



简介

这是我个人用于复习的「*Linear Algebra Done Right 3E/4E, by Sheldon Axler*」笔记，一本习题选答与课文补注。因为使用中文会给我编撰这份笔记带来额外的中英文输入法切换的工作成本，况且对于专业学习者，直接使用英文不会造成任何困扰。但英文词句的冗长性拖慢我复习的效率，所以我对许多常用术语作了简写。这份笔记的内容范围和标识说明，我已经在自述中写得很清楚，不再赘述。这份笔记尚处于缓慢的编撰进度中。

我目前还没有能力和资格评论原书好坏以及线性代数课程教材选用的问题。但作为原书的学习者，我可以说，这本书作为初学线性代数的第一教材，虽然不需要其他辅助教材，但要求学习者有足够的耐心和毅力：课文一次看不懂就多看几遍，一天看不懂就分三天看；习题一个小时做不出来，隔六个小时再尝试，一天做不出来，就隔天再尝试。我虽然没有学过除此以外的其他任何线性代数教材，但我认为这样钻研原书是值得的。

Goto

1	2	3	4	5	6	7	8	9	10
A	A	A		A	A	A	A	A	A
B	B	B		B <sup>I</sup>	B	B	B	B	B
				B <sup>II</sup>					
C	C	C		C	C	C	C		
		D			D	D	D		
		E		E*					
		F				F*			

ABBREVIATION TABLE

def	definition	vec	vector
vecsp	vector space	subsp	subspace
add	addition/additive	multi	multiplication/multiplicative/multiple
assoc	associative/associativity	distr	distributive properties/property
inv	inverse	existns	existence
uniques	uniqueness	linely inde	linearly independent/independence
linely dep	linearly dependent/dependence	dim	dimension(al)
coeff	coefficient	degree	deg
req	require(d)/requiring	$B_V$	basis of $V$
inje	injective	surj	surjective
col	column	with resp	with respect
standard basis	std basis	iso	isomorphism/isomorphic
correspd	correspond(ing)	poly	polynomial
eigval	eigenvalue	eigvec	eigenvector
mini poly	minimal polynomial	char poly	characteristic polynomial

# 1.B

1 Prove that  $\forall v \in V, -(-v) = v$ .

SOLUTION:  $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$ .

OR. Because  $-(-v) + (-v) = 0$  又  $v + (-v) = 0$ . Now by the uniqueness of add inv.  $\square$

2 Suppose  $a \in \mathbf{F}, v \in V$ , and  $av = 0$ . Prove that  $a = 0$  or  $v = 0$ .

SOLUTION: Suppose  $a \neq 0, \exists a^{-1} \in \mathbf{F}, a^{-1}a = 1$ , hence  $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$ .  $\square$

3 Suppose  $v, w \in V$ . Explain why  $\exists! x \in V, v + 3x = w$ .

SOLUTION:  $v + 3x = w \Leftrightarrow 3x = w - v \Leftrightarrow x = \frac{1}{3}(w - v)$ .  $\square$

OR. [Existence] Let  $x = \frac{1}{3}(w - v)$ .

[Uniqueness] If  $v + 3x_1 = w, (I) v + 3x_2 = w (II)$ . Then  $(I) - (II) : 3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$ .  $\square$

5 Show that in the def of a vecsp, the add inv condition can be replaced by [1.29].

Hint: Suppose  $V$  satisfies all conds in the def, except we've replaced the add inv cond with [1.29].

Prove that the add inv is true.

Using [1.31].  $0v = 0$  for all  $v \in V \Leftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$ .  $\square$

6 Let  $\infty$  and  $-\infty$  denote two distinct objects, neither of which is in  $\mathbf{R}$ .

Define an add and scalar multi on  $\mathbf{R} \cup \{\infty, -\infty\}$  as you could guess.

The operations of real numbers is as usual. While for  $t \in \mathbf{R}$  define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

$$(I) t + \infty = \infty + t = \infty + \infty = \infty,$$

$$(II) t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty,$$

$$(III) \infty + (-\infty) = (-\infty) + \infty = 0.$$

With these operations of add and scalar multi, is  $\mathbf{R} \cup \{\infty, -\infty\}$  a vecsp over  $\mathbf{R}$ ? Explain.

SOLUTION: Not a vecsp, since the add and scalar multi is not assoc and distr.

By Assoc:  $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$ .

OR. By Distr:  $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$ .  $\square$

• TIPS: About the Field  $\mathbf{F}$ : Many choices.

EXAMPLE:  $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \dots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$ . [Using Euler's Theorem.]

ENDED

# 1.C

7 8 9 11 12 13 15 16 17 18 21 23 24

• NOTE FOR [1.45]: If  $\mathbf{F} = \{0, 1\}$ . Prove that if  $U + W$  is a direct sum, then  $U \cap W = \{0\}$ .

Because  $\forall v \in U \cap W, \exists! (u, w) \in U \times W, v = u + w$ .

If  $U \cap W \neq \{0\}$ , then  $(u, w)$  can be  $(v, 0)$  or  $(0, v)$ , contradicts the uniqueness.  $\square$

• **TIPS:** Suppose  $U, W \subseteq V$ . And  $U, W, V$  are vecsps  $\Rightarrow U, W$  are subsp of  $V$ .

Then  $U + W$  is also a subsp of  $V$ . Because  $\forall u \in U, w \in U, u + w \in V$  since  $u, w \in V$ .

**7** Give a nonempty  $U \subseteq \mathbb{R}^2$ ,

$U$  is closed under taking add invs and under add, but is not a subsp of  $\mathbb{R}^2$ .

**SOLUTION:** ( $0 \in U$ ;  $v \in U \Rightarrow -v \in U$ . And operations on  $U$  are the same as  $\mathbb{R}^2$ .) Let  $\mathbb{Z}^2, \mathbb{Q}^2$ .

**8** Give a nonempty  $U \subseteq \mathbb{R}^2$ ,  $U$  is closed under scalar multi, but is not a subsp of  $\mathbb{R}^2$ .

**SOLUTION:** Let  $U = \{(x, y) \in \mathbb{R}^2 : x = 0 \vee y = 0\}$ .

**9** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called periodic if  $\exists p \in \mathbb{N}^+, f(x) = f(x + p)$  for all  $x \in \mathbb{R}$ .

Is the set of periodic functions  $\mathbb{R} \rightarrow \mathbb{R}$  a subsp of  $\mathbb{R}^{\mathbb{R}}$ ? Explain.

**SOLUTION:** Denote the set by  $S$ .

Suppose  $h(x) = \cos x + \sin \sqrt{2}x \in S$ , since  $\cos x, \sin \sqrt{2}x \in S$ .

Assume  $\exists p \in \mathbb{N}^+$  such that  $h(x) = h(x + p)$ ,  $\forall x \in \mathbb{R}$ . Let  $x = 0 \Rightarrow h(0) = h(\pm p) = 1$ .

Thus  $1 = \cos p + \sin \sqrt{2}p = \cos p - \sin \sqrt{2}p$

$\Rightarrow \sin \sqrt{2}p = 0, \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbb{Z}$ , while  $p = \frac{m\pi}{\sqrt{2}}, m \in \mathbb{Z}$ .

Hence  $2k = \frac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \frac{m}{2k} \in \mathbb{Q}$ . Contradiction! □

OR. Because [I] :  $\cos x + \sin \sqrt{2}x = \cos(x + p) + \sin(\sqrt{2}x + \sqrt{2}p)$ . By differentiating twice,

[II] :  $\cos x + 2 \sin \sqrt{2}x = \cos(x + p) + 2 \sin(\sqrt{2}x + \sqrt{2}p)$ .

[II] - [I] :  $\sin \sqrt{2}x = \sin(\sqrt{2}x + \sqrt{2}p)$   
 $2[I] - [II] :$   $\cos x = \cos(x + p)$   $\left\} \Rightarrow \text{Let } x = 0, p = \frac{m\pi}{\sqrt{2}} = 2k\pi. \text{ Contradicts.} \right.$  □

• Suppose  $U, W, V_1, V_2, V_3$  are subsp of  $V$ .

**15**  $U + U \ni u + w \in U$ . **16**  $U + W \ni u + w = w + u \in W + U$ . □

**17**  $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3)$ . □

•  $(U + W)_C \ni (u_1 + w_1) + i(u_2 + w_2) = (u_1 + iu_2) + (w_1 + iw_2) \in U_C + W_C$ . □

**18** Does the add on the subsp of  $V$  have an add identity? Which subsp have add invs?

**SOLUTION:** Suppose  $\Omega$  is the unique add identity.

(a) For any subsp  $U$  of  $V$ .  $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$ . Let  $U = \{0\}$ , then  $\Omega = \{0\}$ .

(b) Now suppose  $W$  is an add inv of  $U \Rightarrow U + W = \Omega$ .

Note that  $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$ . Thus  $U = W = \Omega = \{0\}$ . □

**11** Prove that the intersection of every collection of subsp of  $V$  is a subsp of  $V$ .

**SOLUTION:** Suppose  $\{U_\alpha\}_{\alpha \in \Gamma}$  is a collection of subsp of  $V$ ; here  $\Gamma$  is an index set.

We show that  $\bigcap_{\alpha \in \Gamma} U_\alpha$ , which equals the set of vecs that are in  $U_\alpha$  for each  $\alpha \in \Gamma$ , is a subsp of  $V$ .

(一)  $0 \in \bigcap_{\alpha \in \Gamma} U_\alpha$ . Nonempty.

(二)  $u, v \in \bigcap_{\alpha \in \Gamma} U_\alpha \Rightarrow u + v \in U_\alpha, \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha \in \Gamma} U_\alpha$ . Closed under add.

(三)  $u \in \bigcap_{\alpha \in \Gamma} U_\alpha, \lambda \in \mathbb{F} \Rightarrow \lambda u \in U_\alpha, \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha \in \Gamma} U_\alpha$ . Closed under scalar multi.

Thus  $\bigcap_{\alpha \in \Gamma} U_\alpha$  is nonempty subset of  $V$  that is closed under add and scalar multi. □

**12** Suppose  $U, W$  are subsp of  $V$ . Prove that  $U \cup W$  is a subsp of  $V \iff U \subseteq W$  or  $W \subseteq U$ .

**SOLUTION:** (a) Suppose  $U \subseteq W$ . Then  $U \cup W = W$  is a subsp of  $V$ .

(b) Suppose  $U \cup W$  is a subsp of  $V$ . Assume that  $U \not\subseteq W, U \not\supseteq W$  ( $U \cup W \neq U$  and  $W$ ).  
Then  $\forall a \in U \wedge a \notin W, \forall b \in W \wedge b \notin U$ , we have  $a + b \in U \cup W$ .  
 $a + b \in U \Rightarrow b = (a + b) + (-a) \in U$ , contradicts  $\Rightarrow W \subseteq U$ .  
 $a + b \in W \Rightarrow a = (a + b) + (-b) \in W$ , contradicts  $\Rightarrow U \subseteq W$ . | Contradicts the assumption.  $\square$

**13** Prove that the union of three subsp of  $V$  is a subsp of  $V$  if and only if one of the subsp contains the other two.

*This exercise is not true if we replace  $\mathbf{F}$  with a field containing only two elements.*

**SOLUTION:**

Suppose  $U_1, U_2, U_3$  are subsp of  $V$ . Denote  $U_1 \cup U_2 \cup U_3$  by  $\mathcal{U}$ .

(a) Suppose that one of the subsp contains the other two.

Then  $\mathcal{U} = U_1, U_2$  or  $U_3$  is a subsp of  $V$ .

(b) Suppose that  $U_1 \cup U_2 \cup U_3$  is a subsp of  $V$ .

Distinctively notice that  $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$ .

Also note that, if  $U \cup W = V$  is a vecsp, then in general  $U$  and  $W$  are not subsp of  $V$ .

Hence this literal trick is invalid.

(I) If any  $U_j$  is contained in the union of the other two, say  $U_1 \subseteq U_2 \cup U_3$ , then  $\mathcal{U} = U_2 \cup U_3$ .

By applying Problem (12) we conclude that one  $U_j$  contains the other two. Thus we are done.

(II) Assume that no  $U_j$  is contained in the union of the other two,

and no  $U_j$  contains the union of the other two. Say  $U_1 \not\subseteq U_2 \cup U_3$  and  $U_1 \not\supseteq U_2 \cup U_3$ .

$\exists u \in U_1 \wedge u \notin U_2 \cup U_3; v \in U_2 \cup U_3 \wedge v \notin U_1$ . Let  $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$ .

Note that  $W \cap U_1 = \emptyset$ , for if any  $v + \lambda u \in W \cap U_1$  then  $v + \lambda u - \lambda u = v \in U_1$ .

Now  $W \subseteq U_1 \cup U_2 \cup U_3 \Rightarrow W \subseteq U_2 \cup U_3$ .  $\forall v + \lambda u \in W, v + \lambda u \in U_i, i = 2, 3$ .

If  $U_2 \subseteq U_3$  or  $U_2 \supseteq U_3$ , then  $\mathcal{U} = U_1 \cup U_i, i = 2, 3$ . By Problem (12) we are done.

Otherwise, both  $U_2, U_3 \neq \{0\}$ . Because  $W \subseteq U_2 \cup U_3$  has at least three elements.

There must be some  $U_i$  that contains at least two elements of  $W$ .

$\exists$  distinct  $\lambda_1, \lambda_2 \in \mathbf{F}, v + \lambda_1 u, v + \lambda_2 u \in U_i, i \in \{2, 3\}$ .

Then  $u \in U_i$  while  $u \notin U_2 \cup U_3$ . Contradicts.  $\square$

**EXAMPLE:** Let  $\mathbf{F} = \mathbf{Z}_2$ .  $U_1 = \{u, 0\}, U_2 = \{v, 0\}, U_3 = \{v + u, 0\}$ . While  $\mathcal{U} = \{0, u, v, v + u\}$  is a subsp.

• **EXAMPLE:** Suppose  $U = \{(x, x, y, y) \in \mathbf{F}^4\}, W = \{(x, x, x, y) \in \mathbf{F}^4\}$ .

Prove that  $U + W = \{(x, x, y, z) \in \mathbf{F}^4\}$ .

Let  $T$  denote  $\{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$ . By def,  $U + W \subseteq T$ .

And  $T \ni (x, x, y, z) \Rightarrow (0, 0, y - x, y - x) + (x, x, x, -y + x + z) \in U + W$ . Hence  $T \subseteq U + W$ .  $\square$

**21** Suppose  $U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5\}$ . Find a  $W$  such that  $\mathbf{F}^5 = U \oplus W$ .

**SOLUTION:** Let  $W = \{(0, 0, z, w, u) \in \mathbf{F}^5\}$ . Then  $U \cap W = \{0\}$ .

And  $\mathbf{F}^5 \ni (x, y, z, w, u) \Rightarrow (x, y, x + y, x - y, 2x) + (0, 0, z - x - y, w - x - y, u - 2x) \in U + W$ .

**23** Give an example of vecsps  $V_1, V_2, U$  such that  $V_1 \oplus U = V_2 \oplus U$ , but  $V_1 \neq V_2$ .

**SOLUTION:**  $V = \mathbf{F}^2, U = \{(x, x) \in \mathbf{F}^2\}, V_1 = \{(x, 0) \in \mathbf{F}^2\}, V_2 = \{(0, x) \in \mathbf{F}^2\}$ .

• **TIPS:** Suppose  $V_1 \subseteq V_2$  in Exercise (23). Prove or give a counterexample:  $V_1 = V_2$ .

**SOLUTION:**

Because the subset  $V_1$  of vecsp  $V_2$  is closed under add and scalar multi,  $V_1$  is a subspace of  $V_2$ .

Suppose  $W$  is such that  $V_2 = V_1 \oplus W$ . Now  $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$ .

If  $W \neq \{0\}$ , then  $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$ , contradicts. Hence  $W = \{0\}, V_1 = V_2$ .  $\square$

• Suppose  $V_1, V_2, U_1, U_2$  are vecsps,  $V_1 \oplus U_1 = V_2 \oplus U_2, V_1 \subseteq V_2, U_2 \subseteq U_1$ .

Prove or give a counterexample:  $V_1 = V_2, U_1 = U_2$ .

$V_1$	$U_1$
$V_2$	$U_2$

**SOLUTION:** A counterexample: [ Using notations in Chapter 2. ]

Let  $V = \mathbf{F}^3, B_V = (e_1, e_2, e_3), V_1 = \text{span}(e_1), U_1 = \text{span}(e_2, e_3), V_2 = \text{span}(e_1, e_2), U_2 = \text{span}(e_3)$ .

Now  $V_1 \subseteq V_2, U_2 \subseteq U_1$  and  $V_1 \oplus U_1 = V_2 \oplus U_2$ . But  $V_1 \neq V_2, U_1 \neq U_2$ .  $\square$

**24** Let  $V_E = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is even}\}, V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f \text{ is odd}\}$ . Show that  $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$ .

**SOLUTION:** (a)  $V_E \cap V_O = \{f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x)\} = \{0\}$ .

$$(b) \left\{ \begin{array}{l} \text{Let } f_e(x) = \frac{1}{2} [g(x) + g(-x)] \Rightarrow f_e \in V_E \\ \text{Let } f_o(x) = \frac{1}{2} [g(x) - g(-x)] \Rightarrow f_o \in V_O \end{array} \right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}}, g(x) = f_e(x) + f_o(x). \quad \square$$

**ENDED**

## 2.A 1 2 6 10 11 14 16 17 | 4E: 3,14

**2** (a) [P] A list  $(v)$  of length 1 in  $V$  is linely inde  $\iff v \neq 0$ . [Q]

(b) [P] A list  $(v, w)$  of length 2 in  $V$  is linely inde  $\iff \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$ . [Q]

**SOLUTION:**

(a)  $Q \xrightarrow{1} P : v \neq 0 \Rightarrow$  if  $av = 0$  then  $a = 0 \Rightarrow (v)$  linely inde.

$P \xrightarrow{2} Q : (v)$  linely inde  $\Rightarrow v \neq 0$ , for if  $v = 0$ , then  $av = 0 \not\Rightarrow a = 0$ .

OR.  $\left\{ \begin{array}{l} \neg Q \xrightarrow{3} \neg P : v = 0 \Rightarrow av = 0 \text{ while we can let } a \neq 0 \Rightarrow (v) \text{ is linely dep.} \\ \neg P \xrightarrow{4} \neg Q : (v) \text{ linely dep} \Rightarrow av = 0 \text{ while } a \neq 0 \Rightarrow v = 0. \end{array} \right.$

**COMMENT:** (1) with (3) and (2) with (4) will do as well.  $\square$

(b)  $P \xrightarrow{1} Q : (v, w)$  linely inde  $\Rightarrow$  if  $av + bw = 0$ , then  $a = b = 0 \Rightarrow$  no scalar multi.

$Q \xrightarrow{2} P : \text{no scalar multi} \Rightarrow$  if  $av + bw = 0$ , then  $a = b = 0 \Rightarrow (v, w)$  linely inde.

OR.  $\left\{ \begin{array}{l} \neg P \xrightarrow{3} \neg Q : (v, w) \text{ linely dep} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{scalar multi} \\ \neg Q \xrightarrow{4} \neg P : \text{scalar multi} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{linely dep.} \end{array} \right.$

**COMMENT:** (1) with (3) and (2) with (4) will do as well.  $\square$

**1** Prove that  $[P] (v_1, v_2, v_3, v_4)$  spans  $V \iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$  also spans  $V [Q]$ .

**SOLUTION:**

Notice that  $V = \text{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists a_1, \dots, a_n \in \mathbb{F}, v = a_1 v_1 + \dots + a_n v_n$ .

Assume that  $\forall v \in V, \exists a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbb{F}$ , ( that is, if  $\exists a_i$ , then we are to find  $b_i$ , vice versa )

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= b_1 (v_1 - v_2) + b_2 (v_2 - v_3) + b_3 (v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4. \end{aligned}$$

Now we can let  $b_i = \sum_{r=1}^i a_r$  if we are to prove  $Q$  with  $P$  already assumed;

or let  $a_i = b_i - b_{i-1}$  with  $b_0 = 0$ , if we are to prove  $P$  with  $Q$  already assumed.  $\square$

**6** Prove that  $[P] (v_1, v_2, v_3, v_4)$  is linely inde  $\iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$  is linely inde.  $[Q]$

**SOLUTION:**

$$\begin{aligned} P \Rightarrow Q : a_1 (v_1 - v_2) + a_2 (v_2 - v_3) + a_3 (v_3 - v_4) + a_4 v_4 &= 0 \\ \Rightarrow a_1 v_1 + (a_2 - a_1) v_2 + (a_3 - a_2) v_3 + (a_4 - a_3) v_4 &= 0 \Rightarrow a_1 = a_2 = a_3 = a_4 = 0 \end{aligned}$$

$$\begin{aligned} Q \Rightarrow P : a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 &= 0 \\ \Rightarrow a_1 (v_1 - v_2) + (a_1 + a_2) (v_2 - v_3) + (a_1 + a_2 + a_3) (v_3 - v_4) + (a_1 + \dots + a_4) v_4 &= 0 \\ \Rightarrow a_1 = a_1 + a_2 = a_1 + a_2 + a_3 = a_1 + \dots + a_4 = 0. \end{aligned} \quad \square$$

• Suppose  $(v_1, \dots, v_m)$  is a list of vecs in  $V$ . For each  $k$ , let  $w_k = v_1 + \dots + v_k$ .

(a) Show that  $\text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$ .

(b) Show that  $[P] (v_1, \dots, v_m)$  is linely inde  $\iff (w_1, \dots, w_m)$  is linely inde  $[Q]$ .

**SOLUTION:**

(a) Assume  $a_1 v_1 + \dots + a_m v_m = b_1 w_1 + \dots + b_m w_m = b_1 v_1 + \dots + b_k (v_1 + \dots + v_k) + \dots + b_m (v_1 + \dots + v_m)$ .

Then  $a_k = b_k + \dots + b_m$ ;  $a_{k+1} = b_{k+1} + \dots + b_m \Rightarrow b_k = a_k - a_{k+1}$ ;  $b_m = a_m$ . Similar to Problem (1).

(b)  $P \Rightarrow Q$ :  $b_1 w_1 + \dots + b_m w_m = 0 = a_1 v_1 + \dots + a_m v_m$ , where  $0 = a_k = b_k + \dots + b_m$ .

$Q \Rightarrow P$ :  $a_1 v_1 + \dots + a_m v_m = 0 = b_1 w_1 + \dots + b_m w_m = 0$ , where  $0 = b_m = a_m$ ,  $0 = b_k = a_k - a_{k+1}$ .

OR. Because  $W = \text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$ .

By [2.21](b), a list of length  $(m - 1)$  spans  $W$ , then by [2.23],

$(w_1, \dots, w_m)$  linely dep  $\implies (v_1, \dots, v_m)$  linely dep. Conversely it is true as well.  $\square$

**10** Suppose  $(v_1, \dots, v_m)$  is linely inde in  $V$  and  $w \in V$ .

Prove that if  $(v_1 + w, \dots, v_m + w)$  is linely depe, then  $w \in \text{span}(v_1, \dots, v_m)$ .

**SOLUTION:**

Suppose  $a_1 (v_1 + w) + \dots + a_m (v_m + w) = 0, \exists a_i \neq 0 \Rightarrow a_1 v_1 + \dots + a_m v_m = -(a_1 + \dots + a_m) w$ .

Then  $a_1 + \dots + a_m \neq 0$ , for if not,  $a_1 v_1 + \dots + a_m v_m = 0$  while  $a_i \neq 0$  for some  $i$ , contradicts.  $\square$

OR. By contrapositive: Prove that  $w \notin \text{span}(v_1, \dots, v_m) \implies (v_1 + w, \dots, v_m + w)$  is linely inde.

Suppose  $a_1 (v_1 + w) + \dots + a_m (v_m + w) = 0 \Rightarrow a_1 v_1 + \dots + a_m v_m = -(a_1 + \dots + a_m) w$ .

Now by assumption,  $a_1 + \dots + a_m = 0$ . Then  $a_1 v_1 + \dots + a_m v_m = 0 \Rightarrow a_1 = \dots = a_m = 0$ .  $\square$

OR.  $\exists j \in \{1, \dots, m\}, v_j + w \in \text{span}(v_1 + w, \dots, v_{j-1} + w)$ . If  $j = 1$  then  $v_1 + w = 0$  and we are done.

If  $j \geq 2$ , then  $\exists a_i \in \mathbb{F}, v_j + w = a_1 (v_1 + w) + \dots + a_{j-1} (v_{j-1} + w) \iff v_j + \lambda w = a_1 v_1 + \dots + a_{j-1} v_{j-1}$ .

Where  $\lambda = 1 - (a_1 + \dots + a_{j-1})$ . Note that  $\lambda \neq 0$ , for if not,  $v_j + \lambda w = v_j \in \text{span}(v_1, \dots, v_{j-1})$ , contradicts.

Now  $w = \lambda^{-1} (a_1 v_1 + \dots + a_{j-1} v_{j-1} - v_j) \Rightarrow w \in \text{span}(v_1, \dots, v_m)$ .  $\square$



**11** Suppose  $(v_1, \dots, v_m)$  is linely inde in  $V$  and  $w \in V$ .

Show that  $[P] (v_1, \dots, v_m, w)$  is linely inde  $\iff w \notin \text{span}(v_1, \dots, v_m) [Q]$ .

**SOLUTION:**  $\neg Q \Rightarrow \neg P$ : Suppose  $w \in \text{span}(v_1, \dots, v_m)$ . Then  $(v_1, \dots, v_m, w)$  is linely depe.

$\neg P \Rightarrow \neg Q$ : Suppose  $(v_1, \dots, v_m, w)$  is linely dep. Then by [2.21](a),  $w \in \text{span}(v_1, \dots, v_m)$ .  $\square$

**14** Prove that  $[P] V$  is infinite-dim  $\iff [Q] \left| \begin{array}{l} \text{there is a sequence } (v_1, v_2, \dots) \text{ in } V \text{ such that} \\ (v_1, \dots, v_m) \text{ is linely inde for each } m \in \mathbb{N}^+. \end{array} \right.$

**SOLUTION:**

$P \Rightarrow Q$ : Suppose  $V$  is infinite-dim, so that no list spans  $V$ .

**Step 1** Pick a  $v_1 \neq 0$ ,  $(v_1)$  linely inde.

**Step m** Pick a  $v_m \notin \text{span}(v_1, \dots, v_{m-1})$ , by Problem (11),  $(v_1, \dots, v_m)$  is linely inde.

This process recursively defines the desired sequence  $(v_1, v_2, \dots)$ .

$\neg P \Rightarrow \neg Q$ : Suppose  $V$  is finite-dim and  $V = \text{span}(w_1, \dots, w_m)$ .

Let  $(v_1, v_2, \dots)$  be a sequence in  $V$ , then  $(v_1, v_2, \dots, v_{m+1})$  must be linely dep.

OR.  $Q \Rightarrow P$ : Suppose there is such a sequence.

Choose an  $m$ . Suppose a linely inde list  $(v_1, \dots, v_m)$  spans  $V$ .

Similar to [2.16].  $\exists v_{m+1} \in V \setminus \text{span}(v_1, \dots, v_m)$ . Hence no list spans  $V$ .  $\square$

**16** Prove that the vecsp of all continuous functions in  $\mathbb{R}^{[0,1]}$  is infinite-dim.

**SOLUTION:** Denote the vecsp by  $U$ .

Choose one  $m \in \mathbb{N}^+$ . Suppose  $a_0, \dots, a_m \in \mathbb{R}$  are such that  $p(x) = a_0 + a_1x + \dots + a_mx^m = 0, \forall x \in [0, 1]$ .

Then  $p$  has infinitely many roots and hence each  $a_k = 0$ , otherwise  $\deg p \geq 0$ , contradicts [4.12].

Thus  $(1, x, \dots, x^m)$  is linely inde in  $\mathbb{R}^{[0,1]}$ . Similar to [2.16],  $U$  is infinite-dim.  $\square$

OR. Note that  $\frac{1}{1} > \frac{1}{2} > \dots > \frac{1}{m}, \forall m \in \mathbb{N}^+$ . Suppose  $f_m = \begin{cases} x - \frac{1}{m}, & x \in \left(\frac{1}{m}, 1\right] \\ 0, & x \in \left[0, \frac{1}{m}\right] \end{cases}$

Then  $f_1\left(\frac{1}{m}\right) = \dots = f_m\left(\frac{1}{m}\right) = 0 \neq f_{m+1}\left(\frac{1}{m}\right)$ . Hence  $f_{m+1} \notin \text{span}(f_1, \dots, f_m)$ . By Problem (14).  $\square$

**17** Suppose  $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbb{F})$  such that  $p_k(2) = 0$  for each  $k \in \{0, \dots, m\}$ .

Prove that  $(p_0, p_1, \dots, p_m)$  is not linely inde in  $\mathcal{P}_m(\mathbb{F})$ .

**SOLUTION:**

Suppose  $(p_0, p_1, \dots, p_m)$  is linely inde. Define  $p \in \mathcal{P}_m(\mathbb{F})$  by  $p(z) = z$ .

NOTICE that  $\forall a_i \in \mathbb{F}, z \neq a_0p_0(z) + \dots + a_mp_m(z)$ , for if not, let  $z = 2$ . Thus  $z \notin \text{span}(p_0, p_1, \dots, p_m)$ .

Then  $\text{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbb{F})$  while the list  $(p_0, p_1, \dots, p_m)$  has length  $(m+1)$ .

Hence  $(p_0, p_1, \dots, p_m)$  is linely depe in  $\mathcal{P}_m(\mathbb{F})$ .

For if not, then because  $(1, z, \dots, z^m)$  of length  $(m+1)$  spans  $\mathcal{P}_m(\mathbb{F})$ ,

by the steps in [2.23] trivially,  $(p_0, p_1, \dots, p_m)$  of length  $(m+1)$  spans  $\mathcal{P}_m(\mathbb{F})$ . Contradicts.  $\square$

OR. Note that  $\mathcal{P}_m(\mathbb{F}) = \text{span}(\underbrace{1, z, \dots, z^m}_{\text{of length } (m+1)})$ . Then  $(p_0, p_1, \dots, p_m, z)$  of length  $(m+2)$  is linely dep.

As shown above,  $z \notin \text{span}(p_0, p_1, \dots, p_m)$ . And hence by [2.21](a),  $(p_0, p_1, \dots, p_m)$  is linely dep.  $\square$

7 Prove or give a counterexample: If  $(v_1, v_2, v_3, v_4)$  is a basis of  $V$  and  $U$  is a subsp of  $V$  such that  $v_1, v_2 \in U$  and  $v_3 \notin U$  and  $v_4 \notin U$ , then  $(v_1, v_2)$  is a basis of  $U$ .

SOLUTION: A counterexample: Let  $V = \mathbb{R}^4$  and  $B_V = (e_1, e_2, e_3, e_4)$  be std basis.

Let  $v_1 = e_1, v_2 = e_2, v_3 = e_3 + e_4, v_4 = e_4$ . Then  $(v_1, \dots, v_4)$  is a basis of  $\mathbb{R}^4$ .

Let  $U = \text{span}(e_1, e_2, e_3) = \text{span}(v_1, v_2, v_3 - v_4)$ . Then  $v_3 \notin U$  and  $(v_1, v_2)$  is not a basis of  $U$ .  $\square$

• NOTE FOR " $\mathcal{C}_V U \cup \{0\}$ ": " $\mathcal{C}_V U \cup \{0\}$ " is supposed to be a subsp  $W$  such that  $V = U \oplus W$ .

But if we let  $u \in U \setminus \{0\}$  and  $w \in W \setminus \{0\}$ , then  $\left. \begin{array}{l} w \in \mathcal{C}_V U \cup \{0\} \\ u \pm w \in \mathcal{C}_V U \cup \{0\} \end{array} \right\} \Rightarrow u \in \mathcal{C}_V U \cup \{0\}$ . Contradicts.

To fix this, denote the set  $\{W_1, W_2, \dots\}$  by  $\mathcal{S}_V U$ , where for each  $W_i$ ,  $V = U \oplus W_i$ . See also in (1.C.23).

• TIPS: Suppose  $V$  is finite-dim with  $\dim V = n$  and  $U$  is a subsp of  $V$  with  $U \neq V$ .  
Prove that  $\exists B_V = (v_1, \dots, v_n)$  such that each  $v_k \notin U$ .

Note that  $U \neq V \Rightarrow n \geq 1$ . We will construct  $B_V$  via the following process.

Step 1.  $\exists v_1 \in V \setminus U \Rightarrow v_1 \neq 0$ . If  $\text{span}(v_1) = V$  then we stop.

Step k. Suppose  $(v_1, \dots, v_{k-1})$  is linely inde in  $V$ , each of which belongs to  $V \setminus U$ .

Note that  $\text{span}(v_1, \dots, v_{k-1}) \neq V$ . And if  $\text{span}(v_1, \dots, v_{k-1}) \cup U = V$ , then by (1.C.12),

[ because  $\text{span}(v_1, \dots, v_{k-1}) \not\subseteq U$ , ]  $U \subseteq \text{span}(v_1, \dots, v_{k-1}) \Rightarrow \text{span}(v_1, \dots, v_{k-1}) = V$ .

Hence because  $\text{span}(v_1, \dots, v_{k-1}) \neq V$ , it must be case that  $\text{span}(v_1, \dots, v_{k-1}) \cup U \neq V$ .

Thus  $\exists v_k \in V \setminus U$  such that  $v_k \notin \text{span}(v_1, \dots, v_{k-1})$ .

By (2.A.11),  $(v_1, \dots, v_k)$  is linely inde in  $V$ . If  $\text{span}(v_1, \dots, v_k) = V$ , then we stop.

Because  $V$  is finite-dim, this process will stop after  $n$  steps.  $\square$

OR. Suppose  $U \neq \{0\}$ . Let  $B_U = (u_1, \dots, u_m)$ . Extend to a basis  $(u_1, \dots, u_n)$  of  $V$ .

Then let  $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$ .  $\square$

1 Find all vecsp on whatever  $\mathbf{F}$  that have exactly one basis.

SOLUTION: The trivial vecsp  $\{0\}$  will do. Indeed, the only basis of  $\{0\}$  is the empty list  $()$ .

