简介

这是我个人用于复习的「***Linear Algebra Done Right 3E/4E, by Sheldon Axler***」笔记，一本习题选答与课文补注。范围覆盖所有第三版和第四版的课文和习题（除了第一章 A 节、极少数结合上下文太过显而易见的习题、没有任何日后反复推敲价值的当堂/′ 一遍过′ 习题和方法套路过于雷同的习题）。这份笔记尚处于缓慢的编撰进度中。

习题答案中，有我完全独立思考得出的，有抄 https://linearalgebras.com/ 的，有抄 https://math.stackexchange.com/ 的，有抄 LADR2eSolutions（By Axler）.pdf，有抄最新的 LADR4eSolutions 经典最全（By Axler?）.pdf，还有请教别人，乃至请教 AI 得出来的。

这些文档的许可证件，除 LADR4eSolutions 经典最全（By Axler?）.pdf 找不到/没有指明外，都允许复制/引用。

课文补注中，除了我独立思考总结出的易错误区和技巧、难点之外，还（因为我想要兼容那些使用 LADR 第三版纸质书的读者，包括我在内）把 LADR4e 中对课文定理等等的修改也（作了简化和提炼）摘录上去。部分课文内容因为比较简单，比如 3E 节的积空间，所以我做了概念前置，这相当于更改了原书的内容顺序。

题目标为正常数字 **N** 的，为第三版某章某节第 **N** 题（有个别题是第四版又删去的，这里，或直接摘录，或合并简化，仍然作保留；还有个别题是第四版增添条件、设问的，也一并写在第 **N** 题下）。题目标为‘•’的，为第四版。因为要面向以第三版为主要教材的学习者，所以为了避免混淆，故而将题号（部分题目的实心黑点后有标注具体第四版的数字标号）、甚至章节略去（一些变动过大的章节除外）。题目顺序会有调换，在每章大标题处会交代清楚。除了原书第四版新加入的章节外，均使用原书第三版的索引。这也许对第四版的使用者很不友好，我在此欢迎有心人士将我的作品修改后在同样的 **CC BY NC SA** 条款下作为**衍生作品**发布。

因为使用中文会给我编撰这份笔记带来额外的中英文输入法切换的工作成本，况且对于专业学习者，直接使用英文不会造成任何困扰。但英文词句的冗长性拖慢我编撰/复习的效率，所以我对许多常用术语作了简写。

Email: 13012057210@163.com
Bilibili: H-U_O
Gitee/GitHub: Songbingzhi628

作者序

我目前还没有能力和资格评论原书好坏以及线性代数课程教材选用的问题。但作为原书的学习者，我可以说：

相较于（其他课程的）其他教材，以 LADR 作为**自学读本**的**精学**计划，往往在执行中出现一次又一次的时间误判/超时，比如我最开始计划 $40\times 8\text{h}$ 完成 LADR 的精学，差不多是一天（8h）完成一节，还有额外的复习时间。但在实际学习中，（刨去笔记的功夫）完成到一半时，发现已经耗费了约 $35\times 8\text{h}$，于是我不得不重新估计 LADR 精学所需的总时间为 $70\times 8\text{h}$。这一点对于有学时/学期限制/应试要求的线性代数初学者来说很不安全。更主观地讲，这是因为 LADR 更像是一本参考手册，而不是一本细致入微的自学读本；如果把 LADR 作为初学线性代数第一教材和自学读本来学习，会面临不小的困难。

以上或许能劝退相当一部分打算入门的线性代数初学者。S.Axler 说这本书作为第二遍学习线性代数的教材更合适。我认为理由就是，在校的科班生第二遍学习线性代数时，也已经学习过了离散数学、抽象代数、数论、数学分析等课程，这些学习经验统统会化作一个叫“mathematical maturity”的东西，让他们面对 LADR 的课文和习题不再少见多怪、茫然无措。据此，我进一步认为，对于完全的初学者，想要完成 LADR 的精学，要么有很好的天赋，要么有与之相匹配的“mathematical maturity”，再要么，拿出足够的耐心和毅力。幸运的是，在坚持学习 LADR 的过程中，这三样会一同增益。就我个人来说：课文一次看不懂，就多看几遍，一天看不懂，就分三天看；习题一个小时做不出来，就隔六个小时再尝试，一天做不出来，就隔天再尝试。这确实让我收获了独特的学习体验和能力，我迄今也无法在别处得到，因此我很珍视 LADR，我愿意为此编撰一份电子辅助书并免费公开于网络中。这本身并不花费什么，因为实际的时间开销包括了很多不相干的额外项目：初学 LaTeX、调整代码架构、了解许可证选用，诸如此类的各种波折，也不乏戏剧性——时间花销主要在：早期的学习态度还不够主动，导致太多′ 一遍过′ 的习题被摘录到这里；没有独立编撰大型文档的经验和模板，可能会强迫症似地纠结散乱的格式和对齐。

我在学习过程中碰到了很多重大误区：**第一章中**，我一开始误认为 $W=\complement_V U\cup\{0\}$ 是唯一使得 $W\oplus U=V$ 的子空间，但这压根就不是子空间，而且 C 节习题中也提示这样的子空间 W 不唯一。**第二章中**，我随意地将“线性无关的序列”等同于有/无限维向量空间的基，没有任何理论依据，我也并不懂什么选择公理。**第三章 B 到 D 节中**，我总觉得子空间是超脱有限维的存在；因为放不下第二章无限维向量空间的基的情结，我刻意寻找那些避开涉及基的解法，一些臆测的结论和容易就找到反例。**第三章 E 节中**，我似乎对商空间有什么误解，觉得 $v+U=v'+U$ 如同变戏法一样，把 v 中一切带有 U 的部分抹除掉，让 v 变得纯粹独立于 U，为此我还单门发明了 Pure V/U 并试着证明一些命题，甚至用它发现了 F 节 23 题无限维情况下不依赖基的解法。后来我猛然发现我最开始的想法多么荒诞，却仍然放不下 Pure V/U 的情结。这些挫折让我思维变得更加缜密，于是在学习抽象的**第三章 F 节**时比想象中的要顺利。

Abbreviation Table

A B

add	addi(tion)(tive)
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because
bss	basis
bses	bases
B_V	basis of V

C

ch	characteristic
closd	closed under
coeff	coefficient
col	column
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	constrapositive

D

def	definition
deg	degree
dep	dependen(t)(ce)
deri	derivative(s)
diag	diagonal(iza-ble/ility/tion)
diff	differentia(l)(ting)(tion)
diffce	difference
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

E

-ec	-ec(t)(tor)(tion)(tive)
eig-	eigen-
elem	element(s)
ent	entr(y)(ies)
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
generalized eig-	gig-
G disk	Gershgorin disk
homo	homogeneity
hypo	hypothesis

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
invar	invariant
invard	invariant under
invarsp	invariant subspace
invarspd	invariant subspace under
iso	isomorph(ism)(ic)

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
multy	multiplicity
nilp	nilpotent
non0	nonzero
nonC	nonconst
notat	notation(al)

O P Q

optor	operator
othws	otherwise
prod	product
poly	polynomial
quad	quadratic
quotient	quot

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)
rotat	rotation

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

T U V W X Y Z

trig	triangular
trslate	translate
trspose	transpose
uniq	unique
uniqnes	uniqueness
up-	upper-
val	value
vec	vector
-wd	-ward
-ws	-wise
wrto	with respect to

1·B

- **NOTE FOR Fields:** *Many choices.* [*Req Multi Inv Uniq*]
  EXA: $\mathbf{Z}_m = \{K_0, K_1, \dots, K_{m-1}\}$ *is a field* $\Longleftrightarrow m \in \mathbf{N}^+$ *is a prime.*

- (4E 1.B.7) *Supp* $V \neq \emptyset$ *and* W *is a vecsp. Let* $W^V = \{f : V \to W\}$.
  (a) *Define a natural add and scalar multi on* W^V. (b) *Prove* W^V *is a vecsp with these defs.*

SOLUS:

(a) $W^V \ni f + g : x \to f(x) + g(x)$; where $f(x) + g(x)$ is the vec add on W.
$W^V \ni \lambda f : x \to \lambda f(x)$; where $\lambda f(x)$ is the scalar multi on W.

(b) Commu, Assoc, Distr are omitted.
Add Inv: $(f+g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0 = 0(x)$.
Multi Id: $(1f)(x) = 1f(x) = f(x)$. (NOTICE that the smallest $\mathbf{F}$ is $\{0, 1\}$.)
We must have used the same properties in W. [***If*** W^V ***is a vecsp, then*** W ***must be a vecsp.***] ☐

ENDED

1·C 注意：这里我将 3.E 积空间的定义前置；仅涉及概念。

- **NOTE FOR Exe (5):** $\mathbf{C} = \mathbf{R} \oplus \{c\mathrm{i} : c \in \mathbf{R}\} = \{a + b\mathrm{i} : a, b \in \mathbf{R}\}$ if we let $\mathbf{F} = \mathbf{R}$ and $\mathrm{i}^2 = -1$.

- **NOTE FOR Exe (6):** Supp V is a vecsp over $\mathbf{R}$. Then V is not a vecsp over $\mathbf{C}$. See also (9.A.16,17).
- **COMMENT:** Supp V is a vecsp over $\mathbf{C}$ of dim n. Then V is also a vecsp over $\mathbf{R}$ of dim $2n$.

- *Supp* U, W, V_1, V_2, V_3 *are subsps of* V.

15 $U + U \ni u + w \in U$. **16** $U + W \ni u + w = w + u \in W + U$. ☐

17 $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3)$. ☐

- $(U + W)_{\mathbf{C}} \ni (u_1 + w_1) + \mathrm{i}(u_2 + w_2) = (u_1 + \mathrm{i}u_2) + (w_1 + \mathrm{i}w_2) \in U_{\mathbf{C}} + W_{\mathbf{C}}$. ☐
- $(U \cap W)_{\mathbf{C}} \ni u_1 + \mathrm{i}u_2 = w_1 + \mathrm{i}w_2 \in U_{\mathbf{C}} \cap W_{\mathbf{C}}$. ☐
- $U_{\mathbf{C}} = W_{\mathbf{C}} \Longleftrightarrow U = W$. Supp $U_{\mathbf{C}} \ni u + \mathrm{i}v \in W_{\mathbf{C}}$. Then $U \ni u, v \in W$. ☐
- $V_{1\mathbf{C}} \times \cdots \times V_{m\mathbf{C}} = (V_1 \times \cdots \times V_m)_{\mathbf{C}}$. ☐

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18 *Does the add on the subsps of* V *have an add id? Which subsps have add invs?*

SOLUS: Supp Ω is the uniq add id.

(a) For any subsp U of V, $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

(b) Supp $U + W = \Omega$. Becs $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W \Rightarrow U = W = \Omega = \{0\}$. ☐

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11 *Prove the intersec of every collec of subsps of* V *is a subsp of* V.

SOLUS: Supp $\{U_\alpha\}_{\alpha \in \Gamma}$ is a collec of subsps of V; here Γ is an index set.

We show $\bigcap_{\alpha \in \Gamma} U_\alpha$, which equals the set of vecs in each U_α, is a subsp of V.

(a) $0 \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Nonempty.

(b) $u, v \in \bigcap_{\alpha \in \Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Closd add.

(c) $u \in \bigcap_{\alpha \in \Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Closd scalar multi.

Thus $\bigcap_{\alpha \in \Gamma} U_\alpha$ is nonempty subset of V that is closd add and scalar multi. ☐

• **Note For [1.45]:** Another proof: Supp $\forall v \in V, \exists!\,(u,w) \in U \times W, v = u + w$.
Asum non0 $v \in U \cap W$. Then the $(u,w) = (v,0)$ or $(0,v)$, ctradic the uniqnes. $\square$

• **Tips 1:** Supp $U, W \subseteq V$. And U, W, V are vecsps $\Rightarrow U, W$ are subsps of V.
Then $U + W$ is also a subsp of V. Becs $\forall u \in U, w \in U, u + w \in V$ since $u, w \in V$.

• **Note For "$\complement_V U \cup \{0\}$":** "$\complement_V U \cup \{0\}$" is supposed to be a subsp W suth $V = U \oplus W$.

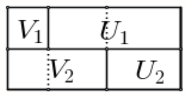
But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cup \{0\} \\ u \pm w \in \complement_V U \cup \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cup \{0\}$. Ctradic.

To fix this, denote the set $\{W_1, W_2, \cdots\}$ by $\mathcal{S}_V U$, where each $W_i \oplus U = V$.

• **Tips 2:** *Supp $V_1 \subseteq V_2$ in Exe (23). Prove $V_1 = V_2$.*

Solus: Becs the subset V_1 of vecsp V_2 is closd add and scalar multi, V_1 is a subspace of V_2.
Supp W is suth $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$.
If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, ctradic. Hence $W = \{0\}, V_1 = V_2$. $\square$

• *Supp V_1, V_2, U_1, U_2 are vecsps, $V_1 \oplus U_1 = V_2 \oplus U_2,\ V_1 \subseteq V_2,\ U_2 \subseteq U_1$.
Prove or give a countexa: $V_1 = V_2,\ U_1 = U_2$.*



Solus: Let $U_2 = \{0\}$. Give an exa that each of V_1, V_2, U_1 is non0. $\square$

• *Supp the intersec of any two of the vecsps U, W, X, Y is $\{0\}$.
Give an exa that $(X \oplus U) \cap (Y \oplus W) \neq \{0\}$.*

Solus: [Using notas in Chapter 2.] Let $B_X = (e_1), B_U = (e_2 - e_1), B_Y = (\ ), B_W = (e_2)$.

• **Tips 3:** *Supp $V = X \oplus Y$, and Z is a subsp of V. Show $X \subseteq Z \Rightarrow Z = X \oplus (Y \cap Z)$.*

Solus: $\forall z \in Z, \exists!\,(x,y) \in X \times Y, z = x + y$.
Becs $x \in Z \Rightarrow z - x = y \in Z \Rightarrow z \in X + (Y \cap Z)$. 又 $X \cap (Y \cap Z) \subseteq X \cap Y$. $\square$

• **Tips 4:** *Let $V = U + W,\ I = U \cap W,\ U = I \oplus X,\ W = I \oplus Y$. Prove $V = I \oplus (X \oplus Y)$.*

Solus: We show $X \cap Y = U \cap Y = W \cap X = \{0\}$ by ctradic.
$X \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap X \neq \{0\}, I \cap Y \neq \{0\}$.
$U \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap Y \neq \{0\}$. Simlr for $W \cap X$.
Thus $I + (X + Y) = (I \oplus X) \oplus Y = I \oplus (X \oplus Y)$.
Now we show $V = I + (X + Y)$. $\forall v \in V, v = u + w, \exists\,(u,w) \in U \times W$
$\Rightarrow \exists\,(i_u, x_u) \in I \times X,\ (i_w, y_w) \in I \times Y,\ v = (i_u + i_w) + x_u + y_w \in I + (X + Y)$. $\square$

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12 *Supp U, W are subsps of V. Prove $U \cup W$ is a subsp of $V \iff U \subseteq W$ or $W \subseteq U$.*

Solus: (a) Supp $U \subseteq W$. Then $U \cup W = W$ is a subsp of V.
(b) Supp $U \cup W$ is a subsp of V. Asum $U \not\subseteq W,\ U \not\supseteq W$ ($U \cup W \neq U$ and W).
Then $\forall a \in U \wedge a \notin W, \forall b \in W \wedge b \notin U$, we have $a + b \in U \cup W$.
$\left.\begin{array}{l} a + b \in U \Rightarrow b = (a + b) + (-a) \in U, \text{ctradic} \Rightarrow W \subseteq U. \\ a + b \in W \Rightarrow a = (a + b) + (-b) \in W, \text{ctradic} \Rightarrow U \subseteq W. \end{array}\right|$ Ctradic asum. $\square$

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13 *Supp U_1, U_2, U_3 are subsps of V, and the union $U_1 \cup U_2 \cup U_3 = \mathcal{U}$ is a subsp of V. Prove one of the subsps contains the other two.*

This exe is not true if we replace $\mathbf{F}$ with a field containing only two elems.

Solus: Exa: Let $\mathbf{F} = \mathbf{Z}_2$. $U_1 = \{u, 0\}, U_2 = \{v, 0\}, U_3 = \{v+u, 0\}$. While $\mathcal{U} = \{0, u, v, v+u\}$ is a subsp.

Notice that, $U \cup W = V$ is vecsp $\not\Rightarrow U, W$ are subsps of V.

This trick is invalid: $(A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.

(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$. By applying Exe (12) we conclude that one U_j contains the other two. Thus done.

(II) Asum no one is contained in the union of other two, and no one contains the other two. Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3;\ v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if any $v + \lambda u \in W \cap U_1$ then $v + \lambda u - \lambda u = v \in U_1$.

Now $W \subseteq U_1 \cup U_2 \cup U_3 \Rightarrow W \subseteq U_2 \cup U_3$. $\forall v + \lambda u \in W, v + \lambda u \in U_i, i = 2, 3$.

If $U_2 \subseteq U_3$ or $U_2 \supseteq U_3$, then $\mathcal{U} = U_1 \cup U_i, i = 2, 3$. By Exe (12) done.

Othws, both $U_2, U_3 \neq \{0\}$. Becs $W \subseteq U_2 \cup U_3$ *has at least three disti elems.*

There must be some U_i that contains at least two disti elems of W.

$\exists\, \lambda_1 \neq \lambda_2,\ v + \lambda_1 u$ and $v + \lambda_2 u$ both in U_2 or $U_3 \Rightarrow u \in U_2 \cap U_3$, ctradic. $\square$

Ended

2·A

1 *Prove* $[P]$ (v_1, v_2, v_3, v_4) *spans* $V \iff \big(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\big)$ *also spans* V $[Q]$.

Solus: Note that $V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists\, a_1, \dots, a_n \in \mathbf{F}, v = a_1v_1 + \cdots + a_nv_n$.

Asum $\forall v \in V, \exists\, a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbf{F}$, (that is, if $\exists\, a_i$, then we are to find b_i, vice versa)

$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4$

$\quad = b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4$

$\quad = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \cdots + a_4)v_4$. $\square$

• (4E 3, 14) *Supp* $(v_1, \dots, v_m)$ *is a list in* V*. For each* k*, let* $w_k = v_1 + \cdots + v_k$.

(a) *Show* $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

(b) *Show* $[P]$ $(v_1, \dots, v_m)$ *is liney indep* $\iff$ $(w_1, \dots, w_m)$ *is liney indep* $[Q]$.

Solus:

(a) Asum $a_1v_1 + \cdots + a_mv_m = b_1w_1 + \cdots + b_mw_m = b_1v_1 + \cdots + b_k(v_1 + \cdots + v_k) + \cdots + b_m(v_1 + \cdots + v_m)$.
Then $a_k = b_k + \cdots + b_m;\ \ a_{k+1} = b_{k+1} + \cdots + b_m \Rightarrow b_k = a_k - a_{k+1};\ b_m = a_m$. Simlr to Exe (1).

(b) $P \Rightarrow Q:\ \ b_1w_1 + \cdots + b_mw_m = 0 = a_1v_1 + \cdots + a_mv_m$, where $0 = a_k = b_k + \cdots + b_m$.
$Q \Rightarrow P:\ \ a_1v_1 + \cdots + a_mv_m = 0 = b_1w_1 + \cdots + b_mw_m = 0$, where $0 = b_m = a_m,\ 0 = b_k = a_k - a_{k+1}$.
Or. By (a), let $W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$. Supp $(v_1, \dots, v_m)$ is liney dep.
By [2.21](b), a list of len $(m-1)$ spans W. 又 By [2.23], $(w_1, \dots, w_m)$ liney indep $\Rightarrow m \leqslant m - 1$.
Thus $(w_1, \dots, w_m)$ is liney dep. Now rev the roles of v and w. $\square$

---

2 (a) $[P]$ *A list* (v) *of len* 1 *in* V *is liney indep* $\iff v \neq 0$. $[Q]$

(b) $[P]$ *A list* (v, w) *of len* 2 *in* V *is liney indep* $\iff \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$. $[Q]$

Solus: (a) $Q \Rightarrow P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ liney indep.
$P \Rightarrow Q : (v)$ liney indep $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.
$\neg Q \Rightarrow \neg P : v = 0 \Rightarrow av = 0$ while we can let $a \neq 0 \Rightarrow (v)$ is liney dep.
$\neg P \Rightarrow \neg Q : (v)$ liney dep $\Rightarrow av = 0$ while $a \neq 0 \Rightarrow v = 0$.

(b) $P \Rightarrow Q : (v, w)$ liney indep $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.
$Q \Rightarrow P :$ no scalar multi $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v, w)$ liney indep.
$\neg P \Rightarrow \neg Q : (v, w)$ liney dep $\Rightarrow$ if $av + bw = 0$, then a or $b \neq 0 \Rightarrow$ scalar multi.
$\neg Q \Rightarrow \neg P :$ scalar multi $\Rightarrow$ if $av + bw = 0$, then a or $b \neq 0 \Rightarrow$ liney dep. $\square$

---

10 *Supp* $(v_1, \dots, v_m)$ *is liney indep in* V *and* $w \in V$.
Prove if $\big(v_1 + w, \dots, v_m + w\big)$ *is linely dep, then* $w \in \operatorname{span}(v_1, \dots, v_m)$.

Solus:

Note that $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0 \Rightarrow a_1v_1 + \cdots + a_mv_m = -(a_1 + \cdots + a_m)w$.
Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1v_1 + \cdots + a_mv_m = 0$ while $a_i \neq 0$ for some i, ctradic.
Or. We prove the ctrapos: Supp $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $a_1 + \cdots + a_m = 0$.
Thus $a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$. Hence $\big(v_1 + w, \dots, v_m + w\big)$ is liney indep. $\square$
Or. $\exists\, j \in \{1, \dots, m\}, v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and done.
If $j \geqslant 2$, then $\exists\, a_i \in \mathbf{F}, v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1v_1 + \cdots + a_{j-1}v_{j-1}$.
Where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, ctradic.
Now $w = \lambda^{-1}(a_1v_1 + \cdots + a_{j-1}v_{j-1} - v_j) \Rightarrow w \in \operatorname{span}(v_1, \dots, v_m)$. $\square$

---

11 *Supp* $(v_1, \dots, v_m)$ *is liney indep in* V *and* $w \in V$.
Show $[P]$ $(v_1, \dots, v_m, w)$ *is liney indep* $\iff w \notin \operatorname{span}(v_1, \dots, v_m)$ $[Q]$.

SOLUS: Equiv to $(v_1, \dots, v_m, w)$ liney dep $\iff w \in \operatorname{span}(v_1, \dots, v_m)$. Using [2.21]. Obviously. $\square$

NOTE: (a) Supp $(v_1, \dots, v_m, w)$ is liney indep. Then $(v_1, \dots, v_m)$ liney indep $\iff w \notin \operatorname{span}(v_1, \dots, v_m)$.
(b) Supp $(v_1, \dots, v_m, w)$ is liney dep. Then $(v_1, \dots, v_m)$ liney indep $\iff w \in \operatorname{span}(v_1, \dots, v_m)$.

---

14 *Prove* $[P]$ V *is infinide* $\iff \exists$ *seq* $(v_1, v_2, \dots)$ *in* V *suth each* $(v_1, \dots, v_m)$ *liney indep.* $[Q]$

SOLUS: $P \Rightarrow Q$: Supp V is infinide, so that no list spans V. Define the desired seq recurly via:
Step 1 Pick a $v_1 \neq 0$, (v_1) liney indep.
Step m Pick a $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$, by Exe (11), $(v_1, \dots, v_m)$ is liney indep.

$\neg P \Rightarrow \neg Q$: Supp V is finide and $V = \operatorname{span}(w_1, ..., w_m)$.
Let $(v_1, v_2, \dots)$ be a seq in V, then $(v_1, v_2, \dots, v_{m+1})$ must be liney dep.

OR. $Q \Rightarrow P$: Supp there is such a seq.
Choose an m. Supp a liney indep list $(v_1, \dots, v_m)$ spans V.
Simlr to [2.16]. $\exists\, v_{m+1} \in V \backslash \operatorname{span}(v_1, \dots, v_m)$. Hence no list spans V. $\square$

---

17 *Prove* $(p_0, p_1, \dots, p_m)$ *cannot be liney indep in* $\mathcal{P}_m(\mathbf{F})$ *with each* $p_k(2) = 0$.

SOLUS:

Supp $(p_0, p_1, \dots, p_m)$ is liney indep. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z$.
NOTICE that $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \cdots + a_m p_m(z)$, for if not, let $z = 2$. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has len $(m+1)$.
Hence $(p_0, p_1, \dots, p_m)$ is linely dep. For if not, then becs $(1, z, \dots, z^m)$ of len $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$,
by the steps in [2.23] trivially, $(p_0, p_1, \dots, p_m)$ of len $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$. Ctradic. $\square$

OR. Becs $(1, z, \dots, z^m)$ of len $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$. Then $(p_0, p_1, \dots, p_m, z)$ of len $(m+2)$ is liney dep.
As shown above, $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$. And hence by [2.21](a), $(p_0, p_1, \dots, p_m)$ is liney dep. $\square$

ENDED

2·B

• **NOTE FOR *liney indep seq and* [2.34]:** "$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expr.
If we allow using "infini list", then we must assure that $(v_1, \dots, v_n, \dots)$ is a spanning "list"
suth $\forall v \in V, \exists$ smallest $n \in \mathbf{N}^+$, $v = a_1 v_1 + \cdots + a_n v_n$. Moreover, given a list $(w_1, \cdots, w_n, \cdots)$ in W,
we can prove $\exists\,!\, T \in \mathcal{L}(V, W)$ with each $T v_k = w_k$, which has less restr than [3.5].
But the key point is, how can we assure that such a "list" exis ? [*See higher courses*]

---

1 *Find all vecsps on whatever* $\mathbf{F}$ *that have exactly one bss.*

SOLUS: The trivial vecsp $\{0\}$ will do. Indeed, the only bss of $\{0\}$ is the empty list ( ).
Now consider the field $\{0, 1\}$ containing only the add id and multi id,
with $1 + 1 = 0$. Then the list (1) is the uniq bss. Now the vecsp $\{0, 1\}$ will do.
COMMENT: All vecsp on such $\mathbf{F}$ of dim 1 will do.
Consider other $\mathbf{F}$. Note that this $\mathbf{F}$ contains at least and strictly more than 0 and 1. Failed. $\square$

• (4E 9) *Supp* $(v_1, \dots, v_m)$ *is a list in* V*. For* $k \in \{1, \dots, m\}$*, let* $w_k = v_1 + \cdots + v_k$*.*
Show $[P]$ $B_V = (v_1, \dots, v_m) \iff B_V = (w_1, \dots, w_m)$. $[Q]$

SOLUS: NOTICE that $B_U = (u_1, \dots, u_n) \iff \forall u \in U, \exists !\, a_i \in \mathbf{F}, u = a_1 u_1 + \cdots + a_n u_n$.

$P \Rightarrow Q : \forall v \in V, \exists !\, a_i \in \mathbf{F},\ v = a_1 v_1 + \cdots + a_m v_m \Rightarrow v = b_1 w_1 + \cdots + b_m w_m, \exists !\, b_k = a_k - a_{k+1}, b_m = a_m$.

$Q \Rightarrow P : \forall v \in V, \exists !\, b_i \in \mathbf{F},\ v = b_1 w_1 + \cdots + b_m w_m \Rightarrow v = a_1 v_1 + \cdots + a_m v_m, \exists !\, a_k = \sum_{j=k}^{m} b_j$. ☐

COMMENT: OR. Using [3.C NOTE FOR [3.30, 32](a)].

---

8 *Supp* $B_U = (u_1, \dots, u_m), B_W = (w_1, \dots, w_n)$*.*
Prove $V = U \oplus W \iff B_V = (u_1, \dots, u_m, w_1, \dots, w_n)$*.*

SOLUS: $\forall v \in V, \exists !\, u \in U, w \in W \Rightarrow \exists !\, a_i, b_i \in \mathbf{F}, v = u + w = \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i$.

OR. $V = \operatorname{span}(u_1, \dots, u_m) \oplus \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.

Note that $\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0 \Rightarrow \sum_{i=1}^{m} a_i u_i = -\sum_{i=1}^{n} b_i w_i \in U \cap W = \{0\}$. ☐

• (9.A.3,4 OR 4E 11) *Supp* V *is on* $\mathbf{R}$*, and* $v_1, \dots, v_n \in V$*. Let* $B = (v_1, \dots, v_n)$*.*

(a) *Show* $[P]$ *B is liney indep in* $V \iff$ *B is liney indep in* $V_{\mathbf{C}}$. $[Q]$

(b) *Show* $[P]$ *B spans* $V \iff$ *B spans* $V_{\mathbf{C}}$. $[Q]$

SOLUS:

(a) $P \Rightarrow Q$: Note that each $v_k \in V_{\mathbf{C}}$. Supp $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ with $\mathbf{F} = \mathbf{C}$.
Then $(\operatorname{Re}\lambda_1) v_1 + \cdots + (\operatorname{Re}\lambda_n) v_n = 0 \Rightarrow$ each $\operatorname{Re}\lambda_i = 0$, simlr for $\operatorname{Im}\lambda_i$.

$Q \Rightarrow P$: If $\lambda_k \in \mathbf{R}$ with $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then each $\operatorname{Re}\lambda_k = \lambda_k = 0$.

$\neg P \Rightarrow \neg Q$: $\exists\, v_j = a_{j-1} v_{j-1} + \cdots + a_1 v_1 \in V_{\mathbf{C}}$.

$\neg Q \Rightarrow \neg P$: $\exists\, v_j = \lambda_{j-1} v_{j-1} + \cdots + \lambda_1 v_1 \in V \Rightarrow v_j = (\operatorname{Re}\lambda_{j-1}) v_{j-1} + \cdots + (\operatorname{Re}\lambda_1) v_1 \in V$.

(b) $P \Rightarrow Q$: $\forall u + \mathrm{i}v \in V_{\mathbf{C}},\ u, v \in V \Rightarrow \exists\, a_i, b_i \in \mathbf{R}, u + \mathrm{i}v = \sum_{i=1}^{n} (a_i + \mathrm{i}b_i) v_i$.

$Q \Rightarrow P$: $\forall v \in V, \exists\, a_i + \mathrm{i}b_i \in \mathbf{C},\ v + \mathrm{i}0 = \left(\sum_{i=1}^{n} a_i v_i\right) + \mathrm{i}\left(\sum_{i=1}^{n} b_i v_i\right) \Rightarrow v \in \operatorname{span}(v_1, \dots, v_m)$.

$\neg P \Rightarrow \neg Q$: $\exists\, v \in V,\ v \notin \operatorname{span} B$ with $\mathbf{F} = \mathbf{R} \Rightarrow v + \mathrm{i}0 \notin \operatorname{span} B$ with $\mathbf{F} = \mathbf{C}$.

$\neg Q \Rightarrow \neg P$: $\exists\, u + \mathrm{i}v \in V_{\mathbf{C}}, u + \mathrm{i}v \notin \operatorname{span} B \Rightarrow (\operatorname{Re} 1) u + (\operatorname{Re} \mathrm{i}) v = u$ or $(\operatorname{Im} 1) u + (\operatorname{Im} \mathrm{i}) v = v \notin \operatorname{span} B$. ☐

• **TIPS:** *Supp* $\dim V = n$*, and* U *is a subsp of* V *with* $U \neq V$*.*
Prove $\exists\, B_V = (v_1, \dots, v_n)$ *suth each* $v_k \notin U$*.*

Note that $U \neq V \Rightarrow n \geqslant 1$. We will construct B_V via the following process.

Step 1. $\exists\, v_1 \in V \setminus U \Rightarrow v_1 \neq 0$. If $\operatorname{span}(v_1) = V$ then we stop.

Step k. Supp $(v_1, \dots, v_{k-1})$ is liney indep in V, each of which belongs to $V \setminus U$.

Note that $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$. And if $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U = V$, then by (1.C.12),
[becs $\operatorname{span}(v_1, \dots, v_{k-1}) \not\subseteq U$,] $U \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) \Rightarrow \operatorname{span}(v_1, \dots, v_{k-1}) = V$.
Hence becs $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$, it must be case that $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U \neq V$.
Thus $\exists\, v_k \in V \setminus U$ suth $v_k \notin \operatorname{span}(v_1, \dots, v_{k-1})$.
By (2.A.11), $(v_1, \dots, v_k)$ is liney indep in V. If $\operatorname{span}(v_1, \dots, v_k) = V$, then we stop.

Becs V is finide, this process will stop after n steps. ☐

OR. Supp $U \neq \{0\}$. Let $B_U = (u_1, \dots, u_m)$. Extend to a bss $(u_1, \dots, u_n)$ of V.
Then let $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$. ☐

ENDED

2·C

• **Note For Exe (15):** *Supp $v \in V\backslash\{0\}$. Prove $\exists\, B_V = (v_1, \dots, v_n), v = v_1 + \cdots + v_n$.*

Solus: If $n = 1$ then let $v_1 = v$ and done. Supp $n > 1$.
Extend (v) to a bss $(v, v_1, \dots, v_{n-1})$ of V. Let $v_n = v - v_1 - \cdots - v_{n-1}$.
又 $\operatorname{span}(v, v_1, \dots, v_{n-1}) = \operatorname{span}(v_1, \dots, v_n)$. Hence $(v_1, \dots, v_n)$ is also a bss of V. □

Comment: Let $B_V = (v_1, \dots, v_n)$ and supp $v = u_1 + \cdots + u_n$, where each $u_i = a_i v_i \in V_i$.
But $(u_1, \dots, u_n)$ might not be a bss, becs there might be some $u_i = 0$.

• Let $v_1, \dots, v_n \in V$ and $\dim \operatorname{span}(v_1, \dots, v_n) = n$. Then $(v_1, \dots, v_n)$ is a bss of $\operatorname{span}(v_1, \dots, v_n)$.
Notice that $(v_1, \dots, v_n)$ is a spanning list of $\operatorname{span}(v_1, \dots, v_n)$ *of len* $n = \dim \operatorname{span}(v_1, \dots, v_n)$.

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9 *Supp $(v_1, \dots, v_m)$ is liney indep in $V, w \in V$. Prove* $\dim \operatorname{span}\big(v_1 + w, \dots, v_m + w\big) \geqslant m - 1$.

Solus: Using (2.A.10, 11).
Note that each $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$.
$(v_1, \dots, v_m)$ liney indep $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ liney indep $\Rightarrow \underbrace{(v_2 - v_1, \dots, v_m - v_1)}_{\text{of len } (m-1)}$ liney indep.
又 If $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1 + w, \dots, v_m + w)$ is liney indep.
Hence $m \geqslant \dim \operatorname{span}\big(v_1 + w, \dots, v_m + w\big) \geqslant m - 1$. □

• (4E 16) *Supp V is finide, U is a subsp of V with $U \neq V$. Let $n = \dim V, m = \dim U$.*
Prove $\exists\, (n - m)$ subsps $U_1, \dots, U_{n-m}$, each of dim $(n - 1)$, suth $\bigcap\limits_{i=1}^{n-m} U_i = U$.

Solus: Let $B_U = (v_1, \dots, v_m)$, $B_V = \big(v_1, \dots, v_m, u_1, \dots, u_{n-m}\big)$.
Define each $U_i = \operatorname{span}\big(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m}\big) \Rightarrow U \subseteq U_i$.
And becs $\forall v \in \bigcap\limits_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \cdots + b_{n-m} u_{n-m} \in U_i \Rightarrow$ each $b_i = 0 \Rightarrow v \in U$.
Hence $\bigcap\limits_{i=1}^{n-m} U_i \subseteq U$. □

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14 *Supp $V_1, \dots, V_m$ are finide. Prove* $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$.

Solus: For each V_i, let $B_{V_i} = \mathcal{E}_i$. Then $V_1 + \cdots + V_m = \operatorname{span}\big(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m\big)$; $\dim V_i = \operatorname{card} \mathcal{E}_i$.
Now $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}\big(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m\big) \leqslant \operatorname{card}\big(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m\big) \leqslant \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.

Coro: $V_1 + \cdots + V_m$ is direct
$\iff$ For each $k \in \{1, \dots, m-1\}$, $(V_1 \oplus \cdots \oplus V_k) \cap V_{k+1} = \{0\}$, $(\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$
$\iff \dim \operatorname{span}\big(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m\big) = \operatorname{card}\big(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m\big) = \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$
$\iff \dim(V_1 \oplus \cdots \oplus V_m) = \dim V_1 + \cdots + \dim V_m$. □

• *Supp $\mathcal{C}$ is a collec of k-dim subsps of V with any two of them have a $(k-1)$-dim intersec.*
Prove either all contain a $(k-1)$-dim intersec, or all contained in a $(k+1)$-dim subsp.

Solus: If V is finide and $\dim V = k$, then $\mathcal{C} = \{V\}$, done. We use induc on k. (i) $k = 1$. Immed.
(ii) $k > 1$. Asum it holds for $k - 1$. If $\exists$ common $(k-1)$-dim intersec, then done.
Othws, we show all $X \in \mathcal{C}$ are contained in a $(k+1)$-dim subsp.
Supp $U, W \in \mathcal{C} \Rightarrow \dim(U + W) = k + 1$. Then for $X \in \mathcal{C}$, $X \cap U, X \cap W$ are $(k-1)$-dim.
Now by asum, $\dim(X \cap U + X \cap W) = k \Rightarrow X = (X \cap U) + (X \cap W) \Rightarrow X \subseteq U + W$. □

---

17 *Supp V_1, V_2, V_3 are subsps of a finide vecsp. Explain and give a countexa*:

$$\dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1\cap V_2) - \dim(V_1\cap V_3) - \dim(V_2\cap V_3) + \dim(V_1\cap V_2\cap V_3).$$

SOLUS:

(1) $\big|(A\cup B)\cup C\big| = |A\cup B| + |C| - \big|(A\cup B)\cap C\big|$.

(2) $\big|(A\cup B)\cap C\big| = \big|(A\cap C)\cup(B\cap C)\big| = |A\cap C| + |B\cap C| - |A\cap B\cap C|$.

Thus $\big|(A\cup B)\cup C\big| = |A| + |B| + |C| + |A\cap B\cap C| - |A\cap B| - |A\cap C| - |B\cap C|$.

Becs $(V_1+V_2)+V_3 = V_1 + (V_2+V_3) = (V_1+V_3)+V_2$.

$$\begin{aligned}\dim(V_1+V_2+V_3) &= \dim(V_1+V_2) + \dim(V_3) - \dim\big((V_1+V_2)\cap V_3\big) && (1)\\ &= \dim(V_2+V_3) + \dim(V_1) - \dim\big((V_2+V_3)\cap V_1\big) && (2)\\ &= \dim(V_1+V_3) + \dim(V_2) - \dim\big((V_1+V_3)\cap V_2\big) && (3).\end{aligned}$$

Generally, $(X+Y)\cap Z \neq (X\cap Z)+(Y\cap Z)$. **EXA:** $X = \{(x,0)\}, Y = \{(0,y)\}, Z = \{(z,z)\} \subseteq \mathbf{F}^2$.

COMMENT: If $X\subseteq Y$, then $(X+Y)\cap Z = Y\cap Z$; $\dim(X+Y+Z) = \dim Y + \dim Z - \dim(Y\cap Z)$, and the wrong formula holds. Simlr for $Y\subseteq Z$, $X\subseteq Z$, and $X, Y\subseteq Z$.

NOTE: However, it's true that $(X+Y)\cap Z \supseteq (X\cap Z)+(Y\cap Z) = \big(X+(Y\cap Z)\big)\cap Z$.
Becs $(X\cap Z)+(Y\cap Z) \ni v = x+y = z_1+z_2 \in \big(X+(Y\cap Z)\big)\cap Z \Rightarrow v\in (X+Y)\cap Z$.

- **TIPS:** Becs $\dim(V_1\cap V_2\cap V_3) = \dim V_1 + \dim(V_2\cap V_3) - \dim\big(V_1+(V_2\cap V_3)\big)$.
  And $\dim(V_2\cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2+V_3)$. We have (1), and (2), (3) simlr.
  (1) $\dim(V_1\cap V_2\cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2+V_3) - \dim\big(V_1+(V_2\cap V_3)\big)$.
  (2) $\dim(V_1\cap V_2\cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1+V_3) - \dim\big(V_2+(V_1\cap V_3)\big)$.
  (3) $\dim(V_1\cap V_2\cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1+V_2) - \dim\big(V_3+(V_1\cap V_2)\big)$.

- *Supp V_1, V_2, V_3 are subsps of V with*
  (a) $\dim V = 10$, $\dim V_1 = \dim V_2 = \dim V_3 = 7$. *Prove* $V_1\cap V_2\cap V_3 \neq \{0\}$.
  By TIPS, $\dim(V_1\cap V_2\cap V_3) \geqslant \dim V_1 + \dim V_2 + \dim V_3 \underline{-2\dim V} > 0$.
  (b) $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. *Prove* $V_1\cap V_2\cap V_3 \neq \{0\}$.
  By TIPS, $\dim(V_1\cap V_2\cap V_3) \underline{\underline{> 2\dim V}} - \dim(V_2+V_3) - \dim\big(V_1+(V_2\cap V_3)\big) \geqslant 0$. □

ENDED

3·A

注意：这里我将 3.B 的值空间、零空间、单满射、和 3.D 的可逆性定义前置；仅涉及概念。

- **TIPS 1:** $T : V \to W$ is liney $\iff \left|\begin{array}{l}(\text{一})\ \forall v,u\in V, T(v+u) = Tv + Tu;\\ (\text{二})\ \forall v,u\in V, \lambda\in\mathbf{F}, T(\lambda v) = \lambda(Tv).\end{array}\right| \iff T(v+\lambda u) = Tv + \lambda Tu.$

NOTE: Supp V is a vecsp. For $U\subseteq V$, U is a subsp of $V \iff \forall u_1, u_2\in U, \lambda\in\mathbf{F}, u_1+\lambda u_2\in U$.

- (3.E.1) *A function $T : V\to W$ is liney $\iff$ The graph of T is a subspace of $V\times W$.*

- (4E 10) **NOTE:** Composition and product are not the same in $\mathcal{P}(\mathbf{F})$.

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11 *Supp U is a subsp of V and $S \in \mathcal{L}(U,W)$.*
Prove $\exists\, T \in \mathcal{L}(V,W), Tu = Su, \forall u \in U$. (Or. $\exists\, T \in \mathcal{L}(V,W), T|_U = S$.)
*In other words, every liney map on a subsp of V can be **extended** to a liney map on the entire V.*

Solus: Supp W is suth $V = U \oplus W$. Then $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.
Define $T \in \mathcal{L}(V,W)$ by $T(u_v + w_v) = Su_v$. ☐
Or. [*Finide Req*] Define by $T\left(\sum_{i=1}^{m} a_i u_i\right) = \sum_{i=1}^{n} a_i S u_i$. Let $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$. ☐

• **Note For Restr:** U is a subsp of V. (a) $(T + \lambda S)\big|_U = T|_U + \lambda S|_U$. (b) $(ST)\big|_U = ST|_U$.

• **Tips 2:** $T \in \mathcal{L}(V,W)$. (a) If U is a subsp of W. Then $\operatorname{range} T \subseteq U \iff T \in \mathcal{L}(V,U) \subseteq \mathcal{L}(V,W)$.
(b) If U is a subsp of V. Then $U \subseteq \operatorname{null} T \iff T|_U = 0$.

• (4E 4.3) *Supp $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V,W)$, $S = \operatorname{Re} \circ T_{\mathrm{C}}$. Show $T_{\mathrm{C}} = S - \mathrm{i}\, S \circ \mathrm{i} I$.*

Solus: $T_{\mathrm{C}} = S + \mathrm{i} \operatorname{Im} T_{\mathrm{C}}$. 又 $\operatorname{Re} \circ (T_{\mathrm{C}}\, \mathrm{i} I) = \operatorname{Re} \circ (\mathrm{i}\, T_{\mathrm{C}}) = -\operatorname{Im} \circ T_{\mathrm{C}} = S \circ \mathrm{i} I$. ☐

Comment: $\operatorname{Re}, \operatorname{Im} : \mathbf{C} \mapsto \mathbf{R}$ is not liney, while they have the add.

• **Note For Complex of Liney Maps:** Supp V, W are vecsps over $\mathbf{R}$. Then $\mathcal{L}(V,W)_{\mathrm{C}} = \mathcal{L}(V_{\mathrm{C}}, W_{\mathrm{C}})$.
For $S, T \in \mathcal{L}(V,W)$, $(S + \lambda T)_{\mathrm{C}} = S_{\mathrm{C}} + \lambda T_{\mathrm{C}}$. For $S \in \mathcal{L}(V,W), T \in \mathcal{L}(U,V)$, $(ST)_{\mathrm{C}} = S_{\mathrm{C}} T_{\mathrm{C}}$.
For $T \in \mathcal{L}(V,W)$, $\operatorname{null}(T_{\mathrm{C}}) = (\operatorname{null} T)_{\mathrm{C}}$, $\operatorname{range}(T_{\mathrm{C}}) = (\operatorname{range} T)_{\mathrm{C}}$.

