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Email: 13012057210@163.com

简介

这是我个人用于复习的「***Linear Algebra Done Right 3E/4E, by Sheldon Axler***」笔记，一本习题选答与课文补注。因为使用中文会给我编撰这份笔记带来额外的中英文输入法切换的工作成本，况且对于专业学习者，直接使用英文不会造成任何困扰。但英文词句的冗长性拖慢我复习的效率，所以我对许多常用术语作了简写。这份笔记的内容范围和标识说明，我已经在自述中写得很清楚，不再赘述。这份笔记尚处于缓慢的编撰进度中。

我目前还没有能力和资格评论原书好坏以及线性代数课程教材选用的问题。但作为原书的学习者，我可以说，这本书作为初学线性代数的第一教材，虽然不需要其他辅助教材，但要求学习者有足够的耐心和毅力：课文一次看不懂就多看几遍，一天看不懂就分三天看；习题一个小时做不出来，隔六个小时再尝试，一天做不出来，就隔天再尝试。我虽然没有学过除此以外的其他任何线性代数教材，但我认为这样钻研原书是值得的。

Goto

1	2	3	4	5	6	7	8	9	10
A	A	A		A	A	A	A	A	A
B	B	B		B^{I}	B	B	B	B	B
				B^{II}					
C	C	C		C	C	C	C		
		D			D	D	D		
		E		E*					
		F				F*			

Abbreviation Table

sup	suppose	asm	assum(e)(ption)	shat	show that
provt	prove that	exe	exercise	becs	because
ele	element(s)	arb	arbitrary	suth	such that
othws	otherwise	notat	notation(al)	solus	solution
exa	example	simlr	similar(ly)	algo	algorithm
div	div(ide)(ision)	conveni	convenience	restr	restrict(ion)(ive)(ing)
stam	statement	ctrapos	constrapositive	ctradic	contradict(s)(ion)
def	definition	closd	closed under	sp	space
val	value	len	length	disti	distinct
min	mini(mal(ity))(mum)	max	maxi(mal(ity))(mum)	add	addi(tion)(tive)
multi	multipl(e)(icati-on/ve)	assoc	associa(tive)(tivity)	distr	distributive propert(ies)(ty)
commu	commut(es)(ing)(ativity)	-ec	-ec(t)(tor)(tion)(tive)	inv	inver(se)(tib-le/ility)
id	identity	existns	existence	uniqnes	uniqueness
finide	finite-dimensional	fini	finite	infily	infinitely
linely inde	linearly independen(t)(ce)	linely dep	linearly dependen(t)(ce)	std basis	standard basis
dim	dimension(al)	poly	polynomial	coeff	coefficient
deg	degree	deri	derivative(s)	diff	differentia(l)(ting)(tion)
req	require(s)(d)/requiring	B_V	basis of V	inje	injective
surj	surjective	col	column	ent	entr(y)(ies)
with resp	with respect	corres	correspond(ing)	iso	isomorph(ism)(ic)
optor	operator				
quotient	quot	tspose	transpose	tslate	translate
invar	invariant	invard	invariant under	invarsp	invariant subspace
eig-	eigen-	ch	characteristic	diag	diagonal(iza-ble/ility/tion)
trig	triangular	G disk(s)	Gershgorin disk(s)		

1·B

1 *Provt* $\forall v \in V, -(-v) = v$.

Solus: $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$.

Or. Becs $-(-v) + (-v) = 0$ 又 $v + (-v) = 0$. Now by the uniqnes of add inv. $\square$

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2 *Sup* $a \in \mathbf{F}, v \in V$, *and* $av = 0$. *Provt* $a = 0$ *or* $v = 0$.

Solus: Sup $a \neq 0, \exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. $\square$

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3 *Sup* $v, w \in V$. *Explain why* $\exists\,!\, x \in V, v + 3x = w$.

Solus: $v + 3x = w \quad\Leftrightarrow\quad 3x = w - v \quad\Leftrightarrow\quad x = \frac{1}{3}(w - v)$. $\square$

Or. [*Existns*] Let $x = \dfrac{1}{3}(w - v)$.

[*Uniqnes*] If $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) $-$ (II) : $3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. $\square$

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5 *Shat in the def of a vecsp, the add inv condition can be replaced by* [1.29].

Hint: *Sup V satisfies all conds in the def, except we've replaced the add inv cond with* [1.29]. *Provt the add inv is true.*

Using [1.31]. $0v = 0$ for all $v \in V \Longleftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. $\square$

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6 *Let* ∞ *and* $-\infty$ *denote two disti objects, neither of which is in* $\mathbf{R}$.
Define an add and scalar multi on $\mathbf{R} \cup \{\infty, -\infty\}$ *as you could guess.*
The operations of real numbers is as usual. While for $t \in \mathbf{R}$ *define*

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

(I) $t + \infty = \infty + t = \infty + \infty = \infty$,
(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,
(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of add and scalar multi, is $\mathbf{R} \cup \{\infty, -\infty\}$ *a vecsp over* $\mathbf{R}$? *Explain.*

Solus: Not a vecpsp, since the add and scalar mult is not assoc and distr.

By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

Or. By Distr: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. $\square$

• **Tips:** *About the Field* $\mathbf{F}$: *Many choices.*

Exa: $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \ldots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$. [Using Euler's Theorem.]

Ended

1·C

7 8 9 11 12 13 15 16 17 18 21 23 24

• **Note For [1.45]:** If $\mathbf{F} = \{0, 1\}$. Provt if $U + W$ is a direct sum, then $U \cap W = \{0\}$.

Becs $\forall v \in U \cap W, \exists\,!\,(u, w) \in U \times W, v = u + w$.

If $U \cap W \neq \{0\}$, then (u, w) can be $(v, 0)$ or $(0, v)$, ctradic the uniqnes. $\square$

• **Tips 1:** Sup $U, W \subseteq V$. And U, W, V are vecsps $\Rightarrow U, W$ are subsps of V.
Then $U + W$ is also a subsp of V. Becs $\forall u \in U, w \in U, u + w \in V$ since $u, w \in V$.

---

7 *Give a nonempty $U \subseteq \mathbf{R}^2$, U is closd taking add invs and add, but is not a subsp of $\mathbf{R}^2$.*

Solus: $\big($ $0 \in U$; $v \in U \Rightarrow -v \in U$. And operations on U are the same as $\mathbf{R}^2$. $\big)$ Let $\mathbf{Z}^2, \mathbf{Q}^2$.

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8 *Give a nonempty $U \subseteq \mathbf{R}^2$, U is closd scalar multi, but is not a subsp of $\mathbf{R}^2$.*

Solus: Let $U = \big\{(x, y) \in \mathbf{R}^2 : x = 0 \vee y = 0\big\}$.

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9 *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if $\exists\, p \in \mathbf{N}^+$, $f(x) = f(x + p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions $\mathbf{R} \to \mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$? Explain.*

Solus: Denote the set by S.
Sup $h(x) = \cos x + \sin\sqrt{2}x \in S$, since $\cos x, \sin\sqrt{2}x \in S$.
Asm $\exists\, p \in \mathbf{N}^+$ suth $h(x) = h(x + p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.
Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$
$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.
Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Ctradic! $\square$

Or. Becs $[\mathrm{I}] : \cos x + \sin\sqrt{2}x = \cos(x + p) + \sin\big(\sqrt{2}x + \sqrt{2}p\big)$. By diff twice,
$[\mathrm{II}] : \cos x + 2\sin\sqrt{2}x = \cos(x + p) + 2\sin\big(\sqrt{2}x + \sqrt{2}p\big)$.

$$\left.\begin{array}{ll} [\mathrm{II}] - [\mathrm{I}] : & \sin\sqrt{2}x = \sin\big(\sqrt{2}x + \sqrt{2}p\big) \\ 2[\mathrm{I}] - [\mathrm{II}] : & \cos x = \cos(x + p) \end{array}\right\} \Rightarrow \text{Let } x = 0,\ p = \frac{m\pi}{\sqrt{2}} = 2k\pi.\ \text{Ctradic.}$$ $\square$

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24 *Let $V_E = \big\{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is even}\big\}$, $V_O = \big\{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is odd}\big\}$. Shat $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$.*

Solus: (a) $V_E \cap V_O = \big\{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x)\big\} = \big\{0\big\}$.

$$\text{(b)} \left|\begin{array}{l} \text{Let } f_e(x) = \frac{1}{2}\big[g(x) + g(-x)\big] \Longrightarrow f_e \in V_E \\ \text{Let } f_o(x) = \frac{1}{2}\big[g(x) - g(-x)\big] \Longrightarrow f_o \in V_O \end{array}\right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}},\ g(x) = f_e(x) + f_o(x).$$ $\square$

• *Sup U, W, V_1, V_2, V_3 are subsps of V.*
15 $U + U \ni u + w \in U$. **16** $U + W \ni u + w = w + u \in W + U$. $\square$
17 $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3)$. $\square$
• $(U + W)_{\mathrm{C}} \ni (u_1 + w_1) + \mathrm{i}(u_2 + w_2) = (u_1 + \mathrm{i}u_2) + (w_1 + \mathrm{i}w_2) \in U_{\mathrm{C}} + W_{\mathrm{C}}$. $\square$

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18 *Does the add on the subsps of V have an add id? Which subsps have add invs?*

Solus: Sup Ω is the unique add id.
(a) For any subsp U of V. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.
(b) Now sup W is an add inv of $U \Rightarrow U + W = \Omega$.
Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. $\square$

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11 *Provt the intersec of every collec of subsps of V is a subsp of V.*

SOLUS: Sup $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collec of subsps of V; here Γ is an index set.

We shat $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vecs that are in U_α for each $\alpha \in \Gamma$, is a subsp of V.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closd add.

(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closd scalar multi.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of V that is closd add and scalar multi. □

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12 *Sup U, W are subsps of V. Provt $U \cup W$ is a subsp of $V \iff U \subseteq W$ or $W \subseteq U$.*

SOLUS: (a) Sup $U \subseteq W$. Then $U \cup W = W$ is a subsp of V.

(b) Sup $U \cup W$ is a subsp of V. Asm $U \not\subseteq W,\ U \not\supseteq W$ ($U \cup W \neq U$ and W).

Then $\forall a \in U \wedge a \notin W, \forall b \in W \wedge b \notin U$, we have $a + b \in U \cup W$.

$a + b \in U \Rightarrow b = (a+b) + (-a) \in U$, ctradic $\Rightarrow W \subseteq U$. | Ctradic asm.

$a + b \in W \Rightarrow a = (a+b) + (-b) \in W$, ctradic $\Rightarrow U \subseteq W$. | □

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13 *Provt the union of three subsps of V is a subsp of V if and only if one of the subsps contains the other two.*

This exe is not true if we replace $\mathbf{F}$ with a field containing only two ele.

SOLUS:

Sup U_1, U_2, U_3 are subsps of V. Denote $U_1 \cup U_2 \cup U_3$ by $\mathcal{U}$.

(a) Sup that one of the subsps contains the other two.

Then $\mathcal{U} = U_1, U_2$ or U_3 is a subsp of V.

(b) Sup that $U_1 \cup U_2 \cup U_3$ is a subsp of V.

Distinctively notice that $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.

Also note that, if $U \cup W = V$ is a vecsp, then in general U and W are not subsps of V.

Hence this literal trick is invalid.

(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.

By applying Exe (12) we conclude that one U_j contains the other two. Thus we are done.

(II) Asm no U_j is contained in the union of the other two,

and no U_j contains the union of the other two. Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3;\ v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if any $v + \lambda u \in W \cap U_1$ then $v + \lambda u - \lambda u = v \in U_1$.

Now $W \subseteq U_1 \cup U_2 \cup U_3 \Rightarrow W \subseteq U_2 \cup U_3$. $\forall v + \lambda u \in W, v + \lambda u \in U_i, i = 2, 3$.

If $U_2 \subseteq U_3$ or $U_2 \supseteq U_3$, then $\mathcal{U} = U_1 \cup U_i, i = 2, 3$. By Exe (12) we are done.

Othws, both $U_2, U_3 \neq \{0\}$. Becs $W \subseteq U_2 \cup U_3$ *has at least three ele.*

There must be some U_i that contains at least two ele of W.

$\exists$ disti $\lambda_1, \lambda_2 \in \mathbf{F}, v + \lambda_1 u, v + \lambda_2 u \in U_i, i \in \{2, 3\}$.

Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Ctradic. □

EXA: Let $\mathbf{F} = \mathbf{Z}_2$. $U_1 = \{u, 0\}, U_2 = \{v, 0\}, U_3 = \{v + u, 0\}$. While $\mathcal{U} = \{0, u, v, v + u\}$ is a subsp.

• *Sup* $U = \{(x,x,y,y)\}, W = \{(x,x,x,y)\} \subseteq \mathbf{F}^4$. *Provt* $U + W = \{(x,x,y,z)\}$.

Solus: Let T denote $\{(x,x,y,z)\}$. By def, $U + W \subseteq T$.

And $T \ni (x,x,y,z) \Rightarrow (0,0,y-x,y-x) + (x,x,x,-y+x+z) \in U + W$. Hence $T \subseteq U + W$. $\square$

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21 *Sup* $U = \{(x,y,x+y,x-y,2x)\}$. *Find a* W *suth* $\mathbf{F}^5 = U \oplus W$.

Solus: Let $W = \{(0,0,z,w,u)\}$. Then $U \cap W = \{0\}$.

And $\mathbf{F}^5 \ni (x,y,z,w,u) \Rightarrow (x,y,x+y,x-y,2x) + (0,0,z-x-y,w-x-y,u-2x) \in U + W$.

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23 *Give an exa of vecsps* V_1, V_2, U *suth* $V_1 \oplus U = V_2 \oplus U$, *but* $V_1 \neq V_2$.

Solus: $V = \mathbf{F}^2, U = \{(x,x)\}, V_1 = \{(x,0)\}, V_2 = \{(0,x)\}$.

• **Tips 2:** *Sup* $V_1 \subseteq V_2$ *in Exe* (23). *Provt* $V_1 = V_2$.

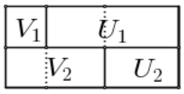
Solus:

Becs the subset V_1 of vecsp V_2 is closd add and scalar multi, V_1 is a subspace of V_2.

Sup W is suth $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$.

If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, ctradic. Hence $W = \{0\}, V_1 = V_2$. $\square$

• *Sup* V_1, V_2, U_1, U_2 *are vecsps*, $V_1 \oplus U_1 = V_2 \oplus U_2$, $V_1 \subseteq V_2$, $U_2 \subseteq U_1$.
Prove or give a counterexa: $V_1 = V_2$, $U_1 = U_2$.



Solus: Let $U_2 = \{0\}$. Give an exa that each of V_1, V_2, U_1 is nonzero. $\square$

• **Tips 3:** *Sup the intersec of any two of the vecsps* U, W, X, Y *is* $\{0\}$.
Give an exa that $(X \oplus U) \cap (Y \oplus W) \neq \{0\}$.

Solus: [Using notas in Chapter 2.] Let $B_X = (e_1), B_U = (e_2 - e_1), B_Y = (\ ), B_W = (e_2)$.

• **Tips 4:** *Let* $V = U + W$, $I = U \cap W$, $U = I \oplus X$, $W = I \oplus Y$. *Provt* $V = I \oplus (X \oplus Y)$.

Solus: We shat $X \cap Y = U \cap Y = W \cap X = \{0\}$ by ctradic.

$X \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap X \neq \{0\}, I \cap Y \neq \{0\}$.

$U \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap Y \neq \{0\}$. Simlr for $W \cap X$.

Thus $I + (X + Y) = (I \oplus X) \oplus Y = I \oplus (X \oplus Y)$.

Now we shat $V = I + (X + Y)$. $\forall v \in V, v = u + w, \exists\, (u,w) \in U \times W$

$\Rightarrow \exists\, (i_u, x_u) \in I \times X,\ (i_w, y_w) \in I \times Y,\ v = (i_u + i_w) + x_u + y_w \in I + (X + Y)$. $\square$

Ended

2·A

1 2 10 11 14 16 17 | 4E: 3,14

1 *Provt [P] (v_1, v_2, v_3, v_4) spans $V \iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V [Q].*

Solus: Note that $V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists a_1, \dots, a_n \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n$.

Asm $\forall v \in V, \exists a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbf{F}$, (that is, if $\exists a_i$, then we are to find b_i, vice versa)

$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4$

$= b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4$

$= a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \cdots + a_4) v_4.$ □

• *Sup $(v_1, \dots, v_m)$ is a list of vecs in V. For each k, let $w_k = v_1 + \cdots + v_k$.*
(a) *Shat $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.*
(b) *Shat [P] $(v_1, \dots, v_m)$ is linely inde $\iff (w_1, \dots, w_m)$ is linely inde [Q].*

Solus:

(a) Asm $a_1 v_1 + \cdots + a_m v_m = b_1 w_1 + \cdots + b_m w_m = b_1 v_1 + \cdots + b_k(v_1 + \cdots + v_k) + \cdots + b_m(v_1 + \cdots + v_m)$.
Then $a_k = b_k + \cdots + b_m$; $a_{k+1} = b_{k+1} + \cdots + b_m \Rightarrow b_k = a_k - a_{k+1}$; $b_m = a_m$. Simlr to Exe (1).

(b) $P \Rightarrow Q$: $b_1 w_1 + \cdots + b_m w_m = 0 = a_1 v_1 + \cdots + a_m v_m$, where $0 = a_k = b_k + \cdots + b_m$.
$Q \Rightarrow P$: $a_1 v_1 + \cdots + a_m v_m = 0 = b_1 w_1 + \cdots + b_m w_m = 0$, where $0 = b_m = a_m$, $0 = b_k = a_k - a_{k+1}$.

Or. By (a), let $W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$. Sup $(w_1, \dots, w_m)$ is linely dep.
By [2.21](b), a list of len $(m-1)$ spans W. 又 By [2.23], $(w_1, \dots, w_m)$ linely inde $\Rightarrow m \leqslant m - 1$.
Thus $(w_1, \dots, w_m)$ is linely dep. Now reversing the roles of v and w. □

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2 (a) [P] *A list (v) of len 1 in V is linely inde $\iff v \neq 0$.* [Q]
(b) [P] *A list (v, w) of len 2 in V is linely inde $\iff \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$.* [Q]

Solus: (a) $Q \Rightarrow P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ linely inde.
$P \Rightarrow Q : (v)$ linely inde $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.
$\neg Q \Rightarrow \neg P : v = 0 \Rightarrow av = 0$ while we can let $a \neq 0 \Rightarrow (v)$ is linely dep.
$\neg P \Rightarrow \neg Q : (v)$ linely dep $\Rightarrow av = 0$ while $a \neq 0 \Rightarrow v = 0$.

(b) $P \Rightarrow Q : (v, w)$ linely inde $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.
$Q \Rightarrow P :$ no scalar multi $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v, w)$ linely inde.
$\neg P \Rightarrow \neg Q : (v, w)$ linely dep $\Rightarrow$ if $av + bw = 0$, then a or $b \neq 0 \Rightarrow$ scalar multi.
$\neg Q \Rightarrow \neg P :$ scalar multi $\Rightarrow$ if $av + bw = 0$, then a or $b \neq 0 \Rightarrow$ linely dep. □

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10 *Sup $(v_1, \dots, v_m)$ is linely inde in V and $w \in V$.*
Provt if $(v_1 + w, \dots, v_m + w)$ is linely depe, then $w \in \operatorname{span}(v_1, \dots, v_m)$.

Solus:

Note that $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0 \Rightarrow a_1 v_1 + \cdots + a_m v_m = -(a_1 + \cdots + a_m) w$.
Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1 v_1 + \cdots + a_m v_m = 0$ while $a_i \neq 0$ for some i, ctradic.
Or. We prove the ctrapos: Sup $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $a_1 + \cdots + a_m = 0$.
Thus $a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$. Hence $(v_1 + w, \dots, v_m + w)$ is linely inde. □
Or. $\exists j \in \{1, \dots, m\}, v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and we are done.
If $j \geqslant 2$, then $\exists a_i \in \mathbf{F}, v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1 v_1 + \cdots + a_{j-1} v_{j-1}$.
Where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, ctradic.
Now $w = \lambda^{-1}(a_1 v_1 + \cdots + a_{j-1} v_{j-1} - v_j) \Rightarrow w \in \operatorname{span}(v_1, \dots, v_m)$. □

---

11 *Sup $(v_1, \dots, v_m)$ is linely inde in V and $w \in V$.*
Shat $[P]$ $(v_1, \dots, v_m, w)$ is linely inde $\Longleftrightarrow$ $w \notin \operatorname{span}(v_1, \dots, v_m)$ $[Q]$.

Solus: $\neg Q \Rightarrow \neg P$: Sup $w \in \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1, \dots, v_m, w)$ is linely depe.
$\neg P \Rightarrow \neg Q$: Sup $(v_1, \dots, v_m, w)$ is linely dep. Then by [2.21](a), $w \in \operatorname{span}(v_1, \dots, v_m)$. □

---

14 *Provt $[P]$ V is infinide $\Longleftrightarrow$ $[Q]$ | there is a sequence $(v_1, v_2, \dots)$ in V suth $(v_1, \dots, v_m)$ is linely inde for each $m \in \mathbf{N}^+$.*

Solus:

$P \Rightarrow Q$: Sup V is infinide, so that no list spans V.
Step 1 Pick a $v_1 \neq 0$, (v_1) linely inde.
Step m Pick a $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$, by Exe (11), $(v_1, \dots, v_m)$ is linely inde.
This process recursively defines the desired sequence $(v_1, v_2, \dots)$.

$\neg P \Rightarrow \neg Q$: Sup V is finide and $V = \operatorname{span}(w_1, ..., w_m)$.
Let $(v_1, v_2, \dots)$ be a sequence in V, then $(v_1, v_2, \dots, v_{m+1})$ must be linely dep.

Or. $Q \Rightarrow P$: Sup there is such a sequence.
Choose an m. Sup a linely inde list $(v_1, \dots, v_m)$ spans V.
Simlr to [2.16]. $\exists\, v_{m+1} \in V \backslash \operatorname{span}(v_1, \dots, v_m)$. Hence no list spans V. □

---

16 *Provt the vecsp of all continuous functions in $\mathbf{R}^{[0,1]}$ is infinide.*

Solus: Denote the vecsp by U.

Choose one $m \in \mathbf{N}^+$. Sup $a_0, \dots, a_m \in \mathbf{R}$ are suth $p(x) = a_0 + a_1 x + \cdots + a_m x^m = 0$, $\forall x \in [0,1]$.
Then p has infily many roots and hence each $a_k = 0$, othws $\deg p \geqslant 0$, ctradic [4.12].
Thus $(1, x, \dots, x^m)$ is linely inde in $\mathbf{R}^{[0,1]}$. Simlr to [2.16], U is infinide. □

Or. Note that $\dfrac{1}{1} > \dfrac{1}{2} > \cdots > \dfrac{1}{m}$, $\forall m \in \mathbf{N}^+$. Sup $f_m = \begin{cases} x - \frac{1}{m}, & x \in (\frac{1}{m}, 1\,] \\ 0, & x \in [\,0, \frac{1}{m}\,] \end{cases}$

Then $f_1\left(\frac{1}{m}\right) = \cdots = f_m\left(\frac{1}{m}\right) = 0 \neq f_{m+1}\left(\frac{1}{m}\right)$. Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. By Exe (14). □

---

17 *Sup $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ suth $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$.*
Provt $(p_0, p_1, \dots, p_m)$ is not linely inde in $\mathcal{P}_m(\mathbf{F})$.

Solus:

Sup $(p_0, p_1, \dots, p_m)$ is linely inde. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z$.
Notice that $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \cdots + a_m p_m(z)$, for if not, let $z = 2$. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has len $(m+1)$.
Hence $(p_0, p_1, \dots, p_m)$ is linely depe. For if not, then becs $(1, z, \dots, z^m)$ of len $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$, by the steps in [2.23] trivially, $(p_0, p_1, \dots, p_m)$ of len $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$. Ctradic. □

Or. Note that $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(\underbrace{1, z, \dots, z^m}_{\text{of len } (m+1)})$. Then $(p_0, p_1, \dots, p_m, z)$ of len $(m+2)$ is linely dep.
As shown above, $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$. And hence by [2.21](a), $(p_0, p_1, \dots, p_m)$ is linely dep. □

Ended

2·B 1 7 8 | 4E: 5 9 11

7 *Prove or give a counterexa: If* (v_1, v_2, v_3, v_4) *is a basis of* V *and* U *is a subsp of* V *suth* $v_1, v_2 \in U$ *and* $v_3 \notin U$ *and* $v_4 \notin U$*, then* (v_1, v_2) *is a basis of* U.

Solus: A counterexa: Let $V = \mathbf{R}^4$ and $B_V = (e_1, e_2, e_3, e_4)$ be std basis.

Let $v_1 = e_1, v_2 = e_2, v_3 = e_3 + e_4, v_4 = e_4$. Then $(v_1, \dots, v_4)$ is a basis of $\mathbf{R}^4$.

Let $U = \operatorname{span}(e_1, e_2, e_3) = \operatorname{span}(v_1, v_2, v_3 - v_4)$. Then $v_3 \notin U$ and (v_1, v_2) is not a basis of U. □

• **Note For "$\complement_V U \cup \{0\}$":** "$\complement_V U \cup \{0\}$" is supd to be a subsp W suth $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cup \{0\} \\ u \pm w \in \complement_V U \cup \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cup \{0\}$. Ctradic.

To fix this, denote the set $\{W_1, W_2, \cdots\}$ by $\mathcal{S}_V U$, where for each W_i, $V = U \oplus W_i$. See also in (1.C.23).

• **Tips:** *Sup* V *is finide with* $\dim V = n$ *and* U *is a subsp of* V *with* $U \neq V$.
Provt $\exists\, B_V = (v_1, \dots, v_n)$ *suth each* $v_k \notin U$.

Note that $U \neq V \Rightarrow n \geqslant 1$. We will construct B_V via the following process.

Step 1. $\exists\, v_1 \in V \backslash U \Rightarrow v_1 \neq 0$. If $\operatorname{span}(v_1) = V$ then we stop.

Step k. Sup $(v_1, \dots, v_{k-1})$ is linely inde in V, each of which belongs to $V \backslash U$.

Note that $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$. And if $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U = V$, then by (1.C.12),

$\big[$ becs $\operatorname{span}(v_1, \dots, v_{k-1}) \not\subseteq U$, $\big]$ $U \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) \Rightarrow \operatorname{span}(v_1, \dots, v_{k-1}) = V$.

Hence becs $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$, it must be case that $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U \neq V$.

Thus $\exists\, v_k \in V \backslash U$ suth $v_k \notin \operatorname{span}(v_1, \dots, v_{k-1})$.

By (2.A.11), $(v_1, \dots, v_k)$ is linely inde in V. If $\operatorname{span}(v_1, \dots, v_k) = V$, then we stop.

Becs V is finide, this process will stop after n steps. □

Or. Sup $U \neq \{0\}$. Let $B_U = (u_1, \dots, u_m)$. Extend to a basis $(u_1, \dots, u_n)$ of V.

Then let $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$. □

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1 *Find all vecsps on whatever* $\mathbf{F}$ *that have exactly one basis.*

Solus: The trivial vecsp $\{0\}$ will do. Indeed, the only basis of $\{0\}$ is the empty list ( ).

Now consider the field $\{0, 1\}$ containing only the add id and multi id,

with $1 + 1 = 0$. Then the list (1) is the unique basis. Now the vecsp $\{0, 1\}$ will do.

Comment: All vecsp on such $\mathbf{F}$ of dim 1 will do.

And more generally, consider $\mathbf{F} = \mathbf{Z}_m$, $\forall m - 1 \in \mathbf{N}^+$. For each $s, t \in \{1, \dots, m\}$,

$\mathbf{F} = \operatorname{span}(K_s) = \operatorname{span}(K_t)$. More than one basis. So are $\mathbf{Q}, \mathbf{R}, \mathbf{C}$ and all vecsps on such $\mathbf{F}$.

Consider other $\mathbf{F}$. Note that this $\mathbf{F}$ contains at least and strictly more than 0 and 1. Failed. □

• (4E 9) *Sup* $(v_1, \dots, v_m)$ *is a list of vecs in* V. *For* $k \in \{1, \dots, m\}$*, let* $w_k = v_1 + \cdots + v_k$.
Shat $[P]$ $B_V = (v_1, \dots, v_m) \iff B_W = (w_1, \dots, w_m)$. $[Q]$

Solus: Notice that $B_U = (u_1, \dots, u_n) \iff \forall u \in U, \exists!\, a_i \in \mathbf{F}, u = a_1 u_1 + \cdots + a_n u_n$.

$P \Rightarrow Q: \forall v \in V, \exists!\, a_i \in \mathbf{F},\ v = a_1 v_1 + \cdots + a_m v_m \Rightarrow v = b_1 w_1 + \cdots + b_m w_m, \exists!\, b_k = a_k - a_{k+1}, b_m = a_m$.

$Q \Rightarrow P: \forall v \in V, \exists!\, b_i \in \mathbf{F},\ v = b_1 w_1 + \cdots + b_m w_m \Rightarrow v = a_1 v_1 + \cdots + a_m v_m, \exists!\, a_k = \sum_{j=k}^m b_j$. □

Comment: See also ??? in (3.F).

- (4E 5) *Sup U, W are finide, $V = U + W$, $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$.*
  Provt $\exists\, B_V$ consisting of vecs in $U \cup W$.

Solus: $V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(\overbrace{u_1, \dots, u_m, w_1, \dots, w_n}^{\text{Reduce}})$. By [2.31]. $\square$

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8 *Sup $V = U \oplus W$, $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$.*
Provt $B_V = \big(u_1, \dots, u_m, w_1, \dots, w_n\big)$.

Solus: $\forall v \in V, \exists!\, u \in U, w \in W \Rightarrow \exists!\, a_i, b_i \in \mathbf{F}, v = u + w = \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i$.
Or. $V = \operatorname{span}(u_1, \dots, u_m) \oplus \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.
Note that $\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0 \Rightarrow \sum_{i=1}^{m} a_i u_i = -\sum_{i=1}^{n} b_i w_i \in U \cap W = \{0\}$. $\square$

- (9.A.3,4 Or 4E 11) *Sup V is on $\mathbf{R}$, and $v_1, \dots, v_n \in V$. Let $B = (v_1, \dots, v_n)$.*
  (a) *Shat $[P]$ B is linely inde in $V \iff B$ is linely inde in $V_{\mathbf{C}}$. $[Q]$*
  (b) *Shat $[P]$ B spans $V \iff B$ spans $V_{\mathbf{C}}$. $[Q]$*

(a) $P \Rightarrow Q:$ Note that each $v_k \in V_{\mathbf{C}}$. $Q \Rightarrow P:$ If $\lambda_k \in \mathbf{R}$ with $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then each $\operatorname{Re}\lambda_k = \lambda_k = 0$.
$\neg P \Rightarrow \neg Q:$ $\exists\, v_j = a_{j-1}v_{j-1} + \cdots + a_1 v_1 \in V_{\mathbf{C}}$.
$\neg Q \Rightarrow \neg P:$ $\exists\, v_j = \lambda_{j-1}v_{j-1} + \cdots + \lambda_1 v_1 \Rightarrow v_j = (\operatorname{Re}\lambda_{j-1})v_{j-1} + \cdots + (\operatorname{Re}\lambda_1)v_1 \in V$.

(b) $P \Rightarrow Q:$ $\forall u + \mathrm{i}v \in V_{\mathbf{C}},\ u, v \in V \Rightarrow \exists\, a_i, b_i \in \mathbf{R}, u + \mathrm{i}v = \sum_{i=1}^{n} (a_i + \mathrm{i}b_i)v_i$.
$Q \Rightarrow P:$ $\forall v \in V, \exists\, a_i + \mathrm{i}b_i \in \mathbf{C},\ v + \mathrm{i}0 = \big(\sum_{i=1}^{n} a_i v_i\big) + \mathrm{i}\,\big(\sum_{i=1}^{n} b_i v_i\big) \Rightarrow v \in \operatorname{span}(v_1, \dots, v_m)$.
$\neg Q \Rightarrow \neg P:$ $\exists\, v \in V, v \notin \operatorname{span}(B) \Rightarrow v + \mathrm{i}0 \notin \operatorname{span}(B)$ while $v + \mathrm{i}0 \in V_{\mathbf{C}}$.
$\neg Q \Rightarrow \neg P:$ $\exists\, u + \mathrm{i}v \in V_{\mathbf{C}}, u + \mathrm{i}v \notin \operatorname{span}(B) \Rightarrow u$ or $v \notin \operatorname{span}(B)$. Note that $u, v \in V$. $\square$

- **Note For *linely inde sequence and* [2.34]:** "$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expression.
  If we allow using "infini list", then we must guarantee that $(v_1, \dots, v_n, \dots)$ is a spanning "list"
  suth $\forall v \in V, \exists$ smallest $n \in \mathbf{N}^+$, $v = a_1 v_1 + \cdots + a_n v_n$. Moreover, given a list $(w_1, \cdots, w_n, \cdots)$ in W,
  we can provt $\exists!\, T \in \mathcal{L}(V, W)$ with each $Tv_k = w_k$, which has less restr than [3.5].
  But the key point is, how can we guarantee that such a "list" exists. TODO: More details.

Ended

2·C

1 7 9 10 14,16 15 17 | 4E: 10 14,15 16

15 *Sup V is finide and $\dim V = n \geqslant 1$.*
Provt $\exists$ one-dim subsps $V_1, \dots, V_n$ of V suth $V = V_1 \oplus \cdots \oplus V_n$.

Solus: Sup $B_V = (v_1, \dots, v_n)$. Define V_i by $V_i = \operatorname{span}(v_i)$ for each $i \in \{1, \dots, n\}$.
Then $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n \Rightarrow \exists!\, u_i \in V_i, v = u_1 + \cdots + u_n$ $\square$

- **Note For Exe (15):**
  Sup $v \in V \backslash \{0\}$, and $\dim V = n \geqslant 1$. Provt $\exists\, B_V = (v_1, \dots, v_n), v = v_1 + \cdots + v_n$.

Solus: If $n = 1$ then let $v_1 = v$ and we are done. Sup $n > 1$.
Extend (v) to a basis $(v, v_1, \dots, v_{n-1})$ of V. Let $v_n = v - v_1 - \cdots - v_{n-1}$.
又 $\operatorname{span}(v, v_1, \dots, v_{n-1}) = \operatorname{span}(v_1, \dots, v_n)$. Hence $(v_1, \dots, v_n)$ is also a basis of V. $\square$

Comment: Let $B_V = (v_1, \dots, v_n)$ and sup $v = u_1 + \cdots + u_n$, where each $u_i = a_i v_i \in V_i$.
But $(u_1, \dots, u_n)$ might not be a basis, becs there might be some $u_i = 0$.

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1 [CORO for [2.38,39]] *Sup U is a subsp of V suth* $\dim V = \dim U$. *Then* $V = U$.

Let $B_U = (u_1, \dots, u_m)$. Then $m = \dim V$. 又 $u_i \in V$. By [2.39], $B_V = (u_1, \dots, u_m)$. $\square$

- Let $v_1, \dots, v_n \in V$ and $\dim \operatorname{span}(v_1, \dots, v_n) = n$. Then $(v_1, \dots, v_n)$ is a basis of $\operatorname{span}(v_1, \dots, v_n)$.
  Notice that $(v_1, \dots, v_n)$ *is a spanning list of* $\operatorname{span}(v_1, \dots, v_n)$ *of len* $n = \dim \operatorname{span}(v_1, \dots, v_n)$.

---

7 (a) *Let* $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. *Find a basis of* U.
(b) *Extend the basis in* (b) *to a basis of* $\mathcal{P}_4(\mathbf{F})$.
(c) *Find a subsp* W *of* $\mathcal{P}_4(\mathbf{F})$ *suth* $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

SOLUS: Using Exe (10).

NOTICE that $\nexists\, p \in \mathcal{P}(\mathbf{F})$ of deg 1 and 2, while $p \in U$. Thus $\dim U \leqslant \dim \mathcal{P}_4(\mathbf{F}) - 2 = 3$.

(a) Consider $B = \big(1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.
Let $a_0 + a_3(z-2)(z-5)(z-6) + a_4 z(z-2)(z-5)(z-6) = 0 \Rightarrow a_0 = a_3 = a_4 = 0$.
Thus the list B is linely inde in U. Now $\dim U \geqslant 3 \Rightarrow \dim U = 3$. Thus $B_U = B$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $\big(1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. $\square$

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9 *Sup* $(v_1, \dots, v_m)$ *is linely inde in V and* $w \in V$. *Provt* $\dim \operatorname{span}(v_1+w, \dots, v_m+w) \geq m-1$.

SOLUS: Using the result of (2.A.10, 11).

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$, for each $i = 1, \dots, m$.
$(v_1, \dots, v_m)$ linely inde $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ linely inde $\Rightarrow \underbrace{(v_2 - v_1, \dots, v_m - v_1)}_{\text{of len } (m-1)}$ linely inde.
又 If $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1 + w, \dots, v_m + w)$ is linely inde.
Hence $m \geqslant \dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$. $\square$

- (4E 16) *Sup V is finide, U is a subsp of V with $U \neq V$. Let* $n = \dim V, m = \dim U$.
  Provt $\exists\, (n-m)$ *subsps* $U_1, \dots, U_{n-m}$, *each of dim* $(n-1)$, *suth* $\bigcap_{i=1}^{n-m} U_i = U$.

SOLUS: Let $B_U = (v_1, \dots, v_m)$, $B_V = (v_1, \dots, v_m, u_1, \dots, v_{n-m})$.
Define $U_i = \operatorname{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m})$ for each i. Then $U \subseteq U_i$ for each i.
And becs $\forall v \in \bigcap_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \cdots + b_{n-m} u_{n-m} \in U_i \Rightarrow$ each $b_i = 0 \Rightarrow v \in U$.
Hence $\bigcap_{i=1}^{n-m} U_i \subseteq U$. $\square$

- **NOTE FOR Exe 10:** For each nonconst $p \in \operatorname{span}(1, z, \dots, z^m)$, $\exists$ smallest $m \in \mathbf{N}^+$, which is $\deg p$.

(a) If $p_0, p_1, \dots, p_m$ are suth all $a_{k,k} \neq 0$, and
$p_0 = a_{0,0}$, each $p_k = a_{0,k} + a_{1,k} z + \cdots + a_{k,k} z^k$.
Then the upper-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m} \\ 0 & a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix}$.

(b) If $p_0, p_1, \dots, p_m$ are suth all $a_{k,k} \neq 0$, and
$p_0 = a_{0,0} + \cdots + a_{m,0} x^m$, each $p_k = a_{k,k} x^k + \cdots + a_{m,k} x^m$.
Then the lower-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,0} & a_{m,1} & \cdots & a_{m,m} \end{pmatrix}$.

COMMENT: Define $\xi_k(p)$ by the coeff of z^k in $p \in \mathcal{P}_m(\mathbf{F})$.
Then $\mathcal{M}\big(\xi_k, (1, z, \dots, z^m), (1)\big) = \mathcal{E}^{(1,k)} \in \mathbf{F}^{1,m+1}$.

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10 *Sup $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ are suth each p_k has deg k.*
Provt $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Solus: Using mathematical induction on m.

(i) $k = 1$. $\deg p_0 = 0$; $\deg p_1 = 1 \Rightarrow \operatorname{span}(p_0, p_1) = \operatorname{span}(1, x)$.

(ii) $1 \leqslant k \leqslant m-1$. Asm $\operatorname{span}(p_0, p_1, \dots, p_k) = \operatorname{span}(1, x, \dots, x^k)$.

Then $\operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \subseteq \operatorname{span}(1, x, \dots, x^k, x^{k+1})$.

又 $\deg p_{k+1} = k+1$, $p_{k+1}(x) = a_{k+1}x^{k+1} + r_{k+1}(x)$; $a_{k+1} \neq 0$, $\deg r_{k+1} \leqslant k$.

$\Rightarrow x^{k+1} = \dfrac{1}{a_{k+1}}\Big(p_{k+1}(x) - r_{k+1}(x)\Big) \in \operatorname{span}(1, x, \dots, x^k, p_{k+1}) = \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

$\therefore x^{k+1} \in \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \Rightarrow \operatorname{span}(1, x, \dots, x^k, x^{k+1}) \subseteq \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m)$. □

Or. 用比较系数法. Denote the coeff of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Sup $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We shat $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is linely inde.

Step 1. For $k = m$, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m$, $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.

Now $L = a_{m-1}p_{m-1}(x) + \cdots + a_0 p_0(x)$.

Step k. For $0 \leqslant k \leqslant m$, we have $a_m = \cdots = a_{k+1} = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = 0$, then we are done. Othws, we have $L = a_{k-1}p_{k-1}(x) + \cdots + a_0 p_0(x)$. □

• **Tips:** *Sup $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ are suth the lowest term of each p_k is of deg k.*
Provt $(p_0, p_1, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Solus: Using mathematical induction on m.

Let each p_k be defined by $p_k(x) = a_{k,k}x^k + \cdots + a_{m,k}x^m$, where $a_{k,k} \neq 0$.

(i) $k = 1$. $p_m(x) = a_{m,m}x^m$; $p_{m-1}(x) = a_{m-1,m-1}x^{m-1} + a_{m,m-1}x^m \Longrightarrow \operatorname{span}(x^m, x^{m-1}) = \operatorname{span}(p_m, p_{m-1})$.

(ii) $1 \leqslant k \leqslant m-1$. Asm $\operatorname{span}(x^m, \dots, x^{m-k}) = \operatorname{span}(p_m, \dots, p_{m-k})$.

Then $\operatorname{span}(p_m, \dots, p_{m-(k+1)}) \subseteq \operatorname{span}(x^m, \dots, x^{m-(k+1)})$.

又 $p_{m-(k+1)}$ has the form $a_{m-(k+1),m-(k+1)}x^{m-(k+1)} + r_{m-(k+1)}(x)$;

where the lowest term of $r_{m-(k+1)} \in \mathcal{P}_m(\mathbf{F})$ is of deg $(m-k)$.

$$\Rightarrow x^{m-(k+1)} = \frac{1}{a_{m-(k+1),m-(k+1)}}\Big(p_{m-(k+1)}(x) - r_{m-(k+1)}(x)\Big) \in \operatorname{span}(x^m, \dots, x^{m-k}, p_{m-(k+1)}) = \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$$

$\therefore x^{m-(k+1)} \in \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$

$\Rightarrow \operatorname{span}(x^m, \dots, x^{m-k}, x^{m-(k+1)}) \subseteq \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}(p_m, \dots, p_1, p_0)$. □

Or. 用比较系数法. Denote the coeff of x^k in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Sup $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We shat $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is linely inde.

Step 1. For $k = 0$, $\xi_0(L) = a_0 \xi_0(p_0) = \xi_0(R) = 0$ 又 $\deg p_0 = 0$, $\xi_0(p_0) \neq 0 \Rightarrow a_0 = 0$.

Now $L = a_1 p_1(x) + \cdots + a_m p_m(x)$.

Step k. For $0 \leqslant k \leqslant m$, we have $a_{k-1} = \cdots = a_0 = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = m$, then we are done. Othws, we have $L = a_{k+1}p_{k+1}(x) + \cdots + a_m p_m(x)$. □

• **Note For** [2.11]: ***Good definition for a general term always aviods undefined behaviours.***
If $\deg p = 0$, then $p(z) = a_0 \neq 0$, but not literally $a_0 z^0$, by which if p is defined, then it comes to 0^0.
To make it clear, we specify that *in* $\mathcal{P}(\mathbf{F})$, $a_0 z^0 = a_0$, where z^0 appears just for nota conveni.
Becs by def, the term $a_0 z^0$ in a poly only represents the const term of the poly, which is a_0.
For conveni, we asm $z^0 = 1$ in formula deduction and poly def. Absolutely without 0^0.

• (4E 10) *Sup m is a positive integer. For $0 \leqslant k \leqslant m$, let $p_k(x) = x^k(1-x)^{m-k}$.*
Shat $(p_0, \dots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Solus: *We may see $p_0 = 1$ and $p_m(x) = x^m$, from the expansion below, by the* Note For [2.11] *above.*

Note that each $p_k(x) = \sum_{j=0}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j = \underbrace{(-1)^0 \cdot x^k \cdot 1^0}_{\text{of deg k}} + \underbrace{\sum_{j=1}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j}_{\text{of deg m; denote it by } q_k(x)}$.
And, each $q_k \in \operatorname{span}(x^{k+1}, \dots, x^m)$. Using Tips above. □

Or. Simlr to the Tips above. We will recursively provt each $x^{m-k} \in \operatorname{span}(p_m, \dots, p_{m-k})$.
(i) $k = 1$. $p_m(x) = x^m \in \operatorname{span}(p_m)$; $p_{m-1}(x) = x^{m-1} - x^m \Rightarrow x^{m-1} \in \operatorname{span}(p_{m-1}, p_m)$.
(ii) $k \in \{1, \dots, m-1\}$. Sup for each $k \in \{0, \dots, k\}$, we have $x^{m-k} \in \operatorname{span}(p_{m-k}, \dots, p_m), \exists!\, a_m \in \mathbf{F}$.
Note that $x^{m-(k+1)} = p_{m-(k+1)}(x) + \sum_{j=1}^{k+1} C_{k+1}^{j}(-1)^{j+1} x^{m-(k+1)+j} \in \operatorname{span}(p_{m-(k+1)}, x^{m-k}, \dots, x^m)$.
Thus $x^{m-(k+1)} \in \operatorname{span}(p_{m-(k+1)}, p_{m-k}, \dots, p_m)$. □

Comment: The base step and the inductive step can be independent.

Or. For any $m, k \in \mathbf{N}^+$ suth $k \leqslant m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k(1-x)^{m-k}$.
Define the stam $S(m)$ by $S(m) : (p_{0,m}, \dots, p_{m,m})$ is linely inde (and therefore is a basis).
We use induction on to shat $S(m)$ holds for all $m \in \mathbf{N}^+$.
(i) $m = 0$. $p_{0,0} = 1$, and $a p_{0,0} = 0 \Rightarrow a = 0$.
$m = 1$. Let $a_0(1-x) + a_1 x = 0, \forall x \in \mathbf{F}$. Then take $x = 1, x = 0 \Rightarrow a_1 = a_0 = 0$.
(ii) $1 \leqslant m$. Asm $S(m)$ and $S(m-1)$ holds. Now we shat $S(m+1)$ holds.
Sup $\sum_{k=0}^{m+1} a_k p_{k,m+1}(x) = \sum_{k=0}^{m+1} a_k \left[x^k(1-x)^{m+1-k}\right] = 0, \forall x \in \mathbf{F}$.
Now $a_0(1-x)^{m+1} + \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k} + a_{m+1} x^{m+1} = 0, \forall x \in \mathbf{F}$.
While $x = 0 \Rightarrow a_0 = 0$; and $x = 1 \Rightarrow a_{m+1} = 0$.
Then $0 = \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k}$
$= x(1-x) \sum_{k=1}^{m} a_k x^{k-1}(1-x)^{m-k}$, note that $m - k = (m-1) - (k-1)$
$= x(1-x) \sum_{k=0}^{m-1} a_{k+1} x^k (1-x)^{m-1-k} = x(1-x) \sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x)$.
Hence $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0, \forall x \in \mathbf{F} \backslash \{0, 1\}$. Which has infily many zeros.
Moreover, $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0$. By asm, $a_1 = \dots = a_{m-1} = a_m = 0$.
Thus $(p_{0,m+1}, \dots, p_{m+1,m+1})$ is linely inde and $S(m+1)$ holds. □

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14 *Sup $V_1, \dots, V_m$ are finide. Provt* $\dim(V_1 + \dots + V_m) \leqslant \dim V_1 + \dots + \dim V_m$.

Solus: For each V_i, let $B_{V_i} = \mathcal{E}_i$. Then $V_1 + \dots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$; $\dim V_i = \operatorname{card} \mathcal{E}_i$.
Now $\dim(V_1 + \dots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leqslant \operatorname{card}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leqslant \operatorname{card} \mathcal{E}_1 + \dots + \operatorname{card} \mathcal{E}_m$.

