1}$ .

(c) The columns of  $\mathcal{M}(T)$  span  $\mathbf{F}^{n,1}$ .

(d) The rows of  $\mathcal{M}(T)$  span  $\mathbf{F}^{1,n}$ .

(e) The rows of  $\mathcal{M}(T)$  are linearly independent in  $\mathbf{F}^{1,n}$ .

Here  $A = \mathcal{M}(T)$  means  $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$ .

**SOLUTION:**

$T$  is injective  $\iff \dim V = \dim \text{range } T + \dim \text{null } T = \dim \text{range } T$

$\iff (Tu_1, \dots, Tu_n)$  is linearly independent in  $V$ , and therefore is a basis of  $V$

$\iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n))$  is linearly independent, as well as  $(A_{\cdot,1}, \dots, A_{\cdot,n})$

$\iff (A_{\cdot,1}, \dots, A_{\cdot,n})$  is a basis of  $\mathbf{F}^{n,1}$ .

$\left( \text{又 } \dim \text{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) = \dim \text{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = n \right)$

$\iff (A_{1,\cdot}, \dots, A_{n,\cdot})$  is a basis of  $\mathbf{F}^{1,n}$ .  $\square$



• Suppose  $A$  is an  $m$ -by- $n$  matrix with  $A \neq 0$ .

Prove that the rank of  $A$  is 1 if and only if there exist  $(c_1, \dots, c_m) \in \mathbf{F}^m$  and  $(d_1, \dots, d_n) \in \mathbf{F}^n$  such that  $A_{j,k} = c_j \cdot d_k$  for every  $j = 1, \dots, m$  and every  $k = 1, \dots, n$ .

**SOLUTION:** Using the notation in CR Factorization.

$$(a) \text{ Suppose } A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix} \cdot (\exists c_j, d_k \in \mathbf{F}, \forall j, k)$$

$$\text{Then } S_c = \text{span} \left( \begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \dots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right) = \text{span} \left( \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right).$$

$$\text{OR. } S_r = \text{span} \left( \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ c_2 d_1 & \cdots & c_2 d_n \\ \vdots & & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix} \right) = \text{span} \left( \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} \right). \quad \text{Hence the rank of } A \text{ is 1.}$$

(b) Suppose the rank of  $A$  is  $\dim S_c = \dim S_r = 1$

$$\text{Let } c_j = \frac{A_{j,1}}{A_{1,1}} = \frac{A_{j,2}}{A_{1,2}} = \cdots = \frac{A_{j,n}}{A_{1,n}}, \quad d'_k = \frac{A_{1,k}}{A_{1,1}} = \frac{A_{2,k}}{A_{2,1}} = \cdots = \frac{A_{m,k}}{A_{m,1}} \quad (\forall j, k)$$

$$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k. \text{ Letting } d_k = d'_k A_{1,1}. \quad \square$$

**1** Suppose  $T \in \mathcal{L}(V, W)$ . Show that with respect to each choice of bases of  $V$  and  $W$ , the matrix of  $T$  has at least  $\dim \text{range } T$  nonzero entries.

**SOLUTION:** Let  $(v_1, \dots, v_n)$  and  $(w_1, \dots, w_m)$  be bases of  $V$  and  $W$  respectively. We prove by contradiction.

Suppose  $A = \mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$  has at most  $(\dim \text{range } T - 1)$  nonzero entries.

Then by Pigeon Hole Principle, at least one of  $A_{\cdot, k} = 0$ .

Thus there are at most  $(\dim \text{range } T - 1)$  nonzero vectors in  $Tv_1, \dots, Tv_n$ .

While  $\text{range } T = \text{span}(Tv_1, \dots, Tv_n) \Rightarrow \dim \text{range } T \leq \dim \text{range } T - 1$ . Hence we get a contradiction.  $\square$

**3** Suppose  $V$  and  $W$  are finite-dim and  $T \in \mathcal{L}(V, W)$ .

Prove that there exist a basis of  $V$  and a basis of  $W$  such that

[ letting  $A = \mathcal{M}(T)$  with respect to these bases ],

$A_{k,k} = 1, A_{i,j} = 0$ , where  $1 \leq k \leq \dim \text{range } T, i \neq j$ .

**SOLUTION:**

Let  $R = (Tv_1, \dots, Tv_n)$  be a basis of  $\text{range } T$ , extend it to the basis of  $W$  as  $(Tv_1, \dots, Tv_n, w_1, \dots, w_p)$ .

Let  $\mathcal{K}_R = \text{span}(v_1, \dots, v_n)$ . Let  $(u_1, \dots, u_m)$  be a basis of  $\text{null } T$ .

Then  $(v_1, \dots, v_n, u_1, \dots, u_m)$  is the basis of  $V$ .

Thus  $T(v_k) = Tv_k, T(u_j) = 0 \Rightarrow A_{k,k} = 1, A_{i,j} = 0$  for each  $k \in \{1, \dots, \dim \text{range } T\}$  and  $j \in \{1, \dots, m\}$ .  $\square$

**4** Suppose  $(v_1, \dots, v_m)$  is a basis of  $V$  and  $W$  is finite-dim. Suppose  $T \in \mathcal{L}(V, W)$ .

Prove that there exists a basis  $(w_1, \dots, w_n)$  of  $W$  such that

all entries in the first column of  $A = \mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_n))$  are 0 except for possibly a 1 in the first row, first column.

**SOLUTION:** If  $Tv_1 = 0$ , then we are done. Otherwise, extend  $(Tv_1)$  to a basis of  $W$ , as desired.  $\square$

**5** Suppose  $(w_1, \dots, w_n)$  is a basis of  $W$  and  $V$  is finite-dim. Suppose  $T \in \mathcal{L}(V, W)$ . Prove that there exists a basis  $(v_1, \dots, v_m)$  of  $V$  such that all entries in the first row of  $\mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_n))$  are 0 except for possibly a 1 in the first row, first column.

**SOLUTION:**

Let  $(u_1, \dots, u_m)$  be a basis of  $V$ . If  $A_{1,\cdot} = 0$ , then let  $v_i = u_i$  for each  $i = 1, \dots, m$ , we are done.

Otherwise,  $\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \end{pmatrix} \neq 0$ , choose one  $A_{1,k} \neq 0$ .

Let  $v_1 = \frac{u_k}{A_{1,k}}$ ;  $v_j = u_{j-1} - A_{1,j-1}v_1$  for  $j = 2, \dots, k$ ;  
 $v_i = u_i - A_{1,i}v_1$  for  $i = k+1, \dots, m$ .  $\square$

**6** Suppose  $V$  and  $W$  are finite-dim and  $T \in \mathcal{L}(V, W)$ . Prove that  $\dim \text{range } T = 1$  if and only if there exist a basis of  $V$  and a basis of  $W$  such that with respect to these bases, all entries of  $A = \mathcal{M}(T)$  equal 1.

**SOLUTION:**

Denote the bases of  $V$  and  $W$  by  $B_V = (v_1, \dots, v_n)$  and  $B_W = (w_1, \dots, w_m)$  respectively.

(a) Suppose  $B_V, B_W$  are the bases such that all entries of  $A$  equal 1.

Then  $Tv_i = w_1 + \cdots + w_m$  for all  $i = 1, \dots, n$ . Hence  $\dim \text{range } T = 1$ .

(b) Suppose  $\dim \text{range } T = 1$ . Then  $\dim \text{null } T = \dim V - 1$ .

Let  $(u_2, \dots, u_n)$  be a basis of  $\text{null } T$ . Extend it to a basis of  $V$  as  $(u_1, u_2, \dots, u_n)$ .

Let  $w_1 = Tv_1 - w_2 - \cdots - w_m$ . Extend it to  $B_W$  the basis of  $W$ .

Let  $v_1 = u_1, v_i = u_i + u_1$ . Extend it to  $B_V$  the basis of  $V$ .  $\square$

**12** Give an example of 2-by-2 matrices  $A$  and  $B$  such that  $AB \neq BA$ .

**SOLUTION:**  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

**13** Prove that the distributive property holds for matrix addition and matrix multiplication.

In other words, suppose  $A, B, C, D, E$  and  $F$  are matrices

whose sizes are such that  $A(B + C)$  and  $(D + E)F$  make sense.

Explain why  $AB + AC$  and  $DF + EF$  both make sense and prove that.

**SOLUTION:** Using [3.36], [3.43].

(a) Left distributive: Suppose  $A \in \mathbf{F}^{m,n}$  and  $B, C \in \mathbf{F}^{n,p}$ .

Because  $[A(B + C)]_{j,k} = \sum_{r=1}^n A_{j,r}(B + C)_{r,k} = \sum_{r=1}^n (A_{j,r}B_{r,k} + A_{j,r}C_{r,k})$ .

Hence we conclude that  $A(B + C) = AB + AC$ .

OR. Let  $(e_1, \dots, e_M)$  be the standard basis of  $\mathbf{F}^M$ , where  $M = \max\{m, n, p\}$ .

Suppose  $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$  such that  $Te_k = \sum_{j=1}^m A_{j,k}e_j$  for each  $k = 1, \dots, n$ . Then  $\mathcal{M}(T) = A$ .

Similarly, define  $S, R$  such that  $\mathcal{M}(S) = B, \mathcal{M}(R) = C$ .

Thus  $T(S + R) = TS + TR$   $\left| \begin{array}{l} \Rightarrow \mathcal{M}(T(S + R)) = \mathcal{M}(TS + TR) \\ \Rightarrow \mathcal{M}(T)[\mathcal{M}(S) + \mathcal{M}(R)] = \mathcal{M}(T)\mathcal{M}(S) + \mathcal{M}(T)\mathcal{M}(R) \\ \Rightarrow A(B + C) = AB + AC. \end{array} \right.$

Suppose  $\mathcal{M}(T) = D, \mathcal{M}(S) = E, \mathcal{M}(R) = F$ .

Then  $(T + S)R = TR + SR$

(b) Right distributive: Similarly.  $\left| \begin{array}{l} \Rightarrow \mathcal{M}((T + S)R) = \mathcal{M}(TR) + \mathcal{M}(SR) \\ \Rightarrow [\mathcal{M}(T) + \mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)\mathcal{M}(R) + \mathcal{M}(S)\mathcal{M}(R) \\ \Rightarrow (D + E)F = DF + EF. \end{array} \right. \square$

**14** Prove that matrix multiplication is associative. In other words, suppose  $A, B$  and  $C$  are matrices whose sizes are such that  $(AB)C$  makes sense. Explain why  $A(BC)$  makes sense and prove that  $(AB)C = A(BC)$ .

Try to find a clean proof that illustrates the following quote from Emil Artin:

“It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.”

**SOLUTION:**

$$\text{Because } [(AB)C]_{j,k} = (AB)_{j,\cdot} C_{\cdot,k} = \sum_{s=1}^n (A_{j,s} B_{s,\cdot}) C_{\cdot,k} = \sum_{s=1}^n A_{j,s} (B_{s,\cdot} C_{\cdot,k}) = \sum_{s=1}^n A_{j,s} (BC)_{s,k} = A(BC)_{j,k}$$

Hence we conclude that  $(AB)C = A(BC)$ .

OR. Suppose  $A \in \mathbf{F}^{m,n}, B \in \mathbf{F}^{n,p}, C \in \mathbf{F}^{p,s}$ .

Let  $(e_1, \dots, e_M)$  be the standard basis of  $\mathbf{F}^M$ , where  $M = \max\{m, n, p, s\}$ .

Suppose  $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$  such that  $Te_k = \sum_{j=1}^m A_{j,k} e_j$  for each  $k = 1, \dots, n$ . Then  $\mathcal{M}(T) = A$ .

Similarly, define  $S, R$  such that  $\mathcal{M}(S) = B, \mathcal{M}(R) = C$ .

$$\begin{aligned} \text{Hence } (TS)R = T(SR) &\Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR)) \\ &\Rightarrow [\mathcal{M}(T)\mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)[\mathcal{M}(S)\mathcal{M}(R)] \\ &\Rightarrow (AB)C = A(BC). \quad \square \end{aligned}$$

**15** Suppose  $A$  is an  $n$ -by- $n$  matrix and  $1 \leq j, k \leq n$ .

Show that the entry in row  $j$ , column  $k$ , of  $A^3$

(which is defined to mean  $AAA$ ) is  $\sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}$ .

$$\text{SOLUTION: } (AAA)_{j,k} = (AA)_{j,\cdot} A_{\cdot,k} = \sum_{p=1}^n (A_{j,p} A_{p,\cdot}) A_{\cdot,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}.$$

$$\begin{aligned} \text{OR. } (AAA)_{j,k} &= \sum_{r=1}^n (AA)_{j,r} A_{r,k} = \sum_{r=1}^n \left( \sum_{p=1}^n A_{j,p} A_{p,r} \right) A_{r,k} \\ &= \sum_{r=1}^n (A_{j,1} A_{1,r} A_{r,k} + \dots + A_{j,n} A_{n,r} A_{r,k}) \\ &= A_{j,1} \sum_{r=1}^n A_{1,r} A_{r,k} + \dots + A_{j,n} \sum_{r=1}^n A_{n,r} A_{r,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}. \quad \square \end{aligned}$$

**ENDED**

### 3.D

• Suppose  $T \in \mathcal{L}(V, W)$  is invertible. Show that  $T^{-1}$  is invertible and  $(T^{-1})^{-1} = T$ .

**SOLUTION:**

$$\left. \begin{array}{l} TT^{-1} = (T^{-1})^{-1}T^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = T^{-1}(T^{-1})^{-1} = I \in \mathcal{L}(W) \end{array} \right\} \Rightarrow T = (T^{-1})^{-1}, \text{ by the uniqueness of inverse. } \square$$

1 Suppose  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$  are both invertible linear maps.

Prove that  $ST \in \mathcal{L}(U, W)$  is invertible and that  $(ST)^{-1} = T^{-1}S^{-1}$ .

**SOLUTION:**

$$\left. \begin{array}{l} (ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(U) \end{array} \right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}, \text{ by the uniqueness of inverse. } \square$$

9 Suppose  $V$  is finite-dim and  $S, T \in \mathcal{L}(V)$ .

Prove that  $ST$  is invertible  $\iff S$  and  $T$  are invertible.

**SOLUTION:**

Suppose  $S, T$  are invertible. Then  $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$ . Hence  $ST$  is invertible.

Suppose  $ST$  is invertible. Let  $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$ .

$$\left. \begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \text{range } S \end{array} \right\} \Rightarrow T \text{ is injective, } S \text{ is surjective.}$$

Notice that  $V$  is finite-dim. Hence  $S, T$  are invertible.  $\square$

10 Suppose  $V$  is finite-dim and  $S, T \in \mathcal{L}(V)$ . Prove that  $ST = I \iff TS = I$ .

**SOLUTION:**

Suppose  $ST = I$ .

$$\left. \begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \text{range } S \end{array} \right\} \Rightarrow T \text{ is injective, } S \text{ is surjective.}$$

Notice that  $V$  is finite-dim. Thus  $T, S$  are invertible.

OR. By Problem (9),  $V$  is finite-dim and  $ST = I$  is invertible  $\Rightarrow S, T$  are invertible.

$$S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v \text{ ( } S \text{ is invertible )}.$$

$$\text{OR. } ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T. \text{ 又 } S = S \Rightarrow TS = S^{-1}S = I.$$

Reversing the roles of  $S$  and  $T$ , we conclude that  $TS = I \Rightarrow ST = I$ .  $\square$

11 Suppose  $V$  is finite-dim and  $S, T, U \in \mathcal{L}(V)$  and  $STU = I$ .

Show that  $T$  is invertible and that  $T^{-1} = US$ .

**SOLUTION:** Using Problem (9) and (10).

$$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I.$$

$$\Rightarrow U^{-1} = ST, \quad T^{-1} = US, \quad S^{-1} = TU. \quad \square$$

12 Show that the result in Exercise 11 can fail without the hypothesis that  $V$  is finite-dim.

**SOLUTION:**

$$\text{Let } V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots), T(a_1, \dots) = (0, a_1, \dots), U = I.$$

Then  $STU = I$  but  $T^{-1}$  is not invertible.

**13** Suppose  $V$  is finite-dim and  $R, S, T \in \mathcal{L}(V)$  are such that  $RST$  is surjective.

Prove that  $S$  is injective.

**SOLUTION:**

By Problem (1) and (9), Notice that  $V$  is finite-dim. Then  $RST$  is invertible.

$$(RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}. \quad \square$$

OR. Let  $X = (RST)^{-1}$ ,  $\left\{ \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is injective, and therefore is invertible.} \\ \forall v \in V, v = (RST)Xv \in \text{range } R \Rightarrow R \text{ is surjective, and therefore is invertible.} \end{array} \right.$

Thus  $S = R^{-1}(RST)T^{-1}$  is invertible.

**15** Prove that every linear map from  $\mathbf{F}^{n,1}$  to  $\mathbf{F}^{m,1}$  is given by a matrix multiplication.

In other words, prove that if  $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$ , then  $\exists A \in \mathbf{F}^{m,n}, Tx = Ax, \forall x \in \mathbf{F}^{n,1}$ .

**SOLUTION:**

Let  $E_i \in \mathbf{F}^{n,1}$  for each  $i = 1, \dots, n$  (where  $M = \max\{m, n\}$ ) be such that  $(E_i)_{j,1} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Then  $(E_1, \dots, E_n)$  is linearly independent and thus is a basis of  $\mathbf{F}^{n,1}$ .

Similarly, let  $(R_1, \dots, R_m)$  be a basis of  $\mathbf{F}^{m,1}$ .

Suppose  $T(E_i) = A_{1,i}R_1 + \dots + A_{m,i}R_m$  for each  $i = 1, \dots, n$ . Hence by letting  $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$ .  $\square$

COMMENT:  $\mathcal{M}(T) = A$ . Conversely it is true as well.

• OR (10.A.2) Suppose  $A, B \in \mathbf{F}^{n,n}$ . Prove that  $AB = I \iff BA = I$ .

**SOLUTION:** Using Problem (10) and (15).

Define  $T, S \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1})$  by  $Tx = Ax, Sx = Bx$  for all  $x \in \mathbf{F}^{n,1}$ . Then  $\mathcal{M}(T) = A, \mathcal{M}(S) = B$ .

Thus  $AB = I \iff A(Bx) = x \iff T(Sx) = x \iff TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I. \square$

• **NOTE FOR [3.60]:** Suppose  $(v_1, \dots, v_n)$  is a basis of  $V$  and  $(w_1, \dots, w_m)$  is a basis of  $W$ .

Define  $E_{i,j} \in \mathcal{L}(V, W)$  by  $E_{i,j}(v_x) = \delta_{ix}w_j$ ;  $\delta_{ix} = \begin{cases} 0, & i \neq x \\ 1, & i = x \end{cases}$  COROLLARY:  $E_{l,k}E_{i,j} = \delta_{jl}E_{i,k}$ .

Denote  $\mathcal{M}(E_{i,j})$  by  $\mathcal{E}^{(j,i)}$ .  $(\mathcal{E}^{(j,i)})_{l,k} = \begin{cases} 0, & i \neq k \text{ or } j \neq l \\ 1, & i = k \text{ and } j = l \end{cases}$

Because  $\mathcal{L}(V, W)$  and  $\mathbf{F}^{m,n}$  are isomorphic. And  $T = \mathcal{M}^{-1}\mathcal{M}(T), E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$ .

Hence  $\forall T \in \mathcal{L}(V, W), \exists! A_{i,j} \in \mathbf{F} (\forall i = \{1, \dots, m\}, j = \{1, \dots, n\}), \mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$ .

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)} + & \cdots & +A_{1,n}\mathcal{E}^{(1,n)} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}\mathcal{E}^{(m,1)} + & \cdots & +A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff T = \begin{pmatrix} A_{1,1}E_{1,1} + & \cdots & +A_{1,n}E_{n,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}E_{1,m} + & \cdots & +A_{m,n}E_{n,m} \end{pmatrix}.$$

$$\therefore \mathcal{L}(V, W) = \text{span} \underbrace{\begin{pmatrix} E_{1,1}, & \cdots, & E_{n,1} \\ \vdots & & \vdots \\ E_{1,m}, & \cdots, & E_{n,m} \end{pmatrix}}_B; \quad \mathbf{F}^{m,n} = \text{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots, & \mathcal{E}^{(1,n)} \\ \vdots & & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots, & \mathcal{E}^{(m,n)} \end{pmatrix}}_{B_M}.$$

Hence by [2.42] and [3.61], we conclude that  $B$  is a basis of  $\mathcal{L}(V, W)$  and that  $B_M$  is a basis of  $\mathbf{F}^{m,n}$ .

◦ Suppose  $V$  is finite-dim and  $S \in \mathcal{L}(V)$ . Define  $A \in \mathcal{L}(\mathcal{L}(V))$  by  $A(T) = ST$  for  $T \in \mathcal{L}(V)$ .

(a) Show that  $\dim \text{null } A = (\dim V)(\dim \text{null } S)$ .

(b) Show that  $\dim \text{range } A = (\dim V)(\dim \text{range } S)$ .

**SOLUTION:** Using NOTE FOR [3.60].

Let  $(w_1, \dots, w_m)$  be a basis of range  $S$ , extend it to a basis of  $V$  as  $(w_1, \dots, w_m, \dots, w_n)$ .

Let  $v_i \in V$  such that  $Sv_i = w_i$  for  $m = 1, \dots, m$ . Extend  $(v_1, \dots, v_m)$  to a basis of  $V$  as  $(v_1, \dots, v_m, \dots, v_n)$ .

Define  $E_{i,j} \in \mathcal{L}(V)$  by  $E_{i,j}(v_x) = \delta_{ix}w_i$ .

$$\text{Thus } S = E_{1,1} + \dots + E_{m,m}; \quad \mathcal{M}(S, (v_1, \dots, v_n), (w_1, \dots, w_n)) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Define  $R_{i,j} \in \mathcal{L}(V)$  by  $R_{i,j}(w_x) = \delta_{ix}v_i$ .

