# 简介

这是我个人用于复习的「***Linear Algebra Done Right 3E/4E, by Sheldon Axler***」笔记，一本习题选答与课文补注。范围覆盖所有第三版和第四版的课文和习题（除了第一章 A 节、极少数结合上下文太过显而易见的习题。没有任何日后反复推敲价值的当堂习题和方法套路过于雷同的习题）。这份笔记尚处于缓慢的编撰进度中。

习题答案中，有我完全独立思考得出的，有抄 https://linearalgebras.com/ 的，有抄 https://math.stackexchange.com/ 的，有抄 LADR2eSolutions（By Axler）.pdf ，有抄最新的 LADR4eSolutions 经典最全（By Axler?）.pdf，还有请教别人，乃至请教 AI 得出来的。

这些文档的许可证件，除 LADR4eSolutions 经典最全（By Axler?）.pdf 找不到/没有指明外，都允许复制/引用。

课文补注中，除了我独立思考总结出的易错误区和技巧、难点之外，还（因为我想要兼容那些使用 LADR 第三版纸质书的读者，包括我在内）把 LADR4e 中对课文定理等等的修改也（作了简化和提炼）摘录上去。部分课文内容因为比较简单，比如 3E 节的积空间，所以我做了概念前置，这相当于更改了原书的内容顺序。

题目标为正常数字 **N** 的，为第三版某章某节第 **N** 题（有个别题是第四版又删去的，这里，或直接摘录，或合并简化，仍然作保留；还有个别题是第四版增添条件、设问的，也一并写在第 **N** 题下）。题目标为 '•' 的，为第四版。因为要面向以第三版为主要教材的学习者，所以为了避免混淆，故而将题号（部分题目的实心黑点后有标注具体第四版的数字标号）、甚至章节略去（一些变动过大的章节除外）。题目顺序会有调换，在每章大标题处会交代清楚。除了原书第四版新加入的章节外，均使用原书第三版的索引。这也许对第四版的使用者很不友好，我在此欢迎有心人士将我的作品修改后在同样的 **CC BY NC SA** 条款下作为**衍生作品**发布。

因为使用中文会给我编撰这份笔记带来额外的中英文输入法切换的工作成本，况且对于专业学习者，直接使用英文不会造成任何困扰。但英文词句的冗长性拖慢我编撰/复习的效率，所以我对许多常用术语作了简写。

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# 作者序

我目前还没有能力和资格评论原书好坏以及线性代数课程教材选用的问题。但作为原书的学习者，我可以说：

相较于（其他课程的）其他教材，以 LADR 作为**自学读本**的**精学**计划，往往在执行中出现一次又一次的时间误判/超时，比如我最开始计划 $40 \times 8$h 完成 LADR 的精学，差不多是一天（8h）完成一节，还有额外的复习时间。但在实际学习中，（刨去笔记的功夫）完成到一半时，发现已经耗费了约 $35 \times 8$h，于是我不得不重新估计 LADR 精学所需的总时间为 $70 \times 8$h。这一点对于有学时/学期限制/应试要求的线性代数初学者来说很不安全。更主观地讲，这是因为 LADR 更像是一本参考手册，而不是一本细致入微的自学读本；如果把 LADR 作为初学线性代数第一教材和自学读本来学习，会面临不小的困难。

以上或许能劝退相当一部分打算入门的线性代数初学者。S.Axler 说这本书作为第二遍学习线性代数的教材更合适。我认为理由就是，在校的科班生第二遍学习线性代数时，也已经学习过了离散数学、抽象代数、数论、数学分析等课程，这些知识储备统统会化作一个叫 "mathematical maturity" 的东西，让他们面对 LADR 的课文和习题不再少见多怪、茫然无措。据此，我进一步认为，对于完全的初学者，想要完成 LADR 的精学，要么有很好的天赋，要么有与之相匹配的 "mathematical maturity"，再要么，拿出足够的耐心和毅力。幸运的是，在坚持学习 LADR 的过程中，这三样会一同增益。就我个人来说：课文一次看不懂，就多看几遍，一天看不懂，就分三天看；习题一个小时做不出来，就隔六个小时再尝试，一天做不出来，就隔天再尝试。这确实让我收获了独特的学习体验和能力，我迄今也无法在别处得到，因此我很珍视 LADR，我愿意为此编撰一份电子辅助书并免费公开于网络中。这本身并不花费什么，因为实际的时间开销包括了很多不相干的额外项目：初学 LaTeX、调整代码架构、了解许可证选用，诸如此类的各种波折，也不乏戏剧性。

