

1.B

- Suppose S is a nonempty set. Let V^S denote the set of functions from S to V . Define a natural addition and scalar multiplication on V^S , and show that V^S is a vector space with these definitions.

SOLUTION:

- Addition on V^S is defined by $(f + g)(x) = f(x) + g(x)$ for any $x \in S$ and $f, g \in V^S$.
- Scalar Multiplication on V^S is defined by $(\lambda f)(x) = \lambda f(x)$ for any $x \in S, \lambda \in \mathbf{F}, f \in V^S$. \square

1 Prove that $-(-v) = v$ for every $v \in V$.

SOLUTION:
$$\left. \begin{array}{l} (-(-v)) + (-v) = 0 \\ v + (-v) = 0 \end{array} \right\} \Rightarrow \text{By the uniqueness of additive inverse. } \square$$

OR. $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$ \square

2 Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

SOLUTION: If $a = 0$, then we are done.

Otherwise, $\exists a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. \square

3 Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

SOLUTION:

[Existence] Let $x = \frac{1}{3}(w - v)$.

[Uniqueness] Suppose $v + 3x_1 = w, (I) v + 3x_2 = w (II)$.

Then (I) - (II) : $3(x_1 - x_2) = 0 \Rightarrow$ By Problem (2), $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$. \square

5 Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that $0v = 0$ for all $v \in V$.

SOLUTION: Using [1.31]. $0v = 0$ for all $v \in V \Leftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. \square

6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in \mathbf{R} .

Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

and (I) $t + \infty = \infty + t = \infty + \infty = \infty$,

(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,

(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over \mathbf{R} ? Explain.

SOLUTION:

Not a vector space. By Associativity: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

OR By Distributive properties: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. \square

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1.C

2 (1.35)

(b) The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$

$$\left. \begin{array}{l} \text{Denote the set by } U. \forall x \in [0, 1] \text{ we have} \\ \text{(一)} 0 \in U; f(x) = 0 \Leftrightarrow f = 0 \\ \text{(二)} \forall f, g \in U, (f + g)(x) = f(x) + g(x) \\ \text{(三)} \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R}, (\lambda f)(x) = \lambda f(x) \end{array} \right\} \Rightarrow \square$$

(c) The set of differentiable real-valued functions on \mathbf{R} is a subspace of $\mathbf{R}^{\mathbf{R}}$

$$\left. \begin{array}{l} \text{Denote the set by } U. \\ \text{(一)} 0 \in U \\ \text{(二)} \forall f, g \in U, (f' + g') = f' + g' \\ \text{(三)} \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R}, (\lambda f)' = \lambda(f)' \end{array} \right\} \Rightarrow \square$$

(d) The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.

Denote the set by U . Suppose $b = 0$. Then

$$\left. \begin{array}{l} \text{(一)} 0 \in U \\ \text{(二)} \forall f, g \in U, (f + g)'(2) = f'(2) + g'(2) = 0 \\ \text{(三)} \forall f \in U, \forall \lambda \in \mathbf{F} = \mathbf{R}, (\lambda f)'(2) = \lambda f'(2) \end{array} \right\} \Rightarrow U \text{ is a subspace of } \mathbf{R}^{(0,3)}.$$

Suppose U is a subspace of $\mathbf{R}^{(0,3)}$. Suppose $f = 0 \Rightarrow f \in U$. Then $f'(2) = 0 = b$. \square

(e) The set of all sequences with limit 0, of complex numbers, is a subspace of \mathbf{C}^{∞} .

Denote the set by A .

$$\left. \begin{array}{l} \text{(一)} (0, 0, \dots) \in A \\ \text{(二)} \forall (a_1, a_2, \dots), (b_1, b_2, \dots) \in A \iff \lim_{n \rightarrow \infty} a_n = 0, \lim_{n \rightarrow \infty} b_n = 0. \\ \text{Thus } \lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = 0 \Rightarrow (a_1 + b_1, a_2 + b_2, \dots) \in A \\ \text{(三)} \forall (a_1, a_2, \dots) \in A, \forall \lambda \in \mathbf{F} = \mathbf{C}, \lim_{n \rightarrow \infty} (\lambda a_n) = \lambda \lim_{n \rightarrow \infty} a_n = 0 \Rightarrow \lambda(a_1, a_2, \dots) \in A \end{array} \right\} \Rightarrow \square$$

4 Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$

SOLUTION: Denote the set by V_b .

(a) Suppose V_b is a subspace of $\mathbf{R}^{[0,1]}$, then $\forall f \in V_b$, we have $\int_0^1 f = b$.

Because $kf \in V_b$ for any $k \in \mathbf{R}$. Hence $\int_0^1 (kf) = k \int_0^1 f = kb = b \Leftrightarrow b = 0$.

OR. Because $g = 0 \in V_b \Rightarrow \int_0^1 g = 0 = b$.

(b) Suppose $b = 0$. $\forall f, g \in V_b = V_0, \lambda \in \mathbf{R}, \int_0^1 (f + \lambda g) = \int_0^1 f + \int_0^1 g = 0$. \square

5 Is \mathbf{R}^2 a subspace of the complex vector space \mathbf{C}^2 ?

ANSWER: No. Because \mathbf{R}^2 is not closed on scalar multiplication on \mathbf{C} .

6 (a) Is $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of \mathbf{R}^3 ?

ANSWER: True. As can be easily checked (note that $a^3 = b^3 \Leftrightarrow a = b$).

(b) Is $\{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of \mathbf{C}^3 ?

ANSWER: No. Note that $(\frac{-1 \pm \sqrt{3}i}{2})^3 = 1$.

7 Prove or give a counterexample: If U is a nonempty subset of \mathbf{R}^2 such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of \mathbf{R}^2 .

SOLUTION: Let $U = \mathbf{Z}^2, (\mathbf{Z}^*)^2, (\mathbf{Q}^*)^2, \mathbf{Q}^2 \setminus \{0\}$, or $\mathbf{R}^2 \setminus \{0\}$.

8 Give an example of a nonempty subset U of \mathbf{R}^2 such that

U is closed under scalar multiplication, but U is not a subspace of \mathbf{R}^2 .

SOLUTION: $U = \{(x, y) \in \mathbf{R}^2 : x = 0 \vee y = 0\}$.

9 A function $f : \mathbf{R} \rightarrow \mathbf{R}$ is called periodic if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbf{R}$.

Is the set of periodic functions from \mathbf{R} to \mathbf{R} a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.

SOLUTION: Denote the set by S .

Suppose $h(x) = \sin \sqrt{2}x + \cos x \in S$, since both $f(x) = \sin \sqrt{2}x, g(x) = \cos x$ are periodic functions $\mathbf{R} \rightarrow \mathbf{R}$.

Assume $\exists p \in \mathbf{N}^+$ such that $h(x) = h(x + p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin \sqrt{2}p = \cos p - \sin \sqrt{2}p$

$\Rightarrow \sin \sqrt{2}p = 0, \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \frac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \frac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \frac{m}{2k} \in \mathbf{Q}$. Contradiction! \square

11 Prove that the intersection of every collection of subspaces of V is a subspace of V .

SOLUTION:

Suppose $\{U_\alpha\}_{\alpha \in \Gamma}$ is a collection of subspaces of V ; here Γ is an arbitrary index set.

We need to prove that $\bigcap_{\alpha \in \Gamma} U_\alpha$, which equals the set of vectors

that are in U_α for each $\alpha \in \Gamma$, is a subspace of V .

(一) $0 \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha \in \Gamma} U_\alpha \Rightarrow u + v \in U_\alpha, \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Closed under addition.

(三) $u \in \bigcap_{\alpha \in \Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha, \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Closed under scalar multiplication.

Thus $\bigcap_{\alpha \in \Gamma} U_\alpha$ is nonempty subset of V that is closed under addition and scalar multiplication.

Hence $\bigcap_{\alpha \in \Gamma} U_\alpha$ is a subspace of V . \square

12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

SOLUTION: Suppose U and W are subspaces of V .

(a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subspace of V .

(b) Suppose $U \cup W$ is a subspace of V . Suppose $U \not\subseteq W$ and $U \not\supseteq W$ ($U \cup W \neq U$ and W).

Then $\forall a \in U$ but $a \notin W$; $b \in W$ but $b \notin U$. $a + b \in U \cup W$.

(1) Consider $a + b \in U \Rightarrow b = (a + b) + (-a) \in U$, contradicts!

(2) Consider $a + b \in W \Rightarrow a = (a + b) + (-b) \in W$, contradicts!

Thus $U \subseteq W$ and $U \supseteq W$. \square

13 Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

This exercise is surprisingly harder than Exercise 12, possibly because this exercise is not true if we replace \mathbf{F} with a field containing only two elements.

SOLUTION: Suppose A, B, C are subspaces of V .

(a) If any two of them are subsets of the third one, then $A \cup B \cup C = A, B$ or C , which is a subspace of V .

(b)* If $A \cup B \cup C$ is a subspace of V , suppose $\left\{ \begin{array}{l} A \not\subseteq B \text{ and } C \\ B \not\subseteq A \text{ and } C \\ C \not\subseteq A \text{ and } B \end{array} \right\} \iff A \cap B \cap C \neq A, B \text{ and } C$.

$\forall a \in A$ but $a \notin B, C$; $\forall b \in B$ but $b \notin A, C$; $\forall c \in C$ but $c \notin A, B$; by assumption, $a + b + c \in A \cup B \cup C$.

(I) $A \cup B$ is a subspace \Rightarrow By Problem (12), $A \subseteq B$ or $A \supseteq B$.

(II) $A \cup C$ is a subspace \Rightarrow By Problem (12), $A \subseteq C$ or $A \supseteq C$.

(III) $B \cup C$ is a subspace \Rightarrow By Problem (12), $B \subseteq C$ or $B \supseteq C$.

Any two of (I), (II) and (III) must be true.

$$(一). \text{ (I) and (II) are true. Then } \left. \begin{array}{l} C \subseteq B \subseteq A \\ \text{or } C \supseteq B \supseteq A \\ \text{or } B \supseteq A, C \\ \text{or } \underbrace{B \subseteq A, C}_{(1)} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array} \right.$$

$$(二). \text{ (II) and (III) are true. Then } \left. \begin{array}{l} A \subseteq C \subseteq B \\ \text{or } A \supseteq C \supseteq B \\ \text{or } C \supseteq A, B \\ \text{or } \underbrace{C \subseteq A, B}_{(2)} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array} \right.$$

$$(三). \text{ (I) and (III) are true. Then } \left. \begin{array}{l} B \subseteq A \subseteq C \\ \text{or } B \supseteq A \supseteq C \\ \text{or } A \supseteq B, C \\ \text{or } \underbrace{A \subseteq B, C}_{(3)} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A \supseteq B, C \\ \text{or } B \supseteq A, C \\ \text{or } C \supseteq A, B \end{array} \right.$$

$$\text{Among these, any two of (1), (2) and (3) must be true. } \left. \begin{array}{l} (1) \\ (2) \\ (2) \\ (3) \\ (1) \\ (3) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} B \subseteq C \subseteq A \\ C \subseteq A \subseteq B \\ B \subseteq A \subseteq C \end{array} \right\} \Rightarrow \square$$

• Suppose $U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$ and $W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$.

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

SOLUTION:

$$(a) U + W = \{(x + y, x - y, 2(x + y)) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} = \{(x', y', 2x') \in \mathbf{F}^3 : x', y' \in \mathbf{F}\}.$$

$$(b) U + W \text{ is a plane of which } (1, 0, 2), (0, 1, 0) \text{ is a basis. } \square$$

15 Suppose U is a subspace of V . What is $U + U$?

$$\text{SOLUTION: } \left. \begin{array}{l} \forall x, y \in U, x + y \in U \Rightarrow U + U \subseteq U \\ \forall x \in U, 0 \in U, x + 0 \in U + U \Rightarrow U \subseteq U + U \end{array} \right\} \Rightarrow U + U = U. \square$$

16 Suppose U and W are subspaces of V . Prove that $U + W = W + U$?

$$\text{SOLUTION: } \left. \begin{array}{l} \forall x \in U, y \in W, x + y = y + x \in W + U \Rightarrow U + W \subseteq W + U \\ y + x = x + y \in U + W \Rightarrow W + U \subseteq U + W \end{array} \right\} \Rightarrow U + W = W + U. \square$$

17 Suppose V_1, V_2, V_3 are subspaces of V . Prove that $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$.

SOLUTION:

Let $x \in V_1, y \in V_2, z \in V_3$. Denote $(V_1 + V_2) + V_3$ by L , $V_1 + (V_2 + V_3)$ by R .

$$\left. \begin{array}{l} \forall u \in L, \exists x, y, z, u = (x + y) + z = x + (y + z) \in R \Rightarrow L \subseteq R \\ \forall u \in R, \exists x, y, z, u = x + (y + z) = (x + y) + z \in L \Rightarrow R \subseteq L \end{array} \right\} \Rightarrow (V_1 + V_2) + V_3 = V_1 + (V_2 + V_3). \square$$

18 Does the operation of addition on the subspaces of V have an additive identity?

Which subspaces have additive inverses?

SOLUTION:

Suppose Ω is the additive identity.

For any subspace U of V . $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

Now suppose W is an additive inverse of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. \square

19 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that $V_1 + U = V_2 + U$, then $V_1 = V_2$.

SOLUTION: A counterexample:

$$V = \mathbf{F}^3, U = \{(x, 0, 0) \in \mathbf{F}^3 : x \in \mathbf{F}\},$$

$$V_1 = \{(x, x, y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}, V_2 = \{(x, y, z) \in \mathbf{F}^3 : x, y, z \in \mathbf{F}\}.$$

EXAMPLE: Suppose $U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}, W = \{(x, x, x, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$.

Prove that $U + W = \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$.

SOLUTION: Let T denote $\{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$.

(a) By definition, $U + W = \{(x_1 + x_2, x_1 + x_2, y_1 + x_2, y_1 + y_2) \in \mathbf{F}^4 : (x_1, x_1, y_1, y_1) \in U, (x_2, x_2, x_2, y_2) \in W\}$.

$$\Rightarrow \forall v \in U + W, \exists t \in T, v = t \Rightarrow U + W \subseteq T.$$

(b) $\forall x, y, z \in \mathbf{F}$, let $u = (0, 0, y - x, y - x) \in U, w = (x, x, x, -y + x + z) \in W$

$$\Rightarrow (x, x, y, z) = u + w \in U + W. \text{ Hence } \forall t \in T, \exists u \in U, w \in W, t = u + w \Rightarrow T \subseteq U + W. \quad \square$$

21 Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}$.

Find a subspace W of \mathbf{F}^5 such that $\mathbf{F}^5 = U \oplus W$.

SOLUTION:

(a) Let $W = \{(0, 0, z, w, u) \in \mathbf{F}^5 : z, w, u \in \mathbf{F}\}$. Then $W \cap U = \{0\}$.

(b) $\forall x, y, z, w, u \in \mathbf{F}$, let $u = (x, y, x + y, x - y, 2x) \in U$,

$$w = (0, 0, z - x - y, w - x - y, u - 2x) \in W$$

$$\Rightarrow (x, y, z, w, u) = u + w \Rightarrow \mathbf{F}^5 \subseteq U + W. \quad \square$$

22 Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}$.

Find three subspaces W_1, W_2, W_3 of \mathbf{F}^5 , none of which equals $\{0\}$,

such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

SOLUTION:

(1) Let $W_1 = \{(0, 0, z, 0, 0) \in \mathbf{F}^5 : z \in \mathbf{F}\}$. Then $W_1 \cap U = \{0\}$.

Let $U_1 = U \oplus W_1$. Then $U_1 = \{(x, y, z, x - y, 2x) \in \mathbf{F}^5 : x, y, z \in \mathbf{F}\}$. (Check it!)

(2) Let $W_2 = \{(0, 0, 0, w, 0) \in \mathbf{F}^5 : w \in \mathbf{F}\}$. Then $W_2 \cap U_1 = \{0\}$.

Let $U_2 = U_1 \oplus W_2$. Then $U_2 = \{(x, y, z, w, 2x) \in \mathbf{F}^5 : x, y, z, w \in \mathbf{F}\}$.

(3) Let $W_3 = \{(0, 0, 0, 0, u) \in \mathbf{F}^5 : u \in \mathbf{F}\}$. Then $W_3 \cap U_2 = \{0\}$.

Let $U_3 = U_2 \oplus W_3$. Then $U_3 = \{(x, y, z, w, u) \in \mathbf{F}^5 : x, y, z, w, u \in \mathbf{F}\}$.

$$\text{Thus } \mathbf{F}^5 = ((U \oplus W_1) \oplus W_2) \oplus W_3. \quad \square$$

23 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$V = V_1 \oplus U$ and $V = V_2 \oplus U$, then $V_1 = V_2$.

HINT: When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in \mathbf{F}^2 .

SOLUTION: A counterexample:

$$V = \mathbf{F}^2, U = \{(x, x) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_1 = \{(x, 0) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_2 = \{(0, x) \in \mathbf{F}^2 : x \in \mathbf{F}\}.$$

24 Let V_e denote the set of real-valued even functions on \mathbf{R}

and let V_o denote the set of real-valued odd functions on \mathbf{R} . Show that $\mathbb{R}^{\mathbf{R}} = V_e \oplus V_o$.

SOLUTION:

$$\begin{aligned} \text{(a)} \quad & V_e \cap V_o = \{f : f(x) = f(-x) = -f(-x)\} = \{0\}. \\ \text{(b)} \quad & \left. \begin{aligned} f_e \in V_e &\Leftrightarrow f_e(x) = f_e(-x) \Leftarrow \text{let } f_e(x) = \frac{g(x) + g(-x)}{2} \\ f_o \in V_o &\Leftrightarrow f_o(x) = -f_o(-x) \Leftarrow \text{let } f_o(x) = \frac{g(x) - g(-x)}{2} \end{aligned} \right\} \Rightarrow \forall g \in \mathbb{R}^{\mathbf{R}}, g(x) = f_e(x) + f_o(x). \square \end{aligned}$$

ENDED

2.A

2 (a) A list (v) of length 1 in V is linearly independent $\Leftrightarrow v \neq 0$.

(b) A list (v, w) of length 2 in V is linearly independent $\Leftrightarrow \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$.

SOLUTION:

- (a) Suppose $v \neq 0$. Then let $av = 0, a \in \mathbf{F}$. Getting $a = 0$. Thus (v) is linearly independent.
 Suppose (v) is linearly independent. $av = 0 \Rightarrow a = 0$. Then $v \neq 0$, for if not, $a \neq 0$ while $av = 0$. Contradicts.
- (b) Denote the list by (v, w) , where $v, w \in V$. If (v, w) is linearly independent, suppose $av + bw = 0 \Rightarrow a = b = 0$.
 Without loss of generality, suppose $v \neq cw \forall c \in \mathbf{F}$. Then let $av + bw = 0$, getting $a = b = 0 \Rightarrow (v, w)$ is linearly independent.

1 Prove that if (v_1, v_2, v_3, v_4) spans V , then the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V .

SOLUTION: Assume that $\forall v \in V, \exists a_1, \dots, a_4 \in \mathbf{F}$,

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4, \text{ letting } b_i = \sum_{r=1}^i a_r. \end{aligned}$$

Thus $\forall v \in V, \exists b_i \in \mathbf{F}, v = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4$.

Hence the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V . \square

6 Suppose (v_1, v_2, v_3, v_4) is linearly independent in V .

Prove that the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ is also linearly independent.

SOLUTION: $a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0$

$$\Rightarrow a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0$$

$$\Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \Rightarrow a_1 = \dots = a_4 = 0 \Rightarrow \square$$

7 Prove that if (v_1, v_2, \dots, v_m) is a linearly independent list of vectors in V , then $(5v_1 - 4v_2, v_2, v_3, \dots, v_m)$ is linearly independent.

SOLUTION: $a_1(5v_1 - 4v_2) + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0$

$$\Rightarrow 5a_1 v_1 + (a_2 - 4a_1)v_2 + a_3 v_3 + a_4 v_4 = 0$$

$$\Rightarrow 5a_1 = a_2 - 4a_1 = a_3 = a_4 = 0 \Rightarrow a_1 = \dots = a_4 = 0 \square$$

• Suppose (v_1, \dots, v_m) is a list of vectors in V . For $k \in \{1, \dots, m\}$, let $w_k = v_1 + \dots + v_k$.

(a) Show that $\text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$.

(b) Show that (v_1, \dots, v_m) is linearly independent if and only if (w_1, \dots, w_m) is linearly independent.

SOLUTION:

(a) Let $\text{span}(v_1, \dots, v_m) = U$. Assume that $\forall v \in U, \exists a_i \in \mathbf{F}$,

$$v = a_1 v_1 + \dots + a_m v_m = b_1 w_1 + \dots + b_m w_m = \sum_{j=1}^m \left(\sum_{i=j}^m b_i \right) v_j$$

$$\Rightarrow b_1 = a_1, \quad b_i = a_i - \sum_{r=1}^{i-1} b_r. \text{ Thus } \exists b_i \in \mathbf{F} \text{ such that } v = b_1 w_1 + \dots + b_m w_m.$$

又 Each $w_i \in U \Rightarrow \text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$.

(b) $a_1 w_1 + \dots + a_m w_m = 0$

$$\Rightarrow (a_1 + \dots + a_m) v_1 + \dots + (a_i + \dots + a_m) v_i + \dots + a_m v_m = 0$$

$$\Rightarrow a_m = \dots = (a_m + \dots + a_i) = \dots = (a_m + \dots + a_1) = 0. \quad \square$$

10 Suppose (v_1, \dots, v_m) is linearly independent in V and $w \in V$.

(a) Prove that if $(v_1 + w, \dots, v_m + w)$ is linearly dependent, then $w \in \text{span}(v_1, \dots, v_m)$.

(b) Show that (v_1, \dots, v_m, w) is linearly independent $\iff w \notin \text{span}(v_1, \dots, v_m)$.

SOLUTION:

(a) Suppose $a_1(v_1 + w) + \dots + a_m(v_m + w) = 0, \exists a_i \neq 0 \Rightarrow a_1 v_1 + \dots + a_m v_m = 0 = -(a_1 + \dots + a_m)w$.

Then $a_1 + \dots + a_m \neq 0$, for if not, $a_1 v_1 + \dots + a_m v_m = 0$ while $a_i \neq 0$ for some i , contradicts.

Hence $w \in \text{span}(v_1, \dots, v_m)$.

(b) Suppose $w \in \text{span}(v_1, \dots, v_m)$. Then (v_1, \dots, v_m, w) is linearly dependent.

Thus have we proven the “ \Rightarrow ” by its contrapositive.

Suppose $w \notin \text{span}(v_1, \dots, v_m)$. Then by [2.23], (v_1, \dots, v_m, w) is linearly independent. \square

14 Prove that V is infinite-dim if and only if there is a sequence (v_1, v_2, \dots) in V such that (v_1, \dots, v_m) is linearly independent for every $m \in \mathbf{N}^+$.

SOLUTION: Similar to [2.16].

Suppose there is a sequence (v_1, v_2, \dots) in V such that (v_1, \dots, v_m) is linearly independent for any $m \in \mathbf{N}^+$.

Choose an m . Suppose a linearly independent list (v_1, \dots, v_m) spans V .

Then there exists $v_{m+1} \in V$ but $v_{m+1} \notin \text{span}(v_1, \dots, v_m)$. Hence no list spans V . Thus V is infinite-dim.

Conversely it is true as well. For if not, V must be finite-dim, contradicting the assumption. \square

15 Prove that \mathbf{F}^∞ is infinite-dim.

SOLUTION: Let $e_i = (0, \dots, 0, 1, 0, \dots) \in \mathbf{F}^\infty$ for every $m \in \mathbf{N}^+$, where ‘1’ is on the i^{th} entry of e_i .

Suppose \mathbf{F}^∞ is finite-dim. Then let $\text{span}(e_1, \dots, e_m) = V$. But $e_{m+1} \notin \text{span}(e_1, \dots, e_m)$. Contradicts. \square

16 Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dim.

SOLUTION: Denote the vec-sp by U . Note that for each $m \in \mathbf{N}^+$, $(1, x, \dots, x^m)$ is linearly independent.

Because if $a_0, \dots, a_m \in \mathbf{R}$ are such that $a_0 + a_1 x + \dots + a_m x^m = 0, \forall x \in [0, 1]$, then the polynomial has infinitely many roots and hence $a_0 = \dots = a_m = 0$. $\left. \begin{array}{l} \text{Similar to [2.16], } U \text{ is infinite-dim.} \end{array} \right\}$

OR. Note that for $a_n = \frac{1}{n}, a_1 < a_2 < \dots < a_m, \forall m \in \mathbf{N}^+$.

Suppose $f_n = \begin{cases} x - \frac{1}{n}, & x \in [\frac{1}{n}, 1) \\ 0, & x \in [0, \frac{1}{n}] \end{cases}$. Then for any $m, f_1(\frac{1}{m}) = \dots = f_m(\frac{1}{m}) = 0$, while $f_{m+1}(\frac{1}{m}) \neq 0$.

Hence $f_{m+1} \notin \text{span}(f_1, \dots, f_m)$. Thus by Problem (14), U is infinite-dim.

17 Suppose $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$.

Prove that (p_0, p_1, \dots, p_m) is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

SOLUTION:

Suppose (p_0, p_1, \dots, p_m) is linearly independent. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z \ \forall z \in \mathbf{F}$.

But $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \dots + a_m p_m(z)$, for if not, let $z = 2$, contradicts. Thus $z \notin \text{span}(p_0, p_1, \dots, p_m)$.

Then $\text{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list (p_0, p_1, \dots, p_m) has length $m + 1$.

Hence (p_0, p_1, \dots, p_m) is linearly dependent in $\mathcal{P}_m(\mathbf{F})$.

For if not, notice that the list $(1, z, \dots, z^m)$ spans $\mathcal{P}_m(\mathbf{F})$,

thus by [2.23], (p_0, p_1, \dots, p_m) spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. \square

ENDED

2.B

NOTE FOR linearly independent sequence and [2.34].

“ $V = \text{span}(v_1, \dots, v_n, \dots)$ ” is an invalid expression.

If we allow using “infinite list”, then we must guarantee that (v_1, \dots, v_n, \dots) is a spanning “list”

such that for all $v \in V$, there exists a certain positive integer such that $v = a_1 v_{\alpha_1} + \dots + a_n v_{\alpha_n}$,

where $\Gamma = \{\alpha_1, \dots, \alpha_n\}$ is an finite index set. The key point is, how do we find such a “list” ?

NOTE FOR “ $\mathbb{C}_V U \cap \{0\}$ ”: “ $\mathbb{C}_V U \cap \{0\}$ ” is supposed to be “ W ”, where $V = U \oplus W$.

But if we let $u \in U \setminus \{0\}$ and $w \in W \setminus \{0\}$, then $\left. \begin{array}{l} w \in \mathbb{C}_V U \cap \{0\} \\ u \pm w \in \mathbb{C}_V U \cap \{0\} \end{array} \right\} \Rightarrow u \in \mathbb{C}_V U \cap \{0\}$. Contradicts.

NEW NOTATION: Denote the set $\{W_1, W_2, \dots\}$ by $\mathcal{S}_V U$, where for each $W_i, V = U \oplus W_i$. See also in (1.C.23).

1 Find all vector spaces that have exactly one basis. SOLUTION: $\mathbf{F} = \mathbf{C}, \mathbf{R}, \mathbf{Q}, \{0, 1\}, \mathcal{P}_0(\mathbf{F})$.

6 Suppose (v_1, v_2, v_3, v_4) is a basis of V . Prove that $(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4)$ is also a basis.

SOLUTION: $\forall v \in V, \exists! a_1, \dots, a_4 \in \mathbf{F}, v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$.

Assume that $v = b_1(v_1 + v_2) + b_2(v_2 + v_3) + b_3(v_3 + v_4) + b_4 v_4$. Then $v = b_1 v_1 + (b_1 + b_2)v_2 + (b_2 + b_3)v_3 + (b_3 + b_4)v_4$.

$\Rightarrow \exists! b_1 = a_1, b_2 = a_2 - b_1, b_3 = a_3 - b_2, b_4 = a_4 - b_3 \in \mathbf{F}$. \square

7 Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \in U$, then v_1, v_2 is a basis of U .

SOLUTION: Let $V = \mathbf{F}^4, v_1 = (1, 0, 0, 0), v_2 = (0, 1, 0, 0), v_3 = (0, 0, 1, 1), v_4 = (0, 0, 0, 1)$.

And $U = \{(x, y, z, 0) \in \mathbf{R}^4 : x, y, z \in \mathbf{F}\}$. We have a counterexample.

• Suppose V is finite-dim and U, W are subspaces of V such that $V = U + W$.

Prove that there exists a basis of V consisting of vectors in $U \cup W$.

SOLUTION: Let (u_1, \dots, u_m) and (w_1, \dots, w_n) be bases of U and W respectively.

Then $V = \text{span}(u_1, \dots, u_m) + \text{span}(w_1, \dots, w_n) = \text{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.

Hence, by [2.31], we get a basis of V consisting of vectors in U or W . \square

8 Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that

(u_1, \dots, u_m) is a basis of U and (w_1, \dots, w_n) is a basis of W .

Prove that $(u_1, \dots, u_m, w_1, \dots, w_n)$ is a basis of V .

SOLUTION:

$\forall v \in V, \exists! a_i, b_i \in \mathbf{F}, v = (a_1 u_1 + \dots + a_m u_m) + (b_1 w_1 + \dots + b_n w_n)$

$\Rightarrow (a_1 u_1 + \dots + a_m u_m) = -(b_1 w_1 + \dots + b_n w_n) \in U \cap W = \{0\}$. Thus $a_1 = \dots = a_m = b_1 = \dots = b_n$. \square

• (OR 9.4) Suppose V is a real vector space.

Show that if (v_1, \dots, v_n) is a basis of V (as a real vector space), then (v_1, \dots, v_n) is also a basis of the complexification $V_{\mathbb{C}}$ (as a complex vector space).

See Section 1B (4e) for the definition of the complexification $V_{\mathbb{C}}$.

SOLUTION:

$$\forall u + iv \in V_{\mathbb{C}}, \exists! u, v \in V, a_i, b_i \in \mathbf{R},$$

$$u + iv = (a_1v_1 + \dots + a_nv_n) + i(b_1v_1 + \dots + b_nv_n) = (a_1 + b_1i)v_1 + \dots + (a_n + b_ni)v_n$$

$$\Rightarrow u + iv = c_1v_1 + \dots + c_nv_n, \exists! c_i = a_i + b_i i \in \mathbf{C}$$

\Rightarrow By the uniqueness of c_i and [2.29], (v_1, \dots, v_n) is a basis of $V_{\mathbb{C}}$. \square

ENDED

2.C

1 Suppose V is finite-dim and U is a subspace of V such that $\dim V = \dim U$.

Let (u_1, \dots, u_m) be a basis of U . Then $n = \dim U = \dim V$. $\forall u_i \in V$.

Then by [2.39], (u_1, \dots, u_m) is a basis of V . Thus $V = U$.

2 Show that the subspaces of \mathbf{R}^2 are precisely $\{0\}$, all lines in \mathbf{R}^2 containing the origin, and \mathbf{R}^2 .

SOLUTION:

Suppose U is a subspace of \mathbf{R}^2 . Let $\dim U = n$.

If $n = 0$, then $U = \{0\}$.

If $n = 1$, then $U = \text{span}(v)$, where v is a vector in \mathbf{R}^2 . Thus U can be any line in \mathbf{R}^2 containing the origin.

If $n = 2$, then $U = \text{span}(v, w)$, where v, w are vectors in \mathbf{R}^2 and (v, w) is linearly independent $\Rightarrow U = \mathbf{R}^2$. \square

3 Show that the subspaces of \mathbf{R}^3 are precisely $\{0\}$, all lines in \mathbf{R}^3 containing the origin, all planes in \mathbf{R}^3 containing the origin, and \mathbf{R}^3 .

SOLUTION:

Suppose U is a subspace of \mathbf{R}^3 . Let $\dim U = n$.

If $n = 0$, then $U = \{0\}$.

If $n = 1$, then $U = \text{span}(v)$, where v is a vector in \mathbf{R}^3 . Thus U can be any line in \mathbf{R}^3 containing the origin.

If $n = 2$, then $U = \text{span}(v, w)$, where v, w are vectors in \mathbf{R}^3 and (v, w) is linearly independent.

Thus U can be any plane in \mathbf{R}^3 containing the origin.

If $n = 3$, then $U = \text{span}(u, v, w)$, where u, v, w are vectors in \mathbf{R}^3 and (u, v, w) is linearly independent

$\Rightarrow U = \mathbf{R}^3$. \square

7 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U .

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

SOLUTION:

Suppose $p(z) = az^4 + bz^3 + cz^2 + dz + e$ and $p(2) = p(5) = p(6)$.

$$\text{Then } \begin{cases} p(2) = 16a + 8b + 4c + 2d + e \text{ (I)} \\ p(5) = 625a + 125b + 25c + 5d + e \text{ (II)} \\ p(6) = 1296a + 216b + 36c + 6d + e \text{ (III)} \end{cases}$$

You don't have to compute to know that the dimension of the set of solutions is 3.

(a) A basis: $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(c) Let $W = \text{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. \square

9 Suppose (v_1, \dots, v_m) is linearly independent in V and $w \in V$.

Prove that $\dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$.

SOLUTION:

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \text{span}(v_1 + w, \dots, v_m + w)$, for each $i = 1, \dots, m$.

(v_1, \dots, v_m) is linearly independent $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ is linearly independent

$\Rightarrow (v_2 - v_1, \dots, v_m - v_1)$ is linearly independent of length $m - 1$.

又 By the contrapositive of (2.A.10), $w \notin \text{span}(v_1, \dots, v_m) \Rightarrow (v_1 + w, \dots, v_m + w)$ is linearly independent.

$\therefore m \geq \dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$. \square

10 Suppose m is a positive integer and $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k . Prove that (p_0, p_1, \dots, p_m) is a basis of $\mathcal{P}_m(\mathbf{F})$.

SOLUTION: Using mathematical induction on m .

(i) For $p_0, \deg p_0 = 0 \Rightarrow \text{span}(p_0) = \text{span}(1)$.

(ii) Suppose for $i \geq 1, \text{span}(p_0, p_1, \dots, p_i) = \text{span}(1, x, \dots, x^i)$.

Then $\text{span}(p_0, p_1, \dots, p_i, p_{i+1}) \subseteq \text{span}(1, x, \dots, x^i, x^{i+1})$.

又 $\deg p_{i+1} = i + 1, p_{i+1}(x) = a_{i+1}x^{i+1} + r_{i+1}(x); a_{i+1} \neq 0, \deg r_{i+1} \leq i$.

$\Rightarrow x^{i+1} = \frac{1}{a_{i+1}}(p_{i+1}(x) - r_{i+1}(x)) \in \text{span}(1, x, \dots, x^i, p_{i+1}) = \text{span}(p_0, p_1, \dots, p_i, p_{i+1})$.

$\therefore x^{i+1} \in \text{span}(p_0, p_1, \dots, p_i, p_{i+1}) \Rightarrow \text{span}(1, x, \dots, x^i, x^{i+1}) \subseteq \text{span}(p_0, p_1, \dots, p_i, p_{i+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \text{span}(1, x, \dots, x^m) = \text{span}(p_0, p_1, \dots, p_m)$. \square

• Suppose m is a positive integer. For $0 \leq k \leq m$, let $p_k(x) = x^k(1 - x)^{m-k}$.

Show that (p_0, \dots, p_m) is a basis of $\mathcal{P}(\mathbf{F})$.

The basis in this exercise leads to what are called Bernstein polynomials. You can do a web search to learn how

Bernstein polynomials are used to approximate continuous functions on $[0, 1]$.

SOLUTION: Using mathematical induction.

(i) $k = 0, 1, 2, p_m(x) = x^m, p_{m-1}(x) = x^{m-1} - x^m, p_{m-2}(x) = x^{m-2} + x^m - 2x^{m-1}$.

(ii) $k \geq 2$. Suppose for $p_{m-k}(x), \exists ! a_i \in \mathbf{F}, x^{m-k} = p_{m-k}(x) + a_m x^m + \dots + a_{m-k+1} x^{m-k+1}$.

Then for $p_{m-k-1}(x), \exists ! c_i \in \mathbf{F}$,

$$x^{m-k-1} = p_{m-k-1}(x) + \mathcal{C}_{k+1}^1(-1)^2 x^{m-k} + \dots + \mathcal{C}_{k+1}^k(-1)^{k+1} x^{m-1} + (-1)^{k-2} x^m$$

$$\Rightarrow c_{m-i} = \mathcal{C}_{k+1}^{k+1-i}(-1)^{k-i}.$$

Thus for each $x^i, \exists ! b_i \in \mathbf{F}, x^i = b_m p_m(x) + \dots + b_{m-i} p_{m-i}(x)$.

$\Rightarrow \text{span}(x^m, \dots, x, 1) = \text{span}(\underbrace{p_m, \dots, p_1, p_0}_{\text{Basis}})$. \square

• Suppose V is finite-dim and V_1, V_2, V_3 are subspaces of V with

$\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

SOLUTION: $\left. \begin{array}{l} \dim V_1 + \dim V_2 > 2 \dim V - \dim V_3 \geq \dim V \Rightarrow V_1 \cap V_2 \neq \{0\} \\ \dim V_2 + \dim V_3 > 2 \dim V - \dim V_1 \geq \dim V \Rightarrow V_2 \cap V_3 \neq \{0\} \\ \dim V_1 + \dim V_3 > 2 \dim V - \dim V_2 \geq \dim V \Rightarrow V_1 \cap V_3 \neq \{0\} \end{array} \right\} \Rightarrow V_1 \cap V_2 \cap V_3 \neq \{0\}. \square$

• Suppose V is finite-dim and U is a subspace of V with $U \neq V$. Let $n = \dim V, m = \dim U$.

Prove that there exist $(n - m)$ subspaces of V , say U_1, \dots, U_{n-m} , each of dimension $(n - 1)$, such that $\bigcap_{i=1}^{n-m} U_i = U$.

SOLUTION: Let (v_1, \dots, v_m) be a basis of U , extend to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.

Define $U_i = \text{span}(v_1, \dots, v_m, v_{m+1}, \dots, v_{m+i-1}, v_{m+i+1}, \dots, v_n)$ for each i . Thus we are done.