Let  $E_{j,k}R_{i,j} = Q_{i,k}$ ,  $R_{j,k}E_{i,j} = G_{i,k}$

$$\text{Because } \forall T \in \mathcal{L}(V), \quad \exists! A_{i,j} \in \mathbf{F} (\forall i, j = 1, \dots, n), \quad T = \begin{pmatrix} A_{1,1}R_{1,1} + & \dots & +A_{1,m}R_{m,1} + & \dots & +A_{1,n}R_{n,1} \\ + & \dots & + & \dots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \dots & + & \dots & + \\ A_{m,1}R_{1,m} + & \dots & +A_{m,m}R_{m,m} + & \dots & +A_{m,n}R_{n,m} \\ + & \dots & + & \dots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \dots & + & \dots & + \\ A_{n,1}R_{1,n} + & \dots & +A_{n,m}R_{m,n} + & \dots & +A_{n,n}R_{n,n} \end{pmatrix}.$$

$$\Rightarrow A(T) = ST = \left( \sum_{r=1}^m E_{r,r} \right) \left( \sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \right) = \sum_{i=1}^m \sum_{j=1}^n A_{i,j} Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1} + & \dots & +A_{1,m}Q_{m,1} + & \dots & +A_{1,n}Q_{n,1} \\ + & \dots & + & \dots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \dots & + & \dots & + \\ A_{m,1}Q_{1,m} + & \dots & +A_{m,m}Q_{m,m} + & \dots & +A_{m,n}Q_{n,m} \end{pmatrix}.$$

$$\text{Thus } \text{null } A = \text{span} \begin{pmatrix} R_{1,m+1}, & \dots, & R_{n,m+1}, \\ \vdots & \ddots & \vdots \\ R_{1,n}, & \dots, & R_{n,n} \end{pmatrix}, \quad \text{range } A = \text{span} \begin{pmatrix} Q_{1,1}, & \dots, & Q_{n,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,m}, & \dots, & Q_{n,m} \end{pmatrix}.$$

Hence (a)  $\dim \text{null } A = n \times (n - m)$ ; (b)  $\dim \text{range } A = n \times m$ .  $\square$

• COMMENT: Define  $B \in \mathcal{L}(\mathcal{L}(V))$  by  $B(T) = TS$  for  $T \in \mathcal{L}(V)$ .

$$\text{Similarly, } B(T) = TS = \left( \sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \right) \left( \sum_{r=1}^m E_{r,r} \right) = \sum_{i=1}^n \sum_{j=1}^m A_{i,j} G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1} + & \dots & +A_{1,m}G_{m,1} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{m,1}G_{1,m} + & \dots & +A_{m,m}G_{m,m} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{n,1}G_{1,n} + & \dots & +A_{n,m}G_{m,n} \end{pmatrix}.$$

• OR (10.A.1) Suppose  $T \in \mathcal{L}(V)$  and  $(v_1, \dots, v_n)$  is a basis of  $V$ .

Prove that  $\mathcal{M}(T, (v_1, \dots, v_n))$  is invertible  $\iff T$  is invertible.

**SOLUTION:** Notice that  $\mathcal{M}$  is an isomorphism of  $\mathcal{L}(V)$  onto  $\mathbf{F}^{n,n}$ .

(a)  $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$ .

(b)  $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I$ .  $\exists! S \in \mathcal{L}(V)$  such that  $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$

$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$

$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$ .  $\square$

- OR (10.A.4) Suppose that  $(u_1, \dots, u_n)$  and  $(v_1, \dots, v_n)$  are bases of  $V$ .

Let  $T \in \mathcal{L}(V)$  be such that  $Tv_k = u_k$  for each  $k = 1, \dots, n$ .

Prove that  $A = \mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)) = B$ .

**SOLUTION:**

$$\forall k \in \{1, \dots, n\}, Iu_k = u_k = B_{1,k}v_1 + \dots + B_{n,k}v_n = Tv_k = A_{1,k}v_1 + \dots + A_{n,k}v_n \Rightarrow A = B. \quad \square$$

OR. Note that  $\mathcal{M}(T, (v_1, \dots, v_n), (u_1, \dots, u_n))$  is the identity matrix.

$$A = \mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)) \underbrace{\mathcal{M}(T, (v_1, \dots, v_n), (u_1, \dots, u_n))}_{=I} = B. \quad \square$$

- COMMENT: Denote  $\mathcal{M}(T, (u_1, \dots, u_n))$  by  $A'$ .

$$u_k = Iu_k = B_{1,k}v_1 + \dots + B_{n,k}v_n, \quad \forall k \in \{1, \dots, n\}.$$

$$\text{又 } Tu_k = T(B_{1,k}v_1 + \dots + B_{n,k}v_n) = B_{1,k}u_1 + \dots + B_{n,k}u_n = A'_{1,k}u_1 + \dots + A'_{n,k}u_n \Rightarrow A' = B.$$

$$\text{OR. } A' = \mathcal{M}(T, (u_1, \dots, u_n)) = \mathcal{M}(T, (v_1, \dots, v_n), (u_1, \dots, u_n))\mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)) = B.$$

## 16 Suppose $V$ is finite-dim and $S \in \mathcal{L}(V)$ .

Prove that  $\exists \lambda \in \mathbf{F}, S = \lambda I \iff ST = TS$  for every  $T \in \mathcal{L}(V)$ .

**SOLUTION:** Using the notation and result in (o).

Suppose  $S = \lambda I$ . Then  $ST = TS = \lambda T$  for every  $T \in \mathcal{L}(V)$ . Conversely, if  $S = 0$ , then we are done.

Suppose  $S \neq 0, ST = TS, \forall T \in \mathcal{L}(V)$ . Let  $S = E_{1,1} + \dots + E_{m,m} \Rightarrow \mathcal{M}(S, (v_1, \dots, v_1)) = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_1))$ .

Then  $\forall k \in \{m+1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$ . Hence  $n = \dim V = \dim \text{range } S = m$ .

Note that  $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$ . Thus  $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \dots + a_{n,i}v_n)$ . Where:

$$a_{i,j} = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \dots + a_{n,i}v_n;$$

For each  $j$ , for all  $i$ . Thus  $a_{i,i} = a_{k,k} = \lambda, \forall k \neq i$ .

$$\text{Hence } w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} = \mathcal{M}(\lambda I, (v_1, \dots, v_n)) \Rightarrow S = \mathcal{M}^{-1}(\mathcal{M}(\lambda I)) = \lambda I. \quad \square$$

- OR (10.A.3) Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

Prove that  $T$  has the same matrix with respect to every basis of  $V$  if and only if  $T$  is a scalar multiple of the identity operator.

**SOLUTION:** [ Compare with the first solution of Problem (16) in (3.A) ]

Suppose  $T = \lambda I$  for some  $\lambda \in \mathbf{F}$ . Then  $T$  has the same matrix with respect to every basis of  $V$ .

Conversely, if  $T = 0$ , then we are done; Suppose  $T \neq 0$ . And  $v$  is a nonzero vector in  $V$ .

Assume that  $(v, Tv)$  is linearly independent.

Extend  $(v, Tv)$  to a basis of  $V$  as  $(v, Tv, u_3, \dots, u_n)$ . Let  $B = \mathcal{M}(T, (v, Tv, u_3, \dots, u_n))$ .

$$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \dots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \forall i \neq 2.$$

By assumption,  $A = \mathcal{M}(T, (v, w_2, \dots, w_n)) = B$  for any basis  $(v, w_2, \dots, w_n)$ .

$$\text{Then } A_{2,1} = 1, A_{i,1} = 0 (i \neq 2) \Rightarrow Tv = w_2,$$

which is not true if we let  $w_2 = u_3, w_3 = Tv, w_j = u_j (j = 4, \dots, n)$ . Contradicts.

Hence  $(v, Tv)$  is linearly dependent  $\Rightarrow \forall v \in V, \exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$ .

Now we show that  $\lambda_v$  is independent of  $v$ , that is,

to show that for any two nonzero distinct vectors  $v, w \in V, \lambda_v = \lambda_w$ . Thus  $T = \lambda I$  for some  $\lambda \in \mathbf{F}$ .

$$\left. \begin{aligned} (v, w) \text{ is linearly independent} &\Rightarrow T(v+w) = \lambda_{v+w}(v+w) \\ &= \lambda_{v+w}v + \lambda_{v+w}w \\ &= \lambda_v v + \lambda_w w = Tv + Tw \Rightarrow \lambda_{v+w} = \lambda_v = \lambda_w \\ (v, w) \text{ is linearly dependent, } w = cv &\Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \Rightarrow \lambda_v = \lambda_w \end{aligned} \right\} \Rightarrow \square$$

**17** Suppose  $V$  is finite-dim. Show that the only two-sided ideals of  $\mathcal{L}(V)$  are  $\{0\}$  and  $\mathcal{L}(V)$ .

A subspace  $\mathcal{E}$  of  $\mathcal{L}(V)$  is called a two-sided ideal of  $\mathcal{L}(V)$  if  $TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$ .

**SOLUTION:** Using NOTE FOR [3.60]. Let  $(v_1, \dots, v_n)$  be a basis of  $V$ . If  $\mathcal{E} = 0$ , then we are done.

Suppose  $\mathcal{E} \neq 0$  and  $\mathcal{E}$  is a two-sided ideal of  $\mathcal{L}(V)$ .

Then for any  $E_{i,j} \in \mathcal{E}, (\forall x, y = 1, \dots, n)$ , by assumption,  $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$ .

Again,  $E_{y,x'}, E_{y',x} \in \mathcal{E}$  for all  $x', y', x, y = 1, \dots, n$ . Thus  $\mathcal{E} = \mathcal{L}(V)$ .  $\square$

**18** Show that  $V$  and  $\mathcal{L}(\mathbf{F}, V)$  are isomorphic vector spaces.

**SOLUTION:**

Define  $\varphi \in \mathcal{L}(V, \mathcal{L}(\mathbf{F}, V))$  by  $\varphi(v) = \varphi_v$ ; where  $\varphi_v \in \mathcal{L}(\mathbf{F}, V)$  and  $\varphi_v(\lambda) = \lambda v$ .

(a)  $\varphi(v) = \varphi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \varphi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$ . Hence  $\varphi$  is injective.  
 (b)  $\forall \psi \in \mathcal{L}(\mathbf{F}, V)$ , let  $v = \psi(1) \Rightarrow \psi(\lambda) = \lambda v = \varphi_v(\lambda), \forall \lambda \in \mathbf{F}$   
 $\Rightarrow \psi = \varphi_{\psi(1)} = \varphi(\psi(1))$ . Hence  $\varphi$  is surjective.  $\square$

• Suppose  $q \in \mathcal{P}(\mathbf{R})$ . Prove that there exists a polynomial  $p \in \mathcal{P}(\mathbf{R})$  such that  $q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$  for all  $x \in \mathbf{R}$ .

**SOLUTION:**

Note that  $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$ .

Define  $T_n : \mathcal{P}_n(\mathbf{R}) \rightarrow \mathcal{P}_n(\mathbf{R})$  by  $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ .

As can be easily checked,  $T_n$  is an operator.

Now how can we prove that  $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3) = 0 \iff p = 0$ ?

Hence  $T_n$  is injective and therefore is surjective.

Thus  $\forall q \in \mathcal{P}(\mathbf{R}), \deg q = m, \exists p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$  for all  $x \in \mathbf{R}$ .

**19** Suppose  $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$  is injective.  $\deg Tp \leq \deg p$  for every nonzero  $p \in \mathcal{P}(\mathbf{R})$ .

(a) Prove that  $T$  is surjective.

(b) Prove that for every nonzero  $p, \deg Tp = \deg p$ .

**SOLUTION:**

(a)  $T$  is injective  $\iff T|_{\mathcal{P}_n(\mathbf{R})} : \mathcal{P}_n(\mathbf{R}) \rightarrow \mathcal{P}_n(\mathbf{R})$  is injective for any  $n \in \mathbf{N}^+$

$\iff T|_{\mathcal{P}_n(\mathbf{R})}$  is surjective for any  $n \in \mathbf{N}^+ \iff T$  is surjective.

(b) Using mathematical induction.

(i)  $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$ .

$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$ .

(ii) Suppose  $\deg f = \deg Tf$  for all  $f \in \mathcal{P}_n(\mathbf{R})$ . Then suppose  $\deg g = n + 1, g \in \mathcal{P}_{n+1}(\mathbf{R})$ .

Assume that  $\deg Tg < \deg g (\Rightarrow \deg Tg \leq n, Tg \in \mathcal{P}_n(\mathbf{R}))$ .

Then by (a),  $\exists f \in \mathcal{P}_n(\mathbf{R}), T(f) = (Tg)$ .  $\text{又 } T$  is injective  $\Rightarrow f = g$ .

While  $\deg f = \deg Tf = \deg Tg < \deg g$ . Contradicts the assumption.

Hence  $\deg Tp = \deg p$  for all  $p \in \mathcal{P}_{n+1}(\mathbf{R})$ .

Thus  $\deg Tp = \deg p$  for all  $p \in \mathcal{P}(\mathbf{R})$ .  $\square$



- Suppose  $T \in \mathcal{L}(V)$  and  $(v_1, \dots, v_m)$  is a list in  $V$  such that  $(Tv_1, \dots, Tv_m)$  spans  $V$ . Prove that  $(v_1, \dots, v_m)$  spans  $V$ .

**SOLUTION:**

$V = \text{span}(Tv_1, \dots, Tv_m) \Rightarrow T$  is surjective,  $\wedge V$  is finite-dim  $\Rightarrow T$  is invertible  $\Rightarrow T^{-1}$  is invertible.  
 $\forall v \in V, \exists a_i \in \mathbf{F}, v = a_1Tv_1 + \dots + a_nTv_n$   
 $\Rightarrow T^{-1}v = a_1v_1 + \dots + a_nv_n \Rightarrow \text{range } T^{-1} \subseteq \text{span}(v_1, \dots, v_n) \wedge \text{range } T^{-1} = V. \quad \square$

OR. Reduce  $(Tv_1, \dots, Tv_n)$  to a basis of  $V$  as  $(Tv_{\alpha_1}, \dots, Tv_{\alpha_m})$ , where  $m = \dim V$  and  $\alpha_i \in \{1, \dots, m\}$ .

Then  $(v_{\alpha_1}, \dots, v_{\alpha_m})$  is linearly independent of length  $m$ , therefore is a basis of  $V$ , contained in the list  $(v_1, \dots, v_m)$ .  $\square$

- Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

$(Tv_1, \dots, Tv_n)$  is a basis of  $V$  for some basis  $(v_1, \dots, v_n)$  of  $V \iff T$  is surjective  
 $(Tv_1, \dots, Tv_n)$  is a basis of  $V$  for every basis  $(v_1, \dots, v_n)$  of  $V \iff T$  is injective  $\left. \vphantom{\begin{matrix} (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1, \dots, v_n) \text{ of } V \\ (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1, \dots, v_n) \text{ of } V \end{matrix}} \right\} \iff T \text{ is invertible.}$

- 2** Suppose  $V$  is finite-dim and  $\dim V > 1$ .

Prove that the set of noninvertible operators on  $V$  is not a subspace of  $\mathcal{L}(V)$ .

**SOLUTION:**

Suppose  $\dim V = n > 1$ . Let  $(v_1, \dots, v_n)$  be a basis of  $V$ .

Define  $S, T \in \mathcal{L}(V)$  by  $S(a_1v_1 + \dots + a_nv_n) = a_1v_1$  and  $T(a_1v_1 + \dots + a_nv_n) = a_2v_1 + \dots + a_nv_n$ .

Hence  $S + T = I$  is invertible.

Thus the set of noninvertible linear maps in  $\mathcal{L}(V)$  is not closed under addition and therefore is not a subspace.  $\square$

COMMENT: If  $\dim V = 1$ , then the set of noninvertible operators on  $V$  equals  $\{0\}$ , which is a subspace of  $\mathcal{L}(V)$ .

- 3** Suppose  $V$  is finite-dim,  $U$  is a subspace of  $V$ , and  $S \in \mathcal{L}(U, V)$ .

Prove that there exists an invertible  $T \in \mathcal{L}(V, V)$  such that

$Tu = Su$  for every  $u \in U$  if and only if  $S$  is injective.

**SOLUTION:** [ Compare this with (3.A.II). ]

(a)  $Tu = Su$  for every  $u \in U \Rightarrow u = T^{-1}Su \Rightarrow S$  is injective.

(b) Suppose  $(u_1, \dots, u_m)$  be a basis of  $U$  and  $S$  is injective  $\Rightarrow (Su_1, \dots, Su_m)$  is linearly independent in  $V$ .

Extend these to bases of  $V$  as  $(u_1, \dots, u_m, v_1, \dots, v_n)$  and  $(Su_1, \dots, Su_m, w_1, \dots, w_n)$ .

Define  $T \in \mathcal{L}(V)$  by  $T(u_i) = Su_i; T v_j = w_j$ , for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ .

- 4** Suppose that  $W$  is finite-dim and  $S, T \in \mathcal{L}(V, W)$ .

Prove that  $\text{null } S = \text{null } T (= U) \iff S = ET, \exists \text{ invertible } E \in \mathcal{L}(W)$ .

**SOLUTION:**

Define  $E \in \mathcal{L}(W)$  by  $E(Tv_i) = Sv_i, E(w_j) = x_j$ , for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ . Where:

<p>Let <math>(Tv_1, \dots, Tv_m)</math> be a basis of <math>\text{range } T</math>, extend it to a basis of <math>W</math> as <math>(Tv_1, \dots, Tv_m, w_1, \dots, w_n)</math>.          Let <math>(u_1, \dots, u_n)</math> be a basis of <math>U</math>. Then by (3.B.TIPS), <math>(v_1, \dots, v_m, u_1, \dots, u_n)</math> is a basis of <math>V</math>.  <math>\wedge \text{null } S = \text{null } T \Rightarrow V = \text{span}(v_1, \dots, v_m) \oplus \text{null } S \Rightarrow \text{span}(Sv_1, \dots, Sv_m) = \text{range } S</math>.          And <math>\dim \text{range } T = \dim \text{range } S = \dim V - \text{null } U = m</math>. Hence <math>(Sv_1, \dots, Sv_m)</math> is a basis of <math>\text{range } S</math>.          Thus we let <math>(Sv_1, \dots, Sv_m, x_1, \dots, x_n)</math> be a basis of <math>W</math>.</p>	<p>Hence <math>E</math> is invertible and <math>S = ET</math>.</p>
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Conversely,  $S = ET \Rightarrow \text{null } S = \text{null } ET$ .

Then  $v \in \text{null } ET \iff ET(v) = 0 \iff Tv = 0 \iff v \in \text{null } T$ . Hence  $\text{null } ET = \text{null } T = \text{null } S. \quad \square$

**5** Suppose that  $W$  is finite-dim and  $S, T \in \mathcal{L}(V, W)$ .

Prove that  $\text{range } S = \text{range } T (= R) \iff S = TE, \exists \text{ invertible } E \in \mathcal{L}(V)$ .

**SOLUTION:**

Define  $E \in \mathcal{L}(V)$  as  $E : v_i \mapsto r_i ; u_j \mapsto s_j ;$  for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ . Where:

<p>Let <math>(Tv_1, \dots, Tv_m)</math> and <math>(Sr_1, \dots, Sr_m)</math> be bases of <math>R</math> such that <math>\forall i, Tv_i = Sr_i</math>.</p> <p>Let <math>(u_1, \dots, u_n)</math> and <math>(s_1, \dots, s_n)</math> be bases of <math>\text{null } T</math> and <math>\text{null } S</math> respectively.</p> <p>Thus <math>(v_1, \dots, v_m, u_1, \dots, u_n)</math> and <math>(r_1, \dots, r_m, s_1, \dots, s_n)</math> are bases of <math>V</math>.</p>	Hence $E$ is invertible and $S = TE$ .
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Conversely,  $S = TE \Rightarrow \text{range } S = \text{range } TE$ .

Then  $w \in \text{range } S \iff \exists v \in V, Sv = TE(v) = T(E(v)) = w \in \text{range } T$ . Hence  $\text{range } S = \text{range } T$ .  $\square$

**6** Suppose  $V$  and  $W$  are finite-dim and  $S, T \in \mathcal{L}(V, W)$ .

[ $\dim \text{null } S = \dim \text{null } T = n$ ]

Prove that  $S = E_2TE_1, \exists \text{ invertible } E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W) \iff \dim \text{null } S = \dim \text{null } T$ .

**SOLUTION:**

Define  $E_1 : v_i \mapsto r_i ; u_j \mapsto s_j ;$  for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ .

Define  $E_2 : Tv_i \mapsto Sr_i ; x_j \mapsto y_j ;$  for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ . Where:

<p>Let <math>(Tv_1, \dots, Tv_m)</math> and <math>(Sr_1, \dots, Sr_m)</math> be bases of <math>\text{range } T</math> and <math>\text{range } S</math>.</p> <p>Let <math>(u_1, \dots, u_n)</math> and <math>(s_1, \dots, s_n)</math> be bases of <math>\text{null } T</math> and <math>\text{null } S</math> respectively.</p> <p>Thus <math>(v_1, \dots, v_m, u_1, \dots, u_n)</math> and <math>(r_1, \dots, r_m, s_1, \dots, s_n)</math> are bases of <math>V</math>.</p> <p>Extend <math>(Tv_1, \dots, Tv_m)</math> and <math>(Sr_1, \dots, Sr_m)</math> to bases of <math>W</math> as</p> <p style="text-align: center;"><math>(Tv_1, \dots, Tv_m, x_1, \dots, x_p)</math> and <math>(Sr_1, \dots, Sr_m, y_1, \dots, y_p)</math>.</p>	Thus $E_1, E_2$ are invertible and $S = E_2TE_1$ .
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Conversely,  $S = E_2TE_1 \Rightarrow \dim \text{null } S = \dim \text{null } E_2TE_1$ .

$v \in \text{null } E_2TE_1 \iff E_2TE_1(v) = 0 \iff TE_1(v) = 0$ . Hence  $\text{null } E_2TE_1 = \text{null } TE_1 = \text{null } S$ .

By (3.B.22.COROLLARY),  $E$  is invertible  $\Rightarrow \dim \text{null } TE_1 = \dim \text{null } T = \dim \text{null } S$ .  $\square$

**8** Suppose  $V$  is finite-dim and  $T : V \rightarrow W$  is a surjective linear map of  $V$  onto  $W$ .

Prove that there is a subspace  $U$  of  $V$  such that  $T|_U$  is an isomorphism of  $U$  onto  $W$ .

$T|_U$  is the function whose domain is  $U$ , with  $T|_U$  defined by  $T|_U(u) = Tu$  for every  $u \in U$ .

**SOLUTION:**

$T$  is surjective  $\Rightarrow \text{range } T = W \Rightarrow \dim \text{range } T = \dim W = \dim V - \dim \text{null } T$ .

Let  $(w_1, \dots, w_m)$  be a basis of  $\text{range } T = W \Rightarrow \forall w_i, \exists ! v_i \in V, Tv_i = w_i$ .