Now consider the field  $\{0, 1\}$  containing only the add identity and multi identity, with  $1 + 1 = 0$ . Then the list  $(1)$  is the unique basis. Now the vecsp  $\{0, 1\}$  will do.

COMMENT: All vecsp on such  $\mathbf{F}$  of dim 1 will do.

And more generally, consider  $\mathbf{F} = \mathbb{Z}_m, \forall m - 1 \in \mathbb{N}^+$ . For each  $s, t \in \{1, \dots, m\}$ ,

$\mathbf{F} = \text{span}(K_s) = \text{span}(K_t)$ . More than one basis. So are  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$  and all vecsp on such  $\mathbf{F}$ .

Consider other  $\mathbf{F}$ . Note that this  $\mathbf{F}$  contains at least and strictly more than 0 and 1. Failed.  $\square$

• (4E 9) Suppose  $(v_1, \dots, v_m)$  is a list of vecs in  $V$ . For  $k \in \{1, \dots, m\}$ , let  $w_k = v_1 + \dots + v_k$ .  
Show that  $[P] B_V = (v_1, \dots, v_m) \iff B_W = (w_1, \dots, w_m)$ .  $[Q]$

SOLUTION: NOTICE that  $B_U = (u_1, \dots, u_n) \iff \forall u \in U, \exists! a_i \in \mathbf{F}, u = a_1 u_1 + \dots + a_n u_n$ .

$P \Rightarrow Q$ :  $\forall v \in V, \exists! a_i \in \mathbf{F}, v = a_1 v_1 + \dots + a_m v_m \Rightarrow v = b_1 w_1 + \dots + b_m w_m, \exists! b_k = a_k - a_{k+1}, b_m = a_m$ .

$Q \Rightarrow P$ :  $\forall v \in V, \exists! b_i \in \mathbf{F}, v = b_1 w_1 + \dots + b_m w_m \Rightarrow v = a_1 v_1 + \dots + a_m v_m, \exists! a_k = \sum_{j=k}^m b_j$ .  $\square$

COMMENT: See also ??? in (3.F).



- (4E 5) Suppose  $U, W$  are finite-dim,  $V = U + W$ ,  $B_U = (u_1, \dots, u_m)$ ,  $B_W = (w_1, \dots, w_n)$ . Prove that  $\exists B_V$  consisting of vecs in  $U \cup W$ .

SOLUTION:  $V = \text{span}(u_1, \dots, u_m) + \text{span}(w_1, \dots, w_n) = \text{span}(\overbrace{u_1, \dots, u_m, w_1, \dots, w_n}^{\text{Reduce}})$ . By [2.31].  $\square$

- 8 Suppose  $V = U \oplus W$ ,  $B_U = (u_1, \dots, u_m)$ ,  $B_W = (w_1, \dots, w_n)$ . Prove that  $B_V = (u_1, \dots, u_m, w_1, \dots, w_n)$ .

SOLUTION:  $\forall v \in V, \exists! u \in U, w \in W \Rightarrow \exists! a_i, b_i \in \mathbb{F}, v = u + w = \sum_{i=1}^m a_i u_i + \sum_{i=1}^n b_i w_i$ .

OR.  $V = \text{span}(u_1, \dots, u_m) \oplus \text{span}(w_1, \dots, w_n) = \text{span}(u_1, \dots, u_m, w_1, \dots, w_n)$ .

Note that  $\sum_{i=1}^m a_i u_i + \sum_{i=1}^n b_i w_i = 0 \Rightarrow \sum_{i=1}^m a_i u_i = -\sum_{i=1}^n b_i w_i \in U \cap W = \{0\}$ .  $\square$

- (9.A.3.4 OR 4E 11) Suppose  $V$  is on  $\mathbb{R}$ , and  $v_1, \dots, v_n \in V$ . Let  $B = (v_1, \dots, v_n)$ .

(a) Show that  $[P] B$  is linely inde in  $V \iff B$  is linely inde in  $V_{\mathbb{C}}$ . [Q]

(b) Show that  $[P] B$  spans  $V \iff B$  spans  $V_{\mathbb{C}}$ . [Q]

(a)  $P \Rightarrow Q$ : Note that each  $v_k \in V_{\mathbb{C}}$ .  $Q \Rightarrow P$ : If  $\lambda_k \in \mathbb{R}$  with  $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ , then each  $\text{Re } \lambda_k = \lambda_k = 0$ .

$\neg P \Rightarrow \neg Q$ :  $\exists v_j = a_{j-1} v_{j-1} + \dots + a_1 v_1 \in V_{\mathbb{C}}$ .

$\neg Q \Rightarrow \neg P$ :  $\exists v_j = \lambda_{j-1} v_{j-1} + \dots + \lambda_1 v_1 \Rightarrow v_j = (\text{Re } \lambda_{j-1}) v_{j-1} + \dots + (\text{Re } \lambda_1) v_1 \in V$ .

(b)  $P \Rightarrow Q$ :  $\forall u + iv \in V_{\mathbb{C}}, u, v \in V \Rightarrow \exists a_i, b_i \in \mathbb{R}, u + iv = \sum_{i=1}^n (a_i + ib_i) v_i$ .

$Q \Rightarrow P$ :  $\forall v \in V, \exists a_i + ib_i \in \mathbb{C}, v + i0 = (\sum_{i=1}^n a_i v_i) + i(\sum_{i=1}^n b_i v_i) \Rightarrow v \in \text{span}(v_1, \dots, v_n)$ .

$\neg Q \Rightarrow \neg P$ :  $\exists v \in V, v \notin \text{span}(B) \Rightarrow v + i0 \notin \text{span}(B)$  while  $v + i0 \in V_{\mathbb{C}}$ .

$\neg Q \Rightarrow \neg P$ :  $\exists u + iv \in V_{\mathbb{C}}, u + iv \notin \text{span}(B) \Rightarrow u$  or  $v \notin \text{span}(B)$ . Note that  $u, v \in V$ .  $\square$

- **NOTE FOR linely inde sequence and [2.34]:** " $V = \text{span}(v_1, \dots, v_n, \dots)$ " is an invalid expression.

If we allow using "infinite list", then we must guarantee that  $(v_1, \dots, v_n, \dots)$  is a spanning "list" such that  $\forall v \in V, \exists$  smallest  $n \in \mathbb{N}^+, v = a_1 v_1 + \dots + a_n v_n$ . Moreover, given a list  $(w_1, \dots, w_n, \dots)$  in  $W$ , we can prove that  $\exists! T \in \mathcal{L}(V, W)$  with each  $Tv_k = w_k$ , which has less restrictions than [3.5].

But the key point is, how can we guarantee that such a "list" exists. **TODO: More details.**

ENDED

## 2.C 1 7 9 10 14,16 15 17 | 4E: 10 14,15 16

- 15 Suppose  $V$  is finite-dim and  $\dim V = n \geq 1$ .

Prove that  $\exists$  one-dim subspcs  $V_1, \dots, V_n$  of  $V$  such that  $V = V_1 \oplus \dots \oplus V_n$ .

SOLUTION: Suppose  $B_V = (v_1, \dots, v_n)$ . Define  $V_i$  by  $V_i = \text{span}(v_i)$  for each  $i \in \{1, \dots, n\}$ .

Then  $\forall v \in V, \exists! a_i \in \mathbb{F}, v = a_1 v_1 + \dots + a_n v_n \Rightarrow \exists! u_i \in V_i, v = u_1 + \dots + u_n$   $\square$

- **NOTE FOR Problem (15):**

Suppose  $v \in V \setminus \{0\}$ , and  $\dim V = n \geq 1$ . Prove that  $\exists B_V = (v_1, \dots, v_n), v = v_1 + \dots + v_n$ .

SOLUTION: If  $n = 1$  then let  $v_1 = v$  and we are done. Suppose  $n > 1$ .

Extend  $(v)$  to a basis  $(v, v_1, \dots, v_{n-1})$  of  $V$ . Let  $v_n = v - v_1 - \dots - v_{n-1}$ .

又  $\text{span}(v, v_1, \dots, v_{n-1}) = \text{span}(v_1, \dots, v_n)$ . Hence  $(v_1, \dots, v_n)$  is also a basis of  $V$ .  $\square$

COMMENT: Let  $B_V = (v_1, \dots, v_n)$  and suppose  $v = u_1 + \dots + u_n$ , where each  $u_i = a_i v_i \in V_i$ .

But  $(u_1, \dots, u_n)$  might not be a basis, because there might be some  $u_i = 0$ .

1 [COROLLARY for [2.38,39]] Suppose  $U$  is a subsp of  $V$  such that  $\dim V = \dim U$ . Then  $V = U$ .

Let  $B_U = (u_1, \dots, u_m)$ . Then  $m = \dim V$ . 又  $u_i \in V$ . By [2.39],  $B_V = (u_1, \dots, u_m)$ .  $\square$

- Let  $v_1, \dots, v_n \in V$  and  $\dim \text{span}(v_1, \dots, v_n) = n$ . Then  $(v_1, \dots, v_n)$  is a basis of  $\text{span}(v_1, \dots, v_n)$ .  
Notice that  $(v_1, \dots, v_n)$  is a spanning list of  $\text{span}(v_1, \dots, v_n)$  of length  $n = \dim \text{span}(v_1, \dots, v_n)$ .

- 7 (a) Let  $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$ . Find a basis of  $U$ .  
 (b) Extend the basis in (b) to a basis of  $\mathcal{P}_4(\mathbf{F})$ .  
 (c) Find a subsp  $W$  of  $\mathcal{P}_4(\mathbf{F})$  such that  $\mathcal{P}_4(\mathbf{F}) = U \oplus W$ .

SOLUTION: Using Problem (10).

NOTICE that  $\nexists p \in \mathcal{P}(\mathbf{F})$  of deg 1 and 2, while  $p \in U$ . Thus  $\dim U \leq \dim \mathcal{P}_4(\mathbf{F}) - 2 = 3$ .

(a) Consider  $B = (1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6))$ .

Let  $a_0 + a_3(z-2)(z-5)(z-6) + a_4z(z-2)(z-5)(z-6) = 0 \Rightarrow a_0 = a_3 = a_4 = 0$ .

Thus the list  $B$  is linely inde in  $U$ . Now  $\dim U \geq 3 \Rightarrow \dim U = 3$ . Thus  $B_U = B$ .

(b) Extend to a basis of  $\mathcal{P}_4(\mathbf{F})$  as  $(1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6))$ .

(c) Let  $W = \text{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$ , so that  $\mathcal{P}_4(\mathbf{F}) = U \oplus W$ .  $\square$

9 Suppose  $(v_1, \dots, v_m)$  is linely inde in  $V$  and  $w \in V$ .

Prove that  $\dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$ .

SOLUTION: Using the result of (2.A.10, 11).

Note that  $v_i - v_1 = (v_i + w) - (v_1 + w) \in \text{span}(v_1 + w, \dots, v_m + w)$ , for each  $i = 1, \dots, m$ .

$(v_1, \dots, v_m)$  linely inde  $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$  linely inde  $\Rightarrow \underbrace{(v_2 - v_1, \dots, v_m - v_1)}_{\text{of length } (m-1)}$  linely inde.

又 If  $w \notin \text{span}(v_1, \dots, v_m)$ . Then  $(v_1 + w, \dots, v_m + w)$  is linely inde. of length  $(m-1)$

Hence  $m \geq \dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$ .  $\square$

• (4E 16) Suppose  $V$  is finite-dim,  $U$  is a subsp of  $V$  with  $U \neq V$ . Let  $n = \dim V, m = \dim U$ .

Prove that  $\exists (n - m)$  subsp  $U_1, \dots, U_{n-m}$ , each of dim  $(n - 1)$ , such that  $\bigcap_{i=1}^{n-m} U_i = U$ .

SOLUTION: Let  $B_U = (v_1, \dots, v_m)$ ,  $B_V = (v_1, \dots, v_m, u_1, \dots, u_{n-m})$ .

Define  $U_i = \text{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m})$  for each  $i$ . Then  $U \subseteq U_i$  for each  $i$ .

And because  $\forall v \in \bigcap_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \dots + b_{n-m} u_{n-m} \in U_i \Rightarrow$  each  $b_i = 0 \Rightarrow v \in U$ .

Hence  $\bigcap_{i=1}^{n-m} U_i \subseteq U$ .  $\square$

• NOTE FOR Problem 10: For each nonconst  $p \in \text{span}(1, z, \dots, z^m)$ ,  $\exists$  smallest  $m \in \mathbf{N}^+$ , which is  $\deg p$ .

(a) If  $p_0, p_1, \dots, p_m$  are such that all  $a_{k,k} \neq 0$ , and

$p_0 = a_{0,0}$ , each  $p_k = a_{0,k} + a_{1,k}z + \dots + a_{k,k}z^k$ .

Then the upper-trig  $\mathcal{M}(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)) = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m} \\ 0 & a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix}$ .

(b) If  $p_0, p_1, \dots, p_m$  are such that all  $a_{k,k} \neq 0$ , and

$p_0 = a_{0,0} + \dots + a_{m,0}x^m$ , each  $p_k = a_{k,k}x^k + \dots + a_{m,k}x^m$ .

Then the lower-trig  $\mathcal{M}(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)) = \begin{pmatrix} a_{0,0} & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,0} & a_{m,1} & \cdots & a_{m,m} \end{pmatrix}$ .

COMMENT: Define  $\xi_k(p)$  by the coeff of  $z^k$  in  $p \in \mathcal{P}_m(\mathbf{F})$ .

Then  $\mathcal{M}(\xi_k, (1, z, \dots, z^m), (1)) = \mathcal{E}^{(1,k)} \in \mathbf{F}^{1,m+1}$ .

**10** Suppose  $m \in \mathbf{N}^+$ ,  $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$  are such that each  $p_k$  has degree  $k$ .

Prove that  $(p_0, p_1, \dots, p_m)$  is a basis of  $\mathcal{P}_m(\mathbf{F})$ .

**SOLUTION:** Using mathematical induction on  $m$ .

(i)  $k = 1$ .  $\deg p_0 = 0$ ;  $\deg p_1 = 1 \Rightarrow \text{span}(p_0, p_1) = \text{span}(1, x)$ .

(ii)  $1 \leq k \leq m-1$ . Assume that  $\text{span}(p_0, p_1, \dots, p_k) = \text{span}(1, x, \dots, x^k)$ .

Then  $\text{span}(p_0, p_1, \dots, p_k, p_{k+1}) \subseteq \text{span}(1, x, \dots, x^k, x^{k+1})$ .

又  $\deg p_{k+1} = k+1$ ,  $p_{k+1}(x) = a_{k+1}x^{k+1} + r_{k+1}(x)$ ;  $a_{k+1} \neq 0$ ,  $\deg r_{k+1} \leq k$ .

$$\Rightarrow x^{k+1} = \frac{1}{a_{k+1}}(p_{k+1}(x) - r_{k+1}(x)) \in \text{span}(1, x, \dots, x^k, p_{k+1}) = \text{span}(p_0, p_1, \dots, p_k, p_{k+1}).$$

$$\therefore x^{k+1} \in \text{span}(p_0, p_1, \dots, p_k, p_{k+1}) \Rightarrow \text{span}(1, x, \dots, x^k, x^{k+1}) \subseteq \text{span}(p_0, p_1, \dots, p_k, p_{k+1}).$$

Thus  $\mathcal{P}_m(\mathbf{F}) = \text{span}(1, x, \dots, x^m) = \text{span}(p_0, p_1, \dots, p_m)$ . □

OR. 用比较系数法. Denote the coeff of  $x^k$  in  $p \in \mathcal{P}(\mathbf{F})$  by  $\xi_k(p)$ .

Suppose  $L = a_m p_m(x) + \dots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \dots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$ .

We show that  $a_m = \dots = a_0 = 0$  via the following process. So that  $(p_0, p_1, \dots, p_m)$  is linely inde.

**Step 1.** For  $k = m$ ,  $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$  又  $\deg p_m = m$ ,  $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$ .

Now  $L = a_{m-1} p_{m-1}(x) + \dots + a_0 p_0(x)$ .

**Step k.** For  $0 \leq k \leq m$ , we have  $a_m = \dots = a_{k+1} = 0$ .

Now  $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$  又  $\deg p_k = k$ ,  $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$ .

Now if  $k = 0$ , then we are done. Otherwise, we have  $L = a_{k-1} p_{k-1}(x) + \dots + a_0 p_0(x)$ . □

• **TIPS:** Suppose  $m \in \mathbf{N}^+$ ,  $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$  are such that the lowest term of each  $p_k$  is of  $\deg k$ . Prove that  $(p_0, p_1, \dots, p_m)$  is a basis of  $\mathcal{P}_m(\mathbf{F})$ .

**SOLUTION:** Using mathematical induction on  $m$ .

Let each  $p_k$  be defined by  $p_k(x) = a_{k,k}x^k + \dots + a_{m,k}x^m$ , where  $a_{k,k} \neq 0$ .

(i)  $k = 1$ .  $p_m(x) = a_{m,m}x^m$ ;  $p_{m-1}(x) = a_{m-1,m-1}x^{m-1} + a_{m,m-1}x^m \Rightarrow \text{span}(x^m, x^{m-1}) = \text{span}(p_m, p_{m-1})$ .

(ii)  $1 \leq k \leq m-1$ . Assume that  $\text{span}(x^m, \dots, x^{m-k}) = \text{span}(p_m, \dots, p_{m-k})$ .

Then  $\text{span}(p_m, \dots, p_{m-(k+1)}) \subseteq \text{span}(x^m, \dots, x^{m-(k+1)})$ .

又  $p_{m-(k+1)}$  has the form  $a_{m-(k+1),m-(k+1)}x^{m-(k+1)} + r_{m-(k+1)}(x)$ ;

where the lowest term of  $r_{m-(k+1)} \in \mathcal{P}_m(\mathbf{F})$  is of  $\deg(m-k)$ .

$$\Rightarrow x^{m-(k+1)} = \frac{1}{a_{m-(k+1),m-(k+1)}}(p_{m-(k+1)}(x) - r_{m-(k+1)}(x)) \in \text{span}(x^m, \dots, x^{m-k}, p_{m-(k+1)})$$

$$= \text{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$$

$$\therefore x^{m-(k+1)} \in \text{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$$

$$\Rightarrow \text{span}(x^m, \dots, x^{m-k}, x^{m-(k+1)}) \subseteq \text{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$$

Thus  $\mathcal{P}_m(\mathbf{F}) = \text{span}(x^m, \dots, x, 1) = \text{span}(p_m, \dots, p_1, p_0)$ . □

OR. 用比较系数法. Denote the coeff of  $x^k$  in  $p \in \mathcal{P}(\mathbf{F})$  by  $\xi_k(p)$ .

Suppose  $L = a_m p_m(x) + \dots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \dots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$ .

We show that  $a_m = \dots = a_0 = 0$  via the following process. So that  $(p_0, p_1, \dots, p_m)$  is linely inde.

**Step 1.** For  $k = 0$ ,  $\xi_0(L) = a_0 \xi_0(p_0) = \xi_0(R) = 0$  又  $\deg p_0 = 0$ ,  $\xi_0(p_0) \neq 0 \Rightarrow a_0 = 0$ .

Now  $L = a_1 p_1(x) + \dots + a_m p_m(x)$ .

**Step k.** For  $0 \leq k \leq m$ , we have  $a_{k-1} = \dots = a_0 = 0$ .

Now  $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$  又  $\deg p_k = k$ ,  $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$ .

Now if  $k = m$ , then we are done. Otherwise, we have  $L = a_{k+1} p_{k+1}(x) + \dots + a_m p_m(x)$ . □

• **NOTE FOR [2.11]:** *Good definition for a general term always avoids undefined behaviours.*

If  $\deg p = 0$ , then  $p(z) = a_0 \neq 0$ , but not literally  $a_0 z^0$ , by which if  $p$  is defined, then it comes to  $0^0$ .

To make it clear, we specify that in  $\mathcal{P}(\mathbb{F})$ ,  $a_0 z^0 = a_0$ , where  $z^0$  appears just for notational convenience.

Because by definition, the term  $a_0 z^0$  in a poly only represents the const term of the poly, which is  $a_0$ .

For convenience, we assume that  $z^0 = 1$  in formula deduction and poly def. Absolutely without  $0^0$ .

• (4E 10) Suppose  $m$  is a positive integer. For  $0 \leq k \leq m$ , let  $p_k(x) = x^k(1-x)^{m-k}$ .

Show that  $(p_0, \dots, p_m)$  is a basis of  $\mathcal{P}_m(\mathbb{F})$ .

**SOLUTION:** We may see  $p_0 = 1$  and  $p_m(x) = x^m$ , from the expansion below, by the NOTE FOR [2.11] above.

Note that each  $p_k(x) = \sum_{j=0}^{m-k} C_{m-k}^j (-1)^j \cdot x^{j+k} \cdot 1^j = \underbrace{(-1)^0 \cdot x^k \cdot 1^0}_{\text{of deg } k} + \underbrace{\sum_{j=1}^{m-k} C_{m-k}^j (-1)^j \cdot x^{j+k} \cdot 1^j}_{\text{of deg } m; \text{ denote it by } q_k(x)}$

And, each  $q_k \in \text{span}(x^{k+1}, \dots, x^m)$ . Using TIPS above. □

OR. Similar to the TIPS above. We will recursively prove that each  $x^{m-k} \in \text{span}(p_m, \dots, p_{m-k})$ .

(i)  $k = 1$ .  $p_m(x) = x^m \in \text{span}(p_m)$ ;  $p_{m-1}(x) = x^{m-1} - x^m \Rightarrow x^{m-1} \in \text{span}(p_{m-1}, p_m)$ .

(ii)  $k \in \{1, \dots, m-1\}$ . Suppose for each  $k \in \{0, \dots, k\}$ , we have  $x^{m-k} \in \text{span}(p_{m-k}, \dots, p_m)$ ,  $\exists ! a_m \in \mathbb{F}$ .

Note that  $x^{m-(k+1)} = p_{m-(k+1)}(x) + \sum_{j=1}^{k+1} C_{k+1}^j (-1)^{j+1} x^{m-(k+1)+j} \in \text{span}(p_{m-(k+1)}, x^{m-k}, \dots, x^m)$ .

Thus  $x^{m-(k+1)} \in \text{span}(p_{m-(k+1)}, p_{m-k}, \dots, p_m)$ . □

**COMMENT:** The base step and the inductive step can be independent.

OR. For any  $m, k \in \mathbb{N}^+$  such that  $k \leq m$ . Define  $p_{k,m}$  by  $p_{k,m}(x) = x^k(1-x)^{m-k}$ .

Define the statement  $S(m)$  by  $S(m) : (p_{0,m}, \dots, p_{m,m})$  is linely inde ( and therefore is a basis ).

We use induction on to show that  $S(m)$  holds for all  $m \in \mathbb{N}^+$ .

(i)  $m = 0$ .  $p_{0,0} = 1$ , and  $ap_{0,0} = 0 \Rightarrow a = 0$ .

$m = 1$ . Let  $a_0(1-x) + a_1x = 0, \forall x \in \mathbb{F}$ . Then take  $x = 1, x = 0 \Rightarrow a_1 = a_0 = 0$ .

(ii)  $1 \leq m$ . Assume that  $S(m)$  and  $S(m-1)$  holds. Now we show that  $S(m+1)$  holds.

Suppose  $\sum_{k=0}^{m+1} a_k p_{k,m+1}(x) = \sum_{k=0}^{m+1} a_k [x^k(1-x)^{m+1-k}] = 0, \forall x \in \mathbb{F}$ .

Now  $a_0(1-x)^{m+1} + \sum_{k=1}^m a_k x^k(1-x)^{m+1-k} + a_{m+1}x^{m+1} = 0, \forall x \in \mathbb{F}$ .

While  $x = 0 \Rightarrow a_0 = 0$ ; and  $x = 1 \Rightarrow a_{m+1} = 0$ .

Then  $0 = \sum_{k=1}^m a_k x^k(1-x)^{m+1-k}$

$= x(1-x) \sum_{k=1}^m a_k x^{k-1}(1-x)^{m-k}$ , note that  $m-k = (m-1) - (k-1)$

$= x(1-x) \sum_{k=0}^{m-1} a_{k+1} x^k(1-x)^{m-1-k} = x(1-x) \sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x)$ .

Hence  $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0, \forall x \in \mathbb{F} \setminus \{0, 1\}$ . Which has infinitely many zeros.

Moreover,  $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0$ . By assumption,  $a_1 = \dots = a_{m-1} = a_m = 0$ .

Thus  $(p_{0,m+1}, \dots, p_{m+1,m+1})$  is linely inde and  $S(m+1)$  holds. □

**14** Suppose  $V_1, \dots, V_m$  are finite-dim. Prove that  $\dim(V_1 + \dots + V_m) \leq \dim V_1 + \dots + \dim V_m$ .

**SOLUTION:** For each  $V_i$ , let  $B_{V_i} = \mathcal{E}_i$ . Then  $V_1 + \dots + V_m = \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m)$ ;  $\dim V_i = \text{card } \mathcal{E}_i$ .

Now  $\dim(V_1 + \dots + V_m) = \dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leq \text{card}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) \leq \text{card } \mathcal{E}_1 + \dots + \text{card } \mathcal{E}_m$ .

**COROLLARY:**  $V_1 + \dots + V_m$  is direct

$\Leftrightarrow$  For each  $k$ ,  $(V_1 \oplus \dots \oplus V_k) \cap V_{k+1} = \{0\}$ ,  $(\mathcal{E}_1 \cap \dots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$

$\Leftrightarrow \dim \text{span}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) = \text{card}(\mathcal{E}_1 \cup \dots \cup \mathcal{E}_m) = \text{card } \mathcal{E}_1 + \dots + \text{card } \mathcal{E}_m$

$\Leftrightarrow \dim(V_1 \oplus \dots \oplus V_m) = \dim V_1 + \dots + \dim V_m$ . □

**17** Suppose  $V_1, V_2, V_3$  are subsp of a finite-dim vecsp, then

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

**SOLUTION:**

[ Similar to ] Given three sets  $A, B$  and  $C$ .

Because  $|X \cup Y| = |X| + |Y| - |X \cap Y|$ ;  $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$ .

Now  $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$ .

And  $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$ .

Hence  $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$ .

Note that  $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$ .

$$\dim(V_1 + V_2 + V_3) = \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \quad (1)$$

$$= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) \quad (2)$$

$$= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) \quad (3).$$

Notice that in general,  $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$ .

For example,  $X = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}, Y = \{(0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}, Z = \{(z, z) \in \mathbb{R}^2 : z \in \mathbb{R}\}$ .

**COMMENT:** If  $X \subseteq Y$ , then  $(X + Y) \cap Z = Y \cap Z$ ;  $\dim(X + Y + Z) = \dim(Y) + \dim(Z) - \dim(Y \cap Z)$ , and the wrong formul above holds. Similar for  $Y \subseteq Z, X \subseteq Z$ , and  $X, Y \subseteq Z$ .

• **COROLLARY:** Suppose  $V_1, V_2$  and  $V_3$  are finite-dim vecsp, then  $\frac{(1) + (2) + (3)}{3}$  :

$$\begin{aligned} \dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\ &\quad - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}. \end{aligned}$$

• **TIPS:** Because  $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim(V_1 + (V_2 \cap V_3))$ .

And  $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$ . We have (1), and (2), (3) similarly.

$$(1) \dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)).$$

$$(2) \dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_3) - \dim(V_2 + (V_1 \cap V_3)).$$

$$(3) \dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim(V_3 + (V_1 \cap V_2)).$$

• Suppose  $V_1, V_2, V_3$  are subsp of  $V$  with

(a)  $\dim V = 10, \dim V_1 = \dim V_2 = \dim V_3 = 7$ . Prove that  $V_1 \cap V_2 \cap V_3 \neq \{0\}$ .

By TIPS,  $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 - 2 \dim V > 0$ .

(b)  $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$ . Prove that  $V_1 \cap V_2 \cap V_3 \neq \{0\}$ .

By TIPS,  $\dim(V_1 \cap V_2 \cap V_3) \geq 2 \dim V - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)) \geq 0$ . □

**ENDED**



• **TIPS 1:**  $T : V \rightarrow W$  is linear  $\iff \left\{ \begin{array}{l} \text{(一)} \forall v, u \in V, T(v + u) = Tv + Tu; \\ \text{(二)} \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv). \end{array} \right. \iff T(v + \lambda u) = Tv + \lambda Tu.$

• (9.A.2,6 OR 4E 3.B.33) Suppose that  $V, W$  are on  $\mathbf{R}$ , and  $T \in \mathcal{L}(V, W)$ . Show that  
 (a)  $T_C \in \mathcal{L}(V_C, W_C)$ . (b)  $\text{null}(T_C) = (\text{null } T)_C, \text{range}(T_C) = (\text{range } T)_C$ . (c)  $T_C$  is inv  $\iff T$  is inv.

**SOLUTION:** (a)  $T_C((u_1 + iv_1) + (x + iy)(u_2 + iv_2)) = T(u_1 + xu_2 - yv_2) + iT(v_1 + xv_2 + yu_2)$   
 $= T_C(u_1 + iv_1) + (x + iy)T_C(u_2 + iv_2).$

(b)  $u + iv \in \text{null}(T_C) \iff u, v \in \text{null } T \iff u + iv \in (\text{null } T)_C.$

$w + ix \in \text{range}(T_C) \iff w, x \in \text{range } T \iff w + ix \in (\text{range } T)_C.$

(c)  $\forall w, x \in W, \exists! u, v \in V, T_C(u + iv) = w + ix \iff Tu = w, Tv = x.$  OR. By (b).  $\square$

• (9.A.5) Suppose  $V$  is on  $\mathbf{R}$ , and  $S, T \in \mathcal{L}(V, W)$ . Prove that  $(S + \lambda T)_C = S_C + \lambda T_C.$

**SOLUTION:**  $(S + \lambda T)_C(u + iv) = (S + \lambda T)(u) + i(S + \lambda T)(v)$   
 $= Su + iSv + \lambda(Tu + iTv) = (S_C + \lambda T_C)(u + iv).$   $\square$

• Suppose  $U, V, W$  are on  $\mathbf{R}$ ,  $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$ . Prove that  $(ST)_C = S_C T_C.$

**SOLUTION:**  $\forall u + ix \in U_C, (ST)_C(u + ix) = STu + iSTx = S_C(Tu + iTx) = (S_C T_C)(u + ix).$   $\square$

• **NOTE FOR Restriction:**  $U$  is a subsp of  $V$ .

(a)  $\forall S, T \in \mathcal{L}(V, W), \lambda \in \mathbf{F}, (T + \lambda S)|_U = T|_U + \lambda S|_U.$

(b)  $\forall S \in \mathcal{L}(W, X), T \in \mathcal{L}(V, W), (ST)|_U = ST|_U.$

• (4E 1.B.7) Suppose  $V \neq \emptyset$  and  $W$  is a vecsp. Let  $W^V = \{f : V \rightarrow W\}.$

(a) Define a natural add and scalar multi on  $W^V$ .

(b) Prove that  $W^V$  is a vecsp with these definitions.

**SOLUTION:**

(a)  $W^V \ni f + g : x \rightarrow f(x) + g(x);$  where  $f(x) + g(x)$  is the vec add on  $W$ .

$W^V \ni \lambda f : x \rightarrow \lambda f(x);$  where  $\lambda f(x)$  is the scalar multi on  $W$ .

(b) Commutativity:  $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$

Associativity:  $((f + g) + h)(x) = (f(x) + g(x)) + h(x)$   
 $= f(x) + (g(x) + h(x)) = (f + (g + h))(x).$

Additive Identity:  $(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x).$

Additive Inverse:  $(f + g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0 = 0(x).$

Distributive Properties:

$(a(f + g))(x) = a(f + g)(x) = a(f(x) + g(x))$   
 $= af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x).$

Similarly,  $((a + b)f)(x) = (af + bf)(x).$

So far, we have used the same properties in  $W$ .

Which means that **if  $W^V$  is a vecsp, then  $W$  must be a vecsp.**

Multiplication Identity:  $(1f)(x) = 1f(x) = f(x).$  ( NOTICE that the smallest  $\mathbf{F}$  is  $\{0, 1\}.$  )  $\square$



• **TIPS 2:**  $T \in \mathcal{L}(V, W) \iff T \in \mathcal{L}(V, \text{range } T) \iff T \in \mathcal{L}(V, U)$ , if  $\text{range } T$  is a subsp of  $U$ .

**COROLLARY:**  $\{T \in \mathcal{L}(V, W) : \text{range } T \subseteq U\} = \{T \in \mathcal{L}(V, U)\} = \mathcal{L}(V, U)$ .

**5** Because  $\mathcal{L}(V, W) = \{T : V \rightarrow W \mid T \text{ is linear}\}$  is a subsp of  $W^V$ ,  $\mathcal{L}(V, W)$  is a vecsp.

**3** Suppose  $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ . Prove that  $\exists A_{j,k} \in \mathbb{F}$  such that for any  $(x_1, \dots, x_n) \in \mathbb{F}^n$ ,

$$T(x_1, \dots, x_n) = \begin{pmatrix} A_{1,1}x_1 + \dots + A_{1,n}x_n \\ \vdots \\ A_{m,1}x_1 + \dots + A_{m,n}x_n \end{pmatrix}$$

**SOLUTION:**

Let  $T(1, 0, 0, \dots, 0, 0) = (A_{1,1}, \dots, A_{m,1})$ , Note that  $(1, 0, \dots, 0, 0), \dots, (0, 0, \dots, 0, 1)$  is a basis of  $\mathbb{F}^n$ .

$T(0, 1, 0, \dots, 0, 0) = (A_{1,2}, \dots, A_{m,2})$ , Then by [3.5], we are done.  $\square$

$T(0, 0, 0, \dots, 0, 1) = (A_{1,n}, \dots, A_{m,n})$ .

**4** Suppose  $T \in \mathcal{L}(V, W)$ , and  $v_1, \dots, v_m \in V$  such that  $(Tv_1, \dots, Tv_m)$  is linely inde in  $W$ .  
Prove that  $(v_1, \dots, v_m)$  is linely inde.

