

简介

这是我个人用于复习的笔记，一本习题补注。由于我个人的复习特点，我把许多对我个人而言没什么复习价值的习题作了省略。为什么我没有用中文？因为我将来要学习的绝大多数数学课本都是全英的，国内目前的专业翻译速度慢、不全面，况且对于专业学习者来说，直接使用英文不会造成任何困扰，并且我不愿意花费额外的时间去翻译，所以我用英文。但我讨厌英文单词的冗长性，这会让我复习起来很不爽，所以我对许多常用词汇适当地作了简写。这份笔记的内容范围和标识说明，我已经在[README](#)中写得很清楚，不再赘述。这份笔记尚处于缓慢的编撰进度中。

Goto

1	2	3	4	5	6	7	8	9	10
A	A	A	/	A	A	A	A	A	A
B	B	B	/	B ^I	B	B	B	B	B
/	/	/	/	B ^{II}	/	/	/	/	/
C	C	C	/	C	C	C	C	/	/
/	/	D	/	/	D	D	D	/	/
/	/	E	/	E*	/	/	/	/	/
/	/	F	/	/	/	F*	/	/	/

Abbreviation Table

def	definition
vec	vector
vecsp	vector space
subsp	subspace
add	addition/additive
multi	multiplication/multiplicative/multiple
assoc	associative/associativity
distr	distributive properties/property
inv	inverse
existns	existence
uniques	uniqueness
linely inde	linearly independent/independence
linely dep	linearly dependent/dependence
dim	dimension(al)
inje	injective
surj	surjective
col	column
with resp	with respect
standard basis	std basis
iso	isomorphism/isomorphic
correspd	correspond(ing)
poly	polynomial
eigval	eigenvalue
eigvec	eigenvector
mini poly	minimal polynomial
char poly	characteristic polynomial

1.B

1 Prove that $\forall v \in V, -(-v) = v$.

SOLUTION:

$$\left. \begin{array}{l} -(-v) + (-v) = 0 \\ v + (-v) = 0 \end{array} \right\} \Rightarrow \text{By the uniqueness of add inv, we are done.}$$

$$\text{OR. } -(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v. \quad \square$$

2 Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

SOLUTION:

$$\text{Suppose } a \neq 0, \exists a^{-1} \in \mathbf{F}, a^{-1}a = 1, \text{ hence } v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0. \quad \square$$

3 Suppose $v, w \in V$. Explain why $\exists! x \in V, v + 3x = w$.

SOLUTION:

$$[\text{Existence}] \text{ Let } x = \frac{1}{3}(w - v).$$

$$[\text{Uniqueness}] \text{ Suppose } v + 3x_1 = w, \text{ (I) } v + 3x_2 = w \text{ (II). Then (I) - (II) : } 3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2. \quad \square$$

$$\text{OR. } v + 3x = w \Leftrightarrow 3x = w - v \Leftrightarrow x = \frac{1}{3}(w - v). \quad \square$$

5 Show that in the def of a vecsp, the add inv condition can be replaced by [1.29].

Hint: Suppose V satisfies all conds in the def, except we've replaced the add inv cond with [1.29].

Prove that the add inv is true.

SOLUTION:

$$\text{Using [1.31]. } 0v = 0 \text{ for all } v \in V \Leftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0. \quad \square$$

6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in \mathbf{R} .

Define an add and scalar multi on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess.

The operations of real numbers is as usual. While for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

$$\text{(I) } t + \infty = \infty + t = \infty + \infty = \infty,$$

$$\text{(II) } t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty,$$

$$\text{(III) } \infty + (-\infty) = (-\infty) + \infty = 0.$$

With these operations of add and scalar multi, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vecsp over \mathbf{R} ? Explain.

SOLUTION:

Not a vecsp, since the add and scalar multi is not assoc and distr.

By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

$$\text{OR. By Distr: } \infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0. \quad \square$$

• TIPS: About the Field \mathbf{F} : Many choices.

EXAMPLE: $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \dots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$.

1.C 7 8 9 11 12 13 15 16 17 18 21 22 23 24

7 Give a nontrivial $U \subseteq \mathbb{R}^2$,

U is closed under taking add invs and under add, but is not a subsp of \mathbb{R}^2 .

SOLUTION: ($0 \in U$; $v \in U \Rightarrow -v \in U$.) Let $U = \{0, 1\}^2, \mathbb{Z}^2, \mathbb{Q}^2$.

8 Give a nontrivial $U \subseteq \mathbb{R}^2$, U is closed under scalar multi, but is not a subsp of \mathbb{R}^2 .

SOLUTION: Let $U = \{(x, y) \in \mathbb{R}^2 : x = 0 \vee y = 0\}$.

9 A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called periodic if $\exists p \in \mathbb{N}^+$, $f(x) = f(x + p)$ for all $x \in \mathbb{R}$.
Is the set of periodic functions $\mathbb{R} \rightarrow \mathbb{R}$ a subsp of $\mathbb{R}^{\mathbb{R}}$? Explain.

SOLUTION: Denote the set by S .

Suppose $h(x) = \cos x + \sin \sqrt{2}x \in S$, since $\cos x, \sin \sqrt{2}x \in S$.

Assume $\exists p \in \mathbb{N}^+$ such that $h(x) = h(x + p)$, $\forall x \in \mathbb{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin \sqrt{2}p = \cos p - \sin \sqrt{2}p$

$\Rightarrow \sin \sqrt{2}p = 0$, $\cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbb{Z}$, while $p = \frac{m\pi}{\sqrt{2}}, m \in \mathbb{Z}$.

Hence $2k = \frac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \frac{m}{2k} \in \mathbb{Q}$. Contradiction! □

OR. Because [I] : $\cos x + \sin \sqrt{2}x = \cos(x + p) + \sin(\sqrt{2}x + \sqrt{2}p)$. By differentiating twice,

[II] : $\cos x + 2 \sin \sqrt{2}x = \cos(x + p) + 2 \sin(\sqrt{2}x + \sqrt{2}p)$.

[II] - [I] : $\sin \sqrt{2}x = \sin(\sqrt{2}x + \sqrt{2}p)$
 $2[I] - [II] :$ $\cos x = \cos(x + p)$ $\left\} \Rightarrow \text{Let } x = 0, p = \frac{m\pi}{\sqrt{2}} = 2k\pi. \text{ Contradicts.} \right.$ □

• Suppose U, W, V_1, V_2, V_3 are subsp of V .

15 $U + U \ni u + w \in U$. □

16 $U + W \ni u + w = w + u \in W + U$. □

17 $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3)$. □

18 Does the add on the subsp of V have an add identity? Which subsp have add invs?

SOLUTION: Suppose Ω is the additive identity.

(a) For any subsp U of V . $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

(b) Now suppose W is an add inv of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. □

11 Prove that the intersection of every collection of subsp of V is a subsp of V .

SOLUTION: Suppose $\{U_\alpha\}_{\alpha \in \Gamma}$ is a collection of subsp of V ; here Γ is an arbitrary index set.

We show that $\bigcap_{\alpha \in \Gamma} U_\alpha$, which equals the set of vecs that are in U_α for each $\alpha \in \Gamma$, is a subsp of V .

(一) $0 \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha \in \Gamma} U_\alpha \Rightarrow u + v \in U_\alpha, \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Closed under add.

(三) $u \in \bigcap_{\alpha \in \Gamma} U_\alpha, \lambda \in \mathbb{F} \Rightarrow \lambda u \in U_\alpha, \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Closed under scalar multi.

Thus $\bigcap_{\alpha \in \Gamma} U_\alpha$ is nonempty subset of V that is closed under add and scalar multi. □

12 Suppose U, W are subsp of V . Prove that $U \cup W$ is a subsp of $V \iff U \subseteq W$ or $W \subseteq U$.

SOLUTION:

(a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subsp of V .

(b) Suppose $U \cup W$ is a subsp of V . Suppose $U \not\subseteq W$ and $U \not\supseteq W$ ($U \cup W \neq U$ and W).

Then $\forall a \in U \wedge a \notin W, b \in W \wedge b \notin U, a + b \in U \cup W$.

$\left. \begin{array}{l} \text{If } a + b \in U \Rightarrow b = (a + b) + (-a) \in U, \text{ contradicts!} \\ \text{If } a + b \in W \Rightarrow a = (a + b) + (-b) \in W, \text{ contradicts!} \end{array} \right\} \Rightarrow U \cup W = U \text{ or } W. \text{ Contradicts!}$

Thus $U \subseteq W$ and $U \supseteq W$. □

13 Prove that the union of three subsp of V is a subsp of V if and only if one of the subsp contains the other two.

This exercise is not true if we replace \mathbf{F} with a field containing only two elements.

SOLUTION:

Suppose U_1, U_2, U_3 are subsp of V . Denote $U_1 \cup U_2 \cup U_3$ by \mathcal{U} .

(a) Suppose that one of the subsp contains the other two.

Then $\mathcal{U} = U_1, U_2$ or U_3 is a subsp of V .

(b) Suppose that $U_1 \cup U_2 \cup U_3$ is a subsp of V .

Distinctively notice that $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.

Also note that, if $U \cup W = V$ is a vecsp, then in general U and W are not subsp of V .

Hence this literal trick is invalid.

(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.

By applying Problem (12) we conclude that one U_j contains the other two. Thus we are done.

(II) Assume that no U_j is contained in the union of the other two,

and no U_j contains the union of the other two.

Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists u \in U_1 \wedge u \notin U_2 \cup U_3; v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if $v + \lambda u \in U_1$ then $v + \lambda u - \lambda u = v \in U_1$.

又 $W \subseteq U_1 \cup U_2 \cup U_3$. Thus $W \subseteq U_2 \cup U_3$.

$\forall v + \lambda u \in W, \exists i \in \{2, 3\}, v + \lambda u \in U_i$.

Because U_2, U_3 are subsp and hence have at least one element (zero).

If $U_2 = U_3$, then $\mathcal{U} = U_1 \cup U_2$ and by Problem (12) we are done.

(Note that at least one of U_2, U_3 is not $\{0\}$, for if not, $U_1 \supseteq U_2 \cup U_3$.)

Otherwise, \exists distinct $\lambda, \mu \in \mathbf{F}, v + \lambda u, v + \mu u \in U_i$ for some $i \in \{2, 3\}$. (Δ)

Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Contradicts.

EXAMPLE: Let $\mathbf{F} = \mathbf{Z}_2$. TODO

□

• **EXAMPLE:** Suppose $U = \{(x, x, y, y) \in \mathbf{F}^4\}$, $W = \{(x, x, x, y) \in \mathbf{F}^4\}$.

Prove that $U + W = \{(x, x, y, z) \in \mathbf{F}^4\}$.

Let T denote $\{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$. By def, $U + W \subseteq T$.

And $T \ni (x, x, y, z) \Rightarrow (0, 0, y - x, y - x) + (x, x, x, -y + x + z) \in U + W$. Hence $T \subseteq U + W$. \square

21 Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5\}$. Find a W such that $\mathbf{F}^5 = U \oplus W$.

SOLUTION: Let $W = \{(0, 0, z, w, u) \in \mathbf{F}^5\}$. Then $U \cap W = \{0\}$.

And $\mathbf{F}^5 \ni (x, y, z, w, u) \Rightarrow (x, y, x + y, x - y, 2x) + (0, 0, z - x - y, w - x - y, u - 2x) \in U + W$.

23 Give an example of vecsps V_1, V_2, U such that $V_1 \oplus U = V_2 \oplus U$, but $V_1 \neq V_2$.

SOLUTION: $V = \mathbf{F}^2$, $U = \{(x, x) \in \mathbf{F}^2\}$, $V_1 = \{(x, 0) \in \mathbf{F}^2\}$, $V_2 = \{(0, x) \in \mathbf{F}^2\}$.

• **TIPS:** Suppose $V_1 \subseteq V_2$ in Exercise (23). Prove or give a counterexample: $V_1 = V_2$.

SOLUTION:

Because the subset V_1 of vecsp V_2 is closed under add and scalar multi, V_1 is a subspace of V_2 .

Suppose W is such that $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$.

If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, contradicts. Hence $W = \{0\}$, $V_1 = V_2$. \square

Suppose $V_1 \oplus U_1 = V_2 \oplus U_2$, $V_1 \subseteq V_2$, $U_2 \subseteq U_1$. Prove or give a counterexample: $V_1 = V_2$.

24 Let $V_E = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is even}\}$, $V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is odd}\}$. Show that $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$.

SOLUTION: (a) $V_E \cap V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x)\} = \{0\}$.

(b) $\left\{ \begin{array}{l} \text{Let } f_e(x) = \frac{1}{2}(g(x) + g(-x)) \Rightarrow f_e \in V_E \\ \text{Let } f_o(x) = \frac{1}{2}(g(x) - g(-x)) \Rightarrow f_o \in V_O \end{array} \right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}}, g(x) = f_e(x) + f_o(x)$. \square

ENDED

2.A 1 2 6 10 11 14 16 17 | 4E: 3,14

2 (a) $[P]$ A list (v) of length 1 in V is linely inde $\iff v \neq 0$. $[Q]$

(b) $[P]$ A list (v, w) of length 2 in V is linely inde $\iff \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$. $[Q]$

SOLUTION:

(a) $Q \xrightarrow{1} P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ linely inde.

$P \xrightarrow{2} Q : (v)$ linely inde $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.

OR. $\left\{ \begin{array}{l} \neg Q \xrightarrow{3} \neg P : v = 0 \Rightarrow av = 0 \text{ while we can let } a \neq 0 \Rightarrow (v) \text{ is linely dep.} \\ \neg P \xrightarrow{4} \neg Q : (v) \text{ linely dep} \Rightarrow av = 0 \text{ while } a \neq 0 \Rightarrow v = 0. \end{array} \right.$

COMMENT: (1) with (3) and (2) with (4) will do as well. \square

(b) $P \xrightarrow{1} Q : (v, w)$ linely inde \Rightarrow if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.

$Q \xrightarrow{2} P : \text{no scalar multi} \Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v, w)$ linely inde.

OR. $\left\{ \begin{array}{l} \neg P \xrightarrow{3} \neg Q : (v, w) \text{ linely dep} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{scalar multi} \\ \neg Q \xrightarrow{4} \neg P : \text{scalar multi} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{linely dep.} \end{array} \right.$

COMMENT: (1) with (3) and (2) with (4) will do as well. \square

1 Prove that $[P] (v_1, v_2, v_3, v_4)$ spans $V \iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans $V [Q]$.

SOLUTION:

Notice that $V = \text{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists a_1, \dots, a_n \in \mathbb{F}, v = a_1 v_1 + \dots + a_n v_n$.

Assume that $\forall v \in V, \exists a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbb{F}$, (that is, if $\exists a_i$, then we are to find b_i , vice versa)

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= b_1 (v_1 - v_2) + b_2 (v_2 - v_3) + b_3 (v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4. \end{aligned}$$

Now we can let $b_i = \sum_{r=1}^i a_r$ if we are to prove Q with P already assumed;

or let $a_i = b_i - b_{i-1}$ with $b_0 = 0$, if we are to prove P with Q already assumed. \square

6 Prove that $[P] (v_1, v_2, v_3, v_4)$ is linely inde $\iff [Q] (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ is linely inde.

SOLUTION:

$$\begin{aligned} P \Rightarrow Q : a_1 (v_1 - v_2) + a_2 (v_2 - v_3) + a_3 (v_3 - v_4) + a_4 v_4 &= 0 \\ \Rightarrow a_1 v_1 + (a_2 - a_1) v_2 + (a_3 - a_2) v_3 + (a_4 - a_3) v_4 &= 0 \\ \Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \end{aligned}$$

$$\begin{aligned} Q \Rightarrow P : a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 &= 0 \\ \Rightarrow a_1 (v_1 - v_2) + (a_1 + a_2) (v_2 - v_3) + (a_1 + a_2 + a_3) (v_3 - v_4) + (a_1 + \dots + a_4) v_4 &= 0 \\ \Rightarrow a_1 = a_1 + a_2 = a_1 + a_2 + a_3 = a_1 + \dots + a_4 = 0. \end{aligned} \quad \square$$

• Suppose (v_1, \dots, v_m) is a list of vecs in V . For each k , let $w_k = v_1 + \dots + v_k$.

(a) Show that $\text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$.

(b) Show that $[P] (v_1, \dots, v_m)$ is linely inde $\iff (w_1, \dots, w_m)$ is linely inde $[Q]$.

SOLUTION:

$$(a) \text{ let } a_k = \sum_{j=1}^k b_j \Leftarrow a_1 v_1 + \dots + a_m v_m = b_1 w_1 + \dots + b_m w_m \Rightarrow \text{let } b_1 = a_1, b_k = a_k - \sum_{j=1}^{k-1} b_j = \sum_{j=1}^k (-1)^{k-j} a_j.$$

$$(b) P \Rightarrow Q : b_1 w_1 + \dots + b_m w_m = 0 = a_1 v_1 + \dots + a_m v_m, \text{ where } 0 = a_k = \sum_{j=1}^k b_j.$$

$$Q \Rightarrow P : a_1 v_1 + \dots + a_m v_m = 0 = b_1 w_1 + \dots + b_m w_m = 0, \text{ where } 0 = b_1 = a_1, 0 = b_k = \sum_{j=1}^k (-1)^{k-j} a_j$$

OR. Because $W = \text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$.

By [2.21](b), a list of length $(m-1)$ spans W , then by [2.23],

(w_1, \dots, w_m) linely dep $\Rightarrow (v_1, \dots, v_m)$ linely dep. Conversely it is true as well. \square

10 Suppose (v_1, \dots, v_m) is linely inde in V and $w \in V$.

Prove that if $(v_1 + w, \dots, v_m + w)$ is linely depe, then $w \in \text{span}(v_1, \dots, v_m)$.

SOLUTION:

Suppose $a_1 (v_1 + w) + \dots + a_m (v_m + w) = 0, \exists a_i \neq 0 \Rightarrow a_1 v_1 + \dots + a_m v_m = 0 = -(a_1 + \dots + a_m) w$.

Then $a_1 + \dots + a_m \neq 0$, for if not, $a_1 v_1 + \dots + a_m v_m = 0$ while $a_i \neq 0$ for some i , contradicts. \square

OR. By contrapositive, $w \notin \text{span}(v_1, \dots, v_m)$, similarly. \square

OR. $\exists j \in \{1, \dots, m\}, v_j + w \in \text{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and we are done.

If $j \geq 2$, then $\exists a_i \in \mathbb{F}, v_j + w = a_1 (v_1 + w) + \dots + a_{j-1} (v_{j-1} + w) \iff v_j + \lambda w = a_1 v_1 + \dots + a_{j-1} v_{j-1}$.

Where $\lambda = 1 - (a_1 + \dots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \text{span}(v_1, \dots, v_{j-1})$, contradicts.

Now $w = \lambda^{-1} (a_1 v_1 + \dots + a_{j-1} v_{j-1} - v_j) \Rightarrow w \in \text{span}(v_1, \dots, v_m)$. \square

11 Suppose (v_1, \dots, v_m) is linely inde in V and $w \in V$.

Show that $[P] (v_1, \dots, v_m, w) \text{ is linely inde} \iff w \notin \text{span}(v_1, \dots, v_m) [Q]$.

SOLUTION: $\neg Q \Rightarrow \neg P$: Suppose $w \in \text{span}(v_1, \dots, v_m)$. Then (v_1, \dots, v_m, w) is linely depe.

$\neg P \Rightarrow \neg Q$: Suppose (v_1, \dots, v_m, w) is linely dep. Then by [2.21] $w \in \text{span}(v_1, \dots, v_m)$. □

14 Prove that $[P] V \text{ is infinite-dim} \iff [Q] \left| \begin{array}{l} \text{there is a sequence } (v_1, v_2, \dots) \text{ in } V \text{ such that} \\ (v_1, \dots, v_m) \text{ is linely inde for each } m \in \mathbf{N}^+ \end{array} \right|$

SOLUTION:

$P \Rightarrow Q$: Suppose V is infinite-dim, so that no list spans V .

Step 1 Pick a $v_1 \neq 0, (v_1)$ linely inde.

Step m Pick a $v_m \notin \text{span}(v_1, \dots, v_{m-1})$, by Problem (10)(b), (v_1, \dots, v_m) is linely inde.

This process recursively defines the desired sequence (v_1, v_2, \dots) .

$\neg P \Rightarrow \neg Q$: Suppose V is finite-dim and $V = \text{span}(w_1, \dots, w_m)$.

Let (v_1, v_2, \dots) be a sequence in V , then $(v_1, v_2, \dots, v_{m+1})$ must be linely dep.

OR. $Q \Rightarrow P$: Suppose there is such a sequence.

Choose an m . Suppose a linely inde list (v_1, \dots, v_m) spans V .

(Similar to [2.16]) Then $\exists v_{m+1} \in V \setminus \text{span}(v_1, \dots, v_m)$.

Hence no list spans V . Thus V is infinite-dim. □

16 Prove that the vecsp of all continuous functions in $\mathbf{R}^{[0,1]}$ is infinite-dim.

SOLUTION: Denote the vecsp by U .

Choose an $m \in \mathbf{N}^+$. Suppose $a_0, \dots, a_m \in \mathbf{R}$ are such that $a_0 + a_1x + \dots + a_mx^m = 0, \forall x \in [0, 1]$.

Then the poly has infinitely many roots and hence $a_0 = \dots = a_m = 0$.

Thus $(1, x, \dots, x^m)$ is linely inde in $\mathbf{R}^{[0,1]}$. Similar to [2.16], U is infinite-dim. □

OR. Note that for $a_n = \frac{1}{n}, a_1 < a_2 < \dots < a_m, \forall m \in \mathbf{N}^+$.

Suppose $f_n = \begin{cases} x - \frac{1}{n}, & x \in \left(\frac{1}{n}, 1\right) \\ 0, & x \in \left[0, \frac{1}{n}\right] \end{cases}$ Then for any $m, f_1\left(\frac{1}{m}\right) = \dots = f_m\left(\frac{1}{m}\right)$, while $f_{m+1}\left(\frac{1}{m}\right) \neq 0$.

Hence $f_{m+1} \notin \text{span}(f_1, \dots, f_m)$. Thus by Problem (14), U is infinite-dim. □

17 Suppose $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$.

Prove that (p_0, p_1, \dots, p_m) is not linely inde in $\mathcal{P}_m(\mathbf{F})$.

SOLUTION:

Suppose (p_0, p_1, \dots, p_m) is linely inde. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z \forall z \in \mathbf{F}$.

But $\forall a_i \in \mathbf{F}, z \neq a_0p_0(z) + \dots + a_mp_m(z)$, for if not, let $z = 2$, contradicts. Thus $z \notin \text{span}(p_0, p_1, \dots, p_m)$.

Then $\text{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list (p_0, p_1, \dots, p_m) has length $(m+1)$.

Hence (p_0, p_1, \dots, p_m) is linely depe in $\mathcal{P}_m(\mathbf{F})$.

For if not, because $(1, z, \dots, z^m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$,

thus by [2.23] trivially, (p_0, p_1, \dots, p_m) spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. □

OR. Note that $\mathcal{P}_m(\mathbf{F}) = \text{span}(\underbrace{1, z, \dots, z^m}_{\text{of length } (m+1)})$. $(p_0, p_1, \dots, p_m, z)$ of length $(m+2)$ is linely dep.

(See the above) Now $z \notin \text{span}(p_0, p_1, \dots, p_m)$ and hence (p_0, p_1, \dots, p_m) is linely depe. □

7 Prove or give a counterexample: If (v_1, v_2, v_3, v_4) is a basis of V and U is a subsp of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then (v_1, v_2) is a basis of U .

SOLUTION: A counterexample:

Let $V = \mathbb{R}^4$ and e_j be the j^{th} standard basis.

Let $v_1 = e_1, v_2 = e_2, v_3 = e_3 + e_4, v_4 = e_4$. Then (v_1, \dots, v_4) is a basis of \mathbb{R}^4 .

Let $U = \text{span}(e_1, e_2, e_3) = \text{span}(v_1, v_2, v_3 - v_4)$. Then $v_3 \notin U$ and (v_1, v_2) is not a basis of U . \square

• NOTE FOR “ $\mathbb{C}_V U \cap \{0\}$ ”:

“ $\mathbb{C}_V U \cap \{0\}$ ” is supposed to be a subsp W such that $V = U \oplus W$.

But if we let $u \in U \setminus \{0\}$ and $w \in W \setminus \{0\}$, then $\left. \begin{array}{l} w \in \mathbb{C}_V U \cap \{0\} \\ u \pm w \in \mathbb{C}_V U \cap \{0\} \end{array} \right\} \Rightarrow u \in \mathbb{C}_V U \cap \{0\}$. Contradicts.

To fix this, denote the set $\{W_1, W_2 \dots\}$ by $\mathcal{S}_V U$, where for each $W_i, V = U \oplus W_i$. See also in (1.C.23).

1 Find all vecsps that have exactly one basis.

SOLUTION: The trivial vecsp $\{0\}$ will do. Indeed, the only basis of $\{0\}$ is the empty list.

Now consider a field containing only the add identity 0 and the multi identity 1,

and we specify that $1 + 1 = 0$. Hence the vecsp $\{0, 1\}$ will do, the list (1) will be the unique basis.

And more generally, consider $\mathbb{F} = \mathbb{Z}_m, \forall m - 1 \in \mathbb{N}^+$. For each $s, t \in \{1, \dots, m\}$,

$\mathbb{F} = \text{span}(K_s) = \text{span}(K_t)$. Hence we fail. Are there other vecsps? Suppose so.

(I) Consider $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . Let (v_1, \dots, v_m) be a basis of $V \neq \{0\}$.

While there are infinitely many bases distinct from this one. Hence we fail.

(II) Consider other \mathbb{F} . Note that a field contains at least 0 and 1

By *some theories or facts* given in the course of Elementary Abstract Algebra, we fail. \square

• Suppose (v_1, \dots, v_m) is a list of vecs in V . For $k \in \{1, \dots, m\}$, let $w_k = v_1 + \dots + v_k$.

Show that $[P] B_V = (v_1, \dots, v_m) \iff [Q] B_W = (w_1, \dots, w_m)$.

SOLUTION: NOTICE that $B_U = (u_1, \dots, u_n) \iff \forall u \in U, \exists! a_i \in \mathbb{F}, u = a_1 u_1 + \dots + a_n u_n$.

$P \Rightarrow Q: \forall v \in V, \exists! a_i \in \mathbb{F}, v = a_1 v_1 + \dots + a_m v_m \Rightarrow v = b_1 w_1 + \dots + b_m w_m, \exists! b_k = \sum_{j=1}^k (-1)^{k-j} a_j$.

$Q \Rightarrow P: \forall v \in V, \exists! b_i \in \mathbb{F}, v = b_1 w_1 + \dots + b_m w_m \Rightarrow v = a_1 v_1 + \dots + a_m v_m, \exists! a_k = \sum_{j=1}^k b_j$. \square

• Suppose V is finite-dim and U, W are subsp of V such that $V = U + W$.

Let $B_U = (u_1, \dots, u_m), B_W = (w_1, \dots, w_n)$. Prove that $\exists B_V$ consisting of vecs in $U \cup W$.

SOLUTION:

$V = \text{span}(u_1, \dots, u_m) + \text{span}(w_1, \dots, w_n) = \text{span}(u_1, \dots, u_m, w_1, \dots, w_n)$. By [2.31], we get the basis \square

8 Suppose U and W are subsp of V such that $V = U \oplus W$.

Let $B_U = (u_1, \dots, u_m), B_W = (w_1, \dots, w_n)$. Prove that $B_V = (u_1, \dots, u_m, w_1, \dots, w_n)$.

SOLUTION:

$\forall v \in V, \exists! u \in U, w \in W, v = u + w = (a_1 u_1 + \dots + a_m u_m) + (b_1 w_1 + \dots + b_n w_n), \exists! a_i, b_i \in \mathbb{F}$

$\Rightarrow (a_1 u_1 + \dots + a_m u_m) = -(b_1 w_1 + \dots + b_n w_n) \in U \cap W = \{0\} \Rightarrow a_1 = \dots = a_m = b_1 = \dots = b_n = 0$. \square

• **NOTE FOR linely inde sequence and [2.34]:**

“ $V = \text{span}(v_1, \dots, v_n, \dots)$ ” is an invalid expression.

If we allow using “infinite list”, then we must guarantee that (v_1, \dots, v_n, \dots) is a spanning “list” such that for all $v \in V$, there exists a smallest positive integer n such that $v = a_1v_1 + \dots + a_nv_n$,

The key point is, how can we guarantee that such a “list” exists?

2.C 1 7 9 10 14,16 15 17 | 4E: 10, 14, 15, 16

1 (COROLLARY for [2.38,39])

Suppose U is a subsp of V such that $\dim V = \dim U$. Then $V = U$.

9 Suppose (v_1, \dots, v_m) is linely inde in V and $w \in V$.

Prove that $\dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$.

SOLUTION: Using the result of Problem (10) and (11) in 2.A.

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \text{span}(v_1 + w, \dots, v_m + w)$, for each $i = 1, \dots, m$.

(v_1, \dots, v_m) linely inde $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ linely inde $\Rightarrow \underbrace{(v_2 - v_1, \dots, v_m - v_1)}_{\text{of length } (m-1)}$ linely inde.

又 $w \notin \text{span}(v_1, \dots, v_m) \Rightarrow (v_1 + w, \dots, v_m + w)$ is linely inde.

Hence $m \geq \dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$. □

10 Suppose m is a positive integer and $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k . Prove that (p_0, p_1, \dots, p_m) is a basis of $\mathcal{P}_m(\mathbf{F})$.

SOLUTION:

Using mathematical induction on m .

(i) For p_0 , $\deg p_0 = 0 \Rightarrow \text{span}(p_0) = \text{span}(1)$.

(ii) Suppose for $i \geq 1$, $\text{span}(p_0, p_1, \dots, p_i) = \text{span}(1, x, \dots, x^i)$.

Then $\text{span}(p_0, p_1, \dots, p_i, p_{i+1}) \subseteq \text{span}(1, x, \dots, x^i, x^{i+1})$.

又 $\deg p_{i+1} = i + 1$, $p_{i+1}(x) = a_{i+1}x^{i+1} + r_{i+1}(x)$; $a_{i+1} \neq 0$, $\deg r_{i+1} \leq i$.

$\Rightarrow x^{i+1} = \frac{1}{a_{i+1}}(p_{i+1}(x) - r_{i+1}(x)) \in \text{span}(1, x, \dots, x^i, p_{i+1}) = \text{span}(p_0, p_1, \dots, p_i, p_{i+1})$.

$\therefore x^{i+1} \in \text{span}(p_0, p_1, \dots, p_i, p_{i+1}) \Rightarrow \text{span}(1, x, \dots, x^i, x^{i+1}) \subseteq \text{span}(p_0, p_1, \dots, p_i, p_{i+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \text{span}(1, x, \dots, x^m) = \text{span}(p_0, p_1, \dots, p_m)$. □

OR. 用比较系数法. Denote the coefficient of x^i in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_i(p)$.

Suppose $L = a_m p_m(x) + \dots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \dots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We use induction on m to show that $a_m = \dots = a_0 = 0$.

(i) $k = m$, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m$, $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.

Now $L = a_{m-1} p_{m-1}(x) + \dots + a_0 p_0(x)$.

(ii) $1 \leq k \leq m$, $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now $L = a_{k-1} p_{k-1}(x) + \dots + a_0 p_0(x)$. □

- (4E 2.C.10) Suppose m is a positive integer. For $0 \leq k \leq m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show that (p_0, \dots, p_m) is a basis of $\mathcal{P}(\mathbb{F})$.

The basis in this exercise leads to what are called Bernstein polys. You can do a web search to learn how Bernstein polys are used to approximate continuous functions on $[0, 1]$.

SOLUTION: Using mathematical induction.

- (i) $k = 0, 1, 2$, $p_m(x) = x^m$, $p_{m-1}(x) = x^{m-1} - x^m$, $p_{m-2}(x) = x^{m-2} + x^m - 2x^{m-1}$.
- (ii) $k \geq 2$. Suppose for $p_{m-k}(x)$, $\exists ! a_i \in \mathbb{F}$, $x^{m-k} = p_{m-k}(x) + a_m x^m + \dots + a_{m-k+1} x^{m-k+1}$.

Then for $p_{m-k-1}(x)$, $\exists ! c_i \in \mathbb{F}$,

$$x^{m-k-1} = p_{m-k-1}(x) + C_{k+1}^1(-1)^2 x^{m-k} + \dots + C_{k+1}^k(-1)^{k+1} x^{m-1} + (-1)^{k-2} x^m \\ \Rightarrow c_{m-i} = C_{k+1}^{k+1-i}(-1)^{k-i}.$$

Thus for each x^i , $\exists ! b_i \in \mathbb{F}$, $x^i = b_m p_m(x) + \dots + b_{m-i} p_{m-i}(x)$
 $\Rightarrow \text{span}(x^m, \dots, x, 1) = \text{span}(\underbrace{p_m, \dots, p_1, p_0}_{\text{Basis}}).$ □

OR. For any $m, k \in \mathbb{N}^+$ such that $k \leq m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k(1-x)^{m-k}$.

Define the statement $S(m)$ by $S(m) : \underbrace{(p_{0,m}, \dots, p_{m,m})}_{\dim \mathcal{P}_m(\mathbb{F}) = m+1}$ is linely inde (and therefore is a basis).

We use induction on to show that $S(m)$ holds for all $m \in \mathbb{N}^+$.

- (i) $m = 1$. Suppose $a_0(1-x) + a_1x = 0, \forall x \in \mathbb{F}$. Then $\begin{cases} x = 0 \Rightarrow a_0 = 0; \\ x = 1 \Rightarrow a_1 = 0. \end{cases}$

$$m = 2. \text{ Suppose } a_0(1-x)^2 + a_1(1-x)x + a_2x^2, \forall x \in \mathbb{F}. \text{ Then } \begin{cases} x = 0 \Rightarrow a_0 + a_1 = 0; \\ x = 1 \Rightarrow a_2 = 0; \\ x = 2 \Rightarrow a_0 + 2a_1 = 0. \end{cases}$$

- (ii) $2 \leq m$. Assume that $S(m)$ holds.

Suppose $\sum_{k=0}^{m+2} a_k p_{k,m+2}(x) = \sum_{k=0}^{m+2} a_k x^k(1-x)^{m+2-k} = 0, \forall x \in \mathbb{F}$.

While $x = 0 \Rightarrow a_0 = 0$; $x = 1 \Rightarrow a_{m+2} = 0$. Then $\sum_{k=1}^{m+1} a_k x^k(1-x)^{m+2-k} = 0$;

$$\text{And note that } \sum_{k=1}^{m+1} a_k x^k(1-x)^{m+2-k} \\ = x(1-x) \sum_{k=1}^{m+1} a_k x^{k-1}(1-x)^{m+1-k} \\ = x(1-x) \sum_{k=0}^m a_{k+1} x^k(1-x)^{m-k} = x(1-x) \sum_{k=0}^m a_{k+1} p_{k,m}(x).$$

Hence $x(1-x) \sum_{k=0}^m a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbb{F} \Rightarrow \sum_{k=0}^m a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbb{F} \setminus \{0, 1\}$.

Because $\sum_{k=0}^m a_{k+1} p_{k,m}(x)$ has infinitely many zeros. We have $\sum_{k=0}^m a_{k+1} p_{k,m}(x) = 0, \forall x \in \mathbb{F}$.

By assumption, $a_1 = \dots = a_m = 0$, while $a_0 = a_{m+2} = 0$,

and also $a_{m+1} = 0$ (because $\sum_{k=0}^m a_{k+1} p_{k,m}(x) = a_{m+1} p_{m,m}(x) = a_{m+1} x^m = 0, \forall x \in \mathbb{F}.$)

Thus $(p_{0,m+2}, \dots, p_{m+2,m+2})$ is linely inde and $S(m+2)$ holds.

Since $\forall m \in \mathbb{N}^+, S(m) \Rightarrow S(m+2)$. We have $\left\{ \begin{array}{l} \forall k \in \mathbb{N}, S(2k+1) \text{ holds} \\ \forall k \in \mathbb{N}^+, S(2k) \text{ holds} \end{array} \right\} \Rightarrow S(m) \text{ holds.}$ □

7 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U .

(b) Extend the basis in (b) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

SOLUTION: Suppose $p(z) = az^4 + bz^3 + cz^2 + dz + e$ such that $p(2) = p(5) = p(6)$.

$$\text{Then } \left\{ \begin{array}{l} p(2) = 16a + 8b + 4c + 2d + e \quad (\text{I}) \\ p(5) = 625a + 125b + 25c + 5d + e \quad (\text{II}) \\ p(6) = 1296a + 216b + 36c + 6d + e \quad (\text{III}) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\text{II}) - (\text{I}) = 0 \\ (\text{III}) - (\text{II}) = 0 \\ (\text{III}) - (\text{I}) = 0 \end{array} \right.$$

You don't have to compute to know that the dimension of the set of solutions is 3.

(Because $\nexists p \in \mathcal{P}_2(\mathbf{F})$ with $1 \leq \deg p \leq 2, p(2) = p(5) = p(6)$.)

(a) A basis: $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(c) Let $W = \text{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. □

• **TIPS:**

(1) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3))$.

(2) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_3) - \dim(V_2 + (V_1 \cap V_3))$.

(3) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim(V_3 + (V_1 \cap V_2))$.

For (1). Because $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim(V_1 + (V_2 \cap V_3))$.

And $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$.

• Suppose V is a 10-dim vecsp and V_1, V_2, V_3 are subsp of V with

(a) $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

(b) $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

SOLUTION:

(a) By TIPS, $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 - 2 \dim V > 0$.

(b) By TIPS, $\dim(V_1 \cap V_2 \cap V_3) > 2 \dim V - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)) \geq 0$. □

• (4E 2.C.16)

Suppose V is finite-dim and U is a subspace of V with $U \neq V$. Let $n = \dim V, m = \dim U$.

Prove that $\exists (n-m)$ subspaces U_1, \dots, U_{n-m} , each of dim $(n-1)$, such that $\bigcap_{i=1}^{n-m} U_i = U$.

SOLUTION:

Let (v_1, \dots, v_m) be a basis of U , extend to a basis of V as $(v_1, \dots, v_m, u_1, \dots, u_{n-m})$.

Define $U_i = \text{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m})$ for each i . Then $U \subseteq U_i$ for each i .

And because $\forall v \in \bigcap_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \dots + b_{n-m} u_{n-m} \in U_i \Rightarrow b_i = 0$ for each $i \Rightarrow v \in U$.

Hence $\bigcap_{i=1}^{n-m} U_i \subseteq U$. □

EXAMPLE: Suppose $\dim V = 6, \dim U = 3$.

$$\left(\underbrace{(v_1, v_2, v_3, v_4, v_5, v_6)}_{\text{Basis of } V}, \text{ define } \left\{ \begin{array}{l} U_1 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_5, v_6) \\ U_2 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_6) \\ U_3 = \text{span}(v_1, v_2, v_3) \oplus \text{span}(v_4, v_5) \end{array} \right\} \Rightarrow \dim U_i = 6-1, i = \underbrace{1, 2, 3}_{6-3=3} \right.$$

□

14 Suppose that V_1, \dots, V_m are finite-dim subsp of V .

Prove that $V_1 + \dots + V_m$ is finite-dim and $\dim(V_1 + \dots + V_m) \leq \dim V_1 + \dots + \dim V_m$.

SOLUTION:

Choose a basis \mathcal{E}_i of $V_i \Rightarrow V_1 + \dots + V_m = \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$; $\dim V_i = \text{card } \mathcal{E}_i$.

Then $\dim(V_1 + \dots + V_m) = \dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$.

又 $\dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leq \text{card}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leq \text{card } \mathcal{E}_1 + \dots + \text{card } \mathcal{E}_m$.

Thus $\dim(V_1 + \dots + V_m) \leq \dim V_1 + \dots + \dim V_m$. □

COMMENT: $\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m \iff V_1 + \dots + V_m$ is a direct sum.

For each i , $(V_1 + \dots + V_i) \cap V_{i+1} = \{0\} \iff V_1 + \dots + V_m$ is a direct sum

$\iff (\mathcal{E}_1 \cap \dots \cap \mathcal{E}_{i-1}) \cap \mathcal{E}_i = \emptyset$ for each i 又 $\dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) = \text{card}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$

$\iff \dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) = \text{card } \mathcal{E}_1 + \dots + \text{card } \mathcal{E}_m$

$\iff \dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$. □

17 Suppose V_1, V_2, V_3 are subsp of a finite-dim vecsp, then

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3$$

$$- \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

SOLUTION:

[Similar to] Given three sets A, B and C .

Because $|X + Y| = |X| + |Y| - |X \cap Y|$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

Now $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$.

And $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$.

Hence $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$.

Because $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\dim(V_1 + V_2 + V_3) = \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \quad (1)$$

$$= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) \quad (2)$$

$$= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) \quad (3)$$

Notice that in general, $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$.

For example, $X = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}, Y = \{(0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}, Z = \{(z, z) \in \mathbb{R}^2 : z \in \mathbb{R}\}$.

• **COROLLARY:** Suppose V_1, V_2 and V_3 are finite-dim vecsp, then $\frac{(1) + (2) + (3)}{3}$:

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3$$

$$- \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3}$$

$$+ \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.$$

The formula above may seem strange because the right side does not look like an integer. □

• **TIPS:** Suppose $v_1, \dots, v_n \in V, \dim \text{span}(v_1, \dots, v_n) = n$. Then (v_1, \dots, v_n) is a basis of $\text{span}(v_1, \dots, v_n)$.

Notice that (v_1, \dots, v_n) is a spanning list of $\text{span}(v_1, \dots, v_n)$ of length $n = \dim \text{span}(v_1, \dots, v_n)$.

15 Suppose V is finite-dim and $\dim V = n \geq 1$.

Prove that \exists one-dim subsp s V_1, \dots, V_n of V such that $V = V_1 \oplus \dots \oplus V_n$.

SOLUTION:

Suppose $B_V = (v_1, \dots, v_n)$. Define V_i by $V_i = \text{span}(v_i)$ for each $i \in \{1, \dots, n\}$.

Then $\forall v \in V, \exists! a_i \in \mathbb{F}, v = a_1 v_1 + \dots + a_n v_n$

$$\Rightarrow \exists! u_i \in V_i, v = u_1 + \dots + u_n \Rightarrow V = V_1 \oplus \dots \oplus V_n. \quad \square$$

• **COROLLARY:**

Suppose W is finite-dim, $\dim W = m$ and $w \in W \setminus \{0\}$.

Prove that $\exists B_W = (w_1, \dots, w_m)$ such that $w = w_1 + \dots + w_m$.

[Proof]

By Problem (15), \exists one-dim subsp s W_1, \dots, W_m of W such that $W = W_1 \oplus \dots \oplus W_m$.

Note that $\dim W_i = \dim \text{span}(w_i) = 1 \Rightarrow \forall x_i \in W_i, \exists! c_i \in \mathbb{F}, x_i = c_i w_i$.

Suppose $w = x_1 + \dots + x_m$, where each $x_i = c_i w_i \in W_i$. Then (x_1, \dots, x_m) is also a basis of W . \square

OR. Note that $w \neq 0 \Rightarrow m \geq 1$. If $m = 1$ then let $w_1 = w$ and we are done. Suppose $m > 1$.

Extend (w) to a basis (w, w_1, \dots, w_{m-1}) of W . Let $w_m = w - w_1 - \dots - w_{m-1}$.

$\text{span}(w, w_1, \dots, w_{m-1}) = \text{span}(w_1, \dots, w_m)$. Hence (w_1, \dots, w_m) is also a basis of W . \square

• **NEW THEOREM:** Suppose V is finite-dim with $\dim V = n$ and U is a subsp of V with $U \neq V$.

Prove that $\exists B_V = (v_1, \dots, v_n)$ such that each $v_k \notin U$.