EXAMPLE: Suppose $\dim V = 6, \dim U = 3$.

$$\left. \begin{array}{l} \underbrace{(v_1, v_2, v_3, v_4, v_5, v_6)}_{\text{Basis of } V}, \text{ define } \left. \begin{array}{l} U_1 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_5, v_6) \\ U_2 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_6) \\ U_3 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_5) \end{array} \right\} \Rightarrow \dim U_i = 6 - 1, \quad i = \underbrace{1, 2, 3}_{6-3=3} \end{array} \right\}$$

□

14 Suppose that V_1, \dots, V_m are finite-dim subspaces of V .

Prove that $V_1 + \dots + V_m$ is finite-dim and $\dim(V_1 + \dots + V_m) \leq \dim V_1 + \dots + \dim V_m$.

SOLUTION:

Choose a basis \mathcal{E}_i of $V_i \Rightarrow V_1 + \dots + V_m = \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$; $\dim U_i = \text{card } \mathcal{E}_i$.

Then $\dim(V_1 + \dots + V_m) = \dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$.

又 $\dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leq \text{card}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leq \text{card } \mathcal{E}_1 + \dots + \text{card } \mathcal{E}_m$.

Thus $\dim(V_1 + \dots + V_m) \leq \dim U_1 + \dots + \dim U_m$.

• The inequality above is an equality if and only if $V_1 + \dots + V_m$ is a direct sum.

For each i , $(V_1 + \dots + V_i) \cap V_{i+1} = \{0\} \iff V_1 + \dots + V_m$ is a direct sum $\iff \square$

17 Suppose V_1, V_2, V_3 are subspaces of a finite-dim vector space, then

$$\begin{aligned} \dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3). \end{aligned}$$

Explain why you might think and prove the formula above or give a counterexample.

SOLUTION:

Looks like: given three sets A, B and C .

Note that: $\text{card}(X \cup Y) = \text{card}(X) + \text{card}(Y) - \text{card}(X \cap Y)$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

Then: $\text{card}((A \cup B) \cup C) = \text{card}(A \cup B) + \text{card } C - \text{card}((A \cup B) \cap C)$.

And: $\text{card}((A \cup B) \cap C) = \text{card}((A \cap C) \cup (B \cap C)) = \text{card}(A \cap C) + \text{card}(B \cap C) - \text{card}(A \cap B \cap C)$.

Thus: $\text{card}((A \cup B) \cup C) = \text{card } A + \text{card } B + \text{card } C + \text{card}(A \cap B \cap C) - \text{card}(A \cap B) - \text{card}(A \cap C) - \text{card}(B \cap C)$.

Because $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\dim(V_1 + V_2 + V_3) = \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \quad (1)$$

$$= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) \quad (2)$$

$$= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) \quad (3)$$

Notice that $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$.

For example, $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

• **COROLLARY:** If V_1, V_2 and V_3 are finite-dim vector spaces, then $\frac{(1) + (2) + (3)}{3}$:

$$\begin{aligned} \dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\ &\quad - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}. \end{aligned}$$

The formula above may seem strange because the right side does not look like an integer. □

3.A

• **TIPS:** $T : V \rightarrow W$ is linear $\iff \begin{cases} \forall v, u \in V, T(v + u) = Tv + Tu \\ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv) \end{cases} \iff T(v + \lambda u) = Tv + \lambda Tu.$

3 Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists A_{j,k} \in \mathbf{F}$ such that

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$$

for any $(x_1, \dots, x_n) \in \mathbf{F}^n$.

SOLUTION:

Let $T(1, 0, 0, \dots, 0, 0) = (A_{1,1}, \dots, A_{m,1})$, Note that $(1, 0, \dots, 0, 0), \dots, (0, 0, \dots, 0, 1)$ is a basis of \mathbf{F}^n .
 $T(0, 1, 0, \dots, 0, 0) = (A_{1,2}, \dots, A_{m,2})$, Then by [3.5], we are done. \square
 \vdots
 $T(0, 0, 0, \dots, 0, 1) = (A_{1,n}, \dots, A_{m,n})$.

4 Suppose $T \in \mathcal{L}(V, W)$ and (v_1, \dots, v_m) is a list of vectors in V such that

(Tv_1, \dots, Tv_m) is linearly independent in W . Prove that (v_1, \dots, v_m) is linearly independent.

SOLUTION: Suppose $a_1v_1 + \dots + a_mv_m = 0$. Then $a_1Tv_1 + \dots + a_mTv_m = 0$. Thus $a_1 = \dots = a_m = 0$. \square

5 Prove that $\mathcal{L}(V, W)$ is a vector space, **SOLUTION:** Note that $\mathcal{L}(V, W)$ is a subspace of W^V . \square

7 Show that every linear map from a one-dim vector space to itself

is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

SOLUTION:

Let u be a nonzero vector in $V \Rightarrow V = \text{span}(u)$.
 Because $Tu \in V \Rightarrow Tu = \lambda u$ for some λ .
 Suppose $v \in V \Rightarrow v = au$, $\exists! a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. \square

8 Give an example of a function $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}$ such that

$\varphi(av) = a\varphi(v)$ for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.

SOLUTION:

Define $T(x, y) = \begin{cases} x + y, & \text{if } (x, y) \in \text{span}(3, 1), \\ 0, & \text{otherwise.} \end{cases}$ OR. Define $T(x, y) = \sqrt[3]{(x^3 + y^3)}$. \square

9 Give an example of a function $\varphi : \mathbf{C} \rightarrow \mathbf{C}$ such that

$\varphi(w + z) = \varphi(w) + \varphi(z)$ for all $w, z \in \mathbf{C}$ but φ is not linear.

(Here \mathbf{C} is thought of as a complex vector space.)

SOLUTION:

Suppose $V_{\mathbb{C}}$ is the complexification of a vector space V . Suppose $\varphi : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$.

Define $\varphi(u + iv) = u = \Re(u + iv)$

OR. Define $\varphi(u + iv) = v = \Im(u + iv)$. \square

• *Prove or give a counterexample:*

If $q \in \mathcal{P}(\mathbf{R})$ and $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$ is defined by $Tp = q \circ p$, then T is linear.

SOLUTION:

Because in general, $q(p_1(x) + \lambda p_2(x)) \neq (q \circ p_1)(x) + \lambda(q \circ p_2)(x)$. \square

• OR (3.D.16) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Suppose $ST = TS$ for every $S \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity.*

SOLUTION:

If $V = \{0\}$, then we are done. Now suppose $V \neq \{0\}$.

Assume that (v, Tv) is linearly dependent for every $v \in V$, then by (2.A.2.(b)), $Tv = \lambda_v v$ for some $\lambda_v \in \mathbf{F}$.

To prove that λ_v is independent of v

(in other words, for any two distinct nonzero vectors v and w in V , we have $\lambda_v \neq \lambda_w$), we discuss in two cases:

$$\left. \begin{array}{l} (-) \text{ If } (v, w) \text{ is linearly independent, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = a_v v + a_w w \\ \quad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=) \text{ Otherwise, suppose } w = cv, a_w w = Tw = cTv = ca_v v = a_v w \Rightarrow (a_w - a_v)w \end{array} \right\} \Rightarrow a_w = a_v.$$

Now we prove the assumption by contradiction.

Suppose (v, Tv) is linearly independent for every nonzero vector $v \in V$.

Fix one v . Extend to (v, Tv, u_1, \dots, u_n) a basis of V .

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \dots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Contradicts. \square

OR. Let (v_1, \dots, v_m) be a basis of V .

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \dots = \varphi(v_m) = 1$. Let $\lambda = \varphi(Tv_1) \in \mathbf{F}$.

For any $v \in V$, define $S_v \in \mathcal{L}(V)$ by $S_v u = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. \square

10 *Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$).*

Define $T : V \rightarrow W$ by $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \setminus U. \end{cases}$ Prove that T is not a linear map on V .

SOLUTION:

Suppose T is a linear map. And $v \in V \setminus U$, $u \in U$ such that $Su \neq 0$.

Then $v + u \in V \setminus U$, (for if not, $v = (v + u) - u \in U$) while $T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$.

Hence we get a contradiction. \square

11 *Suppose V is finite-dim. Prove that every linear map on a subspace of V can be extended to a linear map on V . In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.*

SOLUTION:

Define $T \in \mathcal{L}(V, W)$ by $T(a_1u_1 + \dots + a_nu_n + a_{n+1}u_{n+1} + \dots + a_mu_m) = a_1Su_1 + \dots + a_nSu_n$.

Where we let (u_1, \dots, u_n) be a basis of U , extend to a basis of V as $(u_1, \dots, u_n, \dots, u_m)$. \square

12 Suppose V is finite-dim with $\dim V > 0$, and W is infinite-dim.

Prove that $\mathcal{L}(V, W)$ is infinite-dim.

SOLUTION:

Let (v_1, \dots, v_n) be a basis of V . Let (w_1, \dots, w_m) be linearly independent in W for any $m \in \mathbf{N}^+$.

Define $T_{x,y} \in \mathcal{L}(V, W)$ by $T_{x,y}(v_z) = \delta_{zy}w_y, \forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{zy} = \begin{cases} 0, & z \neq y, \\ 1, & z = y. \end{cases}$

Suppose $a_1T_{x,1} + \dots + a_mT_{x,m} = 0$. Then $(a_1T_{x,1} + \dots + a_mT_{x,m})(v_x) = 0 = a_1w_1 + \dots + a_mw_m$.

$\Rightarrow a_1 = \dots = a_m = 0$. $\forall m$ is arbitrarily chosen.

Thus $(T_{x,1}, \dots, T_{x,m})$ is a linearly independent list in $\mathcal{L}(V, W)$ for any x and length m . Hence by (2.A.14). \square

13 Suppose (v_1, \dots, v_m) is a linearly dependent list of vectors in V .

Suppose also that $W \neq \{0\}$. Prove that there exist $(w_1, \dots, w_m) \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \dots, m$.

SOLUTION:

We show it by contradiction. By linear independence lemma, $\exists j \in \{1, \dots, m\}$ such that $v_j \in \text{span}(v_1, \dots, v_{j-1})$.

Fix j . Let $w_j \neq 0$, while $w_1 = \dots = w_{j-1} = w_{j+1} = \dots = w_m = 0$.

Define T by $Tv_k = w_k$ for all k . Suppose $a_1v_1 + \dots + a_mv_m = 0$ (where $a_j \neq 0$).

Then $T(a_1v_1 + \dots + a_mv_m) = 0 = a_1w_1 + \dots + a_mw_m = a_jw_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. \square

OR. We prove the contrapositive:

Suppose for any list $(w_1, \dots, w_m) \in W, \exists T \in \mathcal{L}(V, W), Tv_k = w_k$ for each w_k .

(We need to) Prove that (v_1, \dots, v_n) is linearly independent.

Suppose $\exists a_i \in \mathbf{F}, a_1v_1 + \dots + a_nv_n = 0$. Choose a nonzero $w \in W$.

By assumption, for the list $(\overline{a_1}w, \dots, \overline{a_m}w), \exists T \in \mathcal{L}(V, W), Tv_k = \overline{a_k}w$ for each v_k .

$0 = T(\sum_{k=1}^m a_kv_k) = \sum_{k=1}^m a_kTv_k = \sum_{k=1}^m a_k\overline{a_k}w = (\sum_{k=1}^m |a_k|^2)w$. Hence $\sum_{k=1}^m |a_k|^2 = 0 \Rightarrow a_k = 0$. \square

• (4E 3.A.16)

Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

A subspace \mathcal{E} of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION:

Let (v_1, \dots, v_n) be a basis of V . If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and \mathcal{E} is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \setminus \{0\}$.

Suppose $Sw_i \neq 0$ and $Sw_i = a_1v_1 + \dots + a_nv_n$, where $a_k \neq 0$.

Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y}(v_x) = v_y, R_{x,y}(v_z) = 0 (z \neq x)$. Then for any $x, y \in \mathbf{N}^+,$

$(R_{k,y}S)(v_i) = a_kv_y \Rightarrow ((R_{k,y}S) \circ R_{x,i})(v_x) = a_kv_y$, and $((R_{k,y}S) \circ R_{x,i})(v_z) = 0$ for $z \neq x$.

Thus $R_{k,y}SR_{x,i} = a_kR_{x,y}$. Denote by $T_{x,y}$.

Getting $(\frac{1}{a_k}T_{1,1} + \dots + \frac{1}{a_k}T_{n,n})v_j = v_j \Rightarrow \sum_{r=1}^n \frac{1}{a_k}T_{r,r} = I$.

\forall By assumption, $T_{x,y} \in \mathcal{E} \Rightarrow I \in \mathcal{E}$.

Hence for any $T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. \square

ENDED

3.B

2 Suppose $S, T \in \mathcal{L}(V)$ are such that $\text{range } S \subseteq \text{null } T$. Prove that $(ST)^2 = 0$.

SOLUTION: $TS = 0 \Rightarrow STST = (ST)^2 = 0$. \square

- 3** Suppose (v_1, \dots, v_m) in V . Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m$.
- (a) The surjectivity of T corresponds to (v_1, \dots, v_m) spanning V .
- (b) The injectivity of T corresponds to (v_1, \dots, v_m) being linearly independent.

4 Show that $U = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \text{null } T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

SOLUTION: Let $(v_1, v_2, v_3, v_4, v_5)$ be a basis of \mathbf{R}^5 , (w_1, w_2, w_3, w_4) be a basis of \mathbf{R}^4 .

Define $T_1, T_2 \in U$ as $T_1 v_1 = 0, T_1 v_2 = 0, T_1 v_3 = 0, T_1 v_4 = w_4, T_1 v_5 = w_1$;

$T_2 v_1 = 0, T_2 v_2 = 0, T_2 v_3 = w_3, T_2 v_4 = 0, T_2 v_5 = w_4$. Thus $T_1 + T_2 \notin U$.

For $U' = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \text{null } T > 0\}$,

define $T_1, T_2 \in U'$ as $T_1 v_1 = 0, T_1 v_2 = w_2, T_1 v_3 = w_3, T_1 v_4 = w_4, T_1 v_5 = w_1$;

$T_2 v_1 = w_1, T_2 v_2 = w_2, T_2 v_3 = 0, T_2 v_4 = w_3, T_2 v_5 = w_4$. Thus $T_1 + T_2 \notin U'$. \square

5 Give an example of $T \in \mathcal{L}(\mathbf{R}^4)$ such that $\text{range } T = \text{null } T$.

SOLUTION: Define such T by $T e_1 = e_3, T e_2 = e_4, T e_3 = T e_4 = 0$. Where (e_1, e_2, e_3, e_4) is a basis. \square

7 Suppose V is finite-dim with $2 \leq \dim V$ and also $\dim V \leq \dim W$, if W is finite-dim. Show that $U = \{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

SOLUTION:

Let (v_1, \dots, v_n) be a basis of V , (w_1, \dots, w_m) be linearly independent in W .

(Let $\dim W = m$, if W is finite, otherwise, we choose $m \in \{n, n+1, \dots\}$ arbitrarily; $2 \leq n \leq m$).

Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 : v_1 \mapsto 0, v_2 \mapsto w_2, v_i \mapsto w_i$.

Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 : v_1 \mapsto w_1, v_2 \mapsto 0, v_i \mapsto w_i, i = 3, \dots, n$.

Thus $T_1 + T_2 \notin U$. \square

COMMENT: If $\dim V = 0$, then $V = \{0\} = \text{span}(\cdot)$. $\forall T \in \mathcal{L}(V, W)$, T is injective. Hence $U = \emptyset$.

If $\dim V = 1$, then $V = \text{span}(v_0)$. Thus $U = \text{span}(T_0)$, where $T_0 v_0 = 0$.

If V is infinite-dim, the result is true as well.

8 Suppose W is finite-dim with $\dim W \geq 2$ and also $\dim V \geq \dim W$, if V is finite-dim. Show that $U = \{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

SOLUTION:

Let (v_1, \dots, v_n) be linearly independent in V , (w_1, \dots, w_m) be a basis of W .

(Let $n = \dim V$, if V is finite, otherwise we choose $n \in \{m, m+1, \dots\}$; $2 \leq m \leq n$).

Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 : v_1 \mapsto 0, v_2 \mapsto w_2, v_j \mapsto w_j, v_{m+i} \mapsto 0$.

Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 : v_1 \mapsto w_1, v_2 \mapsto 0, v_j \mapsto w_j, v_{m+i} \mapsto 0$.

For each $j = 2, \dots, m$; $i = 1, \dots, n - m$, if V is finite, otherwise let $i \in \mathbf{N}^+$.

Thus $T_1 + T_2 \notin U$. \square

COMMENT: If $\dim W = 0$, then $W = \{0\} = \text{span}(\cdot)$. $\forall T \in \mathcal{L}(V, W)$, T is surjective. Hence $U = \emptyset$.

If $\dim W = 1$, then $W = \text{span}(w_0)$. Thus $U = \text{span}(T_0)$, where $T_0 w_0 = 0$.

If W is infinite-dim, the result is true as well.

9 Suppose $T \in \mathcal{L}(V, W)$ is injective and (v_1, \dots, v_n) is linearly independent in V . Prove that $(T v_1, \dots, T v_n)$ is linearly independent in W .

SOLUTION:

$$a_1Tv_1 + \cdots + a_nTv_n = 0 = T\left(\sum_{i=1}^n a_iv_i\right) \iff \sum_{i=1}^n a_iv_i = 0 \iff a_1 = \cdots = a_n = 0. \quad \square$$

10 Suppose (v_1, \dots, v_n) spans V and $T \in \mathcal{L}(V, W)$. Show that (Tv_1, \dots, Tv_n) spans $\text{range } T$.

SOLUTION:

(a) $\text{range } T = \{Tv : v \in V\} = \{Tv : v \in \text{span}(v_1, \dots, v_n)\} \Rightarrow Tv_1, \dots, Tv_n \in \text{range } T \Rightarrow \text{By [2.7].}$

OR. $\text{span}(Tv_1, \dots, Tv_n) \ni a_1Tv_1 + \cdots + a_nTv_n = T(a_1v_1 + \cdots + a_nv_n) \in \text{range } T$.

(b) $\forall w \in \text{range } T, \exists v \in V, w = Tv. (\exists a_i \in \mathbf{F}, v = a_1v_1 + \cdots + a_nv_n) \Rightarrow w = a_1Tv_1 + \cdots + a_nTv_n \Rightarrow \square$

11 Suppose S_1, \dots, S_n are injective linear maps and $S_1S_2 \dots S_n$ makes sense. Prove that $S_1S_2 \dots S_n$ is injective.

SOLUTION: $S_1S_2 \dots S_n(v) = 0 \iff S_2S_3 \dots S_n(v) = 0 \iff \cdots \iff S_n(v) = 0 \iff v = 0. \quad \square$

12 Suppose that V is finite-dim and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \text{null } T = \{0\}$ and $\text{range } T = \{Tu : u \in U\}$.

SOLUTION:

By [2.34], there exists a subspace U of V such that $V = U \oplus \text{null } T$.

$\forall v \in V, \exists! w \in \text{null } T, u \in U, v = w + u$. Then $Tv = T(w + u) = Tu \in \{Tu : u \in U\} \Rightarrow \square$

COMMENT: V can be infinite-dim. See the above of [2.34].

16 Suppose there exists a linear map on V whose null space and range are both finite-dim. Prove that V is finite-dim.

SOLUTION:

Denote the linear map by T . Let (Tv_1, \dots, Tv_n) be a basis of $\text{range } T$, (u_1, \dots, u_m) be a basis of $\text{null } T$.

Then for all $v \in V, T(\underbrace{v - a_1v_1 - \cdots - a_nv_n}_{u \in \text{null } T}) = 0$, where $Tv = a_1Tv_1 + \cdots + a_nTv_n$.

$\Rightarrow u = b_1u_1 + \cdots + b_mu_m \Rightarrow v = a_1v_1 + \cdots + a_nv_n + b_1u_1 + \cdots + b_mu_m$.

Getting $V \subseteq \text{span}(v_1, \dots, v_n, u_1, \dots, u_m)$. Thus V is finite-dim. \square

17 Suppose V and W are both finite-dim. Prove that there exists an injective $T \in \mathcal{L}(V, W)$ if and only if $\dim V \leq \dim W$.

SOLUTION:

(a) Suppose there exists an injective T . Then $\dim V = \dim \text{range } T + \dim \text{null } T = \dim \text{range } T \leq \dim W$.

(b) Suppose $\dim V \leq \dim W$, letting (v_1, \dots, v_n) and (w_1, \dots, w_m) be bases of V and W respectively.

Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i, i = 1, \dots, n (= \dim V)$. \square

18 Suppose V and W are both finite-dim. Prove that there exists a surjective $T \in \mathcal{L}(V, W)$ if and only if $\dim V \geq \dim W$.

SOLUTION:

(a) Suppose there exists a surjective T . Then $\dim V = \dim \text{range } T + \dim \text{null } T = \dim W + \dim \text{null } T \Rightarrow \dim W = \dim V - \dim \text{null } T \leq \dim V$.

(b) Suppose $\dim V \geq \dim W$, letting (v_1, \dots, v_n) and (w_1, \dots, w_m) be bases of V and W respectively.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1v_1 + \cdots + a_mv_m + a_{m+1}v_{m+1} + \cdots + a_nv_n) = a_1w_1 + \cdots + a_mw_m. \quad \square$

19 Suppose V and W are finite-dim and that U is a subspace of V .

Prove that $\exists T \in \mathcal{L}(V, W), \text{null } T = U \iff \dim U \geq \dim V - \dim W$.

SOLUTION:

(a) Suppose $\exists T \in \mathcal{L}(V, W)$, $\text{null } T = U$. Then $\dim \text{null } T = \dim U \geq \dim V - \dim W$.

(b) Suppose $\underbrace{\dim U}_m \geq \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_p (\Rightarrow \dim W = p \geq n = \dim V - \dim U)$.

Let (u_1, \dots, u_m) be a basis of U , extend to a basis of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$.

Let (w_1, \dots, w_p) be a basis of W .

Define $T \in \mathcal{L}(V, W)$ by $T(a_1v_1 + \dots + a_nv_n + b_1u_1 + \dots + b_mu_m) = a_1w_1 + \dots + a_nw_n$. \square

• **TIPS:** Suppose $T \in \mathcal{L}(V, W)$ and $R = (Tv_1, \dots, Tv_n)$ is linearly independent in range T .

(Let $\dim \text{range } T = n$, if $\text{range } T$ is finite, otherwise choose n arbitrarily.)

By (3.A.4), $L = (v_1, \dots, v_n)$ is linearly independent in V .

• **NEW NOTATION:** Denote \mathcal{K}_R by $\text{span } L$, if $\text{range } T$ is finite-dim,
otherwise, denote it by an vector space in the set $\mathcal{S}_V \text{null } T$.

• **NEW THEOREM:**

If $\text{range } T$ is infinite-dim, then choose a vecsp in $\mathcal{S}_V \text{null } T$ and we are done. Now suppose $\text{range } T$ is finite-dim.

$$\mathcal{K}_R \oplus \text{null } T = V \iff \begin{cases} \text{(a) } T(\sum_{i=1}^n a_i v_i) = 0 \Rightarrow \sum_{i=1}^n a_i T v_i = 0 \Rightarrow a_1 = \dots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \text{null } T = \{0\}. \\ \text{(b) } \forall v \in V, T v = \sum_{i=1}^n a_i T v_i \Rightarrow T v - \sum_{i=1}^n a_i T v_i = T(v - \sum_{i=1}^n a_i v_i) = 0 \\ \Rightarrow v - \sum_{i=1}^n a_i v_i \in \text{null } T \Rightarrow v = (v - \sum_{i=1}^n a_i v_i) + (\sum_{i=1}^n a_i v_i) \Rightarrow \mathcal{K}_R + \text{null } T = V. \end{cases}$$

• **COMMENT:** $\text{null } T \in \mathcal{S}_V \mathcal{K}_R$.

• (4E 3.B.21) Suppose V is finite-dim, $T \in \mathcal{L}(V, W)$, and U is a subspace of W .

Prove that $\mathcal{K}_U = \{v \in V : T v \in U\}$ is a subspace of V

and $\dim \mathcal{K}_U = \dim \text{null } T + \dim(U \cap \text{range } T)$.

SOLUTION: For any $u, w \in \mathcal{K}_U$ and $\lambda \in \mathbf{F}$, $T(u + \lambda w) = T u + \lambda T w \in U \Rightarrow \mathcal{K}_U$ is a subspace of V .

Define $S \in \mathcal{L}(\mathcal{K}_U, U)$ as $R v = T v$ for all $v \in \mathcal{K}_U$. Hence $\text{range } R = U \cap \text{range } T$.

Suppose $T v = 0$ for some $v \in V$. $\text{As } 0 \in U \Rightarrow R v = 0$. Thus $\text{null } T \subseteq \text{null } R$. \square

20 Suppose $T \in \mathcal{L}(V, W)$. Prove that T is injective $\iff \exists S \in \mathcal{L}(W, V)$, $ST = I \in \mathcal{L}(V)$.

SOLUTION:

(a) Suppose $\exists S \in \mathcal{L}(W, V)$, $ST = I$. Then if $T v = 0 \Rightarrow ST(v) = 0 = v$.

(b) Suppose T is injective. Let $R = (Tv_1, \dots, Tv_n)$ be linearly independent in $\text{range } T \subseteq W$,

where $n = \dim \text{range } T$ if finite-dim, otherwise $n \in \mathbf{N}^+$.

Then $\mathcal{K}_R \oplus \text{null } T = V$. And suppose $U \oplus \text{range } T = W$.

Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$ and $Su = 0$, $u \in U$. Thus $ST = I$. \square

21 Suppose $T \in \mathcal{L}(V, W)$. Prove that T is surjective $\iff \exists S \in \mathcal{L}(W, V)$, $TS = I \in \mathcal{L}(W)$.

SOLUTION:

(a) Suppose $\exists S \in \mathcal{L}(W, V)$, $TS = I$. Then $\forall w \in W, TS(w) = w \in \text{range } T \Rightarrow \text{range } T = W$.

(b) Suppose T is surjective. Let $R = (Tv_1, \dots, Tv_n)$ be linearly independent in $\text{range } T = W$,

where $n = \dim \text{range } T$ if finite-dim, otherwise $n \in \mathbf{N}^+$.

Then $\mathcal{K}_R \oplus \text{null } T = V$. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$. Then $TS = I$. \square

22 Suppose U and V are finite-dim vec-sps and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$.

Prove that $\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T$.

SOLUTION:

Define $R \in \mathcal{L}(\text{null } ST, V)$ by $Ru = Tu$ for all $u \in \text{null } ST \subseteq U$.

$$\left. \begin{aligned} S(Tu) = 0 = S(Ru) &\Rightarrow \text{range } R \subseteq \text{null } S \Rightarrow \dim \text{range } R \leq \dim \text{null } S \\ Tu = 0 = Ru &\Rightarrow \text{null } R \supseteq \text{null } T \Rightarrow \dim \text{null } R = \dim \text{null } T \end{aligned} \right\} \Rightarrow \square$$

OR. For any $u \in U$, note that $u \in \text{null } ST \iff S(Tu) = 0 \iff Tu \in \text{null } S$.

Thus $\text{null } ST = \mathcal{K}_{\text{null } S \cap \text{range } T} = \{u \in U : Tu \in \text{null } S\}$. By Problem (4E 3B.21),

$$\dim \text{null } ST = \dim \text{null } T + \dim(\text{null } S \cap \text{range } T) \leq \dim \text{null } T + \dim \text{null } S. \quad \square$$

• **COROLLARY:**

- (1) If T is injective, then $\dim \text{null } T = 0 \Rightarrow \dim \text{null } ST \leq \dim \text{null } S$.
- (2) If T is surjective, then $\text{range } R = \text{null } S \Rightarrow \dim \text{null } ST = \dim \text{null } S + \dim \text{null } T$.
- (3) If S is injective, then $\text{range } R = \{0\} \Rightarrow \dim \text{null } ST = \dim \text{null } R = \dim \text{null } T$.

23 Suppose U and V are finite-dim vec-sps and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$.

Prove that $\dim \text{range } ST \leq \min\{\dim \text{range } S, \dim \text{range } T\}$.

SOLUTION:

$$\begin{aligned} \text{range } ST &= \{Sv : v \in \text{range } T\} = \text{span}(Su_1, \dots, Su_{\dim \text{range } T}), \\ &\quad \text{where } \text{span}(u_1, \dots, u_{\dim \text{range } T}) = \text{range } T. \end{aligned}$$

$$\dim \text{range } ST \leq \dim \text{range } T \quad \text{and} \quad \dim \text{range } ST \leq \dim \text{range } S \Rightarrow \square$$

OR. Note that $\text{range}(S|_{\text{range } T}) = \text{range } ST$.

$$\text{Thus } \dim \text{range } ST = \dim \text{range}(S|_{\text{range } T}) = \dim \text{range } T - \dim \text{null}(S|_{\text{range } T}) \leq \dim \text{range } T. \quad \square$$

• **COROLLARY:**

- (1) If S is injective, then $\dim \text{range } ST = \dim \text{range } T$.
- (2) If T is surjective, then $\dim \text{range } ST = \dim \text{range } S$.

• (a) Suppose $\dim V = 5$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \text{range } TS \leq 2$.

(b) Give an example of $S, T \in \mathcal{L}(\mathbf{F}^5)$ with $ST = 0$ and $\dim \text{range } TS = 2$.

SOLUTION:

$$\text{By Problem (23), } \dim \text{range } TS \leq \min\{\overbrace{\dim \text{range } S}^{5 - \dim \text{null } T}, \overbrace{\dim \text{range } T}^{5 - \dim \text{null } S}\}.$$

We show that $\dim \text{range } TS \leq 2$ by contradiction. Assume that $\dim \text{range } TS \geq 3$.

$$\text{Then } \min\{5 - \dim \text{null } T, 5 - \dim \text{null } S\} \geq 3 \Rightarrow \max\{\dim \text{null } T, \dim \text{null } S\} \leq 2.$$

$$\text{and } \dim \text{null } ST = 5 \leq \dim \text{null } S + \dim \text{null } T \leq 4. \text{ Contradicts. } \square$$

$$\text{OR. } \left. \begin{aligned} \dim \text{null } S &= 5 - \dim \text{range } S \\ \dim \text{range } TS &\leq \dim \text{range } S \end{aligned} \right\} \Rightarrow \dim \text{null } S \leq 5 - \dim \text{range } TS.$$

$$\text{And } ST = 0 \Rightarrow \text{range } T \subseteq \text{null } S \Rightarrow \dim \text{range } TS \leq \dim \text{range } T \leq \dim \text{null } S.$$

$$\text{Thus } \dim \text{range } TS \leq 5 - \dim \text{range } TS \Rightarrow \dim \text{range } TS \leq \frac{5}{2}. \quad \square$$

EXAMPLE: Let (v_1, \dots, v_5) be a basis of \mathbf{F}^5 . Define $S, T \in \mathcal{L}(\mathbf{F}^5)$ by:

$$\begin{aligned} T : \quad & v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i \quad ; \\ S : \quad & v_1 \mapsto v_4, \quad v_2 \mapsto v_5, \quad v_i \mapsto 0 \quad ; \quad i = 3, 4, 5. \end{aligned}$$

• Suppose $\dim V = n$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \text{range } TS \leq \lfloor \frac{n}{2} \rfloor$.

SOLUTION:

By Problem (23), $\dim \text{range } TS \leq \min\{\overbrace{\dim \text{range } S}^{n - \dim \text{null } T}, \overbrace{\dim \text{range } T}^{n - \dim \text{null } S}\}$. We prove by contradiction.

Assume that $\dim \text{range } TS \geq \lfloor \frac{n}{2} \rfloor + 1$.

Then $\min\{n - \dim \text{null } T, n - \dim \text{null } S\} \geq \lfloor \frac{n}{2} \rfloor + 1$

$$\Rightarrow \max\{\dim \text{null } T, \dim \text{null } S\} \leq n - \lfloor \frac{n}{2} \rfloor - 1.$$

$$\text{又 } \dim \text{null } ST = n \leq \dim \text{null } S + \dim \text{null } T \leq 2(n - \lfloor \frac{n}{2} \rfloor - 1)$$

$$\Rightarrow \lfloor \frac{n}{2} \rfloor + 1 \leq \frac{n}{2}. \text{ Contradicts. Thus } \dim \text{range } TS \leq \lfloor \frac{n}{2} \rfloor. \quad \square$$

OR. $\dim \text{null } S = n - \dim \text{range } S \leq n - \dim \text{range } TS$.

And $ST = 0 \Rightarrow \dim \text{range } TS \leq \dim \text{range } T \leq \dim \text{null } S \leq n - \dim \text{range } TS$

$$\Rightarrow 2 \dim \text{range } TS \leq n \Rightarrow \dim \text{range } TS \leq \frac{n}{2}$$

$$\Rightarrow \dim \text{range } TS \leq \lfloor \frac{n}{2} \rfloor \text{ (because } \dim \text{range } TS \text{ is an integer)}. \quad \square$$

24 Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.

Prove that $\text{null } S \subseteq \text{null } T \iff \exists E \in \mathcal{L}(W)$ such that $T = ES$.

SOLUTION:

Suppose $\exists E \in \mathcal{L}(W)$ such that $T = ES$. Then $\text{null } T = \text{null } ES \supseteq \text{null } S$.

Suppose $\text{null } S \subseteq \text{null } T$. Let $R = (Sv_1, \dots, Sv_n)$ be a basis of $\text{range } S$

$\Rightarrow (v_1, \dots, v_n)$ is linearly independent.

Let $\mathcal{K}_R = \text{span}(v_1, \dots, v_n) \Rightarrow V = \mathcal{K}_R \oplus \text{null } S$.

Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i$, $Eu = 0$; for each $i = 1 \dots, n$ and $u \in \text{null } S$.

Hence $\forall v \in V$, $(\exists! a_i \in \mathbf{F}, u \in \text{null } S)$, $Tv = a_1Tv_1 + \dots + a_nTv_n = E(a_1Sv_1 + \dots + a_nSv_n) \Rightarrow T = ES. \quad \square$

OR. Extend R to a basis $(Sv_1, \dots, Sv_n, w_1, \dots, w_m)$ of W .

Define $E \in \mathcal{L}(W)$ by $E(Sv_k) = Tv_k$, $EW_j = 0$.

Because $\forall v \in V$, $\exists a_i \in \mathbf{F}$, $Sv = a_1Sv_1 + \dots + a_nSv_n$

$$\Rightarrow S(v - (a_1v_1 + \dots + a_nv_n)) = 0$$

$$\Rightarrow v - (a_1v_1 + \dots + a_nv_n) \in \text{null } S$$

$$\Rightarrow v - (a_1v_1 + \dots + a_nv_n) \in \text{null } T.$$

$$\Rightarrow T(v - (a_1v_1 + \dots + a_nv_n)) = 0$$

Thus $Tv = a_1Tv_1 + \dots + a_nTv_n$. Hence $E(Sv) = a_1E(Sv_1) + \dots + a_nE(Sv_n) = a_1Tv_1 + \dots + a_nTv_n = Tv. \quad \square$

25 Suppose that V is finite-dim and $S, T \in \mathcal{L}(V, W)$.

Prove that $\text{range } S \subseteq \text{range } T \iff \exists E \in \mathcal{L}(V)$ such that $S = TE$.

SOLUTION:

Suppose $\exists E \in \mathcal{L}(V)$ such that $S = TE$. Then $\text{range } S = \text{range } TE \subseteq \text{range } T$.

Suppose $\text{range } S \subseteq \text{range } T$. Let (v_1, \dots, v_m) be a basis of V .

Because $\text{range } S \subseteq \text{range } T \Rightarrow Sv_i \in \text{range } T$ for each i . Suppose $u_i \in V$ for each i such that $Tu_i = Sv_i$.

Thus defining $E \in \mathcal{L}(V)$ by $Ev_i = u_i$ for each $i \Rightarrow S = TE. \quad \square$

26 Prove that the differentiation map $D \in \mathcal{P}(\mathbf{R})$ is surjective.

SOLUTION:

[Informal Proof]

Note that $\deg Dx^n = n - 1$.

Because $\text{span}(Dx, Dx^2, \dots) \subseteq \text{range } D$. \times By (2.C.10), $\text{span}(Dx, Dx^2, \dots) = \text{span}(1, x, \dots) = \mathcal{P}(\mathbf{R})$. \square

[Proper Proof]

We will recursively define a sequence of polynomials $(p_k)_{k=0}^\infty$ where $Dp_k = x^k$.

Because $\dim Dx = (\deg x) - 1 = 0$, we have $Dx = C \in \mathbf{F}$. Define $p_0 = C^{-1}x$. Then $Dp_0 = C^{-1}Dx = 1$.

Suppose we have defined p_0, \dots, p_n such that $Dp_k = x^k$ for each $k \in \{0, \dots, n\}$.

Because $\deg D(x^{n+2}) = n + 1$, we let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_nx^n + \dots + a_1x + a_0$, where $a_{n+1} \neq 0$.

Then $a_{n+1}^{-1}D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_nDp_n + \dots + a_1Dp_1 + a_0Dp_0)$

$\Rightarrow x^{n+1} = D(a_{n+1}^{-1}(x^{n+2} - a_n p_n - \dots - a_1 p_1 - a_0 p_0))$.

Thus defining $p_{n+1} = a_{n+1}^{-1}(x^{n+2} - a_n p_n - \dots - a_1 p_1 - a_0 p_0)$, we have $Dp_{n+1} = x^{n+1}$.

Hence we get the sequence $(p_k)_{k=0}^\infty$ by recursion.

Now it suffices to show that D is surjective. Let $p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R})$.

Then $D\left(\sum_{k=0}^{\deg p} a_k p_k\right) = \sum_{k=0}^{\deg p} a_k Dp_k = \sum_{k=0}^{\deg p} a_k x^k = p$. \square

27 Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.

SOLUTION:

Define $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$.

Note that $\deg Bx^n = n - 1$. Similar to Problem (26), we conclude that B is surjective. \square

28 Suppose $T \in \mathcal{L}(V, W)$ and (w_1, \dots, w_m) is a basis of $\text{range } T$. Prove that

$\exists \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ such that for all $v \in V$, $Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m$.

SOLUTION:

Suppose (v_1, \dots, v_m) in V such that $Tv_i = w_i$ for each i .