**SOLUTION:** Suppose  $a_1v_1 + \dots + a_mv_m = 0$ . Then  $a_1Tv_1 + \dots + a_mTv_m = 0$ . Thus  $a_1 = \dots = a_m = 0$ .  $\square$

**7** Show that every linear map from a one-dim vecsp to itself is a multi by some scalar.

More precisely, prove that if  $\dim V = 1$  and  $T \in \mathcal{L}(V)$ , then  $\exists \lambda \in \mathbb{F}, Tv = \lambda v, \forall v \in V$ .

**SOLUTION:** Let  $u$  be a nonzero vec in  $V \Rightarrow V = \text{span}(u)$ . Because  $Tu \in V \Rightarrow Tu = \lambda u$  for some  $\lambda$ .

Suppose  $v \in V \Rightarrow v = au, \exists! a \in \mathbb{F}$ . Then  $Tv = T(au) = \lambda au = \lambda v$ .  $\square$

**8** Give a map  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $\forall a \in \mathbb{R}, v \in \mathbb{R}^2, \varphi(av) = a\varphi(v)$  but  $\varphi$  is not linear.

**SOLUTION:** Define  $T(x, y) = \begin{cases} x + y, & \text{if } (x, y) \in \text{span}(3, 1), \\ 0, & \text{otherwise.} \end{cases}$  OR. Define  $T(x, y) = \sqrt[3]{(x^3 + y^3)}$ .  $\square$

**9** Give a map  $\varphi : \mathbb{C} \rightarrow \mathbb{C}$  such that  $\forall w, z \in \mathbb{C}, \varphi(w + z) = \varphi(w) + \varphi(z)$  but  $\varphi$  is not linear.

**SOLUTION:** Define  $\varphi(u + iv) = u = \text{Re}(u + iv)$  OR. Define  $\varphi(u + iv) = v = \text{Im}(u + iv)$ .  $\square$

• Prove that if  $q \in \mathcal{P}(\mathbb{R})$  and  $T : \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$  is defined by  $Tp = \underbrace{q \circ p}_{\text{composition}}$ , then  $T$  is not linear.

**SOLUTION:** Composition and product are not the same in  $\mathcal{P}(\mathbb{F})$ .

NOTICE that  $(p \circ q)(x) = p(q(x))$ , while  $(pq)(x) = p(x)q(x) = q(x)p(x)$ .

Because in general,  $[q \circ (p_1 + \lambda p_2)](x) = q(p_1(x) + \lambda p_2(x)) \neq (qp_1)(x) + \lambda(qp_2)(x)$ .

**EXAMPLE:** Let  $q$  be defined by  $q(x) = x^2$ , then  $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$ .  $\square$

**10** Suppose  $U$  is a subsp of  $V$  with  $U \neq V$ . Suppose  $S \in \mathcal{L}(U, W)$  with  $S \neq 0$

( which means that  $\exists u \in U, Su \neq 0$  ). Define  $T : V \rightarrow W$  by  $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \setminus U. \end{cases}$

Prove that  $T$  is not a linear map on  $V$ .

**SOLUTION:** Suppose  $T$  is a linear map. And  $v \in V \setminus U, u \in U$  such that  $Su \neq 0$ .

Then  $v + u \in V \setminus U$ , for if not,  $v = (v + u) - u \in U$ ;

while  $T(v + u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$ . Contradicts.  $\square$

**11** Suppose  $U$  is a subsp of  $V$  and  $S \in \mathcal{L}(U, W)$ .

Prove that  $\exists T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U$ . ( OR.  $\exists T \in \mathcal{L}(V, W), T|_U = S$ .)

In other words, every linear map on a subsp of  $V$  can be **extended** to a linear map on the entire  $V$ .

**SOLUTION:** Suppose  $W$  is such that  $V = U \oplus W$ . Then  $\forall v \in V, \exists! u_v \in U, w_v \in W, v = u_v + w_v$ .

Define  $T \in \mathcal{L}(V, W)$  by  $T(u_v + w_v) = Su_v$ . □

OR. [Finite-dim Req] Define by  $T\left(\sum_{i=1}^m a_i u_i\right) = \sum_{i=1}^m a_i Su_i$ . Let  $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$ . □

**12** Suppose nonzero  $V$  is finite-dim and  $W$  is infinite-dim. Prove that  $\mathcal{L}(V, W)$  is infinite-dim.

**SOLUTION:** Using (2.A.14).

Let  $B_V = (v_1, \dots, v_n)$  be a basis of  $V$ . Let  $(w_1, \dots, w_m)$  be linely inde in  $W$  for any  $m \in \mathbb{N}^+$ .

Define  $T_{x,y} : V \rightarrow W$  by  $T_{x,y}(v_z) = \delta_{z,x} w_y, \forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$ , where  $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$   
 $\forall v = \sum_{i=1}^n a_i v_i, u = \sum_{i=1}^n b_i v_i, \lambda \in \mathbb{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) w_y = T_{x,y}(v) + \lambda T_{x,y}(u)$ .

Linearity checked. Now suppose  $a_1 T_{x,1} + \dots + a_m T_{x,m} = 0$ .

Then  $(a_1 T_{x,1} + \dots + a_m T_{x,m})(v_x) = 0 = a_1 w_1 + \dots + a_m w_m \Rightarrow a_1 = \dots = a_m = 0$ .  $\forall m$  arbitrary.

Thus  $(T_{x,1}, \dots, T_{x,m})$  is a linely inde list in  $\mathcal{L}(V, W)$  for any  $x$  and length  $m$ . Hence by (2.A.14). □

**13** Suppose  $(v_1, \dots, v_m)$  is linely depe in  $V$  and  $W \neq \{0\}$ .

Prove that  $\exists w_1, \dots, w_m \in W, \nexists T \in \mathcal{L}(V, W)$  such that  $Tv_k = w_k, \forall k = 1, \dots, m$ .

**SOLUTION:**

We prove by contradiction. By linear dependence lemma,  $\exists j \in \{1, \dots, m\}, v_j \in \text{span}(v_1, \dots, v_{j-1})$ .

Fix  $j$ . Let  $w_j \neq 0$ , while  $w_1 = \dots = w_{j-1} = w_{j+1} = \dots = w_m = 0$ . Define  $T \in \mathcal{L}(V, W)$  by  $Tv_k = w_k$  for each  $k$ .

Suppose  $a_1 v_1 + \dots + a_m v_m = 0$ , where  $a_j \neq 0$ .

Then  $T(a_1 v_1 + \dots + a_m v_m) = 0 = a_1 w_1 + \dots + a_m w_m = a_j w_j$  while  $a_j \neq 0$  and  $w_j \neq 0$ . Contradicts. □

OR. We prove the contrapositive: Suppose  $\forall w_1, \dots, w_m \in W, \exists T \in \mathcal{L}(V, W), Tv_k = w_k$  for each  $w_k$ .

Now we show that  $(v_1, \dots, v_n)$  is linely inde. Suppose  $\exists a_i \in \mathbb{F}, a_1 v_1 + \dots + a_n v_n = 0$ .

Choose one  $w \in W \setminus \{0\}$ . By assumption, for  $(\overline{a_1} w, \dots, \overline{a_m} w), \exists T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$  for each  $v_k$ .

Now we have  $0 = T\left(\sum_{k=1}^m a_k v_k\right) = \sum_{k=1}^m a_k Tv_k = \sum_{k=1}^m a_k \overline{a_k} w = \left(\sum_{k=1}^m |a_k|^2\right) w$ .

Then  $\sum_{k=1}^m |a_k|^2 = 0 \Rightarrow$  each  $a_k = 0$ . Hence  $(v_1, \dots, v_n)$  is linely inde. □

• (4E 3.A.17) Suppose  $V$  is finite-dim. Show that all two-sided ideals of  $\mathcal{L}(V)$  are  $\{0\}$  and  $\mathcal{L}(V)$ .

A subsp  $\mathcal{E}$  of  $\mathcal{L}(V)$  is called a two-sided ideal of  $\mathcal{L}(V)$  if  $TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$ .

**SOLUTION:** Let  $B_V = (v_1, \dots, v_n)$ . If  $\mathcal{E} = 0$ , then we are done.

Suppose  $\mathcal{E} \neq 0$  and  $\mathcal{E}$  is a two-sided ideal of  $\mathcal{L}(V)$ . Let  $S \in \mathcal{E} \setminus \{0\}$ .

Suppose  $Sv_i \neq 0$  and  $Sv_i = a_1 v_1 + \dots + a_n v_n$ , where  $a_k \neq 0$ .

Define  $R_{x,y} \in \mathcal{L}(V)$  by  $R_{x,y} : v_x \mapsto v_y, v_z \mapsto 0 (z \neq x)$ . OR.  $R_{x,y} v_z = \delta_{z,x} v_y$ .

Then  $(R_{1,1} + \dots + R_{n,n})v_j = v_j \Rightarrow \sum_{r=1}^n R_{r,r} = I$ . Assume that each  $R_{x,y} \in \mathcal{E}$ .

Hence  $\forall T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$ . Now we prove the assumption.

Notice that  $\forall x, y \in \mathbb{N}^+, (R_{k,y} S)(v_i) = a_k v_y \Rightarrow ((R_{k,y} S) \circ R_{x,i})(v_z) = \delta_{z,x} (a_k v_y)$ .

Thus  $R_{k,y} S R_{x,i} = a_k R_{x,y}$ . Now  $S \in \mathcal{E} \Rightarrow R_{k,y} S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}$ . □

- (4E 3.B.32) Suppose  $V$  is finite-dim with  $n = \dim V > 1$ .

Show that if  $\varphi : \mathcal{L}(V) \rightarrow \mathbf{F}$  is linear and  $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$ , then  $\varphi = 0$ .

**SOLUTION:** Using notations in (4E 3.A.17). Using the result in NOTE FOR [3.60].

Suppose  $\varphi \neq 0 \Rightarrow \exists i, j \in \{1, \dots, n\}, \varphi(R_{i,j}) \neq 0$ . Because  $R_{i,j} = R_{x,j} \circ R_{i,x}, \forall x = 1, \dots, n$   
 $\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$  and  $\varphi(R_{i,x}) \neq 0$ .

Again, because  $R_{i,x} = R_{y,x} \circ R_{i,y}, \forall y = 1, \dots, n$ . Thus  $\varphi(R_{y,x}) \neq 0, \forall x, y = 1, \dots, n$ .

Let  $k \neq i, j \neq l$  and then  $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$   
 $\Rightarrow \varphi(R_{l,k}) = 0$  or  $\varphi(R_{i,j}) = 0$ . Contradicts.  $\square$

OR. Note that by (4E 3.A.17),  $\exists S, T \in \mathcal{L}(V), ST - TS \neq 0$ .

Then  $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \text{null } \varphi \neq \{0\}$ .

Note that  $\forall E \in \text{null } \varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \text{null } \varphi$ .

Hence  $\text{null } \varphi$  is a nonzero two-sided ideal of  $\mathcal{L}(V)$ .  $\square$

- Suppose  $V$  is finite-dim.  $T \in \mathcal{L}(V)$  is such that  $\forall S \in \mathcal{L}(V), ST = TS$ .

Prove that  $\exists \lambda \in \mathbf{F}, T = \lambda I$ .

**SOLUTION:** If  $V = \{0\}$ , then we are done. Now suppose  $V \neq \{0\}$ .

Assume that  $\forall v \in V, (v, Tv)$  is linely depe, then by (2.A.2.(b)),  $\exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$ .

To prove that  $\lambda_v$  is independent of  $v$ , we discuss in two cases:

(-) If  $(v, w)$  is linely inde,  $\lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v v + \lambda_w w$   
 $\Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0$   
 (=) Otherwise, suppose  $w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w = 0$   $\Rightarrow \lambda_w = \lambda_v$ .

Now we prove the assumption. Assume that  $\exists v \in V, (v, Tv)$  is linely inde. Let  $B_V = (v, Tv, u_1, \dots, u_n)$ .

Define  $S \in \mathcal{L}(V)$  by  $S(av + bTv + c_1 u_1 + \dots + c_n u_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$ . Contradicts.  $\square$

OR. Let  $B_V = (v_1, \dots, v_m)$ .

Define  $\varphi \in \mathcal{L}(V, \mathbf{F})$  by  $\varphi(v_1) = \dots = \varphi(v_m) = 1$ . Let  $\lambda = \varphi(Tv_1) \in \mathbf{F}$ .

For any  $v \in V$ , define  $S_v \in \mathcal{L}(V)$  by  $S_v u = \varphi(u)v$ .

Then  $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$ .  $\square$

OR. For each  $k \in \{1, \dots, n\}$ , define  $S_k \in \mathcal{L}(V)$  by  $S_k v_j = \begin{cases} v_k, & j = k, \\ 0, & j \neq k. \end{cases}$  OR.  $S_k v_j = \delta_{j,k} v_k$

Note that  $S_k \left( \sum_{i=1}^n a_i v_i \right) = a_k v_k$ . Then  $S_k v = v \iff \exists ! a_k \in \mathbf{F}, v = a_k v_k$ .

Hence  $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$ .

Define  $A^{(j,k)} \in \mathcal{L}(V)$  by  $A^{(j,k)} v_j = v_k, A^{(j,k)} v_k = v_j, A^{(j,k)} v_x = 0, x \neq j, k$ .

Then  $\left\{ \begin{array}{l} A^{(j,k)} T v_j = T A^{(j,k)} v_j = T v_k = a_k v_k \\ A^{(j,k)} T v_j = A^{(j,k)} a_j v_j = a_j A^{(j,k)} v_j = a_j v_k \end{array} \right\} \Rightarrow a_k = a_j$ . Hence  $a_k$  is inde of  $v_k$ .  $\square$

- **TIPS 3:** Suppose  $T \in \mathcal{L}(V, W)$ . Prove that  $Tv \neq 0 \Rightarrow v \neq 0$ .

**SOLUTION:** Assume that  $v = 0$ . Then  $Tv = T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$ .

OR.  $T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$ . Contradicts.  $\square$

- Given the fact that  $\mathcal{L}(V, W)$  is a vecsp. Prove or give a counterexample:  $V, W$  are vecsp.  
We can guarantee that  $\{0\} \subseteq \mathcal{L}(V, W), \{0\} \subseteq V, \{0\} \subseteq W$ .  
And by [3.2], the additivity and homogeneity imply that  $V$  is closed under add and scalar multi.  
( We cannot even guarantee that  $W^V$  is a vecsp. )

**SOLUTION:** **TODO: Too tricky to be answered by AI.**

(I) If  $W^V = \{0\}$ . Then  $\mathcal{L}(V, W) = \{0\}$ .

And  $W = \{0\}$ , for if not,  $\exists w \in W \setminus \{0\}$ , define a map  $f$  by  $f(x) = w, \forall x \in V$ .

And  $V$  might not be a vecsp. Example: ???

(II) If  $W^V$  is a nonzero vecsp. Then  $W$  is a vecsp.

(a) If  $\mathcal{L}(V, W) = \{0\}$ , then we cannot guarantee that  $V$  is a vecsp. Example: ???

(b) If not, then  $\exists T \in \mathcal{L}(V, W), T \neq 0$ . Which means  $\exists v \in V, Tv \neq 0 \Rightarrow v \neq 0$ .

Then both  $W$  and  $V$  have a nonzero element.

(i) If  $\exists$  inje  $T \in \mathcal{L}(V, W)$ , then  $T(u+v) = T(v+u) \Rightarrow u+v = v+u$ . etc. Hence  $V$  is a vecsp.

(ii) If not, then we cannot guarantee that  $V$  is a vecsp. Example: ???

(III) If  $W^V$  is not a vecsp, then  $W$  is not a vecsp. Example: ???

□

ENDED

### 3.B 3 7 8 9 10 11 12 16 17 18 19 20 21 22 23 24 25 26 28 29 30 | 4E: 21 24 27 31 32 33

3 Suppose  $(v_1, \dots, v_m)$  in  $V$ . Define  $T \in \mathcal{L}(\mathbb{F}^m, V)$  by  $T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m$ .

(a) The surj of  $T$  correspds to  $(v_1, \dots, v_m)$  spanning  $V$ .

(b) The inje of  $T$  correspds to  $(v_1, \dots, v_m)$  being linely inde.

**COMMENT:** Let  $(e_1, \dots, e_m)$  be the std basis of  $\mathbb{F}^m$ . Then  $Te_k = v_k$ .

(a)  $\text{range } T = \text{span}(v_1, \dots, v_m) = V$ ; (b)  $(v_1, \dots, v_m)$  is linely inde  $\iff T$  is inje.

7 Suppose  $V$  is finite-dim with  $2 \leq \dim V$ . And  $\dim V \leq \dim W = m$ , if  $W$  is finite-dim.  
Show that  $U = \{ T \in \mathcal{L}(V, W) : \text{null } T \neq \{0\} \}$  is not a subsp of  $\mathcal{L}(V, W)$ .

**SOLUTION:** The set of all inje  $T \in \mathcal{L}(V, W)$  is a not subsp either.

Let  $(v_1, \dots, v_n)$  be a basis of  $V$ ,  $(w_1, \dots, w_m)$  be linely inde in  $W$ . [  $2 \leq n \leq m$ . ]

Define  $T_1 \in \mathcal{L}(V, W)$  as  $T_1: v_1 \mapsto 0, v_2 \mapsto w_2, v_i \mapsto w_i$ .

Define  $T_2 \in \mathcal{L}(V, W)$  as  $T_2: v_1 \mapsto w_1, v_2 \mapsto 0, v_i \mapsto w_i, i = 3, \dots, n$ .

Thus  $T_1 + T_2 \notin U$ . □

**COMMENT:** If  $\dim V = 0$ , then  $V = \{0\} = \text{span}(\ )$ .  $\forall T \in \mathcal{L}(V, W), T$  is inje. Hence  $U = \emptyset$ .

If  $\dim V = 1$ , then  $V = \text{span}(v_0)$ . Thus  $U = \text{span}(T_0)$ , where  $\forall v \in V, T_0 v = 0 \Rightarrow T_0 = 0$ .

8 Suppose  $W$  is finite-dim with  $\dim W \geq 2$ . And  $n = \dim V \geq \dim W$ , if  $V$  is finite-dim.  
Show that  $U = \{ T \in \mathcal{L}(V, W) : \text{range } T \neq W \}$  is not a subsp of  $\mathcal{L}(V, W)$ .

**SOLUTION:** The set of all surj  $T \in \mathcal{L}(V, W)$  is not a subsp either. Using the generalized version of [3.5].

Let  $(v_1, \dots, v_n)$  be linely inde in  $V$ ,  $(w_1, \dots, w_m)$  be a basis of  $W$ . [  $n \in \{m, m+1, \dots\}; 2 \leq m \leq n$ . ]

Define  $T_1 \in \mathcal{L}(V, W)$  as  $T_1: v_1 \mapsto 0, v_2 \mapsto w_2, v_j \mapsto w_j, v_{m+i} \mapsto 0$ .

Define  $T_2 \in \mathcal{L}(V, W)$  as  $T_2: v_1 \mapsto w_1, v_2 \mapsto 0, v_j \mapsto w_j, v_{m+i} \mapsto 0$ .

( For each  $j = 2, \dots, m; i = 1, \dots, n-m$ , if  $V$  is finite, otherwise let  $i \in \mathbb{N}^+$ . ) Thus  $T_1 + T_2 \notin U$ . □

**COMMENT:** If  $\dim W = 0$ , then  $W = \{0\} = \text{span}(\ )$ .  $\forall T \in \mathcal{L}(V, W), T$  is surj. Hence  $U = \emptyset$ .

If  $\dim W = 1$ , then  $W = \text{span}(w_0)$ . Thus  $U = \text{span}(T_0)$ , where each  $T_0 v_i = 0 \Rightarrow T_0 = 0$ .

**9** Suppose  $(v_1, \dots, v_n)$  is linely inde. Prove that  $\forall$  inje  $T$ ,  $(Tv_1, \dots, Tv_n)$  is linely inde.

**SOLUTION:**  $a_1Tv_1 + \dots + a_nTv_n = 0 = T\left(\sum_{i=1}^n a_i v_i\right) \iff \sum_{i=1}^n a_i v_i = 0 \iff a_1 = \dots = a_n = 0.$   $\square$

**10** Suppose  $\text{span}(v_1, \dots, v_n) = V$ . Show that  $\text{span}(Tv_1, \dots, Tv_n) = \text{range } T$ .

**SOLUTION:** (a)  $\text{range } T = \{Tv : v \in \text{span}(v_1, \dots, v_n)\} \Rightarrow Tv_1, \dots, Tv_n \in \text{range } T$ . By [2.7].

OR.  $\text{span}(Tv_1, \dots, Tv_n) \ni a_1Tv_1 + \dots + a_nTv_n = T(a_1v_1 + \dots + a_nv_n) \in \text{range } T$ .

(b)  $\forall w \in \text{range } T, w = Tv, \exists v \in V \Rightarrow \exists a_i \in \mathbf{F}, v = \sum_{i=1}^n a_i v_i, w = a_1Tv_1 + \dots + a_nTv_n.$   $\square$

**11** Suppose  $S_1, \dots, S_n \in \mathcal{L}(V)$  and  $S = S_1S_2 \dots S_n$  makes sense. Then using induction:

(a)  $\text{range } S_1 \supseteq \text{range } (S_1S_2) \supseteq \dots \supseteq \text{range } (S)$ ; (b)  $\text{null } S_n \subseteq \text{null } (S_{n-1}S_n) \subseteq \dots \subseteq \text{null } (S)$ .

• Define  $X_p = \{T \in \mathcal{L}(V) : p(T) \text{ holds}\}$ ;  $P_p : X_p$  is closed under vec multi;  $Q_p : X_p$  is a group.

(1)  $S$  surj  $\iff$  each  $S_k$  surj.  $P_{\text{surj}}$  holds. (2)  $S$  inje  $\iff$  each  $S_k$  inje.  $P_{\text{inje}}$  holds.

(3)  $P_{\text{inv}}$  and  $Q_{\text{inv}}$  hold.  $Q_p$  in (1) and (2) holds  $\iff V$  is finite-dim.

(4)  $P_{\text{inje or surj}}$  holds  $\iff V$  is finite-dim  $\iff Q_{\text{inje or surj}}$  holds.

• Suppose  $S, T \in \mathcal{L}(V)$ . Prove or give a counterexample:

(a)  $\text{null } S \subseteq \text{null } T \Rightarrow \text{range } T \subseteq \text{range } S$ ; (b)  $\text{range } T \subseteq \text{range } S \Rightarrow \text{null } S \subseteq \text{null } T$ .

**SOLUTION:** Let  $B_V = (v_1, v_2, v_3)$ . Counterexamples:

(a) Let  $S : v_1 \mapsto 0; v_2 \mapsto 0; v_3 \mapsto v_2$ . Then  $\text{null } S = \text{null } T$ , but

$T : v_1 \mapsto 0; v_2 \mapsto 0; v_3 \mapsto v_3$ .  $\text{range } T = \text{span}(v_3) \not\subseteq \text{span}(v_2) = \text{null } T$ .

(b) Let  $S : v_1 \mapsto v_2; v_2 \mapsto v_2; v_3 \mapsto v_2$ . Then  $\text{range } T = \text{range } S$ , but

$T : v_1 \mapsto 0; v_2 \mapsto 0; v_3 \mapsto v_2$ .  $\text{null } S = \text{span}(v_1 - v_2, v_2 - v_3, v_3 - v_1) \not\subseteq \text{span}(v_1, v_2) = \text{null } T$ .

**16** Suppose  $T \in \mathcal{L}(V)$  such that  $\text{null } T, \text{range } T$  are finite-dim. Prove that  $V$  is finite-dim.

**SOLUTION:** Let  $B_{\text{range } T} = (Tv_1, \dots, Tv_n), B_{\text{null } T} = (u_1, \dots, u_m)$ .

$\forall v \in V, \exists! a_i \in \mathbf{F}, T(v - a_1v_1 - \dots - a_nv_n) = 0 \Rightarrow \exists! b_i \in \mathbf{F}, v - \sum_{i=1}^n a_i v_i = \sum_{i=1}^m b_i u_i.$   $\square$

**17** Suppose  $V, W$  are finite-dim. Prove that  $\exists$  inje  $T \in \mathcal{L}(V, W) \iff \dim V \leq \dim W$ .

**SOLUTION:** (a) Suppose  $\exists$  inje  $T$ . Then  $\dim V = \dim \text{range } T \leq \dim W$ .

(b) Suppose  $\dim V \leq \dim W$ . Let  $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$ .

Define  $T \in \mathcal{L}(V, W)$  by  $Tv_i = w_i, i = 1, \dots, n (= \dim V)$ .  $\square$

**18** Suppose  $V, W$  are finite-dim. Prove that  $\exists$  surj  $T \in \mathcal{L}(V, W) \iff \dim V \geq \dim W$ .

**SOLUTION:** (a) Suppose  $\exists$  surj  $T$ . Then  $\dim V = \dim W + \dim \text{null } T \Rightarrow \dim W \leq \dim V$ .

(b) Suppose  $\dim V \geq \dim W$ . Let  $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$ .

Define  $T \in \mathcal{L}(V, W)$  by  $T(a_1v_1 + \dots + a_mv_m + \dots + a_nv_n) = a_1w_1 + \dots + a_mw_m.$   $\square$

**19** Suppose  $V, W$  are finite-dim,  $U$  is a subsp of  $V$ .

Prove that  $\exists T \in \mathcal{L}(V, W), \text{null } T = U \iff \underline{\dim U}_m \geq \underline{\dim V}_{m+n} - \underline{\dim W}_p$ .

**SOLUTION:**

(a) Suppose  $\exists T \in \mathcal{L}(V, W), \text{null } T = U$ . Then  $\dim U + \dim \text{range } T = \dim V \leq \dim U + \dim W$ .

(b) Let  $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_W = (w_1, \dots, w_p)$ . Suppose that  $p \geq n$ .

Define  $T \in \mathcal{L}(V, W)$  by  $T(a_1v_1 + \dots + a_nv_n + b_1u_1 + \dots + b_mu_m) = a_1w_1 + \dots + a_nw_n.$   $\square$



• **TIPS 1:** Suppose  $U$  is a subsp of  $V$ . Then  $\forall T \in \mathcal{L}(V, W), U \cap \text{null } T = \text{null } T|_U$ .

• **TIPS 2:** Suppose  $T \in \mathcal{L}(V, W)$  and  $T|_U$  is inje. Let  $V = M + N, U = X + Y$ .

Then  $\text{range } T = \text{range } T|_M + \text{range } T|_N = \text{range } T|_X + \text{range } T|_Y$ .

(a) Show that if  $U = X \oplus Y$ , then  $\text{range } T = \text{range } T|_X \oplus \text{range } T|_Y$ .

(b) Give an example such that  $V = M \oplus N, \text{range } T \neq \text{range } T|_M \oplus \text{range } T|_N$ .

**SOLUTION:** Assume that for some  $v \in V$ , there exist two distinct pairs  $(x_1, y_1), (x_2, y_2)$  in  $X \times Y$  such that  $Tv = Tx_1 + Ty_1 = Tx_2 + Ty_2$ . Because  $\forall v \in X \oplus Y, \exists! (x, y) \in X \times Y, v = x + y$ . Now  $T(x_1 + y_1) = T(x_2 + y_2) \Rightarrow x_1 + y_1 = x_2 + y_2 \Rightarrow x_1 = x_2, y_1 = y_2$ . Contradicts. Thus  $\forall Tv \in \text{range } T, \exists! Tx \in \text{range } T|_X, Ty \in \text{range } T|_Y, Tv = Tx + Ty$ .  $\square$

**EXAMPLE:** Let  $B_V = (v_1, v_2, v_3), B_W = (w_1, w_2), T : v_1 \mapsto 0, v_2 \mapsto w_1, v_3 \mapsto w_2$ .

Let  $B_M = (v_1 - v_2, v_3), B_N = (v_2)$ . Then  $\text{range } T|_M = \text{span}(w_1, w_2), \text{range } T|_N = \text{span}(w_1)$

**COMMENT:** Also  $\text{null } T|_M = \text{null } T|_N = \{0\}$ . Hence  $\text{null } T \neq \text{null } T|_M \oplus \text{null } T|_N$ .

**12** Prove that  $\forall T \in \mathcal{L}(V, W), \exists$  subsp  $U$  of  $V$  such that

$U \cap \text{null } T = \text{null } T|_U = \{0\}, \text{range } T = \{Tu : u \in U\} = \text{range } T|_U$ .

Which is equivalent to  $T|_U : U \rightarrow \text{range } T$  being an iso.

**SOLUTION:** By [2.34] (note that  $V$  can be infinite-dim),  $\exists$  subsp  $U$  of  $V$  such that  $V = U \oplus \text{null } T$ .

$\forall v \in V, \exists! w \in \text{null } T, u \in U, v = w + u$ . Then  $Tv = T(w + u) = Tu \in \{Tu : u \in U\}$ .  $\square$

**COROLLARY:**  $[P] \quad T|_U : U \rightarrow \text{range } T \text{ is an iso} \iff U \oplus \text{null } T = V. \quad [Q]$

We have shown  $Q \Rightarrow P$ . Now we show that  $P \Rightarrow Q$  to complete the proof.

$\forall v \in V, Tv \in \text{range } T = \text{range } T|_U \Rightarrow \exists! u \in U, Tv = Tu \Rightarrow v - u \in \text{null } T$ .

Thus  $v = (v - u) + u \in U + \text{null } T$ .  $\forall u \in U \cap \text{null } T \iff T|_U(u) = 0 \iff u = 0$ .  $\square$

OR.  $\neg Q \Rightarrow \neg P$ : Because  $U \oplus \text{null } T \subsetneq V$ . We show  $\text{range } T \neq \text{range } T|_U$  by contradiction.

Let  $X \oplus (U \oplus \text{null } T) = V$ . Now  $\text{range } T = \text{range } T|_X \oplus \text{range } T|_U$ . And  $X$  is nonzero.

Assume that  $\text{range } T = \text{range } T|_U$ . Then  $\text{range } T|_X = \{0\}$ . While  $T|_X$  is inje. Contradicts.

OR.  $\text{range } T|_X \subseteq \text{range } T|_U \Rightarrow \forall x \in X, Tx \in \text{range } T|_U, \exists u \in U, Tu = Tx \Rightarrow x = 0$ .

Also,  $\neg P \Rightarrow \neg Q$ : (a)  $\text{range } T|_U \subsetneq \text{range } T$ ; OR (b)  $U \cap \text{null } T \neq \{0\}$ .

For (a),  $\exists x \in V \setminus U, Tx \neq 0 \iff x \notin \text{null } T$ . Thus  $U + \text{null } T \subsetneq V$ . For (b), immediately.  $\square$

**COMMENT:** If  $T|_U : U \rightarrow \text{range } T$  is an iso. Let  $R \oplus U = V$ . Then  $R$  might not be  $\text{null } T$ .

OR. Extend  $B_U$  to  $B_V = (u_1, \dots, u_n, r_1, \dots, r_m)$ , then  $(r_1, \dots, r_m)$  might not be a  $B_{\text{null } T}$ .

• **TIPS 3:** Suppose  $T \in \mathcal{L}(V, W)$  and  $U$  is a subsp such that  $V = U \oplus \text{null } T$ . Let  $\text{null } T = X \oplus Y$ .

Now  $\forall v \in V, \exists! u_v \in U, (x_v, y_v) \in X \times Y, v = u_v + x_v + y_v$ . Define  $i \in \mathcal{L}(V, U)$  by  $i(v) = u_v + x_v$ .

Then  $T = T \circ i$ . Because  $\forall v \in V, T(v) = T(u_v + x_v + y_v) = T(u_v) = T(u_v + x_v) = T(i(v)) = (T \circ i)(v)$ .

• **TIPS 4:** Suppose  $T \in \mathcal{L}(V, W), T \neq 0$ . Let  $B_{\text{range } T} = (Tv_1, \dots, Tv_n)$ .

By (3.A.4),  $R = (v_1, \dots, v_n)$  is linely inde in  $V$ . Let  $\text{span } R = U$ . We will prove that  $U \oplus \text{null } T = V$ .

(a)  $T(\sum_{i=1}^n a_i v_i) = 0 \Rightarrow \sum_{i=1}^n a_i Tv_i = 0 \Rightarrow a_1 = \dots = a_n = 0 \Rightarrow U \cap \text{null } T = \{0\}$ .

(b)  $Tv = \sum_{i=1}^n a_i Tv_i \Rightarrow v - \sum_{i=1}^n a_i v_i \in \text{null } T \Rightarrow v = (v - \sum_{i=1}^n a_i v_i) + (\sum_{i=1}^n a_i v_i) \Rightarrow U + \text{null } T = V$ .

OR.  $\text{range } T = \{Tu : u \in U\} = \text{range } T|_U$ . Then by the COROLLARY in Problem (12).  $\square$

**COROLLARY:** Conversely, if  $U \oplus \text{null } T = V$  and  $B_U = (v_1, \dots, v_n)$ , then  $B_{\text{range } T} = (Tv_1, \dots, Tv_n)$ .

Because  $\text{range } T = \text{range } T|_U = \text{span}(Tv_1, \dots, Tv_n)$ ,  $\forall T$  is inje.



- (4E 21) Suppose  $V$  is finite-dim,  $T \in \mathcal{L}(V, W)$ ,  $Y$  is a subsp of  $W$ . Let  $\{v \in V : Tv \in Y\}$ .  
 (a) Prove that  $\{v \in V : Tv \in Y\}$  is a subsp of  $V$ .  
 (b) Prove that  $\dim\{v \in V : Tv \in Y\} = \dim \text{null } T + \dim(Y \cap \text{range } T)$ .

**SOLUTION:** Let  $\mathcal{K}_Y = \{v \in V : Tv \in Y\}$ .

(a)  $\forall u, w \in \mathcal{K}_Y, [Tu, Tw \in Y], \lambda \in \mathbf{F}, T(u + \lambda w) = Tu + \lambda Tw \in Y \Rightarrow \mathcal{K}_Y$  is a subsp of  $V$ .

(b) Define the range-restricted map  $R$  of  $T$  by  $R = T|_{\mathcal{K}_Y} \in \mathcal{L}(\mathcal{K}_Y, Y)$ . Now  $\text{range } R = Y \cap \text{range } T$ .

And  $v \in \text{null } T \Leftrightarrow Tv = 0 \in Y \Leftrightarrow Rv = 0 \in \text{range } T \Leftrightarrow v \in \text{null } R$ . By [3.22].  $\square$

**COMMENT:** Now  $\text{span}(v_1, \dots, v_m) \oplus \text{null } T = \mathcal{K}_Y$ . Where  $B_{Y \cap \text{range } T} = (Tv_1, \dots, Tv_m)$ .