• (9.A.17) *Supp $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$ suth $T^2 = -I$. Define complex scalar multi on V as $(a + b\mathrm{i})v = av + bTv$. Then V itself is already a complex vecsp with these defs. Show the dim of V as a complex vecsp is half of the dim of V as the usual real vecsp.*

Solus: Supp $V \neq \{0\}$. Let $N = \dim V$ as real vecsp. We construct a real B_V via a $(N-1)$-step process.
Let (v_1, Tv_1) be liney indep in V as real vecsp. Let $v_2 \notin \operatorname{span}(v_1, Tv_1) \Rightarrow (v_1, Tv_1, v_2)$ liney indep.
Step 1. We show (v_1, Tv_1, v_2, Tv_2) liney indep in V as real vecsp. Asum $Tv_2 = a_1 v_1 + b_1 Tv_1 + a_2 v_2$.
Then $-v_2 = a_1 Tv_1 - b_1 v_1 + a_2 Tv_2$. Note that $a_2 \neq 0$ and $a_2^2 = -1$ while $a_2 \in \mathbf{R}$, ctradic.
Step k. $[k \leqslant N-1]$ We show $(v_1, Tv_1, \dots, v_k, Tv_k, v_{k+1}, Tv_{k+1})$ liney indep in V as real vecsp. Simlr.
Asum $Tv_{k+1} = a_1 v_1 + b_1 Tv_1 + \cdots + a_{k+1} v_{k+1}$. Then $-v_{k+1} = a_1 Tv_1 - b_1 v_1 + \cdots + a_{k+1} Tv_{k+1}$. ☐

• **Note For $\mathbf{F}^S$:**
Supp $S \neq \emptyset$, $C_S = \{f \in \mathbf{F}^S : \exists$ finily many x, suth $f(x) \neq 0\}$. Then C_S is a subsp of $\mathbf{F}^S$.
(a) *If $S = \{x_1, \dots, x_n\}$. Find a bss of $\mathbf{F}^S$ and conclude $\mathbf{F}^S = C_S$.* $\qquad$ $\mathbf{F}^S$ infinide $\Rightarrow S$ infini.
(b) *If S has infily many elem. Prove $\mathbf{F}^S$ is infinide.* $\qquad$ $\mathbf{F}^S$ finide $\Rightarrow S$ fini.
(c) *Supp V is on $\mathbf{F}$. Prove $\exists$ surj $T \in \mathcal{L}(C_V, V)$.*

Solus: (a) Define each $f_i(x_j) = \delta_{i,j}$. Supp $f \in C_S$, let each $y_k = f(x_k) = (y_1 f_1 + \cdots + y_n f_n)(x_k)$.
Then $f = y_1 f_1 + \cdots + y_n f_n \in \operatorname{span}(f_1, \dots, f_n)$. 又 If $f = 0$, then each $y_k = 0$.
(b) Let $S = \{x_1, \cdots, x_n, \cdots\}$. Define each $f_i(x_j) = \delta_{i,j} \Rightarrow f_i \in C_S$. 又 $(f_1, \cdots, f_n, \cdots)$ liney indep.
Coro: S fini $\iff \mathbf{F}^S$ finide.
(c) Define $T : C_V \to V$ by $T(f) = \sum f(x)x$. Note that $f(x) \neq 0$ for finily many $x \in V$.
<u>Becs for any $v \in V, \exists$ liney indep $(v_1, \dots, v_n)$ suth $v = a_1 v_1 + \cdots + a_n v_n$.</u> [*See higher courses*]
Define each $f(v_k) = a_k$ and $f(x) = 0$ for $x \notin \{v_1, \dots, v_n\}$. Then $T(f) = v$. ☐

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13 *Supp $(v_1,\dots,v_m)$ is linely dep in V and $W \neq \{0\}$.*
Prove $\exists\, w_1,\dots,w_m \in W, \nexists\, T \in \mathcal{L}(V,W)$ suth $Tv_k = w_k$, $\forall k = 1,\dots,m$.

Solus:

We prove by ctradic. By liney dep lemma, $\exists\, j \in \{1,\dots,m\}, v_j \in \operatorname{span}(v_1,\dots,v_{j-1})$.
Supp $a_1v_1 + \cdots + a_mv_m = 0$, where $a_j \neq 0$. Now let $w_j \neq 0$, while $w_1 = \cdots = w_{j-1} = w_{j+1} = w_m = 0$.
Define $T \in \mathcal{L}(V,W)$ with each $Tv_k = w_k$. Then $T(a_1v_1 + \cdots + a_mv_m) = 0 = a_1w_1 + \cdots + a_mw_m$.
And $0 = a_jw_j$ while $a_j \neq 0$ and $w_j \neq 0$. Ctradic. □

Or. We prove the ctrapos: Supp $\forall w_1,\dots,w_m \in W, \exists\, T \in \mathcal{L}(V,W)$, each $Tv_k = w_k$.
Now we show $(v_1,\dots,v_n)$ is liney indep. Supp $\exists\, a_i \in \mathbf{F}, a_1v_1 + \cdots + a_nv_n = 0$.
Choose one $w \in W \backslash \{0\}$. By asum, for $(\overline{a_1}w,\dots,\overline{a_m}w), \exists\, T \in \mathcal{L}(V,W)$, each $Tv_k = \overline{a_k}w$.
Now we have $0 = T\left(\sum_{k=1}^m a_kv_k\right) = \sum_{k=1}^m a_kTv_k = \sum_{k=1}^m a_k\overline{a_k}w = \left(\sum_{k=1}^m |a_k|^2\right)w$.
Then $\sum_{k=1}^m |a_k|^2 = 0$. Thus $a_1 = \cdots = a_m = 0$. Hence $(v_1,\dots,v_n)$ is liney indep. □

• (4E 11) *Supp V is finide, $T \in \mathcal{L}(V)$ is suth $\forall S \in \mathcal{L}(V), ST = TS$. Prove $\exists\, \lambda \in \mathbf{F}, T = \lambda I$.*

Solus: Asum $V \neq \{0\}$ and $\forall v \in V, (v, Tv)$ is linely dep, then $\exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.
To prove λ_v is indep of v, we discuss in two cases:
$$\left.\begin{array}{l} (-)\ \text{If } (v,w) \text{ is liney indep, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \\ (=)\ \text{Othws, supp } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w \end{array}\right\} \Rightarrow \lambda_w = \lambda_v.$$
Now we prove the asum. Asum $\exists\, v \in V, (v, Tv)$ is liney indep. Let $B_V = (v, Tv, u_1,\dots,u_n)$.
Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. □

Or. Let $B_V = (v_1,\dots,v_m)$. Define $\varphi \in \mathcal{L}(V,\mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$.
Supp $v \in V$. Define $S_v \in \mathcal{L}(V)$ by $S_v(u) = \varphi(u)v$.
Then $Tv = T(\varphi(v_1)v) = T(S_vv_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. □

Or. Define $S_k\left(\sum_{i=1}^n a_iv_i\right) = a_kv_k$. Then $S_kv = v \iff \exists\,!\, a_k \in \mathbf{F}, v = a_kv_k$.
Hence $S_k(Tv_k) = T(S_kv_k) = Tv_k \Rightarrow Tv_k = a_kv_k$.
Define $A^{(j,k)} \in \mathcal{L}(V)$ by $A^{(j,k)}v_j = v_k$, $A^{(j,k)}v_k = v_j$, $A^{(j,k)}v_x = 0, x \neq j,k$.
$$\text{Then}\ \left|\begin{array}{r} A^{(j,k)}Tv_j = TA^{(j,k)}v_j = Tv_k = a_kv_k \\ A^{(j,k)}Tv_j = A^{(j,k)}a_jv_j = a_jA^{(j,k)}v_j = a_jv_k \end{array}\right\} \Rightarrow a_k = a_j.\ \text{Hence } a_k \text{ is indep of } v_k.$$ □

• (4E 17) *Supp V is finide. Show all two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}$, $\forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

Solus: If $\mathcal{E} = \{0\}$, then done. Supp $0 \neq S \in \mathcal{E}$, a two-sided ideal of $\mathcal{L}(V)$. Let $B_V = (v_1,\dots,v_n)$.
Define $R_{x,y} \in \mathcal{L}(V) : v_x \mapsto v_y$, $v_z \mapsto 0\ (z \neq x)$. Or. $R_{x,y}v_z = \delta_{z,x}v_y$. Asum each $R_{x,y} \in \mathcal{E}$.
Then $(R_{1,1} + \cdots + R_{n,n})v_j = v_j \Rightarrow \sum_{r=1}^n R_{r,r} = I \Rightarrow \mathcal{L}(V) \ni T = I \circ T = T \circ I \in \mathcal{E}$.
Or. Let each $Tv_j = w_j = A_{1,j}v_1 + \cdots + A_{n,j}v_n \Rightarrow T = \sum_{x=1}^n \sum_{y=1}^n A_{y,x}R_{x,y} \in \mathcal{E}$. Now we prove the asum.
Supp $Sv_i \neq 0$ and $Sv_i = a_1v_1 + \cdots + a_nv_n$, where $a_k \neq 0$. We show $R_{k,y}SR_{x,i} = a_kR_{x,y} \in \mathcal{E}$.
Becs $SR_{x,i} = a_1R_{x,1} + \cdots + a_kR_{x,k} + \cdots + a_nR_{x,n} \in \mathcal{E}$, for all $x \in \{1,\dots,n\}$.
Or. $(R_{k,y}S)v_i = a_kv_y \Rightarrow \left((R_{k,y}S) \circ R_{x,i}\right)v_z = \delta_{z,x}(a_kv_y)$, for all $y \in \{1,\dots,n\}$. Immed. □

Comment: Not true if infinide. Consider the subsp $X = \{T \in \mathcal{L}(V) : \operatorname{range} T \text{ is finide}\}$.
For any $T \in X$, $\forall E \in \mathcal{L}(V), \operatorname{range} TE \subseteq \operatorname{range} T$; $\operatorname{range} ET = \operatorname{span}(Ew_1,\dots,Ew_n) \Rightarrow TE, ET \in X$.

• (4E 3.B.32) *Supp* $\dim V = n$. *Supp* $\varphi : \mathcal{L}(V) \to \mathbf{F}$ *is liney.*
Show if $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$, *then* $\varphi = 0$.

Solus: Using notas in (4E 17) and Note For [3.60].

Supp $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \dots, n\}$, $\varphi(R_{i,j}) \neq 0$. Becs $R_{i,j} = R_{x,j} \circ R_{i,x}$, $\forall x = 1, \dots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, becs $R_{i,x} = R_{y,x} \circ R_{i,y}$, $\forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0$, $\forall x, y = 1, \dots, n$.

Let $k \neq i, j \neq l$ and then $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Ctradic. □

Or. Becs $\exists\, S, T \in \mathcal{L}(V), ST - TS \neq 0$. While $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0$.

Note that $\forall E \in \operatorname{null}\varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \operatorname{null}\varphi$. By (4E 17). □

• *Given the fact that* $\mathcal{L}(V, W)$ *is a vecsp. Prove or give a countexa*: V, W *are vecsps.*

By [3.2], the add and homo imply that V *is closd add and scalar multi. While* W^V *might not be a vecsp.*

Solus: We can assure that $\{0\} \subseteq \mathcal{L}(V, W), \{0\} \subseteq V, \{0\} \subseteq W$.

(I) If $W^V = \{0\}$. Then $\mathcal{L}(V, W) = \{0\}$.

And $W = \{0\}$, for if not, $\exists\, w \in W \backslash \{0\}$, define a map f by $f(x) = w, \forall x \in V$.

And V might not be a vecsp. **Exa:** Let $V = \mathbf{R}$, but with the scalar multi defined by $a \odot v = 0$.

(II) If W^V is a non0 vecsp $\Longleftrightarrow W$ is a non0 vecsp.

(a) If $\mathcal{L}(V, W) = \{0\}$, then by Exa (I), V might not be vecsp.

(b) If not, then $\exists\, T \in \mathcal{L}(V, W)$, $T \neq 0$. Which means $\exists\, v \in V, Tv \neq 0 \Rightarrow v \neq 0$. TODO

Then both W and V have a non0 elem.

(i) If $\exists$ inje $T \in \mathcal{L}(V, W)$, then $T(u + v) = T(v + u) \Rightarrow u + v = v + u$. etc. Hence V is a vecsp.

(ii) If not, then we cannot guarantee that V is a vecsp. Exa: ???

(III) If W^V is not a vecsp $\Longleftrightarrow W$ is not a vecsp.

(a) If $\mathcal{L}(V, W) = \{0\}$, then by Exa (I), V might not be vecsp.

(b) If not. □

Ended

3·B **注意：**这里我将 3.D 可逆性、同构部分前置。

10 *Supp* $\operatorname{span}(v_1, \dots, v_n) = V$. *Show* $\operatorname{span}(Tv_1, \dots, Tv_n) = \operatorname{range} T$.

Solus: (a) $\operatorname{range} T = \{Tv : v \in \operatorname{span}(v_1, \dots, v_n)\} \Rightarrow Tv_1, \dots, Tv_n \in \operatorname{range} T$. By [2.7].

Or. $\operatorname{span}(Tv_1, \dots, Tv_n) \ni a_1 Tv_1 + \dots + a_n Tv_n = T(a_1 v_1 + \dots + a_n v_n) \in \operatorname{range} T$.

(b) $\forall w \in \operatorname{range} T, w = Tv, \exists\, v \in V \Rightarrow \exists\, a_i \in \mathbf{F}, v = \sum_{i=1}^{n} a_i v_i, w = a_1 Tv_1 + \dots + a_n Tv_n$. □

• (4E 5.A.33) *Supp* $T \in \mathcal{L}(V), m \in \mathbf{N}^+$. *Prove* T *inje* $\Longleftrightarrow T^m$ *inje, and* T *surj* $\Longleftrightarrow T^m$ *surj.*

Solus: (a) T^m inje $\Rightarrow$ if $Tv = 0$, then $T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$, thus T inje. Convly immed.

(b) T^m surj $\Rightarrow \forall u \in V, \exists\, v \in V \Rightarrow \exists\, w = T^{m-1}v$, $T^m v = u = Tw$.

T surj $\Rightarrow \forall u \in V, \exists\, v_1, \dots, v_m \in V$, $T(v_1) = T^2 v_2 = \dots = T^m v_m = u$. □

• **Tips 1:** *Supp U is a subsp of V. Then $\forall T \in \mathcal{L}(V,W), U \cap \operatorname{null} T = \operatorname{null} T|_U$.*

• **Tips 2:** *Supp $T \in \mathcal{L}(V,W)$ and $T|_U$ is inje. Let $V = M + N,\ U = X + Y$.*
Then $\operatorname{range} T = \operatorname{range} T|_M + \operatorname{range} T|_N = \operatorname{range} T|_X + \operatorname{range} T|_Y$.
(a) *Show* $U = X \oplus Y \iff \operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_Y$.
(b) *Give an exa suth* $V = M \oplus N,\ \operatorname{range} T \neq \operatorname{range} T|_M \oplus \operatorname{range} T|_N$.

Solus: Supp $U = X \oplus Y$. Asum for some $v \in V$, there exis two disti pairs $(x_1, y_1), (x_2, y_2)$ in $X \times Y$ suth $Tv = Tx_1 + Ty_1 = Tx_2 + Ty_2$. Becs $\forall v \in X \oplus Y, \exists!\,(x, y) \in X \times Y, v = x + y$.
Now $T(x_1 + y_1) = T(x_2 + y_2) \Longrightarrow x_1 + y_1 = x_2 + y_2 \Longrightarrow x_1 = x_2, y_1 = y_2$. Ctradic.
Thus $\forall Tv \in \operatorname{range} T, \exists!\, Tx \in \operatorname{range} T|_X, Ty \in \operatorname{range} T|_Y, Tv = Tx + Ty$. Convly, becs T is inje.$\square$

Exa: Let $B_V = (v_1, v_2, v_3), B_W = (w_1, w_2),\ T : v_1 \mapsto 0,\ v_2 \mapsto w_1,\ v_3 \mapsto w_2$.
Let $B_M = (v_1 - v_2, v_3), B_N = (v_2)$. Then $\operatorname{range} T|_M = \operatorname{span}(w_1, w_2), \operatorname{range} T|_N = \operatorname{span}(w_1)$

Comment: Also $\operatorname{null} T|_M = \operatorname{null} T|_N = \{0\}$. Hence $\operatorname{null} T \neq \operatorname{null} T|_M \oplus \operatorname{null} T|_N$.

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12 *Prove $\forall T \in \mathcal{L}(V,W), \exists$ subsp U of V suth*
$U \cap \operatorname{null} T = \operatorname{null} T|_U = \{0\},\ \operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$.
Which is equiv to $T|_U : U \to \operatorname{range} T$ being iso.

Solus: By [2.34] (note that V can be infinide), $\exists$ subsp U of V suth $V = U \oplus \operatorname{null} T$.
$\forall v \in V, \exists!\, w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w + u) = Tu \in \{Tu : u \in U\}$. $\square$

Coro: $[P]\quad T|_U : U \to \operatorname{range} T$ *is iso* $\iff U \oplus \operatorname{null} T = V.\quad [Q]$
We have shown $Q \Rightarrow P$. Now we show $P \Rightarrow Q$ to complete the proof.
$\forall v \in V, Tv \in \operatorname{range} T = \operatorname{range} T|_U \Rightarrow \exists!\, u \in U, Tv = Tu \Rightarrow v - u \in \operatorname{null} T$.
Thus $v = (v - u) + u \in U + \operatorname{null} T$. 又 $u \in U \cap \operatorname{null} T \iff T|_U(u) = 0 \iff u = 0$. $\square$

Or. $\neg Q \Rightarrow \neg P$: Becs $U \oplus \operatorname{null} T \subsetneq V$. We show $\operatorname{range} T \neq \operatorname{range} T|_U$ by ctradic.
Let $X \oplus (U \oplus \operatorname{null} T) = V$. Now $\operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_U$. And X is non0.
Asum $\operatorname{range} T = \operatorname{range} T|_U$. Then $\operatorname{range} T|_X = \{0\}$. While $T|_X$ is inje. Ctradic.
Or. $\operatorname{range} T|_X \subseteq \operatorname{range} T|_U \Rightarrow \forall x \in X, Tx \in \operatorname{range} T|_U, \exists\, u \in U, Tu = Tx \Rightarrow x = 0$.

Also, $\neg P \Rightarrow \neg Q$: (a) $\operatorname{range} T|_U \subsetneq \operatorname{range} T$; Or (b) $U \cap \operatorname{null} T \neq \{0\}$.
For (a), $\exists\, x \in V \backslash U, Tx \neq 0 \iff x \notin \operatorname{null} T$. Thus $U + \operatorname{null} T \subsetneq V$. For (b), immed. $\square$

Comment: If $T|_U : U \to \operatorname{range} T$ is iso. Let $R \oplus U = V$. Then R might not be $\operatorname{null} T$.
Or. Extend B_U to $B_V = (u_1, \dots, u_n, r_1, \dots, r_m)$, then $(r_1, \dots, r_m)$ might not be a $B_{\operatorname{null} T}$.

• **Tips 3:** Supp $T \in \mathcal{L}(V,W)$ and U is a subsp suth $V = U \oplus \operatorname{null} T$. Let $\operatorname{null} T = X \oplus Y$.
Now $\forall v \in V, \exists!\, u_v \in U, (x_v, y_v) \in X \times Y, v = u_v + x_v + y_v$. Define $i \in \mathcal{L}(V, U \oplus X)$ by $i(v) = u_v + x_v$.
Then $T = T \circ i$. Becs $\forall v \in V, T(v) = T(u_v + x_v + y_v) = T(u_v) = T(u_v + x_v) = T(i(v)) = (T \circ i)(v)$.

• **Tips 4:** Supp $T \in \mathcal{L}(V,W), T \neq 0$. Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
By (3.A.4), $R = (v_1, \dots, v_n)$ is liney indep in V. Let $\operatorname{span} R = U$. We will prove $U \oplus \operatorname{null} T = V$.
(a) $T\left(\sum_{i=1}^n a_i v_i\right) = 0 \iff \sum_{i=1}^n a_i T v_i = 0 \iff a_1 = \cdots = a_n = 0$. Thus $U \cap \operatorname{null} T = \{0\}$.
(b) $Tv = \sum_{i=1}^n a_i T v_i \iff v - \sum_{i=1}^n a_i v_i \in \operatorname{null} T \iff v = \left(v - \sum_{i=1}^n a_i v_i\right) + \left(\sum_{i=1}^n a_i v_i\right)$.
Thus $U + \operatorname{null} T = V$. Or. $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$. Using Exe (12). $\square$

Coro: Convly, if $U \oplus \operatorname{null} T = V$ and $B_U = (v_1, \dots, v_n)$, then $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
Becs $\operatorname{range} T = \operatorname{range} T|_U = \operatorname{span}(Tv_1, \dots, Tv_n)$, 又 T is inje.

• **TIPS 5:** Supp $S \in \mathcal{L}(U,V)$ is surj. Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V,W), \mathcal{L}(U,W))$ by $\mathcal{B}(T) = TS$.
Then $\mathcal{B}$ is inje. Becs $\mathcal{B}(T) = TS = 0 \iff T|_{\operatorname{range} S} = 0$. OR. $\operatorname{range} TS = \operatorname{range} T = \{0\}$.

• (4E 3.D.15) *Supp* $T \in \mathcal{L}(V)$ *and* $V = \operatorname{span}(Tv_1, \dots, Tv_m)$. *Prove* $V = \operatorname{span}(v_1, \dots, v_m)$.

SOLUS: Becs $V = \operatorname{span}(Tv_1, \dots, Tv_m) \Rightarrow T$ surj $\Rightarrow T, T^{-1}$ inv.
$\forall v \in V, \exists a_i \in \mathbf{F}, v = \sum_{i=1}^{m} a_i T v_i \Rightarrow T^{-1}v = \sum_{i=1}^{m} a_i v_i \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \dots, v_m)$.
OR. Reduce to a bss $(Tv_{\alpha_1}, \dots, Tv_{\alpha_k})$, where $k = \dim V$, each $\alpha_i \in \{1, \dots, m\}$. By (4E 3.D.3). ☐

• (4E 27) *Supp* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove* $V = \operatorname{null} P \oplus \operatorname{range} P$.

SOLUS: (a) If $v \in \operatorname{null} P \cap \operatorname{range} P \Rightarrow Pv = 0$, and $\exists\, u \in V, v = Pu$. Then $v = Pu = P^2 u = Pv = 0$.
(b) Note that $\forall v \in V, v = Pv + (v - Pv)$ and $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$.
OR. Becs $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P = \dim(\operatorname{null} P \oplus \operatorname{range} P)$. ☐

OR. Becs $P|_{\operatorname{range} P} : Pv \mapsto Pv^2 = Pv \Rightarrow P|_{\operatorname{range} P} = I$ is iso. By CORO in Exe (12). ☐

• (4E 21) *Supp* V *is finide,* $T \in \mathcal{L}(V,W)$, Y *is a subsp of* W. *Let* $\mathcal{K}_Y = \{v \in V : Tv \in Y\}$.
Then $\mathcal{K}_Y$ *is a subsp. Prove* $\mathcal{K}_Y = \dim \operatorname{null} T + \dim(Y \cap \operatorname{range} T)$.

SOLUS: Define the range-restr map R of T by $R = T|_{\mathcal{K}_Y} \in \mathcal{L}(\mathcal{K}_Y, Y)$. Now $\operatorname{range} R = Y \cap \operatorname{range} T$.
And $v \in \operatorname{null} T \iff Tv = 0 \in Y \iff Rv = 0 \in \operatorname{range} T \iff v \in \operatorname{null} R$. By [3.22]. ☐

COMMENT: Now $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = \mathcal{K}_Y$. Where $B_{Y \cap \operatorname{range} T} = (Tv_1, \dots, Tv_m)$.
In particular, $\dim \mathcal{K}_{\operatorname{range} T} = \dim \operatorname{null} T + \dim \operatorname{range} T \Longrightarrow \mathcal{K}_{\operatorname{range} T} = V$.

• (4E 31) *Supp* V *is finide,* X *is a subsp of* V, *and* Y *is a finide subsp of* W.
Prove if $\dim X + \dim Y = \dim V$, *then* $\exists\, T \in \mathcal{L}(V,W), \operatorname{null} T = X, \operatorname{range} T = Y$.

SOLUS: Let $V = U \oplus X, B_U = (v_1, \dots, v_m)$. Then $\forall v \in V, \exists\,! a_i \in \mathbf{F}, x \in X,\ v = \sum_{i=1}^{m} a_i v_i + x$.
Let $B_Y = (w_1, \dots, w_m)$. Define $T \in \mathcal{L}(V,W)$ with each $Tv_i = w_i$, $Tx = 0$.
Now $v \in \operatorname{null} T \iff Tv = a_1 w_1 + \dots + a_m w_m = 0 \iff v = x \in X$. Hence $\operatorname{null} T = X$.
And $Y \ni w = a_1 w_1 + \dots + a_m w_m = a_1 T v_1 + \dots + a_m T v_m \in \operatorname{range} T$. Hence $\operatorname{range} T = Y$.
OR. NOTICE that $V = U \oplus \operatorname{null} T$. By Exe (12), $\operatorname{range} T = \operatorname{range} T|_U$.
又 $\dim \operatorname{range} T|_U = \dim U = \dim Y$; $\operatorname{range} T \subseteq Y$.
OR. Let $B_X = (x_1, \dots, x_n)$. Now $\operatorname{range} T = \operatorname{span}(Tv_1, \dots, Tv_m, Tx_1, \dots, Tx_n) = \operatorname{span}(w_1, \dots, w_m) = Y$. ☐

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22 *Supp* U, V *are finide,* $S \in \mathcal{L}(V,W), T \in \mathcal{L}(U,V)$.
Prove $\dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$.

SOLUS: We show $\dim \operatorname{null} ST = \dim \operatorname{null} S|_{\operatorname{range} T} + \dim \operatorname{null} T$.
Becs (a) $\underline{\operatorname{range} T|_{\operatorname{null} ST} = \operatorname{range} T \cap \operatorname{null} S = \operatorname{null} S|_{\operatorname{range} T}}$,
(b) $\underline{\operatorname{null} T|_{\operatorname{null} ST} = \operatorname{null} T \cap \operatorname{null} ST = \operatorname{null} T}$. By [3.22] ☐

OR. NOTICE that $\underline{u \in \operatorname{null} ST \iff S(Tu) = 0 \iff Tu \in \operatorname{null} S}$.
Thus $\underline{\{u \in U : Tu \in \operatorname{null} S\} = \mathcal{K}_{\operatorname{null} S \cap \operatorname{range} T} = \operatorname{null} ST}$.
By Exe (4E 21), $\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T)$. ☐

CORO: (1) T surj $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.
(2) T inv $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S$, $\operatorname{null} ST = \operatorname{null} T$.
(3) S inje $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} T$.

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23 *Supp V is finide, $S \in \mathcal{L}(V,W), T \in \mathcal{L}(U,V)$.*
Prove $\dim \operatorname{range} ST \leqslant \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.

COMMENT: If $\dim V = \dim U$. Then $\dim \operatorname{null} ST \geqslant \max\{\dim \operatorname{null} S, \dim \operatorname{null} T\}$.

SOLUS: NOTICE that $\underline{\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{range} S|_{\operatorname{range} T}}$.
Let $\operatorname{range} ST = \operatorname{span}(Su_1, \dots, Su_{\dim \operatorname{range} T})$, where $B_{\operatorname{range} T} = (u_1, \dots, u_{\dim \operatorname{range} T})$.
$\dim \operatorname{range} ST \leqslant \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leqslant \dim \operatorname{range} S$. □

OR. $\underline{\dim \operatorname{range} ST = \dim \operatorname{range} S|_{\operatorname{range} T} = \dim \operatorname{range} T - \dim \operatorname{null} S|_{\operatorname{range} T}} \leqslant \operatorname{range} T$. □

COMMENT: $\dim \operatorname{range} ST = \dim U - \dim \operatorname{null} ST = \dim \operatorname{range} T|_U - \dim \operatorname{range} T|_{\operatorname{null} ST}$.

CORO: (1) $S|_{\operatorname{range} T}$ inje $\iff \dim \operatorname{range} ST = \dim \operatorname{range} T$.
(2) Let $X \oplus \operatorname{null} S = V$. Then $X \subseteq \operatorname{range} T \iff \operatorname{range} ST = \operatorname{range} S$.
And T is surj $\Rightarrow \operatorname{range} ST = \operatorname{range} S$.

• (a) *Supp* $\dim V = n$, *$ST = 0$ where $S, T \in \mathcal{L}(V)$. Prove* $\dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$.
(b) *Give an exa of such S, T with $n = 5$ and* $\dim \operatorname{range} TS = 2$.

SOLUS: Note that $\dim \operatorname{range} TS \leqslant \min\{\dim \operatorname{range} T, \dim \operatorname{range} S\}$. We prove by ctradic.
Asum $\dim \operatorname{range} TS \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$. Then $\min\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\} \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$
又 $\dim \operatorname{null} ST = n \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \mid \Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant \left\lceil \frac{n}{2} \right\rceil - 1$.
Thus $n \leqslant 2\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \Rightarrow \frac{n}{2} \leqslant \left\lceil \frac{n}{2} \right\rceil - 1$. Ctradic. □

OR. $\dim \operatorname{null} S = n - \dim \operatorname{range} S \leqslant n - \dim \operatorname{range} TS$. 又 $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S$.
$\dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S \leqslant n - \dim \operatorname{range} TS$. Thus $2 \dim \operatorname{range} TS \leqslant n$. □

OR. Becs $\dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$, and $\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n$.
We show $\dim \operatorname{null} TS \geqslant \left\lceil \frac{n}{2} \right\rceil$. Note that $\dim \operatorname{null} S + \dim \operatorname{null} T \geqslant n$.
$\dim \operatorname{null} S + \dim \operatorname{null} T|_{\operatorname{range} S} = \dim \operatorname{null} TS$. If $\dim \operatorname{null} S \geqslant \left\lceil \frac{n}{2} \right\rceil$. Then done.
Othws, $\dim \operatorname{null} S \leqslant \left\lceil \frac{n}{2} \right\rceil - 1 \Rightarrow \dim \operatorname{null} T \geqslant n - \dim \operatorname{null} S \geqslant n - \left\lceil \frac{n}{2} \right\rceil + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1 \geqslant \left\lceil \frac{n}{2} \right\rceil$.
Thus $\dim \operatorname{null} TS \geqslant \max\{\dim \operatorname{null} S, \dim \operatorname{null} T\} = \left\lceil \frac{n}{2} \right\rceil$. □

EXA: Define $T:\ v_1 \mapsto 0,\ \ v_2 \mapsto 0,\ \ v_i \mapsto v_i\ ;\ \ S:\ v_1 \mapsto v_4,\ v_2 \mapsto v_5,\ v_i \mapsto 0\ ;\ \ i = 3,4,5.$

20, 21 (a) *Prove if $ST = I \in \mathcal{L}(V)$, then T is inje and S is surj.*
(b) *Supp $T \in \mathcal{L}(V,W)$. Prove if T is inje, then $\exists$ surj $S \in \mathcal{L}(W,V)$, $ST = I$.*
(c) *Supp $S \in \mathcal{L}(W,V)$. Prove if S is surj, then $\exists$ inje $T \in \mathcal{L}(V,W)$, $ST = I$.*

SOLUS:

(a) $Tv = 0 \Rightarrow S(Tv) = 0 = v$. OR. $\operatorname{null} T \subseteq \operatorname{null} ST = \{0\}$.
$\forall v \in V, ST(v) = v \in \operatorname{range} S$. OR. $V = \operatorname{range} ST \subseteq \operatorname{range} S$.

(b) Define $S \in \mathcal{L}(\operatorname{range} T, V)$ by $Sw = T^{-1}w$, where T^{-1} is the inv of $T \in \mathcal{L}(V, \operatorname{range} T)$.
Then extend to $S \in \mathcal{L}(W,V)$ by (3.A.11). Now $\forall v \in V, STv = T^{-1}Tv = v$.

OR. [*Req V Finide*] Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n) \Rightarrow B_V = (v_1, \dots, v_n)$. Let $U \oplus \operatorname{range} T = W$.
Define $S \in \mathcal{L}(W,V)$ with each $S(Tv_i) = v_i$, $Su = 0$ for $u \in U$. Thus $ST = I$.

(c) By Exe (12), $\exists$ subsp U of $W, W = U \oplus \operatorname{null} S$, $\operatorname{range} S = \operatorname{range} S|_U = V$.
Note that $S|_U : U \to V$ is iso. Define $T = (S|_U)^{-1}$, where $(S|_U)^{-1} : V \to U$.
Then $ST = S \circ (S|_U)^{-1} = S|_U \circ (S|_U)^{-1} = I_V$.

OR. [*Req V Finide*] Let $B_{\operatorname{range} S} = B_V = (Sw_1, \dots, Sw_n) \Rightarrow \operatorname{span}(w_1, \dots, w_n) \oplus \operatorname{null} S = W$.
Define $T \in \mathcal{L}(V,W)$ by $T(Sw_i) = w_i$. Now $ST(a_1Sw_1 + \cdots + a_nSw_n) = (a_1Sw_1 + \cdots + a_nSw_n)$. □

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24 *Supp* $S \in \mathcal{L}(V,M), T \in \mathcal{L}(V,W)$, *and* $\operatorname{null} S \subseteq \operatorname{null} T$. *Prove* $\exists\, E \in \mathcal{L}(M,W), T = ES$.

SOLUS:

Let $\underline{V = U \oplus \operatorname{null} S}$

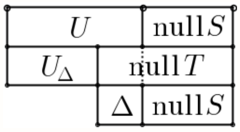
$\underline{\Rightarrow S|_U : U \to \operatorname{range} S \text{ is iso.}}$

Extend $T\big(S|_U\big)^{-1}$ to $E \in \mathcal{L}(M,W)$.

$$\begin{tikzcd} \operatorname{range} T & U \arrow[l, "surj\ T"'] \arrow[d, "inv"', "S"] \\ & \operatorname{range} S \arrow[ul, "surj\ E"] \end{tikzcd}$$

OR. Define $E : \operatorname{range} S \to W$ by $\underline{\underline{E : Sv \mapsto Tv.}}$
Extend $E \in \mathcal{L}\big(\operatorname{range} S, W\big)$ to $E \in \mathcal{L}(M,W)$. $\square$

COMMENT: Let $\Delta \oplus \operatorname{null} S = \operatorname{null} T$, $U_\Delta \oplus (\Delta \oplus \operatorname{null} S) = V = U_\Delta \oplus \operatorname{null} T$. Redefine $U = U_\Delta \oplus \Delta$.



$$\operatorname{range} S \xleftarrow{S} \begin{array}{l} U_\Delta \xrightarrow{T} \operatorname{range} T \\ \oplus \\ \Delta \xrightarrow{T} \{0\} \end{array}$$

Becs $\Delta = \operatorname{null} T|_U = \operatorname{null} T \cap \operatorname{range}\big(S|_U\big)^{-1}$.
Thus $E = T\big(S|_U\big)^{-1}$ is not inje $\iff \Delta \neq \{0\}$.
In other words, $\operatorname{range} S|_\Delta = \operatorname{null} E$,
while $E|_{\dots} : \operatorname{range} S|_{U_\Delta} \to \operatorname{range} T$ is iso.

COMMENT: Let $E_1 \in \mathcal{L}(U_\Delta \oplus \operatorname{null} T, U_\Delta)$, and E_2 be an iso of $\operatorname{range} S|_{U_\Delta}$ onto $\operatorname{range} T$.
Define $E_1|_{U_\Delta} = I|_{U_\Delta}$, and $E_2 = T\big(S|_{U_\Delta}\big)^{-1}$. Then $T = E_2 S E_1$.

CORO: If $\operatorname{null} S = \operatorname{null} T$. Then $\Delta = \{0\}, U_\Delta = U$. $\big[$ *Req W Finide* $\big]$ By (3.D.3),
we can extend inje $T\big(S|_U\big)^{-1} \in \mathcal{L}\big(\operatorname{range} S, W\big)$ to inv $E \in \mathcal{L}(M,W)$.

OR. $\big[$ *Req* range S *Finide* $\big]$ Let $B_{\operatorname{range} S} = (Sv_1, \dots, Sv_n)$. Then $\underline{V = \operatorname{span}(v_1, \dots, v_n) \oplus \operatorname{null} S.}$
Define $E \in \mathcal{L}(\operatorname{range} S, W)$ by $E(Sv_i) = Tv_i$. Extend to $E \in \mathcal{L}(M,W)$.
Hence $\forall v = \sum_{i=1}^n a_i v_i + u \in V$, $\underline{(\exists!\, u \in \operatorname{null} S \subseteq \operatorname{null} T\,),\ Tv = \sum_{i=1}^n a_i T v_i + 0} = E\Big(\sum_{i=1}^n a_i S v_i + 0\Big)$. $\square$

CORO: $\big[$ *Req W Finide* $\big]$ Supp $\operatorname{null} S = \operatorname{null} T$. We show $\exists$ inv $E \in \mathcal{L}(M,W), T = ES$.
Redefine $E \in \mathcal{L}(M,W)$ by $E(Tv_i) = Sv_i$, $E(w_j) = x_j$, for each Tv_i and w_j. Where:
Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m), B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n), B_U = (v_1, \dots, v_m)$.
Now $V = U \oplus \operatorname{null} T = U \oplus \operatorname{null} S \Rightarrow B_{\operatorname{range} S} = \big(Sv_1, \dots, Sv_m\big)$. Let $B_M = \big(Sv_1, \dots, Sv_m, x_1, \dots, x_n\big)$. $\square$

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25 *Supp* $S \in \mathcal{L}(Y,W), T \in \mathcal{L}(V,W)$, *and* $\operatorname{range} T \subseteq \operatorname{range} S$. *Prove* $\exists\, E \in \mathcal{L}(V,Y), T = SE$.

SOLUS: Let $Y = U \oplus \operatorname{null} S$
$\Rightarrow S|_U : U \to \operatorname{range} S$ is iso. Becs $(S|_U)^{-1} : \overset{\operatorname{range} T \subseteq}{\operatorname{range} S} \to U$.
$\underline{\text{Define } E = \big(S|_U\big)^{-1} T = \big(S|_U\big)^{-1}\big|_{\operatorname{range} T} T \in \mathcal{L}(V,U) \subseteq \mathcal{L}(V,Y).}$ $\square$

$$\begin{array}{ccccc} U_1 & \xrightarrow[S]{inv} & \operatorname{range} S & & \\ || & & || & & \\ \Delta & \xrightarrow[S]{inv} & \operatorname{range} S|_\Delta & & \\ \oplus & & \oplus & & \\ U_{1\Delta} & \xrightarrow[S]{inv} & \operatorname{range} T & \xleftarrow[T]{inv} & U_2 \\ \uparrow & & \underset{inv\ \ E|_{U_2}}{\rule{4em}{0.4pt}} & & \rfloor \end{array}$$

COMMENT: Let $U_1 = U$. Let $U_2 \oplus \operatorname{null} T = V$.
Let $U_{1\Delta} = \operatorname{range}\big(S|_{U_1}\big)^{-1}\big|_{\operatorname{range} T} \subseteq U_1 = \Delta \oplus U_{1\Delta}$.
OR. Let $U_{1\Delta} = \operatorname{range} E|_{U_2}$. Let $\Delta \oplus \operatorname{range} E|_{U_2} = U_1$.

$\big[$ *Req* range T *Finide* $\big]$ Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$. Now $B_{U_2} = (v_1, \dots, v_n)$.
$\underline{\text{Let } S(u_i) = Tv_i \text{ for each } Tv_i.}$ Define E with each $Ev_i = u_i, Ex = 0$ for $x \in \operatorname{null} T$. $\square$

COMMENT: $\big[$ *Req V Finide* $\big]$ Note that $\dim U_2 \leqslant \dim U_1 \Longrightarrow \dim \operatorname{null} T = p \geqslant q = \dim \operatorname{null} S$.
Let $B_{\operatorname{null} T} = (x_1, \dots, x_p), B_{\operatorname{null} S} = (y_1, \dots, y_q)$. Redefine $E : v_i \mapsto u_i$, $x_k \mapsto y_k$, $x_j \mapsto 0$,
for each $i \in \{1, \dots, \dim U_2\}, k \in \{1, \dots, \dim \operatorname{null} S\} = K, j \in \{1, \dots, \dim \operatorname{null} T\} \setminus K$.
Note that $(u_1, \dots, u_n)$ is liney indep. Let $X = \operatorname{span}(x_1, \dots, x_q) \oplus \operatorname{span}(v_1, \dots, v_n)$.
Now $E|_X$ is inje, but cannot be re-extend to inv $E \in \mathcal{L}(V,Y)$ suth $T = SE$.

CORO: $\big[$ *Req V Finide* $\big]$ If $\operatorname{range} T = \operatorname{range} S$, then $\dim \operatorname{null} T = \dim \operatorname{null} S = p$.
Redefine E by $Ev_i = u_i$, $Ex_j = y_j$ for each v_i and x_j. Then $E \in \mathcal{L}(V,Y)$ is inv. $\square$

- **NOTE:** $\operatorname{null} T = \operatorname{null} S \Rightarrow E : Sv \mapsto Tv$ and $E^{-1} : Tv \mapsto Sv$ well-defined $\Rightarrow \operatorname{range} T, \operatorname{range} S$ iso.
While $\operatorname{range} T = \operatorname{range} S \not\Rightarrow \operatorname{null} T, \operatorname{null} S$ iso. **EXA:** Backwd shift optor and id optor on $\mathbf{F}^\infty$.

• (3.D.6) *Supp* V, W *are finide, and* $S, T \in \mathcal{L}(V, W)$, *and* $\dim \operatorname{null} S = \dim \operatorname{null} T = n$.
Prove $S = E_2 T E_1, \exists$ *inv* $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$.

Solus: Define $E_1 : \quad v_i \mapsto r_i \ ; \quad u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.
Define $E_2 : Tv_i \mapsto Sr_i \ ; \ x_j \mapsto y_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m)$; $B_{\operatorname{range} S} = (Sr_1, \dots, Sr_m)$. Let $B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p)$; $B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p)$. Let $B_{\operatorname{null} T} = (u_1, \dots, u_n)$; $B_{\operatorname{null} S} = (s_1, \dots, s_n)$. Thus $B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$; $B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)$.	$\therefore E_1, E_2$ are inv and $S = E_2 T E_1$.

□

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28 *Supp* $T \in \mathcal{L}(V, W)$. *Let* $(Tv_1, \dots, Tv_m)$ *be a bss of* $\operatorname{range} T$ *and each* $w_i = Tv_i$.
(a) *Prove* $\exists \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ *suth* $\forall v \in V, Tv = \varphi_1(v) w_1 + \cdots + \varphi_m(v) w_m$.
(b) [4E 3.F.5] $\forall v \in V, \exists! \varphi_i(v) \in \mathbf{F}, Tv = \varphi_1(v) w_1 + \cdots + \varphi_m(v) w_m$.
Thus defining each $\varphi_i : V \to \mathbf{F}$. *Show each* $\varphi_i \in \mathcal{L}(V, \mathbf{F})$.

Solus: *The answer for* (b) *with* (b) *itself is the answer for* (a).

(b) $\sum_{i=1}^m \varphi_i(u + \lambda v) w_i = T(u + \lambda v) = Tu + \lambda Tv = \left(\sum_{i=1}^m \varphi_i(u) w_i\right) + \lambda \left(\sum_{i=1}^m \varphi_i(v) w_i\right)$. □

Or. $\forall v \in V, \exists! a_i \in \mathbf{F}, Tv = a_1 Tv_1 + \cdots + a_m Tv_m$. Let $B_{(\operatorname{range} T)'} = (\psi_1, \dots, \psi_m)$.
Then $[T'(\psi_i)](v) = (\psi_i \circ T)(v) = a_i$. Thus each $\varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$. □

(a) $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = V \Rightarrow \forall v \in V, \exists! a_i \in \mathbf{F}, u \in \operatorname{null} T, \ v = \sum_{i=1}^m a_i v_i + u$.
Define $\varphi_i \in \mathcal{L}(V, \mathbf{F})$ by $\varphi_i(v_j) = \delta_{i,j}, \ \varphi_i(u) = 0$ for all $u \in \operatorname{null} T$.
Linity: $\forall v, w \in V \ [\exists! a_i, b_i \in \mathbf{F}], \lambda \in \mathbf{F}, \varphi_i(v + \lambda w) = a_i + \lambda b_i = \varphi(v) + \lambda \varphi(w)$. □

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30 *Supp* $\varphi, \beta \in \mathcal{L}(V, \mathbf{F})$ *and* $\operatorname{null} \varphi = \operatorname{null} \beta = \eta$. *Prove* $\exists c \in \mathbf{F}, \varphi = c\beta$.

Solus: If $\eta = V$, then $\varphi = \beta = 0$, done. Now by Exe (29),
$\varphi(u) \neq 0 \iff V = \operatorname{null} \varphi \oplus \operatorname{span}(u) \iff V = \operatorname{null} \beta \oplus \operatorname{span}(u) \iff \beta(u) \neq 0$.

Note that $\forall v \in V, \exists! u_0 \in \eta, \ a_v \in \mathbf{F}, v = u_0 + a_v u$ $\Rightarrow \varphi(u_0 + a_v u) = a_v \varphi(u), \ \beta(u_0 + a_v u) = a_v \beta(u)$.	Let $c = \dfrac{\varphi(u)}{\beta(u)} \in \mathbf{F} \backslash \{0\}$.

□

• (4E 3.F.6) *Supp* $\varphi, \beta \in \mathcal{L}(V, \mathbf{F})$. *Prove* $\operatorname{null} \beta \subseteq \operatorname{null} \varphi \iff \varphi = c\beta, \exists c \in \mathbf{F}$.
Coro: $\operatorname{null} \varphi = \operatorname{null} \beta \iff \varphi = c\beta, \ \exists c \in \mathbf{F} \backslash \{0\}$.