Then (v_1, \dots, v_m) is linearly independent, extend it to a basis of V as $(v_1, \dots, v_m, u_1, \dots, u_n)$.

Note that $\forall v \in V, v = a_1v_1 + \dots + a_mv_m + b_1u_1 + \dots + b_nu_n, \exists! a_i, b_i \in \mathbf{F} \Rightarrow Tv = a_1w_1 + \dots + a_mw_m$.

Define $\varphi_i : V \rightarrow \mathbf{F}$ by $\varphi_i(v) = a_i v_i$ for each i . We now check the linearity.

$\forall v, u \in V (\exists! a_i, b_i, c_i, d_i \in \mathbf{F}), \lambda \in \mathbf{F}, \varphi_i(v + \lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda \varphi(u)$. \square

29 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F}) \setminus \{0\}$. Suppose $u \in V \setminus \text{null } \varphi$. Prove that $V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$.

SOLUTION:

(a) $\forall v = cu \in \text{null } \varphi \cap \{au : a \in \mathbf{F}\}, \varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$. Hence $\text{null } \varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$.

(b) $\forall v \in V, v = (v - \frac{\varphi(v)}{\varphi(u)}u) + \frac{\varphi(v)}{\varphi(u)}u$. $\left. \begin{array}{l} v - \frac{\varphi(v)}{\varphi(u)}u \in \text{null } \varphi \\ \frac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array} \right\} \Rightarrow V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$. \square

This may seem strange. Here we explain why.

$\varphi \neq 0 \Rightarrow \exists$ a linearly independent list $(v_1, \dots, v_n \in V)$ such that $\varphi(v_i) = a_i \neq 0$.

Choose a v_k arbitrarily. Then $\varphi(v_k - \frac{\varphi(v_k)}{\varphi(v_j)}v_j) = 0$ for each $j = 1, \dots, k-1, k+1, \dots, n$.

Thus $\text{span}\{v_k - \frac{\varphi(v_k)}{\varphi(v_j)}v_j\}_{j \neq k} \subseteq \text{null } \varphi$. Hence there is only one nonzero vector in every vecsp in $\mathcal{S}_V \text{null } \varphi$.

30 Suppose $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$ and $\text{null } \varphi_1 = \text{null } \varphi_2 = \text{null } \varphi$. Prove that $\exists c \in \mathbf{F}, \varphi_1 = c\varphi_2$

SOLUTION:

If $\text{null } \varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $u \in V/\text{null } \varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$.

By Problem (29), $V = \text{null } \varphi \oplus \text{span}(u)$.

Hence for any $v \in V, v = w + a_v u, \exists! w \in \text{null } \varphi, a_v \in \mathbf{F}$.

$$\varphi_1(v) = a_v \varphi_1(u), \quad \varphi_2(v) = a_v \varphi_2(u) \Rightarrow a_v = \frac{\varphi_1(v)}{\varphi_1(u)} = \frac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \frac{\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}.$$

Thus $\varphi_1 = c\varphi_2$. \square

- Suppose V is finite-dim, X is a subspace of V , and Y is a finite-dim subspace of W . Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\text{null } T = X$ and $\text{range } T = Y$ if and only if $\dim X + \dim Y = \dim V$.

SOLUTION:

(a) Suppose $\dim X + \dim Y = \dim V$. Let (u_1, \dots, u_n) be a basis of X , $R = (w_1, \dots, w_m)$ be a basis of Y .

Extend (u_1, \dots, u_n) to a basis of V as $(u_1, \dots, u_n, v_1, \dots, v_m)$.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \dots + a_m v_m + b_1 u_1 + \dots + b_n u_n) = a_1 w_1 + \dots + a_m w_m$.

Now we show that $\text{null } T = X$ and $\text{range } T = Y$

Suppose $v \in V$. Then $\exists! a_i, b_j \in \mathbf{F}, v = a_1 v_1 + \dots + a_m v_m + b_1 u_1 + \dots + b_n u_n$.

$$\left. \begin{array}{l} v \in \text{null } T \Rightarrow Tv = 0 \\ \Rightarrow a_1 = \dots = a_m = 0 \\ \Rightarrow v \in X \Rightarrow \text{null } T \subseteq X. \\ v \in X \Rightarrow v \in \text{null } T \Rightarrow \text{null } T \supseteq X. \end{array} \right\} \Rightarrow \text{null } T = X.$$

$$\left. \begin{array}{l} w \in \text{range } T \Rightarrow \exists v \in V, Tv = w \Rightarrow \text{let } v = a_1 v_1 + \dots + a_m v_m + b_1 u_1 + \dots + b_n u_n \\ \Rightarrow Tv = w = a_1 w_1 + \dots + a_m w_m \Rightarrow w \in Y \Rightarrow \text{range } T \subseteq Y. \\ w \in Y \Rightarrow w = a_1 T v_1 + \dots + a_m T v_m = T(a_1 v_1 + \dots + a_m v_m) \\ \Rightarrow w \in \text{range } T \Rightarrow \text{range } T \supseteq Y. \end{array} \right\} \Rightarrow \text{range } T = Y.$$

(b) Conversely it is true as well.

\square

- Suppose V is finite-dim and $T \in \mathcal{L}(V, W)$. Let (Tv_1, \dots, Tv_n) be a basis of $\text{range } T$. Extend (v_1, \dots, v_n) to a basis of V as $(v_1, \dots, v_n, u_1, \dots, u_m)$. Prove or give a counterexample: (u_1, \dots, u_m) is a basis of $\text{null } T$.

SOLUTION: A counterexample:

Suppose $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$. Then $\text{span}(Tv_1, Tv_2, Tv_3) = \text{span}(w_1)$.

Extend (v_i) to (v_1, v_2, v_3) for each i . But none of $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is a basis of $\text{null } T$. \square

COMMENT: $(v_2 - v_1, v_3 - v_1), (v_1 - v_2, v_3 - v_2)$ or $(v_1 - v_3, v_2 - v_3)$ are all bases of $\text{null } T$.

- Suppose V is finite-dim and $T \in \mathcal{L}(V, W)$. Let (u_1, \dots, u_m) be a basis of $\text{null } T$. Extend (u_1, \dots, u_m) to a basis of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$. Prove or give a counterexample: (Tv_1, \dots, Tv_n) spans $\text{range } T$.

SOLUTION:

$\forall w \in \text{range } T, \exists v \in V, (\exists! a_i, b_i \in \mathbf{F}), Tv = T(a_1 v_1 + \dots + a_n v_n) = w$

$\Rightarrow w \in \text{span}(Tv_1, \dots, Tv_n) \Rightarrow \text{range } T \subseteq \text{span}(Tv_1, \dots, Tv_n)$. \square

COMMENT: If T is injective, then (Tv_1, \dots, Tv_n) is a basis of $\text{range } T$.

- (OR (5.B.4)) Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \text{null } P \oplus \text{range } P$.

SOLUTION:

Let (P^2v_1, \dots, P^2v_n) be a basis of $\text{range } P^2$. Then (Pv_1, \dots, Pv_n) is linearly independent in V .
 Let $\mathcal{K} = \text{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \text{null } P^2$
 $\text{又 } \mathcal{K} = \text{range } P = \text{range } P^2; \text{ null } P = \text{null } P^2 \quad \left. \vphantom{\begin{array}{l} \text{Let } (P^2v_1, \dots, P^2v_n) \text{ be a basis of } \text{range } P^2. \text{ Then } (Pv_1, \dots, Pv_n) \text{ is linearly independent in } V. \\ \text{Let } \mathcal{K} = \text{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \text{null } P^2 \\ \text{又 } \mathcal{K} = \text{range } P = \text{range } P^2; \text{ null } P = \text{null } P^2 \end{array}} \right\} \Rightarrow \square$

OR. (a) Suppose $v \in \text{null } P \cap \text{range } P$.

Then $\exists u \in V, v = Pu, Pv = 0 \Rightarrow v = Pu = P^2u = Pv = 0$. Hence $\text{null } P \cap \text{range } P = \{0\}$.

(b) Note that $v = Pv + (v - Pv)$ and $P^2v = Pv$ for all $v \in V$.

Then $P(v - Pv) = 0 \Rightarrow v - Pv \in \text{null } P$. Hence $V = \text{range } P + \text{null } P$. \square

• Suppose V is finite-dim with $\dim V > 1$. Show that if $\varphi : \mathcal{L}(V) \rightarrow \mathbf{F}$ is a linear map such that $\varphi(ST) = \varphi(S) \cdot \varphi(T)$ for all $S, T \in \mathcal{L}(V)$, then $\varphi = 0$.

HINT: The description of the two-sided ideals of $\mathcal{L}(V)$ in (4E 3A.16) might be useful.

SOLUTION:

Using notations in (4E 3A.16).

Suppose $\varphi \neq 0 \Rightarrow \exists i, j \in \{1, \dots, n\}, \varphi(R_{i,j}) \neq 0$.

Because $R_{i,j} = R_{x,j} \circ R_{i,x}, \forall x = 1, \dots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, because $R_{i,x} = R_{y,x} \circ R_{i,y}, \forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0$ for any $x, y = 1, \dots, n$.

Let $l \neq i, k \neq j$ and then $\varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. \square

OR. Note that by (4E 3A.16), $\exists S, T \in \mathcal{L}(V), ST - TS \neq 0$.

Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \text{null } \varphi \neq \{0\}$.

Thus $\forall E \in \text{null } \varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \text{null } \varphi$.

Hence $\text{null } \varphi$ is a nonzero two-sided ideal of $\mathcal{L}(V)$. \square

• Suppose that V and W are real vector spaces and $T \in \mathcal{L}(V, W)$.

Define $T_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ by $T_{\mathbb{C}}(u + iv) = Tu + iTv$ for all $u, v \in V$.

(a) Show that $T_{\mathbb{C}}$ is a (complex) linear map from $V_{\mathbb{C}}$ to $W_{\mathbb{C}}$.

(b) Show that $T_{\mathbb{C}}$ is injective $\iff T$ is injective.

(c) Show that $\text{range } T_{\mathbb{C}} = W_{\mathbb{C}} \iff \text{range } T = W$.

See (4E 1B.8) for the definition of the complexification $V_{\mathbb{C}}$. $T_{\mathbb{C}}$ is called the complexification of T .

SOLUTION:

(a) $\forall u_1 + iv_1, u_2 + iv_2 \in V_{\mathbb{C}}, \lambda \in \mathbf{F}$,

$T((u_1 + iv_1) + \lambda(u_2 + iv_2)) = T((u_1 + \lambda u_2) + i(v_1 + \lambda v_2)) = T(u_1 + \lambda u_2) + iT(v_1 + \lambda v_2)$
 $= Tu_1 + iTv_1 + \lambda Tu_2 + i\lambda Tv_2 = T(u_1 + iv_1) + \lambda T(u_2 + iv_2)$. \square

(b) Suppose $T_{\mathbb{C}}$ is injective. Let $T(u) = 0 \Rightarrow T_{\mathbb{C}}(u + i0) = Tu = 0 \Rightarrow u = 0$.
 Suppose T is injective. Let $T_{\mathbb{C}}(u + iv) = Tu + iTv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + iv = 0$. $\left. \vphantom{\begin{array}{l} \text{Suppose } T_{\mathbb{C}} \text{ is injective. Let } T(u) = 0 \Rightarrow T_{\mathbb{C}}(u + i0) = Tu = 0 \Rightarrow u = 0. \\ \text{Suppose } T \text{ is injective. Let } T_{\mathbb{C}}(u + iv) = Tu + iTv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + iv = 0. \end{array}} \right\} \Rightarrow \square$

Suppose $T_{\mathbb{C}}$ is surjective. $\forall w, x \in W, \exists u, v \in V, T(u + iv) = Tu + iTv = w + ix$
 $\Rightarrow Tu = w, Tv = x \Rightarrow T$ is surjective.
 (c) Suppose T is surjective. $\forall w, x \in W, \exists u, v \in V, Tu = w, Tv = x$
 $\Rightarrow \forall w + ix \in W_{\mathbb{C}}, \exists u + iv \in V, T(u + iv) = w + ix \Rightarrow T_{\mathbb{C}}$ is surjective. $\left. \vphantom{\begin{array}{l} \text{Suppose } T_{\mathbb{C}} \text{ is surjective. } \forall w, x \in W, \exists u, v \in V, T(u + iv) = Tu + iTv = w + ix \\ \Rightarrow Tu = w, Tv = x \Rightarrow T \text{ is surjective.} \end{array}} \right\} \Rightarrow \square$

ENDED

• **NOTE FOR [3.47]:** $LHS = (AC)_{j,k} = \sum_{r=1}^n A_{j,r} C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r} (C_{\cdot,k})_{r,1} = (A_{j,\cdot} C_{\cdot,k})_{1,1} = A_{j,\cdot} C_{\cdot,k} = RHS.$

• **NOTE FOR [3.48]:**

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix} \\ = \begin{pmatrix} 1 \times 5 + 2 \times 8 & 1 \times 6 + 2 \times 9 & 1 \times 7 + 2 \times 10 \\ 3 \times 5 + 4 \times 8 & 3 \times 6 + 4 \times 9 & 3 \times 7 + 4 \times 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

• **NOTE FOR [3.49]:** $\because [(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r} C_{r,k} = \sum_{r=1}^n A_{j,r} (C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$
 $\therefore (AC)_{\cdot,k} = A_{\cdot,\cdot} C_{\cdot,k} = AC_{\cdot,k}$

• **EXERCISE 10:** $\because [(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r} C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r} C_{r,k} = (A_{j,\cdot} C)_{1,k}$
 $\therefore (AC)_{j,\cdot} = A_{j,\cdot} C_{\cdot,\cdot} = A_{j,\cdot} C.$

• *Suppose $C \in \mathbf{F}^{m,c}$, $R \in \mathbf{F}^{c,p}$.*

(a) For $k = 1, \dots, p$, $(CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot} R_{\cdot,k} = \sum_{r=1}^c C_{\cdot,r} R_{r,k} = R_{1,k} C_{\cdot,1} + \dots + R_{c,k} C_{\cdot,c}$

(b) For $j = 1, \dots, m$, $(CR)_{j,\cdot} = C_{j,\cdot} R = C_{j,\cdot} R_{\cdot,\cdot} = \sum_{r=1}^c C_{j,r} R_{r,\cdot} = C_{j,1} R_{1,\cdot} + \dots + C_{j,c} R_{c,\cdot}$

EXAMPLE:

$$P = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} \in \mathbf{F}^{2,3}.$$

$$P_{\cdot,1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 8 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 21 \\ 47 \end{pmatrix};$$

$$P_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix};$$

$$P_{\cdot,3} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 10 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 27 \\ 61 \end{pmatrix};$$

$$P_{1,\cdot} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = 1 \begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 2 \begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \end{pmatrix};$$

$$P_{2,\cdot} = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = 3 \begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 4 \begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 47 & 54 & 61 \end{pmatrix};$$

• **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}$, $c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$

$$\because (Ac)_{j,1} = \sum_{r=1}^n A_{j,r} c_{r,1} = \left[\sum_{r=1}^n (A_{\cdot,r} c_{r,1}) \right]_{j,1} = (c_1 A_{\cdot,1} + \dots + c_n A_{\cdot,n})_{j,1}$$

$$\therefore Ac = A_{\cdot,\cdot} c_{\cdot,1} = \sum_{r=1}^n A_{\cdot,r} c_{r,1} = c_1 A_{\cdot,1} + \dots + c_n A_{\cdot,n} \quad \text{OR. By } (Ac)_{\cdot,1} = Ac_{\cdot,1} \text{ Using (a) above.}$$

• **EXERCISE 10:** $a \in \mathbf{F}^{1,n}$, $C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$

$$\because (aC)_{1,k} = \sum_{r=1}^n a_{1,r} C_{r,k} = \left[\sum_{r=1}^n a_{1,r} (C_{r,\cdot}) \right]_{1,k} = (a_1 C_{1,\cdot} + \dots + a_n C_{n,\cdot})_{1,k}$$

$$\therefore aC = a_{1,\cdot} C_{\cdot,\cdot} = \sum_{r=1}^n a_{1,r} C_{r,\cdot} = a_1 C_{1,\cdot} + \dots + a_n C_{n,\cdot} \quad \text{OR. By } (aC)_{1,\cdot} = a_{1,\cdot} C. \text{ Using (b) above.}$$

• **COLUMN-ROW FACTORIZATION** (CR Factorization)

Suppose $A \in \mathbf{F}^{m,n}$, $A \neq 0$. Let $S_c = \text{span}(A_{.,1}, \dots, A_{.,n}) \subseteq \mathbf{F}^{m,1}$, $\dim S_c = c$.

And $S_r = \text{span}(A_{1,.}, \dots, A_{n,.}) \subseteq \mathbf{F}^{1,n}$, $\dim S_r = r$.

Prove that $A = CR$. $\exists C \in \mathbf{F}^{m,c}$, $R \in \mathbf{F}^{c,n}$.

SOLUTION: Notice that $A \neq 0 \Rightarrow c, r \geq 1$.

Let $(C_{.,1}, \dots, C_{.,c})$ be a basis of S_c , forming $C \in \mathbf{F}^{m,c}$.

Then for any $A_{.,k}$, $A_{.,k} = R_{1,k}C_{.,1} + \dots + R_{c,k}C_{.,c} = (CR)_{.,k}$, $\exists! R_{1,k}, \dots, R_{c,k} \in \mathbf{F}$.

Hence, by letting $R = \begin{pmatrix} R_{1,1} & \dots & R_{1,n} \\ \vdots & \ddots & \vdots \\ R_{c,1} & \dots & R_{c,n} \end{pmatrix}$, we have $A = CR$.

OR. Let $(R_{1,.}, \dots, R_{c,.})$ be a basis of S_r , forming $R \in \mathbf{F}^{c,n}$.

For any $A_{j,.}$, $A_{j,.} = C_{j,1}R_{1,.} + \dots + C_{j,c}R_{c,.} = (CR)_{j,.}$, $\exists! C_{j,1}, \dots, C_{j,c} \in \mathbf{F}$. Similarly. \square

EXAMPLE:

$$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \stackrel{(1)}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \stackrel{(2)}{=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

(1) Because $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2 \begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix}$.

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix}$ can be uniquely written as a linear combination of $A_{1,.}, A_{2,.}$.

Hence $\dim S_r = 2$. We choose $(A_{1,.}, A_{2,.})$ as the basis.

$$(2) \text{ Because } \begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2 \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}; \quad \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix} - \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix}.$$

Hence $\dim S_c = 2$. We choose $(A_{.,2}, A_{.,3})$ as the basis.

• **COLUMN RANK EQUALS ROW RANK** (Using the notation above)

For any $A_{j,.} \in S_r$, $A_{j,.} = (CR)_{j,.} = C_{j,1}R_{1,.} + \dots + C_{j,c}R_{c,.}$

$\Rightarrow \text{span}(A_{1,.}, \dots, A_{m,.}) = S_r = \text{span}(R_{1,.}, \dots, R_{c,.}) \Rightarrow \dim S_r = r \leq c = \dim S_c$.

Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leq r = \dim S_r = \dim S_c^t$. \square

• Suppose $T \in \mathcal{L}(V)$, and u_1, \dots, u_n and v_1, \dots, v_n are bases of V .

Prove that the following are equivalent.

(a) T is injective.

(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$.

(c) The columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.

(d) The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.

(e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$.

Here $A = \mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.

SOLUTION:

T is injective $\iff \dim V = \dim \text{range } T + \dim \text{null } T = \dim \text{range } T$

$\iff (Tu_1, \dots, Tu_n)$ is linearly independent in V , and therefore is a basis of V

$\iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n))$ is linearly independent, as well as $(A_{.,1}, \dots, A_{.,n})$

$\iff (A_{.,1}, \dots, A_{.,n})$ is a basis of $\mathbf{F}^{n,1}$.

$\left(\text{又 } \dim \text{span}(A_{.,1}, \dots, A_{.,n}) = \dim \text{span}(A_{1,.}, \dots, A_{n,.}) = n \right)$

$\iff (A_{1,.}, \dots, A_{n,.})$ is a basis of $\mathbf{F}^{1,n}$. \square

• Suppose A is an m -by- n matrix with $A \neq 0$.

Prove that the rank of A is 1 if and only if there exist $(c_1, \dots, c_m) \in \mathbf{F}^m$ and $(d_1, \dots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j \cdot d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$.

SOLUTION: Using the notation in CR Factorization.

$$(a) \text{ Suppose } A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix}. \quad (\exists c_j, d_k \in \mathbf{F}, \forall j, k)$$

$$\text{Then } S_c = \text{span} \left(\begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \dots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right) = \text{span} \left(\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right).$$

$$\text{OR. } S_r = \text{span} \left(\begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ c_2 d_1 & \cdots & c_2 d_n \\ \vdots & & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix} \right) = \text{span} \left(\begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} \right). \quad \text{Hence the rank of } A \text{ is 1.}$$

(b) Suppose the rank of A is $\dim S_c = \dim S_r = 1$

$$\text{Let } c_j = \frac{A_{j,1}}{A_{1,1}} = \frac{A_{j,2}}{A_{1,2}} = \cdots = \frac{A_{j,n}}{A_{1,n}}, \quad d'_k = \frac{A_{1,k}}{A_{1,1}} = \frac{A_{2,k}}{A_{2,1}} = \cdots = \frac{A_{m,k}}{A_{m,1}}. \quad (\forall j, k)$$

$$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k. \text{ Letting } d_k = d'_k A_{1,1}. \quad \square$$

1 Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W , the matrix of T has at least $\dim \text{range } T$ nonzero entries.

SOLUTION: Let (v_1, \dots, v_n) and (w_1, \dots, w_m) be bases of V and W respectively. We prove by contradiction.

Suppose $A = \mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$ has at most $(\dim \text{range } T - 1)$ nonzero entries.

Then by Pigeon Hole Principle, at least one of $A_{\cdot, k} = 0$.

Thus there are at most $(\dim \text{range } T - 1)$ nonzero vectors in Tv_1, \dots, Tv_n .

While $\text{range } T = \text{span}(Tv_1, \dots, Tv_n) \Rightarrow \dim \text{range } T \leq \dim \text{range } T - 1$. Hence we get a contradiction. \square

3 Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$.

Prove that there exist a basis of V and a basis of W such that

[letting $A = \mathcal{M}(T)$ with respect to these bases],

$A_{k,k} = 1, A_{i,j} = 0$, where $1 \leq k \leq \dim \text{range } T, i \neq j$.

SOLUTION:

Let $R = (Tv_1, \dots, Tv_n)$ be a basis of $\text{range } T$, extend it to the basis of W as $(Tv_1, \dots, Tv_n, w_1, \dots, w_p)$.

Let $\mathcal{K}_R = \text{span}(v_1, \dots, v_n)$. Let (u_1, \dots, u_m) be a basis of $\text{null } T$.

Then $(v_1, \dots, v_n, u_1, \dots, u_m)$ is the basis of V .

Thus $T(v_k) = Tv_k, T(u_j) = 0 \Rightarrow A_{k,k} = 1, A_{i,j} = 0$ for each $k \in \{1, \dots, \dim \text{range } T\}$ and $j \in \{1, \dots, m\}$. \square

4 Suppose (v_1, \dots, v_m) is a basis of V and W is finite-dim. Suppose $T \in \mathcal{L}(V, W)$.

Prove that there exists a basis (w_1, \dots, w_n) of W such that

all entries in the first column of $A = \mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_n))$ are 0 except for possibly a 1 in the first row, first column.

SOLUTION: If $Tv_1 = 0$, then we are done. Otherwise, extend (Tv_1) to a basis of W , as desired. \square

5 Suppose (w_1, \dots, w_n) is a basis of W and V is finite-dim. Suppose $T \in \mathcal{L}(V, W)$.

Prove that there exists a basis (v_1, \dots, v_m) of V such that all entries in the first row of $\mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_n))$ are 0 except for possibly a 1 in the first row, first column.

SOLUTION:

Let (u_1, \dots, u_m) be a basis of V . If $A_{1,\cdot} = 0$, then let $v_i = u_i$ for each $i = 1, \dots, m$, we are done.

Otherwise, $(A_{1,1} \ \dots \ A_{1,m}) \neq 0$, choose one $A_{1,k} \neq 0$.

Let $v_1 = \frac{u_k}{A_{1,k}}$; $v_j = u_{j-1} - A_{1,j-1}v_1$ for $j = 2, \dots, k$;
 $v_i = u_i - A_{1,i}v_1$ for $i = k+1, \dots, m$. \square

6 Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$. Prove that $\dim \text{range } T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $A = \mathcal{M}(T)$ equal 1.

SOLUTION:

Denote the bases of V and W by $B_V = (v_1, \dots, v_n)$ and $B_W = (w_1, \dots, w_m)$ respectively.

(a) Suppose B_V, B_W are the bases such that all entries of A equal 1.

Then $Tv_i = w_1 + \dots + w_m$ for all $i = 1, \dots, n$. Hence $\dim \text{range } T = 1$.

(b) Suppose $\dim \text{range } T = 1$. Then $\dim \text{null } T = \dim V - 1$.

Let (u_2, \dots, u_n) be a basis of $\text{null } T$. Extend it to a basis of V as (u_1, u_2, \dots, u_n) .

Let $w_1 = Tv_1 - w_2 - \dots - w_m$. Extend it to B_W the basis of W .

Let $v_1 = u_1, v_i = u_i + u_1$. Extend it to B_V the basis of V . \square

12 Give an example of 2-by-2 matrices A and B such that $AB \neq BA$.

SOLUTION: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

13 Prove that the distributive property holds for matrix addition and matrix multiplication.

In other words, suppose A, B, C, D, E and F are matrices

whose sizes are such that $A(B + C)$ and $(D + E)F$ make sense.

Explain why $AB + AC$ and $DF + EF$ both make sense and prove that.

SOLUTION: Using [3.36], [3.43].

(a) Left distributive: Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Because $[A(B + C)]_{j,k} = \sum_{r=1}^n A_{j,r}(B + C)_{r,k} = \sum_{r=1}^n (A_{j,r}B_{r,k} + A_{j,r}C_{r,k})$.

Hence we conclude that $A(B + C) = AB + AC$.

OR. Let (e_1, \dots, e_M) be the standard basis of \mathbf{F}^M , where $M = \max\{m, n, p\}$.

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum_{j=1}^m A_{j,k}e_j$ for each $k = 1, \dots, n$. Then $\mathcal{M}(T) = A$.

Similarly, define S, R such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

Thus $T(S + R) = TS + TR$	$\Rightarrow \mathcal{M}(T(S + R)) = \mathcal{M}(TS + TR)$
	$\Rightarrow \mathcal{M}(T)[\mathcal{M}(S) + \mathcal{M}(R)] = \mathcal{M}(T)\mathcal{M}(S) + \mathcal{M}(T)\mathcal{M}(R)$
	$\Rightarrow A(B + C) = AB + AC$.

Suppose $\mathcal{M}(T) = D, \mathcal{M}(S) = E, \mathcal{M}(R) = F$.

Then $(T + S)R = TR + SR$

(b) Right distributive: Similarly.	$\Rightarrow \mathcal{M}((T + S)R) = \mathcal{M}(TR + SR)$
	$\Rightarrow [\mathcal{M}(T) + \mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)\mathcal{M}(R) + \mathcal{M}(S)\mathcal{M}(R)$
	$\Rightarrow (D + E)F = DF + EF. \square$

14 Prove that matrix multiplication is associative. In other words,

suppose A, B and C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that $(AB)C = A(BC)$.

Try to find a clean proof that illustrates the following quote from Emil Artin:

“It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.”

SOLUTION:

Because $[(AB)C]_{j,k} = (AB)_{j,\cdot} C_{\cdot,k} = \sum_{s=1}^n (A_{j,s} B_{s,\cdot}) C_{\cdot,k} = \sum_{s=1}^n A_{j,s} (B_{s,\cdot} C_{\cdot,k}) = \sum_{s=1}^n A_{j,s} (BC)_{s,k} = A(BC)_{j,k}$
Hence we conclude that $(AB)C = A(BC)$.

OR. Suppose $A \in \mathbf{F}^{m,n}, B \in \mathbf{F}^{n,p}, C \in \mathbf{F}^{p,s}$.

Let (e_1, \dots, e_M) be the standard basis of \mathbf{F}^M , where $M = \max\{m, n, p, s\}$.

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum_{j=1}^m A_{j,k} e_j$ for each $k = 1, \dots, n$. Then $\mathcal{M}(T) = A$.

Similarly, define S, R such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

Hence $(TS)R = T(SR) \Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR))$

$$\Rightarrow [\mathcal{M}(T)\mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)[\mathcal{M}(S)\mathcal{M}(R)]$$

$$\Rightarrow (AB)C = A(BC). \quad \square$$

15 Suppose A is an n -by- n matrix and $1 \leq j, k \leq n$.

Show that the entry in row j , column k , of A^3

(which is defined to mean AAA) is $\sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}$.

SOLUTION: $(AAA)_{j,k} = (AA)_{j,\cdot} A_{\cdot,k} = \sum_{p=1}^n (A_{j,p} A_{p,\cdot}) A_{\cdot,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}$.

$$\begin{aligned} \text{OR. } (AAA)_{j,k} &= \sum_{r=1}^n (AA)_{j,r} A_{r,k} = \sum_{r=1}^n \left(\sum_{p=1}^n A_{j,p} A_{p,r} \right) A_{r,k} \\ &= \sum_{r=1}^n (A_{j,1} A_{1,r} A_{r,k} + \dots + A_{j,n} A_{n,r} A_{r,k}) \\ &= A_{j,1} \sum_{r=1}^n A_{1,r} A_{r,k} + \dots + A_{j,n} \sum_{r=1}^n A_{n,r} A_{r,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}. \quad \square \end{aligned}$$

ENDED

• Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and $(T^{-1})^{-1} = T$.

SOLUTION:

$$\left. \begin{array}{l} TT^{-1} = (T^{-1})^{-1}T^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = T^{-1}(T^{-1})^{-1} = I \in \mathcal{L}(W) \end{array} \right\} \Rightarrow T = (T^{-1})^{-1}, \text{ by the uniqueness of inverse. } \square$$

1 Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps.

Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

SOLUTION:

$$\left. \begin{array}{l} (ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(U) \end{array} \right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}, \text{ by the uniqueness of inverse. } \square$$

9 Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$.

Prove that ST is invertible $\iff S$ and T are invertible.

SOLUTION:

Suppose S, T are invertible. Then $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$. Hence ST is invertible.

Suppose ST is invertible. Let $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$.

$$\left. \begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \text{range } S \end{array} \right\} \Rightarrow T \text{ is injective, } S \text{ is surjective.}$$

Notice that V is finite-dim. Hence S, T are invertible. \square

OR. Suppose ST is invertible but S or T is not invertible (\Rightarrow not surjective and injective).

If S is not invertible then $\dim \text{range } S < \dim V$ and by Problem (23) in (3.B),

$\dim \text{range } ST \leq \dim \text{range } S < \dim V$. Thus ST is not surjective. Contradicts.

If T is not invertible then $\dim \text{range } T < 0$. Similarly, ST is not surjective. Contradicts. \square

10 Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that $ST = I \iff TS = I$.

SOLUTION:

$$\text{Suppose } ST = I. \left. \begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \text{range } S \end{array} \right\} \Rightarrow T \text{ is injective, } S \text{ is surjective.}$$

Notice that V is finite-dim. Thus T, S are invertible.

OR. By Problem (9), V is finite-dim and $ST = I$ is invertible $\Rightarrow S, T$ are invertible.

$S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v$ (S is invertible).

OR. $ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T$. \times $S = S \Rightarrow TS = S^{-1}S = I$.

Reversing the roles of S and T , we conclude that $TS = I \Rightarrow ST = I$. \square

11 Suppose V is finite-dim and $S, T, U \in \mathcal{L}(V)$ and $STU = I$.

Show that T is invertible and that $T^{-1} = US$.

SOLUTION: Using Problem (9) and (10).

$$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I.$$

$$\Rightarrow U^{-1} = ST, \quad T^{-1} = US, \quad S^{-1} = TU. \quad \square$$

12 Show that the result in Exercise 11 can fail without the hypothesis that V is finite-dim.

SOLUTION:

Let $V = \mathbf{R}^\infty$, $S(a_1, a_2, \dots) = (a_2, \dots)$, $T(a_1, \dots) = (0, a_1, \dots)$, $U = I$.

Then $STU = I$ but T^{-1} is not invertible.

13 Suppose V is finite-dim and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective.

Prove that S is injective.

SOLUTION:

By Problem (1) and (9), Notice that V is finite-dim. Then RST is invertible.

$$(RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}. \quad \square$$

OR. Let $X = (RST)^{-1}$, $\left\{ \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is injective, and therefore is invertible.} \\ \forall v \in V, v = (RST)Xv \in \text{range } R \Rightarrow R \text{ is surjective, and therefore is invertible.} \end{array} \right.$

Thus $S = R^{-1}(RST)T^{-1}$ is invertible.

15 Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multiplication.

In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists A \in \mathbf{F}^{m,n}, Tx = Ax, \forall x \in \mathbf{F}^{n,1}$.

SOLUTION:

Let $E_i \in \mathbf{F}^{n,1}$ for each $i = 1, \dots, n$ (where $M = \max\{m, n\}$) be such that $(E_i)_{j,1} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Then (E_1, \dots, E_n) is linearly independent and thus is a basis of $\mathbf{F}^{n,1}$.

Similarly, let (R_1, \dots, R_m) be a basis of $\mathbf{F}^{m,1}$.

Suppose $T(E_i) = A_{1,i}R_1 + \dots + A_{m,i}R_m$ for each $i = 1, \dots, n$. Hence by letting $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$. \square

COMMENT: $\mathcal{M}(T) = A$. Conversely it is true as well.

• OR (10.A.2) Suppose $A, B \in \mathbf{F}^{n,n}$. Prove that $AB = I \iff BA = I$.

SOLUTION: Using Problem (10) and (15).

Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Then $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.

Thus $AB = I \iff A(Bx) = x \iff T(Sx) = x \iff TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I. \square$

• **NOTE FOR [3.60]:** Suppose (v_1, \dots, v_n) is a basis of V and (w_1, \dots, w_m) is a basis of W .

Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{ix}w_j$; $\delta_{ix} = \begin{cases} 0, & i \neq x \\ 1, & i = x \end{cases}$ COROLLARY: $E_{l,k}E_{i,j} = \delta_{jl}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. $(\mathcal{E}^{(j,i)})_{l,k} = \begin{cases} 0, & i \neq k \text{ or } j \neq l \\ 1, & i = k \text{ and } j = l \end{cases}$

Because $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$ are isomorphic. And $T = \mathcal{M}^{-1}\mathcal{M}(T), E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

Hence $\forall T \in \mathcal{L}(V, W), \exists! A_{i,j} \in \mathbf{F} (\forall i = \{1, \dots, m\}, j = \{1, \dots, n\}), \mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)} + & \cdots & + A_{1,n}\mathcal{E}^{(1,n)} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}\mathcal{E}^{(m,1)} + & \cdots & + A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff T = \begin{pmatrix} A_{1,1}E_{1,1} + & \cdots & + A_{1,n}E_{n,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}E_{1,m} + & \cdots & + A_{m,n}E_{n,m} \end{pmatrix}.$$

$$\therefore \mathcal{L}(V, W) = \text{span} \underbrace{\begin{pmatrix} E_{1,1}, & \cdots, & E_{n,1} \\ \vdots & & \vdots \\ E_{1,m}, & \cdots, & E_{n,m} \end{pmatrix}}_B; \quad \mathbf{F}^{m,n} = \text{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots, & \mathcal{E}^{(1,n)} \\ \vdots & & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots, & \mathcal{E}^{(m,n)} \end{pmatrix}}_{B_M}.$$

Hence by [2.42] and [3.61], we conclude that B is a basis of $\mathcal{L}(V, W)$ and that B_M is a basis of $\mathbf{F}^{m,n}$.

◦ Suppose V is finite-dim and $S \in \mathcal{L}(V)$. Define $A \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T) = ST$ for $T \in \mathcal{L}(V)$.

(a) Show that $\dim \text{null } \mathcal{A} = (\dim V)(\dim \text{null } S)$.

(b) Show that $\dim \text{range } \mathcal{A} = (\dim V)(\dim \text{range } S)$.

SOLUTION:

(a) For all $T \in \mathcal{L}(V)$, $ST = 0 \iff \text{range } T \subseteq \text{null } S$. Thus $\text{null } \mathcal{A} = \mathcal{L}(V, \text{null } S) \Rightarrow \square$

(b) For all $R \in \mathcal{L}(V)$, $\text{range } R \subseteq \text{range } S \iff \exists T \in \mathcal{L}(V), R = ST$. Thus $\text{range } \mathcal{A} = \mathcal{L}(V, \text{range } S) \Rightarrow \square$

OR. Using NOTE FOR [3.60].

Let (w_1, \dots, w_m) be a basis of $\text{range } S$, extend it to a basis of V as $(w_1, \dots, w_m, \dots, w_n)$.

Let $v_i \in V$ such that $Sw_i = w_i$ for $m = 1, \dots, m$. Extend (v_1, \dots, v_m) to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.

Define $E_{i,j} \in \mathcal{L}(V)$ by $E_{i,j}(v_x) = \delta_{ix}w_i$.

$$\text{Thus } S = E_{1,1} + \dots + E_{m,m}; \quad \mathcal{M}(S, (v_1, \dots, v_n), (w_1, \dots, w_n)) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j}(w_x) = \delta_{ix}v_i$.