In particular,  $\dim \mathcal{K}_{\text{range } T} = \dim \text{null } T + \dim \text{range } T \Rightarrow \mathcal{K}_{\text{range } T} = V$ .

**28** Suppose  $T \in \mathcal{L}(V, W)$ . Let  $B_{\text{range } T} = (w_1, \dots, w_m)$ .

Prove that  $\exists \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$  such that  $\forall v \in V, Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m$ .

**SOLUTION:** Suppose  $v_1, \dots, v_m \in V$  such that  $Tv_i = w_i$  for each  $v_i$ . Then  $(v_1, \dots, v_m)$  is linely inde.

And  $\text{span}(v_1, \dots, v_m) \oplus \text{null } T = V$ . Now  $\forall v \in V, \exists! a_i \in \mathbf{F}, u \in \text{null } T, v = \sum_{i=1}^m a_i v_i + u$ .

Define  $\varphi_i \in \mathcal{L}(V, \mathbf{F})$  by  $\varphi_i(v_j) = \delta_{ij}$ ,  $\varphi_i(u) = 0$  for all  $u \in \text{null } T$ .

Linearity:  $\forall v, w \in V [\exists! a_i, b_i \in \mathbf{F}], \lambda \in \mathbf{F}, \varphi_i(v + \lambda w) = a_i + \lambda b_i = \varphi(v) + \lambda \varphi(w)$ .  $\square$

**29** Suppose  $\varphi \in \mathcal{L}(V, \mathbf{F})$ . Suppose  $\varphi(u) \neq 0$ . Prove that  $V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$ .

**SOLUTION:** Let  $B_{\text{range } \varphi} = (\varphi(u))$ . Then by TIPS (4),  $\text{span}(u) \oplus \text{null } \varphi = V$ .  $\square$

OR. (a)  $v = cu \in \text{null } \varphi \cap \text{span}(u) \Rightarrow c\varphi(u) = 0 \Rightarrow v = 0$ . Now  $\text{null } \varphi \cap \text{span}(u) = \{0\}$ .

(b)  $\forall v \in V, v = \underbrace{\left(v - \frac{\varphi(v)}{\varphi(u)}u\right)}_{\in \text{null } \varphi} + \frac{\varphi(v)}{\varphi(u)}u \Rightarrow V = \text{null } \varphi + \text{span}(u)$ .  $\square$

**30** Suppose  $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$  and  $\text{null } \varphi_1 = \text{null } \varphi_2 = \text{null } \varphi$ . Prove that  $\exists c \in \mathbf{F}, \varphi_1 = c\varphi_2$

**SOLUTION:**

If  $\text{null } \varphi = V$ , then  $\varphi_1 = \varphi_2 = 0$ , we are done. Suppose  $\varphi(u) \neq 0 \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$ .

By Problem (29),  $V = \text{null } \varphi \oplus \text{span}(u)$ . Hence  $\forall v \in V, \exists! w \in \text{null } \varphi, a \in \mathbf{F}, v = w + a_u u$ .

Now  $\varphi_1(v) = a\varphi_1(u), \varphi_2(v) = a\varphi_2(u) \Rightarrow a = \frac{\varphi_1(v)}{\varphi_1(u)} = \frac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \frac{\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}$ .  $\square$

- (4E 31) Suppose  $V$  is finite-dim,  $X$  is a subsp of  $V$ , and  $Y$  is a finite-dim subsp of  $W$ .

Prove that if  $\dim X + \dim Y = \dim V$ , then  $\exists T \in \mathcal{L}(V, W), \text{null } T = X, \text{range } T = Y$ .

**SOLUTION:** Let  $V = U \oplus X, B_U = (v_1, \dots, v_m), B_Y = (w_1, \dots, w_m)$ .

Define  $T \in \mathcal{L}(V, W)$  by  $Tv_i = w_i, Tx = 0$  for each  $v_i$  and all  $x \in X$ .

Because  $\forall v \in V, \exists! a_i \in \mathbf{F}, x \in X, v = \sum_{i=1}^m a_i v_i + x$ .

Now  $v \in \text{null } T \Leftrightarrow Tv = a_1 w_1 + \dots + a_m w_m = 0 \Leftrightarrow v = x \in X$ . Hence  $\text{null } T = X$ .

And  $Y \ni w = a_1 w_1 + \dots + a_m w_m = a_1 Tv_1 + \dots + a_m Tv_m \in \text{range } T$ . Hence  $\text{range } T = Y$ .

OR. NOTICE that  $V = U \oplus \text{null } T$ . By the COROLLARY in Problem (12),  $\text{range } T = \text{range } T|_U$ .

又  $\dim \text{range } T|_U = \dim U = \dim Y; \text{range } T \subseteq Y$ .

OR. Let  $B_X = (x_1, \dots, x_n)$ . Now  $\text{range } T = \text{span}(Tv_1, \dots, Tv_m, Tx_1, \dots, Tx_n) = \text{span}(w_1, \dots, w_m) = Y$ .  $\square$

- 20, 21** (a) Prove that if  $ST = I \in \mathcal{L}(V)$ , then  $T$  is inje and  $S$  is surj.  
 (b) Suppose  $T \in \mathcal{L}(V, W)$ . Prove that if  $T$  is inje, then  $\exists S \in \mathcal{L}(W, V)$ ,  $ST = I$ .  
 (c) Suppose  $S \in \mathcal{L}(W, V)$ . Prove that if  $S$  is surj, then  $\exists T \in \mathcal{L}(V, W)$ ,  $ST = I$ .

**SOLUTION:**

- (a)  $Tv = 0 \Rightarrow S(Tv) = 0 = v$ . OR.  $\text{null } T \subseteq \text{null } ST = \{0\}$ .  
 $\forall v \in V, ST(v) = v \in \text{range } S$ . OR.  $V = \text{range } ST \subseteq \text{range } S$ .  
 (b) Define  $S \in \mathcal{L}(\text{range } T, V)$  by  $Sw = T^{-1}w$ , where  $T^{-1}$  is the inv of  $T \in \mathcal{L}(V, \text{range } T)$ .  
 Then extend to  $S \in \mathcal{L}(W, V)$  by (3.A.11). Now  $\forall v \in V, STv = T^{-1}Tv = v$ .  
 OR. [Req  $V$  Finite-dim] Let  $B_{\text{range } T} = (Tv_1, \dots, Tv_n) \Rightarrow B_V = (v_1, \dots, v_n)$ . Let  $U \oplus \text{range } T = W$ .  
 Define  $S \in \mathcal{L}(W, V)$  by  $S(Tv_i) = v_i, Su = 0$  for each  $v_i$  and all  $u \in U$ . Thus  $ST = I$ .  
 (c) By Problem (12),  $\exists$  subsp  $U$  of  $W, W = U \oplus \text{null } S, \text{range } S = \text{range } S|_U = V$ .  
 Note that  $S|_U : U \rightarrow V$  is an iso. Define  $T = (S|_U)^{-1}$ , where  $(S|_U)^{-1} : V \rightarrow U$ .  
 Then  $ST = S \circ (S|_U)^{-1} = S|_U \circ (S|_U)^{-1} = I_V$ .  
 OR. [Req  $V$  Finite-dim] Let  $B_{\text{range } S} = B_V = (Sw_1, \dots, Sw_n) \Rightarrow \text{span}(w_1, \dots, w_n) \oplus \text{null } S = W$ .  
 Define  $T \in \mathcal{L}(V, W)$  by  $T(Sw_i) = w_i$ . Now  $ST(a_1Sw_1 + \dots + a_nSw_n) = (a_1Sw_1 + \dots + a_nSw_n)$ .  $\square$

**COROLLARY:** For (b), if  $T$  is inje and  $\exists S, ST = I$ , then by (a), this  $S$  is surj. Similar for (c).

**22** Suppose  $U, V$  are finite-dim,  $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$ .

Prove that  $\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T$ .

**SOLUTION:** We show that  $\dim \text{null } ST = \dim \text{null } S|_{\text{range } T} + \dim \text{null } T$ .

Because (a)  $\text{range } T|_{\text{null } ST} = \text{range } T \cap \text{null } S = \text{null } S|_{\text{range } T}$ ,

(b)  $\text{null } T|_{\text{null } ST} = \text{null } T \cap \text{null } ST = \text{null } T$ . By [3.22]  $\square$

OR. NOTICE that  $u \in \text{null } ST \iff S(Tu) = 0 \iff Tu \in \text{null } S$ .

Thus  $\text{null } ST = \{u \in U : Tu \in \text{null } S\} = \mathcal{K}_{\text{null } S \cap \text{range } T} = \text{null } ST$ .

By Problem (4E 21),  $\dim \text{null } ST = \dim \text{null } T + \dim(\text{null } S \cap \text{range } T)$ .  $\square$

**COROLLARY:** (1)  $T$  surj  $\Rightarrow \dim \text{null } ST = \dim \text{null } S + \dim \text{null } T$ .

(2)  $T$  inv  $\Rightarrow \dim \text{null } ST = \dim \text{null } S, \text{null } ST = \text{null } T$ .

(3)  $S$  inje  $\Rightarrow \dim \text{null } ST = \dim \text{null } T$ .

**23** Suppose  $U, V$  are finite-dim,  $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$ .

Prove that  $\dim \text{range } ST \leq \min\{\dim \text{range } S, \dim \text{range } T\}$ .

**SOLUTION:** NOTICE that  $\text{range } ST = \{Sv : v \in \text{range } T\} = \text{range } S|_{\text{range } T}$ .

Let  $\text{range } ST = \text{span}(Su_1, \dots, Su_{\dim \text{range } T})$ , where  $B_{\text{range } T} = (u_1, \dots, u_{\dim \text{range } T})$ .

$\dim \text{range } ST \leq \dim \text{range } T$   $\wedge$   $\dim \text{range } ST \leq \dim \text{range } S$ .  $\square$

OR.  $\dim \text{range } ST = \dim \text{range } S|_{\text{range } T} = \dim \text{range } T - \dim \text{null } S|_{\text{range } T} \leq \dim \text{range } T$ .  $\square$

**COMMENT:**  $\dim \text{range } ST = \dim U - \dim \text{null } ST = \dim \text{range } T|_U - \dim \text{range } T|_{\text{null } ST}$ .

**COROLLARY:** (1)  $S|_{\text{range } T}$  inje  $\iff \dim \text{range } ST = \dim \text{range } T$ .

(2) Let  $X \oplus \text{null } S = V$ . Then  $X \subseteq \text{range } T \iff \text{range } ST = \text{range } S$ .

And  $T$  is surj  $\Rightarrow \text{range } ST = \text{range } S$ .

• **TIPS 5:** Suppose  $S \in \mathcal{L}(U, V)$  is surj. Define  $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(U, W))$  by  $\mathcal{B}(T) = TS$ .

Then  $\mathcal{B}$  is inje. Because  $\mathcal{B}(T) = TS = 0 \iff T|_{\text{range } S} = 0$ . OR.  $\text{range } TS = \text{range } T = \{0\}$ .

**24** Suppose  $S, T \in \mathcal{L}(V, W)$ , and  $\text{null } S \subseteq \text{null } T$ . Prove that  $\exists E \in \mathcal{L}(W), T = ES$ .

**SOLUTION:**

Let  $V = U \oplus \text{null } S$   
 $\Rightarrow S|_U : U \rightarrow \text{range } S$  is an iso.  
 Extend  $T(S|_U)^{-1}$  to  $E \in \mathcal{L}(W)$ .

$$\begin{array}{ccc} \text{range } T & \xleftarrow{\text{surj } T} & U \\ & \nwarrow \text{inv } S & \downarrow \\ & & \text{range } S \end{array}$$

OR. Define  $E : \text{range } S \rightarrow W$  by  $E : Sv \mapsto Tv$ .  
 Extend  $E \in \mathcal{L}(\text{range } S, W)$  to  $E \in \mathcal{L}(W)$ . □

**COMMENT:** Let  $\Delta \oplus \text{null } S = \text{null } T$ ,  $U_\Delta \oplus (\Delta \oplus \text{null } S) = V = U_\Delta \oplus \text{null } T$ . Redefine  $U = U_\Delta \oplus \Delta$ .

$U$	$\text{null } S$
$U_\Delta$	$\text{null } T$
$\Delta$	$\text{null } S$

$$\text{range } S \xleftarrow{S} \begin{array}{c} U_\Delta \\ \oplus \\ \Delta \end{array} \xrightarrow{T} \begin{array}{c} \text{range } T \\ \oplus \\ \{0\} \end{array}$$

Because  $\Delta = \text{null } T|_U = \text{null } T \cap \text{range } (S|_U)^{-1}$ .  
 Thus  $E = T(S|_U)^{-1}$  is not inje  $\Leftrightarrow \Delta \neq \{0\}$ .  
 In other words,  $\text{range } S|_\Delta = \text{null } E$ ,  
 while  $E|_{\text{range } S|_{U_\Delta}} : \text{range } S|_{U_\Delta} \rightarrow \text{range } T$  is an iso.

**COROLLARY:** If  $\text{null } S = \text{null } T$ . Then  $\Delta = \{0\}, U_\Delta = U$ .

By (3.D.3), we can extend inje  $T(S|_U)^{-1} \in \mathcal{L}(\text{range } S, W)$  to  $\text{inv } E \in \mathcal{L}(W)$ .

OR. [Req range  $S$  Finite-dim] Let  $B_{\text{range } S} = (Sv_1, \dots, Sv_n)$ . Then  $V = \text{span}(v_1, \dots, v_n) \oplus \text{null } S$ .

Let  $U \oplus \text{range } S = W$ . Define  $E \in \mathcal{L}(W)$  by  $E(Sv_i) = Tv_i$ ,  $Eu = 0$  for all  $u \in U$  and each  $v_i$ .

Hence  $\forall v \in V$ ,  $(\exists! a_i \in \mathbb{F}, u \in \text{null } S \subseteq \text{null } T)$ ,  $Tv = a_1Tv_1 + \dots + a_nTv_n = E(a_1Sv_1 + \dots + a_nSv_n)$  □

**COROLLARY:** [Req  $W$  Finite-dim] Suppose  $\text{null } S = \text{null } T$ . We show that  $\exists \text{inv } E \in \mathcal{L}(W), T = ES$ .

Redefine  $E \in \mathcal{L}(W)$  by  $E(Tv_i) = Sv_i$ ,  $E(w_j) = x_j$ , for each  $Tv_i$  and  $w_j$ . Where:

Let  $B_{\text{range } T} = (Tv_1, \dots, Tv_m), B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n), B_U = (v_1, \dots, v_m)$ .

Now  $V = U \oplus \text{null } T = U \oplus \text{null } S \Rightarrow B_{\text{range } S} = (Sv_1, \dots, Sv_m)$ . Let  $B'_W = (Sv_1, \dots, Sv_m, x_1, \dots, x_n)$ . □

**25** Suppose  $S, T \in \mathcal{L}(V, W)$ , and  $\text{range } T \subseteq \text{range } S$ . Prove that  $\exists E \in \mathcal{L}(V), T = SE$ .

**SOLUTION:**

Let  $V = U \oplus \text{null } S \Rightarrow S|_U : U \rightarrow \text{range } S$  is an iso. Because  $(S|_U)^{-1} : \text{range } S \rightarrow U$ .

Define  $E = (S|_U)^{-1}T = (S|_U)^{-1}|_{\text{range } T}T \in \mathcal{L}(V, U) \subseteq \mathcal{L}(V)$ . □

**COMMENT:** Let  $U_1 = U$ . Let  $U_2 \oplus \text{null } T = V = U_1 \oplus \text{null } S$ .

Let  $U_{1\Delta} = \text{range } (S|_{U_1})|_{\text{range } T} \subseteq U_1 = \Delta \oplus U_{1\Delta}$ .

OR. Let  $U_{1\Delta} = \text{range } E|_{U_2}$ . Let  $\Delta \oplus \text{range } E|_{U_2} = U_1$ .

Thus  $U_1 \oplus \text{null } S = U_{1\Delta} \oplus (\Delta \oplus \text{null } S) = U_2 \oplus \text{null } T$ .

iso, by (3.D.Tirs).

$$\begin{array}{ccc} U_1 & \xrightarrow{\text{inv } S} & \text{range } S \\ || & & || \\ \Delta & \xrightarrow{\text{inv } S} & \text{range } S|_\Delta \\ \oplus & & \oplus \\ U_{1\Delta} & \xrightarrow{\text{inv } S} & \text{range } T \xleftarrow{\text{inv } T} U_2 \\ \uparrow & & \uparrow \\ & \xrightarrow{\text{inv } E|_{U_2}} & \end{array}$$

If  $\Delta \neq \{0\}$ , assume  $\exists \text{inv } E \in \mathcal{L}(V)$  re-extended from  $E|_{U_2}$  still satisfying  $T = SE$ ,

then let  $\Delta \xrightarrow{E^{-1}} \Theta$ ;  $\text{null } S \xrightarrow{E^{-1}} \text{null } T_\Theta$ . Now  $\Theta \oplus \text{null } T_\Theta = \text{null } T$ .

Then  $\Theta \xrightarrow{E} \Delta \neq \{0\}$ , while  $\text{null } S \cap \Delta = \{0\}$ . Thus  $T|_\Theta = SE|_\Theta \neq 0$ , contradicts.

**COROLLARY:** If  $\Delta = \{0\}$ , then  $U_1 = U_{1\Delta} \Rightarrow \text{range } S = \text{range } T$ . 又  $\text{null } S, \text{null } T$  are iso.

By (3.D.3), we can re-extend inje  $E|_{U_2} \in \mathcal{L}(U_2, U_1 \oplus \text{null } S)$  to  $\text{inv } E \in \mathcal{L}(U_2 \oplus \text{null } T, U_1 \oplus \text{null } S)$ .

Thus we have  $\Delta \neq \{0\} \Leftrightarrow E|_{U_2} \in \mathcal{L}(U_2, V)$  cannot be re-extended to  $\text{inv } E \in \mathcal{L}(V)$  freely.

OR. [Req range  $T$  Finite-dim] Let  $B_{\text{range } T} = (Tv_1, \dots, Tv_n)$ . Then  $V = \text{span}(v_1, \dots, v_n) \oplus \text{null } T$ .

Let  $S(u_i) = Tv_i$  for each  $Tv_i$ . Define  $E$  by  $Ev_i = u_i$ ,  $Ex = 0$  for all  $x \in \text{null } T$  and each  $v_i$ . □

**COMMENT:** [Req  $V$  Finite-dim] Note that  $\dim U_2 \leq \dim U_1 \Rightarrow \dim \text{null } T = p \geq q = \dim \text{null } S$ .

Let  $B_{\text{null } T} = (x_1, \dots, x_p), B_{\text{null } S} = (y_1, \dots, y_q)$ . Redefine  $E : v_i \mapsto u_i, x_k \mapsto y_k, x_j \mapsto 0$ ,

for each  $i \in \{1, \dots, \dim U_2\}, k \in \{1, \dots, \dim \text{null } S\}, j \in \{\dim \text{null } S + 1, \dots, \dim \text{null } T\}$ .

Note that  $(u_1, \dots, u_n)$  is linely inde. Let  $X = \text{span}(x_1, \dots, x_q) \oplus \text{span}(v_1, \dots, v_n)$ .

Now  $E|_X$  is inje, but cannot be re-extend to  $\text{inv } E \in \mathcal{L}(V)$  without loss of functionality.

**COROLLARY:** [Req  $V$  Finite-dim] If  $\text{range } T = \text{range } S$ , then  $\dim \text{null } T = \dim \text{null } S = p$ .

Redefine  $E$  by  $Ev_i = u_i$ ,  $Ex_j = y_j$  for each  $v_i$  and  $x_j$ . Then  $E \in \mathcal{L}(V)$  is inv.  $\square$

• OR (5.B.4) Suppose  $P \in \mathcal{L}(V)$  and  $P^2 = P$ . Prove that  $V = \text{null } P \oplus \text{range } P$ .

**SOLUTION:** (a) If  $v \in \text{null } P \cap \text{range } P \Rightarrow Pv = 0$  and  $\exists u \in V, v = Pu$ . Then  $v = Pu = P^2u = Pv = 0$ .

(b) Note that  $\forall v \in V, v = Pv + (v - Pv)$  and  $P(v - Pv) = 0 \Rightarrow v - Pv \in \text{null } P$ .  $\square$

OR. [Only in Finite-dim] Let  $B_{\text{range } P^2} = (P^2v_1, \dots, P^2v_n)$ . Then  $(Pv_1, \dots, Pv_n)$  is linely inde.

Let  $U = \text{span}(Pv_1, \dots, Pv_n) \Rightarrow V = U \oplus \text{null } P^2$ . While  $U = \text{range } P = \text{range } P^2$ ;  $\text{null } P = \text{null } P^2$ .  $\square$

• (a) Suppose  $\dim V = n$ ,  $ST = 0$  where  $S, T \in \mathcal{L}(V)$ . Prove that  $\dim \text{range } TS \leq \left\lfloor \frac{n}{2} \right\rfloor$ .

(b) Give an example of such  $S, T$  with  $n = 5$  and  $\dim \text{range } TS = 2$ .

**SOLUTION:** Using Problem (23).

$\dim \text{range } TS \leq \min\{\dim \text{range } S, \dim \text{range } T\}$ . We prove by contradiction.

Assume that  $\dim \text{range } TS \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$ . Then  $\min\{n - \dim \text{null } T, n - \dim \text{null } S\} \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$   
 $\text{又 } \dim \text{null } ST = n \leq \dim \text{null } S + \dim \text{null } T \mid \Rightarrow \max\{\dim \text{null } T, \dim \text{null } S\} \leq n - \left\lfloor \frac{n}{2} \right\rfloor - 1$ .

Thus  $n \leq 2\left(n - \left\lfloor \frac{n}{2} \right\rfloor - 1\right) \Rightarrow \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \frac{n}{2}$ . Contradicts.  $\square$

OR.  $\dim \text{null } S = n - \dim \text{range } S \leq n - \dim \text{range } TS$ .  $\text{又 } ST = 0 \Rightarrow \text{range } T \subseteq \text{null } S$ .

$\dim \text{range } TS \leq \dim \text{range } T \leq \dim \text{null } S \leq n - \dim \text{range } TS$ . Thus  $2 \dim \text{range } TS \leq n$ .  $\square$

**EXAMPLE:** Let  $B_V = (v_1, \dots, v_5)$ . Define  $T : v_1 \mapsto 0, v_2 \mapsto 0, v_i \mapsto v_i$ ;

$S : v_1 \mapsto v_4, v_2 \mapsto v_5, v_i \mapsto 0; i = 3, 4, 5$ .

**26** Suppose  $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$  and  $\forall p, \deg(Dp) = (\deg p) - 1$ . Prove that  $D \in \mathcal{P}(\mathbb{R})$  is surj.

**SOLUTION:** [D might not be  $D : p \mapsto p'$ .] NOTICE that the following proof is wrong:

Because  $\text{span}(Dx, Dx^2, Dx^3, \dots) \subseteq \text{range } D$ , and  $\deg Dx^n = n - 1$ .

又 By (2.C.10),  $\text{span}(Dx, Dx^2, Dx^3, \dots) = \text{span}(1, x, x^2, \dots) = \mathcal{P}(\mathbb{R})$ .

Let  $D(C) = 0, Dx^k = p_k$  of  $\deg(k - 1)$ , for all  $C \in \mathbb{R} = \mathcal{P}_0(\mathbb{R})$  and for each  $k \in \mathbb{N}^+$ .

Because  $B_{\mathcal{P}_m(\mathbb{R})} = (p_1, \dots, p_m, p_{m+1})$ . And for all  $p \in \mathcal{P}(\mathbb{R}), \exists ! m = \deg p \in \mathbb{N}^+$ .

So that  $\exists ! a_i \in \mathbb{R}, p = \sum_{i=1}^{m+1} a_i p_i \Rightarrow \exists q = \sum_{i=1}^{m+1} a_i x^i, Dq = p$ .  $\square$

OR. We will recursively define a sequence of polys  $(p_k)_{k=0}^\infty$  where  $Dp_0 = 1, Dp_k = x^k$  for each  $k \in \mathbb{N}^+$ .

So that  $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbb{R}), Dq = p, \exists q = \sum_{k=0}^{\deg p} a_k p_k$ .

(i) Because  $\deg Dx = (\deg x) - 1 = 0, Dx = C \in \mathbb{F} \setminus \{0\}$ . Let  $p_0 = C^{-1}x \Rightarrow Dp_0 = C^{-1}Dx = 1$ .

(ii) Suppose we have defined  $Dp_0 = 1, Dp_k = x^k$  for each  $k \in \{1, \dots, n\}$ . Because  $\deg D(x^{n+2}) = n + 1$ .

Let  $D(x^{n+2}) = a_{n+1}x^{n+1} + a_nx^n + \dots + a_1x + a_0$ , with  $a_{n+1} \neq 0$ .

Then  $a_{n+1}^{-1}D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_nDp_n + \dots + a_1Dp_1 + a_0Dp_0)$

$\Rightarrow x^{n+1} = D[a_{n+1}^{-1}(x^{n+2} - a_np_n - \dots - a_1p_1 - a_0p_0)]$ . Thus defining  $p_{n+1}$ , so that  $Dp_{n+1} = x^{n+1}$ .  $\square$

ENDED

- **NOTE FOR Transpose:** [3.F.33] Define  $\mathcal{T} : A \rightarrow A^t$ . By [3.111],  $\mathcal{T}$  is linear. Because  $(A^t)^t = A$ .  
 $\mathcal{T}^2 = I$ ,  $\mathcal{T} = \mathcal{T}^{-1} \Rightarrow \mathcal{T}$  is an iso of  $\mathbf{F}^{m,n}$  onto  $\mathbf{F}^{n,m}$ . Define  $\mathcal{C}_k : A \rightarrow A_{\cdot,k}$ ,  $\mathcal{R}_j : A \rightarrow A_{j,\cdot}$ ,  $\mathcal{P}_{j,k} : A \rightarrow A_{j,k}$ .  
 Now we show that (a)  $\mathcal{T}\mathcal{R}_j = \mathcal{C}_j\mathcal{T}$ , (b)  $\mathcal{T}\mathcal{C}_k = \mathcal{R}_k\mathcal{T}$ , and (c)  $\mathcal{T}\mathcal{P}_{j,k} = \mathcal{P}_{k,j}\mathcal{T}$ .  
 So that furthermore,  $\mathcal{T}\mathcal{C}_k\mathcal{T} = \mathcal{R}_k$ ,  $\mathcal{T}\mathcal{R}_j\mathcal{T} = \mathcal{C}_j$ , and  $\mathcal{T}\mathcal{P}_{j,k}\mathcal{T} = \mathcal{P}_{k,j}$ .

Let  $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{pmatrix}$ .  $\left\{ \begin{array}{l} \text{Note that } (A_{j,k})^t = A_{j,k} = (A^t)_{k,j}. \text{ Thus (c) holds.} \\ \text{And } (A_{\cdot,k})^t = (A_{1,k} \cdots A_{m,k}) = (A_{k,1}^t \cdots A_{k,m}^t) = (A^t)_{k,\cdot} \\ \Rightarrow \text{(b) holds. Similar for (a).} \end{array} \right.$

• **NOTE FOR [3.48]:**  $\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_A \underbrace{\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}}_B = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$

- **NOTE FOR [3.47]:**  $(AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k} \quad \square$
- **NOTE FOR [3.49]:**  $[(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1} \quad \square$
- **EXERCISE 10:**  $[(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k} \quad \square$
- **COMMENT:** For [3.49], let  $B_U = (u_1, \dots, u_p)$ ,  $B_V = (v_1, \dots, v_n)$ ,  $B_W = (w_1, \dots, w_m)$ .  
 And  $C = \mathcal{M}(T, B_U, B_V) \in \mathbf{F}^{n,p}$ ,  $A = \mathcal{M}(S, B_V, B_W) \in \mathbf{F}^{m,n}$ .  
 Then  $\mathcal{M}(Tu_k, B_V) = C_{\cdot,k} \Rightarrow \mathcal{M}(S(Tu_k), B_W) = AC_{\cdot,k}$ , 又  $\mathcal{M}((ST)(u_k), B_W) = (AC)_{\cdot,k} \quad \square$   
 By NOTE FOR Transpose,  $(AC)_{j,\cdot} = [((AC)^t)_{\cdot,j}]^t = (C^t(A^t)_{\cdot,j})^t = ((A^t)_{\cdot,j})^t C = A_{j,\cdot}C \quad \square$

- **NOTE FOR [3.52]:**  $A \in \mathbf{F}^{m,n}$ ,  $c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$ . By [4E 3.51(a)],  $(Ac)_{\cdot,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n} \quad \square$   
 OR.  $\because (Ac)_{j,1} = \sum_{r=1}^n A_{j,r}c_{r,1} = [\sum_{r=1}^n (A_{\cdot,r}c_{r,1})]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$   
 $\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^n A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$  OR.  $(Ac)_{j,1} = (Ac)_{j,\cdot} = A_{j,\cdot}c \in \mathbf{F} \quad \square$   
 OR. Let  $B_V = (v_1, \dots, v_n)$ . Now  $Ac = \mathcal{M}(Tv, B_W) = \mathcal{M}(T(c_1v_1 + \cdots + c_nv_n)) = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n} \quad \square$

- **EXERCISE 11:**  $a \in \mathbf{F}^{1,n}$ ,  $C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$ . By [4E 3.51(b)],  $(aC)_{1,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot} \quad \square$   
 OR.  $\because (aC)_{1,k} = \sum_{r=1}^n a_{1,r}C_{r,k} = [\sum_{r=1}^n a_{1,r}(C_{r,\cdot})]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$   
 $\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^n a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$  OR.  $(aC)_{1,k} = (aC)_{\cdot,k} = aC_{\cdot,k} \in \mathbf{F} \quad \square$   
 OR.  $aC = ((aC)^t)^t = (C^ta^t)^t = [a_1^t(C^t)_{\cdot,1} + \cdots + a_n^t(C^t)_{\cdot,n}]^t = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot} \quad \square$

- [4E 3.51] Suppose  $C \in \mathbf{F}^{m,c}$ ,  $R \in \mathbf{F}^{c,p}$ . [ See also NOTE FOR [3.49] and Problem (10). ]  
 (a) For  $k = 1, \dots, p$ ,  $(CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^c C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$   
 (b) For  $j = 1, \dots, m$ ,  $(CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^c C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
- **EXAMPLE:**  $m = 2, c = 2, p = 3$ .

$$(AB)_{\cdot,2} = AB_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} = A_{\cdot,1}B_{1,2} + A_{\cdot,2}B_{2,2} = 6 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix};$$

$$(AB)_{1,\cdot} = A_{1,\cdot}B = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = A_{1,1}B_{1,\cdot} + A_{1,2}B_{2,\cdot} = 1 \begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 2 \begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \end{pmatrix};$$



• **COLUMN-ROW FACTORIZATION (CR Factorization)** Suppose  $A \in \mathbf{F}^{m,n}, A \neq 0$ .

Prove, with  $p$  specified below, that  $\exists C \in \mathbf{F}^{m,p}, R \in \mathbf{F}^{p,n}, A = CR$ .

(a) Suppose  $S_c = \text{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}, \dim S_c = c$ , the col rank. Let  $p = c$ .

(b) Suppose  $S_r = \text{span}(A_{1,\cdot}, \dots, A_{m,\cdot}) \subseteq \mathbf{F}^{1,n}, \dim S_r = r$ , the row rank. Let  $p = r$ .

**SOLUTION:** Using [4E 3.51]. Notice that  $A \neq 0 \Rightarrow c, r \geq 1$ .

(a) Reduce to basis  $B_C = (C_{\cdot,1}, \dots, C_{\cdot,c})$ , forming  $C \in \mathbf{F}^{m,c}$ . Then  $\forall k \in \{1, \dots, n\}$ ,

$$A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \dots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}, \exists! R_{1,k}, \dots, R_{c,k} \in \mathbf{F}, \text{ forming } R \in \mathbf{F}^{c,n}. \text{ Thus } A = CR.$$

(b) Reduce to basis  $B_R = (R_{1,\cdot}, \dots, R_{r,\cdot})$ , forming  $R \in \mathbf{F}^{r,n}$ . Then  $\forall j \in \{1, \dots, m\}$ ,

$$A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \dots + C_{j,r}R_{r,\cdot} = (CR)_{j,\cdot}, \exists! C_{j,1}, \dots, C_{j,r} \in \mathbf{F}, \text{ forming } C \in \mathbf{F}^{m,r}. \text{ Thus } A = CR. \quad \square$$

**EXAMPLE:**  $A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \xrightarrow{\text{(I)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \xrightarrow{\text{(II)}} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$

$$\text{(I)} \begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2 \begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix}, \text{ using [4E 3.51(b)]}.$$

$$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} \in \text{span}(A_{1,\cdot}, A_{2,\cdot}), \text{ and } (A_{1,\cdot}, A_{2,\cdot}) \text{ is linely inde. Thus } B_R = (A_{1,\cdot}, A_{2,\cdot}).$$

$$\text{(II)} \begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2 \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}; \quad \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = - \begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}. \text{ Thus } B_C = (A_{\cdot,2}, A_{\cdot,3}).$$

• **COLUMN RANK EQUALS ROW RANK** Using notation and result above.

$$\text{For each } A_{j,\cdot} \in S_r, A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,1}R_{1,\cdot} + \dots + C_{j,c}R_{c,\cdot}.$$

$$\text{For each } A_{\cdot,k} \in S_c, A_{\cdot,k} = (CR)_{\cdot,k} = CR_{\cdot,k} = R_{1,k}C_{\cdot,1} + \dots + R_{c,k}C_{\cdot,c}.$$

$$\Rightarrow \text{span}(A_{1,\cdot}, \dots, A_{m,\cdot}) = S_r = \text{span}(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leq c = \dim S_c.$$

$$\Rightarrow \text{span}(A_{\cdot,1}, \dots, A_{\cdot,m}) = S_c = \text{span}(C_{\cdot,1}, \dots, C_{\cdot,r}) \Rightarrow \dim S_c = c \leq r = \dim S_r.$$

$$\text{OR. Apply the result to } A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leq r = \dim S_r = \dim S_c^t. \quad \square$$

• Suppose  $A \in \mathbf{F}^{m,n} \setminus \{0\}$ . Prove that  $[P] \text{ rank } A = 1 \iff \exists c_j, d_k \in \mathbf{F}, \text{ each } A_{j,k} = c_j \cdot d_k. [Q]$

**SOLUTION:**

[ Using CR Factorization ]

$P \Rightarrow Q$ : Immediately.

$$Q \Rightarrow P: \text{ Because } A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \dots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \dots & c_m d_n \end{pmatrix} \Rightarrow S_r = \text{span} \left\{ \begin{pmatrix} \underline{c_1} d_1 & \dots & \underline{c_1} d_n \\ \vdots \\ \underline{c_m} d_1 & \dots & \underline{c_m} d_n \end{pmatrix} \right\}.$$

$$\text{OR. } S_c = \text{span} \left\{ \begin{pmatrix} \underline{c_1} d_1 \\ \vdots \\ \underline{c_m} d_1 \end{pmatrix}, \dots, \begin{pmatrix} \underline{c_1} d_n \\ \vdots \\ \underline{c_m} d_n \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}.$$

$\square$

[ Not Using CR Factorization ]

$$Q \Rightarrow P: \text{ Using [4E 3.51(a)]. Each } A_{\cdot,k} \in \text{span} \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}. \text{ Then rank } A = \dim S_c \leq 1$$

$$\text{又 } A \neq 0 \Rightarrow \dim S_c \geq 1.$$

$$P \Rightarrow Q: \text{ Because } \dim S_c = \dim S_r = 1.$$

$$\text{Let } c_j = \frac{A_{j,1}}{A_{1,1}} = \frac{A_{j,2}}{A_{1,2}} = \dots = \frac{A_{j,n}}{A_{1,n}}, \quad d'_k = \frac{A_{1,k}}{A_{1,1}} = \frac{A_{2,k}}{A_{2,1}} = \dots = \frac{A_{m,k}}{A_{m,1}}.$$

$$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k, \text{ where } d_k = d'_k A_{1,1}. \quad \square$$



- [4E 17, OR 3.F.32] Suppose  $T \in \mathcal{L}(V)$  and  $(u_1, \dots, u_n), (v_1, \dots, v_n)$  are bases of  $V$ . Prove that the following are equi. Here  $A = \mathcal{M}(T) = \mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$ .  
 (a)  $T$  is inje; (b)  $(A_{.,1}, \dots, A_{.,n})$  is a basis of  $\mathbf{F}^{n,1}$ ; (c)  $(A_{1,.}, \dots, A_{n,.})$  is a basis of  $\mathbf{F}^{1,n}$ .