Coro: $V_1 + \dots + V_m$ is direct
$\iff$ For each $k \in \{1, \dots, m-1\}$, $(V_1 \oplus \dots \oplus V_k) \cap V_{k+1} = \{0\}$, $(\mathcal{E}_1 \cap \dots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$
$\iff \dim \operatorname{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) = \operatorname{card}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) = \operatorname{card} \mathcal{E}_1 + \dots + \operatorname{card} \mathcal{E}_m$
$\iff \dim(V_1 \oplus \dots \oplus V_m) = \dim V_1 + \dots + \dim V_m$. □

---

17 *Sup V_1, V_2, V_3 are subsps of a finide vecsp, then*

$$\dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1\cap V_2) - \dim(V_1\cap V_3) - \dim(V_2\cap V_3) + \dim(V_1\cap V_2\cap V_3).$$

Explain why you might think and prove the formula above or give a counterexa.

SOLUS:

$\left[\textit{Simlr to}\right]$ Given three sets A, B and C.

Becs $|X\cup Y| = |X| + |Y| - |X\cap Y|$; $(X\cup Y)\cap Z = (X\cap Z)\cup(Y\cap Z)$.

Now $|(A\cup B)\cup C| = |A\cup B| + |C| - |(A\cup B)\cap C|$.

And $|(A\cup B)\cap C| = |(A\cap C)\cup(B\cap C)| = |A\cap C| + |B\cap C| - |A\cap B\cap C|$.

Hence $|(A\cup B)\cup C| = |A| + |B| + |C| + |A\cap B\cap C| - |A\cap B| - |A\cap C| - |B\cap C|$.

Note that $(V_1+V_2)+V_3 = V_1+(V_2+V_3) = (V_1+V_3)+V_2$.

$$\begin{aligned}\dim(V_1+V_2+V_3) &= \dim(V_1+V_2)+\dim(V_3)-\dim\big((V_1+V_2)\cap V_3\big) \quad (1)\\ &= \dim(V_2+V_3)+\dim(V_1)-\dim\big((V_2+V_3)\cap V_1\big) \quad (2)\\ &= \dim(V_1+V_3)+\dim(V_2)-\dim\big((V_1+V_3)\cap V_2\big) \quad (3).\end{aligned}$$

Notice that in general, $(X+Y)\cap Z \neq (X\cap Z)+(Y\cap Z)$.

For exa, $X = \{(x,0)\in\mathbf{R}^2 : x\in\mathbf{R}\}, Y = \{(0,y)\in\mathbf{R}^2 : y\in\mathbf{R}\}, Z = \{(z,z)\in\mathbf{R}^2 : z\in\mathbf{R}\}$.

COMMENT: If $X\subseteq Y$, then $(X+Y)\cap Z = Y\cap Z$; $\dim(X+Y+Z) = \dim(Y)+\dim(Z)-\dim(Y\cap Z)$, and the wrong formual holds. Simlr for $Y\subseteq Z, X\subseteq Z$, and $X, Y\subseteq Z$.

However, $(X+Y)\cap Z \supseteq (X\cap Z)+(Y\cap Z)$ holds. Becs $\forall v\in(X\cap Z)+(Y\cap Z)$,
$\exists\, u = x_1 = z_1\in X\cap Z, w = y_2 = z_2\in Y\cap Z$, $v = u+w = x_1+y_2 = z_1+z_2\in(X+Y)\cap Z$.

COMMENT: $\dim\big((X+Y)\cap Z\big) \geqslant \dim(X\cap Z)+\dim(Y\cap Z)-\dim(X\cap Y\cap Z)$.

- **CORO:** *Sup V_1, V_2, V_3 are finide, then* $\dfrac{(1)+(2)+(3)}{3}$:

$$\begin{aligned}\dim(V_1+V_2+V_3) = \dim V_1+\dim V_2+\dim V_3 &- \frac{\dim(V_1\cap V_2)+\dim(V_1\cap V_3)+\dim(V_2\cap V_3)}{3}\\ &- \frac{\dim\big((V_1+V_2)\cap V_3\big)+\dim\big((V_1+V_3)\cap V_2\big)+\dim\big((V_2+V_3)\cap V_1\big)}{3}.\end{aligned}$$

- **TIPS:** Becs $\dim(V_1\cap V_2\cap V_3) = \dim V_1+\dim(V_2\cap V_3)-\dim\big(V_1+(V_2\cap V_3)\big)$.
  And $\dim(V_2\cap V_3) = \dim V_2+\dim V_3-\dim(V_2+V_3)$. We have (1), and (2), (3) simlr.
  (1) $\dim(V_1\cap V_2\cap V_3) = \dim V_1+\dim V_2+\dim V_3-\dim(V_2+V_3)-\dim\big(V_1+(V_2\cap V_3)\big)$.
  (2) $\dim(V_1\cap V_2\cap V_3) = \dim V_1+\dim V_2+\dim V_3-\dim(V_1+V_3)-\dim\big(V_2+(V_1\cap V_3)\big)$.
  (3) $\dim(V_1\cap V_2\cap V_3) = \dim V_1+\dim V_2+\dim V_3-\dim(V_1+V_2)-\dim\big(V_3+(V_1\cap V_2)\big)$.

- *Sup V_1, V_2, V_3 are subsps of V with*
  (a) $\dim V = 10$, $\dim V_1 = \dim V_2 = \dim V_3 = 7$. *Provt* $V_1\cap V_2\cap V_3 \neq \{0\}$.
  By TIPS, $\dim(V_1\cap V_2\cap V_3) \geq \dim V_1+\dim V_2+\dim V_3 \underline{- 2\dim V} > 0$.
  (b) $\dim V_1+\dim V_2+\dim V_3 > 2\dim V$. *Provt* $V_1\cap V_2\cap V_3 \neq \{0\}$.
  By TIPS, $\dim(V_1\cap V_2\cap V_3) \underline{> 2\dim V} - \dim(V_2+V_3) - \dim\big(V_1+(V_2\cap V_3)\big) \geq 0$. $\square$

ENDED

3·A 3 4 5 7 8 10 11 12 13 | 4E: 10 11 17

• **Tips 1:** $T : V \to W$ is linear $\iff \left| \begin{array}{l} (一)\ \forall v,u \in V, T(v+u) = Tv + Tu; \\ (二)\ \forall v,u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv). \end{array} \right| \iff T(v+\lambda u) = Tv + \lambda Tu.$

• (9.A.2,6 Or 4E 3.B.33) *Sup that V, W are on $\mathbf{R}$, and $T \in \mathcal{L}(V,W)$. Shat*
(a) $T_{\mathrm{C}} \in \mathcal{L}(V_{\mathrm{C}}, W_{\mathrm{C}})$. (b) null $(T_{\mathrm{C}}) = (\text{null}\, T)_{\mathrm{C}}$, range $(T_{\mathrm{C}}) = (\text{range}\, T)_{\mathrm{C}}$. (c) T_{C} is inv $\iff T$ is inv.

Solus: (a) $T_{\mathrm{C}}\big((u_1 + \mathrm{i}v_1) + (x+\mathrm{i}y)(u_2 + \mathrm{i}v_2)\big) = T(u_1 + xu_2 - yv_2) + \mathrm{i}T(v_1 + xv_2 + yu_2)$
$= T_{\mathrm{C}}(u_1 + \mathrm{i}v_1) + (x+\mathrm{i}y)T_{\mathrm{C}}(u_2 + \mathrm{i}v_2)$.
(b) $u + \mathrm{i}v \in \text{null}\,(T_{\mathrm{C}}) \iff u, v \in \text{null}\, T \iff u + \mathrm{i}v \in (\text{null}\, T)_{\mathrm{C}}$.
$w + \mathrm{i}x \in \text{range}\,(T_{\mathrm{C}}) \iff w, x \in \text{range}\, T \iff w + \mathrm{i}x \in (\text{range}\, T)_{\mathrm{C}}$.
(c) $\forall w, x \in W, \exists!\, u, v \in V, T_{\mathrm{C}}(u + \mathrm{i}v) = w + \mathrm{i}x \iff Tu = w, Tv = x$. Or. By (b). □

• (9.A.5) *Sup V is on $\mathbf{R}$, and $S, T \in \mathcal{L}(V,W)$. Provt $(S+\lambda T)_{\mathrm{C}} = S_{\mathrm{C}} + \lambda T_{\mathrm{C}}$.*

Solus: $(S+\lambda T)_{\mathrm{C}}(u + \mathrm{i}v) = (S+\lambda T)(u) + \mathrm{i}(S+\lambda T)(v)$
$= Su + \mathrm{i}Sv + \lambda(Tu + \mathrm{i}Tv) = (S_{\mathrm{C}} + \lambda T_{\mathrm{C}})(u + \mathrm{i}v)$. □

• *Sup U, V, W are on $\mathbf{R}$, $S \in \mathcal{L}(V,W), T \in \mathcal{L}(U,V)$. Provt $(ST)_{\mathrm{C}} = S_{\mathrm{C}}T_{\mathrm{C}}$.*

Solus: $\forall u + \mathrm{i}x \in U_{\mathrm{C}}$, $(ST)_{\mathrm{C}}(u + \mathrm{i}x) = STu + \mathrm{i}STx = S_{\mathrm{C}}(Tu + \mathrm{i}Tx) = (S_{\mathrm{C}}T_{\mathrm{C}})(u + \mathrm{i}x)$. □

• **Note For Restriction:** *U is a subsp of V.*
(a) $\forall S, T \in \mathcal{L}(V,W), \lambda \in \mathbf{F}$, $(T + \lambda S)\big|_U = T|_U + \lambda S|_U$.
(b) $\forall S \in \mathcal{L}(W,X), T \in \mathcal{L}(V,W)$, $(ST)\big|_U = ST|_U$.

• (4E 1.B.7) *Sup $V \neq \emptyset$ and W is a vecsp. Let $W^V = \{f : V \to W\}$.*
(a) *Define a natural add and scalar multi on W^V.*
(b) *Provt W^V is a vecsp with these defs.*

Solus:
(a) $W^V \ni f + g : x \to f(x) + g(x)$; where $f(x) + g(x)$ is the vec add on W.
$W^V \ni \lambda f : x \to \lambda f(x)$; where $\lambda f(x)$ is the scalar multi on W.
(b) Commu: $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$.
Assoc: $\big((f+g)+h\big)(x) = \big(f(x) + g(x)\big) + h(x)$
$= f(x) + \big(g(x) + h(x)\big) = \big(f + (g+h)\big)(x)$.
Add Id: $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$.
Add Inv: $(f+g)(x) = f(x) + g(x) = f(x) + \big(-f(x)\big) = 0 = 0(x)$.
Distr: $\big(a(f+g)\big)(x) = a(f+g)(x) = a\big(f(x) + g(x)\big)$
$= af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x)$.
Simlr, $\big((a+b)f\big)(x) = (af + bf)(x)$.
So far, we have used the same properties in W.
Which means that ***if** W^V **is a vecsp, then** W **must be a vecsp.***
Multi Id: $(1f)(x) = 1f(x) = f(x)$. (Notice that the smallest $\mathbf{F}$ is $\{0, 1\}$.) □

• **Tips 2**: $T \in \mathcal{L}(V,W) \iff T \in \mathcal{L}(V, \operatorname{range} T) \iff T \in \mathcal{L}(V,U)$, if range T is a subsp of U.
Coro: $\{T \in \mathcal{L}(V,W) : \operatorname{range} T \subseteq U\} = \{T \in \mathcal{L}(V,U)\} = \mathcal{L}(V,U)$.

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5 *Becs* $\mathcal{L}(V,W) = \{T : V \to W \mid T \text{ is linear}\}$ *is a subsp of* W^V, $\mathcal{L}(V,W)$ *is a vecsp.*

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3 *Sup* $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. *Provt* $\exists\, A_{j,k} \in \mathbf{F}$ *suth for any* $(x_1, \dots, x_n) \in \mathbf{F}^n$,

$$T(x_1, \dots, x_n) = \begin{pmatrix} A_{1,1}x_1 + \cdots + A_{1,n}x_n, \\ \vdots \qquad \ddots \qquad \vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n \end{pmatrix}$$

Solus:

Let $T(1,0,0,\dots,0,0) = (A_{1,1}, \dots, A_{m,1})$, Note that $(1,0,\dots,0,0), \cdots, (0,0,\dots,0,1)$ is a basis of $\mathbf{F}^n$.
$T(0,1,0,\dots,0,0) = (A_{1,2}, \dots, A_{m,2})$, Then by [3.5], we are done. $\square$
$\vdots$
$T(0,0,0,\dots,0,1) = (A_{1,n}, \dots, A_{m,n})$.

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4 *Sup* $T \in \mathcal{L}(V,W)$, *and* $v_1, \dots, v_m \in V$ *suth* $(Tv_1, \dots, Tv_m)$ *is linely inde in* W.
Provt $(v_1, \dots, v_m)$ *is linely inde.*

Solus: Sup $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$. $\square$

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7 *Shat every linear map from a one-dim vecsp to itself is a multi by some scalar.*
More precisely, provt if $\dim V = 1$ *and* $T \in \mathcal{L}(V)$, *then* $\exists\, \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$.

Solus: Let u be a nonzero vec in $V \Rightarrow V = \operatorname{span}(u)$. Becs $Tu \in V \Rightarrow Tu = \lambda u$ for some λ.
Sup $v \in V \Rightarrow v = au$, $\exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. $\square$

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8 *Give a map* $\varphi : \mathbf{R}^2 \to \mathbf{R}$ *suth* $\forall a \in \mathbf{R}, v \in \mathbf{R}^2, \varphi(av) = a\varphi(v)$ *but* φ *is not linear.*

Solus: Define $T(x,y) = \begin{cases} x+y, & \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, & \text{othws.} \end{cases}$ **Or**. Define $T(x,y) = \sqrt[3]{(x^3+y^3)}$. $\square$

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9 *Give a map* $\varphi : \mathbf{C} \to \mathbf{C}$ *suth* $\forall w, z \in \mathbf{C}, \varphi(w+z) = \varphi(w) + \varphi(z)$ *but* φ *is not linear.*

Solus: Define $\varphi(u + \mathrm{i}v) = u = \operatorname{Re}(u + \mathrm{i}v)$ **Or**. Define $\varphi(u + \mathrm{i}v) = v = \operatorname{Im}(u + \mathrm{i}v)$. $\square$

• *Provt if* $q \in \mathcal{P}(\mathbf{R})$ *and* $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ *is defined by* $\underbrace{Tp = q \circ p}_{\text{composition}}$, *then* T *is not linear.*

Solus: ***Composition and product are not the same in*** $\mathcal{P}(\mathbf{F})$.
Notice that $(p \circ q)(x) = p(q(x))$, while $(pq)(x) = p(x)q(x) = q(x)p(x)$.
Becs in general, $[q \circ (p_1 + \lambda p_2)](x) = q(p_1(x) + \lambda p_2(x)) \neq (qp_1)(x) + \lambda(qp_2)(x)$.
Exa: Let q be defined by $q(x) = x^2$, then $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$. $\square$

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10 *Sup* U *is a subsp of* V *with* $U \neq V$.
Sup $S \in \mathcal{L}(U,W)$ *with* $S \neq 0$. *Define* $T : V \to W$ *by* $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \backslash U. \end{cases}$
Provt T *is not a linear map on* V.

Solus: Asm T is a linear map. Sup $v \in V \backslash U$, $u \in U$ suth $Su \neq 0$.
Then $v + u \in V \backslash U$, for if not, $v = (v+u) - u \in U$;
while $T(v+u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$. Ctradic. $\square$

---

11 *Sup U is a subsp of V and $S \in \mathcal{L}(U,W)$.*
Provt $\exists\, T \in \mathcal{L}(V,W), Tu = Su, \forall u \in U$. (Or. $\exists\, T \in \mathcal{L}(V,W), T|_U = S$.)
In other words, every linear map on a subsp of V can be ***extended*** *to a linear map on the entire V.*

Solus: Sup W is suth $V = U \oplus W$. Then $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.
Define $T \in \mathcal{L}(V,W)$ by $T(u_v + w_v) = Su_v$. □
Or. [*Finid Req*] Define by $T\Big(\sum_{i=1}^{m} a_i u_i\Big) = \sum_{i=1}^{n} a_i S u_i$. Let $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$. □

---

12 *Sup nonzero V is finide and W is infinide. Provt $\mathcal{L}(V,W)$ is infinide.*

Solus: Using (2.A.14).
Let $B_V = (v_1, \dots, v_n)$ be a basis of V. Let $(w_1, \dots, w_m)$ be linely inde in W for any $m \in \mathbf{N}^+$.
Define $T_{x,y} : V \to W$ by $T_{x,y}(v_z) = \delta_{z,x} w_y, \forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$
$\forall v = \sum_{i=1}^{n} a_i v_i,\ u = \sum_{i=1}^{n} b_i v_i,\ \lambda \in \mathbf{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) w_y = T_{x,y}(v) + \lambda T_{x,y}(u)$.
Linearity checked. Now sup $a_1 T_{x,1} + \cdots + a_m T_{x,m} = 0$.
Then $\big(a_1 T_{x,1} + \cdots + a_m T_{x,m}\big)(v_x) = 0 = a_1 w_1 + \cdots + a_m w_m \Rightarrow a_1 = \cdots = a_m = 0$. 又 m arb.
Thus $(T_{x,1}, \dots, T_{x,m})$ is a linely inde list in $\mathcal{L}(V,W)$ for any x and len m. Hence by (2.A.14). □

---

13 *Sup $(v_1, \dots, v_m)$ is linely depe in V and $W \neq \{0\}$.*
Provt $\exists\, w_1, \dots, w_m \in W, \nexists\, T \in \mathcal{L}(V,W)$ suth $Tv_k = w_k, \forall k = 1, \dots, m$.

Solus:
We prove by ctradic. By linear dependence lemma, $\exists\, j \in \{1, \dots, m\}, v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$.
Sup $a_1 v_1 + \cdots + a_m v_m = 0$, where $a_j \neq 0$. Now let $w_j \neq 0$, while $w_1 = \cdots = w_{j-1} = w_{j+1} = w_m = 0$.
Define $T \in \mathcal{L}(V,W)$ by $Tv_k = w_k$ for each k. Then $T\big(a_1 v_1 + \cdots + a_m v_m\big) = 0 = a_1 w_1 + \cdots + a_m w_m$.
And $0 = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Ctradic. □

Or. We prove the ctrapos: Sup $\forall w_1, \dots, w_m \in W, \exists\, T \in \mathcal{L}(V,W), Tv_k = w_k$ for each w_k.
Now we shat $(v_1, \dots, v_n)$ is linely inde. Sup $\exists\, a_i \in \mathbf{F}, a_1 v_1 + \cdots + a_n v_n = 0$.
Choose one $w \in W \backslash \{0\}$. By asm, for $(\overline{a_1} w, \dots, \overline{a_m} w), \exists\, T \in \mathcal{L}(V,W), Tv_k = \overline{a_k} w$ for each v_k.
Now we have $0 = T\Big(\sum_{k=1}^{m} a_k v_k\Big) = \sum_{k=1}^{m} a_k T v_k = \sum_{k=1}^{m} a_k \overline{a_k} w = \Big(\sum_{k=1}^{m} |a_k|^2\Big) w$.
Then $\sum_{k=1}^{m} |a_k|^2 = 0$. Thus $a_1 = \cdots = a_m = 0$. Hence $(v_1, \dots, v_n)$ is linely inde. □

• (4E 17) *Sup V is finide. Shat all two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

Solus: Let $B_V = (v_1, \dots, v_n)$. If $\mathcal{E} = 0$, then we are done.
Sup $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \backslash \{0\}$.
Sup $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \cdots + a_n v_n$, where $a_k \neq 0$.
Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y} :\ v_x \mapsto v_y,\ v_z \mapsto 0\,(\, z \neq x \,)$. Or. $R_{x,y} v_z = \delta_{z,x} v_y$.
Then $\big(R_{1,1} + \cdots + R_{n,n}\big) v_j = v_j \Rightarrow \sum_{r=1}^{n} R_{r,r} = I$. Asm each $R_{x,y} \in \mathcal{E}$.
Hence $\forall T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. Now we prove the asm.
Notice that $\forall x, y \in \mathbf{N}^+,\ (R_{k,y} S)(v_i) = a_k v_y \Rightarrow \big((R_{k,y} S) \circ R_{x,i}\big)(v_z) = \delta_{z,x}(a_k v_y)$.
Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Now $S \in \mathcal{E} \Rightarrow R_{k,y} S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}$. □

- $(4\text{E } 3.\text{B}.32)$ *Sup V is finide with $n = \dim V > 1$.*
  Shat if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is linear and $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$, then $\varphi = 0$.

Solus: Using notas in (4E 3.A.17). Using the result in Note For [3.60].

Sup $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \dots, n\}$, $\varphi(R_{i,j}) \neq 0$. Becs $R_{i,j} = R_{x,j} \circ R_{i,x}$, $\forall x = 1, \dots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, becs $R_{i,x} = R_{y,x} \circ R_{i,y}$, $\forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0$, $\forall x, y = 1, \dots, n$.

Let $k \neq i, j \neq l$ and then $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Ctradic. □

Or. Note that by (4E 3.A.17), $\exists\, S, T \in \mathcal{L}(V), ST - TS \neq 0$.

Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \operatorname{null} \varphi \neq \{0\}$.

Note that $\forall\, E \in \operatorname{null} \varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \operatorname{null} \varphi$.

Hence $\operatorname{null} \varphi$ is a nonzero two-sided ideal of $\mathcal{L}(V)$. □

- *Sup V is finide. $T \in \mathcal{L}(V)$ is suth $\forall S \in \mathcal{L}(V), ST = TS$.*
  Provt $\exists\, \lambda \in \mathbf{F}, T = \lambda I$.

Solus: If $V = \{0\}$, then we are done. Now sup $V \neq \{0\}$.

Asm $\forall v \in V, (v, Tv)$ is linely depe, then by (2.A.2.(b)), $\exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

To provt λ_v is independent of v, we discuss in two cases:

$$\left.\begin{array}{l} (-) \text{ If } (v, w) \text{ is linely inde, } \lambda_{v+w}(v + w) = T(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \\ \qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=) \text{ Othws, sup } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w \end{array}\right\} \Rightarrow \lambda_w = \lambda_v.$$

Now we prove the asm. Asm $\exists\, v \in V, (v, Tv)$ is linely inde. Let $B_V = (v, Tv, u_1, \dots, u_n)$.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1 u_1 + \cdots + c_n u_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Ctradic. □

Or. Let $B_V = (v_1, \dots, v_m)$. Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$.

Sup $v \in V$. Define $S_v \in \mathcal{L}(V)$ by $S_v(u) = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. □

Or. For each $k \in \{1, \dots, n\}$, define $S_k \in \mathcal{L}(V)$ by $S_k v_j = \begin{cases} v_k, & j = k, \\ 0, & j \neq k. \end{cases}$ Or. $S_k v_j = \delta_{j,k} v_k$

Note that $S_k\left(\sum_{i=1}^{n} a_i v_i \right) = a_k v_k$. Then $S_k v = v \iff \exists\,!\, a_k \in \mathbf{F}, v = a_k v_k$.

Hence $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$.

Define $A^{(j,k)} \in \mathcal{L}(V)$ by $A^{(j,k)} v_j = v_k$, $A^{(j,k)} v_k = v_j$, $A^{(j,k)} v_x = 0, x \neq j, k$.

Then $$\left| \begin{array}{r} A^{(j,k)} Tv_j = TA^{(j,k)} v_j = Tv_k = a_k v_k \\ A^{(j,k)} Tv_j = A^{(j,k)} a_j v_j = a_j A^{(j,k)} v_j = a_j v_k \end{array} \right\} \Rightarrow a_k = a_j.$$ Hence a_k is inde of v_k. □

- **Tips 3:** *Sup $T \in \mathcal{L}(V, W)$. Provt $Tv \neq 0 \Rightarrow v \neq 0$.*

Solus: Asm $v = 0$. Then $Tv = T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$.

Or. $T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$. Ctradic. □

• *Given the fact that $\mathcal{L}(V,W)$ is a vecsp. Prove or give a counterexa: V,W are vecsps.*

We can guarantee that $\{0\} \subseteq \mathcal{L}(V,W), \{0\} \subseteq V, \{0\} \subseteq W$.

And by [3.2], the additivity and homogeneity imply that V is closd add and scalar multi.

(We cannot even guarantee that W^V is a vecsp.)

SOLUS: TODO: Too tricky to be answered by AI.

(I) If $W^V = \{0\}$. Then $\mathcal{L}(V,W) = \{0\}$.

And $W = \{0\}$, for if not, $\exists\, w \in W \backslash \{0\}$, define a map f by $f(x) = w, \forall x \in V$.

And V might not be a vecsp. Example: ???

(II) If W^V is a nonzero vecsp. Then W is a vecsp.

(a) If $\mathcal{L}(V,W) = \{0\}$, then we cannot guarantee that V is a vecsp. Example: ???

(b) If not, then $\exists\, T \in \mathcal{L}(V,W)$, $T \neq 0$. Which means $\exists\, v \in V, Tv \neq 0 \Rightarrow v \neq 0$.

Then both W and V have a nonzero ele.

(i) If $\exists$ inje $T \in \mathcal{L}(V,W)$, then $T(u+v) = T(v+u) \Rightarrow u+v = v+u$. etc. Hence V is a vecsp.

(ii) If not, then we cannot guarantee that V is a vecsp. Example: ???

(III) If W^V is not a vecsp, then W is not a vecsp. Example: ??? □

ENDED

3·B

3 7 8 9 10 11 12 16 17 18 19 20 21 22 23 24 25 26 28 29 30 | 4E: 21 24 27 31 32 33

3 *Sup $(v_1,\dots,v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1,\dots,z_m) = z_1 v_1 + \cdots + z_m v_m$.*

(a) *The surj of T corres to $(v_1,\dots,v_m)$ spanning V.* range $T = \operatorname{span}(v_1,\dots,v_m) = V$.

(b) *The inje of T corres to $(v_1,\dots,v_m)$ being linely inde.* *$(v_1,\dots,v_m)$ linely inde $\iff$ T inje.*

COMMENT: Let $(e_1,\dots,e_m)$ be std basis of $\mathbf{F}^m$. Then $Te_k = v_k$.

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7 *Sup V is finide with $2 \leqslant \dim V$. And $\dim V \leqslant \dim W = m$, if W is finide.*

Shat $U = \{ T \in \mathcal{L}(V,W) : \operatorname{null} T \neq \{0\} \}$ is not a subsp of $\mathcal{L}(V,W)$.

SOLUS: *The set of all inje $T \in \mathcal{L}(V,W)$ is a not subsp either.*

Let $(v_1,\dots,v_n)$ be a basis of V, $(w_1,\dots,w_m)$ be linely inde in W. $[\ 2 \leqslant n \leqslant m.\ ]$

Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 :\quad v_1 \mapsto 0, \qquad v_2 \mapsto w_2, \qquad v_i \mapsto w_i.$

Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 :\quad v_1 \mapsto w_1, \qquad v_2 \mapsto 0, \qquad v_i \mapsto w_i, \quad i = 3,\dots,n.$

Thus $T_1 + T_2 \notin U$. □

COMMENT: If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V,W), T$ is inje. Hence $U = \emptyset$.

If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $\forall v \in V, T_0 v = 0 \Rightarrow T_0 = 0$.

---

8 *Sup W is finide with $\dim W \geqslant 2$. And $n = \dim V \geqslant \dim W$, if V is finide.*

Shat $U = \{ T \in \mathcal{L}(V,W) : \operatorname{range} T \neq W \}$ is not a subsp of $\mathcal{L}(V,W)$.

SOLUS: *The set of all surj $T \in \mathcal{L}(V,W)$ is not a subsp either.* ***Using the generalized version of*** **[3.5].**

Let $(v_1,\dots,v_n)$ be linely inde in V, $(w_1,\dots,w_m)$ be a basis of W. $[\ n \in \{m, m+1, \dots\}; 2 \leqslant m \leqslant n.\ ]$

Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 :\quad v_1 \mapsto 0, \qquad v_2 \mapsto w_2, \qquad v_j \mapsto w_j, \qquad v_{m+i} \mapsto 0.$

Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 :\quad v_1 \mapsto w_1, \qquad v_2 \mapsto 0, \qquad v_j \mapsto w_j, \qquad v_{m+i} \mapsto 0.$

(For each $j = 2,\dots,m;\ i = 1,\dots,n-m$, if V is finide, othws let $i \in \mathbf{N}^+$.) Thus $T_1 + T_2 \notin U$. □

COMMENT: If $\dim W = 0$, then $W = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V,W), T$ is surj. Hence $U = \emptyset$.

If $\dim W = 1$, then $W = \operatorname{span}(w_0)$. Thus $U = \operatorname{span}(T_0)$, where each $T_0 v_i = 0 \Rightarrow T_0 = 0$.

---

9 *Sup* $(v_1,\dots,v_n)$ *is linely inde. Provt* $\forall$ *inje* T, $(Tv_1,\dots,Tv_n)$ *is linely inde.*

Solus: $a_1Tv_1+\cdots+a_nTv_n=0=T\Big(\sum_{i=1}^n a_iv_i\Big)\iff\sum_{i=1}^n a_iv_i=0\iff a_1=\cdots=a_n=0.$ $\square$

---

10 *Sup* $\operatorname{span}(v_1,\dots,v_n)=V$. *Shat* $\operatorname{span}(Tv_1,\dots,Tv_n)=\operatorname{range}T$.

Solus: (a) $\operatorname{range}T=\{Tv:v\in\operatorname{span}(v_1,\dots,v_n)\}\Rightarrow Tv_1,\dots,Tv_n\in\operatorname{range}T$. By [2.7].

Or. $\operatorname{span}(Tv_1,\dots,Tv_n)\ni a_1Tv_1+\cdots+a_nTv_n=T(a_1v_1+\cdots+a_nv_n)\in\operatorname{range}T$.

(b) $\forall w\in\operatorname{range}T, w=Tv,\exists v\in V\Rightarrow\exists a_i\in\mathbf{F}, v=\sum_{i=1}^n a_iv_i, w=a_1Tv_1+\cdots+a_nTv_n$. $\square$

---

11 *Sup* $S_1,\dots,S_n\in\mathcal{L}(V)$ *and* $S=S_1S_2\cdots S_n$ *makes sense.* Then using induction:

(a) $\operatorname{range}S_1\supseteq\operatorname{range}(S_1S_2)\supseteq\cdots\supseteq\operatorname{range}(S)$; (b) $\operatorname{null}S_n\subseteq\operatorname{null}(S_{n-1}S_n)\subseteq\cdots\subseteq\operatorname{null}(S)$.

- Define $X_p=\{T\in\mathcal{L}(V):p(T)\text{ holds}\}$; P_p : X_p is closd vec multi; Q_p : X_p is a group.

(1) S surj $\iff$ each S_k surj. P_{surj} holds. (2) S inje $\iff$ each S_k inje. P_{inje} holds.

(3) P_{inv} and Q_{inv} hold. Q_p in (1) and (2) holds $\iff$ V is finide.

(4) $P_{inje\ or\ surj}$ holds $\iff$ V is finide $\iff$ $Q_{inje\ or\ surj}$ holds.

- *Sup* $S,T\in\mathcal{L}(V)$. *Prove or give a counterexa*:

(a) $\operatorname{null}S\subseteq\operatorname{null}T\Rightarrow\operatorname{range}T\subseteq\operatorname{range}S$; (b) $\operatorname{range}T\subseteq\operatorname{range}S\Rightarrow\operatorname{null}S\subseteq\operatorname{null}T$.

Solus: Let $B_V=(v_1,v_2,v_3)$. Counterexas:

(a) Let $S: v_1\mapsto 0;\ v_2\mapsto 0;\ v_3\mapsto v_2.$ $T: v_1\mapsto 0;\ v_2\mapsto 0;\ v_3\mapsto v_3.$ | Then $\operatorname{null}S=\operatorname{null}T$, but $\operatorname{range}T=\operatorname{span}(v_3)\not\subseteq\operatorname{span}(v_2)=\operatorname{null}T$.

(b) Let $S: v_1\mapsto v_2;\ v_2\mapsto v_2;\ v_3\mapsto v_2.$ $T: v_1\mapsto 0;\ v_2\mapsto 0;\ v_3\mapsto v_2.$ | Then $\operatorname{range}T=\operatorname{range}S$, but $\operatorname{null}S=\operatorname{span}(v_1-v_2,v_2-v_3,v_3-v_1)\not\subseteq\operatorname{span}(v_1,v_2)=\operatorname{null}T$.

---

16 *Sup* $T\in\mathcal{L}(V)$ *suth* $\operatorname{null}T,\operatorname{range}T$ *are finide. Provt* V *is finide.*

Solus: Let $B_{\operatorname{range}T}=(Tv_1,\dots,Tv_n), B_{\operatorname{null}T}=(u_1,\dots,u_m)$.

$\forall v\in V,\exists!\,a_i\in\mathbf{F}, T(v-a_1v_1-\cdots-a_nv_n)=0\Rightarrow\exists!\,b_i\in\mathbf{F}, v-\sum_{i=1}^n a_iv_i=\sum_{i=1}^m b_iu_i$. $\square$

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17 *Sup* V,W *are finide. Provt* $\exists$ *inje* $T\in\mathcal{L}(V,W)\iff\dim V\leqslant\dim W$.

Solus: (a) Sup $\exists$ inje T. Then $\dim V=\dim\operatorname{range}T\leqslant\dim W$.

(b) Sup $\dim V\leqslant\dim W$. Let $B_V=(v_1,\dots,v_n), B_W=(w_1,\dots,w_m)$.

Define $T\in\mathcal{L}(V,W)$ by $Tv_i=w_i,\ i=1,\dots,n\ (=\dim V)$. $\square$

---

18 *Sup* V,W *are finide. Provt* $\exists$ *surj* $T\in\mathcal{L}(V,W)\iff\dim V\geqslant\dim W$.

Solus: (a) Sup $\exists$ surj T. Then $\dim V=\dim W+\dim\operatorname{null}T\Rightarrow\dim W\leqslant\dim V$.

(b) Sup $\dim V\geqslant\dim W$. Let $B_V=(v_1,\dots,v_n), B_W=(w_1,\dots,w_m)$.

Define $T\in\mathcal{L}(V,W)$ by $T\big(a_1v_1+\cdots+a_mv_m+\cdots+a_nv_n\big)=a_1w_1+\cdots+a_mw_m$. $\square$

---

19 *Sup* V,W *are finide,* U *is a subsp of* V.

Provt $\exists T\in\mathcal{L}(V,W)$, $\operatorname{null}T=U\iff\underbrace{\dim U}_{m}\geqslant\underbrace{\dim V}_{m+n}-\underbrace{\dim W}_{p}$.

Solus:

(a) Sup $\exists T\in\mathcal{L}(V,W),\operatorname{null}T=U$. Then $\dim U+\dim\operatorname{range}T=\dim V\leqslant\dim U+\dim W$.

(b) Let $B_U=(u_1,\dots,u_m), B_V=(u_1,\dots,u_m,v_1,\dots,v_n), B_W=(w_1,\dots,w_p)$. Sup that $p\geqslant n$.

Define $T\in\mathcal{L}(V,W)$ by $T(a_1v_1+\cdots+a_nv_n+b_1u_1+\cdots+b_mu_m)=a_1w_1+\cdots+a_nw_n$. $\square$

• **Tips 1:** *Sup U is a subsp of V. Then $\forall T \in \mathcal{L}(V,W), U \cap \operatorname{null} T = \operatorname{null} T|_U$.*

• **Tips 2:** *Sup $T \in \mathcal{L}(V,W)$ and $T|_U$ is inje. Let $V = M + N,\ U = X + Y$.*
Then $\operatorname{range} T = \operatorname{range} T|_M + \operatorname{range} T|_N = \operatorname{range} T|_X + \operatorname{range} T|_Y$.
(a) *Shat if $U = X \oplus Y$, then $\operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_Y$.*
(b) *Give an exa suth $V = M \oplus N,\ \operatorname{range} T \neq \operatorname{range} T|_M \oplus \operatorname{range} T|_N$.*

Solus: Asm for some $v \in V$, there exist two disti pairs $(x_1, y_1), (x_2, y_2)$ in $X \times Y$
suth $Tv = Tx_1 + Ty_1 = Tx_2 + Ty_2$. Becs $\forall v \in X \oplus Y, \exists!\,(x,y) \in X \times Y, v = x + y$.
Now $T(x_1 + y_1) = T(x_2 + y_2) \Longrightarrow x_1 + y_1 = x_2 + y_2 \Longrightarrow x_1 = x_2, y_1 = y_2$. Ctradic.
Thus $\forall Tv \in \operatorname{range} T, \exists!\, Tx \in \operatorname{range} T|_X, Ty \in \operatorname{range} T|_Y, Tv = Tx + Ty$. $\square$

Exa: Let $B_V = (v_1, v_2, v_3), B_W = (w_1, w_2),\ T : v_1 \mapsto 0,\ v_2 \mapsto w_1,\ v_3 \mapsto w_2$.
Let $B_M = (v_1 - v_2, v_3), B_N = (v_2)$. Then $\operatorname{range} T|_M = \operatorname{span}(w_1, w_2), \operatorname{range} T|_N = \operatorname{span}(w_1)$

Comment: Also $\operatorname{null} T|_M = \operatorname{null} T|_N = \{0\}$. Hence $\operatorname{null} T \neq \operatorname{null} T|_M \oplus \operatorname{null} T|_N$.

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12 *Provt $\forall T \in \mathcal{L}(V,W), \exists$ subsp U of V suth*
$U \cap \operatorname{null} T = \operatorname{null} T|_U = \{0\},\ \operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$.
Which is equivalent to $T|_U : U \to \operatorname{range} T$ being iso.

Solus: By [2.34] (note that V can be infinide), $\exists$ subsp U of V suth $V = U \oplus \operatorname{null} T$.
$\forall v \in V, \exists!\, w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w + u) = Tu \in \{Tu : u \in U\}$. $\square$

Coro: $[P] \quad T|_U : U \to \operatorname{range} T$ *is iso* $\Longleftrightarrow U \oplus \operatorname{null} T = V. \quad [Q]$
We have shown $Q \Rightarrow P$. Now we shat $P \Rightarrow Q$ to complete the proof.
$\forall v \in V, Tv \in \operatorname{range} T = \operatorname{range} T|_U \Rightarrow \exists!\, u \in U, Tv = Tu \Rightarrow v - u \in \operatorname{null} T$.
Thus $v = (v - u) + u \in U + \operatorname{null} T$. 又 $u \in U \cap \operatorname{null} T \Longleftrightarrow T|_U(u) = 0 \Longleftrightarrow u = 0$. $\square$

Or. $\neg Q \Rightarrow \neg P$: Becs $U \oplus \operatorname{null} T \subsetneq V$. We show $\operatorname{range} T \neq \operatorname{range} T|_U$ by ctradic.
Let $X \oplus (U \oplus \operatorname{null} T) = V$. Now $\operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_U$. And X is nonzero.
Asm $\operatorname{range} T = \operatorname{range} T|_U$. Then $\operatorname{range} T|_X = \{0\}$. While $T|_X$ is inje. Ctradic.
Or. $\operatorname{range} T|_X \subseteq \operatorname{range} T|_U \Rightarrow \forall x \in X, Tx \in \operatorname{range} T|_U, \exists\, u \in U, Tu = Tx \Rightarrow x = 0$.

Also, $\neg P \Rightarrow \neg Q$: (a) $\operatorname{range} T|_U \subsetneq \operatorname{range} T$; Or (b) $U \cap \operatorname{null} T \neq \{0\}$.
For (a), $\exists\, x \in V \backslash U, Tx \neq 0 \Longleftrightarrow x \notin \operatorname{null} T$. Thus $U + \operatorname{null} T \subsetneq V$. For (b), immediately. $\square$

Comment: If $T|_U : U \to \operatorname{range} T$ is iso. Let $R \oplus U = V$. Then R might not be $\operatorname{null} T$.
Or. Extend B_U to $B_V = (u_1, \dots, u_n, r_1, \dots, r_m)$, then $(r_1, \dots, r_m)$ might not be a $B_{\operatorname{null} T}$.

• **Tips 3:** Sup $T \in \mathcal{L}(V,W)$ and U is a subsp suth $V = U \oplus \operatorname{null} T$. Let $\operatorname{null} T = X \oplus Y$.
Now $\forall v \in V, \exists!\, u_v \in U, (x_v, y_v) \in X \times Y, v = u_v + x_v + y_v$. Define $i \in \mathcal{L}(V, U \oplus X)$ by $i(v) = u_v + x_v$.
<u>Then $T = T \circ i$.</u> Becs $\forall v \in V, T(v) = T(u_v + x_v + y_v) = T(u_v) = T(u_v + x_v) = T\big(i(v)\big) = (T \circ i)(v)$.

• **Tips 4:** Sup $T \in \mathcal{L}(V,W), T \neq 0$. Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
By (3.A.4), $R = (v_1, \dots, v_n)$ is linely inde in V. Let $\operatorname{span} R = U$. We will provt $U \oplus \operatorname{null} T = V$.
(a) $T\big(\sum_{i=1}^n a_i v_i\big) = 0 \Longleftrightarrow \sum_{i=1}^n a_i T v_i = 0 \Longleftrightarrow a_1 = \cdots = a_n = 0$. Thus $U \cap \operatorname{null} T = \{0\}$.
(b) $Tv = \sum_{i=1}^n a_i T v_i \Longleftrightarrow v - \sum_{i=1}^n a_i v_i \in \operatorname{null} T \Longleftrightarrow v = \big(v - \sum_{i=1}^n a_i v_i\big) + \big(\sum_{i=1}^n a_i v_i\big)$.
Thus $U + \operatorname{null} T = V$. Or. $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$. Using Exe (12). $\square$

Coro: Conversely, if $U \oplus \operatorname{null} T = V$ and $B_U = (v_1, \dots, v_n)$, then $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
Becs $\operatorname{range} T = \operatorname{range} T|_U = \operatorname{span}(Tv_1, \dots, Tv_n)$, 又 T is inje.

• $[$4E 27, OR 5.B.4$]$ *Sup* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Provt* $V = \operatorname{null} P \oplus \operatorname{range} P$.

SOLUS: (a) If $v \in \operatorname{null} P \cap \operatorname{range} P \Rightarrow Pv = 0$, and $\exists\, u \in V, v = Pu$. Then $v = Pu = P^2u = Pv = 0$.

(b) Note that $\forall v \in V, v = Pv + (v - Pv)$ and $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$.

OR. Becs $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P = \dim\big(\operatorname{null} P \oplus \operatorname{range} P\big)$. $\square$

OR. $\big[$ *Only in Finid* $\big]$ Let $B_{\operatorname{range} P^2} = (P^2v_1, \dots, P^2v_n)$. Then $(Pv_1, \dots, Pv_n)$ is linely inde.
Let $U = \operatorname{span}(Pv_1, \dots, Pv_n) \Rightarrow V = U \oplus \operatorname{null} P^2$. While $U = \operatorname{range} P = \operatorname{range} P^2$; $\operatorname{null} P = \operatorname{null} P^2$. $\square$

• *Sup* $T \in \mathcal{L}(V), v \in V$, *and* $n \in \mathbf{N}^+$ *suth* $T^{n-1}v \neq 0,\ T^n v = 0$. $\big[$See [5.16]$\big]$
Provt $\big(v, Tv, \dots, T^{n-1}v\big)$ *is linely inde.*

SOLUS: $a_0 v + a_1 Tv + \cdots + a_{n-1}T^{n-1}v = 0 \Rightarrow a_0 T^{n-1}v = 0 \Rightarrow a_0 = 0$. Simlr for $a_1, \dots, a_{n-1}$. $\square$

• (4E 21) *Sup* V *is finide*, $T \in \mathcal{L}(V, W)$, Y *is a subsp of* W. *Let* $\big\{v \in V : Tv \in Y\big\}$.
(a) *Provt* $\big\{v \in V : Tv \in Y\big\}$ *is a subsp of* V.
(b) *Provt* $\dim\big\{v \in V : Tv \in Y\big\} = \dim \operatorname{null} T + \dim\big(Y \cap \operatorname{range} T\big)$.

SOLUS: Let $\mathcal{K}_Y = \big\{v \in V : Tv \in Y\big\}$.

(a) $\forall u, w \in \mathcal{K}_Y, \big[\, Tu, Tw \in Y \,\big], \lambda \in \mathbf{F}, T(u + \lambda w) = Tu + \lambda Tw \in Y \Longrightarrow \mathcal{K}_Y$ is a subsp of V.

(b) Define the range-restr map R of T by $R = T|_{\mathcal{K}_Y} \in \mathcal{L}(\mathcal{K}_Y, Y)$. Now $\operatorname{range} R = Y \cap \operatorname{range} T$.
And $v \in \operatorname{null} T \Longleftrightarrow Tv = 0 \in Y \Longleftrightarrow Rv = 0 \in \operatorname{range} T \Longleftrightarrow v \in \operatorname{null} R$. By [3.22]. $\square$

COMMENT: Now $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = \mathcal{K}_Y$. Where $B_{Y \cap \operatorname{range} T} = (Tv_1, \dots, Tv_m)$.
In particular, $\dim \mathcal{K}_{\operatorname{range} T} = \dim \operatorname{null} T + \dim \operatorname{range} T \Longrightarrow \mathcal{K}_{\operatorname{range} T} = V$.

• (4E 31) *Sup* V *is finide*, X *is a subsp of* V, *and* Y *is a finide subsp of* W.
Provt if $\dim X + \dim Y = \dim V$, *then* $\exists\, T \in \mathcal{L}(V, W), \operatorname{null} T = X, \operatorname{range} T = Y$.

SOLUS: Let $V = U \oplus X, B_U = (v_1, \dots, v_m)$. Then $\forall v \in V, \exists\,!\, a_i \in \mathbf{F}, x \in X,\ v = \sum_{i=1}^{m} a_i v_i + x$.
Let $B_Y = (w_1, \dots, w_m)$. Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i, Tx = 0$ for each v_i and all $x \in X$.
Now $v \in \operatorname{null} T \Longleftrightarrow Tv = a_1 w_1 + \cdots + a_m w_m = 0 \Longleftrightarrow v = x \in X$. Hence $\operatorname{null} T = X$.
And $Y \ni w = a_1 w_1 + \cdots + a_m w_m = a_1 Tv_1 + \cdots + a_m Tv_m \in \operatorname{range} T$. Hence $\operatorname{range} T = Y$.
OR. NOTICE that $V = U \oplus \operatorname{null} T$. By Exe (12), $\operatorname{range} T = \operatorname{range} T|_U$.
又 $\dim \operatorname{range} T|_U = \dim U = \dim Y$; $\operatorname{range} T \subseteq Y$.

OR. Let $B_X = (x_1, \dots, x_n)$. Now $\operatorname{range} T = \operatorname{span}\big(Tv_1, \dots, Tv_m, Tx_1, \dots, Tx_n\big) = \operatorname{span}(w_1, \dots, w_m) = Y$. $\square$

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22 *Sup* U, V *are finide*, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.
Provt $\dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$.

SOLUS: We shat $\dim \operatorname{null} ST = \dim \operatorname{null} S|_{\operatorname{range} T} + \dim \operatorname{null} T$.
Becs (a) $\underline{\operatorname{range} T|_{\operatorname{null} ST} = \operatorname{range} T \cap \operatorname{null} S = \operatorname{null} S|_{\operatorname{range} T}}$,
(b) $\underline{\operatorname{null} T|_{\operatorname{null} ST} = \operatorname{null} T \cap \operatorname{null} ST = \operatorname{null} T}$. By [3.22] $\square$

OR. NOTICE that $\underline{u \in \operatorname{null} ST \Longleftrightarrow S(Tu) = 0 \Longleftrightarrow Tu \in \operatorname{null} S}$.
Thus $\underline{\big\{u \in U : Tu \in \operatorname{null} S\big\} = \mathcal{K}_{\operatorname{null} S \cap \operatorname{range} T} = \operatorname{null} ST}$.
By Exe (4E 21), $\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim\big(\operatorname{null} S \cap \operatorname{range} T\big)$. $\square$

CORO: (1) T surj $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.
(2) T inv $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S,\ \operatorname{null} ST = \operatorname{null} T$.
(3) S inje $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} T$.

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23 *Sup* V *is finide,* $S \in \mathcal{L}(V,W), T \in \mathcal{L}(U,V)$.
Provt $\dim \operatorname{range} ST \leqslant \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.
COMMENT: If $\dim V = \dim U$. Then $\dim \operatorname{null} ST \geqslant \max\{\dim \operatorname{null} S, \dim \operatorname{null} T\}$.