Let $E_{j,k}R_{i,j} = Q_{i,k}, \quad R_{j,k}E_{i,j} = G_{i,k}$

$$\begin{aligned} \text{Because } \forall T \in \mathcal{L}(V), \quad \exists! A_{i,j} \in \mathbf{F} \ (\forall i, j = 1, \dots, n), \quad T &= \begin{pmatrix} A_{1,1}R_{1,1}+ & \dots & +A_{1,m}R_{m,1}+ & \dots & +A_{1,n}R_{n,1} \\ + & \dots & + & \dots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \dots & + & \dots & + \\ A_{m,1}R_{1,m}+ & \dots & +A_{m,m}R_{m,m}+ & \dots & +A_{m,n}R_{n,m} \\ + & \dots & + & \dots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \dots & + & \dots & + \\ A_{n,1}R_{1,n}+ & \dots & +A_{n,m}R_{m,n}+ & \dots & +A_{n,n}R_{n,n} \end{pmatrix}. \\ \Rightarrow \mathcal{A}(T) = ST = \left(\sum_{r=1}^m E_{r,r} \right) \left(\sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \right) &= \sum_{i=1}^m \sum_{j=1}^n A_{i,j} Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1}+ & \dots & +A_{1,m}Q_{m,1}+ & \dots & +A_{1,n}Q_{n,1} \\ + & \dots & + & \dots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \dots & + & \dots & + \\ A_{m,1}Q_{1,m}+ & \dots & +A_{m,m}Q_{m,m}+ & \dots & +A_{m,n}Q_{n,m} \end{pmatrix}. \\ \text{Thus } \text{null } \mathcal{A} = \text{span} \begin{pmatrix} R_{1,m+1}, & \dots, & R_{n,m+1}, \\ \vdots & & \vdots \\ R_{1,n}, & \dots, & R_{n,n} \end{pmatrix}, \quad \text{range } \mathcal{A} = \text{span} \begin{pmatrix} Q_{1,1}, & \dots, & Q_{n,1}, \\ \vdots & & \vdots \\ Q_{1,m}, & \dots, & Q_{n,m} \end{pmatrix}. \end{aligned}$$

Hence (a) $\dim \text{null } \mathcal{A} = n \times (n - m)$; (b) $\dim \text{range } \mathcal{A} = n \times m$. \square

• COMMENT: Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(T) = TS$ for $T \in \mathcal{L}(V)$.

$$\text{Similarly, } \mathcal{B}(T) = TS = \left(\sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \right) \left(\sum_{r=1}^m E_{r,r} \right) = \sum_{i=1}^n \sum_{j=1}^m A_{i,j} G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1}+ & \dots & +A_{1,m}G_{m,1} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{m,1}G_{1,m}+ & \dots & +A_{m,m}G_{m,m} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{n,1}G_{1,n}+ & \dots & +A_{n,m}G_{m,n} \end{pmatrix}.$$

• OR (10.A.1) Suppose $T \in \mathcal{L}(V)$ and (v_1, \dots, v_n) is a basis of V .

Prove that $\mathcal{M}(T, (v_1, \dots, v_n))$ is invertible $\iff T$ is invertible.

SOLUTION: Notice that \mathcal{M} is an isomorphism of $\mathcal{L}(V)$ onto $\mathbf{F}^{n,n}$.

(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I$. $\exists! S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$

$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$

$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. \square

• OR (10.A.4) Suppose that $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bases of V .

Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1, \dots, n$.

Prove that $\mathcal{M}(T, (\alpha_1, \dots, \alpha_n)) = \mathcal{M}(I, (\beta_1, \dots, \beta_n), (\alpha_1, \dots, \alpha_n))$.

SOLUTION:

For ease of notation, write $\mathcal{M}(T, \alpha \rightarrow \beta) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n))$
and $\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n))$.

Denote $\mathcal{M}(T, \alpha \rightarrow \alpha)$ by A and $\mathcal{M}(I, \beta \rightarrow \alpha)$ by B .

$\forall k \in \{1, \dots, n\}, Iu_k = u_k = B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \dots + A_{n,k}\alpha_n \Rightarrow A = B. \quad \square$

OR. Note that $\mathcal{M}(T, \alpha \rightarrow \beta)$ is the identity matrix.

$$\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \beta \rightarrow \alpha) \underbrace{\mathcal{M}(T, \alpha \rightarrow \beta)}_{=\mathcal{M}(I, \beta \rightarrow \beta)} = \mathcal{M}(I, \beta \rightarrow \alpha). \quad \square$$

OR. Note that $\mathcal{M}(T, \beta \rightarrow \beta)\mathcal{M}(I, \alpha \rightarrow \beta) = \mathcal{M}(T, \alpha \rightarrow \beta) = I$.

$$\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \alpha \rightarrow \beta)^{-1} \underbrace{[\mathcal{M}(T, \beta \rightarrow \beta)\mathcal{M}(I, \alpha \rightarrow \beta)]}_{\mathcal{M}(T, \alpha \rightarrow \beta)} = \mathcal{M}(I, \beta \rightarrow \alpha). \quad \square$$

• COMMENT: Denote $\mathcal{M}(T, \beta \rightarrow \beta)$ by A' .

$$u_k = Iu_k = B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n, \quad \forall k \in \{1, \dots, n\}.$$

$$\text{又 } Tu_k = T(B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n) = B_{1,k}\beta_1 + \dots + B_{n,k}\beta_n = A'_{1,k}\beta_1 + \dots + A'_{n,k}\beta_n \Rightarrow A' = B.$$

$$\text{OR. } \mathcal{M}(T, \beta \rightarrow \beta) = \mathcal{M}(T, \alpha \rightarrow \beta)\mathcal{M}(I, \beta \rightarrow \alpha) = B.$$

16 Suppose V is finite-dim and $S \in \mathcal{L}(V)$.

Prove that $\exists \lambda \in \mathbf{F}, S = \lambda I \iff ST = TS$ for every $T \in \mathcal{L}(V)$.

SOLUTION: Using the notation and result in (o).

Suppose $S = \lambda I$. Then $ST = TS = \lambda T$ for every $T \in \mathcal{L}(V)$. Conversely, if $S = 0$, then we are done.

Suppose $S \neq 0, ST = TS, \forall T \in \mathcal{L}(V)$.

Let $S = E_{1,1} + \dots + E_{m,m} \Rightarrow \mathcal{M}(S, (v_1, \dots, v_1)) = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))$.

Then $\forall k \in \{m+1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $n = \dim V = \dim \text{range } S = m$.

Note that $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$. Thus $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \dots + a_{n,i}v_n)$. Where:

$$a_{i,j} = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \dots + a_{n,i}v_n;$$

For each j , for all i . Thus $a_{i,i} = a_{k,k} = \lambda, \forall k \neq i$.

$$\text{Hence } w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} = \mathcal{M}(\lambda I, (v_1, \dots, v_n)) \Rightarrow S = \mathcal{M}^{-1}(\mathcal{M}(\lambda I)) = \lambda I. \quad \square$$

• OR (10.A.3) Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that T has the same matrix with respect to every basis of V
if and only if T is a scalar multiple of the identity operator.

SOLUTION: [Compare with the first solution of Problem (16) in (3.A)]

Suppose $T = \lambda I$ for some $\lambda \in \mathbf{F}$. Then T has the same matrix with respect to every basis of V .

Conversely, if $T = 0$, then we are done; Suppose $T \neq 0$. And v is a nonzero vector in V .

Assume that (v, Tv) is linearly independent.

Extend (v, Tv) to a basis of V as (v, Tv, u_3, \dots, u_n) . Let $B = \mathcal{M}(T, (v, Tv, u_3, \dots, u_n))$.

$$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \dots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \forall i \neq 2.$$

By assumption, $A = \mathcal{M}(T, (v, w_2, \dots, w_n)) = B$ for any basis (v, w_2, \dots, w_n) . Then $A_{2,1} = 1, A_{i,1} = 0 (\dots)$.

$\Rightarrow Tv = w_2$, which is not true if we let $w_2 = u_3, w_3 = Tv, w_j = u_j (j = 4, \dots, n)$. Contradicts.

Hence (v, Tv) is linearly dependent $\Rightarrow \forall v \in V, \exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show that λ_v is independent of v , that is,

to show that for any two nonzero distinct vectors $v, w \in V, \lambda_v = \lambda_w$. Thus $T = \lambda I$ for some $\lambda \in \mathbf{F}$.

$$\left. \begin{aligned} (v, w) \text{ is linearly independent} &\Rightarrow T(v+w) = \lambda_{v+w}(v+w) \\ &= \lambda_{v+w}v + \lambda_{v+w}w \\ &= \lambda_v v + \lambda_w w = Tv + Tw \Rightarrow \lambda_{v+w} = \lambda_v = \lambda_w \\ (v, w) \text{ is linearly dependent, } w = cv &\Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \Rightarrow \lambda_v = \lambda_w \end{aligned} \right\} \Rightarrow \square$$

OR. Conversely, denote $\mathcal{M}(T, (u_1, \dots, u_m))$ by A , where the basis (u_1, \dots, u_m) is arbitrarily chosen.

Fix one basis (v_1, \dots, v_m) and then $(v_1, \dots, \frac{1}{2}v_k, \dots, v_m)$ is also a basis for any given $k \in \{1, \dots, m\}$.

Fix one k . Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \dots + A_{k,k}(\frac{1}{2}v_k) + \dots + A_{m,k}v_m$

$$\Rightarrow Tv_k = 2A_{1,k}v_1 + \dots + A_{k,k}v_k + \dots + 2A_{m,k}v_m = A_{1,k}v_1 + \dots + A_{k,k}v_k + \dots + A_{m,k}v_m.$$

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1, \dots, m\}$.

Now we show that $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j, k arbitrarily but $j \neq k$.

Consider the basis $(v'_1, \dots, v'_j, \dots, v'_k, \dots, v'_m)$, where $v'_j = v_k, v'_k = v_j$ and $v'_i = v_i$ for all $i \in \{1, \dots, m\} \setminus \{j, k\}$.

Remember that $\mathcal{M}(T, (v'_1, \dots, v'_m)) = \mathcal{M}(T, (v_1, \dots, v_m)) = A$.

Hence $T(v'_k) = A_{1,k}v'_1 + \dots + A_{k,k}v'_k + \dots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_j$, while $T(v'_k) = T(v_j) = A_{j,j}v_j$.

Thus $A_{k,k} = A_{j,j}$. \square

17 Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

A subspace \mathcal{E} of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Using NOTE FOR [3.60]. Let (v_1, \dots, v_n) be a basis of V . If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and \mathcal{E} is a two-sided ideal of $\mathcal{L}(V)$.

Then for any $E_{i,j} \in \mathcal{E}$, $(\forall x, y = 1, \dots, n)$, by assumption, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.

Again, $E_{y,x'}, E_{y',x} \in \mathcal{E}$ for all $x', y', x, y = 1, \dots, n$. Thus $\mathcal{E} = \mathcal{L}(V)$. \square

18 Show that V and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces.

SOLUTION:

Define $\Psi \in \mathcal{L}(V, \mathcal{L}(\mathbf{F}, V))$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.

(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is injective.

(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi(T(1))$. Hence Ψ is surjective. \square

OR.

Define $\Phi \in \mathcal{L}(\mathcal{L}(\mathbf{F}, V), V)$ by $\Phi(T) = T(1)$.

(a) Suppose $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is injective.

(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surjective. \square

COMMENT: $\Phi = \Psi^{-1}$.

• Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that

$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3) \text{ for all } x \in \mathbf{R}.$$

SOLUTION:

Note that $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$.

Define $T_n : \mathcal{P}_n(\mathbf{R}) \rightarrow \mathcal{P}_n(\mathbf{R})$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.

As can be easily checked, T_n is an operator.

Because $\deg(T_n p) = \deg p$. If $T_n(p) = 0 \Rightarrow \deg(T_n p) = -\infty$, then $\deg p = -\infty \Rightarrow p = 0$.

Hence T_n is injective and therefore is surjective.

For all $q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $m = 0$; if $q \neq 0$, let $m = \deg q$. We have $q \in \mathcal{P}_m(\mathbf{R})$.

Hence $\exists p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$.

19 Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is injective. $\deg Tp \leq \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.

(a) Prove that T is surjective.

(b) Prove that for every nonzero p , $\deg Tp = \deg p$.

SOLUTION:

(a) T is injective $\iff T|_{\mathcal{P}_n(\mathbf{R})} : \mathcal{P}_n(\mathbf{R}) \rightarrow \mathcal{P}_n(\mathbf{R})$ is injective for any $n \in \mathbf{N}^+$
 $\iff T|_{\mathcal{P}_n(\mathbf{R})}$ is surjective for any $n \in \mathbf{N}^+ \iff T$ is surjective.

(b) Using mathematical induction.

(i) $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$.

$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$.

(ii) Suppose $\deg f = \deg Tf$ for all $f \in \mathcal{P}_n(\mathbf{R})$. Then suppose $\deg g = n + 1, g \in \mathcal{P}_{n+1}(\mathbf{R})$.

Assume that $\deg Tg < \deg g$ ($\Rightarrow \deg Tg \leq n, Tg \in \mathcal{P}_n(\mathbf{R})$).

Then by (a), $\exists f \in \mathcal{P}_n(\mathbf{R}), T(f) = (Tg)$. $\nexists T$ is injective $\Rightarrow f = g$.

While $\deg f = \deg Tf = \deg Tg < \deg g$. Contradicts the assumption.

Hence $\deg Tp = \deg p$ for all $p \in \mathcal{P}_{n+1}(\mathbf{R})$.

Thus $\deg Tp = \deg p$ for all $p \in \mathcal{P}(\mathbf{R})$. \square

• Suppose $T \in \mathcal{L}(V)$ and (v_1, \dots, v_m) is a list in V such that (Tv_1, \dots, Tv_m) spans V .
 Prove that (v_1, \dots, v_m) spans V .

SOLUTION:

$V = \text{span}(Tv_1, \dots, Tv_m) \Rightarrow T$ is surjective, $\nexists V$ is finite-dim $\Rightarrow T$ is invertible $\Rightarrow T^{-1}$ is invertible.

$\forall v \in V, \exists a_i \in \mathbf{F}, v = a_1Tv_1 + \dots + a_nTv_n$

$\Rightarrow T^{-1}v = a_1v_1 + \dots + a_nv_n$

$\Rightarrow \text{range } T^{-1} \subseteq \text{span}(v_1, \dots, v_n) \nexists \text{range } T^{-1} = V. \square$

OR. Reduce (Tv_1, \dots, Tv_n) to a basis of V as $(Tv_{\alpha_1}, \dots, Tv_{\alpha_m})$, where $m = \dim V$ and $\alpha_i \in \{1, \dots, m\}$.

Then $(v_{\alpha_1}, \dots, v_{\alpha_m})$ is linearly independent of length m , therefore is a basis of V , contained in the list (v_1, \dots, v_m) . \square

2 Suppose V is finite-dim and $\dim V > 1$.

Prove that the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.

SOLUTION:

Suppose $\dim V = n > 1$. Let (v_1, \dots, v_n) be a basis of V .

Define $S, T \in \mathcal{L}(V)$ by $S(a_1v_1 + \dots + a_nv_n) = a_1v_1$ and $T(a_1v_1 + \dots + a_nv_n) = a_2v_1 + \dots + a_nv_n$.

Hence $S + T = I$ is invertible.

Thus the set of noninvertible linear maps in $\mathcal{L}(V)$ is not closed under addition and therefore is not a subspace. \square

COMMENT: If $\dim V = 1$, then the set of noninvertible operators on V equals $\{0\}$, which is a subspace of $\mathcal{L}(V)$.

3 Suppose V is finite-dim, U is a subspace of V , and $S \in \mathcal{L}(U, V)$.

Prove that there exists an invertible $T \in \mathcal{L}(V, V)$ such that

$Tu = Su$ for every $u \in U$ if and only if S is injective.

SOLUTION: [Compare this with (3.A.11).]

(a) $Tu = Su$ for every $u \in U \Rightarrow u = T^{-1}Su \Rightarrow S$ is injective. OR. $\text{null } S = \text{null } T \cap U = \{0\} \cap U = \{0\}$.

(b) Suppose (u_1, \dots, u_m) be a basis of U and S is injective $\Rightarrow (Su_1, \dots, Su_m)$ is linearly independent in V .

Extend these to bases of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$ and $(Su_1, \dots, Su_m, w_1, \dots, w_n)$.

Define $T \in \mathcal{L}(V)$ by $T(u_i) = Su_i; Tv_j = w_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.

4 Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.

Prove that $\text{null } S = \text{null } T (= U) \iff S = ET, \exists \text{ invertible } E \in \mathcal{L}(W)$.

SOLUTION:

Define $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i, E(w_j) = x_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let (Tv_1, \dots, Tv_m) be a basis of range T , extend it to a basis of W as $(Tv_1, \dots, Tv_m, w_1, \dots, w_n)$.	Hence E is invertible and $S = ET$.
Let (u_1, \dots, u_n) be a basis of U . Then by (3.B.TIPS), $(v_1, \dots, v_m, u_1, \dots, u_n)$ is a basis of V .	
$\text{null } S = \text{null } T \Rightarrow V = \text{span}(v_1, \dots, v_m) \oplus \text{null } S \Rightarrow \text{span}(Sv_1, \dots, Sv_m) = \text{range } S$.	
And $\dim \text{range } T = \dim \text{range } S = \dim V - \text{null } U = m$. Hence (Sv_1, \dots, Sv_m) is a basis of range S .	
Thus we let $(Sv_1, \dots, Sv_m, x_1, \dots, x_n)$ be a basis of W .	

Conversely, $S = ET \Rightarrow \text{null } S = \text{null } ET$.

Then $v \in \text{null } ET \iff ET(v) = 0 \iff Tv = 0 \iff v \in \text{null } T$. Hence $\text{null } ET = \text{null } T = \text{null } S$. \square

5 Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.

Prove that $\text{range } S = \text{range } T (= R) \iff S = TE, \exists \text{ invertible } E \in \mathcal{L}(V)$.

SOLUTION:

Define $E \in \mathcal{L}(V)$ as $E: v_i \mapsto r_i; u_j \mapsto s_j$; for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let (Tv_1, \dots, Tv_m) and (Sr_1, \dots, Sr_m) be bases of R such that $\forall i, Tv_i = Sr_i$.	Hence E is invertible and $S = TE$.
Let (u_1, \dots, u_n) and (s_1, \dots, s_n) be bases of $\text{null } T$ and $\text{null } S$ respectively.	
Thus $(v_1, \dots, v_m, u_1, \dots, u_n)$ and $(r_1, \dots, r_m, s_1, \dots, s_n)$ are bases of V .	

Conversely, $S = TE \Rightarrow \text{range } S = \text{range } TE$.

Then $w \in \text{range } S \iff \exists v \in V, Sv = TE(v) = T(E(v)) = w \in \text{range } T$. Hence $\text{range } S = \text{range } T$. \square

6 Suppose V and W are finite-dim and $S, T \in \mathcal{L}(V, W)$.

[$\dim \text{null } S = \dim \text{null } T = n$]

Prove that $S = E_2TE_1, \exists \text{ invertible } E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W) \iff \dim \text{null } S = \dim \text{null } T$.

SOLUTION:

Define $E_1: v_i \mapsto r_i; u_j \mapsto s_j$; for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.

Define $E_2: Tv_i \mapsto Sr_i; x_j \mapsto y_j$; for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let (Tv_1, \dots, Tv_m) and (Sr_1, \dots, Sr_m) be bases of range T and range S .	Thus E_1, E_2 are invertible and $S = E_2TE_1$.
Let (u_1, \dots, u_n) and (s_1, \dots, s_n) be bases of $\text{null } T$ and $\text{null } S$ respectively.	
Thus $(v_1, \dots, v_m, u_1, \dots, u_n)$ and $(r_1, \dots, r_m, s_1, \dots, s_n)$ are bases of V .	
Extend (Tv_1, \dots, Tv_m) and (Sr_1, \dots, Sr_m) to bases of W as $(Tv_1, \dots, Tv_m, x_1, \dots, x_p)$ and $(Sr_1, \dots, Sr_m, y_1, \dots, y_p)$.	

Conversely, $S = E_2TE_1 \Rightarrow \dim \text{null } S = \dim \text{null } E_2TE_1$.

$v \in \text{null } E_2TE_1 \iff E_2TE_1(v) = 0 \iff TE_1(v) = 0$. Hence $\text{null } E_2TE_1 = \text{null } TE_1 = \text{null } S$.

$\text{By (3.B.22.COROLLARY), } E \text{ is invertible} \Rightarrow \dim \text{null } TE_1 = \dim \text{null } T = \dim \text{null } S$. \square

8 Suppose V is finite-dim and $T: V \rightarrow W$ is a surjective linear map of V onto W .

Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W .

$T|_U$ is the function whose domain is U , with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.

SOLUTION:

T is surjective $\Rightarrow \text{range } T = W \Rightarrow \dim \text{range } T = \dim W = \dim V - \dim \text{null } T$.

Let (w_1, \dots, w_m) be a basis of range $T = W \Rightarrow \forall w_i, \exists! v_i \in V, Tv_i = w_i$.

$\Rightarrow (v_1, \dots, v_m)$ is a basis of \mathcal{K} . Thus $\dim \mathcal{K} = \dim W$.

Thus $T|_{\mathcal{K}}$ maps a basis of \mathcal{K} to a basis of range $T = W$. Denote \mathcal{K} by U . \square

OR. By Problem (12) in (3.B), there is a subspace U of V such that
 $U \cap \text{null } T = \{0\} = \text{null } T|_U$, $\text{range } T = \{Tu : u \in U\} = \text{range } T|_U$. \square

• Suppose V and W are finite-dim and U is a subspace of V .

Let $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\}$.

(a) Show that \mathcal{E} is a subspace of $\mathcal{L}(V, W)$.

(b) Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$ and $\dim U$.

Hint: Define $\Phi : \mathcal{L}(V, W) \rightarrow \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. What is $\text{null } \Phi$? What is $\text{range } \Phi$?

SOLUTION:

(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.

(b) Define Φ as in the hint.

$T \in \text{null } \Phi \iff \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$. Hence $\text{null } \Phi = \mathcal{E}$.

$S \in \mathcal{L}(U, W) \Rightarrow \exists T \in \mathcal{L}(V, W), \Phi(T) = S$, by (3.B.11) $\Rightarrow S \in \text{range } \Phi$. Hence $\text{range } \Phi = \mathcal{L}(U, W)$.

Thus $\dim \text{null } \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \text{range } \Phi = (\dim V - \dim U) \dim W$. \square

OR. Extend (u_1, \dots, u_m) a basis of U to $(u_1, \dots, u_m, v_1, \dots, v_n)$ a basis of V . Let $p = \dim W$.

(See NOTE FOR [3.60])

$$\forall T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \underbrace{\text{span} \left\{ \begin{matrix} E_{1,1}, & \cdots & E_{m,1}, \\ \vdots & & \vdots \\ E_{1,p}, & \cdots & E_{m,p} \end{matrix} \right\}}_{\text{Denote it by } R} \cap \mathcal{E} = \{0\}.$$

$$\text{Let } W = \text{span} \left\{ \begin{matrix} E_{m+1,1}, & \cdots & E_{n,1}, \\ \vdots & & \vdots \\ E_{m+1,p}, & \cdots & E_{n,p} \end{matrix} \right\} \subseteq \mathcal{E}. \text{ Where } \mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}.$$

Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. \square

ENDED

3.E

2 Suppose V_1, \dots, V_m are vec-sps such that $V_1 \times \dots \times V_m$ is finite-dim.

Prove that every V_j is finite-dim.

SOLUTION: Denote $V_1 \times \dots \times V_m$ by U . Denote $\{0\} \times \dots \times \{0\} \times V_i \times \{0\} \times \dots \times \{0\}$ by U_i .

Let (v_1, \dots, v_M) be a basis of U . Note that $\forall u_i \in V_i, u_i \in U_i \subseteq U$, for each i .

Define $R_i \in \mathcal{L}(V_i, U)$ by $R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0)$. $\left. \begin{matrix} \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{matrix} \right\} \Rightarrow S_i|_{U_i} = R_i^{-1}|_{U_i}$.

Thus U_i and V_i are isomorphic. $\text{Let } U_i \text{ is a subspace of a finite-dim vec-sp } U$. \square

3 Give an example of a vec-sp V and its two subspaces U_1, U_2 such that $U_1 \times U_2$ and $U_1 + U_2$ are isomorphic but $U_1 + U_2$ is not a direct sum.

SOLUTION:

NOTE that at least one of U_1, U_2 must be infinite-dim.

For if not, $U_1 \times U_2$ is finite-dim and $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$.

And V must be infinite-dim. For if not, both U_1 and U_2 are finite-dim subspaces.

Let $V = \mathbf{F}^\infty = U_1, U_2 = \{(x, 0, \dots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$.

Define $T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2)$ by $T((x_1, x_2, \dots), (x, 0, \dots)) = (x, x_1, x_2, \dots)$ $\left. \begin{matrix} \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \dots) = ((x_1, x_2, \dots), (x, 0, \dots)) \end{matrix} \right\} \Rightarrow S = T^{-1}$. \square

4 Suppose V_1, \dots, V_m are vec-sps.

Prove that $\mathcal{L}(V_1 \times \dots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$ are isomorphic.

SOLUTION: Using the notations in Problem (2). Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \dots + T(0, \dots, u_m)$.

Define $\varphi : T \mapsto (T_1, \dots, T_m)$ by $\varphi(T) = (TR_1, \dots, TR_m)$.
 Define $\psi : (T_1, \dots, T_m) \mapsto T$ by $\psi(T_1, \dots, T_m) = T_1S_1 + \dots + T_mS_m$. $\left. \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (TR_1, \dots, TR_m). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1S_1 + \dots + T_mS_m. \end{array} \right\} \Rightarrow \psi = \varphi^{-1}. \square$

5 Suppose W_1, \dots, W_m are vec-sps.

Prove that $\mathcal{L}(V, W_1 \times \dots \times W_m)$ and $\mathcal{L}(V, W_1) \times \dots \times \mathcal{L}(V, W_m)$ are isomorphic.

SOLUTION: Using the notations in Problem (2).

Note that $Tv = (w_1, \dots, w_m)$. Define $T_i \in \mathcal{L}(V, W_i)$ by $T_i(v) = w_i$.

Define $\varphi : T \mapsto (T_1, \dots, T_m)$ by $\varphi(T) = (S_1T, \dots, S_mT)$.
 Define $\psi : (T_1, \dots, T_m) \mapsto T$ by $\psi(T_1, \dots, T_m) = T_1S_1 + \dots + T_mS_m$. $\left. \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1T, \dots, S_mT). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1S_1 + \dots + T_mS_m. \end{array} \right\} \Rightarrow \psi = \varphi^{-1}. \square$

6 For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \dots \times V}_{m \text{ times}}$. Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are isomorphic.

SOLUTION:

Define $T : (v_1, \dots, v_m) \rightarrow \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1v_1 + \dots + a_mv_m$.

Suppose $T(v_1, \dots, v_m) = 0$. Then $\forall (a_1, \dots, a_m) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1v_1 + \dots + a_mv_m = 0$

$\Rightarrow (v_1, \dots, v_m) = 0 \Rightarrow T$ is injective.

Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let (e_1, \dots, e_m) be the standard basis of \mathbf{F}^m . Then $\forall (b_1, \dots, b_m) \in \mathbf{F}^m$,

$(T(\psi(e_1), \dots, \psi(e_m)))(b_1, \dots, b_m) = b_1\psi(e_1) + \dots + b_m\psi(e_m) = \psi(b_1e_1 + \dots + b_me_m) = \psi(b_1, \dots, b_m)$.

Thus $T(\psi(e_1), \dots, \psi(e_m)) = \psi$. Hence T is surjective. \square

7 Suppose $v, x \in V$ (chosen arbitrarily) of which U and W are subspaces.

Suppose $v + U = x + W$. Prove that $U = W$.

SOLUTION:

(a) $\forall u \in U, \exists w \in W, v + u = x + w$, let $u = 0$, getting $v = x + w \Rightarrow v - x \in W$.

(b) $\forall w \in W, \exists u \in U, v + u = x + w$, let $w = 0$, getting $x = v + u \Rightarrow x - v \in U$.

Thus $\pm(v - x) \in U \cap W \Rightarrow \left\{ \begin{array}{l} u = (x - v) + w \in W \Rightarrow U \subseteq W \\ w = (v - x) + u \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W. \square$

• Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$.

Prove that A is a translate of $U \iff \exists c \in \mathbf{R}, A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$.

[Do it in your mind.]

• Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $U = \{x \in V : Tx = c\}$ is either \emptyset or is a translate of null T .

SOLUTION:

If $c \in W$ but $c \notin \text{range } T$, then $U = \emptyset$ and we are done.

Suppose $c \in \text{range } T$, then $\exists u \in V, Tu = c \Rightarrow u \in U$.

Suppose $y \in \text{null } T \Rightarrow y + u \in U \iff T(y + u) = Ty + c = c$. Thus $u + \text{null } T \subseteq U$. Hence $u + \text{null } T = U$,

for if not, suppose $z \notin u + \text{null } T$ but $Tz = c (\Leftrightarrow z \in U)$, then $\forall w \in \text{null } T, z \neq u + w \Leftrightarrow z - u \notin \text{null } T$.

$\text{But } \tilde{T}(z + \text{null } T) = \tilde{T}(u + \text{null } T) \Rightarrow z + \text{null } T = u + \text{null } T \Rightarrow z - u \in \text{null } T$, contradicts. \square

• **COROLLARY:** The set of solutions to a system of linear equations such as [3.28] is either \emptyset or a translate of the null subspace.

8 Prove that a nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$.

SOLUTION:

Suppose $A = a + U$, where U is a subspace of V . $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$,

$$\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + [\lambda(u_1 - u_2) + u_2] \in A.$$

Suppose $\lambda v + (1 - \lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $a \in A$ and let $A' = \{x - a : x \in A\}$.

Then $\forall x - a, y - a \in A', \lambda \in \mathbf{F}$,

(I) $\lambda(x - a) = [\lambda x + (1 - \lambda)a] - a \in A'$. Then let $\lambda = 2$.

(II) $\lambda(x - a) + (1 - \lambda)(y - a) = \frac{1}{2}(x - a) + \frac{1}{2}(y - a) = \frac{1}{2}x + (1 - \frac{1}{2})(y) - a \in A'$.

By (I), $2 \times [\frac{1}{2}(x - a) + \frac{1}{2}(y - a)] = (x - a) + (y - a) \in A'$.

Thus A' is a subspace of V . Hence $a + A' = \{(x - a) + a : x \in A\} = A$ is a translate. \square

9 Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V . Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or is \emptyset .

SOLUTION: Suppose $v + u_1, w + u_2 \in A_1 \cap A_2 \neq \emptyset$. By Problem (8),

$\forall \lambda \in \mathbf{F}, \lambda(v + u_1) + (1 - \lambda)(w + u_2) \in A_1$ and A_2 . Thus $A_1 \cap A_2$ is a translate of some subspace of V . \square

10 Prove that the intersection of any collection of translates of subspaces of V is either a translate of some subspace or \emptyset .

SOLUTION: Suppose $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collection of translates of subspaces of V , where Γ is an arbitrary index set.

Suppose $x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset$, then by Problem (18), $\forall \lambda \in \mathbf{F}, \lambda x + (1 - \lambda)y \in A_\alpha$ for every $\alpha \in \Gamma$.

Thus $\bigcap_{\alpha \in \Gamma} A_\alpha$ is a translate of some subspace of V . \square

11 Suppose $A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.

(a) Prove that A is a translate of some subspace of V : By Problem (8),

$$\forall \sum_{i=1}^m a_i v_i, \sum_{i=1}^m b_i v_i \in A, \lambda \in \mathbf{F}, \quad \lambda \sum_{i=1}^m a_i v_i + (1 - \lambda) \sum_{i=1}^m b_i v_i = (\lambda \sum_{i=1}^m a_i + (1 - \lambda) \sum_{i=1}^m b_i) v_i \in A. \quad \square$$

(b) Prove that if B is a translate of some subspace of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.

(c) Prove that A is a translate of some subspace of V and $\dim V < m$.

SOLUTION:

(b) Let $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on m by k .

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. $\forall v_1 \in B$. Hence $v \in B$.

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. $\forall v_1, v_2 \in B$. By problem (8), $v \in B$.

(ii) $2 \leq k \leq m$, we assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. ($\forall \lambda_i$ such that $\sum_{i=1}^k \lambda_i = 1$)

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. $\forall i = 1, \dots, k, \exists \mu_i \neq 1$, fix one such i by ι .

$$\text{Then } \sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow \left(\sum_{i=1}^{k+1} \frac{\mu_i}{1 - \mu_\iota} \right) - \frac{\mu_\iota}{1 - \mu_\iota} = 1.$$

$$\text{Let } w = \underbrace{\frac{\mu_1}{1 - \mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1 - \mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1 - \mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1 - \mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let $\lambda_i = \frac{\mu_i}{1 - \mu_\iota}$ for $i = 1, \dots, \iota - 1$; $\lambda_j = \frac{\mu_{j+1}}{1 - \mu_\iota}$ for $j = \iota, \dots, k$. Then,

$$\left. \begin{array}{l} \sum_{i=1}^k \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1 - \lambda)v_\iota \in B \end{array} \right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \quad \square$$

(c) $\forall k = 1, \dots, m, \forall \lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_m$, let $\lambda_k = 1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m$

$$\Rightarrow \lambda_1 v_1 + \cdots + \lambda_m v_m$$

$$= \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m) v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_m v_m$$

$$= v_k + \lambda_1(v_1 - v_k) + \cdots + \lambda_{k-1}(v_{k-1} - v_k) + \lambda_{k+1}(v_{k+1} - v_k) + \cdots + \lambda_m(v_m - v_k).$$

Thus $A = v_k + \text{span}(v_1 - v_k, \dots, v_{k-1} - v_k, v_{k+1} - v_k, \dots, v_m - v_k)$. \square

12 Suppose U is a subspace of V such that V/U is finite-dim.

Prove that V is isomorphic to $U \times (V/U)$.

SOLUTION: Let $(v_1 + U, \dots, v_n + U)$ be a basis of V/U . Note that

$$\forall v \in V, \exists ! a_1, \dots, a_n \in \mathbf{F}, v + U = \sum_{i=1}^n a_i(v_i + U) = (\sum_{i=1}^n a_i v_i) + U$$

$$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) = u \in U \text{ for some } u; v = \sum_{i=1}^n a_i v_i + u.$$

Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ by $\varphi(v) = (u, \sum_{i=1}^n a_i v_i + U)$

and $\psi \in \mathcal{L}(U \times (V/U), V)$ by $\psi(u, w + U) = u + w; w = \sum_{i=1}^n b_i v_i + U$.

So that $\psi = \varphi^{-1}$. \square

• Suppose $V = U \oplus W$, (w_1, \dots, w_m) is a basis of W .

Prove that $(w_1 + U, \dots, w_m + U)$ is a basis of V/U .

SOLUTION: Note that for any $v \in V$,

$$\exists ! u \in U, w \in W, v = u + w \text{ 又 } \exists ! c_i \in \mathbf{F} \text{ such that } w = \sum_{i=1}^m c_i w_i \Rightarrow v = u + \sum_{i=1}^m c_i w_i.$$

$$\text{Thus } v + U = \sum_{i=1}^m c_i w_i + U \Rightarrow v + U \in \text{span}(w_1 + U, \dots, w_m + U) \Rightarrow V/U \subseteq \text{span}(w_1 + U, \dots, w_m + U).$$

$$\text{Now suppose } a_1(w_1 + U) + \cdots + a_m(w_m + U) = 0 + U \Rightarrow \sum_{i=1}^m a_i w_i \in U \text{ while } U \cap W = \{0\}.$$

$$\text{Then } \sum_{i=1}^m a_i w_i = 0 \Rightarrow a_1 = \cdots = a_m = 0. \quad \square$$

13 Suppose $(v_1 + U, \dots, v_m + U)$ is a basis of V/U and (u_1, \dots, u_n) is a basis of U .

Prove that $(v_1, \dots, v_m, u_1, \dots, u_n)$ is a basis of V .

SOLUTION: By Problem (12), U and V/U are finite-dim $\Rightarrow U \times (V/U)$ is finite-dim, so is V .

$$\dim V = \dim(U \times (V/U)) = \dim U + \dim V/U = m + n.$$

$$\text{OR. Note that for any } v \in V, v + U = \sum_{i=1}^m a_i v_i + U, \exists ! a_i \in \mathbf{F} \Rightarrow v = \sum_{i=1}^m a_i v_i = \sum_{i=1}^m b_i v_i, \exists ! b_i \in \mathbf{F}.$$

$$\Rightarrow v \in \text{span}(v_1, \dots, v_m, u_1, \dots, u_n) \supseteq V.$$

$$\text{又 Notice that } (\sum_{i=1}^m a_i v_i) + U = 0 + U (\Rightarrow \sum_{i=1}^m a_i v_i \in U) \iff a_1 = \cdots = a_m = 0.$$

$$\text{Hence } \text{span}(v_1, \dots, v_m) \cap U = \{0\} \Rightarrow \text{span}(v_1, \dots, v_m) \oplus U = V$$

Thus $(v_1, \dots, v_m, u_1, \dots, u_n)$ is linearly independent, so is a basis of V . \square

14 Suppose $U = \{(x_1, x_2, \dots) \in \mathbf{F}^\infty : x_k \neq 0 \text{ for only finitely many } k\}$.

(a) Show that U is a subspace of \mathbf{F}^∞ . [Do it in your mind]

(b) Prove that \mathbf{F}^∞/U is infinite-dim.

SOLUTION:

For $u = (x_1, \dots, x_p, \dots) \in \mathbf{F}^\infty$, denote x_p by $u[p]$. For each $r \in \mathbf{N}^+$.

$$\text{Define } e_r[p] = \begin{cases} 1, & (p-1) \equiv 0 \pmod{r} \\ 0, & \text{otherwise} \end{cases}, \text{ simply } e_r = (1, \underbrace{0, \dots, 0}_{(p-1) \text{ times}}, 1, \underbrace{0, \dots, 0}_{(p-1) \text{ times}}, 1, \dots) \in \mathbf{F}^\infty.$$

Choose $m \in \mathbf{N}^+$ arbitrarily.

$$\text{Suppose } a_1(e_1 + U) + \cdots + a_m(e_m + U) = (a_1 e_1 + \cdots + a_m e_m) + U = 0 + U = 0.$$

$$\Rightarrow a_1 e_1 + \cdots + a_m e_m = u \text{ for some } u \in U.$$

$$\text{Then suppose } u = (x_1, \dots, x_t, 0, \dots) \Rightarrow u[t+i] = 0, \forall i \in \mathbf{N}^+,$$

then let $j = s \cdot m! + 1 \geq t$ ($\exists s \in \mathbf{N}^+$) so that $e_1[j] = \cdots = e_m[j] = 1$, $u[j+i] = 0$.