**SOLUTION:**  $T$  is inje  $\iff \dim V = \dim \text{range } T = n$

$$\iff B_V = (Tu_1, \dots, Tu_n) \xLeftrightarrow{\Delta} B_{\mathbf{F}^{n,1}} = (A_{.,1}, \dots, A_{.,n}) \xLeftrightarrow[2.42]{\text{rank}} B_{\mathbf{F}^{1,n}} = (A_{1,.}, \dots, A_{n,.}).$$

**COMMENT:**  $T'$  is inv  $\iff B_{V'} = (T'(\psi_1), \dots, T'(\psi_n))$

$$\xLeftrightarrow{\Delta} B_{\mathbf{F}^{n,1}} = ((A^t)_{.,1}, \dots, (A^t)_{.,n}) \xLeftrightarrow[2.39]{} B_{\mathbf{F}^{1,n}} = (((A^t)_{.,1})^t, \dots, ((A^t)_{.,n})^t).$$

Note that  $\mathcal{M}: Tu_k \rightarrow A_{.,k}$  is an iso of  $V$  onto  $\mathbf{F}^{1,n}$ . Hence  $\Delta$  holds. □

OR. Using TIPS below. We show  $[P] (Tu_1, \dots, Tu_n)$  linely inde  $\iff (A_{.,1}, \dots, A_{.,n})$  linely inde.  $[Q]$

$P \Rightarrow Q$ : Suppose  $b_1 A_{.,1} + \dots + b_n A_{.,n} = 0$ . Let  $u = b_1 u_1 + \dots + b_n u_n$ .

Then  $Tu = (b_1 A_{1,1} + \dots + b_n A_{1,n})v_1 + \dots + (b_1 A_{n,1} + \dots + b_n A_{n,n})v_n = 0v_1 + \dots + 0v_n$ .

Now  $b_1 Tu_1 + \dots + b_n Tu_n = 0$ . Then each  $b_k = 0$ . Thus  $(A_{.,1}, \dots, A_{.,n})$  is linely inde.

$Q \Rightarrow P$ : Because  $b_1 Tu_1 + \dots + b_n Tu_n = 0 \Rightarrow b_1 A_{1,1} + \dots + b_n A_{1,n} = \dots = b_1 A_{n,1} + \dots + b_n A_{n,n} = 0$ .

Which is equi to  $b_1 A_{.,1} + \dots + b_n A_{.,n} = 0$ . Thus each  $b_k = 0 \Rightarrow \text{null } T = \{0\}$ . □

- **TIPS:**  $b_1 Tu_1 + \dots + b_n Tu_n = b_1 (A_{1,1}v_1 + \dots + A_{n,1}v_n) + \dots + b_n (A_{1,n}v_1 + \dots + A_{n,n}v_n)$   
 $= (b_1 A_{1,1} + \dots + b_n A_{1,n})v_1 + \dots + (b_1 A_{n,1} + \dots + b_n A_{n,n})v_n$ .

$$\text{And } b_1 A_{.,1} + \dots + b_n A_{.,n} = b_1 \begin{pmatrix} A_{1,1} \\ \vdots \\ A_{n,1} \end{pmatrix} + \dots + b_n \begin{pmatrix} A_{1,n} \\ \vdots \\ A_{n,n} \end{pmatrix} = \begin{pmatrix} b_1 A_{1,1} + \dots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \dots + b_n A_{n,n} \end{pmatrix}.$$

**1** Suppose  $T \in \mathcal{L}(V, W)$ . Show that for each pair of  $B_V$  and  $B_W$ ,

$A = \mathcal{M}(T, B_V, B_W)$  has at least  $n = \dim \text{range } T$  nonzero entries.

**SOLUTION:**

Using [3.B TIPS (4)]. Let  $U \oplus \text{null } T = V$ ;  $B_U = (v_1, \dots, v_n), B_V = (v_1, \dots, v_m)$ .

For each  $k \in \{1, \dots, n\}, Tv_k \neq 0 \iff A_{.,k} \neq 0$ . Hence every such  $A_{.,k}$  has at least one nonzero entry. □

OR. We prove by contradiction. Suppose  $A$  has at most  $(n-1)$  nonzero entries.

Then by Pigeon Hole Principle, at least one of  $A_{.,1}, \dots, A_{.,n}$  equals 0.

Thus there are at most  $(n-1)$  nonzero vecs in  $Tv_1, \dots, Tv_n$ .

$\text{range } T = \text{span}(Tv_1, \dots, Tv_n) \Rightarrow \dim \text{range } T = \dim \text{span}(Tv_1, \dots, Tv_n) \leq n-1$ . Contradicts. □

**3** Suppose  $V$  and  $W$  are finite-dim and  $T \in \mathcal{L}(V, W)$ . Prove that  $\exists B_V, B_W$  such that

[letting  $A = \mathcal{M}(T, B_V, B_W)$ ]  $A_{k,k} = 1, A_{i,j} = 0$ , where  $1 \leq k \leq \dim \text{range } T, i \neq j$ .

**SOLUTION:** Using [3.B TIPS (4)]. Let  $B_{\text{range } T} = (Tv_1, \dots, Tv_n), B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$ . □

**COMMENT:** Let each  $Tv_k = w_k$ . Extend  $B_{\text{range } T}$  to  $B_W = (w_1, \dots, w_n, \dots, w_p)$ . See [3.D NOTE FOR [3.60]].

**4** Suppose  $B_V = (v_1, \dots, v_m)$  and  $W$  is finite-dim. Suppose  $T \in \mathcal{L}(V, W)$ .

Prove that  $\exists B_W = (w_1, \dots, w_n), \mathcal{M}(T, B_V, B_W)_{.,1} = (1 \ 0 \ \dots \ 0)^t$  or  $(0 \ \dots \ 0)^t$ .

**SOLUTION:** If  $Tv_1 = 0$ , then we are done. If not then extend  $(Tv_1)$  to  $B_W$ . □

**5** Suppose  $B_W = (w_1, \dots, w_n)$  and  $V$  is finite-dim. Suppose  $T \in \mathcal{L}(V, W)$ .

Prove that  $\exists B_V = (v_1, \dots, v_m)$ ,  $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = (0 \ \dots \ 0)$  or  $(1 \ 0 \ \dots \ 0)$ .

**SOLUTION:** See also in (3.F).

Let  $(u_1, \dots, u_n)$  be a basis of  $V$ . Denote  $\mathcal{M}(T, (u_1, \dots, u_n), B_W)$  by  $A$ .

If  $A_{1,\cdot} = 0$ , then  $B_V = (u_1, \dots, u_n)$  and we are done. Otherwise, suppose  $A_{1,k} \neq 0$ .

Let  $v_1 = \frac{u_k}{A_{1,k}}$ , so that  $Tv_1 = 1w_1 + \frac{A_{2,k}}{A_{1,k}}w_2 + \dots + \frac{A_{n,k}}{A_{1,k}}w_n$ .

Let  $v_j = u_{j-1} - A_{1,j-1}v_1$  for each  $j \in \{2, \dots, k\}$ . Let  $v_i = u_i - A_{1,i}v_1$  for  $i \in \{k+1, \dots, n\}$ .

NOTICE that  $Tu_i = A_{1,i}w_1 + \dots + A_{n,i}w_n$ . 又 Each  $u_i \in \text{span}(v_1, \dots, v_n) = V$ . Let  $B_V = (v_1, \dots, v_n)$ .  $\square$

**6** Suppose  $V$  and  $W$  are finite-dim and  $T \in \mathcal{L}(V, W)$ .

Prove that  $\dim \text{range } T = 1 \iff \exists B_V, B_W$ , all entries of  $A = \mathcal{M}(T, B_V, B_W)$  equal 1.

**SOLUTION:**

(a) Suppose  $B_V = (v_1, \dots, v_n)$ ,  $B_W = (w_1, \dots, w_m)$  are the bases such that all entries of  $A$  equal 1.

Then  $Tv_i = w_1 + \dots + w_m$  for all  $i = 1, \dots, n$ . Because  $w_1, \dots, w_m$  is linely inde,  $w_1 + \dots + w_m \neq 0$ .

(b) Suppose  $\dim \text{range } T = 1$ . Then  $\dim \text{null } T = \dim V - 1$ .

Let  $B_{\text{null } T} = (u_2, \dots, u_n)$ . Extend to a basis  $(u_1, u_2, \dots, u_n)$  of  $V$ .

Let  $w_1 = Tv_1 - w_2 - \dots - w_m$ . Extend to  $B_W$ . Let  $v_1 = u_1$ ,  $v_i = u_1 + u_i$ . Extend to  $B_V$ .  $\square$

OR. Suppose  $\text{range } T$  has a basis  $(w)$ .

By [2.C NOTE FOR (15)],  $\exists B_W = (w_1, \dots, w_m)$  such that  $w = w_1 + \dots + w_m$ .

By [2.C TIPS],  $\exists$  a basis  $(u_1, \dots, u_n)$  of  $V$  such that each  $u_k \notin \text{null } T$ .

Now each  $Tu_k \in \text{range } T = \text{span}(w) \Rightarrow Tu_k = \lambda_k w$ ,  $\exists \lambda_k \in \mathbb{F} \setminus \{0\}$ .

Let  $v_k = \lambda_k^{-1}u_k \neq 0$ , so that each  $Tv_k = w = w_1 + \dots + w_m$ . Thus  $B_V = (v_1, \dots, v_n)$  will do.  $\square$

• **TIPS:** Suppose  $p$  is a poly of  $n$  variables in  $\mathbb{F}$ .

Prove that  $\mathcal{M}(p(T_1, \dots, T_n)) = p(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n))$ .

Where the linear maps  $T_1, \dots, T_n$  are such that  $p(T_1, \dots, T_n)$  makes sense. See [5.16,17,20].

**SOLUTION:** Suppose the poly  $p$  is defined by  $p(x_1, \dots, x_n) = \sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n x_i^{k_i}$ .

Note that  $\mathcal{M}(T^x S^y) = \mathcal{M}(T)^x \mathcal{M}(S)^y$ ;  $\mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$ .

Then  $\mathcal{M}(p(T_1, \dots, T_n)) = \mathcal{M}\left(\sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n T_i^{k_i}\right)$   
 $= \sum_{k_1, \dots, k_n} \alpha_{k_1, \dots, k_n} \prod_{i=1}^n \mathcal{M}(T_i^{k_i}) = p(\mathcal{M}(T_1), \dots, \mathcal{M}(T_n))$ .  $\square$

• **COROLLARY:** Suppose  $\tau$  is an algebraic property.

Then  $\tau$  holds for matrices  $\iff \tau$  holds for linear maps.

**13** Prove that the distr holds for matrix add and matrix multi.

Suppose  $A, B, C$  are matrices such that  $A(B+C)$  make sense, we prove the left distr.

**SOLUTION:** Suppose  $A \in \mathbb{F}^{m,n}$  and  $B, C \in \mathbb{F}^{n,p}$ .

Note that  $[A(B+C)]_{j,k} = \sum_{r=1}^n A_{j,r}(B+C)_{r,k} = \sum_{r=1}^n (A_{j,r}B_{r,k} + A_{j,r}C_{r,k}) = (AB+AC)_{j,k}$   $\square$

OR. Define  $T, S, R$  such that  $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$ .

$A(B+C) = \mathcal{M}(T(S+R)) \stackrel{[3.9]}{=} \mathcal{M}(TS+TR) = AB+AC$ .

Or  $T(S+R) = TS+TR \Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR) \Rightarrow A(B+C) = AB+AC$ .  $\square$

#### 14 Prove that matrix multi is associ.

Suppose  $A, B, C$  are matrices such that  $(AB)C$  makes sense, we prove that  $(AB)C = A(BC)$ .

**SOLUTION:** Suppose  $A \in \mathbb{F}^{m,n}$  and  $B, C \in \mathbb{F}^{n,p}$ . We show that  $LHS = [(AB)C]_{j,k} = [A(BC)]_{j,k} = RHS$ .  
 $LHS = (AB)_{j,\cdot} C_{\cdot,k} = \sum_{s=1}^n (A_{j,s} B_{s,\cdot}) C_{\cdot,k} = \sum_{s=1}^n A_{j,s} (B_{s,\cdot} C_{\cdot,k}) = \sum_{s=1}^n A_{j,s} (BC)_{s,k} = RHS. \quad \square$

OR. Define  $T, S, R$  such that  $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$ .

$$(AB)C = \mathcal{M}(T(SR)) \stackrel{[3.9]}{=} \mathcal{M}(TSR) \stackrel{[3.9]}{=} \mathcal{M}((TS)R) = A(BC).$$

OR.  $(TS)R = T(SR) \Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR)) \Rightarrow (AB)C = A(BC). \quad \square$

#### 15 Suppose $A \in \mathbb{F}^{n,n}, j, k \in \{1, \dots, n\}$ . Show that $(A^3)_{j,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}$ .

**SOLUTION:**  $(AAA)_{j,k} = (AA)_{j,\cdot} A_{\cdot,k} = \sum_{p=1}^n (A_{j,p} A_{p,\cdot}) A_{\cdot,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}$ .

$$\text{OR. } (AAA)_{j,k} = \sum_{r=1}^n (AA)_{j,r} A_{r,k} = \sum_{r=1}^n \left( \sum_{p=1}^n A_{j,p} A_{p,r} \right) A_{r,k}$$

$$= \sum_{r=1}^n \left[ A_{j,1} (A_{1,r} A_{r,k}) + \dots + A_{j,n} (A_{n,r} A_{r,k}) \right]$$

$$= A_{j,1} \sum_{r=1}^n A_{1,r} A_{r,k} + \dots + A_{j,n} \sum_{r=1}^n A_{n,r} A_{r,k} = \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}. \quad \square$$

#### • Prove that the commutativity does not hold in $\mathbb{F}^{m,n}$ .

**SOLUTION:** Suppose  $\dim V = n, \dim W = m$  and the commutativity holds in  $\mathbb{F}^{n,m}$ .

$$\forall T \in \mathcal{L}(V, W), S \in \mathcal{L}(W, V), \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST).$$

Hence  $ST = TS$ . Which in general does not hold.  $\square$

**ENDED**

## 3.D

1 2 3 4 5 6 8 9 10 11 12 13 15 16 17 18 19 | 4E: 3 10 15 17 19 20 22 23 24

#### 2 Suppose $V$ is finite-dim and $\dim V > 1$ .

Prove that the set  $U$  of non-inv operators on  $V$  is not a subsp of  $\mathcal{L}(V)$ .

The set of inv operators is not either. Although multi identity/inv, and commutativity for vec multi hold.

**SOLUTION:** Let  $B_V = (v_1, \dots, v_n)$ . [ If  $\dim V = 1$ , then  $U = \{0\}$  is a subsp of  $\mathcal{L}(V)$ . ]

Define  $S, T \in \mathcal{L}(V)$  by  $S(a_1 v_1 + \dots + a_n v_n) = a_1 v_1, T(a_1 v_1 + \dots + a_n v_n) = a_2 v_1 + \dots + a_n v_n$ .

Hence  $S, T \in U$  while  $S + T \notin U$ .  $\square$

#### • TIPS: Suppose $U \oplus X = W \oplus Y$ , and $X, Y$ are iso. Prove that $U, W$ are iso.

**SOLUTION:** Let  $\zeta$  be an iso of  $X$  onto  $Y$ . That is,  $\forall y \in Y, \exists! x \in X, \zeta(x) = y$ .

$\forall u \in U, \exists! w \in W, y \in Y, u = w + y \Rightarrow \exists! x \in X, u = w + \zeta(x)$ . Define  $\pi : u \mapsto w$ .

Now suppose  $u_1, u_2 \in U$ , then each  $u_i = w_i + \zeta(x_i), \exists! w_i \in W, x_i \in X$ .

Linearity:  $\forall \lambda \in \mathbb{F}, \pi(u_1 + \lambda u_2) = w_1 + \lambda w_2 = \pi(u_1) + \lambda \pi(u_2)$ .

Injectivity:  $\pi(u_1) = \pi(u_2) \Rightarrow w_1 = w_2 \Rightarrow \zeta(x_1) = \zeta(x_2) \Rightarrow x_1 = x_2 \Rightarrow u_1 = u_2$ .

Surjectivity:  $\forall w \in W, \pi(w) = w \in \text{range } \pi$ . Thus  $\pi$  is an iso of  $U$  onto  $W$ .  $\square$

**3** Suppose  $V$  and  $W$  are iso,  $U$  is a subsp of  $V$ , and  $S \in \mathcal{L}(U, W)$ .

Prove that  $\exists \text{ inv } T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U \iff S \text{ is inje.}$  [ See also (3.A.11). ]

**SOLUTION:** (a)  $\forall u \in U, u = T^{-1}Su \Rightarrow T^{-1}S = I \in \mathcal{L}(U) \Rightarrow S \text{ is inje, by (3.B.20).}$

OR.  $\text{null } S = \text{null } T|_U = \text{null } T \cap U = \{0\}.$

(b) Let  $X \oplus U = V$ . Because  $S : U \rightarrow W$  is inje. By (3.B.12),  $S : U \rightarrow \text{range } S$  is an iso.

Let  $Y \oplus \text{range } S = W$ . Then by TIPS,  $X$  and  $Y$  are iso. Let  $E : X \rightarrow Y$  be an iso.

Define  $T \in \mathcal{L}(V, W)$  by  $Tu = Su, Tw = Ew$  for all  $u \in U, w \in X$ .

OR. [ Req  $V$  Finite-dim ] Let  $B_U = (u_1, \dots, u_m)$ . Then  $S \text{ inje} \Rightarrow (Su_1, \dots, Su_m)$  linely inde.

Extend to  $B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_W = (Su_1, \dots, Su_m, w_1, \dots, w_n)$ .

Define  $T \in \mathcal{L}(V, W)$  by  $T(u_i) = Su_i; Tv_j = w_j$ , for each  $u_i$  and  $v_j$ .  $\square$

**8** Suppose  $T \in \mathcal{L}(V, W)$  is **surj**. Prove that  $\exists \text{ subsp } U \text{ of } V, T|_U : U \rightarrow W \text{ is an iso.}$

**SOLUTION:** [ Req  $\text{range } T$  Finite-dim ] Let  $B_{\text{range } T} = B_W = (Tv_1, \dots, Tv_m), B_U = (v_1, \dots, v_m).$   $\square$

OR. By (3.B.12). Note that  $\text{range } T = W$ .  $\square$

**18** Show that  $V$  and  $\mathcal{L}(\mathbf{F}, V)$  are iso vecsps.

**SOLUTION:**

Define  $\Psi \in \mathcal{L}(V, \mathcal{L}(\mathbf{F}, V))$  by  $\Psi(v) = \Psi_v$ ; where  $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$  and  $\Psi_v(\lambda) = \lambda v$ .

(a)  $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$ . Hence  $\Psi$  is inje.

(b)  $\forall T \in \mathcal{L}(\mathbf{F}, V)$ , let  $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi(T(1))$ . Hence  $\Psi$  is surj.  $\square$

OR. Define  $\Phi \in \mathcal{L}(\mathcal{L}(\mathbf{F}, V), V)$  by  $\Phi(T) = T(1)$ .

(a) Suppose  $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$ . Thus  $\Phi$  is inje.

(b) For any  $v \in V$ , define  $T \in \mathcal{L}(\mathbf{F}, V)$  by  $T(\lambda) = \lambda v$ . Then  $\Phi(T) = T(1) = v$ . Thus  $\Phi$  is surj.  $\square$

**COMMENT:**  $\Phi = \Psi^{-1}$ .

• Suppose  $S, T \in \mathcal{L}(V, W)$ . [ For Problem (4) and (5), see the COROLLARY in (3.B.24, 25). ]

**6** Suppose  $V$  and  $W$  are finite-dim.  $\dim \text{null } S = \dim \text{null } T = n$ .

Prove that  $S = E_2 T E_1, \exists \text{ inv } E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$ .

**SOLUTION:** Define  $E_1 : v_i \mapsto r_i; u_j \mapsto s_j$ ; for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ .

Define  $E_2 : Tv_i \mapsto Sr_i; x_j \mapsto y_j$ ; for each  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ . Where:

$$\left| \begin{array}{l} \text{Let } B_{\text{range } T} = (Tv_1, \dots, Tv_m); B_{\text{range } S} = (Sr_1, \dots, Sr_m). \\ \text{Let } B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p); B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p). \\ \text{Let } B_{\text{null } T} = (u_1, \dots, u_n); B_{\text{null } S} = (s_1, \dots, s_n). \\ \text{Thus } B_V = (v_1, \dots, v_m, u_1, \dots, u_n); B'_V = (r_1, \dots, r_m, s_1, \dots, s_n). \end{array} \right| \begin{array}{l} \therefore E_1, E_2 \text{ are inv} \\ \text{and } S = E_2 T E_1. \end{array} \quad \square$$

• (a) Suppose  $T = ES$  and  $E \in \mathcal{L}(W)$  is inv. Prove that  $\text{null } S = \text{null } T$ .

(b) Suppose  $T = SE$  and  $E \in \mathcal{L}(V)$  is inv. Prove that  $\text{range } S = \text{range } T$ .

(c) Suppose  $T = E_2 S E_1$  and  $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$  are inv.

Prove that  $\dim \text{null } S = \dim \text{null } T$ .

**SOLUTION:** (a)  $v \in \text{null } T \iff Tv = 0 = E(Sv) \iff Sv = 0 \iff v \in \text{null } S$ .

(b)  $w \in \text{range } T \iff \exists v \in V, Tv = S(Ev) \iff \exists u \in V, w = Su \iff w \in \text{range } S$ .

(c) Using (3.B.22).  $\dim \text{null } E_2 S E_1 \xrightarrow[\text{inv}]{E_2} \dim \text{null } S E_1 \xrightarrow[\text{inv}]{E_1} \dim \text{null } S = \dim \text{null } T$ .  $\square$

• **NOTE FOR [3.69]:** Suppose  $V, W$  are finite-dim and iso,  $T \in \mathcal{L}(V, W)$ . Then  $T \text{ inv} \iff \text{inje} \iff \text{surj}$ .

**9 [OR 1]** Suppose  $U, V, W$  are iso and finite-dim,  $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$ .  
Prove that  $ST$  is inv  $\iff S, T$  are inv.

**COMMENT:** If any two of  $U, V, W$  are not iso or finite-dim, then  $S, T$  are inv  $\implies ST$  is inv.

**SOLUTION:** Suppose  $S, T$  are inv. Then  $(ST)(T^{-1}S^{-1}) = I_W, (T^{-1}S^{-1})(ST) = I_U$ . Hence  $ST$  is inv.

Suppose  $ST$  is inv. Let  $R = (ST)^{-1} \Rightarrow R(ST) = I_U, (ST)R = I_W$ .

$$\begin{array}{l|l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0. & T \text{ is inje, } S \text{ is surj.} \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \text{range } S. & \text{又 } \dim U = \dim V = \dim W. \end{array}$$

OR. By (3.B.23),  $\dim W = \dim \text{range } ST \leq \min\{\text{range } S, \text{range } T\} \Rightarrow S, T$  are surj.  $\square$

**13** Suppose  $U, V, W, X$  are iso and finite-dim,  $R \in \mathcal{L}(W, X), S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$ .  
Suppose  $RST$  is surj. Prove that  $S$  is inje.

**SOLUTION:** Using Problem (9). Notice that  $U, X$  are finite-dim, so that  $RST$  is inv.

$$\text{Let } X = (RST)^{-1} \left\{ \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje.} \\ \forall v \in V, v = (RST)Xv \in \text{range } R \Rightarrow R \text{ is surj.} \end{array} \right\} \Rightarrow S = R^{-1}(RST)T^{-1}. \quad \square$$

$$\text{OR. } (RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}. \quad \square$$

**10** Suppose  $V$  is finite-dim and  $S, T \in \mathcal{L}(V)$ . Prove that  $ST = I \iff TS = I$ .

**SOLUTION:** (a) Suppose  $ST = I$ .

By (3.B 20, 21)(a),  $ST = I \Rightarrow T$  is inje and  $S$  is surj. 又  $V$  is finite-dim.  $S, T$  are inv.

OR. By Problem (9),  $V$  is finite-dim and  $ST = I$  is inv  $\Rightarrow S, T$  are inv.

$$\text{Then } \forall v \in V, S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow TS = I.$$

$$\text{OR. } S^{-1} = T \text{ 又 } S = S \Rightarrow TS = S^{-1}S = I.$$

(b) Reversing the roles of  $S$  and  $T$ , we conclude that  $TS = I \Rightarrow ST = I$ .  $\square$

**11** Suppose  $V$  is finite-dim,  $S, T, U \in \mathcal{L}(V)$  and  $STU = I$ . Show that  $T$  is inv and  $T^{-1} = US$ .

**SOLUTION:** Using Problem (9) and (10). This result can fail without the hypothesis that  $V$  is finite-dim.

$$(ST)U = U(ST) = (US)T = I \Rightarrow T^{-1} = US.$$

$$\text{OR. } (ST)U = S(TU) = I \Rightarrow U, S \text{ are inv} \Rightarrow TU = S^{-1}. \text{ 又 } U^{-1} = U^{-1} \Rightarrow T = S^{-1}U^{-1}. \quad \square$$

**EXAMPLE:**  $V = \mathbb{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots); T(a_1, \dots) = (0, a_1, \dots); U = I \Rightarrow STU = I$  but  $T$  is not inv.

• (4E 3)  $T \in \mathcal{L}(V) \left| \begin{array}{l} (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1, \dots, v_n) \text{ of } V \iff T \text{ is surj} \\ V \text{ is finite-dim} \quad (Tv_1, \dots, Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1, \dots, v_n) \text{ of } V \iff T \text{ is inje} \end{array} \right\} \iff T \text{ is inv.}$

• (4E 15) Suppose  $T \in \mathcal{L}(V)$  and  $V = \text{span}(Tv_1, \dots, Tv_m)$ . Prove that  $V = \text{span}(v_1, \dots, v_m)$ .

**SOLUTION:** Because  $V = \text{span}(Tv_1, \dots, Tv_m) \Rightarrow T$  is surj, and therefore is inv  $\Rightarrow T^{-1}$  is inv.

$$\forall v \in V, \exists a_i \in \mathbb{F}, v = \sum_{i=1}^m a_i Tv_i \Rightarrow T^{-1}v = \sum_{i=1}^m a_i v_i \Rightarrow \text{range } T^{-1} \subseteq \text{span}(v_1, \dots, v_m).$$

OR. Reduce the spanning list  $(Tv_1, \dots, Tv_m)$  of  $V$  to a basis  $(Tv_{\alpha_1}, \dots, Tv_{\alpha_k})$  of  $V$ .

Where  $k = \dim V$  and each  $\alpha_i \in \{1, \dots, m\}$ . Then by Problem (4E 3),

$(v_{\alpha_1}, \dots, v_{\alpha_k})$  is also a basis of  $V$ , contained in the list  $(v_1, \dots, v_m)$ .  $\square$



**15** Prove that every linear map from  $\mathbf{F}^{n,1}$  to  $\mathbf{F}^{m,1}$  is given by a matrix multi.

In other words, prove that if  $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$ , then  $\exists A \in \mathbf{F}^{m,n}, Tx = Ax, \forall x \in \mathbf{F}^{n,1}$ .

**SOLUTION:** Let  $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$  be the std bases of  $\mathbf{F}^{n,1}, \mathbf{F}^{m,1}$ .

$$\forall k = 1, \dots, n, T(E_k) = A_{1,k}R_1 + \dots + A_{m,k}R_m, \exists A_{j,k} \in \mathbf{F}, \text{ forming } A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix}.$$

OR. Let  $A = \mathcal{M}(T, B_1, B_2)$ . Note that  $\mathcal{M}(x, B_1) = x, \mathcal{M}(Tx, B_2) = Tx$ .

Hence  $Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax$ , by [3.65].  $\square$

• **NOTE FOR [3.62]:**  $\mathcal{M}(v) = \mathcal{M}(I, (v), B_V)$ . Where  $I$  is the identity operator restricted to  $\text{span}(v)$ .

• **NOTE FOR [3.65]:**  $\mathcal{M}(Tv) = \mathcal{M}(I, (Tv), B_W) = \mathcal{M}(T, B_V, B_W)\mathcal{M}(I, (v), B_V) = \mathcal{M}(T, (v), B_W)$ .

If  $v = 0$ , then  $\text{span}(v) = \text{span}(\ )$ , we replace  $(v)$  by  $B = (\ )$ ; similar for  $Tv = 0$ .

• (4E 23, OR 10.A.4) Suppose that  $(\beta_1, \dots, \beta_n)$  and  $(\alpha_1, \dots, \alpha_n)$  are bases of  $V$ .

Let  $T \in \mathcal{L}(V)$  be such that  $T\alpha_k = \beta_k, \forall k$ . Prove that  $\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \beta \rightarrow \alpha)$

For ease of notation, let  $\mathcal{M}(T, \alpha \rightarrow \beta) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n))$ ,  $\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n))$ .

**SOLUTION:**

Denote  $\mathcal{M}(T, \alpha \rightarrow \alpha)$  by  $A$  and  $\mathcal{M}(I, \beta \rightarrow \alpha)$  by  $B$ .

$$\forall k \in \{1, \dots, n\}, Iu_k = u_k = B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \dots + A_{n,k}\alpha_n \Rightarrow A = B. \quad \square$$

OR. Note that  $\mathcal{M}(T, \alpha \rightarrow \beta) = I$ . Hence  $\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \beta \rightarrow \alpha) \underbrace{\mathcal{M}(T, \alpha \rightarrow \beta)}_{=\mathcal{M}(I, \beta \rightarrow \beta)} = \mathcal{M}(I, \beta \rightarrow \alpha)$ .  $\square$

OR. Note that  $\mathcal{M}(T, \beta \rightarrow \beta)\mathcal{M}(I, \alpha \rightarrow \beta) = \mathcal{M}(T, \alpha \rightarrow \beta) = I$ .

$$\mathcal{M}(T, \alpha \rightarrow \alpha) = \mathcal{M}(I, \alpha \rightarrow \beta)^{-1} \left( \underbrace{\mathcal{M}(T, \beta \rightarrow \beta)\mathcal{M}(I, \alpha \rightarrow \beta)}_{=\mathcal{M}(T, \alpha \rightarrow \beta)} \right) = \mathcal{M}(I, \beta \rightarrow \alpha). \quad \square$$

**COMMENT:** Let  $A' = \mathcal{M}(T, \beta \rightarrow \beta)$ .

$$u_k = Iu_k = B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n, \forall k \in \{1, \dots, n\}.$$

$$\text{又 } Tu_k = T(B_{1,k}\alpha_1 + \dots + B_{n,k}\alpha_n) = B_{1,k}\beta_1 + \dots + B_{n,k}\beta_n = A'_{1,k}\beta_1 + \dots + A'_{n,k}\beta_n \Rightarrow A' = B.$$

$$\text{OR. } \mathcal{M}(T, \beta \rightarrow \beta) = \mathcal{M}(T, \alpha \rightarrow \beta)\mathcal{M}(I, \beta \rightarrow \alpha) = B.$$

• **TIPS:** When using  $\mathcal{M}^{-1}$ , you must first declare bases and the purpose for using  $\mathcal{M}^{-1}$ .

That is, to declare  $B_U, B_V, B_W, \mathcal{M} : \mathcal{L}(V, W) \mapsto \mathbf{F}^{m,n}$ , or  $\mathcal{M} : v \mapsto \mathbf{F}^{n,1}$ .

So that  $\mathcal{M}^{-1}(AC, B_U, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(C, B_U, B_V)$ ;

Or  $\mathcal{M}^{-1}(Ax, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(x, B_V)$ . Where everything is well-defined.