Solus: Using Exe (29) and (30).

(a) If $\varphi = 0$, then done. Othws, supp $u \notin \operatorname{null} \varphi \supseteq \operatorname{null} \beta$.
Now $V = \operatorname{null} \varphi \oplus \operatorname{span}(u) = \operatorname{null} \beta \oplus \operatorname{span}(u)$. By [1.C Tips (2)], $\operatorname{null} \varphi = \operatorname{null} \beta$. Let $c = \dfrac{\varphi(u)}{\beta(u)}$.
Or. We discuss in two cases. If $\operatorname{null} \beta = \operatorname{null} \varphi$, or if $\varphi = 0$, then done. Othws,
$\exists u' \in \operatorname{null} \varphi \backslash \operatorname{null} \beta, \exists u \notin \operatorname{null} \varphi \supsetneq \operatorname{null} \beta \Rightarrow V = \operatorname{null} \beta \oplus \operatorname{span}(u') = \operatorname{null} \beta \oplus \operatorname{span}(u)$.

$\forall v \in V, v = w + au = w' + bu', \exists! w, w' \in \operatorname{null} \beta$ Thus $\varphi(w + au) = a\varphi(u), \ \beta(w' + bu) = b\beta(u')$.	Let $c = \dfrac{a\varphi(u)}{b\beta(u')} \in \mathbf{F} \backslash \{0\}$. Done.

Notice that by (b) below, we have $\operatorname{null} \varphi \subseteq \operatorname{null} \beta$, ctradic the asum.

(b) If $c = 0$, then $\operatorname{null} \varphi = V \supseteq \operatorname{null} \beta$, done. Othws, becs $v \in \operatorname{null} \beta \iff v \in \operatorname{null} \varphi$. □

Or. By Exe (24), $\operatorname{null} \beta \subseteq \operatorname{null} \varphi \iff \exists E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta$. [If E is inv. Then $\operatorname{null} \beta = \operatorname{null} \varphi$.]
Now $\exists E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta \iff \exists c = E(1) \in \mathbf{F}, \varphi = c\beta$. [E is inv $\iff E(1) \neq 0 \iff c \neq 0$.] □

Ended

3·C

- **Note For [3.30, 32]:** *matrix of span*

  Supp $L_\alpha = (\alpha_1, \dots, \alpha_n)$ and $L_\beta = (\beta_1, \dots, \beta_m)$ are in a vecsp V.
  Let each $\alpha_k = A_{1,k}\beta_1 + \cdots + A_{m,k}\beta_m$, forming $A = \mathcal{M}(\operatorname{span} L_\beta \supseteq L_\alpha) \in \mathbf{F}^{m,n}$.

  Which is *the matrix of span*. Then $\begin{pmatrix}\beta_1 & \cdots & \beta_m\end{pmatrix}\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n}\end{pmatrix} = \begin{pmatrix}\alpha_1 & \cdots & \alpha_n\end{pmatrix}$.

  (a) Supp $m = n$. If $(A_{\cdot,1}, \dots, A_{\cdot,n})$ is a bss of $\mathbf{F}^{n,1}$. We show L_α liney indep $\iff L_\beta$ liney indep.
  ($\Leftarrow$) Immed. ($\Rightarrow$) Asum L_β is liney dep and $\beta_j = c_1\beta_1 + \cdots + c_{j-1}\beta_{j-1}$. By ctradic. $\square$

  (b) Supp $m \geqslant n$. If L_β liney indep. We show $(A_{\cdot,1}, \dots, A_{\cdot,n})$ liney indep $\iff L_\alpha$ liney indep.
  ($\Rightarrow$) Immed. ($\Leftarrow$) By ctradic. $\square$
  Note: $\mathcal{M}(\operatorname{span} L_\beta \supseteq L_\alpha) = \mathcal{M}(I, L_\alpha, L_\beta) \iff L_\alpha, L_\beta$ liney indep $\Rightarrow (A_{\cdot,1}, \dots, A_{\cdot,n})$ liney indep.
  Where I is the id optor retr to $\operatorname{span} L_\alpha \subseteq \operatorname{span} L_\beta$.

  (c) Supp $m < n$. Then $(A_{\cdot,1}, \dots, A_{\cdot,n})$ is liney dep, so is L_α.

  Supp $T \in \mathcal{L}(V, W)$ and $B_V = (v_1, \dots, v_m), B_W = (w_1, \dots, w_n)$.
  Then $\mathcal{M}(T, B_V, B_W) = \mathcal{M}\big(\operatorname{span} B_W \supseteq (Tv_1, \dots, Tv_m)\big)$. See also Exe (4E 23).

- **Note For Trspose:** [3.F.33] Define $\mathcal{T} : A \to A^t$. By [3.111], $\mathcal{T}$ is liney. Becs $(A^t)^t = A$.
  $\mathcal{T}^2 = I$, $\mathcal{T} = \mathcal{T}^{-1} \Rightarrow \mathcal{T}$ is iso of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$. Define $\mathcal{C}_k : A \to A_{\cdot,k}$, $\mathcal{R}_j : A \to A_{j,\cdot}$, $\mathcal{E}_{j,k} : A \to A_{j,k}$.
  Now we show (a) $\underline{\mathcal{T}\mathcal{R}_j = \mathcal{C}_j\mathcal{T}}$, (b) $\underline{\mathcal{T}\mathcal{C}_k = \mathcal{R}_k\mathcal{T}}$, and (c) $\underline{\mathcal{T}\mathcal{E}_{j,k} = \mathcal{E}_{k,j}\mathcal{T}}$.
  So that $\mathcal{T}\mathcal{C}_k\mathcal{T} = \mathcal{R}_k$, $\mathcal{T}\mathcal{R}_j\mathcal{T} = \mathcal{C}_j$, and $\mathcal{T}\mathcal{E}_{j,k}\mathcal{T} = \mathcal{E}_{k,j}$.

  Let $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n}\end{pmatrix} \Rightarrow A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n}\end{pmatrix}$.

  Note that $(A_{j,k})^t = A_{j,k} = (A^t)_{k,j}$. Thus (c) holds.
  And $(A_{\cdot,k})^t = \begin{pmatrix}A_{1,k} & \cdots & A_{m,k}\end{pmatrix} = \begin{pmatrix}A^t_{k,1} & \cdots & A^t_{k,m}\end{pmatrix} = (A^t)_{k,\cdot}$
  $\Longrightarrow$ (b) holds. Simlr for (a).

- **Note For [3.47]:** $(AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k}$ $\square$
- **Note For [3.49]:** $\big[(AC)_{\cdot,k}\big]_{j,1} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$ $\square$
- **Exe 10:** $\big[(AC)_{j,\cdot}\big]_{1,k} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$ $\square$
- **Note:** By (3.A.3), let $C = \mathcal{M}(T) \in \mathbf{F}^{n,p}, A = \mathcal{M}(S) \in \mathbf{F}^{m,n}$ wrto std bses.
  For [3.49], $\mathcal{M}(Te_k, B_V) = C_{\cdot,k} \Rightarrow \mathcal{M}\big(S(Te_k), B_W\big) = AC_{\cdot,k}$, 又 $\mathcal{M}\Big(\big(ST\big)(e_k), B_W\Big) = (AC)_{\cdot,k}$ $\square$
  For Exe (10), $(AC)_{j,\cdot} = \Big[\Big(\big(AC\big)^t\Big)_{\cdot,j}\Big]^t = \big(C^t(A^t)_{\cdot,j}\big)^t = \big((A^t)_{\cdot,j}\big)^t C = A_{j,\cdot}C$ $\square$

- [4E 3.51] Supp $C \in \mathbf{F}^{m,c}$. (a) For $k = 1, \dots, p$, $(CR)_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^c C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$
  $R \in \mathbf{F}^{c,p}$. (b) For $j = 1, \dots, m$, $(CR)_{j,\cdot} = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^c C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

- **Note For [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$. By [4E 3.51(a)], $(Ac)_{\cdot,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$. $\square$
  Or. $\because (Ac)_{j,1} = \sum_{r=1}^n A_{j,r}c_{r,1} = \Big[\sum_{r=1}^n (A_{\cdot,r}c_{r,1})\Big]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$
  $\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^n A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ Or. $(Ac)_{j,1} = (Ac)_{j,\cdot} = A_{j,\cdot}c \in \mathbf{F}$. $\square$
  Or. Let $B_V = (v_1, \dots, v_n)$. Now $Ac = \mathcal{M}(Tv, B_W) = \mathcal{M}\big(T(c_1v_1 + \cdots + c_nv_n)\big) = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$. $\square$

• **EXE 11 :** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$. By $[4E\ 3.51(b)]$, $(aC)_{1,\cdot} = a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot}$. ☐

OR. $\because (aC)_{1,k} = \sum_{r=1}^{n} a_{1,r} C_{r,k} = \left[\sum_{r=1}^{n} a_{1,r} (C_{r,\cdot}) \right]_{1,k} = (a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot})_{1,k}$

$\therefore aC = a_{1,\cdot} C_{\cdot,\cdot} = \sum_{r=1}^{n} a_{1,r} C_{r,\cdot} = a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot}$ OR. $(aC)_{1,k} = (aC)_{\cdot,k} = aC_{\cdot,k} \in \mathbf{F}$. ☐

OR. $aC = \left((aC)^t\right)^t = (C^t a^t)^t = \left[a_1^t (C^t)_{\cdot,1} + \cdots + a_n^t (C^t)_{\cdot,n}\right]^t = a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot}$. ☐

• CR FACTORIZ Supp non0 $A \in \mathbf{F}^{m,n}$. *Prove, with p below, that* $\exists\, C \in \mathbf{F}^{m,p}, R \in \mathbf{F}^{p,n}, A = CR$.

(a) *Supp* $\operatorname{col} A = \operatorname{span}(A_{\cdot,1}, \cdots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}, \dim \operatorname{col} A = c$, *the col rank. Let* $p = c$.

(b) *Supp* $\operatorname{row} A = \operatorname{span}(A_{1,\cdot}, \cdots, A_{m,\cdot}) \subseteq \mathbf{F}^{1,n}, \dim \operatorname{row} A = r$, *the row rank. Let* $p = r$.

SOLUS: Using [4E 3.51]. Notice that $A \neq 0 \Rightarrow c, r \geqslant 1$.

(a) Reduce to bss $B_C = (C_{\cdot,1}, \cdots, C_{\cdot,c})$, forming $C \in \mathbf{F}^{m,c}$. Then $\forall k \in \{1, \ldots, n\}$,

$A_{\cdot,k} = R_{1,k} C_{\cdot,1} + \cdots + R_{c,k} C_{\cdot,c} = (CR)_{\cdot,k}$, $\exists\,!\, R_{1,k}, \cdots, R_{c,k} \in \mathbf{F}$, forming $R \in \mathbf{F}^{c,n}$. Thus $A = CR$.

(b) Reduce to bss $B_R = (R_{1,\cdot}, \cdots, R_{r,\cdot})$, forming $R \in \mathbf{F}^{r,n}$. Then $\forall j \in \{1, \ldots, m\}$,

$A_{j,\cdot} = C_{j,1} R_{1,\cdot} + \cdots + C_{j,r} R_{r,\cdot} = (CR)_{j,\cdot}$, $\exists\,!\, C_{j,1}, \ldots, C_{j,r} \in \mathbf{F}$, forming $C \in \mathbf{F}^{m,r}$. Thus $A = CR$. ☐

EXA: $A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \overset{(\mathrm{I})}{=\!=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \overset{(\mathrm{II})}{=\!=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$.

(I) $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2\begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix}$, using [4E 3.51(b)].

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} \in \operatorname{span}(A_{1,\cdot}, A_{2,\cdot})$, and $(A_{1,\cdot}, A_{2,\cdot})$ is liney indep. Thus $B_R = (A_{1,\cdot}, A_{2,\cdot})$.

(II) $\begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = -\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$. Thus $B_C = (A_{\cdot,2}, A_{\cdot,3})$.

• COL RANK = ROW RANK Using CR Factoriz. Let $A = CY$ by (a) and $A = XR$ by (b).

(a) $A_{j,\cdot} = (CY)_{j,\cdot} = C_{j,\cdot} Y = C_{j,1} Y_{1,\cdot} + \cdots + C_{j,c} Y_{c,\cdot} \in \operatorname{row} A = \operatorname{span}(A_{1,\cdot}, \cdots, A_{n,\cdot}) = \operatorname{span}(Y_{1,\cdot}, \cdots, Y_{c,\cdot})$.

(b) $A_{\cdot,k} = (XR)_{\cdot,k} = XR_{\cdot,k} = R_{1,k} X_{\cdot,1} + \cdots + R_{r,k} X_{\cdot,r} \in \operatorname{col} A = \operatorname{span}(A_{\cdot,1}, \cdots, A_{\cdot,m}) = \operatorname{span}(X_{\cdot,1}, \cdots, X_{\cdot,r})$.

Thus (a) $\dim \operatorname{row} A = r \leqslant c = \dim \operatorname{col} A$, and (b) $\dim \operatorname{col} A = c \leqslant r = \dim \operatorname{row} A$. ☐

OR. Apply (a) to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim \operatorname{row} A^t = \dim \operatorname{col} A = c \leqslant r = \dim \operatorname{row} A = \dim \operatorname{col} A^t$. ☐

• (4E 16) *Supp* $A \in \mathbf{F}^{m,n} \backslash \{0\}$. *Prove* $\operatorname{rank} A = 1 \Rightarrow \exists\, c_j, d_k \in \mathbf{F}$, *each* $A_{j,k} = c_j \cdot d_k$.

SOLUS: Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $d'_k = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$.

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$, where $d_k = d'_k A_{1,1}$. OR. Using CR Factoriz, immed ☐

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6 *Supp V, W are finide and $T \in \mathcal{L}(V, W)$. Supp* $\dim \operatorname{range} T = 1$.

Prove $\exists\, B_V, B_W$, *all ent of* $A = \mathcal{M}(T, B_V, B_W)$ *equal* 1.

SOLUS: Let $B_{\operatorname{null} T} = (u_2, \ldots, u_n)$. Extend to a bss $(u_1, u_2, \ldots, u_n)$ of V.

Extend to $(Tu_1, w_2, \ldots, w_m)$ a bss of W. Let $w_1 = Tu_1 - w_2 - \cdots - w_m \Rightarrow B_W = (w_1, \ldots, w_m)$.

Let $v_1 = u_1$, $v_i = u_1 + u_i \Rightarrow B_V = (v_1, \ldots, v_n)$. ☐

OR. Supp $B_{\operatorname{range} T} = (w)$. By NOTE FOR (2.C.15), $\exists\, B_W = (w_1, \ldots, w_m)$, $w = w_1 + \cdots + w_m$.

By $[2.C\ \text{TIPS}]$, $\exists$ a bss $(u_1, \ldots, u_n)$ of V suth each $u_k \notin \operatorname{null} T$.

Now each $Tu_k \in \operatorname{range} T = \operatorname{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists\, \lambda_k \in \mathbf{F} \backslash \{0\}$. Let each $v_k = \lambda_k^{-1} u_k$. ☐

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5 *Supp $B_W = (w_1, \dots, w_n)$ and V is finide. Supp $T \in \mathcal{L}(V, W)$.*
Prove $\exists\, B_V = (v_1, \dots, v_m)$, $\mathcal{M}\big(T, B_V, B_W\big)_{1,\cdot} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$.

Solus:

Let $(u_1, \dots, u_n)$ be a bss of V. Denote $\mathcal{M}\big(T, (u_1, \dots, u_n), B_W\big)$ by A.

If $A_{1,\cdot} = 0$, then $B_V = (u_1, \dots, u_n)$ and done. Othws, supp $A_{1,k} \neq 0$.

Let $v_1 = \dfrac{u_k}{A_{1,k}} \Rightarrow Tv_1 = 1w_1 + \dfrac{A_{2,k}}{A_{1,k}} w_2 + \dots + \dfrac{A_{n,k}}{A_{1,k}} w_n$. | Let $v_{j+1} = u_j - A_{1,j} v_1$ for each $j \in \{1, \dots, k-1\}$. Let $v_i = u_i - A_{1,i} v_1$ for $i \in \{k+1, \dots, n\}$.

Notice that $Tu_i = A_{1,i} w_1 + \dots + A_{n,i} w_n$. 又 Each $u_i \in \operatorname{span}(v_1, \dots, v_n) = V$. Let $B_V = (v_1, \dots, v_n)$. □

Or. Using Exe (4). Let $B_{W'}$ be the B_V. Now $\exists\, B_{V'}$ suth $\mathcal{M}(T', B_{W'}, B_{V'})_{\cdot,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^t$ or $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}^t$.
Which is equiv to $\exists\, B_V$ [Using (3.F.31)] suth $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$. □

• (10.A.3, Or 4E 3.D.19) *Supp V is finide and $T \in \mathcal{L}(V)$.* [*See also in (3.A).*]
Prove $\forall B_V \neq B_V'$, $\mathcal{M}\big(T, B_V\big) = \mathcal{M}\big(T, B_V'\big) \Longrightarrow T = \lambda I, \exists\, \lambda \in \mathbf{F}$.

Solus: Supp $\forall B_V \neq B_V'$, $\mathcal{M}(T, B_V) = \mathcal{M}(T, B_V')$. If $T = 0$, then done.
Supp $T \neq 0$, and $v \in V \backslash \{0\}$. Asum (v, Tv) is liney indep.
Extend (v, Tv) to $B_V = (v, Tv, u_3, \dots, u_n)$. Let $B = \mathcal{M}(T, B_V)$.
$\Rightarrow Tv = B_{1,1} v + B_{2,1}(Tv) + B_{3,1} u_3 + \dots + B_{n,1} u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \forall i \neq 2$.
By asum, $A = \mathcal{M}(T, B_V') = B, \forall B_V' = (v, w_2, \dots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$.
$\Rightarrow Tv = w_2$, which is not true if $w_2 = u_3,\ w_3 = Tv,\ w_j = u_j, \forall j \in \{4, \dots, n\}$. Ctradic.
Hence (v, Tv) is linely dep $\Rightarrow \forall v \in V, \exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.
Now we show λ_v is indep of v, that is, for all disti $v, w \in V \backslash \{0\}, \lambda_v = \lambda_w$.

$$\left.\begin{array}{l} (v, w) \text{ liney indep} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v, w) \text{ linely dep}, w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \end{array}\right\} \Rightarrow T = \lambda I.$$ □

Or. Let $A = \mathcal{M}(T, B_V)$, where $B_V = (u_1, \dots, u_m)$ is arb.
Fix one $B_V = (v_1, \dots, v_m)$ and then $(v_1, \dots, \frac{1}{2} v_k, \dots, v_m)$ is also a bss for any given $k \in \{1, \dots, m\}$.
Fix one k. Now we have $T(\frac{1}{2} v_k) = A_{1,k} v_1 + \dots + A_{k,k}(\frac{1}{2} v_k) + \dots + A_{m,k} v_m$
$\Rightarrow Tv_k = 2A_{1,k} v_1 + \dots + A_{k,k} v_k + \dots + 2A_{m,k} v_m = A_{1,k} v_1 + \dots + A_{k,k} v_k + \dots + A_{m,k} v_m$.
Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k} v_k, \forall k \in \{1, \dots, m\}$.
Now we show $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j, k suth $j \neq k$.
Consider $B_V' = (v_1', \dots, v_j', \dots, v_k', \dots, v_m')$, where $v_j' = v_k,\ v_k' = v_j$ and $v_i' = v_i$ for all $i \in \{1, \dots, m\} \backslash \{j, k\}$.
Now $T(v_k') = A_{1,k} v_1' + \dots + A_{k,k} v_k' + \dots + A_{m,k} v_m' = A_{k,k} v_k' = A_{k,k} v_j$, while $T(v_k') = T(v_j) = A_{j,j} v_j$. □

• **Tips 1:** *Supp p is a poly of n variables in $\mathbf{F}$. Prove $\mathcal{M}\big(p(T_1, \dots, T_n)\big) = p\big(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n)\big)$.*
Where the liney maps $T_1, \dots, T_n$ are suth $p(T_1, \dots, T_n)$ makes sense. See [5.16,17,20].

Solus: Supp the poly p is defined by $p(x_1, \dots, x_n) = \sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n x_i^{k_i}$.
Note that $\mathcal{M}(T^x S^y) = \mathcal{M}(T)^x \mathcal{M}(S)^y$; $\mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$.
Then $\mathcal{M}\big(p(T_1, \dots, T_n)\big) = \mathcal{M}\Big(\sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n T_i^{k_i}\Big)$
$= \sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n \mathcal{M}(T_i^{k_i}) = p\big(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n)\big)$. □

• **Coro:** Supp τ is an algebraic property. Then τ holds for liney maps $\Longleftrightarrow$ τ holds for matrices.
Supp $\alpha_1, \dots, \alpha_n$ are disti with each $\alpha_k \in \{1, \dots, n\}$.
Now $p(T_1, \dots, T_n) = p(T_{\alpha_1}, \dots, T_{\alpha_n}) \Longleftrightarrow p\big(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n)\big) = p\big(\mathcal{M}(T_{\alpha_1}), \dots, \mathcal{M}(T_{\alpha_n})\big)$.

• **TIPS 2:** *Supp* $T \in \mathcal{L}(V,W)$, $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.
Let $L = \big(Tv_{\alpha_1}, \dots, Tv_{\alpha_k}\big)$, $L_{\mathcal{M}} = \big(A_{\cdot,\alpha_1}, \cdots, A_{\cdot,\alpha_k}\big)$, *where each* $\alpha_i \in \{1, \dots, n\}$.
(a) *Show* $[P]$ L *is liney indep* $\iff$ $L_{\mathcal{M}}$ *is liney indep.* $[Q]$
(b) *Show* $[P]$ $\operatorname{span} L = W \iff \operatorname{span} L_{\mathcal{M}} = \mathbf{F}^{m,1}$. $[Q]$ $\big[$ *Let* $A = \mathcal{M}(T, B_V, B_W)$. $\big]$

SOLUS: (a) Note that $\mathcal{M}\colon Tv_k \to A_{\cdot,k}$ is iso. of $\operatorname{span} L$ onto $\operatorname{span} L_{\mathcal{M}}$. By (3.B.9).
(b) Reduce to liney indep lists. By (a) and [2.39]. □

OR. $$c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} = c_1\big(A_{1,\alpha_1} w_1 + \cdots + A_{m,\alpha_1} w_m\big) + \cdots + c_k\big(A_{1,\alpha_k} w_1 + \cdots + A_{m,\alpha_k} w_m\big)$$
$$= \big(c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k}\big) w_1 + \cdots + \big(c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k}\big) w_m.$$
And $$c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} = c_1 \begin{pmatrix} A_{1,\alpha_1} \\ \vdots \\ A_{m,\alpha_1} \end{pmatrix} + \cdots + c_k \begin{pmatrix} A_{1,\alpha_k} \\ \vdots \\ A_{m,\alpha_k} \end{pmatrix} = \begin{pmatrix} c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k} \\ \vdots \\ c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k} \end{pmatrix}.$$

(a) $P \Rightarrow Q$: Supp $c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} = 0$. Let $v = c_1 v_{\alpha_1} + \cdots + c_k v_{\alpha_k}$.
Then $Tv = \big(c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k}\big) w_1 + \cdots + \big(c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k}\big) w_m = 0 w_1 + \cdots + 0 w_m$.
Now $c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} = 0$. Then each $c_i = 0 \Rightarrow L_{\mathcal{M}}$ liney indep.

$Q \Rightarrow P$: Becs $c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} = 0$. For each $i \in \{1, \dots, m\}$, $c_1 A_{i,\alpha_1} + \cdots + c_k A_{i,\alpha_k} = 0$.
Which is equiv to $c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} = 0$. Thus each $c_i = 0 \Rightarrow L$ liney indep.

OR. $\exists\, A_{\cdot,\alpha_j} = c_1 A_{\cdot,\alpha_1} + \cdots + c_{j-1} A_{\cdot,\alpha_{j-1}}$
$\iff$ For each $i \in \{1, \dots, m\}$, $A_{i,\alpha_j} = c_1 A_{i,\alpha_1} + \cdots + c_{j-1} A_{i,\alpha_{j-1}}$
$\iff Tv_{\alpha_j} = A_{1,\alpha_j} w_1 + \cdots + A_{m,\alpha_j} w_m$
$= \big(c_1 A_{1,\alpha_1} + \cdots + c_{j-1} A_{1,\alpha_{j-1}}\big) w_1 + \cdots + \big(c_1 A_{m,\alpha_1} + \cdots + c_{j-1} A_{m,\alpha_{j-1}}\big) w_m$
$\iff \exists\, Tv_{\alpha_j} = c_1 Tv_{\alpha_1} + \cdots + c_{j-1} Tv_{\alpha_{j-1}}$.

(b) Note that each $\mathcal{M}(Tv_{\alpha_i}) = A_{\cdot,\alpha_i}$
$P \Rightarrow Q$: Supp each $w_i = I w_i = J_{1,i} Tv_{\alpha_1} + \cdots + J_{k,i} Tv_{\alpha_k}$.
$\forall a \in \mathbf{F}^{m,1}, \exists!\, w = a_1 w_1 + \cdots + a_m w_m \in W$, $a = \mathcal{M}(w, B_W)$.
Becs $w = a_1\big(J_{1,1} Tv_{\alpha_1} + \cdots + J_{k,1} Tv_{\alpha_k}\big) + \cdots + a_m\big(J_{1,m} Tv_{\alpha_1} + \cdots + J_{k,m} Tv_{\alpha_k}\big)$
$= \big(a_1 J_{1,1} + \cdots + a_m J_{1,m}\big) Tv_{\alpha_1} + \cdots + \big(a_1 J_{k,1} + \cdots + a_m J_{k,m}\big) Tv_{\alpha_k}$.
Apply $\mathcal{M}$ to both sides, $a = c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k}$, where each $c_i = a_1 J_{i,1} + \cdots + a_m J_{i,m}$.

$Q \Rightarrow P$: $\forall w \in W, \exists\, a = \mathcal{M}(w, B_W) \Rightarrow \exists\, c_k \in \mathbf{F}, a = c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} \in \mathbf{F}^{m,1}$
$\Rightarrow w = \big(c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k}\big) w_1 + \cdots + \big(c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k}\big) w_m = c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k}$.

$\neg Q \Rightarrow \neg P$: $\exists\, w \in W, \exists\, a \in \mathbf{F}^{m,1}, \mathcal{M}(w, B_W) = a$, but $\nexists\, (c_1, \dots, c_k) \in \mathbf{F}^k, a = c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k}$
$\Rightarrow \nexists\, (c_1, \dots, c_k) \in \mathbf{F}^k$, $w = c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k}$. For if not, ctradic. □

NOTE: Let $L = \big(Tv_1, \dots, Tv_n\big)$, $L_{\mathcal{M}} = \big(A_{\cdot,1}, \cdots, A_{\cdot,n}\big)$.
Then (a*) By $\big[$3.B.9, TIPS (4)$\big]$, T is inje $\iff L$ is liney indep, so is $L_{\mathcal{M}}$.
And (b*) T is surj $\iff \operatorname{span} L = W \iff \operatorname{span} L_{\mathcal{M}} = \mathbf{F}^{m,1}$.
CORO: $B_{\mathbf{F}^{n,1}} = \big(A_{\cdot,1}, \cdots, A_{\cdot,n}\big) \iff T$ is inje and surj $\iff B_{\mathbf{F}^{1,n}} = \big(A_{\cdot,1}, \cdots, A_{\cdot,n}\big)$.
COMMENT: If T is inv. Then by (a*, c) or (b*, d), we have another proof of CORO.
OR. If $m = n$. Then by [3.118] and one of (a*, b*, c, d). Yet another proof.
(c) T surj $\iff T'$ inje $\iff \big(T'(\psi_1), \dots, T'(\psi_m)\big)$ liney indep
$\overset{\text{(a)}}{\iff} \big((A^t)_{\cdot,1}, \cdots, (A^t)_{\cdot,m}\big)$ liney indep in $\mathbf{F}^{n,1}$, so is $\big(A_{1,\cdot}, \cdots, A_{m,\cdot}\big)$ in $\mathbf{F}^{1,n}$.
(d) T inje $\iff T'$ surj $\iff V' = \operatorname{span}\big(T'(\psi_1), \dots, T'(\psi_m)\big)$
$\overset{\text{(b)}}{\iff} \mathbf{F}^{n,1} = \operatorname{span}\big((A^t)_{\cdot,1}, \cdots, (A^t)_{\cdot,m}\big) \iff \mathbf{F}^{1,n} = \operatorname{span}\big(A_{1,\cdot}, \cdots, A_{m,\cdot}\big)$.

3·D

• (3.E.2) *Supp $V_1 \times \cdots \times V_m$ is finide. Prove each V_j is finide.*

SOLUS: Define each $S_k \in \mathcal{L}(V_1 \times \cdots \times V_m, V_k)$ by $S_k(v_1, \dots, v_m) = v_k$. By [3.22], range $S_k = V_k$ is finide. □

OR. Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \times \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.
We show each U_i is iso to V_i. Then U is finide $\Longrightarrow$ its subsp U_i is finide, so is V_i.

$$\left.\begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U_i) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{array}\right\} \Rightarrow \left\{\begin{array}{l} R_i S_j|_{U_j} = \delta_{i,j} I_{U_j}, \\ S_i R_j = \delta_{i,j} I_{V_j}. \end{array}\right.$$ □

COMMENT: The key tool for solving (3.E.4,5).

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18 *Show V and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.*

SOLUS: Define $\Psi \in \mathcal{L}\big(V, \mathcal{L}(\mathbf{F}, V)\big)$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.
(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Now Ψ inje.
(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi\big(T(1)\big) \in \operatorname{range} \Psi$. □

OR. Define $\Phi \in \mathcal{L}\big(\mathcal{L}(\mathbf{F}, V), V\big)$ by $\Phi(T) = T(1)$.
(a) Supp $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Now Φ inje.
(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v \in \operatorname{range} \Phi$. □

COMMENT: $\Phi = \Psi^{-1}$. This is a countexa of the stmt that $\mathcal{L}(V, W)$ and $\mathcal{L}(W, V)$ are iso if infinde. See (3.F).

• (3.E.6) *Supp $m \in \mathbf{N}^+$. Prove V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.* By (3.D.18, 3.E.4), immed.

SOLUS: OR. Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto a_1 v_1 + \cdots + a_m v_m$.
(a) Supp $T(v_1, \dots, v_m) = 0$. Then $\forall (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$
For each k, let $a_k = 1, a_j = 0$ for all $j \neq k$. Then each $v_k = 0 \Rightarrow (v_1, \dots, v_m) = 0$. Thus T is inje.
(b) Supp $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be std bss of $\mathbf{F}^m$. Then $\forall (b_1, \dots, b_n) \in \mathbf{F}^m$,
$\Big[T\big(\psi(e_1), \dots, \psi(e_m)\big)\Big](b_1, \dots, b_m) = b_1 \psi(e_1) + \cdots + b_m \psi(e_m) = \psi\big(b_1 e_1 + \cdots + b_m e_m\big) = \psi(b_1, \dots, b_m)$.
Thus $T\big(\psi(e_1), \dots, \psi(e_m)\big) = \psi$. Hence T is surj. □

• (3.E.3) *Give an exa of a vecsp V and its two subsps U_1, U_2 suth $U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum.* [*V must be infinide.*]

SOLUS: NOTE *that at least one of U_1, U_2 must be infinide. Both can be infinide.* [*Req Other Courses.*]
Let $V = \mathbf{F}^\infty = U_1$, $U_2 = \big\{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\big\}$. Then $V = U_1 + U_2$ is not a direct sum.

$$\left.\begin{array}{l} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T\big((x_1, x_2, \cdots), (x, 0, \cdots)\big) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = \big((x_1, x_2, \cdots), (x, 0, \cdots)\big) \end{array}\right\} \Rightarrow S = T^{-1}.$$ □

• *Supp $T \in \mathcal{L}(V)$. Prove $\exists$ inv $T_1, T_2 \in \mathcal{L}(V)$ suth $T = T_1 + T_2$.*

SOLUS: Let $U \oplus \operatorname{null} T = V$, $W \oplus \operatorname{range} T = V$. Let $S : \operatorname{null} T \to W$ be an iso.

$$\left.\begin{array}{l} \text{Define } T_1 \in \mathcal{L}(V) \text{ by } T_1(u) = \tfrac{1}{2} Tu, T_1(w) = Sw \\ \text{Define } T_2 \in \mathcal{L}(V) \text{ by } T_2(u) = \tfrac{1}{2} Tu, T_2(w) = -Sw \end{array}\right\} \Rightarrow T = T_1 + T_2 \text{ and } T_1, T_2 \text{ inv.}$$ □

• *Supp $A, B \in \mathcal{L}(V)$ and $A + B, A - B$ are inv. Supp $C, D \in \mathcal{L}(V)$. Prove $\exists\, X, Y \in \mathcal{L}(V)$ suth $AX + BY = C, BX + AY = D$.*

SOLUS: Asum $AX + BY = C, BX + AY = D$. Then $(A \pm B)(X \pm Y) = C \pm D$.
Let $S = (A + B)^{-1}(C + D), T = (A - B)^{-1}(C - D)$. Now $X = \frac{1}{2}(S + T), Y = \frac{1}{2}(S - T)$. □

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3 *Supp V and W are iso and finide, U is a subsp of V, and $S \in \mathcal{L}(U, W)$.*
Prove $\exists$ inv $T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U \iff S$ is inje. $\big[$ *See also* (3.A.11). $\big]$

Solus: (a) $\forall u \in U, u = T^{-1}Su \Rightarrow T^{-1}S = I \in \mathcal{L}(U)$. Or. $\operatorname{null} S = \operatorname{null} T|_U = \operatorname{null} T \cap U = \{0\}$.
(b) Get a B_U, apply S, then extend to B_V, B_W. $\square$

Exa: Let $V = W = \mathbf{F}^\infty$. Define $S(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots) \Rightarrow S$ inje.
Asum $\exists$ inv $T \in \mathcal{L}(V, W)$ suth $T|_V = S$. Then $T = S$ while S is not surj.

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8 *Supp $T \in \mathcal{L}(V, W)$ is **surj**. Prove $\exists$ subsp U of V, $T|_U : U \to W$ is iso.*

Solus: By (3.B.12). Note that $\operatorname{range} T = W$. Or. $\big[$ *Req* $\operatorname{range} T$ *Finide* $\big]$ By $\big[$3.B Tips (4)$\big]$. $\square$

• **Tips 1:** *Supp $V = U \oplus X = W \oplus X$. Prove U, W are iso.*

Solus: $\forall u \in U, \exists!\,(w, x_1) \in W \times X, u = w + x_1$. While $\exists!\,(u', x_2) \in U \times X, w = u' + x_2$.
Now $x_1 = -x_2$, $u = u'$. Thus $\pi : U \to W$ defined by $\pi(u) = w$, is inje.
$\forall w \in W, \exists!\,(u, x_1) \in U \times X, w = u + x_1$. While $\exists!\,(w', x_2) \in W \times X, u = w' + x_2$.
Now $x_1 = -x_2$, $w = w'$. Thus $\pi : U \to W$ defined by $\pi(u) = w$, is surj. $\square$

Comment: Let $V = \mathbf{F}^\infty$. Let $X = \mathbf{F}^\infty, Y = \{(0, x_1, x_2, \cdots) \in \mathbf{F}^\infty\}$. Now X, Y are iso subsps of V.
But $\nexists$ iso subsps M, N of V, suth $V = M \oplus X = N \oplus Y$.

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9 *Supp U, V, W are iso and finide, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$, and ST inv.*
Prove S, T are inv. **Note:** Not true if U, V, W infinide. Exa: Forwd and backwd shift.

Solus: Let $R = (ST)^{-1}$. Becs $R(ST) = (RS)T = I_U$ Or $(ST)R = S(TR) = I_W$, T inje and S surj. $\square$
Or. $\dim W = \dim \operatorname{range} ST \leqslant \min\{\operatorname{range} S, \operatorname{range} T\} \Rightarrow S, T$ surj. $\square$

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10 *Supp V, W are finide and $T \in \mathcal{L}(V, W), S \in \mathcal{L}(W, V)$. Prove $ST = I \iff TS = I$.*

Solus: Supp $ST = I$. Then S, T inv. We show $TS = I$, then done by rev the roles.
Then $S\big((TS)w\big) = ST(Sw) = Sw \Rightarrow (TS)w = w$. Or. $S^{-1} = T$ 又 $S = S \Rightarrow TS = S^{-1}S = I$. $\square$

• **Tips 2:** Supp each $S_k \in \mathcal{L}(V_k, W_k), W_k \subseteq V_{k+1} \Rightarrow S_m \circ S_{m-1} \circ \cdots \circ S_2 \circ S_1$ makes sense.
(a) By the ctrapos of (3.B.11), $S_m \circ \cdots \circ S_1$ not inje $\Rightarrow \exists\, S_k$ not inje. Convly not true unless $k = 1$.
(b) By Exe (9), if all V_k finide and iso to each other, then $S_m \circ \cdots \circ S_1$ inje $\Rightarrow$ inv, so are all S_k.
(c) $\operatorname{null} S_1 \subseteq \operatorname{null}(S_2 S_1) \subseteq \cdots \subseteq \operatorname{null}(S_m \cdots S_2 S_1)$; $S_m \circ \cdots \circ S_1$ inje $\Rightarrow$ each $S_k \circ \cdots \circ S_1$ inje.
Supp each $W_k = V_{k+1}$, for if $W_k \subsetneq V_{k+1}$, then S_1, S_2 surj $\not\Rightarrow S_2 \circ S_1 \in \mathcal{L}(V_1, W_2)$ surj.
(d) Each S_k surj $\Rightarrow S_m \circ \cdots \circ S_1$ surj. Convly not true unless all V_k finide and iso to each other.
(e) $\operatorname{range} S_m \supseteq \operatorname{range}(S_m S_{m-1}) \supseteq \cdots \supseteq \operatorname{range}(S_m S_{m-1} \cdots S_1)$; $S_m \circ \cdots \circ S_1$ surj $\Rightarrow$ each $S_m \circ \cdots \circ S_k$ surj.

• (4E 23, Or 10.A.4) *Supp $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bses of V.*
Let $T \in \mathcal{L}(V)$ be suth each $T\alpha_k = \beta_k$. Prove $A = \mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha) = B$.

Solus: Each $I\beta_k = \beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = T\alpha_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. $\square$
Or. $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)\underline{\mathcal{M}(T, \alpha \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. $\square$
Or. $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}\big[\underline{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}\big] = \mathcal{M}(I, \beta \to \alpha)$. $\square$

Comment: $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$. Or. Let $A' = \mathcal{M}(T, \beta \to \beta)$.
Simlr. Now each $T\beta_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.

- **NOTE FOR [3.62]:** $\mathcal{M}(v) = \mathcal{M}\big(I, (v), B_V\big)$. Here I is restr to span(v), and $(v) = (\,)$ if $v = 0$.
- **NOTE FOR [3.65]:** $\mathcal{M}(Tv) = \mathcal{M}\big(I, (Tv), B_W\big) = \mathcal{M}(T, B_V, B_W)\mathcal{M}\big(I, (v), B_V\big) = \mathcal{M}\big(T, (v), B_W\big)$.

- **NOTE FOR Exe (15):**
  Supp $A \in \mathbf{F}^{m,n}$. Define $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1}) : x \mapsto Ax \Rightarrow A = \mathcal{M}(T)$ wrto std bses B_1, B_2 of $\mathbf{F}^{m,1}, \mathbf{F}^{n,1}$.
  Becs T have infily many defs of the form $T : x \mapsto C_2AC_1x$, where $C_1 = \mathcal{M}(I, B_1', B_1), C_2 = \mathcal{M}(I, B_2, B_2')$.
  Now we can well define range A = range T and null A = range T, becs A is indep of bses.

- **TIPS 3:** Identify $\mathbf{F}^{m,n}$ with $\mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, due to $Tx = Ax$; or with $\mathcal{L}(\mathbf{F}^{1,m}, \mathbf{F}^{1,n})$, due to $Tx = xA^t$.

- **TIPS 4:** You must first declare bses and the purpose when using $\mathcal{M}^{-1} : \mathbf{F}^{n,1} \mapsto v$, or $\mathbf{F}^{m,n} \mapsto \mathcal{L}(V, W)$.

- **NOTE FOR Exe (3, 4E 22):** $T \in \mathcal{L}(V, W)$ inv $\Longleftrightarrow \mathcal{M}(T)$ inv, wrto some $B_V, B_W \Longleftrightarrow$ wrto any B_V, B_W.
  Supp $\mathcal{M}(T)$ wrto some bses is inv. Let $S \in \mathcal{L}(W, V)$ be suth $\mathcal{M}(S) = \mathcal{M}(T)^{-1}$.
  Then $\mathcal{M}(TS) = I = \mathcal{M}(ST)$ wrto same bses. Apply $\mathcal{M}^{-1}$, now $TS = I, ST = I \Rightarrow S = T^{-1}$. □
  OR. (a) $T \in \mathcal{L}(V, W)$ inje $\Longleftrightarrow$ the cols of $\mathcal{M}(T, B_V, B_W)$ are liney indep $\Longleftrightarrow \mathcal{M}(T)$ inje wrto any bses.
  (b) $T \in \mathcal{L}(V, W)$ surj $\Longleftrightarrow$ the cols of $\mathcal{M}(T, B_V, B_W)$ spans $\mathbf{F}^{\dim W, 1} \Longleftrightarrow \mathcal{M}(T)$ surj wrto any bses.

- **NOTE FOR [3.60]:** Supp $B_V = (v_1, \dots, v_n),\ B_W = (w_1, \dots, w_m)$.
  Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{i,x}w_j$. Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. And $\big(\mathcal{E}^{(j,i)}\big)_{l,k} = \delta_{i,l}\delta_{j,k}$.
  CORO: $E_{l,k}E_{i,j} = \delta_{j,l}E_{i,k},\ \mathcal{E}^{(k,l)}\mathcal{E}^{(j,i)} = \delta_{l,j}\mathcal{E}^{(k,i)}$.
  Becs $\mathcal{M} : \mathcal{L}(V, W) \to \mathbf{F}^{m,n}$ is iso.
  $E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$. By [2.42] and [3.61]:
  $$B_{\mathcal{L}(V,W)} = \begin{pmatrix} E_{1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \cdots & , E_{n,m} \end{pmatrix}; \quad B_{\mathbf{F}^{m,n}} = \begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & , \mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots & , \mathcal{E}^{(m,n)} \end{pmatrix}.$$
- **TIPS:** Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_p), B_V = \big(v_1, \dots, v_p, \dots, v_n\big)$. Let each $w_k = Tv_k$.
  Extend to $B_W = \big(w_1, \dots, w_p, \dots, w_m\big)$. Then $T = E_{1,1} + \cdots + E_{p,p},\ \mathcal{M}(T) = \mathcal{E}^{(1,1)} + \cdots + \mathcal{E}^{(p,p)}$.

- (4E 17) *Supp* U, V, W *finide*, $S \in \mathcal{L}(V, W), \mathcal{A} \in \mathcal{L}\big(\mathcal{L}(U, V), \mathcal{L}(U, W)\big) : T \mapsto ST$.
  Show $\dim \operatorname{null} \mathcal{A} = (\dim U)(\dim \operatorname{null} S)$, $\dim \operatorname{range} \mathcal{A} = (\dim U)(\dim \operatorname{range} S)$.

SOLUS: (a) $\forall T \in \mathcal{L}(U, V), ST = 0 \Longleftrightarrow \operatorname{range} T \subseteq \operatorname{null} S$. Thus $\operatorname{null} \mathcal{A} = \mathcal{L}(U, \operatorname{null} S)$.
(b) $\forall R \in \mathcal{L}(U, W), \operatorname{range} R \subseteq \operatorname{range} S \Longleftrightarrow \exists\, T \in \mathcal{L}(U, V), R = ST$, by (3.B 25).
Thus $\operatorname{range} \mathcal{A} = \mathcal{L}(U, \operatorname{range} S)$. □

OR. Let $B_{\operatorname{range} S} = (w_1, \dots, w_s)$ with each $w_i = Sv_i$. Let $B_W = (w_1, \dots, w_n), B_{\operatorname{null} S} = (v_{s+1}, \dots, v_p)$.
Let $B_U = (u_1, \dots, u_m)$. Define $E_{i,j} \in \mathcal{L}(V, W) : v_x \mapsto \delta_{i,x}w_j$. Now $S = E_{1,1} + \cdots + E_{s,s}$.
Define $R_{i,j} \in \mathcal{L}(U, V) : u_x \mapsto \delta_{i,x}v_j$. Let $E_{k,j}R_{i,k} = Q_{i,j} : u_x \mapsto \delta_{i,x}w_j$.
For any $T \in \mathcal{L}(V), \exists!\, A_{i,j} \in \mathbf{F}, T = \sum_{j=1}^{p}\sum_{i=1}^{m} A_{j,i}R_{i,j} \Longrightarrow \mathcal{M}(T, u \to v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,s} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{s,1} & \cdots & A_{s,s} & \cdots & A_{s,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{p,1} & \cdots & A_{p,s} & \cdots & A_{p,m} \end{pmatrix}$.
$\Longrightarrow \mathcal{A}(T) = ST = \left(\sum_{k=1}^{s} E_{k,k}\right)\left(\sum_{j=1}^{p}\sum_{i=1}^{m} A_{j,i}R_{i,j}\right) = \sum_{j=1}^{s}\sum_{i=1}^{m} A_{i,j}Q_{j,i}$.

$$\begin{aligned} \mathcal{M}(S, v \to w)\mathcal{M}(T, u \to v) = \mathcal{M}(ST, u \to w) &= \\ \mathcal{M}(\mathcal{A}, R \to Q)\mathcal{M}(T, R) = \mathcal{M}\big(\mathcal{A}(T), Q\big) &= \end{aligned} \begin{pmatrix} A_{1,1} & \cdots & A_{1,s} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{s,1} & \cdots & A_{s,s} & \cdots & A_{s,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

又 $\mathcal{M}(T, R) = \mathcal{M}(T, u \to v)$.
If $m = p$, let $\mathcal{M}(T, R) = I$,
$\mathcal{M}(\mathcal{A}, R \to Q) = \mathcal{M}(S, v \to w)$.

$$\operatorname{range} \mathcal{A} = \operatorname{span}\begin{Bmatrix} Q_{1,1}, & \cdots & , Q_{m,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,s}, & \cdots & , Q_{m,s} \end{Bmatrix}, \ \operatorname{null} \mathcal{A} = \operatorname{span}\begin{Bmatrix} R_{1,s+1}, & \cdots & , R_{m,s+1}, \\ \vdots & \ddots & \vdots \\ R_{1,p}, & \cdots & , R_{m,p} \end{Bmatrix}.$$

(a) $\dim \operatorname{null} \mathcal{A} = m \times (p - s)$;
(b) $\dim \operatorname{range} \mathcal{A} = m \times s$. □

• (4E 10) *Supp V, W finide, U is a subsp of V, $\mathcal{E} = \{T \in \mathcal{L}(V,W) : T|_U = 0\}$. Find* $\dim \mathcal{E}$.