Note that $U \neq V \Rightarrow n \geq 1$. We will construct B_V via the following process.

Step 1. $\exists v_1 \in V \setminus U \Rightarrow v_1 \neq 0$. If $\text{span}(v_1) = V$ then we stop.

Step k. Suppose (v_1, \dots, v_{k-1}) is linely inde in V , each of which belongs to $V \setminus U$.

Note that $\text{span}(v_1, \dots, v_{k-1}) \neq V$. And if $\text{span}(v_1, \dots, v_{k-1}) \cup U = V$, then by (1.C.12),

(because $\text{span}(v_1, \dots, v_{k-1}) \not\subseteq U$,) $U \subseteq \text{span}(v_1, \dots, v_{k-1}) \Rightarrow \text{span}(v_1, \dots, v_{k-1}) = V$.

Hence because $\text{span}(v_1, \dots, v_{k-1}) \neq V$, it must be case that $\text{span}(v_1, \dots, v_{k-1}) \cup U \neq V$.

Thus $\exists v_k \in V \setminus U$ such that $v_k \notin \text{span}(v_1, \dots, v_{k-1})$.

By (2.A.11), (v_1, \dots, v_k) is linely inde in V . If $\text{span}(v_1, \dots, v_k) = V$, then we stop.

Because V is finite-dim, this process will stop after n steps. \square

OR. If $U = \{0\}$ then we are done. Suppose $\dim U \geq 1$.

Let (u_1, \dots, u_m) be a basis of U , extend to a basis (u_1, \dots, u_n) of V .

Then let $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$. \square

ENDED

3.A [3](#) [4](#) [5](#) [7](#) [8](#) [10](#) [11](#) [12](#) [13](#) | [4E: 10](#), [11](#), [16](#)

• **TIPS:** $T : V \rightarrow W$ is linear $\Leftrightarrow \left\{ \begin{array}{l} \text{(一)} \forall v, u \in V, T(v+u) = Tv + Tu; \\ \text{(二)} \forall v, u \in V, \lambda \in \mathbb{F}, T(\lambda v) = \lambda(Tv). \end{array} \right. \Leftrightarrow T(v + \lambda u) = Tv + \lambda Tu.$

$$T \in \mathcal{L}(V, W) \Leftrightarrow T \in \mathcal{L}(V, \text{range } T). \text{ And } \{T \in \mathcal{L}(V, W) : \text{range } T \subseteq U\} = \mathcal{L}(V, U).$$

• Suppose $T \in \mathcal{L}(V, W)$. Prove that $Tv \neq 0 \Rightarrow v \neq 0$.

SOLUTION: Assume that $v = 0$. Then $Tv = T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$.

OR. $T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$. Contradicts. \square

- (4E 1.B.7) Suppose $V \neq \emptyset$ and W is a vecsp. Let $W^V = \{f : V \rightarrow W\}$.

(a) Define a natural add and scalar multi on W^V .

(b) Prove that W^V is a vecsp with these definitions.

SOLUTION:

(a) $W^V \ni f + g : x \rightarrow f(x) + g(x)$; where $f(x) + g(x)$ is the vec add on W .

$W^V \ni \lambda f : x \rightarrow \lambda f(x)$; where $\lambda f(x)$ is the scalar multi on W .

(b) Commutativity: $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$.

Associativity: $((f + g) + h)(x) = (f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x)) = (f + (g + h))(x)$.

Additive Identity: $(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$.

Additive Inverse: $(f + g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0 = 0(x)$.

Distributive Properties:

$$(a(f + g))(x) = a(f + g)(x) = a(f(x) + g(x)) = af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x).$$

Similarly, $((a + b)f)(x) = (af + bf)(x)$.

So far, we have used the same properties in W .

Which means that **if W^V is a vecsp, then W must be a vecsp.**

Multiplication Identity: $(1f)(x) = 1f(x) = f(x)$. (NOTICE that the smallest F is $\{0, 1\}$.) □

5 Because $\mathcal{L}(V, W) = \{T : V \rightarrow W \mid T \text{ is linear}\}$ is a subsp of W^V , $\mathcal{L}(V, W)$ is a vecsp.

- Given the fact that $\mathcal{L}(V, W)$ is a vecsp. Prove or give a counterexample: V, W are vecsp.

We can guarantee that $\{0\} \subseteq \mathcal{L}(V, W), \{0\} \subseteq V, \{0\} \subseteq W$.

And by [3.2], the additivity and homogeneity imply that V is closed under add and scalar multi.

(We cannot even guarantee that W^V is a vecsp.)

SOLUTION:

(I) If $W^V = \{0\}$. Then $\mathcal{L}(V, W) = \{0\}$.

And $W = \{0\}$, for if not, $\exists w \in W \setminus \{0\}$, define a map f by $f(x) = w, \forall x \in V$.

And V might not be a vecsp. Example:

(II) If W^V is a nonzero vecsp. Then W is a vecsp.

(a) If $\mathcal{L}(V, W) = \{0\}$, then we cannot guarantee that V is a vecsp.

Example:

(b) If not, then $\exists T \in \mathcal{L}(V, W), T \neq 0$. Which means $\exists v \in V, Tv \neq 0 \Rightarrow v \neq 0$.

Then both W and V have a nonzero element.

(i) If \exists inje $T \in \mathcal{L}(V, W)$, then $T(u + v) = (v + u) \Rightarrow u + v = v + u$. etc. Hence V is a vecsp.

(ii) If not, then we cannot guarantee that V is a vecsp.

Example:

(III) If W^V is not a vecsp, then W is not a vecsp.

Example:

TODO

□

3 Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists A_{j,k} \in \mathbf{F}$ such that for any $(x_1, \dots, x_n) \in \mathbf{F}^n$

$$T(x_1, \dots, x_n) = \begin{pmatrix} A_{1,1}x_1 + \dots + A_{1,n}x_n \\ \vdots \quad \ddots \quad \vdots \\ A_{m,1}x_1 + \dots + A_{m,n}x_n \end{pmatrix}$$

SOLUTION:

Let $T(1, 0, 0, \dots, 0, 0) = (A_{1,1}, \dots, A_{m,1})$, Note that $(1, 0, \dots, 0, 0), \dots, (0, 0, \dots, 0, 1)$ is a basis of \mathbf{F}^n .

$T(0, 1, 0, \dots, 0, 0) = (A_{1,2}, \dots, A_{m,2})$, Then by [3.5], we are done. \square

\vdots

$T(0, 0, 0, \dots, 0, 1) = (A_{1,n}, \dots, A_{m,n})$.

4 Suppose $T \in \mathcal{L}(V, W)$, and $v_1, \dots, v_m \in V$ such that (Tv_1, \dots, Tv_m) is linely inde in W .
Prove that (v_1, \dots, v_m) is linely inde.

SOLUTION: Suppose $a_1v_1 + \dots + a_mv_m = 0$. Then $a_1Tv_1 + \dots + a_mTv_m = 0$. Thus $a_1 = \dots = a_m = 0$. \square

7 Show that every linear map from a one-dim vecsp to itself is a multi by some scalar.
More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then $\exists \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$.

SOLUTION:

Let u be a nonzero vec in $V \Rightarrow V = \text{span}(u)$. Because $Tu \in V \Rightarrow Tu = \lambda u$ for some λ .

Suppose $v \in V \Rightarrow v = au, \exists! a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. \square

8 Give a function $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}$ such that $\forall a \in \mathbf{R}, v \in \mathbf{R}^2, \varphi(av) = a\varphi(v)$ but φ is not linear.

SOLUTION: Define $T(x, y) = \begin{cases} x + y, & \text{if } (x, y) \in \text{span}(3, 1), \\ 0, & \text{otherwise.} \end{cases}$ OR. Define $T(x, y) = \sqrt[3]{(x^3 + y^3)}$. \square

9 Give a function $\varphi : \mathbf{C} \rightarrow \mathbf{C}$ such that $\forall w, z \in \mathbf{C}, \varphi(w + z) = \varphi(w) + \varphi(z)$
but φ is not linear. (Here \mathbf{C} is thought of as a complex vecsp.)

SOLUTION:

Suppose $V_{\mathbf{C}}$ is the complexification of a vecsp V . Suppose $\varphi : V_{\mathbf{C}} \rightarrow V_{\mathbf{C}}$.

Define $\varphi(u + iv) = u = \text{Re}(u + iv)$ OR. Define $\varphi(u + iv) = v = \text{Im}(u + iv)$. \square

• Prove that if $q \in \mathcal{P}(\mathbf{R})$ and $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$ is defined by $Tp = \underbrace{q \circ p}_{\text{composition}}$, then T is not linear.

SOLUTION: Composition and product are not the same in $\mathcal{P}(\mathbf{F})$.

Because in general, $q \circ (p_1 + \lambda p_2)(x) = q(p_1(x) + \lambda p_2(x)) \neq (q \circ p_1)(x) + \lambda(q \circ p_2)(x)$.

EXAMPLE: Let q be defined by $q(x) = x^2$, then $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$. \square

10 Suppose U is a subsp of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ with $S \neq 0$
(which means that $\exists u \in U, Su \neq 0$).

Define $T : V \rightarrow W$ by $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \setminus U. \end{cases}$ Prove that T is not a linear map on V .

SOLUTION:

Suppose T is a linear map. And $v \in V \setminus U, u \in U$ such that $Su \neq 0$.

Then $v + u \in V \setminus U$, (for if not, $v = (v + u) - u \in U$) while $T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$.

Hence we get a contradiction. \square

11 Suppose U is a subsp of V and $S \in \mathcal{L}(U, W)$.

Prove that $\exists T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U$. (OR. $\exists T \in \mathcal{L}(V, W), T|_U = S$)

In other words, every linear map on a subsp of V can be extended to a linear map on the entire V .

SOLUTION: Suppose W is such that $V = U \oplus W$. Then $\forall v \in V, \exists! u_v \in U, w_v \in W, v = u_v + w_v$.

Define $T \in \mathcal{L}(V, W)$ by $T(u_v + w_v) = Su_v$. □

OR. [Finite-dim Req] Define by $T\left(\sum_{i=1}^m a_i u_i\right) = \sum_{i=1}^m a_i Su_i$. Let $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$. □

12 Suppose nonzero V is finite-dim and W is infinite-dim. Prove that $\mathcal{L}(V, W)$ is infinite-dim.

SOLUTION:

Let (v_1, \dots, v_n) be a basis of V . Let (w_1, \dots, w_m) be linely inde in W for any $m \in \mathbb{N}^+$.

Define $T_{x,y} : V \rightarrow W$ by $T_{x,y}(v_z) = \delta_{z,x} w_y, \forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$

$\forall v = \sum_{i=1}^n a_i v_i, u = \sum_{i=1}^n b_i v_i, \lambda \in \mathbb{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) v_y = T_{x,y}(v) + \lambda T_{x,y}(u)$.

Linearity checked. Now suppose $a_1 T_{x,1} + \dots + a_m T_{x,m} = 0$.

Then $(a_1 T_{x,1} + \dots + a_m T_{x,m})(v_x) = 0 = a_1 w_1 + \dots + a_m w_m \Rightarrow a_1 = \dots = a_m = 0$. $\forall m$ arbitrary.

Thus $(T_{x,1}, \dots, T_{x,m})$ is a linely inde list in $\mathcal{L}(V, W)$ for any x and length m . Hence by (2.A.14). □

13 Suppose (v_1, \dots, v_m) is linely depe in V and $W \neq \{0\}$.

Prove that $\exists w_1, \dots, w_m \in W, \nexists T \in \mathcal{L}(V, W)$ such that $Tv_k = w_k, \forall k = 1, \dots, m$.

SOLUTION:

We prove by contradiction. By linear dependence lemma, $\exists j \in \{1, \dots, m\}, v_j \in \text{span}(v_1, \dots, v_{j-1})$.

Fix j . Let $w_j \neq 0$, while $w_1 = \dots = w_{j-1} = w_{j+1} = \dots = w_m = 0$.

Define T by $Tv_k = w_k$ for all k . Suppose $a_1 v_1 + \dots + a_m v_m = 0$ (where $a_j \neq 0$).

Then $T(a_1 v_1 + \dots + a_m v_m) = 0 = a_1 w_1 + \dots + a_m w_m = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. □

OR. We prove the contrapositive: Suppose $\forall w_1, \dots, w_m \in W, \exists T \in \mathcal{L}(V, W), Tv_k = w_k$ for each w_k .

Now we show that (v_1, \dots, v_n) is linely inde. Suppose $\exists a_i \in \mathbb{F}, a_1 v_1 + \dots + a_n v_n = 0$.

Choose one $w \in W \setminus \{0\}$. By assumption, for $(\overline{a_1} w, \dots, \overline{a_m} w), \exists T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$ for each v_k .

Now we have $0 = T\left(\sum_{k=1}^m a_k v_k\right) = \sum_{k=1}^m a_k Tv_k = \sum_{k=1}^m a_k \overline{a_k} w = \left(\sum_{k=1}^m |a_k|^2\right) w$.

Then $\sum_{k=1}^m |a_k|^2 = 0 \Rightarrow a_k = 0$ for each k . Hence (v_1, \dots, v_n) is linely inde. □

• (4E 3.A.16)

Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

A subsp \mathcal{E} of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Let (v_1, \dots, v_n) be a basis of V . If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and \mathcal{E} is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \setminus \{0\}$.

Suppose $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \dots + a_n v_n$, where $a_k \neq 0$.

Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y}(v_x) = v_y, R_{x,y}(v_z) = 0 (z \neq x)$. OR. $R_{x,y} v_z = \delta_{z,x} v_y$.

Then $(R_{1,1} + \dots + R_{n,n})v_j = v_j \Rightarrow \sum_{r=1}^n R_{r,r} = I$. Assume that each $R_{x,y} \in \mathcal{E}$.

Hence $\forall T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. Now we prove the assumption.

Notice that $\forall x, y \in \mathbb{N}^+, (R_{k,y} S)(v_i) = a_k v_y \Rightarrow ((R_{k,y} S) \circ R_{x,i})(v_z) = \delta_{z,x} (a_k v_y)$.

Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Now $S \in \mathcal{E} \Rightarrow R_{k,y} S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}$. □

- (4E 3.B.32) Suppose V is finite-dim with $n = \dim V > 1$.

Show that if $\varphi : \mathcal{L}(V) \rightarrow \mathbf{F}$ is linear and $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$, then $\varphi = 0$.

SOLUTION:

Using notations in (4E 3.A.16). Using the result in NOTE FOR [3.60].

Suppose $\varphi \neq 0 \Rightarrow \exists i, j \in \{1, \dots, n\}, \varphi(R_{i,j}) \neq 0$. Because $R_{i,j} = R_{x,j} \circ R_{i,x}, \forall x = 1, \dots, n$
 $\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, because $R_{i,x} = R_{y,x} \circ R_{i,y}, \forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0, \forall x, y = 1, \dots, n$.

Let $k \neq i, j \neq l$ and then $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$
 $\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. □

OR. Note that by (4E 3.A.16), $\exists S, T \in \mathcal{L}(V), ST - TS \neq 0$.

Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \text{null } \varphi \neq \{0\}$.

Note that $\forall E \in \text{null } \varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \text{null } \varphi$.

Hence $\text{null } \varphi$ is a nonzero two-sided ideal of $\mathcal{L}(V)$. □

- Suppose V is finite-dim. $T \in \mathcal{L}(V)$ is such that $\forall S \in \mathcal{L}(V), ST = TS$.

Prove that $\exists \lambda \in \mathbf{F}, T = \lambda I$.

SOLUTION:

If $V = \{0\}$, then we are done. Now suppose $V \neq \{0\}$.

Assume that (v, Tv) is linely depe for every $v \in V$, then by (2.A.2.(b)), $Tv = \lambda_v v$ for some $\lambda_v \in \mathbf{F}$.

To prove that λ_v is independent of v , we discuss in two cases:

(-) If (v, w) is linely inde, $\lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v v + \lambda_w w$
 $\Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0$
 (=) Otherwise, suppose $w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w = 0$ } $\Rightarrow \lambda_w = \lambda_v$.

Now we show the assumption. Assume that (v, Tv) is linely inde for some v . Let $B_V = (v, Tv, u_1, \dots, u_n)$.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1 u_1 + \dots + c_n u_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Contradicts. □

OR. Let (v_1, \dots, v_m) be a basis of V .

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \dots = \varphi(v_m) = 1$. Let $\lambda = \varphi(Tv_1) \in \mathbf{F}$.

For any $v \in V$, define $S_v \in \mathcal{L}(V)$ by $S_v u = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. □

OR. For each $k \in \{1, \dots, n\}$, define $S_k \in \mathcal{L}(V)$ by $S_k v_j = \begin{cases} v_k, & j = k, \\ 0, & j \neq k. \end{cases}$ OR. $S_k v_j = \delta_{j,k} v_k$

Note that $S_k \left(\sum_{i=1}^n a_i v_i \right) = a_k v_k$. Then $S_k v = v \iff \exists ! a_k \in \mathbf{F}, v = a_k v_k$.

Hence $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$.

Define $A^{(j,k)} \in \mathcal{L}(V)$ by $A^{(j,k)} v_j = v_k, A^{(j,k)} v_k = v_j, A^{(j,k)} v_x = 0, x \neq j, k$.

Then $A^{(j,k)} T v_j = T A^{(j,k)} v_j = Tv_k = a_k v_k; A^{(j,k)} T v_j = A^{(j,k)} a_j v_j = a_j A^{(j,k)} v_j = a_j v_k$.

Hence $a_k = a_j$. Thus a_k is independent of v_k . □

- Suppose that V and W are real vecsps and $T \in \mathcal{L}(V, W)$.
Define $T_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ by $T_{\mathbb{C}}(u + iv) = Tu + iTv$ for all $u, v \in V$.
Show that (a) $T_{\mathbb{C}}$ is linear, (b) $T_{\mathbb{C}}$ is inje $\iff T$ is inje, (c) $T_{\mathbb{C}}$ is surj $\iff T$ is surj.

SOLUTION:

- (a) $\forall u_1 + iv_1, u_2 + iv_2 \in V_{\mathbb{C}}, \lambda \in \mathbb{F}$,

$$T((u_1 + iv_1) + \lambda(u_2 + iv_2)) = T((u_1 + \lambda u_2) + i(v_1 + \lambda v_2)) = T(u_1 + \lambda u_2) + iT(v_1 + \lambda v_2)$$

$$= Tu_1 + iTv_1 + \lambda Tu_2 + i\lambda Tv_2 = T(u_1 + iv_1) + \lambda T(u_2 + iv_2).$$
- (b) $\left\{ \begin{array}{l} \text{Suppose } T_{\mathbb{C}} \text{ is inje. Let } T(u) = 0 \Rightarrow T_{\mathbb{C}}(u + i0) = Tu = 0 \Rightarrow u = 0. \\ \text{Suppose } T \text{ is inje. Let } T_{\mathbb{C}}(u + iv) = Tu + iTv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + iv = 0. \end{array} \right.$
- (c) $\left\{ \begin{array}{l} \text{Suppose } T_{\mathbb{C}} \text{ is surj. } \forall w \in W, \exists u \in V, T(u + i0) = Tu = w + i0 = w \Rightarrow T \text{ is surj.} \\ \text{Suppose } T \text{ is surj. } \forall w, x \in W, \exists u, v \in V, Tu = w, Tv = x \\ \Rightarrow \forall w + ix \in W_{\mathbb{C}}, \exists u + iv \in V, T(u + iv) = w + ix \Rightarrow T_{\mathbb{C}} \text{ is surj.} \end{array} \right.$

- 3** Suppose (v_1, \dots, v_m) in V . Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m$.
 (a) The surj of T correspds to (v_1, \dots, v_m) spanning V .
 (b) The inje of T correspds to (v_1, \dots, v_m) being linely inde.

COMMENT: Let (e_1, \dots, e_m) be the standard basis of \mathbb{F}^m . Then $Te_k = v_k$.

- (a) $\text{range } T = \text{span}(v_1, \dots, v_m) = V$; (b) (v_1, \dots, v_m) is linely inde $\iff T$ is inje.

- 7** Suppose V is finite-dim with $2 \leq \dim V$. And $\dim V \leq \dim W = m$, if W is finite-dim.
 Show that $U = \{T \in \mathcal{L}(V, W) : \text{null } T \neq \{0\}\}$ is not a subsp of $\mathcal{L}(V, W)$.

SOLUTION: The set of all inje $T \in \mathcal{L}(V, W)$ is a not subsp either.

Let (v_1, \dots, v_n) be a basis of V , (w_1, \dots, w_m) be linely inde in W . ($2 \leq n \leq m$.)
 Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 : v_1 \mapsto 0, v_2 \mapsto w_2, v_i \mapsto w_i$.
 Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 : v_1 \mapsto w_1, v_2 \mapsto 0, v_i \mapsto w_i, i = 3, \dots, n$. $\left| \text{Thus } T_1 + T_2 \notin U. \square \right.$

COMMENT: If $\dim V = 0$, then $V = \{0\} = \text{span}(\)$. $\forall T \in \mathcal{L}(V, W)$, T is inje. Hence $U = \emptyset$.

If $\dim V = 1$, then $V = \text{span}(v_0)$. Thus $U = \text{span}(T_0)$, where $T_0 v_0 = 0$.

- 8** Suppose W is finite-dim with $\dim W \geq 2$. And $n = \dim V \geq \dim W$, if V is finite-dim.
 Show that $U = \{T \in \mathcal{L}(V, W) : \text{range } T \neq W\}$ is not a subsp of $\mathcal{L}(V, W)$.

SOLUTION: The set of all surj $T \in \mathcal{L}(V, W)$ is not a subspace either.

Let (v_1, \dots, v_n) be linely inde in V , (w_1, \dots, w_m) be a basis of W . ($n \in \{m, m+1, \dots\}; 2 \leq m \leq n$.)
 Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 : v_1 \mapsto 0, v_2 \mapsto w_2, v_j \mapsto w_j, v_{m+i} \mapsto 0$.
 Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 : v_1 \mapsto w_1, v_2 \mapsto 0, v_j \mapsto w_j, v_{m+i} \mapsto 0$.
 (For each $j = 2, \dots, m; i = 1, \dots, n - m$, if V is finite, otherwise let $i \in \mathbb{N}^+$.) Thus $T_1 + T_2 \notin U. \square$

COMMENT: If $\dim W = 0$, then $W = \{0\} = \text{span}(\)$. $\forall T \in \mathcal{L}(V, W)$, T is surj. Hence $U = \emptyset$.

If $\dim W = 1$, then $W = \text{span}(v_0)$. Thus $U = \text{span}(T_0)$, where $T_0 v_0 = 0$.

- 11** Suppose S_1, \dots, S_n are linear and inje. $S_1 S_2 \dots S_n$ makes sence. Prove that $S_1 S_2 \dots S_n$ is inje.

SOLUTION: $S_1 S_2 \dots S_n(v) = 0 \iff S_2 S_3 \dots S_n(v) = 0 \iff \dots \iff S_n(v) = 0 \iff v = 0. \square$

9 Suppose (v_1, \dots, v_n) is linely inde. Prove that \forall inje $T, (Tv_1, \dots, Tv_n)$ is linely inde.

SOLUTION: $a_1Tv_1 + \dots + a_nTv_n = 0 = T(\sum_{i=1}^n a_iv_i) \iff \sum_{i=1}^n a_iv_i = 0 \iff a_1 = \dots = a_n = 0.$ \square

10 Suppose $\text{span}(v_1, \dots, v_n) = V$. Show that $\text{span}(Tv_1, \dots, Tv_n) = \text{range } T$.

SOLUTION:

(a) $\text{range } T = \{Tv : v \in V\} = \{Tv : v \in \text{span}(v_1, \dots, v_n)\} \Rightarrow Tv_1, \dots, Tv_n \in \text{range } T \Rightarrow$ By [2.7].

OR. $\text{span}(Tv_1, \dots, Tv_n) \ni a_1Tv_1 + \dots + a_nTv_n = T(a_1v_1 + \dots + a_nv_n) \in \text{range } T$.

(b) $\forall w \in \text{range } T, \exists v \in V, w = Tv. (\exists a_i \in \mathbb{F}, v = a_1v_1 + \dots + a_nv_n) \Rightarrow w = a_1Tv_1 + \dots + a_nTv_n.$ \square

16 Suppose $\exists T \in \mathcal{L}(V)$ such that $\text{null } T, \text{range } T$ are finite-dim. Prove that V is finite-dim.

SOLUTION: Let $B_{\text{range } T} = (Tv_1, \dots, Tv_n), B_{\text{null } T} = (u_1, \dots, u_m).$

$\forall v \in V, T(v - a_1v_1 - \dots - a_nv_n) = 0$, letting $Tv = a_1Tv_1 + \dots + a_nTv_n$.

$\Rightarrow v - a_1v_1 - \dots - a_nv_n = b_1u_1 + \dots + b_mu_m$. Hence $V \subseteq \text{span}(v_1, \dots, v_n, u_1, \dots, u_m).$ \square

17 Suppose V, W are finite-dim. Prove that \exists inje $T \in \mathcal{L}(V, W) \iff \dim V \leq \dim W$.

SOLUTION:

(a) Suppose \exists inje T . Then $\dim V = \dim \text{range } T \leq \dim W$.

(b) Suppose $\dim V \leq \dim W$. Let $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m).$

Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i, i = 1, \dots, n (= \dim V).$ \square

18 Suppose V, W are finite-dim. Prove that \exists surj $T \in \mathcal{L}(V, W) \iff \dim V \geq \dim W$.

SOLUTION:

(a) Suppose \exists surj T . Then $\dim V = \dim W + \dim \text{null } T \Rightarrow \dim W \leq \dim V$.

(b) Suppose $\dim V \geq \dim W$. Let $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m).$

Define $T \in \mathcal{L}(V, W)$ by $T(a_1v_1 + \dots + a_mv_m + \dots + a_nv_n) = a_1w_1 + \dots + a_mw_m.$ \square

19 Suppose V, W are finite-dim, U is a subsp of V .

Prove that if $\underbrace{\dim U}_m \geq \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_p$, then $\exists T \in \mathcal{L}(V, W), \text{null } T = U$.

SOLUTION:

Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_W = (w_1, \dots, w_p).$

Define $T \in \mathcal{L}(V, W)$ by $T(a_1v_1 + \dots + a_nv_n + b_1u_1 + \dots + b_mu_m) = a_1w_1 + \dots + a_nw_n.$ \square

• (4E 3.B.21)

Suppose V is finite-dim, $T \in \mathcal{L}(V, W), U$ is a subsp of W . Let $\mathcal{K}_U = \{v \in V : Tv \in U\}.$

Prove that \mathcal{K}_U is a subsp of V and $\dim \mathcal{K}_U = \dim \text{null } T + \dim(U \cap \text{range } T).$

SOLUTION:

$\forall u, w \in \mathcal{K}_U, \lambda \in \mathbb{F}, T(u + \lambda w) = Tu + \lambda Tw \in U \Rightarrow \mathcal{K}_U$ is a subsp of V .

Define $S \in \mathcal{L}(\mathcal{K}_U, U)$ as $Rv = Tv$ for all $v \in \mathcal{K}_U$. Hence $\text{range } R = U \cap \text{range } T$.

Suppose $\exists v, Tv = 0. \nexists 0 \in U \Rightarrow Rv = 0$. Thus $\text{null } T \subseteq \text{null } R.$ \square

• **TIPS:** Suppose U is a subsp of V . Prove that $\forall T \in \mathcal{L}(V, W), U \cap \text{null } T = \text{null } T|_U$.

SOLUTION: Note that $U \cap \text{null } T \subseteq \text{null } T|_U$. On the other hand, suppose $u \in \text{null } T|_U$.

Then $T|_U(u)$ makes sense $\Rightarrow u \in U$. And $T|_U(u) = Tu = 0 \Rightarrow u \in \text{null } T.$ \square

12 Prove that $\forall T \in \mathcal{L}(V, W), \exists \text{ subsp } U \text{ of } V \text{ such that}$

$$U \cap \text{null } T = \text{null } T|_U = \{0\}, \quad \text{range } T = \{Tu : u \in U\} = \text{range } T|_U.$$

Which is equivalent to $T|_U : U \rightarrow \text{range } T$ being an iso.

SOLUTION:

By [2.34] (note that V can be infinite-dim), $\exists \text{ subsp } U \text{ of } V \text{ such that } V = U \oplus \text{null } T$.

$\forall v \in V, \exists ! w \in \text{null } T, u \in U, v = w + u$. Then $Tv = T(w + u) = Tu \in \{Tu : u \in U\}$. □

• **NEW NOTATION:**

Suppose $T \in \mathcal{L}(V, W)$ and $R = (Tv_1, \dots, Tv_n)$ is linely inde in $\text{range } T$.

Where $n = \dim \text{range } T$ if finite-dim, otherwise $n \in \mathbb{N}^+$.

By (3.A.4), $L = (v_1, \dots, v_n)$ is linely inde in V .

Denote \mathcal{K}_R by $\text{span } L$, if $\text{range } T$ is finite-dim, otherwise, denote it by a vecsp in $\mathcal{S}_V \text{null } T$.

Note that if $\text{range } T$ is finite-dim, then $\mathcal{K}_R = \text{range } T$ for any basis R of $\text{range } T$.

• **COMMENT:**

If $\text{range } T$ is infinite-dim, we cannot write $\mathcal{K}_R = \text{range } T$. For if we do so, we must guarantee that $\forall Tv \in \text{range } T, \exists ! n \in \mathbb{N}^+, Tv \in \text{span}(Tv_1, \dots, Tv_n)$, where $(Tv_k)_{k=1}^\infty$ is linely inde.

So that $\text{range } T \subseteq \text{span}(Tv_1, \dots, Tv_n, \dots)$. This would be invalid, as we have shown before.

• **NEW THEOREM:** $\mathcal{K}_R \in \mathcal{S}_V \text{null } T$. **COMMENT:** $\text{null } T \in \mathcal{S}_V \mathcal{K}_R$.

Suppose $\text{range } T$ is finite-dim. Otherwise, we are done immediately.

$$(a) T\left(\sum_{i=1}^n a_i v_i\right) = 0 \Rightarrow \sum_{i=1}^n a_i Tv_i = 0 \Rightarrow a_1 = \dots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \text{null } T = \{0\}.$$

$$(b) \forall v \in V, Tv = \sum_{i=1}^n a_i Tv_i \Rightarrow Tv - \sum_{i=1}^n a_i Tv_i = T(v - \sum_{i=1}^n a_i v_i) = 0$$

$$\Rightarrow v - \sum_{i=1}^n a_i v_i \in \text{null } T \Rightarrow v = (v - \sum_{i=1}^n a_i v_i) + (\sum_{i=1}^n a_i v_i) \Rightarrow \mathcal{K}_R + \text{null } T = V. \quad \square$$

• Suppose V is finite-dim, $T \in \mathcal{L}(V, W)$, $B_{\text{range } T} = (Tv_1, \dots, Tv_n)$, $B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$.
Prove or give a counterexample: (u_1, \dots, u_m) is a basis of $\text{null } T$.

SOLUTION: A counterexample:

Suppose $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$. Then $\text{span}(Tv_1, Tv_2, Tv_3) = \text{span}(w_1)$.

Extend (v_i) to (v_1, v_2, v_3) for each i . But none of $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is a basis of $\text{null } T$. □

COMMENT: $(v_2 - v_1, v_3 - v_1), (v_1 - v_2, v_3 - v_2)$ or $(v_1 - v_3, v_2 - v_3)$ are all bases of $\text{null } T$.

Always notice that $\mathcal{S}_V \text{span}(v_1, \dots, v_n) = \{U_1, \dots, \text{null } T, \dots, U_n, \dots\}$.

• Suppose V is finite-dim, X is a subsp of V , and Y is a finite-dim subsp of W .

Prove that if $\dim X + \dim Y = \dim V$, then $\exists T \in \mathcal{L}(V, W), \text{null } T = X, \text{range } T = Y$.

SOLUTION:

Suppose $\dim X + \dim Y = \dim V$. Let $B_X = (u_1, \dots, u_n), B_Y = (w_1, \dots, w_m), B_V = (u_1, \dots, u_n, v_1, \dots, v_m)$.

Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i, Tu_j = 0$. Notice that $\forall v \in V, \exists ! a_i, b_j \in \mathbb{F}, v = \sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i$.

$$v \in \text{null } T \iff Tv = 0 \iff a_1 = \dots = a_m = 0 \iff v \in X.$$

$$Y \ni w = a_1 w_1 + \dots + a_m w_m = a_1 Tv_1 + \dots + a_m Tv_m \in \text{range } T.$$

$$\text{OR. range } T = \text{span}(Tv_1, \dots, Tv_m, Tu_1, \dots, Tu_n) = \text{span}(Tv_1, \dots, Tv_m) = \text{span}(w_1, \dots, w_m) = Y. \quad \square$$

• OR (5.B.4) Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \text{null } P \oplus \text{range } P$.

SOLUTION:

(a) If $v \in \text{null } P \cap \text{range } P \Rightarrow Pv = 0$ and $\exists u \in V, v = Pu$. Then $v = Pu = P^2u = Pv = 0$.

(b) Note that $\forall v \in V, v = Pv + (v - Pv)$ and $P(v - Pv) = 0 \Rightarrow v - Pv \in \text{null } P$. \square

OR. [Only in Finite-dim] Let (P^2v_1, \dots, P^2v_n) be a basis of $\text{range } P^2$. Then (Pv_1, \dots, Pv_n) is linely inde.

Let $\mathcal{K} = \text{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \text{null } P^2$. While $\mathcal{K} = \text{range } P = \text{range } P^2$; $\text{null } P = \text{null } P^2$. \square

20 Suppose W is finite-dim. Prove that $T \in \mathcal{L}(V, W)$ is inje $\iff \exists S \in \mathcal{L}(W, V), ST = I_V$.

SOLUTION:

(a) Suppose $\exists S \in \mathcal{L}(W, V), ST = I$. Then if $Tv = 0 \Rightarrow ST(v) = 0 = v$. OR. $\text{null } T \subseteq \text{null } ST = \{0\}$.

(b) Suppose T is inje. Let $R = B_{\text{range } T} = (Tv_1, \dots, Tv_n)$. Then $\mathcal{K}_R \oplus \text{null } T = V$. Let $U \oplus \text{range } T = W$.

Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$ and $Su = 0$, where $i \in \{1, \dots, n\}, u \in U$. Thus $ST = I$.

OR. Define $S \in \mathcal{L}(\text{range } T, V)$ by $Sw = T^{-1}w$, where T^{-1} is the inv of $T \in \mathcal{L}(V, \text{range } T)$.

Then extend it to $S \in \mathcal{L}(W, V)$ by (3.A.11). Now $\forall v \in V, STv = T^{-1}Tv = v$. \square

21 Suppose W is finite-dim. Prove that $T \in \mathcal{L}(V, W)$ is surj $\iff \exists S \in \mathcal{L}(W, V), TS = I_W$.

SOLUTION:

(a) Suppose $\exists S \in \mathcal{L}(W, V), TS = I$. Then $\forall w \in W, TS(w) = w \in \text{range } T \Rightarrow \text{range } T = W$.

(b) Suppose T is surj. Let $R = B_{\text{range } T} = B_W = (Tv_1, \dots, Tv_n)$. Then $\mathcal{K}_R \oplus \text{null } T = V$.

Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$. Then $TS = I$.

OR. By Problem (12), \exists subsp U of $V, V = U \oplus \text{null } T, \text{range } T = \{Tu : u \in U\}$.

Note that $T|_U : U \rightarrow W$ is an iso. Define $S = (T|_U)^{-1}$, where $(T|_U)^{-1} : W \rightarrow U$.

Then $TS = T \circ (T|_U)^{-1} = T|_U \circ (T|_U)^{-1}$. \square

24 Suppose $S, T \in \mathcal{L}(V, W)$. Prove that $\text{null } S \subseteq \text{null } T \iff \exists E \in \mathcal{L}(W)$ such that $T = ES$.

SOLUTION:

Suppose $\exists E \in \mathcal{L}(W)$ such that $T = ES$. Then $\text{null } T = \text{null } ES \supseteq \text{null } S$.

Suppose $\text{null } S \subseteq \text{null } T$. Let $W = \text{range } S \oplus U$.

Define $E \in \mathcal{L}(W)$ by $E(Sv + w) = Tv$ for each Sv and each $w \in U$. Now we check that E is linear.

Because $\forall w_1, w_2 \in W, \exists! Sv_1, Sv_2 \in \text{range } S, u_1, u_2 \in U, w_1 = Sv_1 + u_1, w_2 = Sv_2 + u_2$.

Now $E(w_1 + \lambda w_2) = E((Sv_1 + \lambda Sv_2) + (u_1 + \lambda u_2)) = T(v_1 + \lambda v_2) = Tv_1 + \lambda Tv_2 = Ew_1 + \lambda Ew_2$.

OR. Let $V = \mathcal{K} \oplus U$. Then $S|_{\mathcal{K}} : \mathcal{K} \rightarrow \text{range } S$ is an iso.

Now extend $T(S|_{\mathcal{K}})^{-1} \in \mathcal{L}(\text{range } S, W)$ to $E \in \mathcal{L}(W, W)$.

OR. [Requires that $\text{range } S$ is Finite-dim] Let $R = B_{\text{range } S} = (Sv_1, \dots, Sv_n)$. Then $V = \mathcal{K}_R \oplus \text{null } S$.

Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i, Eu = 0$; for each $i = 1, \dots, n$ and each $u \in \text{null } S$.

Hence $\forall v \in V, (\exists! a_i \in \mathbb{F}, u \in \text{null } S), Tv = a_1Tv_1 + \dots + a_nTv_n = E(a_1Sv_1 + \dots + a_nSv_n) \Rightarrow T = ES$.

OR. [Requires that W is Finite-dim] Extend R to a basis $(Sv_1, \dots, Sv_n, w_1, \dots, w_m)$ of W .

Define $E \in \mathcal{L}(W)$ by $E(Sv_k) = Tv_k, Ew_j = 0$. Because $\forall v \in V, \exists a_i \in \mathbb{F}, Sv = a_1Sv_1 + \dots + a_nSv_n$.

Now $v - (a_1v_1 + \dots + a_nv_n) \in \text{null } S \Rightarrow v - (a_1v_1 + \dots + a_nv_n) \in \text{null } T \Rightarrow T(v - (a_1v_1 + \dots + a_nv_n)) = 0$.

Thus $Tv = a_1Tv_1 + \dots + a_nTv_n$. Hence $E(Sv) = a_1E(Sv_1) + \dots + a_nE(Sv_n) = a_1Tv_1 + \dots + a_nTv_n = Tv$ \square

25 Suppose V is finite-dim and $S, T \in \mathcal{L}(V, W)$.

Prove that $\text{range } S \subseteq \text{range } T \iff \exists E \in \mathcal{L}(V)$ such that $S = TE$.

SOLUTION:

Suppose $\exists E \in \mathcal{L}(V)$ such that $S = TE$. Then $\text{range } S = \text{range } TE \subseteq \text{range } T$.

Suppose $\text{range } S \subseteq \text{range } T$. Let (v_1, \dots, v_m) be a basis of V .

Note that each $sv_i \in \text{range } T$. Suppose $u_i \in V$ such that $Tu_i = sv_i$.

Thus defining $E \in \mathcal{L}(V)$ by $Ev_i = u_i$ for each $i \Rightarrow S = TE$. □

22 Suppose U and V are finite-dim vecsps and $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.

Prove that $\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T$.

SOLUTION:

Define $R \in \mathcal{L}(\text{null } ST, V)$ by $Ru = Tu$ for all $u \in \text{null } ST \subseteq U$.

$$\left. \begin{aligned} S(Tu) = 0 = S(Ru) &\Rightarrow \text{range } R \subseteq \text{null } S \Rightarrow \dim \text{range } R \leq \dim \text{null } S \\ Tu = 0 = Ru &\Rightarrow \text{null } R \supseteq \text{null } T \Rightarrow \dim \text{null } R = \dim \text{null } T \end{aligned} \right\} \Rightarrow \text{By [3.22], we are done. } \square$$

OR. For any $u \in U$, note that $u \in \text{null } ST \iff S(Tu) = 0 \iff Tu \in \text{null } S$.

Thus $\text{null } ST = \mathcal{K}_{\text{null } S \cap \text{range } T} = \{u \in U : Tu \in \text{null } S\}$. By Problem (4E 3B.21),

$\dim \text{null } ST = \dim \text{null } T + \dim(\text{null } S \cap \text{range } T) \leq \dim \text{null } T + \dim \text{null } S$. □

COROLLARY: (1) If T is inje, then $\dim \text{null } T = 0 \Rightarrow \dim \text{null } ST \leq \dim \text{null } S$.

(2) If T is surj, then $\text{range } R = \text{null } S \Rightarrow \dim \text{null } ST = \dim \text{null } S + \dim \text{null } T$.

(3) If S is inje, then $\text{range } R = \{0\} \Rightarrow \dim \text{null } ST = \dim \text{null } R = \dim \text{null } T$.

23 Suppose U and V are finite-dim vecsps and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$.

Prove that $\dim \text{range } ST \leq \min\{\dim \text{range } S, \dim \text{range } T\}$.

SOLUTION:

$\text{range } ST = \{Sv : v \in \text{range } T\} = \text{span}(Su_1, \dots, Su_{\dim \text{range } T})$, where $B_{\text{range } T} = (u_1, \dots, u_{\dim \text{range } T})$.

$\dim \text{range } ST \leq \dim \text{range } T$ and $\dim \text{range } ST \leq \dim \text{range } S$. □

OR. Note that $\text{range } S|_{\text{range } T} = \text{range } ST$.

Thus $\dim \text{range } ST = \dim \text{range } S|_{\text{range } T} = \dim \text{range } T - \dim \text{null } S|_{\text{range } T} \leq \dim \text{range } T$. □

COROLLARY: (1) If S is inje, then $\dim \text{range } ST = \dim \text{range } T$.

(2) If T is surj, then $\dim \text{range } ST = \dim \text{range } S$.

• (a) Suppose $\dim V = 5, S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \text{range } TS \leq 2$.

(b) Let $\dim V = n$ in (a). Prove that $\dim \text{range } TS \leq \left\lfloor \frac{n}{2} \right\rfloor$.

(c) Give an example of $S, T \in \mathcal{L}(\mathbb{F}^5)$ with $ST = 0$ and $\dim \text{range } TS = 2$.

SOLUTION:

(a) By Problem (23), $\dim \text{range } TS \leq \min\left\{\frac{5 - \dim \text{null } T}{1}, \frac{5 - \dim \text{null } S}{1}\right\}$.

We show that $\dim \text{range } TS \leq 2$ by contradiction. Assume that $\dim \text{range } TS \geq 3$.

Then $\min\{5 - \dim \text{null } T, 5 - \dim \text{null } S\} \geq 3 \Rightarrow \max\{\dim \text{null } T, \dim \text{null } S\} \leq 2$.

and $\dim \text{null } ST = 5 \leq \dim \text{null } S + \dim \text{null } T \leq 4$. Contradicts.

OR.
$$\left. \begin{aligned} \dim \text{null } S &= 5 - \dim \text{range } S \\ \dim \text{range } TS &\leq \dim \text{range } S \end{aligned} \right\} \Rightarrow \dim \text{null } S \leq 5 - \dim \text{range } TS$$

And $ST = 0 \Rightarrow \text{range } T \subseteq \text{null } S \Rightarrow \dim \text{range } TS \leq \dim \text{range } T \leq \dim \text{null } S$. □

(b) By Problem (23), $\dim \text{range } TS \leq \min \left\{ \overbrace{\dim \text{range } S}^{n - \dim \text{null } T}, \overbrace{\dim \text{range } T}^{n - \dim \text{null } S} \right\}$. We prove by contradiction.

Assume that $\dim \text{range } TS \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Then $\min \{n - \dim \text{null } T, n - \dim \text{null } S\} \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$

$\Rightarrow \max \{ \dim \text{null } T, \dim \text{null } S \} \leq n - \left\lfloor \frac{n}{2} \right\rfloor - 1$.