• (4E 22, OR 10.A.1) Suppose  $T \in \mathcal{L}(V)$ . Prove that  $\mathcal{M}(T, B_V)$  is inv  $\iff T$  itself is inv.

**SOLUTION:** Notice that  $\mathcal{M} : T \mapsto \mathcal{M}(T, B_V)$  is an iso. And that  $\mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(TS)$ .

$$(a) T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(I) = \mathcal{M}(T)\mathcal{M}(T^{-1}) \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}.$$

$$(b) \mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I, \exists! S \in \mathcal{L}(V) \text{ such that } \mathcal{M}(T)^{-1} = \mathcal{M}(S)$$

$$\Rightarrow \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = I = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$$

$$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}. \quad \square$$

• (4E 24, OR 10.A.2) Suppose  $A, B \in \mathbf{F}^{n,n}$ . Prove that  $AB = I \iff BA = I$ . [Using Problem (10, 15).]

**SOLUTION:** Define  $T, S \in \mathcal{L}(\mathbf{F}^{n,1})$  by  $Tx = Ax, Sx = Bx$  for all  $x \in \mathbf{F}^{n,1}$ . Now  $\mathcal{M}(T) = A, \mathcal{M}(S) = B$ .

$$AB = I \iff A(Bx) = x \iff T(Sx) = x \iff TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I.$$

OR. Because  $\mathcal{M} : \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1}) \rightarrow \mathbf{F}^{n,n}$  is an iso.  $\mathcal{M}^{-1}(AB) = TS = ST = \mathcal{M}^{-1}(BA) = I. \quad \square$



• **NOTE FOR [3.60]:** Suppose  $B_V = (v_1, \dots, v_n)$ ,  $B_W = (w_1, \dots, w_m)$ .

Define  $E_{i,j} \in \mathcal{L}(V, W)$  by  $E_{i,j}(v_x) = \delta_{i,x} w_j$ . **COROLLARY:**  $E_{l,k} E_{i,j} = \delta_{j,l} E_{i,k}$ .

Denote  $\mathcal{M}(E_{i,j})$  by  $\mathcal{E}^{(j,i)}$ . And  $(\mathcal{E}^{(j,i)})_{l,k} = \begin{cases} 1, & \text{if } (i,j) = (l,k); \\ 0, & \text{otherwise.} \end{cases}$

NOTICE that  $\mathcal{M} : \mathcal{L}(V, W) \rightarrow \mathbf{F}^{m,n}$  is an iso. And  $E_{i,j} = \mathcal{M}^{-1} \mathcal{E}^{(j,i)}$ .

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)} + \dots + A_{1,n}\mathcal{E}^{(1,n)} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{m,1}\mathcal{E}^{(m,1)} + \dots + A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff \begin{pmatrix} A_{1,1}E_{1,1} + \dots + A_{1,n}E_{n,1} \\ + & \dots & + \\ \vdots & \ddots & \vdots \\ + & \dots & + \\ A_{m,1}E_{1,m} + \dots + A_{m,n}E_{n,m} \end{pmatrix} = T.$$

$$\text{By [2.42] and [3.61], } B_{\mathcal{L}(V,W)} = \begin{pmatrix} E_{1,1}, & \dots, & E_{n,1} \\ \vdots & & \vdots \\ E_{1,m}, & \dots, & E_{n,m} \end{pmatrix}; \quad B_{\mathbf{F}^{m,n}} = \begin{pmatrix} \mathcal{E}^{(1,1)}, & \dots, & \mathcal{E}^{(1,n)} \\ \vdots & & \vdots \\ \mathcal{E}^{(m,1)}, & \dots, & \mathcal{E}^{(m,n)} \end{pmatrix}.$$

• **TIPS:** Let  $B_{\text{range } T} = (Tv_1, \dots, Tv_p)$ ,  $B_V = (v_1, \dots, v_p, \dots, v_n)$ . Let each  $w_k = Tv_k$ ;  $B_W = (w_1, \dots, w_p, \dots, w_m)$ .

Then  $T = E_{1,1} + \dots + E_{p,p}$ ,  $\mathcal{M}(T, B_V, B_W) = \mathcal{E}^{(1,1)} + \dots + \mathcal{E}^{(p,p)}$ .

**17** Suppose  $V$  is finite-dim. Show that the only two-sided ideals of  $\mathcal{L}(V)$  are  $\{0\}$  and  $\mathcal{L}(V)$ .

A subsp  $\mathcal{E}$  of  $\mathcal{L}(V)$  is called a two-sided ideal of  $\mathcal{L}(V)$  if  $TE \in \mathcal{E}, ET \in \mathcal{E}, \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$ .

**SOLUTION:** [ See also in (3.A). ] Using NOTE FOR [3.60].

Let  $B_V = (v_1, \dots, v_n)$ . If  $\mathcal{E} = 0$ , then we are done. Suppose  $\mathcal{E} \neq 0$  and  $\mathcal{E}$  is a two-sided ideal of  $\mathcal{L}(V)$ .

Then  $\forall E_{i,j} \in \mathcal{E}$ , by assumption,  $\forall x, y \in \{1, \dots, n\}, E_{j,x} E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j} E_{y,i} = E_{y,j} \in \mathcal{E}$ .

Again,  $\forall x, x', y, y' \in \{1, \dots, n\}, E_{y,x'}, E_{y',x} \in \mathcal{E}$ . Thus  $\mathcal{E} = \mathcal{L}(V)$ . □

• (4E 10) Suppose  $V, W$  are finite-dim,  $U$  is a subsp of  $V$ .

Let  $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\} = \{T \in \mathcal{L}(V, W) : T|_U = 0\}$ .

(a) Show that  $\mathcal{E}$  is a subsp of  $\mathcal{L}(V, W)$ .

(b) Find a formula for  $\dim \mathcal{E}$  in terms of  $\dim V$ ,  $\dim W$  and  $\dim U$ .

**Hint:** Define  $\Phi : \mathcal{L}(V, W) \rightarrow \mathcal{L}(U, W)$  by  $\Phi(T) = T|_U$ . What is  $\text{null } \Phi$ ? What is  $\text{range } \Phi$ ?

**SOLUTION:**

(a)  $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$ .

(b) Define  $\Phi$  as in the hint.  $\Phi$  is linear, by [3.A NOTE FOR Restriction].

$\forall T \in \text{null } \Phi, \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$ . Thus  $\text{null } \Phi = \mathcal{E}$ .

$\forall S \in \mathcal{L}(U, W)$ , extend to  $T \in \mathcal{L}(V, W)$ , then  $\Phi(T) = S \in \text{range } \Phi$ . Thus  $\text{range } \Phi = \mathcal{L}(U, W)$ .

Thus  $\dim \text{null } \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \text{range } \Phi = (\dim V - \dim U) \dim W$ . □

OR. Let  $B_U = (u_1, \dots, u_m)$ ,  $B_V = (u_1, \dots, u_m, v_1, \dots, v_n)$ . Let  $p = \dim W$ . [ See NOTE FOR [3.60]. ]

$$\forall T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \text{span} \left\{ \begin{matrix} E_{1,1}, \dots, E_{m,1} \\ \vdots & \ddots & \vdots \\ E'_{1,p}, \dots, E'_{m,p} \end{matrix} \right\} \cap \mathcal{E} = \{0\}.$$

$$\text{又 } W = \text{span} \left\{ \begin{matrix} E_{m+1,1}, \dots, E_{n,1} \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, \dots, E_{n,p} \end{matrix} \right\} \subseteq \mathcal{E}.$$

Denote it by  $R$

Where  $\mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}$ .

Then  $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$ . □

• (4E 17) Suppose  $V$  is finite-dim and  $S \in \mathcal{L}(V)$ . Define  $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$  by  $\mathcal{A}(T) = ST$ .

(a) Show that  $\dim \text{null } \mathcal{A} = (\dim V)(\dim \text{null } S)$ .

(b) Show that  $\dim \text{range } \mathcal{A} = (\dim V)(\dim \text{range } S)$ .

**SOLUTION:** (a)  $\forall T \in \mathcal{L}(V), ST = 0 \iff \text{range } T \subseteq \text{null } S$ .

Thus  $\text{null } \mathcal{A} = \{T \in \mathcal{L}(V) : \text{range } T \subseteq \text{null } S\} = \mathcal{L}(V, \text{null } S)$ .

(b)  $\forall R \in \mathcal{L}(V), \text{range } R \subseteq \text{range } S \iff \exists T \in \mathcal{L}(V), R = ST$ , by (3.B 25).

Thus  $\text{range } \mathcal{A} = \{R \in \mathcal{L}(V) : \text{range } R \subseteq \text{range } S\} = \mathcal{L}(V, \text{range } S)$ .  $\square$

OR. Using NOTE FOR [3.60]. Let  $B_{\text{range } S} = (\overline{w_1}, \dots, \overline{w_m})$ ,  $B_U = (v_1, \dots, v_m)$ .

Let  $(w_1, \dots, w_n), (v_1, \dots, v_n)$  be bases of  $V$ . Now  $S = E_{1,1} + \dots + E_{m,m}$ .  $\mathcal{M}(S, v \rightarrow w) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ .

Define  $R_{i,j} \in \mathcal{L}(V)$  by  $R_{i,j} : w_x \mapsto \delta_{i,x} v_i$ . Let  $E_{j,k} R_{i,j} = Q_{i,k}$ ,  $R_{j,k} E_{i,j} = G_{i,k}$ .

Where  $E_{i,k} : v_x \mapsto \delta_{i,x} w_k$ ,  $Q_{i,k} : w_x \mapsto \delta_{i,x} w_k$ , and  $G_{i,k} : v_x \mapsto \delta_{i,x} v_k$ .

For any  $T \in \mathcal{L}(V)$ ,  $\exists! A_{i,j} \in \mathbb{F}, T = \sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \implies \mathcal{M}(T, w \rightarrow v) = \begin{pmatrix} A_{1,1} & \dots & A_{1,m} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,m} & \dots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \dots & A_{n,m} & \dots & A_{n,n} \end{pmatrix}$ .

$\implies \mathcal{A}(T) = ST = \left( \sum_{r=1}^m E_{r,r} \right) \left( \sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \right) = \sum_{i=1}^m \sum_{j=1}^n A_{i,j} Q_{j,i}$ .  $\mathcal{M}(S, v \rightarrow w) \mathcal{M}(T, w \rightarrow v) = \mathcal{M}(ST, w) = \begin{pmatrix} A_{1,1} & \dots & A_{1,m} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,m} & \dots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}$   $\mathcal{M}(T, R) = \mathcal{M}(T, w \rightarrow v)$ .  
Let  $T = I$ , we have  $\mathcal{M}(\mathcal{A}, R \rightarrow Q) = \mathcal{M}(S, v \rightarrow w)$ .

$\text{range } \mathcal{A} = \text{span} \left\{ \begin{pmatrix} Q_{1,1} & \dots & Q_{n,1} \\ \vdots & \ddots & \vdots \\ Q_{1,m} & \dots & Q_{n,m} \end{pmatrix} \right\}$ ,  $\text{null } \mathcal{A} = \text{span} \left\{ \begin{pmatrix} R_{1,m+1} & \dots & R_{n,m+1} \\ \vdots & \ddots & \vdots \\ R_{1,n} & \dots & R_{n,n} \end{pmatrix} \right\}$ . (a)  $\dim \text{null } \mathcal{A} = n \times (n - m)$ ;  
(b)  $\dim \text{range } \mathcal{A} = n \times m$ .  $\square$

• **NOTE FOR Problem (4E 17):** Define  $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$  by  $\mathcal{B}(T) = TS$ .

(a) Show that  $\dim \text{null } \mathcal{B} = (\dim V)(\dim \text{null } S)$ .

(b) Show that  $\dim \text{range } \mathcal{B} = (\dim V)(\dim \text{range } S)$ .

**SOLUTION:** (a)  $\forall T \in \mathcal{L}(V), TS = 0 \iff \text{range } S \subseteq \text{null } T$ .

Thus  $\text{null } \mathcal{B} = \{T \in \mathcal{L}(V) : \text{range } S \subseteq \text{null } T\} = \{T \in \mathcal{L}(V) : T|_{\text{range } S} = 0\}$ .

(b)  $\forall R \in \mathcal{L}(V), \text{null } S \subseteq \text{null } R \iff \exists T \in \mathcal{L}(V), R = TS$ , by (3.B.24).

Thus  $\text{range } \mathcal{B} = \{R \in \mathcal{L}(V) : \text{null } S \subseteq \text{null } R\} = \{R \in \mathcal{L}(V) : R|_{\text{null } S} = 0\}$ .

Using [3.22] and Problem (4E 10).  $\square$

OR. Using NOTE FOR [3.60] and notation in Problem (4E 17).

$\mathcal{B}(T) = TS = \left( \sum_{i=1}^n \sum_{j=1}^n A_{i,j} R_{j,i} \right) \left( \sum_{r=1}^m E_{r,r} \right) = \sum_{i=1}^n \sum_{j=1}^m A_{i,j} G_{j,i} \implies \mathcal{M}(TS, v) = \begin{pmatrix} A_{1,1} & \dots & A_{1,m} & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,m} & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \dots & A_{n,m} & \dots & 0 \end{pmatrix}$ .

$\text{range } \mathcal{B} = \text{span} \left\{ \begin{pmatrix} G_{1,1} & \dots & G_{m,1} \\ \vdots & \ddots & \vdots \\ G_{1,n} & \dots & G_{m,n} \end{pmatrix} \right\}$ ,  $\text{null } \mathcal{B} = \text{span} \left\{ \begin{pmatrix} R_{m+1,1} & \dots & R_{n,1} \\ \vdots & \ddots & \vdots \\ R_{m+1,n} & \dots & R_{n,n} \end{pmatrix} \right\}$ . (a)  $\dim \text{null } \mathcal{B} = n \times (n - m)$ ;  
(b)  $\dim \text{range } \mathcal{B} = n \times m$ .  $\square$

• (4E 20) Suppose  $q \in \mathcal{P}(\mathbb{R})$ . Prove that  $\exists p \in \mathcal{P}(\mathbb{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ .

**SOLUTION:** Note that  $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$ .

Define  $T_n \in \mathcal{L}(\mathcal{P}_n(\mathbb{R}))$  by  $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ .

And note that  $T_n(p) = 0 \implies \deg T_n(p) = -\infty = \deg p \implies p = 0$ . Thus  $T_n$  is inv.

$\forall q \in \mathcal{P}(\mathbb{R})$ , if  $q = 0$ , let  $n = 0$ ; if  $q \neq 0$ , let  $n = \deg q$ , we have  $q \in \mathcal{P}_n(\mathbb{R})$ .

Now  $\exists p \in \mathcal{P}_n(\mathbb{R}), q(x) = T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$  for all  $x \in \mathbb{R}$ .  $\square$

**19** Suppose  $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$  is inje. And  $\deg Tp \leq \deg p$  for every nonzero  $p \in \mathcal{P}(\mathbf{R})$ .

(a) Prove that  $T$  is surj; (b) Prove that for every nonzero  $p$ ,  $\deg Tp = \deg p$ .

**SOLUTION:** (a)  $T$  is inje  $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$  is inje, so is inv  $\iff T$  is surj.

(b) Using mathematical induction.

(i)  $\deg p = -\infty \geq \deg Tp \iff p = 0 = Tp$ . And  $\deg p = 0 \geq \deg Tp \iff p = C \neq 0$ .

(ii) Assume  $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$ . We show  $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$  by contradiction.

Suppose  $\exists r \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tr \leq n < n+1 = \deg r$ . Then by (a),  $\exists s \in \mathcal{P}_n(\mathbf{R}), T(s) = (Tr)$ .

$\wedge T$  is inje  $\Rightarrow s = r$ . While  $\deg s = \deg Ts = \deg Tr < \deg r$ . Contradicts.  $\square$

**16** Suppose  $V$  is finite-dim and  $S \in \mathcal{L}(V)$  such that  $\forall T \in \mathcal{L}(V), ST = TS$ .

Prove that  $\exists \lambda \in \mathbf{F}, S = \lambda I$ .

[Using notation in Problem (4E 17). See also in (3.A).]

**SOLUTION:** If  $S = 0$ , we are done. Now suppose  $S \neq 0$ .

Let  $S = E_{1,1} + \dots + E_{m,m} \Rightarrow \mathcal{M}(S, B_U) = \mathcal{M}(I, B_{\text{range } S}, B_U)$ . Note that  $R_{k,1} : w_x \mapsto \delta_{k,x} v_1$ .

Then  $\forall k \in \{1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$ . Hence  $\dim \text{null } S = 0, \dim \text{range } S = m = n$ .

NOTICE that  $G_{i,j} = R_{i,j}S = SR_{i,j} = Q_{i,j}$ . Where  $G_{i,j} : v_x \mapsto \delta_{i,x} v_j, Q_{i,j} : w_x \mapsto \delta_{i,x} w_j$ .

For each  $w_i, \exists ! a_{k,i} \in \mathbf{F}, w_i = a_{1,i} v_1 + \dots + a_{n,i} v_n$ . Where  $a_{k,i} = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))_{k,i}$ .

Then fix one  $i$ . Now for each  $j \in \{1, \dots, n\}, Q_{i,j}(w_i) = w_j = a_{i,i} v_j = G_{i,j}(\sum_{k=1}^n a_{k,i} v_k)$ .

Let  $\lambda = a_{i,i}$ . Hence each  $w_j = \lambda v_j$ . Now fix one  $j$ , we have  $a_{1,1} v_j = \dots = a_{n,n} v_j$ , then all  $a_{i,i}$  are equal.

Thus each  $w_j = \lambda v_j \Rightarrow \mathcal{M}(S, B_U) = \mathcal{M}(\lambda I)$ .  $\square$

• (10.A.3, OR 4E 19) Suppose  $V$  is finite-dim and  $T \in \mathcal{L}(V)$ .

[See also in (3.A).]

Prove that  $\forall B_V \neq B'_V, \mathcal{M}(T, B_V) = \mathcal{M}(T, B'_V) \implies T = \lambda I, \exists \lambda \in \mathbf{F}$ .

**SOLUTION:** Suppose  $\forall B_V \neq B'_V, \mathcal{M}(T, B_V) = \mathcal{M}(T, B'_V)$ . If  $T = 0$ , then we are done.

Suppose  $T \neq 0$ , and  $v \in V \setminus \{0\}$ . Assume that  $(v, Tv)$  is linely inde.

Extend  $(v, Tv)$  to  $B_V = (v, Tv, u_3, \dots, u_n)$ . Let  $B = \mathcal{M}(T, B_V)$ .

$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \dots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \forall i \neq 2$ .

By assumption,  $A = \mathcal{M}(T, B'_V) = B, \forall B'_V = (v, w_2, \dots, w_n)$ . Then  $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$ .

$\Rightarrow Tv = w_2$ , which is not true if  $w_2 = u_3, w_3 = Tv, w_j = u_j, \forall j \in \{4, \dots, n\}$ . Contradicts.

Hence  $(v, Tv)$  is linely depe  $\Rightarrow \forall v \in V, \exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$ .

Now we show that  $\lambda_v$  is independent of  $v$ , that is, for all distinct  $v, w \in V \setminus \{0\}, \lambda_v = \lambda_w$ .

$(v, w)$  linely inde  $\Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw$   
 $(v, w)$  linely depe,  $w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv)$   $\Rightarrow T = \lambda I$ .  $\square$

OR. Let  $A = \mathcal{M}(T, B_V)$ , where  $B_V = (u_1, \dots, u_m)$  is arbitrary.

Fix one  $B_V = (v_1, \dots, v_m)$  and then  $(v_1, \dots, \frac{1}{2}v_k, \dots, v_m)$  is also a basis for any given  $k \in \{1, \dots, m\}$ .

Fix one  $k$ . Now we have  $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \dots + A_{k,k}(\frac{1}{2}v_k) + \dots + A_{m,k}v_m$

$\Rightarrow Tv_k = 2A_{1,k}v_1 + \dots + A_{k,k}v_k + \dots + 2A_{m,k}v_m = A_{1,k}v_1 + \dots + A_{k,k}v_k + \dots + A_{m,k}v_m$ .

Then  $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$  for all  $j \neq k$ . Thus  $Tv_k = A_{k,k}v_k, \forall k \in \{1, \dots, m\}$ .

Now we show that  $A_{k,k} = A_{j,j}$  for all  $j \neq k$ . Choose  $j, k$  such that  $j \neq k$ .

Consider  $B'_V = (v'_1, \dots, v'_j, \dots, v'_k, \dots, v'_m)$ , where  $v'_j = v_k, v'_k = v_j$  and  $v'_i = v_i$  for all  $i \in \{1, \dots, m\} \setminus \{j, k\}$ .

Now  $T(v'_k) = A_{1,k}v'_1 + \dots + A_{k,k}v'_k + \dots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_j$ , while  $T(v'_j) = T(v_j) = A_{j,j}v_j$ .  $\square$

1 A function  $T : V \rightarrow W$  is linear  $\iff$  The graph of  $T$  is a subspace of  $V \times W$ .

2 Suppose  $V_1 \times \cdots \times V_m$  is finite-dim. Prove that each  $V_j$  is finite-dim.

SOLUTION:

For any  $k \in \{1, \dots, m\}$ , define  $S_k \in \mathcal{L}(V_1 \times \cdots \times V_m, V_k)$  by  $S_k(v_1, \dots, v_m) = v_k$ .

Then  $S_k$  is linear map. By [3.22],  $\text{range } S_k = V_k$  is finite-dim.  $\square$

OR. Denote  $V_1 \times \cdots \times V_m$  by  $U$ . Denote  $\{0\} \times \cdots \times \{0\} \times V_i \times \{0\} \times \cdots \times \{0\}$  by  $U_i$ .

We show that each  $U_i$  is iso to  $V_i$ . Then  $U$  is finite-dim  $\implies$  its subsp  $U_i$  is finite-dim, so is  $V_i$ .

Let  $B_U = (v_1, \dots, v_m) \left\{ \begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U_i) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} R_i S_j|_{U_j} = \delta_{ij} I_{U_j}, \\ S_i R_j = \delta_{ij} I_{V_j}. \end{array} \right. \square$

4 Prove that  $\mathcal{L}(V_1 \times \cdots \times V_m, W)$  and  $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$  are iso.

SOLUTION: Using notation in Problem (2):  $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0)$ ;  $S_i : (u_1, \dots, u_m) \mapsto u_i$ .

Note that  $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$ .

Define  $\varphi : T \mapsto (T_1, \dots, T_m)$  by  $\varphi(T) = (TR_1, \dots, TR_m)$ .

Define  $\psi : (T_1, \dots, T_m) \mapsto T$  by  $\psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m$ .  $\left. \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (TR_1, \dots, TR_m) \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m \end{array} \right\} \Rightarrow \psi = \varphi^{-1}.$   $\square$

5 Prove that  $\mathcal{L}(V, W_1 \times \cdots \times W_m)$  and  $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$  are iso.

SOLUTION: Using notation in Problem (2):  $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0)$ ;  $S_i : (u_1, \dots, u_m) \mapsto u_i$ .

Note that  $T_i : v \mapsto w_i$ ,  $\left\{ \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1 T, \dots, S_m T) \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = R_1 T_1 + \cdots + R_m T_m \end{array} \right\} \Rightarrow \psi = \varphi^{-1}.$

$T : v \mapsto (w_1, \dots, w_m)$ .  $\left. \begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1 T, \dots, S_m T) \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = R_1 T_1 + \cdots + R_m T_m \end{array} \right\} \Rightarrow \psi = \varphi^{-1}.$   $\square$

6 For  $m \in \mathbb{N}^+$ , define  $V^m$  by  $\underbrace{V \times \cdots \times V}_{m \text{ times}}$ . Prove that  $V^m$  and  $\mathcal{L}(\mathbb{F}^m, V)$  are iso.

SOLUTION:

Define  $T : (v_1, \dots, v_m) \mapsto \varphi$ , where  $\varphi : (a_1, \dots, a_m) \mapsto v$  is defined by  $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m$ .

(a) Suppose  $T(v_1, \dots, v_m) = 0$ . Then  $\forall (a_1, \dots, a_m) \in \mathbb{F}^m$ ,  $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$

For each  $k$ , let  $a_k = 1, a_j = 0$  for all  $j \neq k$ . Then each  $v_k = 0 \Rightarrow (v_1, \dots, v_m) = 0$ . Thus  $T$  is inj.

(b) Suppose  $\psi \in \mathcal{L}(\mathbb{F}^m, V)$ . Let  $(e_1, \dots, e_m)$  be the std basis of  $\mathbb{F}^m$ . Then  $\forall (b_1, \dots, b_m) \in \mathbb{F}^m$ ,

$\left[ T(\psi(e_1), \dots, \psi(e_m)) \right] (b_1, \dots, b_m) = b_1 \psi(e_1) + \cdots + b_m \psi(e_m) = \psi(b_1 e_1 + \cdots + b_m e_m) = \psi(b_1, \dots, b_m).$

Thus  $T(\psi(e_1), \dots, \psi(e_m)) = \psi$ . Hence  $T$  is surj.  $\square$

3 Give an example of a vecsp  $V$  and its two subsp  $U_1, U_2$  such that

$U_1 \times U_2$  and  $U_1 + U_2$  are iso but  $U_1 + U_2$  is not a direct sum.  $[V \text{ must be infinite-dim.}]$

SOLUTION: NOTE that at least one of  $U_1, U_2$  must be infinite-dim. And at least one must be finite-dim??

Let  $V = \mathbb{F}^\infty = U_1$ ,  $U_2 = \{(x, 0, \dots) \in \mathbb{F}^\infty : x \in \mathbb{F}\}$ . Then  $V = U_1 + U_2$  is not a direct sum.

Define  $T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2)$  by  $T((x_1, x_2, \dots), (x, 0, \dots)) = (x, x_1, x_2, \dots)$   $\left. \begin{array}{l} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T((x_1, x_2, \dots), (x, 0, \dots)) = (x, x_1, x_2, \dots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \dots) = ((x_1, x_2, \dots), (x, 0, \dots)) \end{array} \right\} \Rightarrow S = T^{-1}.$

Define  $S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2)$  by  $S(x, x_1, x_2, \dots) = ((x_1, x_2, \dots), (x, 0, \dots))$   $\square$

- **NOTE FOR [3.79, 3.83]:** If  $U = \{0\}$ , then  $v + U = v + \{0\} = \{v\}$ ,  $V/U = V/\{0\} = \{\{v\} : v \in V\}$ .  
If  $U = \emptyset$ , then  $v + U = v + \emptyset = \emptyset$ ,  $V/U = V/\emptyset = \{\emptyset\}$ .

- **COMMENT:** If  $U$  is merely a subset of  $V$ , then [3.85, 3.86] do not hold, and  $V/U$  is not a vecsp.  
Because  $((v - w) + u) \in U$  or  $u - u' \in U$  needs that  $U$  is closed under add.  
And because  $(v - \hat{v}) + (w - \hat{w}) \in U$  and  $\lambda(v - \hat{v}) \in U$  assume that  $U$  is a subsp.

- **NOTE FOR [3.85]:**  $v + U = w + U \iff v \in w + U, w \in v + U$   
 $\iff v - w \in U \iff (v + U) \cap (w + U) \neq \emptyset$ .

- (4E 8) Suppose  $T \in \mathcal{L}(V, W), w \in \text{range } T$ . Prove that  $\{u \in V : Tu = w\} = u + \text{null } T$ .

**SOLUTION:** Let  $\mathcal{K}_u = \{u \in V : Tu = w\}$ . [Not a vecsp.] Suppose  $u \in \mathcal{K}_u$ . Then  $u + \text{null } T \subseteq \mathcal{K}_u$ .

And  $\forall u' \in \mathcal{K}_u, u' - u \in \text{null } T \Rightarrow u' \in u + \text{null } T$ . Now  $\mathcal{K}_u \subseteq u + \text{null } T$ . □

- 7 Suppose  $v, x \in V$ , and  $U, W$  are subsp of  $V$ . Prove that  $v + U = x + W \Rightarrow U = W$ .

**SOLUTION:** (a)  $\forall u_1 \in U, \exists w_1 \in W, v + u_1 = x + w_1$ . Let  $u_1 = 0$ , then  $v = x + w'_1 \Rightarrow v - x \in W$ .

(b)  $\forall w_2 \in W, \exists u_2 \in U, v + u_2 = x + w_2$ . Let  $w_2 = 0$ , then  $x = v + u'_2 \Rightarrow x - v \in U$ .

Now  $x + U = v + U = x + W = v + W$ . Thus  $\{v + u : u \in U\} = \{v + w : w \in W\} \Rightarrow U = W$ .

OR.  $\pm(v - x) \in U \cap W \Rightarrow \begin{cases} u_1 = (x - v) + w_1 \in W \Rightarrow U \subseteq W \\ w_2 = (v - x) + u_2 \in U \Rightarrow W \subseteq U \end{cases} \Rightarrow U = W$ . □

- 8 Suppose  $A$  is a nonempty subset of  $V$ .

Prove that  $A$  is a translate of some subsp of  $V \iff \lambda v + (1 - \lambda)w \in A, \forall v, w \in A, \lambda \in \mathbb{F}$ .

**SOLUTION:**

(a) Suppose  $A = a + U$ . Then  $\lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + (\lambda(u_1 - u_2) + u_2) \in A$ .

(b) Suppose  $\lambda v + (1 - \lambda)w \in A, \forall v, w \in A, \lambda \in \mathbb{F}$ . Suppose  $\underline{a \in A}$  and let  $A' = \{x - a : x \in A\}$ .

Then  $0 \in A'$  and  $\forall (v - a), (w - a) \in A', \lambda \in \mathbb{F}$ ,

(I)  $\lambda(v - a) = [\lambda v + (1 - \lambda)a] - a \in A'$ .

(II) Because  $\lambda(v - a) + (1 - \lambda)(w - a) = [\lambda v + (1 - \lambda)w] - a \in A'$ .

Let  $\lambda = \frac{1}{2}$  here and use (I) above by  $\lambda = 2$ , we have  $(v - a) + (w - a) \in A'$ .

OR. Note that  $v, a \in A \Rightarrow \lambda v + (1 - \lambda)a = 2v - a \in A$ . Similarly  $2w - a \in A$ .

Now  $(v - \frac{1}{2}a) + (w - \frac{1}{2}a) = v + w - a \in A \Rightarrow v + w - 2a = (v - a) + (w - a) \in A'$ .

Thus  $A' = -a + A$  is a subsp of  $V$ . Hence  $a + A' = a + \{x - a : x \in A\} = A$  is a translate. □

- 9 Suppose  $A = v + U$  and  $B = x + W$  for some  $v, x \in V$  and some subsp  $U, W$  of  $V$ .

Prove that  $A \cap B$  is either a translate of some subsp of  $V$  or is  $\emptyset$ .

**SOLUTION:**  $\forall v + u, x + w \in A \cap B \neq \emptyset, \lambda \in \mathbb{F}, \lambda(v + u) + (1 - \lambda)(x + w) \in A \cap B$ . By Problem (8). □

OR. Let  $A = v + U, B = x + W$ . Suppose  $\alpha \in (v + U) \cap (x + W) \neq \emptyset$ .

Then  $\alpha - v \in U \Rightarrow \alpha + U = v + U = A$ , and  $\alpha - x \in W \Rightarrow \alpha + W = x + W = B$ .

We show that  $A \cap B = \alpha + (U \cap W)$ . Note that  $\alpha + (U \cap W) \subseteq A \cap B$ .

And  $\forall \beta = \alpha + u = \alpha + w \in A \cap B \Rightarrow u = w \in U \cap W \Rightarrow \beta \in \alpha + (U \cap W)$ . □



**10** Prove that the intersection of any collection of translates of subspaces is either a translate of some subspace or  $\emptyset$ .

**SOLUTION:** Suppose  $\{A_\alpha\}_{\alpha \in \Gamma}$  is a collection of translates of subspaces of  $V$ , where  $\Gamma$  is an index set.

$\forall x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset, \lambda \in \mathbf{F}, \lambda x + (1 - \lambda)y \in A_\alpha$  for each  $\alpha$ . By Problem (8).  $\square$

OR. Let each  $A_\alpha = w_\alpha + V_\alpha$ . Suppose  $x \in \bigcap_{\alpha \in \Gamma} (w_\alpha + V_\alpha) \neq \emptyset$ .

Then  $x - w_\alpha \in V_\alpha \implies x + V_\alpha = w_\alpha + V_\alpha = A_\alpha$ , for each  $\alpha$ .

We show that  $\bigcap_{\alpha \in \Gamma} A_\alpha = \bigcap_{\alpha \in \Gamma} (x + V_\alpha) = x + \bigcap_{\alpha \in \Gamma} V_\alpha$ .

$y \in \bigcap_{\alpha \in \Gamma} A_\alpha \iff$  for each  $\alpha$ ,  $y = x + v_\alpha \in A_\alpha$

$\iff$  each  $v_\alpha = y - x \in \bigcap_{\alpha \in \Gamma} V_\alpha \iff y \in x + \bigcap_{\alpha \in \Gamma} V_\alpha$ .  $\square$

**11** Suppose  $A = \{\lambda_1 v_1 + \dots + \lambda_m v_m : \sum_{i=1}^m \lambda_i = 1\}$ , where each  $v_i \in V, \lambda_i \in \mathbf{F}$ .

(a) Prove that  $A$  is a translate of some subspace of  $V$

(b) Prove that if  $B$  is a translate of some subspace of  $V$  and  $\{v_1, \dots, v_m\} \subseteq B$ , then  $A \subseteq B$ .

(c) Prove that  $A$  is a translate of some subspace of  $V$  of dimension less than  $m$ .

**SOLUTION:**

(a) By Problem (8),  $\forall u, w \in A, \lambda \in \mathbf{F}, \lambda u + (1 - \lambda)w = \left(\lambda \sum_{i=1}^m a_i + (1 - \lambda) \sum_{i=1}^m b_i\right)v_i \in A$ .  $\square$

(b) Suppose  $B = v + U$ , where  $v \in V$  and  $U$  is a subspace of  $V$ . Let each  $v_k = v + u_k \in B, \exists! u_k \in U$ .