我在学习过程中碰到了很多重大误区：**第一章中**，我一开始误认为 $W = \complement_V U \cup \{0\}$ 是唯一使得 $W \oplus U = V$ 的子空间，但这压根就不是子空间，而且 C 节习题中也提示这样的子空间 $W$ 不唯一。**第二章中**，我随意地将"线性无关的序列"等同于有/无限维向量空间的基，没有任何理论依据，我也并不懂什么选择公理。**第三章 B 到 D 节中**，我总觉得子空间是超脱有限维的存在；因为放不下第二章无限维向量空间的基的情结，我刻意寻找那些避开涉及基的解法，一些臆测的结论和容易就找到反例。**第三章 E 节中**，我似乎对商空间有什么误解，觉得 $v + U = v' + U$ 如同变戏法一样，把 $v$ 中一切带有 $U$ 的部分抹除掉，让 $v$ 变得纯粹独立于 $U$，为此我还单门发明了 Pure $V/U$ 并试着证明一些命题，甚至用它发现了 F 节 23 题无限维情况下不依赖基的解法。后来我猛然发现我最开始的想法多么荒诞，却仍然放不下 Pure $V/U$ 的情结。这些挫折让我思维变得更加缜密，于是在学习抽象的**第三章 F 节**时比想象中的要顺利。

# Abbreviation Table

## A B

| | |
|---|---|
| add | addi(tion)(tive) |
| algo | algorithm |
| arb | arbitrary |
| assoc | associa(tive)(tivity) |
| asum | assum(e)(ption) |
| becs | because |
| bss | basis |
| bses | bases |
| $B_V$ | basis of $V$ |

## C

| | |
|---|---|
| ch | characteristic |
| closd | closed under |
| coeff | coefficient |
| col | column |
| commu | commut(es)(ing)(ativity) |
| cond | condition |
| corres | correspond(s)(ing) |
| conveni | convenience |
| convly | conversely |
| count- | counter- |
| ctradic | contradict(s)(ion) |
| ctrapos | constrapositive |

## D

| | |
|---|---|
| def | definition |
| deg | degree |
| dep | dependen(t)(ce) |
| deri | derivative(s) |
| diag | diagonal(iza-ble/ility/tion) |
| diff | differentia(l)(ting)(tion) |
| diffce | difference |
| dim | dimension(al) |
| disti | distinct |
| distr | distributive propert(ies)(ty) |
| div | div(ide)(ision) |

## E

| | |
|---|---|
| -ec | -ec(t)(tor)(tion)(tive) |
| eig- | eigen- |
| elem | element(s) |
| ent | entr(y)(ies) |
| equiv | equivalen(t)(ce) |
| exa | example |
| exe | exercise |
| exis | exist(s)(ing) |
| existns | existence |
| expr | expression |

## F G H

| | |
|---|---|
| factoriz | factorizaion |
| fini | finite |
| finide | finite-dimensional |
| G disk | Gershgorin disk |
| homo | homogeneity |
| hypo | hypothesis |

## I

| | |
|---|---|
| id | identity |
| immed | immediately |
| induc | induct(ion)(ive) |
| infily | infinitely |
| inje | injectiv(e)(ity) |
| inv | inver(se)(tib-le/ility) |
| invar | invariant |
| invard | invariant under |
| invarsp | invariant subspace |
| invarspd | invariant subspace under |
| iso | isomorph(ism)(ic) |

## L

| | |
|---|---|
| liney | linear(ly) |
| linity | linearity |
| len | length |
| low- | lower- |

## M N

| | |
|---|---|
| max | maxi(mal(ity))(mum) |
| min | mini(mal(ity))(mum) |
| multi | multipl(e)(icati-on/ve) |
| non0 | nonzero |
| nonC | nonconst |
| notat | notation(al) |

## O P Q

| | |
|---|---|
| optor | operator |
| othws | otherwise |
| poly | polynomial |
| quotient | quot |