又 $\dim \text{null } ST = n \leq \dim \text{null } S + \dim \text{null } T \leq 2(n - \left\lfloor \frac{n}{2} \right\rfloor - 1)$

$\Rightarrow \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \frac{n}{2}$. Contradicts. Thus $\dim \text{range } TS \leq \left\lfloor \frac{n}{2} \right\rfloor$. □

OR. $\dim \text{null } S = n - \dim \text{range } S \leq n - \dim \text{range } TS$.

And $ST = 0 \Rightarrow \dim \text{range } TS \leq \dim \text{range } T \leq \dim \text{null } S \leq n - \dim \text{range } TS$

$\Rightarrow 2 \dim \text{range } TS \leq n \Rightarrow \dim \text{range } TS \leq \frac{n}{2}$

$\Rightarrow \dim \text{range } TS \leq \left\lfloor \frac{n}{2} \right\rfloor$ (because $\dim \text{range } TS$ is an integer). □

(c) Let (v_1, \dots, v_5) be a basis of \mathbf{F}^5 . Define $S, T \in \mathcal{L}(\mathbf{F}^5)$ by:

$T : \quad v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i ;$

$S : \quad v_1 \mapsto v_4, \quad v_2 \mapsto v_5, \quad v_i \mapsto 0 ; \quad i = 3, 4, 5.$ □

26 Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ such that $\forall p \in \mathcal{P}(\mathbf{R}), \deg(Dp) = (\deg p) - 1$.

Prove that $D \in \mathcal{P}(\mathbf{R})$ is surj.

SOLUTION:

[Informal Proof] $\left| \begin{array}{l} \text{Note that } \deg Dx^n = n - 1. \text{ Because } \text{span}(Dx, Dx^2, \dots) \subseteq \text{range } D. \\ \text{又 By (2.C.10), } \text{span}(Dx, Dx^2, \dots) = \text{span}(1, x, \dots) = \mathcal{P}(\mathbf{R}). \end{array} \right.$

[Proper Proof]

We will recursively define a sequence of polys $(p_k)_{k=0}^\infty$ where $Dp_k = x^k$.

(i) Because $\dim Dx = (\deg x) - 1 = 0$, we have $Dx = C \in \mathbf{F}$.

Define $p_0 = C^{-1}x$. Then $Dp_0 = C^{-1}Dx = 1$.

(ii) Suppose we have defined p_0, \dots, p_n such that $Dp_k = x^k$ for each $k \in \{0, \dots, n\}$.

Because $\deg D(x^{n+2}) = n + 1$. Let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_nx^n + \dots + a_1x + a_0$, where $a_{n+1} \neq 0$.

Then $a_{n+1}^{-1}D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_nDp_n + \dots + a_1Dp_1 + a_0Dp_0)$

$\Rightarrow x^{n+1} = D(a_{n+1}^{-1}(x^{n+2} - a_n p_n - \dots - a_1 p_1 - a_0 p_0))$.

Thus defining $p_{n+1} = a_{n+1}^{-1}(x^{n+2} - a_n p_n - \dots - a_1 p_1 - a_0 p_0)$, we have $Dp_{n+1} = x^{n+1}$.

Now we get $(p_k)_{k=0}^\infty$ by recursion. Hence $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), \exists q = (\sum_{k=0}^{\deg p} a_k p_k), Dq = p$. □

OR. Let $Dx^0 = 0, Dx^k = p_k$ for all $k \in \mathbf{N}^+$. For any $m \in \mathbf{N}^+, (p_1, \dots, p_m)$ is a basis of $\mathcal{P}_{m-1}(\mathbf{R})$.

Because $\forall p' \in \text{range } D, \exists ! m \in \mathbf{N}, \deg p = m - 1 \Rightarrow \exists ! a_k \in \mathbf{R}, p' = a_m p_m + \dots + a_1 p_1$.

Now $Dp = p' = a_m p_m + \dots + a_1 p_1 = D(a_m x^m + \dots + a_1 x)$. Thus $\exists q \in \mathcal{P}_m(\mathbf{R}), Dq = p$. □

27 Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that $\exists q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.

SOLUTION:

Define $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$.

Note that $\deg Bx^n = n - 1$. Similar to Problem (26), we conclude that B is surj. □

28 Suppose $T \in \mathcal{L}(V, W)$, $B_{\text{range } T} = (w_1, \dots, w_m)$.

Prove that $\exists \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ such that $\forall v \in V, Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m$.

SOLUTION:

Suppose $v_1, \dots, v_m \in V$ such that $Tv_i = w_i$ for each v_i . Then (v_1, \dots, v_m) is linearly inde.

Let $B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$. Note that $\forall v \in V, v = \sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i, \exists! a_i, b_i \in \mathbf{F}$.

Define $\varphi_i : V \rightarrow \mathbf{F}$ by $\varphi_i(v) = a_i v_i$ for each i . We now check the linearity.

$\forall v, u \in V (\exists! a_i, b_i, c_i, d_i \in \mathbf{F}), \lambda \in \mathbf{F}, \varphi_i(v + \lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda \varphi(u)$. □

29 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$. Suppose $u \in V \setminus \text{null } \varphi$. Prove that $V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$.

SOLUTION: If $\varphi = 0$ then we are done. Suppose $\varphi \neq 0$.

(a) $\forall v = cu \in \text{null } \varphi \cap \{au : a \in \mathbf{F}\}, \varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$. Hence $\text{null } \varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$.

(b) $\forall v \in V, v = \left(v - \frac{\varphi(v)}{\varphi(u)}u\right) + \frac{\varphi(v)}{\varphi(u)}u$. $\left\{ \begin{array}{l} v - \frac{\varphi(v)}{\varphi(u)}u \in \text{null } \varphi \\ \frac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array} \right\} \Rightarrow V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$. □

COMMENT: $\varphi \neq 0 \Rightarrow \varphi(v_i) = a_i \neq 0$ for each v_i , for some linearly inde list (v_1, \dots, v_k) .

Fix one v_k . Then $\forall j \in \{1, \dots, k-1, k+1, \dots, n\}, \text{span}\{a_j v_k - a_k v_j\} \subseteq \text{null } \varphi$.

Hence every vecsp in $S_V \text{null } \varphi$ is one-dim.

30 Suppose $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$ and $\text{null } \varphi_1 = \text{null } \varphi_2 = \text{null } \varphi$. Prove that $\exists c \in \mathbf{F}, \varphi_1 = c\varphi_2$

SOLUTION:

If $\text{null } \varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $u \in V \setminus \text{null } \varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$.

By Problem (29), $V = \text{null } \varphi \oplus \text{span}(u)$. Hence for any $v \in V, v = w + a_v u, \exists! w \in \text{null } \varphi, a_v \in \mathbf{F}$.

$\varphi_1(v) = a_v \varphi_1(u), \varphi_2(v) = a_v \varphi_2(u) \Rightarrow a_v = \frac{\varphi_1(v)}{\varphi_1(u)} = \frac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \frac{\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}$. □

31 Prove that $\exists T_1, T_2 \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^2), \text{null } T_1 = \text{null } T_2$ and $T_1 \neq cT_2, \forall c \in \mathbf{F}$.

SOLUTION:

Let (v_1, \dots, v_5) be a basis of \mathbf{R}^5 , (w_1, w_2) be a basis of \mathbf{R}^2 . Define $T, S \in \mathcal{L}(V, W)$ by

$\left. \begin{array}{l} Tv_1 = w_1, \quad Tv_2 = w_2, \quad Tv_3 = Tv_4 = Tv_5 = 0 \\ Sv_1 = w_1, \quad Sv_2 = 2w_2, \quad Sv_3 = Sv_4 = Sv_5 = 0 \end{array} \right\} \Rightarrow \text{null } T = \text{null } S$.

Suppose $T = \lambda S$. Then $w_1 = Tv_1 = \lambda Sv_1 = \lambda w_1 \Rightarrow \lambda = 1$.

While $w_2 = Tv_2 = \lambda Sv_2 = 2\lambda w_2 \Rightarrow \lambda = \frac{1}{2}$. Contradicts. □

• **TIPS:** Suppose $T \in \mathcal{L}(V, W)$ and U is a subsp such that $V = U \oplus \text{null } T$.

Now $\forall v \in V, \exists! u_v \in U, w_v \in \text{null } T, v = u_v + w_v$.

Then $T = T \circ i$, where $i : V \rightarrow U$ is defined by $i(v) = u_v$.

Because $\forall v \in V, T(v) = T(u_v + w_v) = T(u_v) = T(i(v)) = (T \circ i)(v)$. □

TODO: More Exercises and Notes.

ENDED

3.C

1 3 4 5 6 9 10 11 12 13 14 15 | 4E: 16, 17

• **NOTE FOR [3.47]:** $LHS = (AC)_{j,k} = \sum_{r=1}^n A_{j,r} C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r} (C_{\cdot,k})_{r,1} = (A_{j,\cdot} C_{\cdot,k})_{1,1} = A_{j,\cdot} C_{\cdot,k} = RHS.$

• **NOTE FOR [3.48]:**

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} &= \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 5 + 2 \times 8 & 1 \times 6 + 2 \times 9 & 1 \times 7 + 2 \times 10 \\ 3 \times 5 + 4 \times 8 & 3 \times 6 + 4 \times 9 & 3 \times 7 + 4 \times 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix} \end{aligned}$$

• [4E 3.51] Suppose $C \in \mathbb{F}^{m,c}, R \in \mathbb{F}^{c,p}$.

(a) For $k = 1, \dots, p$, $(CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot} R_{\cdot,k} = \sum_{r=1}^c C_{\cdot,r} R_{r,k} = R_{1,k} C_{\cdot,1} + \dots + R_{c,k} C_{\cdot,c}$

Which means that each cols CR is a linear combination of the cols of C .

(b) For $j = 1, \dots, m$, $(CR)_{j,\cdot} = C_{j,\cdot} R = C_{j,\cdot} R_{\cdot,\cdot} = \sum_{r=1}^c C_{j,r} R_{r,\cdot} = C_{j,1} R_{1,\cdot} + \dots + C_{j,c} R_{c,\cdot}$

Which means that each rows CR is a linear combination of the rows of R .

• **COLUMN-ROW FACTORIZATION (CR Factorization)** Suppose $A \in \mathbb{F}^{m,n}, A \neq 0$.

(a) Let $S_c = \text{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) \subseteq \mathbb{F}^{m,1}, \dim S_c = c$, the col rank.

Prove that $\exists C \in \mathbb{F}^{m,c}, R \in \mathbb{F}^{c,n}, A = CR$.

(b) Let $S_r = \text{span}(A_{1,\cdot}, \dots, A_{m,\cdot}) \subseteq \mathbb{F}^{1,n}, \dim S_r = r$, the row rank.

Prove that $\exists C \in \mathbb{F}^{m,r}, R \in \mathbb{F}^{r,n}, A = CR$.

SOLUTION: Notice that $A \neq 0 \Rightarrow c, r \geq 1$.

(a) Let $(C_{\cdot,1}, \dots, C_{\cdot,c})$ be a basis of S_c , forming $C \in \mathbb{F}^{m,c}$. Then $\forall k \in \{1, \dots, n\}$,

$$A_{\cdot,k} = R_{1,k} C_{\cdot,1} + \dots + R_{c,k} C_{\cdot,c} = (CR)_{\cdot,k}, \exists! R_{1,k}, \dots, R_{c,k} \in \mathbb{F}, \text{ forming } R \in \mathbb{F}^{c,n}. \text{ Thus } A = CR.$$

(b) Let $(R_{1,\cdot}, \dots, R_{r,\cdot})$ be a basis of S_r , forming $R \in \mathbb{F}^{r,n}$. Then $\forall j \in \{1, \dots, m\}$,

$$A_{j,\cdot} = C_{j,1} R_{1,\cdot} + \dots + C_{j,r} R_{r,\cdot} = (CR)_{j,\cdot}, \exists! C_{j,1}, \dots, C_{j,r} \in \mathbb{F}, \text{ forming } C \in \mathbb{F}^{m,r}. \text{ Thus } A = CR. \quad \square$$

EXAMPLE:

$$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \stackrel{(1)}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \stackrel{(2)}{=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

$$(I) \begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2 \begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix}.$$

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix}$ can be uniquely written as a linear combination of $(A_{1,\cdot}, A_{2,\cdot})$.

Hence $\dim S_r = 2$. $(A_{1,\cdot}, A_{2,\cdot})$ is a basis.

$$(II) \begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2 \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}; \quad \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix} - \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix}. \text{ Hence } \dim S_c = 2. (A_{\cdot,2}, A_{\cdot,3}) \text{ is a basis.}$$

• COLUMN RANK EQUALS ROW RANK (Using the notation and result above)

For each $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$.

For each $A_{\cdot,k} \in S_c$, $A_{\cdot,k} = (CR)_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$.

$\Rightarrow \text{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = S_r = \text{span}(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leq c = \dim S_c$.

$\Rightarrow \text{span}(A_{\cdot,1}, \dots, A_{\cdot,m}) = S_r = \text{span}(C_{\cdot,1}, \dots, C_{\cdot,r}) \Rightarrow \dim S_c = c \leq r = \dim S_r$.

OR. Apply the result to $A^t \in \mathbb{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leq r = \dim S_r = \dim S_c^t$. □

• [4E 3.C.17, OR 3.F.32] Suppose $T \in \mathcal{L}(V)$ and $(u_1, \dots, u_n), (v_1, \dots, v_n)$ are bases of V . Prove that the following are equi. Here $A = \mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.

(a) T is inje.

(b) The cols of $\mathcal{M}(T)$ are linely inde in $\mathbb{F}^{n,1}$.

(c) The cols of $\mathcal{M}(T)$ span $\mathbb{F}^{n,1}$.

(d) The rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$.

(e) The rows of $\mathcal{M}(T)$ are linely inde in $\mathbb{F}^{1,n}$.

SOLUTION: Using TIPS in 2.C.

$$T \text{ is inje} \iff \dim V = \dim \text{range } T + \dim \text{null } T = \dim \text{range } T$$

$$\Delta \begin{cases} \iff (Tu_1, \dots, Tu_n) \text{ is a basis of } V; \dim \text{range } T = \dim \text{span}(\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n)) = n \\ \iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n)) \text{ is a basis of } \mathbb{F}^{n,1}, \text{ as well as } (A_{\cdot,1}, \dots, A_{\cdot,n}) \end{cases}$$

$$\left[\text{又 } \dim S_c = \dim \text{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) = \dim \text{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = \dim S_r = n \right]$$

$$\iff (A_{1,\cdot}, \dots, A_{n,\cdot}) \text{ is a basis of } \mathbb{F}^{1,n}. \quad \square$$

Now we show (Δ) properly, that is $T \text{ is inje} \iff \text{The cols of } \mathcal{M}(T) \text{ are linely inde.}$

(a) \Rightarrow (b) :

$$\text{Suppose } b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n} = \begin{pmatrix} b_1 A_{1,1} + \cdots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \cdots + b_n A_{n,n} \end{pmatrix} = 0. \text{ Let } u = b_1 u_1 + \cdots + b_n u_n.$$

$$\text{Then } Tu = b_1 Tu_1 + \cdots + b_n Tu_n$$

$$= b_1 (A_{1,1}v_1 + \cdots + A_{n,1}v_n) + \cdots + b_n (A_{1,n}v_1 + \cdots + A_{n,n}v_n)$$

$$= (b_1 A_{1,1} + \cdots + b_n A_{1,n})v_1 + \cdots + (b_1 A_{n,1} + \cdots + b_n A_{n,n})v_n$$

$$= 0v_1 + \cdots + 0v_n = 0$$

$$\Rightarrow b_1 = \cdots = b_n = 0.$$

Thus by (2.39), (b) holds.

(b) \Rightarrow (a) :

Suppose $u = b_1 u_1 + \cdots + b_n u_n \in \text{null } T$.

$$\text{Then } Tu = 0 = (b_1 A_{1,1} + \cdots + b_n A_{1,n})v_1 + \cdots + (b_1 A_{n,1} + \cdots + b_n A_{n,n})v_n.$$

$$\text{Thus } b_1 A_{1,1} + \cdots + b_n A_{1,n} = \cdots = b_1 A_{n,1} + \cdots + b_n A_{n,n} = 0.$$

$$\text{Which is equi to } \begin{pmatrix} b_1 A_{1,1} + \cdots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \cdots + b_n A_{n,n} \end{pmatrix} = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n} = 0 \Rightarrow b_1 = \cdots = b_n = 0.$$

Thus by (2.39), (a) holds. □

- [4E 3.C.16, OR 3.E.11] Suppose A is an m -by- n matrix with $A \neq 0$.
Prove that $\text{rank } A = 1 \iff \exists (c_1, \dots, c_m) \in \mathbf{F}^m, (d_1, \dots, d_n) \in \mathbf{F}^n$
such that $A_{j,k} = c_j \cdot d_k$ for every $j = 1, \dots, m$ and $k = 1, \dots, n$.

SOLUTION:

Using the notation in CR Factorization.

$$(a) \text{ Suppose } A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \dots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \dots & c_m d_n \end{pmatrix}. \quad (\exists c_j, d_k \in \mathbf{F}, \forall j, k)$$

$$\text{Then } S_c = \left\{ \begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \dots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}.$$

$$\text{OR. } S_r = \text{span} \left\{ \begin{pmatrix} c_1 d_1 & \dots & c_1 d_n \\ c_2 d_1 & \dots & c_2 d_n \\ \vdots & & \vdots \\ c_m d_1 & \dots & c_m d_n \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} \right\}. \quad \text{Hence rank } A = 1.$$

OR. Using also the result in [4E 3.51(a)].

Every col of A is a scalar multi of C . Then $\text{rank } A \leq 1$ 又 $\text{rank } A \geq 1$ ($A \neq 0$).

$$(b) \text{ By CR Factorization, } \exists C = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbf{F}^{m,1}, R = \begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} \in \mathbf{F}^{1,n} \text{ such that } A = CR.$$

OR. Not using CR Factorization. Suppose $\text{rank } A = \dim S_c = \dim S_r = 1$.

$$\text{Let } c_j = \frac{A_{j,1}}{A_{1,1}} = \frac{A_{j,2}}{A_{1,2}} = \dots = \frac{A_{j,n}}{A_{1,n}}, \quad d'_k = \frac{A_{1,k}}{A_{1,1}} = \frac{A_{2,k}}{A_{2,1}} = \dots = \frac{A_{m,k}}{A_{m,1}}. \quad (\forall j, k)$$

$$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k. \text{ Letting } d_k = d'_k A_{1,1}. \quad \square$$

- 1 Suppose $T \in \mathcal{L}(V, W)$. Show that with resp to each choice of bases of V and W , the matrix of T has at least $\dim \text{range } T$ nonzero entries.

SOLUTION:

Let $B_{\text{null } T} = (v_1, \dots, v_p), B_V = (v_1, \dots, v_n)$. Let $B_W = (w_1, \dots, w_m)$. Denote $\mathcal{M}(T, B_V, B_W)$ by A .

Because at most p of the v_k 's can belong to $\text{null } T \iff$ at least $n - p = q$ of the v_k 's do not.

For $v_k \notin \text{null } T, T v_k = A_{1,k} w_1 + \dots + A_{m,k} w_m \neq 0$. Thus col k has at least one nonzero entry.

Since there are $n - p = q$ choices of such k , A has at least $q = \dim \text{range } T$ nonzero entries. \square

OR. We prove by contradiction.

Suppose A has at most $(\dim \text{range } T - 1)$ nonzero entries.

Then by Pigeon Hole Principle, at least one of $A_{\cdot, p+1}, \dots, A_{\cdot, n}$ equals 0.

Thus there are at most $(\dim \text{range } T - 1)$ nonzero vecs in $T v_{p+1}, \dots, T v_n$.

While $\text{range } T = \text{span}(T v_{p+1}, \dots, T v_n) \Rightarrow \dim \text{range } T = \dim \text{span}(T v_{p+1}, \dots, T v_n)$. Contradicts. \square

3 Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$. Prove that $\exists B_V, B_W$ such that
 [letting $A = \mathcal{M}(T, B_V, B_W)$] $A_{k,k} = 1, A_{i,j} = 0$, where $1 \leq k \leq \dim \text{range } T, i \neq j$.

SOLUTION:

Let $R = (Tv_1, \dots, Tv_n)$ be a basis of $\text{range } T$, extend to $B_W = (Tv_1, \dots, Tv_n, w_1, \dots, w_p)$.

Let $\mathcal{K}_R = \text{span}(v_1, \dots, v_n)$. Let (u_1, \dots, u_m) be a basis of $\text{null } T$. Then $B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$. \square

4 Suppose $B_V = (v_1, \dots, v_m)$ and W is finite-dim. Suppose $T \in \mathcal{L}(V, W)$.

Prove that $\exists B_W = (w_1, \dots, w_n)$, $\mathcal{M}(T, B_V, B_W)_{\cdot,1}^t = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & \dots & 0 \end{pmatrix}$.

SOLUTION: If $Tv_1 = 0$, then we are done. If not then extend (Tv_1) . \square

5 Suppose $B_W = (w_1, \dots, w_n)$ and V is finite-dim. Suppose $T \in \mathcal{L}(V, W)$.

Prove that $\exists B_V = (v_1, \dots, v_m)$, $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$.

SOLUTION:

Let (u_1, \dots, u_n) be a basis of V . Denote $\mathcal{M}(T, (u_1, \dots, u_n), B_W)$ by A .

If $A_{1,\cdot} = 0$, then let $B_V = (u_1, \dots, u_n)$, we are done.

Otherwise, $(A_{1,1} \dots A_{1,m}) \neq 0$, choose one $A_{1,k} \neq 0$.

Let $v_1 = \frac{u_k}{A_{1,k}}$; $v_j = u_{j-1} - A_{1,j-1}v_1$ for $j = 2, \dots, k$;
 $v_i = u_i - A_{1,i}v_1$ for $i = k+1, \dots, n$.

Now because each $u_k \in \text{span}(v_1, \dots, v_n) \Rightarrow V = \text{span}(v_1, \dots, v_n), B_V = (v_1, \dots, v_n)$.

And $Tv_1 = T\left(\frac{u_k}{A_{1,k}}\right) = \frac{1}{A_{1,k}}(A_{1,k}w_1 + \dots + A_{n,k}w_n) = 1w_1 + \dots + \frac{A_{n,k}}{A_{1,k}}w_n$.

$\forall j \in \{2, \dots, k, \underbrace{k+2, \dots, n+1}_{i \in \{k+1, \dots, n\}}\}$, $Tv_j = T(u_{j-1} - A_{1,j-1}v_1) = Tu_{j-1} - T\left(\frac{A_{1,j-1}u_k}{A_{1,k}}\right)$
 $= A_{1,j-1}w_1 + \dots + A_{n,j-1}w_n - A_{1,j-1}\left(1w_1 + \dots + \frac{A_{n,k}}{A_{1,k}}w_n\right) = 0w_1 + \dots + \left(A_{n,j-1} - \frac{A_{1,j-1}A_{n,k}}{A_{1,k}}\right)w_n. \square$

6 Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$.

Prove that $\dim \text{range } T = 1 \iff \exists B_V, B_W$, all entries of $A = \mathcal{M}(T, B_V, B_W)$ equal 1.

SOLUTION:

(a) Suppose $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$ are the bases such that all entries of A equal 1.

Then $Tv_i = w_1 + \dots + w_m$ for all $i = 1, \dots, n$. Because w_1, \dots, w_m is linearly inde, $w_1 + \dots + w_m \neq 0$.

(b) Suppose $\dim \text{range } T = 1$. Then $\dim \text{null } T = \dim V - 1$.

Let (u_2, \dots, u_n) be a basis of $\text{null } T$. Extend it to a basis of V as (u_1, u_2, \dots, u_n) .

Let $w_1 = Tv_1 - w_2 - \dots - w_m$. Extend to a basis of W and we have B_W .

Let $v_1 = u_1, v_i = u_1 + u_i$. Extend to a basis of V and we have B_V . \square

OR. Suppose $\text{range } T$ has a basis (w) .

By (2.C.15 [COROLLARY]), $\exists B_W = (w_1, \dots, w_m)$ such that $w = w_1 + \dots + w_m$.

By (2.C [NEW THEOREM]), \exists a basis (u_1, \dots, u_n) of V such that each $u_k \notin \text{null } T$.

$\forall k \in \{1, \dots, n\}, Tu_k \in \text{range } T = \text{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists \lambda_k \in \mathbf{F} \setminus \{0\}$.

Let $v_k = \lambda_k^{-1}u_k \neq 0 \Rightarrow B_V = (v_1, \dots, v_n)$. Hence for each $v_k, Tv_k = w = w_1 + \dots + w_m$. \square

• **NOTE FOR [3.49]:** $\cdot[(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$
 $\cdot(AC)_{\cdot,k} = A_{\cdot,\cdot}C_{\cdot,k} = AC_{\cdot,k}$ □

• **EXERCISE 10:** $\cdot[(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$
 $\cdot(AC)_{j,\cdot} = A_{j,\cdot}C_{\cdot,\cdot} = A_{j,\cdot}C$ □

• **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$
 $\cdot(AC)_{j,1} = \sum_{r=1}^n A_{j,r}c_{r,1} = (\sum_{r=1}^n (A_{\cdot,r}c_{r,1}))_{j,1} = (c_1A_{\cdot,1} + \dots + c_nA_{\cdot,n})_{j,1}$
 $\cdot Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^n A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \dots + c_nA_{\cdot,n}$ OR. By $(Ac)_{\cdot,1} = Ac_{\cdot,1}$ Using (a) above. □

• **EXERCISE 11:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$
 $\cdot(aC)_{1,k} = \sum_{r=1}^n a_{1,r}C_{r,k} = (\sum_{r=1}^n a_{1,r}(C_{r,\cdot}))_{1,k} = (a_1C_{1,\cdot} + \dots + a_nC_{n,\cdot})_{1,k}$
 $\cdot aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^n a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \dots + a_nC_{n,\cdot}$ OR. By $(aC)_{1,\cdot} = a_{1,\cdot}C$ Using (b) above. □

• Suppose p is a poly of n variables in \mathbf{F} . Prove that $\mathcal{M}(p(T_1, \dots, T_n)) = p(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n))$.
Where the linear maps T_1, \dots, T_n are such that $p(T_1, \dots, T_n)$ makes sense. See [5.B.16,17,20].

SOLUTION:

Suppose the poly p is defined by $p(x_1, \dots, x_n) = \sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n x_i^{k_i}$.

Note that $\mathcal{M}(T^x S^y) = \mathcal{M}(T)^x \mathcal{M}(S)^y$; $\mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$.

Then $\mathcal{M}(p(T_1, \dots, T_n)) = \mathcal{M}(\sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n T_i^{k_i})$
 $= \sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n \mathcal{M}(T_i^{k_i}) = p(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n))$. □

13 Prove that the distr holds for matrix add and matrix multi.

Suppose A, B, C are matrices such that $A(B + C)$ make sense, we prove the left distr.

SOLUTION:

Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Note that $[A(B + C)]_{j,k} = \sum_{r=1}^n A_{j,r}(B + C)_{r,k} = \sum_{r=1}^n (A_{j,r}B_{r,k} + A_{j,r}C_{r,k}) = (AB + AC)_{j,k}$. □

OR. Define T, S, R such that $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$A(B + C) = \mathcal{M}(T(S + R)) \stackrel{[3.9]}{=} \mathcal{M}(TS + TR) = AB + AC$.

Or $T(S + R) = TS + TR \Rightarrow \mathcal{M}(T(S + R)) = \mathcal{M}(TS + TR) \Rightarrow A(B + C) = AB + AC$. □

14 Prove that matrix multi is associ.

Suppose A, B, C are matrices such that $(AB)C$ makes sense, we prove that $(AB)C = A(BC)$.

SOLUTION:

Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$. We will show that $LHS = [(AB)C]_{j,k} = [A(BC)]_{j,k} = RHS$.

$LHS = (AB)_{j,\cdot}C_{\cdot,k} = \sum_{s=1}^n (A_{j,s}B_{s,\cdot})C_{\cdot,k} = \sum_{s=1}^n A_{j,s}(B_{s,\cdot}C_{\cdot,k}) = \sum_{s=1}^n A_{j,s}(BC)_{s,k} = RHS$. □

OR. Define T, S, R such that $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$(AB)C = \mathcal{M}(T(SR)) \stackrel{[3.9]}{=} \mathcal{M}(TSR) \stackrel{[3.9]}{=} \mathcal{M}((TS)R) = A(BC)$.

OR. $(TS)R = T(SR) \Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR)) \Rightarrow (AB)C = A(BC)$. □

15 Suppose $A \in \mathbf{F}^{n,n}$, $j, k \in \{1, \dots, n\}$. Show that $(A^3)_{j,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}$.

SOLUTION: $(AAA)_{j,k} = (AA)_{j,\cdot} A_{\cdot,k} = \sum_{p=1}^n (A_{j,p} A_{p,\cdot}) A_{\cdot,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}$.

$$\begin{aligned} \text{OR. } (AAA)_{j,k} &= \sum_{r=1}^n (AA)_{j,r} A_{r,k} = \sum_{r=1}^n \left(\sum_{p=1}^n A_{j,p} A_{p,r} \right) A_{r,k} \\ &= \sum_{r=1}^n \left[A_{j,1} (A_{1,r} A_{r,k}) + \dots + A_{j,n} (A_{n,r} A_{r,k}) \right] \\ &= A_{j,1} \sum_{r=1}^n A_{1,r} A_{r,k} + \dots + A_{j,n} \sum_{r=1}^n A_{n,r} A_{r,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}. \quad \square \end{aligned}$$

• Prove that the commutativity does not hold in $\mathbf{F}^{m,n}$.

SOLUTION:

Suppose $\dim V = n, \dim W = m$ and the commutativity holds in $\mathbf{F}^{n,m}$.

$$\forall T \in \mathcal{L}(V, W), S \in \mathcal{L}(W, V), \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST).$$

Hence $ST = TS$. Which in general is not true. (See 3.D) □

• [10.A.3, OR 4E 3.D.19] Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that $\forall B_V \neq B'_V, \mathcal{M}(T, B_V) = \mathcal{M}(T, B'_V) \iff T = \lambda \mathcal{M}(I), \exists \lambda \in \mathbf{F}$.

SOLUTION: [Compare with the first solution of (3.D.16) in 3.A]

Suppose $T = \lambda I$ for some $\lambda \in \mathbf{F}$. Then $T = \lambda \mathcal{M}(I)$.

Suppose $\forall B_V \neq B'_V, \mathcal{M}(T, B_V) = \mathcal{M}(T, B'_V)$. If $T = 0$, then we are done.

Suppose $T \neq 0$, and $v \in V \setminus \{0\}$. Assume that (v, Tv) is linely inde.

Extend (v, Tv) to $B_V = (v, Tv, u_3, \dots, u_n)$. Let $B = \mathcal{M}() (T, B_V)$.

$$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \dots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \forall i \neq 2.$$

By assumption, $A = \mathcal{M}(T, B'_V) = B, \forall B'_V = (v, w_2, \dots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$.

$\Rightarrow Tv = w_2$, which is not true if we let $w_2 = u_3, w_3 = Tv, w_j = u_j, \forall j \in \{4, \dots, n\}$. Contradicts.

Hence (v, Tv) is linely depe $\Rightarrow \forall v \in V, \exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show that λ_v is independent of v , that is, to show that for all $v \neq w \in V \setminus \{0\}, \lambda_v = \lambda_w$.

$$\left. \begin{aligned} (v, w) \text{ is linely inde} &\Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v, w) \text{ is linely depe, } w = cv &\Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \end{aligned} \right\} \Rightarrow T = \lambda I, \exists \lambda \in \mathbf{F}. \quad \square$$

OR. Conversely, denote $\mathcal{M}(T, B_V)$ by A , where $B_V = (u_1, \dots, u_m)$ is arbitrary.

Fix one $B_V = (v_1, \dots, v_m)$ and then $(v_1, \dots, \frac{1}{2}v_k, \dots, v_m)$ is also a basis for any given $k \in \{1, \dots, m\}$.

Fix one k . Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \dots + A_{k,k}(\frac{1}{2}v_k) + \dots + A_{m,k}v_m$

$$\Rightarrow Tv_k = 2A_{1,k}v_1 + \dots + A_{k,k}v_k + \dots + 2A_{m,k}v_m = A_{1,k}v_1 + \dots + A_{k,k}v_k + \dots + A_{m,k}v_m.$$

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1, \dots, m\}$.

Now we show that $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j, k such that $j \neq k$.

Consider the basis $B'_V = (v'_1, \dots, v'_j, \dots, v'_k, \dots, v'_m)$,

where $v'_j = v_k, v'_k = v_j$ and $v'_i = v_i$ for all $i \in \{1, \dots, m\} \setminus \{j, k\}$.

Remember that $\mathcal{M}(T, B'_V) = \mathcal{M}(T, B_V) = A$.

Hence $T(v'_k) = A_{1,k}v'_1 + \dots + A_{k,k}v'_k + \dots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_j$, while $T(v'_k) = T(v_j) = A_{j,j}v_j$.

Thus $A_{k,k} = A_{j,j}$. □

3.D

1 2 3 4 5 6 8 9 10 11 12 13 15 16 17 18 19 | 4E: 1, 3, 10, 15, 17, 19, 20, 22, 23, 24

• Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

$$\left. \begin{array}{l} (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1, \dots, v_n) \text{ of } V \iff T \text{ is surj} \\ (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1, \dots, v_n) \text{ of } V \iff T \text{ is inje} \end{array} \right\} \iff T \text{ is inv.}$$

• Suppose $T \in \mathcal{L}(V)$ and $V = \text{span}(Tv_1, \dots, Tv_m)$. Prove that $V = \text{span}(v_1, \dots, v_m)$.

SOLUTION:

Because $V = \text{span}(Tv_1, \dots, Tv_m) \Rightarrow T$ is surj, $\wedge V$ is finite-dim $\Rightarrow T$ is inv $\Rightarrow T^{-1}$ is inv.

$$\forall v \in V, \exists a_i \in \mathbb{F}, v = a_1Tv_1 + \dots + a_mTv_m \Rightarrow T^{-1}v = a_1v_1 + \dots + a_mv_m \Rightarrow \text{range } T^{-1} \subseteq \text{span}(v_1, \dots, v_m).$$

OR. Reduce (Tv_1, \dots, Tv_m) to a basis of V as $(Tv_{\alpha_1}, \dots, Tv_{\alpha_k})$, where $k = \dim V$ and $\alpha_i \in \{1, \dots, m\}$.

Then $(v_{\alpha_1}, \dots, v_{\alpha_k})$ is linely inde of length k , hence is a basis of V , contained in the list (v_1, \dots, v_m) . \square

• OR (10.A.1) Suppose $T \in \mathcal{L}(V)$, $B_V = (v_1, \dots, v_n)$. Prove that $\mathcal{M}(T, B_V)$ is inv $\iff T$ is inv.

SOLUTION: Notice that $\mathcal{M} \in \mathcal{L}(\mathcal{L}(V), \mathbb{F}^{n,n})$ is an iso.

$$(a) T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}.$$

$$(b) \mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I. \quad \exists! S \in \mathcal{L}(V) \text{ such that } \mathcal{M}(T)^{-1} = \mathcal{M}(S)$$

$$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$$

$$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}. \quad \square$$

• Suppose $T \in \mathcal{L}(V, W)$ is inv. Show that T^{-1} is inv and $(T^{-1})^{-1} = T$.

$$\text{SOLUTION: } \left. \begin{array}{l} TT^{-1} = (T^{-1})^{-1}T^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = T^{-1}(T^{-1})^{-1} = I \in \mathcal{L}(W) \end{array} \right\} \Rightarrow T = (T^{-1})^{-1}, \text{ by the uniqueness of inverse.} \quad \square$$

1 Suppose $T \in \mathcal{L}(U, V)$, $S \in \mathcal{L}(V, W)$ are inv. Prove that ST is inv and $(ST)^{-1} = T^{-1}S^{-1}$.

$$\text{SOLUTION: } \left. \begin{array}{l} (ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(U) \end{array} \right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}, \text{ by the uniqueness of inv.} \quad \square$$

2 Suppose V is finite-dim and $\dim V > 1$.

Prove that the set of non-inv operators on V is not a subsp of $\mathcal{L}(V)$.

The set of inv operators is not either, although multi identity/inv, and commutativity for vec multi holds.

SOLUTION:

Denote the set by U . Suppose $\dim V = n > 1$. Let (v_1, \dots, v_n) be a basis of V . Define $S, T \in \mathcal{L}(V)$ by

$$S(a_1v_1 + \dots + a_nv_n) = a_1v_1, T(a_1v_1 + \dots + a_nv_n) = a_2v_1 + \dots + a_nv_n. \text{ Hence } S + T = I \text{ is inv.} \quad \square$$

COMMENT: If $\dim V = 1$, then $U = \{0\}$ is a subsp of $\mathcal{L}(V)$.

3 Suppose V is finite-dim, U is a subsp of V , and $S \in \mathcal{L}(U, V)$.

Prove that \exists inv $T \in \mathcal{L}(V)$, $Tu = Su, \forall u \in U \iff S$ is inje. [Compare this with (3.A.11).]

SOLUTION:

$$(a) Tu = Su \text{ for every } u \in U \Rightarrow u = T^{-1}Su \Rightarrow S \text{ is inje. OR. } \text{null } S = \text{null } T \cap U = \{0\} \cap U = \{0\}.$$

$$(b) \text{ Suppose } (u_1, \dots, u_m) \text{ be a basis of } U \text{ and } S \text{ is inje} \Rightarrow (Su_1, \dots, Su_m) \text{ is linely inde in } V.$$

Extend these to bases of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$ and $(Su_1, \dots, Su_m, w_1, \dots, w_n)$.

Define $T \in \mathcal{L}(V)$ by $T(u_i) = Su_i; T v_j = w_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. \square

4 Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.

Prove that $\text{null } S = \text{null } T (= U) \iff S = ET, \exists \text{ inv } E \in \mathcal{L}(W)$.

SOLUTION:

Define $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i$, $E(w_j) = x_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

$$\left| \begin{array}{l} \text{Let } B_{\text{range } T} = \mathcal{L}(Tv_1, \dots, Tv_m), \text{ extend to } B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n). \\ \text{Let } \mathcal{K} = \text{span}(v_1, \dots, v_m). \text{ } \mathcal{K} \text{ null } S = \text{null } T \implies V = \mathcal{K} \oplus \text{null } S \Leftrightarrow \mathcal{K} \in \mathcal{S}_V \text{null } S. \\ \implies \text{span}(Sv_1, \dots, Sv_m) = \text{range } S \text{ } \mathcal{K} \text{ dim range } T = \text{dim range } S = m. \\ \text{Hence } B_{\text{range } S} = (Sv_1, \dots, Sv_m). \text{ Thus we let } B'_W = (Sv_1, \dots, Sv_m, x_1, \dots, x_n). \end{array} \right| \begin{array}{l} \therefore E \text{ is inv} \\ \text{and } S = ET. \end{array}$$

Conversely, $S = ET \Rightarrow \text{null } S = \text{null } ET$.

Then $v \in \text{null } ET \iff ET(v) = 0 \iff Tv = 0 \iff v \in \text{null } T$. Hence $\text{null } ET = \text{null } T = \text{null } S$.

5 Suppose that V is finite-dim and $S, T \in \mathcal{L}(V, W)$.

Prove that $\text{range } S = \text{range } T(=R) \iff S = TE, \exists \text{ inv } E \in \mathcal{L}(V)$.

SOLUTION:

Define $E \in \mathcal{L}(V)$ as $E: v_i \mapsto r_i; u_j \mapsto s_j$; for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

$$\left| \begin{array}{l} \text{Let } B_R = \mathcal{L}(Tv_1, \dots, Tv_m); B'_R = (Sr_1, \dots, Sr_m) \text{ such that } \forall i, Tv_i = Sr_i. \\ \text{Let } B_{\text{null } T} = (u_1, \dots, u_n); B_{\text{null } S} = (s_1, \dots, s_n). \\ \text{Thus } B_V = (v_1, \dots, v_m, u_1, \dots, u_n); B'_V = (r_1, \dots, r_m, s_1, \dots, s_n). \end{array} \right| \therefore E \text{ is inv and } S = TE.$$

Conversely, $S = TE \Rightarrow \text{range } S = \text{range } TE$.

Then $w \in \text{range } S \iff \exists v \in V, Sv = TE(v) = T(E(v)) = w \in \text{range } T$. Hence $\text{range } S = \text{range } T$. \square

6 Suppose V and W are finite-dim and $S, T \in \mathcal{L}(V, W)$.

Prove that $S = E_2TE_1, \exists \text{ inv } E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W) \iff \dim \text{null } S = \dim \text{null } T = n$.

SOLUTION:

Define $E_1: v_i \mapsto r_i; u_j \mapsto s_j$; for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.

Define $E_2 : Tv_i \mapsto Sr_i ; x_j \mapsto y_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

<p>Let $B_{\text{range } T} = \mathcal{L}(Tv_1, \dots, Tv_m)$; $B_{\text{range } S} = (Sr_1, \dots, Sr_m)$. Extend to $B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p)$; $B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p)$. Let $B_{\text{null } T} = (u_1, \dots, u_n)$; $B_{\text{null } S} = (s_1, \dots, s_n)$. Thus $B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$; $B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)$.</p>	$\therefore E_1, E_2$ are inv and $S = E_2 T E_1$.
---	--

Conversely, $S = E_2TE_1 \Rightarrow \dim \text{null } S = \dim \text{null } E_2TE_1$.

$v \in \text{null } E_2TE_1 \iff E_2TE_1(v) = 0 \iff TE_1(v) = 0$. Hence $\text{null } E_2TE_1 = \text{null } TE_1 = \text{null } S$.

⌘ By (3.B.22.COROLLARY), E is inv $\Rightarrow \dim \text{null } TE_1 = \dim \text{null } T = \dim \text{null } S$.

8 Suppose V is finite-dim and $T : V \rightarrow W$ is a **surj** linear map of V onto W .

Prove that there is a subsp U of V such that $T|_U$ is an iso of U onto W .

SOLUTION:

Let $B_{\text{range } T} = B_W = (w_1, \dots, w_m) \Rightarrow \forall w_i, \exists! v_i \in V, T v_i = w_i$. Let $B_{\mathcal{K}} = (v_1, \dots, v_m)$.

Then $\dim \mathcal{K} = \dim W$. Thus $T|_{\mathcal{K}}$ is an iso of \mathcal{K} onto W .

OR. By Problem (12) in (3.B), there is a subsp U of V such that

$$U \cap \text{null } T = \{0\} = \text{null } T|_U, \text{range } T = \{Tu : u \in U\} = \text{range } T|_U.$$

9 Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that ST is inv $\iff S$ and T are inv.

SOLUTION:

Suppose S, T are inv. Then $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$. Hence ST is inv.

Suppose ST is inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$.

$$\left. \begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \text{range } S \end{array} \right\} \Rightarrow T \text{ is inje, } S \text{ is surj. While } V \text{ is finite-dim.} \quad \square$$

OR. Because by (3.B.23), $\dim V = \dim \text{range } ST \leq \min\{\text{range } T, \text{range } S\}$. \square

10 Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that $ST = I \iff TS = I$.