Solus: Define $\Phi : \mathcal{L}(V,W) \to \mathcal{L}(U,W)$ by $\Phi(T) = T|_U$. By [3.A Note For Restr], Φ is liney.
$\Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$. Thus $\operatorname{null}\Phi = \mathcal{E}$.
Extend $S \in \mathcal{L}(U,W)$ to $T \in \mathcal{L}(V,W) \Rightarrow \Phi(T) = S \in \operatorname{range}\Phi$. Thus $\operatorname{range}\Phi = \mathcal{L}(U,W)$. □

Or. Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, \dots, u_n), B_W = (w_1, \dots, w_p)$.
Define $E_{i,j} \in \mathcal{L}(V,W) : u_x \mapsto \delta_{i,x} w_j$.
$\forall T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \operatorname{span} \underbrace{\begin{Bmatrix} E_{1,1}, & \cdots & , E_{m,1}, \\ \vdots & \ddots & \vdots \\ E_{1,p}, & \cdots & , E_{m,p} \end{Bmatrix}}_{=R} \cap \mathcal{E} = \{0\}$.
又 $C = \operatorname{span}\begin{Bmatrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{Bmatrix} \subseteq \mathcal{E}$. Now $\mathcal{L}(V,W) = \operatorname{span} R \oplus C$
$\Rightarrow \mathcal{L}(V,W) = \operatorname{span} R + \mathcal{E}$. □

• *Supp U, V, W finide, $S \in \mathcal{L}(U,V), \mathcal{B} \in \mathcal{L}\big(\mathcal{L}(V,W), \mathcal{L}(U,W)\big) : T \mapsto TS$.*
Show $\dim \operatorname{null} \mathcal{B} = (\dim W)(\dim \operatorname{null} S)$, $\dim \operatorname{range} \mathcal{B} = (\dim W)(\dim \operatorname{range} S)$.

Solus: (a) $\forall T \in \mathcal{L}(V,W), TS = 0 \iff \operatorname{range} S \subseteq \operatorname{null} T$. Thus $\operatorname{null}\mathcal{B} = \{T \in \mathcal{L}(V,W) : T|_{\operatorname{range} S} = 0\}$.
(b) $\forall R \in \mathcal{L}(U,W), \operatorname{null} S \subseteq \operatorname{null} R \iff \exists T \in \mathcal{L}(V,W), R = TS$, by (3.B.24).
Thus $\operatorname{range}\mathcal{B} = \{R \in \mathcal{L}(U,W) : R|_{\operatorname{null} S} = 0\}$. Now by Exe (4E 10). □

Or. Let $B_{\operatorname{range} S} = (v_1, \dots, v_r)$ with each $u_i = Sv_i$. Let $B_V = (v_1, \dots, v_m), B_{\operatorname{null} S} = (u_{r+1}, \dots, u_n)$.
Let $B_W = (w_1, \dots, w_p)$. Define $E_{i,j} \in \mathcal{L}(U,V) : u_x \mapsto \delta_{i,x} v_j \Rightarrow S = E_{1,1} + \cdots + E_{r,r}$.
Define $R_{i,j} \in \mathcal{L}(V,W) : v_x \mapsto \delta_{i,x} w_j$. Let $R_{k,j}E_{i,k} = Q_{i,j} : u_x \mapsto \delta_{i,x} w_j$.
$\mathcal{B}(T) = TS = \left(\sum_{j=1}^{p} \sum_{i=1}^{m} A_{j,i} R_{i,j}\right)\left(\sum_{k=1}^{r} E_{k,k}\right) = \sum_{j=1}^{p} \sum_{i=1}^{r} A_{j,i} Q_{i,j} \Rightarrow \mathcal{M}(TS, v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,r} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{r,1} & \cdots & A_{r,r} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{p,1} & \cdots & A_{p,r} & \cdots & 0 \end{pmatrix}$.
$\operatorname{range}\mathcal{B} = \operatorname{span}\begin{Bmatrix} Q_{1,1}, & \cdots & , Q_{r,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,p}, & \cdots & , Q_{r,p} \end{Bmatrix}$, $\operatorname{null}\mathcal{B} = \operatorname{span}\begin{Bmatrix} R_{r+1,1}, & \cdots & , R_{n,1}, \\ \vdots & \ddots & \vdots \\ R_{r+1,p}, & \cdots & , R_{n,p} \end{Bmatrix}$. □

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16 *Supp V is finide and $S \in \mathcal{L}(V)$ suth $\forall T \in \mathcal{L}(V), ST = TS$. Prove $\exists \lambda \in \mathbf{F}, S = \lambda I$.*

Solus: If $S = 0$, done. Now supp $S \neq 0$. [*Using nota in Exe (4E 17). See also in (3.A).*]
Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, B_U) = \mathcal{M}(I, B_{\operatorname{range} S}, B_U)$. Note that $R_{k,1} : w_x \mapsto \delta_{k,x} v_1$.
Then $\forall k \in \{1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $\dim \operatorname{null} S = 0$, $\dim \operatorname{range} S = m = n$.
Notice that $G_{i,j} = R_{i,j}S = SR_{i,j} = Q_{i,j}$. Where $G_{i,j} : v_x \mapsto \delta_{i,x} v_j$; $Q_{i,j} : w_x \mapsto \delta_{i,x} w_j$.
For each $w_i, \exists!\, a_{k,i} \in \mathbf{F}, w_i = a_{1,i} v_1 + \cdots + a_{n,i} v_n$. Where $a_{k,i} = \mathcal{M}\big(I, (w_1, \dots, w_n), (v_1, \dots, v_n)\big)_{k,i}$.
Then fix one i. Now for each $j \in \{1, \dots, n\}, Q_{i,j}(w_i) = w_j = a_{i,i} v_j = G_{i,j}\left(\sum_{k=1}^{n} a_{k,i} v_k\right)$.
Let $\lambda = a_{i,i}$. Hence each $w_j = \lambda v_j$. Now fix one j, we have $a_{1,1} v_j = \cdots = a_{n,n} v_j$, then all $a_{i,i}$ are equal. □

• *Supp $A \in \mathbf{F}^{n,n}$. Define $T, S \in \mathcal{L}(\mathbf{F}^{n,n})$ by $T(X) = AX$, $S(Y) = YA^t$. Find* $\dim \operatorname{range} ST$.

Solus: Becs $A\mathcal{E}^{(j,k)} = \left[\sum_{x=1}^{n} A_{x,j} \mathcal{E}^{(x,j)}\right] \mathcal{E}^{(j,k)} = \sum_{x=1}^{n} A_{x,j} \mathcal{E}^{(x,k)}$. Let $B_{\operatorname{col} A} = (C_{\cdot,1}, \dots, C_{\cdot,r})$.
Each $A_{\cdot,j} = R_{1,j} C_{\cdot,1} + \cdots + R_{r,j} C_{\cdot,r} \Rightarrow B_{\operatorname{range} T} = \{\mathcal{C}_{j,k} = \sum_{x=1}^{n} C_{x,j} \mathcal{E}^{(x,k)} : 1 \leqslant j \leqslant r, 1 \leqslant k \leqslant n\}$.
Becs $\mathcal{C}_{j,k} A^t = \mathcal{C}_{j,k}\left[\sum_{y=1}^{n} A^t_{k,y} \mathcal{E}^{(k,y)}\right] = \sum_{x=1}^{n} \sum_{y=1}^{n} C_{x,j} A_{y,k} \mathcal{E}^{(x,y)}$.
Simlr, $B_{\operatorname{range} ST} = \{\mathcal{X}_{j,k} = \sum_{x=1}^{n} \sum_{y=1}^{n} C_{x,j} C_{y,k} \mathcal{E}^{(x,y)} : 1 \leqslant j, k \leqslant r\}$.
Each $\mathcal{X}_{j,k} = C_{1,k} \mathcal{C}_{j,1} + \cdots + C_{n,k} \mathcal{C}_{j,n} = C_{1,j} (\mathcal{C}_{k,1})^t + \cdots + C_{n,j} (\mathcal{C}_{k,n})^t$.
$\mathcal{X}_{j,k} = \begin{pmatrix} C_{1,j} C_{1,k} & \cdots & C_{1,j} C_{n,k} \\ \vdots & \ddots & \vdots \\ C_{n,j} C_{1,k} & \cdots & C_{n,j} C_{n,k} \end{pmatrix}$. □

Or. By Tips (3). Define $\varphi \in \mathcal{L}\big(\mathcal{L}(\mathbf{F}^{n,1})\big) : X \mapsto AX$; $\psi \in \mathcal{L}\big(\mathcal{L}(\mathbf{F}^{n,1}, \operatorname{range} A)\big) : Y \mapsto YA^t$.
Then $\operatorname{range} \psi\varphi = \{X \in \mathcal{L}(\mathbf{F}^{n,1}, \operatorname{range} A) : X|_{\operatorname{null} A^T} = 0\}$ of dim $(\operatorname{range} A)(\operatorname{range} A^t) = (\operatorname{rank} A)^2$. □

3·E

- **Note For [3.79], def of $v+U$:** Given $v+U$, v is already uniqly determined, as a sort of precond. Even though $v+U=v'+U$, where v' is *purer* than v.

- **Note For [3.85]:** $v+U=w+U \iff v\in w+U,\ w\in v+U$
$\iff v-w\in U \iff (v+U)\cap(w+U)\neq\emptyset.$

- **Note For [3.79, 3.83]:**

If U is merely a subset of V, then [3.85, 86] do not hold $\Rightarrow V/U$ not a vecsp.
If V is merely a subset of a vecsp of which U is a subsp, then [3,79, 86] do not hold $\Rightarrow V/U$ not a vecsp.
<u>If U is a vecsp but not a subsp of V, while U, V are subsps of some vecsp, then everything's alright.</u>
Hence if V/U is a vecsp, then V, U are subsps of some vecsp.
Comment: Supp U, V are subsps and U is not a subsp of V. Note that $V/U=(V+U)/U$.
Supp $v+U\in V/U$. Then $v\in V$, or possibly $v\in V+U$ as well. To avoid this ambiguity, you have to specify the precond, what subsp that v belongs to.
Exa: Supp $U+W=V$. Then $V/U=(U+W)/U=W/U$. Let $W\cap U=I, U_I\oplus I=U, W_I\oplus I=W$.
Now $U_I\oplus W_I\oplus I=V$. Thus $W/U=(W_I\oplus I)/U=W_I/U$.
$\forall w_1', w_2'\in W_I$ suth $w_1'+U=w_2'+U\in W_I/U,\ w_1'-w_2'\in U\cap W_I=\{0\}\Rightarrow w_1'=w_2'$.

- ***Trivial Cases***: If $v\in U$, then $v+U=0+U=\{u:u\in U\}=U$. Now $U=0\in V/U$.
If $U=\{0\}$, then $v+U=v+\{0\}=\{v\},\ V/U=V/\{0\}=\{\{v\}:v\in V\}$.
If $U=\emptyset$, then $v+U=v+\emptyset=\emptyset,\ V/U=V/\emptyset=\{\emptyset\}$.
- **Tips 1:** V is a subsp of $U\iff\forall v\in V, v+U=0+U=U\iff V/U=\{0\}=\{U\}$.

- **Note For [3.88]:** If U, V are subsp of some vecsp $\mathcal{V}$. Define the quot map $\pi\in\mathcal{L}(V, V/U)$.
Then π is surj by def, and null $\pi=V\cap U$. Thus if $\mathcal{V}$ is finide, then $\dim V=\dim V/U+\dim(V\cap U)$.
Or. Let $I=V\cap U,\ V_I\oplus I=V$. Becs $V/U=V_I/U$, iso to V_I. Now $\dim V=\dim V_I+\dim I$.

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7 *Supp $\alpha,\beta\in V$, and U, W are subsps of V. Prove $\alpha+U=\beta+W\Rightarrow U=W$.*

Solus: (a) $\alpha\in\alpha+U=\beta+W\Rightarrow\exists w\in W, \alpha=\beta+w\Rightarrow\alpha-\beta\in W\Rightarrow\alpha+W=\beta+W$.
(b) $\beta\in\beta+W=\alpha+U\Rightarrow\exists u\in U, \beta=\alpha+u\Rightarrow\beta-\alpha\in U\Rightarrow\alpha+U=\beta+U$. □

Or. $\pm(\alpha-\beta)\in U\cap W\Rightarrow\left\{\begin{array}{l}U\ni u=(\beta-\alpha)+w\in W\Rightarrow U\subseteq W\\ W\ni w=(\alpha-\beta)+u\in U\Rightarrow W\subseteq U\end{array}\right\}\Rightarrow U=W.$ □

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8 *Supp $\emptyset\neq A\subseteq V$. Prove A is a trslate $\iff\lambda v+(1-\lambda)w\in A$, $\forall v, w\in A, \lambda\in\mathbf{F}$.*

Solus: (a) Supp $A=a+U$. Then $\lambda(a+u_1)+(1-\lambda)(a+u_2)=a+\big(\lambda(u_1-u_2)+u_2\big)\in A$.
(b) Supp $\lambda v+(1-\lambda)w\in A, \forall v, w\in A, \lambda\in\mathbf{F}$. Supp <u>$a\in A$</u> and let $A'=\{x-a:x\in A\}$.
Then $0\in A'$ and $\forall(v-a),(w-a)\in A', \lambda\in\mathbf{F}$, (I) $\lambda(v-a)=[\lambda v+(1-\lambda)a]-a\in A'$.
(II) Becs $\lambda(v-a)+(1-\lambda)(w-a)=[\lambda v+(1-\lambda)w]-a\in A'$.
Let $\lambda=\frac{1}{2}$ here and use (I) above by $\lambda=2$, we have $(v-a)+(w-a)\in A'$.
Or. Note that $v, a\in A\Rightarrow\lambda v+(1-\lambda)a=2v-a\in A$. Simlr $2w-a\in A$.
Now $(v-\frac{1}{2}a)+(w-\frac{1}{2}a)=v+w-a\in A\Rightarrow v+w-2a=(v-a)+(w-a)\in A'$.
Thus $A'=-a+A$ is a subsp of V. Hence $a+A'=a+\{x-a:x\in A\}=A$ is a trslate. □

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9 *Supp $A = \alpha + U$ and $B = \beta + W$ for some $\alpha, \beta \in V$ and some subsps U, W of V. Prove $A \cap B$ is either a trslate of some subsp of V or is $\emptyset$.*

Solus: $\forall \alpha + u, \beta + w \in A \cap B \neq \emptyset, \lambda \in \mathbf{F}, \lambda(\alpha + u) + (1-\lambda)(\beta + w) \in A \cap B$. By Exe (8). $\square$

Or. Let $A = \alpha + U,\ B = \beta + W$. Supp $v \in (\alpha + U) \cap (\beta + W) \neq \emptyset$.
Then $v - \alpha \in U \Rightarrow v + U = \alpha + U = A$, and simlr $v + W = \beta + W = B$.
We show $A \cap B = v + (U \cap W)$. Note that $v + (U \cap W) \subseteq A \cap B$.
And $\forall \gamma = v + u = v + w \in A \cap B \Rightarrow u = w \in U \cap W \Rightarrow \gamma \in v + (U \cap W)$. $\square$

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10 *Prove the intersec of any collec of trslates of subsps is either a trslate of some subsps or $\emptyset$.*

Solus: Supp $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collec of trslates of subsps of V, where Γ is an index set.
$\forall x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset, \lambda \in \mathbf{F}, \lambda x + (1-\lambda) y \in A_\alpha$ for each α. By Exe (8). $\square$

Or. Let each $A_\alpha = w_\alpha + V_\alpha$. Supp $x \in \bigcap_{\alpha \in \Gamma} (w_\alpha + V_\alpha) \neq \emptyset$.
Then $x - w_\alpha \in V_\alpha \Longrightarrow x + V_\alpha = w_\alpha + V_\alpha = A_\alpha$, for each α.
We show $\bigcap_{\alpha \in \Gamma} A_\alpha = \bigcap_{\alpha \in \Gamma} (x + V_\alpha) = x + \bigcap_{\alpha \in \Gamma} V_\alpha$.
$y \in \bigcap_{\alpha \in \Gamma} A_\alpha \Longleftrightarrow$ for each $\alpha,\ y = x + v_\alpha \in A_\alpha$
$\Longleftrightarrow$ each $v_\alpha = y - x \in \bigcap_{\alpha \in \Gamma} V_\alpha \Longleftrightarrow y \in x + \bigcap_{\alpha \in \Gamma} V_\alpha$. $\square$

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11 *Supp $A = \left\{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\right\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.*
(a) *Prove A is a trslate of some subsp of V*
(b) *Prove if B is a trslate of some subsp of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.*
(c) *Prove A is a trslate of some subsp of V of dim $< m$.*

Solus: (a) By Exe (8), $\forall u, w \in A, \lambda \in \mathbf{F}, \lambda u + (1-\lambda) w = \left(\lambda \sum_{i=1}^m a_i + (1-\lambda) \sum_{i=1}^m b_i\right) v_i \in A$.

(b) Supp $B = v + U$, where $v \in V$ and U is a subsp of V. <u>Let each $v_k = v + u_k \in B, \exists!\, u_k \in U$.</u>
$\forall w \in A,\ w = \sum_{i=1}^m \lambda_i v_i = \sum_{i=1}^m \lambda_i (v + u_i) = \underline{\sum_{i=1}^m \lambda_i v} + \sum_{i=1}^m \lambda_i u_i = v + \sum_{i=1}^m \lambda_i u_i \in v + U = B$. $\square$

Or. Let $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$. To show $v \in B$, use induc on m by k.
(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.
$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By Exe (8), $v \in B$.
(ii) $2 \leqslant k < m$. Asum $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. $\left[\forall \lambda_i \text{ suth } \sum_{i=1}^k \lambda_i = 1\right]$
For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. Fix one $\mu_\iota \neq 1$.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Longrightarrow \left[\sum_{i=1}^{k+1} \dfrac{\mu_i}{1-\mu_\iota}\right] - \dfrac{\mu_\iota}{1-\mu_\iota} = 1$.

$$\text{Let } w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let $\lambda_i = \dfrac{\mu_i}{1-\mu_\iota}$ for $i \in \{1, \dots, \iota - 1\}$; $\lambda_j = \dfrac{\mu_{j+1}}{1-\mu_\iota}$ for $j \in \{\iota, \dots, k\}$. Then,

$$\left.\begin{array}{r} \sum_{i=1}^k \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \quad \square$$

(c) If $m = 1$, then let $A = v_1 + \{0\}$ and done. Now supp $m \geqslant 2$. Fix one $k \in \{1, \dots, m\}$.
$$\begin{aligned} A &\ni \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m) v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_m v_m \\ &= v_k + \lambda_1 (v_1 - v_k) + \cdots + \lambda_{k-1}(v_{k-1} - v_k) + \lambda_{k+1}(v_{k+1} - v_k) + \cdots + \lambda_m (v_m - v_k) \\ &\in v_k + \operatorname{span}(v_1 - v_k, \dots, v_m - v_k). \end{aligned}$$
$\square$

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18 *Supp $T \in \mathcal{L}(V,W)$ and $U, \mathcal{V}$ are subsps of $\mathcal{V}$. Let $\pi : V \to V/U$ be the quot map. Prove $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \cap V = \operatorname{null}\pi \subseteq \operatorname{null} T$.*

Solus: Supp $\operatorname{null}\pi \subseteq \operatorname{null} T$. By (3.B.24), done. Or. Define $S : (v+U) \mapsto Tv$.
$\forall v_1, v_2 \in V$ suth $v_1 + U = v_2 + U \iff v_1 - v_2 \in U \cap V \subseteq \operatorname{null} T \iff Tv_1 = Tv_2$.
Thus S is well-defined. Convly true as well. □

Coro: $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V,W)$ with $S \mapsto S \circ \pi$ is inje, $\operatorname{range}\Gamma = \{T \in \mathcal{L}(V,W) : U \subseteq \operatorname{null} T\}$.
Comment: If $T = I_V$. Then $S : v + U \mapsto v$ is not well-defined, unless $U \cap V = \{0\} \subseteq \operatorname{null} I_V$.

- **Note For [3.88, 3.90, 3.91]:** Supp $W \oplus U = V$. Then $V/U = W/U$ is iso to W. [Convly not true.]
Becs $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$. Define $T \in \mathcal{L}(V)$ by $T(v) = w_v$.
Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$, $\operatorname{range} T \oplus \operatorname{null} T = V$.
Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, V)$ is defined by $\tilde{T}(v+U) = \tilde{T}(w_v' + U) = Tw_v' = w_v$. [*See Exa below*]
Now $\pi \circ \tilde{T} = I_{V/U}$, $\tilde{T} \circ \pi|_W = I_W = T|_W$. Hence $\tilde{T} = (\pi|_W)^{-1}$ is iso of V/U onto W.
- **Exa:** Let $V = \mathbf{F}^2, B_U = (e_1), B_W = (e_2 - e_1) \Rightarrow U \oplus W = V$.

Solus: Although $(e_2 - e_1) + U = e_2 + U, \tilde{T}(e_2 + U) = T(e_2) = e_2 - e_1$. Becs $e_2 = e_1 + (e_2 - e_1) \in U \oplus W$.

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17 *Supp V/U is finide. Supp W is finide and $V = U + W$. Show $\dim W \geqslant \dim V/U$.*

Solus: Let $Y \oplus (U \cap W) = W$. Then by [1.C Tips (4)], $V = U \oplus Y$. Note that V/U and Y are iso. □
Or. Let $B_W = (w_1, \dots, w_n)$. Then $V = U + \operatorname{span}(w_1, \dots, w_n)$.
$\forall v \in V, \exists\, u \in U$, $v = u + (a_1 w_1 + \cdots + a_n w_n) \Rightarrow v + U = (a_1 w_1 + \cdots + a_n w_n) + U$. □

Note: If $\dim W = \dim V/U$. Then $B_{V/U} = (w_1 + U, \dots, w_n + U)$. Supp $v = \sum_{i=1}^n a_i w_i \in U \cap W$
$\Rightarrow v + U = 0 = \sum_{i=1}^n a_i (w_i + U) \Rightarrow$ each $a_i = 0$. Thus $V = U \oplus W$.

---

12 *Supp U is a subsp of V. Prove is V is iso to $U \times (V/U)$.*

Solus:
[*Req V/U Finide*] Let $B_{V/U} = (v_1 + U, \dots, v_n + U)$.
Now $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^n a_i v_i + U \Rightarrow v - \sum_{i=1}^n a_i v_i \in U \Rightarrow \exists!\, u \in U, v = \sum_{i=1}^n a_i v_i + u$.
Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ and $\psi \in \mathcal{L}(U \times (V/U), V)$
by $\varphi(v) = \left(u, \sum_{i=1}^n a_i v_i + U\right)$, and $\psi(u, v + U) = \sum_{i=1}^n a_i v_i + u$. Then $\psi = \varphi^{-1}$. □

Or. Let $W \oplus U = V$. Define $Tv = u_v, Sv = w_v \Rightarrow \tilde{T} \in \mathcal{L}(V/W, U), \tilde{S} \in \mathcal{L}(V/U, W)$ are iso.
Define $\psi(u, v + U) = u + \tilde{S}(v + U) = u + w_v$. Define $\varphi(v) = (\tilde{T}(v), v + U)$.
$$\left.\begin{aligned}(\psi \circ \varphi)(u_v + w_v) &= \psi(u_v, w_v + U) = u_v + w_v \\ (\varphi \circ \psi)(u, v + U) &= \varphi(u + w_v) = (u, w_v + U)\end{aligned}\right\} \Rightarrow \psi = \varphi^{-1}.$$
Or Becs ψ or φ is inje and surj. □

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13 *Prove $B_{V/U} = (v_1 + U, \dots, v_m + U), B_U = (u_1, \dots, u_n) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$.*

Solus: $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^m a_i v_i + U \Rightarrow \exists!\, b_i \in \mathbf{F}, v - \sum_{i=1}^m a_i v_i = \sum_{i=1}^n b_i u_i \in U$
$\Rightarrow \forall v \in V, \exists!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^m a_i v_i + \sum_{j=1}^n b_j u_j$. □

Or. $\sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i = 0 \Rightarrow \sum_{i=1}^m a_i (v_i + U) = 0 \Rightarrow$ each $a_i = 0 \Rightarrow$ each $b_i = 0$. □

Or. Note that $B = (v_1, \dots, v_m)$ is liney indep, and $[\operatorname{span}(v_1, \dots, v_m) + U] \subseteq V$.
$v \in \operatorname{span} B \cap U \iff v + U = \sum_{i=1}^m a_i (v_i + U) = 0 + U \iff v = 0$. Hence $\operatorname{span} B \cap U = \{0\}$.
Becs $\dim[\operatorname{span}(v_1, \dots, v_m) \oplus U] = m + n = \dim V$. Now by (2.B.8). □

• (4E 14) *Supp* $V = U \oplus W,\ B_W = (w_1, \dots, w_m)$. *Prove* $B_{V/U} = \big(w_1 + U, \dots, w_m + U\big)$.

SOLUS: $\forall v \in V, \exists!\, u \in U, w \in W, v = u + w$. 又 $\exists!\, c_i \in \mathbf{F}, w = \sum_{i=1}^{m} c_i w_i \Rightarrow v = \sum_{i=1}^{m} c_i w_i + u$.
Hence $\forall v + U \in V/U, \exists!\, c_i \in \mathbf{F}, v + U = \sum_{i=1}^{m} c_i w_i + U$. □
OR. Becs $\pi|_W : W \to W/U$ is inv, and $V/U = W/U$. □

---

16 *Supp* $\dim V/U = 1$. *Prove* $\exists\, \varphi \in \mathcal{L}(V, \mathbf{F}), \operatorname{null} \varphi = U$.

SOLUS: Supp $V_0 \oplus U = V$. Then V_0 is iso to V/U. Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(av_0 + u) = a$. □
OR. Let $B_{V/U} = (w + U)$. Then $\forall v \in V, \exists!\, a \in \mathbf{F}, v + U = aw + U$.
Define $\varphi \in \mathcal{L}(V/U, \mathbf{F})$ by $\varphi(aw + U) = a$. Then $\operatorname{null}(\varphi \circ \pi) = U$. □

• *Supp* U, W *are subsps of* $\mathcal{V}$, *and* X, Y *are subsps of* $\mathcal{W}$.
Supp U, X *are iso,* W, Y *are iso. Prove or give a countexa*: U/W *and* X/Y *are iso.*

SOLUS: A countexa: Let $\mathcal{V} = \mathcal{W} = \mathbf{F}^2$. Let $U = X = Y = \operatorname{span}(e_1), W = \operatorname{span}(e_2)$.
Then $\dim U/W = \dim U - \dim(U \cap W) = 1 \neq 0 = \dim X - \dim(X \cap Y) = \dim X/Y$. □
OR. Let $\mathcal{V} = U = W = \mathbf{F}^\infty = X$, $Y = \big\{(0, x_1, x_2, \cdots)\big\}$. Then $U/W = \big\{0\big\}$, while $\dim X/Y = 1$. □

• **TIPS 2:** *Supp* U, W *are vecsps,* $I = U \cap W$. *Prove* $V = U + W \iff V/I = U/I \oplus W/I$.

SOLUS: (a) Supp $V = U + W$. Then $\forall v + I \in V/I, \exists\, (u_v, w_v) \in U \times W,\ v + I = (u_v + w_v) + I$.
Note that $U/I, W/I \subseteq V/I$. Thus $V/I = U/I + W/I$.
$\forall u + I = w + I \in (U/I) \cap (W/I), \underline{u - w \in I = U \cap W}$
$\underline{\Rightarrow \exists\, w' \in I, u = w + w' \in U \cap W} \Rightarrow u + I = 0 + I = w + I$. Thus $(U/I) \cap (W/I) = \big\{0\big\}$.
(b) Supp $V/I = U/I \oplus W/I$. Then $\forall v \in V, v + I = (u + I) + (w + I)$
$\Rightarrow v - u - w \in I = U \cap W \Rightarrow \exists\, x \in U \cap W, v = u + w + x \in U + W$. □

• Supp $T \in \mathcal{L}(V, W)$, and U, V are subsps of some vecsp, and X, W are subsps of some vecsp.
Define $T/^U_X : V/U \to W/X$ *by* $T/^U_X(v + U) = Tv + X$.
(a) *Prove* $T/^U_X$ *is well-defined* $\iff (\operatorname{range} T|_{U \cap V})/(X \cap W) = \{0\} \iff \operatorname{range} T|_{U \cap V}$ *is a subsp of* $X \cap W$.
Supp $T/^U_X$ is well-defined, and thus is liney. Define $\pi_U \in \mathcal{L}(V, V/U), \pi_X \in \mathcal{L}(W, W/X)$.
Then $T/^U_X \circ \pi_U = \pi_X \circ T$. Define $T/_X \in \mathcal{L}(V, W/X)$ by $T/_X(v) = Tv + X$.
(b) $\operatorname{range} T/^U_X = \operatorname{range}\big(T/^U_X \circ \pi_U\big) = \operatorname{range}(\pi_X \circ T) = (\operatorname{range} T)/X$.
(c) *Prove* $T/^U_X$ *is surj* $\iff W = \operatorname{range} T + X \cap W$.
(d) *Show* $\operatorname{null} T/^U_X = \big(\operatorname{null} T/_X\big)/U$. (e) $T/^U_X$ *is inje* $\iff \operatorname{null} T/_X \subseteq U$.

SOLUS: (a) For $v, w \in V$. If $v + U = w + U \iff v - w \in U \Rightarrow Tv - Tw \in X \cap W \iff Tv + X = Tw + X$.
Then $\forall u \in V \cap U, Tu \in X \Rightarrow \operatorname{range} T|_{U \cap V} \subseteq X \cap W$. Convly true as well.
(c) Supp $T/^U_X$ is surj. $\forall w \in W, w + X \in W/X \Rightarrow \exists\, v + U \in V/U,\ Tv + X = w + X$
$\Rightarrow w - Tv \in X \cap W \Rightarrow w \in \operatorname{range} T + X \cap W$. Hence $W \subseteq \operatorname{range} T + X \cap W$.
Convly, $W = \operatorname{range} T + X \cap W \Rightarrow (\operatorname{range} T)/X = (\operatorname{range} T + X \cap W)/X = W/X$.
(d) $v + U \in \operatorname{null} T/^U_X \iff Tv \in X \iff v \in \operatorname{null} T/_X \iff v + U \in \big(\operatorname{null} T/_X\big)/U$. □

• **COMMENT:** Supp $T \in \mathcal{L}(V)$. Define $T/U \in \mathcal{L}(V/U)$ by $T/U = T/^U_U$. Then
(a) T/U well-defined $\iff U \cap V$ invard T. (b) $\operatorname{range} T/U = \operatorname{range}(\pi \circ T) = (\operatorname{range} T)/U$.
(c) T/U surj $\iff V = \operatorname{range} T + U \cap V$. (d) $\operatorname{null} T/U = \big(\operatorname{null} T/_U\big)/U$. (e) T/U inje $\iff \operatorname{null} T/_U \subseteq U$.

• (5.A.33) *Supp* $T \in \mathcal{L}(V)$. *Prove* $T/\operatorname{range} T = 0$. By (b) or (d) above, immed.

Solus: $v + \operatorname{range} T \in V/\operatorname{range} T \Rightarrow v + \operatorname{range} T \in \operatorname{null}(T/\operatorname{range} T)$. Thus $T/\operatorname{range} T = 0$. □

• (5.A.34) *Supp* $T \in \mathcal{L}(V)$. *Prove* $T/\operatorname{null} T$ *is inje* $\Longleftrightarrow \operatorname{null} T \cap \operatorname{range} T = \{0\}$.

Solus: Notice that $(T/\operatorname{null} T)(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0 \Longleftrightarrow Tu \in \operatorname{null} T \cap \operatorname{range} T$.
Now $T/\operatorname{null} T$ is inje $\Longleftrightarrow u + \operatorname{null} T = 0 \Longleftrightarrow Tu = 0 \Longleftrightarrow \operatorname{null} T \cap \operatorname{range} T = \{0\}$. □

• **Tips 3:** *Supp* U, W *are subsps of* V *and* X *is a subsp of* $U \cap W$.
Prove U/W *and* $(U/X)/(W/X)$ *are iso.*

Solus: Let $U_X \oplus X = U, W_X \oplus X = W$. Becs $U/W = U_X/W$, and $U/X = U_X/X$.
Define $T \in \mathcal{L}\Big((U_X/X)/(W/X), U_X/W\Big)$ by $T\big((u_x + X) + W/X\big) = u_x + W$.
$\forall u_1, u_2 \in U_X$ suth $(u_1 + X) + W/X = (u_2 + X) + W/X \Rightarrow u_1 - u_2 + X \in W/X$
$\Rightarrow u_1 - u_2 \in X + W$ 又 $u_1, u_2 \in U_X \Rightarrow u_1 - u_2 \in W \Rightarrow u_1 + W = u_2 + W$. Now T is well-defined.
Inje: $\forall u_x \in U_X$ suth $u_x + W = 0 \Rightarrow u_x \in W_X \Rightarrow (u_x + X) \in W_X/X$.
Surj: $\forall u_x \in U_X, u_x + W = T\big((u_x + X) + W/X\big)$. Hence T is iso. □

Or. Define $S \in \mathcal{L}(U_X/X, U_X)$ by $S(u_x + X) = u_x$. Becs $\forall u_1 + X = u_2 + X \in U_X/X$,
$u_1 - u_2 \in X$ 又 $u_1, u_2 \in U_X \Rightarrow u_1 = u_2$. Now S well-defined, and $S/^{(W/X)}_{W} = T$ defined above.
Becs $\operatorname{range} S|_{W/X \cap U_X/X} \subseteq W$, and $U_X = \operatorname{range} S \Rightarrow U_X \subseteq \operatorname{range} S + W$. Well-defined. Surj.
For $u_x \in U_X$, $u_x + W = 0 \Longleftrightarrow u_x \in U_X \cap W \Longleftrightarrow u_x + X \in (U_X \cap W)/X = \operatorname{null} S/_W$. Inje. □

Ended

3·F

4 *Supp* U *is a subsp of* $V \neq U$. *Prove* $U^0 \neq \{0\}$.

Solus: Let $X \oplus U = V \Rightarrow X \neq \{0\}$. Supp $s \in X \backslash \{0\}$. Let $Y \oplus \operatorname{span}(s) = X$.
Define $\varphi \in V'$ by $\varphi(u + \lambda s + y) = \lambda$. Hence $\varphi \neq 0$ and $\varphi(u) = 0$ for all $u \in U$. □

Or. [*Req V Finide*] By [3.106], $\dim U^0 = \dim V - \dim U > 0$.
Or. Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1 \dots, v_n)$ with $n \geqslant 1$.
Let $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$. Then each $\varphi \in \operatorname{span}(\varphi_1, \dots, \varphi_n)$ will do. □

Coro: **19** $U^0 = \{0\} = V^0 \Longleftrightarrow U = V$. By the inv and ctrapos of Exe (4).

Comment: ***Another proof of*** **[3.108]**: T is surj $\Longleftrightarrow T'$ is inje.
(a) Supp T' is inje. Notice that $\psi \neq 0 \Longleftrightarrow T'(\psi) \neq 0 \Longleftrightarrow \psi \notin (\operatorname{range} T)^0$.
(b) T is surj $\Rightarrow (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T'$. □

• **Note For [3.102] and Exe (18):** For $U = \emptyset$, $U^0 = \{\varphi \in V' : U \subseteq \operatorname{null} \varphi\} = V'$. While $\{0\}^0_V = V'$.
Not a ctradic becs $\emptyset$ is not a subsp. Now $U^0 = V'$ can be true with $U = \emptyset \neq \{0\}$.

- **Note For Exe (1):** Every liney functional is either surj or is a zero map.
  Which means, for $\varphi \in V'$, $\varphi = 0 \iff \dim \operatorname{span}(\varphi) = 0 \iff \dim \operatorname{range} \varphi = 0$.
  And $\varphi \neq 0 \iff \dim \operatorname{span}(\varphi) = 1 \iff \dim \operatorname{range} \varphi = 1$. Thus $\dim \operatorname{span}(\varphi) = \dim \operatorname{range} \varphi$.

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25 *Supp U is a subsp of V. Explain why $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.*

Solus: Asum $\forall \varphi \in U^0, \varphi(v) = 0$ while $v \in V \backslash U$. Then let $\operatorname{span}(v) \oplus U \oplus X = V$.
$\exists\, \varphi \in V', \operatorname{null} \varphi = U \oplus X \Rightarrow \varphi \in U^0$. 又 $\varphi(v) = 0 \Rightarrow 0 \neq v \in \operatorname{null} \varphi \cap \operatorname{span}(v)$. Ctradic. $\square$

Comment: $X \subseteq W = \{v \in V : \varphi(v) = 0, \forall \varphi \in X^0\}$, the ***promotion*** of the subset X of V.

- *Supp U, W are subsps of V. Prove the promotion of $U \cup W$ is $U + W$.*

Solus: $(U \cup W)^0 = \{\varphi \in V' : \varphi(u) = \varphi(w) = \varphi(u + w) = 0, \forall u \in U, w \in W\} = (U + W)^0$. $\square$

- *Supp $X = \{x_1, \dots, x_m\} \subsetneq V$. Prove the promotion of X is $\operatorname{span}(x_1, \dots, x_m)$.*

Solus: $X^0 = \left\{\varphi \in V' : \varphi(\lambda x_j + \mu x_k) = 0, \forall j, k \in \{1, \dots, m\}, \lambda, \mu \in \mathbf{F}\right\} = \operatorname{span}(x_1, \dots, x_m)^0$. $\square$

Comment: The promotion of every finite subset X of V is the smallest subsp of V containing X.

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21 *Supp U, W are subsps of V. Prove $W^0 \subseteq U^0 \Rightarrow U \subseteq W$.*

Solus: Using Exe (25). Now $v \in U \Rightarrow \forall \varphi \in W^0 \subseteq U^0, \varphi(v) = 0 \Rightarrow v \in W$. $\square$

Note: $\varphi \in W^0 \iff \underline{\operatorname{null} \varphi \supseteq W \Rightarrow \operatorname{null} \varphi \supseteq U} \iff \varphi \in U^0$. But cannot conclude $W \supseteq U$.

Comment: (1) If U is merely a subset and W is a subsp. Promote U as X, let $W = Y$.
Then $Y^0 = W^0 \subseteq U^0 = X^0 \Rightarrow Y = W \supseteq X \supseteq U$. Still true.
(2) If W is merely a subset and U is a subsp. Promote W as Y, let $U = X$. For exa,
Let $W = \{(1,0), (0,1)\} \not\supseteq U = \{(x, 0) \in \mathbf{R}^2\}$. Then $Y = \mathbf{R}^2 \supseteq X = U,\ Y^0 = \{0\} \subseteq X^0$.

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22 *Supp U and W are subsps of V. Prove $(U + W)^0 = U^0 \cap W^0$.*

Solus: (a) $\varphi \in (U + W)^0 \Rightarrow \forall u \in U, w \in W,$ $\varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$. | $U \subseteq U + W \Rightarrow (U + W)^0 \subseteq U^0$; $W \subseteq U + W \Rightarrow (U + W)^0 \subseteq W^0$
(b) $\varphi \in U^0 \cap W^0 \subseteq V' \Rightarrow \forall u \in U, w \in W, \varphi(u + w) = 0 \Rightarrow \varphi \in (U + W)^0$. $\square$

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37 *Supp U is a subsp of V and π is the quot map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.*
(a) *Show π' is inje:* Becs π is surj. Use [3.108].
(b) *Show $\operatorname{range} \pi' = U^0$:* By [3.109](b), $\operatorname{range} \pi' = (\operatorname{null} \pi)^0 = U^0$.
(c) *Conclude that π' is iso from $(V/U)'$ onto U^0:* Immed.

Solus: (a) Or. $\pi'(\varphi) = 0 \iff \forall v \in V\ (\forall v + U \in V), \varphi(\pi(v)) = \varphi(v + U) = 0 \iff \varphi = 0$.
(b) Or. $\psi \in \operatorname{range} \pi' \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null} \psi \supseteq U \iff \psi \in U^0$. $\square$

- *Supp U is a subsp of V. Prove $(V/U)'$ is iso to U^0.* [***Another proof of* [3.106]**]

Solus: Define $\xi : U^0 \to (V/U)'$ by $\xi(\varphi) = \tilde{\varphi}$, where $\tilde{\varphi} \in (V/U)'$ is defined by $\tilde{\varphi}(v + U) = \varphi(v)$.
Inje: $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V\ (\forall v + U \in V/U), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$.
Surj: $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null}(\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$.
Or. Define $\nu : (V/U)' \to U^0$ by $\nu(\Phi) = \Phi \circ \pi$. Now $\nu \circ \xi = I_{U^0},\ \xi \circ \nu = I_{(V/U)'} \Rightarrow \xi = \nu^{-1}$. $\square$

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23 *Supp U and W are subsps of V. Prove $\big(U \cap W\big)^0 = U^0 + W^0$.*

SOLUS:

(a) $\varphi = \psi + \beta \in U^0 + W^0 \Rightarrow \forall v \in U \cap W,$
$\varphi(v) = (\psi + \beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0.$

OR. $U \cap W \subseteq U \Rightarrow (U \cap W)^0 \supseteq U^0$
$U \cap W \subseteq W \Rightarrow (U \cap W)^0 \supseteq W^0$

(b) $\big[$ *Only in Finide* $\big]$ By Exe (22), $\dim\big(U^0 + W^0\big) = \dim U^0 + \dim W^0 - \dim\big(U^0 \cap W^0\big)$
$= 2\dim V - \dim U - \dim W - \big(\dim V - \dim(U + W)\big) = \dim V - \dim(U \cap W).$ □

OR. Let $I = U \cap W$. We show $(U \cap W)^0 \subseteq U^0 + W^0$.
Define $\chi \in \mathcal{L}(V/I, V/U \times V/W)$ by $\chi : v + I \mapsto (v + U, v + W)$.
Well-defined: $v_1 + I = v_2 + I \in V/I \iff v_1 - v_2 \in I$
$\iff v_1 - v_2 \in U$ and $v_1 - v_2 \in W \Rightarrow (v_1 + U, v_1 + W) = (v_2 + U, v_2 + W)$.
Inje: $(v + U, v + W) = 0 \iff v \in U \cap W = I \iff v + I = 0$.
Surj: $\forall v \in V$ suth $(v + U, v + W) \in V/U \times V/W$, becs $\emptyset \neq (v + U) \cap (v + W) = v + I \in V/I$.
Thus $\chi' \in \mathcal{L}\big((V/U \times V/W)', (V/I)'\big)$ is iso. Now we find an iso of $U^0 \times W^0$ onto $(U \cap W)^0$.
By (3.E.4), supp $\xi : (V/U)' \times (V/W)' \to (V/U \times V/W)'$ is iso.
By (c) in Exe (37), supp $\Lambda_1 : U^0 \times W^0 \to (V/U)' \times (V/W)'$ and $\Lambda_2 : (V/I)' \to (U \cap W)^0$ are isos.
Hence $(\Lambda_2 \circ \chi' \circ \xi \circ \Lambda_1) : U^0 \times W^0 \to (U \cap W)^0$ is iso. Now we see how it works:
$\forall (\varphi_U, \varphi_W) \in U^0 \times W^0$, $\operatorname{null} \pi_U \subseteq \operatorname{null} \varphi_U \Rightarrow \exists\, \psi_U \in (V/U)'$, $\psi_U \circ \pi_U = \varphi_U$, simlr for φ_W,
thus $\Lambda_1 : (\varphi_U, \varphi_W) \mapsto (\psi_U, \psi_W)$. Then $\xi : (\psi_U, \psi_W) \mapsto (\psi_U S_U + \psi_W S_W)$, $\big[$ See notas in (3.E.2). $\big]$
Now $(\psi_U S_U + \psi_W S_W) \xmapsto{\chi'} (\psi_U S_U + \psi_W S_W) \circ \chi \xmapsto{\Lambda_2} (\psi_U S_U + \psi_W S_W) \circ \chi \circ \pi_I$,
which sends v to $\psi_U(v + U) + \psi_W(v + W) = (\varphi_U + \varphi_W)(v)$, which is $\varphi_U + \varphi_W$.
Thus $(\Lambda_2 \circ \chi' \circ \xi \circ \Lambda_1)$ is the surj $\Lambda : U^0 \times W^0 \to U^0 + W^0$ defined in [3.77]. □

COMMENT: Not true if U or W is merely a subset. Promote $U \cap W$ as I, U as X, and W as Y.
EXA: Let $U = \big\{(x, x+1) \in \mathbf{R}^2\big\}$, $W = \mathbf{R}^2$. Then $U \cap W = I = U \neq \mathbf{R}^2 = X \cap Y$.