$\forall w \in A, w = \sum_{i=1}^m \lambda_i v_i = \sum_{i=1}^m \lambda_i (v + u_i) = \sum_{i=1}^m \lambda_i v + \sum_{i=1}^m \lambda_i u_i = v + \sum_{i=1}^m \lambda_i u_i \in v + U = B$ .  $\square$

OR. Let  $v = \lambda_1 v_1 + \dots + \lambda_m v_m \in A$ . To show that  $v \in B$ , use induction on  $m$  by  $k$ .

(i)  $k = 1, v = \lambda_1 v_1 \implies \lambda_1 = 1$ .  $\forall v_1 \in B$ . Hence  $v \in B$ .

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \implies \lambda_2 = 1 - \lambda_1$ .  $\forall v_1, v_2 \in B$ . By Problem (8),  $v \in B$ .

(ii)  $2 \leq k < m$ . Assume that  $v = \lambda_1 v_1 + \dots + \lambda_k v_k \in A \subseteq B$ .  $\left(\forall \lambda_i \text{ such that } \sum_{i=1}^k \lambda_i = 1\right)$

For  $u = \mu_1 v_1 + \dots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$ .  $\forall i = 1, \dots, k, \exists \mu_i \neq 1$ , fix one such  $i$  by  $\iota$ .

Then  $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \implies \left(\sum_{i=1}^{k+1} \frac{\mu_i}{1 - \mu_\iota}\right) - \frac{\mu_\iota}{1 - \mu_\iota} = 1$ .

Let  $w = \underbrace{\frac{\mu_1}{1 - \mu_\iota} v_1 + \dots + \frac{\mu_{\iota-1}}{1 - \mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1 - \mu_\iota} v_{\iota+1} + \dots + \frac{\mu_{k+1}}{1 - \mu_\iota} v_{k+1}}_{k \text{ terms}}$ .

Let  $\lambda_i = \frac{\mu_i}{1 - \mu_\iota}$  for  $i = 1, \dots, \iota - 1$ ;  $\lambda_j = \frac{\mu_{j+1}}{1 - \mu_\iota}$  for  $j = \iota, \dots, k$ . Then,

$\left. \begin{array}{l} \sum_{i=1}^k \lambda_i = 1 \implies w \in B \\ v_\iota \in B \implies u' = \lambda w + (1 - \lambda)v_\iota \in B \end{array} \right\} \implies \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \implies A \subseteq B$ .  $\square$

(c) If  $m = 1$ , then let  $A = v_1 + \{0\}$  and we are done.

Fix one  $k \in \{1, \dots, m\}$ . For  $\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_m \in \mathbf{F}$ . Let  $\lambda_k = 1 - \lambda_1 - \dots - \lambda_{k-1} - \lambda_{k+1} - \dots - \lambda_m$   
 $\implies \lambda_1 v_1 + \dots + \lambda_k v_k + \dots + \lambda_m v_m = v_k + \sum_{i=1}^m \lambda_i (v_i - v_k)$ .  $A = v_k + \text{span}(v_1 - v_k, \dots, v_m - v_k)$ .  $\square$

**14** Suppose  $U = \{(x_1, x_2, \dots) \in \mathbf{F}^\infty : x_k \neq 0 \text{ for only finitely many } k\}$ .

(a) Show that  $U$  is a subsp of  $\mathbf{F}^\infty$ . [Do it in your mind]

(b) Prove that  $\mathbf{F}^\infty/U$  is infinite-dim.

**SOLUTION:** For ease of notation, denote the  $p^{\text{th}}$  term of  $u = (x_1, \dots, x_p, \dots) \in \mathbf{F}^\infty$  by  $u[p]$ .

For each  $r \in \mathbf{N}^+$ , let  $e_r[p] = \begin{cases} 1, & (p-1) \equiv 0 \pmod{r} \\ 0, & \text{otherwise} \end{cases}$  simply  $e_r = (1, \underbrace{0, \dots, 0}_{(p-1) \text{ times}}, 1, \underbrace{0, \dots, 0}_{(p-1) \text{ times}}, 1, \dots)$ .

Choose one  $m \in \mathbf{N}^+$ . Let  $a_1(e_1 + U) + \dots + a_m(e_m + U) = 0 + U \Rightarrow \exists u \in U, a_1e_1 + \dots + a_me_m = u$ .

Suppose  $u = (x_1, \dots, x_L, 0, \dots)$ , where  $L$  is the largest such that  $u[L] \neq 0$ .

Let  $s \in \mathbf{N}^+$  be such that  $h = s \cdot m! + 1 > L$  and  $e_1[h] = \dots = e_m[h] = 1$ .

Note that by definition,  $e_r[s \cdot m! + 1 + p] = e_r[p + 1] = 1 \iff p \equiv 0 \pmod{r} \iff r | p$ .

Now for any  $p \in \{1, \dots, m\}$ ,  $u[h + p] = \left( \sum_{r=1}^m a_r e_r \right) [p + 1] = \sum_{k=1}^{\tau(p)} a_{p_k} = 0$  (Δ)

where  $1 = p_1 \leq \dots \leq p_{\tau(p)} = p$  are all the distinct factors of  $p$ .

Let  $q = p_{\tau(p)-1}$ . Notice that  $\tau(q) = \tau(p) - 1$  and  $q_k = p_k, \forall k \in \{1, \dots, \tau(q)\}$ .

Again by (Δ),  $\left( \sum_{r=1}^m a_r e_r \right) [h + q] = \sum_{k=1}^{\tau(q)-1} a_{p_k} = 0$ . Thus  $a_{p_{\tau(p)}} = a_p = 0$  for any  $p \in \{1, \dots, m\}$ .

Hence  $\forall m \in \mathbf{N}^+$ ,  $(e_1, \dots, e_m)$  is linely inde in  $\mathbf{F}^\infty$ , so is  $(e_1 + U, \dots, e_m + U)$  in  $\mathbf{F}^\infty/U$ . By (2.A.14). □

OR. For each  $r \in \mathbf{N}^+$ , let  $e_r[p] = \begin{cases} 1, & \text{if } 2^r | p \\ 0, & \text{otherwise} \end{cases}$ .

Similarly, let  $m \in \mathbf{N}^+$  and  $a_1(e_1 + U) + \dots + a_m(e_m + U) = 0 \Rightarrow a_1e_1 + \dots + a_me_m = u \in U$ .

Suppose  $L$  is the largest such that  $u[L] \neq 0$ . And  $l$  is such that  $2^{ml} > L$ .

Then  $\forall k \in \{1, \dots, m\}, u[2^{ml} + 2^k] = \left( \sum_{r=1}^m a_r e_r \right) [2^k] = a_1 + \dots + a_k = 0$ .

Thus  $a_1 = \dots = a_m = 0$  and  $(e_1, \dots, e_m)$  is linely inde. Similarly. □

• **NOTE FOR [3.88, 3.90, 3.91]:** Suppose  $W \in \mathcal{S}_V U$ . Then  $V/U$  is iso to  $W$ .

Because  $\forall v \in V, \exists! u_v \in U, w_v \in W, v = u_v + w_v$ . Define  $T \in \mathcal{L}(V)$  by  $T(v) = w_v$ .

Hence  $\text{null } T = U$ ,  $\text{range } T = W$ ,  $\text{range } T \oplus \text{null } T = V$ .

Then  $\tilde{T} \in \mathcal{L}(V/\text{null } T, V)$  is defined by  $\tilde{T}(v + U) = \tilde{T}(w'_v + U) = Tw'_v = w_v$ . [See TIPS below]

Now  $\pi \circ \tilde{T} = I_{V/U}$ ,  $\tilde{T} \circ \pi|_W = I_W = T|_W$ . Hence  $\tilde{T}$  is an iso of  $V/U$  onto  $W$ .

• **TIPS:** Suppose  $U$  is a subsp of  $V$ . Define  $S \in \mathcal{L}(V/U, V)$  by  $S(v + U) = v$ .

Then  $\text{range } S$  is the *purest* in  $\mathcal{S}_V U$ . Now  $\text{null } S = \{0\}$ ,  $U \oplus \text{range } S = V$ .

Let  $E = S \circ \pi$ . Because  $S$  is inje and  $\pi$  is surj,  $\text{null } E = \text{null } \pi = U$ ,  $\text{range } E = \text{range } S$ .

Then  $\text{range } E \oplus \text{null } E = V$ . NOTICE that  $E : V \rightarrow W$  is the *purest eraser*. Now we explain why:

**EXAMPLE:** Let  $V = \mathbf{F}^2, B_U = (e_1), B_W = (e_2 - e_1) \Rightarrow U \oplus W = V$ .

Notice that  $T(e_2 - e_1) = (e_2 - e_1)$ , while  $(e_2 - e_1) + U = e_2 + U$ , but

because  $e_2 = v_1 + (e_2 - e_1)$ , now still,  $\tilde{T}((e_2 - e_1) + U) = e_2 - e_1 = Te_2$ .

In contrast,  $S((e_2 - e_1) + U) = S(e_2 + U) = e_2$ ,  $E(e_2 - e_1) = e_2$ .

And  $\text{range } E = \text{range } S = \text{span}(e_2)$  is the *purest* in  $\mathcal{S}_V U$ .

**12** Suppose  $U$  is a subsp of  $V$ . Prove that  $V$  is iso to  $U \times (V/U)$ .

**SOLUTION:**

[Req  $V/U$  Finite-dim] Let  $B_{V/U} = (v_1 + U, \dots, v_n + U)$ .

Note that  $\forall v \in V, \exists! a_i \in \mathbf{F}, v + U = \sum_{i=1}^n a_i(v_i + U) = \left(\sum_{i=1}^n a_i v_i\right) + U$

$\Rightarrow (v - a_1 v_1 - \dots - a_n v_n) \in U \Rightarrow \exists! u \in U, v = \sum_{i=1}^n a_i v_i + u$ .

Thus define  $\varphi \in \mathcal{L}(V, U \times (V/U))$  and  $\psi \in \mathcal{L}(U \times (V/U), V)$

by  $\varphi(v) = (u, v + U)$  and  $\psi(u, v + U) = v + u$ . Then  $\psi = \varphi^{-1}$ .  $\square$

OR. Define  $S \in \mathcal{L}(V/U, V)$  by  $S(v + U) = v$ .

By NOTE FOR [3.88, 90, 91],  $\text{range } S \oplus U = V$ . Thus  $\forall v \in V, \exists! u \in U, w \in \text{range } S, v = u + w$ .

Define  $T \in \mathcal{L}(U \times (V/U), V)$  by  $T(u, v + U) = u + S(v + U) = u + w = v$ . Then  $T$  is surj.

And  $T(u, v + U) = u + S(v + U) = 0 \Rightarrow \pi(T(u, v + U)) = v + U = 0$ , and  $u = -S(v + U) = 0$ .

OR. Define  $R \in \mathcal{L}(V, U \times (V/U))$  by  $R(v) = (u, (w + U))$ . Now  $R \circ T = I_{U \times (V/U)}$ ,  $T \circ R = I_V$ .  $\square$

• (4E 14) Suppose  $V = U \oplus W$ ,  $B_W = (w_1, \dots, w_m)$ . Prove that  $B_{V/U} = (w_1 + U, \dots, w_m + U)$ .

**SOLUTION:**  $\forall v \in V, \exists! u \in U, w \in W, v = u + w$ .  $\text{又 } \exists! c_i \in \mathbf{F}, w = \sum_{i=1}^m c_i w_i \Rightarrow v = \sum_{i=1}^m c_i w_i + u$ .

Hence  $\forall v + U \in V/U, \exists! c_i \in \mathbf{F}, v + U = \sum_{i=1}^m c_i w_i + U$ .  $\square$

**13** Prove that  $B_{V/U} = (v_1 + U, \dots, v_m + U), B_U = (u_1, \dots, u_n) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$ .

**SOLUTION:**

Note that  $\forall v \in V, \exists! a_i \in \mathbf{F}, v + U = \sum_{i=1}^m a_i v_i + U \Rightarrow \exists! b_i \in \mathbf{F}, v - \sum_{i=1}^m a_i v_i = \sum_{i=1}^m b_i u_i \in U$   
 $\Rightarrow \forall v \in V, \exists! a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^m a_i v_i + \sum_{j=1}^n b_j u_j$ .  $\square$

OR. Note that  $B = (v_1, \dots, v_m)$  is linely inde. Now we show that  $\text{span } B \cap U = \{0\}$ .

$v \in \text{span } B \cap U \Leftrightarrow v + U = \sum_{i=1}^m a_i(v_i + U) = 0 + U \Leftrightarrow a_1 = \dots = a_m = 0 \Leftrightarrow v = 0$ .

Then by Problem (12) and (3.76),  $\dim V = \dim(U \times (V/U)) = n + m$ .

While  $\dim[\text{span}(v_1, \dots, v_m) \oplus U] = m + n$  and  $[\text{span}(v_1, \dots, v_m) \oplus U] \subseteq V$ . Hence by (2.B.8).  $\square$

• **NOTE FOR Problem (13) and (4E 14):** Let  $U \oplus W = V$ . Define  $S(w + U) = w$ . [See also the TIPS.]

(a) Let  $B_W = (w_1, \dots, w_m) \Rightarrow B_{V/U} = (w_1 + U, \dots, w_m + U)$ . Then  $S(w_k + U)$  might not equal  $w_k$ .

(b) Let  $B_{V/U} = (w_1 + U, \dots, w_m + U)$ , then let  $B_W = (w_1, \dots, w_m)$ . Now each  $S(w_k + U) = w_k$ .

**15** Suppose  $\varphi \in \mathcal{L}(V, \mathbf{F}) \setminus \{0\}$ . Prove that  $\dim V/(\text{null } \varphi) = 1$ .

**SOLUTION:** By [3.91] (d),  $\dim \text{range } \varphi = 1 = \dim V/(\text{null } \varphi)$ .

OR. By (3.B.29),  $\exists u, \text{span}(u) \oplus \text{null } \varphi = V$ . Then  $B_{V/\text{null } \varphi} = (u + \text{null } \varphi)$ .  $\square$

**16** Suppose  $\dim V/U = 1$ . Prove that  $\exists \varphi \in \mathcal{L}(V, \mathbf{F}), \text{null } \varphi = U$ .

**SOLUTION:** Suppose  $V_0 \oplus U = V$ . Then  $V_0$  is iso to  $V/U$ .  $\dim V_0 = 1$ .

Define  $\varphi \in \mathcal{L}(V, \mathbf{F})$  by  $\varphi(v_0) = 1, \varphi(u) = 0$ , where  $v_0 \in V_0, u \in U$ .  $\square$

OR. Let  $B_{V/U} = (w + U)$ . Then  $\forall v \in V, \exists! a \in \mathbf{F}, v + U = aw + U$ .

Define  $\varphi : V \rightarrow \mathbf{F}$  by  $\varphi(v) = a$ . Then  $\varphi(v_1 + \lambda v_2) = a_1 + \lambda a_2 = \varphi(v_1) + \lambda \varphi(v_2)$ .

Now  $u \in U \Leftrightarrow u + U = 0w + U \Leftrightarrow \varphi(u) = 0$ .  $\square$

**17** Suppose  $V/U$  is finite-dim,  $W$  is a subsp of  $V$ .

(a) Show that if  $V = U + W$ , then  $\dim W \geq \dim V/U$ .

(b) Show that  $\exists W \in \mathcal{S}_V U, \dim W = \dim V/U$ .

**SOLUTION:** Let  $B_W = (w_1, \dots, w_n)$ .

(a)  $\forall v \in V, \exists u \in U, w \in W, v = u + w \implies v + U = w + U = (a_1 w_1 + \dots + a_n w_n) + U, \exists! a_i \in \mathbb{F}$ .

Then  $V/U \subseteq \text{span}(w_1 + U, \dots, w_n + U)$ . Hence  $\dim V/U \leq \dim \text{span}(w_1 + U, \dots, w_n + U)$ .

(b) Reduce  $(w_1 + U, \dots, w_n + U)$  to  $B_{V/U} = (w_1 + U, \dots, w_m + U)$ , and let  $W = \text{span}(w_1, \dots, w_m)$ .  $\square$

OR. Let  $B_{V/U} = (v_1 + U, \dots, v_m + U)$  and define  $\tilde{T} \in \mathcal{L}(V/U, V)$  by  $\tilde{T}(v_k + U) = v_k$ .

Note that  $\pi \circ \tilde{T} = I$ . By (3.B.20),  $\tilde{T}$  is inje. And  $(v_1, \dots, v_m)$  is linely inde.

Let  $W = \text{range } \tilde{T} = \text{span}(v_1, \dots, v_m)$ . Then  $\tilde{T} \in \mathcal{L}(V/U, W)$  is an iso. Thus  $\dim W = \dim V/U$ .

And  $\forall v \in V, \exists! a_i \in \mathbb{F}, v + U = a_1 v_1 + \dots + a_m v_m + U \implies \exists! w \in W, u \in U, v = w + u$ .  $\square$

**18** Suppose  $T \in \mathcal{L}(V, W)$  and  $U$  is a subsp of  $V$ . Let  $\pi : V \rightarrow V/U$  be the quotient map.

Prove that  $\exists S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \subseteq \text{null } T$ .

**SOLUTION:**

(a) Suppose  $\exists S \in \mathcal{L}(V/U, W), T = S \circ \pi$ . Then  $U = \text{null } \pi \subseteq \text{null } (S \circ \pi) = \text{null } T$ .

(b) Suppose  $U = \text{null } \pi \subseteq \text{null } T$ . By (3.B.24), we are done. OR. Define  $S : (v + U) \mapsto Tv$ .

$v_1 + U = v_2 + U \iff v_1 - v_2 \in \text{null } T \iff Tv_1 = Tv_2$ . Thus  $S$  is well-defined. Hence  $S \circ \pi = T$ .  $\square$

**COROLLARY:** Define  $\Gamma : S \mapsto S \circ \pi$ . Then  $\Gamma$  is inje,  $\text{range } \Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\}$ .

**ENDED**

**3.F** [4](#) [5](#) [6](#) [7](#) [8](#) [9](#) [12](#) [13](#) [15](#) [16](#) [17](#) [18](#) [19](#) [20](#) [21](#) [22](#) [23](#) [24](#) [25](#) [26](#)  
[28](#) [29](#) [30](#) [31](#) [32](#) [33](#) [34](#) [35](#) [36](#) [37](#) | [4E: 5](#) [6](#) [8](#) [17](#) [23](#) [24](#) [25](#)

**20, 21** Suppose  $U$  and  $W$  are subsets of  $V$ . Prove that  $U \subseteq W \iff W^0 \subseteq U^0$ .

**SOLUTION:**

(a) Suppose  $U \subseteq W$ . Then  $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(u) = 0 \implies \varphi \in U^0$ . Thus  $W^0 \subseteq U^0$ .

(b) Suppose  $W^0 \subseteq U^0$ . Then  $\varphi \in W^0 \implies \varphi \in U^0$ . Hence  $\text{null } \varphi \supseteq W \implies \text{null } \varphi \supseteq U$ . Thus  $W \supseteq U$ .

OR. For a subsp  $U$  of  $V$ , let  $A_U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = U$ , by Problem (25).

Suppose  $W^0 \subseteq U^0$ . Then  $\forall \varphi \in W^0, v \in A_U, \varphi(v) = 0 \implies v \in A_W$ . Thus  $A_U \subseteq A_W$ .  $\square$

**COROLLARY:**  $W^0 = U^0 \iff U = W$ .

**22** Suppose  $U$  and  $W$  are subspaces of  $V$ . Prove that  $(U + W)^0 = U^0 \cap W^0$ .

**SOLUTION:**

$$(a) \left. \begin{array}{l} U \subseteq U + W \\ W \subseteq U + W \end{array} \right\} \Rightarrow \left. \begin{array}{l} (U + W)^0 \subseteq U^0 \\ (U + W)^0 \subseteq W^0 \end{array} \right\} \Rightarrow (U + W)^0 \subseteq U^0 \cap W^0.$$

OR. Suppose  $\varphi \in (U + W)^0$ . Then  $\forall u \in U, w \in W, \varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$ .

$$(b) \text{ Suppose } \varphi \in U^0 \cap W^0 \subseteq V'. \text{ Then } \forall u \in U, w \in W, \varphi(u + w) = 0 \Rightarrow \varphi \in (U + W)^0. \quad \square$$

**23** Suppose  $U$  and  $W$  are subsets of  $V$ . Prove that  $(U \cap W)^0 = U^0 + W^0$ .

**SOLUTION:**

$$(a) \left. \begin{array}{l} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array} \right\} \Rightarrow \left. \begin{array}{l} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array} \right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \quad [\supseteq U^0 \cap W^0 = (U + W)^0.]$$

OR. Suppose  $\varphi = \psi + \beta \in U^0 + W^0$ . Then  $\forall v \in U \cap W, \varphi(v) = (\psi + \beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0$ .

(b) [Only in Finite-dim; Req  $U, W$  are subspaces] Using Problem (22).

$$\begin{aligned} \dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\ &= 2 \dim V - \dim U - \dim W - (\dim V - \dim(U + W)) = \dim V - \dim(U \cap W). \end{aligned}$$

OR. Suppose  $\varphi \in (U \cap W)^0$ . Let  $X, Y$  be such that  $V = U \oplus X = W \oplus Y$ .

Define  $\psi \in U^0, \beta \in W^0$  by  $\psi(u + x) = \frac{1}{2}\varphi(x), \beta(w + y) = \frac{1}{2}\varphi(y)$ .

$\forall v = u + x = w + y \in V, \varphi(v) = \varphi(x) = \varphi(y)$ . Now  $\varphi(v) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y) = \psi(v) + \beta(v)$ .

Hence  $\varphi \in U^0 + W^0$ . Now  $(U \cap W)^0 \subseteq U^0 + W^0$ .  $\square$

• **COROLLARY:**

(a) Suppose  $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$  is a collection of subsets of  $V$ . Then  $\left(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \sum_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$ .

(b) Suppose  $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$  is a collection of subspaces of  $V$ . Then  $\left(\sum_{\alpha_i \in \Gamma} V_{\alpha_i}\right)^0 = \bigcap_{\alpha_i \in \Gamma} (V_{\alpha_i}^0)$ .

(c) Suppose  $V = U \oplus W$ . Then  $V' = U^0 \oplus W^0$ . And  $U'_V = W^0, W'_V = U^0$ .

Where  $U'_V = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$ . And  $\iota \in \mathcal{L}(V, U)$  is defined by  $\iota(u_v + w_v) = u_v$ .

• (4E 3.F.23) Suppose  $\varphi_1, \dots, \varphi_m \in V'$ . Prove that the following sets are the same.

(a)  $\text{span}(\varphi_1, \dots, \varphi_m)$

(b)  $((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m))^0 \stackrel{(c)}{=} \{\varphi \in V' : (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) \subseteq \text{null } \varphi\}$

**SOLUTION:** By Problem (17), (c) holds.

By Problem (26) [May req Finite-dim] and the COROLLARY in Problem (23),

$$\left. \begin{array}{l} ((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m))^0 = (\text{null } \varphi_1)^0 + \dots + (\text{null } \varphi_m)^0 \\ \text{span}(\varphi_i) = \{v \in V : \forall \psi \in \text{span}(\varphi_i), \psi(v) = 0\}^0 = (\text{null } \varphi_i)^0 \end{array} \right\} \Rightarrow (a) = (b). \quad \square$$

OR. Note that by COROLLARY in Problem (4E 6), for each  $\varphi_i$ , we have

$$\forall c \in \mathbf{F} \setminus \{0\}, \psi = c\varphi_i \in \text{span}(\varphi_i) \iff \text{null } \psi = \text{null } \varphi_i \iff \psi \in (\text{null } \psi)^0 = (\text{null } \varphi_i)^0.$$

And  $0 \in \text{span}(\varphi_i), 0 \in (\text{null } \varphi_i)^0$ . Hence  $\text{span}(\varphi_i) = (\text{null } \varphi_i)^0$ . Similarly.  $\square$

OR. [Only in Finite-dim] Suppose  $\varphi \in V'$ . Note that  $\dim(\text{null } \varphi)^0 = \dim \text{range } \varphi = \dim \text{span}(\varphi)$ .

And because  $\forall c \in \mathbf{F}, v \in \text{null } \varphi, c\varphi(v) = 0 \Rightarrow \text{span}(\varphi) \subseteq (\text{null } \varphi)^0$ . Similarly.  $\square$

**COROLLARY: 30** Suppose  $V$  is finite-dim and  $\varphi_1, \dots, \varphi_m$  is a linearly inde list in  $V'$ .

$$\text{Then } \dim((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)) = (\dim V) - m.$$



**31** Suppose  $V$  is finite-dim and  $B_{V'} = (\varphi_1, \dots, \varphi_n)$ . Show that the correspd  $B_V$  exists.

**SOLUTION:**

Using (3.B.29). Let  $\varphi_i(u_i) = 1$  and then  $V = \text{null } \varphi_i \oplus \text{span}(u_i)$  for each  $\varphi_i$ .

Suppose  $a_1 u_1 + \dots + a_n u_n = 0$ . Then  $0 = \varphi_i(a_1 u_1 + \dots + a_n u_n) = a_i$  for each  $i$ .

Thus  $B_V = (\varphi_1, \dots, \varphi_n)$ . And  $\varphi_i(u_x) = \delta_{i,x}$ . □

OR. For each  $k \in \{1, \dots, n\}$ , define  $\Gamma_k = \{1, \dots, k-1, k+1, \dots, n\}$  and  $U_k = \bigcap_{j \in \Gamma_k} \text{null } \varphi_j$ .

By Problem (30) OR (4E 2.C.16),  $\dim U_k = 1$ . Thus  $\exists u_k \in V, U_k = \text{span}(u_k) \neq 0$ .

又 By Problem (30),  $(\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_n) = \{0\} = U \cap \text{null } \varphi_k$ .

Then if  $\varphi_k(u_k) = 0 \Rightarrow u_k \in \text{null } \varphi_k$  while  $u_k \in U \Rightarrow u_k \in \{0\}$ , contradicts.

Thus  $\varphi_k(u_k) \neq 0$ . Let  $v_k = (\varphi_k(u_k))^{-1} u_k \Rightarrow \varphi_k(v_k) = 1$ . Now for  $j \neq k, u_k \in \text{null } \varphi_j \Rightarrow \varphi_j(v_k) = 0$ .

Similarly, suppose  $a_1 v_1 + \dots + a_n v_n = 0 \Rightarrow a_1 = \dots = a_n = 0$ .  $B_V = (v_1, \dots, v_n)$ . And  $\varphi_j(v_k) = \delta_{j,k}$ . □

**25** Suppose  $U$  is a subsp of  $V$ . Explain why  $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$ .

**SOLUTION:** Note that  $U = \{v \in V : v \in U\}$  is a subsp of  $V$ ; And  $v \in U \iff \varphi(v) = 0, \forall \varphi \in U^0$ . □

**COROLLARY:**  $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}^0$ .

**COMMENT:**  $\{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = ((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) \cap \dots)$ , where  $\varphi_k \in U^0$ , always remains a subsp, whether the subset  $U$  is a subsp or not.

**26** Suppose  $\Omega$  is a subsp of  $V'$ . Prove that  $\Omega = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega^0\}$ .

**SOLUTION:**

Suppose  $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}$ , which is the set of vecs that each  $\varphi \in \Omega$  sends to zero in common.

Then  $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}^0$ . 又  $U^0 = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}^0$ .

Immediately by the COROLLARY in Problem (20,21), we may conclude that  $\Omega = U^0$ . □

OR. [Req  $\Omega$  finite-dim] Let  $(\varphi_1, \dots, \varphi_m)$  be a basis of  $\Omega$ . Then by def,  $U \subseteq (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ .

$\forall \varphi \in \Omega, \exists ! a_i \in \mathbb{F}, \varphi = a_1 \varphi_1 + \dots + a_m \varphi_m \Rightarrow \forall v \in (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m), \varphi(v) = 0 \Rightarrow v \in U$ .

Hence  $(\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) = U$ . 又  $\text{span}(\varphi_1, \dots, \varphi_m) = \Omega$ . By Problem (23), we are done. □

**COROLLARY:** For every subsp  $\Omega$  of  $V'$ ,  $\exists !$  subsp  $U$  of  $V$  such that  $\Omega = U^0$ .

**COMMENT:** [Only in Finite-dim] Using Problem (31) and the COROLLARY(c) in Problem (22, 23).

Let  $B_\Omega = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n), B_V = (v_1, \dots, v_m, \dots, v_n)$ .

$V' = \text{span}(\varphi_1, \dots, \varphi_m) \oplus \text{span}(\varphi_{m+1}, \dots, \varphi_n) \stackrel{(I)}{=} \text{span}(v_{m+1}, \dots, v_n)^0 \oplus \text{span}(v_1, \dots, v_m)^0$ .

$\Omega = \text{span}(\varphi_1, \dots, \varphi_m) \stackrel{(II)}{=} \text{span}(v_{m+1}, \dots, v_n)^0 = U^0; \text{span}(\varphi_{m+1}, \dots, \varphi_n) \stackrel{(III)}{=} \text{span}(v_1, \dots, v_m)^0$ .

$\iff U = \text{span}(v_{m+1}, \dots, v_n) = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ . [Another proof of [3.106] OR. Problem (24)]

(I) Using the COROLLARY(c), immediately.

(II) NOTICE that each  $\text{null } \varphi_k = \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n) = U_k; \dim U_k = \dim V - 1$ .

By (4E 2.C.16),  $U = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) = \bigcap_{k=1}^m U_k = \text{span}(v_{m+1}, \dots, v_n)$ .

Hence  $\text{span}(v_{m+1}, \dots, v_n)^0 = U^0 = \Omega = \text{span}(\varphi_1, \dots, \varphi_m)$ .

(III) NOTICE that  $V' = \Omega \oplus \text{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0 \oplus \text{span}(v_1, \dots, v_m)^0$ .

And that  $\text{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq \text{span}(v_1, \dots, v_m)^0$ .

By (1.C TIPS),  $\text{span}(\varphi_{m+1}, \dots, \varphi_n) = \text{span}(v_1, \dots, v_m)$ .

OR. Similar to (II), let  $\Omega = \text{span}(\varphi_{m+1}, \dots, \varphi_n)$ , immediately. □

• Suppose  $T \in \mathcal{L}(V, W)$ ,  $\varphi_k \in V'$ ,  $\psi_k \in W'$ .

**28** Prove that  $\text{null } T' = \text{span}(\psi_1, \dots, \psi_m) \iff \text{range } T = (\text{null } \psi_1) \cap \dots \cap (\text{null } \psi_m)$ .

**29** Prove that  $\text{range } T' = \text{span}(\varphi_1, \dots, \varphi_m) \iff \text{null } T = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ .

**SOLUTION:** Using [3.107], [3.109], Problem (23) and the COROLLARY in Problem (20, 21).

$$(28) (\text{range } T)^0 = \text{null } T' = \text{span}(\psi_1, \dots, \psi_m) = ((\text{null } \psi_1) \cap \dots \cap (\text{null } \psi_m))^0.$$

$$(29) (\text{null } T)^0 = \text{range } T' = \text{span}(\varphi_1, \dots, \varphi_m) = ((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m))^0. \quad \square$$

**COROLLARY:** Using the COMMENT in Problem (26).

$$\text{null } T = \text{span}(v_1, \dots, v_m) \iff \text{null } T = (\text{null } \varphi_{m+1}) \cap \dots \cap (\text{null } \varphi_n) \iff \text{range } T' = \text{span}(\varphi_{m+1}, \dots, \varphi_n).$$

$$\text{---Where } B_V = (v_1, \dots, v_m, \dots, v_n) \iff B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n).$$

$$\text{range } T = \text{span}(w_1, \dots, w_m) \iff \text{range } T = (\text{null } \psi_{m+1}) \cap \dots \cap (\text{null } \psi_n) \iff \text{null } T' = \text{span}(\psi_{m+1}, \dots, \psi_n).$$

$$\text{---Where } B_W = (w_1, \dots, w_m, \dots, w_n) \iff B_{W'} = (\psi_1, \dots, \psi_m, \dots, \psi_n).$$

**9** Let  $B_V = (v_1, \dots, v_n)$ ,  $B_{V'} = (\varphi_1, \dots, \varphi_n)$ . Then  $\forall \psi \in V'$ ,  $\psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n$ .

**COROLLARY:** For other  $B'_V = (u_1, \dots, u_n)$ ,  $B'_{V'} = (\rho_1, \dots, \rho_n)$ ,  $\forall \psi \in V'$ ,  $\psi = \psi(u_1)\rho_1 + \dots + \psi(u_n)\rho_n$ .

**SOLUTION:**

$$\psi(v) = \psi\left(\sum_{i=1}^n a_i v_i\right) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i) \varphi_i(v) = [\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n](v).$$

$$\text{OR. } [\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n]\left(\sum_{i=1}^n a_i v_i\right) = \psi(v_1)\varphi_1\left(\sum_{i=1}^n a_i v_i\right) + \dots + \psi(v_n)\varphi_n\left(\sum_{i=1}^n a_i v_i\right). \quad \square$$

**13** Define  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  by  $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$ .

Let  $(\varphi_1, \varphi_2)$ ,  $(\psi_1, \psi_2, \psi_3)$  denote the dual basis of the std basis of  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .

(a) Describe the linear functionals  $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$

$$\text{For any } (x, y, z) \in \mathbb{R}^3, (T'(\varphi_1))(x, y, z) = 4x + 5y + 6z, (T'(\varphi_2))(x, y, z) = 7x + 8y + 9z.$$

(b) Write  $T'(\varphi_1)$  and  $T'(\varphi_2)$  as linear combinations of  $\psi_1, \psi_2, \psi_3$ .

$$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3, \quad T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3.$$

(c) What is  $\text{null } T'$ ? What is  $\text{range } T'$ ?

$$T(x, y, z) = 0 \iff \begin{cases} 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases} \iff \begin{cases} x + y + z = 0 \\ y = 2z = 0 \end{cases} \iff (x, y, z) \in \text{span}(e_1 - 2e_2 + e_3).$$

Where  $(e_1, e_2, e_3)$  is std basis of  $\mathbb{R}^3$ .