Now we have: $u[j+i] = (\sum_{r=1}^m a_r e_r)[j+i] = \sum_{r=1}^m a_r e_r[s \cdot m! + 1 + i] = a_{i_1} + \cdots + a_{i_{\tau(i)}} = 0$,

$\Rightarrow (\sum_{r=1}^m a_r e_r)[j+i] = a_{i_1} + \cdots + a_{i_{\tau(i)}} = 0$. (Δ)

where $i_1, \dots, i_{\tau(i)}$ are distinct ordered factors of i ($1 = i_1 \leq \cdots \leq i_{\tau(i)} = i$).

(Note that by definition, $e_r[s \cdot m! + 1 + i] = 1 \iff s \cdot m! + i \equiv 0 \pmod{r} \iff r \mid i$.)

Let $i' = i_{\tau(i)-1}$. Notice that $i'_l = i_l, \forall l \in \{1, \dots, \tau(i')\}$; and $\tau(i') = \tau(i) - 1$.

Again by (Δ), $(\sum_{r=1}^m a_r e_r)[j+i'] = a_{i'_1} + \cdots + a_{i'_{\tau(i')}} = a_{i_1} + \cdots + a_{i_{\tau(i)-1}} = 0$.

Thus $a_{i_{\tau(i)}} = a_i = 0$ for any $i \in \{1, \dots, m\}$.

Hence (e_1, \dots, e_m) is linearly independent in \mathbf{F}^∞ , so is (e_1, \dots, e_m, \dots) , since $m \in \mathbf{N}^+$.

又 $e_i \notin U \Rightarrow (e_1 + U, e_2 + U, \dots)$ is linearly independent in \mathbf{F}^∞/U . By [2.B.14]. \square

15 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F}) \setminus \{0\}$. Prove that $\dim V/(\text{null } \varphi) = 1$.

SOLUTION: By [3.91] (d), $\dim \text{range } \varphi = 1 = \dim V/(\text{null } \varphi)$. \square

NOTE FOR [3.88, 3.90, 3.91]

For any $W \in \mathcal{S}_V U$, because $V = U \oplus W$. $\forall v \in V, \exists! u_v \in U, w_v \in W, v = u_v + w_v$.

Define $T \in \mathcal{L}(V, W)$ by $T(v) = w_v$. Hence $\text{null } T = U$, $\text{range } T = W$.

Then $\tilde{T} \in \mathcal{L}(V/\text{null } T, W)$ is defined as $\tilde{T}(v+U) = Tv = w_v$.

Thus \tilde{T} is injective (by [3.91(b)]) and surjective ($\text{range } \tilde{T} = \text{range } T = W$),

and therefore is an isomorphism. We conclude that V/U and W , namely any vec-sp in \mathcal{S}_V , are isomorphic.

16 Suppose $\dim V/U = 1$. Prove that $\exists \varphi \in \mathcal{L}(V, \mathbf{F})$ such that $\text{null } \varphi = U$.

SOLUTION:

Suppose V_0 is a subspace of V such that $V = U \oplus V_0$. Then V_0 and V/U are isomorphic. $\dim V_0 = 1$.

Define a linear map $\varphi : v \mapsto \lambda$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. \square

17 Suppose V/U is finite-dim. W is a subspace of V .

(a) Show that if $V = U + W$, then $\dim W \geq \dim V/U$.

(b) Suppose $\dim W = \dim V/U$ and $V = U \oplus W$. Find such W .

SOLUTION:

Let (w_1, \dots, w_n) be a basis of W

(a) $\forall v \in V, \exists u \in U, w \in W$ such that $v = u + w \Rightarrow v + U = w + U$

Then $V/U \subseteq \text{span}(w_1 + U, \dots, w_n + U) \Rightarrow V/U = \text{span}(w_1 + U, \dots, w_n + U)$.

Hence $\dim V/U = \dim \text{span}(w_1 + U, \dots, w_n + U) \leq \dim W$.

(b) Let $W \in \mathcal{S}_V U$. In other words,

reduce $(w_1 + U, \dots, w_n + U)$ to a basis of V/U as $(w_{\alpha_1} + U, \dots, w_{\alpha_m} + U)$ and let $W = \text{span}(w_{\alpha_1}, \dots, w_{\alpha_m})$. \square

18 Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V . Let π denote the quotient map.

Prove that $\exists S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subseteq \text{null } T$.

SOLUTION:

(a) Define $S \in \mathcal{L}(V/U, W)$ by $S(v+U) = Tv$. We have to check it is well-defined.

Suppose $v_1 + U = v_2 + U$, while $v_1 \neq v_2$.

Then $(v_1 - v_2) \in U \Rightarrow S((v_1 - v_2) + U) = T(v_1 - v_2) = 0 \Rightarrow Tv_1 = Tv_2$. Checked. \square

(b) Suppose $\exists S \in \mathcal{L}(V/U, W)$, $T = S \circ \pi$. Then $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \text{null } T$. \square

20 Define $\Gamma : \mathcal{L}(V/U, W) \rightarrow \mathcal{L}(V, W)$ by $\Gamma(S) = S \circ \pi (= \pi'(S))$.

(a) Prove that Γ is linear: By [3.9] distributive properties and [3.6]. \square

(b) Prove that Γ is injective:

$$\Gamma(S) = 0$$

$$\iff \forall v \in V, S(\pi(v)) = 0$$

$$\iff \forall v + U \in V/U, S(v + U) = 0$$

$$\iff S = 0. \quad \square$$

(c) Prove that $\text{range } \Gamma (= \text{range } \pi') = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\}$:

By Problem (18). \square

ENDED

3.F

• By (18) in (3.D) we know that $\varphi : V \rightarrow \mathcal{L}(\mathbf{F}, V)$ is an isomorphism. Now we prove that (v_1, \dots, v_m) is linearly independent $\iff (\varphi(v_1), \dots, \varphi(v_m))$ is linearly independent.

SOLUTION:

(a) Suppose (v_1, \dots, v_m) is linearly independent and $\vartheta \in \text{span}(\varphi(v_1), \dots, \varphi(v_m))$.

Let $\vartheta = 0 = a_1\varphi(v_1) + \dots + a_m\varphi(v_m)$. Then $\vartheta(1) = 0 = a_1v_1 + \dots + a_mv_m \Rightarrow a_1 = \dots = a_m = 0$.

OR Because φ is injective. Suppose $a_1\varphi(v_1) + \dots + a_m\varphi(v_m) = 0 = \varphi(a_1v_1 + \dots + a_mv_m)$.

Then $a_1v_1 + \dots + a_mv_m = 0 \Rightarrow a_1 = \dots = a_m = 0$.

Thus $(\varphi(v_1), \dots, \varphi(v_m))$ is linearly independent.

(b) Suppose $(\varphi(v_1), \dots, \varphi(v_m))$ is linearly independent and $v \in \text{span}(v_1, \dots, v_m)$.

Let $v = 0 = a_1v_1 + \dots + a_mv_m$. Then $\varphi(v) = a_1\varphi(v_1) + \dots + a_m\varphi(v_m) = 0 \Rightarrow a_1 = \dots = a_m = 0$.

Thus v_1, \dots, v_m is linearly independent. \square

1 Explain why each linear functional is surjective or is the zero map.

SOLUTION: For any $\varphi \in V'$ and $\varphi \neq 0$, $\exists v \in V$, such that $\varphi(v) \neq 0$. (a) $\left. \begin{array}{l} \text{For any } \varphi \in V' \text{ and } \varphi \neq 0, \exists v \in V, \text{ such that } \varphi(v) \neq 0. \\ \dim \text{range } \varphi = \dim \mathbf{F} = 1. \end{array} \right\} \Rightarrow \square$

4 Suppose V is finite-dim and U is a subspace of V such that $U \neq V$.

Prove that $\exists \varphi \in V'$ and $\varphi \neq 0$ such that $\varphi(u) = 0$ for every $u \in U$.

SOLUTION:

Let (u_1, \dots, u_m) be a basis of U , extend to $(u_1, \dots, u_m, u_{m+1}, \dots, u_{m+n})$ a basis of V .

Choose $k \in \{1, \dots, n\}$ arbitrarily. Define $\varphi \in V'$ by $\varphi(u_i) = \begin{cases} 1, & \text{if } i = m + k. \\ 0, & \text{otherwise.} \end{cases}$

OR: Equivalent to proving that $U^0 \neq \{0\}$. By [3.106], $\dim U^0 = \dim V - \dim U > 0$. \square

• Suppose $T \in \mathcal{L}(V, W)$ and (w_1, \dots, w_m) is a basis of $\text{range } T$.

Hence $\forall v \in V, Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m$, $\exists! \varphi_1(v), \dots, \varphi_m(v)$, thus defining functions $\varphi_1, \dots, \varphi_m$ from V to \mathbf{F} . Show that each $\varphi_i \in V'$.

SOLUTION:

For each $w_i, \exists v_i \in V, Tv_i = w_i$, getting a linearly independent list (v_1, \dots, v_m) .

Now we have $Tv = a_1Tv_1 + \dots + a_mTv_m, \forall v \in V, \exists! a_i \in \mathbf{F}$.

Let (ψ_1, \dots, ψ_m) be the dual basis of $\text{range } T$. Then $(T'(\psi_i))(v) = \psi_i \circ T(v) = a_i$.

Thus letting $\varphi_i = \psi_i \circ T$. \square

• Suppose $\varphi, \beta \in V'$. Prove that $\text{null}\varphi \subseteq \text{null}\beta$ if and only if $\beta = c\varphi$. $\exists c \in \mathbf{F}$.

SOLUTION: Using (3.B.29, 30)

(a) Suppose $\text{null}\varphi \subseteq \text{null}\beta$. Choose a $u \notin \text{null}\beta$. $V = \text{null}\beta \oplus \{au : a \in \mathbf{F}\}$.

If $\text{null}\varphi = \text{null}\beta$, then let $c = \frac{\beta(u)}{\varphi(u)}$, we are done.

Otherwise, suppose $u' \in \text{null}\beta$, but $u' \notin \text{null}\varphi$, then $V = \text{null}\varphi \oplus \{bu' : b \in \mathbf{F}\}$.

$\forall v \in V, v = w + au = w' + bu', \exists! w, w' \in \text{null}\varphi, a, b \in \mathbf{F}$.

Thus $\beta(v) = a\beta(u), \varphi(v) = b\varphi(u')$. Let $c = \frac{a\beta(u)}{b\varphi(u')}$. We are done

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$.

If $c = 0$, then $\text{null}\beta = V \supseteq \text{null}\varphi$, we are done.

Otherwise, $\left. \begin{array}{l} \forall v \in \text{null}\varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \text{null}\varphi \subseteq \text{null}\beta. \\ \forall v \in \text{null}\beta, \beta(v) = 0 = \varphi(v) \Rightarrow \text{null}\beta \subseteq \text{null}\varphi. \end{array} \right\} \Rightarrow \text{null}\varphi = \text{null}\beta$

$\Rightarrow \text{null}\varphi \subseteq \text{null}\beta$. \square

5 Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic.

SOLUTION: Using notations in (3.E.2).

Define $\varphi : (V_1 \times \cdots \times V_m)' \rightarrow V_1' \times \cdots \times V_m'$
 by $\varphi(T) = (T \circ R_1, \dots, T \circ R_m) = (R_1'(T), \dots, R_m'(T))$.
 Define $\psi : V_1' \times \cdots \times V_m' \rightarrow (V_1 \times \cdots \times V_m)'$
 by $\psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m = S_1'(T_1) + \cdots + S_m'(T_m)$. $\left. \vphantom{\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \rightarrow V_1' \times \cdots \times V_m' \\ \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = (R_1'(T), \dots, R_m'(T)) \\ \text{Define } \psi : V_1' \times \cdots \times V_m' \rightarrow (V_1 \times \cdots \times V_m)' \\ \text{by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m = S_1'(T_1) + \cdots + S_m'(T_m) \end{array}} \right\} \Rightarrow \psi = \varphi^{-1}. \square$

• Suppose (v_1, \dots, v_n) is a basis of V and $(\varphi_1, \dots, \varphi_n)$ is the dual basis of V' .

Define $\Gamma : V \rightarrow \mathbf{F}^n$ by $\Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v))$.
 Define $\Lambda : \mathbf{F}^n \rightarrow V$ by $\Lambda(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n$. $\left. \vphantom{\begin{array}{l} \text{Define } \Gamma : V \rightarrow \mathbf{F}^n \text{ by } \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)) \\ \text{Define } \Lambda : \mathbf{F}^n \rightarrow V \text{ by } \Lambda(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n \end{array}} \right\} \Rightarrow \Lambda = \Gamma^{-1}.$

35 Prove that $(\mathcal{P}(\mathbf{R}))'$ and \mathbf{R}^∞ are isomorphic.

SOLUTION:

Define $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{R}))', \mathbf{R}^\infty)$ by $\theta(\varphi) = (\varphi(1), \varphi(x), \dots, \varphi(x^n), \dots)$.

Injectivity: $\theta(\varphi) = 0 \Rightarrow \forall x^k$ in the basis $(1, x, \dots, x^n, \dots)$ of $\mathcal{P}_n(\mathbf{R})$ for any n , $\varphi(x^k) = 0 \Rightarrow \varphi = 0$.

Surjectivity: $\forall (a_0, a_1, \dots, a_n, \dots) \in \mathbf{F}^\infty$, let ψ be such that $\psi(x^k) = a_k$ and thus $\theta(\psi) = (a_0, a_1, \dots, a_n, \dots)$.

Hence θ is an isomorphism from $(\mathcal{P}(\mathbf{R}))'$ onto \mathbf{R}^∞ . \square

7 Suppose m is a positive integer. Show that the dual basis of the basis $(1, x, \dots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \dots, \varphi_m$, where $\varphi_k = \frac{p^{(k)}(0)}{k!}$. Here $p^{(k)}$ denotes the k^{th} derivative of p , with the understanding that the 0^{th} derivative of p is p .

SOLUTION:

For each j and k , $(x^j)^{(k)} = \begin{cases} j(j-1)\dots(j-k+1) \cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\dots(j-j+1) = j!, & j = k. \\ 0, & j \leq k. \end{cases}$ Then $(x^j)^{(k)}(0) = \begin{cases} 0, & j \neq k. \\ k!, & j = k. \end{cases}$

Thus $\varphi_k = \psi_k$, where ψ_1, \dots, ψ_m is the dual basis of $(1, x, \dots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$. \square

8 Suppose m is a positive integer.

(a) By [2.C.10], $B = (1, x - 5, \dots, (x - 5)^m)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) Let $\varphi_k = \frac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \dots, m$. Then $(\varphi_0, \varphi_1, \dots, \varphi_m)$ is the dual basis of B .

9 Suppose (v_1, \dots, v_n) is a basis of V and $(\varphi_1, \dots, \varphi_n)$ is the corresponding dual basis of V' .

Suppose $\psi \in V'$. Prove that $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.

SOLUTION: $\psi(v) = \psi(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i) \varphi_i(v) = [\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n](v) \Rightarrow \square$

COMMENT: For any other basis (u_1, \dots, u_n) of V and the corresponding dual basis of (ρ_1, \dots, ρ_n) ,

$$\psi = \rho(u_1)\rho_1 + \dots + \rho(u_n)\rho_n.$$

12 Show that the dual map of the identity operator on V is the identity operator on V' .

SOLUTION: $I'(\varphi) = \varphi \circ I = \varphi, \quad \forall \varphi \in V'. \quad \square$

• Suppose W is finite-dim and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \iff T = 0$.

SOLUTION: $T = 0 \iff T'(\varphi) = \varphi \circ T = 0$ for all $\varphi \in V' \iff T' = 0. \quad \square$

13 Define $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$.

Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ denote the dual basis of the standard basis of \mathbf{R}^2 and \mathbf{R}^3 .

(a) Describe the linear functionals $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$

For any $(x, y, z) \in \mathbf{R}^3, (T'(\varphi_1))(x, y, z) = 4x + 5y + 6z, (T'(\varphi_2))(x, y, z) = 7x + 8y + 9z.$

(b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3 .

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3, T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3.$

\square

14 Define $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$ by $(Tp)(x) = x^2p(x) + p''(x)$ for each $x \in \mathbf{R}$.

(a) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe $T'(\varphi) \in \mathcal{P}(\mathbf{R})'$.

$(T'(\varphi))(p) = [x^2p(x) + p''(x)]'(4) = [2xp(x) + x^2p'(x) + p'''(x)](4) = 8p(4) + 16p'(4) + p'''(4).$

(b) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\varphi))(x^3)$.

$(T'(\varphi))(x^3) = \int_0^1 (x^5 + 6x)dx = \int_0^1 (\frac{1}{6}x^6 + 3x^2)'dx = \frac{6}{19}.$

\square

• Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$.

Prove that T is invertible if and only if $T' \in \mathcal{L}(W', V')$ is invertible.

SOLUTION: By [3.108] and [3.110]. \square

16 Suppose V and W are finite-dim. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(L, W)$.

Prove that Γ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

SOLUTION:

V, W are finite-dim $\Rightarrow \dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$. And by [3.101], Γ is linear.

又 Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus T is injective $\Rightarrow T$ is invertible. \square

17 Suppose $U \subseteq V$. Explain why $U^0 = \{\varphi \in V' : U \subseteq \text{null} \varphi\}$.

SOLUTION: Because for $\varphi \in V', U \subseteq \text{null} \varphi \iff \forall u \in U, \varphi(u) = 0$. By definition in [3.102]. \square

18 $U \subseteq V$. We have $U = \{0\} \iff \forall \varphi \in V', U \subseteq \text{null} \varphi \iff U^0 = V'$.

19 U is a subspace of V . Prove that $U = V \iff U_V^0 = \{0\} = V_V^0$.

SOLUTION:

Suppose $U_V^0 = \{0\}$. Then $U = V$.

Conversely, suppose $U = V$, then $U_V^0 = \{\varphi \in V' : V \subseteq \text{null} \varphi\}$, therefore $U_V^0 = \{0\}$.

20, 21 Suppose U and W are subsets of V . Prove that $U \subseteq W \iff W^0 \subseteq U^0$.

SOLUTION:

(a) $U \subseteq W \Rightarrow \forall w \in W, u \in U \cap W = U, \quad \forall \varphi \in W^0, \varphi(w) = 0 = \varphi(u) \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.

(b) $W^0 \subseteq U^0 \Rightarrow \forall w \in W, u \in U, \varphi(w) = 0 \Rightarrow \varphi(u) = 0$. Then $\text{null} \varphi \supseteq W \Rightarrow \text{null} \varphi \supseteq U$. Thus $W \supseteq U$. \square .

• **COROLLARY:** $W^0 = U^0 \iff U = W$.

22 Prove that $(U + W)^0 = U^0 \cap W^0$.

SOLUTION:

$$(a) \left. \begin{array}{l} U \subseteq U + W \\ W \subseteq U + W \end{array} \right\} \Rightarrow \left. \begin{array}{l} (U + W)^0 \subseteq U^0 \\ (U + W)^0 \subseteq W^0 \end{array} \right\} \Rightarrow (U + W)^0 \subseteq U^0 \cap W^0.$$

(b) $\forall \varphi \in U^0 \cap W^0, \varphi(u + w) = 0$, where $u \in U, w \in W \Rightarrow \varphi \in (U + W)^0$. Thus $(U + W)^0 \supseteq U^0 \cap W^0$. \square

23 Prove that $(U \cap W)^0 = U^0 + W^0$.

SOLUTION:

$$(a) \left. \begin{array}{l} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array} \right\} \Rightarrow \left. \begin{array}{l} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array} \right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \supseteq U^0 \cap W^0.$$

(b) $\forall \varphi \in U^0, \psi \in W^0$ and $\forall v \in U \cap W, (\varphi + \psi)(v) = \varphi(v) + \psi(v) = 0$. Thus $U^0 + W^0 \subseteq (U \cap W)^0$. \square

• **COROLLARY:**

Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subspaces of V .

Then $(\sum_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \bigcap_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$;

And $(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \sum_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$.

24 Suppose V is finite-dim and U is a subspace of V .

Prove, using the pattern of [3.104], that $\dim U + \dim U^0 = \dim V$.

SOLUTION:

Let (u_1, \dots, u_m) be a basis of U , extend to a basis of V as $(u_1, \dots, u_m, \dots, u_n)$,

and let $(\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$ be the dual basis.

(a) Suppose $\varphi \in \text{span}(\varphi_{m+1}, \dots, \varphi_n)$, then $\exists a_i \in \mathbf{F}, \varphi = a_{m+1}\varphi_{m+1} + \dots + a_n\varphi_n$.

For all $u \in U, \varphi(u) = 0$. Thus $\varphi \in U^0$, getting $\text{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq U^0$.

(b) Suppose $\varphi \in U^0$, then $\exists a_i \in \mathbf{F}, \varphi = a_1\varphi_1 + \dots + a_m\varphi_m + \dots + a_n\varphi_n$.

For all $u_i \in U, 0 = \varphi(u_i) = \sum_{i=1}^n \varphi(u_i) = a_i$. Then $\varphi = a_{m+1}\varphi_{m+1} + \dots + a_n\varphi_n$.

Thus $\varphi \in \text{span}(\varphi_{m+1}, \dots, \varphi_n)$, getting $\text{span}(\varphi_{m+1}, \dots, \varphi_n) \supseteq U^0$.

Hence $\text{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0, \dim U^0 = n - m = \dim V - \dim U$. \square

25 Suppose U is a subspace of V . Explain why $U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}$.

SOLUTION: Note that $U = \{v \in V : v \in U\}$ is a subspace of V and $\varphi(v) = 0$ for every $\varphi \in U^0 \iff v \in U$. \square

26 Suppose V is finite-dim and Ω is a subspace of V' .

Prove that $\Omega = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}^0$.

SOLUTION: Using the corollary in Problem (20, 21).

Suppose $U = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}$.

Getting $U^0 = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}^0$. We need to show that $\Omega = U^0$.

(a) $\forall \varphi \in \Omega, v \in U, \varphi(v) = 0 \Rightarrow \varphi \in U^0 \Rightarrow \Omega \subseteq U^0$.

(b) $v \in U \Leftrightarrow \left\{ \begin{array}{l} \forall \varphi \in \Omega, \varphi(v) = 0 \\ \forall \psi \in U^0, \psi(v) = 0 \end{array} \right. \text{ Thus } \Omega \supseteq U^0. \right\} \Rightarrow \square$

27 Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ and $\text{null}T' = \text{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that $\text{range}T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$.

SOLUTION: By Problem (26), $\text{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \psi(p) = 0, \forall \psi \in \text{span}(\varphi)\}^0$,

Hence $\text{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \varphi(p) = p(8) = 0\}^0$, $\text{span}(\varphi) = \text{null}T' = (\text{range}T)^0$.

By the corollary in Problem (20, 21), $\text{range}T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$. \square

28, 29 Suppose V, W are finite-dim, $T \in \mathcal{L}(V, W)$.

(a) Suppose $\exists \varphi \in W'$ such that $\text{null}T' = \text{span}(\varphi)$. Prove that $\text{range}T = \text{null}\varphi$.

(b) Suppose $\exists \varphi \in V'$ such that $\text{range}T' = \text{span}(\varphi)$. Prove that $\text{null}T = \text{null}\varphi$.

SOLUTION: Using Problem (26), [3.107] and [3.109].

Because $\text{span}(\varphi) = \{v \in V : \forall \psi \in \text{span}(\varphi), \psi(v) = 0\}^0 = \{v \in V : \varphi(v) = 0\}^0 = (\text{null}\varphi)^0$.

$$\left. \begin{array}{l} \text{(a) } (\text{range}T)^0 = \text{null}T' = \text{span}(\varphi) = (\text{null}\varphi)^0 \iff \text{range}T = \text{null}\varphi. \\ \text{(b) } (\text{null}T)^0 = \text{range}T' = \text{span}(\varphi) = (\text{null}\varphi)^0 \iff \text{null}T = \text{null}\varphi. \end{array} \right\} \Rightarrow \square$$

31 Suppose V is finite-dim and $(\varphi_1, \dots, \varphi_n)$ is a basis of V' .

Show that there exists a basis of V whose dual basis is $(\varphi_1, \dots, \varphi_n)$.

SOLUTION: Using (3.B.29,30).

For each φ_i , $\text{null}\varphi_i \oplus \{a u_i : a \in \mathbf{F}\} = V$.

Because $\varphi_1, \dots, \varphi_m$ is linearly independent, $\text{null}\varphi_i \neq \text{null}\varphi_j$ for all $i, j \in \mathbf{N}^+$ such that $i \neq j$.

Thus (u_1, \dots, u_m) is linearly independent, for if not, then $\exists i, j$ such that $\text{null}\varphi_i = \text{null}\varphi_j$, contradicts.

又 $\dim V' = m = \dim V$. Then (u_1, \dots, u_m) is a basis of V whose dual basis is $\varphi_1, \dots, \varphi_n$. \square .

• Suppose \dim and $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following three sets are equal to each other.

(a) $\text{span}(\varphi_1, \dots, \varphi_m)$

(b) $((\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m))^0$

(c) $\{\varphi \in V' : (\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m) \subseteq \text{null}\varphi\}$

SOLUTION: By Problem (17), (b) and (c) are equivalent. By Problem (26) and the corollary in Problem (23),

$$\left. \begin{array}{l} ((\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m))^0 = (\text{null}\varphi_1)^0 + \dots + (\text{null}\varphi_m)^0. \\ \text{又 } \text{span}(\varphi_i) = \{v \in V : \forall \psi \in \text{span}(\varphi_i), \psi(v) = 0\}^0 = (\text{null}\varphi_i)^0. \end{array} \right\} \Rightarrow (a) = (b). \quad \square$$

30 OR COROLLARY:

Suppose V is finite-dim and $\varphi_1, \dots, \varphi_m$ is a linearly independent list in V' .

Then $\dim((\text{null}\varphi_1) \cap \dots \cap (\text{null}\varphi_m)) = (\dim V) - m$.

6 Define $\Gamma : V' \rightarrow \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$, where $v_1, \dots, v_m \in V$.

(a) Show that $\text{span}(v_1, \dots, v_m) = V \iff \Gamma$ is injective.

(b) Show that (v_1, \dots, v_m) is linearly independent $\iff \Gamma$ is surjective.

SOLUTION:

$$\left\{ \begin{array}{l} \text{(a) } \left\{ \begin{array}{l} \text{Suppose } \Gamma \text{ is injective. Then let } \Gamma(\varphi) = 0, \text{ getting } \varphi = 0 \Leftrightarrow \text{null}\varphi = V = \text{span}(v_1, \dots, v_m). \\ \text{Suppose } \text{span}(v_1, \dots, v_m) = V. \text{ Then let } \Gamma(\varphi) = 0, \text{ getting } \varphi(v_i) = 0 \text{ for each } i, \\ \text{null}\varphi = \text{span}(v_1, \dots, v_m) = V, \text{ thus } \varphi = 0, \Gamma \text{ is injective.} \end{array} \right. \\ \text{(b) } \left\{ \begin{array}{l} \text{Suppose } \Gamma \text{ is surjective. Then let } \Gamma(\varphi_i) = e_i \text{ for each } i, \text{ where } (e_1, \dots, e_m) \text{ is the standard basis of } \mathbf{F}^m. \\ \text{Then } (\varphi_1, \dots, \varphi_m) \text{ is linearly independent, suppose } a_1 v_1 + \dots + a_m v_m = 0, \\ \text{then for each } i, \text{ we have } \varphi_i(a_1 v_1 + \dots + a_m v_m) = a_i = 0. \text{ Thus } v_1, \dots, v_m \text{ is linearly independent.} \\ \text{Suppose } (v_1, \dots, v_m) \text{ is linearly independent. Let } (\varphi_1, \dots, \varphi_m) \text{ be the dual basis of } \text{span}(v_1, \dots, v_m). \\ \text{Thus for each } (a_1, \dots, a_m) \in \mathbf{F}^m, \text{ we have } \varphi = a_1 \varphi_1 + \dots + a_m \varphi_m \text{ so that } \Gamma(\varphi) = (a_1, \dots, a_m). \quad \square \end{array} \right. \end{array} \right.$$

• Define $\Gamma : V \rightarrow \mathbf{F}^m$ by $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$, where $\varphi_1, \dots, \varphi_m \in V'$.

(c) Show that $\text{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ is injective.

(d) Show that $(\varphi_1, \dots, \varphi_m)$ is linearly independent $\iff \Gamma$ is surjective.

SOLUTION:

- (c) $\left\{ \begin{array}{l} \text{Suppose } \Gamma \text{ is injective. Then } \Gamma(v) = 0 \Leftrightarrow \forall i, \varphi_i(v) = 0 \Leftrightarrow v \in (\text{null } \varphi_1) \cap \cdots \cap (\text{null } \varphi_m) \Leftrightarrow v = 0. \\ \text{Getting } (\text{null } \varphi_1) \cap \cdots \cap (\text{null } \varphi_m) = \{0\}. \text{ By Problem } (\bullet) \text{ above, } \text{span}(\varphi_1, \dots, \varphi_m) = V' \\ \text{Suppose } \text{span}(\varphi_1, \dots, \varphi_m) = V'. \text{ Again by Problem } (\bullet), (\text{null } \varphi_1) \cap \cdots \cap (\text{null } \varphi_m) = \{0\}. \\ \text{Thus } \Gamma(v) = 0 \Rightarrow \forall i, \varphi_i(v) = 0 \Rightarrow v = 0. \end{array} \right.$
- (d) $\left\{ \begin{array}{l} \text{Suppose } (\varphi_1, \dots, \varphi_m) \text{ is linearly independent. Then by Problem (31), } (v_1, \dots, v_m) \text{ is linearly independent.} \\ \text{Thus for any } (a_1, \dots, a_m) \in \mathbf{F}, \text{ by letting } v = \sum_{i=1}^m a_i v_i, \text{ then } \varphi_i(v) = a_i \Rightarrow \Gamma(v) = (a_1, \dots, a_m). \\ \text{Suppose } \Gamma \text{ is surjective. Let } e_1, \dots, e_m \text{ be a basis of } \mathbf{F}^m. \\ \text{For every } e_i, \exists v_i \in V \text{ such that } \Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i, \\ \text{fix } v_i (\Rightarrow (v_1, \dots, v_m) \text{ is linearly independent}). \text{ Thus } \varphi_i(v_i) = 1, \varphi_i(v_j) = 0. \\ \text{Hence } (\varphi_1, \dots, \varphi_m) \text{ is the dual basis of the basis } v_1, \dots, v_m \text{ of } \text{span}(v_1, \dots, v_m). \quad \square \end{array} \right.$

33 Suppose $A \in \mathbf{F}^{m,n}$. Define $T : A \rightarrow A^t$. Prove that T is an isomorphism of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$

SOLUTION: By [3.111], T is linear. Note that $(A^t)^t = A$.

- (a) For any $B \in \mathbf{F}^{n,m}$, let $A = B^t$ so that $T(A) = B$. Thus T is surjective.
- (b) If $T(A) = 0$ for some $A \in \mathbf{F}^{m,n}$, then $A = 0$. Thus T is injective.
- for if not, $\exists j, k \in \mathbf{N}^+$ such that $A_{j,k} \neq 0$, then $T(A)_{k,j} \neq 0$, contradicts. $\Rightarrow \square$

32 Suppose $T \in \mathcal{L}(V)$, and (u_1, \dots, u_m) and (v_1, \dots, v_m) are bases of V . Prove that T is invertible \iff The rows of $\mathcal{M}(T, (u_1, \dots, u_m), (v_1, \dots, v_m))$ form a basis of $\mathbf{F}^{1,n}$.

SOLUTION: Note that T is invertible $\Rightarrow T'$ is invertible. And $\mathcal{M}(T') = \mathcal{M}(T)^t = A^t$, denote it by B .

Let $(\varphi_1, \dots, \varphi_m)$ be the dual basis of (v_1, \dots, v_m) , (ψ_1, \dots, ψ_m) be the dual basis of (u_1, \dots, u_m) .

- (a) Suppose T is invertible, so is T' . Because $T'(\varphi_1), \dots, T'(\varphi_m)$ is linearly independent.

Noticing that $T'(\varphi_i) = B_{1,i}\psi_1 + \cdots + B_{m,i}\psi_m$.

Thus the columns of B , namely the rows of A , are linearly independent (check it by contradiction).

- (b) Suppose the rows of A are linearly independent, so are the columns of B .

Then $(T'(\varphi_1), \dots, T'(\varphi_m))$ is a basis of range T' , namely V' . Thus T' is surjective.

Hence T' is invertible, so is T . \square

34 The double dual space of V , denoted by V'' , is defined to be the dual space of V' .

In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \rightarrow V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.

- (a) Show that Λ is a linear map from V to V'' .

- (b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

- (c) Show that if V is finite-dim, then Λ is an isomorphism from V onto V'' .

Suppose V is finite-dim. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V . In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

SOLUTION:

- (a) $\forall \varphi \in V', \forall v, w \in V, a \in \mathbf{F}, (\Lambda(v + aw))(\varphi) = \varphi(v + aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.

Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence Λ is linear.

- (b) $(T''(\Lambda v))(\varphi) = ((\Lambda v) \circ (T'))(\varphi) = (\Lambda v)(T'(\varphi)) = (T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = (\Lambda(Tv))(\varphi)$.

Hence $T''(\Lambda v) = (\Lambda(Tv))$, getting $T'' \circ \Lambda = \Lambda \circ T$.

- (c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is injective.

又 Because V is finite-dim. $\dim V = \dim V' = \dim V''$. Hence Λ is an isomorphism. \square

36 Suppose U is a subspace of V . Define $i : U \rightarrow V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.

- (a) Show that $\text{null } i' = U^0$: $\text{null } i' = (\text{range } i)^0 = U^0 \Leftarrow \text{range } i = U$. \square

(b) Prove that if V is finite-dim, then $\text{range } i' = U'$: $\text{range } i' = (\text{null } i)_U^0 = (\{0\})_U^0 = U'$. \square

(c) Prove that if V is finite-dim, then \tilde{i}' is an isomorphism from V'/U^0 onto U' :

Note that $\tilde{i}' : V'/\text{null } i' \rightarrow \text{range } i' \Rightarrow \tilde{i}' : V'/U^0 \rightarrow U'$. By (a), (b) and [3.91(d)]. \square

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

37 Suppose U is a subspace of V and π is the quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.

(a) Show that π' is injective: Because π is surjective. Use [3.108]. \square

(b) Show that $\text{range } \pi' = U^0$.

(c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0 .

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

In fact, there is no assumption here that any of these vector spaces are finite-dim.

SOLUTION: [3.109] is not available. Using (3.E.18), also see (3.E.20).

(b) $\psi \in \text{range } \pi' \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \text{null } \psi \supseteq U \iff \psi \in U^0$. Hence $\text{range } \pi' = U^0$.

(c) $\psi \in U^0 \iff \text{null } \psi \supseteq U \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi = \pi(\varphi)$. Thus π' is surjective. And by (a). \square

ENDED

4

• **NOTE FOR [4.8]:** division algorithm for polynomials

Suppose $p, s \in \mathcal{P}(\mathbf{F})$, with $s \neq 0$. Then $\exists! q, r \in \mathcal{P}(\mathbf{F})$ such that $p = sq + r$ and $\deg r < \deg s$. *Another Proof:*

Suppose $\deg p \geq \deg s$. Then $\underbrace{(1, z, \dots, z^{\deg s-1})}_{\text{of length } \deg s}, \underbrace{s, zs, \dots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)}$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.

Because $q \in \mathcal{P}(\mathbf{F})$, $\exists! a_i, b_j \in \mathbf{F}$,

$$\begin{aligned} q &= a_0 + a_1 z + \dots + a_{\deg s-1} z^{\deg s-1} + b_0 s + b_1 z s + \dots + b_{\deg p - \deg s} z^{\deg p - \deg s} s \\ &= \underbrace{a_0 + a_1 z + \dots + a_{\deg s-1} z^{\deg s-1}}_r + s \underbrace{(b_0 + b_1 z + \dots + b_{\deg p - \deg s} z^{\deg p - \deg s})}_q. \end{aligned}$$

With r, q as defined uniquely above, we are done. \square

• **NOTE FOR [4.11]:** each zero of a polynomial corresponds to a degree-one factor; *Another Proof:*

First suppose $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \dots + a_m z^m$, $\exists! a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.

Then $p(z) = p(z) - p(\lambda) = a_1(z - \lambda) + \dots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.

Hence $\forall k \in \{1, \dots, m\}$, $z^k - \lambda^k = (z - \lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \dots + z^{k-(j+1)}\lambda^j + \dots + z\lambda^{k-2} + z^0\lambda^{k-1})$.

Thus $p(z) = \sum_{j=1}^m a_j(z - \lambda) \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z - \lambda) \sum_{j=1}^m a_j \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z - \lambda)q(z)$.

• **NOTE FOR [4.13]:** fundamental theorem of algebra, first version

Every nonconstant polynomial with complex coefficients has a zero in \mathbf{C} . *Another Proof:*

De Moivre's theorem (which you can prove using induction on k and the addition formulas for cosine and sine), states that if $k \in \mathbf{N}^+$, $\theta \in \mathbf{R}$, then $(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$.

Suppose $w \in \mathbf{C}$, $k \in \mathbf{N}^+$ and using polar coordinates. $\exists r \geq 0, \theta \in \mathbf{R}$ such that $r(\cos \theta + i \sin \theta) = w$.

Hence $(r^{1/k}(\cos \frac{\theta}{k} + i \sin \frac{\theta}{k}))^k = w$. Thus every complex number has a k^{th} root, a fact that we will soon use.

Suppose a nonconstant $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.

Then $|p(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$ (because $\frac{|p(z)|}{|z_m|} \rightarrow |c_m|$ as $|z| \rightarrow \infty$).

Thus the continuous function $z \rightarrow |p(z)|$ has a global minimum at some point $\zeta \in \mathbf{C}$.

To show that $p(\zeta) = 0$, suppose that $p(\zeta) \neq 0$.

Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \frac{p(z + \zeta)}{p(\zeta)}$.

The function $z \rightarrow |q(z)|$ has a global minimum value of 1 at $z = 0$.

Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where k is the smallest positive integer such that $a_k \neq 0$.

Let $\beta \in \mathbf{C}$ be such that $\beta^k = -\frac{1}{a_k}$.

There is a constant $c > 1$ such that if $t \in (0, 1)$,

then $|q(t\beta)| \leq |1 + a_k t^k \beta^k| + t^{k+1} c = 1 - t^k(1 - tc)$.