SOLUS: NOTICE that $\underline{\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{range} S|_{\operatorname{range} T}}$.
Let $\operatorname{range} ST = \operatorname{span}(Su_1, \dots, Su_{\dim \operatorname{range} T})$, where $B_{\operatorname{range} T} = (u_1, \dots, u_{\dim \operatorname{range} T})$.
$\dim \operatorname{range} ST \leqslant \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leqslant \dim \operatorname{range} S$. □

OR. $\underline{\dim \operatorname{range} ST = \dim \operatorname{range} S|_{\operatorname{range} T} = \dim \operatorname{range} T - \dim \operatorname{null} S|_{\operatorname{range} T}} \leqslant \operatorname{range} T$. □
COMMENT: $\dim \operatorname{range} ST = \dim U - \dim \operatorname{null} ST = \dim \operatorname{range} T|_U - \dim \operatorname{range} T|_{\operatorname{null} ST}$.

CORO: (1) $S|_{\operatorname{range} T}$ inje $\iff \dim \operatorname{range} ST = \dim \operatorname{range} T$.
(2) Let $X \oplus \operatorname{null} S = V$. Then $X \subseteq \operatorname{range} T \iff \operatorname{range} ST = \operatorname{range} S$.
And T is surj $\Rightarrow \operatorname{range} ST = \operatorname{range} S$.

• (a) *Sup* $\dim V = n$, $ST = 0$ *where* $S, T \in \mathcal{L}(V)$. *Provt* $\dim \operatorname{range} TS \leqslant \lfloor \frac{n}{2} \rfloor$.
(b) *Give an exa of such* S, T *with* $n = 5$ *and* $\dim \operatorname{range} TS = 2$.

SOLUS: Note that $\dim \operatorname{range} TS \leqslant \min\{\dim \operatorname{range} T, \dim \operatorname{range} S\}$. We prove by ctradic.
Asm $\dim \operatorname{range} TS \geqslant \lfloor \frac{n}{2} \rfloor + 1$. Then $\min\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\} \geqslant \lfloor \frac{n}{2} \rfloor + 1$
又 $\dim \operatorname{null} ST = n \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \,\Big|\Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant \lceil \frac{n}{2} \rceil - 1$.
Thus $n \leqslant 2\left(\lceil \frac{n}{2} \rceil - 1\right) \Rightarrow \frac{n}{2} \leqslant \lceil \frac{n}{2} \rceil - 1$. Ctradic. □

OR. $\dim \operatorname{null} S = n - \dim \operatorname{range} S \leqslant n - \dim \operatorname{range} TS$. 又 $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S$.
$\dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S \leqslant n - \dim \operatorname{range} TS$. Thus $2 \dim \operatorname{range} TS \leqslant n$. □

OR. Becs $\dim \operatorname{range} TS \leqslant \lfloor \frac{n}{2} \rfloor$, and $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$.
We shat $\dim \operatorname{null} TS \geqslant \lceil \frac{n}{2} \rceil$. Note that $\dim \operatorname{null} S + \dim \operatorname{null} T \geqslant n$.
$\dim \operatorname{null} S + \dim \operatorname{null} T|_{\operatorname{range} S} = \dim \operatorname{null} TS$. If $\dim \operatorname{null} S \geqslant \lceil \frac{n}{2} \rceil$. Then we are done.
Othws, $\dim \operatorname{null} S \leqslant \lceil \frac{n}{2} \rceil - 1 \Rightarrow \dim \operatorname{null} T \geqslant n - \dim \operatorname{null} S \geqslant n - \lceil \frac{n}{2} \rceil + 1 = \lfloor \frac{n}{2} \rfloor + 1 \geqslant \lceil \frac{n}{2} \rceil$.
Thus $\dim \operatorname{null} TS \geqslant \max\{\dim \operatorname{null} S, \dim \operatorname{null} T\} = \lceil \frac{n}{2} \rceil$. □

EXA: Define $T: v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i\,; \quad S: v_1 \mapsto v_4, \; v_2 \mapsto v_5, \; v_i \mapsto 0\,; \quad i = 3, 4, 5$.

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26 *Sup* $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ *and* $\forall p, \deg(Dp) = (\deg p) - 1$. *Provt* $D \in \mathcal{P}(\mathbf{R})$ *is surj.*

SOLUS: $[\,D$ might not be $D : p \mapsto p'.\,]$ NOTICE that the following proof is wrong:
Becs $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) \subseteq \operatorname{range} D$, and $\deg Dx^n = n - 1$.
又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) = \operatorname{span}(1, x, x^2, \cdots) = \mathcal{P}(\mathbf{R})$.

$\underline{\text{Let } D(C) = 0, Dx^k = p_k}$ of deg $(k-1)$, for all $C \in \mathbf{R} = \mathcal{P}_0(\mathbf{R})$ and for each $k \in \mathbf{N}^+$.
Becs $B_{\mathcal{P}_m(\mathbf{R})} = (p_1, \dots, p_m, p_{m+1})$. And for all $p \in \mathcal{P}(\mathbf{R}), \exists\, !\, m = \deg p \in \mathbf{N}^+$.
So that $\exists\, !\, a_i \in \mathbf{R}, p = \sum_{i=1}^{m+1} a_i p_i \Rightarrow \exists\, q = \sum_{i=1}^{m+1} a_i x^i, Dq = p$. □

OR. We will $\underline{\text{recursively define a sequence of polys } (p_k)_{k=0}^{\infty} \text{ where } Dp_0 = 1, Dp_k = x^k}$ for each $k \in \mathbf{N}^+$.
So that $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), Dq = p, \exists\, q = \sum_{k=0}^{\deg p} a_k p_k$.
(i) Becs $\deg Dx = (\deg x) - 1 = 0, Dx = C \in \mathbf{F} \backslash \{0\}$. Let $p_0 = C^{-1}x \Rightarrow Dp_0 = C^{-1}Dx = 1$.
(ii) Sup we have defined $Dp_0 = 1, Dp_k = x^k$ for each $k \in \{1, \dots, n\}$. Becs $\deg D(x^{n+2}) = n + 1$.
Let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$, with $a_{n+1} \neq 0$.
Then $a_{n+1}^{-1} D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_n Dp_n + \cdots + a_1 Dp_1 + a_0 Dp_0)$
$\Rightarrow x^{n+1} = D\left[\underline{a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)}\right]$. Thus defining p_{n+1}, so that $Dp_{n+1} = x^{n+1}$. □

20, 21 (a) *Provt if* $ST = I \in \mathcal{L}(V)$*, then* T *is inje and* S *is surj.*
(b) *Sup* $T \in \mathcal{L}(V, W)$*. Provt if* T *is inje, then* $\exists\, S \in \mathcal{L}(W, V),\ ST = I.$
(c) *Sup* $S \in \mathcal{L}(W, V)$*. Provt if* S *is surj, then* $\exists\, T \in \mathcal{L}(V, W),\ ST = I.$

SOLUS:

(a) $Tv = 0 \Rightarrow S(Tv) = 0 = v$. OR. $\operatorname{null} T \subseteq \operatorname{null} ST = \{0\}$.
$\forall v \in V, ST(v) = v \in \operatorname{range} S$. OR. $V = \operatorname{range} ST \subseteq \operatorname{range} S$.

(b) Define $S \in \mathcal{L}(\operatorname{range} T, V)$ by $Sw = T^{-1}w$, where T^{-1} is the inv of $T \in \mathcal{L}(V, \operatorname{range} T)$.
Then extend to $S \in \mathcal{L}(W, V)$ by (3.A.11). Now $\forall v \in V, STv = T^{-1}Tv = v$.

OR. [*Req V Finid*] Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n) \Rightarrow B_V = (v_1, \dots, v_n)$. Let $U \oplus \operatorname{range} T = W$.
Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i, Su = 0$ for each v_i and all $u \in U$. Thus $ST = I$.

(c) By Exe (12), $\exists$ subsp U of $W, W = U \oplus \operatorname{null} S,\ \operatorname{range} S = \operatorname{range} S|_U = V$.
Note that $S|_U : U \to V$ is iso. Define $T = (S|_U)^{-1}$, where $(S|_U)^{-1} : V \to U$.
Then $ST = S \circ (S|_U)^{-1} = S|_U \circ (S|_U)^{-1} = I_V$.

OR. [*Req V Finid*] Let $B_{\operatorname{range} S} = B_V = (Sw_1, \dots, Sw_n) \Rightarrow \operatorname{span}(w_1, \dots, w_n) \oplus \operatorname{null} S = W$.
Define $T \in \mathcal{L}(V, W)$ by $T(Sw_i) = w_i$. Now $ST(a_1Sw_1 + \cdots + a_nSw_n) = (a_1Sw_1 + \cdots + a_nSw_n)$. □

CORO: For (b), if T is inje and $\exists\, S,\ ST = I$, then by (a), this S is surj. Simlr for (c).

• **TIPS 5:** Sup $S \in \mathcal{L}(U, V)$ is surj. Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(U, W))$ by $\mathcal{B}(T) = TS$.
Then $\mathcal{B}$ is inje. Becs $\mathcal{B}(T) = TS = 0 \iff T|_{\operatorname{range} S} = 0$. OR. $\operatorname{range} TS = \operatorname{range} T = \{0\}$.

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24 *Sup* $S, T \in \mathcal{L}(V, W)$*, and* $\operatorname{null} S \subseteq \operatorname{null} T$*. Provt* $\exists\, E \in \mathcal{L}(W), T = ES$.

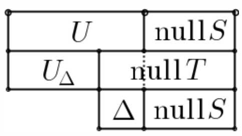
SOLUS:

Let $\underline{V = U \oplus \operatorname{null} S}$
$\underline{\Rightarrow S|_U : U \to \operatorname{range} S \text{ is iso.}}$
Extend $T(S|_U)^{-1}$ to $E \in \mathcal{L}(W)$.

$$\begin{tikzcd} \operatorname{range} T & U \arrow[l, "surj\ T"'] \arrow[d, "inv", "S"'] \\ & \operatorname{range} S \arrow[ul, "surj\ E"] \end{tikzcd}$$

OR. Define $E : \operatorname{range} S \to W$ by $\underline{E : Sv \mapsto Tv.}$
Extend $E \in \mathcal{L}(\operatorname{range} S, W)$ to $E \in \mathcal{L}(W)$. □

COMMENT: Let $\Delta \oplus \operatorname{null} S = \operatorname{null} T,\ U_\Delta \oplus (\Delta \oplus \operatorname{null} S) = V = U_\Delta \oplus \operatorname{null} T$. Redefine $U = U_\Delta \oplus \Delta$.



$$\operatorname{range} S \xleftarrow{S} \begin{matrix} U_\Delta \xrightarrow{T} \operatorname{range} T \\ \oplus \\ \Delta \xrightarrow{T} \{0\} \end{matrix}$$

Becs $\Delta = \operatorname{null} T|_U = \operatorname{null} T \cap \operatorname{range}(S|_U)^{-1}$.
Thus $E = T(S|_U)^{-1}$ is not inje $\iff \Delta \neq \{0\}$.
In other words, $\operatorname{range} S|_\Delta = \operatorname{null} E$,
while $E|_{\dots} : \operatorname{range} S|_{U_\Delta} \to \operatorname{range} T$ is iso.

COMMENT: Let $E_1 \in \mathcal{L}(U_\Delta \oplus \operatorname{null} T, U_\Delta)$, and E_2 be an iso of $\operatorname{range} S|_{U_\Delta}$ onto $\operatorname{range} T$.
Define $E_1|_{U_\Delta} = I|_{U_\Delta}$, and $E_2 = T(S|_{U_\Delta})^{-1}$. Then $T = E_2SE_1$.

CORO: If $\operatorname{null} S = \operatorname{null} T$. Then $\Delta = \{0\}, U_\Delta = U$.
By (3.D.3), we can extend inje $T(S|_U)^{-1} \in \mathcal{L}(\operatorname{range} S, W)$ to inv $E \in \mathcal{L}(W)$.

OR. [*Req* range S *Finid*] Let $B_{\operatorname{range} S} = (Sv_1, \dots, Sv_n)$. Then $\underline{V = \operatorname{span}(v_1, \dots, v_n) \oplus \operatorname{null} S.}$
Let $U \oplus \operatorname{range} S = W$. Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i,\ Eu = 0$ for all $u \in U$ and each v_i.
Hence $\forall v \in V$, $\underline{(\exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} S \subseteq \operatorname{null} T\,), Tv = a_1Tv_1 + \cdots + a_nTv_n} = E(a_1Sv_1 + \cdots + a_nSv_n)$ □

CORO: [*Req W Finid*] Sup $\operatorname{null} S = \operatorname{null} T$. We shat $\exists$ inv $E \in \mathcal{L}(W), T = ES$.
Redefine $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i,\ E(w_j) = x_j$, for each Tv_i and w_j. Where:
Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m), B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n), B_U = (v_1, \dots, v_m)$.
Now $V = U \oplus \operatorname{null} T = U \oplus \operatorname{null} S \Rightarrow B_{\operatorname{range} S} = (Sv_1, \dots, Sv_m)$. Let $B'_W = (Sv_1, \dots, Sv_m, x_1, \dots, x_n)$. □

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25 *Sup* $S, T \in \mathcal{L}(V, W)$, *and* $\operatorname{range} T \subseteq \operatorname{range} S$. *Provt* $\exists\, E \in \mathcal{L}(V), T = SE$.

Solus:

Let $V = U \oplus \operatorname{null} S \Rightarrow S|_U : U \to \operatorname{range} S$ is iso. Becs $(S|_U)^{-1} : \underbrace{\overline{\operatorname{range} S}}_{\supseteq \operatorname{range} T} \to U$.

$\underline{\text{Define } E = (S|_U)^{-1}T = (S|_U)^{-1}\big|_{\operatorname{range} T} T \in \mathcal{L}(V, U) \subseteq \mathcal{L}(V).}$ ☐

Comment: Let $U_1 = U$. Let $U_2 \oplus \operatorname{null} T = V = U_1 \oplus \operatorname{null} S$.
Let $U_{1\Delta} = \operatorname{range}\big(S|_{U_1}\big)\big|_{\operatorname{range} T} \subseteq U_1 = \Delta \oplus U_{1\Delta}$.
Or. Let $U_{1\Delta} = \operatorname{range} E|_{U_2}$. Let $\Delta \oplus \operatorname{range} E|_{U_2} = U_1$.
Thus $U_1 \oplus \operatorname{null} S = U_{1\Delta} \oplus \underbrace{\underline{(\Delta \oplus \operatorname{null} S)} = U_2 \oplus \underline{\operatorname{null} T}}_{\text{iso, by (3.D.Tips)}}$.

$$\begin{tikzcd}
U_1 \arrow[r, "inv", "S"'] \arrow[d, phantom, "||"] & \operatorname{range} S \arrow[d, phantom, "||"] & \\
\Delta \arrow[r, "inv", "S"'] \arrow[d, phantom, "\oplus"] & \operatorname{range} S|_\Delta \arrow[d, phantom, "\oplus"] & \\
U_{1\Delta} \arrow[r, "inv", "S"'] & \operatorname{range} T & U_2 \arrow[l, "inv"', "T"] \arrow[ll, bend left, "inv\ E|_{U_2}"]
\end{tikzcd}$$

If $\Delta \neq \{0\}$, asm $\exists$ inv $E \in \mathcal{L}(V)$ re-extended from $E|_{U_2}$ still satisfying $T = SE$,
then let $\Delta \xrightarrow{E^{-1}} \Theta$; $\operatorname{null} S \xrightarrow{E^{-1}} \operatorname{null} T_\Theta$. Now $\Theta \oplus \operatorname{null} T_\Theta = \operatorname{null} T$.
Then $\Theta \xrightarrow{E} \Delta \neq \{0\}$, while $\operatorname{null} S \cap \Delta = \{0\}$. Thus $T|_\Theta = SE|_\Theta \neq 0$, ctradic.

Coro: If $\Delta = \{0\}$, then $U_1 = U_{1\Delta} \Rightarrow \operatorname{range} S = \operatorname{range} T$. 又 $\operatorname{null} S, \operatorname{null} T$ are iso.
By (3.D.3), we can re-extend inje $E|_{U_2} \in \mathcal{L}\big(U_2, U_1 \oplus \operatorname{null} S\big)$ to inv $E \in \mathcal{L}\big(U_2 \oplus \operatorname{null} T, U_1 \oplus \operatorname{null} S\big)$.
Thus we have $\Delta \neq \{0\} \iff E|_{U_2} \in \mathcal{L}(U_2, V)$ cannot be re-extended to inv $E \in \mathcal{L}(V)$ freely.

Or. $\big[$ *Req* $\operatorname{range} T$ *Finid* $\big]$ Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$. Then $\underline{V = \operatorname{span}(v_1, \dots, v_n) \oplus \operatorname{null} T.}$
$\underline{\text{Let } S(u_i) = Tv_i \text{ for each } Tv_i.}$ Define E by $Ev_i = u_i, Ex = 0$ for all $x \in \operatorname{null} T$ and each v_i. ☐

Comment: $\big[$ *Req* V *Finid* $\big]$ Note that $\dim U_2 \leqslant \dim U_1 \Longrightarrow \dim \operatorname{null} T = p \geqslant q = \dim \operatorname{null} S$.
Let $B_{\operatorname{null} T} = (x_1, \dots, x_p), B_{\operatorname{null} S} = (y_1, \dots, y_q)$. Redefine $E : v_i \mapsto u_i,\ x_k \mapsto y_k,\ x_j \mapsto 0$,
for each $i \in \{1, \dots, \dim U_2\}, k \in \{1, \dots, \dim \operatorname{null} S\}, j \in \{\dim \operatorname{null} S + 1, \dots, \dim \operatorname{null} T\}$.
Note that $(u_1, \dots, u_n)$ is linely inde. Let $X = \operatorname{span}(x_1, \dots, x_q) \oplus \operatorname{span}(v_1, \dots, v_n)$.
Now $E|_X$ is inje, but cannot be re-extend to inv $E \in \mathcal{L}(V)$ without loss of functionality.

Coro: $\big[$ *Req* V *Finid* $\big]$ If $\operatorname{range} T = \operatorname{range} S$, then $\dim \operatorname{null} T = \dim \operatorname{null} S = p$.
Redefine E by $Ev_i = u_i,\ Ex_j = y_j$ for each v_i and x_j. Then $E \in \mathcal{L}(V)$ is inv. ☐

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28 *Sup* $T \in \mathcal{L}(V, W)$. *Let* $B_{\operatorname{range} T} = (w_1, \dots, w_m)$.

(a) *Provt* $\exists\, \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ *suth* $\forall v \in V, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.

(b) [4E 3.F.5] $\forall v \in V, \exists!\, \varphi_i(v) \in \mathbf{F}, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.
Thus defining each $\varphi_i : V \to \mathbf{F}$. *Shat each* $\varphi_i \in \mathcal{L}(V, \mathbf{F})$.

Solus: (a) Using Tips (4). Let each $w_i = Tv_i$. Then $(v_1, \dots, v_m)$ is linely inde.
And $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = V$. Now $\forall v \in V, \exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} T,\ v = \sum_{i=1}^m a_i v_i + u$.
Define $\varphi_i \in \mathcal{L}(V, \mathbf{F})$ by $\varphi_i(v_j) = \delta_{i,j},\ \varphi_i(u) = 0$ for all $u \in \operatorname{null} T$.
Linearity: $\forall v, w \in V\ [\exists!\, a_i, b_i \in \mathbf{F}], \lambda \in \mathbf{F}, \varphi_i(v + \lambda w) = a_i + \lambda b_i = \varphi(v) + \lambda\varphi(w)$. ☐

(b) $\sum_{i=1}^m \varphi_i(u + \lambda v)w_i = T(u + \lambda v) = Tu + \lambda Tv = \big(\sum_{i=1}^m \varphi_i(u)w_i\big) + \lambda\big(\sum_{i=1}^m \varphi_i(v)w_i\big)$. ☐

Or. Using (3.F). Let each $w_i = Tv_i \Rightarrow (v_1, \dots, v_m)$ is linely inde.
Now $\forall v \in V, \exists!\, a_i \in \mathbf{F}, Tv = a_1 Tv_1 + \cdots + a_m Tv_m$. Let $B_{(\operatorname{range} T)'} = (\psi_1, \dots, \psi_m)$.
Then $[T'(\psi_i)](v) = (\psi_i \circ T)(v) = a_i$. Where $T : V \to \operatorname{range} T$; $T' : (\operatorname{range} T)' \to V'$.
Thus each $\varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$. ☐

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29 *Sup* $\varphi \in \mathcal{L}(V, \mathbf{F})$. *Sup* $\varphi(u) \neq 0$. *Provt* $V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}$.

SOLUS: Let $B_{\operatorname{range}\varphi} = (\varphi(u))$. Then by TIPS (4), $\operatorname{span}(u) \oplus \operatorname{null}\varphi = V$. $\square$

OR. (a) $v = cu \in \operatorname{null}\varphi \cap \operatorname{span}(u) \Rightarrow c\varphi(u) = 0 \Rightarrow v = 0$. Now $\operatorname{null}\varphi \cap \operatorname{span}(u) = \{0\}$.

(b) $\forall\, v \in V, v = \left(v - \dfrac{\varphi(v)}{\varphi(u)}u\right) + \dfrac{\varphi(v)}{\varphi(u)}u$. Now $V = \operatorname{null}\varphi + \operatorname{span}(u)$. $\square$

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30 *Sup* $\varphi, \beta \in \mathcal{L}(V, \mathbf{F})$ *and* $\operatorname{null}\varphi = \operatorname{null}\beta = \eta$. *Provt* $\exists\, c \in \mathbf{F}, \varphi = c\beta$.

SOLUS: If $\eta = V$, then $\varphi = \beta = 0$, we are done. Now by Exe (29),

$\varphi(u) \neq 0 \iff V = \operatorname{null}\varphi \oplus \operatorname{span}(u) \iff V = \operatorname{null}\beta \oplus \operatorname{span}(u) \iff \beta(u) \neq 0$.

Note that $\forall v \in V, \exists\,!\, u_0 \in \eta,\ a_v \in \mathbf{F}, v = u_0 + a_v u$
$\Rightarrow \varphi(u_0 + a_v u) = a_v\varphi(u),\ \beta(u_0 + a_v u) = a_v\beta(u).$ | Let $c = \dfrac{\varphi(u)}{\beta(u)} \in \mathbf{F}\backslash\{0\}$. $\square$

- (4E 3.F.6) *Sup* $\varphi, \beta \in \mathcal{L}(V, \mathbf{F})$. *Provt* $\operatorname{null}\beta \subseteq \operatorname{null}\varphi \iff \varphi = c\beta, \exists\, c \in \mathbf{F}$.

  CORO: $\operatorname{null}\varphi = \operatorname{null}\beta \iff \varphi = c\beta,\ \exists\, c \in \mathbf{F}\backslash\{0\}$.

SOLUS: Using Exe (29) and (30).

(a) If $\varphi = 0$, then we are done. Othws, sup $u \notin \operatorname{null}\varphi \supseteq \operatorname{null}\beta$.
Now $V = \operatorname{null}\varphi \oplus \operatorname{span}(u) = \operatorname{null}\beta \oplus \operatorname{span}(u)$. By [1.C TIPS (2)], $\operatorname{null}\varphi = \operatorname{null}\beta$. Let $c = \dfrac{\varphi(u)}{\beta(u)}$.

OR. We discuss in two cases. If $\operatorname{null}\beta = \operatorname{null}\varphi$, or if $\varphi = 0$, then we are done. Othws,
$\exists\, u' \in \operatorname{null}\varphi \backslash \operatorname{null}\beta$, $\exists\, u \notin \operatorname{null}\varphi \supsetneq \operatorname{null}\beta \Rightarrow V = \operatorname{null}\beta \oplus \operatorname{span}(u') = \operatorname{null}\beta \oplus \operatorname{span}(u)$.
$\forall v \in V, v = w + au = w' + bu',\ \exists\,!\, w, w' \in \operatorname{null}\beta$
Thus $\varphi(w + au) = a\varphi(u),\ \beta(w' + bu) = b\beta(u')$. | Let $c = \dfrac{a\varphi(u)}{b\beta(u')} \in \mathbf{F}\backslash\{0\}$. We are done.

NOTICE that by (b) below, we have $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$, ctradic the asm.

(b) If $c = 0$, then $\operatorname{null}\varphi = V \supseteq \operatorname{null}\beta$, we are done. Othws, becs $v \in \operatorname{null}\beta \iff v \in \operatorname{null}\varphi$. $\square$

OR. By Exe (24), $\operatorname{null}\beta \subseteq \operatorname{null}\varphi \iff \exists\, E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta$. [If E is inv. Then $\operatorname{null}\beta = \operatorname{null}\varphi$.]
Now $\exists\, E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta \iff \exists\, c = E(1) \in \mathbf{F}, \varphi = c\beta$. [E is inv $\iff E(1) \neq 0 \iff c \neq 0$.] $\square$

ENDED

3·C   1 3 4 5 6 9 10 11 13 | 4E: 16 17

- **Note For Transpose:** [3.F.33] Define $\mathcal{T} : A \to A^t$. By [3.111], $\mathcal{T}$ is linear. Becs $(A^t)^t = A$.
 $\mathcal{T}^2 = I$, $\mathcal{T} = \mathcal{T}^{-1} \Rightarrow \mathcal{T}$ is iso of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$. Define $\mathcal{C}_k : A \to A_{\cdot,k}$, $\mathcal{R}_j : A \to A_{j,\cdot}$, $\mathcal{E}_{j,k} : A \to A_{j,k}$.
 Now we shat (a) $\underline{\mathcal{T}\mathcal{R}_j = \mathcal{C}_j\mathcal{T}}$, (b) $\underline{\mathcal{T}\mathcal{C}_k = \mathcal{R}_k\mathcal{T}}$, and (c) $\underline{\mathcal{T}\mathcal{E}_{j,k} = \mathcal{E}_{k,j}\mathcal{T}}$.
 So that furthermore, $\mathcal{T}\mathcal{C}_k\mathcal{T} = \mathcal{R}_k$, $\mathcal{T}\mathcal{R}_j\mathcal{T} = \mathcal{C}_j$, and $\mathcal{T}\mathcal{E}_{j,k}\mathcal{T} = \mathcal{E}_{k,j}$.

 Let $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{pmatrix}$.
 Note that $(A_{j,k})^t = A_{j,k} = (A^t)_{k,j}$. Thus (c) holds.
 And $(A_{\cdot,k})^t = (A_{1,k} \cdots A_{m,k}) = (A^t_{k,1} \cdots A^t_{k,m}) = (A^t)_{k,\cdot}$
 $\Longrightarrow$ (b) holds. Simlr for (a).

- **Note For [3.48]:**

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}}_{B} = \begin{pmatrix} (1\ 2)\begin{pmatrix}5\\8\end{pmatrix} & (1\ 2)\begin{pmatrix}6\\9\end{pmatrix} & (1\ 2)\begin{pmatrix}7\\10\end{pmatrix} \\ (3\ 4)\begin{pmatrix}5\\8\end{pmatrix} & (3\ 4)\begin{pmatrix}6\\9\end{pmatrix} & (3\ 4)\begin{pmatrix}7\\10\end{pmatrix} \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

- **Note For [3.47]:** $(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k}$ ☐
- **Note For [3.49]:** $\left[(AC)_{\cdot,k}\right]_{j,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$ ☐
- **Exe 10:** $\left[(AC)_{j,\cdot}\right]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$ ☐
- **Comment:** For [3.49], let $B_U = (u_1, \dots, u_p), B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.
 And $C = \mathcal{M}(T, B_U, B_V) \in \mathbf{F}^{n,p}, A = \mathcal{M}(S, B_V, B_W) \in \mathbf{F}^{m,n}$.
 Then $\mathcal{M}(Tu_k, B_V) = C_{\cdot,k} \Rightarrow \mathcal{M}(S(Tu_k), B_W) = AC_{\cdot,k}$, 又 $\mathcal{M}((ST)(u_k), B_W) = (AC)_{\cdot,k}$ ☐
 By Note For Transpose, $(AC)_{j,\cdot} = \left[\left((AC)^t\right)_{\cdot,j}\right]^t = \left(C^t(A^t)_{\cdot,j}\right)^t = \left((A^t)_{\cdot,j}\right)^t C = A_{j,\cdot}C$ ☐

- **Note For [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$. By [4E 3.51(a)], $(Ac)_{\cdot,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ ☐
 Or. $\because (Ac)_{j,1} = \sum_{r=1}^{n} A_{j,r}c_{r,1} = \left[\sum_{r=1}^{n} (A_{\cdot,r}c_{r,1})\right]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$
 $\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^{n} A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ Or. $(Ac)_{j,1} = (Ac)_{j,\cdot} = A_{j,\cdot}c \in \mathbf{F}$. ☐
 Or. Let $B_V = (v_1, \dots, v_n)$. Now $Ac = \mathcal{M}(Tv, B_W) = \mathcal{M}(T(c_1v_1 + \cdots + c_nv_n)) = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$. ☐

- **Exe 11:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$. By [4E 3.51(b)], $(aC)_{1,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ ☐
 Or. $\because (aC)_{1,k} = \sum_{r=1}^{n} a_{1,r}C_{r,k} = \left[\sum_{r=1}^{n} a_{1,r}(C_{r,\cdot})\right]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$
 $\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^{n} a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ Or. $(aC)_{1,k} = (aC)_{\cdot,k} = aC_{\cdot,k} \in \mathbf{F}$. ☐
 Or. $aC = \left((aC)^t\right)^t = \left(C^ta^t\right)^t = \left[a_1^t(C^t)_{\cdot,1} + \cdots + a_n^t(C^t)_{\cdot,n}\right]^t = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$. ☐

- [4E 3.51] Sup $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}$. [See also Note For [3.49] and Exe (10).]
 (a) For $k = 1, \dots, p$, $(CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^{c} C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$
 (b) For $j = 1, \dots, m$, $(CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
- **Exa:** $m = 2, c = 2, p = 3$.
 $(AB)_{\cdot,2} = AB_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} = A_{\cdot,1}B_{1,2} + A_{\cdot,2}B_{2,2} = 6\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix}$;
 $(AB)_{1,\cdot} = A_{1,\cdot}B = (1\ \ 2)\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = A_{1,1}B_{1,\cdot} + A_{1,2}B_{2,\cdot} = 1(5\ \ 6\ \ 7) + 2(8\ \ 9\ \ 10) = (21\ \ 24\ \ 27)$;

• **Column-Row Factorization** (CR Factorization) *Sup* $A \in \mathbf{F}^{m,n}, A \neq 0$.

Prove, with p specified below, that $\exists\, C \in \mathbf{F}^{m,p}, R \in \mathbf{F}^{p,n}, A = CR$.

(a) *Sup* $S_c = \operatorname{span}\left(A_{\cdot,1}, \cdots, A_{\cdot,n}\right) \subseteq \mathbf{F}^{m,1}, \dim S_c = c$, *the col rank. Let* $p = c$.

(b) *Sup* $S_r = \operatorname{span}\left(A_{1,\cdot}, \cdots, A_{m,\cdot}\right) \subseteq \mathbf{F}^{1,n}, \dim S_r = r$, *the row rank. Let* $p = r$.

Solus: Using [4E 3.51]. Notice that $A \neq 0 \Rightarrow c, r \geqslant 1$.

(a) Reduce to basis $B_C = \left(C_{\cdot,1}, \cdots, C_{\cdot,c}\right)$, forming $C \in \mathbf{F}^{m,c}$. Then $\forall k \in \{1, \dots, n\}$,

$A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}\,, \exists!\, R_{1,k}, \cdots, R_{c,k} \in \mathbf{F}$, forming $R \in \mathbf{F}^{c,n}$. Thus $A = CR$.

(b) Reduce to basis $B_R = \left(R_{1,\cdot}, \cdots, R_{r,\cdot}\right)$, forming $R \in \mathbf{F}^{r,n}$. Then $\forall j \in \{1, \dots, m\}$,

$A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,r}R_{r,\cdot} = (CR)_{j,\cdot}\,, \exists!\, C_{j,1}, \dots, C_{j,r} \in \mathbf{F}$, forming $C \in \mathbf{F}^{m,r}$. Thus $A = CR$. ☐

Exa: $$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \overset{\text{(I)}}{=\!=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \overset{\text{(II)}}{=\!=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

(I) $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2\begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix}$, using [4E 3.51(b)].

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} \in \operatorname{span}(A_{1,\cdot}, A_{2,\cdot})$, and $(A_{1,\cdot}, A_{2,\cdot})$ is linely inde. Thus $B_R = \left(A_{1,\cdot}, A_{2,\cdot}\right)$.

(II) $\begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = -\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$. Thus $B_C = \left(A_{\cdot,2}, A_{\cdot,3}\right)$.

• **Column Rank Equals Row Rank** Using nota and result above.

For each $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

For each $A_{\cdot,k} \in S_c$, $A_{\cdot,k} = (CR)_{\cdot,k} = CR_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$

$\Rightarrow \operatorname{span}\left(A_{1,\cdot}, \cdots, A_{n,\cdot}\right) = S_r = \operatorname{span}\left(R_{1,\cdot}, \cdots, R_{c,\cdot}\right) \Rightarrow \dim S_r = r \leqslant c = \dim S_c$.

$\Rightarrow \operatorname{span}\left(A_{\cdot,1}, \cdots, A_{\cdot,m}\right) = S_c = \operatorname{span}\left(C_{\cdot,1}, \cdots, C_{\cdot,r}\right) \Rightarrow \dim S_c = c \leqslant r = \dim S_r$.

Or. Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leqslant r = \dim S_r = \dim S_c^t$. ☐

• *Sup* $A \in \mathbf{F}^{m,n} \backslash \{0\}$. *Provt* $[P]$ $\operatorname{rank} A = 1 \iff \exists\, c_j, d_k \in \mathbf{F}$, *each* $A_{j,k} = c_j \cdot d_k$. $[Q]$

Solus:

[*Using CR Factorization*]

$P \Rightarrow Q$: Immediately.

$Q \Rightarrow P$: Becs $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1d_1 & \cdots & c_1d_n \\ \vdots & \ddots & \vdots \\ c_md_1 & \cdots & c_md_n \end{pmatrix} \Longrightarrow S_r = \operatorname{span}\left\{ \begin{matrix} \left(\underline{c_1}d_1 \cdots \underline{c_1}d_n\right), \\ \vdots \\ \left(\underline{c_m}d_1 \cdots \underline{c_m}d_n\right) \end{matrix} \right\}$.

Or. $S_c = \operatorname{span}\left\{ \begin{pmatrix} c_1\underline{d_1} \\ \vdots \\ c_m\underline{d_1} \end{pmatrix}, \dots, \begin{pmatrix} c_1\underline{d_n} \\ \vdots \\ c_m\underline{d_n} \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$. ☐

[*Not Using CR Factorization*]

$Q \Rightarrow P$: Using [4E 3.51(a)]. Each $A_{\cdot,k} \in \operatorname{span}\left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$. Then $\operatorname{rank} A = \dim S_c \leqslant 1$

又 $A \neq 0 \Rightarrow \dim S_c \geqslant 1$.

$P \Rightarrow Q$: Becs $\dim S_c = \dim S_r = 1$.

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $\quad d'_k = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$.

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$, where $d_k = d'_k A_{1,1}$. ☐

• **TIPS 1:** *Sup* $T \in \mathcal{L}(V,W)$, $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.

Let $L = (Tv_{\alpha_1}, \dots, Tv_{\alpha_k})$, $M = (A_{\cdot,\alpha_1}, \cdots, A_{\cdot,\alpha_k})$, *where each* $\alpha_i \in \{1, \dots, n\}$.

(a) *Shat* $[P]$ L *is linely inde* $\iff$ M *is linely inde.* $[Q]$

(b) *Shat* $[P]$ $\operatorname{span} L = W \iff \operatorname{span} M = \mathbf{F}^{m,1}$. $[Q]$ $\quad$ $\big[$ *Let* $A = \mathcal{M}(T, B_V, B_W)$. $\big]$

SOLUS:

(a) Note that $\mathcal{M}\colon Tv_k \to A_{\cdot,k}$ is iso of W onto $\mathbf{F}^{m,1}$. (b) Reduce L to B'_W, M to $B_{\mathbf{F}^{m,1}}$. Simlr. □

OR. $$c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} = c_1(A_{1,\alpha_1} w_1 + \cdots + A_{m,\alpha_1} w_m) + \cdots + c_k(A_{1,\alpha_k} w_1 + \cdots + A_{m,\alpha_k} w_m)$$
$$= (c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k}) w_1 + \cdots + (c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k}) w_m.$$

And $$c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} = c_1 \begin{pmatrix} A_{1,\alpha_1} \\ \vdots \\ A_{m,\alpha_1} \end{pmatrix} + \cdots + c_k \begin{pmatrix} A_{1,\alpha_k} \\ \vdots \\ A_{m,\alpha_k} \end{pmatrix} = \begin{pmatrix} c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k} \\ \vdots \\ c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k} \end{pmatrix}.$$

(a) $P \Rightarrow Q$: Sup $c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} = 0$. Let $v = c_1 v_{\alpha_1} + \cdots + c_k v_{\alpha_k}$.

Then $Tv = (c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k}) w_1 + \cdots + (c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k}) w_m = 0w_1 + \cdots + 0w_m$.

Now $c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} = 0$. Then each $c_i = 0 \Rightarrow M$ linely inde.

$Q \Rightarrow P$: Becs $c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k} = 0$. For each $i \in \{1, \dots, m\}$, $c_1 A_{i,\alpha_1} + \cdots + c_k A_{i,\alpha_k} = 0$.

Which is equi to $c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} = 0$. Thus each $c_i = 0 \Rightarrow L$ linely inde.

OR. $\exists A_{\cdot,\alpha_j} = c_1 A_{\cdot,\alpha_1} + \cdots + c_{j-1} A_{\cdot,\alpha_{j-1}}$

$\iff$ For each $i \in \{1, \dots, m\}$, $A_{i,\alpha_j} = c_1 A_{i,\alpha_1} + \cdots + c_{j-1} A_{i,\alpha_{j-1}}$

$\iff Tv_{\alpha_j} = A_{1,\alpha_j} w_1 + \cdots + A_{m,\alpha_j} w_m$

$= (c_1 A_{1,\alpha_1} + \cdots + c_{j-1} A_{1,\alpha_{j-1}}) w_1 + \cdots + (c_1 A_{m,\alpha_1} + \cdots + c_{j-1} A_{m,\alpha_{j-1}}) w_m$

$\iff \exists Tv_{\alpha_j} = c_1 Tv_{\alpha_1} + \cdots + c_{j-1} Tv_{\alpha_{j-1}}$.

(b) Note that each $\mathcal{M}(Tv_{\alpha_i}) = A_{\cdot,\alpha_i}$

$P \Rightarrow Q$: Sup each $w_i = Iw_i = J_{1,i} Tv_{\alpha_1} + \cdots + J_{k,i} Tv_{\alpha_k}$.

$\forall a \in \mathbf{F}^{m,1}, \exists w = a_1 w_1 + \cdots + a_m w_m \in W$, $a = \mathcal{M}(w, B_W)$.

Becs $w = a_1(J_{1,1} Tv_{\alpha_1} + \cdots + J_{k,1} Tv_{\alpha_k}) + \cdots + a_m(J_{1,m} Tv_{\alpha_1} + \cdots + J_{k,m} Tv_{\alpha_k})$

$= (a_1 J_{1,1} + \cdots + a_m J_{1,m}) Tv_{\alpha_1} + \cdots + (a_1 J_{k,1} + \cdots + a_m J_{k,m}) Tv_{\alpha_k}$.

Apply $\mathcal{M}$ to both sides, $a = c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k}$, where each $c_i = a_1 J_{i,1} + \cdots + a_m J_{i,m}$.

$Q \Rightarrow P$: $\forall w \in W, \exists a = c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k} \in \mathbf{F}^{m,1}$, $\mathcal{M}(w, B_W) = a$

$\Rightarrow w = (c_1 A_{1,\alpha_1} + \cdots + c_k A_{1,\alpha_k}) w_1 + \cdots + (c_1 A_{m,\alpha_1} + \cdots + c_k A_{m,\alpha_k}) w_m = c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k}$.

$\neg Q \Rightarrow \neg P$: $\exists w \in W, \exists a \in \mathbf{F}^{m,1}, \mathcal{M}(w, B_W) = a$, but $\nexists c_i \in \mathbf{F}, a = c_1 A_{\cdot,\alpha_1} + \cdots + c_k A_{\cdot,\alpha_k}$

$\Rightarrow \nexists c_i \in \mathbf{F}$, $w = c_1 Tv_{\alpha_1} + \cdots + c_k Tv_{\alpha_k}$. □

CORO: Let $L = (Tv_1, \dots, Tv_n)$, $M = (A_{\cdot,1}, \cdots, A_{\cdot,n})$.

Then (a*) By $[$3.B.9, TIPS (4)$]$, T is inje $\iff$ L is linely inde, so is M.

And (b*) T is surj $\iff \operatorname{span} L = W \iff \operatorname{span} M = \mathbf{F}^{m,1}$.

CORO: $B_{\mathbf{F}^{n,1}} = (A_{\cdot,1}, \cdots, A_{\cdot,n}) \iff T$ is inje and surj $\iff B_{\mathbf{F}^{1,n}} = (A_{\cdot,1}, \cdots, A_{\cdot,n})$.

COMMENT: If T is inv. Then by (a*, c) or (b*, d), we have another proof of CORO.

OR. If $m = n$. Then by [3.118] and one of (a*, b*, c, d). Yet another proof.

(c) T surj $\iff T'$ inje $\iff (T'(\psi_1), \dots, T'(\psi_m))$ linely inde

$\overset{(a)}{\iff} ((A^t)_{\cdot,1}, \cdots, (A^t)_{\cdot,m})$ linely inde in $\mathbf{F}^{n,1}$, so is $(A_{1,\cdot}, \cdots, A_{m,\cdot})$ in $\mathbf{F}^{1,n}$.

(d) T inje $\iff T'$ surj $\iff V' = \operatorname{span}(T'(\psi_1), \dots, T'(\psi_m))$

$\overset{(b)}{\iff} \mathbf{F}^{n,1} = \operatorname{span}((A^t)_{\cdot,1}, \cdots, (A^t)_{\cdot,m}) \iff \mathbf{F}^{1,n} = \operatorname{span}(A_{1,\cdot}, \cdots, A_{m,\cdot})$.

• **TIPS 2:** *Sup p is a poly of n variables in $\mathbf{F}$. Provt $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big)$.*

Where the linear maps $T_1,\dots,T_n$ are suth $p(T_1,\dots,T_n)$ makes sense. See [5.16,17,20].

SOLUS: Sup the poly p is defined by $p(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} \prod_{i=1}^{n} x_i^{k_i}$.

Note that $\mathcal{M}(T^x S^y) = \mathcal{M}(T)^x \mathcal{M}(S)^y$; $\mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$.

Then $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = \mathcal{M}\Big(\sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} \prod_{i=1}^{n} T_i^{k_i}\Big)$

$= \sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} \prod_{i=1}^{n} \mathcal{M}\big(T_i^{k_i}\big) = p\Big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\Big)$. □

• **CORO:** Sup τ is an algebraic property. Then τ holds for linear maps $\Longleftrightarrow$ τ holds for matrices.

Each $\alpha_k \in \{1,\dots,n\}$. Now $p(T_1,\dots,T_n) = p(T_{\alpha_1},\dots,T_{\alpha_n})$

$\Longleftrightarrow p\Big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\Big) = p\Big(\mathcal{M}(T_{\alpha_1}),\dots,\mathcal{M}(T_{\alpha_n})\Big)$.

---

13 *Provt the distr holds for matrix add and matrix multi.*

Sup A, B, C are matrices suth $A(B+C)$ make sense, we prove the left distr.

SOLUS: Sup $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Note that $\big[A(B+C)\big]_{j,k} = \sum_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum_{r=1}^{n}\big(A_{j,r}B_{r,k} + A_{j,r}C_{r,k}\big) = \big(AB+AC\big)_{j,k}$.

OR. Define T, S, R suth $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$A(B+C) = \mathcal{M}(T(S+R)) \xlongequal{[3.9]} \mathcal{M}(TS+TR) = AB + AC$.

OR. $T(S+R) = TS + TR \Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR) \Rightarrow A(B+C) = AB + AC$. □

---

1 *Sup $T \in \mathcal{L}(V,W)$. Shat for each pair of B_V and B_W,*
$A = \mathcal{M}(T, B_V, B_W)$ has at least $n = \dim \operatorname{range} T$ nonzero ent.

SOLUS:

Using $\big[$3.B TIPS (4)$\big]$. Let $U \oplus \operatorname{null} T = V$; $B_U = (v_1,\dots,v_n), B_V = (v_1,\dots,v_m)$.

For each $k \in \{1,\dots,n\}, Tv_k \neq 0 \Longleftrightarrow A_{\cdot,k} \neq 0$. Hence every such $A_{\cdot,k}$ has at least one nonzero ent. □

OR. We prove by ctradic. Sup A has at most $(n-1)$ nonzero ent.

Then by Pigeon Hole Principle, at least one of $A_{\cdot,1},\dots,A_{\cdot,n}$ equals 0.

Thus there are at most $(n-1)$ nonzero vecs in $Tv_1,\dots,Tv_n$.

又 $\operatorname{range} T = \operatorname{span}(Tv_1,\dots,Tv_n) \Rightarrow \dim \operatorname{range} T = \dim \operatorname{span}(Tv_1,\dots,Tv_n) \leqslant n-1$. Ctradic. □

---

6 *Sup V and W are finide and $T \in \mathcal{L}(V,W)$.*
Provt $\dim \operatorname{range} T = 1 \Longleftrightarrow \exists\, B_V, B_W$, all ent of $A = \mathcal{M}\big(T, B_V, B_W\big)$ equal 1.

SOLUS:

(a) Sup $B_V = (v_1,\dots,v_n), B_W = (w_1,\dots,w_m)$ are the bases suth all ent of A equal 1.

Then $Tv_i = w_1 + \cdots + w_m$ for all $i = 1,\dots,n$. Becs $w_1,\dots,w_n$ is linely inde, $w_1 + \cdots + w_n \neq 0$.

(b) Sup $\dim \operatorname{range} T = 1$. Then $\dim \operatorname{null} T = \dim V - 1$.

Let $B_{\operatorname{null} T} = (u_2,\dots,u_n)$. Extend to a basis $(u_1,u_2,\dots,u_n)$ of V.

Let $w_1 = Tv_1 - w_2 - \cdots - w_m$. Extend to B_W. Let $v_1 = u_1,\ v_i = u_1 + u_i$. Extend to B_V. □

OR. Sup $B_{\operatorname{range} T} = (w)$. By $\big[$2.C NOTE FOR (15)$\big]$, $\exists\, B_W = (w_1,\dots,w_m),\ w = w_1 + \cdots + w_m$.

By $\big[$2.C TIPS$\big]$, $\exists$ a basis $(u_1,\dots,u_n)$ of V suth each $u_k \notin \operatorname{null} T$.

Now each $Tu_k \in \operatorname{range} T = \operatorname{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists\, \lambda_k \in \mathbf{F}\backslash\{0\}$.

Let $v_k = \lambda_k^{-1} u_k \neq 0$, so that each $Tv_k = w = w_1 + \cdots + w_m$. Thus $B_V = (v_1,\dots,v_n)$ will do. □

---

3 *Sup V and W are finide and $T \in \mathcal{L}(V,W)$. Provt $\exists\, B_V, B_W$ suth*
$[$ letting $A = \mathcal{M}(T, B_V, B_W)$ $]$ $A_{k,k} = 1, A_{i,j} = 0$, where $1 \leqslant k \leqslant \operatorname{dim} \operatorname{range} T, i \neq j$.

SOLUS: Using $[$3.B TIPS (4)$]$. Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n), B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$. □

COMMENT: Let each $Tv_k = w_k$. Extend $B_{\operatorname{range} T}$ to $B_W = (w_1, \dots, w_n, \dots, w_p)$. See $[$3.D NOTE FOR [3.60]$]$.

---

4 *Sup $B_V = (v_1, \dots, v_m)$ and W is finide. Sup $T \in \mathcal{L}(V,W)$.*
Provt $\exists\, B_W = (w_1, \dots, w_n)$, $\mathcal{M}(T, B_V, B_W)_{\cdot,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^t$ or $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}^t$.

SOLUS: If $Tv_1 = 0$, then we are done. If not then extend (Tv_1) to B_W. □

---

5 *Sup $B_W = (w_1, \dots, w_n)$ and V is finide. Sup $T \in \mathcal{L}(V,W)$.*
Provt $\exists\, B_V = (v_1, \dots, v_m)$, $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$.