$\Rightarrow (v_1, \dots, v_m)$  is a basis of  $\mathcal{K}$ . Thus  $\dim \mathcal{K} = \dim W$ .

Thus  $T|_{\mathcal{K}}$  maps a basis of  $\mathcal{K}$  to a basis of  $\text{range } T = W$ . Denote  $\mathcal{K}$  by  $U$ .  $\square$

• Suppose  $V$  and  $W$  are finite-dim and  $U$  is a subspace of  $V$ .

Let  $\mathcal{E} = \{ T \in \mathcal{L}(V, W) : U \subseteq \text{null } T \}$ .

(a) Show that  $\mathcal{E}$  is a subspace of  $\mathcal{L}(V, W)$ .

(b) Find a formula for  $\dim \mathcal{E}$  in terms of  $\dim V$ ,  $\dim W$  and  $\dim U$ .

Hint: Define  $\Phi : \mathcal{L}(V, W) \rightarrow L(U, W)$  by  $\Phi(T) = T|_U$ . What is  $\text{null } \Phi$ ? What is  $\text{range } \Phi$ ?

**SOLUTION:**

(a)  $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, Su = Tu = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$ .

(b) Define  $\Phi$  as in the hint.

$T \in \text{null } \Phi \iff \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$ . Hence  $\text{null } \Phi = \mathcal{E}$ .

$S \in \mathcal{L}(U, W) \Rightarrow \exists T \in \mathcal{L}(V, W), \Phi(T) = S$ , by (3.B.11)  $\Rightarrow S \in \text{range } \Phi$ . Hence  $\text{range } \Phi = \mathcal{L}(U, W)$ .

Thus  $\dim \text{null } \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \text{range } \Phi = (\dim V - \dim U) \dim W$ .  $\square$

OR. Extend  $(u_1, \dots, u_m)$  a basis of  $U$  to  $(u_1, \dots, u_m, v_1, \dots, v_n)$  a basis of  $V$ . Let  $p = \dim W$ .  
( See NOTE FOR [3.60])

$$\forall T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \text{span} \underbrace{\left\{ \begin{matrix} E_{1,1}, & \cdots & E_{m,1}, \\ \vdots & & \vdots \\ E_{1,p}, & \cdots & E_{m,p} \end{matrix} \right\}}_{\text{Denote it by } R} \cap \mathcal{E} = \{0\}.$$

$$\forall W = \text{span} \left\{ \begin{matrix} E_{m+1,1}, & \cdots & E_{n,1}, \\ \vdots & & \vdots \\ E_{m+1,p}, & \cdots & E_{n,p} \end{matrix} \right\} \subseteq \mathcal{E}. \text{ Where } \mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}.$$

$$\text{Then } \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W. \quad \square$$

ENDED

### 3.E

2 Suppose  $V_1, \dots, V_m$  are vec-sps such that  $V_1 \times \dots \times V_m$  is finite-dim.

Prove that every  $V_j$  is finite-dim.

SOLUTION: Denote  $V_1 \times \dots \times V_m$  by  $U$ . Denote  $\{0\} \times \dots \times \{0\} \times V_i \times \{0\} \times \dots \times \{0\}$  by  $U_i$ .

Let  $(v_1, \dots, v_M)$  be a basis of  $U$ . Note that  $\forall u_i \in V_i, u_i \in U_i \subseteq U$ , for each  $i$ .

Define  $R_i \in \mathcal{L}(V_i, U)$  by  $R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0)$ .  
Define  $S_i \in \mathcal{L}(U, V_i)$  by  $S_i(u_1, \dots, u_i, \dots, u_m) = u_i$

Thus  $U_i$  and  $V_i$  are isomorphic.  $\forall U_i$  is a subspace of a finite-dim vec-sp  $U$ .  $\square$

3 Give an example of a vec-sp  $V$  and its two subspaces  $U_1, U_2$  such that  $U_1 \times U_2$  and  $U_1 + U_2$  are isomorphic but  $U_1 + U_2$  is not a direct sum.

SOLUTION:

NOTE that at least one of  $U_1, U_2$  must be infinite-dim.

For if not,  $U_1 \times U_2$  is finite-dim and  $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$ .

And  $V$  must be infinite-dim. For if not, both  $U_1$  and  $U_2$  are finite-dim subspaces.

Let  $V = \mathbf{F}^\infty = U_1, U_2 = \{(x, 0, \dots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$ .

Define  $T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2)$  by  $T((x_1, x_2, \dots), (x, 0, \dots)) = (x, x_1, x_2, \dots)$   
Define  $S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2)$  by  $S(x, x_1, x_2, \dots) = ((x_1, x_2, \dots), (x, 0, \dots))$   $\Rightarrow S = T^{-1}$ .  $\square$

4 Suppose  $V_1, \dots, V_m$  are vec-sps.

Prove that  $\mathcal{L}(V_1 \times \dots \times V_m, W)$  and  $\mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$  are isomorphic.

SOLUTION: Using the notations in Problem (2). Note that  $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \dots + T(0, \dots, u_m)$ .

Define  $\varphi : T \mapsto (T_1, \dots, T_m)$  by  $\varphi(T) = (TR_1, \dots, TR_m)$ .  
Define  $\psi : (T_1, \dots, T_m) \mapsto T$  by  $\psi(T_1, \dots, T_m) = T_1S_1 + \dots + T_mS_m$ .  $\Rightarrow \psi = \varphi^{-1}$ .  $\square$

5 Suppose  $W_1, \dots, W_m$  are vec-sps.

Prove that  $\mathcal{L}(V, W_1 \times \dots \times W_m)$  and  $\mathcal{L}(V, W_1) \times \dots \times \mathcal{L}(V, W_m)$  are isomorphic.

SOLUTION: Using the notations in Problem (2).

Note that  $Tv = (w_1, \dots, w_m)$ . Define  $T_i \in \mathcal{L}(V, W_i)$  by  $T_i(v) = w_i$ .

Define  $\varphi : T \mapsto (T_1, \dots, T_m)$  by  $\varphi(T) = (S_1T, \dots, S_mT)$ .  
Define  $\psi : (T_1, \dots, T_m) \mapsto T$  by  $\psi(T_1, \dots, T_m) = T_1S_1 + \dots + T_mS_m$ .  $\Rightarrow \psi = \varphi^{-1}$ .  $\square$

**6** For  $m \in \mathbf{N}^+$ , define  $V^m$  by  $\underbrace{V \times \cdots \times V}_{m \text{ times}}$ . Prove that  $V^m$  and  $\mathcal{L}(\mathbf{F}^m, V)$  are isomorphic.

**SOLUTION:**

Define  $T : (v_1, \dots, v_m) \rightarrow \varphi$ , where  $\varphi : (a_1, \dots, a_m) \mapsto v$  is defined by  $\varphi(a_1, \dots, a_m) = a_1v_1 + \cdots + a_mv_m$ .

Suppose  $T(v_1, \dots, v_m) = 0$ . Then  $\forall (a_1, \dots, a_m) \in \mathbf{F}^m$ ,  $\varphi(a_1, \dots, a_m) = a_1v_1 + \cdots + a_mv_m = 0$

$\Rightarrow (v_1, \dots, v_m) = 0 \Rightarrow T$  is injective.

Suppose  $\psi \in \mathcal{L}(\mathbf{F}^m, V)$ . Let  $(e_1, \dots, e_m)$  be the standard basis of  $\mathbf{F}^m$ . Then  $\forall (b_1, \dots, b_m) \in \mathbf{F}^m$ ,

$(T(\psi(e_1), \dots, \psi(e_m)))(b_1, \dots, b_m) = b_1\psi(e_1) + \cdots + b_m\psi(e_m) = \psi(b_1e_1 + \cdots + b_me_m) = \psi(b_1, \dots, b_m)$ .

Thus  $T(\psi(e_1), \dots, \psi(e_m)) = \psi$ . Hence  $T$  is surjective.  $\square$

**7** Suppose  $v, x \in V$  (chosen arbitrarily) of which  $U$  and  $W$  are subspaces.

Suppose  $v + U = x + W$ . Prove that  $U = W$ .

**SOLUTION:**

(a)  $\forall u \in U, \exists w \in W, v + u = x + w$ , let  $u = 0$ , getting  $v = x + w \Rightarrow v - x \in W$ .

(b)  $\forall w \in W, \exists u \in U, v + u = x + w$ , let  $w = 0$ , getting  $x = v + u \Rightarrow x - v \in U$ .

Thus  $\pm(v - x) \in U \cap W \Rightarrow \begin{cases} u = (x - v) + w \in W \Rightarrow U \subseteq W \\ w = (v - x) + u \in U \Rightarrow W \subseteq U \end{cases} \Rightarrow U = W. \square$

• Let  $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$ . Suppose  $A \subseteq \mathbf{R}^3$ .

Prove that  $A$  is a translate of  $U \iff \exists c \in \mathbf{R}, A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$ .

[Do it in your mind.]

• Suppose  $T \in \mathcal{L}(V, W)$  and  $c \in W$ . Prove that  $U = \{x \in V : Tx = c\}$  is either  $\emptyset$  or is a translate of null  $T$ .

**SOLUTION:**

If  $c \in W$  but  $c \notin \text{range } T$ , then  $U = \emptyset$  and we are done.

Suppose  $c \in \text{range } T$ , then  $\exists u \in V, Tu = c \Rightarrow u \in U$ .

Suppose  $y \in \text{null } T \Rightarrow y + u \in U \iff T(y + u) = Ty + c = c$ . Thus  $u + \text{null } T \subseteq U$ . Hence  $u + \text{null } T = U$ ,

for if not, suppose  $z \notin u + \text{null } T$  but  $Tz = c (\Leftrightarrow z \in U)$ , then  $\forall w \in \text{null } T, z \neq u + w \Leftrightarrow z - u \notin \text{null } T$ .

又  $\tilde{T}(z + \text{null } T) = \tilde{T}(u + \text{null } T) \Rightarrow z + \text{null } T = u + \text{null } T \Rightarrow z - u \in \text{null } T$ , contradicts.  $\square$

• **COROLLARY:** The set of solutions to a system of linear equations such as [3.28] is either  $\emptyset$  or a translate of the null subspace.

**8** Prove that a nonempty subset  $A$  of  $V$  is a translate of some subspace of  $V$  if and only if

$\lambda v + (1 - \lambda)w \in A$  for all  $v, w \in A$  and all  $\lambda \in \mathbf{F}$ .

**SOLUTION:**

Suppose  $A = a + U$ , where  $U$  is a subspace of  $V$ .  $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$ ,

$\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + [\lambda(u_1 - u_2) + u_2] \in A$ .

Suppose  $\lambda v + (1 - \lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$ . Suppose  $a \in A$  and let  $A' = \{x - a : x \in A\}$ .

Then  $\forall x - a, y - a \in A', \lambda \in \mathbf{F}$ ,

(I)  $\lambda(x - a) = [\lambda x + (1 - \lambda)a] - a \in A'$ . Then let  $\lambda = 2$ .

(II)  $\lambda(x - a) + (1 - \lambda)(y - a) = \frac{1}{2}(x - a) + \frac{1}{2}(y - a) = \frac{1}{2}x + (1 - \frac{1}{2})(y) - a \in A'$ .

By (I),  $2 \times [\frac{1}{2}(x - a) + \frac{1}{2}(y - a)] = (x - a) + (y - a) \in A'$ .

Thus  $A'$  is a subspace of  $V$ . Hence  $a + A' = \{(x - a) + a : x \in A\} = A$  is a translate.  $\square$

**9** Suppose  $A_1 = v + U_1$  and  $A_2 = w + U_2$  for some  $v, w \in V$  and some subspaces  $U_1, U_2$  of  $V$ .

Prove that the intersection  $A_1 \cap A_2$  is either a translate of some subspace of  $V$  or is  $\emptyset$ .

**SOLUTION:** Suppose  $v + u_1, w + u_2 \in A_1 \cap A_2 \neq \emptyset$ . By Problem (8),

$\forall \lambda \in \mathbf{F}, \lambda(v + u_1) + (1 - \lambda)(w + u_2) \in A_1 \text{ and } A_2$ . Thus  $A_1 \cap A_2$  is a translate of some subspace of  $V$ .  $\square$

**10** Prove that the intersection of any collection of translates of subspaces of  $V$  is either a translate of some subspace or  $\emptyset$ .

**SOLUTION:** Suppose  $\{A_\alpha\}_{\alpha \in \Gamma}$  is a collection of translates of subspaces of  $V$ , where  $\Gamma$  is an arbitrary index set.

Suppose  $x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset$ , then by Problem (18),  $\forall \lambda \in \mathbf{F}, \lambda x + (1 - \lambda)y \in A_\alpha$  for every  $\alpha \in \Gamma$ .

Thus  $\bigcap_{\alpha \in \Gamma} A_\alpha$  is a translate of some subspace of  $V$ .  $\square$

**11** Suppose  $A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\}$ , where each  $v_i \in V, \lambda_i \in \mathbf{F}$ .

(a) Prove that  $A$  is a translate of some subspace of  $V$ : By Problem (8),

$$\forall \sum_{i=1}^m a_i v_i, \sum_{i=1}^m b_i v_i \in A, \lambda \in \mathbf{F}, \quad \lambda \sum_{i=1}^m a_i v_i + (1 - \lambda) \sum_{i=1}^m b_i v_i = (\lambda \sum_{i=1}^m a_i + (1 - \lambda) \sum_{i=1}^m b_i) v_i \in A. \quad \square$$

(b) Prove that if  $B$  is a translate of some subspace of  $V$  and  $\{v_1, \dots, v_m\} \subseteq B$ , then  $A \subseteq B$ .

(c) Prove that  $A$  is a translate of some subspace of  $V$  and  $\dim V < m$ .

**SOLUTION:**

(b) Let  $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$ . To show that  $v \in B$ , use induction on  $m$  by  $k$ .

(i)  $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$ .  $\forall v_1 \in B$ . Hence  $v \in B$ .

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$ .  $\forall v_1, v_2 \in B$ . By problem (8),  $v \in B$ .

(ii)  $2 \leq k \leq m$ , we assume that  $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$ . ( $\forall \lambda_i$  such that  $\sum_{i=1}^k \lambda_i = 1$ )

For  $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$ .  $\forall i = 1, \dots, k, \exists \mu_i \neq 1$ , fix one such  $i$  by  $\iota$ .

$$\text{Then } \sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow \left( \sum_{i=1}^{k+1} \frac{\mu_i}{1 - \mu_\iota} \right) - \frac{\mu_\iota}{1 - \mu_\iota} = 1.$$

$$\text{Let } w = \underbrace{\frac{\mu_1}{1 - \mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1 - \mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1 - \mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1 - \mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let  $\lambda_i = \frac{\mu_i}{1 - \mu_\iota}$  for  $i = 1, \dots, \iota - 1$ ;  $\lambda_j = \frac{\mu_{j+1}}{1 - \mu_\iota}$  for  $j = \iota, \dots, k$ . Then,

$$\left. \begin{array}{l} \sum_{i=1}^k \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1 - \lambda)v_\iota \in B \end{array} \right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \quad \square$$

(c)  $\forall k = 1, \dots, m, \forall \lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_m$ , let  $\lambda_k = 1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m$

$$\Rightarrow \lambda_1 v_1 + \cdots + \lambda_m v_m$$

$$= \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m) v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_m v_m$$

$$= v_k + \lambda_1(v_1 - v_k) + \cdots + \lambda_{k-1}(v_{k-1} - v_k) + \lambda_{k+1}(v_{k+1} - v_k) + \cdots + \lambda_m(v_m - v_k).$$

Thus  $A = v_k + \text{span}(v_1 - v_k, \dots, v_{k-1} - v_k, v_{k+1} - v_k, \dots, v_m - v_k)$ .  $\square$

**12** Suppose  $U$  is a subspace of  $V$  such that  $V/U$  is finite-dim.

Prove that  $V$  is isomorphic to  $U \times (V/U)$ .

**SOLUTION:** Let  $(v_1 + U, \dots, v_n + U)$  be a basis of  $V/U$ . Note that

$$\forall v \in V, \exists ! a_1, \dots, a_n \in \mathbf{F}, v + U = \sum_{i=1}^n a_i(v_i + U) = \left( \sum_{i=1}^n a_i v_i \right) + U$$

$$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) = u \in U \text{ for some } u; v = \sum_{i=1}^n a_i v_i + u.$$

Thus define  $\varphi \in \mathcal{L}(V, U \times (V/U))$  by  $\varphi(v) = (u, \sum_{i=1}^n a_i v_i + U)$

$$\text{and } \psi \in \mathcal{L}(U \times (V/U), V) \text{ by } \psi(u, w + U) = u + w; w = \sum_{i=1}^n b_i v_i + U.$$

So that  $\psi = \varphi^{-1}$ .  $\square$

• Suppose  $V = U \oplus W$ ,  $(w_1, \dots, w_m)$  is a basis of  $W$ .

Prove that  $(w_1 + U, \dots, w_m + U)$  is a basis of  $V/U$ .

**SOLUTION:** Note that for any  $v \in V$ ,

$$\exists ! u \in U, w \in W, v = u + w \text{ 又 } \exists ! c_i \in \mathbf{F} \text{ such that } w = \sum_{i=1}^m c_i w_i \Rightarrow v = u + \sum_{i=1}^m c_i w_i.$$

$$\text{Thus } v + U = \sum_{i=1}^m c_i w_i + U \Rightarrow v + U \in \text{span}(w_1 + U, \dots, w_m + U) \Rightarrow V/U \subseteq \text{span}(w_1 + U, \dots, w_m + U).$$

$$\text{Now suppose } a_1(w_1 + U) + \dots + a_m(w_m + U) = 0 + U \Rightarrow \sum_{i=1}^m a_i w_i \in U \text{ while } U \cap W = \{0\}.$$

$$\text{Then } \sum_{i=1}^m a_i w_i = 0 \Rightarrow a_1 = \dots = a_m = 0. \quad \square$$

**13** Suppose  $(v_1 + U, \dots, v_m + U)$  is a basis of  $V/U$  and  $(u_1, \dots, u_n)$  is a basis of  $U$ .

Prove that  $(v_1, \dots, v_m, u_1, \dots, u_n)$  is a basis of  $V$ .

**SOLUTION:** By Problem (12),  $U$  and  $V/U$  are finite-dim  $\Rightarrow U \times (V/U)$  is finite-dim, so is  $V$ .

$$\dim V = \dim(U \times (V/U)) = \dim U + \dim V/U = m + n.$$

$$\text{OR. Note that for any } v \in V, v + U = \sum_{i=1}^m a_i v_i + U, \exists ! a_i \in \mathbf{F} \Rightarrow v = \sum_{i=1}^m a_i v_i = \sum_{i=1}^m b_i v_i, \exists ! b_i \in \mathbf{F}.$$

$$\Rightarrow v \in \text{span}(v_1, \dots, v_m, u_1, \dots, u_n) \supseteq V.$$

$$\text{又 Notice that } (\sum_{i=1}^m a_i v_i) + U = 0 + U (\Rightarrow \sum_{i=1}^m a_i v_i \in U) \iff a_1 = \dots = a_m = 0.$$

$$\text{Hence } \text{span}(v_1, \dots, v_m) \cap U = \{0\} \Rightarrow \text{span}(v_1, \dots, v_m) \oplus U = V$$

Thus  $(v_1, \dots, v_m, u_1, \dots, u_n)$  is linearly independent, so is a basis of  $V$ .  $\square$

**14** Suppose  $U = \{(x_1, x_2, \dots) \in \mathbf{F}^\infty : x_k \neq 0 \text{ for only finitely many } k\}$ .

(a) Show that  $U$  is a subspace of  $\mathbf{F}^\infty$ . [Do it in your mind]

(b) Prove that  $\mathbf{F}^\infty/U$  is infinite-dim.

**SOLUTION:**

For  $u = (x_1, \dots, x_p, \dots) \in \mathbf{F}^\infty$ , denote  $x_p$  by  $u[p]$ . For each  $r \in \mathbf{N}^+$ .

$$\text{Define } e_r[p] = \begin{cases} 1, & (p-1) \equiv 0 \pmod{r} \\ 0, & \text{otherwise} \end{cases}, \text{ simply } e_r = (1, \underbrace{0, \dots, 0}_{(p-1) \text{ times}}, 1, \underbrace{0, \dots, 0}_{(p-1) \text{ times}}, 1, \dots) \in \mathbf{F}^\infty.$$

Choose  $m \in \mathbf{N}^+$  arbitrarily.

$$\text{Suppose } a_1(e_1 + U) + \dots + a_m(e_m + U) = (a_1 e_1 + \dots + a_m e_m) + U = 0 + U = 0.$$

$$\Rightarrow a_1 e_1 + \dots + a_m e_m = u \text{ for some } u \in U.$$

$$\text{Then suppose } u = (x_1, \dots, x_t, 0, \dots) \Rightarrow u[t+i] = 0, \forall i \in \mathbf{N}^+,$$

$$\text{then let } j = s \cdot m! + 1 \geq t \text{ (} \exists s \in \mathbf{N}^+ \text{) so that } e_1[j] = \dots = e_m[j] = 1, u[j+i] = 0.$$

$$\text{Now we have: } u[j+i] = (\sum_{r=1}^m a_r e_r)[j+i] = \sum_{r=1}^m a_r e_r[s \cdot m! + 1 + i] = a_{i_1} + \dots + a_{i_{\tau(i)}} = 0,$$

$$\Rightarrow (\sum_{r=1}^m a_r e_r)[j+i] = a_{i_1} + \dots + a_{i_{\tau(i)}} = 0. \quad (\Delta)$$

where  $i_1, \dots, i_{\tau(i)}$  are distinct ordered factors of  $i$  ( $1 = i_1 \leq \dots \leq i_{\tau(i)} = i$ ).

( Note that by definition,  $e_r[s \cdot m! + 1 + i] = 1 \iff s \cdot m! + i \equiv 0 \pmod{r} \iff r \mid i$  )

Let  $i' = i_{\tau(i)-1}$ . Notice that  $i'_l = i_l, \forall l \in \{1, \dots, \tau(i')\}$ ; and  $\tau(i') = \tau(i) - 1$ .

$$\text{Again by } (\Delta), (\sum_{r=1}^m a_r e_r)[j+i'] = a_{i'_1} + \dots + a_{i'_{\tau(i')}} = a_{i_1} + \dots + a_{i_{\tau(i)-1}} = 0.$$

Thus  $a_{i_{\tau(i)}} = a_i = 0$  for any  $i \in \{1, \dots, m\}$ .