## R

| | |
|---|---|
| recurly | recursively |
| repeti | repetition(s) |
| req | require(s)(d)/requiring |
| respectly | respectively |
| restr | restrict(ion)(ive)(ing) |
| rev | revers(e(s))(ed)(ing) |
| rotat | rotation |

## S

| | |
|---|---|
| seq | sequence |
| simlr | similar(ly) |
| solus | solution |
| sp | space |
| stmt | statement |
| std | standard |
| supp | suppose |
| surj | surjectiv(e)(ity) |
| suth | such that |

## T U V W X Y Z

| | |
|---|---|
| trig | triangular |
| trslate | translate |
| trspose | transpose |
| uniq | unique |
| uniqnes | uniqueness |
| up- | upper- |
| val | value |
| -wd | -ward |
| -ws | -wise |
| wrto | with respect to |

# 1·B

**1** *Prove* $\forall v \in V, -(-v) = v$.

**Solus:** $-(-v) = (-1)\big((-1)v\big) = \big((-1)(-1)\big)v = 1 \cdot v = v$.

Or. Becs $-(-v) + (-v) = 0$ 又 $v + (-v) = 0$. Now by the uniqnes of add inv. ☐

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**2** *Supp* $a \in \mathbf{F}, v \in V$, *and* $av = 0$. *Prove* $a = 0$ *or* $v = 0$.

**Solus:** Supp $a \neq 0, \exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. ☐

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**3** *Supp* $v, w \in V$. *Explain why* $\exists\,!\, x \in V, v + 3x = w$.

**Solus:** $v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \frac{1}{3}(w - v)$. ☐

Or. [*Existns*] Let $x = \dfrac{1}{3}(w - v)$.

[*Uniqnes*] If $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) $-$ (II) : $3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. ☐

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**5** *Show in the def of a vecsp, the add inv cond can be replaced by* [1.29].

***Hint***: *Supp V satisfies all conds in the def, except we've replaced the add inv cond with* [1.29]. *Prove the add inv is true.*

**Solus:** Using [1.31]. $0v = 0 \iff (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. ☐

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**6** *Let* $\infty$ *and* $-\infty$ *denote two disti objects, neither of which is in* $\mathbf{R}$.
*Define an add and scalar multi on* $\mathbf{R} \cup \{\infty, -\infty\}$ *as you could guess.*
*The operations of real numbers is as usual. While for* $t \in \mathbf{R}$ *define*

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

(I) $t + \infty = \infty + t = \infty + \infty = \infty$,
(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,
(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

*With these operations of add and scalar multi, is* $\mathbf{R} \cup \{\infty, -\infty\}$ *a vecsp over* $\mathbf{R}$? *Explain.*

**Solus:** Not a vecpsp, since the add and scalar mult is not assoc and distr.

By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

Or. By Distr: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. ☐

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• **Note For Fields:** *Many choices.* [ *Req Multi Inv Uniq* ]

**Exa:** $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \dots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$ *suth* $(m-1)$ *is a prime.*

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**Ended**

# 1·C

**7** *Give a nonempty $U \subseteq \mathbf{R}^2$, $U$ is closd taking add invs and add, but is not a subsp of $\mathbf{R}^2$.*

**Solus:** $\big( 0 \in U;\ v \in U \Rightarrow -v \in U$. And operations on $U$ are the same as $\mathbf{R}^2$. $\big)$ Let $\mathbf{Z}^2, \mathbf{Q}^2$.

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**8** *Give a nonempty $U \subseteq \mathbf{R}^2$, $U$ is closd scalar multi, but is not a subsp of $\mathbf{R}^2$.*

**Solus:** Let $U = \{(x,y) \in \mathbf{R}^2 : x = 0 \vee y = 0\}$.

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• *Supp $U, W, V_1, V_2, V_3$ are subsps of $V$.*

**15** $U + U \ni u + w \in U$. **16** $U + W \ni u + w = w + u \in W + U$. □

**17** $(V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3)$. □

• $(U + W)_{\mathbf{C}} \ni (u_1 + w_1) + \mathrm{i}(u_2 + w_2) = (u_1 + \mathrm{i}u_2) + (w_1 + \mathrm{i}w_2) \in U_{\mathbf{C}} + W_{\mathbf{C}}$. □

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**18** *Does the add on the subsps of $V$ have an add id? Which subsps have add invs?*

**Solus:** Supp $\Omega$ is the uniq add id.

(a) For any subsp $U$ of $V$. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

(b) Now supp $W$ is an add inv of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. □

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**11** *Prove the intersec of every collec of subsps of $V$ is a subsp of $V$.*

**Solus:** Supp $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collec of subsps of $V$; here $\Gamma$ is an index set.