Thus taking t to be $1/(2c)$ in the inequality above, we have $|q(t\beta)| < 1$,

which contradicts the assumption that the global minimum of $z \rightarrow |q(z)|$ is 1.

Hence $p(\zeta) = 0$, as desired. \square

• *Prove that if $w, z \in \mathbf{C}$, then $||w| - |z|| \leq |w - z|$. The inequality here is called the **reverse triangle inequality**.*

SOLUTION:

$$\begin{aligned} |w - z|^2 &= (w - z)(\overline{w} - \overline{z}) \\ &= |w|^2 + |z|^2 - (w\overline{z} + \overline{w}z) \\ &= |w|^2 + |z|^2 - (\overline{\overline{w}z} + \overline{w}z) \\ &= |w|^2 + |z|^2 - 2\operatorname{Re}(\overline{w}z) \\ &\geq |w|^2 + |z|^2 - 2|\overline{w}z| \\ &= |w|^2 + |z|^2 - 2|w||z| = ||w| - |z||^2. \end{aligned} \quad \square$$

Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides.

• *Suppose V is a complex vector space and $\varphi \in V'$.*

Define $\sigma : V \rightarrow \mathbf{R}$ by $\sigma(v) = \Re\varphi(v)$ for each $v \in V$.

Show that $\varphi(v) = \sigma(v) - i\sigma(iv)$ for all $v \in V$.

SOLUTION:

Notice that $\varphi(v) = \Re\varphi(v) + i\Im\varphi(v) = \sigma(v) + i\Im\varphi(v)$. 又 $\Re\varphi(iv) = \Re[i\varphi(v)] = -\Im\varphi(v) = \sigma(iv)$.

Hence $\varphi(v) = \sigma(v) - i\sigma(iv)$. \square

2 *Suppose m is a positive integer. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subspace of $\mathcal{P}(\mathbf{F})$?*

SOLUTION:

$x^m, x^m + x^{m-1} \in U$ but $\deg[(x^m + x^{m-1}) - (x^m)] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$.

Hence U is not closed under addition, and therefore is not a subspace. \square

3 *Suppose m is a positive integer. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p \text{ is even}\}$ a subspace of $\mathcal{P}(\mathbf{F})$?*

SOLUTION:

$x^2, x^2 + x \in U$ but $\deg[(x^2 + x) - (x^2)]$ is odd and hence $(x^2 + x) - (x^2) \notin U$.

Thus U is not closed under addition, and therefore is not a subspace. \square

4 Suppose that m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \dots, \lambda_m \in \mathbf{F}$.

Prove that $\exists p \in \mathcal{P}(\mathbf{F})$ such that $\deg p = n$, the zeros of p are $\lambda_1, \dots, \lambda_m$.

SOLUTION: Let $p(z) = (z - \lambda_1)^{n-(m-1)}(z - \lambda_2) \cdots (z - \lambda_m)$. \square

5 Suppose that $m \in \mathbf{N}$, z_1, \dots, z_{m+1} are distinct elements of \mathbf{F} , and $w_1, \dots, w_{m+1} \in \mathbf{F}$.

Prove that $\exists! p \in \mathcal{P}_m(\mathbf{F})$ such that $p(z_k) = w_k$ for each $k = 1, \dots, m+1$.

This result can be proved without using linear algebra. However, try to find the clearer, shorter proof that uses some linear algebra.

SOLUTION:

Define $T : \mathcal{P}_m(\mathbf{F}) \rightarrow \mathbf{F}^{m+1}$ by $Tq = (q(z_1), \dots, q(z_m), q(z_{m+1}))$. As can be easily checked, T is linear.

We need to show that T is surjective, so that such p exists; and that T is injective, so that such p is unique.

$$Tq = 0 \iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0$$

$$\iff q \in \mathcal{P}_m(\mathbf{F}) \text{ is the zero polynomial, for if not,}$$

q has at least $m+1$ distinct roots, while $\deg q = m$. Contradicts (by [4.12]). Hence T is injective.

$$\dim \text{range } T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \text{null } T = m+1 = \dim \mathbf{F}^{m+1}. \text{ } \nexists \text{ range } T \subseteq \mathbf{F}^{m+1}. \text{ Hence } T \text{ is surjective. } \square$$

6 Suppose $p \in \mathcal{P}_m(\mathbf{C})$ has degree m . Prove that

p has m distinct zeros $\iff p$ and its derivative p' have no zeros in common.

SOLUTION:

(a) Suppose p has m distinct zeros. By [4.14] and $\deg p = m$, let $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$, $\exists! c, \lambda_i \in \mathbf{C}$.

For each $j \in \{1, \dots, m\}$, let $\frac{p(z)}{(z - \lambda_j)} = q_j \in \mathcal{P}_{m-1}(\mathbf{C})$, then $p(z) = (z - \lambda_j)q_j(z)$ and $q_j(\lambda_j) \neq 0$.

$$p'(z) = (z - \lambda_j)q'_j(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0, \text{ as desired.}$$

(b) To prove the implication on the other direction, we prove the contrapositive:

Suppose p has less than m distinct roots.

We must show that p and its derivative p' have at least one zero in common.

Let λ be a zero of p , then write $p(z) = (z - \lambda)^n q(z)$, $\exists! n \in \mathbf{N}^+, q \in \mathcal{P}_{m-n}(\mathbf{C})$.

$$p'(z) = (z - \lambda)^n q'(z) + n(z - \lambda)^{n-1} q(z) \Rightarrow p'(\lambda) = 0, \lambda \text{ is a common root of } p' \text{ and } p. \quad \square$$

7 Prove that every polynomial of odd degree with real coefficients has a real zero.

SOLUTION:

Using the notation proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence λ_1 exists. \square

OR. Using calculus but not using [4.17].

Suppose $p \in \mathcal{P}_m(\mathbf{F})$, $\deg p = m$, m is odd.

Let $p(x) = a_0 + a_1x + \cdots + a_mx^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1}a_m$ by δ

$$\text{Write } p(x) = x^m \left(\frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \cdots + \frac{a_{m-1}}{x} + a_m \right).$$

Thus $p(x)$ is continuous, and $\lim_{x \rightarrow -\infty} p(x) = -\delta\infty$; $\lim_{x \rightarrow \infty} p(x) = \delta\infty$.

Hence we conclude that p has at least one real zero. \square

8 For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \rightarrow \mathbf{R}$ by $(Tp)(x) = \begin{cases} \frac{p(x) - p(3)}{x - 3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3 \end{cases} \text{ for all } x \in \mathbf{R}.$

Show that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$ and that $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$ is a linear map.

SOLUTION:

$$\text{For } x \neq 3, T(x^n) = \frac{x^n - 3^n}{x - 3} = \sum_{i=1}^n 3^{i-1} x^{n-i}.$$

$$\text{For } x = 3, T(x^n) = 3^{n-1} \cdot n. \text{ Note that if } x = 3, \text{ then } \sum_{i=1}^n 3^{i-1} x^{n-i} = \sum_{i=1}^n 3^{n-1} = 3^{n-1} \cdot n.$$

$$\text{Hence for all } x \in \mathbf{R} \text{ and for all } n \in \mathbf{N}, T(x^n) = \sum_{i=1}^n 3^{i-1} x^{n-i} \in \mathcal{P}(\mathbf{R}).$$

Because T is linear, we conclude that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$.

Now we show that T is linear:

$$\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, T(p + \lambda q)(x) = \begin{cases} \frac{(p + \lambda q)(x) - (p + \lambda q)(3)}{x - 3} & \text{if } x \neq 3, \\ (p + \lambda q)'(3) & \text{if } x = 3 \end{cases} \quad \text{for all } x \in \mathbf{R}.$$

$$\text{Notice that } (p + \lambda q)(x) - (p + \lambda q)(3) = (p(x) - p(3)) + (\lambda q(x) - \lambda q(3));$$

$$(p + \lambda q)'(3) = p'(3) + \lambda q'(3).$$

Thus $T(p + \lambda q)(x) = (T(p) + \lambda T(q))(x)$ for all $x \in \mathbf{R}$. \square

9 Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \rightarrow \mathbf{C}$ by $q(z) = p(z)\overline{p(\bar{z})}$.

Prove that q is a polynomial with real coefficients.

SOLUTION:

$$p(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow p(\bar{z}) = \overline{a_n z^n + \cdots + a_1 z + a_0} \Rightarrow \overline{p(\bar{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}.$$

$$\text{Note that } q(z) = p(z)\overline{p(\bar{z})} = \overline{p(\bar{z})}p(z) = \overline{p(\bar{z})p(z)} = \overline{q(\bar{z})}.$$

$$\text{Hence letting } q(z) = c_m z^m + \cdots + c_1 z + c_0 \Rightarrow \overline{c_k} = c_k, c_k \in \mathbf{R} \text{ for each } k. \quad \square$$

10 Suppose $m \in \mathbf{N}$ and $p \in \mathcal{P}_m(\mathbf{C})$ is such that

there are $(m + 1)$ distinct real numbers x_0, x_1, \dots, x_m with $p(x_k) \in \mathbf{R}$ for each x_k .

Prove that all coefficients of p are real.

SOLUTION: Let $p(x_k) = y_k$ for each k . By Problem (5), $\exists! q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_k) = y_k$. Hence $p = q$. \square

OR. Using the Lagrange Interpolating Polynomial.

$$\text{Define } q(x) = \sum_{j=0}^m \frac{(x - x_0)(x - x_1) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_m)}{(x_j - x_0)(x_j - x_1) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_m)} p(x_j).$$

$$\text{又 For each } j, x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R}) \subseteq \mathcal{P}_m(\mathbf{C}).$$

$$\text{Notice that } q(x_k) = 1 \cdot p(x_k) \Rightarrow (q - p)(x_k) = 0 \text{ for each } k \in \{0, 1, \dots, m\}.$$

$$\text{Then } (q - p) \text{ has } (m + 1) \text{ distinct zeros, while } (q - p) \in \mathcal{P}_m(\mathbf{C}). \text{ Hence by [4.12], } q - p = 0 \Rightarrow p = q. \quad \square$$

11 Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.

(a) Show that $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) Find a basis of $\mathcal{P}(\mathbf{F})/U$.

SOLUTION:

$$U \text{ is a subspace of } \mathcal{P}(\mathbf{F}) \text{ because } \forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, pf + \lambda pg = p(f + \lambda g) \in U.$$

$$\text{NOTE: Define } P : \mathcal{P}(\mathbf{F}) \rightarrow \mathcal{P}(\mathbf{F}) \text{ by } (Pq)(x) = p(q(x)) = (p \circ q)(x) (\neq p(x)q(x)). P \text{ is not linear.}$$

$$\text{(a) By [4.8], } \forall f \in \mathcal{P}(\mathbf{F}), \exists! q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r); \deg r < \deg p.$$

$$\text{Hence } \forall f \in \mathcal{P}(\mathbf{F}), \exists! pq \in U, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), f = (pq) + (r); r \notin U.$$

$$\text{Thus } \mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p - 1}(\mathbf{F}). \text{ Therefore } \mathcal{P}(\mathbf{F})/U \text{ and } \mathcal{P}_{\deg p - 1}(\mathbf{F}) \text{ are isomorphic.}$$

$$\text{OR. } \forall f \in \mathcal{P}(\mathbf{F}), \exists! q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r); \deg r < \deg p.$$

Define $R : \mathcal{P}(\mathbf{F}) \rightarrow \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $(Rf)(z) = r(z)$ for each $z \in \mathbf{F}$.

$\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, R(f + \lambda g)(z) = R(f) + \lambda R(g)$.

BECAUSE: $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}$,

$$\exists! q_1, r_1 \in \mathcal{P}(\mathbf{F}), f = (p)q_1 + (r_1), \deg r_1 < \deg p;$$

$$\exists! q_2, r_2 \in \mathcal{P}(\mathbf{F}), g = (p)q_2 + (r_2), \deg r_2 < \deg p;$$

$$\exists! q_3, r_3 \in \mathcal{P}(\mathbf{F}), \lambda g = (p)q_3 + (r_3) = (p)(\lambda q_2) + (\lambda r_2), \deg r_3 < \deg p \text{ and } \deg \lambda r_2 < \deg p.$$

$$\Rightarrow q_3 = \lambda q_2, r_3 = \lambda r_2.$$

$$\exists! q_0, r_0 \in \mathcal{P}(\mathbf{F}), (f + \lambda g) = (p)q_0 + (r_0)$$

$$= (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2), \deg r_0 < \deg p \text{ and } \deg(r_1 + \lambda r_2) < \deg p.$$

$$\Rightarrow q_1 + \lambda q_2 = q_0; r_1 + \lambda r_2 = r_0.$$

Hence R is linear.

$R(f) = 0 \iff f = pq, \exists! q \in \mathcal{P}(\mathbf{F})$. Thus $\text{null } R = U$.

$\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F})$, let $f = p + r$, then $R(f) = r$. Thus $\text{range } R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Finally, by [3.91(d)], $\mathcal{P}(\mathbf{F})/\text{null } R$, namely $\mathcal{P}(\mathbf{F})/U$, and $\text{range } R$, namely $\mathcal{P}_{\deg p-1}(\mathbf{F})$, are isomorphic.

(b) $(1 + U, x + U, \dots, x^{\deg p-1} + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. \square

• Suppose nonconstant $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p, n = \deg q$.

Use (a)—(c) below to prove that $\exists! r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.

(a) Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \rightarrow \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r, s) = rp + sq$. Show that the linear map T is injective.

(b) Show that the linear map T in (a) is surjective.

(c) Use (b) to conclude that $\exists! r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.

SOLUTION:

(a) T is linear because $\forall r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,

$$T((r_1, s_1) + \lambda(r_2, s_2)) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1, s_1) + \lambda T(r_2, s_2).$$

Suppose $T(r, s) = rp + sq = 0$. Notice that p, q have no zeros in common.

Then $r = s = 0$, for if not, write $\frac{q(z)}{r(z)} = \frac{p(z)}{s(z)}$, while for any zero λ of q , $\frac{q(\lambda)}{r(\lambda)} = 0 \neq \frac{p(\lambda)}{s(\lambda)}$. Hence \square

(b) $\dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) = \dim \mathcal{P}_{n-1}(\mathbf{C}) + \dim \mathcal{P}_{m-1}(\mathbf{C}) = n + m = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.

又 T is injective. Hence $\dim \text{range } T = \dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) - \dim \text{null } T = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.

Thus $\text{range } T = \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$ is surjective, and therefore is an isomorphism. \square

(c) Immediately. \square

ENDED

5.A

[1]: 31; [2]: 1, 2, 3, 15, 21; [3]: 23, (2E Ch5.20), (4E.5.A.37), 4, 5; [4]: 6, (4E.5.A.17, 18) OR 16, (4E.5.A.15); [5]: 7, 8, (4E.5.A.8), 22, 9, 10; [6]: 11, 12, 14, 30, 13, (4E.5.A.11); [7]: 17, (4E.5.A.16), 18; [8]: 19, 20, 24; [9]: 24', 25, 26, 27, 28; [10]: (4E.5.A.39), 29; [11]: 32, (4E.5.A.35), (4E.5.A.38) OR 35, 36; [12] 32, 34.

• NOTE FOR [5.6]:

More generally, suppose we do not know whether V is finite-dim. Then $(a) \iff (b)$.

Suppose (a) λ is an eigenvalue of T with an eigenvector v . Then $(T - \lambda I)v = 0$.

Hence we get (b), $(T - \lambda I)$ is not injective. And then (d), $(T - \lambda I)$ is not invertible.

But $(d) \not\Rightarrow (b)$ (because S is not invertible $\iff S$ is not injective or S is not surjective).

• **NOTE FOR [5.10]:** linearly independent eigenvectors Suppose $T \in \mathcal{L}(V)$. Prove that every list of eigenvectors of T corresponding to distinct eigenvalues is linearly independent.

SOLUTION: OR Another Proof.

Suppose the desired result is false. Then ($m \neq 1$ because eigenvectors are nonzero)

\exists smallest $N^+ \ni m > 1$ such that $\exists (v_1, \dots, v_m)$ of eigenvectors of T linearly dependent corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_m$ of T .

Suppose $a_1 v_1 + \dots + a_{m-1} v_{m-1} + a_m v_m = 0$. Then each a_j is zero, for if not, contradicts the minimality of m .

Apply $T - \lambda_m I$ to both sides, getting $a_1(\lambda_1 - \lambda_m)v_1 + \dots + a_{m-1}(\lambda_{m-1} - \lambda_m)v_{m-1} = 0$.

Because the eigenvalues $\lambda_1, \dots, \lambda_m$ are distinct, and $a_j \neq 0$ for all a_j .

Thus (v_1, \dots, v_{m-1}) of length $(m - 1)$ is linearly dependent corresponding to distinct eigenvalues.

Contradicts the minimality of m . \square

31 Suppose V is finite-dim and $v_1, \dots, v_m \in V$. Prove that (v_1, \dots, v_m) is linearly independent $\iff \exists T \in \mathcal{L}(V)$, v_1, \dots, v_m are eigenvectors of T corresponding to distinct eigenvalues.

SOLUTION: Suppose (v_1, \dots, v_m) is linearly independent, extend it to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.

Define $T \in \mathcal{L}(V)$ by $Tv_k = kv_k$ for each $k \in \{1, \dots, m, \dots, n\}$. Conversely by [5.10]. \square

1 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V .

(a) Prove that if $U \subseteq \text{null } T$, then U is invariant under T . $\forall u \in U \subseteq \text{null } T, Tu = 0 \in U$. \square

(b) Prove that if $\text{range } T \subseteq U$, then U is invariant under T . $\forall u \in U, Tu \in \text{range } T \subseteq U$. \square

• Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.

(a) Prove that $\text{null } (T - \lambda I)$ is invariant under S , where λ is chosen arbitrarily.

(b) Prove that $\text{range } (T - \lambda I)$ is invariant under S , where λ is chosen arbitrarily.

SOLUTION:

Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.

(a) Suppose $v \in \text{null } (T - \lambda I)$, then $(T - \lambda I)(Sv) = S(T - \lambda I)v = S(0) = 0$.

Hence $Sv \in \text{null } (T - \lambda I)$ and therefore $\text{null } (T - \lambda I)$ is invariant under S .

(b) Suppose $v \in \text{range } (T - \lambda I)$, therefore $\exists u \in V, (T - \lambda I)u = v$.

Then $Sv = S(T - \lambda I)u = (T - \lambda I)(Su) \in \text{range } (T - \lambda I)$.

Hence $Sv \in \text{range } (T - \lambda I)$ and therefore $\text{range } (T - \lambda I)$ is invariant under S . \square

COMMENT: Reversing the roles of S and T , letting $\lambda = 0$, we can conclude that

$\text{null } S$ and $\text{range } S$ is invariant under T , which is what we will prove in Problem (2) and (3) below.

• Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.

2 Show that $W = \text{null } S$ is invariant under T . $\forall u \in W, Su = 0 \Rightarrow TSu = 0 = STu \Rightarrow Tu \in W$. \square

3 Show that $U = \text{range } S$ is invariant under T . $\forall w \in U, \exists v \in V, Sv = w, STv = TSv = Tw \in U$. \square

15 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector v .

Then $S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v$.

Thus λ is also an eigenvalue of $S^{-1}TS$ with an eigenvector $S^{-1}v$.

Suppose λ is an eigenvalue of $S^{-1}TS$ with an eigenvector v .

Then $S(S^{-1}TS)v = TSv = \lambda Sv$.

Thus λ is also an eigenvalue of T with an eigenvector Sv . \square

OR. Note that $S(S^{-1}TS)S^{-1} = T$. Hence every eigenvalue of $S^{-1}TS$ is an eigenvalue of $S(S^{-1}TS)S^{-1} = T$.
And every eigenvector v of $S^{-1}TS$ is $S^{-1}v$, every eigenvector u of T is Su . \square

21 Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$.

Prove that λ is an eigenvalue of $T \iff \frac{1}{\lambda}$ is an eigenvalue of T^{-1} .

(b) Prove that T and T^{-1} have the same eigenvectors.

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector v .

Then $T^{-1}Tv = \lambda T^{-1}v = v \Rightarrow T^{-1}v = \frac{1}{\lambda}v$. Hence $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} .

(b) Suppose $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} with an eigenvector v .

Then $TT^{-1}v = v = \frac{1}{\lambda}Tv \Rightarrow Tv = \lambda v$. Hence λ is an eigenvalue of T .

OR. Note that $(T^{-1})^{-1} = T$ and $\frac{1}{\frac{1}{\lambda}} = \lambda$. \square

23 Suppose V is finite-dim, $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

SOLUTION:

Suppose λ is an eigenvalue of ST with an eigenvector v . Then $T(STv) = \lambda Tv = TS(Tv)$.

If $Tv \neq 0$, then λ is an eigenvalue of TS .

Otherwise, $\lambda = 0$, ($v \neq 0, \lambda v = 0 = STv$), then T is not invertible

$\Rightarrow TS$ is not invertible $\Rightarrow (TS - 0I)$ is not invertible $\Rightarrow \lambda = 0$ is an eigenvalue of TS .

Reversing the roles of T and S , we conclude that ST and TS have the same eigenvalues. \square

• (2E Ch5.20)

Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and $S \in \mathcal{L}(V)$ has the same eigenvectors (but might not with the same eigenvalues). Prove that $ST = TS$.

SOLUTION:

Let $n = \dim V$. For each $j \in \{1, \dots, n\}$, let v_j be an eigenvector with eigenvalue λ_j of T and α_j of S .

Then (v_1, \dots, v_n) is a basis of V . Because $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$ for each j . Hence $ST = TS$. \square

• Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(S) = TS$ for each $S \in \mathcal{L}(V)$.

Prove that the set of eigenvalues of T equals the set of eigenvalues of \mathcal{A} .

SOLUTION:

(a) Suppose v_1, \dots, v_m are all linearly independent eigenvectors of T

with corresponding eigenvalues $\lambda_1, \dots, \lambda_m$ respectively (possibly with repetitions).

Extend to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.

Then for each $k \in \{1, \dots, m\}$, $\text{span}(v_k) \subseteq \text{null}(T - \lambda_k I)$.

Define $S_k \in \mathcal{L}(V)$ by $S_k(v_j) = v_k$ for each $j \in \{1, \dots, n\}$,

so that $\text{range } S_k = \text{span}(v_k)$ for each $k \in \{1, \dots, m\}$, then $\mathcal{A}(S_k) = TS_k = \lambda_k S_k$.

Thus the eigenvalues of T are eigenvalues of \mathcal{A} .

(b) Suppose $\lambda_1, \dots, \lambda_m$ are all eigenvalues of \mathcal{A} with eigenvectors S_1, \dots, S_m respectively.

Then for each $k \in \{1, \dots, m\}$, $\exists v \in V, 0 \neq u = S_k(v) \in V \Rightarrow Tu = (TS_k)v = (\lambda_k S_k)v = \lambda_k u$.

Thus the eigenvalues of \mathcal{A} are eigenvalues of T . \square

OR.

(a) Suppose λ is an eigenvalue of T with an eigenvector v .

Let $v_1 = v$ and extend to a basis (v_1, \dots, v_m) of V .

Define $S \in \mathcal{L}(V)$ by $Sv_1 = v_1$, $Sv_k = 0$ for $k \geq 2$.

Then $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$.

Hence $(T - \lambda I)S = 0 \Rightarrow TS = \lambda S$ while $S \neq 0$. Thus λ is also an eigenvalue of \mathcal{A} .

(b) Suppose λ is an eigenvalue of \mathcal{A} with an eigenvector S . Then $(T - \lambda I)S = 0$ while $S \neq 0$.

Hence $(T - \lambda I)$ is not injective. Thus λ is also an eigenvalue of T . \square

• COMMENT: Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(S) = ST$, $\forall S \in \mathcal{L}(V)$. Then the eigenvalues of \mathcal{B} are not the eigenvalues of T .

4 Suppose $T \in \mathcal{L}(V)$ and V_1, \dots, V_m are subspaces of V invariant under T .

Prove that $V_1 + \dots + V_m$ is invariant under T .

SOLUTION:

For each $i = 1, \dots, m$, $\forall v_i \in V_i$, $Tv_i \in V_i$

Hence $\forall v = v_1 + \dots + v_m \in V_1 + \dots + V_m$, $Tv = Tv_1 + \dots + Tv_m \in V_1 + \dots + V_m$. \square

5 Suppose $T \in \mathcal{L}(V)$. *Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T .*

SOLUTION:

Suppose $\{V_\alpha\}_{\alpha \in \Gamma}$ is a collection of subspaces of V invariant under T ; here Γ is an arbitrary index set.

We need to prove that $\bigcap_{\alpha \in \Gamma} V_\alpha$, which equals the set of vectors

that are in V_α for each $\alpha \in \Gamma$, is invariant under T .

For each $\alpha \in \Gamma$, $\forall v_\alpha \in V_\alpha$, $Tv_\alpha \in V_\alpha$.

Hence $\forall v \in \bigcap_{\alpha \in \Gamma} V_\alpha$, $Tv \in V_\alpha$, $\forall \alpha \in \Gamma \Rightarrow Tv \in \bigcap_{\alpha \in \Gamma} V_\alpha$. Thus $\bigcap_{\alpha \in \Gamma} V_\alpha$ is invariant under T . \square

6 *Prove or give a counterexample:*

If V is finite-dim and U is a subspace of V that is invariant under every operator on V , then $U = \{0\}$ or $U = V$.

SOLUTION:

Notice that V might be $\{0\}$. In this case we are done. Suppose $\dim V \geq 1$. We prove by contrapositive:

Suppose $U \neq \{0\}$ and $U \neq V$, then $\exists T \in \mathcal{L}(V)$ such that U is not invariant under T .

Let W be such that $V = U \oplus W$.

Let (u_1, \dots, u_m) be a basis of U and (w_1, \dots, w_n) be a basis of W .

Hence $(u_1, \dots, u_m, w_1, \dots, w_n)$ is a basis of V .

Define $T \in \mathcal{L}(V)$ by $T(a_1u_1 + \dots + a_mu_m + b_1w_1 + \dots + b_nw_n) = b_1w_1 + \dots + b_nw_n$. \square

• Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.

(a) (OR (9.11)) $\lambda \in \mathbf{R}$. *Prove that λ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of $T_{\mathbb{C}}$.*

(b) (OR Problem (16)) $\lambda \in \mathbf{C}$. *Prove that λ is an eigenvalue of $T_{\mathbb{C}} \iff \bar{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$.*

SOLUTION:

(a) Suppose $v \in V$ is an eigenvector corresponding to the eigenvalue λ .

Then $Tv = \lambda v \Rightarrow T_{\mathbb{C}}(v + i0) = Tv + iT0 = \lambda v$.

Thus λ is an eigenvalue of T .

Suppose $v + iu \in V_{\mathbb{C}}$ is an eigenvector corresponding to the eigenvalue λ .

Then $T_{\mathbb{C}}(v + iu) = \lambda v + i\lambda u \Rightarrow Tv = \lambda v, Tu = \lambda u$. (Note that v or u might be zero).

Thus λ is an eigenvalue of $T_{\mathbb{C}}$.

(b) Suppose λ is an eigenvalue of $T_{\mathbb{C}}$ with an eigenvector $v + iu$.

Let (v_1, \dots, v_n) be a basis of V . Write $v = \sum_{i=1}^n a_i v_i, u = \sum_{i=1}^n b_i v_i$, where $a_i, b_i \in \mathbf{R}$.

Then $T_{\mathbb{C}}(v + iu) = Tv + iTu = \lambda v + i\lambda u = \lambda \sum_{i=1}^n (a_i + ib_i)v_i$. Conjugating two sides, we have:

$$\overline{T_{\mathbb{C}}(v + iu)} = \overline{Tv + iTu} = \overline{Tv} - i\overline{Tu} = Tv - iTu = T_{\mathbb{C}}(\overline{v + iu}) = \lambda \sum_{i=1}^n (a_i + ib_i)v_i = \overline{\lambda} \sum_{i=1}^n (a_i - ib_i)v_i.$$

Hence $\overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$. To prove the other direction, notice that $\overline{\overline{\lambda}} = \lambda$. \square

• Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.

Show that λ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector v .

Then $(T - \lambda I_V)$ is not invertible. $\times V$ is finite-dim.

Thus by [3.108, 110], [3.101] and Problem (12) in (3.F), $(T - \lambda I_V)' = T' - \lambda I_{V'}$ is not invertible.

Hence λ is an eigenvalue of T' .

(b) Suppose λ is an eigenvalue T' with an eigenvector ψ . Then $T'(\psi) = \psi \circ T = \lambda\psi$.

$\times \psi \neq 0 \Rightarrow \exists v \in V$ such that $\psi(v) \neq 0$. Note that $\psi(Tv) = \lambda\psi(v)$.

Thus $\lambda = \frac{\psi(Tv)}{\psi(v)} \Rightarrow Tv = \frac{\psi(Tv)}{\psi(v)}v = \lambda v$. Hence λ is an eigenvalue of T . \square

7 Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T .

SOLUTION:

Suppose $\lambda \in \mathbf{R}$ and $(x, y) \in \mathbf{R}^2 \setminus \{0\}$ such that $T(x, y) = (-3y, x) = \lambda(x, y)$. Then $-3y = \lambda x$ and $x = \lambda y$.

Thus $-3y = \lambda^2 y \Rightarrow \lambda^2 = -3$, ignoring the possibility of $y = 0$ (because if $y = 0$, then $x = 0$).

Hence the set of solution for this equation is \emptyset , and therefore T has no eigenvalues in \mathbf{R} . \square

8 Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T .

SOLUTION:

Suppose $\lambda \in \mathbf{F}$ and $(w, z) \in \mathbf{F}^2$ such that $T(w, z) = (z, w) = \lambda(w, z)$. Then $z = \lambda w$ and $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, ignoring the possibility of $z = 0$ ($z = 0 \Rightarrow w = 0$).

Hence $\lambda_1 = -1$ and $\lambda_2 = 1$ are all eigenvalues of T . For $\lambda_1 = -1, z = -w, w = -z$; For $\lambda_2 = 1, z = w$.

Thus the set of all eigenvectors is $\{(z, -z), (z, z) : z \in \mathbf{F} \wedge z \neq 0\}$. \square

• Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$.

Prove that if λ is an eigenvalue of P , then $\lambda = 0$ or $\lambda = 1$.

SOLUTION: (See also at (3.B), just below Problem (25), where (5.B.4) is answered.)

Suppose λ is an eigenvalue with an eigenvector v . Then $P(Pv) = Pv \Rightarrow \lambda^2 v = \lambda v$. Thus $\lambda = 1$ or 0 . \square

22 Suppose $T \in \mathcal{L}(V)$ and \exists nonzero vectors u, w in V such that $Tu = 3w$ and $Tw = 3u$.

Prove that 3 or -3 is an eigenvalue of T .

SOLUTION: COMMENT: $Tu = 3w, Tw = 3u \Rightarrow T(Tu) = 9u \Rightarrow T^2$ has an eigenvalue 9 .

$Tu = 3w, Tw = 3u \Rightarrow T(u + w) = 3(u + w), T(u - w) = 3(w - u) = -3(u - w)$. \square

9 Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$.

Find all eigenvalues and eigenvectors of T .

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector $(z_1, z_2, z_3) \in \mathbf{F}^3$.

Then $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$. Thus $2z_2 = \lambda z_1$, $0 = \lambda z_2$, $5z_3 = \lambda z_3$.

We discuss in two cases:

For $\lambda = 0$, $z_2 = z_3 = 0$ and z_1 can be arbitrary ($z_1 \neq 0$).

For $\lambda \neq 0$, $z_2 = 0 = z_1$, and z_3 can be arbitrary ($z_3 \neq 0$), then $\lambda = 5$.

The set of all eigenvectors is $\{(0, 0, z), (z, 0, 0) : z \in \mathbf{F} \wedge z \neq 0\}$. \square

10 Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$

(a) Find all eigenvalues and eigenvectors of T .

(b) Find all invariant subspaces of V under T .

SOLUTION:

(a) Suppose $v = (x_1, x_2, x_3, \dots, x_n)$ is an eigenvector of T with an eigenvalue λ .

Then $Tv = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$.

Hence $1, \dots, n$ are eigenvalues of T .

And $\{(0, \dots, 0, x_\lambda, 0, \dots, 0) \in \mathbf{F}^n : \lambda = 1, \dots, n, x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$ is the set of all eigenvectors of T .

(b) Let $V_\lambda = \{(0, \dots, 0, x_\lambda, 0, \dots, 0) \in \mathbf{F}^n : x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$. Then V_1, \dots, V_n are invariant under T .

Hence by Problem (4), every sum of V_1, \dots, V_n is a invariant subspace of V under T . \square

11 Define $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T .

SOLUTION:

Note that in general, $\deg p' < \deg p$ ($\deg 0 = -\infty$).

Suppose λ is an eigenvalue of T with an eigenvector p .

Suppose $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p \neq p'$. Contradicts. Thus $\lambda = 0$.

Therefore $\deg \lambda p = -\infty = \deg p \Rightarrow p$ is a nonzero constant polynomial.

Hence the set of all eigenvectors is $\{C : C \in \mathbf{R} \wedge C \neq 0\} = \mathcal{P}_0(\mathbf{R}) \setminus \{0\}$. \square

12 Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$.

Find all eigenvalues and eigenvectors of T .

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector p , then $(Tp)(x) = xp'(x) = \lambda p(x)$.

Let $p = a_0 + a_1x + \dots + a_nx^n$.

Then $xp'(x) = a_1x + 2a_2x^2 + \dots + na_nx^n = \lambda a_0 + \lambda a_1x + \lambda a_2x^2 + \dots + \lambda a_nx^n$.

Similar to Problem (10), $0, 1, \dots, n$ are eigenvalues of T .

The set of all eigenvectors of T is $\{cx^\lambda : \lambda = 0, 1, \dots, n, c \in \mathbf{F} \wedge c \neq 0\}$. \square

30 Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigenvalues of T .

Prove that $\exists x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

SOLUTION: Because 9 is not an eigenvalue. Hence $(T - 9I)$ is surjective. \square

14 Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V .

Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.

Find all eigenvalues and eigenvectors of P .

SOLUTION:

Suppose λ is an eigenvalue of P with an eigenvector $(u + w)$.

Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$. By [1.44] and $V = U \oplus W$, $(\lambda - 1)u = \lambda w = 0$.

Thus if $\lambda = 1$, then $w = 0$; if $\lambda = 0$, then $u = 0$.

Hence the eigenvalues of P are 0 and 1, the set of all eigenvectors in P is $U \cup W$. \square

13 Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.

Prove that $\exists \alpha \in \mathbf{F}$, $|\alpha - \lambda| < \frac{1}{1000}$ and $(T - \alpha I)$ is invertible.

SOLUTION:

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \frac{1}{1000 + k}$ for each $k = 1, \dots, \dim V + 1$.

Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigenvalues.

Hence $\exists k = 1, \dots, \dim V + 1$ such that α_k is not an eigenvalue of T and therefore $(T - \alpha_k I)$ is invertible. \square

• Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.

Prove that $\exists \delta > 0$ such that $(T - \alpha I)$ is invertible for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.

SOLUTION:

If T has no eigenvalues, then $(T - \alpha I)$ is injective for all $\alpha \in \mathbf{F}$ and we are done.

Let $\delta > 0$ be such that, for each eigenvalue λ_k , $\lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.

So that for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$, $(T - \alpha I)$ is not injective. \square

17 Give an example of an operator on \mathbf{R}^4 that has no (real) eigenvalues.

SOLUTION:

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$. Where (e_1, e_2, e_3, e_4) is the standard basis of \mathbf{R}^4 .

Suppose λ is an eigenvalue of T with an eigenvector (x, y, z, w) .

$$\text{Then } T(x, y, z, w) = \lambda(x, y, z, w) \Rightarrow \begin{cases} (1 - \lambda)x + y + z + w = 0 \\ -x + (1 - \lambda)y - z - w = 0 \\ 3x + 8y + (11 - \lambda)z + 5w = 0 \\ 3x - 8y - 11z + (5 - \lambda)w = 0 \end{cases}$$

This linear equation has no solutions.

(You can type it on <https://zh.numberempire.com/equationsolver.php> to check.)

OR. Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Suppose λ is an eigenvalue of T with an eigenvector (x, y, z, w) .

$$\text{Then } T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \Rightarrow \begin{cases} -y = \lambda x \\ x = \lambda y \\ -w = \lambda z \\ z = \lambda w \end{cases} \Rightarrow \begin{cases} -xy = \lambda^2 xy \\ -zw = \lambda^2 zw \end{cases}$$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Otherwise, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, contradicts.

Similarly, $y = z = w = 0$. Then we fail.

Thus T has no eigenvalues. \square

• TODO Suppose (v_1, \dots, v_n) is a basis of V and $T \in \mathcal{L}(V)$, $\mathcal{M}(T, (v_1, \dots, v_n)) = A$.

Prove that if λ is an eigenvalue of T , then $|\lambda| \leq n \max\{|A_{j,k}| : 1 \leq j, k \leq n\}$.

SOLUTION:

First we show that $|\lambda| = n \max\{|A_{j,k}| : 1 \leq j, k \leq n\}$ for some cases.

Consider $A = \begin{pmatrix} k & \dots & k \\ \vdots & \ddots & \vdots \\ k & \dots & k \end{pmatrix}$. Then nk is an eigenvalue of T with an eigenvector $v_1 + \dots + v_n$.

Now we show that if $|\lambda| \neq n \max\{|A_{j,k}| : 1 \leq j, k \leq n\}$, then $|\lambda| < n \max\{|A_{j,k}| : 1 \leq j, k \leq n\}$.

18 Show that the forward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$ has no eigenvalues.

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector (z_1, z_2, \dots) .

Then $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$.

Thus $\lambda z_1 = 0, \lambda z_2 = z_1, \dots, \lambda z_k = z_{k-1}, \dots$.

Let $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \lambda z_k = z_{k-1}$, therefore $(z_1, z_2, \dots) = 0$ is not an eigenvector.

Suppose $\lambda \neq 0$. Then $\lambda z_1 = 0 \Rightarrow z_1 = 0 \Rightarrow z_2 = 0 = z_k$ for all $k \in \mathbf{N}^+$.

And then $(z_1, z_2, \dots) = 0$ is not an eigenvector. Hence T has no eigenvalues. \square

19 Suppose $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by

$$T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n).$$

In other words, the entries of $\mathcal{M}(T)$ with respect to the standard basis are all 1's.

Find all eigenvalues and eigenvectors of T .