Hence  $(e_1, \dots, e_m)$  is linearly independent in  $\mathbf{F}^\infty$ , so is  $(e_1, \dots, e_m, \dots)$ , since  $m \in \mathbf{N}^+$ .

又  $e_i \notin U \Rightarrow (e_1 + U, e_2 + U, \dots)$  is linearly independent in  $\mathbf{F}^\infty/U$ . By [2.B.14].  $\square$

**15** Suppose  $\varphi \in \mathcal{L}(V, \mathbf{F}) \setminus \{0\}$ . Prove that  $\dim V / (\text{null } \varphi) = 1$ .

**SOLUTION:** By [3.91] (d),  $\dim \text{range } \varphi = 1 = \dim V / (\text{null } \varphi)$ .  $\square$

**NOTE FOR [3.88, 3.90, 3.91]**

For any  $W \in \mathcal{S}_V U$ , because  $V = U \oplus W$ .  $\forall v \in V, \exists! u_v \in U, w_v \in W, v = u_v + w_v$ .

Define  $T \in \mathcal{L}(V, W)$  by  $T(v) = w_v$ . Hence  $\text{null } T = U$ ,  $\text{range } T = W$ .

Then  $\tilde{T} \in \mathcal{L}(V/\text{null } T, W)$  is defined as  $\tilde{T}(v + U) = Tv = w_v$ .

Thus  $\tilde{T}$  is injective ( by [3.91(b)] ) and surjective (  $\text{range } \tilde{T} = \text{range } T = W$  ),

and therefore is an isomorphism. We conclude that  $V/U$  and  $W$ , namely any vec-sp in  $\mathcal{S}_V$ , are isomorphic.

**16** Suppose  $\dim V/U = 1$ . Prove that  $\exists \varphi \in \mathcal{L}(V, \mathbf{F})$  such that  $\text{null } \varphi = U$ .

**SOLUTION:**

Suppose  $V_0$  is a subspace of  $V$  such that  $V = U \oplus V_0$ . Then  $V_0$  and  $V/U$  are isomorphic.  $\dim V_0 = 1$ .

Define a linear map  $\varphi : v \mapsto \lambda$  by  $\varphi(v_0) = 1, \varphi(u) = 0$ , where  $v_0 \in V_0, u \in U$ .  $\square$

**17** Suppose  $V/U$  is finite-dim.  $W$  is a subspace of  $V$ .

(a) Show that if  $V = U + W$ , then  $\dim W \geq \dim V/U$ .

(b) Suppose  $\dim W = \dim V/U$  and  $V = U \oplus W$ . Find such  $W$ .

**SOLUTION:**

Let  $(w_1, \dots, w_n)$  be a basis of  $W$

(a)  $\forall v \in V, \exists u \in U, w \in W$  such that  $v = u + w \Rightarrow v + U = w + U$

Then  $V/U \subseteq \text{span}(w_1 + U, \dots, w_n + U) \Rightarrow V/U = \text{span}(w_1 + U, \dots, w_n + U)$ .

Hence  $\dim V/U = \dim \text{span}(w_1 + U, \dots, w_n + U) \leq \dim W$ .

(b) Let  $W \in \mathcal{S}_V U$ . In other words,

reduce  $(w_1 + U, \dots, w_n + U)$  to a basis of  $V/U$  as  $(w_{\alpha_1} + U, \dots, w_{\alpha_m} + U)$  and let  $W = \text{span}(w_{\alpha_1}, \dots, w_{\alpha_m})$ .  $\square$

**18** Suppose  $T \in \mathcal{L}(V, W)$  and  $U$  is a subspace of  $V$ . Let  $\pi$  denote the quotient map.

Prove that  $\exists S \in \mathcal{L}(V/U, W)$  such that  $T = S \circ \pi$  if and only if  $U \subseteq \text{null } T$ .

**SOLUTION:**

(a) Define  $S \in \mathcal{L}(V/U, W)$  by  $S(v + U) = Tv$ . We have to check it is well-defined.

Suppose  $v_1 + U = v_2 + U$ , while  $v_1 \neq v_2$ .

Then  $(v_1 - v_2) \in U \Rightarrow S((v_1 - v_2) + U) = T(v_1 - v_2) = 0 \Rightarrow Tv_1 = Tv_2$ . Checked.  $\square$

(b) Suppose  $\exists S \in \mathcal{L}(V/U, W)$ ,  $T = S \circ \pi$ . Then  $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \text{null } T$ .  $\square$

**20** Define  $\Gamma : \mathcal{L}(V/U, W) \rightarrow \mathcal{L}(V, W)$  by  $\Gamma(S) = S \circ \pi (= \pi'(S))$ .

(a) Prove that  $\Gamma$  is linear: By [3.9] distributive properties and [3.6].  $\square$

(b) Prove that  $\Gamma$  is injective:

$$\Gamma(S) = 0$$

$$\iff \forall v \in V, S(\pi(v)) = 0$$

$$\iff \forall v + U \in V/U, S(v + U) = 0$$

$$\iff S = 0. \quad \square$$

(c) Prove that  $\text{range } \Gamma (= \text{range } \pi') = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\}$ :

By Problem (18).  $\square$

### 3.F

- By (18) in (3.D) we know that  $\varphi : V \rightarrow \mathcal{L}(\mathbf{F}, V)$  is an isomorphism. Now we prove that  $(v_1, \dots, v_m)$  is linearly independent  $\iff (\varphi(v_1), \dots, \varphi(v_m))$  is linearly independent.

**SOLUTION:**

(a) Suppose  $(v_1, \dots, v_m)$  is linearly independent and  $\vartheta \in \text{span}(\varphi(v_1), \dots, \varphi(v_m))$ .

Let  $\vartheta = 0 = a_1\varphi(v_1) + \dots + a_m\varphi(v_m)$ . Then  $\vartheta(1) = 0 = a_1v_1 + \dots + a_mv_m \Rightarrow a_1 = \dots = a_m = 0$ .

OR Because  $\varphi$  is injective. Suppose  $a_1\varphi(v_1) + \dots + a_m\varphi(v_m) = 0 = \varphi(a_1v_1 + \dots + a_mv_m)$ .

Then  $a_1v_1 + \dots + a_mv_m = 0 \Rightarrow a_1 = \dots = a_m = 0$ .

Thus  $(\varphi(v_1), \dots, \varphi(v_m))$  is linearly independent.

(b) Suppose  $(\varphi(v_1), \dots, \varphi(v_m))$  is linearly independent and  $v \in \text{span}(v_1, \dots, v_m)$ .

Let  $v = 0 = a_1v_1 + \dots + a_mv_m$ . Then  $\varphi(v) = a_1\varphi(v_1) + \dots + a_m\varphi(v_m) = 0 \Rightarrow a_1 = \dots = a_m = 0$ .

Thus  $v_1, \dots, v_m$  is linearly independent.  $\square$

**1 Explain why each linear functional is surjective or is the zero map.**

**SOLUTION:** For any  $\varphi \in V'$  and  $\varphi \neq 0$ ,  $\exists v \in V$ , such that  $\varphi(v) \neq 0$ . (a)  $\left. \begin{array}{l} \dim \text{range } \varphi = \dim \mathbf{F} = 1. \end{array} \right\} \Rightarrow \square$

**4 Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$  such that  $U \neq V$ .**

*Prove that  $\exists \varphi \in V'$  and  $\varphi \neq 0$  such that  $\varphi(u) = 0$  for every  $u \in U$ .*

**SOLUTION:**

Let  $(u_1, \dots, u_m)$  be a basis of  $U$ , extend to  $(u_1, \dots, u_m, u_{m+1}, \dots, u_{m+n})$  a basis of  $V$ .

Choose  $k \in \{1, \dots, n\}$  arbitrarily. Define  $\varphi \in V'$  by  $\varphi(u_i) = \begin{cases} 1, & \text{if } i = m+k. \\ 0, & \text{otherwise.} \end{cases}$

OR: Equivalent to proving that  $U^0 \neq \{0\}$ . By [3.106],  $\dim U^0 = \dim V - \dim U > 0$ .  $\square$

• Suppose  $T \in \mathcal{L}(V, W)$  and  $(w_1, \dots, w_m)$  is a basis of  $\text{range } T$ .

Hence  $\forall v \in V$ ,  $Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m$ ,  $\exists! \varphi_1(v), \dots, \varphi_m(v)$ , thus defining functions  $\varphi_1, \dots, \varphi_m$  from  $V$  to  $\mathbf{F}$ . Show that each  $\varphi_i \in V'$ .

**SOLUTION:**

For each  $w_i$ ,  $\exists v_i \in V$ ,  $Tv_i = w_i$ , getting a linearly independent list  $(v_1, \dots, v_m)$ .

Now we have  $Tv = a_1Tv_1 + \dots + a_mTv_m$ ,  $\forall v \in V$ ,  $\exists! a_i \in \mathbf{F}$ .

Let  $(\psi_1, \dots, \psi_m)$  be the dual basis of  $\text{range } T$ . Then  $(T'(\psi_i))(v) = \psi_i \circ T(v) = a_i$ .

Thus letting  $\varphi_i = \psi_i \circ T$ .  $\square$

• Suppose  $\varphi, \beta \in V'$ . Prove that  $\text{null } \varphi \subseteq \text{null } \beta$  if and only if  $\beta = c\varphi$ .  $\exists c \in \mathbf{F}$ .

**SOLUTION:** Using (3.B.29, 30)

(a) Suppose  $\text{null } \varphi \subseteq \text{null } \beta$ . Choose a  $u \notin \text{null } \beta$ .  $V = \text{null } \beta \oplus \{au : a \in \mathbf{F}\}$ .

If  $\text{null } \varphi = \text{null } \beta$ , then let  $c = \frac{\beta(u)}{\varphi(u)}$ , we are done.

Otherwise, suppose  $u' \in \text{null } \beta$ , but  $u' \notin \text{null } \varphi$ , then  $V = \text{null } \varphi \oplus \{bu' : b \in \mathbf{F}\}$ .

$\forall v \in V$ ,  $v = w + au = w' + bu'$ ,  $\exists! w, w' \in \text{null } \varphi$ ,  $a, b \in \mathbf{F}$ .

Thus  $\beta(v) = a\beta(u)$ ,  $\varphi(v) = b\varphi(u')$ . Let  $c = \frac{a\beta(u)}{b\varphi(u')}$ . We are done

(b) Suppose  $\beta = c\varphi$  for some  $c \in \mathbf{F}$ .

If  $c = 0$ , then  $\text{null } \beta = V \supseteq \text{null } \varphi$ , we are done.

Otherwise,  $\left. \begin{array}{l} \forall v \in \text{null } \varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \text{null } \varphi \subseteq \text{null } \beta. \\ \forall v \in \text{null } \beta, \beta(v) = 0 = \varphi(v) \Rightarrow \text{null } \beta \subseteq \text{null } \varphi. \end{array} \right\} \Rightarrow \text{null } \varphi = \text{null } \beta$

$\Rightarrow \text{null } \varphi \subseteq \text{null } \beta$ .  $\square$



**5** Prove that  $(V_1 \times \cdots \times V_m)'$  and  $V_1' \times \cdots \times V_m'$  are isomorphic.

**SOLUTION:** Using notations in (3.E.2).

$$\left. \begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \rightarrow V_1' \times \cdots \times V_m' \\ \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = (R_1'(T), \dots, R_m'(T)). \\ \text{Define } \psi : V_1' \times \cdots \times V_m' \rightarrow (V_1 \times \cdots \times V_m)' \\ \text{by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m = S_1'(T_1) + \cdots + S_m'(T_m). \end{array} \right\} \Rightarrow \psi = \varphi^{-1}. \quad \square$$

• Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $\varphi_1, \dots, \varphi_n$  is the dual basis of  $V'$ .

$$\left. \begin{array}{l} \text{Define } \Gamma : V \rightarrow \mathbf{F}^n \text{ by } \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)). \\ \text{Define } \Lambda : \mathbf{F}^n \rightarrow V \text{ by } \Lambda(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n. \end{array} \right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

**35** Prove that  $(\mathcal{P}(\mathbf{R}))'$  and  $\mathbf{R}^\infty$  are isomorphic.

**SOLUTION:**

Define  $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{R}))', \mathbf{R}^\infty)$  by  $\theta(\varphi) = (\varphi(1), \varphi(x), \dots, \varphi(x^n), \dots)$ .

Injectivity:  $\theta(\varphi) = 0 \Rightarrow \forall x^k$  in the basis  $(1, x, \dots, x^n, \dots)$  of  $\mathcal{P}_n(\mathbf{R})$  for any  $n$ ,  $\varphi(x^k) = 0 \Rightarrow \varphi = 0$ .

Surjectivity:  $\forall (a_0, a_1, \dots, a_n, \dots) \in \mathbf{F}^\infty$ , let  $\psi$  be such that  $\psi(x^k) = a_k$  and thus  $\theta(\psi) = (a_0, a_1, \dots, a_n, \dots)$ .

Hence  $\theta$  is an isomorphism from  $(\mathcal{P}(\mathbf{R}))'$  onto  $\mathbf{R}^\infty$ .  $\square$

**7** Suppose  $m$  is a positive integer. Show that the dual basis of the basis  $(1, x, \dots, x_m)$  of  $\mathcal{P}_m(\mathbf{R})$  is  $\varphi_0, \varphi_1, \dots, \varphi_m$ , where  $\varphi_k = \frac{p^{(k)}(0)}{k!}$ . Here  $p^{(k)}$  denotes the  $k^{\text{th}}$  derivative of  $p$ , with the understanding that the  $0^{\text{th}}$  derivative of  $p$  is  $p$ .

**SOLUTION:**

$$\text{For each } j \text{ and } k, (x^j)^{(k)} = \begin{cases} j(j-1)\dots(j-k+1) \cdot x^{j-k}, & j \geq k. \\ j(j-1)\dots(j-j+1) = j!, & j = k. \\ 0, & j \leq k. \end{cases} \quad \text{Then } (x^j)^{(k)}(0) = \begin{cases} 0, & j \neq k. \\ k!, & j = k. \end{cases}$$

Thus  $\varphi_k = \psi_k$ , where  $\psi_1, \dots, \psi_m$  is the dual basis of  $(1, x, \dots, x_m)$  of  $\mathcal{P}_m(\mathbf{R})$ .  $\square$

**8** Suppose  $m$  is a positive integer.

(a) By [2.C.10],  $B = (1, x - 5, \dots, (x-5)^m)$  is a basis of  $\mathcal{P}_m(\mathbf{R})$ .

(b) Let  $\varphi_k = \frac{p^{(k)}(5)}{k!}$  for each  $k = 0, 1, \dots, m$ . Then  $(\varphi_0, \varphi_1, \dots, \varphi_m)$  is the dual basis of  $B$ .

**9** Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $\varphi_1, \dots, \varphi_n$  is the corresponding dual basis of  $V'$ .

Suppose  $\psi \in V'$ . Prove that  $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$ .

**SOLUTION:**  $\psi(v) = \psi(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i) \varphi_i(v) = [\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n](v) \Rightarrow \square$

**COMMENT:** For any other basis  $(u_1, \dots, u_n)$  of  $V$  and the corresponding dual basis of  $(\rho_1, \dots, \rho_n)$ ,

$$\psi = \rho(u_1)\rho_1 + \cdots + \rho(u_n)\rho_n.$$

• Show that the dual map of the identity operator on  $V$  is the identity operator on  $V'$ .

**SOLUTION:**  $I'(\varphi) = \varphi \circ I, \forall \varphi \in V'$ .  $\square$

• Suppose  $W$  is finite-dim and  $T \in \mathcal{L}(V, W)$ . Prove that  $T' = 0 \iff T = 0$ .

**SOLUTION:**  $T = 0 \Leftrightarrow T'(\varphi) = \varphi \circ T = 0$  for all  $\varphi \in V' \Leftrightarrow T' = 0$ .  $\square$

**13** Define  $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$  by  $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$ .

Let  $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$  denote the dual basis of the standard basis of  $\mathbf{R}^2$  and  $\mathbf{R}^3$ .

(a) Describe the linear functionals  $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$

For any  $(x, y, z) \in \mathbf{R}^3$ ,  $(T'(\varphi_1))(x, y, z) = 4x + 5y + 6z, (T'(\varphi_2))(x, y, z) = 7x + 8y + 9z$ .

(b) Write  $T'(\varphi_1)$  and  $T'(\varphi_2)$  as linear combinations of  $\psi_1, \psi_2, \psi_3$ .

$$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3, \quad T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3. \quad \square$$

**14** Define  $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$  by  $(Tp)(x) = x^2p(x) + p''(x)$  for each  $x \in \mathbf{R}$ .

(a) Suppose  $\varphi \in \mathcal{P}(\mathbf{R})'$  is defined by  $\varphi(p) = p'(4)$ . Describe  $T'(\varphi) \in \mathcal{P}(\mathbf{R})'$ .

$$(T'(\varphi))(p) = [x^2p(x) + p''(x)]'(4) = [2xp(x) + x^2p'(x) + p'''(x)](4) = 8p(4) + 16p'(4) + p'''(4).$$

(b) Suppose  $\varphi \in \mathcal{P}(\mathbf{R})'$  is defined by  $\varphi(p) = \int_0^1 p(x)dx$ . Evaluate  $(T'(\varphi))(x^3)$ .

$$(T'(\varphi))(x^3) = \int_0^1 (x^5 + 6x)dx = \int_0^1 (\frac{1}{6}x^6 + 3x^2)'dx = \frac{6}{19}.$$

□

• Suppose  $V$  and  $W$  are finite-dim and  $T \in \mathcal{L}(V, W)$ .

Prove that  $T$  is invertible if and only if  $T' \in \mathcal{L}(W', V')$  is invertible.

**SOLUTION:** By [3.108] and [3.110]. □

**16** Suppose  $V$  and  $W$  are finite-dim. Define  $\Gamma$  by  $\Gamma(T) = T'$  for any  $T \in \mathcal{L}(L, W)$ .

Prove that  $\Gamma$  is an isomorphism of  $\mathcal{L}(V, W)$  onto  $\mathcal{L}(W', V')$ .

**SOLUTION:**

$V, W$  are finite-dim  $\Rightarrow \dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$ . And by [3.101],  $\Gamma$  is linear.

又 Suppose  $\Gamma(T) = T' = 0$ . By Problem (15),  $T = 0$ . Thus  $T$  is injective  $\Rightarrow T$  is invertible. □

**17** Suppose  $U \subseteq V$ . Explain why  $U^0 = \{\varphi \in V' : U \subseteq \text{null}\varphi\}$ .

**SOLUTION:** Because for  $\varphi \in V'$ ,  $U \subseteq \text{null}\varphi \iff \forall u \in U, \varphi(u) = 0$ . By definition in [3.102]. □

**18**  $U \subseteq V$ . We have  $U = \{0\} \iff \forall \varphi \in V', U \subseteq \text{null}\varphi \iff U^0 = V'$ .

**19**  $U$  is a subspace of  $V$ . Prove that  $U = V \iff U_V^0 = \{0\} = V_V^0$ .

**SOLUTION:**

Suppose  $U_V^0 = \{0\}$ . Then  $U = V$ .

Conversely, suppose  $U = V$ , then  $U_V^0 = \{\varphi \in V' : V \subseteq \text{null}\varphi\}$ , therefore  $U_V^0 = \{0\}$ .

**20, 21** Suppose  $U$  and  $W$  are subsets of  $V$ . Prove that  $U \subseteq W \iff W^0 \subseteq U^0$ .

**SOLUTION:**

(a)  $U \subseteq W \Rightarrow \forall w \in W, u \in U \cap W = U, \forall \varphi \in W^0, \varphi(w) = 0 = \varphi(u) \Rightarrow \varphi \in U^0$ . Thus  $W^0 \subseteq U^0$ .

(b)  $W^0 \subseteq U^0 \Rightarrow \forall w \in W, u \in U, \varphi(w) = 0 \Rightarrow \varphi(u) = 0$ . Then  $\text{null}\varphi \supseteq W \Rightarrow \text{null}\varphi \supseteq U$ . Thus  $W \supseteq U$ . □

• **COROLLARY:**  $W^0 = U^0 \iff U = W$ .

**22** Prove that  $(U + W)^0 = U^0 \cap W^0$ .

**SOLUTION:**

$$(a) \left. \begin{array}{l} U \subseteq U + W \\ W \subseteq U + W \end{array} \right\} \Rightarrow \left. \begin{array}{l} (U + W)^0 \subseteq U^0 \\ (U + W)^0 \subseteq W^0 \end{array} \right\} \Rightarrow (U + W)^0 \subseteq U^0 \cap W^0.$$

(b)  $\forall \varphi \in U^0 \cap W^0, \varphi(u + w) = 0$ , where  $u \in U, w \in W \Rightarrow \varphi \in (U + W)^0$ . Thus  $(U + W)^0 \supseteq U^0 \cap W^0$ . □

**23** Prove that  $(U \cap W)^0 = U^0 + W^0$ .

**SOLUTION:**

$$(a) \left. \begin{array}{l} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array} \right\} \Rightarrow \left. \begin{array}{l} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array} \right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \supseteq U^0 \cap W^0.$$

(b)  $\forall \varphi \in U^0, \psi \in W^0$  and  $\forall v \in U \cap W, (\varphi + \psi)(v) = \varphi(v) + \psi(v) = 0$ . Thus  $U^0 + W^0 \subseteq (U \cap W)^0$ . □

• **COROLLARY:**

Suppose  $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$  is a collection of subspaces of  $V$ .

Then  $(\sum_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \bigcap_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$ ;

And  $(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \sum_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$ .

**24** Suppose  $V$  is finite-dim and  $U$  is a subspace of  $V$ .

Prove, using the pattern of [3.104], that  $\dim U + \dim U^0 = \dim V$ .

**SOLUTION:**

Let  $(u_1, \dots, u_m)$  be a basis of  $U$ , extend to a basis of  $V$  as  $(u_1, \dots, u_m, \dots, u_n)$ , and let  $(\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$  be the dual basis.

(a) Suppose  $\varphi \in \text{span}(\varphi_{m+1}, \dots, \varphi_n)$ , then  $\exists a_i \in \mathbf{F}$ ,  $\varphi = a_{m+1}\varphi_{m+1} + \dots + a_n\varphi_n$ .

For all  $u \in U$ ,  $\varphi(u) = 0$ . Thus  $\varphi \in U^0$ , getting  $\text{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq U^0$ .

(b) Suppose  $\varphi \in U^0$ , then  $\exists a_i \in \mathbf{F}$ ,  $\varphi = a_1\varphi_1 + \dots + a_m\varphi_m + \dots + a_n\varphi_n$ .