SOLUTION:

$$\text{Suppose } ST = I. \left. \begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \text{range } S \end{array} \right\} \Rightarrow T \text{ is inje, } S \text{ is surj. While } V \text{ is finite-dim.}$$

OR. By Problem (9), V is finite-dim and $ST = I$ is inv $\Rightarrow S, T$ are inv.

$$S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow S \text{ is inv.}$$

$$\text{OR. } ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T. \text{ } \text{ } S = S \Rightarrow TS = S^{-1}S = I.$$

Reversing the roles of S and T , we conclude that $TS = I \Rightarrow ST = I$. \square

11 Suppose V is finite-dim, $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is inv and $T^{-1} = US$.

SOLUTION: Using Problem (9) and (10). This result can fail without the hypothesis that V is finite-dim.

$$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I.$$

$$\Rightarrow U^{-1} = ST, \quad T^{-1} = US, \quad S^{-1} = TU. \quad \square$$

EXAMPLE: $V = \mathbb{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots); T(a_1, \dots) = (0, a_1, \dots); U = I \Rightarrow STU = I$ but T is not inv.

13 Suppose V is finite-dim, $R, S, T \in \mathcal{L}(V)$ are such that RST is surj. Prove that S is inje.

SOLUTION: By Problem (1) and (9), Notice that V is finite-dim. Then RST is inv.

$$\text{Let } X = (RST)^{-1} \left\{ \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje.} \\ \forall v \in V, v = (RST)Xv \in \text{range } R \Rightarrow R \text{ is surj.} \end{array} \right\} \Rightarrow S = R^{-1}(RST)^{-1} \text{ is inv.} \quad \square$$

$$\text{OR. } (RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}. \quad \square$$

15 Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multi.

In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then $\exists A \in \mathbb{F}^{m,n}, Tx = Ax, \forall x \in \mathbb{F}^{n,1}$.

SOLUTION:

Let $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$ be the standard bases of $\mathbb{F}^{n,1}, \mathbb{F}^{m,1}$.

$$\forall k = 1, \dots, n, \text{ suppose } T(E_k) = A_{1,k}R_1 + \dots + A_{m,k}R_m, \exists A_{j,k} \in \mathbb{F}, \text{ forming } A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix}. \quad \square$$

OR. Let $A = \mathcal{M}(T, B_1, B_2)$. Note that $\mathcal{M}(x, B_1) = x, \mathcal{M}(y, B_2) = y$.

$$\text{Hence } Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax, \text{ by [3.65]}. \quad \square$$

• OR (10.A.2) Suppose $A, B \in \mathbb{F}^{n,n}$. Prove that $AB = I \iff BA = I$.

SOLUTION: Using Problem (10) and (15).

Define $T, S \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbb{F}^{n,1}$. Then $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.

$$\text{Thus } AB = I \iff A(Bx) = x \iff T(Sx) = x \iff TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I. \quad \square$$

• **NOTE FOR [3.60]:** Suppose $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.

Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{i,x} w_j$; See (3.A.12). **COROLLARY:** $E_{l,k} E_{i,j} = \delta_{j,l} E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. And $(\mathcal{E}^{(j,i)})_{l,k} = \begin{cases} 0, & i \neq k \vee j \neq l \\ 1, & i = k \wedge j = l \end{cases}$

Because $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$ are iso. And $T = \mathcal{M}^{-1} \mathcal{M}(T)$; $E_{i,j} = \mathcal{M}^{-1} \mathcal{E}^{(j,i)}$.

Hence $\forall T \in \mathcal{L}(V, W)$, $\exists! A_{i,j} \in \mathbf{F} \left(\forall i \in \{1, \dots, m\}, j \in \{1, \dots, n\} \right)$, $\mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1} \mathcal{E}^{(1,1)} + & \dots & + A_{1,n} \mathcal{E}^{(1,n)} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{m,1} \mathcal{E}^{(m,1)} + & \dots & + A_{m,n} \mathcal{E}^{(m,n)} \end{pmatrix} \iff \begin{pmatrix} A_{1,1} E_{1,1} + & \dots & + A_{1,n} E_{n,1} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{m,1} E_{1,m} + & \dots & + A_{m,n} E_{n,m} \end{pmatrix} = T.$$

$$\therefore \mathcal{L}(V, W) = \text{span} \underbrace{\begin{pmatrix} E_{1,1}, & \dots, & E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \dots, & E_{n,m} \end{pmatrix}}_B; \quad \mathbf{F}^{m,n} = \text{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \dots, & \mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \dots, & \mathcal{E}^{(m,n)} \end{pmatrix}}_{B_{\mathcal{M}}}.$$

Hence by [2.42] and [3.61], we conclude that B is a basis of $\mathcal{L}(V, W)$ and that $B_{\mathcal{M}}$ is a basis of $\mathbf{F}^{m,n}$.

• Suppose V, W are finite-dim, U is a subsp of V .

Let $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\} = \{T \in \mathcal{L}(V, W) : T|_U = 0\}$.

(a) Show that \mathcal{E} is a subsp of $\mathcal{L}(V, W)$.

(b) Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$ and $\dim U$.

Hint: Define $\Phi : \mathcal{L}(V, W) \rightarrow \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. What is $\text{null } \Phi$? What is $\text{range } \Phi$?

SOLUTION:

(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.

(b) Define Φ as in the hint.

Because $T \in \text{null } \Phi \iff \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$.

Hence $\text{null } \Phi = \mathcal{E}$.

Because $S \in \mathcal{L}(U, W) \Rightarrow \exists T \in \mathcal{L}(V, W), \Phi(T) = S$, by (3.A.11) $\Rightarrow S \in \text{range } \Phi$.

Hence $\text{range } \Phi = \mathcal{L}(U, W)$.

Thus $\dim \text{null } \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \text{range } \Phi = (\dim V - \dim U) \dim W$. □

OR. Extend (u_1, \dots, u_m) a basis of U to $(u_1, \dots, u_m, v_1, \dots, v_n)$ a basis of V . Let $p = \dim W$.

(See NOTE FOR [3.60])

$$\forall T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \text{span} \underbrace{\begin{pmatrix} E_{1,1}, & \dots, & E_{m,1}, \\ \vdots & \ddots & \vdots \\ E_{1,p}, & \dots, & E_{m,p} \end{pmatrix}}_{\text{Denote it by } R} \cap \mathcal{E} = \{0\}.$$

$$\text{又 } W = \text{span} \begin{pmatrix} E_{m+1,1}, & \dots, & E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, & \dots, & E_{n,p} \end{pmatrix} \subseteq \mathcal{E}. \text{ Where } \mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}.$$

Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. □

◦ Suppose V is finite-dim and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T) = ST$.

(a) Show that $\dim \text{null } \mathcal{A} = (\dim V)(\dim \text{null } S)$.

(b) Show that $\dim \text{range } \mathcal{A} = (\dim V)(\dim \text{range } S)$.

SOLUTION:

(a) $\forall T \in \mathcal{L}(V), ST = 0 \iff \text{range } T \subseteq \text{null } S$.

Thus $\text{null } \mathcal{A} = \{T \in \mathcal{L}(V) : \text{range } T \subseteq \text{null } S\} = \mathcal{L}(V, \text{null } S)$.

(b) $\forall R \in \mathcal{L}(V), \text{range } R \subseteq \text{range } S \iff \exists T \in \mathcal{L}(V), R = ST$, by (3.B 25).

Thus $\text{range } \mathcal{A} = \{R \in \mathcal{L}(V) : \text{range } R \subseteq \text{range } S\} = \mathcal{L}(V, \text{range } S)$.

□

OR. Using NOTE FOR [3.60].

Let $B_{\text{range } S} = (\underbrace{w_1, \dots, w_m}_{Sv_i=w_i}), B_{\mathcal{K}} = (v_1, \dots, v_m); (w_1, \dots, w_n), (v_1, \dots, v_n)$ are bases of V .

Define $E_{i,j} \in \mathcal{L}(V)$ by $E_{i,j}(v_x) = \delta_{i,x}w_i$.

Thus $S = E_{1,1} + \dots + E_{m,m}; \quad \mathcal{M}(S, (v_1, \dots, v_n), (w_1, \dots, w_n)) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$

Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j}(w_x) = \delta_{i,x}v_i$.

Let $E_{j,k}R_{i,j} = Q_{i,k}, \quad R_{j,k}E_{i,j} = G_{i,k}.$

Because $\forall T \in \mathcal{L}(V), \exists ! A_{i,j} \in \mathbb{F}, \quad T = \begin{pmatrix} A_{1,1}R_{1,1} + \dots + A_{1,m}R_{m,1} + \dots + A_{1,n}R_{n,1} \\ + \dots + \dots + \dots + \dots + \dots \\ \vdots \quad \ddots \quad \vdots \quad \ddots \quad \vdots \\ + \dots + \dots + \dots + \dots + \dots \\ A_{m,1}R_{1,m} + \dots + A_{m,m}R_{m,m} + \dots + A_{m,n}R_{n,m} \\ + \dots + \dots + \dots + \dots + \dots \\ \vdots \quad \ddots \quad \vdots \quad \ddots \quad \vdots \\ + \dots + \dots + \dots + \dots + \dots \\ A_{n,1}R_{1,n} + \dots + A_{n,m}R_{m,n} + \dots + A_{n,n}R_{n,n} \end{pmatrix}.$

$$\begin{aligned} \Rightarrow \mathcal{A}(T) = ST &= \left(\sum_{r=1}^m E_{r,r} \right) \left(\sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \right) \\ &= \sum_{i=1}^m \sum_{j=1}^n A_{i,j} Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1} + \dots + A_{1,m}Q_{m,1} + \dots + A_{1,n}Q_{n,1} \\ + \dots + \dots + \dots + \dots + \dots \\ \vdots \quad \ddots \quad \vdots \quad \ddots \quad \vdots \\ + \dots + \dots + \dots + \dots + \dots \\ A_{m,1}Q_{1,m} + \dots + A_{m,m}Q_{m,m} + \dots + A_{m,n}Q_{n,m} \end{pmatrix}. \end{aligned}$$

Thus $\text{null } \mathcal{A} = \text{span} \begin{pmatrix} R_{1,m+1}, & \dots, & R_{n,m+1}, \\ \vdots & \ddots & \vdots \\ R_{1,n}, & \dots, & R_{n,n} \end{pmatrix}, \quad \text{range } \mathcal{A} = \text{span} \begin{pmatrix} Q_{1,1}, & \dots, & Q_{n,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,m}, & \dots, & Q_{n,m} \end{pmatrix}.$

Hence (a) $\dim \text{null } \mathcal{A} = n \times (n - m); \quad$ (b) $\dim \text{range } \mathcal{A} = n \times m.$

□

• **COMMENT:** Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(T) = TS$. Similarly to Problem (◦),

(a) $\forall T \in \mathcal{L}(V), TS = 0 \iff \text{range } S \subseteq \text{null } T$.

Thus $\text{null } \mathcal{B} = \{T \in \mathcal{L}(V) : \text{range } S \subseteq \text{null } T\} = \{T \in \mathcal{L}(V) : T|_{\text{range } S} = 0\}$.

(b) $\forall R \in \mathcal{L}(V), \text{null } S \subseteq \text{null } R \iff \exists T \in \mathcal{L}(V), R = TS$, by (3.B.24).

Thus $\text{range } \mathcal{B} = \{R \in \mathcal{L}(V) : \text{null } S \subseteq \text{null } R\} = \{R \in \mathcal{L}(V) : R|_{\text{null } S} = 0\}$.

Hence $\dim \text{null } \mathcal{B} = (\dim V - \dim \text{range } S)(\dim V)$;

$\dim \text{range } \mathcal{B} = (\dim V - \dim \text{null } S)(\dim V)$. □

OR. Using NOTE FOR [3.60] and the notation in Problem (◦).

$$\mathcal{B}(T) = TS = \left(\sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \right) \left(\sum_{r=1}^m E_{r,r} \right)$$

$$\text{Thus null } \mathcal{B} = \text{span} \left(\begin{matrix} R_{m+1,1}, & \dots, & R_{n,1} \\ \vdots & \ddots & \vdots \\ R_{m+1,n}, & \dots, & R_{n,n} \end{matrix} \right),$$

$$\text{range } \mathcal{B} = \text{span} \left(\begin{matrix} G_{1,1}, & \dots, & G_{m,1} \\ \vdots & \ddots & \vdots \\ G_{1,n}, & \dots, & G_{m,n} \end{matrix} \right).$$

$$= \sum_{i=1}^n \sum_{j=1}^m A_{i,j} G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1} + & \dots & +A_{1,m}G_{m,1} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{m,1}G_{1,m} + & \dots & +A_{m,m}G_{m,m} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{n,1}G_{1,n} + & \dots & +A_{n,m}G_{m,n} \end{pmatrix}.$$

Hence (a) $\dim \text{null } \mathcal{B} = n \times (n - m)$;
(b) $\dim \text{range } \mathcal{B} = n \times m$. □

17 Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

A subsp \mathcal{E} of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Using NOTE FOR [3.60]. Let (v_1, \dots, v_n) be a basis of V . If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and \mathcal{E} is a two-sided ideal of $\mathcal{L}(V)$.

Then $\forall E_{i,j} \in \mathcal{E}, (\forall x, y = 1, \dots, n)$, by assumption, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.

Again, $E_{y,x'}, E_{y',x} \in \mathcal{E}$ for all $x', y', x, y = 1, \dots, n$. Thus $\mathcal{E} = \mathcal{L}(V)$. □

• **OR (10.A.4)** Suppose that $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bases of V .

Let $T \in \mathcal{L}(V)$ be such that $T\alpha_k = \beta_k, \forall k$. Prove that $\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \beta \rightarrow \alpha)$

For ease of notation, let $\mathcal{M}(T, \alpha \rightarrow \beta) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n))$, $\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n))$.

SOLUTION:

Denote $\mathcal{M}(T, \alpha \rightarrow \alpha)$ by A and $\mathcal{M}(I, \beta \rightarrow \alpha)$ by B .

$\forall k \in \{1, \dots, n\}, Iu_k = u_k = B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \dots + A_{n,k}\alpha_n \Rightarrow A = B$. □

OR. Note that $\mathcal{M}(T, \alpha \rightarrow \beta) = I$. Hence $\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \beta \rightarrow \alpha) \underbrace{\mathcal{M}(T, \alpha \rightarrow \beta)}_{=\mathcal{M}(I, \beta \rightarrow \beta)} = \mathcal{M}(I, \beta \rightarrow \alpha)$. □

OR. Note that $\mathcal{M}(T, \beta \rightarrow \beta)\mathcal{M}(I, \alpha \rightarrow \beta) = \mathcal{M}(T, \alpha \rightarrow \beta) = I$.

$\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \alpha \rightarrow \beta)^{-1} \left(\underbrace{\mathcal{M}(T, \beta \rightarrow \beta)\mathcal{M}(I, \alpha \rightarrow \beta)}_{=\mathcal{M}(T, \alpha \rightarrow \beta)} \right) = \mathcal{M}(I, \beta \rightarrow \alpha)$. □

COMMENT: Denote $\mathcal{M}(T, \beta \rightarrow \beta)$ by A' .

$u_k = Iu_k = B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n, \forall k \in \{1, \dots, n\}$.

又 $Tu_k = T(B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n) = B_{1,k}\beta_1 + \dots + B_{n,k}\beta_n = A'_{1,k}\beta_1 + \dots + A'_{n,k}\beta_n \Rightarrow A' = B$.

OR. $\mathcal{M}(T, \beta \rightarrow \beta) = \mathcal{M}(T, \alpha \rightarrow \beta)\mathcal{M}(I, \beta \rightarrow \alpha) = B$.

16 Suppose V is finite-dim and $S \in \mathcal{L}(V)$ such that $\forall T \in \mathcal{L}(V), ST = TS$.

Prove that $\exists \lambda \in \mathbf{F}, S = \lambda I$.

SOLUTION: Using the notation and result in ().

Suppose $ST = TS$ for every $T \in \mathcal{L}(V)$. If $S = 0$, we are done. Now suppose $S \neq 0$.

Let $S = E_{1,1} + \dots + E_{m,m} \Rightarrow \mathcal{M}(S, B_{\mathcal{K}}) = \mathcal{M}(I, B_{\text{range } S}, B_{\mathcal{K}})$.

Then $\forall k \in \{m+1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $n = \dim V = \dim \text{range } S = m$.

NOTICE that $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$. Thus $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \dots + a_{n,i}v_n)$.

Where $a_{i,j} = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \dots + a_{n,i}v_n$;

And For each j , for all i . Thus $a_{i,i} = a_{k,k} = \lambda, \forall k \neq i$.

Hence $w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \mathcal{M}(\lambda I, (v_1, \dots, v_n)) \Rightarrow S = \mathcal{M}^{-1}(\mathcal{M}(\lambda I))\lambda I$. □

18 Show that V and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.

SOLUTION:

Define $\Psi \in \mathcal{L}(V, \mathcal{L}(\mathbf{F}, V))$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.

(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is inje.

(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi(T(1))$. Hence Ψ is surj. □

OR. Define $\Phi \in \mathcal{L}(\mathcal{L}(\mathbf{F}, V), V)$ by $\Phi(T) = T(1)$.

(a) Suppose $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is inje.

(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surj. □

COMMENT: $\Phi = \Psi^{-1}$.

• Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that $\exists p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.

SOLUTION:

Note that $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$.

Define $T_n : \mathcal{P}_n(\mathbf{R}) \rightarrow \mathcal{P}_n(\mathbf{R})$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$. Then $T_n \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$.

And note that $T_n(p) = 0 \Rightarrow \deg(T_n p) = -\infty = \deg p \Rightarrow p = 0$. Thus T_n is inv.

$\forall q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $m = 0$; if $q \neq 0$, let $m = \deg q$, we have $q \in \mathcal{P}_m(\mathbf{R})$.

Hence $\exists p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$. □

19 Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is inje. $\deg Tp \leq \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.

(a) Prove that T is surj; (b) Prove that for every nonzero p , $\deg Tp = \deg p$.

SOLUTION:

(a) T is inje $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} : \mathcal{P}_n(\mathbf{R}) \rightarrow \mathcal{P}_n(\mathbf{R})$ is inje and therefore is inv $\iff T$ is surj.

(b) Using mathematical induction.

(i) $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$;

$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$.

(ii) Assume that $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$.

Suppose $\exists r \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tr \leq n < \deg r = n + 1$.

Then by (a), $\exists s \in \mathcal{P}_n(\mathbf{R}), T(s) = (Tr)$.

又 T is inje $\Rightarrow s = r$. While $\deg s = \deg Ts = \deg Tr < \deg r$.

Contradicts. Thus $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$. □

3.E

1 A function $T : V \rightarrow W$ is linear $\iff T$ is a subspace of $V \times W$.

2 Suppose $V_1 \times \cdots \times V_m$ is finite-dim. Prove that each V_j is finite-dim.

SOLUTION:

For any $k \in \{1, \dots, m\}$, define $p_k : V_1 \times \cdots \times V_m \rightarrow V_k$ by $p_k(v_1, \dots, v_m) = v_k$.

Then p_k is a surj linear map. By [3.22], $\text{range } p_k = V_k$ is finite-dim. □

OR. Denote $V_1 \times \cdots \times V_m$ by U . Denote $\{0\} \times \cdots \times \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i .

Let (v_1, \dots, v_m) be a basis of U . Note that $\forall u_i \in V_i, u_i \in U_i \subseteq U$, for each i .

Define $R_i \in \mathcal{L}(V_i, U)$ by $R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0)$ $\left. \vphantom{\begin{matrix} \text{Define } R_i \in \mathcal{L}(V_i, U) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \end{matrix}} \right\} \Rightarrow S_i|_{U_i} = R_i^{-1}|_{U_i}$.

Define $S_i \in \mathcal{L}(U, V_i)$ by $S_i(u_1, \dots, u_i, \dots, u_m) = u_i$

Thus U_i and V_i are iso. $\forall U_i$ is a subsp of a finite-dim vecsp U . □

3 Give an example of a vecsp V and its two subsp U_1, U_2 such that $U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum.

SOLUTION: V must be infinite-dim. For if not, both U_1 and U_2 are finite-dim subsp. By [3.76, 3.78].

NOTE that at least one of U_1, U_2 must be infinite-dim. And at least one must be infinite-dim??? TODO

For if not, $U_1 \times U_2$ is finite-dim and $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$.

Let $V = \mathbb{F}^\infty = U_1$, $U_2 = \{(x, 0, \dots) \in \mathbb{F}^\infty : x \in \mathbb{F}\}$.

Define $T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2)$ by $T((x_1, x_2, \dots), (x, 0, \dots)) = (x, x_1, x_2, \dots)$ $\left. \vphantom{\begin{matrix} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T((x_1, x_2, \dots), (x, 0, \dots)) = (x, x_1, x_2, \dots) \end{matrix}} \right\} \Rightarrow S = T^{-1}$.

Define $S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2)$ by $S(x, x_1, x_2, \dots) = ((x_1, x_2, \dots), (x, 0, \dots))$ □

4 Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.

SOLUTION: Using the notation in Problem (2).

Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$.

Define $\varphi : T \mapsto (T_1, \dots, T_m)$ by $\varphi(T) = (TR_1, \dots, TR_m)$.

Define $\psi : (T_1, \dots, T_m) \mapsto T$ by $\psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m$. $\left. \vphantom{\begin{matrix} \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m \end{matrix}} \right\} \Rightarrow \psi = \varphi^{-1}$. □

5 Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.

SOLUTION: Using the notation in Problem (2).

Note that $Tv = (w_1, \dots, w_m)$. Define $T_i \in \mathcal{L}(V, W_i)$ by $T_i(v) = w_i$.

Define $\varphi : T \mapsto (T_1, \dots, T_m)$ by $\varphi(T) = (S_1T, \dots, S_mT)$.

Define $\psi : (T_1, \dots, T_m) \mapsto T$ by $\psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m$. $\left. \vphantom{\begin{matrix} \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m \end{matrix}} \right\} \Rightarrow \psi = \varphi^{-1}$. □

6 For $m \in \mathbb{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove that V^m and $\mathcal{L}(\mathbb{F}^m, V)$ are iso.

SOLUTION:

Define $T : (v_1, \dots, v_m) \mapsto \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1v_1 + \cdots + a_mv_m$.

(a) Suppose $T(v_1, \dots, v_m) = 0$. Then $\forall (a_1, \dots, a_m) \in \mathbb{F}^m$, $\varphi(a_1, \dots, a_m) = a_1v_1 + \cdots + a_mv_m = 0$

$\Rightarrow (v_1, \dots, v_m) = 0 \Rightarrow T$ is inje.

(b) Suppose $\psi \in \mathcal{L}(\mathbb{F}^m, V)$. Let (e_1, \dots, e_m) be the standard basis of \mathbb{F}^m . Then $\forall (b_1, \dots, b_m) \in \mathbb{F}^m$,

$\left[T(\psi(e_1), \dots, \psi(e_m)) \right] (b_1, \dots, b_m) = b_1\psi(e_1) + \cdots + b_m\psi(e_m) = \psi(b_1e_1 + \cdots + b_me_m) = \psi(b_1, \dots, b_m)$.

Thus $T(\psi(e_1), \dots, \psi(e_m)) = \psi$. Hence T is surj. □

14 Suppose $U = \{(x_1, x_2, \dots) \in \mathbf{F}^\infty : x_k \neq 0 \text{ for only finitely many } k\}$.

(a) Show that U is a subspace of \mathbf{F}^∞ . [Do it in your mind]

(b) Prove that \mathbf{F}^∞/U is infinite-dim.

SOLUTION: For ease of notation, denote the p^{th} term of $u = (x_1, \dots, x_p, \dots) \in \mathbf{F}^\infty$ by $u[p]$.

For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, & (p-1) \equiv 0 \pmod{r} \\ 0, & \text{otherwise} \end{cases}$ | simply $e_r = (1, \underbrace{0, \dots, 0}_{(p-1) \text{ times}}, 1, \underbrace{0, \dots, 0}_{(p-1) \text{ times}}, 1, \dots)$.

Choose one $m \in \mathbf{N}^+$. Let $a_1(e_1 + U) + \dots + a_m(e_m + U) = 0 + U \Rightarrow \exists u \in U, a_1e_1 + \dots + a_me_m = u$.

Suppose $u = (x_1, \dots, x_L, 0, \dots)$, where L is the largest such that $u[L] \neq 0$.

Let $s \in \mathbf{N}^+$ be such that $h = s \cdot m! + 1 > L$ and $e_1[h] = \dots = e_m[h] = 1$.

Note that by definition, $e_r[s \cot m! + 1 + p] = e_r[p + 1] = 1 \iff p \equiv 0 \pmod{r} \iff r | p$.

Now for any $p \in \{1, \dots, m\}$, $u[h + p] = \left(\sum_{r=1}^m a_r e_r \right)[p + 1] = \sum_{k=1}^{\tau(p)} a_{p_k} = 0$ (Δ)

where $1 = p_1 \leq \dots \leq p_{\tau(p)} = p$ are all the distinct factors of p .

Let $q = p_{\tau(p)-1}$. Notice that $\tau(q) = \tau(p) - 1$ and $q_k = p_k, \forall k \in \{1, \dots, \tau(q)\}$.

Again by (Δ), $\left(\sum_{r=1}^m a_r e_r \right)[h + q] = \sum_{k=1}^{\tau(q)-1} a_{p_k} = 0$. Thus $a_{p_{\tau(p)}} = a_p = 0$ for any $p \in \{1, \dots, m\}$.

Hence $\forall m \in \mathbf{N}^+$, (e_1, \dots, e_m) is linearly inde in \mathbf{F}^∞ , so is $(e_1 + U, \dots, e_m + U)$ in \mathbf{F}^∞/U . By (2.A.14). \square

OR. For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, & \text{if } 2^r | p \\ 0, & \text{otherwise} \end{cases}$.

Similarly, let $m \in \mathbf{N}^+$ and $a_1(e_1 + U) + \dots + a_m(e_m + U) = 0 \Rightarrow a_1e_1 + \dots + a_me_m = u \in U$.

Suppose L is the largest such that $u[L] \neq 0$. And l is such that $2^{ml} > L$.

Then $\forall k \in \{1, \dots, m\}, u[2^{ml} + 2^k] = \left(\sum_{r=1}^m a_r e_r \right)[2^k] = a_1 + \dots + a_k = 0$.

Thus $a_1 = \dots = a_m = 0$ and (e_1, \dots, e_m) is linearly inde. Similarly. \square

7 Suppose $v, x \in V$ and U and W are subspaces of V . Prove that $v + U = x + W \Rightarrow U = W$.

SOLUTION:

(a) $\forall u_1 \in U, \exists w_1 \in W, v + u_1 = x + w_1$, let $u_1 = 0$, now $v = x + w'_1 \Rightarrow v - x \in W$.

(b) $\forall w_2 \in W, \exists u_2 \in U, v + u_2 = x + w_2$, let $w_2 = 0$, now $x = v + u'_2 \Rightarrow x - v \in U$.

Thus $\pm(v - x) \in U \cap W \Rightarrow \begin{cases} u_1 = (x - v) + w_1 \in W \Rightarrow U \subseteq W \\ w_2 = (v - x) + u_2 \in U \Rightarrow W \subseteq U \end{cases} \Rightarrow U = W$. \square

• Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$.

Then A is a translate of $U \iff \exists c \in \mathbf{R}, A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$.

• Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $U = \{x \in V : Tx = c\}$ is either \emptyset or is a translate of $\text{null } T$.

SOLUTION:

If $c \in W$ but $c \notin \text{range } T$, then $U = \emptyset$, we are done. Now suppose $c \in \text{range } T$ and $x \in U$.

$\forall x + y \in x + \text{null } T$ ($\forall y \in \text{null } T$), $x + y \in U$. Hence $x + \text{null } T \subseteq U$.

$\forall u \in U, u - x \in \text{null } T \Rightarrow u = x + (u - x) \in x + \text{null } T$. Hence $U \subseteq x + \text{null } T$. \square

COROLLARY: The set of solutions to a system of linear equations such as [3.28] is either \emptyset or a translate.

8 Suppose A is a nonempty subset of V .

Prove that A is a translate of some subsp of $V \iff \lambda v + (1 - \lambda)w \in A, \forall v, w \in A, \lambda \in \mathbb{F}$.

SOLUTION:

Suppose $A = a + U$. Then $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbb{F}$,
 $\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + (\lambda(u_1 - u_2) + u_2) \in A$.

Suppose $\lambda v + (1 - \lambda)w \in A, \forall v, w \in A, \lambda \in \mathbb{F}$. Suppose $a \in A$ and let $A' = \{x - a : x \in A\}$.

Then $0 \in A'$ and $\forall x - a, y - a \in A', (\forall x, y \in A), \lambda \in \mathbb{F}$,

(I) $\lambda(x - a) = [\lambda x + (1 - \lambda)a] - a \in A'$.

(II) $\lambda(x - a) + (1 - \lambda)(y - a) = \frac{1}{2}(x - a) + \frac{1}{2}(y - a) = \frac{1}{2}x + (1 - \frac{1}{2})y - a \in A'$.

OR. By (I), $2 \times [\frac{1}{2}(x - a) + \frac{1}{2}(y - a)] = (x - a) + (y - a) \in A'$.

Thus A' is a subsp of V . Hence $a + A' = \{(x - a) + a : x \in A\} = A$ is a translate. \square

OR. Suppose $x - a, y - a \in A', \lambda \in \mathbb{F}$.

Note that $x, a \in A \Rightarrow \lambda x + (1 - \lambda)a = 2x - a \in A$. Similarly $2y - a \in A$.

(I) $(x - \frac{1}{2}a) + (y - \frac{1}{2}a) = x + y - a \in A \Rightarrow x + y - 2a = (x - a) + (y - a) \in A'$.

(II) $\lambda(x - a) = (\lambda x + (1 - \lambda)a) - a \in A'$.

Thus $-x + A$ is a subsp of V . Hence $A = x + (-x + A)$ is a translate of the subsp $(-x + A)$. \square

9 Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subsp U_1, U_2 of V .

Prove that the intersection $A_1 \cap A_2$ is either a translate of some subsp of V or is \emptyset .

SOLUTION:

Suppose $v + u_1, w + u_2 \in A_1 \cap A_2 \neq \emptyset$. By Problem (8),

$\forall \lambda \in \mathbb{F}, \lambda(v + u_1) + (1 - \lambda)(w + u_2) \in A_1 \cap A_2$. Thus $A_1 \cap A_2$ is a translate of some subsp of V . \square

OR. Let $A_1 = v + U_1, A_2 = w + U_2$. Suppose $x \in (v + U_1) \cap (w + U_2) \neq \emptyset$.

Then $\exists u_1 \in U_1, x = v + u_1 \Rightarrow x - v \in U_1, \exists u_2 \in U_2, x = w + u_2 \Rightarrow x - w \in U_2$.

Note that by [3.85], $A_1 = v + U_1 = x + U_1, A_2 = w + U_2 = x + U_2$. We show that $A_1 \cap A_2 = x + (U_1 \cap U_2)$.

(a) $y \in A_1 \cap A_2 \Rightarrow \exists u_1 \in U_1, u_2 \in U_2, y = x + u_1 = x + u_2 \Rightarrow u_1 = u_2 \in U_1 \cap U_2 \Rightarrow y \in x + (U_1 \cap U_2)$.

(b) $y = x + u \in x + (U_1 \cap U_2) = (x + U_1) \cap (x + U_2) \Rightarrow y \in A_1 \cap A_2$. \square

10 Prove that the intersection of any collection of translates of subsp of V is either a translate of some subsp or \emptyset .

SOLUTION:

Suppose $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collection of translates of subsp of V , where Γ is an arbitrary index set.

Suppose $x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset$, then by Problem (8), $\forall \lambda \in \mathbb{F}, \lambda x + (1 - \lambda)y \in A_\alpha$ for every $\alpha \in \Gamma$.

Thus $\bigcap_{\alpha \in \Gamma} A_\alpha$ is a translate of some subsp of V . \square

OR. Let $A_\alpha = w_\alpha + V_\alpha$ for each $\alpha \in \Gamma$. Suppose $x \in \bigcap_{\alpha \in \Gamma} (w_\alpha + V_\alpha) \neq \emptyset$.

Then for each $A_\alpha, \exists v_\alpha \in V_\alpha, x = w_\alpha + v_\alpha \Rightarrow x - w_\alpha \in V_\alpha \Rightarrow A_\alpha = w_\alpha + V_\alpha = x + V_\alpha$.

(a) $y \in \bigcap_{\alpha \in \Gamma} A_\alpha \Rightarrow \forall \alpha \in \Gamma, \exists v_\alpha, y = x + v_\alpha \Rightarrow \forall \alpha, \beta \in \Gamma, v_\alpha = v_\beta \Rightarrow y \in x + \bigcap_{\alpha \in \Gamma} V_\alpha$.

(b) $y = x + v \in x + \bigcap_{\alpha \in \Gamma} V_\alpha = \bigcap_{\alpha \in \Gamma} (x + V_\alpha) \Rightarrow y \in \bigcap_{\alpha \in \Gamma} A_\alpha$. Hence $\bigcap_{\alpha \in \Gamma} A_\alpha = x + \bigcap_{\alpha \in \Gamma} V_\alpha$. \square

11 Suppose $A = \{\lambda_1 v_1 + \dots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.

(a) Prove that A is a translate of some subsp of V

(b) Prove that if B is a translate of some subsp of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.

(c) Prove that A is a translate of some subsp of V of dim less than m .

SOLUTION:

(a) By Problem (8), $\forall u, w \in A, \lambda \in \mathbf{F}, \exists a_i, b_i \in \mathbf{F}$,

$$\lambda u + (1 - \lambda)w = \left(\lambda \sum_{i=1}^m a_i + (1 - \lambda) \sum_{i=1}^m b_i \right) v_i \in A. \quad \square$$

(b) Suppose $B = v + U$, where $v \in V$ and U is a subsp of V . Suppose $\exists ! u_k \in U, v_k = v + u_k \in B$.

$$\text{Then for all } v = \sum_{i=1}^m \lambda_i v_i \in A, v = \sum_{i=1}^m \lambda_i (v + u_i) = v + \sum_{i=1}^m \lambda_i u_i \in v + U = B. \quad \square$$

OR. Let $v = \lambda_1 v_1 + \dots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on m by k .

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. $\forall v_1 \in B$. Hence $v \in B$.

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. $\forall v_1, v_2 \in B$. By Problem (8), $v \in B$.

(ii) $2 \leq k \leq m$, we assume that $v = \lambda_1 v_1 + \dots + \lambda_k v_k \in A \subseteq B$. ($\forall \lambda_i$ such that $\sum_{i=1}^k \lambda_i = 1$)

For $u = \mu_1 v_1 + \dots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. $\forall i = 1, \dots, k, \exists \mu_i \neq 1$, fix one such i by ι .

$$\text{Then } \sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow \left(\sum_{i=1}^{k+1} \frac{\mu_i}{1 - \mu_\iota} \right) - \frac{\mu_\iota}{1 - \mu_\iota} = 1.$$

$$\text{Let } w = \underbrace{\frac{\mu_1}{1 - \mu_\iota} v_1 + \dots + \frac{\mu_{\iota-1}}{1 - \mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1 - \mu_\iota} v_{\iota+1} + \dots + \frac{\mu_{k+1}}{1 - \mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let $\lambda_i = \frac{\mu_i}{1 - \mu_\iota}$ for $i = 1, \dots, \iota - 1$; $\lambda_j = \frac{\mu_{j+1}}{1 - \mu_\iota}$ for $j = \iota, \dots, k$. Then,

$$\left. \begin{array}{l} \sum_{i=1}^k \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1 - \lambda) v_\iota \in B \end{array} \right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \quad \square$$

(c) If $m = 1$, then let $A = v_1 + \{0\}$ and we are done.

Choose one $k \in \{1, \dots, m\}$. Given $\lambda_i \in \mathbf{F}$, where $i \in \{1, \dots, k-1, k+1, \dots, m\}$.

Let $\lambda_k = 1 - \lambda_1 - \dots - \lambda_{k-1} - \lambda_{k+1} - \dots - \lambda_m$

Then $\lambda_1 v_1 + \dots + \lambda_k v_k + \dots + \lambda_m v_m = v_k + \sum_{i=1}^m \lambda_i (v_i - v_k)$.

Thus $A = v_k + \text{span}(v_1 - v_k, \dots, v_{k-1} - v_k, v_{k+1} - v_k, \dots, v_m - v_k)$. \square

18 Suppose $T \in \mathcal{L}(V, W)$ and U is a subsp of V . Let π denote the quotient map.

Prove that $\exists S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \subseteq \text{null } T$.

SOLUTION:

(a) Suppose $U \subseteq \text{null } T$. Define $S \in \mathcal{L}(V/U, W)$ by $S(v + U) = Tv$. Then $S \circ \pi = T$.

Now we show that this map is well-defined.

$$v_1 + U = v_2 + U \iff (v_1 - v_2) \in U \iff S((v_1 - v_2) + U) = T(v_1 - v_2) = 0 \iff Tv_1 = Tv_2.$$

(b) Suppose $\exists S, T = S \circ \pi$. Then $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \text{null } T$. \square

20 Define $\Gamma : \mathcal{L}(V/U, W) \rightarrow \mathcal{L}(V, W)$ by $\Gamma(S) = S \circ \pi$. Prove that:

(a) Γ is linear: By [3.9] distr and [3.6].

(b) Γ is inje: $\Gamma(S) = 0 = S \circ \pi \iff \forall v \in V, S(\pi(v)) = 0 \iff \forall v + U \in V/U, S(v + U) = 0 \iff S = 0$.

(c) $\text{range } \Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\}$: By Problem (18). \square

• **NOTE FOR [3.79, 3.83]:**

Suppose U is a subsp of V . If $U = \{0\}$, then $v + U = v + \{0\} = \{v\}$, $V/U = V/\{0\} = \{\{v\} : v \in V\}$.

• **NOTE FOR [3.88, 3.90, 3.91]:** Suppose $W \in \mathcal{S}_V U$. Then V/U and W are iso.

For any $W \in \mathcal{S}_V U$, because $V = U \oplus W$, $\forall v \in V, \exists! u_v \in U, w_v \in W$ such that $v = u_v + w_v$.

Define $T \in \mathcal{L}(V, W)$ by $T(v) = w_v$. Hence $\text{null } T = U$, $\text{range } T = W$.

Then $\tilde{T} \in \mathcal{L}(V/\text{null } T, W)$ is defined by $\tilde{T}(v + U) = T v = w_v$.

Now $\pi \circ \tilde{T} = I_{V/U}$, $\tilde{T} \circ \pi = I_W = T|_W$ Hence \tilde{T} is an iso of V/U onto W .

12 Suppose U is a subsp of V such that V/U is finite-dim. Prove that V is iso to $U \times (V/U)$.

SOLUTION:

Let $(v_1 + U, \dots, v_n + U)$ be a basis of V/U .

Note that $\forall v \in V, \exists! a_i \in \mathbf{F}, v + U = \sum_{i=1}^n a_i(v_i + U) = \left(\sum_{i=1}^n a_i v_i \right) + U$

$\Rightarrow (v - a_1 v_1 - \dots - a_n v_n) \in U \Rightarrow \exists! u \in U, v = \sum_{i=1}^n a_i v_i + u$.

Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ by $\varphi(v) = (u, v + U)$,

and $\psi \in \mathcal{L}(U \times (V/U), V)$ by $\psi(u, v + U) = v + u$, where $\exists! a_i \in \mathbf{F}, v = \sum_{i=1}^n a_i v_i + U$. \square

OR. $[V/U, U \text{ and } V \text{ can be infinite-dim}]$

Let W be such that $V = U \oplus W$. Thus $\forall v \in V, \exists! u_v \in U, w_v \in W, v = u_v + w_v$.

Define $\tilde{T} \in \mathcal{L}(V/U, W)$ by $\tilde{T}(v + U) = w_v \in W$. Notice that $v + U = w_v + U$.

Define $S \in \mathcal{L}(U \times (V/U), V)$ by $S(u, v + U) = u + \tilde{T}(v + U)$.

Inje: $S(u, v + U) = u + \tilde{T}(v + U) = 0 \Rightarrow \pi(S(u, v + U)) = v + U = 0 \Rightarrow u = -\tilde{T}(v + U) = 0$.

Surj: $\forall v \in V, \exists! u \in U, w \in W, v = u + w \Rightarrow S(u, w + U) = u + \tilde{T}(w + U) = v$.

OR. Define $R \in \mathcal{L}(V, U \times (V/U))$ by $R(v) = (u_v, (w_v + U))$. Now $R \circ S = I_{U \times (V/U)}$, $S \circ R = I_V$. \square

• (4E 3.E.14) Suppose $V = U \oplus W$, (w_1, \dots, w_m) is a basis of W .

Prove that $(w_1 + U, \dots, w_m + U)$ is a basis of V/U .

SOLUTION:

Note that $\forall v \in V, \exists! u \in U, w \in W, v = u + w$. $\text{又 } \exists! c_i \in \mathbf{F}, w = \sum_{i=1}^m c_i w_i \Rightarrow v = \sum_{i=1}^m c_i w_i + u$.

Hence $\forall v + U \in V/U, \exists! c_i \in \mathbf{F}, v + U = \sum_{i=1}^m c_i w_i + U = c_1(w_1 + U) + \dots + c_m(w_m + U)$. \square

13 Suppose $(v_1 + U, \dots, v_m + U)$ is a basis of V/U and (u_1, \dots, u_n) is a basis of U .

Prove that $(v_1, \dots, v_m, u_1, \dots, u_n)$ is a basis of V .

SOLUTION: Notice that (v_1, \dots, v_m) is linely inde.

By Problem (12), U and V/U are finite-dim $\Rightarrow U \times (V/U)$ is finite-dim, so is V .

$\dim V = \dim(U \times (V/U)) = \dim U + \dim V/U = m + n$.

又 Each $v_i \notin U \Rightarrow (v_1, \dots, v_m, u_1, \dots, u_n)$ is linely inde. \square

OR. Note that $\forall v \in V, \exists! a_i \in \mathbf{F}, v + U = \sum_{i=1}^m a_i v_i + U \Rightarrow \exists! b_i \in \mathbf{F}, v - \sum_{i=1}^m a_i v_i = \sum_{i=1}^m b_i u_i \in U$
 $\Rightarrow \forall v \in V, \exists! a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^m a_i v_i + \sum_{j=1}^n b_j u_j$. \square

OR. Note that $\sum_{i=1}^m a_i v_i \in U \iff \left(\sum_{i=1}^m a_i v_i \right) + U = 0 + U \iff a_1 = \dots = a_m = 0$.

Hence $\text{span}(v_1, \dots, v_m) \cap U = \{0\} \Rightarrow \text{span}(v_1, \dots, v_m) \oplus U = V$. By (2.B.8), we are done. \square

15 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F}) \setminus \{0\}$. Prove that $\dim V/(\text{null } \varphi) = 1$.

SOLUTION:

By (3.B.29), $\exists u \in V, V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$. By (4E 3.E.14), $(u + \text{null } \varphi)$ is a basis of $V/\text{null } \varphi$.

OR. By [3.91] (d), $\dim \text{range } \varphi = 1 = \dim V/(\text{null } \varphi)$. □

16 Suppose $\dim V/U = 1$. Prove that $\exists \varphi \in \mathcal{L}(V, \mathbf{F})$ such that $\text{null } \varphi = U$.

SOLUTION:

Suppose V_0 is a subsp of V such that $V = U \oplus V_0$. Then V_0 and V/U are iso. $\dim V_0 = 1$.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. □

OR. Let $(w + U)$ be a basis of V/U . Then $\forall v \in V, \exists! a \in \mathbf{F}, v + U = aw + U$.

Define $\varphi : V \rightarrow \mathbf{F}$ by $\varphi(v) = a$. Assume that φ is linear.

Then $u \in U \iff u + U = 0w + U \iff \varphi(u) = 0 \iff u \in \text{null } \varphi$. Thus $U = \text{null } \varphi$. □

Now we prove the assumption.

$\forall x, y \in V, \lambda \in \mathbf{F}, \exists! a, b \in \mathbf{F}, x + U = aw + U, \lambda y + U = \lambda bw + U \Rightarrow (x + \lambda y) + U = (a + \lambda b)w + U$.

Then $\varphi(x + \lambda y) = a + \lambda b = \varphi(x) + \lambda \varphi(y)$.