• **CORO:** $V = U \oplus W \iff V' = U^0 \oplus W^0$.

• *Supp $V = U \oplus W$. Prove $U^0 = \big\{\varphi \in V' : \varphi = \varphi \circ \iota\big\}$, where $\iota \in \mathcal{L}(V, W) : u_v + w_v \to w_v$.*

SOLUS: $\varphi \in U^0 \iff U \subseteq \operatorname{null} \varphi \iff \varphi = \varphi \circ \iota$, by $\big[$3.B TIPS (3)$\big]$. □

TIPS 1: The nota $W'_V = \big\{\varphi \in V' : \varphi = \varphi \circ \iota\big\} = U^0$ is not well-defined $\big[$without a bss$\big]$.
Simply becs W'_V have no info about the given U. Here is an informal explanation:
Each liney map $T \in \mathcal{L}(V, W)$ that vanishes on a given nontrivial U has its 'P'
(though not uniq) suth '$U \oplus P = V$' with $T : P \mapsto \operatorname{range} T$ being surj.
Hence $\forall W \in \mathcal{S}_V U$, $U^0 = W'_V$. But given nontrivial 'P', the corres 'U' is not uniq.
Fix one W'_V, then U^0 is not uniq, with each U_k not equal to each other while each $U_k^0 = W'_V$.

EXA: Let $B_V = (e_1, e_2)$. Let $B_U = (e_1)$, $B_X = (e_2 - e_1)$, $B_Y = (e_2)$.
Then $\iota_X : ae_1 + b(e_2 - e_1) \mapsto b(e_2 - e_1)$, $\iota_Y : ae_1 + be_2 \mapsto be_2$. Now $X'_V = Y'_V = U^0$.
(1) For $V = U \oplus X$, let $B_{U'_V} = (\varphi)$ with $\varphi : e_1 \mapsto 1,\ e_2 - e_1 \mapsto 0 \Rightarrow e_2 \mapsto 1$.
(2) For $V = U \oplus Y$, let $B_{U'_V} = (\psi)$ with $\psi : e_1 \mapsto 1, e_2 \mapsto 0$.
Thus $X^0 = U'_V$ while $Y^0 = U'_V \Rightarrow X^0 = Y^0 \Rightarrow X = Y$, ctradic.
To fix this, we must have a bss of V' as precond, which we'll see in the NOTE FOR Exa (31).

NOTE: *Supp U is a subsp of V. Then finding the corres subsp in V' firstly req another 'half' $W \in \mathcal{S}_V U$, while finding the corres subsp of V for a subsp of V' must have the another 'half' asumed as precond.*

• *Supp* $V = U \oplus W$. *Define* $\iota : V \to U$ *by* $\iota(u+w) = u$. *Thus* $\iota' \in \mathcal{L}(U', V')$.

(a) *Show* $\operatorname{null}\iota' = \{0\}$: $\operatorname{null}\iota' = (\operatorname{range}\iota)^0_U = U^0_U = \{0\}$. OR. $\iota'(\psi) = \psi\circ\iota = 0 \iff U \subseteq \operatorname{null}\psi$.

(b) *Prove* $\operatorname{range}\iota' = W^0_V$: $\operatorname{range}\iota' = (\operatorname{null}\iota)^0_V = W^0_V$. Now $\tilde{\iota'}$ is iso from $U'/\{0\}$ onto W^0

SOLUS: (b) OR. Note that $W = \operatorname{null}\iota \subseteq \operatorname{null}(\psi\circ\iota)$. Then $\psi\circ\iota \in W^0 \Rightarrow \operatorname{range}\iota' \in W^0$.

Supp $\varphi \in W^0$. Becs $\operatorname{null}\iota = W \subseteq \operatorname{null}\varphi$. By [3.B TIPS (3)], $\varphi = \varphi\circ\iota = \iota'(\varphi)$. □

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31 *Supp* V *is finide and* $B_{V'} = (\varphi_1, \dots, \varphi_n)$. *Show* $\exists!\, B_V$ *whose dual bss is the* $B_{V'}$.

SOLUS: For each $k \in \{1, \dots, n\}$, let $\Gamma_k = \{1, \dots, n\}\setminus\{k\}$. Let each $U_k = \bigcap_{j\in\Gamma_k} \operatorname{null}\varphi_j$.

By Exe (4E 23), $V' = \operatorname{span}(\varphi_1, \dots, \varphi_n) = (\operatorname{null}\varphi_1 \cap \dots \cap \operatorname{null}\varphi_n)^0 \Rightarrow U_k \cap \varphi_k = \{0\}$.

Thus $\forall x_k \in U_k\setminus\{0\}$, $x_k \notin \operatorname{null}\varphi_k$ while $x_k \in \operatorname{null}\varphi_j$ for all $j \in \Gamma_k$.

Fix one x_k and let $v_k = [\varphi_k(x_k)]^{-1}x_k \Rightarrow \varphi_k(v_k) = 1$, $\varphi_j(v_k) = 0$ for all $j \neq k$.

Simply for each v_k, $\varphi_j(v_k) = \delta_{j,k}$ for all $j \iff$ for each φ_j, $\varphi_j(v_k) = \delta_{j,k}$ for all k.

又 $a_1v_1 + \dots + a_nv_n = 0 \Rightarrow$ each $\varphi_k(0) = a_k$.

Now we prove the uniqnes part. Supp the dual bss of $B'_V = (u_1, \dots, u_n)$ is the $B_{V'}$.

For each k, we have $\varphi_j(v_k) = \varphi_j(u_k)$ for all $k \Rightarrow v_k - u_k \in \bigcap \operatorname{null}\varphi_j = \{0\}$. □

• **NOTE FOR Exe (31):** Supp V is finide, and Ω is a subsp of V' with $B_\Omega = (\varphi_1, \dots, \varphi_m)$.

The 'W' is not clear when we are to find one suth $W'_V = \Omega$, becs the another 'half' is undefined.

Extend to $B_{V'} = (\varphi_1, \dots, \varphi_n)$. By Exe (31), $\exists!$ corres $B_V = (v_1, \dots, v_n)$. Let $B_U = (v_{m+1}, \dots, v_n)$.

Let $B_W = (v_1, \dots, v_m)$. Thus $W'_V = \Omega$. Now W is well-defined with B_V as precond.

• **TIPS 2:** Supp $\varphi_1, \dots, \varphi_m \in V'$. Denote $[\operatorname{null}\varphi_a \cap \dots \cap \operatorname{null}\varphi_b]$ by $\bigcap_a^b \operatorname{null}\varphi_I$.

Supp Ω is a subsp of V'. Denote $\{v \in V : \varphi(v) = 0, \forall\varphi \in \Omega\}$ by $C^0\,\Omega$.

If Ω is infinide, then by def, $\bigcap_{\in\Omega} \operatorname{null}\varphi = C^0\,\Omega$. If $\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$,

then $v \in \bigcap_1^m \operatorname{null}\varphi_I \iff$ each $\varphi_k(v) = 0 \iff \forall\varphi = \sum_{i=1}^n a_i\varphi_i \in \Omega, \varphi(v) = 0 \iff v \in C^0\,\Omega$.

• (4E 23) *Supp* V *is finide,* $\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m) \subseteq V'$. *Prove* $\Omega = (\operatorname{null}\varphi_1 \cap \dots \cap \operatorname{null}\varphi_m)^0$.

SOLUS: Becs each $\operatorname{span}(\varphi_k) \subseteq (\operatorname{null}\varphi_k)^0$. By NOTE FOR Exe (1) and Exe (23), Immed. □

OR. Reduce to $B_\Omega = (\beta_1, \dots, \beta_p)$. We show $\Omega = (\operatorname{null}\beta_1 \cap \dots \cap \operatorname{null}\beta_p)^0$, then done by TIPS (3).

Let $B_{V'} = (\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_q)$. By Exe (31), let $B_V = (v_1, \dots, v_p, u_1, \dots, u_q)$.

Define each $\Gamma_k = \{1, \dots, p\}\setminus\{k\}$. Then $\operatorname{null}\beta_k = \operatorname{span}\{v_j\}_{j\in\Gamma_k} \oplus \operatorname{span}(u_1, \dots, u_q)$.

Now $(\operatorname{null}\beta_1 \cap \dots \cap \operatorname{null}\beta_p) = \operatorname{span}(u_1, \dots, u_q)$. Simlr to (4E 2.C.16).

Supp $\varphi = \sum_{i=1}^p a_i\beta_i + \sum_{j=1}^q b_j\gamma_j \in \operatorname{span}(u_1, \dots, u_q)^0$. Then each $\varphi(u_k) = 0 = b_k$

Thus $\operatorname{span}(u_1, \dots, u_q)^0 \subseteq \operatorname{span}(\beta_1, \dots, \beta_p) = \Omega$. □

• **TIPS 3:** *Supp each* $\varphi_i, \beta_j \in \mathcal{L}(V, W)$. *Supp* $\operatorname{span}(\varphi_1, \dots, \varphi_m) = \operatorname{span}(\beta_1, \dots, \beta_n)$.
Prove $\operatorname{null}\varphi_1 \cap \dots \cap \operatorname{null}\varphi_m = \operatorname{null}\beta_1 \cap \dots \cap \operatorname{null}\beta_n$.

SOLUS: Becs each $\beta_k \in \operatorname{span}(\varphi_1, \dots, \varphi_m)$.

$\forall v \in \bigcap_1^m \operatorname{null}\varphi_I, \beta_k(v) = 0$. Thus $\bigcap_1^m \operatorname{null}\varphi_I \subseteq \bigcap_1^n \operatorname{null}\beta_I$. Rev the roles and done. □

Supp $(\varphi_1, \dots, \varphi_j)$ is a bss of $\operatorname{span}(\varphi_1, \dots, \varphi_m)$. Let $N_k \oplus \bigcap_1^j \operatorname{null}\varphi_I = \operatorname{null}\varphi_k$.

Now $\bigcap_1^j \operatorname{null}\varphi_I \cap (\operatorname{null}\varphi_k) = \bigcap_1^j \operatorname{null}\varphi_I$. Thus $\bigcap_1^m \operatorname{null}\varphi_I = \bigcap_1^j \operatorname{null}\varphi_I$.

又 $\beta_k \in \operatorname{span}(\varphi_1, \dots, \varphi_j)$. Let $M_k \oplus \bigcap_1^j \operatorname{null}\varphi_I = \operatorname{null}\beta_k$. Simlr, $\bigcap_1^n \operatorname{null}\beta_I = \bigcap_1^j \operatorname{null}\varphi_I$. □

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26 Supp V is finide, Ω is a subsp of V'. Then get a B_Ω and by TIPS (2) and Exe (4E 23), $\Omega = \left(C^0\,\Omega\right)^0$.

EXA: Immed, $\Omega \subseteq (C^0\,\Omega)^0$. Now we give a countexa for $\Omega \supseteq (C^0\,\Omega)^0$.

Let $V = \left\{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0 \text{ for only finily many } k\right\}$. Then $V' = (\mathbf{F}^\infty)'$.

Let $\Omega = \left\{\varphi \in \operatorname{span}(\varphi_{\alpha_1}, \dots, \varphi_{\alpha_m}) : \exists\, m, \alpha_k \in \mathbf{N}^+\right\} \subsetneq V'$. Then $C^0\,\Omega = \{0\} \Rightarrow (C^0\,\Omega)^0 = V'$.

CORO: Supp V is finide. For every subsp Ω of V', $\exists\,!$ subsp U of V suth $\Omega = U^0$.

• *Supp* $\operatorname{span}(\varphi_1, \dots, \varphi_m) \subseteq V'$. *Let each* $U_k \oplus \operatorname{null}\varphi_k = V$.
Prove or give a countexa: $(U_1 + \cdots + U_m) \oplus \left(\operatorname{null}\varphi_1 \cap \cdots \cap \operatorname{null}\varphi_m\right) = V$.

SOLUS: Let $V = \mathbf{R}^2$. Define $\varphi_1 = \varphi_2 : (x, y) \mapsto x$. Let $B_{U_1} = (e_1), B_{U_2} = (e_1 + e_2) \Rightarrow U_1 + U_2 = V$.
OR. Let $B_{V'} = (\varphi_1, \varphi_2)$ be corres to the std bss. Let $B_{U_1} = B_{U_2} = (e_1 + e_2) \Rightarrow U_1 + U_2 \subsetneq V$. ☐

• **TIPS 4:** Let $B_{U^0} = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_n) \Rightarrow B_V = (v_1, \dots, v_n)$.
We show (a) $B_U = (v_{m+1}, \dots, v_n)$; (b) $U = \operatorname{null}\varphi_1 \cap \cdots \cap \operatorname{null}\varphi_m$.
(a) Becs $\operatorname{span}(v_{m+1}, \dots, v_n)^0 = \operatorname{span}(\varphi_1, \dots, \varphi_m) = U^0$. Now by Exe (20, 21).
OR. Becs by (b), $U = \bigcap_1^m \operatorname{null}\varphi_I = \operatorname{span}(v_{m+1}, \dots, v_n)$.
(b) Each $\operatorname{null}\varphi_k = \operatorname{span}\{B_V \backslash \{v_k\}\} \Rightarrow \bigcap_1^m \operatorname{null}\varphi_I = \operatorname{span}(v_{m+1}, \dots, v_n)$. Now by (a).
OR. Becs $\operatorname{span}(\varphi_1, \dots, \varphi_m) = U^0 = \left(\operatorname{null}\varphi_1 \cap \cdots \cap \operatorname{null}\varphi_m\right)^0$. Now by Exe (20, 21). ☐

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24 *Prove, using the pattern of* [3.104], *that* $\dim U + \dim U^0 = \dim V$.

SOLUS: By TIPS (4). OR. Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$.
Supp $\psi = \sum_{i=1}^m a_i \psi_i + \sum_{j=1}^n b_j \varphi_j \in U^0 \Rightarrow$ each $\psi(u_k) = a_k = 0$. Thus $U^0 \subseteq \operatorname{span}(\varphi_1, \dots, \varphi_n)$. ☐

• *Supp* $T \in \mathcal{L}(V, W)$, *each* $\varphi_k \in V'$, *and each* $\psi_k \in W'$.

28 $\operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) \iff \operatorname{range} T = (\operatorname{null}\psi_1) \cap \cdots \cap (\operatorname{null}\psi_m)$.

29 $\operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \iff \operatorname{null} T = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.

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34 *Define* $\Lambda : V \to \mathbf{F}^{V'}$ *by* $\Lambda v = \overline{v}$, *and* $\overline{v} : V' \to \mathbf{F}$ *by* $\overline{v}(\varphi) = \varphi(v)$.
(a) *Show* $\overline{v} \in V''$ *and* $\Lambda \in \mathcal{L}(V, V'')$.
(b) *Show if* $T \in \mathcal{L}(V)$, *then* $T'' \circ \Lambda = \Lambda \circ T$, *where* $T'' = (T')'$.
(c) *Show if* V *is finide, then* Λ ***is iso from*** V ***onto*** V''.

SOLUS: (a) $\overline{v}(\varphi + \lambda\psi) = (\varphi + \lambda\psi)(v) = \varphi(v) + \lambda\psi(v) = \overline{v}(\varphi) + \lambda\overline{v}(\psi)$.
$\overline{v + \lambda w}(\varphi) = \varphi(v + \lambda w) = \varphi(v) + \lambda\varphi(w) = \overline{v}(\varphi) + \lambda\overline{w}(\varphi)$.
(b) $\left(T''\overline{v}\right)(\varphi) = \left(\overline{v} \circ T'\right)(\varphi) = \overline{v}\left(T'(\varphi)\right) = \left(T'(\varphi)\right)(v) = (\varphi \circ T)(v) = \varphi(Tv) = \overline{Tv}(\varphi)$.
(c) $\overline{v} = 0 \Rightarrow \forall \varphi \in V', \overline{v}(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Inje. Now becs V finide. ☐

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36 *Supp* U *is a subsp of* V. *Define* $i : U \to V$ *by* $i(u) = u$. *Thus* $i' \in \mathcal{L}(V', U')$.
(a) *Show* $\operatorname{null} i' = U^0$: $\operatorname{null} i' = (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$.
(b) *Prove* $\operatorname{range} i' = U'$: $\operatorname{range} i' = (\operatorname{null} i)^0_U = \{0\}^0_U = U'$.
(c) *Prove* $\widetilde{i'}$ *is iso from* V'/U^0 *onto* U': Immed.

SOLUS: (a) OR. $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$. Thus $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$.
(b) OR. Supp $\psi \in U'$. By (3.A.11), $\exists\, \varphi \in V', \varphi|_U = \psi$. Then $i'(\varphi) = \psi$. ☐

• *Supp* $T \in \mathcal{L}(V, W)$. *Prove* $\operatorname{range} T' \supseteq (\operatorname{null} T)^0$. $\Big[$ ***Another proof of*** **[3.109](b)** $\Big]$

Solus: Let $V = U \oplus \operatorname{null} T$. Let $R = (T|_U)^{-1}|_{\operatorname{range} T}$. Define $\iota \in \mathcal{L}(V, U)$ by $\iota(u + w) = u$.

$\forall \Phi \in (\operatorname{null} T)^0$, let $\psi = \Phi \circ R$, then $T'(\psi) = \psi \circ T = \Phi \circ (R \circ T|_V) = \Phi \circ \iota = \Phi \in \operatorname{range} T'$. $\square$

Coro: [3.108] and [3.110] hold without the hypo of finide. Now T inv $\Longleftrightarrow T'$ inv.

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15 *Supp* $T \in \mathcal{L}(V, W)$. *Prove* $T' = 0 \Rightarrow T = 0$. **Coro:** If V, W finide, then $\Gamma : T \mapsto T'$ is iso.

Solus: Supp $T' = 0$. Then $\operatorname{null} T' = \{0\} = (\operatorname{range} T)^0$. $\square$

Or. By Exe (25), $\operatorname{range} T = \{w \in W : \varphi(w) = 0, \forall \varphi \in (\operatorname{range} T)^0 = \operatorname{null} T' = W'\} = \{0\}$. $\square$

• **Note For Exe (16):**

Let $B_V = (v_1, \dots, v_n), B_{V'} = (\varphi_1, \dots, \varphi_n), B_W = (w_1, \dots, w_m), B_{W'} = (\psi_1, \dots, \psi_m)$.

Define each $E_{j,k} \in \mathcal{L}(V, W) : v_x \mapsto \delta_{j,x} w_k$, and each $\exists_{k,j} \in \mathcal{L}(W', V') : \psi_x \mapsto \delta_{k,x} \varphi_j$.

Note that each $E'_{j,k}(\psi_x) = \psi_x \circ E_{j,k} = \delta_{k,x}\varphi_j = \exists_{k,j}(\psi_x) \Rightarrow E'_{j,k} = \exists_{k,j}$.

$\mathcal{L}(V, W) \ni T = \sum_{j=1}^{n} \sum_{k=1}^{m} A_{k,j} E_{j,k} \Longleftrightarrow \mathcal{T} = \sum_{j=1}^{n} \sum_{k=1}^{m} A_{k,j} \exists_{k,j} \in \mathcal{L}(W', V')$. Uniqly by Exe (16).

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6 *Define* $\Gamma : V' \to \mathbf{F}^m$ *by* $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$, *where* $v_1, \dots, v_m \in V$.

Show (a) $\operatorname{span}(v_1, \dots, v_m) = V \Longleftrightarrow \Gamma$ *inje.* (b) $(v_1, \dots, v_m)$ *liney indep* $\Longleftrightarrow \Gamma$ *surj.*

Solus: Let $(e_1, \dots, e_m)$ be the std bss of $\mathbf{F}^m$.

(a) Becs $\Gamma(\varphi) = 0 \Longleftrightarrow \varphi(v_1) = \cdots = \varphi(v_m) = 0 \Longleftrightarrow \operatorname{null} \varphi = \operatorname{span}(v_1, \dots, v_m)$. Immed.

(b) Supp Γ is surj. Let each $e_k = \Gamma(\varphi_k) \Rightarrow \varphi_k(v_j) = \delta_{j,k}$. Now $a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow$ each $a_k = \varphi_k(0)$.

Supp $(v_1, \dots, v_m)$ is liney indep. Let $U = \operatorname{span}(v_1, \dots, v_m), B_{U'} = (\psi_1, \dots, \psi_m)$. Let $W \oplus U = V$.

Define $\iota : u_v + w_v \mapsto u_v$. Each $\psi_k \circ \iota = \varphi_k \in V' \Rightarrow \varphi_k(v_j) = \psi_k(v_j) = \delta_{j,k} \Rightarrow$ each $e_k = \Gamma(\varphi_k)$. $\square$

Or. Let $(\psi_1, \dots, \psi_m)$ be dual bss of the std bss of $\mathbf{F}^m$. Define an iso $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$.

Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $Te_k = v_k$. Now $T(x_1, \dots, x_m) = x_1 v_1 + \cdots + x_m v_m$.

$\forall \varphi \in V', k \in \{1, \dots, m\}, [T'(\varphi)](e_k) = \varphi(Te_k) = \varphi(v_k) = [\varphi(v_1)\psi_1 + \cdots + \varphi(v_m)\psi_m](e_k)$

Now $T'(\varphi) = \varphi(v_1)\psi_1 + \cdots + \varphi(v_m)\psi_m = \Psi(\Gamma(\varphi))$. Hence $T' = \Psi \circ \Gamma$.

By (3.B.3), (a) $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V \Longleftrightarrow T'$ inje $\Longleftrightarrow \Gamma$ inje.

(b) $(v_1, \dots, v_m)$ is liney indep $\Longleftrightarrow T$ is inje $\Longleftrightarrow T'$ surj $\Longleftrightarrow \Gamma$ surj. $\square$

• (4E 25) *Define* $\Gamma : V \to \mathbf{F}^m$ *by* $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$, *where* $\varphi_1, \dots, \varphi_m \in V'$.

Show (a) $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \Longleftrightarrow \Gamma$ *inje.* (b) $(\varphi_1, \dots, \varphi_m)$ *liney indep* $\Longleftrightarrow \Gamma$ *surj.*

Solus: Let $(e_1, \dots, e_m)$ be the std bss of $\mathbf{F}^m$.

(c) Becs $\Gamma(v) = 0 \Longleftrightarrow \varphi_1(v) = \cdots = \varphi_m(v) = 0 \Longleftrightarrow v \in (\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)$.

By Exe (4E 23), $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \Longleftrightarrow \operatorname{null} \Gamma = (\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m) = \{0\}$.

(d) Supp $(\varphi_1, \dots, \varphi_m)$ is liney indep. $\big[$ *Req Finide* $\big]$ Extend to $B_{V'} = (\varphi_1, \dots, \varphi_n)$.

Then by Exe (31), $B_V = (v_1, \dots, v_n)$ and each $\varphi_k(v_j) = \delta_{j,k} \Rightarrow$ each $e_k = \Gamma(\varphi_k)$.

Convly, let each v_k be suth $e_k = \Gamma(v_k) = (\varphi_1(v_k), \dots, \varphi_m(v_k))$. If $a_1\varphi_1 + \cdots + a_m\varphi_m = 0$. Immed.

Or. Let $U = \operatorname{span}(v_1, \dots, v_m)$. Then $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U) \Rightarrow (\varphi_1, \dots, \varphi_m)$ liney indep. $\square$

Or. Let $(\psi_1, \dots, \psi_m)$ be dual bss of the std bss of $\mathbf{F}^m$. Define an iso $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$.

$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'(\Psi(x_1, \dots, x_m)) = x_1\varphi_1 + \cdots + x_m\varphi_m$. Define $\Phi = \Gamma' \circ \Psi$. Thus by (3.B.3),

(c) Γ inje $\Longleftrightarrow \Gamma'$ surj $\Longleftrightarrow \Phi$ surj $\Longleftrightarrow (\varphi_1, \dots, \varphi_m)$ spanning V'.

(d) Γ surj $\Longleftrightarrow \Gamma'$ inje $\Longleftrightarrow \Phi$ inje $\Longleftrightarrow (\varphi_1, \dots, \varphi_m)$ being liney indep. $\square$

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9 *Show* $\forall \psi \in V', \psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$, *where* $B_V = (v_1, \dots, v_n), B_{V'} = (\varphi_1, \cdots, \varphi_n)$.

SOLUS: $\psi(v) = a_1\psi(v_1) + \cdots + a_n\psi(v_n) = \psi(v_1)\varphi_1(v) + \cdots + \psi(v_n)\varphi_n(v)$. ☐

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13 *Define* $T : \mathbf{R}^3 \to \mathbf{R}^2$ *by* $T(x,y,z) = (4x+5y+6z, 7x+8y+9z)$.

Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ *denote the dual bss of std bss of* $\mathbf{R}^2$ *and* $\mathbf{R}^3$.

(a) *Describe the liney functionals* $T'(\varphi_1), T'(\varphi_2)$.

For any $(x,y,z) \in \mathbf{R}^3$, $\big(T'(\varphi_1)\big)(x,y,z) = 4x+5y+6z$, $\big(T'(\varphi_2)\big)(x,y,z) = 7x+8y+9z$.

(b) *Write* $T'(\varphi_1)$ *and* $T'(\varphi_2)$ *as liney combinas of* ψ_1, ψ_2, ψ_3.

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$.

(c) *What is* $\operatorname{null} T'$? *What is* $\operatorname{range} T'$?

$$T(x,y,z) = 0 \iff \begin{cases} 4x+5y+6z = 0 \\ 7x+8y+9z = 0 \end{cases} \iff \begin{cases} x = z, \\ y = -2z. \end{cases} \left| \begin{array}{l} \text{Thus } \operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3), \\ \text{where } (e_1, e_2, e_3) \text{ is std bss of } \mathbf{R}^3. \end{array} \right.$$

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a bss, with corres dual bss $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Thus $\operatorname{span}(e_1 - 2e_2 + e_3) = \operatorname{null} T \Rightarrow \operatorname{span}(e_1 - 2e_2 + e_3)^0 = \operatorname{span}(\varepsilon_2, \varepsilon_3) = \operatorname{range} T'$.

Note that $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$.

$$\text{And} \left| \begin{array}{l} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{array} \right.$$

Hence $\varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2$, $\varepsilon_3 = -\psi_1 + \psi_3$. Now $\operatorname{range} T' = \operatorname{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3)$.

OR. $\operatorname{range} T' = \operatorname{span}\big(T'(\varphi_1), T'(\varphi_2)\big) = \operatorname{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3)$.

Supp $T'(x\varphi_1 + y\varphi_2) = (4x+7y)\psi_1 + (5x+8y)\psi_2 + (6x+9y)\psi_3 = 0$.

Then $x + y = 4x + 7y = x = y = 0$. Hence $\operatorname{null} T' = \{0\}$.

OR. $\operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \operatorname{span}(-2e_2, e_3) \oplus \operatorname{null} T$.

$\Rightarrow \operatorname{range} T = \{Tx : x \in \operatorname{span}(-2e_2, e_3)\} = \operatorname{span}\big(T(-2e_2), T(e_3)\big)$

$= \operatorname{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \operatorname{span}(f_1, f_2) = \mathbf{R}^2$. Now $\operatorname{null} T' = (\operatorname{range} T)^0 = \{0\}$.

OR. For any $A, B \in \mathbf{R}$, asum (x,y,z) is suth $A = 4x+5y+6z$, $B = 7x+8y+9z$.

By computing $x = z + 4/3(b-a)$, $y = -2z + (7a - 4b)/3$, $z = z$. *An exa for (4E 3.E.8).*

Hence (x,y,z) exis $\Rightarrow (A,B) \in \operatorname{range} T$. Now T surj $\Rightarrow T'$ inje. ☐

ENDED

Exes about Sequences and Number Theory before Chapter 4

• (2.A.16) *Prove the vecsp U of all continuous functions in $\mathbf{R}^{[0,1]}$ is infinide.*

Solus: By $\big[$3.A Note For $\mathbf{F}^S\big]$, immed. $\square$

Or. Choose $m \in \mathbf{N}^+$. Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m = 0 \in \mathbf{R}^{[0,1]}$.

Then p has infily many roots and hence each $a_k = 0$, othws $\deg p \geqslant 0$, ctradic [4.12].

Thus $(1, x, \dots, x^m)$ is liney indep in $\mathbf{R}^{[0,1]}$. Simlr to [2.16], U is infinide. $\square$

Or. Note that $\dfrac{1}{1} > \dfrac{1}{2} > \cdots > \dfrac{1}{m}$, $\forall m \in \mathbf{N}^+$. Supp $f_m = \begin{cases} x - \frac{1}{m}, & x \in \left(\frac{1}{m}, 1\right] \\ 0, & x \in \left[0, \frac{1}{m}\right] \end{cases}$

Then $f_1\left(\frac{1}{m}\right) = \cdots = f_m\left(\frac{1}{m}\right) = 0 \neq f_{m+1}\left(\frac{1}{m}\right)$. Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. By (2.A.14). $\square$

• (3.F.35) *Prove $\big(\mathcal{P}(\mathbf{F})\big)'$ is iso to $\mathbf{F}^\infty$.*

Solus: Define $\theta \in \mathcal{L}\big[\big(\mathcal{P}(\mathbf{F})\big)', \mathbf{F}^\infty\big]$ by $\theta(\varphi) = \big(\varphi(1), \varphi(z), \cdots, \varphi(z^m), \cdots\big)$.

Notice that $\forall p \in \mathcal{P}(\mathbf{R}), \exists!\, c_i \in \mathbf{F}, m = \deg p,\ p(z) = c_0 + c_1 z + \cdots + c_m z^m \in \mathcal{P}_m(\mathbf{F})$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall p \in \mathcal{P}(\mathbf{F}), \varphi(p) = c_0\varphi(1) + c_1\varphi(z) + \cdots + c_m\varphi(z^m) = 0$.

Surj: Supp $x = (x_0, x_1, \cdots) \in \mathbf{F}^\infty$. Define $\psi_x(p) = x_0 c_0 + \cdots + x_m c_m \Rightarrow$ each $\psi_x(z^k) = x_k$.

$\forall p, q \in \mathcal{P}(\mathbf{F})$, supp $\deg p = m \geqslant n = \deg q$, $\big[$*which is why we do not write* $(p + \lambda q)$.$\big]$

$\psi_x(\lambda p + \mu q) = \sum_{j=0}^{n} x_j(\lambda a_j + \mu b_j) + \sum_{k=1}^{m-n} x_{n+k}\lambda a_{n+k} = \lambda\psi_x(p) + \mu\psi_x(q)$. $\square$

Comment: $\mathcal{P}(\mathbf{F}), \mathbf{F}^\infty$ not iso $\Longrightarrow \mathcal{P}(\mathbf{F}), \big(\mathcal{P}(\mathbf{F})\big)'$ not iso. But $\mathcal{P}(\mathbf{F})$ is iso to $\mathbf{F}^{\mathbf{N}}$, *see the 'U' in (3.E.14).*

• (3.E.14) *Supp $U = \big\{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0$ for only finily many $k\big\}$.* *Denote it by* $\mathbf{F}^{\mathbf{N}}$.

(a) *Show U is a subsp of $\mathbf{F}^\infty$.* $\big[$*Do it in your mind*$\big]$ (b) *Prove $\mathbf{F}^\infty / U$ is infinide.*

Solus: *For ease of nota, denote the p^{th} term of $u = (x_1, \cdots, x_p, \cdots) \in \mathbf{F}^\infty$ by $u[p]$.*

For each $r \in \mathbf{N}^+$, let $e_r[k] = \begin{cases} 1, & (k-1) \equiv 0 \pmod{r} \\ 0, & \text{othws} \end{cases}$ simply $e_r = \big(1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \cdots\big)$.

For $m \in \mathbf{N}^+$. Let $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 + U \Rightarrow \exists\, u \in U, a_1 e_1 + \cdots + a_m e_m = u$.

Supp $u = (x_1, \cdots, x_L, 0, \cdots)$, where L is the largest suth $u[L] \neq 0$.

Let $s \in \mathbf{N}^+$ be suth $h = s \cdot m! + 1 > L$, and $e_1[h] = \cdots = e_m[h] = 1$.

Notice that for any $p, r \in \{1, \dots, m\}$, $e_r[s \cdot m! + 1 + p] = e_r[p+1] = 1 \Longleftrightarrow p \equiv 0 \pmod{r} \Longleftrightarrow r \,|\, p$.

Let $1 = p_1 \leqslant \cdots \leqslant p_{\tau(p)} = p$ be the disti factors of p. Moreover, $r \,|\, p \Longleftrightarrow r = p_k$ for some k.

Now $u[h+p] = 0 = \sum_{r=1}^{m} a_r e_r[p+1] = \sum_{k=1}^{\tau(p)} a_{p_k}$.

Let $q = p_{\tau(p)-1}$. Then $\tau(q) = \tau(p) - 1$, and each $q_k = p_k$. Again, $\sum_{r=1}^{m} a_r e_r[h+q] = 0 = \sum_{k=1}^{\tau(p)-1} a_{p_k}$.

Thus $a_{p_{\tau(p)}} = a_p = 0$ for all $p \in \{1, \dots, m\} \Rightarrow (e_1, \dots, e_m)$ is liney indep in $\mathbf{F}^\infty$. $\square$

Or. For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, \text{if } 2^r \,|\, p \\ 0, \text{othws} \end{cases}$ Simlr, let $m \in \mathbf{N}^+$ and $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0$ $\Rightarrow a_1 e_1 + \cdots + a_m e_m = u \in U$.

Supp L is the largest suth $u[L] \neq 0$. And l is suth $2^{ml} > L$. Then for each $k \in \{1, \dots, m\}$,

$u[2^{ml} + 2^k] = 0 = \sum_{r=1}^{m} a_r e_r[2^k] = a_1 + \cdots + a_k$. Thus each $a_k = 0$. Simlr. $\square$

Ended

Exes about Polys before Chapter 4

- (1.C.9) *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if $\exists\, p \in \mathbf{N}^+$, $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions $\mathbf{R} \to \mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$? Explain.*

Solus: Denote the set by S.

Supp $h(x) = \cos x + \sin\sqrt{2}x \in S$, since $\cos x, \sin\sqrt{2}x \in S$.

Asum $\exists\, p \in \mathbf{N}^+$ suth $h(x) = h(x+p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$

$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Ctradic! $\square$

Or. Becs $\cos x + \sin\sqrt{2}x = \cos(x+p) + \sin\big(\sqrt{2}x + \sqrt{2}p\big)$. By diff twice,

$$\cos x + 2\sin\sqrt{2}x = \cos(x+p) + 2\sin\big(\sqrt{2}x + \sqrt{2}p\big).$$

$$\left.\begin{aligned} \sin\sqrt{2}x &= \sin\big(\sqrt{2}x + \sqrt{2}p\big) \\ \cos x &= \cos(x+p) \end{aligned}\right\} \Rightarrow \text{Let } x = 0,\ p = \frac{m\pi}{\sqrt{2}} = 2k\pi.\ \text{Ctradic.}$$ $\square$

- (1.C.24) *Let $V_E = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is even}\}$, $V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is odd}\}$. Show $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$.*

Solus: (a) $V_E \cap V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x)\} = \{0\}$.

(b) $$\left|\begin{aligned} &\text{Let } f_e(x) = \frac{1}{2}\big[g(x) + g(-x)\big] \Longrightarrow f_e \in V_E \\ &\text{Let } f_o(x) = \frac{1}{2}\big[g(x) - g(-x)\big] \Longrightarrow f_o \in V_O \end{aligned}\right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}},\ g(x) = f_e(x) + f_o(x).$$ $\square$

- (2.C.7) (a) *Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a bss of U.*
  (b) *Extend the bss in (a) to a bss of $\mathcal{P}_4(\mathbf{F})$, and find a W suth $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.*

Solus: Using (2.C.10).

Notice that $\nexists\, p \in \mathcal{P}(\mathbf{F})$ of deg 1 and 2, while $p \in U$. Thus $\dim U \leqslant \dim \mathcal{P}_4(\mathbf{F}) - 2 = 3$.

(a) Consider $B = \big(1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

Let $a_0 + a_3(z-2)(z-5)(z-6) + a_4 z(z-2)(z-5)(z-6) = 0 \Rightarrow a_0 = a_3 = a_4 = 0$.

Thus the list B is liney indep in U. Now $\dim U \geqslant 3 \Rightarrow \dim U = 3$. Thus $B_U = B$.

(b) Extend to a bss of $\mathcal{P}_4(\mathbf{F})$ as $\big(1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. $\square$

- **Note For (2.C.10):** For each nonC $p \in \operatorname{span}(1, z, \dots, z^m)$, $\exists$ smallest $m \in \mathbf{N}^+$, which is $\deg p$.

(a) If $p_0, p_1, \dots, p_m$ are suth all $a_{k,k} \neq 0$, and

$p_0 = a_{0,0}$, each $p_k = a_{0,k} + a_{1,k}z + \cdots + a_{k,k}z^k$.

Then the upper-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m} \\ 0 & a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix}$.

(b) If $p_0, p_1, \dots, p_m$ are suth all $a_{k,k} \neq 0$, and

$p_0 = a_{0,0} + \cdots + a_{m,0}x^m$, each $p_k = a_{k,k}x^k + \cdots + a_{m,k}x^m$.

Then the lower-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,0} & a_{m,1} & \cdots & a_{m,m} \end{pmatrix}$.

Comment: Define $\xi_k(p)$ by the coeff of z^k in $p \in \mathcal{P}_m(\mathbf{F})$.
Then $\mathcal{M}\big(\xi_k, (1, z, \dots, z^m), (1)\big) = \mathcal{E}^{(1,k)} \in \mathbf{F}^{1,m+1}$.

• (2.C.10) *Supp* $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ *are suth each* $\deg p_k = k$.
Prove $(p_0, p_1, \dots, p_m)$ *is a bss of* $\mathcal{P}_m(\mathbf{F})$.

Solus: Using induc on m.

(i) $k = 1$. $\deg p_0 = 0$; $\deg p_1 = 1 \Rightarrow \operatorname{span}(p_0, p_1) = \operatorname{span}(1, x)$.

(ii) $1 \leqslant k \leqslant m-1$. Asum $\operatorname{span}(p_0, p_1, \dots, p_k) = \operatorname{span}(1, x, \dots, x^k)$.

Then $\operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \subseteq \operatorname{span}(1, x, \dots, x^k, x^{k+1})$.

又 $\deg p_{k+1} = k+1$, $p_{k+1}(x) = a_{k+1}x^{k+1} + r_{k+1}(x)$; $a_{k+1} \neq 0$, $\deg r_{k+1} \leqslant k$.

$\Rightarrow x^{k+1} = \dfrac{1}{a_{k+1}}\Big(p_{k+1}(x) - r_{k+1}(x)\Big) \in \operatorname{span}(1, x, \dots, x^k, p_{k+1}) = \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

$\therefore x^{k+1} \in \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \Rightarrow \operatorname{span}(1, x, \dots, x^k, x^{k+1}) \subseteq \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m)$. □

Or. By comparing coeffs. Denote the coeff of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Supp $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is liney indep.

Step 1. For $k = m$, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m$, $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.

Now $L = a_{m-1}p_{m-1}(x) + \cdots + a_0 p_0(x)$.

Step k. For $0 \leqslant k \leqslant m$, we have $a_m = \cdots = a_{k+1} = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = 0$, then done. Othws, we have $L = a_{k-1}p_{k-1}(x) + \cdots + a_0 p_0(x)$. □

• **Tips:** *Supp* $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ *are suth the lowest term of each* p_k *is of deg* k.
Prove $(p_0, p_1, \dots, p_m)$ *is a bss of* $\mathcal{P}_m(\mathbf{F})$.

Solus: Using induc on m.

Let each p_k be defined by $p_k(x) = a_{k,k}x^k + \cdots + a_{m,k}x^m$, where $a_{k,k} \neq 0$.

(i) $k = 1$. $p_m(x) = a_{m,m}x^m$; $p_{m-1}(x) = a_{m-1,m-1}x^{m-1} + a_{m,m-1}x^m \Longrightarrow \operatorname{span}(x^m, x^{m-1}) = \operatorname{span}(p_m, p_{m-1})$.

(ii) $1 \leqslant k \leqslant m-1$. Asum $\operatorname{span}(x^m, \dots, x^{m-k}) = \operatorname{span}(p_m, \dots, p_{m-k})$.

Then $\operatorname{span}(p_m, \dots, p_{m-(k+1)}) \subseteq \operatorname{span}(x^m, \dots, x^{m-(k+1)})$.

又 $p_{m-(k+1)}$ has the form $a_{m-(k+1),m-(k+1)}x^{m-(k+1)} + r_{m-(k+1)}(x)$;

where the lowest term of $r_{m-(k+1)} \in \mathcal{P}_m(\mathbf{F})$ is of deg $(m-k)$.

$$\Rightarrow x^{m-(k+1)} = \frac{1}{a_{m-(k+1),m-(k+1)}}\Big(p_{m-(k+1)}(x) - r_{m-(k+1)}(x)\Big) \in \operatorname{span}(x^m, \dots, x^{m-k}, p_{m-(k+1)}) = \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$$

$\therefore x^{m-(k+1)} \in \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$

$\Rightarrow \operatorname{span}(x^m, \dots, x^{m-k}, x^{m-(k+1)}) \subseteq \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}(p_m, \dots, p_1, p_0)$. □

Or. By comparing coeffs. Denote the coeff of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Supp $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is liney indep.

Step 1. For $k = 0$, $\xi_0(L) = a_0 \xi_0(p_0) = \xi_0(R) = 0$ 又 $\deg p_0 = 0$, $\xi_0(p_0) \neq 0 \Rightarrow a_0 = 0$.

Now $L = a_1 p_1(x) + \cdots + a_m p_m(x)$.

Step k. For $0 \leqslant k \leqslant m$, we have $a_{k-1} = \cdots = a_0 = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = m$, then done. Othws, we have $L = a_{k+1}p_{k+1}(x) + \cdots + a_m p_m(x)$. □

- **NOTE FOR [2.11]:** ***Good definition for a general term always aviods undefined behaviours.***
  If $\deg p = 0$, then $p(z) = a_0 \neq 0$, but not literally $a_0 z^0$, by which if p is defined, then it comes to 0^0.
  To make it clear, we specify that *in* $\mathcal{P}(\mathbf{F})$, $a_0 z^0 = a_0$, where z^0 appears just for nota conveni.
  Becs by def, the term $a_0 z^0$ in a poly only represents the const term of the poly, which is a_0.
  For conveni, we asum $z^0 = 1$ in formula deduction and poly def. Absolutely without 0^0.

- (4E 2.C.10) *Supp m is a positive integer. For $0 \leqslant k \leqslant m$, let $p_k(x) = x^k(1-x)^{m-k}$.*
  Show $(p_0, \dots, p_m)$ is a bss of $\mathcal{P}_m(\mathbf{F})$.

SOLUS: *We may see $p_0 = 1$ and $p_m(x) = x^m$, from the expansion below, by the* NOTE FOR [2.11] *above.*

Note that each $p_k(x) = \sum_{j=0}^{m-k} C_{m-k}^j (-1)^j \cdot x^{j+k} \cdot 1^j = \underbrace{(-1)^0 \cdot x^k \cdot 1^0}_{\text{of deg k}} + \underbrace{\sum_{j=1}^{m-k} C_{m-k}^j (-1)^j \cdot x^{j+k} \cdot 1^j}_{\text{of deg m; denote it by } q_k(x)}$.
And, each $q_k \in \operatorname{span}(x^{k+1}, \dots, x^m)$. Using TIPS above. $\square$

OR. Simlr to the TIPS above. We will recurly prove each $x^{m-k} \in \operatorname{span}(p_m, \dots, p_{m-k})$.
(i) $k = 1$. $p_m(x) = x^m \in \operatorname{span}(p_m)$; $\quad p_{m-1}(x) = x^{m-1} - x^m \Rightarrow x^{m-1} \in \operatorname{span}(p_{m-1}, p_m)$.
(ii) $k \in \{1, \dots, m-1\}$. Supp for each $j \in \{0, \dots, k\}$, we have $x^{m-j} \in \operatorname{span}(p_{m-j}, \dots, p_m), \exists\,!\, a_m \in \mathbf{F}$.
Note that $x^{m-(k+1)} = p_{m-(k+1)}(x) + \sum_{j=1}^{k+1} C_{k+1}^j (-1)^{j+1} x^{m-(k+1)+j} \in \operatorname{span}(p_{m-(k+1)}, x^{m-k}, \dots, x^m)$.
Thus $x^{m-(k+1)} \in \operatorname{span}(p_{m-(k+1)}, p_{m-k}, \dots, p_m)$. $\square$

COMMENT: The base step and the induc step can be indep.