SOLUS:

Let $(u_1, \dots, u_n)$ be a basis of V. Denote $\mathcal{M}(T, (u_1, \dots, u_n), B_W)$ by A.

If $A_{1,\cdot} = 0$, then $B_V = (u_1, \dots, u_n)$ and we are done. Othws, sup $A_{1,k} \neq 0$.

Let $v_1 = \dfrac{u_k}{A_{1,k}} \Rightarrow Tv_1 = 1w_1 + \dfrac{A_{2,k}}{A_{1,k}} w_2 + \cdots + \dfrac{A_{n,k}}{A_{1,k}} w_n$. | Let $v_j = u_{j-1} - A_{1,j-1} v_1$ for each $j \in \{2, \dots, k\}$. Let $v_i = u_i - A_{1,i} v_1$ for $i \in \{k+1, \dots, n\}$.

NOTICE that $Tu_i = A_{1,i} w_1 + \cdots + A_{n,i} w_n$. 又 Each $u_i \in \operatorname{span}(v_1, \dots, v_n) = V$. Let $B_V = (v_1, \dots, v_n)$. □

OR. Using Exe (4). Let $B_{W'}$ be the B_{V}.

Now $\exists\, B_{V'}$ suth $\mathcal{M}(T', B_{W'}, B_{V'})_{\cdot,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^t$ or $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}^t$.

Which is equiv to $\exists\, B_V$ $[$Using (3.F.31)$]$ suth $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$. □

ENDED

3·D

1 2 3 4 5 6 8 9 10 11 12 13 15 16 17 18 19 | 4E: 3 10 15 17 19 20 22 23 24

2 *Sup V is finide and* $\operatorname{dim} V > 1$.
Provt the set U of non-inv optors on V is not a subsp of $\mathcal{L}(V)$.
The set of inv optors is not either. Although multi id/inv, and commu for vec multi hold.

SOLUS: Let $B_V = (v_1, \dots, v_n)$. $[$ *If* $\operatorname{dim} V = 1$, *then $U = \{0\}$ is a subsp of $\mathcal{L}(V)$.* $]$
Define $S, T \in \mathcal{L}(V)$ by $S(a_1 v_1 + \cdots + a_n v_n) = a_1 v_1$, $T(a_1 v_1 + \cdots + a_n v_n) = a_2 v_1 + \cdots + a_n v_n$.
Hence $S, T \in U$ while $S + T \notin U$. □

• **TIPS:** *Sup $U \oplus X = W \oplus Y$, and X, Y are iso. Provt U, W are iso.*

SOLUS: Let ξ be an iso of X onto Y. That is, $\forall y \in Y, \exists!\, x \in X, \xi(x) = y$.
$\forall u \in U, \exists!\, w \in W, y \in Y, u = w + y \Rightarrow \exists!\, x \in X, u = w + \xi(x)$. Define $\pi : u \mapsto w$.
Now sup $u_1, u_2 \in U$, then each $u_i = w_i + \xi(x_i), \exists!\, w_i \in W, x_i \in X$.
Linearity: $\forall, \lambda \in \mathbf{F}, \pi(u_1 + \lambda u_2) = w_1 + \lambda w_2 = \pi(u_1) + \lambda \pi(u_2)$.
Injectivity: $\pi(u_1) = \pi(u_2) \Rightarrow w_1 = w_2 \Rightarrow \xi(x_1) = \xi(x_2) \Rightarrow x_1 = x_2 \Rightarrow u_1 = u_2$.
Surjectivity: $\forall w \in W, \pi(w) = w \in \operatorname{range} \pi$. Thus π is iso of U onto W. □

---

3 *Sup V and W are iso, U is a subsp of V, and $S \in \mathcal{L}(U,W)$.*
Provt $\exists$ inv $T \in \mathcal{L}(V,W), Tu = Su, \forall u \in U \iff S$ is inje. [*See also (3.A.11).*]

Solus: (a) $\forall u \in U, u = T^{-1}Su \Rightarrow T^{-1}S = I \in \mathcal{L}(U) \Longrightarrow S$ is inje, by (3.B.20).
Or. $\operatorname{null} S = \operatorname{null} T|_U = \operatorname{null} T \cap U = \{0\}$.

(b) Let $X \oplus U = V$. Becs $S : U \to V$ is inje. By (3.B.12), $S : U \to \operatorname{range} S$ is iso.
Let $Y \oplus \operatorname{range} S = V$. Then by Tips, X and Y are iso. Let $E : X \to Y$ be an iso.
Define $T \in \mathcal{L}(V,W)$ by $Tu = Su, Tw = Ew$ for all $u \in U, w \in X$.

Or. [*Req V Finid*] Let $B_U = (u_1, \dots, u_m)$. Then S inje $\Rightarrow (Su_1, \dots, Su_m)$ linely inde.
Extend to $B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_W = (Su_1, \dots, Su_m, w_1, \dots, w_n)$.
Define $T \in \mathcal{L}(V,W)$ by $T(u_i) = Su_i;\ \ Tv_j = w_j$, for each u_i and v_j. ☐

---

8 *Sup $T \in \mathcal{L}(V,W)$ is **surj**. Provt $\exists$ subsp U of V, $T|_U : U \to W$ is iso.*

Solus: By (3.B.12). Note that $\operatorname{range} T = W$. Or. [*Req* range T *Finid*] By [3.B Tips (4)]. ☐

---

18 *Shat V and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.*

Solus:
Define $\Psi \in \mathcal{L}\big(V, \mathcal{L}(\mathbf{F}, V)\big)$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.
(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is inje.
(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi\big(T(1)\big)$. Hence Ψ is surj. ☐

Or. Define $\Phi \in \mathcal{L}\big(\mathcal{L}(\mathbf{F}, V), V\big)$ by $\Phi(T) = T(1)$.
(a) Sup $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is inje.
(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surj. ☐
Comment: $\Phi = \Psi^{-1}$.

• *Sup $S, T \in \mathcal{L}(V,W)$.* [*For Exe (4) and (5), see the* Coro *in (3.B.24, 25).*]
6 *Sup V and W are finide.* $\dim \operatorname{null} S = \dim \operatorname{null} T = n$.
Provt $S = E_2 T E_1, \exists$ inv $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$.

Solus: Define $E_1 :\ \ v_i \mapsto r_i\ ;\ \ \ u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.
Define $E_2 : Tv_i \mapsto Sr_i\ ;\ \ x_j \mapsto y_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m);\ B_{\operatorname{range} S} = (Sr_1, \dots, Sr_m)$. Let $B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p);\ B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p)$. Let $B_{\operatorname{null} T} = (u_1, \dots, u_n);\ B_{\operatorname{null} S} = (s_1, \dots, s_n)$. Thus $B_V = (v_1, \dots, v_m, u_1, \dots, u_n);\ B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)$.	$\therefore E_1, E_2$ are inv and $S = E_2 T E_1$.

☐

• (a) *Sup $T = ES$ and $E \in \mathcal{L}(W)$ is inv. Provt* $\operatorname{null} S = \operatorname{null} T$.
(b) *Sup $T = SE$ and $E \in \mathcal{L}(V)$ is inv. Provt* $\operatorname{range} S = \operatorname{range} T$.
(c) *Sup $T = E_2 S E_1$ and $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$ are inv.*
Provt $\dim \operatorname{null} S = \dim \operatorname{null} T$.

Solus: (a) $v \in \operatorname{null} T \iff Tv = 0 = E(Sv) \iff Sv = 0 \iff v \in \operatorname{null} S$.
(b) $w \in \operatorname{range} T \iff \exists\, v \in V, Tv = S(Ev) \iff \exists\, u \in V, w = Su \iff w \in \operatorname{range} S$.
(c) Using (3.B.22). $\dim \operatorname{null} E_2 S E_1 \xlongequal[\text{inv}]{E_2} \dim \operatorname{null} S E_1 \xlongequal[\text{inv}]{E_1} \dim \operatorname{null} S = \dim \operatorname{null} T$. ☐

• **NOTE FOR [3.69]:** Sup V, W are finide and iso, $T \in \mathcal{L}(V, W)$. Then T inv $\Longleftrightarrow$ inje $\Longleftrightarrow$ surj.

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9 [OR 1] *Sup U, V, W are iso and finide, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Provt ST is inv $\Longleftrightarrow$ S, T are inv.*

COMMENT: If any two of U, V, W are not iso or finide, then S, T are inv $\Longrightarrow$ ST is inv.

SOLUS: Sup S, T are inv. Then $(ST)(T^{-1}S^{-1}) = I_W, (T^{-1}S^{-1})(ST) = I_U$. Hence ST is inv.

Sup ST is inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = I_U, (ST)R = I_W$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0. \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S. \end{array}\right| \begin{array}{l} T \text{ is inje, } S \text{ is surj.} \\ \text{又 } \dim U = \dim V = \dim W. \end{array}$$

OR. By (3.B.23), $\dim W = \dim \operatorname{range} ST \leqslant \min\{\operatorname{range} S, \operatorname{range} T\} \Rightarrow S, T$ are surj. ☐

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13 *Sup U, V, W, X are iso and finide, $R \in \mathcal{L}(W, X), S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Sup RST is surj. Provt S is inje.*

SOLUS: Using Exe (9). Notice that U, X are finide, so that RST is inv.

$$\text{Let } X = (RST)^{-1} \left| \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ is surj.} \end{array} \right\} \Rightarrow S = R^{-1}(RST)T^{-1}.$$ ☐

OR. $(RST)^{-1} = \big((RS)T\big)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. ☐

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10 *Sup V is finide and $S, T \in \mathcal{L}(V)$. Provt $ST = I \Longleftrightarrow TS = I$.*

SOLUS: (a) Sup $ST = I$.

By (3.B 20, 21)(a), $ST = I \Rightarrow T$ is inje and S is surj. 又 V is finide. S, T are inv.

OR. By Exe (9), V is finide and $ST = I$ is inv $\Rightarrow S, T$ are inv.

Then $\forall v \in V, S\big((TS)v\big) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow TS = I$.

OR. $S^{-1} = T$ 又 $S = S \Rightarrow TS = S^{-1}S = I$.

(b) Reversing the roles of S and T, we conclude that $TS = I \Rightarrow ST = I$. ☐

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11 *Sup V is finide, $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Shat T is inv and $T^{-1} = US$.*

SOLUS: Using Exe (9) and (10). *This result can fail without the hypothesis that V is finide.*

$(ST)U = U(ST) = (US)T = I \Rightarrow T^{-1} = US$.

OR. $(ST)U = S(TU) = I \Rightarrow U, S$ are inv $\Rightarrow TU = S^{-1}$. 又 $U^{-1} = U^{-1} \Rightarrow T = S^{-1}U^{-1}$. ☐

EXA: $V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots)$; $T(a_1, \dots) = (0, a_1, \dots)$; $U = I \Rightarrow STU = I$ but T is not inv.

$$\bullet \text{ (4E 3) } \begin{array}{l} T \in \mathcal{L}(V) \\ V \text{ is finide} \end{array} \left| \begin{array}{l} (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1, \dots, v_n) \text{ of } V \Longleftrightarrow T \text{ is surj} \\ (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1, \dots, v_n) \text{ of } V \Longleftrightarrow T \text{ is inje} \end{array} \right\} \Longleftrightarrow T \text{ is inv.}$$

• (4E 15) *Sup $T \in \mathcal{L}(V)$ and $V = \operatorname{span}(Tv_1, \dots, Tv_m)$. Provt $V = \operatorname{span}(v_1, \dots, v_m)$.*

SOLUS: Becs $V = \operatorname{span}(Tv_1, \dots, Tv_m) \Rightarrow T$ is surj, and therefore is inv $\Rightarrow T^{-1}$ is inv.

$\forall v \in V, \exists a_i \in \mathbf{F}, v = \sum_{i=1}^m a_i Tv_i \Rightarrow T^{-1}v = \sum_{i=1}^m a_i v_i \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \dots, v_m)$.

OR. Reduce the spanning list $(Tv_1, \dots, Tv_m)$ of V to a basis $(Tv_{\alpha_1}, \dots, Tv_{\alpha_k})$ of V.

Where $k = \dim V$ and each $\alpha_i \in \{1, \dots, k\}$. Then by Exe (4E 3),

$(v_{\alpha_1}, \dots, v_{\alpha_k})$ is also a basis of V, contained in the list $(v_1, \dots, v_m)$. ☐

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15 *Provt every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multi.*
In other words, provt if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists\, A \in \mathbf{F}^{m,n}, Tx = Ax,\ \forall x \in \mathbf{F}^{n,1}$.

SOLUS: Let $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$ be std bases of $\mathbf{F}^{n,1}, \mathbf{F}^{m,1}$.
$\forall k = 1, \dots, n,\ T(E_k) = A_{1,k}R_1 + \cdots + A_{m,k}R_m, \exists\, A_{j,k} \in \mathbf{F}$, forming $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.
OR. Let $A = \mathcal{M}(T, B_1, B_2)$. Note that $\mathcal{M}(x, B_1) = x, \mathcal{M}(Tx, B_2) = Tx$.
Hence $Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax$, by [3.65]. $\square$

- **NOTE FOR [3.62]:** $\mathcal{M}(v) = \mathcal{M}\big(I, (v), B_V\big)$. Where I is the id optor restr to $\operatorname{span}(v)$.
- **NOTE FOR [3.65]:** $\mathcal{M}(Tv) = \mathcal{M}\big(I, (Tv), B_W\big) = \mathcal{M}(T, B_V, B_W)\mathcal{M}\big(I, (v), B_V\big) = \mathcal{M}\big(T, (v), B_W\big)$.
  If $v = 0$, then $\operatorname{span}(v) = \operatorname{span}(\ )$, we replace (v) by $B = (\ )$; simlr for $Tv = 0$.

- (4E 23, OR 10.A.4) *Sup that $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bases of V.*
  Let $T \in \mathcal{L}(V)$ be suth each $T\alpha_k = \beta_k$. Provt $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)$.
  For ease of nota, let $\mathcal{M}(T, \alpha \to \beta) = \mathcal{M}\big(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)\big)$.

SOLUS:

Denote $\mathcal{M}(T, \alpha \to \alpha)$ by A and $\mathcal{M}(I, \beta \to \alpha)$ by B.
$\forall k \in \{1, \dots, n\}, I\beta_k = \beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. $\square$

OR. Note that $\mathcal{M}(T, \alpha \to \beta) = I$. Hence $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)\underbrace{\mathcal{M}(T, \alpha \to \beta)}_{=\mathcal{M}(I, \beta \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. $\square$

OR. Note that $\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I$.
$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}\Big(\underbrace{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}_{=\mathcal{M}(T, \alpha \to \beta)}\Big) = \mathcal{M}(I, \beta \to \alpha)$. $\square$

COMMENT: Let $A' = \mathcal{M}(T, \beta \to \beta)$.
$\beta_k = I\beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n,\ \forall\, k \in \{1, \dots, n\}$.
又 $T\beta_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.
OR. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$.

- **TIPS:** When using $\mathcal{M}^{-1}$, you must first declare bases and the purpose for using $\mathcal{M}^{-1}$.
  That is, to declare $B_U, B_V, B_W,\ \mathcal{M} : \mathcal{L}(V, W) \mapsto \mathbf{F}^{m,n}$, or $\mathcal{M} : v \mapsto \mathbf{F}^{n,1}$.
  So that $\mathcal{M}^{-1}(AC, B_U, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(C, B_U, B_V)$;
  Or $\mathcal{M}^{-1}(Ax, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(x, B_V)$. Where everything is well-defined.

- (4E 22, OR 10.A.1) *Sup $T \in \mathcal{L}(V)$. Provt $\mathcal{M}(T, B_V)$ is inv $\iff$ T itself is inv.*

SOLUS: Notice that $\mathcal{M} : T \mapsto \mathcal{M}(T, B_V)$ is iso. And that $\mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(TS)$.

(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(I) = \mathcal{M}(T)\mathcal{M}(T^{-1}) \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I,\ \exists!\, S \in \mathcal{L}(V)$ suth $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$
$\Rightarrow \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = I = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$
$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. $\square$

- (4E 24, OR 10.A.2) *Sup $A, B \in \mathbf{F}^{n,n}$. Provt $AB = I \iff BA = I$.* [*Using Exe (10, 15).*]

SOLUS: Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Now $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.
$AB = I \Leftrightarrow A(Bx) = x \iff T(Sx) = x \Leftrightarrow TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I$.
OR. Becs $\mathcal{M} : \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1}) \to \mathbf{F}^{n,n}$ is iso. $\mathcal{M}^{-1}(AB) = TS = ST = \mathcal{M}^{-1}(BA) = I$. $\square$

• **NOTE FOR [3.60]:** Sup $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.

Define $E_{i,j} \in \mathcal{L}(V,W)$ by $E_{i,j}(v_x) = \delta_{i,x} w_j$. **CORO:** $E_{l,k}E_{i,j} = \delta_{j,l}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. And $\left(\mathcal{E}^{(j,i)}\right)_{l,k} = \begin{cases} 1, & \text{if } (i,j) = (l,k); \\ 0, & \text{othws.} \end{cases}$

NOTICE that $\mathcal{M}\colon \mathcal{L}(V,W) \to \mathbf{F}^{m,n}$ is iso. And $E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)} & + & \cdots & + & A_{1,n}\mathcal{E}^{(1,n)} \\ + & & \cdots & & + \\ \vdots & & \ddots & & \vdots \\ + & & \cdots & & + \\ A_{m,1}\mathcal{E}^{(m,1)} & + & \cdots & + & A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff \begin{pmatrix} A_{1,1}E_{1,1} & + & \cdots & + & A_{1,n}E_{n,1} \\ + & & \cdots & & + \\ \vdots & & \ddots & & \vdots \\ + & & \cdots & & + \\ A_{m,1}E_{1,m} & + & \cdots & + & A_{m,n}E_{n,m} \end{pmatrix} = T.$$

$$\text{By [2.42] and [3.61], } B_{\mathcal{L}(V,W)} = \begin{pmatrix} E_{1,1}, & \cdots & ,E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \cdots & ,E_{n,m} \end{pmatrix}; \quad B_{\mathbf{F}^{m,n}} = \begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & ,\mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots & ,\mathcal{E}^{(m,n)} \end{pmatrix}.$$

• **TIPS:** Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_p)$, $B_V = (v_1, \dots, v_p, \dots, v_n)$. Let each $w_k = Tv_k$; $B_W = (w_1, \dots, w_p, \dots, w_m)$.
Then $T = E_{1,1} + \cdots + E_{p,p}$, $\mathcal{M}(T, B_V, B_W) = \mathcal{E}^{(1,1)} + \cdots + \mathcal{E}^{(p,p)}$.

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17 *Sup V is finide. Shat the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}$, $\forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUS: [See also in (3.A).] Using NOTE FOR [3.60].

Let $B_V = (v_1, \dots, v_n)$. If $\mathcal{E} = 0$, then we are done. Sup $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.

Then $\forall E_{i,j} \in \mathcal{E}$, by asm, $\forall x, y \in \{1, \dots, n\}$, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}$, $E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.

Again, $\forall x, x', y, y' \in \{1, \dots, n\}$, $E_{y,x'}, E_{y',x} \in \mathcal{E}$. Thus $\mathcal{E} = \mathcal{L}(V)$. □

• (4E 10) *Sup V, W are finide, U is a subsp of V.*

Let $\mathcal{E} = \{T \in \mathcal{L}(V,W) : U \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V,W) : T|_U = 0\}$.

(a) *Shat $\mathcal{E}$ is a subsp of $\mathcal{L}(V,W)$.*

(b) *Find a formula for* $\dim \mathcal{E}$ *in terms of* $\dim V$, $\dim W$ *and* $\dim U$.

Hint: *Define $\Phi : \mathcal{L}(V,W) \to \mathcal{L}(U,W)$ by $\Phi(T) = T|_U$. What is $\operatorname{null}\Phi$? What is $\operatorname{range}\Phi$?*

SOLUS:

(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.

(b) Define Φ as in the hint. Φ is linear, by [3.A NOTE FOR Restriction].

$\Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$. Thus $\operatorname{null}\Phi = \mathcal{E}$.

Extend $S \in \mathcal{L}(U,W)$ to $T \in \mathcal{L}(V,W) \Rightarrow \Phi(T) = S \in \operatorname{range}\Phi$. Thus $\operatorname{range}\Phi = \mathcal{L}(U,W)$.

Thus $\dim \operatorname{null}\Phi = \dim \mathcal{E} = \dim \mathcal{L}(V,W) - \dim \operatorname{range}\Phi = (\dim V - \dim U)\dim W$. □

OR. Let $B_U = (u_1, \dots, u_m)$, $B_V = (u_1, \dots, u_m, v_1, \dots, v_n)$. Let $p = \dim W$. [See NOTE FOR [3.60].]

$$\forall T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \underbrace{\operatorname{span}\left\{\begin{matrix} E_{1,1}, & \cdots & ,E_{m,1}, \\ \vdots & \ddots & \vdots \\ E_{1,p}, & \cdots & ,E_{m,p} \end{matrix}\right\}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

$$又\ W = \operatorname{span}\left\{\begin{matrix} E_{m+1,1}, & \cdots & ,E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, & \cdots & ,E_{n,p} \end{matrix}\right\} \subseteq \mathcal{E}.$$

Where $\mathcal{L}(V,W) = R \oplus W \Rightarrow \mathcal{L}(V,W) = R + \mathcal{E}$.

Then $\dim \mathcal{E} = \dim \mathcal{L}(V,W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U)\dim W$. □

• (4E 17) *Sup V is finide and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(T) = ST$.*
(a) *Shat* $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Shat* $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

SOLUS: (a) $\forall T \in \mathcal{L}(V), ST = 0 \iff \operatorname{range} T \subseteq \operatorname{null} S$.
Thus $\operatorname{null} \mathcal{A} = \{T \in \mathcal{L}(V) : \operatorname{range} T \subseteq \operatorname{null} S\} = \mathcal{L}(V, \operatorname{null} S)$.
(b) $\forall R \in \mathcal{L}(V), \operatorname{range} R \subseteq \operatorname{range} S \iff \exists\, T \in \mathcal{L}(V), R = ST$, by (3.B 25).
Thus $\operatorname{range} \mathcal{A} = \{R \in \mathcal{L}(V) : \operatorname{range} R \subseteq \operatorname{range} S\} = \mathcal{L}(V, \operatorname{range} S)$. □

OR. Using NOTE FOR [3.60]. Let $B_{\operatorname{range} S} = (\overbrace{w_1, \dots, w_m}^{Sv_i = w_i}), B_U = (v_1, \dots, v_m)$.
Let $(w_1, \dots, w_n), (v_1, \dots, v_n)$ be bases of V. Now $S = E_{1,1} + \cdots + E_{m,m}$. $\mathcal{M}(S, v \to w) = \begin{pmatrix} 1 & \ddots & 0 & \ddots & 0 \\ 0 & & 1 & & 0 \\ 0 & \ddots & 0 & \ddots & 0 \end{pmatrix}$.
Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j} : w_x \mapsto \delta_{i,x} v_i$. Let $E_{j,k} R_{i,j} = Q_{i,k},\ R_{j,k} E_{i,j} = G_{i,k}$.
Where $E_{i,k} : v_x \mapsto \delta_{i,x} w_k,\ Q_{i,k} : w_x \mapsto \delta_{i,x} w_k,$ and $G_{i,k} : v_x \mapsto \delta_{i,x} v_k$.
For any $T \in \mathcal{L}(V), \exists !\, A_{i,j} \in \mathbf{F}, T = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i} \Longrightarrow \mathcal{M}(T, w \to v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} & \cdots & A_{n,m} \end{pmatrix}$.
$\Longrightarrow \mathcal{A}(T) = ST = \Big(\sum_{r=1}^{m} E_{r,r} \Big) \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i} \Big) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j} Q_{j,i}$.

$$\begin{aligned} \mathcal{M}(S, v \to w) \mathcal{M}(T, w \to v) = \mathcal{M}(ST, w) &= \\ \mathcal{M}(\mathcal{A}, R \to Q) \mathcal{M}(T, R) = \mathcal{M}\big(\mathcal{A}(T), Q\big) &= \end{aligned} \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

又 $\mathcal{M}(T, R) = \mathcal{M}(T, w \to v)$.
Let $T = I$, we have
$\mathcal{M}(\mathcal{A}, R \to Q) = \mathcal{M}(S, v \to w)$.

$$\operatorname{range} \mathcal{A} = \operatorname{span} \begin{Bmatrix} Q_{1,1}, & \cdots & , Q_{n,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,m}, & \cdots & , Q_{n,m} \end{Bmatrix}, \ \operatorname{null} \mathcal{A} = \operatorname{span} \begin{Bmatrix} R_{1,m+1}, & \cdots & , R_{n,m+1}, \\ \vdots & \ddots & \vdots \\ R_{1,n}, & \cdots & , R_{n,n} \end{Bmatrix}.$$

(a) $\dim \operatorname{null} \mathcal{A} = n \times (n - m)$;
(b) $\dim \operatorname{range} \mathcal{A} = n \times m$. □

• **NOTE FOR Exe (4E 17):** *Define $\mathcal{B} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{B}(T) = TS$.*
(a) *Shat* $\dim \operatorname{null} \mathcal{B} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Shat* $\dim \operatorname{range} \mathcal{B} = (\dim V)(\dim \operatorname{range} S)$.

SOLUS: (a) $\forall T \in \mathcal{L}(V), TS = 0 \iff \operatorname{range} S \subseteq \operatorname{null} T$.
Thus $\operatorname{null} \mathcal{B} = \{T \in \mathcal{L}(V) : \operatorname{range} S \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V) : T|_{\operatorname{range} S} = 0\}$.
(b) $\forall R \in \mathcal{L}(V), \operatorname{null} S \subseteq \operatorname{null} R \iff \exists\, T \in \mathcal{L}(V), R = TS$, by (3.B.24).
Thus $\operatorname{range} \mathcal{B} = \{R \in \mathcal{L}(V) : \operatorname{null} S \subseteq \operatorname{null} R\} = \{R \in \mathcal{L}(V) : R|_{\operatorname{null} S} = 0\}$.
Using [3.22] and Exe (4E 10). □

OR. Using NOTE FOR [3.60] and nota in Exe (4E 17).
$\mathcal{B}(T) = TS = \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i} \Big) \Big(\sum_{r=1}^{m} E_{r,r} \Big) = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{i,j} G_{j,i} \Longrightarrow \mathcal{M}(TS, v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} & \cdots & 0 \end{pmatrix}$.

$$\operatorname{range} \mathcal{B} = \operatorname{span} \begin{Bmatrix} G_{1,1}, & \cdots & , G_{m,1}, \\ \vdots & \ddots & \vdots \\ G_{1,n}, & \cdots & , G_{m,n} \end{Bmatrix}, \ \operatorname{null} \mathcal{B} = \operatorname{span} \begin{Bmatrix} R_{m+1,1}, & \cdots & , R_{n,1}, \\ \vdots & \ddots & \vdots \\ R_{m+1,n}, & \cdots & , R_{n,n} \end{Bmatrix}.$$

(a) $\dim \operatorname{null} \mathcal{B} = n \times (n - m)$;
(b) $\dim \operatorname{range} \mathcal{B} = n \times m$. □

• (4E 20) *Sup $q \in \mathcal{P}(\mathbf{R})$. Provt $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.*

SOLUS: Note that $\deg\big[(x^2 + x)p''(x) + 2xp'(x) + p(3)\big] = \deg p$.
Define $T_n \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.
And note that $T_n(p) = 0 \Rightarrow \deg T_n(p) = -\infty = \deg p \Rightarrow p = 0$. Thus T_n is inv.
$\forall q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $n = 0$; if $q \neq 0$, let $n = \deg q$, we have $q \in \mathcal{P}_n(\mathbf{R})$.
Now $\exists\, p \in \mathcal{P}_n(\mathbf{R}), q(x) = T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$. □

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19 *Sup* $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ *is inje. And* $\deg Tp \leqslant \deg p$ *for every nonzero* $p \in \mathcal{P}(\mathbf{R})$.
(a) *Provt* T *is surj;* (b) *Provt for every nonzero* p, $\deg Tp = \deg p$.

Solus: (a) T is inje $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ is inje, so is inv $\iff$ T is surj.

(b) Using mathematical induction.

(i) $\deg p = -\infty \geqslant \deg Tp \iff p = 0 = Tp$. And $\deg p = 0 \geqslant \deg Tp \iff p = C \neq 0$.

(ii) Asm $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$. We show $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$ by ctradic.
Sup $\exists\, r \in \mathcal{P}_{n+1}(\mathbf{R}),\ \deg Tr \leqslant n < n+1 = \deg r$. Then by (a), $\exists\, s \in \mathcal{P}_n(\mathbf{R}),\ T(s) = (Tr)$.
又 T is inje $\Rightarrow s = r$. While $\deg s = \deg Ts = \deg Tr < \deg r$. Ctradic. □

---

16 *Sup* V *is finide and* $S \in \mathcal{L}(V)$ *suth* $\forall T \in \mathcal{L}(V), ST = TS$. *Provt* $\exists\, \lambda \in \mathbf{F}, S = \lambda I$.

Solus: If $S = 0$, we are done. Now sup $S \neq 0$. [Using nota in Exe (4E 17). See also in (3.A).]

Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, B_U) = \mathcal{M}(I, B_{\operatorname{range} S}, B_U)$. Note that $R_{k,1} : w_x \mapsto \delta_{k,x} v_1$.

Then $\forall k \in \{1, \ldots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $\dim \operatorname{null} S = 0,\ \dim \operatorname{range} S = m = n$.

Notice that $G_{i,j} = R_{i,j}S = SR_{i,j} = Q_{i,j}$. Where $G_{i,j} : v_x \mapsto \delta_{i,x} v_j$; $Q_{i,j} : w_x \mapsto \delta_{i,x} w_j$.

For each $w_i, \exists\,!\, a_{k,i} \in \mathbf{F}, w_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n$. Where $a_{k,i} = \mathcal{M}(I, (w_1, \ldots, w_n), (v_1, \ldots, v_n))_{k,i}$.

Then fix one i. Now for each $j \in \{1, \ldots, n\}, Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}\left(\sum_{k=1}^{n} a_{k,i}v_k\right)$.

Let $\lambda = a_{i,i}$. Hence each $w_j = \lambda v_j$. Now fix one j, we have $a_{1,1}v_j = \cdots = a_{n,n}v_j$, then all $a_{i,i}$ are equal.

Thus each $w_j = \lambda v_j \Longrightarrow \mathcal{M}(S, B_U) = \mathcal{M}(\lambda I)$. □

• (10.A.3, Or 4E 19) *Sup* V *is finide and* $T \in \mathcal{L}(V)$. [*See also in* (3.A).]
Provt $\forall B_V \neq B_V^{'}$, $\mathcal{M}(T, B_V) = \mathcal{M}(T, B_V^{'}) \Longrightarrow T = \lambda I, \exists\, \lambda \in \mathbf{F}$.

Solus: Sup $\forall B_V \neq B'_V$, $\mathcal{M}(T, B_V) = \mathcal{M}(T, B'_V)$. If $T = 0$, then we are done.

Sup $T \neq 0$, and $v \in V \backslash \{0\}$. Asm (v, Tv) is linely inde.

Extend (v, Tv) to $B_V = (v, Tv, u_3, \ldots, u_n)$. Let $B = \mathcal{M}(T, B_V)$.

$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0,\ \forall i \neq 2$.

By asm, $A = \mathcal{M}(T, B'_V) = B, \forall B'_V = (v, w_2, \ldots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$.

$\Rightarrow Tv = w_2$, which is not true if $w_2 = u_3,\ w_3 = Tv,\ w_j = u_j, \forall j \in \{4, \ldots, n\}$. Ctradic.

Hence (v, Tv) is linely depe $\Rightarrow \forall v \in V, \exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we shat λ_v is independent of v, that is, for all disti $v, w \in V \backslash \{0\}, \lambda_v = \lambda_w$.

$$\left.\begin{array}{l}(v,w) \text{ linely inde} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v,w) \text{ linely depe, } w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv)\end{array}\right\} \Rightarrow T = \lambda I.$$ □

Or. Let $A = \mathcal{M}(T, B_V)$, where $B_V = (u_1, \ldots, u_m)$ is arb.

Fix one $B_V = (v_1, \ldots, v_m)$ and then $(v_1, \ldots, \frac{1}{2}v_k, \ldots, v_m)$ is also a basis for any given $k \in \{1, \ldots, m\}$.

Fix one k. Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \cdots + A_{k,k}(\frac{1}{2}v_k) + \cdots + A_{m,k}v_m$

$\Rightarrow Tv_k = 2A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + 2A_{m,k}v_m = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{m,k}v_m$.

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1, \ldots, m\}$.

Now we shat $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j, k suth $j \neq k$.

Consider $B'_V = (v'_1, \ldots, v'_j, \ldots, v'_k, \ldots, v'_m)$, where $v'_j = v_k,\ v'_k = v_j$ and $v'_i = v_i$ for all $i \in \{1, \ldots, m\} \backslash \{j, k\}$.

Now $T(v'_k) = A_{1,k}v'_1 + \cdots + A_{k,k}v'_k + \cdots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_j$, while $T(v'_k) = T(v_j) = A_{j,j}v_j$. □

Ended

3·E

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 20 | 4E: 8 14

1 *A function $T : V \to W$ is linear $\iff$ The graph of T is a subspace of $V \times W$.*

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2 *Sup $V_1 \times \cdots \times V_m$ is finide. Provt each V_j is finide.*

SOLUS:

For any $k \in \{1, \dots, m\}$, define $S_k \in \mathcal{L}(V_1 \times \cdots \times V_m, V_k)$ by $S_k(v_1, \dots, v_m) = v_k$.
Then S_k is linear map. By [3.22], range $S_k = V_k$ is finide. $\square$

OR. Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.
We shat each U_i is iso to V_i. Then U is finide $\Longrightarrow$ its subsp U_i is finide, so is V_i.

$$\text{Let } B_U = (v_1, \dots, v_M) \left| \begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U_i) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} R_i S_j|_{U_j} = \delta_{i,j} I_{U_j}, \\ S_i R_j = \delta_{i,j} I_{V_j}. \end{array} \right. \square$$

---

4 *Provt $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.*

SOLUS: Using nota in Exe (2): $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0)$; $S_i : (u_1, \dots, u_m) \mapsto u_i$.

Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$.

$$\left. \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (TR_1, \dots, TR_m). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array} \right\} \Rightarrow \psi = \varphi^{-1}. \quad \square$$

---

5 *Provt $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.*

SOLUS: Using nota in Exe (2): $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0)$; $S_i : (u_1, \dots, u_m) \mapsto u_i$.

$$\begin{array}{r} \text{Note that } T_i : v \mapsto w_i, \\ T : v \mapsto (w_1, \dots, w_m). \end{array} \left| \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1 T, \dots, S_m T). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = R_1 T_1 + \cdots + R_m T_m. \end{array} \right\} \Rightarrow \psi = \varphi^{-1}. \quad \square$$

---

6 *For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Provt V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.*

SOLUS:

Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto v$ is defined by $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m$.

(a) Sup $T(v_1, \dots, v_m) = 0$. Then $\forall (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$
For each k, let $a_k = 1, a_j = 0$ for all $j \neq k$. Then each $v_k = 0 \Rightarrow (v_1, \dots, v_m) = 0$. Thus T is inje.

(b) Sup $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be std basis of $\mathbf{F}^m$. Then $\forall (b_1, \dots, b_n) \in \mathbf{F}^m$,
$$\Big[T\big(\psi(e_1), \dots, \psi(e_m)\big)\Big](b_1, \dots, b_m) = b_1 \psi(e_1) + \cdots + b_m \psi(e_m) = \psi(b_1 e_1 + \cdots + b_m e_m) = \psi(b_1, \dots, b_m).$$
Thus $T\big(\psi(e_1), \dots, \psi(e_m)\big) = \psi$. Hence T is surj. $\square$

---

3 *Give an exa of a vecsp V and its two subsps U_1, U_2 suth*
$U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum. [*V must be infinide.*]

SOLUS: NOTE *that at least one of U_1, U_2 must be infinide. And at least one must be finide??*

Let $V = \mathbf{F}^\infty = U_1$, $U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$. Then $V = U_1 + U_2$ is not a direct sum.

$$\left. \begin{array}{l} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T\big((x_1, x_2, \cdots), (x, 0, \cdots)\big) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = \big((x_1, x_2, \cdots), (x, 0, \cdots)\big) \end{array} \right\} \Rightarrow S = T^{-1}. \quad \square$$

• **NOTE FOR [3.79, 3.83]:** If $U = \{0\}$, then $v + U = v + \{0\} = \{v\}$, $V/U = V/\{0\} = \{\{v\} : v \in V\}$.
If $U = V$, then $v + V = 0 + V$, $V/V = \{v + V : v \in V\} = \{0\}$.
If $U = \emptyset$, then $v + U = v + \emptyset = \emptyset$, $V/U = V/\emptyset = \{\emptyset\}$.

• **COMMENT:** If U is merely a subset of V, then [3.85, 3.86] do not hold, and V/U is not a vecsp.
Becs $\big((v-w)+u\big) \in U$ or $u - u' \in U$ needs that U is closd add.
And becs $(v-\hat{v}) + (w - \hat{w}) \in U$ and $\lambda(v - \hat{v}) \in U$ asm U is a subsp.

If U is a vecsp but not a subsp of V, then everything will be all right.
If U is a vecsp and $U \cap V = \{0\}$, then $v + U = w + U \Rightarrow v = w$.

• **NOTE FOR [3.85]:** $v + U = w + U \iff v \in w + U,\ w \in v + U$
$\iff v - w \in U \iff (v+U) \cap (w+U) \neq \emptyset$.

• (4E 8) *Sup* $T \in \mathcal{L}(V, W), w \in \operatorname{range} T$. *Provt* $\{v \in V : Tv = w\} = u + \operatorname{null} T$.

SOLUS: Let $\mathcal{K}_u = \{v \in V : Tv = w\}$. [Not a vecsp.] Sup $u \in \mathcal{K}_u$. Then $u + \operatorname{null} T \subseteq \mathcal{K}_u$.
And $\forall u' \in \mathcal{K}_u,\ u' - u \in \operatorname{null} T \Rightarrow u' \in u + \operatorname{null} T$. Now $\mathcal{K}_u \subseteq u + \operatorname{null} T$. $\square$

---

7 *Sup* $v, x \in V$, *and* U, W *are subsps of* V. *Provt* $v + U = x + W \Rightarrow U = W$.

SOLUS: (a) $v \in v + U = x + W \Rightarrow \exists\, w_v \in W, v = x + w_v \Rightarrow v - x \in W$.
(b) $x \in x + W = v + U \Rightarrow \exists\, u_x \in U, x = v + u_x \Rightarrow x - v \in U$.
Now $x + U = v + U = x + W = v + W$. Thus $\{v + u : u \in U\} = \{v + w : w \in W\} \Rightarrow U = W$.

OR. $\pm(v - x) \in U \cap W \Rightarrow \left\{ \begin{array}{l} u_1 = (x-v) + w_1 \in W \Rightarrow U \subseteq W \\ w_2 = (v - x) + u_2 \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W$. $\square$

---

8 *Sup* A *is a nonempty subset of* V.
Provt A *is a tslate of some subsp of* $V \iff \lambda v + (1-\lambda) w \in A\ , \forall v, w \in A, \lambda \in \mathbf{F}$.

SOLUS:
(a) Sup $A = a + U$. Then $\lambda(a + u_1) + (1-\lambda)(a + u_2) = a + \big(\lambda(u_1 - u_2) + u_2\big) \in A$.
(b) Sup $\lambda v + (1-\lambda) w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Sup $\underline{a \in A}$ and let $A' = \{x - a : x \in A\}$.
Then $0 \in A'$ and $\forall\, (v-a), (w-a) \in A', \lambda \in \mathbf{F}$,
(I) $\lambda(v - a) = [\lambda v + (1-\lambda)a] - a \in A'$.
(II) Becs $\lambda(v-a) + (1-\lambda)(w-a) = [\lambda v + (1-\lambda)w] - a \in A'$.
Let $\lambda = \frac{1}{2}$ here and use (I) above by $\lambda = 2$, we have $(v-a) + (w-a) \in A'$.
OR. Note that $v, a \in A \Rightarrow \lambda v + (1-\lambda)a = 2v - a \in A$. Simlr $2w - a \in A$.
Now $(v - \frac{1}{2}a) + (w - \frac{1}{2}a) = v + w - a \in A \Rightarrow v + w - 2a = (v-a) + (w-a) \in A'$.
Thus $A' = -a + A$ is a subsp of V. Hence $a + A' = a + \{x - a : x \in A\} = A$ is a tslate. $\square$

---

9 *Sup $A = v + U$ and $B = x + W$ for some $v, x \in V$ and some subsps U, W of V. Provt $A \cap B$ is either a tslate of some subsp of V or is $\emptyset$.*

SOLUS: $\forall v + u, x + w \in A \cap B \neq \emptyset, \lambda \in \mathbf{F}, \lambda(v+u) + (1-\lambda)(x+w) \in A \cap B$. By Exe (8). $\square$

OR. Let $A = v + U,\ B = x + W$. Sup $\alpha \in (v+U) \cap (x+W) \neq \emptyset$.

Then $\alpha - v \in U \Rightarrow \alpha + U = v + U = A$, and $\alpha - x \in W \Rightarrow \alpha + W = x + W = B$.

We shat $A \cap B = \alpha + (U \cap W)$. Note that $\alpha + (U \cap W) \subseteq A \cap B$.

And $\forall \beta = \alpha + u = \alpha + w \in A \cap B \Rightarrow u = w \in U \cap W \Rightarrow \beta \in \alpha + (U \cap W)$. $\square$

---

10 *Provt the intersec of any collec of tslates of subsps is either a tslate of some subsps or $\emptyset$.*

SOLUS: Sup $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collec of tslates of subsps of V, where Γ is an index set.

$\forall x, y \in \bigcap_{\alpha\in\Gamma} A_\alpha \neq \emptyset, \lambda \in \mathbf{F}, \lambda x + (1-\lambda) y \in A_\alpha$ for each α. By Exe (8). $\square$

OR. Let each $A_\alpha = w_\alpha + V_\alpha$. Sup $x \in \bigcap_{\alpha\in\Gamma}(w_\alpha + V_\alpha) \neq \emptyset$.

Then $x - w_\alpha \in V_\alpha \Longrightarrow x + V_\alpha = w_\alpha + V_\alpha = A_\alpha$, for each α.

We shat $\bigcap_{\alpha\in\Gamma} A_\alpha = \bigcap_{\alpha\in\Gamma}(x + V_\alpha) = x + \bigcap_{\alpha\in\Gamma} V_\alpha$.

$y \in \bigcap_{\alpha\in\Gamma} A_\alpha \Longleftrightarrow$ for each $\alpha,\ y = x + v_\alpha \in A_\alpha$

$\Longleftrightarrow$ each $v_\alpha = y - x \in \bigcap_{\alpha\in\Gamma} V_\alpha \Longleftrightarrow y \in x + \bigcap_{\alpha\in\Gamma} V_\alpha$. $\square$

---

11 *Sup $A = \left\{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\right\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.*

(a) *Provt A is a tslate of some subsp of V*

(b) *Provt if B is a tslate of some subsp of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.*

(c) *Provt A is a tslate of some subsp of V of dim less than m.*

SOLUS: (a) By Exe (8), $\forall u, w \in A, \lambda \in \mathbf{F}, \lambda u + (1-\lambda) w = \left(\lambda \sum_{i=1}^m a_i + (1-\lambda)\sum_{i=1}^m b_i\right) v_i \in A$.

(b) Sup $B = v + U$, where $v \in V$ and U is a subsp of V. <u>Let each $v_k = v + u_k \in B, \exists!\, u_k \in U$.</u>

$\forall w \in A,\ w = \sum_{i=1}^m \lambda_i v_i = \sum_{i=1}^m \lambda_i (v + u_i) = \underline{\sum_{i=1}^m \lambda_i v} + \sum_{i=1}^m \lambda_i u_i = v + \sum_{i=1}^m \lambda_i u_i \in v + U = B$. $\square$

OR. Let $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$. To shat $v \in B$, use induction on m by k.

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By Exe (8), $v \in B$.

(ii) $2 \leqslant k < m$. Asm $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. $\left[\forall \lambda_i \text{ suth } \sum_{i=1}^k \lambda_i = 1\right]$

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. Fix one $\mu_\iota \neq 1$.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Longrightarrow \left(\sum_{i=1}^{k+1} \dfrac{\mu_i}{1-\mu_\iota}\right) - \dfrac{\mu_\iota}{1-\mu_\iota} = 1.$

$$\text{Let } w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let $\lambda_i = \dfrac{\mu_i}{1-\mu_\iota}$ for $i \in \{1, \dots, \iota-1\}$; $\lambda_j = \dfrac{\mu_{j+1}}{1-\mu_\iota}$ for $j \in \{\iota, \dots, k\}$. Then,

$$\left.\begin{array}{r} \sum_{i=1}^k \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \quad \square$$

(c) If $m = 1$, then let $A = v_1 + \{0\}$ and we are done. Now sup $m \geqslant 2$. Fix one $k \in \{1, \dots, m\}$.

$A \ni \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m) v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_m v_m$

$= v_k + \lambda_1 (v_1 - v_k) + \cdots + \lambda_{k-1}(v_{k-1} - v_k) + \lambda_{k+1}(v_{k+1} - v_k) + \cdots + \lambda_m (v_m - v_k)$

$\in v_k + \operatorname{span}(v_1 - v_k, \dots, v_m - v_k)$. $\square$

• **NOTE FOR [3.88, 3.90, 3.91]:** Sup $W \in \mathcal{S}_V U$. Then V/U is iso to W.

Becs $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$. Define $T \in \mathcal{L}(V)$ by $T(v) = w_v$.

Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$, $\operatorname{range} T \oplus \operatorname{null} T = V$.

Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, V)$ is defined by $\tilde{T}(v+U) = \tilde{T}(w'_v + U) = Tw'_v = w_v$. [See TIPS (1) below]

Now $\pi \circ \tilde{T} = I_{V/U}$, $\tilde{T} \circ \pi|_W = I_W = T|_W$. Hence $\tilde{T}$ is iso of V/U onto W.

• **TIPS 1:** Sup U is a subsp of V. Define $S \in \mathcal{L}(V/U, V)$ by $S(v+U) = v$.

Then $\operatorname{range} S$ is the *purest* in $\mathcal{S}_V U$. Now $\operatorname{null} S = \{0\}$, $U \oplus \operatorname{range} S = V$.

Let $E = S \circ \pi$. Becs S is inje and π is surj, $\operatorname{null} E = \operatorname{null} \pi = U$, $\operatorname{range} E = \operatorname{range} S$.

Then $\operatorname{range} E \oplus \operatorname{null} E = V$. NOTICE that $E : V \to W$ is the *purest eraser*. Now we explain why:

EXA: Let $V = \mathbf{F}^2, B_U = (e_1), B_W = (e_2 - e_1) \Rightarrow U \oplus W = V$.

Notice that $T(e_2 - e_1) = (e_2 - e_1)$, while $(e_2 - e_1) + U = e_2 + U$, but

becs $e_2 = e_1 + (e_2 - e_1)$, now still, $\tilde{T}\big((e_2 - e_1) + U\big) = e_2 - e_1 = Te_2$.

In contrast, $S\big((e_2 - e_1) + U\big) = S(e_2 + U) = e_2$, $E(e_2 - e_1) = e_2$.

And $\operatorname{range} E = \operatorname{range} S = \operatorname{span}(e_2)$ is the *purest* in $\mathcal{S}_V U$.

---

12 *Sup U is a subsp of V. Provt is V is iso to $U \times (V/U)$.*

SOLUS:

[*Req V/U Finid*] Let $B_{V/U} = (v_1 + U, \dots, v_n + U)$.

Note that $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^n a_i(v_i + U) = \left(\sum_{i=1}^n a_i v_i\right) + U$

$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) \in U \Rightarrow \exists!\, u \in U, v = \sum_{i=1}^n a_i v_i + u$.

Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ and $\psi \in \mathcal{L}(U \times (V/U), V)$

by $\varphi(v) = (u,\ v + U)$ and $\psi(u,\ v+U) = v + u$. Then $\psi = \varphi^{-1}$. □

OR. Define $S \in \mathcal{L}(V/U, V)$ by $S(v+U) = v$.

By NOTE FOR [3.88, 90, 91], $\operatorname{range} S \oplus U = V$. Thus $\forall v \in V, \exists!\, u \in U, w \in \operatorname{range} S, v = u + w$.