For all  $u_i \in U$ ,  $0 = \varphi(u_i) = \sum_{i=1}^n \varphi(u_i) = a_i$ . Then  $\varphi = a_{m+1}\varphi_{m+1} + \dots + a_n\varphi_n$ .

Thus  $\varphi \in \text{span}(\varphi_{m+1}, \dots, \varphi_n)$ , getting  $\text{span}(\varphi_{m+1}, \dots, \varphi_n) \supseteq U^0$ .

Hence  $\text{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0$ ,  $\dim U^0 = n - m = \dim V - \dim U$ .  $\square$

**25** Suppose  $U$  is a subspace of  $V$ . Explain why  $U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}$ .

**SOLUTION:** Note that  $U = \{v \in V : v \in U\}$  is a subspace of  $V$  and  $\varphi(v) = 0$  for every  $\varphi \in U^0 \iff v \in U$ .  $\square$

**26** Suppose  $V$  is finite-dim and  $\Omega$  is a subspace of  $V'$ .

Prove that  $\Omega = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}^0$ .

**SOLUTION:** Using the corollary in Problem (20, 21).

Suppose  $U = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}$ .

Getting  $U^0 = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}^0$ . We need to show that  $\Omega = U^0$ .

$$\left. \begin{array}{l} \text{(a) } \forall \varphi \in \Omega, v \in U, \varphi(v) = 0 \Rightarrow \varphi \in U^0 \Rightarrow \Omega \subseteq U^0. \\ \text{(b) } v \in U \Leftrightarrow \left\{ \begin{array}{l} \forall \varphi \in \Omega, \varphi(v) = 0 \\ \forall \psi \in U^0, \psi(v) = 0 \end{array} \right. \text{ Thus } \Omega \supseteq U^0. \end{array} \right\} \Rightarrow \square$$

**27** Suppose  $T \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$  and  $\text{null}T' = \text{span}(\varphi)$ , where  $\varphi$  is the linear functional on  $\mathcal{P}_5(\mathbf{R})$  defined by  $\varphi(p) = p(8)$ . Prove that  $\text{range}T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$ .

**SOLUTION:** By Problem (26),  $\text{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \psi(p) = 0, \forall \psi \in \text{span}(\varphi)\}^0$ ,

Hence  $\text{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \varphi(p) = p(8) = 0\}^0$ ,  $\text{span}(\varphi) = \text{null}T' = (\text{range}T)^0$ .

By the corollary in Problem (20, 21),  $\text{range}T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$ .  $\square$

**28, 29** Suppose  $V, W$  are finite-dim,  $T \in \mathcal{L}(V, W)$ .

(a) Suppose  $\exists \varphi \in W'$  such that  $\text{null}T' = \text{span}(\varphi)$ . Prove that  $\text{range}T = \text{null}\varphi$ .

(b) Suppose  $\exists \varphi \in V'$  such that  $\text{range}T' = \text{span}(\varphi)$ . Prove that  $\text{null}T = \text{null}\varphi$ .

**SOLUTION:** Using Problem (26), [3.107] and [3.109].

Because  $\text{span}(\varphi) = \{v \in V : \forall \psi \in \text{span}(\varphi), \psi(v) = 0\}^0 = \{v \in V : \varphi(v) = 0\}^0 = (\text{null}\varphi)^0$ .

$$\left. \begin{array}{l} \text{(a) } (\text{range}T)^0 = \text{null}T' = \text{span}(\varphi) = (\text{null}\varphi)^0 \iff \text{range}T = \text{null}\varphi. \\ \text{(b) } (\text{null}T)^0 = \text{range}T' = \text{span}(\varphi) = (\text{null}\varphi)^0 \iff \text{null}T = \text{null}\varphi. \end{array} \right\} \Rightarrow \square$$

**31** Suppose  $V$  is finite-dim and  $(\varphi_1, \dots, \varphi_n)$  is a basis of  $V'$ .

Show that there exists a basis of  $V$  whose dual basis is  $(\varphi_1, \dots, \varphi_n)$ .

**SOLUTION:** Using (3.B.29,30).

For each  $\varphi_i$ ,  $\text{null}\varphi_i \oplus \{a u_i : a \in \mathbf{F}\} = V$ .

Because  $\varphi_1, \dots, \varphi_m$  is linearly independent.  $\text{null}\varphi_i \neq \text{null}\varphi_j$  for all  $i, j \in \mathbf{N}^+$  such that  $i \neq j$ .

Thus  $(u_1, \dots, u_m)$  is linearly independent, for if not, then  $\exists i, j$  such that  $\text{null}\varphi_i = \text{null}\varphi_j$ , contradicts.

$\text{dim } V' = m = \dim V$ . Then  $(u_1, \dots, u_m)$  is a basis of  $V$  whose dual basis is  $\varphi_1, \dots, \varphi_n$ .  $\square$

• Suppose  $\dim$  and  $\varphi_1, \dots, \varphi_m \in V'$ . Prove that the following three sets are equal to each other.

- (a)  $\text{span}(\varphi_1, \dots, \varphi_m)$
- (b)  $((\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m))^0$
- (c)  $\{\varphi \in V' : (\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m) \subseteq \text{null}\varphi\}$

**SOLUTION:** By Problem (17), (b) and (c) are equivalent. By Problem (26) and the corollary in Problem (23),

$$\left. \begin{aligned} & ((\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m))^0 = (\text{null}\varphi_1)^0 + \dots + (\text{null}\varphi_m)^0. \\ & \text{又 } \text{span}(\varphi_i) = \{v \in V : \forall \psi \in \text{span}(\varphi_i), \psi(v) = 0\}^0 = (\text{null}\varphi_i)^0. \end{aligned} \right\} \Rightarrow (a) = (b). \quad \square$$

### 30 OR COROLLARY:

Suppose  $V$  is finite-dim and  $\varphi_1, \dots, \varphi_m$  is a linearly independent list in  $V'$ .

Then  $\dim((\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m)) = (\dim V) - m$ .

6 Define  $\Gamma : V' \rightarrow \mathbf{F}^m$  by  $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$ , where  $v_1, \dots, v_m \in V$ .

- (a) Show that  $\text{span}(v_1, \dots, v_m) = V \iff \Gamma$  is injective.
- (b) Show that  $v_1, \dots, v_m$  is linearly independent  $\iff \Gamma$  is surjective.

**SOLUTION:**

- (a)  $\left\{ \begin{array}{l} \text{Suppose } \Gamma \text{ is injective. Then let } \Gamma(\varphi) = 0, \text{ getting } \varphi = 0 \Leftrightarrow \text{null}\varphi = V = \text{span}(v_1, \dots, v_m). \\ \text{Suppose } \text{span}(v_1, \dots, v_m) = V. \text{ Then let } \Gamma(\varphi) = 0, \text{ getting } \varphi(v_i) = 0 \text{ for each } i, \\ \text{null}\varphi = \text{span}(v_1, \dots, v_m) = V, \text{ thus } \varphi = 0, \Gamma \text{ is injective.} \end{array} \right.$
- (b)  $\left\{ \begin{array}{l} \text{Suppose } \Gamma \text{ is surjective. Then let } \Gamma(\varphi_i) = e_i \text{ for each } i, \text{ where } e_1, \dots, e_m \text{ is the standard basis of } \mathbf{F}^m. \\ \text{Then } \varphi_1, \dots, \varphi_m \text{ is linearly independent, suppose } a_1 v_1 + \dots + a_m v_m = 0, \\ \text{then for each } i, \text{ we have } \varphi_i(a_1 v_1 + \dots + a_m v_m) = a_i = 0. \text{ Thus } v_1, \dots, v_m \text{ is linearly independent.} \\ \text{Suppose } v_1, \dots, v_m \text{ is linearly independent. Let } (\varphi_1, \dots, \varphi_m) \text{ be the dual basis of } \text{span}(v_1, \dots, v_m). \\ \text{Thus for each } (a_1, \dots, a_m) \in \mathbf{F}^m, \text{ we have } \varphi = a_1 \varphi_1 + \dots + a_m \varphi_m \text{ so that } \Gamma(\varphi) = (a_1, \dots, a_m). \quad \square \end{array} \right.$

• Define  $\Gamma : V \rightarrow \mathbf{F}^m$  by  $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$ , where  $\varphi_1, \dots, \varphi_m \in V'$ .

- (c) Show that  $\text{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$  is injective.
- (d) Show that  $\varphi_1, \dots, \varphi_m$  is linearly independent  $\iff \Gamma$  is surjective.

**SOLUTION:**

- (c)  $\left\{ \begin{array}{l} \text{Suppose } \Gamma \text{ is injective. Then } \Gamma(v) = 0 \Leftrightarrow \forall i, \varphi_i(v) = 0 \Leftrightarrow v \in (\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m) \Leftrightarrow v = 0. \\ \text{Getting } (\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m) = \{0\}. \text{ By Problem } (\bullet) \text{ above, } \text{span}(\varphi_1, \dots, \varphi_m) = V'. \\ \text{Suppose } \text{span}(\varphi_1, \dots, \varphi_m) = V'. \text{ Again by Problem } (\bullet), (\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m) = \{0\}. \\ \text{Thus } \Gamma(v) = 0 \Rightarrow \forall i, \varphi_i(v) = 0 \Rightarrow v = 0. \end{array} \right.$
- (d)  $\left\{ \begin{array}{l} \text{Suppose } \varphi_1, \dots, \varphi_m \text{ is linearly independent. Then by Problem (31), } (v_1, \dots, v_m) \text{ is linearly independent.} \\ \text{Thus for any } (a_1, \dots, a_m) \in \mathbf{F}, \text{ by letting } v = \sum_{i=1}^m a_i v_i, \text{ then } \varphi_i(v) = a_i \Rightarrow \Gamma(v) = (a_1, \dots, a_m). \\ \text{Suppose } \Gamma \text{ is surjective. Let } e_1, \dots, e_m \text{ be a basis of } \mathbf{F}^m. \\ \text{For every } e_i, \exists v_i \in V \text{ such that } \Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i, \\ \text{fix } v_i (\Rightarrow v_1, \dots, v_m \text{ is linearly independent}). \text{ Thus } \varphi_i(v_i) = 1, \varphi_i(v_j) = 0. \\ \text{Hence } (\varphi_1, \dots, \varphi_m) \text{ is the dual basis of the basis } v_1, \dots, v_m \text{ of } \text{span}(v_1, \dots, v_m). \quad \square \end{array} \right.$

33 Suppose  $A \in \mathbf{F}^{m,n}$ . Define  $T : A \rightarrow A^t$ . Prove that  $T$  is an isomorphism of  $\mathbf{F}^{m,n}$  onto  $\mathbf{F}^{n,m}$ .

**SOLUTION:** By [3.111],  $T$  is linear. Note that  $(A^t)^t = A$ .

- (a) For any  $B \in \mathbf{F}^{n,m}$ , let  $A = B^t$  so that  $T(A) = B$ . Thus  $T$  is surjective.
  - (b) If  $T(A) = 0$  for some  $A \in \mathbf{F}^{n,m}$ , then  $A = 0$ . Thus  $T$  is injective.
- for if not,  $\exists j, k \in \mathbf{N}^+$  such that  $A_{j,k} \neq 0$ , then  $T(A)_{k,j} \neq 0$ , contradicts.  $\Rightarrow \square$

**32** Suppose  $T \in \mathcal{L}(V)$ , and  $(u_1, \dots, u_m)$  and  $(v_1, \dots, v_m)$  are bases of  $V$ . Prove that  $T$  is invertible  $\iff$  The rows of  $\mathcal{M}(T, (u_1, \dots, u_m), (v_1, \dots, v_m))$  form a basis of  $\mathbf{F}^{1,n}$ .

**SOLUTION:** Note that  $T$  is invertible  $\Rightarrow T'$  is invertible. And  $\mathcal{M}(T') = \mathcal{M}(T)^t = A^t$ , denote it by  $B$ .

Let  $(\varphi_1, \dots, \varphi_m)$  be the dual basis of  $(v_1, \dots, v_m)$ ,  $(\psi_1, \dots, \psi_m)$  be the dual basis of  $(u_1, \dots, u_m)$ .

(a) Suppose  $T$  is invertible, so is  $T'$ . Because  $T'(\varphi_1), \dots, T'(\varphi_m)$  is linearly independent.

Noticing that  $T'(\varphi_i) = B_{1,i}\psi_1 + \dots + B_{m,i}\psi_m$ .

Thus the columns of  $B$ , namely the rows of  $A$ , are linearly independent (check it by contradiction).

(b) Suppose the rows of  $A$  are linearly independent, so are the columns of  $B$ .

Then  $(T'(\varphi_1), \dots, T'(\varphi_m))$  is a basis of range  $T'$ , namely  $V'$ . Thus  $T'$  is surjective.

Hence  $T'$  is invertible, so is  $T$ .  $\square$

**34** The double dual space of  $V$ , denoted by  $V''$ , is defined to be the dual space of  $V'$ .

In other words,  $V'' = \mathcal{L}(V', \mathbf{F})$ . Define  $\Lambda : V \rightarrow V''$  by  $(\Lambda v)(\varphi) = \varphi(v)$ .

(a) Show that  $\Lambda$  is a linear map from  $V$  to  $V''$ .

(b) Show that if  $T \in \mathcal{L}(V)$ , then  $T'' \circ \Lambda = \Lambda \circ T$ , where  $T'' = (T')'$ .

(c) Show that if  $V$  is finite-dim, then  $\Lambda$  is an isomorphism from  $V$  onto  $V''$ .

Suppose  $V$  is finite-dim. Then  $V$  and  $V'$  are isomorphic, but finding an isomorphism from  $V$  onto  $V'$  generally requires choosing a basis of  $V$ . In contrast, the isomorphism  $\Lambda$  from  $V$  onto  $V''$  does not require a choice of basis and thus is considered more natural.

**SOLUTION:**

(a)  $\forall \varphi \in V', \forall v, w \in V, a \in \mathbf{F}, (\Lambda(v + aw))(\varphi) = \varphi(v + aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$ .

Thus  $\Lambda(v + aw) = \Lambda v + a\Lambda w$ . Hence  $\Lambda$  is linear.

(b)  $(T''(\Lambda v))(\varphi) = ((\Lambda v) \circ (T'))(\varphi) = (\Lambda v)(T'(\varphi)) = (T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = (\Lambda(Tv))(\varphi)$ .

Hence  $T''(\Lambda v) = (\Lambda(Tv))$ , getting  $T'' \circ \Lambda = \Lambda \circ T$ .

(c) Suppose  $\Lambda v = 0$ . Then  $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$ . Thus  $\Lambda$  is injective.

又 Because  $V$  is finite-dim.  $\dim V = \dim V' = \dim V''$ . Hence  $\Lambda$  is an isomorphism.  $\square$

**36** Suppose  $U$  is a subspace of  $V$ . Define  $i : U \rightarrow V$  by  $i(u) = u$ . Thus  $i' \in \mathcal{L}(V', U')$ .

(a) Show that  $\text{null } i' = U^0$ :  $\text{null } i' = (\text{range } i)^0 = U^0 \Leftarrow \text{range } i = U$ .  $\square$

(b) Prove that if  $V$  is finite-dim, then  $\text{range } i' = U'$ :  $\text{range } i' = (\text{null } i)_U^0 = (\{0\})_U^0 = U'$ .  $\square$

(c) Prove that if  $V$  is finite-dim, then  $\tilde{i}'$  is an isomorphism from  $V'/U^0$  onto  $U'$ :

Note that  $\tilde{i}' : V'/\text{null } i' \rightarrow \text{range } i' \Rightarrow \tilde{i}' : V'/U^0 \rightarrow U'$ . By (a), (b) and [3.91(d)].  $\square$

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

**37** Suppose  $U$  is a subspace of  $V$  and  $\pi$  is the quotient map. Thus  $\pi' \in \mathcal{L}((V/U)', V')$ .

(a) Show that  $\pi'$  is injective: Because  $\pi$  is surjective. Use [3.108].  $\square$

(b) Show that  $\text{range } \pi' = U^0$ .

(c) Conclude that  $\pi'$  is an isomorphism from  $(V/U)'$  onto  $U^0$ .

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

In fact, there is no assumption here that any of these vector spaces are finite-dimensional.

**SOLUTION:** [3.109] is not available. Using (3.E.18), also see (3.E.20).

(b)  $\psi \in \text{range } \pi' \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \text{null } \psi \supseteq U \iff \psi \in U^0$ . Hence  $\text{range } \pi' = U^0$ .

(c)  $\psi \in U^0 \iff \text{null } \psi \supseteq U \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi = \pi(\varphi)$ . Thus  $\pi'$  is surjective. And by (a).  $\square$

• **NOTE FOR [4.8]:** *division algorithm for polynomials*

Suppose  $p, s \in \mathcal{P}(\mathbf{F})$ , with  $s \neq 0$ . Then  $\exists! q, r \in \mathcal{P}(\mathbf{F})$  such that  $p = sq + r$  and  $\deg r < \deg s$ . *Another Proof:*

Suppose  $\deg p \geq \deg s$ . Then  $\underbrace{(1, z, \dots, z^{\deg s-1})}_{\text{of length } \deg s}, \underbrace{(s, zs, \dots, z^{\deg p-\deg s}s)}_{\text{of length } (\deg p-\deg s+1)}$  is a basis of  $\mathcal{P}_{\deg p}(\mathbf{F})$ .

Because  $q \in \mathcal{P}(\mathbf{F})$ ,  $\exists! a_i, b_j \in \mathbf{F}$ ,

$$\begin{aligned} q &= a_0 + a_1z + \dots + a_{\deg s-1}z^{\deg s-1} + b_0s + b_1zs + \dots + b_{\deg p-\deg s}s z^{\deg p-\deg s} \\ &= \underbrace{a_0 + a_1z + \dots + a_{\deg s-1}z^{\deg s-1}}_r + s \underbrace{(b_0 + b_1z + \dots + b_{\deg p-\deg s}s z^{\deg p-\deg s})}_q. \end{aligned}$$

With  $r, q$  as defined uniquely above, we are done.  $\square$

• **NOTE FOR [4.11]:** *each zero of a polynomial corresponds to a degree-one factor; Another Proof:*

First suppose  $p(\lambda) = 0$ . Write  $p(z) = a_0 + a_1z + \dots + a_mz^m$ ,  $\exists! a_0, a_1, \dots, a_m \in \mathbf{F}$  for all  $z \in \mathbf{F}$ .

Then  $p(z) = p(z) - p(\lambda) = a_1(z - \lambda) + \dots + a_m(z^m - \lambda^m)$  for all  $z \in \mathbf{F}$ .

Hence for each  $k \in \{1, \dots, m\}$ ,  $z^k - \lambda^k = (z - \lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \dots + \dots + z\lambda^{k-2} + z^0\lambda^{k-1})$ .

Thus  $p(z) = \sum_{j=1}^m a_j(z - \lambda) \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z - \lambda) \sum_{j=1}^m a_j \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z - \lambda)q(z)$ .

• **NOTE FOR [4.13]:** *fundamental theorem of algebra, first version*

*Every nonconstant polynomial with complex coefficients has a zero in  $\mathbf{C}$ . Another Proof:*

De Moivre's theorem (which you can prove using induction on  $k$  and the addition formulas for cosine and sine), states that if  $k \in \mathbf{N}^+$ ,  $\theta \in \mathbf{R}$ , then  $(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$ .

Suppose  $w \in \mathbf{C}$ ,  $k \in \mathbf{N}^+$  and using polar coordinates.  $\exists r \geq 0, \theta \in \mathbf{R}$  such that  $r(\cos \theta + i \sin \theta) = w$ .

Hence  $(r^{1/k}(\cos \frac{\theta}{k} + i \sin \frac{\theta}{k}))^k = w$ . Thus every complex number has a  $k^{\text{th}}$  root, a fact that we will soon use.

Suppose a nonconstant  $p \in \mathcal{P}(\mathbf{C})$  with highest-order nonzero term  $c_m z^m$ .

Then  $|p(z)| \rightarrow \infty$  as  $|z| \rightarrow \infty$  (because  $\frac{|p(z)|}{|z_m|} \rightarrow |c_m|$  as  $|z| \rightarrow \infty$ ).

Thus the continuous function  $z \rightarrow |p(z)|$  has a global minimum at some point  $\zeta \in \mathbf{C}$ .

To show that  $p(\zeta) = 0$ , suppose that  $p(\zeta) \neq 0$ .

Define  $q \in \mathcal{P}(\mathbf{C})$  by  $q(z) = \frac{p(z + \zeta)}{p(\zeta)}$ .

The function  $z \rightarrow |q(z)|$  has a global minimum value of 1 at  $z = 0$ .

Write  $q(z) = 1 + a_k z^k + \dots + a_m z^m$ , where  $k$  is the smallest positive integer such that  $a_k \neq 0$ .

Let  $\beta \in \mathbf{C}$  be such that  $\beta^k = -\frac{1}{a_k}$ .

There is a constant  $c > 1$  such that if  $t \in (0, 1)$ ,

then  $|q(t\beta)| \leq |1 + a_k t^k \beta^k| + t^{k+1}c = 1 - t^k(1 - tc)$ .

Thus taking  $t$  to be  $1/(2c)$  in the inequality above, we have  $|q(t\beta)| < 1$ ,

which contradicts the assumption that the global minimum of  $z \rightarrow |q(z)|$  is 1.

Hence  $p(\zeta) = 0$ , as desired.  $\square$

• Prove that if  $w, z \in \mathbf{C}$ , then  $||w| - |z|| \leq |w - z|$ . The inequality here is called the **reverse triangle inequality**.

**SOLUTION:**

$$\begin{aligned}
 |w - z|^2 &= (w - z)(\overline{w} - \overline{z}) \\
 &= |w|^2 + |z|^2 - (w\overline{z} + \overline{w}z) \\
 &= |w|^2 + |z|^2 - (\overline{wz} + \overline{wz}) \\
 &= |w|^2 + |z|^2 - 2\operatorname{Re}(\overline{w}z) \\
 &\geq |w|^2 + |z|^2 - 2|\overline{w}z| \\
 &= |w|^2 + |z|^2 - 2|w||z| = ||w| - |z||^2. \quad \square
 \end{aligned}$$

*Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides.*

• Suppose  $V$  is a complex vector space and  $\varphi \in V'$ .

Define  $\sigma : V \rightarrow \mathbf{R}$  by  $\sigma(v) = \Re\varphi(v)$  for each  $v \in V$ .

Show that  $\varphi(v) = \sigma(v) - i\sigma(iv)$  for all  $v \in V$ .