OR. For any $m, k \in \mathbf{N}^+$ suth $k \leqslant m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k(1-x)^{m-k}$.
Define the stmt $S(m) : (p_{0,m}, \dots, p_{m,m})$ is liney indep (and therefore is a bss).
We use induc on to show $S(m)$ holds for all $m \in \mathbf{N}^+$.
(i) $m = 0$. $p_{0,0} = 1$, and $ap_{0,0} = 0 \Rightarrow a = 0$.
$m = 1$. Let $a_0(1-x) + a_1 x = 0, \forall x \in \mathbf{F}$. Then take $x = 1, x = 0 \Rightarrow a_1 = a_0 = 0$.
(ii) $1 \leqslant m$. Asum $S(m)$ and $S(m-1)$ holds. Now we show $S(m+1)$ holds.
Supp $\sum_{k=0}^{m+1} a_k p_{k,m+1}(x) = \sum_{k=0}^{m+1} a_k \left[x^k(1-x)^{m+1-k}\right] = 0, \forall x \in \mathbf{F}$.
Now $a_0(1-x)^{m+1} + \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k} + a_{m+1} x^{m+1} = 0, \forall x \in \mathbf{F}$.
While $x = 0 \Rightarrow a_0 = 0$; and $x = 1 \Rightarrow a_{m+1} = 0$.
Then $0 = \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k}$
$= x(1-x) \sum_{k=1}^{m} a_k x^{k-1} (1-x)^{m-k}$, note that $m - k = (m-1) - (k-1)$
$= x(1-x) \sum_{k=0}^{m-1} a_{k+1} x^k (1-x)^{m-1-k} = x(1-x) \sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x)$.
Hence $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0, \forall x \in \mathbf{F} \backslash \{0, 1\}$. Which has infily many zeros.
Moreover, $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0$. By asum, $a_1 = \cdots = a_{m-1} = a_m = 0$.
Thus $(p_{0,m+1}, \dots, p_{m+1,m+1})$ is liney indep and $S(m+1)$ holds. $\square$

- (4E 3.D.20) *Supp $q \in \mathcal{P}(\mathbf{R})$. Prove $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.*

SOLUS: Note that $\deg\left[(x^2 + x)p''(x) + 2xp'(x) + p(3)\right] = \deg p$.
Define $T_n \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.
And note that $T_n(p) = 0 \Rightarrow \deg T_n(p) = -\infty = \deg p \Rightarrow p = 0$. Thus T_n is inv.
$\forall q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $n = 0$; if $q \neq 0$, let $n = \deg q$, we have $q \in \mathcal{P}_n(\mathbf{R})$.
Now $\exists\, p \in \mathcal{P}_n(\mathbf{R}), q(x) = T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$. $\square$

• (3.D.19) *Supp* $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ *is inje. And* $\deg Tp \leqslant \deg p$ *for every non0* $p \in \mathcal{P}(\mathbf{R})$.
(a) *Prove* T *is surj.* (b) *Prove for every non0* p, $\deg Tp = \deg p$.

Solus: (a) T is inje $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ is inje, so is inv $\iff T$ is surj.
(b) Using induc.
(i) $\deg p = -\infty \geqslant \deg Tp \iff p = 0 = Tp$. And $\deg p = 0 \geqslant \deg Tp \iff p = C \neq 0$.
(ii) Asum $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$. We show $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$ by ctradic.
Supp $\exists\, r \in \mathcal{P}_{n+1}(\mathbf{R}),\ \deg Tr \leqslant n < n+1 = \deg r$. By (a), $\exists\, s \in \mathcal{P}_n(\mathbf{R}),\ T(s) = (Tr)$.
又 T is inje $\Rightarrow s = r$. While $\deg s = \deg Ts = \deg Tr < \deg r$. Ctradic. □

• (3.B.26) *Supp* $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ *and* $\forall p, \deg(Dp) = (\deg p) - 1$. *Prove* $D \in \mathcal{P}(\mathbf{R})$ *is surj.*

Solus: [D might not be $D : p \mapsto p'$.] Notice that the following proof is wrong:
Becs $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) \subseteq \operatorname{range} D$, and $\deg Dx^n = n - 1$.
又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) = \operatorname{span}(1, x, x^2, \cdots) = \mathcal{P}(\mathbf{R})$.

Let $D(C) = 0, Dx^k = p_k$ of deg $(k-1)$, for all $C \in \mathcal{P}_0(\mathbf{R})$ and each $k \in \mathbf{N}^+$. Notice that $\mathbf{R} \neq \mathcal{P}_0(\mathbf{R})$.
Becs $B_{\mathcal{P}_m(\mathbf{R})} = (p_1, \dots, p_m, p_{m+1})$. And for all $p \in \mathcal{P}(\mathbf{R}), \exists\,!\, m = \deg p \in \mathbf{N}^+$.
So that $\exists\,!\, a_i \in \mathbf{R}, p = \sum_{i=1}^{m+1} a_i p_i \Rightarrow \exists\, q = \sum_{i=1}^{m+1} a_i x^i, Dq = p$. □

Or. We will recurly define a seq of polys $(p_k)_{k=0}^{\infty}$ where $Dp_0 = 1, Dp_k = x^k$ for each $k \in \mathbf{N}^+$.
So that $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), Dq = p, \exists\, q = \sum_{k=0}^{\deg p} a_k p_k$.
(i) Becs $\deg Dx = (\deg x) - 1 = 0, Dx = C \in \mathbf{F}\backslash\{0\}$. Let $p_0 = C^{-1}x \Rightarrow Dp_0 = C^{-1}Dx = 1$.
(ii) Supp we have defined $Dp_0 = 1, Dp_k = x^k$ for each $k \in \{1, \dots, n\}$. Becs $\deg D(x^{n+2}) = n+1$.
Let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$, with $a_{n+1} \neq 0$.
Then $a_{n+1}^{-1} D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_n Dp_n + \cdots + a_1 Dp_1 + a_0 Dp_0)$
$\Rightarrow x^{n+1} = D\left[a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)\right]$. Thus defining p_{n+1}, so that $Dp_{n+1} = x^{n+1}$. □

• *Supp* $V = \mathbf{R}^{\mathbf{R}}$ *with a subsp* $U = \{f \in \mathbf{R}^{\mathbf{R}} : f(x_1) = \cdots = f(x_m) = 0\}$, *where each* $x_k \in \mathbf{R}$.
Prove if $W \in \mathcal{S}_V U$, *then* $\dim W = m$. ***Hint***: *Find an iso from* V/U *onto* $\mathbf{R}^m$.

Solus: Define $T \in \mathcal{L}(V/U, \mathbf{R}^m)$ by $T(f + U) = (f(x_1), \dots, f(x_m))$.
$\forall f + U = g + U \in V/U, f - g \in U \Rightarrow f(x_k) = g(x_k)$. Well-defined.
Inje: Each $f(x_k) = 0 \Rightarrow f + U = 0$. Surj: Let $S = T \circ \pi \Rightarrow \tilde{S} = T$. Becs S is surj. □

• (3.F.7) *Show the dual bss of* $(1, x, \dots, x^m)$ *of* $\mathcal{P}_m(\mathbf{R})$ *is* $(\varphi_0, \varphi_1, \dots, \varphi_m)$, *where* $\varphi_k(p) = \dfrac{p^{(k)}(0)}{k!}$.

Solus: *The uniqnes of dual bss is guaranteed by [3.5].*

$$\text{For } j,k \in \mathbf{N},\ (x^j)^{(k)} = \begin{cases} j(j-1)\cdots(j-k+1)\cdot x^{(j-k)}, & j \geqslant k. \\ j(j-1)\cdots(j-j+1) = j! & j = k. \\ 0, & j \leqslant k. \end{cases} \Bigg| \Rightarrow (x^j)^{(k)}(0) = \begin{cases} 0\ , & j \neq k. \\ k!\ , & j = k. \end{cases}$$ □

Exa: By [2.C.10], $B_m = (1, 7x-5, \dots, (7x-5)^m)$ is a bss of $\mathcal{P}_m(\mathbf{R})$. Let each $\varphi_k = \dfrac{p^{(k)}(5/7)}{7 \cdot k!}$.

Ended

4

- **Tips 1:** *Supp $p \in \mathcal{P}_n(\mathbf{F})$ has at least $n+1$ disti zeros. Then by the ctrapos of [4.12], $\deg p < 0 \Rightarrow p = 0$.*

  Or. We show if $p \in \mathcal{P}(\mathbf{F})$ has at least m disti zeros, then either $p = 0$ or $\deg p \geqslant m$.

  If $p = 0$ then done. If not, then supp p has exactly m disti zeros $\lambda_1, \dots, \lambda_m$.

  Becs $\exists\,!\,\alpha_i \geqslant 1, q \in \mathcal{P}(\mathbf{F})$, and $q \neq 0$, suth $p(z) = \left[(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_m)^{\alpha_m}\right] q(z)$. ☐

- **Comment:** Notice *that by [4.17], some term of the poly factoriz might not be in the form $(x-\lambda_k)^{\alpha_k}$.*

- **Note For [4.7]:** *the uniqnes of coeffs of polys* [***Another proof***]

  If a poly had two different sets of coeffs, then subtracting the two exprs would give a poly with some non0 coeffs but infily many zeros. By Tips. ☐

- **Note For [4.8]:** *div algo for polys* [***Another proof***]

  Supp $\deg p \geqslant \deg s$. Then $\Big(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of len } \deg s}, \overbrace{s, zs, \cdots, z^{\deg p - \deg s} s}^{\text{of len } (\deg p - \deg s + 1)} \Big)$ is a bss of $\mathcal{P}_{\deg p}(\mathbf{F})$.

  Becs $q \in \mathcal{P}(\mathbf{F}), \exists\,!\,a_i, b_j \in \mathbf{F}$,

  $$q = a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s$$
  $$= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1}}_{r} + s \underbrace{\left(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} \right)}_{q}$$
  . Note that r, q are uniq. ☐

- **Note For [4.11]:** *each zero of a poly corres to a deg-one factor;* [***Another proof***]

  First supp $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m, \exists\,!\,a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.

  Then $p(z) = p(z) - p(\lambda) = a_1(z-\lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.

  Hence $\forall k \in \{1, \dots, m\}, z^k - \lambda^k = (z-\lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.

  Thus $p(z) = \sum_{j=1}^m a_j (z-\lambda) \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z-\lambda) \sum_{j=1}^m a_j \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z-\lambda) q(z)$. ☐

- (4E 2) *Prove if $w, z \in \mathbf{C}$, then $\big||w| - |z|\big| \leqslant |w - z|$.*

Solus: $|w-z|^2 = (w-z)(\overline{w} - \overline{z}) = |w|^2 + |z|^2 - 2Re(w\overline{z}) \geqslant |w|^2 + |z|^2 - 2|w\overline{z}| = \big||w| - |z|\big|^2$.

Or. $|w| = |w - z + z| \leqslant |w-z| + |z| \Rightarrow |w| - |z| \leqslant |w - z|$.

$|z| = |z - w + w| \leqslant |z - w| + |w| \Rightarrow |z| - |w| \leqslant |w - z|$. ☐

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5 *Supp $m \in \mathbf{N}$, and $z_1, \dots, z_{m+1}$ are disti in $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$.*
Prove $\exists\,!\,p \in \mathcal{P}_m(\mathbf{F}), p(z_k) = w_k$ for each $k \in \{1, \dots, m+1\}$.

Solus:

Define $T \in \mathcal{L}(\mathcal{P}_m(\mathbf{F}), \mathbf{F}^{m+1})$ by $Tq = \big(q(z_1), \dots, q(z_m), q(z_{m+1})\big)$.

Becs $Tq = 0 \Longleftrightarrow q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0 \Longleftrightarrow q = 0$, by Tips. Now T iso. Immed. ☐

Or. Let $p_1 = 1$, $p_k(z) = \prod_{i=1}^{k-1}(z - z_i) = (z - z_1)\cdots(z - z_{k-1})$ for each $k \in \{2, \dots, m+1\}$.

By (2.C.10), $B_p = (p_1, \dots, p_{m+1})$ is a bss of $\mathcal{P}_m(\mathbf{F})$. Let $B_e = (e_1, \dots, e_{m+1})$ be the std bss of $\mathbf{F}^{m+1}$.

Now $Tp_1 = (1, \dots, 1)$, $Tp_k = \Big(\prod_{i=1}^{k-1}(z_1 - z_i), \dots, \prod_{i=1}^{k-1}(z_j - z_i), \dots, \prod_{i=1}^{k-1}(z_{m+1} - z_i) \Big)$;

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & A_{2,2} & 0 & \cdots & 0 \\ 1 & A_{3,2} & A_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{m+1,2} & A_{m+1,3} & \cdots & A_{m+1,m+1} \end{pmatrix}$$

And $\prod_{i=1}^{k-1}(z_j - z_i) = 0 \Longleftrightarrow j \leqslant k-1$, becs $z_1, \dots, z_{m+1}$ are disti.

$= \mathcal{M}(T, B_p, B_e)$. Where $A_{j,k} = \prod_{i=1}^{k-1}(z_j - z_i) \neq 0$ for all $j > k - 1 \geqslant 1$.

Now the rows of $\mathcal{M}(T)$ liney indep. By (4E 3.C.17) Or (3.F.32). ☐

• **TIPS 2:** *Supp non0 $p, q \in \mathcal{P}(\mathbf{F})$ are multi of each other. Prove $p = cq$ for a $c \neq 0$.*

SOLUS: Let $p = rq, q = sp \Rightarrow p = rsp \Rightarrow r(z)s(z) = 1$ for all z with $p(z) \neq 0$, while such z is fini.
Thus $(rs)(z) = 1$ for infily many z, so for all z. Now $\deg rs = 1 = \deg r + \deg s$. $\square$

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6 *Supp non0 $p \in \mathcal{P}_m(\mathbf{F})$ has deg m. Prove*
$[P]$ p has m disti zeros $\Longleftrightarrow$ p and its deri p' have no common zeros. $[Q]$

SOLUS: (a) Supp p of deg m has m disti zeros. By [4.14], $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$.
If $m = 0$, then $p = c \neq 0 \Rightarrow p$ has no zeros, and $p' = 0$, done.
If $m = 1$, then $p(z) = c(z-\lambda_1)$, and $p' = c$ has no zeros, done.
For each $j \in \{1, \dots, m\}$, let $q_j(z-\lambda_j) = p(z) \Rightarrow q_j(\lambda_j) \neq 0$.
Now $p'(z) = (z-\lambda_j)q_j{}'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$.
OR. $\neg Q \Rightarrow \neg P$: Supp $p(z) = (z-\lambda)q(z),\ p'(z) = (z-\lambda)r(z)$.
Becs $p'(z) = (z-\lambda)q'(z) + q(z) \Rightarrow p'(\lambda) = q(\lambda) = 0 \Rightarrow q(z) = (z-\lambda)s(z)$.
Now $p(z) = (z-\lambda)^2 s(z)$. Hence p has strictly less than m disti zeros.

(b) $\neg P \Rightarrow \neg Q$: Becs $0 \neq p \in \mathcal{P}_m(\mathbf{F})$. Supp all disti zeros are $\lambda_1, \dots, \lambda_M$, with $M < m$.
By Pigeon Hole Principle, $(z-\lambda_k)^2 q(z) = p(z)$ for some $\lambda_k \Rightarrow p'(\lambda_k) = 0 = p(\lambda_k)$. $\square$

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7 *Prove every $p \in \mathcal{P}(\mathbf{R})$ of odd deg has a zero.*

SOLUS: Using [4.17], $\deg p = 2M + m \Rightarrow m$ is odd, done. $\square$
OR. Write $p(x) = x^m\left(\frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \cdots + \frac{a_{m-1}}{x} + a_m\right) \Rightarrow p$ continuous on $(-\infty, 0] \cup [0, \infty)$.
Let $\delta = |a_m|^{-1}a_m$. Becs $\lim\limits_{x\to-\infty} p(x) = -\delta\infty$; $\lim\limits_{x\to\infty} p(x) = \delta\infty \Rightarrow p$ has at least one real zero. $\square$

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8 *Supp $p \in \mathcal{P}(\mathbf{R})$. Define $Tp : \mathbf{R} \to \mathbf{R}$ by $(Tp)(x) = \begin{cases} \dfrac{p(x)-p(3)}{x-3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}$*
Show (a) $Tp \in \mathcal{P}(\mathbf{R})$; (b) $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$.

SOLUS:

(a) For $x \neq 3, T(x^n) = \frac{x^n - 3^n}{x-3} = \sum_{i=1}^n 3^{i-1}x^{n-i}$.
For $x = 3, T(x^n) = n3^{n-1} = \sum_{i=1}^n 3^{n-1} = \sum_{i=1}^n 3^{i-1}x^{n-i}$. Now each $T(x^n) = \sum_{i=1}^n 3^{i-1}x^{n-i} \in \mathcal{P}(\mathbf{R})$.

(b) $T(p+\lambda q)(x) = \left\{ \begin{array}{ll} \dfrac{(p+\lambda q)(x) - (p+\lambda q)(3)}{x-3}, & \text{if } x \neq 3, \\ (p+\lambda q)'(3), & \text{if } x = 3 \end{array} \right| = [T(p) + \lambda T(q)](x)$ for all $x \in \mathbf{R}$. $\square$

OR. (a) Becs $\exists!\, q \in \mathcal{P}(\mathbf{R}), p(x) - p(3) = (x-3)q(x)$. For $x \neq 3,\ q(x) = \frac{p(x)-p(3)}{x-3}$.
$p'(x) = (p(x) - p(3))' = q(x) + (x-3)q'(x)$. For $x = 3,\ p'(3) = q(3)$. Now $Tp = q$.
(b) Let $q_k(x)(x-3) = p_k(x) - p_k(3)$. Now by (a), $Tp_k = q_k$.
Then $(p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x-3)(q_1 + \lambda q_2)(x)$. By the uniqnes of $q_1 + \lambda q_2$. $\square$

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11 *Supp $p \in \mathcal{P}(\mathbf{F})$ with $\deg p = m \in \mathbf{N}$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*
(a) Show $\dim \mathcal{P}(\mathbf{F})/U = \deg p$; (b) Find a bss of $\mathcal{P}(\mathbf{F})/U$.

SOLUS: If $\deg p = 0$, then $U = \mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F})/U = \{0 + U\}$, with the uniq bss $()$. Supp $\deg p \geqslant 1$.
(a) Becs $\forall s \in \mathcal{P}(\mathbf{F}), \exists!\, r \in \mathcal{P}_{m-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}) \Rightarrow \exists!\, pq \in U,\ s = (p)q + (r) \Rightarrow \mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{m-1}(\mathbf{F})$.
By NOTE FOR [3.88, 90, 91] OR Define $R(s) = r \Rightarrow \operatorname{null} R = U$, and R surj. Immed.
(b) Let $(1, z, \dots, z^{m-1})$ be a bss of $\mathcal{P}_{m-1}(\mathbf{F})$. By (4E 3.E.14) OR $\tilde{R}^{-1} : \mathcal{P}_{m-1}(\mathbf{F}) \to \mathcal{P}(\mathbf{F})/U$, immed. $\square$

9 *Supp $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\bar{z})}$. Prove $q \in \mathcal{P}(\mathbf{R})$.*

Solus: By [4.5], $\bar{z}^n = \overline{z^n}$. For any $f(z) = a_n z^n + \cdots + a_1 z + a_0$, $\overline{f(\bar{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.
Becs $q(z) = p(z)\overline{p(\bar{z})} = \overline{p(\bar{z})}p(z) = \overline{q(\bar{z})}$. Each $c_k = \overline{c_k} \Rightarrow c_k \in \mathbf{R}$. $\square$

Or. Becs $q(z) = p(z)\overline{p(\bar{z})} = \sum_{k=0}^{2n} \left(\sum_{i+j=k} c_i \overline{c_j} \right) z^k$. For each $k \in \{0, \dots, 2n\}$,
$\overline{\sum_{i+j=k} c_i \overline{c_j}} = \sum_{i+j=k} \overline{c_i \overline{c_j}} = \sum_{i+j=k} c_j \overline{c_i} = \sum_{i+j=k} c_i \overline{c_j} \in \mathbf{R}$. $\square$

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10 *Supp disti $x_0, x_1, \dots, x_m \in \mathbf{R}$, and $p \in \mathcal{P}_m(\mathbf{C})$ suth each $p(x_k) \in \mathbf{R}$. Prove $p \in \mathcal{P}(\mathbf{R})$.*

Solus: By Tips (1) and Exe (5), $\exists !\, q \in \mathcal{P}_m(\mathbf{R})$ suth $q(x_k) = p(x_k)$. Hence $p = q$. $\square$

Or. Define $q(x) = \sum_{j=0}^{m} \dfrac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j)$.
又 Each x_j, $p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R})$. Becs each $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$.
$(q-p)$ has $(m+1)$ zeros. By Tips, $q - p = 0 \Rightarrow p = q \in \mathcal{P}(\mathbf{R})$. $\square$

• (4E 13) *Supp nonC $p, q \in \mathcal{P}(\mathbf{C})$ have no common zeros. Let $m = \deg p$, $n = \deg q$.
Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r,s) = rp + sq$. Prove T is inje.*

Coro: $\exists !\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ *suth* $rp + sq = 1$.

Solus: Immed, T is liney. Supp $T(r,s) = rp + sq = 0$.
Then $rp = -sq$. Becs p, q are <u>coprime</u> $\Rightarrow p \mid s$, while $\deg s \leqslant m - 1 \Rightarrow s = 0 \Rightarrow r = 0$. $\square$

Or. Let $\lambda_1, \dots, \lambda_M$ and $\mu_1, \dots, \mu_N$ be the disti zeros of p and q respectly. Notice that $M \leqslant m, N \leqslant n$.
By the ctrapos of [4.13], $M = 0 \iff m = 0 \Rightarrow s = 0 \iff r = 0 \Leftarrow n = 0 \iff N = 0$.
Now supp $M, N \geqslant 1$. We show $s = 0$. Simlr for $r = 0$. Or. $s = 0 \Rightarrow r = 0$.
Write $p(z) = a(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}$. $(\exists !\, \alpha_j \geqslant 1, a \in \mathbf{F}.)$ Let $\max\{\alpha_1, \dots, \alpha_M\} = A$.
For each $D \in \{0, 1, \dots, A-1\}$, let $I_{>D} = \{I_{D,1}, \dots, I_{D,J_D}\}$ be suth each $\alpha[I_{D,j}] = \alpha_{I_{D,j}} \geqslant D + 1$.
Now $\{M\} = I_{>A-1} \subseteq \cdots \subseteq I_{>0} = \{1, \dots, M\}$. Becs $rp + sq = 0 \Rightarrow (rp+sq)^{(k)} = 0$ for all $k \in \mathbf{N}^+$.
We use induc by D to show $s^{(D)}(\lambda[I_{D,j}]) = 0$ for each $D \in \{0, \dots, A-1\}$.
Notice that $p^{(D)}(\lambda[I_{D,j}]) = 0$ for each $D \in \{0, \dots, A-1\}$ and each $I_{D,j} \in I_{>D}$. (L2)
(i) $D = 0$. Each $(rp+sq)(\lambda[I_{0,j}]) = (sq)(\lambda[I_{0,j}]) = s(\lambda[I_{0,j}]) = 0$. Where $q(\lambda[I_{0,j}]) \neq 0$.
$D = 1$. Each $(r'p + rp')(\lambda[I_{1,j}]) + (s'q + sq')(\lambda[I_{1,j}]) = (s'q)(\lambda[I_{1,j}]) = s'(\lambda[I_{1,j}]) = 0$.
Where $p'(\lambda[I_{1,j}]) = 0$, and each $I_{1,j} \subseteq I_{0,j} \Rightarrow s(\lambda[I_{1,j}]) = 0$.
(ii) $2 \leqslant D \leqslant A - 1$. Asum $s^{(d)}(\lambda[I_{d,j}]) = 0$ for each $d \in \{0, 1, \dots, D-1\}$ and each $\lambda[I_{d,j}] \in I_{>d}$.

$$\begin{aligned}\text{Each } [rp+sq]^{(D)}(\lambda[I_{D,j}]) &= [C_D^D r^{(D)} p^{(0)} + \cdots + C_D^d r^{(d)} p^{(D-d)} + \cdots + C_D^0 r^{(0)} p^{(D)}](\lambda[I_{D,j}]) \qquad \text{(L1)}\\ &\quad + [C_D^D s^{(D)} q^{(0)} + \cdots + C_D^d s^{(d)} q^{(D-d)} + \cdots + C_D^0 s^{(0)} q^{(D)}](\lambda[I_{D,j}])\\ &= [C_D^D s^{(D)} q^{(0)}](\lambda[I_{D,j}]).\ \text{Where each } \lambda[I_{D,j}] \in I_{>D} \subseteq I_{D-1,\alpha}.\end{aligned}$$

Hence $s^{(D)}(\lambda[I_{D,j}]) = 0$. The asum holds for all $D \in \{0, \dots, A-1\}$.

Notice that $\forall k = \{0, \dots, A-2\}, s^{(k)}$ and $s^{(k+1)}$ have zeros $\{\lambda[I_{k+1,1}], \dots, \lambda[I_{k+1,J_{k+1}}]\}$ in common.
Now $\forall D \in \{1, \dots, A-1\}, s = s^{(0)}, \dots, s^{(D)}$ have zeros $\{\lambda[I_{D,1}], \dots, \lambda[I_{D,J_D}]\}$ in common.
Thus $s(z)$ is divisible by $(z - \lambda[I_{D,1}])^{\alpha[I_{D,1}]} \cdots (z - \lambda[I_{D,J_D}])^{\alpha[I_{D,J_D}]}$, for each $D \in \{0, \dots, A-1\}$.
Hence $s(z) = [(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}]\, s_0(z)$, while $\deg s < m = \alpha_1 + \cdots + \alpha_M$. Now by Tips. $\square$

L1 *Prove* $\forall p,q\in\mathcal{P}(\mathbf{F}),k\in\mathbf{N}^{+},(pq)^{(k)}=C_k^k p^{(k)}q^{(0)}+\cdots+C_k^j p^{(j)}q^{(k-j)}+\cdots+C_k^0 p^{(0)}q^{(k)}$.

Solus: We use induc by $k\in\mathbf{N}^{+}$. (i) $k=1$. $(pq)^{(1)}=(pq)'=C_1^1p^{(1)}q^{(0)}+C_1^0p^{(0)}q^{(1)}$. (ii) $k\geqslant 2$.

Asum for $(pq)^{(k-1)}=C_{k-1}^{k-1}p^{(k-1)}q^{(0)}+\cdots+C_{k-1}^{j}p^{(j)}q^{(k-1-j)}+\cdots+C_{k-1}^{0}p^{(0)}q^{(k-1)}$.

Now $(pq)^{(k)}=\left((pq)^{(k-1)}\right)'=\left(\sum_{j=0}^{k-1}C_{k-1}^{j}p^{(j)}q^{(k-j-1)}\right)'=\sum_{j=0}^{k-1}\left[C_{k-1}^{j}\left(p^{(j+1)}q^{(k-j-1)}+p^{(j)}q^{(k-j)}\right)\right]$.

$$
\begin{aligned}
&=\left[C_{k-1}^{0}\left(p^{(1)}q^{(k-1)}+\boxed{p^{(0)}q^{(k)}}\right)\right]+\left[C_{k-1}^{1}\left(p^{(2)}q^{(k-2)}+p^{(1)}q^{(k-1)}\right)\right]\\
&\quad+\cdots+\left[C_{k-1}^{j-2}\left(\underline{p^{(j-1)}q^{(k-j+1)}}+p^{(j-2)}q^{(k-j+2)}\right)\right]+\left[C_{k-1}^{j-1}\left(\underline{\underline{p^{(j)}q^{(k-j)}}}+\underline{p^{(j-1)}q^{(k-j+1)}}\right)\right]\\
&\quad+\left[C_{k-1}^{j}\left(p^{(j+1)}q^{(k-j-1)}+\underline{\underline{p^{(j)}q^{(k-j)}}}\right)\right]+\left[C_{k-1}^{j+1}\left(p^{(j+2)}q^{(k-j-2)}+p^{(j+1)}q^{(k-j-1)}\right)\right]\\
&\quad+\cdots+\left[C_{k-1}^{k-2}\left(\underline{p^{(k-1)}q^{(1)}}+p^{(k-2)}q^{(2)}\right)\right]+\left[C_{k-1}^{k-1}\left(\boxed{p^{(k)}q^{(0)}}+\underline{p^{(k-1)}q^{(1)}}\right)\right].
\end{aligned}
$$

Hence $(pq)^{(k)}=C_k^0p^{(0)}q^{(k)}+\cdots+\left[C_{k-1}^{j}+C_{k-1}^{j-1}\right]\left(p^{(j)}q^{(k-j)}\right)+\cdots+C_k^kp^{(k)}q^{(0)}$. □

L2 *Supp* $\alpha\in\mathbf{N}^{+}$ *suth* $p(z)=(z-\lambda)^{\alpha}q(z)$. *Prove* $p^{(\alpha-1)}(\lambda)=0$.

Solus: $\left[(z-\lambda)^{\alpha}q(z)\right]^{(\alpha-1)}=\sum_{j=1}^{\alpha-1}C_{\alpha-1}^{j}\left[(z-\lambda)^{\alpha}\right]^{(j)}\left[q(z)\right]^{(\alpha-1-j)}$.

Note that $\left[(z-\lambda)^{\alpha}\right]^{(j)}=\alpha(\alpha-1)\cdots(\alpha-j+1)\cdot(z-\lambda)^{(\alpha-j)}$. □

Ended

凭借我的经验，我认为，好的自学教材，除了提供足够的一级知识外，还能通过各种方式，将二级、三级知识顺理成章地经过学科思维的浓缩喻于习题或课文中。在 *LADR* 的熏陶下，我渐渐认为，自学教材带来的长期收益更重要——所谓素养一类的隐形东西，无论堆砌多少知识记忆都难以学到；外在的选拔，表面上看都是知识竞赛，但真正有含金量的选拔，往往十二分地注重隐形能力；实际的工作表现也是如此；客观上看这确实是在当今公共信息过剩的时代下人与人拉开差距的核心原因之一，也是我相信最能仅通过自身努力耕耘获得长期稳定回报的地方。现在看看速学速成应付选拔竞争的选择有多愚蠢吧：考不上放弃吧，因为几乎没有习得那些隐性能力，就确实是除了知识和解题技巧之外啥也没得到，这些知识中实用的那些内容怎么着都能学到，不具有不可替代性，实际工作更需要隐形的素养；再考再战吧，就得辛苦刷题，总归不如在学的过程中把“和习题的挣扎”当作练习对学科思维的启发最好。考上了吧又要和更“拔尖创新人才”竞争隐形的能力，一样难以优胜，只不过这个情况下可以做一个更“优越”的平庸之人罢了，除了短期速成而来的外在“纪念品”之外再也没有什么学习成果可长期变现——和质量至上、不怕耽误时间进度的学习者相比又能有什么优越之处呢？

下面第 5 章中，3e 和 4e 差距过大。我认为是因为 4e 将原来 3e 第 8 章的极小多项式和第 2 章线性无关最小性和第 4 章的多项式的原理结合，以极小多项式为工具重写了第 5 章几个核心定理，并引入一些结论承接读者一些很自然的想法，让第 5 章的定理和习题更加富有动机和系统性。

这份笔记主要面向 3e 纸质书的读者，所以题号和定理索引都采用 3e（除少数 4e 新增章节）。因为 3e 读者可能会对第 5 章这样的 4e 变化感到茫然无措，所以为了内容的紧密性，我决定将 3e 第 8 章提前到第 5 章后，对应到 4e 只有第 8 章前三节。3e 第 8 章个别涉及第 6、7 章的习题，会拆散塞入对应章节的笔记中。

5.A

注意： 这里将 5.B 节多项式作用于算子部分与 5.C 节的本征空间的定义前置.

• **Tips 1:** *Supp* $V = U \oplus W$ *and* U, W *invard* $T \in \mathcal{L}(V)$. *Prove* $\operatorname{null} T|_U \oplus \operatorname{null} T|_W = \operatorname{null} T$.

Solus: $\forall v = u + w \in \operatorname{null} T, Tv = Tu + Tw = 0 \Rightarrow Tu, Tw = 0 \Rightarrow v \in \operatorname{null} T|_U \oplus \operatorname{null} T|_W$. ☐

Coro: $E(\lambda, T) = E(\lambda, T|_U) \oplus E(\lambda, T|_W)$. Replace T with $T - \lambda I$, immed.

• **Note For Exe (2, 3):** $ST = TS \Rightarrow p(S)q(T) = q(T)p(S)$. And $\operatorname{null} q(T), \operatorname{range} q(T)$ invard $p(S)$.

• (5.E.1) *Give* $S, T \in \mathbf{F}^4$ *suth* $ST = TS$ *while* $\exists$ *invarspd* S *but not* T, *invarspd* T *but not* S.

Solus: Define $S : (x, y, z, w) \mapsto (y, x, 0, 0)$ and $T : (x, y, z, w) \mapsto (0, 0, w, z) \Rightarrow TS = ST = 0$.

Thus e_1, e_2 are eigvecs of T but not of S, and e_3, e_4 are eigvecs of S but not of T.

---

10 *Define* $T \in \mathcal{L}(\mathbf{F}^n)$ *by* $T(x_1, x_2, x_3, \dots, x_n) = \big(x_1, 2x_2, 3x_3, \dots, nx_n\big)$.

(a) *Find all eigvals and eigvecs;* (b) *Find all invarsps of* V *under* T.

Solus: Let $(e_1, \dots, e_n)$ be the std bss of $\mathbf{F}^n$. The eigvals are $\{1, \dots, n\}$ of len $\dim \mathbf{F}^n$.

Let each $E_k = \operatorname{span}(e_k)$. The set of all eigvecs is $(E_1 \cup \dots \cup E_n) \setminus \{0\}$.

Supp U is invarsp. Then $u = (x_1, x_2, \dots, x_n) \in U \Rightarrow Tu = (x_1, 2x_2, \dots, x_n) \in U$.

And $Tu - u = \big(0, x_2, 2x_3, \dots, (n-1)x_n\big) \in U \Rightarrow \dots \Rightarrow (0, \dots, 0, x_n) \Rightarrow$ each $x_k e_k \in U$.

Get a B_U and pick all non0 x_k. Forming $\operatorname{span}(e_{k_1}, \dots, e_{k_m}) = U$. ☐

Comment: The result (b) holds generally where $\exists B_V$ consists of eigvecs of T.

• *Supp* $T \in \mathcal{L}(V), \lambda_1, \dots, \lambda_m$ *are the disti eigvals corres* $v_1, \dots, v_m$, *and* U *invarspd* T.

• **Tips 2:** *Supp* $v_1 + \dots + v_m \in U$. *Prove each* $v_k \in U$.

Solus: Consider the stmt $P(k)$: if $v_1 + \dots + v_k \in U$, then each $v_j \in U$.

(i) $v_1 \in U$. $P(1)$ holds. (ii) For $2 \leqslant k \leqslant m$. Asum $P(k-1)$ holds. Supp $v = v_1 + \dots + v_k \in U$.

Then $Tv = \lambda_1 v_1 + \dots + \lambda_k v_k \in U \Longrightarrow Tv - \lambda_k v = (\lambda_1 - \lambda_k)v_1 + \dots + (\lambda_{k-1} - \lambda_k)v_{k-1} \in U$.

For each $j \in \{1, \dots, k-1\}, \lambda_j - \lambda_k \neq 0 \Rightarrow (\lambda_j - \lambda_k)v_j = v_j'$ is an eigvec of T corres λ_j.

By asum, each $v_j' \in U$. Thus $v_1, \dots, v_{k-1} \in U$. So that $v_k = v - v_1 - \dots - v_{k-1} \in U$. ☐

• **Tips 3:** *Supp* $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$. *Prove* $U = E(\lambda_1, T|_U) \oplus \dots \oplus E(\lambda_m, T|_U)$.

Solus: Becs $\forall u \in U, \exists!\, v_j \in E(\lambda_j, T), v = v_1 + \dots + v_m$. By Tips (2), each $v_j \in U$. ☐

---

19 *Supp* $n \in \mathbf{N}^+$. *Define* $T \in \mathcal{L}(\mathbf{F}^n)$ *by* $T(x_1, \dots, x_n) = \big(x_1 + \dots + x_n, \cdots, x_1 + \dots + x_n\big)$.

In other words, the ent of $\mathcal{M}(T)$ *wrto the std bss are all* 1's. *Find all eigvals and eigvecs of* T.

Solus: Supp $x_k \neq 0$ and $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = \big(x_1 + \dots + x_n, \dots, x_1 + \dots + x_n\big)$.

Then (I) $\lambda = 0 \Rightarrow x_1 + \dots + x_n = 0$. If $n > 1$, then $\lambda = 0$ is eigval; othws not, becs $T = I$.

(II) $\lambda \neq 0 \Rightarrow x_1 = \dots = x_n \Rightarrow \lambda x_k = n x_k$. Now n is eigval. ☐

Or. Becs $\operatorname{range} T = \big\{(x, \dots, x) \in \mathbf{F}^n\big\}$ of dim 1. By Exe (29). Simlr. ☐

Or. Supp $n > 1$. Becs $\operatorname{null} T = \big\{(-x_2 - \dots - x_n, x_2, \dots, x_n)\big\}$ of dim $n - 1 > 0 \Rightarrow 0$ is eigval.

Notice that n is also eigval corres $(x, \dots, x) \neq 0$. We show $0, n$ are the only eigvals.

Supp non0 $x \in \mathbf{F}^n$ and $\lambda \in \mathbf{F}$ with $Tx = \lambda x$. Becs $\operatorname{range} T = \operatorname{span}\big((1, \dots, 1)\big), \exists!\, \alpha \in \operatorname{range} T$,

$\lambda x = \alpha \Rightarrow x$ corres λ and α corres n are liney dep. By the ctrapos of [5.10], $\lambda = n$. ☐

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20 *Define $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \cdots) = (z_2, z_3, \cdots)$.*
Show every elem of $\mathbf{F}$ is an eigval of S, and find all eigvecs of S.

Solus: Supp $z_k \neq 0$ and $S(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$. Then each $\lambda z_k = z_{k+1}$.
(I) $\lambda = 0 \Rightarrow$ each$z_k = \cdots = z_2 = \lambda z_1 = 0$. Let $z_1 \neq 0 \Rightarrow E(0, S) = \operatorname{span}(e_1)$.
(II) $\lambda \neq 0 \Rightarrow \lambda^k z_1 = \lambda^{k-1} z_2 = \cdots = \lambda z_k = z_{k+1}$, let $z_1 \neq 0 \Rightarrow E(\lambda, S) = \operatorname{span}\big[(1, \lambda^1, \cdots, \lambda^k, \cdots)\big]$.□

• **Tips 4:** Supp $T \in \mathcal{L}(\mathbf{R}^2)$ is the countclockws rotat by $\theta \in \mathbf{R}$. Define $\mathcal{C}(a, b) = a + \mathrm{i}b$.
Becs $(\cos\theta + \mathrm{i}\sin\theta)(a + \mathrm{i}b) = r\big(\cos(\alpha+\theta) + \mathrm{i}\sin(\alpha+\theta)\big)$.
Hence $T(a, b) = \big(a\cos\theta - b\sin\theta,\, a\sin\theta + b\cos\theta\big)$. Now $\mathcal{M}(T) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

• *Supp V is finide, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$.*

13 *Prove $\exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}$ suth $(T - \alpha I)$ is inv.*

Solus: Let each $|\alpha_k - \lambda| = \frac{1}{1000+k}$, where $k \in \{1, \dots, \underline{\dim V + 1}\}$. Then $\exists\, \alpha_k$ not an eigval. □

• (4E 11) *Prove $\exists\, \delta > 0$ suth $(T - \alpha I)$ is inv for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$.*

Solus: If T has no eigvals, then $(T - \alpha I)$ is inje for all $\alpha \in \mathbf{F}$, done.
Supp $\lambda_1, \dots, \lambda_m$ are all the disti eigvals of T unequal to λ.
Let $\delta > 0$ be suth, for each eigval λ_k, $\lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.
So that for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$, $(T - \alpha I)$ is inv. □

Or. Let $\delta = \min\{|\lambda - \lambda_k| : k \in \{1, \dots, m\}, \lambda_k \neq \lambda\}$.
Then $\delta > 0$ and each $\lambda_k \neq \alpha$ $\big[\iff (T - \alpha I)$ is inv $\big]$ for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$. □

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15 *Supp $T \in \mathcal{L}(V)$. Supp $S \in \mathcal{L}(V)$ is inv.*
(a) *Prove T and $S^{-1}TS$ have the same eigvals.*
(b) *Describe the relationship between eigvecs of T and eigvecs of $S^{-1}TS$.*

Solus: (a) λ is an eigval of T with an eigvec $v \Rightarrow S^{-1}TS(\underline{S^{-1}v}) = S^{-1}Tv = S^{-1}(\lambda v) = \underline{\lambda S^{-1}v}$.
λ is an eigval of $S^{-1}TS$ with an eigvec $v \Rightarrow S(S^{-1}TS)v = T\underline{Sv} = \underline{\lambda Sv}$.
Or. Note that $S(S^{-1}TS)S^{-1} = T$. Every eigval of $S^{-1}TS$ is an eigval of $S(S^{-1}TS)S^{-1} = T$.
Or. $Tv = \lambda v \iff TSu = \lambda Su \iff (S^{-1}TS)u = \lambda u$. Where $v = Su$.
$(S^{-1}TS)u = \lambda u \iff S^{-1}Tv = \lambda S^{-1}v \iff Tv = \lambda v$. Where $u = S^{-1}v$.
(b) $E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\}$; $E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}$. □

• (4E 15) *Show λ is eigval of $T \iff$ of T'.*

Solus: [*Req Finide; For [5.6]*] $T - \lambda I_V$ not inv $\iff (T - \lambda I_V)' = T' - \lambda I_{V'}$ not inv. □
(a) Supp λ is eigval with v. Let U be invar with $U \oplus \operatorname{span}(v) = V$, by Exe (4E 39).
Define $\psi \in V'$ by $\psi(cv + u) = c$. Then $\big[T'(\psi)\big](cv + u) = \psi(c\lambda v + Tu) = \lambda c = \lambda\psi(cv + u)$.
(b) A countexa: Let T be the forwd shift optor on $V = \mathbf{F}^\infty$. No eigvals for T, by Exe (18).
Define $\psi \in V'$ by $\psi(x_1, x_2, \cdots) = x_1$. Then $\big[T'(\psi)\big](x_1, x_2, \cdots) = \psi(0, x_1, x_2, \cdots) = 0$. □

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23 *Supp V is finide, and $S, T \in \mathcal{L}(V)$. Prove ST and TS have the same eigvals.*

Solus: [*False if infinide. See Exe (18, 20).*] Supp $v \neq 0$ and $STv = \lambda v \Rightarrow T(STv) = \lambda Tv = TS(Tv)$.
If $Tv = 0$, then T not inje, so are TS, ST. Othws, λ is eigval of TS. Rev the roles in asum. □

• (4E 37) *Supp V is finide, $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(S) = TS$.*
Prove the set of eigvals of T equals the set of eigvals of $\mathcal{A}$.

SOLUS: (a) For $v \neq 0$ and $Tv = \lambda v$, let $v_1 = v \Rightarrow B_V = (v_1, \dots, v_n)$.
Define $S \in \mathcal{L}(V) : v_j \mapsto v$, OR $v_j \mapsto \delta_{1,j} v_1$. Then each $(T - \lambda I) S v_j = 0$.
Thus $(T - \lambda I)S = 0 \Rightarrow \mathcal{A}(S) = TS = \lambda S$ with $S \neq 0$.
(b) Supp $S \neq 0$ and $TS = \lambda S$. Then $\exists\, v \in V \backslash \operatorname{null} S$. Let $u = Sv \Rightarrow Tu = TSv = \lambda Sv = \lambda u$.
OR. $TS - \lambda S = (T - \lambda I)S = 0 \Rightarrow \{0\} \neq \operatorname{range} S \subseteq \operatorname{null}(T - \lambda I) \Rightarrow (T - \lambda I)$ not inje. □

• **TIPS 5:** *Supp $S, T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{F})$. Prove $Sp(TS) = p(ST)S$.*

SOLUS: We prove each $S(TS)^m = (ST)^m S$ by induc. (i) $m = 0, 1$. Immed.
(ii) $m > 1$. $S(TS)^{m-1} = (ST)^{m-1} S \Rightarrow S(TS)^m = S(TS)^{m-1}(TS) = (ST)^{m-1}(ST)S = (ST)^m S$. □

COMMENT: If S is inv. Then $p(TS) = S^{-1} p(ST) S$, $p(ST) = Sp(TS)S^{-1}$.

CORO: Becs S is inv, $T \in \mathcal{L}(V)$ is arb $\iff ST = R \in \mathcal{L}(V)$ is arb. Hence $p(S^{-1}RS) = S^{-1}p(R)S$.

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26 *Supp $T \in \mathcal{L}(V)$ is suth $\forall v \in V, \exists\,! \lambda_v \in \mathbf{F}, Tv = \lambda_v v$. Prove $T = \lambda I$.* By Exe (25).

SOLUS: Supp V non0. Becs $\forall v \in V, \exists\,! \lambda_v \in \mathbf{F}, Tv = \lambda_v v$. For any disti non0 $v, w \in V$,
$T(v + w) = \lambda_{v+w}(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$. □

27, 28 *Supp* $\dim V > 1$, $k \in \{1, \dots, \dim V - 1\}$.
Supp every subsp of dim k is invard a $T \in \mathcal{L}(V)$. Prove $T = \lambda I$.