Define $T \in \mathcal{L}(U \times (V/U), V)$ by $T(u, v+U) = u + S(v+U) = u + w = v$. Then T is surj.

And $T(u, v+U) = u + S(v+U) = 0 \Longrightarrow \pi\big(T(u, v+U)\big) = v + U = 0$, and $u = -S(v+U) = 0$.

OR. Define $R \in \mathcal{L}(V, U \times (V/U))$ by $R(v) = \big(u, (w+U)\big)$. Now $R \circ T = I_{U \times (V/U)}$, $T \circ R = I_V$. □

• (4E 14) *Sup $V = U \oplus W$, $B_W = (w_1, \dots, w_m)$. Provt $B_{V/U} = (w_1 + U, \dots, w_m + U)$.*

SOLUS: $\forall v \in V, \exists!\, u \in U, w \in W, v = u + w$. 又 $\exists!\, c_i \in \mathbf{F}, w = \sum_{i=1}^m c_i w_i \Rightarrow v = \sum_{i=1}^m c_i w_i + u$.

Hence $\forall v + U \in V/U, \exists!\, c_i \in \mathbf{F}, v + U = \sum_{i=1}^m c_i w_i + U$. □

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13 *Provt $B_{V/U} = (v_1 + U, \dots, v_m + U), B_U = (u_1, \dots, u_n) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$.*

SOLUS: $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^m a_i v_i + U \Rightarrow \exists!\, b_i \in \mathbf{F}, v - \sum_{i=1}^m a_i v_i = \sum_{i=1}^n b_i u_i \in U$

$\Rightarrow \forall v \in V, \exists!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^m a_i v_i + \sum_{j=1}^n b_j u_j$. □

OR. $\sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i = 0 \Rightarrow \left(\sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i\right) + U = 0 \Rightarrow \sum_{i=1}^m a_i(v_i + U) = 0$

$\Rightarrow a_1 = \cdots = a_m = 0 \Rightarrow \sum_{i=1}^n b_i u_i \Rightarrow b_1 = \cdots = b_n = 0$. 又 $\dim V = m + n$. □

OR. Note that $B = (v_1, \dots, v_m)$ is linely inde, and $[\operatorname{span}(v_1, \dots, v_m) + U] \subseteq V$.

$v \in \operatorname{span} B \cap U \Longleftrightarrow v + U = \sum_{i=1}^m a_i(v_i + U) = 0 + U \Longleftrightarrow v = 0$. Hence $\operatorname{span} B \cap U = \{0\}$.

Becs $\dim[\operatorname{span}(v_1, \dots, v_m) \oplus U] = m + n = \dim V$. Now by (2.B.8). □

• **NOTE FOR Exe (13) and (4E 14):** Let $U \oplus W = V$. Define $S(w+U) = w$. $\big[$ See also TIPS (1). $\big]$

(a) Let $B_W = (w_1, \dots, w_m) \Rightarrow B_{V/U} = \big(w_1 + U, \dots, w_m + U\big)$. Then $S(w_k + U)$ might not equal w_k.

(b) Let $B_{V/U} = \big(w_1 + U, \dots, w_m + U\big)$, then let $B_W = (w_1, \dots, w_m)$. Now each $S(w_k + U) = w_k$.

• **NEW NOTA:** Pure $V/U = W \iff V = U \oplus W,\ W = \operatorname{range} S$.

• **NEW THEO:** The uniqnes of Pure V/U follows from range S.

• **TIPS 2:** *Sup U, W are subsps of V. Let $I = U \cap W$. Provt $V = U + W \iff V/I = U/I \oplus W/I$.*

SOLUS: (a) Sup $U + W$. Then $\forall x \in V/I, \exists\, v \in V, (u_v, w_v) \in U \times W, x = v + I = (u_v + w_v) + I$.

Note that $U/I, W/I \subseteq V/I$. Thus $V/I = U/I + W/I$.

$\forall x \in (U/I) \cap (W/I), \exists\, u + I \in U/I, w + I \in W/I, x = u + I = w + I \Rightarrow u - w \in I = U \cap W$

$\Rightarrow \exists\, w' \in I, u = w + w' \in U \cap W \Rightarrow x = u + I = 0 + I$. Thus $(U/I) \cap (W/I) = \{0\}$.

(b) Sup $V/I = U/I \oplus W/I$. Then $\forall v \in V, v + I = (u + I) + (w + I)$

$\Rightarrow v - u - w \in I = U \cap W \Rightarrow \exists\, x \in U \cap W, v = u + w + x \in U + W$. $\square$

• **TIPS 3:** *Sup I is a subsp of U. Sup U is a subsp of V.*

Let $V = S_V I \oplus I = S_V U \oplus U$. Let $U = S_U I \oplus I$. Then $V = S_V U \oplus S_U I \oplus I$.

Sup $S_V I =$ Pure V/I, simlr for $S_V U, S_U I$. Provt $S_V I = S_V U \oplus S_U I$.

SOLUS: $\forall v_i \in S_V I, v_i = v_u + u, \exists!\, v_u \in S_V U, u \in U \Rightarrow \exists!\, u_i \in S_U I, i \in I, v_i = v_u + u_i + i$.

又 $v_i \in$ Pure V/I. Hence $i = 0$, and $v_i \in S_V U \oplus S_U I$. Now becs $S_V U, U \subseteq S_V I$. $\square$

---

15 *Sup $\varphi \in \mathcal{L}(V, \mathbf{F}) \backslash \{0\}$. Provt* $\dim V/(\operatorname{null}\varphi) = 1$.

SOLUS: By [3.91] (d), $\dim \operatorname{range}\varphi = 1 = \dim V/(\operatorname{null}\varphi)$.

OR. By (3.B.29), $\exists\, u$, $\operatorname{span}(u) \oplus \operatorname{null}\varphi = V$. Then $B_{V/\operatorname{null}\varphi} = (u + \operatorname{null}\varphi)$. $\square$

---

16 *Sup* $\dim V/U = 1$. *Provt* $\exists\, \varphi \in \mathcal{L}(V, \mathbf{F}), \operatorname{null}\varphi = U$.

SOLUS: Sup $V_0 \oplus U = V$. Then V_0 is iso to V/U. $\dim V_0 = 1$.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. $\square$

OR. Let $B_{V/U} = (w + U)$. Then $\forall v \in V, \exists!\, a \in \mathbf{F}, v + U = aw + U$.

Define $\varphi : V \to \mathbf{F}$ by $\varphi(v) = a$. Then $\varphi(v_1 + \lambda v_2) = a_1 + \lambda a_2 = \varphi(v_1) + \lambda\varphi(v_2)$.

Now $u \in U \iff u + U = 0w + U \iff \varphi(u) = 0$. $\square$

---

17 *Sup V/U is finide, W is a subsp of V.*

(a) *Shat if $V = U + W$, then* $\dim W \geqslant \dim V/U$.

(b) *Shat* $\exists\, W \in \mathcal{S}_V U, \dim W = \dim V/U$.

SOLUS: Let $B_W = (w_1, \dots, w_n)$.

(a) $\forall v \in V, \exists\, u \in U, w \in W,\ v = u + w \Longrightarrow v + U = w + U = (a_1 w_1 + \cdots + a_n w_n) + U, \exists!\, a_i \in \mathbf{F}$.

Then $V/U \subseteq \operatorname{span}\big(w_1 + U, \dots, w_n + U\big)$. Hence $\dim V/U \leqslant \dim \operatorname{span}\big(w_1 + U, \dots, w_n + U\big)$.

(b) Reduce $\big(w_1 + U, \dots, w_n + U\big)$ to $B_{V/U} = \big(w_1 + U, \dots, w_m + U\big)$, and let $W = \operatorname{span}(w_1, \dots, w_m)$. $\square$

OR. Let $B_{V/U} = \big(v_1 + U, \dots, v_m + U\big)$ and define $\tilde{T} \in \mathcal{L}(V/U, V)$ by $\tilde{T}(v_k + U) = v_k$.

Note that $\pi \circ \tilde{T} = I$. By (3.B.20), $\tilde{T}$ is inje. And $(v_1, \dots, v_m)$ is linely inde.

Let $W = \operatorname{range}\tilde{T} = \operatorname{span}(v_1, \dots, v_m)$. Then $\tilde{T} \in \mathcal{L}(V/U, W)$ is iso. Thus $\dim W = \dim V/U$.

And $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = a_1 v_1 + \cdots + a_m v_m + U \Rightarrow \exists!\, w \in W, u \in U, v = w + u$. $\square$

---

18 *Sup $T \in \mathcal{L}(V,W)$ and U is a subsp of V. Let $\pi : V \to V/U$ be the quot map.*
Provt $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \subseteq \operatorname{null} T$.

SOLUS:

(a) Sup $\exists\, S \in \mathcal{L}(V/U,W), T = S \circ \pi$. Then $U = \operatorname{null}\pi \subseteq \operatorname{null}(S \circ \pi) = \operatorname{null} T$.

(b) Sup $U = \operatorname{null}\pi \subseteq \operatorname{null} T$. By (3.B.24), we are done. OR. Define $S : (v + U) \mapsto Tv$.
$v_1 + U = v_2 + U \iff v_1 - v_2 \in \operatorname{null} T \iff Tv_1 = Tv_2$. Thus S is well-defined. Hence $S \circ \pi = T$. □

CORO: Define $\Gamma : S \mapsto S \circ \pi$. Then Γ is inje, $\operatorname{range}\Gamma = \{T \in \mathcal{L}(V,W) : U \subseteq \operatorname{null} T\}$.

---

14 *Sup $U = \{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0$ for only finily many $k\}$.*
(a) *Shat U is a subsp of $\mathbf{F}^\infty$. [Do it in your mind]* (b) *Provt $\mathbf{F}^\infty/U$ is infinide.*

SOLUS: For ease of nota, denote the p^{th} term of $u = (x_1, \cdots, x_p, \cdots) \in \mathbf{F}^\infty$ by $u[p]$.

For each $r \in \mathbf{N}^+$, let $e_r[k] = \begin{cases} 1, & (k-1) \equiv 0 \pmod r \\ 0, & \text{othws} \end{cases}$ | simply $e_r = (1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \cdots)$.

For $m \in \mathbf{N}^+$. Let $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 + U \Rightarrow \exists\, u \in U, a_1 e_1 + \cdots + a_m e_m = u$.

Sup $u = (x_1, \cdots, x_L, 0, \cdots)$, where L is the largest suth $u[L] \neq 0$.

Let $s \in \mathbf{N}^+$ be suth $h = s \cdot m! + 1 > L$, and $e_1[h] = \cdots = e_m[h] = 1$.

NOTICE that for any $p, r \in \{1, \dots, m\}$, $e_r[s \cdot m! + 1 + p] = e_r[p+1] = 1 \iff p \equiv 0 \pmod r \iff r \mid p$.

Let $1 = p_1 \leqslant \cdots \leqslant p_{\tau(p)} = p$ be the disti factors of p. Moreover, $r \mid p \iff r = p_k$ for some k.

Now $u[h+p] = 0 = \left(\sum_{r=1}^m a_r e_r\right)[p+1] = \sum_{k=1}^{\tau(p)} a_{p_k}$.

Let $q = p_{\tau(p)-1}$. Then $\tau(q) = \tau(p) - 1$, and each $q_k = p_k$. Again, $\left(\sum_{r=1}^m a_r e_r\right)[h+q] = 0 = \sum_{k=1}^{\tau(p)-1} a_{p_k}$.

Thus $a_{p_{\tau(p)}} = a_p = 0$ for all $p \in \{1, \dots, m\} \Rightarrow (e_1, \dots, e_m)$ is linely inde in $\mathbf{F}^\infty$.

So is $(e_1 + U, \dots, e_m + U)$ in $\mathbf{F}^\infty/U$. Becs m is arb. By (2.A.14). □

OR. For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, \text{if } 2^r \mid p \\ 0, \text{othws} \end{cases}$.

Simlr, let $m \in \mathbf{N}^+$ and $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 \Rightarrow a_1 e_1 + \cdots + a_m e_m = u \in U$.

Sup L is the largest suth $u[L] \neq 0$. And l is suth $2^{ml} > L$.

Then for each $k \in \{1, \dots, m\}$, $u[2^{ml} + 2^k] = 0 = \left(\sum_{r=1}^m a_r e_r\right)[2^k] = a_1 + \cdots + a_k$.

Thus $a_1 = \cdots = a_m = 0$ and $(e_1, \dots, e_m)$ is linely inde. Simlr. □

ENDED

3·F

4 5 6 7 8 9 12 13 15 16 17 18 19 20 21 22 23 24 25 26 28 29 30 31 32 33 34 35 36 37 | 4E: 5 6 8 17 23 24 25

4 *Sup U is a subsp of V and $U \neq V$. Provt $\exists\, \varphi \in V' \backslash \{0\}, \varphi(u) = 0$ for all $u \in U$.*

Solus: Let $X \oplus U = V \Rightarrow X \neq \{0\}$. Sup $s \in X \backslash \{0\}$. Let $Y \oplus \operatorname{span}(s) = X$.
Define $\varphi \in V'$ by $\varphi(u + \lambda s + y) = \lambda$. Hence $\varphi \neq 0$ and $\varphi(u) = 0$ for all $u \in U$. □

Or. [*Req V Finid*] By [3.106], $\dim U^0 = \dim V - \dim U > 0$.
Or. Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1 \dots, v_n)$ with $n \geqslant 1$.
Let $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$. Then each $\varphi \in \operatorname{span}(\varphi_1, \dots, \varphi_n)$ will do. □

Coro: (1) $U \neq V \Rightarrow U^0 \neq \{0\}$. (2) $U^0 = \{0\} \Rightarrow U = V$.

Comment: ***Another proof of*** **[3.108]**: T is surj $\iff T'$ is inje.
(a) Sup T' is inje. Notice that $\psi \neq 0 \iff T'(\psi) \neq 0 \iff \psi \notin (\operatorname{range} T)^0$.
(b) T is surj $\Rightarrow (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T'$. □

• *Sup V is a vecsp and U is a subsp of V.*

18 $U^0 = V' \iff \forall \varphi \in V', U \subseteq \operatorname{null}\varphi \iff U = \{0\}$. [*Which means* $\{0\}^0_V = V'$.]

19 $U^0_V = \{0\} = V^0_V \iff U = V$. By the inverse and ctrapos of Exe (4).

• **Note For [3.102]:** For $U = \emptyset$, U^0 is undefined. If U^0 is in the context, then certainly U is nonempty.

---

25 *Sup U is a subsp of V. Explain why $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.*

Solus: Note that $U = \{v \in V : v \in U\}$ is a subsp. Now we show $\forall \varphi \in U^0, \varphi(v) = 0 \Rightarrow v \in U$.
Asm $v \in V \backslash U$. Then let $\operatorname{span}(v) \oplus U \oplus X = V$. $\exists\, \psi \in V', \operatorname{null}\psi = U \oplus X$.
又 $\psi \in U^0 \Rightarrow \psi(v) = 0$. Ctradic. Hence $v \in U \iff \forall \varphi \in U^0, \varphi(v) = 0$. □

Comment: $W \subseteq X = \{v \in V : \varphi(v) = 0, \forall \varphi \in W^0\}$, the ***promotion*** of the subset W of V.
The promotion of every nonempty subset of V is a subsp of V.

---

20 *Sup U, W are nonempty subsets of V. Provt $U \subseteq W \Rightarrow W^0 \subseteq U^0$.*

Solus: $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(u) = 0 \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$. □

---

21 *Sup U, W are subsps of V. Provt $W^0 \subseteq U^0 \Rightarrow U \subseteq W$.*

Solus: Using Exe (25). Now $v \in U \Rightarrow \forall \varphi \in W^0 \subseteq U^0, \varphi(v) = 0 \Rightarrow v \in W$. □

Comment: $\varphi \in W^0 \iff \underline{\operatorname{null}\varphi \supseteq W \Rightarrow \operatorname{null}\varphi \supseteq U} \iff \varphi \in U^0$. But cannot conclude $W \supseteq U$.

Comment: (1) If U is merely a subset and W is a subsp. Promote U as X, let $W = Y$.
Then $Y^0 = W^0 \subseteq U^0 = X^0 \Rightarrow Y = W \supseteq X \supseteq U$. Still true.
(2) If W is merely a subset and U is a subsp. Promote W as Y, let $U = X$. For exa,
Let $W = \{(1,0), (0,1)\} \not\supseteq U = \{(x, 0) \in \mathbf{R}^2\}$. Then $Y = \mathbf{R}^2 \supseteq X = U$, $Y^0 = \{0\} \subseteq X^0$.

---

22 *Sup U and W are subsps of V. Provt $(U + W)^0 = U^0 \cap W^0$.*

Solus: (a) $\varphi \in (U + W)^0 \Rightarrow \forall u \in U, w \in W,$ $\varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$. | $U \subseteq U + W \Rightarrow (U + W)^0 \subseteq U^0$; $W \subseteq U + W \Rightarrow (U + W)^0 \subseteq W^0$

(b) $\varphi \in U^0 \cap W^0 \subseteq V' \Rightarrow \forall u \in U, w \in W, \varphi(u + w) = 0 \Rightarrow \varphi \in (U + W)^0$. □

23 *Sup U and W are subsps of V. Provt $(U \cap W)^0 = U^0 + W^0$.*

Solus:

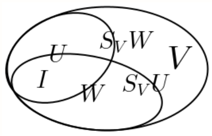
(a) $\varphi = \psi + \beta \in U^0 + W^0 \Rightarrow \forall v \in U \cap W,$
$\varphi(v) = (\psi + \beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0.$

Or. $U \cap W \subseteq U \Rightarrow (U \cap W)^0 \supseteq U^0$
$U \cap W \subseteq W \Rightarrow (U \cap W)^0 \supseteq W^0$

(b) [*Only in Finid; Req U, W Subsps*] Using Exe (22).

$$\dim(U^0 + W^0) = \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0)$$
$$= 2\dim V - \dim U - \dim W - \big(\dim V - \dim(U + W)\big) = \dim V - \dim(U \cap W).$$

Or. [*Req U, W Subsps*] Let $I = U \cap W$. Using [3E Tips (3)].
Now $S_V I = S_V U \oplus S_U I = S_V W \oplus S_W I$. For $\varphi \in (U \cap W)^0 = I^0$.
Let $\operatorname{span}(x) = $ Pure $V/\operatorname{null}\varphi$. If $x = 0$ then we are done.



Now $0 \neq x \in S_V I \Rightarrow \exists!\,(u_v, i_u, w_v, i_w) \in S_V U \times S_U I \times S_V W \times S_W I$,
$x = u_v + i_u = w_v + i_w$. Define $\varphi \in U^0, \beta \in W^0$ by $\varphi : u_v \mapsto 1, u \mapsto 0$, and $\beta : i_u \mapsto 1, i \mapsto 0$,
for all $u \in$ Pure $V/\operatorname{span}(u_v)$ and $i \in$ Pure $V/\operatorname{span}(i_u)$. Or Define $\psi \in W^0, \gamma \in U^0$, simlr.
Then $\varphi = \varphi + \beta = \psi + \gamma \in U^0 + W^0$. ☐

Comment: Not true if U or W is merely a subset. Promote $U \cap W$ as I, U as X, and W as Y.
Exa: Let $U = \{(x, x+1) \in \mathbf{R}^2\}, W = \mathbf{R}^2$. Then $U \cap W = I = U \neq \mathbf{R}^2 = X \cap Y$.

• **Tips 1:** (a) *Provt $V = U \oplus W \iff V' = U^0 \oplus W^0$.*
(b) *Sup $U \oplus W = V$. Provt $U^0 = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$,*
where $\iota \in \mathcal{L}(V, W) : u_v + w_v \to u_v$. **New Nota:** Denote W^0 by U'_V, and U^0 by W'_V.

Solus: (a) $U \cap W = \{0\} \iff (U \cap W)^0 = \{0\}^0_V = V' = U^0 + W^0$.
$V = U + W \iff (U + W)^0 = V^0_V = \{0\} = U^0 \cap W^0$.
(b) Notice that by [3.B Tips (3)], $\varphi \in W^0 \iff W \subseteq \operatorname{null}\varphi \iff \varphi = \varphi \circ \iota$. ☐

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31 *Sup U is a subsp of V. Let $B_{U'_V} = (\varphi_1, \dots, \varphi_n)$. Shat corres B_U exists.*

Solus: Let each $\underline{\operatorname{null}\varphi_i \oplus \operatorname{span}(u_i) = V}$ with $\varphi_i(u_i) = 1$.
Now $a_1 u_1 + \cdots + a_n u_n = 0 \Rightarrow$ $\underline{\text{Each } a_i = \varphi_i(a_1 u_1 + \cdots + a_n u_n) = 0, \text{by def of dual basis.}}$ ☐

Exa: Cannot extend B_U freely. Let $B_V = (e_1, e_2 - e_1)$. Let corres $B_{V'} = (\varphi_1, \varphi_2)$.
Let $U'_V = \operatorname{span}(\varphi_1)$. Then extend to $B_U = (e_1)$ to $B'_V = (e_1, e_2)$. Corres $B'_{V'} \neq B_{V'}$.

• **Tips 2:** Sup $\varphi_1, \dots, \varphi_m \in V'$. Let $\operatorname{null}_I = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.
Sup Ω is a subsp of V'. Let $\operatorname{null}_C = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}$.
If $\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$. Then $\operatorname{null}_I = \operatorname{null}_C$.
Becs $v \in \operatorname{null}_I \iff$ each $\varphi_i(v) = 0 \iff \forall \varphi \in \Omega, \varphi_i(v) = 0 \iff v \in \operatorname{null}_C$.

Comment: If Ω is infinide. Then $\operatorname{null}_I = \bigcap_{\varphi \in \Omega} \operatorname{null}\varphi = \operatorname{null}_C$.

• **Tips 3:** *Let $\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m) \subseteq V'$. Provt* (a) $\Omega = (\operatorname{null}_I)^0$; (b) $\Omega = (\operatorname{null}_C)^0$.

Solus: Here (a) is [4E 23], (b) is Exe (26).

(a) For each $\varphi_k = 0$, $\operatorname{span}(\varphi_k) = \{0\} = (\operatorname{null}\varphi_k)^0$.
For each $\varphi_k \neq 0$. Using (3.B.29) and Tips (1). Let $\varphi(v_k) \neq 0 \Rightarrow \operatorname{null}\varphi_k \oplus \operatorname{span}(v_k) = V$.
Then $(\operatorname{null}\varphi_k)^0 = (\operatorname{span}(v_k))'_V = \{\varphi \in V' : \varphi = \varphi \circ \iota\} = \operatorname{span}(\varphi_k)$, where $\iota : cv_k + u_0 \to cv_k$.
Thus $\Omega = \operatorname{span}(\varphi_1) + \cdots + \operatorname{span}(\varphi_m) = (\operatorname{null}\varphi_1)^0 + \cdots + (\operatorname{null}\varphi_m)^0$
$= \big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big)^0 = (\operatorname{null}_I)^0$. ☐

OR. $\dim(\operatorname{null}\varphi)^0 = \underline{\dim \operatorname{range}\varphi = \dim \operatorname{span}(\varphi)}$. 又 $\operatorname{span}(\varphi) \subseteq (\operatorname{null}\varphi)^0$. OR. By Exe (26). ☐

OR. $c \in \mathbf{F}\backslash\{0\} \iff \operatorname{null}(c\varphi_i) = \operatorname{null}\varphi_i \iff c\varphi_i \in \big(\operatorname{null}(c\varphi_i)\big)^0 = (\operatorname{null}\varphi_i)^0$.

And $0 \in \operatorname{span}(\varphi_i), 0 \in (\operatorname{null}\varphi_i)^0$. Hence $\operatorname{span}(\varphi_i) = (\operatorname{null}\varphi_i)^0$. ☐

(b) $\forall \varphi \in \Omega, \operatorname{null}_C \subseteq \operatorname{null}\varphi \Rightarrow \varphi \in (\operatorname{null}_C)^0$. Hence $\Omega = (\operatorname{null}_I)^0 \subseteq (\operatorname{null}_C)^0$. OR. By TIPS (2). ☐

• **NOTE FOR Exe (26):** For every subsp Ω of V', $\exists!$ subsp U of V suth $\Omega = U^0$.

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24 *Sup V is finide and U is a subsp of V.*
Prove, using the pattern of [3.104], that $\dim U + \dim U^0 = \dim V$.

SOLUS: Let $B_{U^0} = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$. Let $B_{W^0} = (\varphi_{m+1}, \dots, \varphi_n)$.
And let corres (I) $B_U = (v_{m+1}, \dots, v_n)$, (II) $B_W = (v_1, \dots, v_m)$.

(I) NOTICE that each $\operatorname{null}\varphi_k = \operatorname{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n) = U_k$; $\dim U_k = \dim V - 1$.
By (4E 2.C.16), $U = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = \bigcap_{k=1}^m U_k = \operatorname{span}(v_{m+1}, \dots, v_n)$.
Hence $\operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0 = \Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$.

(II) NOTICE that $V' = \Omega \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.
And that $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq \operatorname{span}(v_1, \dots, v_m)^0$.
By [1.C TIPS (2)] OR (2.C.1), $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = \operatorname{span}(v_1, \dots, v_m)^0$.
OR. Simlr to (II), let $\Omega = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, immediately. ☐

• *Sup $T \in \mathcal{L}(V, W), \varphi_k \in V', \psi_k \in W'$.*
28 *Provt* $\operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) \iff \operatorname{range} T = (\operatorname{null}\psi_1) \cap \cdots \cap (\operatorname{null}\psi_m)$.
29 *Provt* $\operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \iff \operatorname{null} T = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.

SOLUS: Using [3.107], [3.109], Exe (23) and the CORO in Exe (20, 21).
(28) $(\operatorname{range} T)^0 = \operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) = \big((\operatorname{null}\psi_1) \cap \cdots \cap (\operatorname{null}\psi_m)\big)^0$.
(29) $(\operatorname{null} T)^0 = \operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) = \big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big)^0$. ☐

CORO: Using the COMMENT in Exe (26).
$\operatorname{null} T = \operatorname{span}(v_1, \dots, v_m) \iff \operatorname{null} T = (\operatorname{null}\varphi_{m+1}) \cap \cdots \cap (\operatorname{null}\varphi_n) \iff \operatorname{range} T' = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$.
——Where $B_V = (v_1, \dots, v_m, \dots, v_n) \iff B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$.
$\operatorname{range} T = \operatorname{span}(w_1, \dots, w_m) \iff \operatorname{range} T = (\operatorname{null}\psi_{m+1}) \cap \cdots \cap (\operatorname{null}\psi_n) \iff \operatorname{null} T' = \operatorname{span}(\psi_{m+1}, \dots, \psi_n)$.
——Where $B_W = (w_1, \dots, w_m, \dots, w_n) \iff B_{W'} = (\psi_1, \dots, \psi_m, \dots, \psi_n)$.

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9 *Let $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \cdots, \varphi_n)$. Then $\forall \psi \in V', \psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.*
CORO: *For other $B'_V = (u_1, \dots, u_n), B'_{V'} = (\rho_1, \dots, \rho_n), \forall \psi \in V', \psi = \psi(u_1)\rho_1 + \cdots + \psi(u_n)\rho_n$.*

SOLUS:
$\psi(v) = \psi\Big(\sum_{i=1}^n a_i v_i\Big) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i)\varphi_i(v) = [\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n](v)$.
OR. $[\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n]\Big(\sum_{i=1}^n a_i v_i\Big) = \psi(v_1)\varphi_1\Big(\sum_{i=1}^n a_i v_i\Big) + \cdots + \psi(v_n)\varphi_n\Big(\sum_{i=1}^n a_i v_i\Big)$. ☐

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13 *Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$.*
Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ denote the dual basis of std basis of $\mathbf{R}^2$ and $\mathbf{R}^3$.

(a) *Describe the linear functionals $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$*

For any $(x, y, z) \in \mathbf{R}^3$, $\big(T'(\varphi_1)\big)(x, y, z) = 4x + 5y + 6z$, $\big(T'(\varphi_2)\big)(x, y, z) = 7x + 8y + 9z$.

(b) *Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.*

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3,\ T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3.$

(c) *What is* null T'? *What is* range T'?

$T(x,y,z) = 0 \iff \begin{cases} 4x+5y+6z=0 \\ 7x+8y+9z=0 \end{cases} \iff \begin{cases} x+y+z=0 \\ y=2z=0 \end{cases} \iff (x,y,z) \in \operatorname{span}(e_1 - 2e_2 + e_3).$

Where (e_1, e_2, e_3) is std basis of $\mathbf{R}^3$.

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a basis, with corres dual basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Thus $\operatorname{span}(e_1 - 2e_2 + e_3) = \operatorname{null} T \Rightarrow \operatorname{span}(e_1 - 2e_2 + e_3)^0 = \operatorname{span}(\varepsilon_2, \varepsilon_3) = \operatorname{range} T'$.

Note that $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$.

And $\left| \begin{array}{l} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{array} \right.$

Hence $\varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2,\ \varepsilon_3 = -\psi_1 + \psi_3$. Now $\operatorname{range} T' = \operatorname{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3)$.

OR. $\operatorname{range} T' = \operatorname{span}(T'(\varphi_1), T'(\varphi_2)) = \operatorname{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3)$.

Sup $T'(x\varphi_1 + y\varphi_2) = (4x+7y)\varphi_1 + (5x+8y)\varphi_2 + (6x+9y)\varphi_3 = 0$.
Then $x + y = 4x + 7y = x = y = 0$. Hence $\operatorname{null} T' = \{0\}$.

OR. $\operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \operatorname{span}(-2e_2, e_3) \oplus \operatorname{null} T$.

$\Rightarrow \operatorname{range} T = \{Tx : x \in \operatorname{span}(-2e_2, e_3)\} = \operatorname{span}(T(-2e_2), T(e_3))$

$= \operatorname{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \operatorname{span}(f_1, f_2) = \mathbf{R}^2$. Now $\operatorname{null} T' = (\operatorname{range} T)^0 = \{0\}$. ☐

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37 *Sup U is a subsp of V and π is the quot map. Thus $\pi' \in \mathcal{L}\big((V/U)', V'\big)$.*

(a) *Shat π' is inje*: Becs π is surj. Use [3.108].

(b) *Shat* range $\pi' = U^0$: By [3.109](b), $\operatorname{range} \pi' = (\operatorname{null} \pi)^0 = U^0$.

(c) *Conclude that π' is iso from $(V/U)'$ onto U^0*: Immediately.

Solus: OR. Using (3.E.18), also see (3.E.20).

(a) $\pi'(\varphi) = 0 \iff \forall v \in V\ (\forall v + U \in V), \varphi(\pi(v)) = \varphi(v + U) = 0 \iff \varphi = 0$.

(b) $\psi \in \operatorname{range} \pi' \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null} \psi \supseteq U \iff \psi \in U^0$. Hence $\operatorname{range} \pi' = U^0$. ☐

• *Sup U is a subsp of V. Provt $(V/U)'$ is iso to U^0.* [***Another proof of* [3.106]**]

Solus:

Define $\xi : U^0 \to (V/U)'$ by $\xi(\varphi) = \tilde{\varphi}$, where $\tilde{\varphi} \in (V/U)'$ is defined by $\tilde{\varphi}(v + U) = \varphi(v)$.

We shat ξ is inje and surj.

Inje: $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V\ (\forall v + U \in V/U), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$.

Surj: $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null}(\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$. ☐

OR. Define $\nu : (V/U)' \to U^0$ by $\nu(\Phi) = \Phi \circ \pi$. Now $\nu \circ \xi = I_{U^0},\ \xi \circ \nu = I_{(V/U)'} \Rightarrow \xi = \nu^{-1}$. ☐

• *Sup $V = U \oplus W$. Define $\iota : V \to U$ by $\iota(u + w) = u$. Thus $\iota' \in \mathcal{L}(U', V')$.*

(a) *Shat* null $\iota' = U_U^0 = \{0\}$: $\operatorname{null} \iota' = (\operatorname{range} \iota)_U^0 = U_U^0 = \{0\}$.

(b) *Provt* range $\iota' = W_V^0$: $\operatorname{range} \iota' = (\operatorname{null} \iota)_V^0 = W_V^0$.

(c) *Provt $\tilde{\iota'}$ is iso from $U'/\{0\}$ onto W^0*: By (a), (b) and [3.91](d).

Solus:

(a) $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \operatorname{null} \psi$.

(b) Note that $W = \operatorname{null}(\iota) \subseteq \operatorname{null}(\psi \circ \iota)$. Then $\psi \circ \iota \in W^0 \Rightarrow \operatorname{range} \iota' \in W^0$.

Sup $\varphi \in W^0$. Becs $\operatorname{null} \iota = W \subseteq \operatorname{null} \varphi$. By $\left[3.\text{B Tips }(3)\right]$, $\varphi = \varphi \circ \iota = \iota'(\varphi)$. $\square$

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36 *Sup U is a subsp of V. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*

(a) *Shat* $\operatorname{null} i' = U^0$: $\operatorname{null} i' = (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$.

(b) *Provt* $\operatorname{range} i' = U'$: $\operatorname{range} i' = (\operatorname{null} i)^0_U = \{0\}^0_U = U'$.

(c) *Provt $\widetilde{i'}$ is iso from V'/U^0 onto U':* By (a), (b) and [3.91](d).

Solus:

(a) $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$. Thus $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$.

(b) Sup $\psi \in U'$. By (3.A.11), $\exists\, \varphi \in V', \varphi|_U = \psi$. Then $i'(\varphi) = \psi$. $\square$

• *Sup $T \in \mathcal{L}(V, W)$. Provt* $\operatorname{range} T' = \big(\operatorname{null} T\big)^0$. $\quad\left[\,\textbf{\textit{Another proof of}}\ \textbf{[3.109](b)}\,\right]$

Solus:

Sup $\Phi \in \big(\operatorname{null} T\big)^0$. Becs by (3.B.12), $T|_U : U \to \operatorname{range} T$ is iso; $V = U \oplus \operatorname{null} T$.

And $\forall v \in V, \exists\,! u_v \in U, w_v \in \operatorname{null} T, v = u_v + w_v$. Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$.

Let $\psi = \Phi \circ \big(T^{-1}|_{\operatorname{range} T}\big)$. Then $T'(\psi) = \psi \circ T = \Phi \circ \big(T^{-1}|_{\operatorname{range} T} \circ T|_V\big)$.

Where $T^{-1}|_{\operatorname{range} T} : \operatorname{range} T \to U$; $\ T : V \to \operatorname{range} T$. Note that $T^{-1}|_{\operatorname{range} T} \circ T|_V = \iota$.

By $\left[3.\text{B Tips }(3)\right]$, $\Phi = \Phi \circ \iota$. Thus $T'(\psi) = \psi \circ T = \Phi \circ \iota = \Phi$. $\square$

• *Sup $T \in \mathcal{L}(V, W)$. Using [3.108], [3.110].*

$$\textit{Now } T \textit{ is inv} \iff \left|\begin{array}{c} \operatorname{null} T = \{0\} \iff (\operatorname{null} T)^0 = V' = \operatorname{range} T' \\ \operatorname{range} T = W \iff (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T' \end{array}\right| \iff T' \textit{ is inv.}$$

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15 *Sup $T \in \mathcal{L}(V, W)$. Provt $T' = 0 \iff T = 0$.*

Solus:

Sup $T = 0$. Then $\forall \varphi \in W', T'(\varphi) = \varphi \circ T = 0$. Hence $T' = 0$.

Sup $T' = 0$. Then $\operatorname{null} T' = W' = (\operatorname{range} T)^0$, by [3.107](a).

$\left[\,\textit{W can be infinide}\,\right]$ By Exe (25),

$\operatorname{range} T = \big\{w \in W : \varphi(w) = 0, \forall \varphi \in (\operatorname{range} T)^0\big\} = \big\{w \in W : \varphi(w) = 0, \forall \varphi \in W'\big\}$.

Now we provt if $\forall \varphi \in W', \varphi(w) = 0$, then $w = 0$. So that $\operatorname{range} T = \{0\}$ and we are done.

Asm $w \neq 0$. Then let U be suth $W = U \oplus \operatorname{span}(w)$.

Define $\psi \in W'$ by $\psi(u + \lambda w) = \lambda$. So that $\psi(w) = 1 \neq 0$. $\square$

Or. $\left[\,\textit{Only if W is finide}\,\right]$ By [3.106], $\dim \operatorname{range} T = \dim W - \dim(\operatorname{range} T)^0 = 0$. $\square$

---

12 Notice that $I_{V'} : V' \to V'$. *Now $\forall \varphi \in V', I_{V'}(\varphi) = \varphi = \varphi \circ I_V = I_V{}'(\varphi)$. Thus $I_{V'} = I_V{}'$.*

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16 *Sup V, W are finide. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(V, W)$.*
Provt Γ is iso of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

Solus: By [3.101], Γ is linear.

Sup $\Gamma(T) = T' = 0$. By Exe (15), $T = 0$. Thus Γ is inje.

Becs V, W are finide. $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$. Now Γ inje $\Rightarrow$ inv. $\square$

Comment: Let $X = \big\{T \in \mathcal{L}(V, W) : \operatorname{range} T \text{ is finide}\big\}$.

Let $Y = \big\{\mathcal{T} \in \mathcal{L}(W', V') : \operatorname{range} \mathcal{T} \text{ is finide}\big\}$.

Then $\Gamma|_X$ is iso of X onto Y, even if V and W are infinide.

The inje of $\Gamma|_X$ is equiv to the inje of Γ, as shown before.

Now we shat $\Gamma|_X$ is surj without the cond that V or W is finide.

Sup $\mathcal{T} \in Y$. Let $B_{\operatorname{range}\mathcal{T}} = (\varphi_1, \dots, \varphi_m)$, with corres $(v_1, \dots, v_m)$. Let $\varphi_k = \mathcal{T}(\psi_k)$.

Let $\mathcal{K}$ be suth $W' = \mathcal{K} \oplus \operatorname{null}\mathcal{T}$. Let $B_{\mathcal{K}} = (\psi_1, \dots, \psi_m)$, with corres $(w_1, \dots, w_m)$.

Define $T \in \mathcal{L}(V, W)$ by $Tv_k = w_k, Tu = 0;\ k \in \{1, \dots, m\}, u \in U$.

$\forall \psi \in \operatorname{null}\mathcal{T}, [T'(\psi)](v) = \psi(Tv) = \psi(a_1w_1 + \cdots + a_pw_p) = 0 = [\mathcal{T}(\psi)](v)$.

$\forall k \in \{1, \dots, m\}, [T'(\psi_k)](v) = \psi_k(Tv) = \psi_k(a_1w_1 + \cdots + a_mw_m) = a_k = \varphi_k(v) = [\mathcal{T}(\psi)](v)$. ☐

Comment: This is another proof of [3.109(a)]: $\dim\operatorname{range}T = \dim\operatorname{range}T'$.

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5 *Provt $(V_1 \times \cdots \times V_m)'$ and ${V'}_1 \times \cdots \times {V'}_m$ are iso.* [Using notas in (3.E.2).]

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to {V'}_1 \times \cdots \times {V'}_m \\ \qquad \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = ({R'}_1(T), \dots, {R'}_m(T)). \\ \text{Define } \psi : {V'}_1 \times \cdots \times {V'}_m \to (V_1 \times \cdots \times V_m)' \\ \qquad \text{by } \psi(T_1, \dots, T_m) = T_1S_1 + \cdots + T_mS_m = {S'}_1(T_1) + \cdots + {S'}_m(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$

☐

• (4E 8) *Sup $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \dots, \varphi_n)$.*

$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \dots, a_n) = a_1v_1 + \cdots + a_nv_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

---

6 *Define $\Gamma : V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = \big(\varphi(v_1), \dots, \varphi(v_m)\big)$, where $v_1, \dots, v_m \in V$.*

(a) *Shat $\operatorname{span}(v_1, \dots, v_m) = V \iff \Gamma$ is inje.*

(b) *Shat $(v_1, \dots, v_m)$ is linely inde $\iff \Gamma$ is surj.*

Solus:

(a) Notice that $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \cdots = \varphi(v_m) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.

If Γ is inje, then $\Gamma(\varphi) = 0 \iff V = \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.

If $V = \operatorname{span}(v_1, \dots, v_m)$, then $\Gamma(\varphi) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$, thus Γ is inje.

(b) Sup Γ is surj. Then let $\Gamma(\varphi_i) = e_i$ for each i, where $(e_1, \dots, e_m)$ is std basis of $\mathbf{F}^m$.

Then by (3.A.4), $(\varphi_1, \dots, \varphi_m)$ is linely inde.

Now $a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow 0 = \varphi_i(a_1v_1 + \cdots + a_mv_m) = a_i$ for each i.

Sup $(v_1, \dots, v_m)$ is linely inde. Let $U = \operatorname{span}(\varphi_1, \dots, \varphi_m), B_{U'} = (\varphi_1, \dots, \varphi_m)$.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists!\, \varphi = a_1\varphi_1 + \cdots + a_m\varphi_m$.

Let W be suth $V = U \oplus W$. Now $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.

Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$. So that $\Gamma(\varphi \circ i -) = (a_1, \dots, a_m)$. ☐

Or. Let $(e_1, \dots, e_m)$ be std basis of $\mathbf{F}^m$ and let $(\psi_1, \dots, \psi_m)$ be corres dual basis.

Define $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$. Then Ψ is iso.

Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $Te_k = v_k$. Now $T(x_1, \dots, x_m) = T(x_1e_1 + \cdots + x_me_m) = x_1v_1 + \cdots + x_mv_m$.

$\forall \varphi \in V', k \in \{1, \dots, m\}, [T'(\varphi)](e_k) = \varphi(Te_k) = \varphi(v_k) = [\varphi(v_1) \circ \psi_1 + \cdots + \varphi(v_m) \circ \psi_m](e_k)$

Now $T'(\varphi) = \varphi(v_1) \circ \psi_1 + \cdots + \varphi(v_m) \circ \psi_m = \Psi\big(\varphi(v_1), \dots, \varphi(v_m)\big) = \Psi\big(\Gamma(\varphi)\big)$. Hence $T' = \Psi \circ \Gamma$.

By (3.B.3), (a) $\operatorname{range}T = \operatorname{span}(v_1, \dots, v_m) = V \iff T' = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.

(b) $(v_1, \dots, v_m)$ is linely inde $\iff T$ is inje $\iff T' = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. ☐

• (4E 25) *Define $\Gamma : V \to \mathbf{F}^m$ by $\Gamma(v) = \big(\varphi_1(v), \dots, \varphi_m(v)\big)$, where $\varphi_1, \dots, \varphi_m \in V'$.*

(c) *Shat $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ is inje.*

(d) *Shat* $(\varphi_1, \dots, \varphi_m)$ *is linely inde* $\iff \Gamma$ *is surj.*

SOLUS:

(c) NOTICE that $\Gamma(v) = 0 \iff \varphi_1(v) = \cdots = \varphi_m(v) = 0 \iff v \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.

By Exe (4E 23) and (18), $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = \{0\}$.

And $\operatorname{null}\Gamma = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$. Hence Γ inje $\iff \operatorname{null}\Gamma = \{0\} \iff \operatorname{span}(\varphi_1, \dots, \varphi_m) = V'$.

(d) Sup $(\varphi_1, \dots, \varphi_m)$ is linely inde. Then by Exe (31), $(v_1, \dots, v_m)$ is linely inde.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}, \exists!\, v = \sum_{i=1}^{m} a_i v_i \in V \Rightarrow \varphi_i(v) = a_i, \Gamma(v) = (a_1, \dots, a_m)$. Hence Γ is surj.

Sup Γ is surj. Let $(e_1, \dots, e_m)$ be std basis of $\mathbf{F}^m$.

Sup $v_i \in V$ suth $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i$, for each i.

Then $(v_1, \dots, v_m)$ is linely inde. And $\varphi_j(v_k) = \delta_{j,k}$.

Now $a_1\varphi_1 + \cdots + a_m\varphi_m = 0 \Rightarrow 0(v_i) = a_i$ for each i. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde.

OR. Let $\operatorname{span}(v_1, \dots, v_m) = U$. Then $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U)$. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde. □

OR. Simlr to Exe (6), we get $(e_1, \dots, e_m), (\psi_1, \dots, \psi_m)$ and the iso Ψ.

$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'(\Psi(x_1, \dots, x_m)) = \Gamma'(\Psi(x_1e_1 + \cdots + x_me_m)) = (x_1\psi_1 + \cdots + x_m\psi_m) \circ \Gamma$.

$\forall v \in V, [\Gamma'(\Psi(x_1, \dots, x_m))](v) = [x_1\psi_1 + \cdots + x_m\psi_m](\Gamma(v)) = [x_1\varphi_1 + \cdots + x_m\varphi_m](v)$.

Now $\Gamma'(\Psi(x_1, \dots, x_m)) = x_1\varphi_1 + \cdots + x_m\varphi_m$.

Define $\Phi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Phi = \Psi \circ \Gamma$. $\Phi(x_1, \dots, x_m) = x_1\varphi_1 + \cdots + x_m\varphi_m$. Thus by (4E 3.B.3),

(c) the inje of Φ corres to $(\varphi_1, \dots, \varphi_m)$ spanning V'; 又 $\Phi = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.

(d) the surj of Φ corres to $(\varphi_1, \dots, \varphi_m)$ being linely inde; 又 $\Phi = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

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35 *Provt* $(\mathcal{P}(\mathbf{F}))'$ *is iso to* $\mathbf{F}^\infty$.

SOLUS:

Define $\theta \in \mathcal{L}\big((\mathcal{P}(\mathbf{F}))', \mathbf{F}^\infty\big)$ by $\theta(\varphi) = \big(\varphi(1), \varphi(z), \cdots, \varphi(z^n), \cdots\big)$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall z^k$ in the basis $(1, z, \dots, z^n)$ of $\mathcal{P}_n(\mathbf{F})$ $(\forall n)$, $\varphi(z^k) = 0 \Rightarrow \varphi = 0$.

[NOTICE that $\forall p \in \mathcal{P}(\mathbf{R}), \exists!\, a_i \in \mathbf{F}, m = \deg p,\ p = a_0 z + a_1 z + \cdots + a_m z^m \in \mathcal{P}_m(\mathbf{F})$.]

Surj: $\forall (a_k)_{k=1}^\infty \in \mathbf{F}^\infty$, let ψ be suth $\forall k, \psi(z^k) = a_k$ [by [3.5]] and thus $\theta(\psi) = (a_k)_{k=1}^\infty$. □

COMMENT: NOTICE that $\mathcal{P}(\mathbf{F})$ is not iso to $\mathbf{F}^\infty$, so is $\mathcal{P}(\mathbf{F})$ to $(\mathcal{P}(\mathbf{F}))'$

But if we let $\mathbf{F}^\infty = \{(a_1, \cdots, a_n, \underbrace{0, \cdots, 0, \cdots}_{\text{all zero}}) \in \mathbf{F}^\infty \mid \exists!\, n \in \mathbf{N}^+\}$. Then $\mathcal{P}(\mathbf{F})$ is iso to $\mathbf{F}^\infty$.

---

7 *Shat the dual basis of* $(1, x, \dots, x^m)$ *of* $\mathcal{P}_m(\mathbf{R})$ *is* $(\varphi_0, \varphi_1, \dots, \varphi_m)$, *where* $\varphi_k(p) = \dfrac{p^{(k)}(0)}{k!}$.

Here $p^{(k)}$ *denotes the* k^{th} *deri of* p, *with the understanding that the* 0^{th} *deri of* p *is* p.

SOLUS:

$$\forall j,k \in \mathbf{N},\ (x^j)^{(k)} = \begin{cases} j(j-1)\dots(j-k+1)\cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\dots(j-j+1) = j! & j = k. \\ 0, & j \leqslant k. \end{cases} \quad \Bigg| \quad \text{Then } (x^j)^{(k)}(0) = \begin{cases} 0\ , & j \neq k. \\ k!\ , & j = k. \end{cases}$$ □

OR. Becs $\forall j,k \in \{1, \dots, m\}$ suth $j \neq k$, $\varphi_k(x^j) = \frac{(x^j)^{(k)}(0)}{k!} = \frac{0}{k!} = 0$; $\varphi_k(x^k) = \frac{(x^k)^{(k)}(0)}{k!} = 1$.

Thus $\frac{p^{(k)}(0)}{k!}$ act exactly the same as φ_k on the same basis $(1, \dots, x^m)$, hence is just another def of φ_k. □

EXA: Sup $m \in \mathbf{N}^+$. By [2.C.10], $B = \big(1, x-5, \dots, (x-5)^m\big)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

Let $\varphi_k = \dfrac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \dots, m$. Then $(\varphi_0, \varphi_1, \dots, \varphi_m)$ is the dual basis of B.