**SOLUTION:**

Notice that  $\varphi(v) = \Re\varphi(v) + i\Im\varphi(v) = \sigma(v) + i\Im\varphi(v)$ . 又  $\Re\varphi(iv) = \Re[i\varphi(v)] = -\Im\varphi(v) = \sigma(iv)$ .

Hence  $\varphi(v) = \sigma(v) - i\sigma(iv)$ .  $\square$

2 Suppose  $m$  is a positive integer. Is the set  $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$  a subspace of  $\mathcal{P}(\mathbf{F})$ ?

**SOLUTION:**

$x^m, x^m + x^{m-1} \in U$  but  $\deg[(x^m + x^{m-1}) - (x^m)] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$ .

Hence  $U$  is not closed under addition, and therefore is not a subspace.  $\square$

3 Suppose  $m$  is a positive integer. Is the set  $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p \text{ is even}\}$  a subspace of  $\mathcal{P}(\mathbf{F})$ ?

**SOLUTION:**

$x^2, x^2 + x \in U$  but  $\deg[(x^2 + x) - (x^2)]$  is odd and hence  $(x^2 + x) - (x^2) \notin U$ .

Thus  $U$  is not closed under addition, and therefore is not a subspace.  $\square$

4 Suppose that  $m$  and  $n$  are positive integers with  $m \leq n$ , and suppose  $\lambda_1, \dots, \lambda_m \in \mathbf{F}$ .

Prove that  $\exists p \in \mathcal{P}(\mathbf{F})$  such that  $\deg p = n$ , the zeros of  $p$  are  $\lambda_1, \dots, \lambda_m$ .

**SOLUTION:** Let  $p(z) = (z - \lambda_1)^{n-(m-1)}(z - \lambda_2) \cdots (z - \lambda_m)$ .  $\square$

5 Suppose that  $m \in \mathbf{N}$ ,  $z_1, \dots, z_{m+1}$  are distinct elements of  $\mathbf{F}$ , and  $w_1, \dots, w_{m+1} \in \mathbf{F}$ .

Prove that  $\exists! p \in \mathcal{P}_m(\mathbf{F})$  such that  $p(z_k) = w_k$  for each  $k = 1, \dots, m+1$ .

*This result can be proved without using linear algebra. However, try to find the clearer, shorter proof that uses some linear algebra.*

**SOLUTION:**

Define  $T : \mathcal{P}_m(\mathbf{F}) \rightarrow \mathbf{F}^{m+1}$  by  $Tq = (q(z_1), \dots, q(z_m), q(z_{m+1}))$ . As can be easily checked,  $T$  is linear.

We need to show that  $T$  is surjective, so that such  $p$  exists; and that  $T$  is injective, so that such  $p$  is unique.

$$Tq = 0 \iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0$$

$$\iff q \in \mathcal{P}_m(\mathbf{F}) \text{ is the zero polynomial, for if not,}$$

$q$  has at least  $m+1$  distinct roots, while  $\deg q = m$ . Contradicts (by [4.12]). Hence  $T$  is injective.

$\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m+1 = \dim \mathbf{F}^{m+1}$ . 又  $\operatorname{range} T \subseteq \mathbf{F}^{m+1}$ . Hence  $T$  is surjective.  $\square$

**6** Suppose  $p \in \mathcal{P}_m(\mathbf{C})$  has degree  $m$ . Prove that

$p$  has  $m$  distinct zeros  $\iff p$  and its derivative  $p'$  have no zeros in common.

**SOLUTION:**

(a) Suppose  $p$  has  $m$  distinct zeros. By [4.14] and  $\deg p = m$ , let  $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$ ,  $\exists! c, \lambda_i \in \mathbf{C}$ .

For each  $j \in \{1, \dots, m\}$ , let  $\frac{p(z)}{(z - \lambda_j)} = q_j \in \mathcal{P}_{m-1}(\mathbf{C})$ , then  $p(z) = (z - \lambda_j)q_j(z)$  and  $q_j(\lambda_j) \neq 0$ .

$p'(z) = (z - \lambda_j)q_j'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$ , as desired.

(b) To prove the implication on the other direction, we prove the contrapositive:

Suppose  $p$  has less than  $m$  distinct roots.

We must show that  $p$  and its derivative  $p'$  have at least one zero in common.

Let  $\lambda$  be a zero of  $p$ , then write  $p(z) = (z - \lambda)^n q(z)$ ,  $\exists! n \in \mathbf{N}^+, q \in \mathcal{P}_{m-n}(\mathbf{C})$ .

$p'(z) = (z - \lambda)^n q'(z) + n(z - \lambda)^{n-1} q(z) \Rightarrow p'(\lambda) = 0$ ,  $\lambda$  is a common root of  $p'$  and  $p$ .  $\square$

**7** Prove that every polynomial of odd degree with real coefficients has a real zero.

**SOLUTION:**

Using the notation proof of [4.17].  $\deg p = 2M + m$  is odd  $\Rightarrow m$  is odd. Hence  $\lambda_1$  exists.  $\square$

OR. Using calculus but not using [4.17].

Suppose  $p \in \mathcal{P}_m(\mathbf{F})$ ,  $\deg p = m$ ,  $m$  is odd.

Let  $p(x) = a_0 + a_1x + \cdots + a_mx^m$ . Then  $a_m \neq 0$ . Denote  $|a_m|^{-1}a_m$  by  $\delta$

Write  $p(x) = x^m \left( \frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \cdots + \frac{a_{m-1}}{x} + a_m \right)$ .

Thus  $p(x)$  is continuous, and  $\lim_{x \rightarrow -\infty} p(x) = -\delta\infty$ ;  $\lim_{x \rightarrow \infty} p(x) = \delta\infty$ .

Hence we conclude that  $p$  has at least one real zero.  $\square$

**8** For  $p \in \mathcal{P}(\mathbf{R})$ , define  $Tp : \mathbf{R} \rightarrow \mathbf{R}$  by  $(Tp)(x) = \begin{cases} \frac{p(x) - p(3)}{x - 3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3 \end{cases}$  for all  $x \in \mathbf{R}$ .

Show that  $Tp \in \mathcal{P}(\mathbf{R})$  for all  $p \in \mathcal{P}(\mathbf{R})$  and that  $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$  is a linear map.

**SOLUTION:**

For  $x \neq 3$ ,  $T(x^n) = \frac{x^n - 3^n}{x - 3} = \sum_{i=1}^n 3^{i-1}x^{n-i}$ .

For  $x = 3$ ,  $T(x^n) = 3^{n-1} \cdot n$ . Note that if  $x = 3$ , then  $\sum_{i=1}^n 3^{i-1}x^{n-i} = \sum_{i=1}^n 3^{n-1} = 3^{n-1} \cdot n$ .

Hence for all  $x \in \mathbf{R}$  and for all  $n \in \mathbf{N}$ ,  $T(x^n) = \sum_{i=1}^n 3^{i-1}x^{n-i} \in \mathcal{P}(\mathbf{R})$ .

Because  $T$  is linear, we conclude that  $Tp \in \mathcal{P}(\mathbf{R})$  for all  $p \in \mathcal{P}(\mathbf{R})$ .

Now we show that  $T$  is linear:

$\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, T(p + \lambda q)(x) = \begin{cases} \frac{(p + \lambda q)(x) - (p + \lambda q)(3)}{x - 3} & \text{if } x \neq 3, \\ (p + \lambda q)'(3) & \text{if } x = 3 \end{cases}$  for all  $x \in \mathbf{R}$ .

Notice that  $(p + \lambda q)(x) - (p + \lambda q)(3) = (p(x) - p(3)) + (\lambda q(x) - \lambda q(3))$ ;

$(p + \lambda q)'(3) = p'(3) + \lambda q'(3)$ .

Thus  $T(p + \lambda q)(x) = (T(p) + \lambda T(q))(x)$  for all  $x \in \mathbf{R}$ .  $\square$



**9** Suppose  $p \in \mathcal{P}(\mathbf{C})$ . Define  $q : \mathbf{C} \rightarrow \mathbf{C}$  by  $q(z) = p(z)\overline{p(\bar{z})}$ .

Prove that  $q$  is a polynomial with real coefficients.

**SOLUTION:**

$$p(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow p(\bar{z}) = \overline{a_n z^n + \cdots + a_1 z + a_0} = \overline{a_n} \bar{z}^n + \cdots + \overline{a_1} \bar{z} + \overline{a_0}.$$

$$\text{Note that } q(z) = p(z)\overline{p(\bar{z})} = \overline{p(\bar{z})}p(z) = \overline{p(\bar{z})p(z)} = \overline{p(\bar{z})p(\bar{\bar{z}})} = \overline{p(\bar{z})p(\bar{z})} = \overline{p(\bar{z})}p(z).$$

$$\text{Hence letting } q(z) = c_m z^m + \cdots + c_1 z + c_0 \Rightarrow \overline{c_k} = c_k, c_k \in \mathbf{R} \text{ for each } k. \quad \square$$

**10** Suppose  $m \in \mathbf{N}$  and  $p \in \mathcal{P}_m(\mathbf{C})$  is such that

there are  $(m+1)$  distinct real numbers  $x_0, x_1, \dots, x_m$  with  $p(x_k) \in \mathbf{R}$  for each  $x_k$ .

Prove that all coefficients of  $p$  are real.

**SOLUTION:** Let  $p(x_k) = y_k$  for each  $k$ . By Problem (5),  $\exists! q \in \mathcal{P}_m(\mathbf{R})$  such that  $q(x_k) = y_k$ . Hence  $p = q$ .  $\square$

OR. Using the Lagrange Interpolating Polynomial.

$$\text{Define } q(x) = \sum_{j=0}^m \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j).$$

$$\text{又 For each } j, x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R}) \subseteq \mathcal{P}_m(\mathbf{C}).$$

$$\text{Notice that } q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0 \text{ for each } k \in \{0, 1, \dots, m\}.$$

$$\text{Then } (q-p) \text{ has } (m+1) \text{ distinct zeros, while } (q-p) \in \mathcal{P}_m(\mathbf{C}). \text{ Hence by [4.12], } q-p=0 \Rightarrow p=q. \quad \square$$

**11** Suppose  $p \in \mathcal{P}(\mathbf{F})$  with  $p \neq 0$ . Let  $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$ .

(a) Show that  $\dim \mathcal{P}(\mathbf{F})/U = \deg p$ .

(b) Find a basis of  $\mathcal{P}(\mathbf{F})/U$ .

**SOLUTION:**

$$U \text{ is a subspace of } \mathcal{P}(\mathbf{F}) \text{ because } \forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, pf + \lambda pg = p(f + \lambda g) \in U.$$

$$\text{NOTE: Define } P : \mathcal{P}(\mathbf{F}) \rightarrow \mathcal{P}(\mathbf{F}) \text{ by } (Pq)(x) = p(q(x)) = (p \circ q)(x) (\neq p(x)q(x)). P \text{ is not linear.}$$

$$(a) \text{ By [4.8], } \forall f \in \mathcal{P}(\mathbf{F}), \exists! q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r); \deg r < \deg p.$$

$$\text{Hence } \forall f \in \mathcal{P}(\mathbf{F}), \exists! pq \in U, r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), f = (pq) + (r); r \notin U.$$

$$\text{Thus } \mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F}). \text{ Therefore } \mathcal{P}(\mathbf{F})/U \text{ and } \mathcal{P}_{\deg p-1}(\mathbf{F}) \text{ are isomorphic.}$$

$$\text{OR. } \forall f \in \mathcal{P}(\mathbf{F}), \exists! q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r); \deg r < \deg p.$$

$$\text{Define } R : \mathcal{P}(\mathbf{F}) \rightarrow \mathcal{P}_{\deg p-1}(\mathbf{F}) \text{ by } (Rf)(z) = r(z) \text{ for each } z \in \mathbf{F}.$$

$$\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, R(f + \lambda g)(z) = R(f)(z) + \lambda R(g)(z).$$

$$\text{BECAUSE: } \forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F},$$

$$\exists! q_1, r_1 \in \mathcal{P}(\mathbf{F}), f = (p)q_1 + (r_1), \deg r_1 < \deg p;$$

$$\exists! q_2, r_2 \in \mathcal{P}(\mathbf{F}), g = (p)q_2 + (r_2), \deg r_2 < \deg p;$$

$$\exists! q_3, r_3 \in \mathcal{P}(\mathbf{F}), \lambda g = (p)q_3 + (r_3) = (p)(\lambda q_2) + (\lambda r_2), \deg r_3 < \deg p \text{ and } \deg \lambda r_2 < \deg p.$$

$$\Rightarrow q_3 = \lambda q_2, r_3 = \lambda r_2.$$

$$\exists! q_0, r_0 \in \mathcal{P}(\mathbf{F}), (f + \lambda g) = (p)q_0 + (r_0)$$

$$= (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2), \deg r_0 < \deg p \text{ and } \deg(r_1 + \lambda r_2) < \deg p.$$

$$\Rightarrow q_1 + \lambda q_2 = q_0; r_1 + \lambda r_2 = r_0.$$

Hence  $R$  is linear.

$$R(f) = 0 \iff f = pq, \exists! q \in \mathcal{P}(\mathbf{F}). \text{ Thus null } R = U.$$

$$\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), \text{ let } f = p + r, \text{ then } R(f) = r. \text{ Thus range } R = \mathcal{P}_{\deg p-1}(\mathbf{F}).$$

$$\text{Finally, by [3.91(d)], } \mathcal{P}(\mathbf{F})/\text{null } R, \text{ namely } \mathcal{P}(\mathbf{F})/U, \text{ and range } R, \text{ namely } \mathcal{P}_{\deg p-1}(\mathbf{F}), \text{ are isomorphic.}$$

$$(b) (1 + U, x + U, \dots, x^{\deg p-1} + U) \text{ can be a basis of } \mathcal{P}(\mathbf{F})/U. \quad \square$$

- Suppose nonconstant  $p, q \in \mathcal{P}(\mathbf{C})$  have no zeros in common. Let  $m = \deg p$ ,  $n = \deg q$ . Use (a)—(c) below to prove that  $\exists! r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$  such that  $rp + sq = 1$ .
  - (a) Define  $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \rightarrow \mathcal{P}_{m+n-1}(\mathbf{C})$  by  $T(r, s) = rp + sq$ . Show that the linear map  $T$  is injective.
  - (b) Show that the linear map  $T$  in (a) is surjective.
  - (c) Use (b) to conclude that  $\exists! r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$  such that  $rp + sq = 1$ .

**SOLUTION:**

- (a)  $T$  is linear because  $\forall r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$ ,  

$$T((r_1, s_1) + \lambda(r_2, s_2)) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1, s_1) + \lambda T(r_2, s_2).$$

Suppose  $T(r, s) = rp + sq = 0$ . Notice that  $p, q$  have no zeros in common.  
 Then  $r = s = 0$ , for if not, write  $\frac{q(z)}{r(z)} = \frac{p(z)}{s(z)}$ , while for any zero  $\lambda$  of  $q$ ,  $\frac{q(\lambda)}{r(\lambda)} = 0 \neq \frac{p(\lambda)}{s(\lambda)}$ . Hence  $\square$
- (b)  $\dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) = \dim \mathcal{P}_{n-1}(\mathbf{C}) + \dim \mathcal{P}_{m-1}(\mathbf{C}) = n + m = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$ .  
 $\bowtie$   $T$  is injective. Hence  $\dim \text{range } T = \dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) - \dim \text{null } T = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$ .  
 Thus  $\text{range } T = \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$  is surjective, and therefore is an isomorphism.  $\square$
- (c) Immediately.  $\square$

**ENDED**

## 5.A

**1** Suppose  $T \in \mathcal{L}(V)$  and  $U$  is a subspace of  $V$ .

- (a) Prove that if  $U \subseteq \text{null } T$ , then  $U$  is invariant under  $T$ .
- (b) Prove that if  $\text{range } T \subseteq U$ , then  $U$  is invariant under  $T$ .

**SOLUTION:**

- (a)  $\forall u \in U \subseteq \text{range } T, Tu = 0 \in U$ .  $\square$
- (b)  $\forall u \in U \subseteq V, Tu \in \text{range } T \subseteq U$ .  $\square$

• Suppose  $S, T \in \mathcal{L}(V)$  are such that  $ST = TS$ .

- (a) Prove that  $\text{null } (T - \lambda I)$  is invariant under  $S$ , where  $\lambda$  is chosen arbitrarily.
- (b) Prove that  $\text{range } (T - \lambda I)$  is invariant under  $S$ , where  $\lambda$  is chosen arbitrarily.

**SOLUTION:**

Note that  $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$ .

- (a) Suppose  $v \in \text{null } (T - \lambda I)$ , then  $(T - \lambda I)(Sv) = S(T - \lambda I)v = S(0) = 0$ .

Hence  $Sv \in \text{null } (T - \lambda I)$  and therefore  $\text{null } (T - \lambda I)$  is invariant under  $S$ .

- (b) Suppose  $v \in \text{range } (T - \lambda I)$ , therefore  $\exists u \in V, (T - \lambda I)u = v$ .

Then  $Sv = S(T - \lambda I)u = (T - \lambda I)(Su) \in \text{range } (T - \lambda I)$ .

Hence  $Sv \in \text{range } (T - \lambda I)$  and therefore  $\text{range } (T - \lambda I)$  is invariant under  $S$ .  $\square$

COMMENT: Reversing the roles of  $S$  and  $T$ , letting  $\lambda = 0$ , we can conclude that

$\text{null } S$  and  $\text{range } S$  is invariant under  $T$ , which is what we will prove in Problem (2) and (3) below.

**2** Suppose  $S, T \in \mathcal{L}(V)$  are such that  $ST = TS$ . Prove that  $\text{null } S$  is invariant under  $T$ .

**SOLUTION:**  $\forall u \in \text{null } S, Su = 0 \Rightarrow TSu = 0 = STu \Rightarrow Tu \in \text{null } S$ .  $\square$

**3** Suppose  $S, T \in \mathcal{L}(V)$  are such that  $ST = TS$ . Prove that  $\text{range } S$  is invariant under  $T$ .

**SOLUTION:**  $\forall w \in \text{range } S, \exists v \in V, Sv = w, STv = TSv = Tw \in \text{range } S$ .  $\square$

**4** Suppose  $T \in \mathcal{L}(V)$  and  $V_1, \dots, V_m$  are subspaces of  $V$  invariant under  $T$ .  
 Prove that  $V_1 + \dots + V_m$  is invariant under  $T$ .

**SOLUTION:**

For each  $i = 1, \dots, m, \forall v_i \in V_i, Tv_i \in V_i$

Hence  $\forall v = v_1 + \dots + v_m \in V_1 + \dots + V_m, Tv = Tv_1 + \dots + Tv_m \in V_1 + \dots + V_m$ .  $\square$

**5** Suppose  $T \in \mathcal{L}(V)$ . Prove that the intersection of every collection of subspaces of  $V$  invariant under  $T$  is invariant under  $T$ .

**SOLUTION:**

Suppose  $\{V_\alpha\}_{\alpha \in \Gamma}$  is a collection of subspaces of  $V$  invariant under  $T$ ; here  $\Gamma$  is an arbitrary index set.

We need to prove that  $\bigcap_{\alpha \in \Gamma} V_\alpha$ , which equals the set of vectors

that are in  $V_\alpha$  for each  $\alpha \in \Gamma$ , is invariant under  $T$ .

For each  $\alpha \in \Gamma, \forall v_\alpha \in V_\alpha, Tv_\alpha \in V_\alpha$ .

Hence  $\forall v \in \bigcap_{\alpha \in \Gamma} V_\alpha, Tv \in V_\alpha, \forall \alpha \in \Gamma \Rightarrow Tv \in \bigcap_{\alpha \in \Gamma} V_\alpha$ . Thus  $\bigcap_{\alpha \in \Gamma} V_\alpha$  is invariant under  $T$ .  $\square$

**6** Prove or give a counterexample:

If  $V$  is finite-dim and  $U$  is a subspace of  $V$  that is invariant under every operator on  $V$ , then  $U = \{0\}$  or  $U = V$ .

**SOLUTION:**

Notice that  $V$  might be  $\{0\}$ . In this case we are done.

Suppose  $\dim V \geq 1$ . We prove by contrapositive:

Suppose  $U \neq \{0\}$  and  $U \neq V$ , then  $\exists T \in \mathcal{L}(V)$  such that  $U$  is not invariant under  $T$ .

Let  $W$  be such that  $V = U \oplus W$ .

Let  $(u_1, \dots, u_m)$  be a basis of  $U$  and  $(w_1, \dots, w_n)$  be a basis of  $W$ .

Hence  $(u_1, \dots, u_m, w_1, \dots, w_n)$  is a basis of  $V$ .

Define  $T \in \mathcal{L}(V)$  by  $T(a_1u_1 + \dots + a_mu_m + b_1w_1 + \dots + b_nw_n) = b_1w_1 + \dots + b_nw_n$ .  $\square$

**7** Suppose  $T \in \mathcal{L}(\mathbf{R}^2)$  is defined by  $T(x, y) = (-3y, x)$ . Find the eigenvalues of  $T$ .

**SOLUTION:**

Suppose  $\lambda \in \mathbf{R}$  and  $(x, y) \in \mathbf{R}^2 \setminus \{0\}$  such that  $T(x, y) = (-3y, x) = \lambda(x, y)$ . Then  $-3y = \lambda x$  and  $x = \lambda y$ .

Thus  $-3y = \lambda^2 y \Rightarrow \lambda^2 = -3$ , ignoring the possibility of  $y = 0$  (because if  $y = 0$ , then  $x = 0$ ).

Hence the set of solution for this equation is  $\emptyset$ , and therefore  $T$  has no eigenvalues in  $\mathbf{R}$ .  $\square$

**8** Define  $T \in \mathcal{L}(\mathbf{F}^2)$  by  $T(w, z) = (z, w)$ . Find all eigenvalues and eigenvectors of  $T$ .

**SOLUTION:**

Suppose  $\lambda \in \mathbf{F}$  and  $(w, z) \in \mathbf{F}^2$  such that  $T(w, z) = (z, w) = \lambda(w, z)$ . Then  $z = \lambda w$  and  $w = \lambda z$ .

Thus  $z = \lambda^2 z \Rightarrow \lambda^2 = 1$ , ignoring the possibility of  $z = 0$  ( $z = 0 \Rightarrow w = 0$ ).

Hence  $\lambda_1 = -1$  and  $\lambda_2 = 1$  are all eigenvalues of  $T$ .

For  $\lambda_1 = -1, z = -w, w = -z$ ; For  $\lambda_2 = 1, z = w$ .