SOLUS: We prove the ctrapos. Supp $\exists\, v \in V \backslash \{0\}$ not eigvec.
Then (v, Tv) liney indep $\Rightarrow B_V = (v, Tv, u_1, \dots, u_n)$. Let $U = \operatorname{span}(v, u_1, \dots, u_{k-1})$. □

OR. Supp $v = v_1 \in V \backslash \{0\} \Rightarrow B_V = (v_1, \dots, v_n)$. Let $Tv_1 = c_1 v_1 + \dots + c_n v_n$.
Let $B_U = (v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$. Becs every such U invar. Now $Tv_1 \in U \Rightarrow Tv_1 = c_1 v_1$.
By Exe (26), done. $\big[$For $0 \neq c_j \in \{c_2, \dots, c_n\}$, let $B_W = (v_1, v_{\beta_1}, \dots, v_{\beta_{k-1}})$ with each $\beta_i \neq j$.$\big]$ □

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29 *Supp $T \in \mathcal{L}(V)$,* $\operatorname{range} T$ *is finide. Prove T has at most* $1 + \dim \operatorname{range} T$ *disti eigvals.*

SOLUS: Becs $\operatorname{range} T$ finide $\Rightarrow$ not too many. Let $\lambda_1, \dots, \lambda_m$ be the disti eigvals of T with corres $v_1, \dots, v_m$.
Then $(v_1, \dots, v_m)$ liney indep $\Rightarrow (\lambda_1 v_1, \dots, \lambda_m v_m)$ liney indep, if each $\lambda_k \neq 0$. Othws,
$\exists\,! \lambda_k = 0$. Now $\{\lambda_j v_j : j \neq k\}$ liney indep. Thus $m - 1 \leqslant \dim \operatorname{range} T$. □

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35 *Supp V is finide, $T \in \mathcal{L}(V)$, and U is invard T. Show λ is eigval of $T/U \Rightarrow$ of T.*

SOLUS:

Supp $v + U \neq 0$ and $Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v = u \in U$. If $u = 0$, done. Othws, two cases.
If $(T - \lambda I)\big|_U$ inje $\Rightarrow$ surj. Then $(T - \lambda I)v = u = (T - \lambda I)\big|_U(w), \exists\, w \in U \Rightarrow T(v + w) = \lambda(v + w)$.
If $(T - \lambda I)\big|_U = T|_U - \lambda I_U$ not inje. Then λ is eigval of $T|_U \Rightarrow$ of T. □

OR. Let $B_U = (u_1, \dots, u_m) \Rightarrow \big(Tv - \lambda v,\ Tu_1 - \lambda u_1, \cdots, Tu_m - \lambda u_m\big)$ of len $(m + 1)$ liney dep in U.
So that $a_0(T - \lambda I)v + a_1(T - \lambda I)u_1 + \dots + a_m(T - \lambda I)u_m = 0, \exists\, a_k \neq 0$.
Then $Tw = \lambda w$, where $w = a_0 v + a_1 u_1 + \dots + a_m u_m \neq 0 \Leftarrow w \notin U \Leftarrow v \notin U$. □

EXA: Let $V = \{f \in \mathbf{R}^{\mathbf{R}} : \exists\,! m \in \mathbf{N}, f \in \operatorname{span}(1, e^x, \dots, e^{mx})\}$.
Let $U = \{f \in \mathbf{R}^{\mathbf{R}} : \exists\,! m \in \mathbf{N}^+, f \in \operatorname{span}(e^x, \dots, e^{mx})\}$.
Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then $(T/U)(1 + U) = e^x + U = 0$ while T inje.

• (4E 39) *Supp* $T \in \mathcal{L}(V)$, V *is finide. Prove* $\exists$ *eigval of* $T \iff \exists$ *invarsp of dim* $\dim V - 1$.

Solus: (a) Supp λ is eigval with v. Becs $\dim \operatorname{null}(T - \lambda I) \geqslant 1 \iff \dim \operatorname{range}(T - \lambda I) \leqslant \dim V - 1 = N$.
Let $B_{\operatorname{range}(T-\lambda I)} = (w_1, \dots, w_m)$, $B_{\operatorname{null}(T-\lambda I)} = (u_1, \dots, u_n)$, $B_U = (w_1, \dots, w_m, u_1, \dots, u_{N-m})$.
Note: $U \notin \mathcal{S}_V \operatorname{span}(v)$ unless $u_n = v$.

(b) Supp U is invarspd T with $\dim U = \dim V - 1 \Rightarrow \dim V/U = 1$. By (3.A.7), Exe (35). $\square$

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24 *Supp* $A \in \mathbf{F}^{n,n}$. *Define* $T \in \mathcal{L}(\mathbf{F}^{n,1})$ *by* $Tx = Ax$. *Prove* 1 *is eigval of* T *if*:
(a) *the sum of the ent in each row of* A *equals* 1. (b) *each col of* A.

Solus: Supp $x \neq 0$ and $Ax = \big(A_{j,1}x_1 + \cdots + A_{j,n}x_n\big)_{j=1}^n = \lambda (x_j)_{j=1}^n = \lambda x$.

(a) Supp $A_{R,1} + \cdots + A_{R,n} = 1$. Let $x_1 = \cdots = x_n$. Immed.

(b) Supp $A_{1,C} + \cdots + A_{n,C} = 1$. Then $\left[\sum_{R=1}^n A_{R,\cdot}\right] x = \sum_{k=1}^n \big(A_{1,k} + \cdots + A_{n,k}\big) x_k$.
Now each $(Ax)_{R,1} = (x)_{R,1} = (\lambda x)_{R,1}$. Thus for x with $\sum_{k=1}^n x_k \neq 0$, $\lambda = 1$ is the corres eigval. $\square$

Or. Becs $(T - I)x = (A - I)x = \big((A_{j,1}x_1 + \cdots + A_{j,n}x_n) - x_j\big)_{j=1}^n = (y_j)_{j=1}^n$.
Now $y_1 + \cdots + y_n = \sum_{j=1}^n \sum_{k=1}^n \big(A_{j,k}x_k - x_j\big) = \sum_{k=1}^n x_k \left[\sum_{j=1}^n A_{j,k}\right] - \sum_{j=1}^n x_j = 0$.
Thus $\operatorname{range}(T - I) \subseteq \big\{(y_1, \dots, y_n) : y_1 + \cdots + y_n = 0\big\}$. Now $(T - I)$ is not inv. $\square$

Or. Let $(e_1, \dots, e_n)$ be the std bss of $\mathbf{F}^{n,1}$. Define $\psi \in \big(\mathbf{F}^{n,1}\big)'$ with each $\psi(e_k) = 1$.
Becs $Ae_k = A_{\cdot,k} = \sum_{j=1}^n A_{j,k}e_j \Rightarrow \psi(T - I)e_k = \psi\Big(\sum_{j=1}^n A_{j,k}e_j - e_k\Big) = \sum_{j=1}^n A_{j,k} - 1 = 0$. $\square$

Or. Define $S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Sx = A^t x$. Becs the rows of $\mathcal{M}(S) = A^t$ are the cols of $\mathcal{M}(T) = A$.
Let $(\varphi_1, \dots, \varphi_n)$ be the dual bss of $(e_1, \dots, e_n)$. Define $\Phi \in \mathcal{L}\big[\mathbf{F}^{n,1}, (\mathbf{F}^{1,n})'\big]$ by $\Phi(e_k) = \varphi_k$.
Now $\big(\Phi^{-1}T'\Phi\big)e_k = \big(\Phi^{-1}T'\big)\varphi_k = \Phi^{-1}\Big(\sum_{j=1}^n A_{j,k}^t \varphi_j\Big) = \sum_{j=1}^n A_{j,k}^t e_j = A^t e_k = Se_k$.
Becs by (a), 1 is eigval of $S = \Phi^{-1}T'\Phi$. So of T', by Exe (15). So of T, by Exe (4E 15). $\square$

• *Supp* $A \in \mathbf{F}^{n,n}$. *Define* $T \in \mathcal{L}(\mathbf{F}^{1,n})$ *by* $Tx = xA$. *Prove* 1 *is eigval of* T *if*:
(a) *the sum of the ent in each col of* A *equals* 1. (b) *each row of* A.

Solus: Supp $x \neq 0$ and $xA = \big(x_1A_{1,k} + \cdots + x_nA_{n,k}\big)_{k=1}^n = \lambda (x_k)_{k=1}^n = \lambda x$.

(a) Supp $A_{1,C} + \cdots + A_{n,C} = 1$. Let $x_1 = \cdots = x_n$. Immed.

(b) Supp $A_{R,1} + \cdots + A_{R,n} = 1$. Then $\sum_{C=1}^n xA_{\cdot,C} = \sum_{j=1}^n \big(A_{j,1} + \cdots + A_{j,n}\big)x_j$.
Now each $(xA)_{1,C} = (x)_{1,C} = (\lambda x)_{1,C}$. Thus for x suth $\sum_{k=1}^n x_k \neq 0$, $\lambda = 1$ is the corres eigval. $\square$

Or. Becs $(T - I)x = x\big(A - I\big) = \big((x_1A_{1,k} + \cdots + x_nA_{n,k}) - x_k\big)_{k=1}^n = (y_k)_{k=1}^n$.
Now $y_1 + \cdots + y_n = \sum_{k=1}^n \sum_{j=1}^n (x_jA_{j,k} - x_k) = \sum_{j=1}^n x_j \left[\sum_{k=1}^n A_{j,k}\right] - \sum_{k=1}^n x_k = 0$.
Thus $\operatorname{range}(T - I) \subseteq \big\{(y_1, \dots, y_n) : y_1 + \cdots + y_n = 0\big\}$. Now $(T - I)$ is not inv. $\square$

Or. Simlr in Exe (24). Becs $e_jA = A_{j,\cdot} = \sum_{k=1}^n A_{j,k}e_k \Rightarrow \psi(T - I)e_j = \sum_{k=1}^n A_{j,k} - 1 = 0$. $\square$

Or. Define $S \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Sx = xA^t$. Notice that $\mathcal{M}(S) \neq A$ and $\mathcal{M}(T) \neq A^t$. [*Noted by* AI.]
Let $(\varphi_1, \dots, \varphi_n)$ be the dual bss. Define Φ by $\Phi(e_k) = \varphi_k$.
Becs $\big[T'(\varphi_k)\big](e_j) = \varphi_k\Big(\sum_{i=1}^n A_{j,i}e_i\Big) = A_{j,k}$. By (3.F.9), $T'(\varphi_k) = \sum_{j=1}^n A_{j,k}\varphi_j$.
Now $\big(\Phi^{-1}T'\Phi\big)e_k = \big(\Phi^{-1}T'\big)\varphi_k = \Phi^{-1}\Big(\sum_{j=1}^n A_{j,k}\varphi_j\Big) = \sum_{j=1}^n A_{j,k}e_j = e_kA^t = Se_k$. Simlr. $\square$

Ended

5.B

(I) 覆盖 4e 的 5.B 节全部、3e 前半部分与之相关的所有习题。(II) 覆盖 3e 的 5.B 节后半部分「上三角矩阵」和 4e 的 5.C 节。
注意：4e 的 5.B 节和 3e 的 8.C 节、9.A 节许多结论和习题有交集。5.B(II) 的题号使用 4e 的 5.C 节.

I.1 *Supp $T \in \mathcal{L}(V)$ and $T^n = 0$. Prove $(I - T)$ is inv and $(I-T)^{-1} = I + T + \cdots + T^{n-1}$.*

Solus: Becs $p(z) = 1 - x^n = (1-x)(1 + x + \cdots + x^{n-1})$. Consider $p(T) = I$, by [5.20]. □

• *Supp $T \in \mathcal{L}(V)$ has no eigvals and $T^4 = I$. Prove $T^2 = -I$.*

Solus: Becs $T^4 - I = (T^2 - I)(T^2 + I) = 0$ not inje, so is $(T^2 - I)$ or $(T^2 + I)$, while T has no eigvals.
$(T - I), (I + T)$ inje, so is $(T^2 - I) \Rightarrow \forall v \in V, 0 = (T^2 - I)(T^2 + I)v \Longleftrightarrow 0 = (T^2 + I)v$. □

Or. Note that $\forall v \in V$, $v = (I - T^2)v/2 + (I + T^2)v/2$. 又 $I - T^4 = (I \pm T^2)(I \mp T^2)$.
Then $\operatorname{range}(I \mp T^2) \subseteq \operatorname{null}(I \pm T^2) \Rightarrow V = \operatorname{null}(I - T^2) + \operatorname{null}(I - T^2)$.
又 T has no eigvals $\Longleftrightarrow (I - T^2)$ inje $\Longleftrightarrow \operatorname{null}(I - T^2) = \{0\} \supseteq \operatorname{range}(I + T^2)$. □

I.8 *Give an exa of $T \in \mathcal{L}(\mathbf{R}^2)$ suth $T^4 = -I$.*

Solus: Define $\mathrm{i}^n \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}^n(x,y) = \Big(\operatorname{Re}(\mathrm{i}^n x + \mathrm{i}^{n+1} y), \operatorname{Im}(\mathrm{i}^n x + \mathrm{i}^{n+1} y)\Big)$.
$T^4 + I = (T^2 + \mathrm{i}I)(T^2 - \mathrm{i}I) = (T + \mathrm{i}^{1/2} I)(T - \mathrm{i}^{1/2} I)\big(T - (-\mathrm{i})^{1/2} I\big)\big(T + (-\mathrm{i})^{1/2} I\big)$.
Note that $\mathrm{i}^{1/2} = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}$, $(-\mathrm{i})^{1/2} = \mathrm{i}^{3/2} = -\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}$. Hence $T = \pm(\pm\mathrm{i})^{1/2} I$. □

Or. Becs $\mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix}$. Using $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}$. □

I.9 *Supp V finide, $T \in \mathcal{L}(V)$, and non0 $v \in V$. Let $p \in \mathcal{P}(\mathbf{F})$ be non0 of smallest deg with $p(T)v = 0$. Show every zero of p is eigval of T.*

Solus: Let $p(z) = (z - \lambda)q(z) \Rightarrow p(T)v = 0 = (T - \lambda I)q(T)v \Rightarrow T\big(q(T)v\big) = \lambda q(T)v$. □

• **I.Tips 1:** *Supp V is finide, and $v \in V$.*
(a) *Prove $\exists!$ monic p_v of smallest deg suth $p_v(T)v = 0$.*
(b) *Prove p_v is the min q of $T|_{\operatorname{null} p_v(T)}$.* So that the min of T is a multi of p_v.

Solus: (a) [*Existns*] If $v = 0$, then let $p_v(z) = 1$. Supp $v \neq 0$. Then $(v, Tv, \dots, T^{\dim V} v)$ liney dep.
$\exists$ smallest m suth $-T^m v = c_0 v + c_1 Tv + \cdots + c_{m-1} T^{m-1} v$. Thus define p_v.
Or. Let $\underline{U = \operatorname{span}(v, Tv, \dots, T^{m-1}v)}$ of dim m invard T. Let p_v be the min of $T|_U$.
[*Uniqnes*] Supp q_v is monic of smallest deg $[= \deg p_v]$ and $q_v(T)v = 0$.
Then $(p_v - q_v)(T)v = 0$, while $\deg p_v = m = \deg q_v \Rightarrow \deg(p_v - q_v) < m$.
(b) Becs $p_v(T|_{\operatorname{null} p_v(T)}) = 0 \Rightarrow p_v$ is multi of q. 又 $q(T)v = 0 \Rightarrow q = p_v$, by the min of $\deg p_v$. □

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11 *Supp $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, nonC $p \in \mathcal{P}(\mathbf{F})$.*
Prove α is eigval of $p(T) \Longleftrightarrow \alpha = p(\lambda)$ for some eigval λ of T.

Solus: Supp $p(T) - \alpha I$ not inje. Let $p(z) - \alpha = c(z - \lambda_1)\cdots(z - \lambda_m)$, with $c \neq 0$, becs p nonC.
Then $\exists\, (T - \lambda_j I)$ not inje. Now $p(\lambda_j) - \alpha = 0$. Convly true immed. □

• *Supp non0 $v \in V$. Prove [5.21] using the given map below.*
I.16 *Define $S : \mathcal{P}_{\dim V}(\mathbf{C}) \to V$ by $S(p) = p(T)v$. Then S not inje $\Rightarrow \exists$ non0 $p \in \operatorname{null} S$.*
I.17 *Define $S : \mathcal{P}_{\dim V^2}(\mathbf{C}) \to \mathcal{L}(V)$ by $S(p) = p(T)$. Then S not inje $\Rightarrow \exists$ non0 $p \in \operatorname{null} S$.*
Note: Another proof of [4E 5.22] by Exe (I.17).

I.18 [4E I.15] *Supp* $\mathbf{F} = \mathbf{C}$, *V finide and non0,* $T \in \mathcal{L}(V)$.
Define $f : \mathbf{C} \to \mathbf{N}$ *by* $f(\lambda) = \dim \operatorname{range}(T - \lambda I)$. *Prove f is not continuous.*

Solus: Let λ_0 be eigval of T. Then $(T - \lambda_0 I)$ is not surj. Hence $\dim \operatorname{range}(T - \lambda_0 I) < \dim V$.
Becs T has finily many eigvals. $\exists$ seq $\{\lambda_n\}$ with each λ_n not eigval of T, suth $\lim\limits_{n\to\infty} \lambda_n = \lambda_0$
Becs each $f(\lambda_n) = \dim \operatorname{range}(T - \lambda_n I) = \dim V \neq f(\lambda_0) \Rightarrow f(\lambda_0) \neq \lim\limits_{n\to\infty} f(\lambda_n)$. □

I.19 *Supp V is finide,* $\dim V > 1, T \in \mathcal{L}(V)$. *Prove* $\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V)$.

Solus: If $\forall S \in \mathcal{L}(V), \exists p \in \mathcal{P}(\mathbf{F}), S = p(T)$. Then by [5.20], $\forall S_1, S_2 \in \mathcal{L}(V), S_1 S_2 = S_2 S_1$.
Note that $\dim V \geqslant 2$. By (3.A.14) Or (3.D.16 Or 4E 3.A.11). □

• (4E I.7) *Supp* $S, T \in \mathcal{L}(V)$ *and p, q are mins of ST, TS respectly. Prove S or T is inv* $\Rightarrow p = q$.

Solus: S inv $\Rightarrow p(TS) = S^{-1}p(ST)S = 0$ and $q(ST) = Sq(TS)S^{-1} = 0 \Rightarrow p = q$. Rev the roles. □

• (4E I.21) *Supp V finide,* $T \in \mathcal{L}(V)$. *Prove the min p has deg at most* $1 + \dim \operatorname{range} T$.

Solus: Let q be the min of $T|_{\operatorname{range} T}$. Then $q(T)Tv = 0 \Rightarrow zq(z)$ of deg $< 1 + \dim \operatorname{range} T$ is multi of p. □

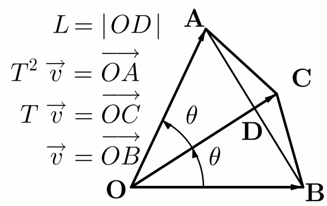
• (4E I.28) *Supp V is finide and* $T \in \mathcal{L}(V)$. *Prove the min p of T' equals the min q of T.*

Solus: (a) $\forall \varphi \in V', p(T')(\varphi) = \varphi \circ p(T) = 0 \Rightarrow p(T) \in \operatorname{null} \varphi$. Thus $p(T) = 0$. And $q(T') = 0$. □
Or. By (3.F.15), for any $s \in \mathcal{P}(\mathbf{F})$, $s(T') = s(T)' = 0 \iff s(T) = 0$. Simlr. □

• (8.C.18 Or 4E I.16) *Define* $T \in \mathcal{L}(\mathbf{F}^n) : (x_1, \dots, x_n) \mapsto \big(-a_0 x_n,\ x_1 - a_1 x_n,\ \cdots,\ x_{n-1} - a_{n-1} x_n\big)$.
Show the min p of T is $q(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$.

Solus: Becs $Te_1 = e_2, T^2 e_1 = e_3, \cdots, T^{n-1} e_1 = e_n, T^n e_1 = T^{n-k} e_{k+1} = Te_n = -(a_0 e_1 + \cdots + a_{n-1} e_n)$.
Let $-T^n = c_0 I + c_1 T + \cdots + c_{n-1} T^{n-1} \Rightarrow$ each $c_k = a_k$. Becs $n = \dim V$. No greater deg. □

• (4E I.8) *Find the min p of* $T \in \mathcal{L}(\mathbf{R}^2)$, *the countclockws rotat optor by* $\theta \in \mathbf{R}^+$.



Solus: If $\theta = 2k\pi$, then $p(z) = z - 1$. If $\theta = \pi + 2k\pi$, then $p(z) = z + 1$.
Othws, let $\operatorname{span}(v, Tv) = \mathbf{R}^2$. Let $L = x^2 + y^2$, where $v = (x, y)$.
Supp $p(z) = z^2 + bz + c$. Let $P = L\cos\theta \Rightarrow L/2P = 1/(2\cos\theta)$.
Then $Tv = (L/2P)(T^2 v + v) \Rightarrow T = (L/2P)(T^2 + I)$.
Hence $p(T) = T^2 - 2\cos\theta\, T + I = 0$. □

Or. Let (e_1, e_2) be the std bss. Becs $Te_1 = \cos\theta\, e_1 + \sin\theta\, e_2$, $T^2 e_1 = \cos 2\theta\, e_1 + \sin 2\theta\, e_2$.
$ce_1 + bTe_1 = -T^2 e_1 \iff \begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix} \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$. Now $\det = \sin\theta \neq 0$, $c = 1$, $b = -2\cos\theta$. □

• (4E I.11) *Supp V is 2-dim,* $T \in \mathcal{L}(V)$ *with the min p, and* $\mathcal{M}\big(T, (v, w)\big) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.
(a) *Show* $q(z) = z^2 - (a + d)z + (ad - bc)$ *is a multi of p.*
(b) *Show if* $b = c = 0$ *and* $a = d$, *then* $p(z) = z - a$; *othws* $p = q$.

Solus: (a) $Tv = av + bw \Rightarrow (T - aI)v = bw \Rightarrow (T - dI)(T - aI)v = bTw - bdw = bcv$.
$Tw = cv + dw \Rightarrow (T - dI)w = cv \Rightarrow (T - aI)(T - dI)w = cTv - acv = bcw$.
(b) If $b = c = 0$ and $a = d$. Then $\mathcal{M}(T) = a\mathcal{M}(I) \Rightarrow T = aI$. Othws, we show $T \notin \operatorname{span}(I)$,
so that $\deg p = \dim V$. Let (1) $a = d$, (2) $b = 0$, (3) $c = 0$. Then (1), (2) and (3) cannot be all true.
(I) Asum (1) is true, with (2) or (3) not true. Then $Tv = av + bw$, or $Tw = cv + aw \notin \operatorname{span}(w)$.
(II) Asum (2) or (3) are true, with (1) not true. Then $Tv = av + bw$, or $Tw = cv + dw$. □

• (4E I.29) *Supp V is finide,* $\dim V = n \geqslant 2$, *and* $T \in \mathcal{L}(V)$. *Show T has a 2-dim invarsp.*

Solus: See [9.8] for a graceful proof. Or. Let each V_k be an arb vecsp of dim k with an arb $T_k \in \mathcal{L}(V_k)$.
Define the stmt $P(k)$: every optor on a V_k has invarsp of dim 2. (i) $k = 2$. Immed.
(ii) $k \geqslant 2$. Asum $P(k)$ holds. Let p be the min of $T_{k+1} = T$. Note that V_{k+1} non0 $\Rightarrow p$ nonC, $\deg p \geqslant 1$.
(a) If $p(z) = (z - \lambda)q(z)$, then by (4E 5.A.39), $\exists\, U$ invarspd T of dim k.
By asum, the optor $T|_U$ on a k-dim vecsp has invarsp of dim 2, so has T.
(b) Othws, T_{k+1} has no eigvals $\Rightarrow p$ of deg $\geqslant 1$ has no zeros, thus $\mathbf{F} = \mathbf{R}$, and $\deg p$ is even.
Let $p(z) = (z^2 + b_1 z + c_1) \cdots (z^2 + b_m z + c_m) \Rightarrow \exists\, (T^2 + b_j T + c_j)$ not inje
$\Rightarrow \exists\, v \neq 0, (T^2 + b_j T + c_j)v = 0 \Rightarrow T^2 v \in \operatorname{span}(v, Tv)$, invard T, while $\dim\operatorname{span}(v, Tv) = 2$. ☐

• **Note For [4E 5.33]:** *Supp* $\mathbf{F} = \mathbf{R}$, *V is finide,* $T \in \mathcal{L}(V)$, *and* $b^2 < 4c$ *for* $b, c \in \mathbf{F}$.
Prove $\dim\operatorname{null}(T^2 + bT + cI)^j$ *is even for each* $j \in \mathbf{N}^+$.

Solus: Using induc on j. (i) Immed. (ii) $j > 1$. Asum it holds for $j - 1$.
Replace V with $\operatorname{null}(T^2 + bT + cI)^j$ and T with T restr to $\operatorname{null}(T^2 + bT + cI)^j$.
Then $(T^2 + bT + cI)^j = 0 \Rightarrow (z^2 + bz + c)^j$ is a multi of the min of $T \Rightarrow$ no eigvecs for T.
Let U be invarspd T and has the largest even dim of all such invarsp. If $V = U$, done. Othws,
for $w \in V \backslash U \Rightarrow W = (w, Tw)$ invard T of dim $2 \Rightarrow U + W$ of dim $(\dim U + 2)$ invard T. ☐
Or. Let $q(z) = z^2 + bz + c$. Note that the min of T restr to $\operatorname{null} q(T)^j$ have no real zeros.
If some $\dim\operatorname{null} q(T)^j$ is odd. Then T restr to $\operatorname{null} q(T)^j$ must have a real eigval, ctradic. ☐

• *Supp V finide,* $T \in \mathcal{L}(V)$ *with the min p.*

• (4E I.13) *Prove* $\forall q \in \mathcal{P}(\mathbf{F}), \exists!\, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), q(T) = r(T)$.

Solus: Becs $p \neq 0$. By the div algo, immed. $\big[r = 0$ if $q = p.\big]$ Or. By Exe (4E I.19). ☐
Or. Let $\deg p = m$. Becs $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$. For $\deg q < m$, the repres of $q(T)$ is uniq.
If $\deg q \geqslant m$. For each $k \in \mathbf{N}, \exists!\, b_{j,k} \in \mathbf{F}$, $T^{m+k} = b_{0,k} I + b_{1,k} T + \cdots + b_{m-1,k} T^{m-1}$. ☐

• (4E I.19) *Let* $\mathcal{E} = \big\{q(T) : q \in \mathcal{P}(\mathbf{F})\big\}$, *a subsp of* $\mathcal{L}(V)$. *Prove* $\dim \mathcal{E} = \deg p$.

Solus: Becs $\mathcal{E} = \operatorname{span}\big(I, T, \dots, T^{\dim\mathcal{L}(V) - 1}\big) = \operatorname{span}\big(I, T, \dots, T^{\deg p - 1}\big)$, by Exe (4E I.13). Immed. ☐
Or. Define $\Phi \in \mathcal{L}\big(\mathcal{P}(\mathbf{F}), \mathcal{L}(V)\big)$ by $\Phi(q) = q(T) \Rightarrow \operatorname{range}\Phi = \mathcal{E}$.
Becs $\Phi(q) = q(T) = 0 \iff q$ is a multi of the min $p \iff q \in \big\{ps : s \in \mathcal{P}(\mathbf{F})\big\} = \operatorname{null}\Phi$.
Now by (4.11), $\dim \mathcal{P}(\mathbf{F})/\operatorname{null}\Phi = \deg p$. By [3.91](d). ☐

• (8.C.11) *Supp* $T \in \mathcal{L}(V)$ *is inv. Prove* $\exists\, q \in \mathcal{P}(\mathbf{F}), T^{-1} = q(T)$.

Solus: Becs the const term of p is non0. Let $I = a_1 T + \cdots + a_m T^m \Rightarrow T^{-1} = a_1 I + a_2 T + \cdots + a_m T^{m-1}$. ☐

• (4E I.14) *Supp* $p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m$, *and* $a_0 \neq 0$.
Give a repres of s, the min of T^{-1}.

Solus: Define $q(z) = z^m + \frac{a_1}{a_0} z^{m-1} + \cdots + \frac{a_{m-1}}{a_0} z + \frac{1}{a_0} \Rightarrow q(T^{-1}) = T^{-m} p(T) = 0$. ☐
Becs $\deg s \leqslant \deg q$, while $(T^{-1})^{-1} = T \Rightarrow \deg q \leqslant \deg s$. Or. By (8.C.11), immed. ☐
Or. Becs each $T^{-k} \notin \operatorname{span}\big(I, T^{-1}, \dots, T^{-(k-1)}\big)$ for $k \in \big\{1, \dots, m-1\big\}$. Done.
For if not, supp $T^{-k} = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. Note that T inv $\Rightarrow \exists\, b_j \neq 0$.
Now $T^k(T^{-k}) = I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T \Rightarrow T^j \in \operatorname{span}(I, T, \dots, T^{k-1})$. ☐

• (4E I.17) *Show the min s of $(T-\lambda I)$ is $q(z)=p(z+\lambda)$.*

Solus: Becs $q(T-\lambda I)=p(T)=0 \Rightarrow q$ a multi of $s \Rightarrow \deg q=\deg p \geqslant \deg s$.
Define $r(z)=s(z-\lambda) \Rightarrow r(T)=s(T-\lambda I)=0 \Rightarrow \deg r=\deg s \geqslant \deg p$. $\square$
Or. Becs $T^k \in \operatorname{span}(I,T,\dots,T^{k-1}) \iff (T-\lambda I)^k \in \operatorname{span}\big(I,(T-\lambda I),\dots,(T-\lambda I)^{k-1}\big)$. $\square$

• (4E I.18) *Supp $\deg p=m$, and $\lambda \neq 0$. Show the min s of λT is $q(z)=\lambda^m p\big(z/\lambda\big)$.*

Solus: Becs $q(\lambda T)=\lambda^m p(T)=0 \Rightarrow q$ is multi $s \Rightarrow \deg q=\deg p \geqslant \deg s$.
Define $r(z)=s(\lambda z) \Rightarrow r(T)=s(\lambda T)=0 \Rightarrow \deg r=\deg s \geqslant \deg p$. $\square$
Or. Becs $(\lambda T)^k \in \operatorname{span}\big(\lambda I,\lambda T,\dots,(\lambda T)^{k-1}\big) \iff T^k \in \operatorname{span}(I,T\dots,T^{k-1})$. $\square$

• (4E I.10,23) *Supp $\deg p=m$, and non0 $v \in V$. Let each $U_k=\operatorname{span}\big(v,Tv,\dots,T^k v\big)$.
Prove $\exists\, j \in \{1,\dots,m\}$, $U_{j-1}=U_n$ for all $n \geqslant j-1$.*

Solus: Supp j is the smallest suth $T^j v=a_0 v+a_1 Tv+\cdots+a_{j-1}T^{j-1}v \in U_{j-1} \Rightarrow j \leqslant m$.
Then U_{j-1} is invard T, so is each $U_n=\operatorname{span}(v,Tv,\dots,T^{j-1}v,\dots,T^n v)$. $\square$

II.2 *Supp A and B are up-trig (and square) matrices of the same size,
with $\alpha_1,\dots,\alpha_n$ on the diag of A and $\beta_1,\dots,\beta_n$ on the diag of B.*
(a) *Show $A+B$ up-trig with $\alpha_1+\beta_1,\dots,\alpha_n+\beta_n$ on the diag.*
(b) *Show AB up-trig with $\alpha_1\beta_1,\dots,\alpha_n\beta_n$ on the diag.*

Solus: (a) By def, immed. (b) Becs $A_{j,k}=B_{j,k}=0$ for $j>k$. By def, for each $p \in \{1,\dots,n\}$,
$(AB)_{p,p}=A_{p,1}B_{1,p}+\cdots+A_{p,p-1}B_{p-1,p}+A_{p,p}B_{p,p}+A_{p,p+1}B_{p+1,p}+\cdots+A_{p,n}B_{n,p}=A_{p,p}B_{p,p}$. $\square$

II.3 *Supp T inv, $B_V=(v_1,\dots,v_n)$, $\mathcal{M}(T)=A$ is up-trig,
with $\lambda_1,\dots,\lambda_n$ on diag. Show A^{-1} is also up-trig, with $\lambda_1^{-1},\dots,\lambda_n^{-1}$ on diag.*

Solus: Becs λ_k on diag of $A \iff \lambda_k$ eigval of $T \iff \lambda_k^{-1}$ eigval of $T^{-1} \iff \lambda_k^{-1}$ on diag of A^{-1}. $\square$

Or. Let each $Tv_k=u_k+\lambda_k v_k$, where $u_k \in \operatorname{span}(v_1,\dots,v_{k-1})$. We use induc on k.
(i) $k=1$. $Tv_1=\lambda_1 v_1 \Rightarrow T^{-1}v_1=\lambda_1^{-1}v_1 \in \operatorname{span}(v_1)$, invard T^{-1}; and λ_1^{-1} is the 1st ent on diag.
(ii) $k \geqslant 2$. Asum $\operatorname{span}(v_1,\dots,v_{k-1})$ invard T^{-1}.
Note that $Tv_k=u_k+\lambda_k v_k \Rightarrow v_k=T^{-1}(c_1v_1+\cdots+c_{k-1}v_{k-1})+\lambda_k T^{-1}v_k$.
Thus $T^{-1}v_k=\lambda_k^{-1}v_k-\lambda_k^{-1}T^{-1}u_k \in \operatorname{span}(v_1,\dots,v_k)$, invard T; and λ_k^{-1} is the k^{th} ent on diag. $\square$

II.8 *Supp V is finide, and $v \in V$ is non0 suth $q(T)v=0$, where $q(z)=z^2+2z+2$.*
(a) *Supp $\mathbf{F}=\mathbf{R}$. Prove $\nexists\, B_V$ suth $\mathcal{M}(T)$ up-trig.*
(b) *Supp $\mathbf{F}=\mathbf{C}$, and $\exists\, B_V$ suth $A=\mathcal{M}(T)$ up-trig. Prove $-1+\mathrm{i}$ or $-1-\mathrm{i}$ on diag.*

Solus: Define p_v as (4E 3.C.7). Note that $\deg p_v \geqslant 1$ becs $v \neq 0$. 又 $q(T|_{\operatorname{null} p_v(v)})=0$.
Now q of deg 2 is a multi of the min of $T|_{\operatorname{null} p_v(v)}$, which is p_v, of which the min of T is a multi.
(a) Note that q has no 1-deg factors $\Rightarrow \deg p_v \geqslant 2$. By [4E 5.44].
(b) $q(z)=(z+1+\mathrm{i})(z+1-\mathrm{i}) \Rightarrow -1-\mathrm{i}$ or $-1+\mathrm{i}$ zero of $p_v \Rightarrow$ is eigval $\Rightarrow$ on diag. $\square$

• **II.Tips 1:** Supp $B_V=(v_1,\dots,v_n), B_{V'}=(\varphi_1,\dots,\varphi_n), T \in \mathcal{L}(V), A=\mathcal{M}(T,B_V)$.
(a) A up-trig $\iff T=\sum_{k=1}^n\sum_{j=1}^k A_{j,k}E_{k,j} \iff T'=\sum_{k=1}^n\sum_{j=1}^k A_{k,j}^t \Xi_{j,k} \iff A^t$ low-trig.
(b) A low-trig $\iff T=\sum_{k=1}^n\sum_{j=1}^k A_{k,j}E_{j,k} \iff T'=\sum_{k=1}^n\sum_{j=1}^k A_{j,k}^t \Xi_{k,j} \iff A^t$ up-trig.

• **II.Tips 2**: *Supp $(\alpha_1,\dots,\alpha_n),(\beta_1,\dots,\beta_n)$ are bses of V, with each $\alpha_k=\beta_{n-k+1}$.*
Prove $\mathcal{M}(T,\alpha\to\alpha)$ up-trig $\iff \mathcal{M}(T,\beta\to\beta)$ low-trig.

Solus: For each $k\in\{1,\dots,n\}$, $T\beta_{n-k+1}=T\alpha_k\in\operatorname{span}(\alpha_1,\dots,\alpha_k)=\operatorname{span}(\beta_n,\dots,\beta_{n-k+1})$. ☐

Coro: (a) Supp $\mathbf{F}=\mathbf{C}$. Then $\exists\, B_V$ suth $\mathcal{M}(T,B_V)$ low-trig. (b) T up-trig $\iff T'$ up-trig.

II.12,13 *Supp V finide, $T\in\mathcal{L}(V)$. Prove $T|_U, T/U$ up-trig for some invarsp $U \iff T$ up-trig.*

Solus: Supp $B_U=(u_1,\dots,u_p), B_{V/U}=(w_1+U,\dots,w_q+U)$ suth $\mathcal{M}(T|_U),\mathcal{M}(T/U)$ up-trig.
Then each $Tu_k\in\operatorname{span}(u_1,\dots,u_k)$ and each $Tw_j+U\in\operatorname{span}(w_1+U,\dots,w_j+U)$.
By (3.E.13), $B_V=(u_1,\dots,u_p,w_1,\dots,w_q)$. Now each $Tw_j\in\operatorname{span}(u_1,\dots,u_p,w_1,\dots,w_j)$. ☐
Or. By (4E 5.B.25)(b) and [4E 5.44], immed. Convly, by [4E 5.44], immed. ☐

Ended

5.C & [4E] 5.D

注意: 这一节的题号主要使用第四版 5.D 节.

3 *Supp $T\in\mathcal{L}(V)$ is diag. Prove $V=\operatorname{null}T\oplus\operatorname{range}T$.*

Solus: Let $U=E(\lambda_1,T)\oplus\cdots\oplus E(\lambda_m,T)$, where each $\lambda_k\neq 0$ and $B_{E(\lambda_k,T)}=(v_{1,k},\dots,v_{M_k,k})$.
By (3.B.12), $\operatorname{range}T=\{Tu: u\in U\}=\left\{\sum_{k=1}^m\lambda_k(a_{1,k}v_{1,k}+\cdots+a_{M_k,k}v_{M_k,k}) : a_{j,k}\in\mathbf{F}\right\}=U$. ☐

Exa: Convly not true. Define the inv $T\in\mathcal{L}(\mathbf{R}^2):(x,y)\mapsto(-y,x)$. No eigvals.

L1 *Supp $T\in\mathcal{L}(V),\alpha,\beta\in\mathbf{F}$ and $\alpha\neq\beta$. Prove $\operatorname{null}(T-\alpha I)\subseteq\operatorname{range}(T-\beta I)$.*

Solus: $\forall v\in\operatorname{null}(T-\alpha I), Tv=\alpha v\Rightarrow (T-\beta I)\big[v/(\alpha-\beta)\big]=v\in\operatorname{range}(T-\beta I)$. ☐

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5 *Supp $\mathbf{F}=\mathbf{C}$, V is finide, and $T\in\mathcal{L}(V)$.*
Supp $V=\operatorname{null}(T-\lambda I)\oplus\operatorname{range}(T-\lambda I)$ for all $\lambda\in\mathbf{C}$. Prove T is diag.

Solus: (i) $\dim V=1$. Immed. (ii) $\dim V>1$. Asum it holds for vecsps of smaller dim.
$\exists$ eigval $\lambda_0\Rightarrow U=\operatorname{range}(T-\lambda_0 I)$ invard $T\Rightarrow U=\operatorname{null}(T|_U-\lambda I)\oplus\operatorname{range}(T|_U-\lambda I)$.
While $V=E(\lambda_0,T)\oplus U\Rightarrow\dim U<\dim V$. By asum, $T|_U$ is diag wrto B_U of eigvecs. ☐

Or. Supp T not diag. We show $\exists\,\lambda\in\mathbf{C}$, $\operatorname{null}(T-\lambda I)\cap\operatorname{range}(T-\lambda I)\neq\{0\}$.
Let the min of T be $p(z)=(z-\lambda_1)^{\alpha_1}\cdots(z-\lambda_m)^{\alpha_m}$, where each $\alpha_k\geqslant 1$ and $\exists\,\alpha_j>1$.
Let $q(z)(z-\lambda_j)=p(z)\Rightarrow 0=p(T)=(T-\lambda_j I)q(T)\Rightarrow\operatorname{range}q(T)\subseteq\operatorname{null}(T-\lambda_j I)$.
Let $q(z)=(z-\lambda_j)s(z)\Rightarrow\operatorname{range}q(T)\subseteq\operatorname{range}(T-\lambda_j I)$. Note that $q(T)\neq 0$. ☐

Or. Let $\lambda_1,\dots,\lambda_m$ be disti eigvals. Now $V=\operatorname{null}(T-\lambda_k I)\oplus\operatorname{range}(T-\lambda_k I)$ for each λ_k.
Asum $V=\left[\bigoplus_{i=1}^j\operatorname{null}(T-\lambda_j I)\right]\oplus\left[\bigcap_{i=1}^j\operatorname{range}(T-\lambda_j I)\right]$ for $j\in\{1,\dots,m-1\}$.
Becs $\bigcap_{i=1}^j\operatorname{range}(T-\lambda_i I)\supseteq\operatorname{null}(T-\lambda_{j+1}I)$. By (L1) and [1.C Tips (3)],
$\bigcap_{i=1}^j\operatorname{range}(T-\lambda_i I)=\operatorname{null}(T-\lambda_{j+1}I)\oplus\left[\bigcap_{i=1}^j\operatorname{range}(T-\lambda_j I)\cap\operatorname{range}(T-\lambda_{j+1}I)\right]$.
By induc, $V=\big[\operatorname{null}(T-\lambda_1 I)\oplus\cdots\oplus\operatorname{null}(T-\lambda_m I)\big]\oplus\big[\operatorname{range}(T-\lambda_1 I)\cap\cdots\cap\operatorname{range}(T-\lambda_m I)\big]$.
Asum $U=\bigcap_{k=1}^m\operatorname{range}(T-\lambda_k I)\neq\{0\}$. Becs U invard T. Thus $\exists\,\mu=\lambda_j$ eigval of $T|_U$. Ctradic. ☐

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13 *Supp $A, B \in \mathbf{F}^{n,n}$ and A is diag with **dist** ents on diag. Prove $AB = BA \iff B$ is diag.*

SOLUS: NOTICE that for any diag C, each $C_{j,k} = 0$ for $j \neq k$.

Becs (I) $A_{j,j}B_{j,k} = A_{j,1}B_{1,k} + \cdots + \left[A_{j,j}B_{j,k}\right] + \cdots + A_{j,n}B_{n,k} = (AB)_{j,k}$.

And (II) $B_{j,k}A_{k,k} = B_{j,1}A_{1,k} + \cdots + \left[B_{j,k}A_{k,k}\right] + \cdots + B_{j,n}A_{n,k} = (BA)_{j,k}$.

Supp B diag. If $j = k$, then $(BA)_{j,k} = (AB)_{j,k}$, othws true as well.

Supp $AB = BA \Rightarrow A_{j,j}B_{j,k} = A_{k,k}B_{j,k}$. Asum $B_{j,k} \neq 0$ with $j \neq k$. Then $A_{j,j} = A_{k,k}$, ctradic. ☐

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14 *Supp $\mathbf{F} = \mathbf{C}$, $k \in \mathbf{N}^+$, and $T \in \mathcal{L}(V)$ is inv. Prove T^k diag $\Rightarrow T$ diag.*

SOLUS: Let the min of T^k be $p(z) = (z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow$ each λ_k non0 and disti.

Becs any non0 $\lambda \in \mathbf{C}$ has k disti k^{th} roots. Let $\{\mu_{1,j}, \dots, \mu_{k,j}\}$ be the roots of $z^k = \lambda_j$.

For $x, y \in \{1, \dots, n\}$, $x \neq y \iff \mu_{p,x}^k = \lambda_x \neq \lambda_y = \mu_{q,y}^k$ for each $p, q \in \{1, \dots, k\} \Rightarrow \mu_{p,x} \neq \mu_{q,y}$.

Thus all μ's are dist. Let $s(z) = (z^k - \lambda_1) \cdots (z^k - \lambda_m) = \prod_{j=1}^{m} \prod_{i=1}^{k} (z - \mu_{i,j}) \Rightarrow s(T) = 0$. ☐

EXA: Not true if $\mathbf{F} = \mathbf{R}$. Define $T \in \mathcal{L}(\mathbf{R}^2) : (x, y) \mapsto (-y, x)$. No eigvals.

• *Supp $\mathbf{F} = \mathbf{C}$, $n \in \mathbf{N}^+$, $p \in \mathcal{P}(\mathbf{F})$. Prove $T \in \mathcal{L}(V)$ is diag $\iff \operatorname{null} p(T) = \operatorname{null}\left[p(T)\right]^n$.*

SOLUS: (a) Supp T diag. Let $p(z) = (z - \alpha_1) \cdots (z - \alpha_m)$. We show each $\operatorname{null}(T - \alpha_k I)^n = \operatorname{null}(T - \alpha_k I)$.

Done if $T - \alpha_k I = S$ inje. Supp S not inje. NOTICE that $\operatorname{null} S|_{\operatorname{range} S} = \operatorname{null} S \cap \operatorname{range} S = \{0\}$.