Let  $(e_1 - 2e_2 + e_3, -2e_2, e_3)$  be a basis, with the correspd dual basis  $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ .

$$\text{Thus } \text{span}(e_1 - 2e_2 + e_3) = \text{null } T \Rightarrow \text{span}(e_1 - 2e_2 + e_3)^0 = \text{span}(\varepsilon_2, \varepsilon_3) = \text{range } T'.$$

Note that  $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$ .

$$\text{And } \begin{cases} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{cases}$$

$$\text{Hence } \varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2, \quad \varepsilon_3 = -\psi_1 + \psi_3. \text{ Now } \text{range } T' = \text{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3).$$

$$\text{OR. } \text{range } T' = \text{span}(T'(\varphi_1), T'(\varphi_2)) = \text{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3).$$

$$\text{Suppose } T'(x\varphi_1 + y\varphi_2) = (4x + 7y)\varphi_1 + (5x + 8y)\varphi_2 + (6x + 9y)\varphi_3 = 0.$$

$$\text{Then } x + y = 4x + 7y = x = y = 0. \text{ Hence } \text{null } T' = \{0\}.$$

$$\text{OR. } \text{null } T = \text{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \text{span}(-2e_2, e_3) \oplus \text{null } T.$$

$$\Rightarrow \text{range } T = \{Tx : x \in \text{span}(-2e_2, e_3)\} = \text{span}(T(-2e_2), T(e_3))$$

$$= \text{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \text{span}(f_1, f_2) = \mathbb{R}^2. \text{ Now } \text{null } T' = (\text{range } T)^0 = \{0\}. \quad \square$$

**24** Suppose  $V$  is finite-dim and  $U$  is a subsp of  $V$ .

Prove, using the pattern of [3.104], that  $\dim U + \dim U^0 = \dim V$ .

**SOLUTION:**

By Problem (31) and the COMMENT in Problem (26),  $B_U = (v_1, \dots, v_m) \iff B_{U^0} = (\varphi_{m+1}, \dots, \varphi_n)$ .  $\square$

**37** Suppose  $U$  is a subsp of  $V$  and  $\pi$  is the quotient map. Thus  $\pi' \in \mathcal{L}((V/U)', V')$ .

(a) Show that  $\pi'$  is inje: Because  $\pi$  is surj. Use [3.108].

(b) Show that  $\text{range } \pi' = U^0$ : By [3.109](b),  $\text{range } \pi' = (\text{null } \pi)^0 = U^0$ .

(c) Conclude that  $\pi'$  is an iso from  $(V/U)'$  onto  $U^0$ : Immediately.

**SOLUTION:** OR. Using (3.E.18), also see (3.E.20).

(a)  $\pi'(\varphi) = 0 \iff \forall v \in V (\forall v + U \in V), \varphi(\pi(v)) = \varphi(v + U) = 0 \iff \varphi = 0$ .

(b)  $\psi \in \text{range } \pi' \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \text{null } \psi \supseteq U \iff \psi \in U^0$ . Hence  $\text{range } \pi' = U^0$ .  $\square$

• Suppose  $U$  is a subsp of  $V$ . Prove that  $(V/U)'$  is iso to  $U^0$ . [ Another proof of [3.106] ]

**SOLUTION:**

Define  $\xi : U^0 \rightarrow (V/U)'$  by  $\xi(\varphi) = \tilde{\varphi}$ , where  $\tilde{\varphi} \in (V/U)'$  is defined by  $\tilde{\varphi}(v + U) = \varphi(v)$ .

We show that  $\xi$  is inje and surj.

Inje:  $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V (\forall v + U \in V/U), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$ .

Surj:  $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \text{null } (\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$ .  $\square$

OR. Define  $\nu : (V/U)' \rightarrow U^0$  by  $\nu(\Phi) = \Phi \circ \pi$ . Now  $\nu \circ \xi = I_{U^0}$ ,  $\xi \circ \nu = I_{(V/U)'}$ ,  $\Rightarrow \xi = \nu^{-1}$ .  $\square$

**4** Suppose  $U$  is a subsp of  $V$  and  $U \neq V$ . Prove that  $\underbrace{\exists \varphi \in V' \setminus \{0\}, \varphi(u) = 0 \text{ for all } u \in U}_{\iff U_V^0 \neq \{0\}}$ .

**SOLUTION:**

Let  $X$  be such that  $V = U \oplus X$ . Then  $X \neq \{0\}$ . Suppose  $s \in X$  and  $s \neq 0$ .

Let  $Y$  be such that  $X = \text{span}(s) \oplus Y$ . Now  $V = U \oplus (\text{span}(s) \oplus Y)$ .

Define  $\varphi \in V'$  by  $\varphi(u + \lambda s + y) = \lambda$ . Hence  $\varphi \neq 0$  and  $\varphi(u) = 0$  for all  $u \in U$ .  $\square$

OR. [Req  $V$  Finite-dim] By [3.106],  $\dim U^0 = \dim V - \dim U > 0$ . Then  $U^0 \neq \{0\}$ .

OR. Let  $B_V = (\underbrace{u_1, \dots, u_m}_{B_U}, v_1, \dots, v_n)$  with  $n \geq 1$ . Let  $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$ . Let  $\varphi = \varphi_i$ .

OR. Define  $\varphi \in V'$  by  $\varphi(u_1) = \dots = \varphi(u_m) = 0$  and  $\varphi(v_1) = \dots = \varphi(v_n) = 1$ .  $\square$

**COMMENT:** Another proof of [3.108]:  $T$  is surj  $\iff T'$  is inje.

(a) Suppose  $T'$  is inje. Note that  $T'(\psi) = 0 \Rightarrow \psi = 0$ .

Then  $\nexists \psi \in W' \setminus \{0\}, (T'(\psi))(v) = \psi(Tv) = 0$  for all  $w \in \text{range } T (\forall v \in V)$ .

Thus if we assume that  $\text{range } T \neq W$  then contradicts. Hence  $\text{range } T = W$ .

(b) Suppose  $T$  is surj. Then  $(\text{range } T)^0 = W_W^0 = \{0\} = \text{null } T'$ .  $\square$

• Suppose  $V$  is a vecsp and  $U$  is a subsp of  $V$ .

**17**  $U^0 = \{\varphi \in V' : U \subseteq \text{null } \varphi\}$ . Noticing  $\varphi \in V', U \subseteq \text{null } \varphi \iff \forall u \in U, \varphi(u) = 0$ .

**18**  $U^0 = V' \iff \forall \varphi \in V', U \subseteq \text{null } \varphi \iff U = \{0\}$ . [Which means  $\{0\}_V^0 = V'$ .]

OR.  $U^0 = V' \iff \dim U^0 = \dim V' = \dim V \iff \dim U = 0 \iff U = \{0\}$ .

**19**  $U_V^0 = \{0\} = V_V^0 \iff U = V$ . By the inverse and contrapositive of Problem (4). OR. By [3.106].

- Suppose  $V = U \oplus W$ . Define  $\iota : V \rightarrow U$  by  $\iota(u + w) = u$ . Thus  $\iota' \in \mathcal{L}(U', V')$ .
  - (a) Show that  $\text{null } \iota' = U_U^0 = \{0\}$ :  $\text{null } \iota' = (\text{range } \iota)_U^0 = U_U^0 = \{0\}$ .
  - (b) Prove that  $\text{range } \iota' = W_V^0$ :  $\text{range } \iota' = (\text{null } \iota)_V^0 = W_V^0$ .
  - (c) Prove that  $\tilde{\iota}'$  is an iso from  $U'/\{0\}$  onto  $W^0$ : By (a), (b) and [3.91](d).

**SOLUTION:**

- (a)  $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \text{null } \psi$ .
- (b) Note that  $W = \text{null } (\iota) \subseteq \text{null } (\psi \circ \iota)$ . Then  $\psi \circ \iota \in W^0 \Rightarrow \text{range } \iota' \in W^0$ .  
Suppose  $\varphi \in W^0$ . Because  $\text{null } \iota = W \subseteq \text{null } \varphi$ . By [3.B TIPS (3)],  $\varphi = \varphi \circ \iota = \iota'(\varphi)$ . □

**36** Suppose  $U$  is a subsp of  $V$ . Define  $i : U \rightarrow V$  by  $i(u) = u$ . Thus  $i' \in \mathcal{L}(V', U')$ .

- (a) Show that  $\text{null } i' = U^0$ :  $\text{null } i' = (\text{range } i)^0 = U^0 \Leftarrow \text{range } i = U$ .
- (b) Prove that  $\text{range } i' = U'$ :  $\text{range } i' = (\text{null } i)_U^0 = \{0\}_U^0 = U'$ .
- (c) Prove that  $\tilde{i}'$  is an iso from  $V'/U^0$  onto  $U'$ : By (a), (b) and [3.91](d).

**SOLUTION:**

- (a)  $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$ . Thus  $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$ .
- (b) Suppose  $\psi \in U'$ . By (3.A.11),  $\exists \varphi \in V', \varphi|_U = \psi$ . Then  $i'(\varphi) = \psi$ . □

• Suppose  $T \in \mathcal{L}(V, W)$ . Prove that  $\text{range } T' = (\text{null } T)^0$ . [Another proof of [3.109](b)]

**SOLUTION:**

Suppose  $\Phi \in (\text{null } T)^0$ . Because by (3.B.12),  $T|_U : U \rightarrow \text{range } T$  is an iso;  $V = U \oplus \text{null } T$ .  
And  $\forall v \in V, \exists ! u_v \in U, w_v \in \text{null } T, v = u_v + w_v$ . Define  $\iota \in \mathcal{L}(V, U)$  by  $\iota(v) = u_v$ .  
Let  $\psi = \Phi \circ (T^{-1}|_{\text{range } T})$ . Then  $T'(\psi) = \psi \circ T = \Phi \circ (T^{-1}|_{\text{range } T} \circ T|_V)$ .  
Where  $T^{-1}|_{\text{range } T} : \text{range } T \rightarrow U$ ;  $T : V \rightarrow \text{range } T$ . Note that  $T^{-1}|_{\text{range } T} \circ T|_V = I$ .  
By [3.B TIPS (3)],  $\Phi = \Phi \circ \iota$ . Thus  $T'(\psi) = \psi \circ T = \Phi \circ \iota = \Phi$ . □

• Suppose  $T \in \mathcal{L}(V, W)$ . Using [3.108], [3.110].

$$\text{Now } T \text{ is inv} \iff \left\{ \begin{array}{l} \text{null } T = \{0\} \iff (\text{null } T)^0 = V' = \text{range } T' \\ \text{range } T = W \iff (\text{range } T)^0 = \{0\} = \text{null } T' \end{array} \right\} \iff T' \text{ is inv.}$$

**15** Suppose  $T \in \mathcal{L}(V, W)$ . Prove that  $T' = 0 \iff T = 0$ .

**SOLUTION:**

Suppose  $T = 0$ . Then  $\forall \varphi \in W', T'(\varphi) = \varphi \circ T = 0$ . Hence  $T' = 0$ .

Suppose  $T' = 0$ . Then  $\text{null } T' = W' = (\text{range } T)^0$ , by [3.107](a).

[  $W$  can be infinite-dim ] By Problem (25),

$$\text{range } T = \{w \in W : \varphi(w) = 0, \forall \varphi \in (\text{range } T)^0\} = \{w \in W : \varphi(w) = 0, \forall \varphi \in W'\}.$$

Now we prove that if  $\forall \varphi \in W', \varphi(w) = 0$ , then  $w = 0$ . So that  $\text{range } T = \{0\}$  and we are done.

Assume that  $w \neq 0$ . Then let  $U$  be such that  $W = U \oplus \text{span}(w)$ .

Define  $\psi \in W'$  by  $\psi(u + \lambda w) = \lambda$ . So that  $\psi(w) = 1 \neq 0$ . □

OR. [ Only if  $W$  is finite-dim ] By [3.106],  $\dim \text{range } T = \dim W - \dim (\text{range } T)^0 = 0$ . □

**12** NOTICE that  $I_{V'} : V' \rightarrow V'$ . Now  $\forall \varphi \in V', I_{V'}(\varphi) = \varphi = \varphi \circ I_V = I_V'(\varphi)$ . Thus  $I_{V'} = I_V'$ .

**16** Suppose  $V, W$  are finite-dim. Define  $\Gamma$  by  $\Gamma(T) = T'$  for any  $T \in \mathcal{L}(V, W)$ .

Prove that  $\Gamma$  is an iso of  $\mathcal{L}(V, W)$  onto  $\mathcal{L}(W', V')$ .

**SOLUTION:** By [3.101],  $\Gamma$  is linear.

Suppose  $\Gamma(T) = T' = 0$ . By Problem (15),  $T = 0$ . Thus  $\Gamma$  is inje.

Because  $V, W$  are finite-dim.  $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$ . Now  $\Gamma$  inje  $\Rightarrow$  inv. □

**COMMENT:** Let  $X = \{T \in \mathcal{L}(V, W) : \text{range } T \text{ is finite-dim}\}$ .

Let  $Y = \{\mathcal{T} \in \mathcal{L}(W', V') : \text{range } \mathcal{T} \text{ is finite-dim}\}$ .

Then  $\Gamma|_X$  is an iso of  $X$  onto  $Y$ , even if  $V$  and  $W$  are infinite-dim.

The inje of  $\Gamma|_X$  is equiv to the inje of  $\Gamma$ , as shown before.

Now we show that  $\Gamma|_X$  is surj without the cond that  $V$  or  $W$  is finite-dim.

Suppose  $\mathcal{T} \in Y$ . Let  $B_{\text{range } \mathcal{T}} = (\varphi_1, \dots, \varphi_m)$ , with the correspd  $(v_1, \dots, v_m)$ . Let  $\varphi_k = \mathcal{T}(\psi_k)$ .

Let  $\mathcal{K}$  be such that  $W' = \mathcal{K} \oplus \text{null } \mathcal{T}$ . Let  $B_{\mathcal{K}} = (\psi_1, \dots, \psi_m)$ , with the correspd  $(w_1, \dots, w_m)$ .

Define  $T \in \mathcal{L}(V, W)$  by  $Tv_k = w_k, Tu = 0; k \in \{1, \dots, m\}, u \in U$ .

$\forall \psi \in \text{null } \mathcal{T}, [T'(\psi)](v) = \psi(Tv) = \psi(a_1w_1 + \dots + a_pw_p) = 0 = [\mathcal{T}(\psi)](v)$ .

$\forall k \in \{1, \dots, m\}, [T'(\psi_k)](v) = \psi_k(Tv) = \psi_k(a_1w_1 + \dots + a_mw_m) = a_k = \varphi_k(v) = [\mathcal{T}(\psi)](v)$ . □

**COMMENT:** This is another proof of [3.109(a)]:  $\dim \text{range } T = \dim \text{range } T'$ .

• (4E 3.F.6) Suppose  $\varphi, \beta \in V'$ . Prove that  $\text{null } \varphi \subseteq \text{null } \beta \iff \beta = c\varphi, \exists c \in \mathbf{F}$ .

**COROLLARY:**  $\text{null } \varphi = \text{null } \beta \iff \beta = c\varphi, \exists c \in \mathbf{F} \setminus \{0\}$ .

**SOLUTION:**

Using (3.B.29, 30).

(a) Suppose  $\text{null } \varphi \subseteq \text{null } \beta$ . Suppose  $u \notin \text{null } \beta$ , then  $u \notin \text{null } \varphi$ .

Now  $V = \text{null } \beta \oplus \text{span}(u) = \text{null } \varphi \oplus \text{span}(u)$ . By (1.C TIPS),  $\text{null } \beta = \text{null } \varphi$ . Let  $c = \frac{\beta(u)}{\varphi(u)}$ .

OR. We discuss in two cases. If  $\text{null } \varphi = \text{null } \beta$ , then we are done.

Otherwise,  $\text{null } \beta \neq \text{null } \varphi$ . Then  $\exists u' \in \text{null } \beta \setminus \text{null } \varphi$ .

Now  $V = \text{null } \varphi \oplus \text{span}(u') = \text{null } \varphi \oplus \text{span}(u)$ .  $\forall v \in V, v = w + au = w' + bu', \exists! w, w' \in \text{null } \varphi$ .

Thus  $\beta(v) = a\beta(u), \varphi(v) = b\varphi(u')$ . Let  $c = \frac{a\beta(u)}{b\varphi(u')}$ . We are done.

NOTICE that by (b) below, we have  $\text{null } \beta \subseteq \text{null } \varphi, u = u'$ . Thus contradicts the assumption.

(b) Suppose  $\beta = c\varphi$  for some  $c \in \mathbf{F}$ . If  $c = 0$ , then  $\text{null } \beta = V \supseteq \text{null } \varphi$ , we are done.

Otherwise,  $\left. \begin{array}{l} \forall v \in \text{null } \varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \text{null } \varphi \subseteq \text{null } \beta \\ \forall v \in \text{null } \beta, \beta(v) = 0 = \varphi(v) \Rightarrow \text{null } \beta \subseteq \text{null } \varphi \end{array} \right\} \Rightarrow \text{null } \varphi = \text{null } \beta$ . □

OR. By (3.B.24),  $\text{null } \varphi \subseteq \text{null } \beta \iff \exists E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi$ . ( if  $E$  is inv, then  $\text{null } \varphi = \text{null } \beta$  )

Now we show that  $[P] \exists E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi \iff \exists c \in \mathbf{F}, \beta = c\varphi$ . [Q].

$[P] \Rightarrow [Q]$ : Let  $c = E(1)$ . Then  $\forall v \in V, \beta(v) = E(\varphi(v)) = \varphi(v)E(1) = c\varphi(v)$ . (  $E(1) \neq 0$  )

$[Q] \Rightarrow [P]$ : Define  $E \in \mathcal{L}(\mathbf{F})$  by  $E(x) = cx$ . Then  $\forall v \in V, \beta(v) = c\varphi(v) = E(\varphi(v))$ . (  $c \neq 0$  ) □

**5** Prove that  $(V_1 \times \dots \times V_m)'$  and  $V'_1 \times \dots \times V'_m$  are iso.

[ Using notations in (3.E.2). ]

Define  $\varphi : (V_1 \times \dots \times V_m)' \rightarrow V'_1 \times \dots \times V'_m$

by  $\varphi(T) = (T \circ R_1, \dots, T \circ R_m) = (R'_1(T), \dots, R'_m(T))$ .

Define  $\psi : V'_1 \times \dots \times V'_m \rightarrow (V_1 \times \dots \times V_m)'$

by  $\psi(T_1, \dots, T_m) = T_1S_1 + \dots + T_mS_m = S'_1(T_1) + \dots + S'_m(T_m)$ .

$\left. \begin{array}{l} \text{Define } \varphi : (V_1 \times \dots \times V_m)' \rightarrow V'_1 \times \dots \times V'_m \\ \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = (R'_1(T), \dots, R'_m(T)) \\ \text{Define } \psi : V'_1 \times \dots \times V'_m \rightarrow (V_1 \times \dots \times V_m)' \\ \text{by } \psi(T_1, \dots, T_m) = T_1S_1 + \dots + T_mS_m = S'_1(T_1) + \dots + S'_m(T_m) \end{array} \right\} \Rightarrow \psi = \varphi^{-1}$ . □



- (4E 3.F.8) Suppose  $B_V = (v_1, \dots, v_n)$ ,  $B_{V'} = (\varphi_1, \dots, \varphi_n)$ .  
Define  $\Gamma : V \rightarrow \mathbf{F}^n$  by  $\Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v))$ .  
Define  $\Lambda : \mathbf{F}^n \rightarrow V$  by  $\Lambda(a_1, \dots, a_n) = a_1 v_1 + \dots + a_n v_n$ .  
 $\left. \begin{array}{l} \text{Define } \Gamma : V \rightarrow \mathbf{F}^n \text{ by } \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)). \\ \text{Define } \Lambda : \mathbf{F}^n \rightarrow V \text{ by } \Lambda(a_1, \dots, a_n) = a_1 v_1 + \dots + a_n v_n. \end{array} \right\} \Rightarrow \Lambda = \Gamma^{-1}.$
- 

- (4E 3.F.5) Suppose  $T \in \mathcal{L}(V, W)$ .  $B_{\text{range } T} = (w_1, \dots, w_m)$ .  
Hence  $\forall v \in V$ ,  $Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m$ ,  $\exists! \varphi_1(v), \dots, \varphi_m(v)$ ,  
thus defining  $\varphi_i : V \rightarrow \mathbf{F}$  for each  $i \in \{1, \dots, m\}$ . Show that each  $\varphi_i \in V'$ .

**SOLUTION:**

$$\begin{aligned} \forall u, v \in V, \lambda \in \mathbf{F}, T(u + \lambda v) &= \sum_{i=1}^m \varphi_i(u + \lambda v)w_i \\ &= Tu + \lambda Tv = \left( \sum_{i=1}^m \varphi_i(u)w_i \right) + \lambda \left( \sum_{i=1}^m \varphi_i(v)w_i \right) = \sum_{i=1}^m (\varphi_i(u) + \lambda \varphi_i(v))w_i. \quad \square \end{aligned}$$

OR. For each  $w_i, \exists v_i \in V$ ,  $Tv_i = w_i$ , then  $(v_1, \dots, v_m)$  is linely inde.

Now we have  $Tv = a_1 Tv_1 + \dots + a_m Tv_m$ ,  $\forall v \in V$ ,  $\exists! a_i \in \mathbf{F}$ . Let  $B_{(\text{range } T)'} = (\psi_1, \dots, \psi_m)$ .

Then  $(T'(\psi_i))(v) = \psi_i \circ T(v) = a_i$ . Where  $T : V \rightarrow \text{range } T$ ;  $T' : (\text{range } T)' \rightarrow V'$ .

Thus for each  $i \in \{1, \dots, m\}$ ,  $\varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$ . □

---

6 Define  $\Gamma : V' \rightarrow \mathbf{F}^m$  by  $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$ , where  $v_1, \dots, v_m \in V$ .

(a) Show that  $\text{span}(v_1, \dots, v_m) = V \iff \Gamma$  is inje.

(b) Show that  $(v_1, \dots, v_m)$  is linely inde  $\iff \Gamma$  is surj.

**SOLUTION:**

(a) NOTICE that  $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \dots = \varphi(v_m) = 0 \iff \text{null } \varphi = \text{span}(v_1, \dots, v_m)$ .

If  $\Gamma$  is inje, then  $\Gamma(\varphi) = 0 \iff V = \text{null } \varphi = \text{span}(v_1, \dots, v_m)$ .

If  $V = \text{span}(v_1, \dots, v_m)$ , then  $\Gamma(\varphi) = 0 \iff \text{null } \varphi = \text{span}(v_1, \dots, v_m)$ , thus  $\Gamma$  is inje.

(b) Suppose  $\Gamma$  is surj. Then let  $\Gamma(\varphi_i) = e_i$  for each  $i$ , where  $(e_1, \dots, e_m)$  is the std basis of  $\mathbf{F}^m$ .

Then by (3.A.4),  $(\varphi_1, \dots, \varphi_m)$  is linely inde.

Now  $a_1 v_1 + \dots + a_m v_m = 0 \Rightarrow 0 = \varphi_i(a_1 v_1 + \dots + a_m v_m) = a_i$  for each  $i$ .

Suppose  $(v_1, \dots, v_m)$  is linely inde. Let  $U = \text{span}(\varphi_1, \dots, \varphi_m)$ ,  $B_{U'} = (\varphi_1, \dots, \varphi_m)$ .

Thus  $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists! \varphi = a_1 \varphi_1 + \dots + a_m \varphi_m$ .

Let  $W$  be such that  $V = U \oplus W$ . Now  $\forall v \in V, \exists! u_v \in U, w_v \in W, v = u_v + w_v$ .

Define  $\iota \in \mathcal{L}(V, U)$  by  $\iota(v) = u_v$ . So that  $\Gamma(\varphi \circ \iota -) = (a_1, \dots, a_m)$ . □

OR. Let  $(e_1, \dots, e_m)$  be the std basis of  $\mathbf{F}^m$  and let  $(\psi_1, \dots, \psi_m)$  be the correspd dual basis.

Define  $\Psi : \mathbf{F}^m \rightarrow (\mathbf{F}^m)'$  by  $\Psi(e_k) = \psi_k$ . Then  $\Psi$  is an iso.

Define  $T \in \mathcal{L}(\mathbf{F}^m, V)$  by  $T e_k = v_k$ . Now  $T(x_1, \dots, x_m) = T(x_1 e_1 + \dots + x_m e_m) = x_1 v_1 + \dots + x_m v_m$ .

$\forall \varphi \in V', k \in \{1, \dots, m\}, [T'(\varphi)](e_k) = \varphi(T e_k) = \varphi(v_k) = [\varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m](e_k)$

Now  $T'(\varphi) = \varphi(v_1) \circ \psi_1 + \dots + \varphi(v_m) \circ \psi_m = \Psi(\varphi(v_1), \dots, \varphi(v_m)) = \Psi(\Gamma(\varphi))$ . Hence  $T' = \Psi \circ \Gamma$ .

By (3.B.3), (a)  $\text{range } T = \text{span}(v_1, \dots, v_m) = V \iff T' = \Psi \circ \Gamma$  inje  $\iff \Gamma$  inje.

(b)  $(v_1, \dots, v_m)$  is linely inde  $\iff T$  is inje  $\iff T' = \Psi \circ \Gamma$  surj  $\iff \Gamma$  surj. □

• (4E 3.F.25) Define  $\Gamma : V \rightarrow \mathbf{F}^m$  by  $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$ , where  $\varphi_1, \dots, \varphi_m \in V'$ .

(c) Show that  $\text{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$  is inje.

(d) Show that  $(\varphi_1, \dots, \varphi_m)$  is linely inde  $\iff \Gamma$  is surj.

**SOLUTION:**

(c) NOTICE that  $\Gamma(v) = 0 \iff \varphi_1(v) = \dots = \varphi_m(v) = 0 \iff v \in (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ .

By Problem (4E 23) and (18),  $\text{span}(\varphi_1, \dots, \varphi_m) = V' \iff (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) = \{0\}$ .

And  $\text{null } \Gamma = (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ . Hence  $\Gamma$  inje  $\iff \text{null } \Gamma = \{0\} \iff \text{span}(\varphi_1, \dots, \varphi_m) = V'$ .

(d) Suppose  $(\varphi_1, \dots, \varphi_m)$  is linely inde. Then by Problem (31),  $(v_1, \dots, v_m)$  is linely inde.

Thus  $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists! v = \sum_{i=1}^m a_i v_i \in V \Rightarrow \varphi_i(v) = a_i, \Gamma(v) = (a_1, \dots, a_m)$ . Hence  $\Gamma$  is surj.

Suppose  $\Gamma$  is surj. Let  $(e_1, \dots, e_m)$  be the std basis of  $\mathbf{F}^m$ .

Suppose  $v_i \in V$  such that  $\Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v_i)) = e_i$ , for each  $i$ .

Then  $(v_1, \dots, v_m)$  is linely inde. And  $\varphi_j(v_k) = \delta_{j,k}$ .

Now  $a_1 \varphi_1 + \dots + a_m \varphi_m = 0 \Rightarrow 0(v_i) = a_i$  for each  $i$ . Hence  $(\varphi_1, \dots, \varphi_m)$  is linely inde.

OR. Let  $\text{span}(v_1, \dots, v_m) = U$ . Then  $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U)$ . Hence  $(\varphi_1, \dots, \varphi_m)$  is linely inde. □

OR. Similar to Problem (6), we get  $(e_1, \dots, e_m), (\psi_1, \dots, \psi_m)$  and the iso  $\Psi$ .

$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'(\Psi(x_1, \dots, x_m)) = \Gamma'(\Psi(x_1 e_1 + \dots + x_m e_m)) = (x_1 \psi_1 + \dots + x_m \psi_m) \circ \Gamma$ .

$\forall v \in V, [\Gamma'(\Psi(x_1, \dots, x_m))](v) = [x_1 \psi_1 + \dots + x_m \psi_m](\Gamma(v)) = [x_1 \varphi_1 + \dots + x_m \varphi_m](v)$ .

Now  $\Gamma'(\Psi(x_1, \dots, x_m)) = x_1 \varphi_1 + \dots + x_m \varphi_m$ .

Define  $\Phi : \mathbf{F}^m \rightarrow (\mathbf{F}^m)'$  by  $\Phi = \Psi \circ \Gamma$ .  $\Phi(x_1, \dots, x_m) = x_1 \varphi_1 + \dots + x_m \varphi_m$ . Thus by (4E 3.B.3),

(c) the inje of  $\Phi$  correspds to  $(\varphi_1, \dots, \varphi_m)$  spanning  $V'$ ;  $\text{又 } \Phi = \Psi \circ \Gamma$  inje  $\iff \Gamma$  inje.

(d) the surj of  $\Phi$  correspds to  $(\varphi_1, \dots, \varphi_m)$  being linely inde;  $\text{又 } \Phi = \Psi \circ \Gamma$  surj  $\iff \Gamma$  surj. □

**35** Prove that  $(\mathcal{P}(\mathbf{F}))'$  is iso to  $\mathbf{F}^\infty$ .

**SOLUTION:**

Define  $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{F}))', \mathbf{F}^\infty)$  by  $\theta(\varphi) = (\varphi(1), \varphi(z), \dots, \varphi(z^n), \dots)$ .

Inje:  $\theta(\varphi) = 0 \Rightarrow \forall z^k$  in the basis  $(1, z, \dots, z^n)$  of  $\mathcal{P}_n(\mathbf{F})$  ( $\forall n$ ),  $\varphi(z^k) = 0 \Rightarrow \varphi = 0$ .

[ NOTICE that  $\forall p \in \mathcal{P}(\mathbf{R}), \exists ! a_i \in \mathbf{F}, m = \deg p, p = a_0 z + a_1 z^2 + \dots + a_m z^m \in \mathcal{P}_m(\mathbf{F})$ . ]

Surj:  $\forall (a_k)_{k=1}^\infty \in \mathbf{F}^\infty$ , let  $\psi$  be such that  $\forall k, \psi(z^k) = a_k$  [ by [3.5] ] and thus  $\theta(\psi) = (a_k)_{k=1}^\infty$ .  $\square$

**COMMENT:** NOTICE that  $\mathcal{P}(\mathbf{F})$  is not iso to  $\mathbf{F}^\infty$ , so is  $\mathcal{P}(\mathbf{F})$  to  $(\mathcal{P}(\mathbf{F}))'$

But if we let  $\mathbf{F}^\infty = \{(a_1, \dots, a_n, \underbrace{0, \dots, 0}_{\text{all zero}}, \dots) \in \mathbf{F}^\infty \mid \exists ! n \in \mathbf{N}^+\}$ . Then  $\mathcal{P}(\mathbf{F})$  is iso to  $\mathbf{F}^\infty$ .

**7** Show that the dual basis of  $(1, x, \dots, x^m)$  of  $\mathcal{P}_m(\mathbf{R})$  is  $(\varphi_0, \varphi_1, \dots, \varphi_m)$ , where  $\varphi_k(p) = \frac{p^{(k)}(0)}{k!}$ .

Here  $p^{(k)}$  denotes the  $k^{\text{th}}$  derivative of  $p$ , with the understanding that the  $0^{\text{th}}$  derivative of  $p$  is  $p$ .

**SOLUTION:**

$$\forall j, k \in \mathbf{N}, (x^j)^{(k)} = \begin{cases} j(j-1) \dots (j-k+1) \cdot x^{(j-k)}, & j \geq k. \\ j(j-1) \dots (j-j+1) = j! & j = k. \\ 0, & j \leq k. \end{cases} \quad \text{Then } (x^j)^{(k)}(0) = \begin{cases} 0, & j \neq k. \\ k!, & j = k. \end{cases} \quad \square$$

OR. Because  $\forall j, k \in \{1, \dots, m\}$  such that  $j \neq k$ ,  $\varphi_k(x^j) = \frac{(x^j)^{(k)}(0)}{k!} = \frac{0}{k!} = 0$ ;  $\varphi_k(x^k) = \frac{(x^k)^{(k)}(0)}{k!} = 1$ .

Thus  $\frac{p^{(k)}(0)}{k!}$  act exactly the same as  $\varphi_k$  on the same basis  $(1, \dots, x^m)$ , hence is just another def of  $\varphi_k$ .  $\square$

**EXAMPLE:** Suppose  $m \in \mathbf{N}^+$ . By [2.C.10],  $B = (1, x-5, \dots, (x-5)^m)$  is a basis of  $\mathcal{P}_m(\mathbf{R})$ .

Let  $\varphi_k = \frac{p^{(k)}(5)}{k!}$  for each  $k = 0, 1, \dots, m$ . Then  $(\varphi_0, \varphi_1, \dots, \varphi_m)$  is the dual basis of  $B$ .

**34** The double dual space of  $V$ , denoted by  $V''$ , is defined to be the dual space of  $V'$ .

In other words,  $V'' = \mathcal{L}(V', \mathbf{F})$ . Define  $\Lambda : V \rightarrow V''$  by  $(\Lambda v)(\varphi) = \varphi(v)$ .

(a) Show that  $\Lambda$  is a linear map from  $V$  to  $V''$ .

(b) Show that if  $T \in \mathcal{L}(V)$ , then  $T'' \circ \Lambda = \Lambda \circ T$ , where  $T'' = (T')'$ .

(c) Show that if  $V$  is finite-dim, then  $\Lambda$  is an iso from  $V$  onto  $V''$ .

Suppose  $V$  is finite-dim. Then  $V$  and  $V'$  are iso, and finding an iso from  $V$  onto  $V'$  generally requires choosing a basis of  $V$ . In contrast, the iso  $\Lambda$  from  $V$  onto  $V''$  does not require a choice of basis and thus is considered more natural.

**SOLUTION:**

(a)  $\forall \varphi \in V', v, w \in V, a \in \mathbf{F}, (\Lambda(v+aw))(\varphi) = \varphi(v+aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$ .

Thus  $\Lambda(v+aw) = \Lambda v + a\Lambda w$ . Hence  $\Lambda$  is linear.

(b)  $(T''(\Lambda v))(\varphi) = ((\Lambda v) \circ T')(\varphi) = (\Lambda v)(T'(\varphi))$   
 $= (T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = (\Lambda(Tv))(\varphi)$ .

Hence  $T''(\Lambda v) = (\Lambda(Tv)) \Rightarrow T'' \circ \Lambda = \Lambda \circ T$ .

(c) Suppose  $\Lambda v = 0$ . Then  $\forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$ . Thus  $\Lambda$  is inje.

又 Because  $V$  is finite-dim.  $\dim V = \dim V' = \dim V''$ . Hence  $\Lambda$  is an iso.  $\square$

**ENDED**