17 Suppose V/U is finite-dim. W is a subsp of V .

(a) Show that if $V = U + W$, then $\dim W \geq \dim V/U$.

(b) Find a W such that $\dim W = \dim V/U$ and $V = U \oplus W$.

SOLUTION: Let (w_1, \dots, w_n) be a basis of W

(a) $\forall v \in V, \exists u \in U, w \in W$ such that $v = u + w \Rightarrow v + U = w + U$

And $\exists! a_i \in \mathbf{F}, v + U = (a_1 w_1 + \dots + a_n w_n) + U$. Then $V/U \subseteq \text{span}(w_1 + U, \dots, w_n + U)$.

Hence $\dim V/U = \dim \text{span}(w_1 + U, \dots, w_n + U) \leq \dim W$.

(b) Let $W \in \mathcal{S}_V U$. In other words, reduce $(w_1 + U, \dots, w_n + U)$

to a basis $(w_1 + U, \dots, w_m + U)$ of V/U and let $W = \text{span}(w_1, \dots, w_m)$. □

OR. Let $(v_1 + U, \dots, v_m + U)$ be a basis of V/U and define $\tilde{T} \in \mathcal{L}(V/U, V)$ by $\tilde{T}(v_k + U) = v_k$.

Note that $\pi \circ \tilde{T} = I$. By (3.B.20), \tilde{T} is inje. And (v_1, \dots, v_m) is linely inde.

Let $W = \text{range } \tilde{T} = \text{span}(v_1, \dots, v_m)$. Then $\tilde{T} \in \mathcal{L}(V/U, W)$ is an iso. Thus $\dim W = \dim V/U$.

And $\forall v \in V, \exists! a_i \in \mathbf{F}, v + U = a_1 v_1 + \dots + a_m v_m + U$

$\Rightarrow v - (a_1 v_1 + \dots + a_m v_m) \in U \Rightarrow \exists! w \in W, u \in U, v = w + u$. □

ENDED

3.F [4](#) [5](#) [6](#) [7](#) [8](#) [9](#) [12](#) [13](#) [15](#) [16](#) [17](#) [18](#) [19](#) [20](#) [21](#) [22](#) [23](#) [24](#) [25](#) [26](#)
[28](#) [29](#) [30](#) [31](#) [33](#) [34](#) [35](#) [36](#) [37](#) | [4E: 5, 6, 8, 17, 23, 24, 25](#)

20, 21 Suppose U and W are subsets of V . Prove that $U \subseteq W \iff W^0 \subseteq U^0$.

SOLUTION:

(a) Suppose $U \subseteq W$. Then $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(u) = 0 \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.

(b) Suppose $W^0 \subseteq U^0$. Then $\varphi \in W^0 \Rightarrow \varphi \in U^0$. Hence $\text{null } \varphi \supseteq W \Rightarrow \text{null } \varphi \supseteq U$. Thus $W \supseteq U$.

OR. For a subsp U of V , let $A_U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = U$, by Problem (25).

Suppose $W^0 \subseteq U^0$. Then $\forall \varphi \in W^0, v \in A_U, \varphi(v) = 0 \Rightarrow v \in A_W$. Thus $A_U \subseteq A_W$. □

COROLLARY: $W^0 = U^0 \iff U = W$.

22 Suppose U and W are subspaces of V . Prove that $(U + W)^0 = U^0 \cap W^0$.

SOLUTION:

$$(a) \left. \begin{array}{l} U \subseteq U + W \\ W \subseteq U + W \end{array} \right\} \Rightarrow \left. \begin{array}{l} (U + W)^0 \subseteq U^0 \\ (U + W)^0 \subseteq W^0 \end{array} \right\} \Rightarrow (U + W)^0 \subseteq U^0 \cap W^0.$$

OR. Suppose $\varphi \in (U + W)^0$. Then $\forall u \in U, w \in W, \varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$.

(b) Suppose $\varphi \in U^0 \cap W^0 \subseteq V'$. Then $\forall u \in U, w \in W, \varphi(u + w) = 0 \Rightarrow \varphi \in (U + W)^0$. \square

23 Suppose U and W are subsets of V . Prove that $(U \cap W)^0 = U^0 + W^0$.

SOLUTION:

$$(a) \left. \begin{array}{l} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array} \right\} \Rightarrow \left. \begin{array}{l} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array} \right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 [\supseteq U^0 \cap W^0 = (U + W)^0.]$$

OR. Suppose $\varphi = \psi + \beta \in U^0 + W^0$. Then $\forall v \in U \cap W, \varphi(v) = (\psi + \beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0$.

(b) [Only in Finite-dim; Requires that U, W are subspaces] Using Problem (22).

$$\begin{aligned} \dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\ &= 2 \dim V - \dim U - \dim W - (\dim V - \dim(U + W)) = \dim V - \dim(U \cap W). \end{aligned}$$

OR. Suppose $\varphi \in (U \cap W)^0$. Let X, Y be such that $V = U \oplus X = W \oplus Y$.

Define $\psi \in U^0, \beta \in W^0$ by $\psi(u + x) = \frac{1}{2}\varphi(x), \beta(w + y) = \frac{1}{2}\varphi(y)$.

$\forall v = u + x = w + y \in V, \varphi(v) = \varphi(x) = \varphi(y)$. Now $\varphi(v) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y) = \psi(v) + \beta(v)$.

Hence $\varphi \in U^0 + W^0$. Now $(U \cap W)^0 \subseteq U^0 + W^0$. \square

• **COROLLARY:**

(a) Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subsets of V . Then $\left(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \sum_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$.

(b) Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subspaces of V . Then $\left(\sum_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \bigcap_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$.

(c) Suppose $V = U \oplus W$. Then $V' = U^0 \oplus W^0$. And $U'_V = W^0, W'_V = U^0$.

Where $U'_V = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$. And $\iota \in \mathcal{L}(V, U)$ is defined by $\iota(u_v + w_v) = u_v$.

• (4E 3.F.23) Suppose $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following sets are the same.

(a) $\text{span}(\varphi_1, \dots, \varphi_m)$

(b) $((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m))^0 \stackrel{(c)}{=} \{\varphi \in V' : (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) \subseteq \text{null } \varphi\}$

SOLUTION: By Problem (17), (c) holds.

By Problem (26) [May require finite-dim] and the COROLLARY in Problem (23),

$$\left. \begin{array}{l} ((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m))^0 = (\text{null } \varphi_1)^0 + \dots + (\text{null } \varphi_m)^0 \\ \text{span}(\varphi_i) = \{v \in V : \forall \psi \in \text{span}(\varphi_i), \psi(v) = 0\}^0 = (\text{null } \varphi_i)^0 \end{array} \right\} \Rightarrow (a) = (b). \quad \square$$

OR. Note that by COROLLARY in Problem (4E 6), for each φ_i , we have

$\forall c \in \mathbf{F} \setminus \{0\}, \psi = c\varphi_i \in \text{span}(\varphi_i) \iff \text{null } \psi = \text{null } \varphi_i \iff \psi \in (\text{null } \psi)^0 = (\text{null } \varphi_i)^0$.

And $0 \in \text{span}(\varphi_i), 0 \in (\text{null } \varphi_i)^0$. Hence $\text{span}(\varphi_i) = (\text{null } \varphi_i)^0$. Similarly. \square

OR. [Only in Finite-dim] Suppose $\varphi \in V'$. Note that $\dim(\text{null } \varphi)^0 = \dim \text{range } \varphi = \dim \text{span}(\varphi)$.

And because $\forall c \in \mathbf{F}, v \in \text{null } \varphi, c\varphi(v) = 0 \Rightarrow \text{span}(\varphi) \subseteq (\text{null } \varphi)^0$. Similarly. \square

COROLLARY: 30 Suppose V is finite-dim and $\varphi_1, \dots, \varphi_m$ is a linearly inde list in V' .

Then $\dim((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)) = (\dim V) - m$.

31 Suppose V is finite-dim and $B_V = (\varphi_1, \dots, \varphi_n)$. Show that the correspd B_V exists.

SOLUTION:

Using (3.B.29). Let $\varphi_i(u_i) = 1$ and then $V = \text{null } \varphi_i \oplus \text{span}(u_i)$ for each φ_i .

Suppose $a_1 u_1 + \dots + a_n u_n = 0$. Then $0 = \varphi_i(a_1 u_1 + \dots + a_n u_n) = a_i$ for each i .

Thus $B_V = (\varphi_1, \dots, \varphi_n)$. And $\varphi_i(u_x) = \delta_{i,x}$. □

OR. For each $k \in \{1, \dots, n\}$, define $\Gamma_k = \{1, \dots, k-1, k+1, \dots, n\}$ and $U_k = \bigcap_{j \in \Gamma_k} \text{null } \varphi_j$.

By Problem (30) OR (4E 2.C.16), $\dim U_k = 1$. Thus $\exists u_k \in V, U_k = \text{span}(u_k) \neq 0$.

又 By Problem (30), $(\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_n) = \{0\} = U \cap \text{null } \varphi_k$.

Then if $\varphi_k(u_k) = 0 \Rightarrow u_k \in \text{null } \varphi_k$ while $u_k \in U \Rightarrow u_k \in \{0\}$, contradicts.

Thus $\varphi_k(u_k) \neq 0$. Let $v_k = (\varphi_k(u_k))^{-1} u_k \Rightarrow \varphi_k(v_k) = 1$. Now for $j \neq k, u_k \in \text{null } \varphi_j \Rightarrow \varphi_j(v_k) = 0$.

Similarly, suppose $a_1 v_1 + \dots + a_n v_n = 0 \Rightarrow a_1 = \dots = a_n = 0$. $B_V = (v_1, \dots, v_n)$. And $\varphi_j(v_k) = \delta_{j,k}$. □

25 Suppose U is a subsp of V . Explain why $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.

SOLUTION: Note that $U = \{v \in V : v \in U\}$ is a subsp of V ; And $v \in U \iff \varphi(v) = 0, \forall \varphi \in U^0$. □

COROLLARY: $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}^0$.

COMMENT: $\{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = ((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) \cap \dots)$, where $\varphi_k \in U^0$, always remains a subsp, whether the subset U is a subsp or not.

26 Suppose Ω is a subsp of V' . Prove that $\Omega = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega^0\}$.

SOLUTION:

Suppose $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}$, which is the set of vecs that each $\varphi \in \Omega$ sends to zero in common.

Then $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}^0$. 又 $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}^0$.

Immediately by the COROLLARY in Problem (20,21), we may conclude that $\Omega = U^0$. □

OR. [Requires Ω finite-dim] Let $(\varphi_1, \dots, \varphi_m)$ be a basis of Ω . Then by def, $U \subseteq (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$.

$\forall \varphi \in \Omega, \exists ! a_i \in \mathbb{F}, \varphi = a_1 \varphi_1 + \dots + a_m \varphi_m \Rightarrow \forall v \in (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m), \varphi(v) = 0 \Rightarrow v \in U$.

Hence $(\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) = U$. 又 $\text{span}(\varphi_1, \dots, \varphi_m) = \Omega$. By Problem (23), we are done. □

COROLLARY: For every subsp Ω of V' , $\exists !$ subsp U of V such that $\Omega = U^0$.

COMMENT: [Only in Finite-dim] Using Problem (31) and the COROLLARY(c) in Problem (22, 23).

Let $B_\Omega = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n), B_V = (v_1, \dots, v_m, \dots, v_n)$.

$V' = \text{span}(\varphi_1, \dots, \varphi_m) \oplus \text{span}(\varphi_{m+1}, \dots, \varphi_n) \stackrel{(I)}{=} \text{span}(v_{m+1}, \dots, v_n)^0 \oplus \text{span}(v_1, \dots, v_m)^0$.

$\Omega = \text{span}(\varphi_1, \dots, \varphi_m) \stackrel{(II)}{=} \text{span}(v_{m+1}, \dots, v_n)^0 = U^0; \text{span}(\varphi_{m+1}, \dots, \varphi_n) \stackrel{(III)}{=} \text{span}(v_1, \dots, v_m)^0$.

$\iff U = \text{span}(v_{m+1}, \dots, v_n) = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$. [Another proof of [3.106] OR. Problem (24)]

(I) Using the COROLLARY(c), immediately.

(II) NOTICE that each $\text{null } \varphi_k = \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n) = U_k; \dim U_k = \dim V - 1$.

By (4E 2.C.16), $U = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) = \bigcap_{k=1}^m U_k = \text{span}(v_{m+1}, \dots, v_n)$.

Hence $\text{span}(v_{m+1}, \dots, v_n)^0 = U^0 = \Omega = \text{span}(\varphi_1, \dots, \varphi_m)$.

(III) NOTICE that $V' = \Omega \oplus \text{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0 \oplus \text{span}(v_1, \dots, v_m)^0$.

And that $\text{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq \text{span}(v_1, \dots, v_m)^0$.

By the TIPS in (1.C), $\text{span}(\varphi_{m+1}, \dots, \varphi_n) = \text{span}(v_1, \dots, v_m)$.

OR. Similar to (II), let $\Omega = \text{span}(\varphi_{m+1}, \dots, \varphi_n)$, immediately. □

• Suppose $T \in \mathcal{L}(V, W)$, $\varphi_k \in V'$, $\psi_k \in W'$.

28 Prove that $\text{null } T' = \text{span}(\psi_1, \dots, \psi_m) \iff \text{range } T = (\text{null } \psi_1) \cap \dots \cap (\text{null } \psi_m)$.

29 Prove that $\text{range } T' = \text{span}(\varphi_1, \dots, \varphi_m) \iff \text{null } T = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$.

SOLUTION: Using [3.107], [3.109], Problem (23) and the COROLLARY in Problem (20, 21).

$$(28) (\text{range } T)^0 = \text{null } T' = \text{span}(\psi_1, \dots, \psi_m) = ((\text{null } \psi_1) \cap \dots \cap (\text{null } \psi_m))^0.$$

$$(29) (\text{null } T)^0 = \text{range } T' = \text{span}(\varphi_1, \dots, \varphi_m) = ((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m))^0. \quad \square$$

COROLLARY: Using the COMMENT in Problem (26).

$$\text{null } T = \text{span}(v_1, \dots, v_m) \iff \text{null } T = (\text{null } \varphi_{m+1}) \cap \dots \cap (\text{null } \varphi_n) \iff \text{range } T' = \text{span}(\varphi_{m+1}, \dots, \varphi_n).$$

$$\text{---Where } B_V = (v_1, \dots, v_m, \dots, v_n) \iff B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n).$$

$$\text{range } T = \text{span}(w_1, \dots, w_m) \iff \text{range } T = (\text{null } \psi_{m+1}) \cap \dots \cap (\text{null } \psi_n) \iff \text{null } T' = \text{span}(\psi_{m+1}, \dots, \psi_n).$$

$$\text{---Where } B_W = (w_1, \dots, w_m, \dots, w_n) \iff B_{W'} = (\psi_1, \dots, \psi_m, \dots, \psi_n).$$

9 Let $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \dots, \varphi_n)$. Then $\forall \psi \in V'$, $\psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n$.

COROLLARY: For other $B'_V = (u_1, \dots, u_n)$, $B'_{V'} = (\rho_1, \dots, \rho_n)$, $\forall \psi \in V'$, $\psi = \psi(u_1)\rho_1 + \dots + \psi(u_n)\rho_n$.

SOLUTION:

$$\psi(v) = \psi\left(\sum_{i=1}^n a_i v_i\right) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i) \varphi_i(v) = [\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n](v).$$

$$\text{OR. } [\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n]\left(\sum_{i=1}^n a_i v_i\right) = \psi(v_1)\varphi_1\left(\sum_{i=1}^n a_i v_i\right) + \dots + \psi(v_n)\varphi_n\left(\sum_{i=1}^n a_i v_i\right). \quad \square$$

13 Define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$.

Let (φ_1, φ_2) , (ψ_1, ψ_2, ψ_3) denote the dual basis of the standard basis of \mathbb{R}^2 and \mathbb{R}^3 .

(a) Describe the linear functionals $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$

$$\text{For any } (x, y, z) \in \mathbb{R}^3, (T'(\varphi_1))(x, y, z) = 4x + 5y + 6z, (T'(\varphi_2))(x, y, z) = 7x + 8y + 9z.$$

(b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3 .

$$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3, \quad T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3.$$

(c) What is $\text{null } T'$? What is $\text{range } T'$?

$$T(x, y, z) = 0 \iff \begin{cases} 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases} \iff \begin{cases} x + y + z = 0 \\ y = 2z = 0 \end{cases} \iff (x, y, z) \in \text{span}(e_1 - 2e_2 + e_3).$$

Where (e_1, e_2, e_3) is standard basis of \mathbb{R}^3 .

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a basis, with the correspond dual basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

$$\text{Thus } \text{span}(e_1 - 2e_2 + e_3) = \text{null } T \Rightarrow \text{span}(e_1 - 2e_2 + e_3)^0 = \text{span}(\varepsilon_2, \varepsilon_3) = \text{range } T'.$$

$$\text{Note that } \varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3.$$

$$\text{And } \begin{cases} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{cases}$$

$$\text{Hence } \varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2, \quad \varepsilon_3 = -\psi_1 + \psi_3. \text{ Now } \text{range } T' = \text{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3).$$

$$\text{OR. } \text{range } T' = \text{span}(T'(\varphi_1), T'(\varphi_2)) = \text{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3).$$

$$\text{Suppose } T'(x\varphi_1 + y\varphi_2) = (4x + 7y)\varphi_1 + (5x + 8y)\varphi_2 + (6x + 9y)\varphi_3 = 0.$$

$$\text{Then } x + y = 4x + 7y = x = y = 0. \text{ Hence } \text{null } T' = \{0\}.$$

$$\text{OR. } \text{null } T = \text{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \text{span}(-2e_2, e_3) \oplus \text{null } T.$$

$$\Rightarrow \text{range } T = \{Tx : x \in \text{span}(-2e_2, e_3)\} = \text{span}(T(-2e_2), T(e_3))$$

$$= \text{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \text{span}(f_1, f_2) = \mathbb{R}^2. \text{ Now } \text{null } T' = (\text{range } T)^0 = \{0\}. \quad \square$$

24 Suppose V is finite-dim and U is a subsp of V .

Prove, using the pattern of [3.104], that $\dim U + \dim U^0 = \dim V$.

SOLUTION:

By Problem (31) and the COMMENT in Problem (26), $B_U = (v_1, \dots, v_m) \iff B_{U^0} = (\varphi_{m+1}, \dots, \varphi_n)$. \square

37 Suppose U is a subsp of V and π is the quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.

(a) Show that π' is inje: Because π is surj. Use [3.108].

(b) Show that $\text{range } \pi' = U^0$: By [3.109](b), $\text{range } \pi' = (\text{null } \pi)^0 = U^0$.

(c) Conclude that π' is an iso from $(V/U)'$ onto U^0 : Immediately.

SOLUTION: OR. Using (3.E.18), also see (3.E.20).

(a) $\pi'(\varphi) = 0 \iff \forall v \in V (\forall v + U \in V), \varphi(\pi(v)) = \varphi(v + U) = 0 \iff \varphi = 0$.

(b) $\psi \in \text{range } \pi' \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \text{null } \psi \supseteq U \iff \psi \in U^0$. Hence $\text{range } \pi' = U^0$. \square

• Suppose U is a subsp of V . Prove that $(V/U)'$ and U^0 are iso. [Another proof of [3.106]]

SOLUTION:

Define $\xi : U^0 \rightarrow (V/U)'$ by $\xi(\varphi) = \tilde{\varphi}$, where $\tilde{\varphi} \in (V/U)'$ is defined by $\tilde{\varphi}(v + U) = \varphi(v)$.

We show that ξ is inje and surj.

Inje: $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V (\forall v + U \in V/U), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$.

Surj: $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \text{null } (\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$. \square

OR. Define $\nu : (V/U)' \rightarrow U^0$ by $\nu(\Phi) = \Phi \circ \pi$. Now $\nu \circ \xi = I_{U^0}$, $\xi \circ \nu = I_{(V/U)'}$, $\Rightarrow \xi = \nu^{-1}$. \square

4 Suppose U is a subsp of V and $U \neq V$. Prove that $\exists \varphi \in V' \setminus \{0\}, \varphi(u) = 0$ for all $u \in U$.

SOLUTION: $\iff U_V^0 \neq \{0\}$.

Let X be such that $V = U \oplus X$. Then $X \neq \{0\}$. Suppose $s \in X$ and $s \neq 0$.

Let Y be such that $X = \text{span}(s) \oplus Y$. Now $V = U \oplus (\text{span}(s) \oplus Y)$.

Define $\varphi \in V'$ by $\varphi(u + \lambda s + y) = \lambda$. Hence $\varphi \neq 0$ and $\varphi(u) = 0$ for all $u \in U$. \square

OR. [Requires that V is finite-dim] By [3.106], $\dim U^0 = \dim V - \dim U > 0$. Then $U^0 \neq \{0\}$.

OR. Let $B_V = (\underbrace{u_1, \dots, u_m}_{B_U}, v_1, \dots, v_n)$ with $n \geq 1$. Let $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$. Let $\varphi = \varphi_i$.

OR. Define $\varphi \in V'$ by $\varphi(u_1) = \dots = \varphi(u_m) = 0$ and $\varphi(v_1) = \dots = \varphi(v_n) = 1$. \square

COMMENT: [Another proof of [3.108]]: T is surj $\iff T'$ is inje.

(a) Suppose T' is inje. Note that $T'(\psi) = 0 \Rightarrow \psi = 0$.

Then $\nexists \psi \in W' \setminus \{0\}, (T'(\psi))(v) = \psi(Tv) = 0$ for all $w \in \text{range } T (\forall v \in V)$.

Thus if we assume that $\text{range } T \neq W$ then contradicts. Hence $\text{range } T = W$.

(b) Suppose T is surj. Then $(\text{range } T)^0 = W_W^0 = \{0\} = \text{null } T'$. \square

• Suppose V is a vecsp and U is a subsp of V .

17 $U^0 = \{\varphi \in V' : U \subseteq \text{null } \varphi\}$. Noticing $\varphi \in V', U \subseteq \text{null } \varphi \iff \forall u \in U, \varphi(u) = 0$.

18 $U^0 = V' \iff \forall \varphi \in V', U \subseteq \text{null } \varphi \iff U = \{0\}$. [Which means $\{0\}_V^0 = V'$.]

OR. $U^0 = V' \iff \dim U^0 = \dim V' = \dim V \iff \dim U = 0 \iff U = \{0\}$.

19 $U_V^0 = \{0\} = V_V^0 \iff U = V$. By the inverse and contrapositive of Problem (4). OR. By [3.106].

- Suppose $V = U \oplus W$. Define $\iota : V \rightarrow U$ by $\iota(u + w) = u$. Thus $\iota' \in \mathcal{L}(U', V')$.
 - (a) Show that $\text{null } \iota' = U_U^0 = \{0\}$: $\text{null } \iota' = (\text{range } \iota)_U^0 = U_U^0 = \{0\}$.
 - (b) Prove that $\text{range } \iota' = W_V^0$: $\text{range } \iota' = (\text{null } \iota)_V^0 = W_V^0$.
 - (c) Prove that $\tilde{\iota}'$ is an iso from $U' / \{0\}$ onto W^0 : By (a), (b) and [3.91](d).

SOLUTION:

- (a) $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \text{null } \psi$.
- (b) Note that $W = \text{null } (\iota) \subseteq \text{null } (\psi \circ \iota)$. Then $\psi \circ \iota \in W^0 \Rightarrow \text{range } \iota' \in W^0$.
Suppose $\varphi \in W^0$. Because $\text{null } \iota = W \subseteq \text{null } \varphi$. By TIPS in (3.B), $\varphi = \varphi \circ \iota = \iota'(\varphi)$. □

36 Suppose U is a subsp of V . Define $i : U \rightarrow V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.

- (a) Show that $\text{null } i' = U^0$: $\text{null } i' = (\text{range } i)^0 = U^0 \Leftarrow \text{range } i = U$.
- (b) Prove that $\text{range } i' = U'$: $\text{range } i' = (\text{null } i)_U^0 = \{0\}_U^0 = U'$.
- (c) Prove that \tilde{i}' is an iso from V' / U^0 onto U' : By (a), (b) and [3.91](d).

SOLUTION:

- (a) $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$. Thus $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$.
- (b) Suppose $\psi \in U'$. By (3.A.11), $\exists \varphi \in V', \varphi|_U = \psi$. Then $i'(\varphi) = \psi$. □

• Suppose $T \in \mathcal{L}(V, W)$. Prove that $\text{range } T' = (\text{null } T)^0$. [Another proof of [3.109](b)]

SOLUTION:

Suppose $\Phi \in (\text{null } T)^0$. Because by (3.B.12), $T|_U : U \rightarrow \text{range } T$ is an iso; $V = U \oplus \text{null } T$.
And $\forall v \in V, \exists! u_v \in U, w_v \in \text{null } T, v = u_v + w_v$. Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$.
Let $\psi = \Phi \circ (T|_{\text{range } T}^{-1})$. Then $T'(\psi) = \psi \circ T = \Phi \circ (T^{-1}|_{\text{range } T} \circ T|_V)$.
Where $T^{-1}|_{\text{range } T} : \text{range } T \rightarrow U$; $T : V \rightarrow \text{range } T$. Note that $T^{-1}|_{\text{range } T} \circ T|_V = \iota$.
By TIPS in (3.B), $\Phi = \Phi \circ \iota$. Thus $T'(\psi) = \psi \circ T = \Phi \circ \iota = \Phi$. □

• Suppose $T \in \mathcal{L}(V, W)$. Using [3.108], [3.110].

$$\text{Now } T \text{ is inv} \iff \left| \begin{array}{l} \text{null } T = \{0\} \iff (\text{null } T)^0 = V' = \text{range } T' \\ \text{range } T = W \iff (\text{range } T)^0 = \{0\} = \text{null } T' \end{array} \right| \iff T' \text{ is inv.}$$

15 Suppose $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \iff T = 0$.

SOLUTION:

Suppose $T = 0$. Then $\forall \varphi \in W', T'(\varphi) = \varphi \circ T = 0$. Hence $T' = 0$.

Suppose $T' = 0$. Then $\text{null } T' = W' = (\text{range } T)^0$, by [3.107](a).

[W can be infinite-dim] By Problem (25),

$$\text{range } T = \{w \in W : \varphi(w) = 0, \forall \varphi \in (\text{range } T)^0\} = \{w \in W : \varphi(w) = 0, \forall \varphi \in W'\}.$$

Now we prove that if $\forall \varphi \in W', \varphi(w) = 0$, then $w = 0$. So that $\text{range } T = \{0\}$ and we are done.

Assume that $w \neq 0$. Then let U be such that $W = U \oplus \text{span}(w)$.

Define $\psi \in W'$ by $\psi(u + \lambda w) = \lambda$. So that $\psi(w) = 1 \neq 0$. □

OR. [Only if W is finite-dim] By [3.106], $\dim \text{range } T = \dim W - \dim(\text{range } T)^0 = 0$. □

12 NOTICE that $I_V : V' \rightarrow V'$. Now $\forall \varphi \in V', I_V(\varphi) = \varphi = \varphi \circ I_V = I_V'(\varphi)$. Thus $I_V = I_V'$.

16 Suppose V, W are finite-dim. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(V, W)$.

Prove that Γ is an iso of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

SOLUTION: By [3.101], Γ is linear.

Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus Γ is inje.

Because V, W are finite-dim. $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$. Now Γ inje \Rightarrow inv. □

COMMENT: Let $X = \{T \in \mathcal{L}(V, W) : \text{range } T \text{ is finite-dim}\}$.

Let $Y = \{\mathcal{T} \in \mathcal{L}(W', V') : \text{range } \mathcal{T} \text{ is finite-dim}\}$.

Then $\Gamma|_X$ is an iso of X onto Y , even if V and W are infinite-dim.

The inje of $\Gamma|_X$ is equiv to the inje of Γ , as shown before.

Now we show that $\Gamma|_X$ is surj without the cond that V or W is finite-dim.

Suppose $\mathcal{T} \in Y$. Let $B_{\text{range } \mathcal{T}} = (\varphi_1, \dots, \varphi_m)$, with the correspd (v_1, \dots, v_m) .

Let $\varphi_k = \mathcal{T}(\psi_k)$. Let \mathcal{K} be such that $W' = \mathcal{K} \oplus \text{null } \mathcal{T}$.

Let $B_{\mathcal{K}} = (\psi_1, \dots, \psi_m)$, with the correspd (w_1, \dots, w_m) .

Define $T \in \mathcal{L}(V, W)$ by $Tv_k = w_k, Tu = 0; k \in \{1, \dots, m\}, u \in U$.

$\forall \psi \in \text{null } \mathcal{T}, [T'(\psi)](v) = \psi(Tv) = \psi(a_1w_1 + \dots + a_pw_p) = 0 = [\mathcal{T}(\psi)](v)$.

$\forall k \in \{1, \dots, m\}, [T'(\psi_k)](v) = \psi_k(Tv) = \psi_k(a_1w_1 + \dots + a_mw_m) = a_k = \varphi_k(v) = [\mathcal{T}(\psi)](v)$. □

COMMENT: This is another proof of [3.109(a)]: $\dim \text{range } T = \dim \text{range } T'$.

• (4E 3.F.6) Suppose $\varphi, \beta \in V'$. Prove that $\text{null } \varphi \subseteq \text{null } \beta \iff \beta = c\varphi, \exists c \in \mathbf{F}$.

COROLLARY: $\text{null } \varphi = \text{null } \beta \iff \beta = c\varphi, \exists c \in \mathbf{F} \setminus \{0\}$.

SOLUTION:

Using (3.B.29, 30).

(a) Suppose $\text{null } \varphi \subseteq \text{null } \beta$. Suppose $u \notin \text{null } \beta$, then $u \notin \text{null } \varphi$.

Now $V = \text{null } \beta \oplus \text{span}(u) = \text{null } \varphi \oplus \text{span}(u)$.

Then by TIPS in (1.C), $\text{null } \beta = \text{null } \varphi$. Hence let $c = \frac{\beta(u)}{\varphi(u)}$.

OR. We discuss in two cases. If $\text{null } \varphi = \text{null } \beta$, then we are done.

Otherwise, $\text{null } \beta \neq \text{null } \varphi$. Suppose $u' \in \text{null } \beta \setminus \text{null } \varphi$, then $V = \text{null } \varphi \oplus \text{span}(u')$.

$\forall v \in V, v = w + au = w' + bu', \exists ! w, w' \in \text{null } \varphi, a, b \in \mathbf{F}$.

Thus $\beta(v) = a\beta(u), \varphi(v) = b\varphi(u')$. Let $c = \frac{a\beta(u)}{b\varphi(u')}$. We are done.

NOTICE that by (b) below, we have $\text{null } \beta \subseteq \text{null } \varphi$, thus contradicts the assumption.

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$.

If $c = 0$, then $\text{null } \beta = V \supseteq \text{null } \varphi$, we are done.

Otherwise, $\left. \begin{array}{l} \forall v \in \text{null } \varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \text{null } \varphi \subseteq \text{null } \beta. \\ \forall v \in \text{null } \beta, \beta(v) = 0 = \varphi(v) \Rightarrow \text{null } \beta \subseteq \text{null } \varphi. \end{array} \right\} \Rightarrow \text{null } \varphi = \text{null } \beta$
 $\Rightarrow \text{null } \varphi \subseteq \text{null } \beta$. □

OR. By (3.B.24), $\text{null } \varphi \subseteq \text{null } \beta \iff \exists E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi$. (if E is inv, then $\text{null } \varphi = \text{null } \beta$)

Now we show that $[P] \exists E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi \iff \exists c \in \mathbf{F}, \beta = c\varphi$. [Q].

$[P] \Rightarrow [Q]$: Let $c = E(1)$. Then $\forall v \in V, \beta(v) = E(\varphi(v)) = \varphi(v)E(1) = c\varphi(v)$. ($E(1) \neq 0$)

$[Q] \Rightarrow [P]$: Define $E \in \mathcal{L}(\mathbf{F})$ by $E(x) = cx$. Then $\forall v \in V, \beta(v) = c\varphi(v) = E(\varphi(v))$. ($c \neq 0$) □

- In (3.D.18), $\varphi : V \rightarrow \mathcal{L}(\mathbf{F}, V)$ is an iso. Now we prove that
 $[P] (v_1, \dots, v_m) \text{ is linely inde} \iff (\varphi(v_1), \dots, \varphi(v_m)) \text{ is linely inde. } [Q]$

SOLUTION:

$[P] \Rightarrow [Q]$: Notice that φ is inje and by (3.B.9).

OR. Suppose $\vartheta \in \text{span}(\varphi(v_1), \dots, \varphi(v_m))$. Let $\vartheta = 0 = a_1\varphi(v_1) + \dots + a_m\varphi(v_m)$.

Then $\vartheta(1) = 0 = a_1v_1 + \dots + a_mv_m \Rightarrow a_1 = \dots = a_m = 0$.

$[Q] \Rightarrow [P]$: Suppose $v \in \text{span}(v_1, \dots, v_m)$. Let $v = 0 = a_1v_1 + \dots + a_mv_m$.

Then $\varphi(v) = 0 = a_1\varphi(v_1) + \dots + a_m\varphi(v_m) \Rightarrow a_1 = \dots = a_m = 0$. □

33 Suppose $A \in \mathbf{F}^{m,n}$. Define $T : A \rightarrow A^t$. Prove that T is an iso of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$

SOLUTION: By [3.111], T is linear. Note that $(A^t)^t = A$.

Inje: If $T(A) = 0$ for some $A \in \mathbf{F}^{n,m}$, then $A = (T(A))^t = 0$.

Surj: For any $B \in \mathbf{F}^{n,m}$, let $A = B^t$ so that $T(A) = B$. □

32 Let $B_\alpha = (\alpha_1, \dots, \alpha_m)$, $B_\alpha' = (\varphi_1, \dots, \varphi_m)$, $B_\beta = (v_1, \dots, v_m)$, $B_\beta' = (\psi_1, \dots, \psi_m)$.

Prove that $\forall T \in \mathcal{L}(V)$, T is inv \iff the rows of $A = \mathcal{M}(T, B_\alpha, B_\beta)$ form a basis of $\mathbf{F}^{1,n}$.

SOLUTION: Note that T is invertible $\iff T'$ is inv. And $A^t = \mathcal{M}(T', B_\beta', B_\alpha')$.

(a) Suppose T is inv, so is T' . Because $(T'(\varphi_1), \dots, T'(\varphi_m))$ is linely inde.

NOTICE that $T'(\varphi_i) = A_{1,i}^t\psi_1 + \dots + A_{m,i}^t\psi_m$. By the (Δ) part in (4E 3.C.17),

the cols of A^t , namely the rows of A , are linely inde.

(b) Suppose the rows of A are linely inde, so are the cols of A^t . NOTICE that A^t has $\dim V'$ cols.

Then $B_{\text{range } T'} = B_{V'} = (T'(\varphi_1), \dots, T'(\varphi_m))$. Thus T' is surj. Hence T' is inv, so is T . □

5 Prove that $(V_1 \times \dots \times V_m)'$ and $V'_1 \times \dots \times V'_m$ are iso.

[Using notations in (3.E.2).]

Define $\varphi : (V_1 \times \dots \times V_m)' \rightarrow V'_1 \times \dots \times V'_m$

by $\varphi(T) = (T \circ R_1, \dots, T \circ R_m) = (R'_1(T), \dots, R'_m(T))$.

Define $\psi : V'_1 \times \dots \times V'_m \rightarrow (V_1 \times \dots \times V_m)'$

by $\psi(T_1, \dots, T_m) = T_1S_1 + \dots + T_mS_m = S'_1(T_1) + \dots + S'_m(T_m)$.

$\Rightarrow \psi = \varphi^{-1}$. □

• (4E 3.F.8) Suppose $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \dots, \varphi_n)$.

Define $\Gamma : V \rightarrow \mathbf{F}^n$ by $\Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v))$.

Define $\Lambda : \mathbf{F}^n \rightarrow V$ by $\Lambda(a_1, \dots, a_n) = a_1v_1 + \dots + a_nv_n$.

$\Rightarrow \Lambda = \Gamma^{-1}$.

• (4E 3.F.5) Suppose $T \in \mathcal{L}(V, W)$. $B_{\text{range } T} = (w_1, \dots, w_m)$.

Hence $\forall v \in V$, $Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m$, $\exists! \varphi_1(v), \dots, \varphi_m(v)$,

thus defining $\varphi_i : V \rightarrow \mathbf{F}$ for each $i \in \{1, \dots, m\}$. Show that each $\varphi_i \in V'$.

SOLUTION:

$\forall u, v \in V, \lambda \in \mathbf{F}, T(u + \lambda v) = \sum_{i=1}^m \varphi_i(u + \lambda v)w_i$

$= Tu + \lambda Tv = \left(\sum_{i=1}^m \varphi_i(u)w_i \right) + \lambda \left(\sum_{i=1}^m \varphi_i(v)w_i \right) = \sum_{i=1}^m (\varphi_i(u) + \lambda \varphi_i(v))w_i$. □

OR. For each $w_i, \exists v_i \in V, Tv_i = w_i$, then (v_1, \dots, v_m) is linely inde.

Now we have $Tv = a_1Tv_1 + \dots + a_mTv_m, \forall v \in V, \exists! a_i \in \mathbf{F}$. Let $B_{(\text{range } T)'} = (\psi_1, \dots, \psi_m)$.

Then $(T'(\psi_i))(v) = \psi_i \circ T(v) = a_i$. Where $T : V \rightarrow \text{range } T; T' : (\text{range } T)' \rightarrow V'$.

Thus for each $i \in \{1, \dots, m\}, \varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$. □

6 Define $\Gamma : V' \rightarrow \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$, where $v_1, \dots, v_m \in V$.

(a) Show that $\text{span}(v_1, \dots, v_m) = V \iff \Gamma$ is inje.

(b) Show that (v_1, \dots, v_m) is linely inde $\iff \Gamma$ is surj.

SOLUTION:

(a) NOTICE that $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \dots = \varphi(v_m) = 0 \iff \text{null } \varphi = \text{span}(v_1, \dots, v_m)$.

If Γ is inje, then $\Gamma(\varphi) = 0 \iff V = \text{null } \varphi = \text{span}(v_1, \dots, v_m)$.

If $V = \text{span}(v_1, \dots, v_m)$, then $\Gamma(\varphi) = 0 \iff \text{null } \varphi = \text{span}(v_1, \dots, v_m)$, thus Γ is inje.

(b) Suppose Γ is surj. Then let $\Gamma(\varphi_i) = e_i$ for each i , where (e_1, \dots, e_m) is the standard basis of \mathbf{F}^m .

Then by (3.A.4), $(\varphi_1, \dots, \varphi_m)$ is linely inde.

Now $a_1 v_1 + \dots + a_m v_m = 0 \Rightarrow 0 = \varphi_i(a_1 v_1 + \dots + a_m v_m) = a_i$ for each i .

Suppose (v_1, \dots, v_m) is linely inde. Let $U = \text{span}(\varphi_1, \dots, \varphi_m)$, $B_{U'} = (\varphi_1, \dots, \varphi_m)$.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists! \varphi = a_1 \varphi_1 + \dots + a_m \varphi_m$.

Let W be such that $V = U \oplus W$. Now $\forall v \in V, \exists! u_v \in U, w_v \in W, v = u_v + w_v$.

Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$. So that $\Gamma(\varphi \circ \iota -) = (a_1, \dots, a_m)$. □

OR. Let (e_1, \dots, e_m) be the standard basis of \mathbf{F}^m and let (ψ_1, \dots, ψ_m) be the correspd dual basis.

Define $\Psi : \mathbf{F}^m \rightarrow (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$. Then Ψ is an iso.

Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T e_k = v_k$. Now $T(x_1, \dots, x_m) = T(x_1 e_1 + \dots + x_m e_m) = x_1 v_1 + \dots + x_m v_m$.

$\forall \varphi \in V', k \in \{1, \dots, m\}, [T'(\varphi)](e_k) = \varphi(T e_k) = \varphi(v_k) = [\varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m](e_k)$

Now $T'(\varphi) = \varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m = \Psi(\varphi(v_1), \dots, \varphi(v_m)) = \Psi(\Gamma(\varphi))$. Hence $T' = \Psi \circ \Gamma$.

By (3.B.3), (a) $\text{range } T = \text{span}(v_1, \dots, v_m) = V \iff T' = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.

(b) (v_1, \dots, v_m) is linely inde $\iff T$ is inje $\iff T' = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

• (4E 3.F.25) Define $\Gamma : V \rightarrow \mathbf{F}^m$ by $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$, where $\varphi_1, \dots, \varphi_m \in V'$.

(c) Show that $\text{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ is inje.

(d) Show that $(\varphi_1, \dots, \varphi_m)$ is linely inde $\iff \Gamma$ is surj.

SOLUTION:

(c) NOTICE that $\Gamma(v) = 0 \iff \varphi_1(v) = \dots = \varphi_m(v) = 0 \iff v \in (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$.

By Problem (4E 23) and (18), $\text{span}(\varphi_1, \dots, \varphi_m) = V' \iff (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) = \{0\}$.

And $\text{null } \Gamma = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$. Hence Γ inje $\iff \text{null } \Gamma = \{0\} \iff \text{span}(\varphi_1, \dots, \varphi_m) = V'$.

(d) Suppose $(\varphi_1, \dots, \varphi_m)$ is linely inde. Then by Problem (31), (v_1, \dots, v_m) is linely inde.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists! v = \sum_{i=1}^m a_i v_i \in V \Rightarrow \varphi_i(v) = a_i, \Gamma(v) = (a_1, \dots, a_m)$. Hence Γ is surj.

Suppose Γ is surj. Let (e_1, \dots, e_m) be the standard basis of \mathbf{F}^m .

Suppose $v_i \in V$ such that $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i$, for each i .

Then (v_1, \dots, v_m) is linely inde. And $\varphi_j(v_k) = \delta_{j,k}$.

Now $a_1 \varphi_1 + \dots + a_m \varphi_m = 0 \Rightarrow 0(v_i) = a_i$ for each i . Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde.

OR. Let $\text{span}(v_1, \dots, v_m) = U$. Then $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U)$. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde. □

OR. Similar to Problem (6), we get $(e_1, \dots, e_m), (\psi_1, \dots, \psi_m)$ and the iso Ψ .

$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'(\Psi(x_1, \dots, x_m)) = \Gamma'(\Psi(x_1 e_1 + \dots + x_m e_m)) = (x_1 \psi_1 + \dots + x_m \psi_m) \circ \Gamma$.

$\forall v \in V, [\Gamma'(\Psi(x_1, \dots, x_m))](v) = [x_1 \psi_1 + \dots + x_m \psi_m](\Gamma(v)) = [x_1 \varphi_1 + \dots + x_m \varphi_m](v)$.

Now $\Gamma'(\Psi(x_1, \dots, x_m)) = x_1 \varphi_1 + \dots + x_m \varphi_m$.

Define $\Phi : \mathbf{F}^m \rightarrow (\mathbf{F}^m)'$ by $\Phi = \Psi \circ \Gamma$. $\Phi(x_1, \dots, x_m) = x_1 \varphi_1 + \dots + x_m \varphi_m$. Thus by (4E 3.B.3),

(c) the inje of Φ correspds to $(\varphi_1, \dots, \varphi_m)$ spanning V' ; 又 $\Phi = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.

(d) the surj of Φ correspds to $(\varphi_1, \dots, \varphi_m)$ being linely inde; 又 $\Phi = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

35 Prove that $(\mathcal{P}(\mathbf{F}))'$ and \mathbf{F}^∞ are iso.

SOLUTION:

Define $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{F}))', \mathbf{F}^\infty)$ by $\theta(\varphi) = (\varphi(1), \varphi(z), \dots, \varphi(z^n), \dots)$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall z^k$ in the basis $(1, z, \dots, z^n)$ of $\mathcal{P}_n(\mathbf{F})$ ($\forall n$), $\varphi(z^k) = 0 \Rightarrow \varphi = 0$.