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34 *The double dual space of* V, *denoted by* V'', *is defined to be the dual space of* V'.

In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \to V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.

(a) *Shat Λ is a linear map from V to V''.*

(b) *Shat if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*

(c) *Shat if V is finide, then Λ **is iso from** V **onto** V''.*

Sup V is finide. Then V and V' are iso, and finding iso from V onto V' generally requires choosing a basis of V. In contrast, the iso Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

Solus:

(a) $\forall \varphi \in V', v, w \in V, a \in \mathbf{F}$, $\big(\Lambda(v+aw)\big)(\varphi) = \varphi(v+aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.
Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence Λ is linear.

(b) $$\begin{aligned}\big(T''(\Lambda v)\big)(\varphi) &= \big((\Lambda v) \circ T'\big)(\varphi) = (\Lambda v)\big(T'(\varphi)\big) \\ &= \big(T'(\varphi)\big)(v) = (\varphi \circ T)(v) = \varphi(Tv) = \big(\Lambda(Tv)\big)(\varphi).\end{aligned}$$

Hence $T''(\Lambda v) = \big(\Lambda(Tv)\big) \Rightarrow T'' \circ \Lambda = \Lambda \circ T$.

(c) Sup $\Lambda v = 0$. Then $\forall \varphi \in V'$, $(\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is inje.
又 Becs V is finide. $\dim V = \dim V' = \dim V''$. Hence Λ is iso. $\square$

Ended

- **TIPS:** *Sup* $p \in \mathcal{P}(\mathbf{F}), \deg p \leqslant m$ *and* p *has at least* $(m+1)$ *disti zeros.*
  Then by the ctrapos of $[4.12]$, 又 $\deg p = m$, *we conclude that* $m < 0$. *Hence* $p = 0$.

  OR. We shat if p has at least m disti zeros, then either $p = 0$ or $\deg p \geqslant m$.
  If $p = 0$ then we are done. If not, then sup p has exactly n disti zeros $\lambda_1, \dots, \lambda_n$.
  Becs $\exists\,!\,\alpha_i \geqslant 1, q \in \mathcal{P}(\mathbf{F})$, and $q \neq 0$, suth $p(z) = \left[(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_n)^{\alpha_n}\right] q(z)$. □
- **COMMENT:** NOTICE *that by* $[4.17]$, *some term of the poly factorization might not be in the form* $(x - \lambda_k)^{\alpha_k}$.

- **NOTE FOR [4.7]:** *the uniqness of coeffs of polys* [*Another proof*]

  If a poly had two different sets of coeffs, then subtracting the two representations would give a poly with some nonzero coeffs but infinly many zeros. By TIPS. □

- **NOTE FOR [4.8]:** *division algo for polys* [*Another proof*]

  Sup $\deg p \geqslant \deg s$. Then $\Big(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of len } \deg s}, \underbrace{s, zs, \cdots, z^{\deg p - \deg s} s}_{\text{of len } (\deg p - \deg s + 1)} \Big)$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.

  Becs $q \in \mathcal{P}(\mathbf{F}), \exists\,!\,a_i, b_j \in \mathbf{F}$,
  $q = a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 z s + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s$
  $= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1}}_{r} + s \underbrace{\left(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s}\right)}_{q}$. Note that r, q are unique. □

- **NOTE FOR [4.11]:** *each zero of a poly corresponds to a deg-one factor;* [*Another proof*]

  First sup $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m, \exists\,!\,a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.
  Then $p(z) = p(z) - p(\lambda) = a_1(z - \lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.
  Hence $\forall k \in \{1, \dots, m\}, z^k - \lambda^k = (z - \lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.
  Thus $p(z) = \sum_{j=1}^{m} a_j (z - \lambda) \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) \sum_{j=1}^{m} a_j \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z - \lambda) q(z)$. □

- **NOTE FOR [4.13]:** *Every nonconst poly with complex coeffs has a zero in* $\mathbf{C}$. [*Another proof*]

  For any $w \in \mathbf{C}, k \in \mathbf{N}^+$, by polar coordinates, $\exists\, r \geqslant 0, \theta \in \mathbf{R}, r(\cos\theta + \mathrm{i}\sin\theta) = w$.
  By De Moivre' theorem, $w^k = \left[r(\cos\theta + \mathrm{i}\sin\theta)\right]^k = r^k(\cos k\theta + \mathrm{i}\sin k\theta)$.
  Hence $\left(r^{1/k}(\cos\frac{\theta}{k} + \mathrm{i}\sin\frac{\theta}{k})\right)^k = w$. Thus every complex number has a k^{th} *root*.

  Sup a nonconst $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.
  Then $|p(z)| \to \infty$ as $|z| \to \infty$ $\left(\text{becs } \dfrac{|p(z)|}{|z_m|} \to |c_m| \text{ as } |z| \to \infty\right)$.
  Thus the continuous function $z \to |p(z)|$ has a global min at some point $\zeta \in \mathbf{C}$.
  To shat $p(\zeta) = 0$, asm $p(\zeta) \neq 0$. Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \dfrac{p(z + \zeta)}{p(\zeta)}$.
  The function $z \to |q(z)|$ has a global min value of 1 at $z = 0$.
  Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where $k \in \mathbf{N}^+$ is the smallest suth $a_k \neq 0$.
  Let $\beta \in \mathbf{C}$ be suth $\beta^k = -\dfrac{1}{a_k}$.
  There is a const $c > 1$ so that if $t \in (0, 1)$, then $|q(t\beta)| \leqslant |1 + a_k t^k \beta^k| + t^{k+1} c = 1 - t^k(1 - tc)$.
  Now letting $t = 1/(2c)$, we get $|q(t\beta)| < 1$. Ctradic. Hence $p(\zeta) = 0$, as desired. □

• (4E 4.2) *Provt if $w, z \in \mathbf{C}$, then $||w|-|z|| \leqslant |w-z|$.*

Solus:

$$\begin{aligned} |w-z|^2 &= (w-z)(\overline{w}-\overline{z}) \\ &= |w|^2+|z|^2-(w\overline{z}+\overline{w}z) \\ &= |w|^2+|z|^2-(\overline{\overline{w}z}+\overline{w}z) \\ &= |w|^2+|z|^2-2Re(\overline{w}z) \\ &\geqslant |w|^2+|z|^2-2|\overline{w}z| \\ &= |w|^2+|z|^2-2|w||z| = ||w|-|z||^2. \end{aligned}$$

Or. $\left.\begin{aligned} |w| = |w-z+z| \leqslant |w-z|+|z| \Rightarrow |w|-|z| \leqslant |w-z| \\ |z| = |z-w+w| \leqslant |z-w|+|w| \Rightarrow |z|-|w| \leqslant |w-z| \end{aligned}\right\}$

Geometric interpretation: The len of each side of a triangle is greater than or equal to the difference of the lens of the two other sides.

□

• (4E 4.3) *Sup $\mathbf{F} = \mathbf{C}, \varphi \in V'$. Define $\sigma : V \to \mathbf{R}$ by $\sigma(v) = \operatorname{Re}\varphi(v)$ for each $v \in V$. Shat $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$ for all $v \in V$.*

Solus: Notice that $\varphi(v) = \operatorname{Re}\varphi(v) + \mathrm{i}\,\mathfrak{Im}\,\varphi(v) = \sigma(v) + \mathrm{i}\,\mathfrak{Im}\,\varphi(v)$.

又 $\operatorname{Re}\varphi(\mathrm{i}\,v) = \operatorname{Re}(\mathrm{i}\,\varphi(v)) = -\mathfrak{Im}\,\varphi(v) = \sigma(\mathrm{i}\,v)$. Hence $\varphi(v) = \sigma(v) - \mathrm{i}\,\sigma(\mathrm{i}\,v)$. □

---

4 *Sup $m, n \in \mathbf{N}^+$ with $m \leqslant n$, $\lambda_1, \dots, \lambda_m \in \mathbf{F}$. Provt $\exists\, p \in \mathcal{P}(\mathbf{F}), \deg p = n$, the zeros of p are $\lambda_1, \dots, \lambda_m$.*

Solus: Let $p(z) = (z-\lambda_1)^{n-(m-1)}(z-\lambda_2)\cdots(z-\lambda_m)$. □

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5 *Sup $m \in \mathbf{N}$, and $z_1, \dots, z_{m+1}$ are disti in $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$. Provt $\exists\,!\,p \in \mathcal{P}_m(\mathbf{F}), p(z_k) = w_k$ for each $k \in \{1, \dots, m+1\}$.*

Solus:

Define $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by $Tq = \big(q(z_1), \dots, q(z_m), q(z_{m+1})\big)$. Moreover, T is linear.

We now shat T is surj, so that such p exists; and that T is inje, so that such p is unique.

Inje: $Tq = 0 \Longleftrightarrow q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0 \Longleftrightarrow q = 0$, by Tips.

Surj: $\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m+1 = \dim \mathbf{F}^{m+1}$ 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1} \Rightarrow T$ is surj. □

Or. Let $p_1 = 1$, $p_k(z) = \prod_{i=1}^{k-1}(z-z_i) = (z-z_1)\cdots(z-z_{k-1})$ for each $k \in \{2, \dots, m+1\}$.

By (2.C.10), $B_p = (p_1, \dots, p_{m+1})$ is a basis of $\mathcal{P}_m(\mathbf{F})$. Let $B_e = (e_1, \dots, e_{m+1})$ be the std basis of $\mathbf{F}^{m+1}$.

Notice that $Tp_1 = (1, \dots, 1)$, $Tp_k = \Big(\prod_{i=1}^{k-1}(z_1-z_i), \dots, \underbrace{\prod_{i=1}^{k-1}(z_j-z_i)}_{j^{th}\text{ ent}}, \dots, \prod_{i=1}^{k-1}(z_{m+1}-z_i)\Big)$.

And that $\prod_{i=1}^{k-1}(z_j-z_i) = 0 \Longleftrightarrow j \leqslant k-1$, becs $z_1, \dots, z_{m+1}$ are disti.

$$\text{Thus } \mathcal{M}(T, B_p, B_e) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & A_{2,2} & 0 & \cdots & 0 \\ 1 & A_{3,2} & A_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{m+1,2} & A_{m+1,3} & \cdots & A_{m+1,m+1} \end{pmatrix}.$$

Where $A_{j,k} = \prod_{i=1}^{k-1}(z_j-z_i) \neq 0$ for all $j > k-1 \geqslant 1$. The rows of $\mathcal{M}(T)$ is linely inde.

By (4E 3.C.17) 又 $\dim \mathcal{P}_m(\mathbf{F}) = \dim \mathbf{F}^{m+1}$; Or By (3.F.32); T is inv. □

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2 *Sup $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

Solus: $x^m, x^m + x^{m-1} \in U$ but $\deg\big[(x^m+x^{m-1})-(x^m)\big] \neq m \Rightarrow (x^m+x^{m-1})-(x^m) \notin U$. □

---

3 *Sup $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : 2 \mid \deg p\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

Solus: $x^2, x^2 + x \in U$ but $\deg\big[(x^2+x) - (x^2)\big]$ is odd and hence $(x^2+x)-(x^2) \notin U$. $\square$

---

6 *Sup nonzero $p \in \mathcal{P}_m(\mathbf{F})$ has deg m. Provt*
$[P]$ p has m disti zeros $\Longleftrightarrow$ p and its deri p' have no zeros in common $[Q]$.

Solus:

(a) Sup p has m disti zeros. And $\deg p = m$. By [4.14], $\exists\, !\, c, \lambda_i \in \mathbf{R}, p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$.
If $m = 0$, then $p = c \neq 0 \Rightarrow p$ has no zeros, and $p' = 0$, we are done.
If $m = 1$, then $p(z) = c(z-\lambda_1)$, and $p' = c$ has no zeros, we are done.
For each $j \in \{1, \dots, m\}$, let $q_j \in \mathcal{P}_{m-1}(\mathbf{F})$ be suth $p(z) = (z-\lambda_j)q_j \Rightarrow q_j(\lambda_j) \neq 0$.
Now $p'(z) = (z-\lambda_j){q_j}'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

Or. To prove $[P] \Rightarrow [Q]$, we prove $\neg[Q] \Rightarrow \neg[P]$:
Sup $p(z) = (z-\lambda)q(z),\ p'(z) = (z-\lambda)r(z)$. 又 $p'(z) = (z-\lambda)q'(z) + q(z)$.
Now $p'(\lambda) = q(\lambda) = 0 \Rightarrow q(z) = (z-\lambda)s(z), p(z) = (z-\lambda)^2 s(z)$.
Hence p has strictly less than m disti zeros.

(b) To prove $[Q] \Rightarrow [P]$, we prove $\neg[P] \Rightarrow \neg[Q]$:
Becs nonzero $p \in \mathcal{P}_m(\mathbf{F})$, we sup $\lambda_1, \dots, \lambda_M$ are all the disti zeros of p, where $M < m$.
By Pigeon Hole Principle, $\exists\, \lambda_k$ suth $p(z) = (z-\lambda_k)^2 q(z)$ for some $q \in \mathcal{P}(\mathbf{F})$.
Hence $p'(z) = 2(z-\lambda_k)q(z) + (z-\lambda_k)^2 q'(z) \Rightarrow p'(\lambda_k) = 0 = p(\lambda_k)$. $\square$

---

7 *Provt every $p \in \mathcal{P}(\mathbf{R})$ of odd deg has a zero.*

Solus:

Using the nota and proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence λ_1 exists. $\square$

Or. Using calculus only. Sup $p \in \mathcal{P}_m(\mathbf{F}), \deg p = m, m$ is odd.
Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1}a_m$ by δ.
Write $p(x) = x^m\left(\dfrac{a_0}{x^m} + \dfrac{a_1}{x^{m-1}} + \cdots + \dfrac{a_{m-1}}{x} + a_m\right)$.
Thus $p(x)$ is continuous, and $\lim\limits_{x\to-\infty} p(x) = -\delta\infty$; $\quad \lim\limits_{x\to\infty} p(x) = \delta\infty$.
Hence we conclude that p has at least one real zero. $\square$

---

9 *Sup $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\bar{z})}$. Provt $q \in \mathcal{P}(\mathbf{R})$.*

Solus:

Notice that by [4.5], $\bar{z}^n = \overline{z^n}$.
Sup $q(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow q(\bar{z}) = a_n \bar{z}^n + \cdots + a_1 \bar{z} + a_0 \Rightarrow \overline{q(\bar{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\bar{z})} = \overline{p(\bar{z})}p(z) = \overline{p(\bar{z})\overline{\overline{p(\bar{\bar{z}})}}} = \overline{q(\bar{z})}$. Hence for each $a_k, \overline{a_k} = a_k \Rightarrow a_k \in \mathbf{R}$. $\square$

Or. Sup $p(z) = a_m z^m + \cdots + a_1 z + a_0$. Now $\overline{p(\bar{z})} = \overline{a_m} z^m + \cdots + \overline{a_1} z + \overline{a_0}$.
Notice that $q(z) = p(z)\overline{p(\bar{z})} = \sum_{k=0}^{2} m\Big(\sum_{i+j=k} a_i \overline{a_j}\Big) z^k$.
Notice that by [4.5], $z - \bar{z} = 2(\mathfrak{Im}\, z) \Rightarrow z = \bar{z} + 2(\mathfrak{Im}\, z)$. So that $z = \bar{z} \Longleftrightarrow \mathfrak{Im}\, z = 0 \Longleftrightarrow z \in \mathbf{R}$.
Now for each $k \in \{0, \dots, 2m\}$, $\overline{\sum_{i+j=k} a_i \overline{a_j}} = \sum_{i+j=k} \overline{a_i \overline{a_j}} = \sum_{i+j=k} a_j \overline{a_i} = \sum_{i+j=k} a_i \overline{a_j} \in \mathbf{R}$. $\square$

---

8 *For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \to \mathbf{R}$ by* $(Tp)(x) = \begin{cases} \dfrac{p(x)-p(3)}{x-3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}$

Shat (a) *$Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$ and that* (b) *$T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is linear.*

Solus:

(a) For $x \neq 3, T(x^n) = \dfrac{x^n - 3^n}{x-3} = \sum_{i=1}^{n} 3^{i-1}x^{n-i}$. For $x = 3, T(x^n) = 3^{n-1} \cdot n$.

Note that if $x = 3$, then $\sum_{i=1}^{n} 3^{i-1}x^{n-i} = \sum_{i=1}^{n} 3^{n-1} = 3^{n-1} \cdot n$.

Hence $T(x^n) = \sum_{i=1}^{n} 3^{i-1}x^{n-i} \Rightarrow T(x^n) \in \mathcal{P}(\mathbf{R})$.

(b) Now we shat T is linear: $\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}$,

$$T(p+\lambda q)(x) = \left\{ \begin{array}{ll} \dfrac{(p+\lambda q)(x) - (p+\lambda q)(3)}{x-3}, & \text{if } x \neq 3, \\ (p+\lambda q)'(3), & \text{if } x = 3 \end{array} \right| = \big[T(p) + \lambda T(q)\big](x) \text{ for all } x \in \mathbf{R}. \qquad \square$$

Or. (a) Note that $\exists!\, q \in \mathcal{P}(\mathbf{R}), p(x) - p(3) = (x-3)q(z) \Rightarrow q(x) = \dfrac{p(x)-p(3)}{x-3}$.

$p'(x) = \big(p(x) - p(3)\big)' = \big((x-3)q(x)\big)' = q(x) + (x-3)q'(x)$.

Hence $p'(3) = q(3)$. Now $Tp = q \in \mathcal{P}(\mathbf{R})$.

(b) $\forall p_1, p_2 \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, \exists!\, q_1, q_2 \in \mathcal{P}(\mathbf{R})$,

$p_1(x) - p_1(3) = (x-3)q_1(x)$ and $p_2(x) - p_2(3) = (x-3)q_2(x)$.

By (a), $Tp_1 = q_1, Tp_2 = q_2$. Note that $(p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x-3)(q_1 + \lambda q_2)(x)$.

Hence by the uniqnes of $q_1 + \lambda q_2$ for $p_1 + \lambda p_2$, we must have $T(p_1 + \lambda p_2) = q_1 + \lambda q_2$. $\square$

---

11 *Sup $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*

(a) *Shat* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) *Find a basis of $\mathcal{P}(\mathbf{F})/U$.*

Solus: Notice that $pq \neq p \circ q$, see (4E 3.A.10).

U is a subsp of $\mathcal{P}(\mathbf{F})$ becs $\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, ps_1 + \lambda ps_2 = p(s_1 + \lambda s_2) \in U$.

If $\deg p = 0$, then $U = \mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F})/U = \{0\}$, with the unique basis $()$. Sup $\deg p \geqslant 1$.

(a) By [4.8], $\forall s \in \mathcal{P}(\mathbf{F}), \exists!\, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F})\ \big[\exists!\, pq \in U\big], s = (p)q + (r)$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F})$. By the Note For [3.91] in (3.E), $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p-1}(\mathbf{F})$ are iso.

Or. Define $R : \mathcal{P}(\mathbf{F}) \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $R(s) = r$ for all $s \in \mathcal{P}(\mathbf{F})$ We shat R is linear.

$\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, \exists!\, r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q_1, q_2 \in \mathcal{P}(\mathbf{F}), s_1 = (p)q_1 + (r_1);\ s_2 = (p)q_2 + (r_2)$.

又 $\exists!\, r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}), (s_1 + \lambda s_2) = (p)q + (r) = (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2)$.

Note that $r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}) \Rightarrow r_1 + \lambda r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Or Note that $\deg(r_1 + \lambda r_2) \leqslant \max\{\deg r_1, \deg(\lambda r_2)\} \leqslant \max\{\deg r_1, \deg r_2\} < \deg p$.

By the uniqnes part of [4.8], $s = s_1 + \lambda s_2;\ r = r_1 + \lambda r_2$. Thus $R(s_1 + \lambda s_2) = R(s_1) + \lambda R(s_2)$.

Becs $Rs = 0 \iff s = pq, \exists!\, q \in \mathcal{P}(\mathbf{F}) \iff s \in U$. And $\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), Rr = r$.

Now null $R = U$, range $R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Hence $\tilde{R} : \mathcal{P}(\mathbf{F})/U \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ is defined by $\tilde{R}(s + U) = Rs$. By [3.91(d)], $\tilde{R}$ is iso.

(b) For each $k \in \{0, 1, \dots, \deg p - 1\}, \tilde{R}(z^k + U) = R(z^k) = z^k \Rightarrow \tilde{R}^{-1}(z^k) = z^k + U$.

Thus $(1 + U, z + U, \dots, z^{\deg p-1} + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. $\square$

---

10 *Sup $m \in \mathbf{N}, p \in \mathcal{P}_m(\mathbf{C})$ is suth $p(x_k) \in \mathbf{R}$ for each of disti $x_0, x_1, \dots, x_m \in \mathbf{R}$. Provt $p \in \mathcal{P}(\mathbf{R})$.*

Solus:

By Tips and Exe (5), $\exists\,!\, q \in \mathcal{P}_m(\mathbf{R})$ suth $q(x_k) = p(x_k)$. Hence $p = q$. $\square$

Or. Using the Lagrange Interpolating Polynomial.

Define $q(x) = \sum_{j=0}^{m} \dfrac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j)$.

又 Each x_j, $p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R})$. Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$ for each x_k.
Then $(q-p)$ has $(m+1)$ zeros, while $(q-p) \in \mathcal{P}_m(\mathbf{C})$. By Tips, $q - p = 0 \Rightarrow p = q \in \mathcal{P}(\mathbf{R})$. $\square$

- (4E 4 13) *Sup nonconst $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p$, $n = \deg q$. Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r,s) = rp + sq$. Provt T is iso.*
  Coro: *$\exists\,!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ suth $rp + sq = 1$.*

Solus:

T is linear becs $\forall\, r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,
$T\big((r_1,s_1) + \lambda(r_2,s_2)\big) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1,s_1) + \lambda T(r_2,s_2)$.

Let $\lambda_1, \dots, \lambda_M$ and $\mu_1, \dots, \mu_N$ be the disti zeros of p and q respectively. Notice that $M \leqslant m, N \leqslant n$.

Note that the ctrapos of [4.13], $M = 0 \iff m = 0 \Rightarrow s = 0 \iff r = 0 \Leftarrow n = 0 \iff N = 0$.

Now sup $M, N \geqslant 1$. We shat $s = 0$. Showing $r = 0$ is almost the same.

Write $p(z) = a(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}$. $\big(\, \exists\,!\, \alpha_j \geqslant 1, a \in \mathbf{F}. \,\big)$ Let $\max\{\alpha_1, \dots, \alpha_M\} = A$.

For each $D \in \{0, 1, \dots, A-1\}$, let $I_{D,\alpha} = \{\gamma_{D,1}, \dots, \gamma_{D,J}\}$ be suth each $\alpha_{\gamma_{D,j}} \geqslant D+1$.

Note that $I_{A-1,\alpha} \subseteq \cdots \subseteq I_{0,\alpha} = \{1, \dots, M\}$. Becs $rp + sq = 0 \Rightarrow (rp+sq)^{(k)} = 0$ for all $k \in \mathbf{N}^+$.

We use induction by D to shat $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$ for each $D \in \{0, \dots, A-1\}$.

Notice that $p^{(D)}(\lambda_\gamma) = 0$ for each $D \in \{0, \dots, A-1\}$ and each $\lambda_\gamma \in I_{D,\alpha}$. (Δ)

(i) $D = 0$. $(rp+sq)(\lambda_{\gamma_{0,j}}) = (sq)(\lambda_{\gamma_{0,j}}) = s(\lambda_{\gamma_{0,j}}) = 0$.
  $D = 1$. $(rp+sq)'(\lambda_{\gamma_{1,j}}) = (r'p + rp')(\lambda_{\gamma_{1,j}}) + (s'q + sq')(\lambda_{\gamma_{1,j}}) = (s'q)(\lambda_{\gamma_{1,j}}) = s'(\lambda_{\gamma_{1,j}}) = 0$.

(ii) $2 \leqslant D \leqslant A-1$. Asm $s^{(d)}(\lambda_{\gamma_{d,j}}) = 0$ for each $d \in \{1, \dots, D-1\}$ and each $\lambda_{\gamma_{d,j}} \in I_{d,\alpha}$.

$\Big($ Becs $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}.\, \Big)$ (Δ)

$$\begin{aligned} \text{Now } [rp+sq]^{(D)}(\lambda_{\gamma_{D,j}}) &= \big[C_D^D r^{(D)} p^{(0)} + \cdots + C_D^d r^{(d)} p^{(D-d)} + \cdots + C_D^0 r^{(0)} p^{(D)}\big](\lambda_{\gamma_{D,j}}) \\ &\quad + \big[C_D^D s^{(D)} q^{(0)} + \cdots + C_D^d s^{(d)} q^{(D-d)} + \cdots + C_D^0 s^{(0)} q^{(D)}\big](\lambda_{\gamma_{D,j}}) \\ &= \big[C_D^D s^{(D)} q^{(0)}\big](\lambda_{\gamma_{D,j}}). \text{ Where each } \lambda_{\gamma_{D,j}} \in I_{D,\alpha} \subseteq I_{D-1,\alpha}. \end{aligned}$$

Hence $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$. The asm holds for all $D \in \{0, \dots, A-1\}$.

Notice that $\forall k = \{0, \dots, A-2\}, s^{(k)}$ and $s^{(k+1)}$ have zeros $\{\lambda_{\gamma_{k+1,1}}, \dots, \lambda_{\gamma_{k+1,J}}\}$ in common.

Now $\forall D \in \{1, A-1\}, s = s^{(0)}, \dots, s^{(D)}$ have zeros $\{\lambda_{\gamma_{D,1}}, \dots, \lambda_{\gamma_{D,J}}\}$ in common.

Thus $\forall D \in \{0, A-1\}$, $s(z)$ is divisible by $(z - \lambda_{\gamma_{D,1}})^{\alpha_{\gamma_{D,1}}} \cdots (z - \lambda_{\gamma_{D,J}})^{\alpha_{\gamma_{D,J}}}$.

Hence we write $s(z) = \Big((z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}\Big) s_0(z)$, while $\deg s \leqslant m - 1 < m = \alpha_1 + \cdots + \alpha_M$.

Thus by Tips, $s = 0$. Following the same pattern, we conclude that $r = 0$.

Hence T is inje. And $\dim\big(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})\big) = \dim \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$ is surj. Thus T is iso. $\square$

COMMENT: We now prove the stam that marked by (Δ) above.

L1: *Provt* $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^{+}, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}$.

SOLUS:

We use induction by $k \in \mathbf{N}^{+}$.

(i) $k = 1$. $(pq)^{(1)} = pq = C_1^1 p^{(1)} q^{(0)} + C_1^0 p^{(0)} q^{(1)}$.

(ii) $k \geqslant 2$. Asm for $(pq)^{(k-1)} = C_{k-1}^{k-1} p^{(k-1)} q^{(0)} + \cdots + C_{k-1}^j p^{(j)} q^{(k-1-j)} + \cdots + C_{k-1}^0 p^{(0)} q^{(k-1)}$.

Now $(pq)^{(k)} = \left((pq)^{(k-1)}\right)' = \left(\sum_{j=0}^{k-1} C_{k-1}^j p^{(j)} q^{(k-j-1)}\right)' = \sum_{j=0}^{k-1} \left[C_{k-1}^j \left(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)}\right)\right]$.

$$\begin{aligned}
&= \left[C_{k-1}^0 \left(p^{(1)} q^{(k-1)} + \boxed{p^{(0)} q^{(k)}}\right)\right] + \left[C_{k-1}^1 \left(p^{(2)} q^{(k-2)} + p^{(1)} q^{(k-1)}\right)\right] \\
&\quad + \cdots + \left[C_{k-1}^{j-2} \left(p^{(j-1)} q^{(k-j+1)} + p^{(j-2)} q^{(k-j+2)}\right)\right] + \left[C_{k-1}^{j-1} \left(p^{(j)} q^{(k-j)} + p^{(j-1)} q^{(k-j+1)}\right)\right] \\
&\quad + \left[C_{k-1}^{j} \left(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)}\right)\right] + \left[C_{k-1}^{j+1} \left(p^{(j+2)} q^{(k-j-2)} + p^{(j+1)} q^{(k-j-1)}\right)\right] \\
&\quad + \cdots + \left[C_{k-1}^{k-2} \left(p^{(k-1)} q^{(1)} + p^{(k-2)} q^{(2)}\right)\right] + \left[C_{k-1}^{k-1} \left(\boxed{p^{(k)} q^{(0)}} + p^{(k-1)} q^{(1)}\right)\right].
\end{aligned}$$

Hence $(pq)^{(k)} = C_k^0 p^{(0)} q^{(k)} + \cdots + \left[C_{k-1}^j + C_{k-1}^{j-1}\right] \left(p^{(j)} q^{(k-j)}\right) + \cdots + C_k^k p^{(k)} q^{(0)}$. □

L2: *Sup* $p(z) = (z-\lambda)^{\alpha} q(z)$ *and* $\alpha \in \mathbf{N}^{+}$. *Provt* $p^{(\alpha-1)}(\lambda) = 0$.

SOLUS:

Sup $p \in \mathcal{P}(\mathbf{F})$. Write $p(z) = (z-\lambda)^{A} q(z)$, where $A \in \mathbf{N}^{+}, q(\lambda) \neq 0$.

We use induction to shat for all $\alpha \in \{1, \ldots, A\}, p^{(\alpha-1)}(\lambda) = 0$.

(i) $\alpha = 1$. $p^{(0)}(\lambda) = 0$.

(ii) $2 \leqslant \alpha \leqslant A$. Asm $p^{(a-2)}(\lambda) = 0$ for all $a \in \{1, \ldots, \alpha\}$.

NOTICE that $p(z) = (z-\lambda)^{\alpha-1} q_{\alpha-1}(z) = (z-\lambda)^{\alpha} q_{\alpha}(z)$, where $q_{\alpha}(z) = (z-\lambda) q_{\alpha-1}(z)$.

Becs $p^{(\alpha-1)}(z) = \Big[C_{\alpha-1}^{\alpha-1} (z-\lambda)^0 q_{\alpha-1}(z) + \cdots + C_{\alpha-1}^{k} (z-\lambda)^{\alpha-1-k} q_{\alpha-1-k}(z)$

$+ \cdots + C_{\alpha-1}^{0} (z-\lambda)^{\alpha-1} {q_{\alpha-1}}^{(\alpha-1)}(z)\Big]$. Now $p^{(\alpha-1)}(\lambda) = C_{\alpha-1}^{\alpha-1} q_{\alpha-1}(\lambda) = 0$. □

ENDED

5.A

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28
29 30 31 32 33 34 35 36 | 2E: Ch5.20 | 4E: 8 11 15 16 17 36 37 38 39

- **NOTE FOR [5.6]:**

More generally, sup we do not know whether V is finid. We shat $(a) \Longleftrightarrow (b)$.

Sup (a) λ is an eigval of T with an eigvec v. Then $(T - \lambda I)v = 0$.

Hence we get (b), $(T - \lambda I)$ is not inje. And then (d), $(T - \lambda I)$ is not inv.

But (d) $\Rightarrow$ (b) fails, becs S is not inv $\Longleftrightarrow$ S is not inje OR S is not surj.

- **TIPS:** For $T_1, \ldots, T_m \in \mathcal{L}(V)$:

(a) Sup $T_1, \ldots, T_m$ are all inje. Then $(T_1 \circ \cdots \circ T_m)$ is inje.

(b) Sup $(T_1 \circ \cdots \circ T_m)$ is not inje. Then at least one of $T_1, \ldots, T_m$ is not inje.

(c) At least one of $T_1, \ldots, T_m$ is not inje $\nRightarrow$ $(T_1 \circ \cdots \circ T_m)$ is not inje.

EXA: In infinid only. Let $V = \mathbf{F}^{\infty}$.

$$\left.\begin{array}{l}\text{Let } S \text{ be the backward shift (surj but not inje)} \\ \text{Let } T \text{ be the forward shift (inje but not surj)}\end{array}\right\} \Rightarrow \text{Then } ST = I.$$ □

• **NOTE FOR [5.2]:** Sup $T \in \mathcal{L}(V)$. Then U is invarsp of V under $T \iff \operatorname{range} T|_U \subseteq U$.

• *Sup V is finid, $T \in \mathcal{L}(V)$, and U is invarsp of V under T.*
Provt there exists invarsp W of dimension $\dim V - dimU$.

SOLUS:

Using the NOTE FOR [3.88,90,91]. Define the eraser S. Now $V = \operatorname{range} S \oplus U$.
Define E_1 by $E_1(u+w) = u$. Define E_2 by $E_2(u+w) = w$. $(\ E_2 = S \circ \pi.\ )$
Note that $T - TE_1 = T(I - E_1) = TE_2$. And $\operatorname{null} TE_2 = \operatorname{null} T \oplus U,\ \operatorname{range} T = \operatorname{range} TE_2 \oplus U$.
Becs $\dim \operatorname{null} TE_2 \geqslant \dim U \iff \dim \operatorname{range} TE_2 \leqslant \dim V - \dim U$.
Let $B_U = (u_1, \dots, u_n),\ B_{\operatorname{range} TE_2} = (v_1, \dots, v_m) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n, \dots, u_p)$.
Let $X = \operatorname{span}(v_1, \dots, v_m, u_{\alpha_1}, \dots, u_{\alpha_{p-\dim U}})$. Where $\alpha_1, \dots, \alpha_{p-\dim U} \in \{1, \dots, p\}$ are disti.
Then $\dim X = \dim V - \dim U$. $[\ \operatorname{range} TE_2 \subseteq\ ]$ X is invard TE_2, by Exe (1)(b).
We have $x \in X \Rightarrow TE_2(x) \in X \Rightarrow Tx - TE_1(x) \in X \Rightarrow Tx \in X$. Hence X is invard T. □
$($ Note that $E_1(x) \in \operatorname{span}(u_{\beta_1}, \dots, u_{\beta_t})$, where $\beta_1, \dots, \beta_t \in \{\alpha_1, \dots, \alpha_{p-\dim U}\}$ and each $u_{\beta_i} \in U.$ $)$
COMMENT: Conversely, by reversing the roles of U and W, we conclude that it is true as well.

• *Sup $T \in \mathcal{L}(V)$ and U is invarsp of V under T.*
Sup $\lambda_1, \dots, \lambda_m$ are the disti eigvals of T corres eigvecs $v_1, \dots, v_m$.

• **TIPS 1:** *Provt $v_1 + \cdots + v_m \in U \iff$ each $v_k \in U$.*

SOLUS:

Sup each $v_k \in U$. Then becs U is a subsp, $v_1 + \cdots + v_m \in U$.
Define the stam $P(k)$: if $v_1 + \cdots + v_k \in U$, then each $v_j \in U$. We use induction on m.
(i) For $k = 1$, $v_1 \in U$.
(ii) For $2 \leqslant k \leqslant m$. Asm $P(k-1)$ holds. Sup $v = v_1 + \cdots + v_k \in U$.
Then $Tv = \lambda_1 v_1 + \cdots + \lambda_k v_k \in U \Longrightarrow Tv - \lambda_k v = (\lambda_1 - \lambda_k)v_1 + \cdots + (\lambda_{k-1} - \lambda_k)v_{k-1} \in U$.
For each $j \in \{1, \dots, k-1\}, \lambda_j - \lambda_k \neq 0 \Rightarrow (\lambda_j - \lambda_k)v_j = v_j'$ is an eigvec of T corres λ_j.
By asm, each $v_j' \in U$. Thus $v_1, \dots, v_{k-1} \in U$. So that $v_k = v - v_1 - \cdots - v_{k-1} \in U$. □

• **TIPS 2:** *If* $\dim V = m$. *Provt $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m)$, where $E_k = \operatorname{span}(v_k)$.*

SOLUS:

Becs $V = E_1 \oplus \cdots \oplus E_m$. $\forall u \in U, \exists!\, e_j \in E_j, u = e_1 + \cdots + e_m$.
If $e_j \neq 0$, then e_j is an eigvec corres λ_j. Othws $e_j = 0 \in U$. By TIPS (1), each nonzero $e_j \in U$.
Thus $u \in (U \cap E_1) + \cdots + (U \cap E_m) = U$. Becs each $(U \cap E_j) \subseteq E_j$.
For each $k \in \{2, \dots, n\},\ \big((U \cap E_1) + \cdots + (U \cap E_{k-1})\big) \bigcap (U \cap E_k) \subseteq (E_1 + \cdots + E_{k-1}) \cap E_k = \{0\}$. □

• **TIPS 3:** *Sup W is a nonzero invarsp of V under T. If* $\dim V = m \geqslant 1$.
Provt $W = \operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ for some disti $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$.

SOLUS:

Each $\operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ is invard T.
By TIPS (2), $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m)$. Becs each $\dim E_k = 1$, $U \cap E_k = \{0\}$ or E_k.
There must be at least one k suth $E_k = U \cap E_k$, for if not, $U = \{0\}$ since $V = E_1 \oplus \cdots \oplus E_m$.
Let $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$ be all the disti indices for which $E_k = U \cap E_k$.
Thus $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m) = E_{\alpha_1} \oplus \cdots E_{\alpha_A} = \operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$. □

---

1 *Sup $T \in \mathcal{L}(V)$ and U is a subsp of V.*

(a) *If $U \subseteq \operatorname{null} T$, then U is invard T.* $\forall u \in U \subseteq \operatorname{null} T, Tu = 0 \in U$. $\square$

(b) *If $\operatorname{range} T \subseteq U$, then U is invard T.* $\forall u \in U, Tu \in \operatorname{range} T \subseteq U$. $\square$

• *Sup $S, T \in \mathcal{L}(V)$ are suth $ST = TS$.*

(a) *Provt $\operatorname{null}(T - \lambda I)$ is invard S for any $\lambda \in \mathbf{F}$.*

(b) *Provt $\operatorname{range}(T - \lambda I)$ is invard S for any $\lambda \in \mathbf{F}$.*

SOLUS:

Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.

(a) $(T - \lambda I)(v) = 0 \Rightarrow (T - \lambda I)(Sv) = \big(S(T - \lambda I)\big)(v) = 0$.

(b) $(T - \lambda I)(u) = v \in \operatorname{range}(T - \lambda I) \Rightarrow Sv = \big(S(T - \lambda I)\big)(u) = (T - \lambda I)(Su) \in \operatorname{range}(T - \lambda I)$. $\square$

• *Sup $S, T \in \mathcal{L}(V)$ are suth $ST = TS$.*

2 *Shat $W = \operatorname{null} T$ is invard S.* $\forall u \in W, Tu = 0 \Rightarrow STu = 0 = TSu \Rightarrow Su \in W$. $\square$

3 *Shat $U = \operatorname{range} T$ is invard S.* $\forall w \in U, \exists\, v \in V, Tv = w, TSv = STv = Sw \in U$. $\square$

• *Sup $T \in \mathcal{L}(V)$ and $V_1, \dots, V_m$ are invarsps of V under T.*

4 $\forall v_i \in V_i, Tv_i \in V_i \Rightarrow \forall v = v_1 + \cdots + v_m \in V_1 + \cdots + V_m, Tv = Tv_1 + \cdots + Tv_m \in V_1 + \cdots + V_m$. $\square$

5 $\forall v \in \bigcap\limits_{i=1}^{m} V_i, Tv \in V_i, \forall i \in \{1, \dots, m\} \Rightarrow Tv \in \bigcap\limits_{i=1}^{m} V_i$. Thus $\bigcap\limits_{i=1}^{m} V_i$ is invard T. $\square$

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6 *Sup U is invarsp of V under each $T \in \mathcal{L}(V)$. Shat $U = \{0\}$ or $U = V$.*

SOLUS: If $V = \{0\}$. Then we are done. Sup $V \neq \{0\}$. We show the ctrapos:

Sup $U \neq \{0\}$ and $U \neq V$. Provt $\exists\, T \in \mathcal{L}(V)$ suth U is not invard T.

Let W be suth $V = U \oplus W$. Define $T \in \mathcal{L}(V)$ by $T(u + w) = w$. $\square$

• **TIPS:** Sup $T \in \mathcal{L}(\mathbf{R}^2)$ is the counterclockwise rotation by the angle $\theta \in \mathbf{R}$.

Define $\mathcal{C} \in \mathcal{L}(\mathbf{R}^2, \mathbf{C})$ by $\mathcal{C}(a, b) = a + \mathrm{i}b = r(\cos\alpha + \mathrm{i}\sin\alpha) \Rightarrow a = r\cos\alpha, b = r\sin\alpha$, where $r = a^2 + b^2$.

Then $(\cos\theta + \mathrm{i}\sin\theta)(a + \mathrm{i}b) = r\big(\cos(\alpha + \theta) + \mathrm{i}\sin(\alpha + \theta)\big) = \mathcal{C}^{-1}T(a, b)$.

Hence $T(a, b) = \big(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta\big)$. Now $\mathcal{M}(T) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

EXA: OR **7** Sup $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find all eigvals of T.

NOTICE that $\mathcal{M}(T) = \begin{pmatrix} \cos 90^\circ & -3\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$. By [5.8](a), we conclude that T has no eigvals.

OR. Sup λ is an eigval with an eigvec (x, y). Then $(\lambda x, \lambda y) = (-3y, x) \Rightarrow -3y = \lambda^2 y \Rightarrow \lambda^2 = -3$.

$\big[$ Ignoring the possibility of $y = 0$, becs $x = 0 \iff y = 0$. $\big]$ $\square$

---

8 *Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigvals and eigvecs.*

SOLUS: Sup λ is an eigval with an eigvec (w, z). Then $z = \lambda w$ and $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, ignoring the possibility of $z = 0$ ($z = 0 \iff w = 0$).

Hence $\lambda_1 = -1$ and $\lambda_2 = 1$ are all the eigvals of T. And $T(z, z) = (z, z), T(z, -z) = (-z, z)$.

又 $\dim \mathbf{F}^2 = 2$. Thus the set of all eigvecs is $\{(z, z), (z, -z) : z \neq 0\}$. $\square$

---

9 *Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigvals and eigvecs.*

SOLUS: Sup λ is an eigval with an eigvec (z_1, z_2, z_3).
Then $(2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$. We discuss in two cases:
For $\lambda = 0$, $z_2 = z_3 = 0$ and z_1 can be arb ($z_1 \neq 0$).
For $\lambda \neq 0$, $z_2 = 0 = z_1$, and z_3 can be arb ($z_3 \neq 0$), then $\lambda = 5$.
The set of all eigvecs is $\{(0, 0, w), (w, 0, 0) : w \neq 0\}$. □

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10 *Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$*
(a) *Find all eigvals and eigvecs;* (b) *Find all invarsps of V under T.*

SOLUS:
(a) Sup $x = (x_1, x_2, x_3, \dots, x_n)$ is an eigvec with an eigval λ.
Then $Tx = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$.
Hence $1, \dots, n$ of len $\dim \mathbf{F}^n$ are all the eigvals.
And $\{(0, \dots, 0, x_k, 0, \dots, 0) \in \mathbf{F}^n : x_k \neq 0, k = 1, \dots, n\}$ is the set of all eigvecs.
(b) Let $(e_1, \dots, e_n)$ be the std basis of $\mathbf{F}^n$. Let $V_k = \operatorname{span}(e_k)$. Then $V_1, \dots, V_n$ are invard T.
Hence by TIPS (3), every sum of $V_1, \dots, V_n$ is a invarsp of V under T. □

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18 *Define the forward shift optor $T \in \mathcal{L}(\mathbf{F}^\infty)$ by $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$.*
Shat T has no eigvals.

SOLUS: Sup λ is an eigval of T with an eigvec $(z_1, z_2, \dots)$.
Then $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$. Thus $\lambda z_1 = 0, \lambda z_k = z_{k-1}$.
If $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \dots = z_k \Rightarrow (z_1, z_2, \dots) = 0 \Longrightarrow 0$ is not an eigval.
If $\lambda \neq 0$, then $\lambda z_1 = 0 \Rightarrow z_1 = \dots = z_k = 0 \Longrightarrow \lambda$ is not an eigval. Now no $\lambda \in \mathbf{F}$ is an eigval. □

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19 *Sup $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$.*
In other words, the ent of $\mathcal{M}(T)$ with resp to the std basis are all 1's.
Find all eigvals and eigvecs of T.

SOLUS:
Sup λ is an eigval of T with an eigvec $(x_1, \dots, x_n)$.
Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$.
Thus $\lambda x_1 = \dots = \lambda x_n = x_1 + \dots + x_n$.

$$\left.\begin{array}{r} \text{For } \lambda = 0, x_1 + \dots + x_n = 0 \\ \text{For } \lambda \neq 0, x_1 = \dots = x_n \Longrightarrow \lambda x_k = n x_k \end{array}\right\} \Rightarrow 0, n \text{ are the eigvals of } T.$$

And the set of all eigvecs of T is $\{(x_1, \dots, x_n) \in \mathbf{F}^n \setminus \{0\} : x_1 + \dots + x_n = 0 \vee x_1 = \dots = x_n\}$. □

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20 *Define the backward shift optor $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.*
(a) *Shat every ele of $\mathbf{F}$ is an eigval of S;* (b) *Find all eigvecs of S.*

SOLUS:
Sup λ is an eigval of S with an eigvec $(z_1, z_2, \dots)$.
Then $S(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$. Thus for each $k \in \mathbf{N}^+, \lambda z_k = z_{k+1}$.
If $\lambda = 0$, then $\lambda z_1 = z_2 = \dots = z_k = 0$ for all k, while z_1 can be nonzero. Thus 0 is an eigval.
If $\lambda \neq 0$, then $\lambda^k z_1 = \lambda^{k-1} z_2 = \dots = \lambda z_k = z_{k+1}$, let $z_1 \neq 0 \Longrightarrow (1, \lambda, \lambda^2, \dots, \lambda^k, \dots)$ is an eigvec.
Now each $\lambda \in \mathbf{F}$ is an eigval of T, with corres eigvecs in $\operatorname{span}((1, \lambda, \lambda^2, \dots, \lambda^k, \dots))$. □

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11 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigvals and eigvecs.*

Solus:

Note that $\forall p \in \mathcal{P}(\mathbf{R}) \setminus \{0\}, \deg p' < \deg p$. And $\deg 0 = -\infty$. Sup λ is an eigval with an eigvec p.

Asm $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p = p'$. Ctradic. Thus $\lambda = 0$.

Therefore $\deg \lambda p = -\infty = \deg p' \Rightarrow p \in \mathcal{P}_0(\mathbf{R})$. Hence the eigvecs are all the nonzero consts. $\square$

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12 *Define $T \in \mathcal{L}\big(\mathcal{P}_n(\mathbf{R})\big)$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigvals and eigvecs.*

Solus:

Sup λ is an eigval of T with an eigvec p, then $(Tp)(x) = xp'(x) = \lambda p(x)$.

Let $p = a_0 + a_1 x + \cdots + a_n x^n$. Then $xp'(x) = a_1 x + 2a_2 x^2 + \cdots + n a_n x^n = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n$.

Define $S \in \mathcal{L}(\mathbf{F}^{n+1}, \mathcal{P}_n(\mathbf{R}))$ by $S(a_0, a_1, \dots, a_n) = a_0 + a_1 x + \cdots + a_n x^n$.

Then $(S^{-1}TS)(a_0, a_1, \dots, a_n) = (0 \cdot a_0, 1 \cdot a_1, 2 \cdot a_2, \dots, n \cdot a_n)$. Thus $0, 1, \dots, n$ are the eigvals of $S^{-1}TS$.

By Exe (15), $0, 1, \dots, n$ are the eigvals of T. The set of all eigvecs is $\{cx^\lambda : c \neq 0, \lambda = 0, 1, \dots, n\}$. $\square$

• *Sup V is finid, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$.*

13 *Provt $\forall \lambda \in \mathbf{F}, \exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}, (T - \alpha I)$ is inv.*

Solus:

Let $\alpha_k \in \mathbf{F}$ be suth $|\alpha_k - \lambda| = \frac{1}{1000+k}$ for each $k = 1, \dots, \dim V + 1$.

Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ disti eigvals.

Hence $\exists\, k = 1, \dots, \dim V + 1$ suth α_k is not an eigval of T and therefore $(T - \alpha_k I)$ is inv. $\square$

• (4E 5.A.11) *Provt $\exists\, \delta > 0$ suth $(T - \alpha I)$ is inv for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$.*

Solus:

If T has no eigvals, then $(T - \alpha I)$ is inje for all $\alpha \in \mathbf{F}$ and we are done.