Thus the set of all eigenvectors is  $\{(z, -z), (z, z) : z \in \mathbf{F} \wedge z \neq 0\}$ .  $\square$

**9** Define  $T \in \mathcal{L}(\mathbf{F}^3)$  by  $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$ .

Find all eigenvalues and eigenvectors of  $T$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $(z_1, z_2, z_3) \in \mathbf{F}^3$ .

Then  $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$ .

Thus  $2z_2 = \lambda z_1$ ,  $0 = \lambda z_2$ ,  $5z_3 = \lambda z_3$ .

We discuss in two cases:

For  $\lambda = 0$ ,  $z_2 = z_3 = 0$  and  $z_1$  can be arbitrary ( $z_1 \neq 0$ ).

For  $\lambda \neq 0$ ,  $z_2 = 0 = z_1$ , and  $z_3$  can be arbitrary ( $z_3 \neq 0$ ), then  $\lambda = 5$ .

The set of all eigenvectors is  $\{(0, 0, z), (z, 0, 0) : z \in \mathbf{F} \wedge z \neq 0\}$ .  $\square$

---

• Suppose  $P \in \mathcal{L}(V)$  is such that  $P^2 = P$ .

Prove that if  $\lambda$  is an eigenvalue of  $P$ , then  $\lambda = 0$  or  $\lambda = 1$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue,  $v \in V \setminus \{0\}$  such that  $Pv = \lambda v$ , then  $P(Pv) = \lambda^2 v = \lambda v = Pv$ . Thus  $\lambda^2 = \lambda$ .  $\square$

---

**10** Define  $T \in \mathcal{L}(\mathbf{F}^n)$  by  $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$

(a) Find all eigenvalues and eigenvectors of  $T$ .

(b) Find all invariant subspaces of  $V$  under  $T$ .

**SOLUTION:**

(a) Suppose  $v = (x_1, x_2, x_3, \dots, x_n)$  is an eigenvector of  $T$  with an eigenvalue  $\lambda$ .

Then  $Tv = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$ .

Hence  $1, \dots, n$  are eigenvalues of  $T$ .

And  $\{(0, \dots, 0, x_\lambda, 0, \dots, 0) \in \mathbf{F}^n : \lambda = 1, \dots, n, x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$  is the set of all eigenvectors of  $T$ .

(b) Let  $V_\lambda = \{(0, \dots, 0, x_\lambda, 0, \dots, 0) \in \mathbf{F}^n : x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$ . Then  $V_1, \dots, V_n$  are invariant under  $T$ .

Hence by Problem (4), every sum of  $V_1, \dots, V_n$  is a invariant subspace of  $V$  under  $T$ .  $\square$

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**11** Define  $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$  by  $Tp = p'$ . Find all eigenvalues and eigenvectors of  $T$ .

**SOLUTION:**

Note that in general,  $\deg p' < \deg p$  ( $\deg 0 = -\infty$ ).

Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $p$ .

Suppose  $\lambda \neq 0$ . Then  $\deg \lambda p > \deg p'$  while  $\lambda p \neq p'$ . Contradicts. Thus  $\lambda = 0$ .

Therefore  $\deg \lambda p = -\infty = \deg p \Rightarrow p$  is a nonzero constant polynomial.

Hence the set of all eigenvectors is  $\{C : C \in \mathbf{R} \wedge C \neq 0\} = \mathcal{P}_0(\mathbf{R}) \setminus \{0\}$ .  $\square$

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**12** Define  $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$  by  $(Tp)(x) = xp'(x)$  for all  $x \in \mathbf{R}$ .

Find all eigenvalues and eigenvectors of  $T$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $p$ , then  $(Tp)(x) = xp'(x) = \lambda p(x)$ .

Let  $p = a_0 + a_1x + \dots + a_nx^n$ .

Then  $xp'(x) = a_1x + 2a_2x^2 + \dots + na_nx^n = \lambda a_0 + \lambda a_1x + \lambda a_2x^2 + \dots + \lambda a_nx^n$ .

Similar to Problem (10),  $0, 1, \dots, n$  are eigenvalues of  $T$ .

The set of all eigenvectors of  $T$  is  $\{cx^\lambda : \lambda = 0, 1, \dots, n, c \in \mathbf{F} \wedge c \neq 0\}$ .  $\square$

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**13** Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ , and  $\lambda \in \mathbf{F}$ .

Prove that  $\exists \alpha \in \mathbf{F}$ ,  $|\alpha - \lambda| < \frac{1}{1000}$  and  $(T - \alpha I)$  is invertible.

**SOLUTION:**

Let  $\alpha_k \in \mathbf{F}$  be such that  $|\alpha_k - \lambda| = \frac{1}{1000 + k}$  for each  $k = 1, \dots, \dim V + 1$ .

Note that each  $T \in \mathcal{L}(V)$  has at most  $\dim V$  distinct eigenvalues.

Hence  $\exists k = 1, \dots, \dim V + 1$  such that  $\alpha_k$  is not an eigenvalue of  $T$  and therefore  $(T - \alpha_k I)$  is invertible.  $\square$

• Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ , and  $\alpha \in \mathbf{F}$ .

Prove that  $\exists \delta > 0$  such that  $(T - \lambda I)$  is invertible for all  $\lambda \in \mathbf{F}$  such that  $0 < |\alpha - \lambda| < \delta$ .

**SOLUTION:**

Choose  $\delta > 0$  arbitrarily.

Let  $\alpha_k \in \mathbf{F}$  be such that  $|\alpha_k - \lambda| = \frac{\delta}{k}$  for each  $k = 1, \dots, \dim V + 1$ .

Similar to Problem (13),  $\exists k$  such that  $\alpha_k$  is not an eigenvalue.  $\square$

**14** Suppose  $V = U \oplus W$ , where  $U$  and  $W$  are nonzero subspaces of  $V$ .

Define  $P \in \mathcal{L}(V)$  by  $P(u + w) = u$  for each  $u \in U$  and each  $w \in W$ .

Find all eigenvalues and eigenvectors of  $P$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue of  $P$  with an eigenvector  $(u + w)$ .

Then  $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$ . By [1.44] and  $V = U \oplus W$ ,  $(\lambda - 1)u = \lambda w = 0$ .

Thus if  $\lambda = 1$ , then  $w = 0$ ; if  $\lambda = 0$ , then  $u = 0$ .

Hence the eigenvalues of  $P$  are 0 and 1, the set of all eigenvectors in  $P$  is  $U \cup W$ .  $\square$

**15** Suppose  $T \in \mathcal{L}(V)$ . Suppose  $S \in \mathcal{L}(V)$  is invertible.

(a) Prove that  $T$  and  $S^{-1}TS$  have the same eigenvalues.

(b) What is the relationship between the eigenvectors of  $T$  and the eigenvectors of  $S^{-1}TS$ ?

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $v$ .

Then  $S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v$ .

Thus  $\lambda$  is also an eigenvalue of  $S^{-1}TS$  with an eigenvector  $S^{-1}v$ .

Suppose  $\lambda$  is an eigenvalue of  $S^{-1}TS$  with an eigenvector  $v$ .

Then  $S(S^{-1}TS)v = TSv = \lambda Sv$ .

Thus  $\lambda$  is also an eigenvalue of  $T$  with an eigenvector  $Sv$ .  $\square$

OR. Note that  $S(S^{-1}TS)S^{-1} = T$ .

Hence every eigenvalue of  $S^{-1}TS$  is an eigenvalue of  $S(S^{-1}TS)S^{-1} = T$ .

And every eigenvector  $v$  of  $S^{-1}TS$  is  $S^{-1}v$ , every eigenvector  $u$  of  $T$  is  $Su$ .  $\square$

**17** Give an example of an operator on  $\mathbf{R}^4$  that has no (real) eigenvalues.

**SOLUTION:**

Define  $T \in \mathcal{L}(\mathbf{R}^4)$  by  $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$ . Where  $(e_1, e_2, e_3, e_4)$  is the standard basis of  $\mathbf{R}^4$ .

Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $(x, y, z, w)$ .

$$\text{Then } T(x, y, z, w) = \lambda(x, y, z, w) \Rightarrow \begin{cases} (1 - \lambda)x + y + z + w = 0 \\ -x + (1 - \lambda)y - z - w = 0 \\ 3x + 8y + (11 - \lambda)z + 5w = 0 \\ 3x - 8y - 11z + (5 - \lambda)w = 0 \end{cases}$$

This linear equation has no solutions.

( You can type it on <https://zh.numberempire.com/equationsolver.php> to check.)

OR. Define  $T \in \mathcal{L}(\mathbf{R}^4)$  by  $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$ .

Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $(x, y, z, w)$ .

$$\text{Then } T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \Rightarrow \begin{cases} -y = \lambda x \\ x = \lambda y \\ -w = \lambda z \\ z = \lambda w \end{cases} \Rightarrow \begin{cases} -xy = \lambda^2 xy \\ -zw = \lambda^2 zw \end{cases}$$

If  $xy \neq 0$  or  $zw \neq 0$ , then  $\lambda^2 = -1$ , we fail.

Otherwise,  $xy = 0 \Rightarrow x = y = 0$ , for if  $x \neq 0$ , then  $\lambda = 0 \Rightarrow x = 0$ , contradicts.

Similarly,  $y = z = w = 0$ . Then we fail.

Thus  $T$  has no eigenvalues.  $\square$

• Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ , and  $\lambda \in \mathbf{F}$ .

Show that  $\lambda$  is an eigenvalue of  $T \iff \lambda$  is an eigenvalue of the dual operator  $T' \in \mathcal{L}(V')$ .

**SOLUTION:**

(a) Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $v$ .

Let  $v_1 = v$  and let  $(Tv_1, \dots, Tv_n)$  be a basis of  $V$ , so is  $(v_1, \dots, v_n)$ .

Define  $\varphi \in \mathcal{L}(V, \mathbf{F})$  by  $\varphi(v) = 1, \varphi(Tv_j) = 0$ , for each  $j = 2, \dots, n$ .

Then  $T'(\varphi) = \varphi \circ T$ .

Thus  $\forall u \in V, (\varphi \circ T)(a_1 v_1 + \dots + a_n v_n) = \varphi(\lambda a_1 v + Tv_2 + \dots + Tv_n) = \lambda a_1 = \lambda \varphi(\lambda a_1 v + Tv_2 + \dots + Tv_n)$ .

Hence  $\lambda$  is an eigenvalue of  $T'$ .

(b) Suppose  $\lambda$  is an eigenvalue of  $T'$  with an eigenvector  $\psi$ . Then  $T'(\psi) = \psi \circ T = \lambda \psi$ .

又  $\psi \neq 0 \Rightarrow \exists v \in V \setminus \{0\}$  such that  $\psi(v) \neq 0$ . Note that  $\psi(Tv) = \lambda \psi(v)$ .

Thus  $\lambda = \frac{\psi(Tv)}{\psi(v)} \Rightarrow Tv = \frac{\psi(Tv)}{\psi(v)}v = \lambda v$ . Hence  $\lambda$  is an eigenvalue of  $T$ .  $\square$

• TODO Suppose  $(v_1, \dots, v_n)$  is a basis of  $V$  and  $T \in \mathcal{L}(V)$ .

Prove that if  $\lambda$  is an eigenvalue of  $T$ , then

$|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\}$ , where  $\mathcal{M}(T, (v_1, \dots, v_n))$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue of  $T$ , and therefore is an eigenvalue of  $\mathcal{M}(T)$ , with an eigenvector  $v$ .

We discuss in two cases:

If  $\mathcal{M}(T)$  is invertible ( $\iff T$  is invertible), then  $\mathcal{M}(Tv) = \mathcal{M}(\lambda v) \Rightarrow \frac{1}{\lambda} \mathcal{M}(v) = \mathcal{M}(T^{-1}v)$ .

Otherwise,  $(T - 0I)$  is not invertible and therefore  $\lambda = 0$  is an eigenvalue. And other  $\lambda$ s?

• Suppose  $\mathbf{F} = \mathbf{R}$ ,  $T \in \mathcal{L}(V)$ .

(a) (OR (9.11))  $\lambda \in \mathbf{R}$ . Prove that  $\lambda$  is an eigenvalue of  $T \iff \lambda$  is an eigenvalue of  $T_{\mathbb{C}}$ .

(b) (OR Problem (16))  $\lambda \in \mathbf{C}$ . Prove that  $\lambda$  is an eigenvalue of  $T_{\mathbb{C}} \iff \bar{\lambda}$  is an eigenvalue of  $T_{\mathbb{C}}$ .

**SOLUTION:**

(a) Suppose  $v \in V$  is an eigenvector corresponding to the eigenvalue  $\lambda$ .

$$\text{Then } Tv = \lambda v \Rightarrow T_{\mathbb{C}}(v + i0) = Tv + iT0 = \lambda v.$$

Thus  $\lambda$  is an eigenvalue of  $T$ .

Suppose  $v + iu \in V_{\mathbb{C}}$  is an eigenvector corresponding to the eigenvalue  $\lambda$ .

$$\text{Then } T_{\mathbb{C}}(v + iu) = \lambda v + i\lambda u \Rightarrow Tv = \lambda v, Tu = \lambda u. \text{ (Note that } v \text{ or } u \text{ might be zero).}$$

Thus  $\lambda$  is an eigenvalue of  $T_{\mathbb{C}}$ .

(b) Suppose  $\lambda$  is an eigenvalue of  $T_{\mathbb{C}}$  with an eigenvector  $v + iu$ .

Let  $(v_1, \dots, v_n)$  be a basis of  $V$ . Write  $v = \sum_{i=1}^n a_i v_i, u = \sum_{i=1}^n b_i v_i$ , where  $a_i, b_i \in \mathbf{R}$ .

Then  $T_{\mathbb{C}}(v + iu) = Tv + iTu = \lambda v + i\lambda u = \lambda \sum_{i=1}^n (a_i + ib_i) v_i$ . Conjugating two sides, we have:

$$\overline{T_{\mathbb{C}}(v + iu)} = \overline{Tv + iTu} = \overline{Tv} - i\overline{Tu} = Tv - iTu = T_{\mathbb{C}}(\overline{v + iu}) = \overline{\lambda \sum_{i=1}^n (a_i + ib_i) v_i} = \bar{\lambda} \sum_{i=1}^n (a_i - ib_i) v_i.$$

Hence  $\bar{\lambda}$  is an eigenvalue of  $T_{\mathbb{C}}$ . To prove the other direction, notice that  $\overline{\bar{\lambda}} = \lambda$ .  $\square$

**18** Show that the forward shift operator  $T \in \mathcal{L}(\mathbf{F}^{\infty})$

defined by  $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$  has no eigenvalues.

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $(z_1, z_2, \dots)$ .

$$\text{Then } T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots).$$

$$\text{Thus } \lambda z_1 = 0, \lambda z_2 = z_1, \dots, \lambda z_k = z_{k-1}, \dots$$

Let  $\lambda = 0$ , then  $\lambda z_2 = z_1 = 0 = \lambda z_k = z_{k-1}$ , therefore  $(z_1, z_2, \dots) = 0$  is not an eigenvector.

Suppose  $\lambda \neq 0$ . Then  $\lambda z_1 = 0 \Rightarrow z_1 = 0 \Rightarrow z_2 = 0 = z_k$  for all  $k \in \mathbf{N}^+$ .

And then  $(z_1, z_2, \dots) = 0$  is not an eigenvector. Hence  $T$  has no eigenvalues.  $\square$

**19** Suppose  $n$  is a positive integer.

Define  $T \in \mathcal{L}(\mathbf{F}^n)$  by  $T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$ .

In other words, the entries of  $\mathcal{M}(T)$  with respect to the standard basis are all 1's.

Find all eigenvalues and eigenvectors of  $T$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $(x_1, \dots, x_n)$ .

$$\text{Then } T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n).$$

$$\text{Thus } \lambda x_1 = \dots = \lambda x_n = x_1 + \dots + x_n.$$

$$\text{For } \lambda = 0, x_1 + \dots + x_n = 0.$$

$$\text{For } \lambda \neq 0, x_1 = \dots = x_n \text{ and then } \lambda x_k = n x_k \text{ for each } k.$$

Hence  $0, n$  are eigenvalues of  $T$ .

And the set of all eigenvectors of  $T$  is  $\{(x_1, \dots, x_n) \in \mathbf{F}^n : x_1 + \dots + x_n = 0 \vee x_1 = \dots = x_n\}$ .  $\square$

**20** Define the backward shift operator  $S \in \mathcal{L}(\mathbf{F}^\infty)$  by  $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$ .

(a) Show that every element of  $\mathbf{F}$  is an eigenvalue of  $S$ .

(b) Find all eigenvectors of  $S$ .

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue of  $S$  with an eigenvector  $(z_1, z_2, \dots)$ .

Then  $S(z_1, z_2, z_3, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$ .

Thus  $\lambda z_1 = z_2, \lambda z_2 = z_3, \dots, \lambda z_k = z_{k+1}, \dots$ .

For  $\lambda = 0$ ,  $\lambda z_1 = z_2 = 0 = \lambda z_2 = z_3 = \dots = z_k$  for all  $k$ .

While  $z_1$  can be arbitrary, so that  $(z_1, 0, \dots)$  is an eigenvector with  $z_1 \neq 0$ .

For  $\lambda \neq 0$ ,  $\lambda^k z_1 = \lambda^{k-1} z_2 = \dots = \lambda z_k = z_{k+1}$  for all  $k$ .

Then  $(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots)$  is an eigenvector with  $z_1 \neq 0$ .

Hence (a) each element of  $\lambda \in \mathbf{F}$  is an eigenvalue of  $T$ .

And (b) the set of all eigenvectors of  $T$  is  $\{(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots) \in \mathbf{F}^\infty : \lambda \in \mathbf{F}, z_1 \neq 0\}$   $\square$

**21** Suppose  $T \in \mathcal{L}(V)$  is invertible.

(a) Suppose  $\lambda \in \mathbf{F}$  with  $\lambda \neq 0$ .

Prove that  $\lambda$  is an eigenvalue of  $T \iff \frac{1}{\lambda}$  is an eigenvalue of  $T^{-1}$ .

(b) Prove that  $T$  and  $T^{-1}$  have the same eigenvectors.

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $v$ .

Then  $T^{-1}Tv = \lambda T^{-1}v = v \Rightarrow T^{-1}v = \frac{1}{\lambda}v$ .

Hence  $\frac{1}{\lambda}$  is an eigenvalue of  $T^{-1}$ .

Suppose  $\frac{1}{\lambda}$  is an eigenvalue of  $T^{-1}$  with an eigenvector  $v$ .

Then  $TT^{-1}v = v = \frac{1}{\lambda}Tv \Rightarrow Tv = \lambda v$ .

Hence  $\lambda$  is an eigenvalue of  $T$ .

OR. Note that  $(T^{-1})^{-1} = T$  and  $\frac{1}{\frac{1}{\lambda}} = \lambda$ .  $\square$

**22** Suppose  $T \in \mathcal{L}(V)$  and there exist nonzero vectors  $u, w$  in  $V$

such that  $Tu = 3w$  and  $Tw = 3u$ . Prove that 3 or  $-3$  is an eigenvalue of  $T$ .

**SOLUTION:**

COMMENT:  $Tu = 3w, Tw = 3u \Rightarrow T(Tu) = 9u \Rightarrow T^2$  has an eigenvalue 9.

$Tu = 3w, Tw = 3u \Rightarrow T(u + w) = 3(u + w), T(u - w) = 3(w - u) = -3(u - w)$ .

Hence 3 or  $-3$  is an eigenvalue of  $T$ .  $\square$

**23** Suppose  $V$  is finite-dim and  $S, T \in \mathcal{L}(V)$ .

Prove that  $ST$  and  $TS$  have the same eigenvalues.

**SOLUTION:**

Suppose  $\lambda$  is an eigenvalue of  $ST$  with an eigenvector  $v$ .

Then  $T(STv) = \lambda Tv = TS(Tv)$ .

If  $Tv \neq 0$ , then  $\lambda$  is an eigenvalue of  $TS$ .

Otherwise,  $\lambda = 0$ , ( $v \neq 0, \lambda v = 0 = STv$ ), then  $T$  is not invertible

$\Rightarrow TS$  is not invertible  $\Rightarrow (TS - 0I)$  is not invertible  $\Rightarrow \lambda = 0$  is an eigenvalue of  $TS$ .

Reversing the roles of  $T$  and  $S$ , we conclude that  $ST$  and  $TS$  have the same eigenvalues.  $\square$



**24** Suppose  $A \in \mathbf{F}^{n,n}$ . Define  $T \in \mathcal{L}(\mathbf{F}^n)$  by  $Tx = Ax$ , where elements of  $\mathbf{F}^n$  are thought of as  $n$ -by-1 column vectors.

(a) Suppose the sum of the entries in each row of  $A$  equals 1.

Prove that 1 is an eigenvalue of  $T$ .

(b) Suppose the sum of the entries in each column of  $A$  equals 1.

Prove that 1 is an eigenvalue of  $T$ .

**SOLUTION:**

(a) Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ .

Then  $Tx = Ax = \begin{pmatrix} \sum_{c=1}^n A_{1,c}x_c \\ \vdots \\ \sum_{c=1}^n A_{n,c}x_c \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ . While  $\sum_{r=1}^n A_{R,c} = 1$  for each  $R = 1, \dots, n$ .

Thus if we let  $x_1 = \dots = x_n$ , then  $\lambda = 1$ , and hence 1 is an eigenvalue of  $T$ .

(b) Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ .

Then  $Tx = Ax = \begin{pmatrix} \sum_{r=1}^n A_{1,r}x_r \\ \vdots \\ \sum_{r=1}^n A_{n,r}x_r \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ . While  $\sum_{r=1}^n A_{r,C} = 1$  for each  $C = 1, \dots, n$ .

$$\begin{aligned} \text{Thus } \sum_{r=1}^n (Ax)_{r,\cdot} &= \sum_{r=1}^n (Ax)_{r,1} \\ &= \sum_{c=1}^n (A_{1,c} + \dots + A_{n,c})x_c = \sum_{c=1}^n x_c = \lambda \begin{pmatrix} x_1 \\ + \\ \vdots \\ + \\ x_n \end{pmatrix}. \end{aligned}$$

Hence  $\lambda = 1$ , for all  $x$  such that  $\sum_{c=1}^n x_{c,1} \neq 0$ .  $\square$

OR. Prove that  $(T - I)$  is not invertible, so that we can conclude  $\lambda = 1$  is an eigenvalue.

$$\text{Because } (T - I)x = (A - \mathcal{M}(I))x = \begin{pmatrix} \sum_{r=1}^n A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^n A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

$$\text{Then } y_1 + \dots + y_n = \sum_{r=1}^n \sum_{c=1}^n (A_{r,c}x_c - x_r) = \sum_{c=1}^n x_c \sum_{r=1}^n A_{r,c} - \sum_{r=1}^n x_r = 0.$$

Thus  $\text{range}(T - I) \subseteq \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbf{F}^n : y_1 + \dots + y_n = 0 \right\}$ . Hence  $(T - I)$  is not surjective.  $\square$

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- Suppose  $A \in \mathbf{F}^{n,n}$ . Define  $T \in \mathcal{L}(\mathbf{F}^n)$  by  $Tx = xA$ , where elements of  $\mathbf{F}^n$  are thought of as 1-by- $n$  row vectors.
- (a) Suppose the sum of the entries in each column of  $A$  equals 1.  
Prove that 1 is an eigenvalue of  $T$ .
- (b) Suppose the sum of the entries in each row of  $A$  equals 1.  
Prove that 1 is an eigenvalue of  $T$ .