[NOTICE that $\forall p \in \mathcal{P}(\mathbf{R}), \exists ! a_i \in \mathbf{F}, m = \deg p, p = a_0 z^0 + a_1 z^1 + \dots + a_m z^m \in \mathcal{P}_m(\mathbf{F})$.]

Surj: $\forall (a_k)_{k=1}^\infty \in \mathbf{F}^\infty$, let ψ be such that $\forall k, \psi(z^k) = a_k$ and thus $\theta(\psi) = (a_k)_{k=1}^\infty$. \square

COMMENT: NOTICE that $\mathcal{P}(\mathbf{F})$ and \mathbf{F}^∞ are not iso, so are $\mathcal{P}(\mathbf{F})$ and $(\mathcal{P}(\mathbf{F}))'$

But if we let $\mathbf{F}^\infty = \{(a_1, \dots, a_n, \underbrace{0, \dots, 0}_{\text{all zero}}) \in \mathbf{F}^\infty \mid \exists ! n \in \mathbf{N}^+\}$. Then $\mathcal{P}(\mathbf{F})$ and \mathbf{F}^∞ are iso.

7 Show that the dual basis of $(1, x, \dots, x^m)$ of $\mathcal{P}_m(\mathbf{R})$ is $(\varphi_0, \varphi_1, \dots, \varphi_m)$, where $\varphi_k(p) = \frac{p^{(k)}(0)}{k!}$.

Here $p^{(k)}$ denotes the k^{th} derivative of p , with the understanding that the 0^{th} derivative of p is p .

SOLUTION:

$$\forall j, k \in \mathbf{N}, (x^j)^{(k)} = \begin{cases} j(j-1) \dots (j-k+1) \cdot x^{(j-k)}, & j \geq k. \\ j(j-1) \dots (j-j+1) = j! & j = k. \\ 0, & j \leq k. \end{cases} \quad \text{Then } (x^j)^{(k)}(0) = \begin{cases} 0, & j \neq k. \\ k!, & j = k. \end{cases} \quad \square$$

OR. Because $\forall j, k \in \{1, \dots, m\}$ such that $j \neq k$, $\varphi_k(x^j) = \frac{(x^j)^{(k)}(0)}{k!} = \frac{0}{k!} = 0$; $\varphi_k(x^k) = \frac{(x^k)^{(k)}(0)}{k!} = 1$.

Thus $\frac{p^{(k)}(0)}{k!}$ act exactly the same as φ_k on the same basis $(1, \dots, x^m)$, hence is just another def of φ_k \square

EXAMPLE: Suppose $m \in \mathbf{N}^+$. By [2.C.10], $B = (1, x-5, \dots, (x-5)^m)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

Let $\varphi_k = \frac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \dots, m$. Then $(\varphi_0, \varphi_1, \dots, \varphi_m)$ is the dual basis of B .

34 The double dual space of V , denoted by V'' , is defined to be the dual space of V' .

In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \rightarrow V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.

(a) Show that Λ is a linear map from V to V'' .

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

(c) Show that if V is finite-dim, then Λ is an iso from V onto V'' .

Suppose V is finite-dim. Then V and V' are iso, and finding an iso from V onto V' generally requires choosing a basis of V . In contrast, the iso Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

SOLUTION:

(a) $\forall \varphi \in V', v, w \in V, a \in \mathbf{F}, (\Lambda(v+aw))(\varphi) = \varphi(v+aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.

Thus $\Lambda(v+aw) = \Lambda v + a\Lambda w$. Hence Λ is linear.

(b) $(T''(\Lambda v))(\varphi) = ((\Lambda v) \circ T')(\varphi) = (\Lambda v)(T'(\varphi))$
 $= (T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = (\Lambda(Tv))(\varphi)$.

Hence $T''(\Lambda v) = (\Lambda(Tv)) \Rightarrow T'' \circ \Lambda = \Lambda \circ T$.

(c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is inje.

又 Because V is finite-dim. $\dim V = \dim V' = \dim V''$. Hence Λ is an iso. \square

ENDED

- **TIPS:** Suppose $p \in \mathcal{P}(\mathbf{F})$, $\deg p \leq m$ and p has at least $(m+1)$ distinct zeros.

Then by the contrapositive of [4.12], 又 $\deg p = m$, we conclude that $m < 0$. Hence $p = 0$.

OR. We show that if p has at least m distinct zeros, then either $p = 0$ or $\deg p \geq m$.

If $p = 0$ then we are done. If not, then suppose p has exactly n distinct zeros $\lambda_1, \dots, \lambda_n$.

Because $\exists! \alpha_i \geq 1, q \in \mathcal{P}(\mathbf{F})$, and $q \neq 0$, such that $p(z) = [(z - \lambda_1)^{\alpha_1} \dots (z - \lambda_n)^{\alpha_n}] q(z)$. \square

- **COMMENT:** NOTICE that by [4.17], some term of the poly factorization might not in the form $(x - \lambda_k)^{\alpha_k}$.

- **NOTE FOR [4.7]:** the uniqueness of coeffs of polys

[Another proof]

If a poly had two different sets of coeffs, then subtracting the two representations would give a poly with some nonzero coeffs but infinitely many zeros. By TIPS. \square

- **NOTE FOR [4.8]:** division algorithm for polys

[Another proof]

Suppose $\deg p \geq \deg s$. Then $\left(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of length } \deg s}, \underbrace{s, zs, \dots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)} \right)$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.

Because $q \in \mathcal{P}(\mathbf{F})$, $\exists! a_i, b_j \in \mathbf{F}$,

$$\begin{aligned} q &= a_0 + a_1 z + \dots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \dots + b_{\deg p - \deg s} z^{\deg p - \deg s} s \\ &= \underbrace{a_0 + a_1 z + \dots + a_{\deg s - 1} z^{\deg s - 1}}_r + \underbrace{(b_0 + b_1 z + \dots + b_{\deg p - \deg s} z^{\deg p - \deg s}) s}_q. \end{aligned}$$

Note that r, q are unique. \square

- **NOTE FOR [4.11]:** each zero of a poly corresponds to a degree-one factor;

[Another proof]

First suppose $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \dots + a_m z^m$, $\exists! a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.

Then $p(z) = p(z) - p(\lambda) = a_1(z - \lambda) + \dots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.

Hence $\forall k \in \{1, \dots, m\}, z^k - \lambda^k = (z - \lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \dots + z^{k-(j+1)}\lambda^j + \dots + z\lambda^{k-2} + z^0\lambda^{k-1})$.

Thus $p(z) = \sum_{j=1}^m a_j(z - \lambda) \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z - \lambda) \sum_{j=1}^m a_j \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z - \lambda)q(z)$. \square

- **NOTE FOR [4.13]:** Every nonconst poly with complex coefficients has a zero in \mathbf{C} .

[Another proof]

For any $w \in \mathbf{C}, k \in \mathbf{N}^+$, by polar coordinates, $\exists r \geq 0, \theta \in \mathbf{R}, r(\cos \theta + i \sin \theta) = w$.

By De Moivre' theorem, $w^k = [r(\cos \theta + i \sin \theta)]^k = r^k(\cos k\theta + i \sin k\theta)$.

Hence $(r^{1/k}(\cos \frac{\theta}{k} + i \sin \frac{\theta}{k}))^k = w$. Thus every complex number has a k^{th} root.

Suppose a nonconst $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z^m$.

Then $|p(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$ (because $\frac{|p(z)|}{|z_m|} \rightarrow |c_m|$ as $|z| \rightarrow \infty$).

Thus the continuous function $z \rightarrow |p(z)|$ has a global minimum at some point $\zeta \in \mathbf{C}$.

To show that $p(\zeta) = 0$, assume $p(\zeta) \neq 0$. Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \frac{p(z + \zeta)}{p(\zeta)}$.

The function $z \rightarrow |q(z)|$ has a global minimum value of 1 at $z = 0$.

Write $q(z) = 1 + a_k z^k + \dots + a_m z^m$, where $k \in \mathbf{N}^+$ is the smallest such that $a_k \neq 0$.

Let $\beta \in \mathbf{C}$ be such that $\beta^k = -\frac{1}{a_k}$.

There is a const $c > 1$ so that if $t \in (0, 1)$, then $|q(t\beta)| \leq |1 + a_k t^k \beta^k| + t^{k+1} c = 1 - t^k(1 - tc)$.

Now letting $t = 1/(2c)$, we get $|q(t\beta)| < 1$. Contradicts. Hence $p(\zeta) = 0$, as desired. \square

- (4E 4 2) Prove that if $w, z \in \mathbb{C}$, then $||w| - |z|| \leq |w - z|$.

SOLUTION:

$$\begin{aligned} |w - z|^2 &= (w - z)(\bar{w} - \bar{z}) \\ &= |w|^2 + |z|^2 - (w\bar{z} + \bar{w}z) \\ &= |w|^2 + |z|^2 - (\overline{wz} + \overline{wz}) \\ &= |w|^2 + |z|^2 - 2\operatorname{Re}(\overline{wz}) \\ &\geq |w|^2 + |z|^2 - 2|\overline{wz}| \\ &= |w|^2 + |z|^2 - 2|w||z| = ||w| - |z||^2. \end{aligned} \quad \left\{ \begin{array}{l} \text{OR. } |w| = |w - z + z| \leq |w - z| + |z| \Rightarrow |w| - |z| \leq |w - z| \\ |z| = |z - w + w| \leq |z - w| + |w| \Rightarrow |z| - |w| \leq |w - z| \end{array} \right\}$$

Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides. □

- (4E 4 3) Suppose $\mathbf{F} = \mathbb{C}$, $\varphi \in V'$. Define $\sigma : V \rightarrow \mathbb{R}$ by $\sigma(v) = \operatorname{Re} \varphi(v)$ for each $v \in V$.

Show that $\varphi(v) = \sigma(v) - i\sigma(iv)$ for all $v \in V$.

SOLUTION: Notice that $\varphi(v) = \operatorname{Re} \varphi(v) + i\operatorname{Im} \varphi(v) = \sigma(v) + i\operatorname{Im} \varphi(v)$.

又 $\operatorname{Re} \varphi(iv) = \operatorname{Re}(i\varphi(v)) = -\operatorname{Im} \varphi(v) = \sigma(iv)$. Hence $\varphi(v) = \sigma(v) - i\sigma(iv)$. □

- 4 Suppose $m, n \in \mathbb{N}^+$ with $m \leq n$, $\lambda_1, \dots, \lambda_m \in \mathbf{F}$.

Prove that $\exists p \in \mathcal{P}(\mathbf{F})$, $\deg p = n$, the zeros of p are $\lambda_1, \dots, \lambda_m$.

SOLUTION: Let $p(z) = (z - \lambda_1)^{n-(m-1)}(z - \lambda_2) \cdots (z - \lambda_m)$. □

- 5 Suppose $m \in \mathbb{N}$, and z_1, \dots, z_{m+1} are distinct in \mathbf{F} , and $w_1, \dots, w_{m+1} \in \mathbf{F}$.

Prove that $\exists! p \in \mathcal{P}_m(\mathbf{F})$, $p(z_k) = w_k$ for each $k \in \{1, \dots, m+1\}$.

SOLUTION:

Define $T : \mathcal{P}_m(\mathbf{F}) \rightarrow \mathbf{F}^{m+1}$ by $Tq = (q(z_1), \dots, q(z_m), q(z_{m+1}))$. Moreover, T is linear.

We now show that T is surj, so that such p exists; and that T is inje, so that such p is unique.

Inje: $Tq = 0 \iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0 \iff q = 0$, by TIPS.

Surj: $\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m + 1 = \dim \mathbf{F}^{m+1}$ 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1} \Rightarrow T$ is surj. □

OR. Let $p_1 = 1, p_k(z) = \prod_{i=1}^{k-1} (z - z_i) = (z - z_1) \cdots (z - z_{k-1})$ for each $k \in \{2, \dots, m+1\}$.

By (2.C.10), $B_p = (p_1, \dots, p_{m+1})$ is a basis of $\mathcal{P}_m(\mathbf{F})$. Let $B_e = (e_1, \dots, e_{m+1})$ be the std basis of \mathbf{F}^{m+1} .

NOTICE that $Tp_1 = (1, \dots, 1), Tp_k = \left(\prod_{i=1}^{k-1} (z_1 - z_i), \dots, \underbrace{\prod_{i=1}^{k-1} (z_j - z_i)}_{j^{\text{th}} \text{ entry}}, \dots, \prod_{i=1}^{k-1} (z_{m+1} - z_i) \right)$.

And that $\prod_{i=1}^{k-1} (z_j - z_i) = 0 \iff j \leq k - 1$, because z_1, \dots, z_{m+1} are distinct.

$$\text{Thus } \mathcal{M}(T, B_p, B_e) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & A_{2,2} & 0 & \cdots & 0 \\ 1 & A_{3,2} & A_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{m+1,2} & A_{m+1,3} & \cdots & A_{m+1,m+1} \end{pmatrix}.$$

Where $A_{j,k} = \prod_{i=1}^{k-1} (z_j - z_i) \neq 0$ for all $j > k - 1 \geq 1$. The rows of $\mathcal{M}(T)$ is linely inde.

By (4E 3.C.17) 又 $\dim \mathcal{P}_m(\mathbf{F}) = \dim \mathbf{F}^{m+1}$; OR By (3.F.32); T is inv. □

- 2 Suppose $m \in \mathbb{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subsp of $\mathcal{P}(\mathbf{F})$?

SOLUTION: $x^m, x^m + x^{m-1} \in U$ but $\deg[(x^m + x^{m-1}) - (x^m)] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$. □

3 Suppose $m \in \mathbb{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbb{F}) : 2 \mid \deg p\}$ a subsp of $\mathcal{P}(\mathbb{F})$?

SOLUTION: $x^2, x^2 + x \in U$ but $\deg[(x^2 + x) - (x^2)]$ is odd and hence $(x^2 + x) - (x^2) \notin U$. \square

6 Suppose nonzero $p \in \mathcal{P}_m(\mathbb{F})$ has degree m . Prove that

$[P] \ p \text{ has } m \text{ distinct zeros} \iff p \text{ and its derivative } p' \text{ have no zeros in common } [Q].$

SOLUTION:

(a) Suppose p has m distinct zeros. And $\deg p = m$. By [4.14], $\exists! c, \lambda_i \in \mathbb{R}, p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$.

If $m = 0$, then $p = c \neq 0 \Rightarrow p$ has no zeros, and $p' = 0$, we are done.

If $m = 1$, then $p(z) = c(z - \lambda_1)$, and $p' = c$ has no zeros, we are done.

For each $j \in \{1, \dots, m\}$, let $q_j \in \mathcal{P}_{m-1}(\mathbb{F})$ be such that $p(z) = (z - \lambda_j)q_j \Rightarrow q_j(\lambda_j) \neq 0$.

Now $p'(z) = (z - \lambda_j)q_j'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

OR. To prove $[P] \Rightarrow [Q]$, we prove $\neg[Q] \Rightarrow \neg[P]$:

Suppose $p(z) = (z - \lambda)q(z)$, $p'(z) = (z - \lambda)r(z)$.

又 $p'(z) = (z - \lambda)q'(z) + q(z)$. Now $p'(\lambda) = q(\lambda) = 0 \Rightarrow p(z) = (z - \lambda)^2 s(z)$.

Hence p has strictly less than m distinct zeros.

(b) To prove $[Q] \Rightarrow [P]$, we prove $\neg[P] \Rightarrow \neg[Q]$:

Because nonzero $p \in \mathcal{P}_m(\mathbb{F})$, we suppose $\lambda_1, \dots, \lambda_M$ are the distinct zeros of p , where $M < m$.

By Pigeon Hole Principle, $\exists \lambda_k$ such that $p(z) = (z - \lambda_k)^2 q(z)$ for some $q \in \mathcal{P}(\mathbb{F})$.

Hence $p'(z) = 2(z - \lambda_k)q(z) + (z - \lambda_k)^2 q'(z) \Rightarrow p'(\lambda_k) = 0 = p(\lambda_k)$. \square

7 Prove that every $p \in \mathcal{P}(\mathbb{R})$ of odd degree has a zero.

SOLUTION:

Using the notation and proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence λ_1 exists. \square

OR. Using calculus only.

Suppose $p \in \mathcal{P}_m(\mathbb{F})$, $\deg p = m$, m is odd.

Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1} a_m$ by δ

Write $p(x) = x^m \left(\frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \cdots + \frac{a_{m-1}}{x} + a_m \right)$.

Thus $p(x)$ is continuous, and $\lim_{x \rightarrow -\infty} p(x) = -\delta\infty$; $\lim_{x \rightarrow \infty} p(x) = \delta\infty$.

Hence we conclude that p has at least one real zero. \square

9 Suppose $p \in \mathcal{P}(\mathbb{C})$. Define $q : \mathbb{C} \rightarrow \mathbb{C}$ by $q(z) = p(z)\overline{p(\bar{z})}$. Prove that $q \in \mathcal{P}(\mathbb{R})$.

SOLUTION:

NOTICE that by [4.5], $\bar{\bar{z}}^n = \bar{z}^n$.

Suppose $q(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow q(\bar{z}) = a_n \bar{z}^n + \cdots + a_1 \bar{z} + a_0 \Rightarrow \overline{q(\bar{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\bar{z})} = \overline{\overline{p(\bar{z})p(z)}} = \overline{p(\bar{z})\overline{p(z)}} = \overline{p(\bar{z})p(\bar{\bar{z}})} = \overline{q(\bar{z})}$. Hence for each $a_k, \overline{a_k} = a_k \Rightarrow a_k \in \mathbb{R}$. \square

OR. Suppose $p(z) = a_m z^m + \cdots + a_1 z + a_0$. Now $\overline{p(\bar{z})} = \overline{a_m} z^m + \cdots + \overline{a_1} z + \overline{a_0}$.

NOTICE that $q(z) = p(z)\overline{p(\bar{z})} = \sum_{k=0}^2 m \left(\sum_{i+j=k} a_i \overline{a_j} \right) z^k$.

NOTICE that by [4.5], $z - \bar{z} = 2(\text{Im } z) \Rightarrow z = \bar{z} + 2(\text{Im } z)$. So that $z = \bar{z} \iff \text{Im } z = 0 \iff z \in \mathbb{R}$.

Now for each $k \in \{0, \dots, 2m\}$, $\overline{\sum_{i+j=k} a_i \overline{a_j}} = \sum_{i+j=k} \overline{a_i} a_j = \sum_{i+j=k} a_j \overline{a_i} = \sum_{i+j=k} a_i \overline{a_j} \in \mathbb{R}$. \square

8 For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \rightarrow \mathbf{R}$ by $(Tp)(x) = \begin{cases} \frac{p(x) - p(3)}{x - 3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}$

Show that (a) $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$ and that (b) $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$ is linear.

SOLUTION:

$$(a) \text{ For } x \neq 3, T(x^n) = \frac{x^n - 3^n}{x - 3} = \sum_{i=1}^n 3^{i-1} x^{n-i}. \text{ For } x = 3, T(x^n) = 3^{n-1} \cdot n.$$

Note that if $x = 3$, then $\sum_{i=1}^n 3^{i-1} x^{n-i} = \sum_{i=1}^n 3^{n-1} = 3^{n-1} \cdot n$.

Hence $T(x^n) = \sum_{i=1}^n 3^{i-1} x^{n-i} \Rightarrow T(x^n) \in \mathcal{P}(\mathbf{R})$.

(b) Now we show that T is linear: $\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}$,

$$T(p + \lambda q)(x) = \begin{cases} \frac{(p + \lambda q)(x) - (p + \lambda q)(3)}{x - 3}, & \text{if } x \neq 3, \\ (p + \lambda q)'(3), & \text{if } x = 3 \end{cases} = [T(p) + \lambda T(q)](x) \text{ for all } x \in \mathbf{R}. \quad \square$$

OR. (a) Note that $\exists! q \in \mathcal{P}(\mathbf{R}), p(x) - p(3) = (x - 3)q(x) \Rightarrow q(x) = \frac{p(x) - p(3)}{x - 3}$.

$$p'(x) = (p(x) - p(3))' = ((x - 3)q(x))' = q(x) + (x - 3)q'(x).$$

Hence $p'(3) = q(3)$. Now $Tp = q \in \mathcal{P}(\mathbf{R})$.

(b) $\forall p_1, p_2 \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, \exists! q_1, q_2 \in \mathcal{P}(\mathbf{R})$,

$$p_1(x) - p_1(3) = (x - 3)q_1(x) \text{ and } p_2(x) - p_2(3) = (x - 3)q_2(x).$$

By (a), $Tp_1 = q_1, Tp_2 = q_2$. Note that $(p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x - 3)(q_1 + \lambda q_2)(x)$.

Hence by the uniqueness of $q_1 + \lambda q_2$ for $p_1 + \lambda p_2$, we must have $T(p_1 + \lambda p_2) = q_1 + \lambda q_2$. \square

11 Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.

(a) Show that $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) Find a basis of $\mathcal{P}(\mathbf{F})/U$.

SOLUTION: NOTICE that $pq \neq p \circ q$, see (4E 3.A.10).

U is a subsp of $\mathcal{P}(\mathbf{F})$ because $\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, ps_1 + \lambda ps_2 = p(s_1 + \lambda s_2) \in U$.

If $\deg p = 0$, then $U = \mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F})/U = \{0\}$, with the unique basis $()$. Suppose $\deg p \geq 1$.

(a) By [4.8], $\forall s \in \mathcal{P}(\mathbf{F}), \exists! r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}) [\exists! pq \in U], s = (p)q + (r)$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F})$. By the NOTE FOR [3.91] in (3.E), $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p-1}(\mathbf{F})$ are iso.

OR. Define $R : \mathcal{P}(\mathbf{F}) \rightarrow \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $R(s) = r$ for all $s \in \mathcal{P}(\mathbf{F})$. We show that R is linear.

$$\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, \exists! r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q_1, q_2 \in \mathcal{P}(\mathbf{F}), s_1 = (p)q_1 + (r_1); s_2 = (p)q_2 + (r_2).$$

$$\text{又 } \exists! r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}), (s_1 + \lambda s_2) = (p)q + (r) = (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2).$$

Note that $r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}) \Rightarrow r_1 + \lambda r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F})$.

OR Note that $\deg(r_1 + \lambda r_2) \leq \max\{\deg r_1, \deg(\lambda r_2)\} \leq \max\{\deg r_1, \deg r_2\} < \deg p$.

By the uniqueness part of [4.8], $s = s_1 + \lambda s_2; r = r_1 + \lambda r_2$. Thus $R(s_1 + \lambda s_2) = R(s_1) + \lambda R(s_2)$.

Because $Rs = 0 \iff s = pq, \exists! q \in \mathcal{P}(\mathbf{F}) \iff s \in U$. And $\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), Rr = r$.

Now $\text{null } R = U, \text{ range } R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Hence $\tilde{R} : \mathcal{P}(\mathbf{F})/U \rightarrow \mathcal{P}_{\deg p-1}(\mathbf{F})$ is defined by $\tilde{R}(s + U) = Rs$. By [3.91(d)], \tilde{R} is an iso.

(b) For each $k \in \{0, 1, \dots, \deg p - 1\}, \tilde{R}(z^k + U) = R(z^k) = z^k \Rightarrow \tilde{R}^{-1}(z^k) = z^k + U$.

Thus $(1 + U, z + U, \dots, z^{\deg p-1} + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. \square

10 Suppose $m \in \mathbf{N}$, $p \in \mathcal{P}_m(\mathbf{C})$ is such that $p(x_k) \in \mathbf{R}$ for each of distinct $x_0, x_1, \dots, x_m \in \mathbf{R}$. Prove that $p \in \mathcal{P}(\mathbf{R})$.

SOLUTION:

By TIPS and Problem (5), $\exists ! q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_k) = p(x_k)$. Hence $p = q$. \square

OR. Using the Lagrange Interpolating Polynomial.

$$\text{Define } q(x) = \sum_{j=0}^m \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j).$$

又 Each $x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R})$. Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$ for each x_k .

Then $(q-p)$ has $(m+1)$ zeros, while $(q-p) \in \mathcal{P}_m(\mathbf{C})$. By TIPS, $q-p=0 \Rightarrow p=q \in \mathcal{P}(\mathbf{R})$. \square

• (4E 4 13) Suppose nonconst $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p, n = \deg q$. Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \rightarrow \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r, s) = rp + sq$. Prove that T is an iso.

COROLLARY: $\exists ! r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.

SOLUTION:

T is linear because $\forall r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,

$$T((r_1, s_1) + \lambda(r_2, s_2)) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1, s_1) + \lambda T(r_2, s_2).$$

Let $\lambda_1, \dots, \lambda_M$ and μ_1, \dots, μ_N be the distinct zeros of p and q respectively. NOTICE that $M \leq m, N \leq n$.

Note that the contrapositive of [4.13], $M = 0 \Leftrightarrow m = 0 \Rightarrow s = 0 \Leftrightarrow r = 0 \Leftarrow n = 0 \Leftrightarrow N = 0$.

Now suppose $M, N \geq 1$. We show that $s = 0$. Showing $r = 0$ is almost the same.

Write $p(z) = a(z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_M)^{\alpha_M}$. ($\exists ! \alpha_j \geq 1, a \in \mathbf{F}$.) Let $\max\{\alpha_1, \dots, \alpha_M\} = A$.

For each $D \in \{0, 1, \dots, A-1\}$, let $I_{D, \alpha} = \{\gamma_{D,1}, \dots, \gamma_{D,J}\}$ be such that each $\alpha_{\gamma_{D,j}} \geq D+1$.

Note that $I_{A-1, \alpha} \subseteq \cdots \subseteq I_{0, \alpha} = \{1, \dots, M\}$. Because $rp + sq = 0 \Rightarrow (rp + sq)^{(k)} = 0$ for all $k \in \mathbf{N}^+$.

We use induction by D to show that $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$ for each $D \in \{0, \dots, A-1\}$.

NOTICE that $p^{(D)}(\lambda_{\gamma}) = 0$ for each $D \in \{0, \dots, A-1\}$ and each $\lambda_{\gamma} \in I_{D, \alpha}$. (Δ)

(i) $D = 0$. $(rp + sq)(\lambda_{\gamma_{0,j}}) = (sq)(\lambda_{\gamma_{0,j}}) = s(\lambda_{\gamma_{0,j}}) = 0$.

$$D = 1. (rp + sq)'(\lambda_{\gamma_{1,j}}) = (r'p + rp')(\lambda_{\gamma_{1,j}}) + (s'q + sq')(\lambda_{\gamma_{1,j}}) = (s'q)(\lambda_{\gamma_{1,j}}) = s'(\lambda_{\gamma_{1,j}}) = 0.$$

(ii) $2 \leq D \leq A-1$. Assume that $s^{(d)}(\lambda_{\gamma_{d,j}}) = 0$ for each $d \in \{1, \dots, D-1\}$ and each $\lambda_{\gamma_{d,j}} \in I_{d, \alpha}$.

(Because $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}$.) (Δ)

$$\begin{aligned} \text{Now } [rp + sq]^{(D)}(\lambda_{\gamma_{D,j}}) &= [C_D^D r^{(D)} p^{(0)} + \cdots + C_D^d r^{(d)} p^{(D-d)} + \cdots + C_D^0 r^{(0)} p^{(D)}](\lambda_{\gamma_{D,j}}) \\ &\quad + [C_D^D s^{(D)} q^{(0)} + \cdots + C_D^d s^{(d)} q^{(D-d)} + \cdots + C_D^0 s^{(0)} q^{(D)}](\lambda_{\gamma_{D,j}}) \\ &= [C_D^D s^{(D)} q^{(0)}](\lambda_{\gamma_{D,j}}). \text{ Where each } \lambda_{\gamma_{D,j}} \in I_{D, \alpha} \subseteq I_{D-1, \alpha}. \end{aligned}$$

Hence $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$. The assumption holds for all $D \in \{0, \dots, A-1\}$.

NOTICE that $\forall k = \{0, \dots, A-2\}$, $s^{(k)}$ and $s^{(k+1)}$ have zeros $\{\lambda_{\gamma_{k+1,1}}, \dots, \lambda_{\gamma_{k+1,J}}\}$ in common.

Now $\forall D \in \{1, A-1\}$, $s = s^{(0)}, \dots, s^{(D)}$ have zeros $\{\lambda_{\gamma_{D,1}}, \dots, \lambda_{\gamma_{D,J}}\}$ in common.

Thus $\forall D \in \{0, A-1\}$, $s(z)$ is divisible by $(z - \lambda_{\gamma_{D,1}})^{\alpha_{\gamma_{D,1}}} \cdots (z - \lambda_{\gamma_{D,J}})^{\alpha_{\gamma_{D,J}}}$.

Hence we write $s(z) = \left((z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_M)^{\alpha_M} \right) s_0(z)$, while $\deg s \leq m-1 < m = \alpha_1 + \cdots + \alpha_M$.

Thus by TIPS, $s = 0$. Following the same pattern, we conclude that $r = 0$.

Hence T is inje. And $\dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) = \dim \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$ is surj. Thus T is an iso. \square

COMMENT: We now prove the statement that marked by (Δ) above.

L1: Prove that $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \dots + C_k^j p^{(j)} q^{(k-j)} + \dots + C_k^0 p^{(0)} q^{(k)}$.

SOLUTION:

We use induction by $k \in \mathbf{N}^+$.

(i) $k = 1$. $(pq)^{(1)} = pq = C_1^1 p^{(1)} q^{(0)} + C_1^0 p^{(0)} q^{(1)}$.

(ii) $k \geq 2$. Assume that for $(pq)^{(k-1)} = C_{k-1}^{k-1} p^{(k-1)} q^{(0)} + \dots + C_{k-1}^j p^{(j)} q^{(k-1-j)} + \dots + C_{k-1}^0 p^{(0)} q^{(k-1)}$.

$$\begin{aligned} \text{Now } (pq)^{(k)} &= ((pq)^{(k-1)})' = \left(\sum_{j=0}^{k-1} C_{k-1}^j p^{(j)} q^{(k-1-j)} \right)' = \sum_{j=0}^{k-1} \left[C_{k-1}^j \left(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)} \right) \right] \\ &= \left[C_{k-1}^0 \left(\underbrace{p^{(1)} q^{(k-1)}}_{\text{}} + \boxed{p^{(0)} q^{(k)}} \right) \right] + \left[C_{k-1}^1 \left(p^{(2)} q^{(k-2)} + \underbrace{p^{(1)} q^{(k-1)}}_{\text{}} \right) \right] \\ &\quad + \dots + \left[C_{k-1}^{j-2} \left(\underbrace{p^{(j-1)} q^{(k-j+1)}}_{\text{}} + p^{(j-2)} q^{(k-j+2)} \right) \right] + \left[C_{k-1}^{j-1} \left(\underbrace{p^{(j)} q^{(k-j)}}_{\text{}} + \underbrace{p^{(j-1)} q^{(k-j+1)}}_{\text{}} \right) \right] \\ &\quad + \left[C_{k-1}^j \left(\underbrace{p^{(j+1)} q^{(k-j-1)}}_{\text{}} + \underbrace{p^{(j)} q^{(k-j)}}_{\text{}} \right) \right] + \left[C_{k-1}^{j+1} \left(p^{(j+2)} q^{(k-j-2)} + \underbrace{p^{(j+1)} q^{(k-j-1)}}_{\text{}} \right) \right] \\ &\quad + \dots + \left[C_{k-1}^{k-2} \left(\underbrace{p^{(k-1)} q^{(1)}}_{\text{}} + p^{(k-2)} q^{(2)} \right) \right] + \left[C_{k-1}^{k-1} \left(\boxed{p^{(k)} q^{(0)}} + \underbrace{p^{(k-1)} q^{(1)}}_{\text{}} \right) \right]. \end{aligned}$$

$$\text{Hence } (pq)^{(k)} = C_k^0 p^{(0)} q^{(k)} + \dots + \left[C_{k-1}^j + C_{k-1}^{j-1} \right] p^{(j)} q^{(k-j)} + \dots + C_k^k p^{(k)} q^{(0)}. \quad \square$$

L2: Suppose $p(z) = (z - \lambda)^\alpha q(z)$ and $\alpha \in \mathbf{N}^+$. Prove that $p^{(\alpha-1)}(\lambda) = 0$.

SOLUTION:

Suppose $p \in \mathcal{P}(\mathbf{F})$. Write $p(z) = (z - \lambda)^A q(z)$, where $A \in \mathbf{N}^+, q(\lambda) \neq 0$.

We use induction to show that for all $\alpha \in \{1, \dots, A\}, p^{(\alpha-1)}(\lambda) = 0$.

(i) $\alpha = 1$. $p^{(0)}(\lambda) = 0$.

(ii) $2 \leq \alpha \leq A$. Assume that $p^{(a-2)}(\lambda) = 0$ for all $a \in \{1, \dots, \alpha\}$.

NOTICE that $p(z) = (z - \lambda)^{\alpha-1} q_{\alpha-1}(z) = (z - \lambda)^\alpha q_\alpha(z)$, where $q_\alpha(z) = (z - \lambda) q_{\alpha-1}(z)$.

$$\begin{aligned} \text{Because } p^{(\alpha-1)}(z) &= \left[C_{\alpha-1}^{\alpha-1} (z - \lambda)^0 q_{\alpha-1}(z) + \dots + C_{\alpha-1}^k (z - \lambda)^{\alpha-1-k} q_{\alpha-1-k}(z) \right. \\ &\quad \left. + \dots + C_{\alpha-1}^0 (z - \lambda)^{\alpha-1} q_{\alpha-1}^{(\alpha-1)}(z) \right]. \text{ Now } p^{(\alpha-1)}(\lambda) = C_{\alpha-1}^{\alpha-1} q_{\alpha-1}(\lambda) = 0. \quad \square \end{aligned}$$

ENDED

5.A [1](#) [2](#) [3](#) [4](#) [5](#) [6](#) [7](#) [8](#) [9](#) [10](#) [11](#) [12](#) [13](#) [14](#) [15](#) [16](#) [17](#) [18](#) [19](#) [20](#) [22](#) [23](#) [24](#) [25](#) [26](#) [27](#) [28](#)
[29](#) [30](#) [31](#) [32](#) [33](#) [34](#) [35](#) [36](#) | 2E: Ch5.20 | 4E 8, 11, 15, 16, 17, 36, 37, 38, 39

• NOTE FOR [5.6]:

More generally, suppose we do not know whether V is finite-dim. We show that $(a) \iff (b)$.

Suppose (a) λ is an eigval of T with an eigvec v . Then $(T - \lambda I)v = 0$.

Hence we get (b), $(T - \lambda I)$ is not inje. And then (d), $(T - \lambda I)$ is not inv.

But $(d) \Rightarrow (b)$ fails, because S is not inv $\iff S$ is not inje OR S is not surj.

• TIPS: For $T_1, \dots, T_m \in \mathcal{L}(V)$:

(a) Suppose T_1, \dots, T_m are all inje. Then $(T_1 \circ \dots \circ T_m)$ is inje.

(b) Suppose $(T_1 \circ \dots \circ T_m)$ is not inje. Then at least one of T_1, \dots, T_m is not inje.

(c) At least one of T_1, \dots, T_m is not inje $\nRightarrow (T_1 \circ \dots \circ T_m)$ is not inje.

EXAMPLE: On infinite-dim only. Let $V = \mathbf{F}^\infty$.

Let S be the backward shift (surj but not inje)
 Let T be the forward shift (inje but not surj) $\Bigg\} \Rightarrow \text{Then } ST = I.$

\square

31 Suppose V is finite-dim, and $v_1, \dots, v_m \in V$.

Prove that (v_1, \dots, v_m) is linely inde

$\iff \exists T \in \mathcal{L}(V), v_1, \dots, v_m$ are eigvecs of T correspd to distinct eigvals.

SOLUTION:

Suppose (v_1, \dots, v_m) is linely inde. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.

Define $T \in \mathcal{L}(V)$ by $Tv_k = k \cdot v_k$ for each $k \in \{1, \dots, m, \dots, n\}$. Conversely by [5.10]. \square

1 Suppose $T \in \mathcal{L}(V)$ and U is a subsp of V .

(a) If $U \subseteq \text{null } T$, then U is invar under T . $\forall u \in U \subseteq \text{null } T, Tu = 0 \in U$. \square

(b) If $\text{range } T \subseteq U$, then U is invar under T . $\forall u \in U, Tu \in \text{range } T \subseteq U$. \square

• Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.

(a) Prove that $\text{null } (T - \lambda I)$ is invar under S for any $\lambda \in \mathbb{F}$.

(b) Prove that $\text{range } (T - \lambda I)$ is invar under S for any $\lambda \in \mathbb{F}$.

SOLUTION:

Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.

(a) $(T - \lambda I)(v) = 0 \Rightarrow (T - \lambda I)(Sv) = (S(T - \lambda I))(v) = 0$.

(b) $(T - \lambda I)(u) = v \in \text{range } (T - \lambda I) \Rightarrow Sv = (S(T - \lambda I))(u) = (T - \lambda I)(Su) \in \text{range } (T - \lambda I)$. \square

• Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.

2 Show that $W = \text{null } T$ is invar under S . $\forall u \in W, Tu = 0 \Rightarrow STu = 0 = TSu \Rightarrow Su \in W$. \square

3 Show that $U = \text{range } T$ is invar under S . $\forall w \in U, \exists v \in V, Tv = w, TSv = STv = Sw \in U$. \square

15 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is inv.

(a) Prove that T and $S^{-1}TS$ have the same eigvals.

(b) What is the relationship between the eigvecs of T and the eigvecs of $S^{-1}TS$?

SOLUTION:

(a) λ is an eigval of T with an eigvec $v \Rightarrow S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v$.

λ is an eigval of $S^{-1}TS$ with an eigvec $v \Rightarrow S(S^{-1}TS)v = TSv = \lambda Sv$.

OR. Note that $S(S^{-1}TS)S^{-1} = T$. Hence every eigval of $S^{-1}TS$ is an eigval of $S(S^{-1}TS)S^{-1} = T$.

OR. $Tv = \lambda v \iff (TS)(u) = \lambda Su \iff (S^{-1}TS)(u) = \lambda u$. Where $v = Su$.

$(S^{-1}TS)(u) = \lambda u \iff (S^{-1}T)(v) = \lambda S^{-1}v \iff Tv = \lambda v$. Where $u = S^{-1}v$.

(b) Because λ is an eigval of $T \iff \lambda$ is an eigval of $S^{-1}TS$.

(See [5.36].) Now $E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\}; E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}$. \square

21 Suppose $T \in \mathcal{L}(V)$ is inv.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigval of $T \iff \frac{1}{\lambda}$ is an eigval of T^{-1} .

(b) Prove that T and T^{-1} have the same eigvecs.

SOLUTION:

(a) $Tv = \lambda v \iff v = \lambda T^{-1}v \iff \lambda^{-1}v = T^{-1}v$. Where $v \neq 0$.

(b) Note that T is inv. Hence 0 is not an eigval of T and T^{-1} .

Now by (a), v is an eigvec of $T \iff v$ is an eigvec of T^{-1} . \square

23 Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigvals.

SOLUTION:

Suppose λ is an eigval of ST with an eigvec v . Then $T(STv) = \lambda Tv = TS(Tv)$.

If $Tv = 0$ (while $v \neq 0$), then T is not inje $\Rightarrow (TS - 0I)$ and $(ST - 0I)$ are not inje.

Thus $\lambda = 0$ is an eigval of ST and TS with the same eigvec v .

Otherwise, $Tv \neq 0$, then λ is an eigval of TS . Reversing the roles of T and S . □

• (2E Ch5.20)

Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigvals and $S \in \mathcal{L}(V)$ has the same eigvecs (but might not with the same eigvals). Prove that $ST = TS$.

SOLUTION:

Let $n = \dim V$. For each $j \in \{1, \dots, n\}$, let v_j be an eigvec with eigval λ_j of T and α_j of S .

Then (v_1, \dots, v_n) is a basis of V . Because $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$ for each j . Hence $ST = TS$. □

• (4E 5.A.37) Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(S) = TS$ for each $S \in \mathcal{L}(V)$.

Prove that the set of eigvals of T equals the set of eigvals of \mathcal{A} .

SOLUTION:

(a) Suppose v_1, \dots, v_m are all linely inde eigvecs of T

with correspd eigvals $\lambda_1, \dots, \lambda_m$ respectively (possibly with repetitions).

Extend to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.

Then for each $k \in \{1, \dots, m\}$, $\text{span}(v_k) \subseteq \text{null}(T - \lambda_k I)$.

Define $S_k \in \mathcal{L}(V)$ by $S_k(v_j) = v_k$ for each $j \in \{1, \dots, n\}$,

so that $\text{range } S_k = \text{span}(v_k)$ for each $k \in \{1, \dots, m\}$, then $\mathcal{A}(S_k) = TS_k = \lambda_k S_k$.

Thus the eigvals of T are eigvals of \mathcal{A} .

(b) Suppose $\lambda_1, \dots, \lambda_m$ are all eigvals of \mathcal{A} with eigvecs S_1, \dots, S_m respectively.

Then for each $k \in \{1, \dots, m\}$, $\exists v \in V, 0 \neq u = S_k(v) \in V \Rightarrow Tu = (TS_k)v = (\lambda_k S_k)v = \lambda_k u$.

Thus the eigvals of \mathcal{A} are eigvals of T . □

OR.

(a) Suppose λ is an eigval of T with an eigvec v .

Let $v_1 = v$ and extend to a basis (v_1, \dots, v_m) of V .

Define $S \in \mathcal{L}(V)$ by $Sv_1 = v_1, Sv_k = 0$ for $k \geq 2$.

Then $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$.

Hence $(T - \lambda I)S = 0 \Rightarrow TS = \lambda S$ while $S \neq 0$. Thus λ is also an eigval of \mathcal{A} .

(b) Suppose λ is an eigval of \mathcal{A} with an eigvec S . Then $(T - \lambda I)S = 0$ while $S \neq 0$.

Hence $(T - \lambda I)$ is not inje. Thus λ is also an eigval of T . □

COMMENT: Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(S) = ST, \forall S \in \mathcal{L}(V)$. Then the eigvals of \mathcal{B} are not the eigvals of T .

• Suppose $T \in \mathcal{L}(V)$ and V_1, \dots, V_m are invar subsp of V under T .

4 $\forall v_i \in V_i, Tv_i \in V_i \Rightarrow \forall v = v_1 + \dots + v_m \in V_1 + \dots + V_m, Tv = Tv_1 + \dots + Tv_m \in V_1 + \dots + V_m$. □

5 $\forall v \in \bigcap_{i=1}^m V_i, Tv \in V_i, \forall i \in \{1, \dots, m\} \Rightarrow Tv \in \bigcap_{i=1}^m V_i$. Thus $\bigcap_{i=1}^m V_i$ is invar under T . □

6 Suppose U is an invar subsp of V under each $T \in \mathcal{L}(V)$. Show that $U = \{0\}$ or $U = V$.

SOLUTION:

If $V = \{0\}$. Then we are done. Suppose $V \neq \{0\}$. We show the contrapositive:

Suppose $U \neq \{0\}$ and $U \neq V$. Prove that $\exists T \in \mathcal{L}(V)$ such that U is not invariant under T .

Let W be such that $V = U \oplus W$. Define $T \in \mathcal{L}(V)$ by $T(u + w) = w$. □

• Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.

(a) [OR (9.11)] $\lambda \in \mathbf{R}$. Prove that λ is an eigval of $T \iff \lambda$ is an eigval of $T_{\mathbf{C}}$.

(b) [OR 16 OR [9.16]] $\lambda \in \mathbf{C}$. Prove that λ is an eigval of $T_{\mathbf{C}} \iff \bar{\lambda}$ is an eigval of $T_{\mathbf{C}}$.

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec v .

Then $Tv = \lambda v \implies T_{\mathbf{C}}(v + i0) = Tv + iT0 = \lambda v$. Thus λ is an eigval of $T_{\mathbf{C}}$.