By (3.B.22), $\dim \operatorname{null} S^2 = \dim \operatorname{null} S \Rightarrow \operatorname{null} S^2 = \operatorname{null} S$. Asum $\operatorname{null} S^j = \operatorname{null} S$ for $j \geqslant 2$.

Becs $\dim \operatorname{null}(S^j S) = \dim(\operatorname{null} S^j \cap \operatorname{range} S) + \dim \operatorname{null} S$. By induc.

(b) Supp $\operatorname{null}(T - \lambda I) = \operatorname{null}(T - \lambda I)^n$ for all $\lambda \in \mathbf{C}$. Let $\lambda_1, \dots, \lambda_m$ be disti eigvals of T.

Define $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$. Then $\left[p(T)\right]^{\dim V} = 0 \Rightarrow p(T) = 0 \Rightarrow p$ is the min. ☐

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18 *Supp $T \in \mathcal{L}(V)$ is diag. Prove $T/U \in \mathcal{L}(V/U)$ is diag for any U invarspd T.*

SOLUS: By [5.A TIPS (2)], $\exists\, B_U = (v_1, \dots, v_m)$ consists of eigvecs of T.

Extend to eigvecs $B_V = (v_1, \dots, v_m, w_1, \dots, w_p) \Rightarrow B_{V/U} = (w_1 + U, \dots, w_p + U)$.

Becs for each w_k, $\exists$ eigval λ of T, $Tw_k = \lambda w_k \Rightarrow (T/U)(w_k + U) = \lambda w_k + U$. ☐

OR. Becs the min of T is multi of that of T/U. By [4E 5.62]. ☐

COMMENT: Exa [5.15] gives a exa of $T \in \mathcal{L}(V)$ not diag while $T|_U, T/U$ diag. ☐

ENDED

5.E [4E]

8 *Find a bss of $\mathcal{P}_m(\mathbf{R}^2)$ suth D_x, D_y up-trig in [5.72].*

Solus: Let $B = (1, x, y, x^2, xy, y^2, \cdots, \cdots, x^m, x^{m-1}y, \cdots, xy^{m-1}, y^m)$ in $\mathcal{P}_m(\mathbf{R}^2)$.
Supp a liney combina of B is 0; $\sum_{j=0}^{m}\sum_{k=0}^{m-j} a_{j,k}x^j y^k = 0$.
Let $x = 0 \Rightarrow$ each $a_{0,k} = 0$, and $y = 0 \Rightarrow$ each $a_{k,0} = 0$. Now $\sum_{j=1}^{m-1}\sum_{k=1}^{m-1-j} a_{j,k}x^j y^k = 0$.
Take $\big((x_1, y_1), \cdots, (x_q, y_q)\big)$ $\big[$where $q = 1 + \cdots + m\big]$ suth all $\sum_{j=1}^{m-1}\sum_{k=1}^{m-1-j} x_s^j y_t^k a_{j,k} = 0$
form a system of q equations having uniq solus $(0, \dots, 0)$. Thus B is liney indep.
Apply D_x to each vec in $B \Rightarrow B_x = \big(0, 1, 0, 2x, y, 0, \cdots, \cdots, mx^{m-1}, (m-1)x^{m-2}y, \cdots, y^{m-1}, 0\big)$.
Apply D_y to each vec in $B \Rightarrow B_y = \big(0, 0, 1, 0, x, 2y, \cdots, \cdots, 0, x^{m-1}, \cdots, (m-1)xy^{m-2}, my^{m-1}\big)$. □

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6 *Supp $\mathbf{F} = \mathbf{C}$, V is finide, and $S, T \in \mathcal{L}(V)$ commu.*
Prove $\exists\, \alpha, \lambda \in \mathbf{C}$ suth $\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V$.

Solus: Supp $A, C \in \mathbf{F}^{n,n}$ are up-trig matrices of S, T wrto a $B_V = (v_1, \dots, v_n)$ suth A, C commu.
Let $\alpha = A_{n,n}$, $\lambda = C_{n,n}$. Then $\operatorname{range}(S - \alpha I), \operatorname{range}(T - \lambda I) \subseteq \operatorname{span}(v_1, \dots, v_{n-1})$. □

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7 *Supp $\mathbf{F} = \mathbf{C}$, and $S, T \in \mathcal{L}(V)$ commu, S diag. Prove $\exists\, B_V$ suth S diag and T up-trig.*

Solus: Let $\lambda_1, \dots, \lambda_m$ be disti eigvals of $S \Rightarrow V = E(\lambda_1, S) \oplus \cdots \oplus E(\lambda_m, S)$.
Becs each $E_k = E(\lambda_k, S)$ invard T. Let each $T|_{E_k}$ be up-trig with $B_{E_k} = (v_{1,k}, \dots, v_{M_k,k})$.
Then S diag while T up-trig with the same $B_V = (v_{1,1}, \dots, v_{M_n,n})$. □

Or. Using induc on $n = \dim V$. (i) $n = 1$. Immed. (ii) $n > 1$. Asum it holds for smaller V.
$\exists$ eigval λ of S, $U = \operatorname{null}(S - \lambda I), W = \operatorname{range}(S - \lambda I) \Rightarrow V = \operatorname{null}(S - \lambda I) \oplus \operatorname{range}(S - \lambda I)$.
Apply the asum to $T|_U, S|_U$ and $T|_W, S|_W$, then put B_U, B_W together. □

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2 *Supp $\mathcal{E} \subseteq \mathcal{L}(V)$ and every elem of $\mathcal{E}$ diag.*
Prove each pair of elems of $\mathcal{E}$ commu $\Rightarrow \exists\, B_V$ suth all elem of $\mathcal{E}$ diag.

Solus: Let $\dim V = n \Rightarrow \dim \mathcal{L}(V) = n^2$.
$\exists\, \{T_1, \dots, T_m\} \subseteq \mathcal{E}$ with each elem of $\mathcal{E}$ in $\operatorname{span}(T_1, \dots, T_m)$ and $m \leqslant n^2$
For each T_k, becs $V = \bigoplus_{\lambda_k} E(\lambda_k, T_k)$ and $E(\lambda_k, T_k)$ non0 for finily many $\lambda_k \in \mathbf{F}$.
Becs $U_k = E(\lambda_1, T_1) \cap \cdots \cap E(\lambda_k, T_k) = E(\lambda_k, T_k|_{U_{k-1}}) = \bigoplus_{\lambda_{k+1}} E(\lambda_{k+1}, T_{k+1}|_{U_k}) = \bigoplus_{\lambda_{k+1}} U_{k+1}$.
Hence $V = \bigoplus_{\lambda_1} E(\lambda_1, T_1) = \bigoplus_{\lambda_1, \dots, \lambda_m}\big[E(\lambda_1, T_1) \cap \cdots \cap E(\lambda_m, T_m)\big]$. Take bss of each summand.
Then we form B_V. For any $T \in \mathcal{E}$, $\mathcal{M}(T, B_V) = c_1\mathcal{M}(T_1, B_V) + \cdots + c_m\mathcal{M}(T_m, B_V)$. □

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9 *Supp $\mathbf{F} = \mathbf{C}$, V finide and non0. Supp $\mathcal{E} \subseteq \mathcal{L}(V)$ is suth all $S, T \in \mathcal{E}$ commu.*
(a) *Prove $\exists$ eigvec $v \in V$ of all elem of $\mathcal{E}$.* (b) *$\exists\, B_V$ suth all elem of $\mathcal{E}$ has up-trig matrix.*

Solus: Simlr to Exe (2). $\exists\, \{T_1, \dots, T_m\} \subseteq \mathcal{E}$. Let $U_0 = V, U_k = E(\lambda_1, T_1) \cap \cdots \cap E(\lambda_k, T_k)$.
(a) Let $\lambda_1, \dots, \lambda_m$ be eigvals of $T_1, \dots, T_m$ respectly with each λ_k eigval of $T_k|_{U_k} \Rightarrow U_k \neq 0$
Now for non0 $v \in U_m$, $\forall T = c_1T_1 + \cdots + c_mT_m \in \mathcal{E}, Tv = (c_1\lambda_1 + \cdots + c_m\lambda_m)v$.
(b) Using induc on $\dim V$. (i) Immed. (ii) $\dim V > 1$. Asum it holds for smaller V.
Let v_1 be a common eigvec of all T_k. Let $W \oplus \operatorname{span}(v_1) = V, P : av_1 + w \mapsto w$.
Simlr in [4E 5.80], each pair of $\{\hat{T}_1, \dots, \hat{T}_m\}$ commu. By asum, $\exists\, B_W \Rightarrow \exists\, B_V$.
Now each $\mathcal{M}(T_k, B_V)$ up-trig $\Rightarrow \forall T \in \mathcal{E}, \mathcal{M}(T) = c_1\mathcal{M}(T_1) + \cdots + c_m\mathcal{M}(T_m)$, wrto B_V. □

Ended

 NOTE: V denotes a finide non0 vecsp over $\mathbf{F}$. An Exe marked by ■ is true if infinide or partially finide.

• **NEW NOTA:** *m_T denotes the min poly of T.*

A.3 *Supp $T \in \mathcal{L}(V)$ inv. Prove $G(\lambda, T) = G(\lambda^{-1}, T^{-1})$ for any non0 $\lambda \in \mathbf{F}$.* ■

SOLUS: $(T - \lambda I)^j v = 0 = \sum_{i=0}^{j} C_j^i(-\lambda)^{j-i} T^i v$. Apply $(-\lambda)^{-j} T^{-j}$ to both sides. $(T^{-1} - \lambda^{-1})^j v = 0$. □

OR. We use induc on j to show each $\operatorname{null}(T - \lambda I)^j = \operatorname{null}(T^{-1} - \lambda^{-1})^j$. (i) Immed. (ii) $j > 1$.
Asum true for $(j-1)$. $\forall v \in \operatorname{null}(T - \lambda I)^j$, $(T - \lambda I)v \in \operatorname{null}(T - \lambda I)^{j-1} = \operatorname{null}(T^{-1} - \lambda^{-1} I)^{j-1}$.
Thus $0 = (T^{-1} - \lambda^{-1} I)^{j-1}(T - \lambda I)v = \underline{(T - \lambda I)}(T^{-1} - \lambda^{-1} I)^{j-1} v$. By (i) and rev the roles. □

A.5 *Supp $T \in \mathcal{L}(V)$, $T^{n-1}v \neq 0$, $T^n v = 0$. Prove $(v, Tv, \dots, T^{n-1}v)$ is liney indep.*

SOLUS: $a_0 v + a_1 Tv + \cdots + a_{n-1} T^{n-1} v = 0 \Rightarrow a_0 T^{n-1} v = 0 \Rightarrow a_0 = 0$. Simlr for $a_1, \dots, a_{n-1}$. ■

• **NOTE FOR [8.19] OR [4E 8.18]:** If $m_T(z) = z^m$. Then $\exists\, v$ suth $T^{m-1}v \neq 0$. If $m = \dim V$.
Now $B_V = (T^{m-1}v, \dots, Tv, v)$. Let each $w_k = T^{m-k}v$. Then $Tw_1 = 0$ and each $T(w_k) = w_{k-1}$.

A.6 *Supp $T \in \mathcal{L}(V)$ nilp, $n = \dim V$, $T^{n-1} \neq 0$. Prove $\nexists\, S \in \mathcal{L}(V)$, $S^k = T$ for all $k > 1$.*

SOLUS: Asum $\exists$ suth S. Then $\operatorname{null} S^{kn} = \operatorname{null} T^n = V = \operatorname{null} S^{kn-1} = \cdots = \operatorname{null} S^n$.
Note that $\exists\, j$ suth $\operatorname{null} S^{kn-j} = \operatorname{null} T^m$ for some $m \in \{1, \dots, n-1\}$. □

• (4E A.4) *Supp $T \in \mathcal{L}(V)$, $\lambda \in \mathbf{F}$, and m_T is a multi of $(z - \lambda)^m$ with $m \in \mathbf{N}^+$.*
Prove $\dim \operatorname{null}(T - \lambda I)^m \geqslant m$. **CORO:** $\dim G(\lambda, T) \geqslant m$. ■

SOLUS: Becs λ is eigval of T. We show z^m is the min of $(T - \lambda I)\big|_{\operatorname{null}(T-\lambda I)^m}$.
Using induc on m. (i) $m = 1$. Becs $\dim E(\lambda, T) \geqslant 1$. (ii) $m > 1$. Asum it holds for $(m-1)$.
$\dim \operatorname{null}(T - \lambda I)(T - \lambda I)^{m-1} = \dim \operatorname{null}(T - \lambda I)\big|_{\operatorname{null}(T-\lambda I)^{m-1}} + \underline{\dim \operatorname{null}(T - \lambda I)^{m-1}}$. □

OR. Let $m_T(z) = (z - \lambda)^m q(z)$. We show $\{0\} \subsetneq \operatorname{null}(T - \lambda I) \subsetneq \cdots \subsetneq \operatorname{null}(T - \lambda I)^m$ by ctradic.
Asum $\operatorname{null}(T - \lambda I)^k = \operatorname{null}(T - \lambda I)^{k+1}$ for $k \in \{1, \dots, m-1\}$.
Then $\operatorname{null}(T - \lambda I)^k = \operatorname{null}(T - \lambda I)^m \Rightarrow (T - \lambda I)^m q(T)v = 0 = (T - \lambda I)^k q(T)v$. □

• (4E A.3) *Supp $T \in \mathcal{L}(V)$. Prove $V = \operatorname{null} T \oplus \operatorname{range} T \iff \operatorname{null} T^2 = \operatorname{null} T$.*

SOLUS: (a) $\operatorname{null} T^2 = \operatorname{null} T = \operatorname{null} T^{\dim V} \Rightarrow \dim \operatorname{range} T^{\dim V} = \dim \operatorname{range} T$.
(b) $V = \operatorname{null} T \oplus U$, $U = \operatorname{range} T$, 又 $\dim \operatorname{null} T^2 = \dim \operatorname{null} T + \dim \operatorname{null} T|_{\operatorname{range} T}$. □

OR. (a) Supp $\operatorname{null} T^2 = \operatorname{null} T$. Then $Tu \in \operatorname{null} T \cap \operatorname{range} T \iff T^2 u = 0 \iff Tu = 0$.
(b) Supp $\operatorname{null} T \cap \operatorname{range} T = \{0\}$. Then $T^2 u = 0 \iff Tu \in \operatorname{null} T \iff Tu = 0$. ■

A.17 *Supp $T \in \mathcal{L}(V)$, $\operatorname{range} T^m = \operatorname{range} T^{m+1}$. Show $\operatorname{range} T^m = \operatorname{range} T^{m+1} = \cdots$.*

SOLUS: By Exe (A.19), $\operatorname{null} T^m = \operatorname{null} T^{m+1} = \cdots \Rightarrow \dim \operatorname{range} T^m = \dim \operatorname{range} T^{m+1} = \cdots$. □

OR. Supp $w = T^{m+k}v$. Then becs $T^m v \in \operatorname{range} T^{m+1}$, $\exists\, T^{m+1}u = T^m v$. Thus $w = T^{m+k+1}u$. ■

A.18 *Supp $T \in \mathcal{L}(V)$, $\dim V = n$. Show $\operatorname{range} T^n = \operatorname{range} T^{n+1} = \cdots$.*

SOLUS: By Exe (A.19), becs $\operatorname{null} T^{\dim V} = \operatorname{null} T^{\dim V + 1} = \cdots$. Simlr. □

OR. Asum $\operatorname{range} T^n \supsetneq \operatorname{range} T^{n+1}$. By Exe (A.17), $V = \operatorname{range} T^0 \supsetneq \operatorname{range} T \supsetneq \cdots \supsetneq \operatorname{range} T^{n+1}$.
Now each $\dim \operatorname{range} T^{k+1} \leqslant \dim \operatorname{range} T^k - 1 \Rightarrow \dim \operatorname{range} T^{n+1} \leqslant \dim \operatorname{range} T^0 - (n+1)$. □

A.10 *Supp $T \in \mathcal{L}(V)$ not nilp, $n = \dim V$. Show $V = \operatorname{null} T^{n-1} \oplus \operatorname{range} T^{n-1}$.*

SOLUS: NOTICE that $\operatorname{null} T^{n-1} \neq \operatorname{null} T^n \Rightarrow \dim \operatorname{null} T^n = n \iff T^n = 0$. Thus $\operatorname{null} T^{n-1} = \operatorname{null} T^n$.
又 $V = \operatorname{null} T^n \oplus \operatorname{range} T^n, \operatorname{range} T^n \subseteq \operatorname{range} T^{n-1} \Rightarrow V = \operatorname{null} T^{n-1} + \operatorname{range} T^{n-1}$.
OR. Then $\dim \operatorname{range} T^{n-1} = \dim \operatorname{range} T^n \Rightarrow \operatorname{range} T^{n-1} = \operatorname{range} T^n$. □
OR. By Exe (4E A.3), $\operatorname{null} T^{2(n-1)} = \operatorname{null} T^{n-1} \iff V = \operatorname{null} T^{n-1} \oplus \operatorname{range} T^{n-1}$. □

• (4E A.18) *Supp $T \in \mathcal{L}(V)$ nilp. Prove $T^{1+\dim \operatorname{range} T} = 0$.* ■

SOLUS: Let $U \oplus \operatorname{null} T = V$. Then $\operatorname{range} T^m|_U = \operatorname{range}(T|_U)^m = \operatorname{range} T^m$. While $U = \dim \operatorname{range} T$. □
OR. Let $\dim \operatorname{range} T = k$. Asum $T^{k+1} \neq 0$. Let m be suth $T^m = 0 \neq T^{m-1}$. Then $k+2 \leqslant m$.
Let v be suth $T^{m-1}v \neq 0 = T^m v \Rightarrow (v, \underline{Tv, \dots, T^{m-1}v})$ liney indep $\Rightarrow k \geqslant m-1 \geqslant k+1$. □

• (4E A.12) *Supp $T \in \mathcal{L}(V)$ and all $v \in V$ is a gigvec of T. Prove $V = G(\lambda, T)$.*

SOLUS: Becs for any liney indep (v, w), $(v, w, v+w)$ of gigvecs is liney dep; say corres α, β, γ repectly.
If $\alpha = \beta$ then done. If $\alpha = \gamma, v, v+w \in G(\alpha, T) \Rightarrow w \in G(\alpha, T)$. If $\beta = \gamma$, then simlr.
Thus $\alpha = \beta = \gamma$. Any two liney indep v, w corres one eigval. ■

B.5 [4E A.15] *Supp $T \in \mathcal{L}(V)$. Prove non0 T diag $\Rightarrow$ each $G(\lambda, T) = E(\lambda, T)$.*

SOLUS: Supp $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$; $\lambda_1 = 0$ if possible, in this case $m > 1$.
Supp $w \in G(\lambda_j, T)$. Then $w = v_1 + \cdots + v_m$, where each $v_i \in E(\lambda_i, T)$.
Becs $(T - \lambda_j I)^k w = 0 = \sum_{i=1}^{m} (\lambda_i - \lambda_j)^k v_i \Rightarrow w = v_j \in E(\lambda_j, T)$. □
OR. Supp $G(\lambda_j, T) \supsetneq E(\lambda_j, T)$. Let $w \in G(\lambda_j, T) \backslash E(\lambda_j, T)$
Then $Iw \neq 0 \neq (T - \lambda_j I)w$. Let $(T - \lambda_j I)^k w = 0 \neq (T - \lambda_j I)^{k-1} w$.
By [5.B(I) TIPS (1)], m_T is a multi of $(z - \lambda_j)^k$. 又 $k \geqslant 2$. □

• (4E A.16) *Supp $S, T \in \mathcal{L}(V)$ nilp and commu. Prove $S+T, ST$ are nilp*

SOLUS: By [4E 5.80], $\exists B_V$ suth S, T up-trig (with only 0's on diags). By (4E 5.C.2). □
OR. Let $S^p = T^q = 0$. Becs S, T commu, $(ST)^{\max\{p,q\}} = 0 = (S+T)^{p+q} = \sum_{i=0}^{p+q} C_{p+q}^i S^i T^{p+q-i}$. ■

B.10 *Supp $\mathbf{F} = \mathbf{C}, T \in \mathcal{L}(V)$. Prove $\exists$ commu $D, N \in \mathcal{L}(V), T = D + N$, D diag, N nilp.*

SOLUS: **NOTE:** D diag, N nilp $\not\Rightarrow D, N$ commu. **EXA:** $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
We use induc on $\dim V = n$. (i) Immed. (ii) $n > 1$. Asum it holds for smaller V.
Becs $V = G_1 \oplus U$, where $U = G_2 \oplus \cdots \oplus G_m$, and each $G_k = G(\lambda_k, T)$.
$\exists B_{G_1}$ suth $T|_{G_1} = (T - \lambda_1 I)\big|_{G_1} + \lambda_1 I|_{G_1} = N_1 + D_1$ up-trig and N_1, D_1 commu.
$\exists$ commu $D_2, N_2 \in \mathcal{L}(U), T|_U = D_2 + N_2$, D_2 diag, N_2 nilp; wrto some B_U, by (4E 5.E.7).
Let $B_V = B_{G_1} \cup B_U$. Define $P_1, P_2 \in \mathcal{L}(V)$ by $P_1(v_1 + u) = v_1, P_2(v_1 + u) = u$.
Let $D = D_1 P_1 + D_2 P_2, N = N_1 P_1 + N_2 P_2$. Becs $P_j D_k = \delta_{j,k} D_j, P_j N_k = \delta_{j,k} N_j$.
Thus $D + N = (D_1 + N_1)P_1 + (D_2 + N_2)P_2 = T$, and $DN = D_1 N_1 P_1 + D_2 N_2 P_2 = NP$. □
OR. Let $V = G_1 \oplus \cdots \oplus G_m \Rightarrow \forall v \in V, \exists!\, v_k \in G_k, v = v_1 + \cdots + v_m$.
Define $D \in \mathcal{L}(V) : v \mapsto (\lambda_1 v_1 + \cdots + \lambda_m v_m) \Rightarrow D|_{G_k} = \lambda_k I$.
Let $N = T - D \Rightarrow N|_{G_k} = (T - D)\big|_{G_k} = (T - \lambda_k I)\big|_{G_k}$ is nilp.
Then $N^M v = N^M v_1 + \cdots + N^M v_m = 0$, where $M = \max\{d_1, \dots, d_m\}$. Now N is nilp.
Becs $DN = DT - D^2, ND = TD - D^2$, 又 each $TDv_k = \lambda_k T v_k = DTv_k \Rightarrow TD = DT$. □

• (4E B.7) *Supp $T \in \mathcal{L}(V)$ and λ is an eigval with multy d. Prove $G(\lambda, T) = \operatorname{null}(T - \lambda I)^d$.*

Solus: Let $N = T - \lambda I$, and $\operatorname{null} N \subsetneq \cdots \subsetneq \operatorname{null} N^m = \operatorname{null} N^{m+1}$. Choose $B_{\operatorname{null} N}$.
Extend to $B_{\operatorname{null} N^2} \Rightarrow \cdots \Rightarrow B_{\operatorname{null} N^m}$, with each time adding at least one bss vec. Thus $m \leqslant d$. ☐

Or. Let $m_T(z) = (z - \lambda)^m q(z)$ with $q(\lambda) \neq 0$. By (4E B.6,4) $d \geqslant m$. ☐

Or. Let the min of $N = (T - \lambda I)^m\big|_{G(\lambda, T)}$ be z^m.
By (4E 5.B.17), the min of $N + \lambda I = T|_{G(\lambda, T)}$ is $s(z) = (z - \lambda)^m$.
Becs the char of T $\big[$See [9.21] for $\mathbf{F} = \mathbf{R}\big]$ is a multi of m_T, which is a multi of s. ☐

• (4E B.6) *Supp $T \in \mathcal{L}(V)$ and λ is an eigval. Explain why the expo of $z - \lambda$ in the factoriz of m_T is the smallest $m \in \mathbf{N}^+$ suth $(T - \lambda I)^m\big|_{G(\lambda, T)} = 0$.*

Solus: Let $G = G(\lambda, T), N = (T - \lambda I)\big|_G$, and $N^m = 0 \neq N^{m-1} \Rightarrow$ the min of $T|_G$ is $s(z) = (z - \lambda)^m$.
Thus m_T is a multi of s. Now we show the expo of $(z - \lambda)$ in m_T is no more than m.
Let $\lambda_1 = \lambda$ and $V_{\mathrm{C}} = G_{\mathrm{C}} \oplus U$, where $G = G(\lambda_1, T), U = G(\lambda_2, T_{\mathrm{C}}) \oplus \cdots \oplus G(\lambda_n, T_{\mathrm{C}})$.
Asum $m_T(z) = (z - \lambda_1)^{m+k}(z - \lambda_2)^{\alpha_2} \cdots (z - \lambda_n)^{\alpha_n}$, where $k \in \mathbf{N}^+$.
Let $r(z) = (z - \lambda)^m (z - \lambda_2)^{\alpha_2} \cdots (z - \lambda_n)^{\alpha_n} \Rightarrow r(T) = 0$. Ctradic the min of m_T. ☐

Or. Let $m_T(z) = (z - \lambda)^m q(z)$, with $q(\lambda) \neq 0$. We show $\operatorname{null}(T - \lambda I)^m \supseteq \operatorname{null}(T - \lambda I)^{m+1}$.
Supp $v \in \operatorname{null}(T - \lambda I)^{m+1} \iff (T - \lambda I)^m v \in \operatorname{null}(T - \lambda I) = E(\lambda, T)$.
Then $0 = m_T(T)v = q(T)\big[(T - \lambda I)^m v\big] = q(\lambda)\big[(T - \lambda I)^m v\big] \Rightarrow v \in \operatorname{null}(T - \lambda I)^m$.
We show m is the smallest. Let k be suth $\operatorname{null}(T - \lambda I)^k = G(\lambda, T)$, $s(z) = (z - \lambda)^k q(z)$.
We show $s(T) = 0$ and done. Consider $0 = m_T(T)v = (T - \lambda I)^m q(T)v$.
If $q(T)v = 0 \Rightarrow s(T)v = 0$. Othws, $q(T)v \in \operatorname{null}(T - \lambda I)^m = \operatorname{null}(T - \lambda I)^k \Rightarrow s(T)v = 0$. ☐

B.9 *Supp A, C are block diag matrices, and A_k, C_k are of the same size n_k for $k \in \{1, \dots, m\}$. Show AC is block diag and the k^{th} block on the diag of AC is $A_k C_k$.*

Solus: Let $A = \mathcal{M}(S), C = \mathcal{M}(T), AC = \mathcal{M}(ST) \in \mathbf{F}^{n,n}$, where $n = n_1 + \cdots + n_m$.
Let $B_1 = (e_1, \cdots, e_{n_1})$, and $B_k = (e_{n_1 + \cdots + n_{k-1} + 1}, \cdots, e_{n_1 + \cdots + n_k})$ for $k \in \{2, \dots, m\}$.
Let each $U_k = \operatorname{span} B_k$ invard S, T. Becs $\mathcal{M}(S|_{U_k}, B_k) = A_k, \mathcal{M}(T|_{U_k}, B_k) = C_k$.
Now $\mathcal{M}\big[(ST)\big|_{U_k}\big] = \mathcal{M}(S|_{U_k} T|_{U_k}) = A_k C_k$. ☐

• *Supp $T \in \mathcal{L}(V)$ and U is invarspd T. Supp $\lambda_1, \dots, \lambda_m$ are the disti eigvals.*

• **B.Tips 1:** *Supp $\mathbf{F} = \mathbf{C}$. Prove $U = G(\lambda_1, T|_U) \oplus \cdots \oplus G(\lambda_m, T|_U)$.*

Solus: We use induc on $\dim U = N$. (i) Immed. (ii) $N > 1$. Asum it holds for smaller U.
Supp λ_1 is an eigval of $T|_U$. Let $W \oplus G(\lambda_1, T|_U) = U$, where $W = \operatorname{range}(T|_U - \lambda_1 I)^N$ invard $T|_U$.
Note that $T|_U|_W = T|_W$. By asum, $W = G(\lambda_2, T|_W) \oplus \cdots \oplus G(\lambda_m, T|_W)$.
Now we show $G(\lambda_k, T|_U) \subseteq G(\lambda_k, T|_W)$ for each $k \in \{2, \dots, m\}$. Supp $v \in G(\lambda_k, T|_U)$.
Then $\exists!\, u_1 \in G(\lambda_1, T|_U), w_k \in G(\lambda_k, T|_W)$, $v = u_1 + w_2 + \cdots + w_m$. By [8.13]. ☐

Comment: Note that generally, $X \oplus Y \supseteq U \neq (X \cap U) \oplus (Y \cap U)$, and $(X + U) \cap (Y + U) \neq U$.

• *Supp $T \in \mathcal{L}(V)$, and $W \oplus E(\lambda, T) = G(\lambda, T)$. Prove W is not invard T.*

Solus: Let $w \in W$ and $(T - \lambda I)^j w = 0 \neq (T - \lambda I)^{j-1} w \in E(\lambda, T) \Rightarrow W$ is not invar, by the min of j. ☐

• **B.Tips 2:** *Supp $V = U \oplus W$, and U, W invard T. Then $G(\lambda, T) = G(\lambda, T|_U) \oplus G(\lambda, T|_W)$.*

• **B.Tips 3:** *Supp $p = sq$ is the min of $T \in \mathcal{L}(V)$ and s, q have no common zeros.*
Prove $V = \operatorname{null} s(T) \oplus \operatorname{null} q(T)$. **Coro:** $\operatorname{null} s(T) = \operatorname{range} q(T)$.

Solus: By Exe (4E 4.13), $\forall v \in V, v = u + w$, where $u = s(T)a(T)v$, $w = q(T)b(T)v$. Immed. ☐

Or. Let $V_{\mathbf{C}} = G(\lambda_1, T_{\mathbf{C}}) \oplus \cdots \oplus G(\lambda_m, T_{\mathbf{C}})$, and $p(z) = (z-\lambda_1)^{k_1} \cdots (z-\lambda_m)^{k_m}$.

Let $s(z) = (z-\lambda_{\alpha_1})^{k_{\alpha_1}} \cdots (z-\lambda_{\alpha_A})^{k_{\alpha_A}}$, $q(z) = (z-\lambda_{\beta_1})^{k_{\beta_1}} \cdots (z-\lambda_{\beta_B})^{k_{\beta_B}}$, with all $\alpha_j \neq \beta_i$.

Note that $\forall v \in V_{\mathbf{C}}, \exists!\, v_i \in G(\lambda_i, T_{\mathbf{C}})$, $v = v_1 + \cdots + v_m = (v_{\alpha_1} + \cdots + v_{\alpha_A}) + (v_{\beta_1} + \cdots + v_{\beta_B})$.

Thus $V_{\mathbf{C}} = \operatorname{null} s(T_{\mathbf{C}}) \oplus \operatorname{null} q(T_{\mathbf{C}})$. And simlr, $\operatorname{null} s(T_{\mathbf{C}}) = G(\lambda_{\alpha_1}, T_{\mathbf{C}}) \oplus \cdots \oplus G(\lambda_{\alpha_A}, T_{\mathbf{C}})$.

Comment: If $\lambda_{\alpha_j} \notin \mathbf{R}$, then $\exists\, \lambda_{\alpha_i} = \overline{\lambda_{\alpha_j}}$. Simlr for λ_{β_j}. Now $V = \operatorname{null} s(T) \oplus \operatorname{null} q(T)$.

New Nota: We call such s or q a ***poly block***. A factor q of p is a block $\iff$ the other half p/q is a block.

Coro: (1) If q is a block of m_T. Then $V = \operatorname{null} q(T) \oplus \operatorname{range} q(T)$.

(2) If s, q are blocks with no common zeros. Then $V = \operatorname{range} s(T) + \operatorname{range} q(T)$.

(3) Supp $p = p_1 \cdots p_k$ and $p_1, \dots, p_k$ have no common zeros, so are poly blocks of p.
Then $V = \operatorname{null} p_1(T) \oplus \cdots \oplus \operatorname{null} p_k(T)$. Let $\Gamma_j = \{1, \dots, j-1, j+1, \dots, k\}$.
Then $\operatorname{range} p_j(T) = \bigoplus_{i \in \Gamma_j} \operatorname{null} p_i(T)$, $\operatorname{null} p_j(T) = \bigcap_{i \in \Gamma_j} \operatorname{range} p_i(T)$.

C.20 [4E B.20] *Supp $\mathbf{F} = \mathbf{C}$, and each V_k non0 invarspd $T \in \mathcal{L}(V)$ of $V = V_1 \oplus \cdots \oplus V_m$.*
Let p_k be the char of $T|_{V_k}$. Prove the char of T is $p_1 \cdots p_m$.

Solus: By [B Tips (1)], $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_n, T) \Rightarrow V_k = G(\lambda_1, T|_{V_k}) \oplus \cdots \oplus G(\lambda_m, T|_{V_k})$.

By [B Tips (2)], each $G(\lambda_j, T) = G(\lambda_j, T|_{V_1}) \oplus \cdots \oplus G(\lambda_j, T|_{V_m})$.

Let $d_{j,k}$ be the multy of λ_j of $T|_{V_k}$. Then $d_{j,1} + \cdots + d_{j,n} = d_j$, the multy of λ_j of T.

Thus each $p_k(z) = (z-\lambda_1)^{d_{1,k}} \cdots (z-\lambda_n)^{d_{n,k}}$. While the char of T is $(z-\lambda_1)^{d_1} \cdots (z-\lambda_n)^{d_n}$. ☐

Or. Let A be a block diag matrix of T, with each $A_k = \mathcal{M}(T|_{V_k})$ up-trig. By Exe (B.11). ☐

• (4E C.12) *Supp $T \in \mathcal{L}(V)$ diag. Show $\mathcal{M}(T)$ diag wrto any Jordan B_V.*

Solus: By Exe (4E C.11), each v_k of a Jordan B_V is a gigvec; so is an eigvec, by Exe (4E A.15). ☐

Or. Let A be a Jordan block diag matrix of T. By Exe (D.3) and [4E 5.62]. ☐

D.8 *Supp $\mathbf{F} = \mathbf{C}$, and $T \in \mathcal{L}(V)$.*
Prove [P] $\nexists$ non0 invarsps U, W suth $U \oplus W = V \iff m_T(z) = (z-\lambda)^{\dim V}$. [Q]

Solus: $Q \Rightarrow P$: Let $N = T - \lambda I \Rightarrow$ the min of N is $z^{\dim V}$.
Then by Exe (D.3), the line directly above the diag of any Jordan $\mathcal{M}(N)$ is all 1.
Thus the only Jordan block of $\mathcal{M}(N)$ is $\mathcal{M}(N)$ itself.

$\neg P \Rightarrow \neg Q$: If $\exists$ two or more eigvals of $T|_U$ or $T|_W$, then m_T has two or more disti factors, done.
Now supp $\exists$ only one eigval λ for $T|_U$, $T|_W$, and T. Supp $m_T(z) = (z-\lambda)^m$.
Let $M = \max\{\dim U, \dim W\}$. Let $S = (T-\lambda I)^M \Rightarrow \operatorname{null} S|_U \oplus \operatorname{null} S|_W = \operatorname{null} S$.
Becs $G(\lambda, T|_U) = U$, $G(\lambda, T|_W) = W$, $G(\lambda, T) = V$, $(T-\lambda I)^M = 0$. Now by Exe (4E B.6).
Or. Becs $\exists$ Jordan $\mathcal{M}(T|_U), \mathcal{M}(T|_W) \Rightarrow$ Jordan $\mathcal{M}(T)$. Consider z^{M+1} by Exe (D.3).

$\neg Q \Rightarrow \neg P$: Supp T has only one eigval. Let $m_T(z) = (z-\lambda)^m$ with $m < \dim V$.
Becs $\exists$ Jordan $B_V = (\underbrace{v_{1,1}, \cdots, v_{m_1,1}}_{\text{bss for } U}, \underbrace{v_{1,2}, \cdots, v_{m_2,2}, \cdots, v_{1,k}, \cdots, v_{m_k,k}}_{\text{bss for } W})$ for T. ☐

Ended

9.A **NOTE:** V denotes a finide non0 vecsp over $\mathbf{F}$.

- **NOTE FOR [9.12]:** Another proof: $\overline{T_C(u+iv)} = \overline{Tu + iTv} = Tu - iTv = T_C(u - iv) = T_C(\overline{u+iv})$.
  $\overline{(T_C - \lambda I)(u+iv)} = \overline{T_C(u+iv) - \lambda(u+iv)} = T_C(u-iv) - \overline{\lambda}(u-iv) = (T_C - \overline{\lambda}I)(u-iv)$.
  We use induc on m to show $\overline{(T_C - \lambda I)^m(u+iv)} = (T_C - \overline{\lambda}I)^m(u-iv)$. (i) Immed. (ii) $m > 1$.
  Asum it holds for $k \leqslant m$. Let $(T_C - \lambda I)^{m-1}(u+iv) = x + iy$. Becs $\overline{(T_C - \lambda I)^{m-1}(u+iv)} = x - iy$. □

- **NOTE FOR [9.17]:** Detailed proof:
  Let $B = (u_1 + iv_1, \dots, u_m + iv_m)$ be a bss of $G(\lambda, T_C)$. By [9.12], $\overline{B} = (u_1 - iv_1, \dots, u_m - iv_m)$ in $G(\overline{\lambda}, T_C)$.
  (a) If $a_1(u_1 - iv_1) + \cdots + a_m(u_m - iv_m) = 0$. Conjugating, now each $\overline{a_k} = 0$. Liney indep.
  (b) $\forall u - iv \in G(\overline{\lambda}, T_C), u + iv \in G(\lambda, T_C) \Rightarrow u + iv \in \operatorname{span} B \Rightarrow u - iv \in \operatorname{span} \overline{B}$. □

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13 *Supp* $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$*, and* $b^2 < 4c$*. Let* $q(z) = z^2 + bz + c = (z-\lambda)(z-\overline{\lambda})$.
Prove $\dim \operatorname{null} q(T)^j$ *is even for each* $j \in \mathbf{N}^+$. [See also NOTE FOR [4E 5.33] in (5.BI).]

SOLUS: (a) Replace V with $\operatorname{null} q(T)^j$ and T with T restr to V. Becs $m_T(z) = q(z)^k$.
By $[$8.B TIPS (3)$]$, $V_C = \operatorname{null}(T_C - \lambda I)^k \oplus \operatorname{null}(T_C - \overline{\lambda}I)$. By [9.17] and [9.4]. □

NOTE: Let $Q(\lambda, T) = \operatorname{null} q(T)^{\dim V}$. Then by (4E 8.B.6,7) for T_C, by [9.10,20], and by $[$8.B TIPS (3)$]$,
(a) $Q(\lambda, T) = \operatorname{null} q(T)^d$, where $d = \dim G(\lambda, T_C)$.
(b) The expo of q in the factoriz of m_T is the smallest $m \in \mathbf{N}^+$ suth $q(T)^m|_{Q(\lambda,T)} = 0$.
(c) $m_T = p_1^{\alpha_1} \cdots p_m^{\alpha_m} q_1^{\beta_1} \cdots q_M^{\beta_M} \iff V = \left[\bigoplus_{j=1}^m G(\mu_j, T)\right] \oplus \left[\bigoplus_{k=1}^M Q(\lambda_k, T)\right]$.
Where each $p_j(z) = z - \mu_j$, $q_k(z) = z^2 - 2(\operatorname{Re}\lambda_k)z + |\lambda_k|^2 = z^2 + b_k z + c_k$.

Fix one k. Let $q(z) = q_k(z) = (z-\lambda)(z-\overline{\lambda}), \lambda = a + bi, G = G(\lambda, T_C), \overline{G} = G(\overline{\lambda}, T_C)$.
Replace T with $T|_Q$. Let $Q = Q(\lambda, T)$ of dim β, and $Q_C = G \oplus \overline{G}$, and Jordan bss B_J of Q_C.
Now $\mathcal{M}(T_C) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \mathcal{M}(T_C - \lambda I) = \begin{pmatrix} \overline{R_1} & 0 \\ 0 & \overline{R_2} \end{pmatrix}$, $\mathcal{M}(T_C - \overline{\lambda}I) = \begin{pmatrix} R_2 & 0 \\ 0 & R_1 \end{pmatrix}$ wrto Jordan bss.

So then $\mathcal{M}(T_C^2 + bT_C + cI) = \mathcal{M}(T_C - \lambda I)\mathcal{M}(\overline{T_C - \lambda I}) = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$, where $R = R_1 R_2$.
Where A_1, R_1, R_2, R are block diag matrices, and $A_1 = \mathcal{M}(T_C|_G)$, $A_2 = \mathcal{M}(T_C|_{\overline{G}}) = \overline{\mathcal{M}(T_C|_G)}$.

Each $A_{1,k} = \begin{pmatrix} \lambda & 1 & 0 \\ & \ddots & \ddots \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}, R_{1,k} = \begin{pmatrix} 0 & 1 & 0 \\ & \ddots & \ddots \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}, R_{2,k} = \begin{pmatrix} 2bi & 1 & 0 \\ & \ddots & \ddots \\ & & \ddots & 1 \\ 0 & & & 2bi \end{pmatrix}, R_k = \begin{pmatrix} 0 & 2bi & 1 & & 0 \\ 0 & & \ddots & \ddots & \\ \vdots & & & \ddots & 1 \\ 0 & 0 & & & 2bi \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$.

Let the Jordan bss Q_C for T_C be $(u_1 + iv_1, \dots, u_\beta + iv_\beta, u_1 - iv_1, \dots, u_\beta - iv_\beta)$.
Now due to $\mathcal{M}(T_C)$, $T(u_1 \pm iv_1) = (a \pm ib)(u_1 \pm iv_1) = (au_1 - bv_1) \pm i(bu_1 + av_1)$,
$T(u_j \pm iv_j) = (a \pm ib)(u_j \pm iv_j) + (u_{j-1} \pm iv_{j-1}) = (au_j - bv_j + u_{j-1}) \pm i(bu_j + av_j + v_{j-1})$.
Hence $Tu_1 = au_1 - bv_1, Tv_1 = bu_1 + av_1$, and $Tu_j = u_{j-1} + au_j - bv_j, Tv_j = v_{j-1} + bu_j + av_j$.

Let $B_Q = (u_1, v_1, \dots, u_\beta, v_\beta) \Rightarrow \mathcal{M}(T, B_Q) = \begin{pmatrix} \mathcal{R} & I_2 & 0 \\ & \ddots & \ddots \\ & & \ddots & I_2 \\ 0 & & & \mathcal{R} \end{pmatrix}$, where $\mathcal{R} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
OR. $B_Q = (v_1, u_1, \dots, v_\beta, u_\beta) \Rightarrow \mathcal{R} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

ENDED

6.A

3

Solus:

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6 *Supp* $u, v \in V$. *Prove* $\langle u, v\rangle = 0 \iff \|u\| \leqslant \|u + av\|$ *for all* $a \in \mathbf{F}$.

Solus: Supp $\langle u, v\rangle = 0 = \langle u, av\rangle \Rightarrow \|u + av\|^2 = \langle u, u + av\rangle + \langle av, u + av\rangle = \|u\|^2 + |a|^2\|v\|^2 \geqslant \|u\|^2$.
Supp $\|u\|^2 \leqslant \|u + av\|^2$. Let $\langle u - cv, cv\rangle = 0 \Rightarrow \|u - cv\|^2 = \langle u, u - cv\rangle = \|u\|^2 - \bar{c}\langle u, v\rangle$.
Thus $\|u\|^2 \leqslant \|u - cv\|^2 = \|u\|^2 - \left|\langle u, v\rangle\right| / \|v\|^2$, by [6.14]. □

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9 *Supp* $u, v \in V$, *and* $\|u\|, \|v\| \leqslant 1$. *Prove* $\sqrt{1 - \|u\|^2}\sqrt{1 - \|v\|^2} \leqslant 1 - \left|\langle u, v\rangle\right|$.

Solus: Becs $\left|\langle u, v\rangle\right| \leqslant 1$. We show $\left(1 - \|u\|^2\right)\left(1 - \|v\|^2\right) \leqslant \left(1 - \left|\langle u, v\rangle\right|\right)^2$.
Equiv to $\|u\|^2\|v\|^2 - \left|\langle u, v\rangle\right|^2 \leqslant \|u\|^2 + \|v\|^2 - 2\left|\langle u, v\rangle\right|$. Note that $2\left|\langle u, v\rangle\right| \leqslant \left|\langle u, v\rangle\right|^2$.

• (4E 19) *Supp* $T \in \mathcal{L}(V), A = \mathcal{M}\big(T, (v_1, \dots, v_n)\big)$, *and* λ *is eigval.*
Prove $|\lambda|^2 \leqslant \sum_{j=1}^{n} \sum_{k=1}^{n} |A_{j,k}|^2$.

Solus:

• (4E 23) *Supp* $v_1, \dots, v_m \in V$ *with each* $\|v_k\| \leqslant 1$.
Show $\exists\, a_k \in \{1, -1\}$ *suth* $\|a_1 v_1 + \cdots + a_m v_m\| \leqslant \sqrt{m}$.

Solus: We use induc on m. (i) $m = 1$. Immed. (ii) $m > 1$. Asum it holds for smaller m.
Let $u = a_1 v_1, w = a_2 v_2 + \cdots + a_m v_m \Rightarrow \|u\| \leqslant 1, \|w\| \leqslant \sqrt{m - 1}$.
Then $\|u + w\|^2 = \|u\|^2 + \|w\|^2 + 2\mathrm{Re}\langle u, w\rangle \leqslant m$.

Ended

6.B

ENDED

6.C

ENDED

7.A

ENDED