SOLUTION:

Suppose λ is an eigenvalue of T with an eigenvector (x_1, \dots, x_n) .

Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$.

Thus $\lambda x_1 = \dots = \lambda x_n = x_1 + \dots + x_n$.

For $\lambda = 0$, $x_1 + \dots + x_n = 0$.

For $\lambda \neq 0$, $x_1 = \dots = x_n$ and then $\lambda x_k = n x_k$ for each k .

Hence 0, n are eigenvalues of T .

And the set of all eigenvectors of T is $\{(x_1, \dots, x_n) \in \mathbf{F}^n : x_1 + \dots + x_n = 0 \vee x_1 = \dots = x_n\}$. \square

20 Define the backward shift operator $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.

(a) Show that every element of \mathbf{F} is an eigenvalue of S .

(b) Find all eigenvectors of S .

SOLUTION:

Suppose λ is an eigenvalue of S with an eigenvector (z_1, z_2, \dots) .

Then $S(z_1, z_2, z_3, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$.

Thus $\lambda z_1 = z_2, \lambda z_2 = z_3, \dots, \lambda z_k = z_{k+1}, \dots$.

For $\lambda = 0$, $\lambda z_1 = z_2 = 0 = \lambda z_2 = z_3 = \dots = z_k$ for all k .

While z_1 can be arbitrary, so that $(z_1, 0, \dots)$ is an eigenvector with $z_1 \neq 0$.

For $\lambda \neq 0$, $\lambda^k z_1 = \lambda^{k-1} z_2 = \dots = \lambda z_k = z_{k+1}$ for all k .

Then $(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots)$ is an eigenvector with $z_1 \neq 0$.

Hence (a) each element of $\lambda \in \mathbf{F}$ is an eigenvalue of T .

And (b) the set of all eigenvectors of T is $\{(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots) \in \mathbf{F}^\infty : \lambda \in \mathbf{F}, z_1 \neq 0\}$ \square

24 Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = Ax$, where elements of \mathbf{F}^n are thought of as n -by-1 column vectors.

(a) Suppose the sum of the entries in each row of A equals 1.

Prove that 1 is an eigenvalue of T .

(b) Suppose the sum of the entries in each column of A equals 1.

Prove that 1 is an eigenvalue of T .

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

$$\text{Then } Tx = Ax = \begin{pmatrix} \sum_{c=1}^n A_{1,c}x_c \\ \vdots \\ \sum_{c=1}^n A_{n,c}x_c \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \text{ While } \sum_{r=1}^n A_{R,r} = 1 \text{ for each } R = 1, \dots, n.$$

Thus if we let $x_1 = \dots = x_n$, then $\lambda = 1$, and hence 1 is an eigenvalue of T .

(b) Suppose λ is an eigenvalue of T with an eigenvector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

$$\text{Then } Tx = Ax = \begin{pmatrix} \sum_{r=1}^n A_{1,r}x_r \\ \vdots \\ \sum_{r=1}^n A_{n,r}x_r \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \text{ While } \sum_{r=1}^n A_{r,C} = 1 \text{ for each } C = 1, \dots, n.$$

$$\begin{aligned} \text{Thus } \sum_{r=1}^n (Ax)_{r,C} &= \sum_{r=1}^n (Ax)_{r,1} \\ &= \sum_{c=1}^n (A_{1,c} + \dots + A_{n,c})x_c = \sum_{c=1}^n x_c = \lambda \begin{pmatrix} x_1 \\ + \\ \vdots \\ + \\ x_n \end{pmatrix}. \end{aligned}$$

Hence $\lambda = 1$, for all x such that $\sum_{c=1}^n x_{c,1} \neq 0$. \square

OR. Prove that $(T - I)$ is not invertible, so that we can conclude $\lambda = 1$ is an eigenvalue.

$$\text{Because } (T - I)x = (A - \mathcal{M}(I))x = \begin{pmatrix} \sum_{r=1}^n A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^n A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

$$\text{Then } y_1 + \dots + y_n = \sum_{r=1}^n \sum_{c=1}^n (A_{r,c}x_c - x_r) = \sum_{c=1}^n x_c \sum_{r=1}^n A_{r,c} - \sum_{r=1}^n x_r = 0.$$

Thus $\text{range}(T - I) \subseteq \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbf{F}^n : y_1 + \dots + y_n = 0 \right\}$. Hence $(T - I)$ is not surjective. \square

• Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = xA$,

where elements of \mathbf{F}^n are thought of as 1-by- n row vectors.

(a) Suppose the sum of the entries in each column of A equals 1.

Prove that 1 is an eigenvalue of T .

(b) Suppose the sum of the entries in each row of A equals 1.

Prove that 1 is an eigenvalue of T .

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector $x = (x_1 \ \cdots \ x_n)$.

Then $Tx = xA = \left(\sum_{r=1}^n x_r A_{r,1} \ \cdots \ \sum_{r=1}^n x_r A_{r,n} \right) = \lambda (x_1 \ \cdots \ x_n)$. While $\sum_{r=1}^n A_{r,C} = 1$ for each $C = 1, \dots, n$.

Thus if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, hence is an eigenvalue of T .

(b) Suppose λ is an eigenvalue of T with an eigenvector $x = (x_1 \ \cdots \ x_n)$.

Then $Tx = xA = \left(\sum_{c=1}^n x_c A_{c,1} \ \cdots \ \sum_{c=1}^n x_c A_{c,n} \right) = \lambda (x_1 \ \cdots \ x_n)$. While $\sum_{c=1}^n A_{R,c} = 1$ for each $R = 1, \dots, n$.

Thus $\sum_{c=1}^n (xA)_{.,c} = \sum_{c=1}^n (xA)_{1,c} = \sum_{c=1}^n (A_{c,1} + \cdots + A_{c,n})x_c = \sum_{c=1}^n x_c = \lambda(x_1 + \cdots + x_n)$.

Hence $\lambda = 1$, for all x such that $\sum_{r=1}^n x_{1,r} \neq 0$. \square

OR. Prove that $(T - I)$ is not invertible, so that we can conclude $\lambda = 1$ is an eigenvalue.

Because $(T - I)x = x(A - \mathcal{M}(I)) = \left(\sum_{c=1}^n x_c A_{c,1} - x_1 \ \cdots \ \sum_{c=1}^n x_c A_{c,n} - x_n \right) = (y_1 \ \cdots \ y_n)$.

Then $y_1 + \cdots + y_n = \sum_{c=1}^n \sum_{r=1}^n (x_r A_{r,c} - x_c) = \sum_{r=1}^n x_r \sum_{c=1}^n A_{r,c} - \sum_{c=1}^n x_c = 0$.

Thus $\text{range}(T - I) \subseteq \{ (y_1 \ \cdots \ y_n) \in \mathbf{F}^n : y_1 + \cdots + y_n = 0 \}$. Hence $(T - I)$ is not surjective. \square

25 Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T

such that $u + w$ is also an eigenvector of T .

Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

SOLUTION:

Suppose $\lambda_1, \lambda_2, \lambda_0$ are eigenvalues of T corresponding to $u, w, u + w$ respectively.

Then $T(u + w) = \lambda_0(u + w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$.

Notice that $u, w, u + w$ are nonzero.

If (u, w) is linearly dependent, then let $w = cu$, therefore

$$\lambda_2 cu = Tw = cTu = \lambda_1 cu \Rightarrow \lambda_2 = \lambda_1.$$

$$\lambda_0(u + w) = T(u + w) = \lambda_1 u + \lambda_1 cu = \lambda_1(u + w) \Rightarrow \lambda_0 = \lambda_1.$$

Otherwise, $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. \square

26 Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T .

Prove that T is a scalar multiple of the identity operator.

SOLUTION:

Because $\forall v \in V, \exists! \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Then for any two distinct nonzero vectors $v, w \in V$,

$$T(v + w) = \lambda_{v+w}(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w.$$

If (v, w) is linearly independent, then let $w = cv$, therefore

$$\lambda_v cv = cTv = Tw = \lambda_w w \Rightarrow \lambda_w = \lambda_v.$$

$$\lambda_{v+w}(v + w) = T(v + w) = Tv + Tw = \lambda_v(v + cv) \Rightarrow \lambda_{v+w} = \lambda_v.$$

Otherwise, $\lambda_v = \lambda_{v+w} = \lambda_w$. \square

27, 28 Suppose V is finite-dim and $k \in \{1, \dots, \dim V - 1\}$.

Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V of dim k is invariant under T .

Prove that T is a scalar multiple of the identity operator.

SOLUTION:

We prove the contrapositive:

If $T \neq \lambda I, \forall \lambda \in \mathbf{F}$, then \exists a subspace U of V such that $\dim U = k$, and U is invariant under T .

By Problem (26), $\exists v \in V$ and $v \neq 0$ such that v is not an eigenvector of T .

Thus (v, Tv) is linearly independent. Extend to a basis of V as (v, Tv, u_1, \dots, u_n) .

Let $U = \text{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$ is not an invariant subspace of V under T .

OR. Suppose $0 \neq v = v_1 \in V$ and extend to a basis of V as (v_1, \dots, v_n) .

Suppose $Tv_1 = c_1v_1 + \dots + c_nv_n$, $\exists! c_i \in \mathbb{F}$.

Consider a k -dim subspace $U = \text{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$,

where $\alpha_j \in \{2, \dots, n\}$ for each j , and $\alpha_1, \dots, \alpha_{k-1}$ are distinct and are chosen arbitrarily.

Because every subspace such U is invariant.

Thus $Tv_1 = c_1v_1 + \dots + c_nv_n \in U$

$\Rightarrow c_2 = \dots = c_n = 0$,

for if not, for each $c_i \neq 0$, choose U_i such that $\alpha_j \in \underbrace{\{2, \dots, i-1, i+1, \dots, n\}}_{\text{length}(n-2)}$ for each j ,

hence for $Tv_1 = c_1v_1 + \dots + c_{i-1}v_{i-1} + c_{i+1}v_{i+1} + \dots + c_nv_n \in U_i$, we conclude that $c_i = 0$.

$\Rightarrow Tv_1 = c_1v_1$, $\forall v_1 = v \in V$ is arbitrarily chosen $\Rightarrow T = \lambda I$ for some λ . \square

• Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Prove that

T has an eigenvalue $\iff \exists$ a subspace U of V

such that $\dim U = \dim V - 1$, U is invariant under T .

SOLUTION:

(a) Suppose λ is an eigenvalue of T with an eigenvector v .

(If $\dim V = 1$, then $U = \{0\}$ and we are done.)

Extend $v_1 = v$ to a basis of V as (v_1, v_2, \dots, v_n) .

Step 1 If $\exists w_1 \in \text{span}(v_2, \dots, v_n)$ such that $0 \neq Tw_1 \in \text{span}(v_1)$,

then extend $w_1 = \alpha_{1,1}$ to a basis of $\text{span}(v_2, \dots, v_n)$ as $(\alpha_{1,1}, \dots, \alpha_{1,n-1})$.

Otherwise, we stop at step 1.

\vdots

Step k If $\exists w_k \in \text{span}(\alpha_{k-1,2}, \dots, \alpha_{k-1,n-k+1})$ such that $0 \neq Tw_k \in \text{span}(v_1, w_1, \dots, w_{k-1})$,

then extend $w_k = \alpha_{k,1}$ to a basis of $\text{span}(\alpha_{k-1,2}, \dots, \alpha_{k-1,n-k+1})$ as $(\alpha_{k,1}, \dots, \alpha_{k,n-k})$.

Otherwise, we stop at step k .

\vdots

Finally, we stop at step m , thus we get $(v_1, w_1, \dots, w_{m-1})$ and $(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1})$,

$\text{range } T|_{\text{span}(w_1, \dots, w_{m-1})} = \text{span}(v_1, w_1, \dots, w_{m-2}) \Rightarrow \dim \text{null } T|_{\text{span}(w_1, \dots, w_{m-1})} = 0$,

$\text{span}(\underbrace{v_1, w_1, \dots, w_{m-1}}_{\text{length dim } m})$ and $\text{span}(\underbrace{\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}}_{\text{length dim}(n-m)})$ are invariant under T .

Let $U = \text{span}(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}) \oplus \text{span}(v_1, w_1, \dots, w_{m-2})$ and we are done. \square

COMMENT: Both $\text{span}(v_2, \dots, v_n)$ and $\text{span}(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}) \oplus \text{span}(w_1, \dots, w_{m-1})$ are in $\mathcal{S}_V \text{span}(v_1)$.

(b) Suppose U is an invariant subspace of V under T with $\dim U = m = \dim V - 1$.

(If $m = 0$, then $\dim V = 1$ and we are done) .

Let (u_1, \dots, u_m) be a basis of U , extend to a basis of V as (u_0, u_1, \dots, u_m) .

We discuss in cases:

For $Tu_0 \in U$, then $\text{range } T = U$ so that T is not surjective $\iff \text{null } T \neq \{0\} \iff 0$ is an eigenvalue of T .

For $Tu_0 \notin U$, then $Tu_0 = a_0u_0 + a_1u_1 + \dots + a_mu_m$.

(1) If $Tu_0 \in \text{span}(u_0)$, then we are done.

(2) Otherwise, if $\text{range } T|_U = U$, then $Tu_0 = a_0u_0$ and we are done;

otherwise, $T|_U : U \rightarrow U$ is not surjective (\Rightarrow not injective), suppose $\text{range } T|_U \neq \{0\}$

(Suppose $\text{range } T|_U = \{0\}$. If $\dim U = 0$ then we are done.

Otherwise $\exists u \in U \setminus \{0\}, Tu = 0$ and we are done.)

then $\exists u \in U \setminus \{0\}, Tu = 0$, we are done. \square

29 Suppose $T \in \mathcal{L}(V)$ and $\text{range } T$ is finite-dim.

Prove that T has at most $1 + \dim \text{range } T$ distinct eigenvalues.

SOLUTION:

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T and let v_1, \dots, v_m be the corresponding eigenvectors.

(Because $\text{range } T$ is finite-dim. Let (v_1, \dots, v_n) be a list of all the linearly independent eigvecs of T , so that the corresponding eigvals are finite.)

For every $\lambda_k \neq 0, T(\frac{1}{\lambda_k}v_k) = v_k$. And if $T = T - 0I$ is not injective, then $\exists! \lambda_A = 0$ and $Tv_A = \lambda_A v_A = 0$.

Thus for $\lambda_k \neq 0, \forall k, (Tv_1, \dots, Tv_m)$ is a linearly independent list of length m in $\text{range } T$.

And for $\lambda_A = 0$, there is a linearly independent list of length at most $(m - 1)$ in $\text{range } T$.

Hence, by [2.23], $m \leq \dim \text{range } T + 1$. \square

32 Suppose that $\lambda_1, \dots, \lambda_n$ are distinct real numbers.

Prove that $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

HINT: Let $V = \text{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$, and define an operator $D \in \mathcal{L}(V)$ by $Df = f'$.

Find eigenvalues and eigenvectors of D .

SOLUTION:

Define V and $D \in \mathcal{L}(V)$ as in HINT. Then because for each $k, D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.

Thus $\lambda_1, \dots, \lambda_n$ are distinct eigenvalues of D . By [5.10], $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$. \square

• Suppose $\lambda_1, \dots, \lambda_n$ are distinct positive numbers.

Prove that $(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

SOLUTION:

Let $V = \text{span}(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.

Then because $D(\cos(\lambda_k x)) = -\lambda_k \sin(\lambda_k x)$. $\forall D(\sin(\lambda_k x)) = \lambda_k \cos(\lambda_k x)$.

Thus $D^2(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$.

Notice that $\lambda_1, \dots, \lambda_n$ are distinct $\Rightarrow -\lambda_1^2, \dots, -\lambda_n^2$ are distinct.

Hence $-\lambda_1^2, \dots, -\lambda_n^2$ are distinct eigenvalues of D^2

with the corresponding eigenvectors $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ respectively.

And then $(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$. \square

• Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T .

The quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U \text{ for each } v \in V.$$

(a) *Show that the definition of T/U makes sense*

(which requires using the condition that U is invariant under T)

and show that T/U is an operator on V/U .

(b) (OR Problem 35) *Show that each eigenvalue of T/U is an eigenvalue of T .*

SOLUTION:

(a) Suppose $v + U = w + U$ ($\iff v - w \in U$).

Then because U is invariant under T , $T(v - w) \in U \iff Tv + U = Tw + U$.

Hence the definition of T/U makes sense.

Now we show that T/U is linear.

$$\begin{aligned} \forall v + U, w + U \in V/U, \lambda \in \mathbf{F}, (T/U)((v + U) + \lambda(w + U)) \\ = T(v + \lambda w) + U = (Tv + U) + \lambda(Tw + U) \\ = (T/U)(v + U) + \lambda(T/U)(w). \end{aligned}$$

(b) Suppose λ is an eigenvalue of T/U with an eigenvector $v + U$.

Then $(T/U)(v + U) = \lambda(v + U) = Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v \in U$.

If $(T - \lambda I)v = 0 \Rightarrow Tv = \lambda v$, then we are done.

Otherwise, then $(T|_U - \lambda I) : U \rightarrow U$ is invertible,

$$\text{hence } \exists! w \in U, (T|_U - \lambda I)(w) = (T - \lambda I)v \Rightarrow T(v - w) = \lambda(v - w).$$

Note that $v - w \neq 0$ (for if not, $v \in U \Rightarrow v + U = 0 + U$ is not an eigenvector). \square

36 Prove or give a counterexample:

The result of (b) in Exercise 35 is still true if V is infinite-dim.

SOLUTION: A counterexample:

Consider $V = \text{span}(1, e^x, e^{2x}, \dots)$ in $\mathbb{R}^{\mathbb{R}}$, and a subspace $U = \text{span}(e^x, e^{2x}, \dots)$ of V .

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then $\text{range } T = U$ is invariant under T .

Consider $(T/U)(1 + U) = e^x + U = 0$

$\Rightarrow 0$ is an eigenvalue of T/U but is not an eigenvalue of T

($\text{null } T = \{0\}$, for if not, $\exists f \in V \setminus \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, contradicts). \square

33 Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{range } T) = 0$.

SOLUTION:

$$\forall v + \text{range } T \in V/\text{range } T, v + \text{range } T \in \text{null } (T/(\text{range } T))$$

$$\Rightarrow \text{null } (T/(\text{range } T)) = V/\text{range } T \Rightarrow T/(\text{range } T) \text{ is a zero map. } \square$$

34 Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null } T)$ is injective $\iff (\text{null } T) \cap (\text{range } T) = \{0\}$.

SOLUTION:

(a) Suppose $T/(\text{null } T)$ is injective.

$$\text{Then } (T/(\text{null } T))(u + \text{null } T) = Tu + \text{null } T = 0$$

$$\iff Tu \in \text{null } T \text{ } \not\iff Tu \in \text{range } T \iff u + \text{null } T = 0 \iff u \in \text{null } T \iff Tu = 0.$$

Thus $(\text{null } T) \cap (\text{range } T) = \{0\}$.

(b) Suppose $(\text{null } T) \cap (\text{range } T) = \{0\}$.

$$\text{Then } (T/(\text{null } T))(u + \text{null } T) = Tu + \text{null } T = 0$$

$$\iff Tu \in \text{null } T \text{ } \not\iff Tu \in \text{range } T \iff Tu = 0 \iff u \in \text{null } T \iff u + \text{null } T = 0.$$

Thus $T/(\text{null } T)$ is injective. \square

5.B: I [See 5.B: II below.]

COMMENT: 下面是第 5 章 B 节。为了照顾 5.B 节两版过大的差距，特别将 5.B 补注分成 I 和 II 两部分。

又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本质征值」(相当一部分是从原第 3 版 8.C 节挪过来的)是对原第 3 版「多项式作用于算子」与「本征值的存在性」(也即第 3 版 5.B 前半部分)的极大扩充，这一扩充也大大改变了原第 3 版后半部分的「上三角矩阵」这一小节，故而将第 4 版 5.B 节放在第 3 版 5.B 节前面。

I 部分除了覆盖第 4 版 5.B 节和第 3 版 5.B 节前半部分与之相关的所有习题，还会覆盖第 4 版 5.A 节末。

II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节，还会覆盖第 4 版 5.C 节；并且，下面 5.C 还会覆盖第 4 版 5.D 节。

[注: [8.40] OR (4E 5.22) —— minimal polynomial;
[8.44,8.45] OR (4E 5.25,5.26) —— how to find the minimal polynomial;
[8.49] OR (4E 5.27) —— eigenvalues are the zeros of the minimal polynomial;
[8.46] OR (4E 5.29) —— $q(T) = 0 \Leftrightarrow q$ is a poly multiple of the mini poly.]

[1]: (4E.5.A.33), 13; [2]: (4E.5.B.25, 26, 27, 28, 22); [3]: 6, (4E.5.B.10, 23, 21), 19; [4]: (4E 5.B.13, 14);
[5]: (4E.5.B.20, 24), 10; [6]: 1, 2, 7, 3, (4E.5.A.32); [7]: 8, (4E 5.B.12, 3, 8); [8]: (4E.5.B.11), 5, (4E.5.B.7);
[9]: 11, 12, (4E.5.B.17, 18); [10]: 18 OR (4E.5.B.15), (4E.5.B.9), (4E.5.B.16); [11]: (2E Ch5.24), (4E.5.B.29).

• Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

(a) Prove that T is injective $\Leftrightarrow T^m$ is injective.

(b) Prove that T is surjective $\Leftrightarrow T^m$ is surjective.

SOLUTION:

(a) Suppose T^m is injective. Then $Tv = 0 \Rightarrow T^{m-1}Tv = T^mv = 0 \Rightarrow v = 0$. \square

Suppose T is injective.

Then $T^mv = T(T^{m-1}v) = 0$

$$\Rightarrow T^{m-1}v = 0 = T(T^{m-2}v) \Rightarrow \dots$$

$$\Rightarrow T^2v = TTv = 0$$

$$\Rightarrow Tv = 0 \Rightarrow v = 0. \quad \square$$

(b) Suppose T^m is surjective. $\forall u \in V, \exists v \in V, T^mv = u = Tw$, let $w = T^{m-1}v$. \square

Suppose T is surjective.

Then $\forall u \in V, \exists v \in V, T(\underline{v}) = u$

$$\Rightarrow \exists v_2 \in V, Tv_2 = \underline{v}, T^2(\underline{v_2}) = u$$

\vdots

$$\Rightarrow \exists v_k \in V, Tv_k = \underline{v_{k-1}}, T^k(\underline{v_k}) = u$$

\vdots

$$\Rightarrow \exists v_{m-1} \in V, Tv_{m-1} = \underline{v_{m-2}}, T^{m-1}(\underline{v_{m-1}}) = u$$

$$\Rightarrow \exists v_m \in V, Tv_m = \underline{v_{m-1}}, T^{m-1}(Tv_m) = u. \quad \square$$

• NOTE FOR [5.17]: Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$.

Prove that $\text{null } p(T)$ and $\text{range } p(T)$ are invariant under T .

SOLUTION: Using the commutativity in [5.10].

(a) Suppose $u \in \text{null } p(T)$. Then $p(T)u = 0$.

Thus $p(T)(Tu) = (p(T)T)u = T(p(T)u) = T(0) = 0$. Hence $Tu \in \text{null } p(T)$. \square

(b) Suppose $u \in \text{range } p(T)$. Then $\exists v \in V$ such that $u = p(T)v$.

Thus $Tu = T(p(T)v) = p(T)(Tv) \in \text{range } p(T)$. \square

• **NOTE FOR [5.21]:** Every operator on a finite-dim nonzero complex vector space has an eigval.

Suppose V is a finite-dim complex vector space of $\dim n > 0$ and $T \in \mathcal{L}(V)$.

Choose a nonzero $v \in V$. $(v, Tv, T^2v, \dots, T^nv)$ of length $n + 1$ is linearly dependent.

Suppose $a_0I + a_1T + \dots + a_nT^n = 0$. Then $\exists a_j \neq 0$.

Thus \exists nonconst p of smallest degree ($\deg p > 0$) such that $p(T)v = 0$.

Because $\exists \lambda \in \mathbf{C}$ such that $p(\lambda) = 0 \Rightarrow \exists q \in \mathcal{P}(\mathbf{C}), p(z) = (z - \lambda)q(z), \forall z \in \mathbf{C}$.

Thus $0 = p(T)v = (T - \lambda I)(q(T)v)$. By the minimality of $\deg p$ and $\deg q < \deg p$, $q(T)v \neq 0$.

Then $(T - \lambda I)$ is not injective. Thus λ is an eigval of T with eigvec $q(T)v$.

• **EXAMPLE:** an operator on a complex vector space with no eigvals

Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by $(Tp)(z) = zp(z)$.

Suppose $p \in \mathcal{P}(\mathbf{C})$ is a nonzero poly. Then $\deg Tp = \deg p + 1$, and thus $Tp \neq \lambda p, \forall \lambda \in \mathbf{C}$.

Hence T has no eigvals. Because $\mathcal{P}(\mathbf{C})$ is infinite-dim, this example does not contradict the result above.

13 Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ has no eigvals.

Prove that every subspace of V invariant under T is either $\{0\}$ or infinite-dim.

SOLUTION: Suppose U is a finite-dim nonzero invariant subspace on \mathbf{C} . Then by [5.21], $T|_U$ has an eigval. \square

• **TIPS:** For $T_1, \dots, T_m \in \mathcal{L}(V)$:

(a) Suppose T_1, \dots, T_m are all injective. Then $(T_1 \circ \dots \circ T_m)$ is injective.

(b) Suppose $(T_1 \circ \dots \circ T_m)$ is not injective. Then at least one of T_1, \dots, T_m is not injective.

(c) At least one of T_1, \dots, T_m is not injective $\nRightarrow (T_1 \circ \dots \circ T_m)$ is not injective.

EXAMPLE: On infinite-dim only. Let $V = \mathbf{F}^\infty$.

Let S be the backward shift (surj but not inje), T be the forward shift (inje but not surj). Then $ST = I$. \square

16 Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}(\mathcal{P}_{\dim V}(\mathbf{C}), V)$ by $S(p) = p(T)v$. Prove [5.21].

SOLUTION:

Because $\dim \mathcal{P}_{\dim V}(\mathbf{C}) = \dim V + 1$. Then S is not injective. Hence $\exists 0 \neq p \in \mathcal{P}_{\dim V}(\mathbf{C}), p(T)v = 0$.

Using [4.14], write $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$. Apply T to both sides: $p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.

Thus at least one of $(T - \lambda_j I)$ is not injective (because $p(T)$ is not injective). \square

17 Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}(\mathcal{P}_{(\dim V)^2}(\mathbf{C}), \mathcal{L}(V))$ by $S(p) = p(T)$. Prove [5.21].

SOLUTION:

Because $\dim \mathcal{P}_{(\dim V)^2}(\mathbf{C}) = (\dim V)^2 + 1$. Then S is not injective. Hence $\exists 0 \neq p \in \mathcal{P}_{(\dim V)^2}(\mathbf{C}), p(T) = 0$.

Using [4.14], write $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$. Applying T , we have $0 = p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.

Thus $(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0 \Rightarrow \exists j, (T - \lambda_j)$ is not injective. \square

COMMENT: \exists monic $q \in \text{null } S \neq \{0\}$ of smallest degree, $S(q) = q(T) = 0$, then q is the *minimal polynomial*.

• **NOTE FOR [8.40]:** definition for minimal polynomial

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that $\exists!$ monic poly $p \in \mathcal{P}(\mathbf{F})$ of smallest degree, $p(T) = 0$. Moreover, $\deg p \leq \dim V$.

SOLUTION OR Another Proof :

[Existence Part] We use induction on $\dim V$.

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, we are done.

(ii) Suppose $\dim V \geq 1$.

Assume that $\dim V > 0$ and that the desired result is true for all operators on all vecsps of smaller dim.

Let $u \in V, u \neq 0$. The list $(u, Tu, \dots, T^{\dim V} u)$ of length $(1 + \dim V)$ is linearly dependent.

Then $\exists ! T^m$ of smallest degree such that $T^m u \in \text{span}(u, Tu, \dots, T^{m-1} u)$.

Thus $\exists c_j \in \mathbf{F}, c_0 u + c_1 Tu + \dots + c_{m-1} T^{m-1} u + T^m u = 0$.

Define q by $q(z) = c_0 + c_1 z + \dots + c_{m-1} z^{m-1} + z^m$.

Then $0 = T^k(q(T)u) = q(T)(T^k u), \forall k \in \{1, \dots, m-1\} \subseteq \mathbf{N}$.

Because $(u, Tu, \dots, T^{m-1} u)$ is linearly independent.

Thus $\dim \text{null } q(T) \geq m \Rightarrow \dim \text{range } q(T) = \dim V - \dim \text{null } q(T) \leq \dim V - m$.

Let $W = \text{range } q(T)$.

By assumption, \exists monic $s \in \mathcal{P}(\mathbf{F})$ and $\deg s \leq \dim W$, so that $s(T|_W) = 0$.

Hence $\forall v \in V, ((sq)(T))(v) = s(T)(q(T)v) = 0$.

Thus sq is a monic poly such that $\deg sq \leq \dim V$ and $(sq)(T) = 0$.

[Uniqueness Part]

Let $p, q \in \mathcal{P}(\mathbf{F})$ be monic polys of smallest degree such that $p(T) = q(T) = 0$

$\Rightarrow (p - q)(T) = 0 \wedge \deg(p - q) < \deg p$.

If $p - q = a_m z^m + \dots + a_1 z_1 + a_0 \neq 0$, then $\frac{1}{a_m}(p - q)$ is a monic poly of smaller degree than p .

Hence contradicts the minimality of $\deg p$. Thus $p - q = 0$ and we are done. \square

• (4E 5.31, 4E 5.B.25 and 26) *mini poly of restriction operator and mini poly of quotient operator*

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invariant subspace of V under T .

Let p be the mini poly of T .

(a) *Prove that p is a polynomial multiple of the mini poly of $T|_U$.*

(b) *Prove that p is a polynomial multiple of the mini poly of T/U .*

(c) *Prove that (mini poly of $T|_U$) \times (mini poly of T/U) is a polynomial multiple of p .*

(d) *Prove that the set of eigvals of T equals*

the union of the set of eigvals of $T|_U$ and the set of eigvals of T/U .

SOLUTION:

(a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46]. \square

(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v + U) = p(T)v + U = 0$. \square

(c) Suppose r is the mini poly of $T|_U$, s is the mini poly of T/U .

Because $\forall v \in V, s(T/U)(v + U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.

$\wedge \forall u \in U, r(T|_U)u = r(T)u = 0$.

Thus $\forall v \in V$ but $v \notin U, (rs)(T)v = r(s(T)v) = 0$.

And $\forall u \in U, (rs)(T)u = r(s(T)u) = 0$ (because $s(T)u = s(T|_U)u \in U$).

Hence $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0$. \square

(d) By [8.49], immediately. \square

• (4E 5.B.27)

Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, and $T \in \mathcal{L}(V)$.

Prove that the mini poly p of $T_{\mathbb{C}}$ equals the mini poly q of T .

SOLUTION:

$\left. \begin{array}{l} \forall u + i0 \in V_{\mathbb{C}}, p(T_{\mathbb{C}})(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p \text{ is a polynomial multiple of } q. \\ q(T) = 0 \Rightarrow \forall u + iv \in V_{\mathbb{C}}, q(T_{\mathbb{C}})(u + iv) = q(T)u + iq(T)v = 0 \Rightarrow q \text{ is a polynomial multiple of } p. \end{array} \right\} \Rightarrow \square$

• (4E 5.B.28)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that the mini poly p of $T' \in \mathcal{L}(V')$ equals the mini poly q of T .

SOLUTION:

$$\left. \begin{aligned} \forall \varphi \in V', p(T')\varphi = \varphi \circ (p(T)) = 0 &\Rightarrow \forall \varphi \in V', p(T) \in \text{null } \varphi \Rightarrow p(T) = 0 \\ &\Rightarrow p(T) = 0 \Rightarrow p \text{ is a polynomial multiple of } q. \\ q(T) = 0 &\Rightarrow \forall \varphi \in V', \varphi \circ (q(T)) = q(T')\varphi = 0 \Rightarrow q \text{ is a polynomial multiple of } p. \end{aligned} \right\} \Rightarrow \square$$

• (4E 5.32) Suppose $T \in \mathcal{L}(V)$ and p is the mini poly.

Prove that T is not injective \iff the const term of p is 0.

SOLUTION: T is not injective $\iff 0$ is an eigval of $T \iff 0$ is a zero of $p \iff$ the const term of p is 0. \square

OR. Because $p(0) = (z - 0)(z - \lambda_1) \cdots (z - \lambda_m) = 0 \Rightarrow T(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0$

$\wedge p$ is the mini poly $\Rightarrow q$ define by $q(z) = (z - \lambda_1) \cdots (z - \lambda_m)$ is such that $q(T) \neq 0$.

Hence $0 = p(T) = Tq(T) \Rightarrow T$ is not injective.

Conversely, suppose $(T - 0I)$ is not injective, then 0 is a zero of p , so that the const term is 0. \square

• (4E 5.B.22)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$. Prove that T is invertible $\iff I \in \text{span}(T, T^2, \dots, T^{\dim V})$.

SOLUTION: Denote the mini poly by p , where for all $z \in \mathbf{F}$, $p(z) = a_0 + a_1 z + \cdots + z^m$.

Notice that V is finite-dim. T is invertible $\iff T$ is injective $\iff p(0) \neq 0$.

Hence $p(T) = 0 = a_0 I + a_1 T + \cdots + T^m$, where $a_0 \neq 0$ and $m \leq \dim V$. \square

6 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T .

Prove that U is invariant under $p(T)$ for every poly $p \in \mathcal{P}(\mathbf{F})$.

SOLUTION:

$$\forall u \in U, Tu \in U \Rightarrow \forall u \in U, Iu, Tu, T(Tu), \dots, T^m u \in U \Rightarrow \forall u \in U, (a_0 I + a_1 T + \cdots + a_m T^m)u \in U. \quad \square$$

• (4E 5.B.10, 5.B.23)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$ and p is the mini poly with degree m . Suppose $v \in V$.

(a) Prove that $\text{span}(v, Tv, \dots, T^{m-1}v) = \text{span}(v, Tv, \dots, T^{j-1}v)$ for some $j \leq m$.

(b) Prove that $\text{span}(v, Tv, \dots, T^{m-1}v) = \text{span}(v, Tv, \dots, T^{m-1}v, \dots, T^n v)$.

COMMENT: By NOTE FOR [8.40], j has an upper bound $m - 1$, m has an upper bound $\dim V$.

SOLUTION: Write $p(z) = a_0 + a_1 z + \cdots + z^m$ ($m \leq \dim V$). If $v = 0$, then we are done. Suppose $v \neq 0$.

(a) Suppose $j \in \mathbf{N}^+$ is the smallest such that $T^j v \in \text{span}(v, Tv, \dots, T^{j-1}v) = U_0$. Then $j \leq m$.

Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1} T^{j-1} v$. And because $T(T^k v) = T^{k+1} v \in U_0$. U_0 is invariant under T .

By Problem (6), $\forall k \in \mathbf{N}$, $T^{j+k} v = T^k(T^j v) \in U_0$.

Thus $U_0 = \text{span}(v, Tv, \dots, T^{j-1}v, \dots, T^n v)$ for all $n \geq j - 1$. Let $n = m - 1$ and we are done.

(b) Let $U = \text{span}(v, Tv, \dots, T^{m-1}v)$.

By (a), $U = U_0 = \text{span}(v, Tv, \dots, T^{j-1}v, \dots, T^{m-1}v, \dots, T^n v)$ for all $n \geq m - 1$. \square

• (4E 5.B.21)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that the mini poly p has degree at most $1 + \dim \text{range } T$.

If $\dim \text{range } T < \dim V - 1$, then this result gives a better upper bound for the degree of mini poly.

SOLUTION:

If T is injective, then $\text{range } T = V$ and we are done. Now choose $0 \neq v \in \text{null } T$, then $Tv + 0 \cdot v = 0$.

1 is the smallest positive integer such that $T^1 v \in \text{span}(v, \dots, T^0 v)$. Define q by $q(z) = z \Rightarrow q(T)v = 0$.

Let $W = \text{range } q(T) = \text{range } T$. \exists monic $s \in \mathcal{P}(\mathbf{F})$ of smallest degree ($\deg s \leq \dim W$), $s(T|_W) = 0$.

Hence sq is the mini poly (see NOTE FOR [8.40]) and $\deg(sq) = \deg s + \deg q \leq \dim \text{range } T + 1$. \square

19 Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.

Prove that $\dim \mathcal{E}$ equals the degree of the minimal polynomial of T .

SOLUTION:

Because the list $(I, T, \dots, T^{(\dim V)^2})$ of length $\dim \mathcal{L}(V) + 1$ is linearly dependent in $\dim \mathcal{L}(V)$.

Suppose $m \in \mathbf{N}^+$ is the smallest such that $T^m = a_0 I + \dots + a_{m-1} T^{m-1}$.

Then q defined by $q(z) = z^m - a_{m-1} z^{m-1} - \dots - a_0$ is the mini poly (see [8.40]).

For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k(T^m) \in \text{span}(I, T, \dots, T^{m-1}) = U$.

Hence $\text{span}(I, T, \dots, T^{(\dim V)^2}) = \text{span}(I, T, \dots, T^{(\dim V)^2-1}) = U$.

Note that by the minimality of m , the list (I, T, \dots, T^{m-1}) is linearly independent.

Thus $\dim U = m = \dim \text{span}(I, T, \dots, T^{(\dim V)^2-1}) = \dim \text{span}(I, T, \dots, T^n)$ for all $m < n \in \mathbf{N}^+$.

Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.

(a) Suppose $p(T) = 0$. $\text{deg } p \leq m-1 \Rightarrow p = 0$. Then φ is injective.

(b) $\forall S = a_0 I + a_1 T + \dots + a_{m-1} T^{m-1} \in \mathcal{E}$, define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by

$$p(z) = a_0 + a_1 z + \dots + a_{m-1} z^{m-1} \Rightarrow \varphi(p) = S. \text{ Then } \varphi \text{ is surjective.}$$

Hence \mathcal{E} and $\mathcal{P}_{m-1}(\mathbf{F})$ are isomorphic. $\text{dim } \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. \square

• (4E 5.B.13)

Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$ is defined by

$$q(z) = a_0 + a_1 z + \dots + a_n z^n, \text{ where } a_n \neq 0, \text{ for all } z \in \mathbf{F}.$$

Denote the mini poly of T by p defined by

$$p(z) = c_0 + c_1 z + \dots + c_{m-1} z^{m-1} + z^m \text{ for all } z \in \mathbf{F}.$$

Prove that $\exists ! r \in \mathcal{P}(\mathbf{F})$ such that $q(T) = r(T)$, $\text{deg } r < \text{deg } p$.