Sup $\lambda_1, \dots, \lambda_m$ are all the disti eigvals of T.

Let $\delta > 0$ be suth, for each eigval λ_k, $\lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.

So that for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$, $(T - \alpha I)$ is not inje. $\square$

Or. Let $\delta = \min\{|\lambda - \lambda_k| : k \in \{1, \dots, m\}, \lambda_k \neq \lambda\}$.

Then $\delta > 0$ and each $\lambda_k \neq \alpha$ $[\iff (T - \alpha I)$ is inv $]$ for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$. $\square$

• (5.B.4 Or 4E 3.B.27) *Sup λ is an eigval of $P \in \mathcal{L}(V), P^2 = P$. Provt $\lambda = 0$ or $\lambda = 1$.*

Solus: Sup λ is an eigval with an eigvec v. Then $P(Pv) = Pv \Rightarrow \lambda^2 v = \lambda v$. Thus $\lambda = 1$ or 0. $\square$

---

14 *Sup $V = U \oplus W$, where U and W are nonzero subsps of V.*
Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.
Find all eigvals and eigvecs of P.

Solus:

Sup λ is an eigval of P with an eigvec $(u + w)$.

Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$.

Or. Note that $P|_{\operatorname{range} P} = I|_{\operatorname{range} P} \iff P^2 = P$. By (4E 5.A.8), 1 and 0 are the eigvals.

By [1.44], $(\lambda - 1)u = \lambda w = 0$, hence $\lambda = 0 \iff u = 0$, and $\lambda = 1 \iff w = 0$.

Thus $Pu = u, Pw = 0$. Hence the eigvals are 0 and 1, the set of all eigvecs of P is $U \cup W$. $\square$

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15 *Sup $T \in \mathcal{L}(V)$. Sup $S \in \mathcal{L}(V)$ is inv.*

(a) *Provt T and $S^{-1}TS$ have the same eigvals.*

(b) *What is the relationship between the eigvecs of T and the eigvecs of $S^{-1}TS$?*

Solus:

(a) λ is an eigval of T with an eigvec $v \Rightarrow S^{-1}TS(\underline{S^{-1}v}) = S^{-1}Tv = S^{-1}(\lambda v) = \underline{\lambda S^{-1}v}$.

λ is an eigval of $S^{-1}TS$ with an eigvec $v \Rightarrow S(S^{-1}TS)v = T\underline{Sv} = \underline{\lambda Sv}$.

Or. Note that $S(S^{-1}TS)S^{-1} = T$. Hence every eigval of $S^{-1}TS$ is an eigval of $S(S^{-1}TS)S^{-1} = T$.

Or. $Tv = \lambda v \iff (TS)(u) = \lambda Su \iff (S^{-1}TS)(u) = \lambda u$. Where $v = Su$.

$(S^{-1}TS)(u) = \lambda u \iff (S^{-1}T)(v) = \lambda S^{-1}v \iff Tv = \lambda v$. Where $u = S^{-1}v$.

(b) Becs λ is an eigval of $T \iff \lambda$ is an eigval of $S^{-1}TS$.

(See [5.36].) Now $E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\}$; $E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}$. ☐

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17 *Give an exa of an optor on $\mathbf{R}^4$ that has no real eigvals.*

Solus:

Let (e_1, e_2, e_3, e_4) be the std basis of $\mathbf{R}^4$.

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$.

Sup λ is an eigval of T with an eigvec (x, y, z, w). Then we get $\begin{cases} (1-\lambda)x + y + z + w = 0, \\ -x + (1-\lambda)y - z - w = 0, \\ 3x + 8y + (11-\lambda)z + 5w = 0, \\ 3x - 8y - 11z + (5-\lambda)w = 0. \end{cases}$

This set of linear equations has no solutions.

[You can type it on `https://zh.numberempire.com/equationsolver.php` to check.]

Or. Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Sup λ is an eigval of T with an eigvec (x, y, z, w).

Then $T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \implies \begin{cases} -y = \lambda x, x = \lambda y \implies -xy = \lambda^2 xy \\ -w = \lambda z, z = \lambda w \implies -zw = \lambda^2 zw \end{cases}$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Othws, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, ctradic.

Simlr, $y = z = w = 0$. Then we fail. Thus T has no eigvals. ☐

• (4E 5.A.16) *Sup $B_V = (v_1, \dots, v_n)$, $T \in \mathcal{L}(V)$, $\mathcal{M}(T, (v_1, \dots, v_n)) = A$.*
Provt if λ is an eigval of T, then $|\lambda| \leqslant n \max\{|A_{j,k}| : 1 \leqslant j, k \leqslant n\}$.

Solus:

Sup v is an eigval of T corres to λ. Let $v = c_1 v_1 + \cdots + c_n v_n$.

Becs $\lambda c_1 v_1 + \cdots + \lambda c_n v_n = c_1 T v_1 + \cdots + c_n T v_n = \sum_k^n c_k \left(\sum_j^n A_{j,k} v_j \right)$.

We have $\lambda c_j = \sum_{k=1}^n c_k A_{j,k} \implies |\lambda||c_j| = \sum_{k=1}^n |c_k||A_{j,k}|$ for each $j \in \{1, \dots, n\}$

Let $|c_j| = \max\{|c_1|, \dots, |c_n|\}$. Note that $|c_j| \neq 0$, for if not, $c_1 = \cdots = c_n = 0 \Rightarrow v = 0$, ctradic.

Let $M = \max\{|A_{j,k}| : 1 \leqslant j, k \leqslant n\}$. Note that for each j, $\sum_{k=1}^n |A_{j,k}| \leqslant \sum_{k=1}^n M = nM$.

Thus $|\lambda||c_j| = \sum_{k=1}^n |c_k||A_{j,k}| \implies |\lambda| \leqslant \sum_{k=1}^n |A_{j,k}| \dfrac{|c_k|}{|c_j|} \leqslant \sum_{k=1}^n |A_{j,k}| \leqslant nM$. ☐

• (4E 5.A.15) *Sup* $T \in \mathcal{L}(V)$, *and* $\lambda \in \mathbf{F}$.
Shat λ *is an eigval of* $T \iff \lambda$ *is an eigval of the dual optor* $T' \in \mathcal{L}(V')$.

Solus:

(a) Sup λ is an eigval of T with an eigvec v.
Let U be invar suth $V = \operatorname{span}(v) \oplus U$ $\big[$ by (4E 5.A.39) $\big]$.
Define $\psi \in V'$ by $\psi(cv + u) = c$.
Now $\big[T'(\psi)\big](cv+u) = \psi(cv + Tu) = \lambda cv = \lambda\psi(cv+u)$. Hence $T'(\psi) = \lambda\psi$.

(b) Sup λ is an eigval T' with an eigvec ψ. Then $T'(\psi) = \psi \circ T = \lambda\psi$.
Note that $\psi \neq 0$, $\psi(Tv) = \lambda\psi(v)$ Thus $\exists\, v \in V\backslash\{0\}, Tv = \frac{\psi(Tv)}{\psi(v)}v = \lambda v$. $\square$

Or. [*Only in Finid*] Using [5.6], (4E 3.F.17), [3.101] and (3.F.12).
λ is an eigval of $T \iff (T - \lambda I_V)$ is not inv
$\iff (T-\lambda I_V)' = T' - \lambda I_{V'}$ is not inv $\iff \lambda$ is an eigval of T'. $\square$

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24 *Sup* $A \in \mathbf{F}^{n,n}$. *Define* $T \in \mathcal{L}(\mathbf{F}^{n,1})$ *by* $Tx = Ax$.
(a) *Sup the sum of the ent in each row of* A *equals* 1. *Provt* 1 *is an eigval of* T.
(b) *Sup the sum of the ent in each col of* A *equals* 1. *Provt* 1 *is an eigval of* T.

Solus:

Sup λ is an eigval of T with an eigvec x. Then $Tx = Ax = \begin{pmatrix} \sum_{k=1}^n A_{1,k}x_k \\ \vdots \\ \sum_{k=1}^n A_{n,k}x_k \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

(a) Sup $\sum_{r=1}^n A_{R,c} = 1$ for each $R \in \{1,\dots,n\}$.
Then if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, and hence is an eigval of T.

(b) Sup $\sum_{r=1}^n A_{r,C} = 1$ for each $C \in \{1,\dots,n\}$.
Then $\sum_{r=1}^n (Ax)_{r,\cdot} = \sum_{r=1}^n (Ax)_{r,1} = \sum_{c=1}^n (A_{1,c} + \cdots + A_{n,c})x_c = \sum_{c=1}^n x_c = \lambda(x_1 + \cdots + x_n)$.
Hence $\lambda = 1$ for all $x \in \mathbf{F}^{n,1}$ suth $\sum_{c=1}^n x_{c,1} \neq 0$. $\square$

Or. We shat $(T - I)$ is not inv, so that $\lambda = 1$ is an eigval.
Becs $(T-I)x = (A-I)x = \begin{pmatrix} \sum_{r=1}^n A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^n A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.
Then $y_1 + \cdots + y_n = \sum_{r=1}^n \sum_{c=1}^n (A_{r,c}x_c - x_r) = \sum_{c=1}^n x_c \sum_{r=1}^n A_{r,c} - \sum_{r=1}^n x_r = 0$.
Thus range $(T-I) \subseteq \Big\{ \big(y_1 \ \cdots \ y_n\big)^t \in \mathbf{F}^{n,1} : y_1 + \cdots + y_n = 0 \Big\}$. Hence $(T - I)$ is not surj. $\square$

Or. Let $(e_1,\dots,e_n)$ be the std basis of $\mathbf{F}^{n,1}$. Define $\psi \in (\mathbf{F}^{n,1})'$ by $\psi(e_k) = 1$.
Thus $\big(\psi \circ (T-I)\big)(e_k) = \psi\Big(\big(\sum_{j=1}^n A_{j,k}e_j\big) - e_k\Big) = \big(\sum_{j=1}^n A_{j,k}\big) - 1 = 0$.
Which means that $\psi \circ (T - I) = 0$. 又 $\psi \neq 0$. Hence $(T-I)$ is not inje. $\square$

Or. Define $S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Sx = A^t x$. Becs the rows of A^t are the cols of A.
Now by (a), 1 is an eigval of S. Let $(\varphi_1,\dots,\varphi_n)$ be the dual basis of $(e_1,\dots,e_n)$.
Define $\Phi \in \mathcal{L}\Big(\mathbf{F}^{n,1}, (\mathbf{F}^{1,n})'\Big)$ by $\Phi(e_k) = \varphi_k$. Note that $\mathcal{M}(T') = A^t$.
Now $(\Phi^{-1}T'\Phi)(e_k) = (\Phi^{-1}T')(\varphi_k) = \Phi^{-1}\big(\sum_{j=1}^n A_{k,j}\varphi_j\big) = \sum_{j=1}^n A_{k,j}e_j = A^t e_k = Se_k$.
Thus 1 is an eigval of $S = \Phi^{-1}T'\Phi$, so of T', $\big[$ by Exe (15) $\big]$, so of T, $\big[$ by (4E 5.A.15) $\big]$. $\square$

- *Sup $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Tx = xA$.*
  (a) *Sup the sum of the ent in each col of A equals 1. Provt 1 is an eigval of T.*
  (b) *Sup the sum of the ent in each row of A equals 1. Provt 1 is an eigval of T.*

Solus:

Sup λ is an eigval with an eigvec x. Then $\left(\sum_{r=1}^{n} x_r A_{r,1} \quad \cdots \quad \sum_{r=1}^{n} x_r A_{r,n}\right) = \lambda\left(x_1 \quad \cdots \quad x_n\right)$.

(a) Sup $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C \in \{1, \dots, n\}$.
Thus if $x_1 = \cdots = x_n$, then $\lambda = 1$, hence is an eigval of T.

(b) Sup $\sum_{c=1}^{n} A_{R,c} = 1$ for each $R \in \{1, \dots, n\}$.
Thus $\sum_{c=1}^{n} (xA)_{\cdot,c} = \sum_{c=1}^{n} (A_{c,1} + \cdots + A_{c,n}) x_c = \sum_{c=1}^{n} x_c = \lambda(x_1 + \cdots + x_n)$.
Hence $\lambda = 1$, for all x suth $\sum_{r=1}^{n} x_{1,r} \neq 0$. ☐

Or. We shat $(T - I)$ is not inv, so that $\lambda = 1$ is an eigval.
Becs $(T - I)x = x(A - \mathcal{M}(I)) = \left(\sum_{c=1}^{n} x_c A_{c,1} - x_1 \quad \cdots \quad \sum_{c=1}^{n} x_c A_{c,n} - x_n\right) = \left(y_1 \quad \cdots \quad y_n\right)$.
Then $y_1 + \cdots + y_n = \sum_{c=1}^{n} \sum_{r=1}^{n} (x_r A_{r,c} - x_c) = \sum_{r=1}^{n} x_r \sum_{c=1}^{n} A_{r,c} - \sum_{c=1}^{n} x_c = 0$.
Thus range $(T - I) \subseteq \left\{ \left(y_1 \quad \cdots \quad y_n\right) \in \mathbf{F}^{1,n} : y_1 + \cdots + y_n = 0 \right\}$. Hence $(T - I)$ is not surj. ☐

Or. Let $(e_1, \dots, e_n)$ be the std basis of $\mathbf{F}^{1,n}$. Define $\psi \in (\mathbf{F}^{n,1})'$ by $\psi(e_k) = 1$.
Becs $Te_k = e_k A = \left(A_{k,1} \quad \cdots \quad A_{k,n}\right) = \sum_{i=1}^{n} A_{k,i} e_i$. **Coro:** $\mathcal{M}(T) = A^t$.
$\left(\psi \circ (T - I)\right)(e_k) = \left(\sum_{i=1}^{n} A_{k,i}\right) - 1 = 0$. Then $\psi \circ (T - I) = 0$. 又 $\psi \neq 0$. $(T - I)$ is not inje. ☐

Or. Define $S \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Sx = xA^t$. Becs the rows of A are the cols of A^t.
Now by (a), 1 is an eigval of S. Let $(\varphi_1, \dots, \varphi_n)$ be the dual basis of $(e_1, \dots, e_n)$.
Define $\Phi \in \mathcal{L}\left(\mathbf{F}^{1,n}, (\mathbf{F}^{1,n})'\right)$ by $\Phi(e_k) = \varphi_k$. Becs $\left[T'(\varphi_k)\right](e_j) = \varphi_k\left(\sum_{i=1}^{n} A_{j,i} e_i\right) = A_{j,k}$.
By (3.F.9), $T'(\varphi_k) = \sum_{j=1}^{n} A_{j,k} \varphi_j$. **Coro:** $\mathcal{M}(T') = A = \mathcal{M}(T)^t$. FIXME: $\mathcal{M}(T) e_k = A^t e_k = e_k A$
Now $(\Phi^{-1} T' \Phi)(e_k) = (\Phi^{-1} T')(\varphi_k) = \Phi^{-1}\left(\sum_{j=1}^{n} A_{j,k} \varphi_j\right) = \sum_{j=1}^{n} A_{j,k} e_j = e_k A^t = S e_k$.
Thus 1 is an eigval of $S = \Phi^{-1} T' \Phi$, so of T', [by Exe (15)], so of T, [by (4E 5.A.15)]. ☐

- *Sup $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.*
  (a) [Or (9.11)] *$\lambda \in \mathbf{R}$. Provt λ is an eigval of $T \iff \lambda$ is an eigval of $T_{\mathbf{C}}$.*
  (b) [Or **16** Or [9.16]] *$\lambda \in \mathbf{C}$. Provt λ is an eigval of $T_{\mathbf{C}} \iff \overline{\lambda}$ is an eigval of $T_{\mathbf{C}}$.*

Solus:

(a) Sup λ is an eigval of T with an eigvec v.
Then $Tv = \lambda v \implies T_{\mathbf{C}}(v + \mathrm{i}0) = Tv + \mathrm{i}T0 = \lambda v$. Thus λ is an eigval of $T_{\mathbf{C}}$.
Sup λ is an eigval of $T_{\mathbf{C}}$ with an eigvec $v + \mathrm{i}u$.
Then $T_{\mathbf{C}}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \implies Tv = \lambda v, Tu = \lambda u$. Thus λ is an eigval of T.
(Note that $v + \mathrm{i}u$ is nonzero $\iff$ at least one of v, u is nonzero).

(b) Sup λ is an eigval of $T_{\mathbf{C}}$ with an eigvec $v + \mathrm{i}u$. Then $T_{\mathbf{C}}(v + \mathrm{i}u) = Tv + \mathrm{i}Tu = \lambda(v + \mathrm{i}u)$.
Note that $\overline{T_{\mathbf{C}}(v + \mathrm{i}u)} = \overline{Tv + \mathrm{i}Tu} = Tv - \mathrm{i}Tu = T_{\mathbf{C}}(v - \mathrm{i}u) = T_{\mathbf{C}}(\overline{v + \mathrm{i}u})$.
And that $\overline{\lambda(v + \mathrm{i}u)} = \overline{\lambda}v - \mathrm{i}\overline{\lambda}u = \overline{\lambda}(v - \mathrm{i}u) = \overline{\lambda}(\overline{v + \mathrm{i}u})$.
Hence $\overline{\lambda}$ is an eigval of $T_{\mathbf{C}}$. To prove the other direction, notice that $\overline{\overline{\lambda}} = \lambda$. ☐

Or. Sup $\lambda = a + ib$ is an eigval of $T_{\mathbf{C}}$ with an eigvec $v + \mathrm{i}u$.
Becs $T_{\mathbf{C}}(v + \mathrm{i}u) = \lambda(v + \mathrm{i}u) = (av - bu) + \mathrm{i}(au + bv) = Tv + \mathrm{i}Tu \implies Tv = av - bu, Tu = au + bv$.
Now $T_{\mathbf{C}}(\overline{v + \mathrm{i}u}) = Tv - \mathrm{i}Tu = (av - bu) - \mathrm{i}(au + bv) = (a - \mathrm{i}b)(v - \mathrm{i}u) = \overline{\lambda}(\overline{v - \mathrm{i}u})$. Simlr. ☐

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21 *Sup $T \in \mathcal{L}(V)$ is inv.*

(a) *Sup $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Provt λ is an eigval of $T \iff \lambda^{-1}$ is an eigval of T^{-1}.*

(b) *Provt T and T^{-1} have the same eigvecs.*

Solus: (a) $Tv = \lambda v \iff v = \lambda T^{-1} v \iff \lambda^{-1} v = T^{-1} v$. Where $v \neq 0$.

(b) Notice that T is inv $\implies 0$ is not an eigval of T or T^{-1}. By (a), immediately. ☐

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22 *Sup $T \in \mathcal{L}(V)$ and $\exists$ nonzero vecs u, w in V suth $Tu = 3w,\ Tw = 3u$.*
Provt 3 or -3 is an eigval of T.

Solus: $T(u+w) = 3(u+w),\ T(u-w) = 3(w-u) = -3(u-w)$. Note that $u - w \neq 0$ or $u + w \neq 0$.

Or. $T(Tu) = 9u \Rightarrow T^2 - 9 = (T-3I)(T+3I)$ is not injective $\Rightarrow$ 3 or -3 is an eigval. ☐

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23 *Sup $S, T \in \mathcal{L}(V)$. Provt ST and TS have the same eigvals.*

Solus: Sup λ is an eigval of ST with an eigvec v. Then $T(STv) = \lambda Tv = TS(Tv)$.

If $Tv = 0$ (while $v \neq 0$), then T is not inje $\Rightarrow (TS - 0I)$ and $(ST - 0I)$ are not inje.

Thus $\lambda = 0$ is an eigval of ST and TS with the same eigvec v.

Othws, $Tv \neq 0$, then λ is an eigval of TS. Reversing the roles of T and S. ☐

• (2E 20) *Sup $T \in \mathcal{L}(V)$ has $\dim V$ disti eigvals and $S \in \mathcal{L}(V)$ has the same eigvecs (but might not with the same eigvals). Provt $ST = TS$.*

Solus: Let $n = \dim V$. For each $j \in \{1, \dots, n\}$, let v_j be an eigvec with eigval λ_j of T and α_j of S.

Then $B_V = (v_1, \dots, v_n)$. Becs $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$ for each j. Hence $ST = TS$. ☐

• (4E 5.A.37) *Sup V is finid and $T \in \mathcal{L}(V)$.*
Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(S) = TS$ for each $S \in \mathcal{L}(V)$.
Provt the set of eigvals of T equals the set of eigvals of $\mathcal{A}$.

Solus:

(a) Sup λ is an eigval of T with an eigvec $v = v_1$. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.

Note that $\operatorname{span}(v) \subseteq \operatorname{null}(T - \lambda I)$. Define $S \in \mathcal{L}(V)$ by $S(v_j) = v$ for each $j \in \{1, \dots, n\}$.

Or. Define $S \in \mathcal{L}(V)$ by $Sv_1 = v_1,\ Sv_j = 0$ for $j \geqslant 2$. Then $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$.

Then $(T - \lambda I)S = 0$. Thus $\mathcal{A}(S) = TS = \lambda S$ while $S \neq 0$. Hence λ is an eigval of $\mathcal{A}$.

(b) Sup λ is an eigval of $\mathcal{A}$ with an eigvec S.

Then $\exists\, v \in V, 0 \neq u = S(v) \in V \Rightarrow Tu = (TS)v = (\lambda S)v = \lambda u$. Thus λ is an eigval T.

Or. Becs $TS - \lambda S = (T - \lambda I)S = 0 \Rightarrow \{0\} \subsetneq \operatorname{range} S \subseteq \operatorname{null}(T - \lambda I)$. $(T - \lambda I)$ is not inje. ☐

Comment: If $\mathcal{A}(S) = ST, \forall S \in \mathcal{L}(V)$. Then the eigvals of $\mathcal{A}$ are not the eigvals of T.

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25 *Sup $T \in \mathcal{L}(V)$ and u, w are eigvecs of T suth $u + w$ is also an eigvec of T.*
Provt u and w corres to the same eigval.

Solus: Sup $\lambda_1, \lambda_2, \lambda_0$ are eigvals of T with eigvecs to $u, w, u + w$ respectively.

Then $T(u+w) = \lambda_0(u+w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$.

If (u, w) is linely depe, then let $w = cu$, therefore $\lambda_2 cu = Tw = cTu = \lambda_1 cu \Rightarrow \lambda_2 = \lambda_1$.

Othws, (u, w) is linely inde. Then $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. ☐

Or. Asm $\lambda_1 \neq \lambda_2$. Then (u, w) is linely inde. Thus $\lambda_0 - \lambda_1 = \lambda_0 - \lambda_2$. Ctradic. ☐

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26 *Sup $T \in \mathcal{L}(V)$ is suth every nonzero vec in V is an eigvec of T.*
Provt T is a scalar multi of the id optor.

Solus: If $\dim V = 0, 1$ then we are done. Sup $\dim V \geqslant 2$.
Becs $\forall v \in V, \exists!\lambda_v \in \mathbf{F}, Tv = \lambda_v v$. For any two disti nonzero vecs $v, w \in V$,
$T(v+w) = \lambda_{v+w}(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$. □

Or. For any two nonzero vecs $u, v \in V$, u, v are eigvecs.
If $u + v \neq 0$, then $u + v$ is also an eigvec. Othws, $u + v = 0$, then $Tu = -Tv = \lambda u = -\lambda v$.
Thus by Exe (25), $\forall u, v \in V, Tu = \lambda u, Tv = \lambda v \Rightarrow \forall v \in V, Tv = \lambda v$. □

27, 28 *Sup V is finid and $k \in \{1, \dots, \dim V - 1\}$.*
Sup $T \in \mathcal{L}(V)$ is suth every subsp of V of dim k is invard T.
Provt T is a scalar multi of the id optor.

Solus: If $\dim V \leqslant 1$ then we are done. Sup $\dim V \geqslant 2$.
We prove the ctrapos: *If T is not a scalar multi of I. Then $\exists$ subsp U of dim k not invard T.*
By Exe (26), $\exists\, v \in V$ and $v \neq 0$ suth v is not an eigvec of T.
Thus (v, Tv) is linely inde. Extend to $B_V = (v, Tv, u_1, \dots, u_n)$.
Let $U = \operatorname{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$ is not invarsp of V under T. □

Or. Sup $0 \neq v = v_1 \in V$. Extend to $B_V = (v_1, \dots, v_n)$. Sup $Tv_1 = c_1 v_1 + \cdots + c_n v_n, \exists!\, c_i \in \mathbf{F}$.
Consider a k-dim subsp $U = \operatorname{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$. Where $\alpha_1, \dots, \alpha_{k-1} \in \{2, \dots, n\}$ are disti.
Becs every subsp such U is invar. $Tv_1 = c_1 v_1 + \cdots + c_n v_n \in U \Longrightarrow c_2 = \cdots = c_n = 0$.
For if not, $\exists\, c_i \neq 0$, let $W = \operatorname{span}(v_1, v_{\beta_1}, \dots, v_{\beta_{k-1}})$, where each $\beta_j \in \{2, \dots, i-1, i+1, \dots, n\}$.
Hence $Tv_1 = c_1 v_1$. Becs $v_1 = v \in V$ is arb. We conclude that $T = \lambda I$ for some $\lambda \in \mathbf{F}$. □

Or. For each $k \in \{1, \dots, \dim V - 1\}$, define $P(k)$: if every subsp of dim k is invar, then $T = \lambda I$.
(i) If every subsp of dim 1 is invar, then by Exe (26), $T = \lambda I$. Thus $P(1)$ holds.
(ii) Asm $P(k)$ holds for $k \in \{1, \dots, \dim V - 1\}$. And every subsp of dim $k+1$ is invar.
Let U be a subsp of dim k. If $\dim U = \dim V - 1$ then extend B_U to B_V and we are done.
Sup $\dim U \in \{1, \dots, \dim V - 2\}$. Choose two linely inde vecs $v, w \notin U$.
Becs $U \oplus \operatorname{span}(v)$ and $U \oplus \operatorname{span}(w)$ of dim $k+1$ are invar.
Sup $u \in U$. Let $Tu = a_1 u_1 + bv = a_2 u_2 + cw, \exists!\, u_1, u_2 \in U, a_1, a_2, b, c \in \mathbf{F}$.
Now $a_1 u_1 - a_2 u_2 = cw - bv \in U \cap \operatorname{span}(v) = \{0\} \Rightarrow b = c = 0$. Thus $Tu \in U$.
Becs $P(k)$ holds, we conclude that $T = \lambda I$. Thus $P(k+1)$ holds. □

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29 *Sup $T \in \mathcal{L}(V)$ and* range *T is finid.*
Provt T has at most $1 + \dim \operatorname{range} T$ disti eigvals.

Solus:
Let $\lambda_1, \dots, \lambda_m$ be the disti eigvals of T with corres eigvecs $v_1, \dots, v_m$.
(Becs range T is finid. The corres eigvals are finite.)
Then $(v_1, \dots, v_m)$ linely inde $\Longrightarrow (\lambda_1 v_1, \dots, \lambda_m v_m)$ linely inde, if each $\lambda_k \neq 0$.
Othws, $\exists!\, \lambda_k = 0$. Now $(\lambda_1 v_1, \dots, \lambda_{k-1} v_{k-1}, \lambda_{k+1} v_{k+1}, \dots, \lambda_m v_m)$ is linely inde.
Hence, by [2.23], $m - 1 \leqslant \dim \operatorname{range} T$. □

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30 *Sup $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigvals. Provt $\exists\, x, Tx - 9x = (-4, 5, \sqrt{7})$.*

Solus: T has $\dim \mathbf{R}^3$ eigvals not including $9 \Rightarrow (T - 9I)$ is inv. $x = (T - 9I)^{-1}(-4, 5, \sqrt{7})$. □

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31 *Sup V is finid, and $v_1, \dots, v_m \in V$. Provt*
$(v_1, \dots, v_m)$ is linely inde $\Longleftrightarrow$ $v_1, \dots, v_m$ are eigvecs of some T corres to disti eigvals.

SOLUS: Sup $(v_1, \dots, v_m)$ is linely inde. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.
Define $T \in \mathcal{L}(V)$ by $Tv_k = k \cdot v_k$ for each $k \in \{1, \dots, m, \dots, n\}$. Conversely by [5.10]. $\square$

• *Sup $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ are disti.*
(a) **32** *Provt $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linely inde in $\mathbf{R}^{\mathbf{R}}$.*
HINT: Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$. Define $D \in \mathcal{L}(V)$ by $Df = f'$. Find eigvals and eigvecs of D.
(b) [4E 36] *Shat $\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$ is linely inde in $\mathbf{R}^{\mathbf{R}}$.*

SOLUS:
(a) Define V and $D \in \mathcal{L}(V)$ as in HINT. Then becs for each k, $D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.
Thus $\lambda_1, \dots, \lambda_n$ are disti eigvals of D. By [5.10], $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linely inde in $\mathbf{R}^{\mathbf{R}}$. $\square$
(b) Let $V = \operatorname{span}\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.
Then becs $D\big(\cos(\lambda_k x)\big) = -\lambda_k \sin(\lambda_k x)$. 又 $D\big(\sin(\lambda_k x)\big) = \lambda_k \cos(\lambda_k x)$.
Thus $D^2\big(\cos(\lambda_k x)\big) = -\lambda_k^2 \cos(\lambda_k x)$.
Notice that $\lambda_1, \dots, \lambda_n$ are disti $\Longrightarrow -\lambda_1^2, \dots, -\lambda_n^2$ are disti. And $\dim V = n$.
Hence $-\lambda_1^2, \dots, -\lambda_n^2$ are all the eigvals of D^2 with corres eigvecs $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$.
And then $\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$ is linely inde in $\mathbf{R}^{\mathbf{R}}$. $\square$

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33 *Sup $T \in \mathcal{L}(V)$. Provt $T/(\operatorname{range} T) = 0$.*

SOLUS: $v + \operatorname{range} T \in V/\operatorname{range} T \Longrightarrow v + \operatorname{range} T \in \operatorname{null}\big(T/(\operatorname{range} T)\big)$. Hence $T/(\operatorname{range} T) = 0$. $\square$

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34 *Sup $T \in \mathcal{L}(V)$. Provt $T/(\operatorname{null} T)$ is inje $\Longleftrightarrow$ $(\operatorname{null} T) \bigcap (\operatorname{range} T) = \{0\}$.*

SOLUS: NOTICE that $\big(T/(\operatorname{null} T)\big)(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0 \Longleftrightarrow Tu \in (\operatorname{null} T) \bigcap (\operatorname{range} T)$.
Now $T/(\operatorname{null} T)$ is inje $\Longleftrightarrow u + \operatorname{null} T = 0 \Longleftrightarrow Tu = 0 \Longleftrightarrow (\operatorname{null} T) \bigcap (\operatorname{range} T) = \{0\}$. $\square$

• *Sup V is finid, $T \in \mathcal{L}(V)$, and U is invarsp of V under T.*
Define $T/U : V/U \to V/U$ by $(T/U)(v + U) = Tv + U$ for each $v \in V$.
(a) *Shat T/U is well-defined and is linear.* Requires that U is invard T.
(b) [OR **35**] *Shat each eigval of T/U is an eigval of T.*

SOLUS:
(a) $v + U = w + U \Longleftrightarrow v - w \in U \Longrightarrow T(v - w) \in U \Longleftrightarrow Tv + U = Tw + U$.
Hence T/U is well-defined. Now we shat T/U is linear.
$(T/U)\big((v + U) + \lambda(w + U)\big) = T(v + \lambda w) + U = (T/U)(v + U) + \lambda(T/U)(w)$. Checked.
(b) Sup λ is an eigval of T/U with an eigvec $v + U$. Then $Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v = u \in U$.
If $u = 0 \Rightarrow Tv = \lambda v$, then we are done. Othws, we discuss in two cases.
If $(T - \lambda I)|_U$ is inv. Then $\exists!\, w \in U, (T - \lambda I)(w) = u = (T - \lambda I)v \Rightarrow T(v + w) = \lambda(v + w)$.
Note that $v + w \neq 0$, for if not, $v \in U \Rightarrow v + U = 0$, ctradic. Thus λ is an eigval of T.
If $(T - \lambda I)|_U$ is not inv. Then becs V is finid, $(T - \lambda I)|_U$ is not inje,
so that $\exists\, w \in \operatorname{null}(T - \lambda I)|_U, w \neq 0, (T - \lambda I)w = 0 \Rightarrow Tw = \lambda w$. $\square$

OR. Let $B_U = (u_1, \dots, u_m)$. Then $\big((T - \lambda I)v, (T - \lambda I)u_1, \dots, (T - \lambda I)u_m\big)$ is linely inde in U.
So that $a_0(T - \lambda I)v + a_1(T - \lambda I)u_1 + \cdots + a_m(T - \lambda I)u_m = 0, \exists\, a_0, a_1, \dots, a_m \in \mathbf{F}$ with some $a_i \neq 0$.
Let $w = a_0 v + a_1 u_1 + \cdots + a_m u_m \Longrightarrow Tw = \lambda w$. Note that $w \neq 0$, for if not, $a_0 v \in U$, each $a_i = 0$. $\square$

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36 *Prove or give a counterexa: The result in Exercise 35 is still true if V is infinid.*

Solus: A counterexa:

Consider $V = \{ f \in \mathbf{R}^{\mathbf{R}} : \exists!\, m \in \mathbf{N}, f \in \operatorname{span}(1, e^x, \dots, e^{mx}) \}$. Note that V is infinid.

And a subsp $U = \{ f \in \mathbf{R}^{\mathbf{R}} : \exists!\, m \in \mathbf{N}^+, f \in \operatorname{span}(e^x, \dots, e^{mx}) \}$.

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then range $T = U$ is invard T.

Consider $(T/U)(1+U) = e^x + U = 0 \Longrightarrow 0$ is an eigval of T/U but is not an eigval of T.

$\big[$ null $T = \{0\}$, for if not, $\exists f \in V \backslash \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, ctradic. $\big]$ $\square$

• (4E 5.A.39) *Sup V is finid and $T \in \mathcal{L}(V)$.*
Provt T has an eigval $\Longleftrightarrow \exists$ invarsp U under T of dimension $\dim V - 1$.

Solus:

(a) Sup λ is an eigval of T with an eigvec v. (If $\dim V = 1$, then $U = \{0\}$ and we are done.)

Extend $v_1 = v$ to $B_V = (v_1, v_2 \dots, v_n)$.

Step 1. If $\exists\, w_1 \in \operatorname{span}(v_2, \dots, v_n)$ suth $0 \neq Tw_1 \in \operatorname{span}(v_1)$.
Then extend $w_1 = \alpha_{1,2}$ to a basis of $\operatorname{span}(v_2, \dots, v_n)$ as $(\alpha_{1,2}, \dots, \alpha_{1,n})$.
Othws, we stop at step 1.

Step 2. If $\exists\, w_2 \in \operatorname{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$ suth $0 \neq Tw_2 \in \operatorname{span}(v_1, w_1)$.
Then extend $w_2 = \alpha_{2,3}$ to a basis of $\operatorname{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$ as $(\alpha_{2,3}, \dots, \alpha_{2,n})$.
Othws, we stop at step 2.

Step k. If $\exists\, w_k \in \operatorname{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$ suth $0 \neq Tw_k \in \operatorname{span}(v_1, w_1, \dots, w_{k-1})$,
Then extend $w_k = \alpha_{k,k+1}$ to a basis of $\operatorname{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$ as $(\alpha_{k,k+1}, \dots, \alpha_{k,n})$.
Othws, we stop at step k.

Finally, we stop at step m, thus we get $(v_1, w_1, \dots, w_{m-1})$ and $(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})$,
range $T|_{\operatorname{span}(w_1,\dots,w_{m-1})} = \operatorname{span}(v_1, w_1, \dots, w_{m-2}) \Rightarrow \dim \operatorname{null} T|_{\operatorname{span}(w_1,\dots,w_{m-1})} = 0$,
$\underbrace{\operatorname{span}(v_1, w_1, \dots, w_{m-1})}_{\dim m}$ and $\underbrace{\operatorname{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})}_{\dim (n-m)}$ are invard T.

Let $U = \operatorname{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n}) \oplus \operatorname{span}(v_1, w_1, \dots, w_{m-2})$ and we are done. $\square$

Comment: Both $\operatorname{span}(v_2, \dots, v_n)$ and $U \oplus \operatorname{span}(w_{m-1})$ are in $\mathcal{S}_V \operatorname{span}(v_1)$.
If $T|_U$ is inv, then by the simlr algo, we can extend U to invarsp.

Or. Note that $\dim \operatorname{null}(T - \lambda I) \geqslant 1$. And $\dim \operatorname{range}(T - \lambda I) \leqslant \dim V - 1$.
Let $B_{\operatorname{range}(T-\lambda I)} = (w_1, \dots, w_m)$, $B_V = (w_1, \dots, w_m, u_1, \dots, u_n)$.
If $m = \dim V - 1$. $\big[\Longleftrightarrow n = 0. \big]$ Then range $(T - \lambda I)$ is invarsp of dim $\dim V - 1$.
Othws, choose $k \in \{1, \dots, n\}$ and then let $U = \operatorname{span}(w_1, \dots, w_m, u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n)$.
By Exe (1)(b), U is invard $(T - \lambda I)$. Now $u \in U \Rightarrow (T - \lambda I)(u) \in U \Rightarrow Tu \in U$.

(b) Sup U is invarsp under T of dim $m = \dim V - 1$. (If $m = 0$, then we are done.)
Let $B_U = (u_1, \dots, u_m), B_V = (u_0, u_1, \dots, u_m)$. We discuss in cases:
(I) If $Tu_0 \in U$, then range $T = U$ so that T is not surj $\Longleftrightarrow$ null $T \neq \{0\} \Longleftrightarrow 0$ is an eigval of T.
(II) If $Tu_0 \notin U$, then $Tu_0 = a_0 u_0 + a_1 u_1 + \dots + a_m u_m$.
If range $T|_U = U \Longleftrightarrow a_1 = \dots = a_m = 0 \Longleftrightarrow Tu_0 \in \operatorname{span}(u_0)$ then we are done.
Othws, $T|_U : U \to U$ is not surj, so is not inje. Thus 0 is an eigval of $T|_U$, so of T. $\square$

Or. Consider $T/U \in \mathcal{L}(V/U)$. Becs $\dim V/U = 1$. $\exists \lambda \in \mathbf{F}, T/U = \lambda I$. By Exe (35). $\square$

Ended

5.B: I [*See 5.B: II below.*]

Comment: 下面，为了照顾原书 5.B 节两版过大的差距，特别将此节补注分成 I 和 II 两部分。
又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本征值」
(相当一部分是从原第 3 版 8.C 节挪过来的) 是对原第 3 版「多项式作用于算子」与
「本征值的存在性」(也即第 3 版 5.B 前半部分) 的极大扩充，这一扩充也大大改变了
原第 3 版后半部分的「上三角矩阵」这一小节，故而将第 4 版 5.B 节放在第 3 版前面。

I 部分除了覆盖第 4 版 5.B 节全部和第 3 版 5.B 节前半部分与之相关的所有习题，
还会覆盖第 4 版 5.A 节末。
II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节，还会覆盖第 4 版 5.C 节；
并且，下面 5.C 还会覆盖第 4 版 5.D 节。
「注：[8.40] Or (4E 5.22) —— min poly;
[8.44,8.45] Or (4E 5.25,5.26) —— how to find the min poly;
[8.49] Or (4E 5.27) —— eigvals are the zeros of the min poly;
[8.46] Or (4E 5.29) —— $q(T) = 0 \Leftrightarrow q$ is a poly multi of the min poly.」

1 2 3 5 6 7 8 10 11 12 13 18 19 | 2E: Ch5.24
4E: 5.A.32 5.A.33 3 7 8 9 10 11 12 13 14 15
16 17 18 19 20 21 22 23 24 25 26 27 28 29

- (4E 5.A.33) *Sup $T \in \mathcal{L}(V)$ and m is a positive integer.*
  (a) *Provt T is inje $\iff T^m$ is inje.*
  (b) *Provt T is surj $\iff T^m$ is surj.*

Solus:

(a) Sup T^m is inje. Then $Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$.
Sup T is inje. Then $T^m v = T^{m-1}v = \cdots = T^2 v = Tv = v = 0$.

(b) Sup T^m is surj. $\forall u \in V, \exists v \in V, T^m v = u = Tw$, let $w = T^{m-1}v$.
Sup T is surj. Then $\forall u \in V, \exists v_1, \dots, v_m \in V, T(v_1) = T^2 v_2 = \cdots = T^m v_m = u$. □

- **Note For [5.17]:**
  Sup $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{F})$. Provt null $p(T)$ *and* range $p(T)$ *are invard T.*
  Solus: Using the commu in [5.10].
  (a) Sup $u \in \operatorname{null} p(T)$. Then $p(T)u = 0$.
  Thus $p(T)(Tu) = (p(T)T)u(Tp(T))u = T(p(T)u) = 0$. Hence $Tu \in \operatorname{null} p(T)$. □
  (b) Sup $u \in \operatorname{range} p(T)$. Then $\exists v \in V$ suth $u = p(T)v$.
  Thus $Tu = T(p(T)v) = p(T)(Tv) \in \operatorname{range} p(T)$. □

- **Note For [5.21]:** *Every optor on a finid nonzero complex vecsp has an eigval.*
  Sup V is a finid complex vecsp of dim $n > 0$ and $T \in \mathcal{L}(V)$.
  Choose a nonzero $v \in V$. $(v, Tv, T^2v, \dots, T^n v)$ of len $n+1$ is linely depe.
  Sup $a_0 I + a_1 T + \cdots + a_n T^n = 0$. Then $\exists a_j \neq 0$.
  Thus $\exists$ nonconst p of smallest deg ($\deg p > 0$) *suth* $p(T)v = 0$.
  Becs $\exists \lambda \in \mathbf{C}$ suth $p(\lambda) = 0 \Rightarrow \exists q \in \mathcal{P}(\mathbf{C}), p(z) = (z-\lambda)q(z), \forall z \in \mathbf{C}$.
  Thus $0 = p(T)v = (T - \lambda I)(q(T)v)$. By the min of $\deg p$ and $\deg q < \deg p$, $q(T)v \neq 0$.
  Then $(T - \lambda I)$ is not inje. Thus λ is an eigval of T with eigvec $q(T)v$.
- **Exa:** *an optor on a complex vecsp with no eigvals*
  Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by $(Tp)(z) = zp(z)$.

Sup $p \in \mathcal{P}(\mathbf{C})$ is a nonzero poly. Then $\deg Tp = \deg p + 1$, and thus $Tp \neq \lambda p$, $\forall \lambda \in \mathbf{C}$.
Hence T has no eigvals.

---

13 *Sup V is a complex vecsp and $T \in \mathcal{L}(V)$ has no eigvals.*
Provt every subsp of V invard T is either $\{0\}$ or infinid.

Solus: Sup U is a finid nonzero invarsp on $\mathbf{C}$. Then by [5.21], $T|_U$ has an eigval. □

---

16 *Sup $0 \neq v \in V$. Define $S \in \mathcal{L}\big(\mathcal{P}_{\dim V}(\mathbf{C}), V\big)$ by $S(p) = p(T)v$. Prove [5.21].*

Solus:

Becs $\dim \mathcal{P}_{\dim V}(\mathbf{C}) = \dim V + 1$. Then S is not inje. Hence $\exists\, 0 \neq p \in \mathcal{P}_{\dim V}(\mathbf{C}), p(T)v = 0$.
Using [4.14], write $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$. Apply T to both sides: $p(T) = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$.
Thus at least one of $(T - \lambda_j I)$ is not inje (becs $p(T)$ is not inje). □

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17 *Sup $0 \neq v \in V$. Define $S \in \mathcal{L}\big(\mathcal{P}_{(\dim V)^2}(\mathbf{C}), \mathcal{L}(V)\big)$ by $S(p) = p(T)$. Prove [5.21].*

Solus:

Becs $\dim \mathcal{P}_{(\dim V)^2}(\mathbf{C}) = (\dim V)^2 + 1$. Then S is not inje.
Hence $\exists\, p \in \mathcal{P}_{(\dim V)^2}(\mathbf{C})\backslash\{0\}, 0 = S(p) = p(T) = c(T - \lambda_1 I)\cdots(T-\lambda_m I)$, where $c \neq 0$.
Thus $(T - \lambda_1 I)\cdots(T - \lambda_m I) = 0 \Longrightarrow \exists\, j, (T - \lambda_j)$ is not inje. □

Comment: $\exists$ monic $q \in \operatorname{null} S \neq \{0\}$ of smallest deg, $S(q) = q(T) = 0$, then q is the *min poly.*

- **Note For [8.40]:** *def for min poly*

Sup V is finid and $T \in \mathcal{L}(V)$.
Sup $M_T^0 = \{p_j\}_{j\in\Gamma}$ is the set of all monic poly that give 0 whenever T is applied.
Provt $\exists\,!\, p_k \in M_T^0, \deg p_k = \min\{\deg p_j\}_{j\in\Gamma} \leqslant \dim V$.

Solus: Or. Another Proof :

[*Existns Part*] We use induction on $\dim V$.

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, we are done.

(ii) Sup $\dim V \geqslant 1$.

Asm $\dim V > 0$ *and that the desired result is true for all optors on all vecsps of smaller dim.*
Let $u \in V, u \neq 0$. The list $(u, Tu, \dots, T^{\dim V} u)$ of len $(1 + \dim V)$ is linely depe.
Then $\exists\,!\, T^m$ of smallest deg suth $T^m u \in \operatorname{span}(u, Tu, \dots, T^{m-1}u)$.
Thus $\exists\, c_j \in \mathbf{F}, c_0 u + c_1 Tu + \cdots + c_{m-1}T^{m-1}u + T^m u = 0$.
Define q by $q(z) = c_0 + c_1 z + \cdots + c_{m-1}z^{m-1} + z^m$.
Then $0 = T^k\big(q(T)u\big) = q(T)(T^k u), \forall k \in \{1, \dots, m-1\} \subseteq \mathbf{N}$.
Becs $(u, Tu, \dots, T^{m-1}u)$ is linely inde.
Thus $\dim \operatorname{null} q(T) \geq m \Rightarrow \dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leqslant \dim V - m$.

Let $W = \operatorname{range} q(T)$.
By *asm*, $\exists\, s \in M_T^0$ of smallest deg (and $\deg s \leqslant \dim W$,) so that $s(T|_W) = 0$.
Hence $\forall v \in V, \big((sq)(T)\big)(v) = s(T)\big(q(T)v\big) = 0$.
Thus $sq \in M_T^0$ and $\deg sq \leqslant \dim V$.

[*Uniqnes Part*]

Sup $p, q \in M_T^0$ are of the smallest deg. Then $(p - q)(T) = 0$. 又 $\deg(p-q) = m < \min\{\deg p_j\}_{j\in\Gamma}$.
Hence $p - q = 0$, for if not, $\exists\,!\, c \in \mathbf{F}, c(p-q) \in M_T^0$. Ctradic. □

• (4E 5.31, 4E 5.B.25 and 26) *min poly of restr optor and min poly of quot optor*
Sup V is finid, $T \in \mathcal{L}(V)$, and U is invarsp of V under T.
Let p be the min poly of T.
(a) *Provt p is a poly multi of the min poly of $T|_U$.*
(b) *Provt p is a poly multi of the min poly of T/U.*
(c) *Provt (min poly of $T|_U$) $\times$ (min poly of T/U) is a poly multi of p.*
(d) *Provt the set of eigvals of T equals*
the union of the set of eigvals of $T|_U$ and the set of eigvals of T/U.

SOLUS:

(a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46]. ☐

(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v+U) = p(T)v + U = 0.$ ☐

(c) Sup r is the min poly of $T|_U$, s is the min poly of T/U.
Becs $\forall v \in V, s(T/U)(v+U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.
又 $\forall u \in U, r(T|_U)u = r(T)u = 0$.
Thus $\forall v \in V$ but $v \notin U, (rs)(T)v = r\big(s(T)v\big) = 0$.
And $\forall u \in U, (rs)(T)u = r\big(s(T)u\big) = 0$ (becs $s(T)u = s(T|_U)u \in U$).
Hence $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0.$ ☐

(d) By [8.49], immediately. ☐

• (4E 5.B.27) *Sup $\mathbf{F} = \mathbf{R}$, V is finid, and $T \in \mathcal{L}(V)$.*
Provt the min poly p of T_C equals the min poly q of T.

SOLUS:

(a) $\forall u + \mathrm{i}0 \in V_C, p(T_C)(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p$ is a poly multi of q.