Suppose λ is an eigval of $T_{\mathbf{C}}$ with an eigvec $v + iu$.

Then $T_{\mathbf{C}}(v + iu) = \lambda v + i\lambda u \implies Tv = \lambda v, Tu = \lambda u$. Thus λ is an eigval of T .

(Note that $v + iu$ is nonzero \iff at least one of v, u is nonzero).

(b) Suppose λ is an eigval of $T_{\mathbf{C}}$ with an eigvec $v + iu$. Then $T_{\mathbf{C}}(v + iu) = Tv + iTu = \lambda(v + iu)$.

Note that $\overline{T_{\mathbf{C}}(v + iu)} = \overline{Tv + iTu} = \overline{Tv} - i\overline{Tu} = T_{\mathbf{C}}(v - iu) = T_{\mathbf{C}}(\overline{v + iu})$.

And that $\overline{\lambda(v + iu)} = \bar{\lambda}v - i\bar{\lambda}u = \bar{\lambda}(v - iu) = \bar{\lambda}(\overline{v + iu})$.

Hence $\bar{\lambda}$ is an eigval of $T_{\mathbf{C}}$. To prove the other direction, notice that $\overline{\bar{\lambda}} = \lambda$. □

OR. Suppose $\lambda = a + ib$ is an eigval of $T_{\mathbf{C}}$ with an eigvec $v + iu$.

Because $T_{\mathbf{C}}(v + iu) = \lambda(v + iu) = (av - bu) + i(au + bv) = Tv + iTu \implies Tv = av - bu, Tu = au + bv$.

Now $T_{\mathbf{C}}(\overline{v + iu}) = Tv - iTu = (av - bu) - i(au + bv) = (a - ib)(v - iu) = \bar{\lambda}(\overline{v + iu})$. Similarly □

• (4E 5.A.15) Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.

Show that λ is an eigval of $T \iff \lambda$ is an eigval of the dual operator $T' \in \mathcal{L}(V')$.

SOLUTION:

Using [3.69], [3.108, 110], [3.101] and (3.F.12).

λ is an eigval of $T \iff (T - \lambda I_V)$ is not inj

$\iff (T - \lambda I_V)' = T' - \lambda I_{V'}$ is not surj

$\iff T' - \lambda I_{V'}$ is not inje $\iff \lambda$ is an eigval of T' . □

OR. Suppose λ is an eigval of T with an eigvec v .

Let $V = \text{span}(v) \oplus U$. **TODO** Define $\psi \in V'$ by $\psi(cv + u) = c$.

Now $[T'(\psi)](cv + u) = \psi(cv + Tu) = \lambda cv = \lambda\psi(cv + u)$.

Suppose λ is an eigval T' with an eigvec ψ . Then $T'(\psi) = \psi \circ T = \lambda\psi$.

Note that $\psi(Tv) = \lambda\psi(v)$ and $\exists v \in V \setminus \text{null } T$.

Thus $\lambda = \frac{\psi(Tv)}{\psi(v)} \implies Tv = \frac{\psi(Tv)}{\psi(v)}v = \lambda v$. Hence λ is an eigval of T . □

• **TIPS:** Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is the counterclockwise rotation by the angle $\theta \in \mathbb{R}$.

Define $\mathcal{C} \in \mathcal{L}(\mathbb{R}^2, \mathbb{C})$ by $\mathcal{C}(a, b) = a + ib = r(\cos \alpha + i \sin \alpha) \Rightarrow a = r \cos \alpha, b = r \sin \alpha$, where $r = a^2 + b^2$.

Then $(\cos \theta + i \sin \theta)(a + ib) = r(\cos(\alpha + \theta) + i \sin(\alpha + \theta)) = \mathcal{C}^{-1}T(a, b)$.

Hence $T(a, b) = (a \cos \theta - b \sin \theta, a \sin \theta + b \cos \theta)$. Now $\mathcal{M}(T) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$.

EXAMPLE: OR **7** Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find all eigvals of T .

NOTICE that $\mathcal{M}(T) = \begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$. By [5.8](a), we conclude that T has no eigvals.

OR. Suppose λ is an eigval with an eigvec (x, y) . Then $(\lambda x, \lambda y) = (-3y, x) \Rightarrow -3y = \lambda^2 y \Rightarrow \lambda^2 = -3$.

[Ignoring the possibility of $y = 0$, because $x = 0 \iff y = 0$.] □

8 Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(w, z) = (z, w)$. Find all eigvals and eigvecs.

SOLUTION:

Suppose λ is an eigval with an eigvec (w, z) . Then $z = \lambda w$ and $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, ignoring the possibility of $z = 0$ ($z = 0 \iff w = 0$).

Hence $\lambda_1 = -1$ and $\lambda_2 = 1$ are all eigvals of T . And $T(z, z) = (z, z), T(z, -z) = (-z, z)$.

又 $\dim \mathbb{F}^2 = 2$. Thus the set of all eigvecs is $\{(z, z), (z, -z) : z \neq 0\}$. □

• (5.B.4 OR 4E 3.B.27) Suppose λ is an eigval of $P \in \mathcal{L}(V), P^2 = P$. Prove that $\lambda = 0$ or $\lambda = 1$.

SOLUTION: Suppose λ is an eigval with an eigvec v . Then $P(Pv) = Pv \Rightarrow \lambda^2 v = \lambda v$. Thus $\lambda = 1$ or 0 . □

22 Suppose $T \in \mathcal{L}(V)$ and \exists nonzero vecs u, w in V such that $Tu = 3w, Tw = 3u$.

Prove that 3 or -3 is an eigval of T .

SOLUTION:

$T(u + w) = 3(u + w), T(u - w) = 3(w - u) = -3(u - w)$. Note that $u - w$ or $u + w$ is nonzero.

OR. $T(Tu) = 9u \Rightarrow T^2 - 9 = (T - 3I)(T + 3I)$ is not injective $\Rightarrow 3$ or -3 is an eigval. □

9 Define $T \in \mathcal{L}(\mathbb{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigvals and eigvecs.

SOLUTION:

Suppose λ is an eigval with an eigvec (z_1, z_2, z_3) .

Then $(2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$. We discuss in two cases:

For $\lambda = 0$, $z_2 = z_3 = 0$ and z_1 can be arbitrary ($z_1 \neq 0$).

For $\lambda \neq 0$, $z_2 = 0 = z_1$, and z_3 can be arbitrary ($z_3 \neq 0$), then $\lambda = 5$.

The set of all eigvecs is $\{(0, 0, w), (w, 0, 0) : w \neq 0\}$. □

10 Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$

(a) Find all eigvals and eigvecs; (b) Find all invar subsp of V under T .

SOLUTION:

(a) Suppose $x = (x_1, x_2, x_3, \dots, x_n)$ is an eigvec with an eigval λ .

Then $Tx = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$. Hence $1, \dots, n$ are eigvals.

And $\{(0, \dots, 0, x_\lambda, 0, \dots, 0) \in \mathbb{F}^n : x_\lambda \neq 0, \lambda = 1, \dots, n\}$ is the set of all eigvecs.

(b) Let $V_\lambda = \{(0, \dots, 0, x_\lambda, 0, \dots, 0) \in \mathbb{F}^n : x_\lambda \neq 0\}$. Then V_1, \dots, V_n are invar under T .

Hence by Problem (4), every sum of V_1, \dots, V_n is a invar subsp of V under T . □

11 Define $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigvals and eigvecs.

SOLUTION:

Note that $\forall p \in \mathcal{P}(\mathbf{R}) \setminus \{0\}, \deg p' < \deg p$. And $\deg 0 = -\infty$. Suppose λ is an eigval with an eigvec p . Assume that $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p = p'$. Contradicts. Thus $\lambda = 0$.
Therefore $\deg \lambda p = -\infty = \deg p' \Rightarrow p \in \mathcal{P}_0(\mathbf{R})$. Hence the eigvecs are all nonzero consts. \square

12 Define $T \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigvals and eigvecs.

SOLUTION:

Suppose λ is an eigval of T with an eigvec p , then $(Tp)(x) = xp'(x) = \lambda p(x)$.
Let $p = a_0 + a_1x + \dots + a_nx^n$. Then $xp'(x) = a_1x + 2a_2x^2 + \dots + na_nx^n = \lambda a_0 + \lambda a_1x + \lambda a_2x^2 + \dots + \lambda a_nx^n$.
Thus $0, 1, \dots, n$ are eigvals. And the set of all eigvecs is $\{cx^\lambda : c \neq 0, \lambda = 0, 1, \dots, n\}$. \square

30 Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigvals. Prove that $\exists x, Tx - 9x = (-4, 5, \sqrt{7})$.

SOLUTION: T has $\dim \mathbf{R}^3$ eigvals not including 9 $\Rightarrow (T - 9I)$ is surj. $x = (T - 9I)^{-1}(-4, 5, \sqrt{7})$. \square

14 Suppose $V = U \oplus W$, where U and W are nonzero subsp of V .

Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.

Find all eigvals and eigvecs of P .

SOLUTION:

Suppose λ is an eigval of P with an eigvec $(u + w)$.
Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$.
OR. Note that $P|_{\text{range } P} = I|_{\text{range } P} \iff P^2 = P$. By (4E 5.A.8), 1 and 0 are the eigvals.
By [1.44], $(\lambda - 1)u = \lambda w = 0$, hence $\lambda = 0 \iff u = 0$, and $\lambda = 1 \iff w = 0$.
Thus $Pu = u, Pw = 0$. Hence the eigvals are 0 and 1, the set of all eigvecs in P is $U \cup W$. \square

13 Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.

Prove that $\exists \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}$ and $(T - \alpha I)$ is inv.

SOLUTION:

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \frac{1}{1000 + k}$ for each $k = 1, \dots, \dim V + 1$.

Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigvals.

Hence $\exists k = 1, \dots, \dim V + 1$ such that α_k is not an eigval of T and therefore $(T - \alpha_k I)$ is inv. \square

• (4E 5.A.11) Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.

Prove that $\exists \delta > 0$ such that $(T - \alpha I)$ is inv for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.

SOLUTION:

If T has no eigvals, then $(T - \alpha I)$ is inje for all $\alpha \in \mathbf{F}$ and we are done.

Let $\delta > 0$ be such that, for each eigval $\lambda_k, \lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.

So that for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta, (T - \alpha I)$ is not inje. \square

17 Give an example of an operator on \mathbf{R}^4 that has no (real) eigvals.

SOLUTION: Where (e_1, e_2, e_3, e_4) is the standard basis of \mathbf{R}^4 .

Define $T \in \mathcal{L}(\mathbb{R}^4)$ by $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$.

Suppose λ is an eigval of T with an eigvec (x, y, z, w) .

$$\text{Then } T(x, y, z, w) = \lambda(x, y, z, w) \Rightarrow \begin{cases} (1 - \lambda)x + y + z + w = 0 \\ -x + (1 - \lambda)y - z - w = 0 \\ 3x + 8y + (11 - \lambda)z + 5w = 0 \\ 3x - 8y - 11z + (5 - \lambda)w = 0 \end{cases}$$

This linear equation has no solutions.

(You can type it on <https://zh.numberempire.com/equationsolver.php> to check.)

OR. Define $T \in \mathcal{L}(\mathbb{R}^4)$ by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Suppose λ is an eigval of T with an eigvec (x, y, z, w) .

$$\text{Then } T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \Rightarrow \begin{cases} -y = \lambda x \\ x = \lambda y \\ -w = \lambda z \\ z = \lambda w \end{cases} \Rightarrow \begin{cases} -xy = \lambda^2 xy \\ -zw = \lambda^2 zw \end{cases}$$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Otherwise, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, contradicts.

Similarly, $y = z = w = 0$. Then we fail. Thus T has no eigvals. □

- (4E 5.A.16) Suppose (v_1, \dots, v_n) is a basis of V and $T \in \mathcal{L}(V)$, $\mathcal{M}(T, (v_1, \dots, v_n)) = A$. Prove that if λ is an eigval of T , then $|\lambda| \leq n \max\{|A_{j,k}| : 1 \leq j, k \leq n\}$.

SOLUTION:

First we show that $|\lambda| = n \max\{|A_{j,k}| : 1 \leq j, k \leq n\}$ for some cases.

Consider $A = \begin{pmatrix} k & \dots & k \\ \vdots & \ddots & \vdots \\ k & \dots & k \end{pmatrix}$. Then nk is an eigval of T with an eigvec $v_1 + \dots + v_n$.

Now we show that if $|\lambda| \neq n \max\{|A_{j,k}| : 1 \leq j, k \leq n\}$, then $|\lambda| < n \max\{|A_{j,k}| : 1 \leq j, k \leq n\}$.

18 Define the forward shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ by $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$.

Show that T has no eigvals.

SOLUTION:

Suppose λ is an eigval of T with an eigvec (z_1, z_2, \dots) .

Then $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$. Thus $\lambda z_1 = 0, \lambda z_k = z_{k-1}$.

If $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \dots = z_k \Rightarrow (z_1, z_2, \dots) = 0 \Rightarrow 0$ is not an eigval.

If $\lambda \neq 0$, then $\lambda z_1 = 0 \Rightarrow z_1 = \dots = z_k = 0 \Rightarrow \lambda$ is not an eigval. Now no $\lambda \in \mathbb{F}$ is an eigval. □

19 Suppose $n \in \mathbb{N}^+$. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n).$$

In other words, the entries of $\mathcal{M}(T)$ with resp to the standard basis are all 1's.

Find all eigvals and eigvecs of T .

SOLUTION:

Suppose λ is an eigval of T with an eigvec (x_1, \dots, x_n) .

Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$. Thus $\lambda x_1 = \dots = \lambda x_n = x_1 + \dots + x_n$.

For $\lambda = 0, x_1 + \dots + x_n = 0$.
 For $\lambda \neq 0, x_1 = \dots = x_n$ and then $\lambda x_k = nx_k$ for each k .
 And the set of all eigvecs of T is $\{(x_1, \dots, x_n) \in \mathbf{F}^n : x_1 + \dots + x_n = 0 \vee x_1 = \dots = x_n\}$. \square

20 Define the backward shift operator $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.

(a) Show that every element of \mathbf{F} is an eigval of S ; (b) Find all eigvecs of S .

SOLUTION:

Suppose λ is an eigval of S with an eigvec (z_1, z_2, \dots) .

Then $S(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$. Thus for each $k \in \mathbf{N}^+, \lambda z_k = z_{k+1}$.

If $\lambda = 0$, then $\lambda z_1 = z_2 = \dots = z_k = 0$ for all k , while z_1 can be nonzero. Thus 0 is an eigval.

If $\lambda \neq 0$, then $\lambda^k z_1 = \lambda^{k-1} z_2 = \dots = \lambda z_k = z_{k+1}$, let $z_1 \neq 0 \implies (1, \lambda, \lambda^2, \dots, \lambda^k, \dots)$ is an eigvec.

Now each $\lambda \in \mathbf{F}$ is an eigval of T , with the corresp eigvecs in $\text{span}((1, \lambda, \lambda^2, \dots, \lambda^k, \dots))$. \square

24 Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = Ax$,
 where elements of \mathbf{F}^n are thought of as n -by-1 col vecs.

(a) Suppose the sum of the entries in each row of A equals 1.

Prove that 1 is an eigval of T .

(b) Suppose the sum of the entries in each col of A equals 1.

Prove that 1 is an eigval of T .

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

$$\text{Then } Tx = Ax = \begin{pmatrix} \sum_{c=1}^n A_{1,c} x_c \\ \vdots \\ \sum_{c=1}^n A_{n,c} x_c \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \text{ While } \sum_{r=1}^n A_{R,c} = 1 \text{ for each } R = 1, \dots, n.$$

Thus if we let $x_1 = \dots = x_n$, then $\lambda = 1$, and hence is an eigval of T .

(b) Suppose λ is an eigval of T with an eigvec $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

$$\text{Then } Tx = Ax = \begin{pmatrix} \sum_{r=1}^n A_{1,r} x_r \\ \vdots \\ \sum_{r=1}^n A_{n,r} x_r \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \text{ While } \sum_{r=1}^n A_{r,C} = 1 \text{ for each } C = 1, \dots, n.$$

$$\text{Thus } \sum_{r=1}^n (Ax)_{r,c} = \sum_{r=1}^n (Ax)_{r,1}$$

$$= \sum_{c=1}^n (A_{1,c} + \dots + A_{n,c}) x_c = \sum_{c=1}^n x_c = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Hence $\lambda = 1$, for all x such that $\sum_{c=1}^n x_{c,1} \neq 0$. \square

OR. Prove that $(T - I)$ is not inv, so that we can conclude $\lambda = 1$ is an eigval.

$$\text{Because } (T - I)x = (A - \mathcal{M}(I))x = \begin{pmatrix} \sum_{r=1}^n A_{1,r} x_r - x_1 \\ \vdots \\ \sum_{r=1}^n A_{n,r} x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

$$\text{Then } y_1 + \cdots + y_n = \sum_{r=1}^n \sum_{c=1}^n (A_{r,c}x_c - x_r) = \sum_{c=1}^n x_c \sum_{r=1}^n A_{r,c} - \sum_{r=1}^n x_r = 0.$$

Thus $\text{range}(T - I) \subseteq \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbf{F}^n : y_1 + \cdots + y_n = 0 \right\}$. Hence $(T - I)$ is not surj. \square

- Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = xA$, where elements of \mathbf{F}^n are thought of as 1-by- n row vecs.
 - (a) Suppose the sum of the entries in each col of A equals 1. Prove that 1 is an eigval of T .
 - (b) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigval of T .

SOLUTION:

- (a) Suppose λ is an eigval of T with an eigvec $x = (x_1 \ \cdots \ x_n)$.

Then $Tx = xA = \left(\sum_{r=1}^n x_r A_{r,1} \ \cdots \ \sum_{r=1}^n x_r A_{r,n} \right) = \lambda (x_1 \ \cdots \ x_n)$. While $\sum_{r=1}^n A_{r,C} = 1$ for each $C = 1, \dots, n$. Thus if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, hence 1 is an eigval of T .

- (b) Suppose λ is an eigval of T with an eigvec $x = (x_1 \ \cdots \ x_n)$.

Then $Tx = xA = \left(\sum_{c=1}^n x_c A_{c,1} \ \cdots \ \sum_{c=1}^n x_c A_{c,n} \right) = \lambda (x_1 \ \cdots \ x_n)$. While $\sum_{c=1}^n A_{R,c} = 1$ for each $R = 1, \dots, n$.

$$\text{Thus } \sum_{c=1}^n (xA)_{.,c} = \sum_{c=1}^n (xA)_{1,c} = \sum_{c=1}^n (A_{c,1} + \cdots + A_{c,n})x_c = \sum_{c=1}^n x_c = \lambda(x_1 + \cdots + x_n).$$

Hence $\lambda = 1$, for all x such that $\sum_{r=1}^n x_{1,r} \neq 0$. \square

OR. Prove that $(T - I)$ is not inv, so that we can conclude $\lambda = 1$ is an eigval.

$$\text{Because } (T - I)x = x(A - \mathcal{M}(I)) = \left(\sum_{c=1}^n x_c A_{c,1} - x_1 \ \cdots \ \sum_{c=1}^n x_c A_{c,n} - x_n \right) = (y_1 \ \cdots \ y_n).$$

$$\text{Then } y_1 + \cdots + y_n = \sum_{c=1}^n \sum_{r=1}^n (x_r A_{r,c} - x_c) = \sum_{r=1}^n x_r \sum_{c=1}^n A_{r,c} - \sum_{c=1}^n x_c = 0.$$

Thus $\text{range}(T - I) \subseteq \{ (y_1 \ \cdots \ y_n) \in \mathbf{F}^n : y_1 + \cdots + y_n = 0 \}$. Hence $(T - I)$ is not surj. \square

- 25** Suppose $T \in \mathcal{L}(V)$ and u, w are eigvecs of T such that $u + w$ is also an eigvec of T . Prove that u and w are eigvecs of T correspd to the same eigval.

SOLUTION:

Suppose $\lambda_1, \lambda_2, \lambda_0$ are eigvals of T correspd to $u, w, u + w$ respectively.

$$\text{Then } T(u + w) = \lambda_0(u + w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w.$$

Note that $u, w, u + w$ are nonzero. If (u, w) is linely depe, then let $w = cu$, therefore

$$(-) \ \lambda_2 cu = Tw = cTu = \lambda_1 cu \Rightarrow \lambda_2 = \lambda_1.$$

$$(=) \ \lambda_0(u + w) = T(u + w) = \lambda_1 u + \lambda_1 cu = \lambda_1(u + w) \Rightarrow \lambda_0 = \lambda_1.$$

Otherwise, (u, w) is linely inde. Then $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. \square

OR. Assume that $\lambda_1 \neq \lambda_2$. Then (u, w) is linely inde. Thus $\lambda_0 - \lambda_1 = \lambda_0 - \lambda_2$. Contradicts. \square

- 26** Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vec in V is an eigvec of T . Prove that T is a scalar multi of the identity operator.

SOLUTION:

Because $\forall v \in V, \exists! \lambda_v \in \mathbf{F}, Tv = \lambda_v v$. For any two distinct nonzero vecs $v, w \in V$,

$$T(v + w) = \lambda_{v+w}(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w. \quad \square$$

OR. If $\dim V = 0$ or 1 . We are done.

Suppose $\dim V \geq 2$. For any two nonzero vecs $u, v \in V$, u, v are eigvecs.

If $u + v \neq 0$, then $u + v$ is also an eigvec. Otherwise, $u + v = 0$, then $Tu = -Tv = \lambda u = -\lambda v$.

Thus by Problem (25), $\forall u, v \in V, Tu = \lambda u, Tv = \lambda v \Rightarrow \forall v \in V, Tv = \lambda v$. □

27, 28 Suppose V is finite-dim and $k \in \{1, \dots, \dim V - 1\}$.

Suppose $T \in \mathcal{L}(V)$ is such that every subsp of V of dim k is invar under T .

Prove that T is a scalar multi of the identity operator.

SOLUTION: We prove the contrapositive:

Suppose T is not a scalar multi of I . Prove that \exists an invar subsp U of V under T such that $\dim U = k$.

By Problem (26), $\exists v \in V$ and $v \neq 0$ such that v is not an eigvec of T .

Thus (v, Tv) is linely inde. Extend to a basis of V as (v, Tv, u_1, \dots, u_n) .

Let $U = \text{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$ is not an invar subsp of V under T .

OR. Suppose $0 \neq v = v_1 \in V$ and extend to a basis of V as (v_1, \dots, v_n) .

Suppose $Tv_1 = c_1v_1 + \dots + c_nv_n, \exists! c_i \in \mathbb{F}$.

Consider a k -dim subsp $U = \text{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$,

where $\alpha_j \in \{2, \dots, n\}$ for each j , and $\alpha_1, \dots, \alpha_{k-1}$ are distinct.

Because every subsp such U is invar.

Thus $Tv_1 = c_1v_1 + \dots + c_nv_n \in U \Rightarrow c_2 = \dots = c_n = 0$,

for if not, for each $c_i \neq 0$, choose U_i such that $\alpha_j \in \overbrace{\{2, \dots, i-1, i+1, \dots, n\}}^{\text{length}(n-2)}$ for each j ,

hence for $Tv_1 = c_1v_1 + \dots + c_{i-1}v_{i-1} + c_{i+1}v_{i+1} + \dots + c_nv_n \in U_i$, we conclude that $c_i = 0$.

$\Rightarrow Tv_1 = c_1v_1$, 又 $v_1 = v \in V$ is arbitrary $\Rightarrow T = \lambda I$ for some λ . □

• (4E 5.A.39) Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Prove that

T has an eigval $\iff \exists$ an invar subsp U of V under T such that $\dim U = \dim V - 1$.

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec v .

(If $\dim V = 1$, then $U = \{0\}$ and we are done.)

Extend $v_1 = v$ to a basis of V as (v_1, v_2, \dots, v_n) .

Step 1. If $\exists w_1 \in \text{span}(v_2, \dots, v_n)$ such that $0 \neq Tw_1 \in \text{span}(v_1)$,

then extend $w_1 = \alpha_{1,1}$ to a basis of $\text{span}(v_2, \dots, v_n)$ as $(\alpha_{1,1}, \dots, \alpha_{1,n-1})$.

Otherwise, we stop at step 1.

\vdots

Step k. If $\exists w_k \in \text{span}(\alpha_{k-1,2}, \dots, \alpha_{k-1,n-k+1})$ such that $0 \neq Tw_k \in \text{span}(v_1, w_1, \dots, w_{k-1})$,

then extend $w_k = \alpha_{k,1}$ to a basis of $\text{span}(\alpha_{k-1,2}, \dots, \alpha_{k-1,n-k+1})$ as $(\alpha_{k,1}, \dots, \alpha_{k,n-k})$.

Otherwise, we stop at step k .

\vdots

Finally, we stop at step m , thus we get $(v_1, w_1, \dots, w_{m-1})$ and $(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1})$,

$\text{range } T|_{\text{span}(w_1, \dots, w_{m-1})} = \text{span}(v_1, w_1, \dots, w_{m-2}) \Rightarrow \dim \text{null } T|_{\text{span}(w_1, \dots, w_{m-1})} = 0$,

$\underbrace{\text{span}(v_1, w_1, \dots, w_{m-1})}_{\dim m}$ and $\underbrace{\text{span}(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1})}_{\dim(n-m)}$ are invar under T .

Let $U = \text{span}(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}) \oplus \text{span}(v_1, w_1, \dots, w_{m-2})$ and we are done. □

COMMENT: Both $\text{span}(v_2, \dots, v_n)$ and $\text{span}(\alpha_{m-1,2}, \dots, \alpha_{m-1,n-m+1}) \oplus \text{span}(w_1, \dots, w_{m-1})$ are in $\mathcal{S}_V \text{span}(v_1)$.

(b) Suppose U is an invariant subspace of V under T with $\dim U = m = \dim V - 1$.

(If $m = 0$, then $\dim V = 1$ and we are done.)

Let (u_1, \dots, u_m) be a basis of U , extend to a basis of V as (u_0, u_1, \dots, u_m) .

We discuss in cases:

For $Tu_0 \in U$, then $\text{range } T = U$ so that T is not surj $\iff \text{null } T \neq \{0\} \iff 0$ is an eigval of T .

For $Tu_0 \notin U$, then $Tu_0 = a_0u_0 + a_1u_1 + \dots + a_mu_m$.

(1) If $Tu_0 \in \text{span}(u_0)$, then we are done.

(2) Otherwise, if $\text{range } T|_U = U$, then $Tu_0 = a_0u_0$ and we are done;

otherwise, $T|_U : U \rightarrow U$ is not surj (\Rightarrow not inje), suppose $\text{range } T|_U \neq \{0\}$

(Suppose $\text{range } T|_U = \{0\}$. If $\dim U = 0$ then we are done.

Otherwise $\exists u \in U \setminus \{0\}, Tu = 0$ and we are done.)

then $\exists u \in U \setminus \{0\}, Tu = 0$, we are done. □

29 Suppose $T \in \mathcal{L}(V)$ and $\text{range } T$ is finite-dim.

Prove that T has at most $1 + \dim \text{range } T$ distinct eigvals.

SOLUTION:

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigvals of T with correspond eigvecs v_1, \dots, v_m .

(Because $\text{range } T$ is finite-dim. The correspond eigvals are finite.)

Then (v_1, \dots, v_m) linely inde $\implies (\lambda_1v_1, \dots, \lambda_mv_m)$ linely inde, if each $\lambda_k \neq 0$.

Otherwise, $\exists! \lambda_k = 0$. Now $(\lambda_1v_1, \dots, \lambda_{k-1}v_{k-1}, \lambda_{k+1}v_{k+1}, \dots, \lambda_mv_m)$ is linely inde.

Hence, by [2.23], $m - 1 \leq \dim \text{range } T$. □

• Suppose $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ are distinct.

(a) **32** Prove that $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linely inde in $\mathbb{R}^{\mathbb{R}}$.

HINT: Let $V = \text{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$. Define $D \in \mathcal{L}(V)$ by $Df = f'$. Find eigvals and eigvecs of D .

(b) [4E 36] Show that $(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$ is linely inde in $\mathbb{R}^{\mathbb{R}}$.

SOLUTION:

(a) Define V and $D \in \mathcal{L}(V)$ as in HINT. Then because for each $k, D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.

Thus $\lambda_1, \dots, \lambda_n$ are distinct eigvals of D . By [5.10], $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linely inde in $\mathbb{R}^{\mathbb{R}}$. □

(b) Let $V = \text{span}(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.

Then because $D(\cos(\lambda_k x)) = -\lambda_k \sin(\lambda_k x)$. $\times D(\sin(\lambda_k x)) = \lambda_k \cos(\lambda_k x)$.

Thus $D^2(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$.

Notice that $\lambda_1, \dots, \lambda_n$ are distinct $\implies -\lambda_1^2, \dots, -\lambda_n^2$ are distinct. And $\dim V = n$.

Hence $-\lambda_1^2, \dots, -\lambda_n^2$ are all the eigvals of D^2

with the correspond eigvecs $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ respectively.

And then $(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$ is linely inde in $\mathbb{R}^{\mathbb{R}}$. □

• Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is a subsp of V invariant under T .

The quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U \text{ for each } v \in V.$$

(a) Show that the definition of T/U makes sense

(which requires using the condition that U is invariant under T)

and show that T/U is an operator on V/U .

(b) [OR 35] Show that each eigval of T/U is an eigval of T .

SOLUTION:

(a) Suppose $v + U = w + U$ ($\Leftrightarrow v - w \in U$).

Then because U is invar under T , $T(v - w) \in U \Leftrightarrow Tv + U = Tw + U$.

Hence the definition of T/U makes sense.

Now we show that T/U is linear.

$$\begin{aligned} \forall v + U, w + U \in V/U, \lambda \in \mathbb{F}, (T/U)((v + U) + \lambda(w + U)) \\ = T(v + \lambda w) + U = (Tv + U) + \lambda(Tw + U) \\ = (T/U)(v + U) + \lambda(T/U)(w). \end{aligned}$$

(b) Suppose λ is an eigval of T/U with an eigvec $v + U$.

Then $(T/U)(v + U) = \lambda(v + U) = Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v \in U$.

If $(T - \lambda I)v = 0 \Rightarrow Tv = \lambda v$, then we are done.

Otherwise, then $(T|_U - \lambda I) : U \rightarrow U$ is inv,

hence $\exists! w \in U, (T|_U - \lambda I)(w) = (T - \lambda I)v \Rightarrow T(v - w) = \lambda(v - w)$.

Note that $v - w \neq 0$ (for if not, $v \in U \Rightarrow v + U = 0 + U$ is not an eigvec). □

36 Prove or give a counterexample:

The result of (b) in Exercise 35 is still true if V is infinite-dim.

SOLUTION: A counterexample:

Consider $V = \text{span}(1, e^x, e^{2x}, \dots)$ in $\mathbb{R}^{\mathbb{R}}$, and a subsp $U = \text{span}(e^x, e^{2x}, \dots)$ of V .

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then $\text{range } T = U$ is invar under T .

Consider $(T/U)(1 + U) = e^x + U = 0$

$\Rightarrow 0$ is an eigval of T/U but is not an eigval of T .

($\text{null } T = \{0\}$, for if not, $\exists f \in V \setminus \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbb{R} \Rightarrow f = 0$, contradicts.) □

33 Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{range } T) = 0$.

SOLUTION:

$\forall v + \text{range } T \in V/\text{range } T, v + \text{range } T \in \text{null } (T/(\text{range } T))$

$\Rightarrow \text{null } (T/(\text{range } T)) = V/\text{range } T \Rightarrow T/(\text{range } T)$ is a zero map. □

34 Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null } T)$ is inje $\Leftrightarrow (\text{null } T) \cap (\text{range } T) = \{0\}$.

SOLUTION:

(a) Suppose $T/(\text{null } T)$ is inje.

Then $(T/(\text{null } T))(u + \text{null } T) = Tu + \text{null } T = 0$

$\Leftrightarrow Tu \in \text{null } T \text{ \& } Tu \in \text{range } T \Leftrightarrow u + \text{null } T = 0 \Leftrightarrow u \in \text{null } T \Leftrightarrow Tu = 0$.

Thus $(\text{null } T) \cap (\text{range } T) = \{0\}$.

(b) Suppose $(\text{null } T) \cap (\text{range } T) = \{0\}$.

Then $(T/(\text{null } T))(u + \text{null } T) = Tu + \text{null } T = 0$

$\Leftrightarrow Tu \in \text{null } T \text{ \& } Tu \in \text{range } T \Leftrightarrow Tu = 0 \Leftrightarrow u \in \text{null } T \Leftrightarrow u + \text{null } T = 0$.

Thus $T/(\text{null } T)$ is inje. □

ENDED

5.B: I [See 5.B: II below.]

COMMENT: 下面, 为了照顾原书 5.B 节两版过大的差距, 特别将此节补注分成 I 和 II 两部分。
又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本征值」(相当一部分是从原第 3 版 8.C 节挪过来的) 是对原第 3 版「多项式作用于算子」与「本征值的存在性」(也即第 3 版 5.B 前半部分) 的极大扩充, 这一扩充也大大改变了原第 3 版后半部分的「上三角矩阵」这一小节, 故而将第 4 版 5.B 节放在第 3 版前面。

I 部分除了覆盖第 4 版 5.B 节全部和第 3 版 5.B 节前半部分与之相关的所有习题, 还会覆盖第 4 版 5.A 节末。

II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节, 还会覆盖第 4 版 5.C 节; 并且, 下面 5.C 还会覆盖第 4 版 5.D 节。

[注: [8.40] OR (4E 5.22) — mini poly;
[8.44,8.45] OR (4E 5.25,5.26) — how to find the mini poly;
[8.49] OR (4E 5.27) — eigvals are the zeros of the mini poly;
[8.46] OR (4E 5.29) — $q(T) = 0 \Leftrightarrow q$ is a poly multi of the mini poly.]

1 2 3 5 6 7 8 10 11 12 13 18 19 | 2E Ch5.24
4E: 5.A.32, 5.A.33; 3, 7, 8, 9, 10, 11, 12, 13, 14, 15,
16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29.

- (4E 5.A.33) Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.
 - (a) Prove that T is inje $\Leftrightarrow T^m$ is inje.
 - (b) Prove that T is surj $\Leftrightarrow T^m$ is surj.

SOLUTION:

(a) Suppose T^m is inje. Then $Tv = 0 \Rightarrow T^{m-1}Tv = T^mv = 0 \Rightarrow v = 0$.

Suppose T is inje. Then $T^mv = T^{m-1}v = \dots = T^2v = Tv = v = 0$.

(b) Suppose T^m is surj. $\forall u \in V, \exists v \in V, T^mv = u = Tw$, let $w = T^{m-1}v$. □

Suppose T is surj. Then $\forall u \in V, \exists v_1, \dots, v_m \in V, T(v_1) = T^2v_2 = \dots = T^mv_m = u$. □

• **NOTE FOR [5.17]:**

Suppose $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{F})$. Prove that $\text{null } p(T)$ and $\text{range } p(T)$ are invar under T .

SOLUTION: Using the commutativity in [5.10].

(a) Suppose $u \in \text{null } p(T)$. Then $p(T)u = 0$.

Thus $p(T)(Tu) = (p(T)T)u(Tp(T))u = T(p(T)u) = 0$. Hence $Tu \in \text{null } p(T)$. □

(b) Suppose $u \in \text{range } p(T)$. Then $\exists v \in V$ such that $u = p(T)v$.

Thus $Tu = T(p(T)v) = p(T)(Tv) \in \text{range } p(T)$. □

• **NOTE FOR [5.21]:** Every operator on a finite-dim nonzero complex vecsp has an eigval.

Suppose V is a finite-dim complex vecsp of $\dim n > 0$ and $T \in \mathcal{L}(V)$.

Choose a nonzero $v \in V$. $(v, Tv, T^2v, \dots, T^nv)$ of length $n + 1$ is linely depe.

Suppose $a_0I + a_1T + \dots + a_nT^n = 0$. Then $\exists a_j \neq 0$.

Thus \exists nonconst p of smallest degree ($\deg p > 0$) such that $p(T)v = 0$.

Because $\exists \lambda \in \mathbf{C}$ such that $p(\lambda) = 0 \Rightarrow \exists q \in \mathcal{P}(\mathbf{C}), p(z) = (z - \lambda)q(z), \forall z \in \mathbf{C}$.

Thus $0 = p(T)v = (T - \lambda I)(q(T)v)$. By the minimality of $\deg p$ and $\deg q < \deg p, q(T)v \neq 0$.

Then $(T - \lambda I)$ is not inje. Thus λ is an eigval of T with eigvec $q(T)v$.

• **EXAMPLE:** an operator on a complex vecsp with no eigvals

Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by $(Tp)(z) = zp(z)$.

Suppose $p \in \mathcal{P}(\mathbf{C})$ is a nonzero poly. Then $\deg Tp = \deg p + 1$, and thus $Tp \neq \lambda p, \forall \lambda \in \mathbf{C}$.

Hence T has no eigvals.

13 Suppose V is a complex vecsp and $T \in \mathcal{L}(V)$ has no eigvals.

Prove that every subsp of V invar under T is either $\{0\}$ or infinite-dim.

SOLUTION: Suppose U is a finite-dim nonzero invar subsp on C . Then by [5.21], $T|_U$ has an eigval. \square

16 Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}(\mathcal{P}_{\dim V}(C), V)$ by $S(p) = p(T)v$. Prove [5.21].

SOLUTION:

Because $\dim \mathcal{P}_{\dim V}(C) = \dim V + 1$. Then S is not inje. Hence $\exists 0 \neq p \in \mathcal{P}_{\dim V}(C), p(T)v = 0$.

Using [4.14], write $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$. Apply T to both sides: $p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.

Thus at least one of $(T - \lambda_j I)$ is not inje (because $p(T)$ is not inje). \square

17 Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}(\mathcal{P}_{(\dim V)^2}(C), \mathcal{L}(V))$ by $S(p) = p(T)$. Prove [5.21].

SOLUTION:

Because $\dim \mathcal{P}_{(\dim V)^2}(C) = (\dim V)^2 + 1$. Then S is not inje. Hence $\exists 0 \neq p \in \mathcal{P}_{(\dim V)^2}(C), p(T) = 0$.

Using [4.14], write $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$. Applying T , we have $0 = p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.

Thus $(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0 \Rightarrow \exists j, (T - \lambda_j I)$ is not inje. \square

COMMENT: \exists monic $q \in \text{null } S \neq \{0\}$ of smallest degree, $S(q) = q(T) = 0$, then q is the *mini poly*.

• **NOTE FOR [8.40]:** def for *mini poly*

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Suppose $M_T^0 = \{p_j\}_{j \in \Gamma}$ is the set of all monic poly that give 0 whenever T is applied.

Prove that $\exists ! p_k \in M_T^0, \deg p_k = \min\{\deg p_j\}_{j \in \Gamma} \leq \dim V$.

SOLUTION: OR. Another Proof :

[Existns Part] We use induction on $\dim V$.

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, we are done.

(ii) Suppose $\dim V \geq 1$.

Assume that $\dim V > 0$ and that the desired result is true for all operators on all vecsp of smaller dim.

Let $u \in V, u \neq 0$. The list $(u, Tu, \dots, T^{\dim V} u)$ of length $(1 + \dim V)$ is linely depe.

Then $\exists ! T^m$ of smallest degree such that $T^m u \in \text{span}(u, Tu, \dots, T^{m-1} u)$.

Thus $\exists c_j \in F, c_0 u + c_1 Tu + \dots + c_{m-1} T^{m-1} u + T^m u = 0$.

Define q by $q(z) = c_0 + c_1 z + \dots + c_{m-1} z^{m-1} + z^m$.

Then $0 = T^k(q(T)u) = q(T)(T^k u), \forall k \in \{1, \dots, m-1\} \subseteq N$.

Because $(u, Tu, \dots, T^{m-1} u)$ is linely inde.

Thus $\dim \text{null } q(T) \geq m \Rightarrow \dim \text{range } q(T) = \dim V - \dim \text{null } q(T) \leq \dim V - m$.

Let $W = \text{range } q(T)$.

By assumption, $\exists s \in M_T^0$ of smallest degree (and $\deg s \leq \dim W$,) so that $s(T|_W) = 0$.

Hence $\forall v \in V, ((sq)(T))(v) = s(T)(q(T)v) = 0$.

Thus $sq \in M_T^0$ and $\deg sq \leq \dim V$.

[Uniques Part]

Suppose $p, q \in M_T^0$ are of the smallest degree. Then $(p-q)(T) = 0$. 又 $\deg(p-q) = m < \min\{\deg p_j\}_{j \in \Gamma}$.

Hence $p - q = 0$, for if not, $\exists ! c \in F, c(p - q) \in M_T^0$. Contradicts. \square

- (4E 5.31, 4E 5.B.25 and 26) *mini poly of restriction operator and mini poly of quotient operator*
Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invar subsp of V under T .

Let p be the mini poly of T .

- Prove that p is a poly multi of the mini poly of $T|_U$.
- Prove that p is a poly multi of the mini poly of T/U .
- Prove that (mini poly of $T|_U$) \times (mini poly of T/U) is a poly multi of p .
- Prove that the set of eigvals of T equals
the union of the set of eigvals of $T|_U$ and the set of eigvals of T/U .

SOLUTION:

- $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46]. □
- $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v + U) = p(T)v + U = 0$. □
- Suppose r is the mini poly of $T|_U$, s is the mini poly of T/U .
Because $\forall v \in V, s(T/U)(v + U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.
又 $\forall u \in U, r(T|_U)u = r(T)u = 0$.
Thus $\forall v \in V$ but $v \notin U, (rs)(T)v = r(s(T)v) = 0$.
And $\forall u \in U, (rs)(T)u = r(s(T)u) = 0$ (because $s(T)u = s(T|_U)u \in U$).
Hence $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0$. □
- By [8.49], immediately. □

- (4E 5.B.27) Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, and $T \in \mathcal{L}(V)$.
Prove that the mini poly p of T_C equals the mini poly q of T .

SOLUTION:

- $\forall u + i0 \in V_C, p(T_C)(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p$ is a poly multi of q .
- $q(T) = 0 \Rightarrow \forall u + iv \in V_C, q(T_C)(u + iv) = q(T)u + iq(T)v = 0 \Rightarrow q$ is a poly multi of p . □

- (4E 5.B.28) Suppose V is finite-dim and $T \in \mathcal{L}(V)$.
Prove that the mini poly p of $T' \in \mathcal{L}(V')$ equals the mini poly q of T .

SOLUTION:

- $\forall \varphi \in V', p(T')\varphi = \varphi \circ (p(T)) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \text{null } \varphi \Rightarrow p(T) = 0, p$ is a poly multi of q .
- $q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ (q(T)) = q(T')\varphi = 0 \Rightarrow q(T) = 0, q$ is a poly multi of p . □

- (4E 5.32) Suppose $T \in \mathcal{L}(V)$ and p is the mini poly.
Prove that T is not inje \iff the const term of p is 0.

SOLUTION:

- T is not inje $\iff 0$ is an eigval of $T \iff 0$ is a zero of $p \iff$ the const term of p is 0. □
- OR. Because $p(0) = (z - 0)(z - \lambda_1) \cdots (z - \lambda_m) = 0 \Rightarrow T(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0$
又 p is the mini poly $\Rightarrow q$ define by $q(z) = (z - \lambda_1) \cdots (z - \lambda_m)$ is such that $q(T) \neq 0$.
Hence $0 = p(T) = Tq(T) \Rightarrow T$ is not inje.
Conversely, suppose $(T - 0I)$ is not inje, then 0 is a zero of p , so that the const term is 0. □

- (4E 5.B.22)
Suppose V is finite-dim, $T \in \mathcal{L}(V)$. Prove that T is inv $\iff I \in \text{span}(T, T^2, \dots, T^{\dim V})$.