SOLUTION:

If $\text{deg } q < \text{deg } p$, then we are done.

If $\text{deg } q = \text{deg } p$, notice that $p(T) = 0 = c_0 I + c_1 T + \dots + c_{m-1} T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0 I - c_1 T - \dots - c_{m-1} T^{m-1},$$

$$\text{define } r \text{ by } r(z) = q(z) + [-a_m z^m + a_m(-c_0 - c_1 z - \dots - c_{m-1} z^{m-1})]$$

$$= (a_0 - a_m c_0) + (a_1 - a_m c_1)z + \dots + (a_{m-1} - a_m c_{m-1})z^{m-1},$$

hence $r(T) = 0$, $\text{deg } r < m$ and we are done.

Now suppose $\text{deg } q \geq \text{deg } p$. We use induction on $\text{deg } q$.

(i) $\text{deg } q = \text{deg } p$, then the desired result is true, as shown above.

(ii) $\text{deg } q > \text{deg } p$, assume that the desired result is true for $\text{deg } q = n$.

Suppose $f \in \mathcal{P}(\mathbf{F})$ such that $f(z) = b_0 + b_1 z + \dots + b_n z^n + b_{n+1} z^{n+1}$.

Apply the assumption to g defined by $g(z) = b_0 + b_1 z + \dots + b_n z^n$,

getting s defined by $s(z) = d_0 + d_1 z + \dots + d_{m-1} z^{m-1}$.

Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1} T^{n+1} = s(T) + b_{n+1} T^{n+1}$.

Apply the assumption to t defined by $t(z) = z^n$,

getting δ defined by $\delta(z) = c'_0 + c'_1 z + \dots + c'_{m-1} z^{m-1}$.

Thus $t(T) = T^n = c'_0 + c'_1 T + \dots + c'_{m-1} T^{m-1} = \delta(T)$.

$\text{span}(v, Tv, \dots, T^{m-1}v)$ is invariant under T .

Hence $\exists ! k_j \in \mathbf{F}$, $T^{n+1} = T(T^n) = k_0 + k_1 T + \dots + k_{m-1} T^{m-1}$.

And $f(T) = s(T) + b_{n+1}(k_0 + k_1 T + \dots + k_{m-1} T^{m-1})$

$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1)z + \dots + (d_{m-1} + k_{m-1})z^{m-1} = h(T)$, thus defining h . \square

-
- (4E 5.B.14) Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has mini poly p defined by $p(z) = a_0 + a_1z + \cdots + a_{m-1}z^{m-1} + z^m$, $a_0 \neq 0$. Find the mini poly of T^{-1} .

SOLUTION: Notice that V is finite-dim. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is invertible.

Then $p(T) = a_0I + a_1T + \cdots + T^m = 0$. Apply T^{-m} to both sides,

$$a_0(T^{-1})^m + a_1(T^{-1})^{m-1} + \cdots + a_{m-1}T^{-1} + I = 0.$$

Define q by $q(z) = z^m + \frac{a_1}{a_0}z^{m-1} + \cdots + \frac{a_{m-1}}{a_0}z + \frac{1}{a_0}$ for all $z \in \mathbf{F}$.

We now show that $(T^{-1})^k \notin \text{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$

for every $k \in \{1, \dots, m-1\}$ by contradiction, so that q is exactly the mini poly of T^{-1} .

Suppose $(T^{-1})^k \in \text{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$.

Then let $(T^{-1})^k = b_0I + b_1T^{-1} + \cdots + b_{k-1}T^{k-1}$. Apply T^k to both sides,

$$\text{getting } I = b_0T^k + b_1T^{k-1} + \cdots + b_{k-1}T, \text{ hence } T^k \in \text{span}(I, T, \dots, T^{k-1}).$$

Thus f defined by $f(z) = z^k + \frac{b_1}{b_0}z^{k-1} + \cdots + \frac{b_{k-1}}{b_0}z - \frac{1}{b_0}$ is a polynomial multiple of p .

While $\deg f < \deg p$. Contradicts. \square

- **NOTE FOR [8.49]:** Suppose V is a finite-dim complex vecsp, $T \in \mathcal{L}(V)$.

By [4.14], the mini poly has the form $(z - \lambda_1) \cdots (z - \lambda_m)$,

where $\lambda_1, \dots, \lambda_m$ is a list of all eigvals of T , possibly with repetitions.

- **COMMENT:** A nonzero poly has at most as many distinct zeros as its degree (see [4.12]). Thus by the upper bound for the deg of mini poly given in NOTE FOR [8.40], and by [8.49,] we can **give an alternative proof of [5.13]**.
-

- **NOTICE:** (See also 4E 5.B.20,24)

Suppose $\alpha_1, \dots, \alpha_n$ are all the distinct eigvals of T ,

and therefore are all the distinct zeros of the mini poly.

Also, the mini poly of T is a polynomial multiple of, but not equal to, $(z - \alpha_1) \cdots (z - \alpha_n)$.

If we define q by $q(z) = (z - \alpha_1)^{\dim V - (n-1)} \cdots (z - \alpha_n)^{\dim V - (n-1)}$,

then q is a polynomial multiple of the char poly (see [8.34] and [8.26])

(Because $\dim V > n$ and $n - 1 > 0$, $n[\dim V - (n - 1)] > \dim V$.)

The char poly has the form $(z - \alpha_1)^{\gamma_1} \cdots (z - \alpha_n)^{\gamma_n}$, where $\gamma_1 + \cdots + \gamma_n = \dim V$.

The mini poly has the form $(z - \alpha_1)^{\delta_1} \cdots (z - \alpha_n)^{\delta_n}$, where $0 \leq \delta_1 + \cdots + \delta_n \leq \dim V$.

- 10** Suppose $T \in \mathcal{L}(V)$, λ is an eigval of T with an eigvec v .

Prove that for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.

SOLUTION:

Suppose p is defined by $p(z) = a_0 + a_1z + \cdots + a_mz^m$ for all $z \in \mathbf{F}$. Because for any $n \in \mathbf{N}^+$, $T^n v = \lambda^n v$.

Thus $p(T)v = a_0v + a_1Tv + \cdots + a_mT^mv = a_0v + a_1\lambda v + \cdots + a_m\lambda^mv = p(\lambda)v$. \square

- **COMMENT:** For any $p \in \mathcal{P}(\mathbf{F})$ such that $p(z) = (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_m)^{\alpha_m}$, the result is true as well.

Now we prove that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v = (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v$.

Define q_i by $q_i(z) = (z - \lambda_i)^{\alpha_i}$ for all $z \in \mathbf{F}$.

Because $(a + b)^n = a^n + \mathcal{C}_n^1 a^{n-1} b + \cdots + \mathcal{C}_n^k a^{n-k} b^k + \cdots + \mathcal{C}_n^n b^n$.

Let $a = z, b = \lambda_i, n = \alpha_i$, so we can write $q_i(z)$ in the form $a_0 + a_1 z + \cdots + a_m z^m$.

Hence $q_i(T)v = q_i(\lambda)v \Rightarrow (T - \lambda_i I)^{\alpha_i} v = (\lambda - \lambda_i)^{\alpha_i} v$.

Then for each $k \in \{2, \dots, m\}$, $(T - \lambda_{k-1} I)^{\alpha_{k-1}} (T - \lambda_k I)^{\alpha_k} v$

$$\begin{aligned} &= q_{k-1}(T)(q_k(T)v) \\ &= q_{k-1}(T)(q_k(\lambda)v) \\ &= q_{k-1}(\lambda)(q_k(\lambda)v) \\ &= (\lambda - \lambda_{k-1})^{\alpha_{k-1}} (\lambda - \lambda_k)^{\alpha_k} v. \end{aligned}$$

So that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v$

$$\begin{aligned} &= q_1(T)(q_2(T)(\cdots (q_m(T)v) \cdots)) \\ &= q_1(\lambda)(q_2(\lambda)(\cdots (q_m(\lambda)v) \cdots)) \\ &= (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v. \quad \square \end{aligned}$$

1 Suppose $T \in \mathcal{L}(V)$ and $\exists n \in \mathbf{N}^+$ such that $T^n = 0$.

Prove that $(I - T)$ is invertible and $(I - T)^{-1} = I + T + \cdots + T^{n-1}$.

SOLUTION: Note that $1 - x^n = (1 - x)(1 + x + \cdots + x^{n-1})$.

$$\left. \begin{aligned} (I - T)(1 + T + \cdots + T^{n-1}) &= I - T^n = I \\ (1 + T + \cdots + T^{n-1})(I - T) &= I - T^n = I \end{aligned} \right\} \Rightarrow (I - T)^{-1} = 1 + T + \cdots + T^{n-1}. \quad \square$$

2 Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$.

Suppose λ is an eigval of T . Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

SOLUTION:

Suppose v is an eigvec corresponding to λ . Then for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.

Hence $0 = (T - 2I)(T - 3I)(T - 4I)v = (\lambda - 2)(\lambda - 3)(\lambda - 4)v$ while $v \neq 0 \Rightarrow \lambda = 2$ or $\lambda = 3$ or $\lambda = 4$. \square

OR. Because $(T - 2I)(T - 3I)(T - 4I) = 0$ is not injective. By TIPS. \square

7 Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigval of $T^2 \iff 3$ or -3 is an eigval of T .

SOLUTION: COMMENT: Note that V can be infinite-dim. See also in (5.A.22).

(a) Suppose 9 is an eigval of T^2 . Then $(T^2 - 9I)v = (T - 3I)(T + 3I)v = 0$ for some v . By TIPS.

(b) Suppose 3 or -3 is an eigval of T with an eigenvector v . Then $Tv = \pm 3v \Rightarrow T^2 v = T(Tv) = 9v$ \square

3 Suppose $T \in \mathcal{L}(V)$, $T^2 = I$ and -1 is not an eigval of T . Prove that $T = I$.

SOLUTION:

$T^2 - I = (T + I)(T - I)$ is not injective, $\text{又 } -1$ is not an eigval of $T \Rightarrow$ By TIPS. \square

OR. Note that $v = [\frac{1}{2}(I - T)v] + [\frac{1}{2}(I + T)v]$ for all $v \in V$.

And $(I - T^2)v = (I - T)(I + T)v = 0$ for all $v \in V$,

$$\left. \begin{aligned} (I + T)(\frac{1}{2}(I - T)v) &= \frac{1}{2}(I - T^2)v = 0 \Rightarrow \frac{1}{2}(I - T)v \in \text{null}(I + T) \\ (I - T)(\frac{1}{2}(I + T)v) &= \frac{1}{2}(I - T^2)v = 0 \Rightarrow \frac{1}{2}(I + T)v \in \text{null}(I - T) \end{aligned} \right\} \Rightarrow V = \text{null}(I + T) + \text{null}(I - T).$$

$\text{又 } -1$ is not an eigval of $T \Rightarrow (I + T)$ is injective $\Rightarrow \text{null}(I + T) = \{0\}$.

Hence $V = \text{null}(I - T) \Rightarrow \text{range}(I - T) = \{0\}$. Thus $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$. \square

• (4E 5.A.32) Suppose $T \in \mathcal{L}(V)$ has no eigvals and $T^4 = I$. Prove that $T^2 = -I$.

SOLUTION:

Because $T^4 - I = (T^2 - I)(T^2 + I) = 0$ is not injective $\Rightarrow (T^2 - I)$ or $(T^2 + I)$ is not injective.

又 T has no eigvals $\Rightarrow (T^2 - I) = (T - I)(T + I)$ is injective,

for if not, $(T - I)$ or $(T + I)$ is not injective. Contradicts.

Hence $T^2 + I = 0 \in \mathcal{L}(V)$, for if not, $\exists v \in V, (T^2 + I)v \neq 0$ while $(T^2 - I)((T^2 + I)v) = 0$. Contradicts. \square

OR. Note that $v = [\frac{1}{2}(I - T^2)v] + [\frac{1}{2}(I + T^2)v]$ for all $v \in V$.

And $(I - T^4)v = (I - T^2)(I + T^2)v = 0$ for all $v \in V$,

$$\left. \begin{aligned} (I + T^2)(\frac{1}{2}(I - T^2)v) &= 0 \Rightarrow \frac{1}{2}(I - T^2)v \in \text{null}(I + T^2) \\ (I - T^2)(\frac{1}{2}(I + T^2)v) &= 0 \Rightarrow \frac{1}{2}(I + T^2)v \in \text{null}(I - T^2) \end{aligned} \right\} \Rightarrow V = \text{null}(I + T^2) + \text{null}(I - T^2).$$

又 T has no eigvals $\Rightarrow (I - T^2)$ is injective $\Rightarrow \text{null}(I - T^2) = \{0\}$.

Hence $V = \text{null}(I + T^2) \Rightarrow \text{range}(I + T^2) = \{0\}$. Thus $I + T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$. \square

8 (OR 4E 5.A.31) Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$.

SOLUTION:

Simply by computing: $p(z) = z^4 + 1 = (z^2 + i)(z^2 - i) = (z + i^{1/2})(z - i^{1/2})(z - (-i)^{1/2})(z + (-i)^{1/2})$.

Note that $i^{1/2} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, $(-i)^{1/2} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.

Hence $T = \pm(\pm i)^{1/2}$.

Define T by $T(x, y) = (\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y)$.

$$\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix} \Rightarrow \mathcal{M}(T)^4 = \mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathcal{M}(I). \quad \square$$

$$\left(\text{Using } \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix} \right)$$

• (4E 5.B.12 See also at 5.A.9)

Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$. Find the mini poly.

SOLUTION:

$T(x_1, \dots, 0) = 0$ By (5.A.9) and [8.49], $1, 2, \dots, n$ are zeros of the mini poly of T .

(又 Each eigvals of T corresponds to exact one-dim subspace of \mathbf{F}^n .)

Define a poly q by $q(z) = (z - 1)(z - 2) \cdots (z - n)$, for all $z \in \mathbf{F}$. (Then q is the char poly of T .)

Because $q(T)e_j = [(T - I) \cdots (T - (j - 1)I)(T - (j + 1)I) \cdots (T - nI)](T - jI)e_j = 0$ for each j ,

where (e_1, \dots, e_n) is the standard basis. Thus $\forall v \in \mathbf{F}^n, q(T)v = 0$. Hence q is the mini poly of T . \square

• Suppose $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.

[See also at (5.A.19)] Find the mini poly of T .

SOLUTION:

Because n and 0 are all eigvals of T , 又 For all $e_k, Te_k = e_1 + \cdots + e_n$; $T^2e_k = n(e_1 + \cdots + e_n)$.

Hence $T^2e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T - n)$. Thus $z(z - n)$ is the mini poly of T . \square

• (4E 5.B.8)

Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is the operator of counterclockwise rotation by the angle θ , where $x \in \mathbf{R}^+$. Find the minimal polynomial of T .

SOLUTION:

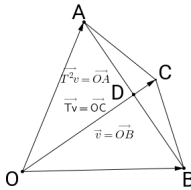
If $\theta = \pi$, then $T(w, z) = (-w, -z)$, $T^2 = I$ and the mini poly is $z + 1$.

If $2\pi|\theta$, then $T = I$ and the mini poly is $z - 1$.

Now suppose (v, Tv) is linearly independent.

Then $\text{span}(v, Tv) = \mathbf{R}^2$.

Suppose the mini poly p is defined by $p(z) = z^2 + bz + c$ for all $z \in \mathbf{R}$.

Because 

$$\left\{ \begin{array}{l} Tv = \frac{|\vec{v}|}{2L}(T^2v + v) \Rightarrow T = \frac{|\vec{v}|}{2L}(T^2 + I) \\ L = |\vec{v}| \cos \theta \Rightarrow \frac{|\vec{v}|}{2L} = \frac{1}{2 \cos \theta} \end{array} \right.$$

Hence $p(T) = T^2 - 2 \cos \theta T + I = 0$. $z^2 - 2 \cos \theta z + 1$ is the mini poly of T . \square

• (4E 5.B.11)

Suppose V is a two-dim vecsp, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) Show that $T^2 - (a + d)T + (ad - bc)I = 0$.

(b) Show that the mini poly of T equals

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a + d)z + (ad - bc) & \text{otherwise.} \end{cases}$$

SOLUTION:

(a) Suppose the basis is (v, w) . Because $\begin{cases} Tv = av + bw \Rightarrow (T - aI)v = bw, \text{ then apply } (T - dI) \text{ to both sides.} \\ Tw = cv + dw \Rightarrow (T - dI)w = cv, \text{ then apply } (T - aI) \text{ to both sides.} \end{cases}$

Hence $(T - aI)(T - dI) = bcI \Rightarrow T^2 - (a + d)T + (ad - bc)I = 0$.

(b) If $b = c = 0$ and $a = d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the mini poly is $z - a$.

Otherwise, by (a), $z^2 - (a + d)z + (ad - bc)$ is a polynomial multiple of the mini poly.

Now we prove that $T \notin \text{span}(I)$, so that then the mini poly of T has exactly degree 2.

(At least one of the assumption of (I),(II) below is true.)

(I) Suppose $a = d$, then $Tv = av + bw \notin \text{span}(v)$, $Tw = cv + aw \notin \text{span}(w)$.

(II) Suppose at most one of b, c is not 0. If $b = 0$, then $Tw \notin \text{span}(w)$; If $c = 0$, then $Tv \notin \text{span}(v)$. \square

5 Suppose $S, T \in \mathcal{L}(V)$, S is invertible, and $p \in \mathcal{P}(\mathbf{F})$. Prove that $p(TS) = S^{-1}p(ST)S$.

SOLUTION: We prove $(TS)^m = S^{-1}(ST)^mS$ for each $m \in \mathbf{N}$ by induction.

(i) $m = 0, 1$. $TS^0 = I = S^{-1}(ST)^0S$; $TS = S^{-1}(ST)S$.

(ii) $m > 1$. Assume that $(TS)^m = S^{-1}(ST)^mS$.

Then $(TS)^{m+1} = (TS)^m(TS) = S^{-1}(ST)^mSTS = S^{-1}(ST)^{m+1}S$.

Hence $\forall p \in \mathcal{P}(\mathbf{F})$, $p(TS) = a_0(TS)^0 + a_1(TS) + \cdots + a_m(TS)^m$

$$= a_0[S^{-1}(ST)^0S] + a_1[S^{-1}(ST)S] + \cdots + a_m[S^{-1}(ST)^mS]$$

$$= S^{-1}[a_0(ST)^0 + a_1(ST) + \cdots + a_m(ST)^m]S = S^{-1}p(ST)S. \quad \square$$

• (4E 5.B.7)

- (a) Give an example of $S, T \in \mathcal{L}(\mathbf{F}^2)$ such that the mini poly of ST does not equal the mini poly of TS .
- (b) Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that if S or T is invertible, then the mini poly of ST equals the mini poly of TS .

SOLUTION:

- (a) Define S by $S(x, y) = (x, x)$. Define T by $T(x, y) = (0, y)$.

Then $ST(x, y) = 0$, $TS(x, y) = (0, x)$ for all $(x, y) \in \mathbf{F}^2$.

Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.

Hence the mini poly of ST does not equal to the mini poly of TS .

- (b) Denote the mini poly of ST by p , and the mini poly TS by q .

Suppose S is invertible.

$$\left. \begin{aligned} p(ST) = 0 &= Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a polynomial multiple of } q. \\ q(TS) = 0 &= S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a polynomial multiple of } p. \end{aligned} \right\} \Rightarrow p = q.$$

Reversing the roles of S and T , we conclude that if T is invertible, then $p = q$ as well. \square

11 Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$, and $\alpha \in \mathbf{C}$.

Prove that α is an eigval of $p(T) \iff \alpha = p(\lambda)$ for some eigval λ of T .

SOLUTION:

- (a) Suppose α is an eigval of $p(T) \iff (p(T) - \alpha I)$ is not injective.

Write $p(z) - \alpha = c(z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow p(T) - \alpha I = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.

By TIPS, $\exists (T - \lambda_j I)$ not injective. Thus $p(\lambda_j) - \alpha = 0$. \square

- (b) Suppose $\alpha = p(\lambda)$ and λ is an eigval of T with an eigvec v . Then $p(T)v = p(\lambda)v = \alpha v$. \square

OR. Define q by $q(z) = p(z) - \alpha$. λ is a zero of q .

Because $q(T)v = (p(T) - \alpha I)v = q(\lambda)v = (p(\lambda) - \alpha)v = 0$.

Hence $q(T)$ is not injective $\Rightarrow (p(T) - \alpha I)$ is not injective. \square

12 (OR 4E.5.B.6) Give an example of an operator on \mathbf{R}^2

that shows the result above does not hold if \mathbf{C} is replaced with \mathbf{R} .

SOLUTION:

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(w, z) = (-z, w)$.

By Problem (4E 5.B.11), $\mathcal{M}(T, ((1, 0), (0, 1))) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ the mini poly of T is $z^2 + 1$.

Define p by $p(z) = z^2$. Then $p(T) = T^2 = -I$. Thus $p(T)$ has eigval -1 .

While $\nexists \lambda \in \mathbf{R}$ such that $-1 = p(\lambda) = \lambda^2$. \square

• (4E 5.B.17)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and p is the mini poly of T . Suppose $\lambda \in \mathbf{F}$.

Show that the mini poly of $(T - \lambda I)$ is the polynomial q defined by $q(z) = p(z + \lambda)$.

SOLUTION:

$q(T - \lambda I) = 0 \Rightarrow q$ is polynomial multiple of the mini poly of $(T - \lambda I)$.

Suppose the degree of the mini poly of $(T - \lambda I)$ is n , and the degree of the mini poly of T is m .

By definition of mini poly,

n is the smallest such that $(T - \lambda I)^n \in \text{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{n-1})$;

m is the smallest such that $T^m \in \text{span}(I, T, \dots, T^{m-1})$.

$\text{又 } T^k \in \text{span}(I, T, \dots, T^{k-1}) \iff (T - \lambda)^k \in \text{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{k-1})$.

Thus $n = m$. 又 q is monic. By the uniqueness of mini poly. \square

• (4E 5.B.18)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and p is the mini poly of T . Suppose $\lambda \in \mathbf{F} \setminus \{0\}$. Show that the mini poly of λT is the polynomial q defined by $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.

SOLUTION:

$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow q$ is a polynomial multiple of the mini poly of λT .

Suppose the degree of the mini poly of λT is n , and the degree of the mini poly of T is m .

By definition of mini poly,

n is the smallest such that $(\lambda T)^n \in \text{span}(\lambda I, \lambda T, \dots, (\lambda T)^{n-1})$;

m is the smallest such that $T^m \in \text{span}(I, T, \dots, T^{m-1})$.

$\text{又 } (\lambda T)^k \in \text{span}(\lambda I, \lambda T, \dots, (\lambda T)^{k-1}) \iff T^k \in \text{span}(I, T, \dots, T^{k-1})$.

Thus $n = m$. 又 q is monic. By the uniqueness of mini poly. \square

18 (OR 4E 5.B.15)

Suppose V is a finite-dim complex vector space with $\dim V > 0$ and $T \in \mathcal{L}(V)$.

Define $f : \mathbf{C} \rightarrow \mathbf{R}$ by $f(\lambda) = \dim \text{range}(T - \lambda I)$. Prove that f is not a continuous function.

SOLUTION: Note that V is finite-dim.

Let λ_0 be an eigval of T . Then $(T - \lambda_0 I)$ is not surjective. Hence $\dim \text{range}(T - \lambda_0 I) < \dim V$.

Because T has finitely many eigvals. There exist a sequence of number $\{\lambda_n\}$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda_0$.

And λ_n is not an eigval of T for each $n \Rightarrow \dim \text{range}(T - \lambda_n I) = \dim V \neq \dim \text{range}(T - \lambda_0 I)$.

Thus $f(\lambda_0) \neq \lim_{n \rightarrow \infty} f(\lambda_n)$. \square

• (4E 5.B.9)

Suppose $T \in \mathcal{L}(V)$ is such that with respect to some basis of V , all entries of the matrix of T are rational numbers.

Explain why all coefficients of the mini poly of T are rational numbers.

SOLUTION:

Let (v_1, \dots, v_n) denote the basis such that $\mathcal{M}(T, (v_1, \dots, v_n))_{j,k} = A_{j,k} \in \mathbf{Q}$ for all $j, k = 1, \dots, n$.

Denote $\mathcal{M}(v_j, (v_1, \dots, v_n))$ by x_j for each v_j .

Suppose p is the mini poly of T and $p(z) = z^m + \dots + c_1 z + c_0$. Now we show that each $c_j \in \mathbf{Q}$.

Note that $\forall s \in \mathbf{N}^+$, $\mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbf{Q}^{n,n}$ and $T^s v_k = A_{1,k}^s v_1 + \dots + A_{n,k}^s v_n$ for all $k \in \{1, \dots, n\}$.

$$\text{Thus } \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \dots + c_1 A + c_0 I)x_1 = \sum_{j=1}^n (A^m + \dots + c_1 A + c_0 I)_{j,1} x_j = 0; \\ \vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \dots + c_1 A + c_0 I)x_n = \sum_{j=1}^n (A^m + \dots + c_1 A + c_0 I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly, } \begin{cases} (A^m + \dots + c_1 A + c_0 I)_{1,1} = \dots = (A^m + \dots + c_1 A + c_0 I)_{n,1} = 0; \\ \vdots \quad \ddots \quad \vdots \\ (A^m + \dots + c_1 A + c_0 I)_{1,n} = \dots = (A^m + \dots + c_1 A + c_0 I)_{n,n} = 0; \end{cases}$$

Hence we get a system of n^2 linear equations in m unknowns c_0, c_1, \dots, c_{m-1} .

We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbf{Q}$. \square

- OR (4E 5.B.16), OR (8.C.18)

Suppose $a_0, \dots, a_{n-1} \in \mathbf{F}$. Let T be the operator on \mathbf{F}^n such that

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & -a_0 \\ 1 & 0 & & -a_1 \\ & 1 & \ddots & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ with respect to the standard basis } (e_1, \dots, e_n).$$

Show that the mini poly of T is p defined by $p(z) = a_0 + a_1z + \dots + a_{n-1}z^{n-1} + z^n$.

$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the mini poly of some operator. Hence a formula or an algorithm that could produce exact eigvals for each operator on each \mathbf{F}^n could then produce exact zeros for each poly [by 8.36(b)]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an operator.

SOLUTION: Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ is linearly independent. 又 The deg of mini poly is at most n .

$$\begin{aligned} T^n e_1 &= \dots = T^{n-k} e_{1+k} = \dots = T e_n = -a_0 e_1 - a_1 e_2 - a_2 e_3 - \dots - a_{n-1} e_n \\ &= (-a_0 I - a_1 T - a_2 T^2 - \dots - a_{n-1} T^{n-1}) e_1. \text{ Thus } p(T) e_1 = 0 = p(T) e_j \text{ for each } e_j = T^{j-1} e_1. \quad \square \end{aligned}$$

- EIGENVALUES ON ODD-DIMENSIONAL REAL VECTOR SPACES (Eigvals on Odd-dim Real Vecsps)
- EVEN-DIMENSIONAL NULL SPACE

Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$.

Prove that $\dim \text{null}(T^2 + bT + cI)$ is an even number.

SOLUTION:

Denote $\text{null}(T^2 + bT + cI)$ by R . Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.

Suppose λ is an eigval of T_R with an eigvec $v \in R$.

$$\text{Then } 0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \left((\lambda + b)^2 + c - \frac{b^2}{4} \right) v.$$

Because $c - \frac{b^2}{4} > 0$ and we have $v = 0$. Thus T_R has no eigvals.

Let U be an invariant subspace of R that has the largest, even dim among all invariant subspaces.

Assume that $U \neq R$. Then $\exists w \in R$ but $w \notin U$. Let W be such that $(w, T|_R w)$ is a basis of W .

Because $T|_R^2 w = -bT|_R w - cw \in W$. Hence W is an invariant subspace of dim 2.

Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,

for if not, because $w \notin U$, $T|_R w \in U$,

$U \cap W$ is invariant under $T|_R$ of one dim (impossible because $T|_R$ has no eigvecs).

Hence $U + W$ is even-dim invariant subspace under $T|_R$, contradicting the maximality of $\dim U$.

Thus the assumption was incorrect. Hence $R = \text{null}(T^2 + bT + cI) = U$ has even dim. \square

- OPERATORS ON ODD-DIMENSIONAL VECTOR SPACES HAVE EIGENVALUES

(a) Suppose $\mathbf{F} = \mathbf{C}$. Then by [5.21], we are done.

(b) Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, and $\dim V = n \neq 0$ is an odd number.

Let $T \in \mathcal{L}(V)$ and the mini poly is p . Prove that T has an eigval.

SOLUTION:

(i) If $n = 1$, then we are done.

(ii) Suppose $n \geq 3$. Assume that every operator, on odd-dim vecsps of dim less than n , has an eigval.

If p is a polynomial multiple of $(x - \lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] λ is an eigval of T and we are done.

Now suppose $b, c \in \mathbf{R}$ such that $b^2 < 4c$ and p is a polynomial multiple of $x^2 + bx + c$ (see [4.17]).

Then $\exists q \in \mathcal{P}(\mathbf{R})$ such that $p(x) = q(x)(x^2 + bx + c)$ for all $x \in \mathbf{R}$.

Now $0 = p(T) = (q(T))(T^2 + bT + cI)$, which means that $q(T)|_{\text{range}(T^2+bT+cI)} = 0$.

Because $\deg q < \deg p$ and p is the mini poly of T , hence $\text{range}(T^2 + bT + cI) \neq V$.

又 $\dim V$ is odd and $\dim \text{null}(T^2 + bT + cI)$ is even (by our previous result).

Thus $\dim V - \dim \text{null}(T^2 + bT + cI) = \dim \text{range}(T^2 + bT + cI)$ is odd.

By [5.18], $\text{range}(T^2 + bT + cI)$ is an invariant subspace of V under T that has odd dim less than n .

Our induction hypothesis now implies that $T|_{\text{range}(T^2+bT+cI)}$ has an eigenvalue.

By mathematical induction. \square

• (2E Ch5.24)

Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$ has no eigvals.

Prove that every invariant subspace of V under T is even-dim.

SOLUTION:

Suppose U is such a subspace. Then $T|_U \in \mathcal{L}(U)$. We prove by contradiction.

If $\dim U$ is odd, then $T|_U$ has an eigval and so is T , so that \exists invariant subspace of 1 dim, contradicts. \square

• (4E 5.B.29)

Show that every operator on a finite-dim vecsp of $\dim \geq 2$

has an invariant subspace of dim 2.

Exercise (4E 5.C.6) will give an improvement of this result when $\mathbf{F} = \mathbf{C}$.

SOLUTION:

Using induction on $\dim V$.

(i) $\dim V = 2$, we are done.

(ii) $\dim V > 2$. Assume that the desired result is true for vecsp of smaller dim.

Suppose p is the mini poly of degree m and $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.

If $T = \lambda I$ ($\Leftrightarrow m = 1 \vee m = -\infty$), then we are done. ($m \neq 0$ because $\dim V \neq 0$.)

Now define a q by $q(z) = (z - \lambda_1)(z - \lambda_2)$.

By assumption, $T|_{\text{null } q(T)}$ has an invariant subspace of dim 2. \square

ENDED

5.B: II

• NOTE FOR []:

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• (4E 5.C.1)

Prove or give a counterexample:

If $T \in \mathcal{L}(V)$ and T^2 has an upper-trig matrix with respect to some basis of V , then T has an upper-trig matrix with respect to some basis of V .

SOLUTION:

• (4E 5.C.2)

Suppose A and B are upper-trig matrices of the same size,

with $\alpha_1, \dots, \alpha_n$ on the diagonal of A and β_1, \dots, β_n on the diagonal of B .

(a) Show that $A + B$ is an upper-trig matrix with $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ on the diagonal.

(b) Show that AB is an upper-trig matrix with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diagonal.

SOLUTION:

• (4E 5.C.3)

Suppose $T \in \mathcal{L}(V)$ is invertible and (v_1, \dots, v_n) is a basis of V with respect to which the matrix of T is upper trig, with $\lambda_1, \dots, \lambda_n$ on the diagonal.

Show that the matrix of T^{-1} is also upper trig with respect to the same basis, with $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ on the diagonal.

SOLUTION:

9 (4E 5.C.7)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $v \in V$.

(a) Prove that $\exists!$ monic poly p_v of smallest degree such that $p_v(T)v = 0$.

(b) Prove that the mini poly of T is a polynomial multiple of p_v .

SOLUTION:

14 (OR 4E 5.C.4)

Give an operator T such that with respect to some basis, $\mathcal{M}(T)_{k,k} = 0$ for each k , while T is invertible.

SOLUTION:

15 (OR 4E 5.C.5)

Give an operator T such that with respect to some basis, $\mathcal{M}(T)_{k,k} \neq 0$ for each k , while T is not invertible.

SOLUTION:

20 (OR 4E 5.C.6)

Suppose $\mathbf{F} = \mathbf{C}$, V is finite-dim, and $T \in \mathcal{L}(V)$.

Prove that if $k \in \{1, \dots, \dim V\}$, then V has a k dim subspace invariant under T .

SOLUTION:

• (4E 5.C.8)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\exists v \in V \setminus \{0\}$ such that $T^2v + 2Tv = -2v$.

(a) Prove that if $\mathbf{F} = \mathbf{R}$, then \nexists a basis of V with respect to which T has an upper-trig matrix.

(b) Prove that if $\mathbf{F} = \mathbf{C}$ and A is an upper-trig matrix that equals the matrix of T with respect to some basis of V , then $-1 + i$ or $-1 - i$ appears on the diagonal of A .

SOLUTION:

• (4E 5.C.9)

Suppose $B \in \mathbf{F}^{n,n}$ with complex entries.

*Prove that \exists invertible $A \in \mathbf{F}^{n,n}$ with complex entries
such that $A^{-1}BA$ is an upper-trig matrix.*

SOLUTION:

• (4E 5.C.10)

Suppose $T \in \mathcal{L}(V)$ and (v_1, \dots, v_n) is a basis of V .

Show that the following are equivalent.

(a) *The matrix of T with respect to (v_1, \dots, v_n) is lower trig.*

(b) *$\text{span}(v_k, \dots, v_n)$ is invariant under T for each $k = 1, \dots, n$.*

(c) *$Tv_k \in \text{span}(v_k, \dots, v_n)$ for each $k = 1, \dots, n$.*

A square matrix is called lower trig if all entries above the diagonal are 0.

SOLUTION:

• (4E 5.C.11)

Suppose $\mathbf{F} = \mathbf{C}$ and V is finite-dim.

Prove that if $T \in \mathcal{L}(V)$, then T has a lower-trig matrix with respect to some basis.

SOLUTION:

• (4E 5.C.12)

*Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has an upper-trig matrix with respect to some basis,
and U is a subspace of V that is invariant under T .*

(a) *Prove that $T|_U$ has an upper-trig matrix with respect to some basis of U .*

(b) *Prove that T/U has an upper-trig matrix with respect to some basis of V/U .*

SOLUTION:

• (4E 5.C.13)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Suppose $\exists U$ of V that is invariant under T such that

$T|_U$ has an upper-trig matrix with respect to some basis of U

and also T/U has an upper-trig matrix with respect to some basis of V/U .

Prove that T has an upper-trig matrix with respect to some basis of V .

SOLUTION:

• (4E 5.C.14)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

*Prove that T has an upper-trig matrix with respect to some basis of V
 $\iff T'$ has an upper-trig matrix with respect to some basis of V' .*

SOLUTION:

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5.E* (4E)

1 Give an example of two commuting operators $S, T \in \mathbf{F}^4$ such that there is a subspace of \mathbf{F}^4 that is invariant under S but not under T and there is a subspace of \mathbf{F}^4 that is invariant under T but not under S .

SOLUTION:

2 Suppose \mathcal{E} is a subset of $\mathcal{L}(V)$ and every element of \mathcal{E} is diagonalizable.

Prove that \exists a basis of V with respect to which

every element of \mathcal{E} has a diagonal matrix \iff every pair of elements of \mathcal{E} commutes.

This exercise extends [5.76], which considers the case in which \mathcal{E} contains only two elements.

For this exercise, \mathcal{E} may contain any number of elements, and \mathcal{E} may even be an infinite set.

SOLUTION:

3 Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.

(a) Prove that $\text{null } p(S)$ is invariant under T .

(b) Prove that $\text{range } p(S)$ is invariant under T .

See NOTE FOR [5.17] for the special case $S = T$.

SOLUTION:

4 Prove or give a counterexample:

If A is a diagonal matrix and B is an upper-trig matrix of the same size as A , then A and B commute.

SOLUTION:

5 Prove that a pair of operators on a finite-dim vecsp commute

\iff their dual operators commute.

SOLUTION:

6 Suppose V is a finite-dim complex vecsp and $S, T \in \mathcal{L}(V)$ commute.

Prove that $\exists \alpha, \lambda \in \mathbf{C}$ such that $\text{range } (S - \alpha I) + \text{range } (T - \lambda I) \neq V$.

SOLUTION:

7 Suppose V is a complex vecsp, $S \in \mathcal{L}(V)$ is diagonalizable, and T commutes with S .

Prove that \exists basis B of V such that S has a diagonal matrix with respect to B and T has an upper-trig matrix with respect to B .

SOLUTION:

8 Suppose $m = 3$ in Example [5.72]

and D_x, D_y are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbf{R}^2)$ from that example.

Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ with respect to which D_x and D_y each have an upper-trig matrix.

SOLUTION:

9 Suppose V is a finite-dim nonzero complex vecsp.

Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.

(a) Prove that $\exists v \in V$ is an eigvec for every element of \mathcal{E} .

(b) Prove that \exists a basis of V with respect to which every element of \mathcal{E} has an upper-trig matrix.

SOLUTION:

10 Give an example of two commuting operators S, T on a finite-dim real vecsp such that $S + T$ has a eigval that does not equal an eigval of S plus an eigval of T and ST has a eigval that does not equal an eigval of S times an eigval of T .

SOLUTION:

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SOLUTION:

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