(b) $q(T) = 0 \Rightarrow \forall u + \mathrm{i}v \in V_C, q(T_C)(u + \mathrm{i}v) = q(T)u + \mathrm{i}q(T)v = 0 \Rightarrow q$ is a poly multi of p. ☐

• (4E 5.B.28) *Sup V is finid and $T \in \mathcal{L}(V)$.*
Provt the min poly p of $T' \in \mathcal{L}(V')$ equals the min poly q of T.

SOLUS:

(a) $\forall \varphi \in V', p(T')\varphi = \varphi \circ \big(p(T)\big) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \operatorname{null} \varphi \Rightarrow p(T) = 0, p$ is a poly multi of q.

(b) $q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ \big(q(T)\big) = q(T')\varphi = 0 \Rightarrow q(T) = 0, q$ is a poly multi of p. ☐

• (4E 5.32) *Sup $T \in \mathcal{L}(V)$ and p is the min poly.*
Provt T is not inje $\iff$ the const term of p is 0.

SOLUS:

T is not inje $\iff$ 0 is an eigval of $T \iff$ 0 is a zero of $p \iff$ the const term of p is 0. ☐

OR. Becs $p(0) = (z-0)(z-\lambda_1)\cdots(z-\lambda_m) = 0 \Rightarrow T(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0$
又 p is the min poly $\Rightarrow q$ define by $q(z) = (z-\lambda_1)\cdots(z-\lambda_m)$ is suth $q(T) \neq 0$.
Hence $0 = p(T) = Tq(T) \Rightarrow T$ is not inje.
Conversely, sup $(T - 0I)$ is not inje, then 0 is a zero of p, so that the const term is 0. ☐

• (4E 5.B.22)
Sup V is finid, $T \in \mathcal{L}(V)$. Provt T is inv $\iff I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$.

SOLUS: Denote the min poly by p, where for all $z \in \mathbf{F}, p(z) = a_0 + a_1 z + \cdots + z^m$.
Notice that V is finid. T is inv $\iff T$ is inje $\iff p(0) \neq 0$.

Hence $p(T) = 0 = a_0 I + a_1 T + \cdots + T^m$, where $a_0 \neq 0$ and $m \leqslant \dim V$. $\square$

---

6 *Sup $T \in \mathcal{L}(V)$ and U is a subsp of V invard T.*
Provt U is invard $p(T)$ for every poly $p \in \mathcal{P}(\mathbf{F})$.

SOLUS:

$\forall u \in U, Tu \in U \Rightarrow Iu, Tu, T(Tu), \ldots, T^m u \in U \Longrightarrow \forall a_k \in \mathbf{F}, (a_0 I + a_1 T + \cdots + a_m T^m)u \in U.$ $\square$

• (4E 5.B.10, 23) *Sup V is finid, $T \in \mathcal{L}(V)$ and p is the min poly with deg m. Sup $v \in V$.*
(a) *Provt* $\operatorname{span}(v, Tv, \ldots, T^{m-1}v) = \operatorname{span}(v, Tv, \ldots, T^{j-1}v)$ *for some $j \leqslant m$.*
(b) *Provt* $\operatorname{span}(v, Tv, \ldots, T^{m-1}v) = \operatorname{span}(v, Tv, \ldots, T^{m-1}v, \ldots, T^n v)$.

SOLUS:

COMMENT: By NOTE FOR[8.40], j has an upper bound $m-1$, m has an upper bound $\dim V$.

Write $p(z) = a_0 + a_1 z + \cdots + z^m$ ($m \leqslant \dim V$). If $v = 0$, then we are done. Sup $v \neq 0$.

(a) Sup $j \in \mathbf{N}^+$ is the smallest suth $T^j v \in \operatorname{span}(v, Tv, \ldots, T^{j-1}v) = U_0$. Then $j \leqslant m$.
Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1} T^{j-1} v$. And becs $T(T^k v) = T^{k+1} \in U_0$. U_0 is invard T.
By Exe (6), $\forall k \in \mathbf{N}$, $T^{j+k} v = T^k(T^j v) \in U_0$.
Thus $U_0 = \operatorname{span}(v, Tv, \ldots, T^{j-1}v, \ldots, T^n v)$ for all $n \geqslant j-1$. Let $n = m-1$ and we are done.

(b) Let $U = \operatorname{span}(v, Tv, \ldots, T^{m-1}v)$.
By (a), $U = U_0 = \operatorname{span}(v, Tv, \ldots, T^{j-1}, \ldots, T^{m-1}, \ldots, T^n)$ for all $n \geqslant m-1$. $\square$

• (4E 5.B.21) *Sup V is finid and $T \in \mathcal{L}(V)$.*
Provt the min poly p has deg at most $1 + \dim \operatorname{range} T$.
If $\dim \operatorname{range} T < \dim V - 1$, then this result gives a better upper bound for the deg of min poly.

SOLUS:

If T is inje, then range $T = V$ and we are done. Now choose $0 \neq v \in \operatorname{null} T$, then $Tv + 0 \cdot v = 0$.
1 is the smallest positive integer suth $T^1 v \in \operatorname{span}(v, \ldots, T^0 v)$. Define q by $q(z) = z \Rightarrow q(T)v = 0$.
Let $W = \operatorname{range} q(T) = \operatorname{range} T$. $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ of smallest deg ($\deg s \leqslant \dim W$), $s(T|_W) = 0$.
Hence sq is the min poly (see NOTE FOR[8.40]) and $\deg(sq) = \deg s + \deg q \leqslant \dim \operatorname{range} T + 1$. $\square$

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19 *Sup V is finid,* $\dim V > 1, T \in \mathcal{L}(V)$. *Provt* $\{ p(T) : p \in \mathcal{P}(\mathbf{F}) \} \neq \mathcal{L}(V)$.

SOLUS: If $\forall S \in \mathcal{L}(V), \exists p \in \mathcal{P}(\mathbf{F}), S = p(T)$. Then by [5.20], $\forall S_1, S_2 \in \mathcal{L}(V), S_1 S_2 = S_2 S_1$.
Note that $\dim \geqslant 2$. By (3.A.14), $\exists S_1, S_2 \in \mathcal{L}(V), S_1 S_2 \neq S_2 S_1$. Ctradic. $\square$

• *Sup V is finid and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{ q(T) : q \in \mathcal{P}(\mathbf{F}) \}$.*
Provt $\dim \mathcal{E}$ *equals the deg of the min poly of T.*

SOLUS:

Becs the list $(I, T, \ldots, T^{(\dim V)^2})$ of len $\dim \mathcal{L}(V) + 1$ is linely depe in $\dim \mathcal{L}(V)$.
Sup $m \in \mathbf{N}^+$ is the smallest suth $T^m = a_0 I + \cdots + a_{m-1} T^{m-1}$.
Then q defined by $q(z) = z^m - a_{m-1} z^{m-1} - \cdots - a_0$ is the min poly (see [8.40]).
For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k(T^m) \in \operatorname{span}(I, T, \ldots, T^{m-1}) = U$.
Hence $\operatorname{span}(I, T, \ldots, T^{(\dim V)^2}) = \operatorname{span}(I, T, \ldots, T^{(\dim V)^2 - 1}) = U$.
Note that by the min of m, $(I, T, \ldots, T^{m-1})$ is linely inde.
Thus $\dim U = m = \dim \operatorname{span}(I, T, \ldots, T^{(\dim V)^2 - 1}) = \dim \operatorname{span}(I, T, \ldots, T^n)$ for all $m < n \in \mathbf{N}^+$.

Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.

(a) Sup $p(T) = 0$. 又 $\deg p \leqslant m-1 \Rightarrow p = 0$. Then φ is inje.

(b) $\forall S = a_0 I + a_1 T + \cdots + a_{m-1}T^{m-1} \in \mathcal{E}$, define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by

$p(z) = a_0 + a_1 z + \cdots + a_{m-1}z^{m-1} \Rightarrow \varphi(p) = S$. Then φ is surj.

Hence $\mathcal{E}$ and $\mathcal{P}_{m-1}(\mathbf{F})$ are iso. 又 $\dim \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. □

• (4E 5.B.13) *Sup $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$ is defined by*

$$q(z) = a_0 + a_1 z + \cdots + a_n z^n, \text{ where } a_n \neq 0, \text{ for all } z \in \mathbf{F}.$$

Denote the min poly of T by p defined by

$$p(z) = c_0 + c_1 z + \cdots + c_{m-1}z^{m-1} + z^m \text{ for all } z \in \mathbf{F}.$$

Provt $\exists!\, r \in \mathcal{P}(\mathbf{F})$ suth $q(T) = r(T), \deg r < \deg p$.

Solus:

If $\deg q < \deg p$, then we are done.

If $\deg q = \deg p$, notice that $p(T) = 0 = c_0 I + c_1 T + \cdots + c_{m-1}T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0 I - c_1 T - \cdots - c_{m-1}T^{m-1},$$

define r by $r(z) = q(z) + \left[-a_m z^m + a_m(-c_0 - c_1 z - \cdots - c_{m-1}z^{m-1}) \right]$

$= (a_0 - a_m c_0) + (a_1 - a_m c_1)z + \cdots + (a_{m-1} - a_m c_{m-1})z^{m-1}$,

hence $r(T) = 0, \deg r < m$ and we are done.

Now sup $\deg q \geqslant \deg p$. We use induction on $\deg q$.

(i) $\deg q = \deg p$, then the desired result is true, as shown above.

(ii) $\deg q > \deg p$, asm the desired result is true for $\deg q = n$.

Sup $f \in \mathcal{P}(\mathbf{F})$ suth $f(z) = b_0 + b_1 z + \cdots + b_n z^n + b_{n+1}z^{n+1}$.

Apply the asm to g defined by $g(z) = b_0 + b_1 z + \cdots + b_n z^n$,

getting s defined by $s(z) = d_0 + d_1 z + \cdots + d_{m-1}z^{m-1}$.

Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1}T^{n+1} = s(T) + b_{n+1}T^{n+1}$.

Apply the asm to t defined by $t(z) = z^n$,

getting δ defined by $\delta(z) = c_0{}' + c_1{}'z + \cdots + c_{m-1}{}'z^{m-1}$.

Thus $t(T) = T^n = c_0{}' + c_1{}'z + \cdots + c_{m-1}{}'z^{m-1} = \delta(T)$.

又 $\operatorname{span}(v, Tv, \dots, T^{m-1}v)$ is invard T.

Hence $\exists!\, k_j \in \mathbf{F}, T^{n+1} = T(T^n) = k_0 + k_1 z + \cdots + k_{m-1}z^{m-1}$.

And $f(T) = s(T) + b_{n+1}(k_0 + k_1 T + \cdots + k_{m-1}T^{m-1})$

$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1)z + \cdots + (d_{m-1} + k_{m-1})z^{m-1} = h(T)$, thus defining h. □

• (4E 5.B.14) *Sup V is finid, $T \in \mathcal{L}(V)$ has min poly p*

$$\textit{defined by } p(z) = a_0 + a_1 z + \cdots + a_{m-1}z^{m-1} + z^m, a_0 \neq 0.$$

Find the min poly of T^{-1}.

Solus:

Notice that V is finid. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is inv.

Then $p(T) = a_0 I + a_1 T + \cdots + T^m = 0$. Apply T^{-m} to both sides,

$a_0 (T^{-1})^m + a_1 (T^{-1})^{m-1} + \cdots + a_{m-1}T^{-1} + I = 0$.

Define q by $q(z) = z^m + \dfrac{a_1}{a_0}z^{m-1} + \cdots + \dfrac{a_{m-1}}{a_0}z + \dfrac{1}{a_0}$ for all $z \in \mathbf{F}$.

We now shat $(T^{-1})^k \notin \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$

for every $k \in \{1, \dots, m-1\}$ by ctradic, so that q is exactly the min poly of T^{-1}.

Sup $(T^{-1})^k \in \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$.

Then let $\left(T^{-1}\right)^k = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. Apply T^k to both sides,
getting $I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T$, hence $T^k \in \operatorname{span}(I, T, \ldots, T^{k-1})$.

Thus f defined by $f(z) = z^k + \dfrac{b_1}{b_0} z^{k-1} + \cdots + \dfrac{b_{k-1}}{b_0} z - \dfrac{1}{b_0}$ is a poly multi of p.

While $\deg f < \deg p$. Ctradic. □

• **Note For [8.49]:**

Sup V is a finid complex vecsp and $T \in \mathcal{L}(V)$.
By [4.14], the min poly has the form $(z - \lambda_1) \cdots (z - \lambda_m)$,
*where $\lambda_1, \ldots, \lambda_m$ are all the eigvals of T, **possibly with repetitions**.*

• **Comment:**

A nonzero poly has at most as many disti zeros as its deg (see [4.12]).
Thus by the upper bound for the deg of min poly given in Note For[8.40]*, and by [8.49,]*
we can ***give an alternative proof of*** [**5.13**].

• Notice (See also 4E 5.B.20,24)

Sup $\alpha_1, \ldots, \alpha_n$ are all the disti eigvals of T,
and therefore are all the disti zeros of the min poly.
Also, the min poly of T is a poly multi of, but not equal to, $(z - \alpha_1) \cdots (z - \alpha_n)$.
If we define q by $q(z) = (z - \alpha_1)^{\dim V - (n-1)} \cdots (z - \alpha_n)^{\dim V - (n-1)}$,
then q is a poly multi of the ch poly (see [8.34] and [8.26])
(Becs $\dim V > n$ and $n - 1 > 0$, $n\left[\dim V - (n-1)\right] > \dim V$.)
The ch poly has the form $(z - \alpha_1)^{\gamma_1} \cdots (z - \alpha_n)^{\gamma_n}$, where $\gamma_1 + \cdots + \gamma_n = \dim V$.
The min poly has the form $(z - \alpha_1)^{\delta_1} \cdots (z - \alpha_n)^{\delta_n}$, where $0 \leqslant \delta_1 + \cdots + \delta_n \leqslant \dim V$.

---

10 *Sup $T \in \mathcal{L}(V)$, λ is an eigval of T with an eigvec v.*
Provt for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.

Solus:

Sup p is defined by $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbf{F}$. Becs for any $n \in \mathbf{N}^+$, $T^n v = \lambda^n v$.
Thus $p(T)v = a_0 v + a_1 T v + \cdots + a_m T^m v = a_0 v + a_1 \lambda v + \cdots + a_m \lambda^m v = p(\lambda) v$. □

Comment: For any $p \in \mathcal{P}(\mathbf{F})$ suth $p(z) = (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_m)^{\alpha_m}$, the result is true as well.

Now we prove that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v = (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v$.
Define q_i by $q_i(z) = (z - \lambda_i)^{\alpha_i}$ for all $z \in \mathbf{F}$.
Becs $(a + b)^n = a^n + C_n^1 a^{n-1} b + \cdots + C_n^k a^{n-k} b^k + \cdots + C_n^n b^n$.
Let $a = z, b = \lambda_i, n = \alpha_i$, so we can write $q_i(z)$ in the form $a_0 + a_1 z + \cdots + a_m z^m$.
Hence $q_i(T)v = q_i(\lambda)v \Rightarrow (T - \lambda_i I)^{\alpha_i} v = (\lambda - \lambda_i)^{\alpha_i} v$.
Then for each $k \in \{2, \ldots, m\}$, $(T - \lambda_{k-1} I)^{\alpha_{k-1}} (T - \lambda_k I)^{\alpha_k} v$

$$
\begin{aligned}
&= q_{k-1}(T)\left(q_k(T)v\right) \\
&= q_{k-1}(T)\left(q_k(\lambda)v\right) \\
&= q_{k-1}(\lambda)\left(q_k(\lambda)v\right) \\
&= (\lambda - \lambda_{k-1})^{\alpha_{k-1}} (\lambda - \lambda_k)^{\alpha_k} v.
\end{aligned}
$$

So that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v$

$$
\begin{aligned}
&= q_1(T)\Big(q_2(T)\Big(\ldots\left(q_m(T)v\right)\ldots\Big)\Big) \\
&= q_1(\lambda)\left(q_2(\lambda)\left(\ldots\left(q_m(\lambda)v\right)\ldots\right)\right)
\end{aligned}
$$

$$= (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v. \qquad \square$$

1 *Sup $T \in \mathcal{L}(V)$ and $\exists\, n \in \mathbf{N}^+$ suth $T^n = 0$.*
Provt $(I - T)$ is inv and $(I - T)^{-1} = I + T + \cdots + T^{n-1}$.

SOLUS: Note that $1 - x^n = (1 - x)(1 + x + \cdots + x^{n-1})$.

$$\left.\begin{array}{l}(I - T)(1 + T + \cdots + T^{n-1}) = I - T^n = I \\ (1 + T + \cdots + T^{n-1})(I - T) = I - T^n = I\end{array}\right\} \Rightarrow (I - T)^{-1} = 1 + T + \cdots + T^{n-1}.$$ □

---

2 *Sup $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$.*
Sup λ is an eigval of T. Provt $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

SOLUS:

Sup v is an eigvec corres to λ. Then for any $p \in \mathcal{P}(\mathbf{F}), p(T)v = p(\lambda)v$.
Hence $0 = (T - 2I)(T - 3I)(T - 4I)v = (\lambda - 2)(\lambda - 3)(\lambda - 4)v$ while $v \neq 0 \Rightarrow \lambda = 2, 3$ or 4. □

COMMENT: Note that $(T - 2I)(T - 3I)(T - 4I) = 0$ is not inje, so that $2, 3, 4$ are eigvals of T.
But it doesn't mean that all the eigvals of T are exactly $2, 3, 4$.

---

7 [See 5.A.22] *Sup $T \in \mathcal{L}(V)$. Provt 9 is an eigval of $T^2 \Longleftrightarrow 3$ or -3 is an eigval of T.*

SOLUS:

(a) Sup λ is an eigval of T with an eigvec v.
Then $(T - 3I)(T + 3I)v = (\lambda - 3)(\lambda + 3)v = 0 \Rightarrow \lambda = \pm 3$.
(b) Sup 3 or -3 is an eigval of T with an eigvec v. Then $Tv = \pm 3v \Rightarrow T^2 v = T(Tv) = 9v$ □

OR. 9 is an eigval of $T^2 \Longleftrightarrow (T^2 - 9I) = (T - 3I)(T + 3I)$ is not inje $\Longleftrightarrow \pm 3$ is an eigval. □

---

3 *Sup $T \in \mathcal{L}(V), T^2 = I$ and -1 is not an eigval of T. Provt $T = I$.*

SOLUS:

$T^2 - I = (T + I)(T - I)$ is not inje, 又 -1 is not an eigval of $T \Longrightarrow$ By TIPS. □

OR. Note that $\forall v \in V, v = [\frac{1}{2}(I - T)v] + [\frac{1}{2}(I + T)v]$.

$$\left.\begin{array}{l}(I + T)\big((I - T)v\big) = 0 \Longrightarrow (I - T)v \in \operatorname{null}(I + T) \\ (I - T)\big((I + T)v\big) = 0 \Longrightarrow (I + T)v \in \operatorname{null}(I - T)\end{array}\right\} \Rightarrow V = \operatorname{null}(I + T) + \operatorname{null}(I - T).$$

又 -1 is not an eigval of $T \Longleftrightarrow (I + T)$ is inje $\Longleftrightarrow \operatorname{null}(I + T) = \{0\}$.
Hence $V = \operatorname{null}(I - T) \Rightarrow \operatorname{range}(I - T) = \{0\}$. Thus $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$. □

• (4E 5.A.32) *Sup $T \in \mathcal{L}(V)$ has no eigvals and $T^4 = I$. Provt $T^2 = -I$.*

SOLUS:

Becs $T^4 - I = (T^2 - I)(T^2 + I) = 0$ is not inje $\Rightarrow (T^2 - I)$ or $(T^2 + I)$ is not inje.
又 T has no eigvals $\Rightarrow (T^2 - I) = (T - I)(T + I)$ is inje. Hence $T^2 + I = 0 \in \mathcal{L}(V)$, for if not,
$\exists\, v \in V, (T^2 + I)v \neq 0$ while $(T^2 - I)\big((T^2 + I)v\big) = 0$ but $(T^2 - I)$ is inje. Ctradic.
OR. $\forall v \in V, 0 = (T^2 - I)(T^2 + I)v \Longleftrightarrow 0 = (T^2 + I)v$. Hence $T^2 + I = 0$. □

OR. Note that $\forall v \in V, v = \left[\frac{1}{2}(I - T^2)v\right] + \left[\frac{1}{2}(I + T^2)v\right]$.

$$\left.\begin{array}{l}(I + T^2)\big((I - T^2)v\big) = 0 \Longrightarrow (I - T^2)v \in \operatorname{null}(I + T^2) \\ (I - T^2)\big((I + T^2)v\big) = 0 \Longrightarrow (I + T^2)v \in \operatorname{null}(I - T^2)\end{array}\right\} \Rightarrow V = \operatorname{null}(I + T^2) + \operatorname{null}(I - T^2).$$

又 T has no eigvals $\Longleftrightarrow (I - T^2)$ is inje $\Longleftrightarrow \operatorname{null}(I - T^2) = \{0\}$.
Hence $V = \operatorname{null}(I + T^2) \Rightarrow \operatorname{range}(I + T^2) = \{0\}$. Thus $I + T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$. □

---

8 [OR (4E 5.A.31)] *Give an exa of $T \in \mathcal{L}(\mathbf{R}^2)$ suth $T^4 = -I$.*

SOLUS:

Define $\mathrm{i} \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y) = (-y,x)$. Just like $\mathrm{i} : \mathbf{C} \to \mathbf{C}$ defined by $\mathrm{i}(x+\mathrm{i}y) = -y + \mathrm{i}x$.

Define $\mathrm{i}^n \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y) = \Big(\mathrm{Re}(\mathrm{i}^n x + \mathrm{i}^{n+1} y), \mathfrak{Im}(\mathrm{i}^n x + \mathrm{i}^{n+1} y)\Big)$.

$T^4 + I = (T^2 + \mathrm{i}I)(T^2 - \mathrm{i}I) = (T + \mathrm{i}^{1/2} I)(T - \mathrm{i}^{1/2} I)(T - (-\mathrm{i})^{1/2} I)(T + (-\mathrm{i})^{1/2} I)$.

Note that $\mathrm{i}^{1/2} = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}$, $(-\mathrm{i})^{1/2} = \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}$. Hence $T = \pm(\pm \mathrm{i})^{1/2} I$.

Let $T = \mathrm{i}^{1/2} I$ defined by $\mathrm{i}^{1/2}(x,y) = \Big(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\Big)$. $\square$

OR. Becs $\mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix}$. Using $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}$.

We define $T \in \mathcal{L}(\mathbf{R}^2)$ suth $\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix}$. $\square$

• (4E 5.B.12) *Find the min poly of T defined in (5.A.10).*

SOLUS: By (5.A.9) and [8.40, 8.49], $1, 2, \dots, n$ are all the zeros of the min poly of T. $\square$

• (4E 5.B.3) *Find the min poly of T defined in (5.A.19).*

SOLUS:

If $n = 1$ then 1 is the only eigval of T, and $(z-1)$ is the min poly.

Becs n and 0 are all the eigvals of T, 又 $\forall k \in \{1, \dots, n\}, Te_k = e_1 + \cdots + e_n$; $T^2 e_k = n(e_1 + \cdots + e_n)$.

Hence $T^2 e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T-n) = 0$. Thus $\big(z(z-n)\big)$ is the min poly. $\square$

• (4E 5.B.8) *Find the min poly of T. Where $T \in \mathcal{L}(\mathbf{R}^2)$ is the optor of counterclockwise rotation by θ, where $\theta \in \mathbf{R}^+$.*

SOLUS:

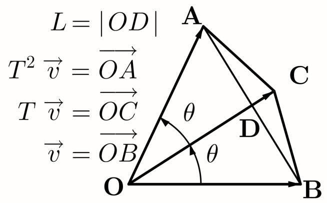
If $\theta = \pi + 2k\pi$, then $T(w,z) = (-w,-z), T^2 = I$ and the min poly is $z+1$.

If $\theta = 2k\pi$, then $T = I$ and the min poly is $z - 1$.

Othws (v, Tv) is linely inde. Then $\operatorname{span}(v, Tv) = \mathbf{R}^2$. Note that $\nexists\, b \in \mathbf{F}, T - bI = 0$.

Thus sup the min poly p is defined by $p(z) = z^2 + bz + c$ for all $z \in \mathbf{R}$.

Becs



$Tv = \frac{|\vec{v}|}{2L}(T^2 v + v) \Rightarrow T = \frac{|\vec{v}|}{2L}(T^2 + I)$

$L = |\vec{v}|\cos\theta \Rightarrow \frac{|\vec{v}|}{2L} = \frac{1}{2\cos\theta}$

Hence $p(T) = T^2 - 2\cos\theta\, T + I = 0$ and $z^2 - 2\cos\theta\, z + 1$ is the min poly of T. $\square$

OR. Let (e_1, e_2) be the std basis of $\mathbf{R}^2$. We use the pattern shown in [8.44].

Becs $Te_1 = \cos\theta\, e_1 + \sin\theta\, e_2$, $T^2 e_1 = \cos 2\theta\, e_1 + \sin 2\theta\, e_2$.

Thus $ce_1 + bTe_1 = -T^2 e_1 \iff \begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix}\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$. Now $\det = \sin\theta \neq 0, c = 1, b = 2\cos\theta$. $\square$

OR. $\mathcal{M}(T, (e_1, e_2)) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. By (4E 5.B.11), the min poly is $(z \pm 1)$ or $(z^2 - 2\cos\theta\, z + 1)$. $\square$

• (4E 5.B.11) *Sup V is a two-dim vecsp, $T \in \mathcal{L}(V)$, and the matrix of T with resp to some B_V is* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) *Shat $T^2 - (a+d)T + (ad-bc)I = 0$.*

(b) *Shat the min poly of T equals*

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad-bc) & \text{othws.} \end{cases}$$

Solus:

(a) Sup the basis is (v, w). Becs $\begin{cases} Tv = av + bw \Rightarrow (T - aI)v = bw, \text{ then apply } (T - dI) \text{ to both sides.} \\ Tw = cv + dw \Rightarrow (T - dI)w = cv, \text{ then apply } (T - aI) \text{ to both sides.} \end{cases}$

Hence $(T - aI)(T - dI) = bcI \Rightarrow T^2 - (a+d)T + (ad-bc)I = 0$.

(b) If $b = c = 0$ and $a = d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the min poly is $z - a$.

Othws, by (a), $z^2 - (a+d)z + (ad-bc)$ is a poly multi of the min poly.

Now we prove that $T \notin \operatorname{span}(I)$, so that then the min poly of T has exactly deg 2.

(At least one of the asm of (I),(II) below is true.)

(I) Sup $a = d$, then $Tv = av + bw \notin \operatorname{span}(v)$, $Tw = cv + aw \notin \operatorname{span}(w)$.

(II) Sup at most one of b, c is not 0. If $b = 0$, then $Tw \notin \operatorname{span}(w)$; If $c = 0$, then $Tv \notin \operatorname{span}(v)$. □

• *Sup $S, T \in \mathcal{L}(V)$, S is inv, and $p \in \mathcal{P}(\mathbf{F})$. Provt $Sp(TS) = p(ST)S$.*

Solus:

We prove $S(TS)^m = (ST)^m S$ for each $m \in \mathbf{N}$ by induction.

(i) If $m = 0, 1$. Then $S(TS)^0 = I = (ST)^0 S$; $S(TS)^1 = (ST)S$.

(ii) If $m > 1$. Asm $S(TS)^m = (ST)^m S$.

Then $S(TS)^{m+1} = S(TS)^m(TS) = (ST)^m STS = (ST)^{m+1}S$.

Hence $\forall p \in \mathcal{P}(\mathbf{F})$, $Sp(TS) = \sum_{k=1}^{m} a_k S(TS)^k = \sum_{k=1}^{m} a_k p(ST)^k S = \left[\sum_{k=1}^{m} a_k (TS)^k\right] S$. □

Comment: $p(TS) = S^{-1}p(ST)S$, $p(ST) = Sp(TS)S^{-1}$.

Coro: **5** Becs S is inv, $T \in \mathcal{L}(V)$ is arb $\Longleftrightarrow R = ST$ is arb.

Hence $\forall R \in \mathcal{L}(V)$, inv $S \in \mathcal{L}(V)$, $p(S^{-1}RS) = S^{-1}p(R)S$.

• (4E 5.B.7) *Sup $S, T \in \mathcal{L}(V)$. Let p, q be the min polys of ST, TS respectively.*

(a) *If $V = \mathbf{F}^2$. Give an exa suth $p \neq q$;* (b) *If S or T is inv. Provt $p = q$.*

Solus:

(a) Define S by $S(x,y) = (x,x)$. Define T by $T(x,y) = (0,y)$.

Then $ST(x,y) = 0$, $TS(x,y) = (0,x)$ for all $(x,y) \in \mathbf{F}^2$. Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.

Hence the min poly of ST does not equal to the min poly of TS.

(b) Sup S is inv. Becs p, q are monic.

$$\left.\begin{array}{l} p(ST) = 0 = Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a poly multi of } q \\ q(TS) = 0 = S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a poly multi of } p \end{array}\right\} \Rightarrow p = q.$$

Reversing the roles of S and T, we conclude that if T is inv, then $p = q$ as well. □

---

11 *Sup $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$, and $\alpha \in \mathbf{C}$.*

Provt α is an eigval of $p(T) \Longleftrightarrow \alpha = p(\lambda)$ for some eigval λ of T.

Solus:

(a) Sup α is an eigval of $p(T) \Leftrightarrow \big(p(T) - \alpha I\big)$ is not inje.

Write $p(z) - \alpha = c(z-\lambda_1)\cdots(z-\lambda_m) \Rightarrow p(T) - \alpha I = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$.
By TIPS, $\exists\,(T-\lambda_j I)$ not inje. Thus $p(\lambda_j) - \alpha = 0$.

(b) Sup $\alpha = p(\lambda)$ and λ is an eigval of T with an eigvec v. Then $p(T)v = p(\lambda)v = \alpha v$. $\square$

OR. Define q by $q(z) = p(z) - \alpha$. λ is a zero of q.
Becs $q(T)v = \big(p(T) - \alpha I\big)v = q(\lambda)v = \big(p(\lambda) - \alpha\big)v = 0$.
Hence $q(T)$ is not inje $\Rightarrow \big(p(T) - \alpha I\big)$ is not inje. $\square$

---

12 [OR (4E.5.B.6)] *Give an exa of an optor on* $\mathbf{R}^2$
that shows the result above does not hold if $\mathbf{C}$ *is replaced with* $\mathbf{R}$.

SOLUS:
Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(w,z) = (-z,w)$.
By Exe (4E 5.B.11), $\mathcal{M}\big(T,((1,0),(0,1))\big) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ the min poly of T is z^2+1.
Define p by $p(z) = z^2$. Then $p(T) = T^2 = -I$. Thus $p(T)$ has eigval -1.
While $\nexists\,\lambda \in \mathbf{R}$ suth $-1 = p(\lambda) = \lambda^2$. $\square$

• (4E 5.B.17) *Sup* V *is finid,* $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$, *and* p *is the min poly of* T.
Shat the min poly of $(T-\lambda I)$ *is the poly* q *defined by* $q(z) = p(z+\lambda)$.

SOLUS:
$q(T-\lambda I) = 0 \Rightarrow q$ is poly multi of the min poly of $(T-\lambda I)$.
Sup the deg of the min poly of $(T-\lambda I)$ is n, and the deg of the min poly of T is m.
By definition of min poly,
n is the smallest suth $(T-\lambda I)^n \in \operatorname{span}(I,(T-\lambda I),\ldots,(T-\lambda I)^{n-1})$;
m is the smallest suth $T^m \in \operatorname{span}(I,T,\ldots,T^{m-1})$.
又 $T^k \in \operatorname{span}(I,T,\ldots,T^{k-1}) \iff (T-\lambda)^k \in \operatorname{span}(I,(T-\lambda I),\ldots,(T-\lambda I)^{k-1})$.
Thus $n = m$. 又 q is monic. By the uniqnes of min poly. $\square$

• (4E 5.B.18) *Sup* V *is finid,* $T \in \mathcal{L}(V), \lambda \in \mathbf{F}\backslash\{0\}$, *and* p *is the min poly of* T.
Shat the min poly of λT *is the poly* q *defined by* $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.

SOLUS:
$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow q$ is a poly multi of the min poly of λT.
Sup the deg of the min poly of λT is n, and the deg of the min poly of T is m.
By definition of min poly,
n is the smallest suth $(\lambda T)^n \in \operatorname{span}(\lambda I,\lambda T,\ldots,(\lambda T)^{n-1})$;
m is the smallest suth $T^m \in \operatorname{span}(I,T,\ldots,T^{m-1})$.
又 $(\lambda T)^k \in \operatorname{span}(\lambda I,\lambda T,\ldots,(\lambda T)^{k-1}) \iff T^k \in \operatorname{span}(I,T\ldots,T^{k-1})$.
Thus $n = m$. 又 q is monic. By the uniqnes of min poly. $\square$

---

18 [OR (4E 5.B.15)] *Sup* V *is a finid complex vecsp with* $\dim V > 0$ *and* $T \in \mathcal{L}(V)$.
Define $f : \mathbf{C} \to \mathbf{R}$ *by* $f(\lambda) = \dim \operatorname{range}(T-\lambda I)$.
Provt f *is not a continuous function.*

SOLUS: Note that V is finid.
Let λ_0 be an eigval of T. Then $(T-\lambda_0 I)$ is not surj. Hence $\dim\operatorname{range}(T-\lambda_0 I) < \dim V$.
Becs T has finily many eigvals. There exist a sequence of number $\{\lambda_n\}$ suth $\lim\limits_{n\to\infty}\lambda_n = \lambda_0$.

And λ_n is not an eigval of T for each $n \Rightarrow \dim \operatorname{range}(T-\lambda_n I) = \dim V \neq \dim \operatorname{range}(T-\lambda_0 I)$.
Thus $f(\lambda_0) \neq \lim_{n\to\infty} f(\lambda_n)$. $\square$

- (4E 5.B.9) *Sup $T \in \mathcal{L}(V)$ is suth with resp to some basis of V,*
all ent of the matrix of T are rational numbers.
Explain why all coeffs of the min poly of T are rational numbers.

Solus:

Let $(v_1,\dots,v_n)$ denote the basis suth $\mathcal{M}(T,(v_1,\dots,v_n))_{j,k} = A_{j,k} \in \mathbf{Q}$ for all $j,k = 1,\dots,n$.
Denote $\mathcal{M}(v_j,(v_1,\dots,v_n))$ by x_j for each v_j.
Sup p is the min poly of T and $p(z) = z^m + \cdots + c_1 z + c_0$. Now we shat each $c_j \in \mathbf{Q}$.
Note that $\forall s \in \mathbf{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbf{Q}^{n,n}$ and $T^s v_k = A^s_{1,k}v_1 + \cdots + A^s_{n,k}v_n$ for all $k \in \{1,\dots,n\}$.

$$\text{Thus} \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \cdots + c_1A + c_0I)x_1 = \sum_{j=1}^{n} (A^m + \cdots + c_1A + c_0I)_{j,1} x_j = 0; \\ \qquad\vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \cdots + c_1A + c_0I)x_n = \sum_{j=1}^{n} (A^m + \cdots + c_1A + c_0I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly,} \begin{cases} (A^m + \cdots + c_1A + c_0I)_{1,1} = \cdots = (A^m + \cdots + c_1A + c_0I)_{n,1} = 0; \\ \qquad\qquad\vdots \quad \ddots \quad \vdots \\ (A^m + \cdots + c_1A + c_0I)_{1,n} = \cdots = (A^m + \cdots + c_1A + c_0I)_{n,n} = 0; \end{cases}$$

Hence we get a system of n^2 linear equations in m unknowns $c_0, c_1, \dots, c_{m-1}$.
We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbf{Q}$. $\square$

- [Or (4E 5.B.16), Or (8.C.18)] *Sup $a_0,\dots,a_{n-1} \in \mathbf{F}$. Let T be the optor on $\mathbf{F}^n$ suth*

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ with resp to the std basis } (e_1,\dots,e_n).$$

Shat the min poly of T is p defined by $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$.

*$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the min poly of some optor. Hence a formula or an algo that could produce exact eigvals for each optor on each $\mathbf{F}^n$ could then produce exact zeros for each poly [by 8.36(b)]. Thus there is no such formula or algo. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an optor.*

Solus: Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ is linely inde. 又 The deg of min poly is at most n.
$T^n e_1 = \cdots = T^{n-k}e_{1+k} = \cdots = Te_n = -a_0e_1 - a_1e_2 - a_2e_3 - \cdots - a_{n-1}e_n$
$= (-a_0I - a_1T - a_2T^2 - \cdots - a_{n-1}T^{n-1})e_1$. Thus $p(T)e_1 = 0 = p(T)e_j$ for each $e_j = T^{j-1}e_1$. $\square$

- Eigenvalues On Odd-Dimensional Real Vector Spaces
- Even-Dimensional Null Space

Sup $\mathbf{F} = \mathbf{R}$, V is finid, $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$.
Provt $\dim \operatorname{null}(T^2 + bT + cI)$ is an even number.

Solus:

Denote $\operatorname{null}(T^2 + bT + cI)$ by R. Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.
Sup λ is an eigval of T_R with an eigvec $v \in R$.
Then $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \left((\lambda + b)^2 + c - \frac{b^2}{4}\right)v$.

Becs $c-\dfrac{b^2}{4}>0$ and we have $v=0$. Thus T_R has no eigvals.

Let U be invarsp of R that has the largest, even dim among all invarsps.

Asm $U \neq R$. Then $\exists\, w \in R$ but $w \notin U$. Let W be suth $(w, T|_R w)$ is a basis of W.

Becs $T|_R^2 w = -bT|_R w - cw \in W$. Hence W is invarsp of dim 2.

Thus $\dim(U+W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,

for if not, becs $w \notin U, T|_R w \in U$,

$U \cap W$ is invard $T|_R$ of one dim (impossible becs $T|_R$ has no eigvecs).

Hence $U+W$ is even-dim invarsp under $T|_R$, ctradic the max of dim U.

Thus the asm was incorrect. Hence $R = \operatorname{null}(T^2+bT+cI) = U$ has even dim. □

• Operators On Odd-Dimensional Vector Spaces Have Eigenvalues

(a) *Sup* $\mathbf{F}=\mathbf{C}$. Then by [5.21], we are done.

(b) *Sup* $\mathbf{F}=\mathbf{R}$, *V is finid, and* $\dim V = n$ *is an odd number.*

Let $T \in \mathcal{L}(V)$ and the min poly is p. Provt T has an eigval.

Solus:

(i) If $n=1$, then we are done.

(ii) Sup $n \geqslant 3$. Asm every optor, on odd-dim vecsps of dim less than n, has an eigval.

If p is a poly multi of $(x-\lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] λ is an eigval of T and we are done.

Now sup $b, c \in \mathbf{R}$ suth $b^2 < 4c$ and p is a poly multi of x^2+bx+c (see [4.17]).

Then $\exists\, q \in \mathcal{P}(\mathbf{R})$ suth $p(x) = q(x)(x^2+bx+c)$ for all $x \in \mathbf{R}$.

Now $0 = p(T) = \big(q(T)\big)(T^2+bT+cI)$, which means that $q(T)|_{\operatorname{range}(T^2+bT+cI)} = 0$.

Becs $\deg q < \deg p$ and p is the min poly of T, hence range $(T^2+bT+cI) \neq V$.

又 $\dim V$ is odd and $\dim\operatorname{null}(T^2+bT+cI)$ is even (by our previous result).

Thus $\dim V - \dim\operatorname{null}(T^2+bT+cI) = \dim\operatorname{range}(T^2+bT+cI)$ is odd.

By [5.18], range $(T^2+bT+cI)$ is invarsp of V under T that has odd dim less than n.

Our induction hypothesis now implies that $T|_{\operatorname{range}(T^2+bT+cI)}$ has an eigval.

By mathematical induction. □

• (2E Ch5.24) *Sup* $\mathbf{F}=\mathbf{R}, T \in \mathcal{L}(V)$ *has no eigvals.*

Provt every invarsp of V under T is even-dim.

Solus:

Sup U is such a subsp. Then $T|_U \in \mathcal{L}(U)$. We prove by ctradic.

If $\dim U$ is odd, then $T|_U$ has an eigval and so is T, so that $\exists$ invarsp of 1 dim, ctradic. □

• (4E 5.B.29) *Shat every optor on a finid vecsp of dim ≥ 2 has a 2-dim invarsp.*

Solus:

Using induction on $\dim V$.

(i) $\dim V = 2$, we are done.

(ii) $\dim V > 2$. Asm the desired result is true for vecsp of smaller dim.

Sup p is the min poly of deg m and $p(z) = (z-\lambda_1)\cdots(z-\lambda_m)$.

If $T = \lambda I$ ($\Leftrightarrow m = 1 \vee m = -\infty$), then we are done. ($m \neq 0$ becs $\dim V \neq 0$.)

Now define a q by $q(z) = (z-\lambda_1)(z-\lambda_2)$.

By asm, $T|_{\operatorname{null} q(T)}$ has invarsp of dim 2. □

5.B: II

9 14 15 20 | 4E: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 11, 12, 13, 14

- (4E 5.C.1) *Prove or give a counterexa*:
  If $T \in \mathcal{L}(V)$ and T^2 has an upper-trig matrix, then T has an upper-trig matrix.

SOLUS:

- (4E 5.C.2) *Sup A and B are upper-trig matrices of the same size, with $\alpha_1, \dots, \alpha_n$ on the diag of A and $\beta_1, \dots, \beta_n$ on the diag of B.*
  (a) *Shat $A + B$ is an upper-trig matrix with $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ on the diag.*
  (b) *Shat AB is an upper-trig matrix with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diag.*

SOLUS:

- (4E 5.C.3)
  Sup $T \in \mathcal{L}(V)$ is inv and $B = (v_1, \dots, v_n)$ is a basis of V suth $\mathcal{M}(T, B) = A$ is upper trig, with $\lambda_1, \dots, \lambda_n$ on the diag.
  Shat the matrix of $\mathcal{M}(T^{-1}, B) = A^{-1}$ is also upper trig, with $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ on the diag.

SOLUS:

---

9 [4E 5.C.7] *Sup V is finid, $T \in \mathcal{L}(V)$, and $v \in V$.*
(a) *Provt $\exists\,!$ monic poly p_v of smallest deg suth $p_v(T)v = 0$.*
(b) *Provt the min poly of T is a poly multi of p_v.*

SOLUS:

---

14 [OR (4E 5.C.4)] *Give an optor T suth with resp to some basis, $\mathcal{M}(T)_{k,k} = 0$ for each k, while T is inv.*

SOLUS:

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15 [OR (4E 5.C.5)] *Give an optor T suth with resp to some basis, $\mathcal{M}(T)_{k,k} \neq 0$ for each k, while T is not inv.*

SOLUS:

---

20 [OR (OR 4E 5.C.6)]
Sup $\mathbf{F} = \mathbf{C}$, V is finid, and $T \in \mathcal{L}(V)$.
Provt if $k \in \{1, \dots, \dim V\}$, then V has a k dim subsp invard T.

SOLUS:

- (4E 5.C.8) *Sup V is finid, $T \in \mathcal{L}(V)$, and $\exists\, v \in V \backslash \{0\}$ suth $T^2 v + 2Tv = -2v$.*
  (a) *Provt if $\mathbf{F} = \mathbf{R}$, then $\nexists$ a basis of V with resp to which T has an upper-trig matrix.*
  (b) *Provt if $\mathbf{F} = \mathbf{C}$ and A is an upper-trig matrix that equals the matrix of T with resp to some basis of V, then $-1 + \mathrm{i}$ or $-1 - \mathrm{i}$ appears on the diag of A.*

SOLUS:

- (4E 5.C.9) *Sup $B \in \mathbf{F}^{n,n}$ with complex ent.*

Provt $\exists$ inv $A \in \mathbf{F}^{n,n}$ with complex ent suth $A^{-1}BA$ is an upper-trig matrix.

Solus:

- (4E 5.C.10) *Sup $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V.*
  Shat the following are equi.
  (a) *The matrix of T with resp to $(v_1, \ldots, v_n)$ is lower trig.*
  (b) $\operatorname{span}(v_k, \ldots, v_n)$ *is invard T for each $k = 1, \ldots, n$.*
  (c) $Tv_k \in \operatorname{span}(v_k, \ldots, v_n)$ *for each $k = 1, \ldots, n$.*

Solus:

- (4E 5.C.11) *Sup $\mathbf{F} = \mathbf{C}$ and V is finid.*
  Provt if $T \in \mathcal{L}(V)$, then T has a lower-trig matrix with resp to some basis.

Solus:

- (4E 5.C.12)
  Sup V is finid, $T \in \mathcal{L}(V)$ has an upper-trig matrix with resp to some basis,
  and U is a subsp of V that is invard T.
  (a) *Provt $T|_U$ has an upper-trig matrix with resp to some basis of U.*
  (b) *Provt T/U has an upper-trig matrix with resp to some basis of V/U.*

Solus:

- (4E 5.C.13) *Sup V is finid, $T \in \mathcal{L}(V)$. Sup U is invarsp of V under T*
  suth $T|_U, T/U$ have upper-trig matrix.
  Provt T has upper-trig matrix.

Solus:

- (4E 5.C.14) *Sup V is finid and $T \in \mathcal{L}(V)$.*
  Provt T has upper-trig matrix $\iff$ T' has upper-trig matrix.

Solus:

Ended

5.C

XXXX

Ended

5.E* [4E] 1 2 3 4 5 6 7 8 9 10

1 *Give commu optors $S, T \in \mathbf{F}^4$ suth*
$\exists$ subsp of $\mathbf{F}^4$ invard S but not T and $\exists$ subsp of $\mathbf{F}^4$ invard T but not S.

Solus:

---

2 *Sup $\mathcal{E}$ is a subset of $\mathcal{L}(V)$ and every ele of $\mathcal{E}$ is diag.*
Provt $\exists$ a basis of V with resp to which
every ele of $\mathcal{E}$ has a diag matrix $\iff$ every pair of ele of $\mathcal{E}$ commu.

This exercise extends [5.76], which considers the case in which $\mathcal{E}$ contains only two ele.
For this exercise, $\mathcal{E}$ may contain any number of ele, and $\mathcal{E}$ may even be an infini set.

Solus:

---

3 *Sup* $S, T \in \mathcal{L}(V)$ *are suth* $ST = TS$. *Sup* $p \in \mathcal{P}(\mathbf{F})$.
(a) *Provt* $\operatorname{null} p(S)$ *is invard* T. (b) *Provt* $\operatorname{range} p(S)$ *is invard* T.
See Note For[5.17] *for the special case* $S = T$.

Solus:

---

4 *Prove or give a counterexa*: *A diag matrix A and upper-trig matrix B of the same size commu.*

Solus:

---

5 *Provt a pair of optors on a finid vecsp commu* $\Longleftrightarrow$ *their dual optors commu.*

Solus:

---

6 *Sup* V *is a finid complex vecsp and* $S, T \in \mathcal{L}(V)$ *commu.*
Provt $\exists\, \alpha, \lambda \in \mathbf{C}$ *suth* $\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V$.

Solus:

---

7 *Sup* V *is a complex vecsp,* $S \in \mathcal{L}(V)$ *is diag, and* T *commu with* S.
Provt $\exists$ *basis* B *of* V *suth* S *has a diag matrix with resp to* B
and T *has upper-trig matrix with resp to* B.

Solus:

---

8 *Sup* $m = 3$ *in* [5.72] *and* D_x, D_y *are the commu partial diff optors on* $\mathcal{P}_3(\mathbf{R}^2)$ *from* [5.72].
Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ *with resp to which* D_x *and* D_y *each have upper-trig matrix.*

Solus:

---

9 *Sup* V *is a finid nonzero complex vecsp.*
Sup that $\mathcal{E} \subseteq \mathcal{L}(V)$ *is suth* S *and* T *commu for all* $S, T \in \mathcal{E}$.
(a) *Provt* $\exists$ *eigvec* $v \in V$ *for every ele of* $\mathcal{E}$.
(b) *Provt* $\exists$ *a basis of* V *with resp to which every ele of* $\mathcal{E}$ *has upper-trig matrix.*

Solus:

---

10 *Give commu optors* S, T *on a finid real vecsp suth*
$S + T$ *has a eigval that does not equal an eigval of* S *plus an eigval of* T
and ST *has a eigval that does not equal an eigval of* S *times an eigval of* T.

Solus:

Ended