SOLUTION: Denote the mini poly by p , where for all $z \in \mathbf{F}, p(z) = a_0 + a_1 z + \cdots + z^m$.

Notice that V is finite-dim. T is inv $\iff T$ is inje $\iff p(0) \neq 0$.

Hence $p(T) = 0 = a_0I + a_1T + \cdots + T^m$, where $a_0 \neq 0$ and $m \leq \dim V$. □

6 Suppose $T \in \mathcal{L}(V)$ and U is a subsp of V invar under T .

Prove that U is invar under $p(T)$ for every poly $p \in \mathcal{P}(\mathbf{F})$.

SOLUTION:

$$\forall u \in U, Tu \in U \Rightarrow \forall u \in U, Iu, Tu, T(Tu), \dots, T^m u \in U \Rightarrow \forall u \in U, (a_0I + a_1T + \cdots + a_m T^m)u \in U \quad \square$$

• (4E 5.B.10, 23) Suppose V is finite-dim, $T \in \mathcal{L}(V)$ and p is the mini poly with degree m . Suppose $v \in V$.

(a) Prove that $\text{span}(v, Tv, \dots, T^{m-1}v) = \text{span}(v, Tv, \dots, T^{j-1}v)$ for some $j \leq m$.

(b) Prove that $\text{span}(v, Tv, \dots, T^{m-1}v) = \text{span}(v, Tv, \dots, T^{m-1}v, \dots, T^n v)$.

SOLUTION:

COMMENT: By NOTE FOR[8.40], j has an upper bound $m - 1$, m has an upper bound $\dim V$.

Write $p(z) = a_0 + a_1z + \cdots + z^m$ ($m \leq \dim V$). If $v = 0$, then we are done. Suppose $v \neq 0$.

(a) Suppose $j \in \mathbf{N}^+$ is the smallest such that $T^j v \in \text{span}(v, Tv, \dots, T^{j-1}v) = U_0$. Then $j \leq m$.

Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1} T^{j-1}v$. And because $T(T^k v) = T^{k+1}v \in U_0$. U_0 is invar under T .

By Problem (6), $\forall k \in \mathbf{N}$, $T^{j+k}v = T^k(T^j v) \in U_0$.

Thus $U_0 = \text{span}(v, Tv, \dots, T^{j-1}v, \dots, T^n v)$ for all $n \geq j - 1$. Let $n = m - 1$ and we are done.

(b) Let $U = \text{span}(v, Tv, \dots, T^{m-1}v)$.

By (a), $U = U_0 = \text{span}(v, Tv, \dots, T^{j-1}v, \dots, T^{m-1}v, \dots, T^n v)$ for all $n \geq m - 1$. □

• (4E 5.B.21) Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that the mini poly p has degree at most $1 + \dim \text{range } T$.

If $\dim \text{range } T < \dim V - 1$, then this result gives a better upper bound for the degree of mini poly.

SOLUTION:

If T is inje, then $\text{range } T = V$ and we are done. Now choose $0 \neq v \in \text{null } T$, then $Tv + 0 \cdot v = 0$.

1 is the smallest positive integer such that $T^1 v \in \text{span}(v, \dots, T^0 v)$. Define q by $q(z) = z \Rightarrow q(T)v = 0$.

Let $W = \text{range } q(T) = \text{range } T$. \exists monic $s \in \mathcal{P}(\mathbf{F})$ of smallest degree ($\deg s \leq \dim W$), $s(T|_W) = 0$.

Hence sq is the mini poly (see NOTE FOR[8.40]) and $\deg(sq) = \deg s + \deg q \leq \dim \text{range } T + 1$. □

19 Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.

Prove that $\dim \mathcal{E}$ equals the degree of the mini poly of T .

SOLUTION:

Because the list $(I, T, \dots, T^{(\dim V)^2})$ of length $\dim \mathcal{L}(V) + 1$ is linely depe in $\dim \mathcal{L}(V)$.

Suppose $m \in \mathbf{N}^+$ is the smallest such that $T^m = a_0I + \cdots + a_{m-1}T^{m-1}$.

Then q defined by $q(z) = z^m - a_{m-1}z^{m-1} - \cdots - a_0$ is the mini poly (see [8.40]).

For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k(T^m) \in \text{span}(I, T, \dots, T^{m-1}) = U$.

Hence $\text{span}(I, T, \dots, T^{(\dim V)^2}) = \text{span}(I, T, \dots, T^{(\dim V)^2-1}) = U$.

Note that by the minimality of m , (I, T, \dots, T^{m-1}) is linely inde.

Thus $\dim U = m = \dim \text{span}(I, T, \dots, T^{(\dim V)^2-1}) = \dim \text{span}(I, T, \dots, T^n)$ for all $m < n \in \mathbf{N}^+$.

Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.

(a) Suppose $p(T) = 0$. $\text{又 } \deg p \leq m - 1 \Rightarrow p = 0$. Then φ is inje.

(b) $\forall S = a_0I + a_1T + \cdots + a_{m-1}T^{m-1} \in \mathcal{E}$, define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by

$p(z) = a_0 + a_1z + \cdots + a_{m-1}z^{m-1} \Rightarrow \varphi(p) = S$. Then φ is surj.
Hence \mathcal{E} and $\mathcal{P}_{m-1}(\mathbf{F})$ are iso. 又 $\dim \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. □

- (4E 5.B.13) Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$ is defined by

$$q(z) = a_0 + a_1z + \cdots + a_nz^n, \text{ where } a_n \neq 0, \text{ for all } z \in \mathbf{F}.$$

Denote the mini poly of T by p defined by

$$p(z) = c_0 + c_1z + \cdots + c_{m-1}z^{m-1} + z^m \text{ for all } z \in \mathbf{F}.$$

Prove that $\exists ! r \in \mathcal{P}(\mathbf{F})$ such that $q(T) = r(T)$, $\deg r < \deg p$.

SOLUTION:

If $\deg q < \deg p$, then we are done.

If $\deg q = \deg p$, notice that $p(T) = 0 = c_0I + c_1T + \cdots + c_{m-1}T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0I - c_1T - \cdots - c_{m-1}T^{m-1},$$

$$\begin{aligned} \text{define } r \text{ by } r(z) &= q(z) + [-a_mz^m + a_m(-c_0 - c_1z - \cdots - c_{m-1}z^{m-1})] \\ &= (a_0 - a_m c_0) + (a_1 - a_m c_1)z + \cdots + (a_{m-1} - a_m c_{m-1})z^{m-1}, \end{aligned}$$

hence $r(T) = 0$, $\deg r < m$ and we are done.

Now suppose $\deg q \geq \deg p$. We use induction on $\deg q$.

(i) $\deg q = \deg p$, then the desired result is true, as shown above.

(ii) $\deg q > \deg p$, assume that the desired result is true for $\deg q = n$.

Suppose $f \in \mathcal{P}(\mathbf{F})$ such that $f(z) = b_0 + b_1z + \cdots + b_nz^n + b_{n+1}z^{n+1}$.

Apply the assumption to g defined by $g(z) = b_0 + b_1z + \cdots + b_nz^n$,

getting s defined by $s(z) = d_0 + d_1z + \cdots + d_{m-1}z^{m-1}$.

Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1}T^{n+1} = s(T) + b_{n+1}T^{n+1}$.

Apply the assumption to t defined by $t(z) = z^n$,

getting δ defined by $\delta(z) = c_0' + c_1'z + \cdots + c_{m-1}'z^{m-1}$.

Thus $t(T) = T^n = c_0' + c_1'T + \cdots + c_{m-1}'T^{m-1} = \delta(T)$.

又 $\text{span}(v, Tv, \dots, T^{m-1}v)$ is invar under T .

Hence $\exists ! k_j \in \mathbf{F}, T^{n+1} = T(T^n) = k_0 + k_1T + \cdots + k_{m-1}T^{m-1}$.

And $f(T) = s(T) + b_{n+1}(k_0 + k_1T + \cdots + k_{m-1}T^{m-1})$

$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1)T + \cdots + (d_{m-1} + k_{m-1})T^{m-1} = h(T)$, thus defining h . □

- (4E 5.B.14) Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has mini poly p
defined by $p(z) = a_0 + a_1z + \cdots + a_{m-1}z^{m-1} + z^m, a_0 \neq 0$.
Find the mini poly of T^{-1} .

SOLUTION:

Notice that V is finite-dim. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is inv.

Then $p(T) = a_0I + a_1T + \cdots + T^m = 0$. Apply T^{-m} to both sides,

$$a_0(T^{-1})^m + a_1(T^{-1})^{m-1} + \cdots + a_{m-1}T^{-1} + I = 0.$$

Define q by $q(z) = z^m + \frac{a_1}{a_0}z^{m-1} + \cdots + \frac{a_{m-1}}{a_0}z + \frac{1}{a_0}$ for all $z \in \mathbf{F}$.

We now show that $(T^{-1})^k \notin \text{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$

for every $k \in \{1, \dots, m-1\}$ by contradiction, so that q is exactly the mini poly of T^{-1} .

Suppose $(T^{-1})^k \in \text{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$.

Then let $(T^{-1})^k = b_0I + b_1T^{-1} + \cdots + b_{k-1}T^{k-1}$. Apply T^k to both sides,

getting $I = b_0T^k + b_1T^{k-1} + \cdots + b_{k-1}T$, hence $T^k \in \text{span}(I, T, \dots, T^{k-1})$.

Thus f defined by $f(z) = z^k + \frac{b_1}{b_0}z^{k-1} + \dots + \frac{b_{k-1}}{b_0}z - \frac{1}{b_0}$ is a poly multi of p .

While $\deg f < \deg p$. Contradicts. □

• **NOTE FOR [8.49]:**

Suppose V is a finite-dim complex vecsp and $T \in \mathcal{L}(V)$.

By [4.14], the mini poly has the form $(z - \lambda_1) \cdots (z - \lambda_m)$,

where $\lambda_1, \dots, \lambda_m$ is a list of all eigvals of T , **possibly with repetitions**.

• **COMMENT:**

A nonzero poly has at most as many distinct zeros as its degree (see [4.12]).

Thus by the upper bound for the deg of mini poly given in NOTE FOR [8.40], and by [8.49,] we can **give an alternative proof of [5.13]**.

• **NOTICE** (See also 4E 5.B.20,24)

Suppose $\alpha_1, \dots, \alpha_n$ are all the distinct eigvals of T ,

and therefore are all the distinct zeros of the mini poly.

Also, the mini poly of T is a poly multi of, but not equal to, $(z - \alpha_1) \cdots (z - \alpha_n)$.

If we define q by $q(z) = (z - \alpha_1)^{\dim V - (n-1)} \cdots (z - \alpha_n)^{\dim V - (n-1)}$,

then q is a poly multi of the char poly (see [8.34] and [8.26])

(Because $\dim V > n$ and $n - 1 > 0$, $n[\dim V - (n - 1)] > \dim V$.)

The char poly has the form $(z - \alpha_1)^{\gamma_1} \cdots (z - \alpha_n)^{\gamma_n}$, where $\gamma_1 + \dots + \gamma_n = \dim V$.

The mini poly has the form $(z - \alpha_1)^{\delta_1} \cdots (z - \alpha_n)^{\delta_n}$, where $0 \leq \delta_1 + \dots + \delta_n \leq \dim V$.

10 Suppose $T \in \mathcal{L}(V)$, λ is an eigval of T with an eigvec v .

Prove that for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.

SOLUTION:

Suppose p is defined by $p(z) = a_0 + a_1z + \dots + a_mz^m$ for all $z \in \mathbf{F}$. Because for any $n \in \mathbf{N}^+$, $T^n v = \lambda^n v$.

Thus $p(T)v = a_0v + a_1Tv + \dots + a_mT^mv = a_0v + a_1\lambda v + \dots + a_m\lambda^mv = p(\lambda)v$. □

COMMENT: For any $p \in \mathcal{P}(\mathbf{F})$ such that $p(z) = (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_m)^{\alpha_m}$, the result is true as well.

Now we prove that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v = (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v$.

Define q_i by $q_i(z) = (z - \lambda_i)^{\alpha_i}$ for all $z \in \mathbf{F}$.

Because $(a + b)^n = a^n + C_n^1 a^{n-1}b + \dots + C_n^k a^{n-k}b^k + \dots + C_n^n b^n$.

Let $a = z, b = \lambda_i, n = \alpha_i$, so we can write $q_i(z)$ in the form $a_0 + a_1z + \dots + a_mz^m$.

Hence $q_i(T)v = q_i(\lambda)v \Rightarrow (T - \lambda_i I)^{\alpha_i} v = (\lambda - \lambda_i)^{\alpha_i} v$.

Then for each $k \in \{2, \dots, m\}$, $(T - \lambda_{k-1} I)^{\alpha_{k-1}} (T - \lambda_k I)^{\alpha_k} v$

$$\begin{aligned} &= q_{k-1}(T)(q_k(T)v) \\ &= q_{k-1}(T)(q_k(\lambda)v) \\ &= q_{k-1}(\lambda)(q_k(\lambda)v) \\ &= (\lambda - \lambda_{k-1})^{\alpha_{k-1}} (\lambda - \lambda_k)^{\alpha_k} v. \end{aligned}$$

So that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v$

$$\begin{aligned} &= q_1(T) \left(q_2(T) \left(\dots (q_m(T)v) \dots \right) \right) \\ &= q_1(\lambda) (q_2(\lambda) (\dots (q_m(\lambda)v) \dots)) \\ &= (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v. \end{aligned}$$

□

1 Suppose $T \in \mathcal{L}(V)$ and $\exists n \in \mathbf{N}^+$ such that $T^n = 0$.

Prove that $(I - T)$ is inv and $(I - T)^{-1} = I + T + \dots + T^{n-1}$.

SOLUTION:

Note that $1 - x^n = (1 - x)(1 + x + \dots + x^{n-1})$.

$$\left. \begin{aligned} (I - T)(1 + T + \dots + T^{n-1}) &= I - T^n = I \\ (1 + T + \dots + T^{n-1})(I - T) &= I - T^n = I \end{aligned} \right\} \Rightarrow (I - T)^{-1} = 1 + T + \dots + T^{n-1}. \quad \square$$

2 Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$.

Suppose λ is an eigval of T . Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

SOLUTION:

Suppose v is an eigvec correspd to λ . Then for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.

Hence $0 = (T - 2I)(T - 3I)(T - 4I)v = (\lambda - 2)(\lambda - 3)(\lambda - 4)v$ while $v \neq 0 \Rightarrow \lambda = 2$ or $\lambda = 3$ or $\lambda = 4$. \square

OR. Because $(T - 2I)(T - 3I)(T - 4I) = 0$ is not inje. By TIPS. \square

7 [See 5.A.22] Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigval of $T^2 \iff 3$ or -3 is an eigval of T .

SOLUTION:

(a) Suppose 9 is an eigval of T^2 . Then $(T^2 - 9I)v = (T - 3I)(T + 3I)v = 0$ for some v . By TIPS.

OR. Suppose λ is an eigval with an eigvec v . Then $(T - 3I)(T + 3I)v = (\lambda - 3)(\lambda + 3)v = 0 \Rightarrow \lambda = \pm 3$.

(b) Suppose 3 or -3 is an eigval of T with an eigvec v . Then $Tv = \pm 3v \Rightarrow T^2v = T(Tv) = 9v$ \square

3 Suppose $T \in \mathcal{L}(V)$, $T^2 = I$ and -1 is not an eigval of T . Prove that $T = I$.

SOLUTION:

$T^2 - I = (T + I)(T - I)$ is not inje, $\text{又 } -1$ is not an eigval of $T \Rightarrow$ By TIPS. \square

OR. Note that $v = [\frac{1}{2}(I - T)v] + [\frac{1}{2}(I + T)v]$ for all $v \in V$.

And $(I - T^2)v = (I - T)(I + T)v = 0$ for all $v \in V$,

$$\left. \begin{aligned} (I + T)(\frac{1}{2}(I - T)v) &= \frac{1}{2}(I - T^2)v = 0 \Rightarrow \frac{1}{2}(I - T)v \in \text{null}(I + T) \\ (I - T)(\frac{1}{2}(I + T)v) &= \frac{1}{2}(I - T^2)v = 0 \Rightarrow \frac{1}{2}(I + T)v \in \text{null}(I - T) \end{aligned} \right\} \Rightarrow V = \text{null}(I + T) + \text{null}(I - T).$$

$\text{又 } -1$ is not an eigval of $T \Rightarrow (I + T)$ is inje $\Rightarrow \text{null}(I + T) = \{0\}$.

Hence $V = \text{null}(I - T) \Rightarrow \text{range}(I - T) = \{0\}$. Thus $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$. \square

• (4E 5.A.32) Suppose $T \in \mathcal{L}(V)$ has no eigvals and $T^4 = I$. Prove that $T^2 = -I$.

SOLUTION:

Because $T^4 - I = (T^2 - I)(T^2 + I) = 0$ is not inje $\Rightarrow (T^2 - I)$ or $(T^2 + I)$ is not inje.

$\text{又 } T$ has no eigvals $\Rightarrow (T^2 - I) = (T - I)(T + I)$ is inje.

Hence $T^2 + I = 0 \in \mathcal{L}(V)$, for if not,

$\exists v \in V, (T^2 + I)v \neq 0$ while $(T^2 - I)((T^2 + I)v) = 0$ but $(T^2 - I)$ is inje. Contradicts. \square

OR. Note that $v = [\frac{1}{2}(I - T^2)v] + [\frac{1}{2}(I + T^2)v]$ for all $v \in V$.

And $(I - T^4)v = (I - T^2)(I + T^2)v = 0$ for all $v \in V$,

$$\left. \begin{aligned} (I + T^2)(\frac{1}{2}(I - T^2)v) &= 0 \Rightarrow \frac{1}{2}(I - T^2)v \in \text{null}(I + T^2) \\ (I - T^2)(\frac{1}{2}(I + T^2)v) &= 0 \Rightarrow \frac{1}{2}(I + T^2)v \in \text{null}(I - T^2) \end{aligned} \right\} \Rightarrow V = \text{null}(I + T^2) + \text{null}(I - T^2).$$

又 T has no eigvals $\Rightarrow (I - T^2)$ is inje $\Rightarrow \text{null}(I - T^2) = \{0\}$.

Hence $V = \text{null}(I + T^2) \Rightarrow \text{range}(I + T^2) = \{0\}$. Thus $I + T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$. \square

8 [OR (4E 5.A.31)] Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^4 = -I$.

SOLUTION:

$$T^4 + I = (T^2 + iI)(T^2 - iI) = (T + i^{1/2}I)(T - i^{1/2}I)(T - (-i)^{1/2}I)(T + (-i)^{1/2}I).$$

Note that $i^{1/2} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, $(-i)^{1/2} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. Hence $T = \pm(\pm i)^{1/2}I$.

Define T by $T(x, y) = (\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y)$.

$$\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix} \Rightarrow \mathcal{M}(T)^4 = \mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathcal{M}(I). \quad \square$$

$$\left(\text{Using } \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix} \right)$$

• (4E 5.B.12. See also at 5.A.9)

Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$. Find the mini poly.

SOLUTION:

$T(x_1, \dots, 0) = 0$ By (5.A.9) and [8.49], $1, 2, \dots, n$ are zeros of the mini poly of T .

(又 Each eigvals of T corresponds to exact one-dim subsp of \mathbb{F}^n .)

Define a poly q by $q(z) = (z - 1)(z - 2) \cdots (z - n)$, for all $z \in \mathbb{F}$. (Then q is the char poly of T .)

Because $q(T)e_j = [(T - I) \cdots (T - (j - 1)I)(T - (j + 1)I) \cdots (T - nI)](T - jI)e_j = 0$ for each j , where (e_1, \dots, e_n) is the standard basis. Thus $\forall v \in \mathbb{F}^n, q(T)v = 0$. Hence q is the mini poly of T . \square

• Suppose $n \in \mathbb{N}^+$. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.
[See also at (5.A.19)] Find the mini poly of T .

SOLUTION:

Because n and 0 are all the eigvals of T , 又 For all $e_k, Te_k = e_1 + \cdots + e_n$; $T^2e_k = n(e_1 + \cdots + e_n)$.

Hence $T^2e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T - n)$. Thus $z(z - n)$ is the mini poly of T . \square

• (4E 5.B.8)

Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is the operator of counterclockwise rotation by θ , where $\theta \in \mathbb{R}^+$. Find the mini poly of T .

SOLUTION:

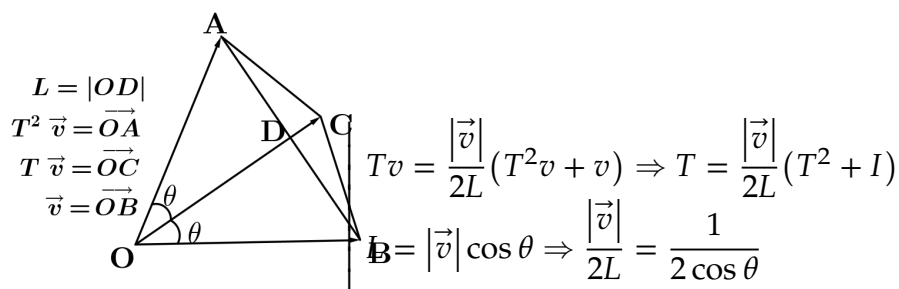
If $\theta = \pi + 2k\pi$, then $T(w, z) = (-w, -z)$, $T^2 = I$ and the mini poly is $z + 1$.

If $\theta = 2k\pi$, then $T = I$ and the mini poly is $z - 1$.

Now suppose (v, Tv) is linely inde. Then $\text{span}(v, Tv) = \mathbb{R}^2$.

Suppose the mini poly p is defined by $p(z) = z^2 + bz + c$ for all $z \in \mathbb{R}$.

Because



Hence $p(T) = T^2 - 2 \cos \theta T + I = 0$ and $z^2 - 2 \cos \theta z + 1$ is the mini poly of T . \square

OR. $\mathcal{M}(T, (e_1, e_2)) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$. By (4E 5.B.11), the mini poly is $z \pm 1$ or $z^2 - 2 \cos \theta z + 1$. \square

- (4E 5.B.11) Suppose V is a two-dim vecsp, $T \in \mathcal{L}(V)$, and the matrix of T with resp to some basis of V is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) Show that $T^2 - (a + d)T + (ad - bc)I = 0$.

(b) Show that the mini poly of T equals

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a + d)z + (ad - bc) & \text{otherwise.} \end{cases}$$

SOLUTION:

(a) Suppose the basis is (v, w) . Because $\begin{cases} Tv = av + bw \Rightarrow (T - aI)v = bw, \text{ then apply } (T - dI) \text{ to both sides} \\ Tw = cv + dw \Rightarrow (T - dI)w = cv, \text{ then apply } (T - aI) \text{ to both sides} \end{cases}$

$$\text{Hence } (T - aI)(T - dI) = bcI \Rightarrow T^2 - (a + d)T + (ad - bc)I = 0.$$

(b) If $b = c = 0$ and $a = d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the mini poly is $z - a$.

Otherwise, by (a), $z^2 - (a + d)z + (ad - bc)$ is a poly multi of the mini poly.

Now we prove that $T \notin \text{span}(I)$, so that then the mini poly of T has exactly degree 2.

(At least one of the assumption of (I),(II) below is true.)

(I) Suppose $a = d$, then $Tv = av + bw \notin \text{span}(v)$, $Tw = cv + aw \notin \text{span}(w)$.

(II) Suppose at most one of b, c is not 0. If $b = 0$, then $Tw \notin \text{span}(w)$; If $c = 0$, then $Tv \notin \text{span}(v)$ \square

5 Suppose $S, T \in \mathcal{L}(V)$, S is inv, and $p \in \mathcal{P}(\mathbf{F})$. Prove that $p(TS) = S^{-1}p(ST)S$.

SOLUTION:

We prove $(TS)^m = S^{-1}(ST)^m S$ for each $m \in \mathbf{N}$ by induction.

(i) $m = 0, 1$. $TS^0 = I = S^{-1}(ST)^0 S$; $TS = S^{-1}(ST)S$.

(ii) $m > 1$. Assume that $(TS)^m = S^{-1}(ST)^m S$.

$$\text{Then } (TS)^{m+1} = (TS)^m (TS) = S^{-1}(ST)^m STS = S^{-1}(ST)^{m+1} S.$$

Hence $\forall p \in \mathcal{P}(\mathbf{F})$, $p(TS) = a_0(TS)^0 + a_1(TS) + \dots + a_m(TS)^m$

$$= a_0[S^{-1}(ST)^0 S] + a_1[S^{-1}(ST)S] + \dots + a_m[S^{-1}(ST)^m S]$$

$$= S^{-1}[a_0(ST)^0 + a_1(ST) + \dots + a_m(ST)^m]S$$

$$= S^{-1}p(ST)S.$$

\square

- (4E 5.B.7)

(a) Give an example of $S, T \in \mathcal{L}(\mathbf{F}^2)$ such that

the mini poly of ST does not equal the mini poly of TS .

(b) Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that if S or T is inv,

then the mini poly of ST equals the mini poly of TS .

SOLUTION:

(a) Define S by $S(x, y) = (x, x)$. Define T by $T(x, y) = (0, y)$.

Then $ST(x, y) = 0$, $TS(x, y) = (0, x)$ for all $(x, y) \in \mathbb{F}^2$.

Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.

Hence the mini poly of ST does not equal to the mini poly of TS .

(b) Denote the mini poly of ST by p , and the mini poly TS by q .

Suppose S is inv.

$$\left. \begin{array}{l} p(ST) = 0 = Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a poly multi of } q. \\ q(TS) = 0 = S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a poly multi of } p. \end{array} \right\} \Rightarrow p = q.$$

Reversing the roles of S and T , we conclude that if T is inv, then $p = q$ as well. □

11 Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$, and $\alpha \in \mathbf{C}$.

Prove that α is an eigval of $p(T) \iff \alpha = p(\lambda)$ for some eigval λ of T .

SOLUTION:

(a) Suppose α is an eigval of $p(T) \iff (p(T) - \alpha I)$ is not inje.

Write $p(z) - \alpha = c(z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow p(T) - \alpha I = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.

By TIPS, $\exists (T - \lambda_j I)$ not inje. Thus $p(\lambda_j) - \alpha = 0$.

(b) Suppose $\alpha = p(\lambda)$ and λ is an eigval of T with an eigvec v . Then $p(T)v = p(\lambda)v = \alpha v$. □

OR. Define q by $q(z) = p(z) - \alpha$. λ is a zero of q .

Because $q(T)v = (p(T) - \alpha I)v = q(\lambda)v = (p(\lambda) - \alpha)v = 0$.

Hence $q(T)$ is not inje $\Rightarrow (p(T) - \alpha I)$ is not inje. □

12 [OR (4E.5.B.6)] Give an example of an operator on \mathbf{R}^2

that shows the result above does not hold if \mathbf{C} is replaced with \mathbf{R} .

SOLUTION:

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(w, z) = (-z, w)$.

By Problem (4E 5.B.11), $\mathcal{M}(T, ((1, 0), (0, 1))) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ the mini poly of T is $z^2 + 1$.

Define p by $p(z) = z^2$. Then $p(T) = T^2 = -I$. Thus $p(T)$ has eigval -1 .

While $\nexists \lambda \in \mathbf{R}$ such that $-1 = p(\lambda) = \lambda^2$. □

• (4E 5.B.17) Suppose V is finite-dim, $T \in \mathcal{L}(V)$, $\lambda \in \mathbf{F}$, and p is the mini poly of T .

Show that the mini poly of $(T - \lambda I)$ is the poly q defined by $q(z) = p(z + \lambda)$.

SOLUTION:

$q(T - \lambda I) = 0 \Rightarrow q$ is poly multi of the mini poly of $(T - \lambda I)$.

Suppose the degree of the mini poly of $(T - \lambda I)$ is n , and the degree of the mini poly of T is m .

By definition of mini poly,

n is the smallest such that $(T - \lambda I)^n \in \text{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{n-1})$;

m is the smallest such that $T^m \in \text{span}(I, T, \dots, T^{m-1})$.

$\forall T^k \in \text{span}(I, T, \dots, T^{k-1}) \iff (T - \lambda)^k \in \text{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{k-1})$.

Thus $n = m$. $\forall q$ is monic. By the uniqueness of mini poly. □

• (4E 5.B.18) Suppose V is finite-dim, $T \in \mathcal{L}(V)$, $\lambda \in \mathbf{F} \setminus \{0\}$, and p is the mini poly of T .

Show that the mini poly of λT is the poly q defined by $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.

SOLUTION:

$$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow q \text{ is a poly multi of the mini poly of } \lambda T.$$

Suppose the degree of the mini poly of λT is n , and the degree of the mini poly of T is m .

By definition of mini poly,

n is the smallest such that $(\lambda T)^n \in \text{span}(\lambda I, \lambda T, \dots, (\lambda T)^{n-1})$;

m is the smallest such that $T^m \in \text{span}(I, T, \dots, T^{m-1})$.

$$\text{又 } (\lambda T)^k \in \text{span}(\lambda I, \lambda T, \dots, (\lambda T)^{k-1}) \iff T^k \in \text{span}(I, T, \dots, T^{k-1}).$$

Thus $n = m$. 又 q is monic. By the uniqueness of mini poly. □

18 [OR (4E 5.B.15)] Suppose V is a finite-dim complex vecsp with $\dim V > 0$ and $T \in \mathcal{L}(V)$.

Define $f : \mathbb{C} \rightarrow \mathbb{R}$ by $f(\lambda) = \dim \text{range}(T - \lambda I)$.

Prove that f is not a continuous function.

SOLUTION: Note that V is finite-dim.

Let λ_0 be an eigval of T . Then $(T - \lambda_0 I)$ is not surj. Hence $\dim \text{range}(T - \lambda_0 I) < \dim V$.

Because T has finitely many eigvals. There exist a sequence of number $\{\lambda_n\}$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda_0$.

And λ_n is not an eigval of T for each $n \Rightarrow \dim \text{range}(T - \lambda_n I) = \dim V \neq \dim \text{range}(T - \lambda_0 I)$.

Thus $f(\lambda_0) \neq \lim_{n \rightarrow \infty} f(\lambda_n)$. □

- (4E 5.B.9) Suppose $T \in \mathcal{L}(V)$ is such that with resp to some basis of V , all entries of the matrix of T are rational numbers.

Explain why all coefficients of the mini poly of T are rational numbers.

SOLUTION:

Let (v_1, \dots, v_n) denote the basis such that $\mathcal{M}(T, (v_1, \dots, v_n))_{j,k} = A_{j,k} \in \mathbb{Q}$ for all $j, k = 1, \dots, n$.

Denote $\mathcal{M}(v_j, (v_1, \dots, v_n))$ by x_j for each v_j .

Suppose p is the mini poly of T and $p(z) = z^m + \dots + c_1 z + c_0$. Now we show that each $c_j \in \mathbb{Q}$.

Note that $\forall s \in \mathbb{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbb{Q}^{n,n}$ and $T^s v_k = A_{1,k}^s v_1 + \dots + A_{n,k}^s v_n$ for all $k \in \{1, \dots, n\}$.

$$\text{Thus } \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \dots + c_1 A + c_0 I)x_1 = \sum_{j=1}^n (A^m + \dots + c_1 A + c_0 I)_{j,1} x_j = 0; \\ \vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \dots + c_1 A + c_0 I)x_n = \sum_{j=1}^n (A^m + \dots + c_1 A + c_0 I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly, } \begin{cases} (A^m + \dots + c_1 A + c_0 I)_{1,1} = \dots = (A^m + \dots + c_1 A + c_0 I)_{n,1} = 0; \\ \vdots \quad \ddots \quad \vdots \\ (A^m + \dots + c_1 A + c_0 I)_{1,n} = \dots = (A^m + \dots + c_1 A + c_0 I)_{n,n} = 0; \end{cases}$$

Hence we get a system of n^2 linear equations in m unknowns c_0, c_1, \dots, c_{m-1} .

We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbb{Q}$. □

- [OR (4E 5.B.16), OR (8.C.18)] Suppose $a_0, \dots, a_{n-1} \in \mathbb{F}$. Let T be the operator on \mathbb{F}^n such that

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & -a_0 \\ 1 & 0 & & -a_1 \\ & 1 & \ddots & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ with resp to the standard basis } (e_1, \dots, e_n).$$

Show that the mini poly of T is p defined by $p(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + z^n$.

$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the mini poly of some operator. Hence a formula or an algorithm that could produce exact eigvals for each operator on each \mathbf{F}^n could then produce exact zeros for each poly [by 8.36(b)]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an operator.

SOLUTION: Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ is linely inde. \times The deg of mini poly is at most n .

$$\begin{aligned} T^n e_1 &= \cdots = T^{n-k} e_{1+k} = \cdots = T e_n = -a_0 e_1 - a_1 e_2 - a_2 e_3 - \cdots - a_{n-1} e_n \\ &= (-a_0 I - a_1 T - a_2 T^2 - \cdots - a_{n-1} T^{n-1}) e_1. \end{aligned}$$

Thus $p(T)e_1 = 0 = p(T)e_j$ for each $e_j = T^{j-1}e_1$. \square

• EIGENVALUES ON ODD-DIMENSIONAL REAL VECTOR SPACES

• EVEN-DIMENSIONAL NULL SPACE

Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$.

Prove that $\dim \text{null}(T^2 + bT + cI)$ is an even number.

SOLUTION:

Denote $\text{null}(T^2 + bT + cI)$ by R . Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.

Suppose λ is an eigval of T_R with an eigvec $v \in R$.

$$\text{Then } 0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \left((\lambda + b)^2 + c - \frac{b^2}{4}\right)v.$$

Because $c - \frac{b^2}{4} > 0$ and we have $v = 0$. Thus T_R has no eigvals.

Let U be an invar subsp of R that has the largest, even dim among all invar subsp.

Assume that $U \neq R$. Then $\exists w \in R$ but $w \notin U$. Let W be such that $(w, T|_R w)$ is a basis of W .

Because $T|_R^2 w = -bT|_R w - cw \in W$. Hence W is an invar subsp of dim 2.

Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,

for if not, because $w \notin U, T|_R w \in U$,

$U \cap W$ is invar under $T|_R$ of one dim (impossible because $T|_R$ has no eigvecs).

Hence $U + W$ is even-dim invar subsp under $T|_R$, contradicting the maximality of $\dim U$.

Thus the assumption was incorrect. Hence $R = \text{null}(T^2 + bT + cI) = U$ has even dim. \square

• OPERATORS ON ODD-DIMENSIONAL VECTOR SPACES HAVE EIGENVALUES

(a) Suppose $\mathbf{F} = \mathbf{C}$. Then by [5.21], we are done.

(b) Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, and $\dim V = n$ is an odd number.

Let $T \in \mathcal{L}(V)$ and the mini poly is p . Prove that T has an eigval.

SOLUTION:

(i) If $n = 1$, then we are done.

(ii) Suppose $n \geq 3$. Assume that every operator, on odd-dim vecsp of dim less than n , has an eigval.

If p is a poly multi of $(x - \lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] λ is an eigval of T and we are done.

Now suppose $b, c \in \mathbf{R}$ such that $b^2 < 4c$ and p is a poly multi of $x^2 + bx + c$ (see [4.17]).

Then $\exists q \in \mathcal{P}(\mathbf{R})$ such that $p(x) = q(x)(x^2 + bx + c)$ for all $x \in \mathbf{R}$.

Now $0 = p(T) = (q(T))(T^2 + bT + cI)$, which means that $q(T)|_{\text{range}(T^2 + bT + cI)} = 0$.

Because $\deg q < \deg p$ and p is the mini poly of T , hence $\text{range}(T^2 + bT + cI) \neq V$.

\times $\dim V$ is odd and $\dim \text{null}(T^2 + bT + cI)$ is even (by our previous result).

Thus $\dim V - \dim \text{null}(T^2 + bT + cI) = \dim \text{range}(T^2 + bT + cI)$ is odd.

By [5.18], $\text{range}(T^2 + bT + cI)$ is an invar subsp of V under T that has odd dim less than n .

Our induction hypothesis now implies that $T|_{\text{range}(T^2 + bT + cI)}$ has an eigval.

By mathematical induction. \square

-
- (2E Ch5.24) Suppose $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$ has no eigvals.
Prove that every invar subsp of V under T is even-dim.

SOLUTION:

Suppose U is such a subsp. Then $T|_U \in \mathcal{L}(U)$. We prove by contradiction.

If $\dim U$ is odd, then $T|_U$ has an eigval and so is T , so that \exists invar subsp of 1 dim, contradicts. \square

- (4E 5.B.29) Show that every operator on a finite-dim vecsp of $\dim \geq 2$ has a 2-dim invar subsp.

SOLUTION:

Using induction on $\dim V$.

(i) $\dim V = 2$, we are done.

(ii) $\dim V > 2$. Assume that the desired result is true for vecsp of smaller dim.

Suppose p is the mini poly of degree m and $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.

If $T = \lambda I$ ($\Leftrightarrow m = 1 \vee m = -\infty$), then we are done. ($m \neq 0$ because $\dim V \neq 0$.)

Now define a q by $q(z) = (z - \lambda_1)(z - \lambda_2)$.

By assumption, $T|_{\text{null } q(T)}$ has an invar subsp of dim 2. \square

ENDED

5.B: II 9 14 15 20 | 4E: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 11, 12, 13, 14

- (4E 5.C.1) Prove or give a counterexample:

If $T \in \mathcal{L}(V)$ and T^2 has an upper-trig matrix, then T has an upper-trig matrix.

SOLUTION:

- (4E 5.C.2) Suppose A and B are upper-trig matrices of the same size,
with $\alpha_1, \dots, \alpha_n$ on the diag of A and β_1, \dots, β_n on the diag of B .
(a) Show that $A + B$ is an upper-trig matrix with $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ on the diag.
(b) Show that AB is an upper-trig matrix with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diag.

SOLUTION:

- (4E 5.C.3)

Suppose $T \in \mathcal{L}(V)$ is inv and $B = (v_1, \dots, v_n)$ is a basis of V such that
 $\mathcal{M}(T, B) = A$ is upper trig, with $\lambda_1, \dots, \lambda_n$ on the diag.

Show that the matrix of $\mathcal{M}(T^{-1}, B) = A^{-1}$ is also upper trig, with $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ on the diag.

SOLUTION:

- 9** [4E 5.C.7] Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $v \in V$.

(a) Prove that $\exists!$ monic poly p_v of smallest degree such that $p_v(T)v = 0$.

(b) Prove that the mini poly of T is a poly multi of p_v .

SOLUTION:

- 14** [OR (4E 5.C.4)] Give an operator T such that with resp to some basis,
 $\mathcal{M}(T)_{k,k} = 0$ for each k , while T is inv.

SOLUTION:

15 [OR (4E 5.C.5)] Give an operator T such that with resp to some basis,
 $\mathcal{M}(T)_{k,k} \neq 0$ for each k , while T is not inv.

SOLUTION:

20 [OR (OR 4E 5.C.6)]

Suppose $\mathbf{F} = \mathbf{C}$, V is finite-dim, and $T \in \mathcal{L}(V)$.

Prove that if $k \in \{1, \dots, \dim V\}$, then V has a k dim subsp invar under T .

SOLUTION:

- (4E 5.C.8) Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\exists v \in V \setminus \{0\}$ such that $T^2v + 2Tv = -2v$.
 - (a) Prove that if $\mathbf{F} = \mathbf{R}$, then \nexists a basis of V with resp to which T has an upper-trig matrix.
 - (b) Prove that if $\mathbf{F} = \mathbf{C}$ and A is an upper-trig matrix that equals the matrix of T with resp to some basis of V , then $-1 + i$ or $-1 - i$ appears on the diag of A .

SOLUTION:

- (4E 5.C.9) Suppose $B \in \mathbf{F}^{n,n}$ with complex entries.
Prove that \exists inv $A \in \mathbf{F}^{n,n}$ with complex entries such that $A^{-1}BA$ is an upper-trig matrix.

SOLUTION:

- (4E 5.C.10) Suppose $T \in \mathcal{L}(V)$ and (v_1, \dots, v_n) is a basis of V .
Show that the following are equi.
 - (a) The matrix of T with resp to (v_1, \dots, v_n) is lower trig.
 - (b) $\text{span}(v_k, \dots, v_n)$ is invar under T for each $k = 1, \dots, n$.
 - (c) $Tv_k \in \text{span}(v_k, \dots, v_n)$ for each $k = 1, \dots, n$.

SOLUTION:

- (4E 5.C.11) Suppose $\mathbf{F} = \mathbf{C}$ and V is finite-dim.
Prove that if $T \in \mathcal{L}(V)$, then T has a lower-trig matrix with resp to some basis.

SOLUTION:

- (4E 5.C.12)
Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has an upper-trig matrix with resp to some basis, and U is a subsp of V that is invar under T .
 - (a) Prove that $T|_U$ has an upper-trig matrix with resp to some basis of U .
 - (b) Prove that T/U has an upper-trig matrix with resp to some basis of V/U .

SOLUTION:

- (4E 5.C.13) Suppose V is finite-dim, $T \in \mathcal{L}(V)$. Suppose U is an invar subsp of V under T such that $T|_U$ has an upper-trig matrix and also T/U has an upper-trig matrix.
Prove that T has an upper-trig matrix.

SOLUTION:

- (4E 5.C.14) Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that T has an upper-trig matrix $\iff T'$ has an upper-trig matrix.

SOLUTION:

ENDED

5.C

XXXX

ENDED

5.E* (4E) 1 2 3 4 5 6 7 8 9 10

- 1 Give an example of two commuting operators $S, T \in \mathbf{F}^4$ such that there is an invar subsp of \mathbf{F}^4 under S but not under T and an invar subsp of \mathbf{F}^4 under T but not under S .

SOLUTION:

- 2 Suppose \mathcal{E} is a subset of $\mathcal{L}(V)$ and every element of \mathcal{E} is diagable. Prove that \exists a basis of V with resp to which every element of \mathcal{E} has a diag matrix \iff every pair of elements of \mathcal{E} commutes. This exercise extends [5.76], which considers the case in which \mathcal{E} contains only two elements. For this exercise, \mathcal{E} may contain any number of elements, and \mathcal{E} may even be an infinite set.

SOLUTION:

- 3 Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.
(a) Prove that $\text{null } p(S)$ is invar under T .
(b) Prove that $\text{range } p(S)$ is invar under T .
See NOTE FOR[5.17] for the special case $S = T$.

SOLUTION:

- 4 Prove or give a counterexample:
A diag matrix A and an upper-trig matrix B of the same size commute.

SOLUTION:

- 5 Prove that a pair of operators on a finite-dim vecsp commute \iff their dual operators commute.

SOLUTION:

- 6 Suppose V is a finite-dim complex vecsp and $S, T \in \mathcal{L}(V)$ commute. Prove that $\exists \alpha, \lambda \in \mathbf{C}$ such that $\text{range}(S - \alpha I) + \text{range}(T - \lambda I) \neq V$.

SOLUTION:

- 7 Suppose V is a complex vecsp, $S \in \mathcal{L}(V)$ is diagable, and T commutes with S . Prove that \exists basis B of V such that S has a diag matrix with resp to B and T has an upper-trig matrix with resp to B .

SOLUTION:

- 8** Suppose $m = 3$ in Example [5.72]
and D_x, D_y are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbb{R}^2)$ from that example.
Find a basis of $\mathcal{P}_3(\mathbb{R}^2)$ with resp to which D_x and D_y each have an upper-trig matrix.

SOLUTION:

- 9** Suppose V is a finite-dim nonzero complex vecsp.
Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.
(a) Prove that $\exists v \in V$ is an eigvec for every element of \mathcal{E} .
(b) Prove that \exists a basis of V with resp to which every element of \mathcal{E} has an upper-trig matrix.

SOLUTION:

- 10** Give an example of two commuting operators S, T on a finite-dim real vecsp such that
 $S + T$ has a eigval that does not equal an eigval of S plus an eigval of T
and ST has a eigval that does not equal an eigval of S times an eigval of T .

SOLUTION:

ENDED