**SOLUTION:**

- (a) Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $x = (x_1 \ \cdots \ x_n)$ .

Then  $Tx = xA = \left( \sum_{r=1}^n x_r A_{r,1} \ \cdots \ \sum_{r=1}^n x_r A_{r,n} \right) = \lambda (x_1 \ \cdots \ x_n)$ . While  $\sum_{r=1}^n A_{r,C} = 1$  for each  $C = 1, \dots, n$ .

Thus if we let  $x_1 = \cdots = x_n$ , then  $\lambda = 1$ , hence is an eigenvalue of  $T$ .

- (b) Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $x = (x_1 \ \cdots \ x_n)$ .

Then  $Tx = xA = \left( \sum_{c=1}^n x_c A_{c,1} \ \cdots \ \sum_{c=1}^n x_c A_{c,n} \right) = \lambda (x_1 \ \cdots \ x_n)$ . While  $\sum_{c=1}^n A_{R,c} = 1$  for each  $R = 1, \dots, n$ .

Thus  $\sum_{c=1}^n (xA)_{.,c} = \sum_{c=1}^n (xA)_{1,c} = \sum_{c=1}^n (A_{c,1} + \cdots + A_{c,n})x_c = \sum_{c=1}^n x_c = \lambda(x_1 + \cdots + x_n)$ .

Hence  $\lambda = 1$ , for all  $x$  such that  $\sum_{r=1}^n x_{1,r} \neq 0$ .  $\square$

OR. Prove that  $(T - I)$  is not invertible, so that we can conclude  $\lambda = 1$  is an eigenvalue.

Because  $(T - I)x = x(A - \mathcal{M}(I)) = \left( \sum_{c=1}^n x_c A_{c,1} - x_1 \ \cdots \ \sum_{c=1}^n x_c A_{c,n} - x_n \right) = (y_1 \ \cdots \ y_n)$ .

Then  $y_1 + \cdots + y_n = \sum_{c=1}^n \sum_{r=1}^n (x_r A_{r,c} - x_c) = \sum_{r=1}^n x_r \sum_{c=1}^n A_{r,c} - \sum_{c=1}^n x_c = 0$ .

Thus  $\text{range}(T - I) \subseteq \{ (y_1 \ \cdots \ y_n) \in \mathbf{F}^n : y_1 + \cdots + y_n = 0 \}$ . Hence  $(T - I)$  is not surjective.  $\square$

**25** Suppose  $T \in \mathcal{L}(V)$  and  $u, w$  are eigenvectors of  $T$

such that  $u + w$  is also an eigenvector of  $T$ .

Prove that  $u$  and  $w$  are eigenvectors of  $T$  corresponding to the same eigenvalue.

**SOLUTION:**

Suppose  $\lambda_1, \lambda_2, \lambda_0$  are eigenvalues of  $T$  corresponding to  $u, w, u + w$  respectively.

Then  $T(u + w) = \lambda_0(u + w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$ .

Notice that  $u, w, u + w$  are nonzero.

If  $(u, w)$  is linearly dependent, then let  $w = cu$ , therefore

$$\lambda_2 cu = Tw = cTu = \lambda_1 cu \Rightarrow \lambda_2 = \lambda_1.$$

$$\lambda_0(u + w) = T(u + w) = \lambda_1 u + \lambda_1 cu = \lambda_1(u + w) \Rightarrow \lambda_0 = \lambda_1.$$

Otherwise,  $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$ .  $\square$

**26** Suppose  $T \in \mathcal{L}(V)$  is such that every nonzero vector in  $V$  is an eigenvector of  $T$ .

Prove that  $T$  is a scalar multiple of the identity operator.

**SOLUTION:**

Because  $\forall v \in V, \exists! \lambda_v \in \mathbf{F}, Tv = \lambda_v v$ .

Then for any two distinct nonzero vectors  $v, w \in V$ ,

$T(v + w) = \lambda_{v+w}(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$ .

If  $(v, w)$  is linearly independent, then let  $w = cv$ , therefore

$$\lambda_v cv = cTv = Tw = \lambda_w w \Rightarrow \lambda_w = \lambda_v.$$

$$\lambda_{v+w}(v + w) = T(v + w) = Tv + Tw = \lambda_v(v + cv) \Rightarrow \lambda_{v+w} = \lambda_v.$$

Otherwise,  $\lambda_v = \lambda_{v+w} = \lambda_w$ .  $\square$

**27, 28** Suppose  $V$  is finite-dim and  $k \in \{1, \dots, \dim V - 1\}$ .

Suppose  $T \in \mathcal{L}(V)$  is such that every subspace of  $V$  of dim  $k$  is invariant under  $T$ .

Prove that  $T$  is a scalar multiple of the identity operator.

**SOLUTION:**

We prove the contrapositive:

If  $T \neq \lambda I, \forall \lambda \in \mathbb{F}$ , then  $\exists$  a subspace  $U$  of  $V$  such that  $\dim U = k$ , and  $U$  is invariant under  $T$ .

By Problem (26),  $\exists v \in V$  and  $v \neq 0$  such that  $v$  is not an eigenvector of  $T$ .

Thus  $(v, Tv)$  is linearly independent. Extend to a basis of  $V$  as  $(v, Tv, u_1, \dots, u_n)$ .

Let  $U = \text{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$  is not an invariant subspace of  $V$  under  $T$ .

OR. Suppose  $0 \neq v = v_1 \in V$  and extend to a basis of  $V$  as  $(v_1, \dots, v_n)$ .

Suppose  $Tv_1 = c_1v_1 + \dots + c_nv_n, \exists! c_i \in \mathbb{F}$ .

Consider a  $k$ -dim subspace  $U = \text{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$ ,

where  $\alpha_j \in \{2, \dots, n\}$  for each  $j$ , and  $\alpha_1, \dots, \alpha_{k-1}$  are distinct and are chosen arbitrarily.

Because every subspace such  $U$  is invariant.

Thus  $Tv_1 = c_1v_1 + \dots + c_nv_n \in U$

$\Rightarrow c_2 = \dots = c_n = 0$ ,

for if not, for each  $c_i \neq 0$ , choose  $U_i$  such that  $\alpha_j \in \underbrace{\{2, \dots, i-1, i+1, \dots, n\}}_{\text{length } (n-2)}$  for each  $j$ ,

hence for  $Tv_1 = c_1v_1 + \dots + c_{i-1}v_{i-1} + c_{i+1}v_{i+1} + \dots + c_nv_n \in U_i$ , we conclude that  $c_i = 0$ .

$\Rightarrow Tv_1 = c_1v_1, \forall v_1 = v \in V$  is arbitrarily chosen  $\Rightarrow T = \lambda I$  for some  $\lambda$ .  $\square$

• Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ . Prove that

$T$  has an eigenvalue  $\iff \exists$  a subspace  $U$  of  $V$

such that  $\dim U = \dim V - 1$ ,  $U$  is invariant under  $T$ .

**SOLUTION:**

(a) Suppose  $\lambda$  is an eigenvalue of  $T$  with an eigenvector  $v$ .

( If  $\dim V = 1$ , then  $U = \{0\}$  and we are done. )

Extend  $v_1 = v$  to a basis of  $V$  as  $(v_1, v_2, \dots, v_n)$ .

**Step 1** If  $\exists w_1 \in \text{span}(v_2, \dots, v_n)$  such that  $0 \neq Tw_1 \in \text{span}(v_1)$ ,

then extend  $w_1 = \alpha_{1,1}$  to a basis of  $\text{span}(v_2, \dots, v_n)$  as  $(\alpha_{1,1}, \dots, \alpha_{1,n-1})$ .

Otherwise, we stop at step 1.

**Step 2** If  $\exists w_2 \in \text{span}(\alpha_{1,2}, \dots, \alpha_{1,n-1})$  such that  $0 \neq Tw_2 \in \text{span}(v_1, w_1)$ ,

then extend  $w_2 = \alpha_{2,1}$  to a basis of  $\text{span}(\alpha_{1,2}, \dots, \alpha_{1,n-1})$  as  $(\alpha_{2,1}, \dots, \alpha_{2,n-2})$ .

Otherwise, we stop at step 2.

$\vdots$

**Step  $k$**  If  $\exists w_k \in \text{span}(\alpha_{k-1,2}, \dots, \alpha_{k-1,n-k+1})$  such that  $0 \neq Tw_k \in \text{span}(v_1, w_1, \dots, w_{k-1})$ ,

then extend  $w_k = \alpha_{k,1}$  to a basis of  $\text{span}(\alpha_{k-1,2}, \dots, \alpha_{k-1,n-k+1})$  as  $(\alpha_{k,1}, \dots, \alpha_{k,n-k})$ .

Otherwise, we stop at step  $k$ .

$\vdots$

Finally, we stop at step  $m$ , thus we get  $(v_1, w_1, \dots, w_{m-1})$  and  $(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1})$ ,

$\text{range } T|_{\text{span}(w_1, \dots, w_{m-1})} = \text{span}(v_1, w_1, \dots, w_{m-2}) \Rightarrow \dim \text{null } T|_{\text{span}(w_1, \dots, w_{m-1})} = 0$ ,

$\text{span}(\underbrace{v_1, w_1, \dots, w_{m-1}}_{\text{length dim } m})$  and  $\text{span}(\underbrace{\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}}_{\text{length dim } (n-m)})$  are invariant under  $T$ .

Let  $U = \text{span}(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}) \oplus \text{span}(v_1, w_1, \dots, w_{m-2})$  and we are done.  $\square$

COMMENT: Both  $\text{span}(v_2, \dots, v_n)$  and  $\text{span}(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}) \oplus \text{span}(w_1, \dots, w_{m-1})$  are in  $\mathcal{S}_V \text{span}(v_1)$ .

(b) Suppose  $U$  is an invariant subspace of  $V$  under  $T$  with  $\dim U = m = \dim V - 1$ .

( If  $m = 0$ , then  $\dim V = 1$  and we are done ).

Let  $(u_1, \dots, u_m)$  be a basis of  $U$ , extend to a basis of  $V$  as  $(u_0, u_1, \dots, u_m)$ .

We discuss in cases:

For  $Tu_0 \in U$ , then  $\text{range } T = U$  so that  $T$  is not surjective  $\iff \text{null } T \neq \{0\} \iff 0$  is an eigenvalue of  $T$ .

For  $Tu_0 \notin U$ , then  $Tu_0 = a_0u_0 + a_1u_1 + \dots + a_mu_m$ .

(1) If  $Tu_0 \in \text{span}(u_0)$ , then we are done.

(2) Otherwise, if  $\text{range } T|_U = U$ , then  $Tu_0 = a_0u_0$  and we are done;

otherwise,  $T|_U : U \rightarrow U$  is not surjective ( $\Rightarrow$  not injective ), suppose  $\text{range } T|_U \neq \{0\}$

( Suppose  $\text{range } T|_U = \{0\}$ . If  $\dim U = 0$  then we are done.

Otherwise  $\exists u \in U \setminus \{0\}, Tu = 0$  and we are done. )

then  $\exists u \in U \setminus \{0\}, Tu = 0$ , we are done.  $\square$

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**29** Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

*Prove that  $T$  has at most  $1 + \dim \text{range } T$  distinct eigenvalues.*

**SOLUTION:**

Let  $\lambda_1, \dots, \lambda_m$  be the distinct eigenvalues of  $T$  and let  $v_1, \dots, v_m$  be the corresponding eigenvectors.

For every  $\lambda_k \neq 0$ ,  $T(\frac{1}{\lambda_k}v_k) = v_k$ . And if  $T = T - 0I$  is not invertible, then  $\exists! \lambda_A = 0$  and  $Tv_A = \lambda_A v_A = 0$ .

Thus for  $\lambda_k \neq 0, \forall k$ ,  $(Tv_1, \dots, Tv_m)$  is a linearly independent list of length  $m$  in  $\text{range } T$ .

And for  $\lambda_A = 0$ , there is a linearly independent list of length at most  $(m - 1)$  in  $\text{range } T$ .

Hence, by [2.23],  $m \leq \dim \text{range } T + 1$ .  $\square$

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**30** Suppose  $T \in \mathcal{L}(\mathbf{R}^3)$  and  $-4, 5, \sqrt{7}$  are eigenvalues of  $T$ .

*Prove that  $\exists x \in \mathbf{R}^3$  such that  $Tx - 9x = (-4, 5, \sqrt{7})$ .*

**SOLUTION:** Because 9 is not an eigenvalue. Hence  $(T - 9I)$  is surjective.  $\square$

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**31** Suppose  $V$  is finite-dim and  $v_1, \dots, v_m \in V$ .

*Prove that  $(v_1, \dots, v_m)$  is linearly independent*

$\iff \exists T \in \mathcal{L}(V)$  such that  $v_1, \dots, v_m$  are eigenvectors of  $T$   
corresponding to distinct eigenvalues.

**SOLUTION:**

Suppose  $(v_1, \dots, v_m)$  is linearly independent, extend it to a basis of  $V$  as  $(v_1, \dots, v_m, \dots, v_n)$ .

Then define  $T \in \mathcal{L}(V)$  by  $Tv_k = kv_k$  for each  $k \in \{1, \dots, m, \dots, n\}$ .

Conversely by [5.10] it is true as well.  $\square$

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**32** Suppose that  $\lambda_1, \dots, \lambda_n$  are distinct real numbers.

*Prove that  $(e^{\lambda_1}x, \dots, e^{\lambda_n}x)$  is linearly independent in  $\mathbb{R}^{\mathbb{R}}$ .*

HINT: Let  $V = \text{span}(e^{\lambda_1}x, \dots, e^{\lambda_n}x)$ , and define an operator  $D \in \mathcal{L}(V)$  by  $Df = f'$ .

*Find eigenvalues and eigenvectors of  $D$ .*

**SOLUTION:**

Define  $V$  and  $D \in \mathcal{L}(V)$  as in HINT. Then because for each  $k$ ,  $D(e^{\lambda_k}x) = \lambda_k e^{\lambda_k}x$ .

Thus  $\lambda_1, \dots, \lambda_n$  are distinct eigenvalues of  $D$ . By [5.10],  $(e^{\lambda_1}x, \dots, e^{\lambda_n}x)$  is linearly independent in  $\mathbb{R}^{\mathbb{R}}$ .  $\square$

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- Suppose  $\lambda_1, \dots, \lambda_n$  are distinct positive numbers.

Prove that  $(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$  is linearly independent in  $\mathbb{R}^{\mathbb{R}}$ .

**SOLUTION:**

Let  $V = \text{span}(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$ . Define  $D \in \mathcal{L}(V)$  by  $Df = f'$ .

Then because  $D(\cos(\lambda_k x)) = -\lambda_k \sin(\lambda_k x)$  and  $D(\sin(\lambda_k x)) = \lambda_k \cos(\lambda_k x)$ .

Thus  $D^2(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$ .

Notice that  $\lambda_1, \dots, \lambda_n$  are distinct  $\Rightarrow -\lambda_1^2, \dots, -\lambda_n^2$  are distinct.

Hence  $-\lambda_1^2, \dots, -\lambda_n^2$  are distinct eigenvalues of  $D^2$

with the corresponding eigenvectors  $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$  respectively.

And then  $(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$  is linearly independent in  $\mathbb{R}^{\mathbb{R}}$ .  $\square$

- Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

Define  $A \in \mathcal{L}(\mathcal{L}(V))$  by  $A(S) = TS$  for each  $S \in \mathcal{L}(V)$ .

Prove that the set of eigenvalues of  $T$  equals the set of eigenvalues of  $A$ .

**SOLUTION:**

- (a) Suppose  $\lambda_1, \dots, \lambda_m$  are all eigenvalues of  $T$  with eigenvectors  $v_1, \dots, v_m$  respectively.

Extend to a basis of  $V$  as  $(v_1, \dots, v_m, \dots, v_n)$ .

Then for each  $k \in \{1, \dots, m\}$ ,  $\text{span}(v_k) \subseteq \text{null}(T - \lambda_k I)$ .

Define  $S_k \in \mathcal{L}(V)$  by  $S_k(v_j) = v_k$  for each  $j \in \{1, \dots, n\}$ ,

so that  $\text{range } S_k = \text{span}(v_k)$  for each  $k \in \{1, \dots, m\}$ , then  $A(S_k) = TS_k = \lambda_k S_k$ .

Thus the eigenvalues of  $T$  are eigenvalues of  $A$ .

- (b) Suppose  $\lambda_1, \dots, \lambda_m$  are all eigenvalues of  $A$  with eigenvectors  $S_1, \dots, S_m$  respectively.

Then for each  $k \in \{1, \dots, m\}$ , because  $\forall v \in V, u = S_k(v) \in V \Rightarrow Tu = \lambda_k u$ .

Thus the eigenvalues of  $A$  are eigenvalues of  $T$ .  $\square$

- COMMENT: Define  $B \in \mathcal{L}(\mathcal{L}(V))$  by  $B(S) = ST, \forall S \in \mathcal{L}(V)$ . Then the eigenvalues of  $B$  are not the eigenvalues of  $T$ .

- Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V)$ , and  $U$  is a subspace of  $V$  invariant under  $T$ .

The quotient operator  $T/U \in \mathcal{L}(V/U)$  is defined by

$$(T/U)(v + U) = Tv + U \text{ for each } v \in V.$$

- (a) Show that the definition of  $T/U$  makes sense

(which requires using the condition that  $U$  is invariant under  $T$ )

and show that  $T/U$  is an operator on  $V/U$ .

- (b) (OR Problem 35) Show that each eigenvalue of  $T/U$  is an eigenvalue of  $T$ .

**SOLUTION:**

- (a) Suppose  $v + U = w + U$  ( $\iff v - w \in U$ ).

Then because  $U$  is invariant under  $T$ ,  $T(v - w) \in U \iff Tv + U = Tw + U$ .

Hence the definition of  $T/U$  makes sense.

- (b) Suppose  $\lambda$  is an eigenvalue of  $T/U$  with an eigenvector  $v + U$ .

Then  $(T/U)(v + U) = \lambda(v + U) = Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v \in U$ .

If  $(T - \lambda I)v = 0 \Rightarrow Tv = \lambda v$ , then we are done.

Otherwise, then  $(T|_U - \lambda I) : U \rightarrow U$  is invertible,

hence  $\exists! w \in U, (T|_U - \lambda I)(w) = (T - \lambda I)v \Rightarrow T(v - w) = \lambda(v - w)$ .

Note that  $v - w \neq 0$  (for if not,  $v \in U \Rightarrow v + U = 0 + U$  is not an eigenvector).

Thus  $\lambda$  is an eigenvalue of  $T$ .  $\square$

**36** Prove or give an counterexample:

The result of (b) in Exercise 35 is still true if  $V$  is infinite-dim.

**SOLUTION:** An counterexample:

Consider  $V = \text{span}(1, e^x, e^{2x}, \dots)$  in  $\mathbb{R}^{\mathbb{R}}$ , and a subspace  $U = \text{span}(e^x, e^{2x}, \dots)$  of  $V$ .

Define  $T \in \mathcal{L}(V)$  by  $Tf = e^x f$ . Then  $\text{range } T = U$  is invariant under  $T$ .

Consider  $(T/U)(1 + U) = e^x + U = 0$

$\Rightarrow 0$  is an eigenvalue of  $T/U$  but is not an eigenvalue of  $T$

( $\text{null } T = \{0\}$ , for if not,  $\exists f \in V \setminus \{0\}$ ,  $(Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$ , contradicts ).  $\square$

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**33** Suppose  $T \in \mathcal{L}(V)$ . Prove that  $T/(\text{range } T) = 0$ .

**SOLUTION:**

$\forall v + \text{range } T \in V/\text{range } T, v + \text{range } T \in \text{null } (T/(\text{range } T))$

$\Rightarrow \text{null } (T/(\text{range } T)) = V/\text{range } T \Rightarrow T/(\text{range } T)$  is a zero map.  $\square$

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**34** Suppose  $T \in \mathcal{L}(V)$ . Prove that  $T/(\text{null } T)$  is injective  $\iff (\text{null } T) \cap (\text{range } T) = \{0\}$ .

**SOLUTION:**

(a) Suppose  $T/(\text{null } T)$  is injective.

Then  $(T/(\text{null } T))(u + \text{null } T) = Tu + \text{null } T = 0$

$\iff Tu \in \text{null } T \text{ } \text{and} \text{ } Tu \in \text{range } T \iff u + \text{null } T = 0 \iff u \in \text{null } T \iff Tu = 0$ .

Thus  $(\text{null } T) \cap (\text{range } T) = \{0\}$ .

(b) Suppose  $(\text{null } T) \cap (\text{range } T) = \{0\}$ .

Then  $(T/(\text{null } T))(u + \text{null } T) = Tu + \text{null } T = 0$

$\iff Tu \in \text{null } T \text{ } \text{and} \text{ } Tu \in \text{range } T \iff Tu = 0 \iff u \in \text{null } T \iff u + \text{null } T = 0$ .

Thus  $T/(\text{null } T)$  is injective.  $\square$

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**ENDED**

## 5.B

• Give an example of  $T \in \mathcal{L}(\mathbf{R}^2)$  such that  $T^4 = -I$ .

**SOLUTION:**

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• Suppose  $T \in \mathcal{L}(V)$  has no eigenvalues and  $T^4 = I$ . Prove that  $T^2 = -I$ .

**SOLUTION:**

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• Suppose  $T \in \mathcal{L}(V)$  and  $m$  is a positive integer.

(a) Prove that  $T$  is injective  $\iff T^m$  is injective.

(b) Prove that  $T$  is surjective  $\iff T^m$  is surjective.

**SOLUTION:**

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