We show $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vecs in each $U_\alpha$, is a subsp of $V$.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closd add.

(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closd scalar multi.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of $V$ that is closd add and scalar multi. □

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• **Note For [1.45]:** If $\mathbf{F} = \{0,1\}$. Prove if $U + W$ is a direct sum, then $U \cap W = \{0\}$.

Becs $\forall v \in U \cap W, \exists !\, (u,w) \in U \times W, v = u + w$.

If $U \cap W \neq \{0\}$, then $(u,w)$ can be $(v,0)$ or $(0,v)$, ctradic the uniqnes. □

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• **Tips 1:** Supp $U, W \subseteq V$. And $U, W, V$ are vecsps $\Rightarrow U, W$ are subsps of $V$.

Then $U + W$ is also a subsp of $V$. Becs $\forall u \in U, w \in U, u + w \in V$ since $u, w \in V$.

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• *Supp $U = \{(x,x,y,y)\}, W = \{(x,x,x,y)\} \subseteq \mathbf{F}^4$. Prove $U + W = \{(x,x,y,z)\}$.*

**Solus:** Let T denote $\{(x,x,y,z)\}$. By def, $U + W \subseteq T$.

And $T \ni (x,x,y,z) \Rightarrow (0,0,y-x,y-x) + (x,x,x,-y+x+z) \in U + W$. Hence $T \subseteq U + W$. □

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**21** *Supp $U = \{(x,y,x+y,x-y,2x)\}$. Find a $W$ suth $\mathbf{F}^5 = U \oplus W$.*

**Solus:** Let $W = \{(0,0,z,w,u)\}$. Then $U \cap W = \{0\}$.

And $\mathbf{F}^5 \ni (x,y,z,w,u) \Rightarrow (x,y,x+y,x-y,2x) + (0,0,z-x-y,w-x-y,u-2x) \in U + W$.

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**22** *Supp* $U = \{(x, y, x+y, x-y, 2x) \in \mathbf{F}^5\}$.
*Find non0 subsps* $W_1, W_2, W_3$ *of* $\mathbf{F}^5$ *suth* $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

**Solus:**

Let $W_1 = \{(0,0,z,0,0) \in \mathbf{F}^5\} \Rightarrow W_1 \cap U = \{0\}$. Now $U \oplus W_1 = \{(x,y,z,x-y,2x) \in \mathbf{F}^5\} = U_1$.
Let $W_2 = \{(0,0,0,w,0) \in \mathbf{F}^5\} \Rightarrow W_2 \cap U_1 = \{0\}$. Now $U_1 \oplus W_2 = \{(x,y,z,w,2x) \in \mathbf{F}^5\} = U_2$.
Let $W_3 = \{(0,0,0,0,u) \in \mathbf{F}^5\} \Rightarrow W_3 \cap U_2 = \{0\}$. Now $U_2 \oplus W_3 = \{(x,y,z,w,u) \in \mathbf{F}^5\} = U_3$.
Thus $\mathbf{F}^5 = \big((U \oplus W_1) \oplus W_2\big) \oplus W_3$. ☐

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**23** *Give an exa of vecsps* $V_1, V_2, U$ *suth* $V_1 \oplus U = V_2 \oplus U$, *but* $V_1 \neq V_2$.

**Solus:** $V = \mathbf{F}^2, U = \{(x,x)\}, V_1 = \{(x,0)\}, V_2 = \{(0,x)\}$.

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• **Note For "$\complement_V U \cup \{0\}$":** "$\complement_V U \cup \{0\}$" is supposed to be a subsp $W$ suth $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cup \{0\} \\ u \pm w \in \complement_V U \cup \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cup \{0\}$. Ctradic.

To fix this, denote the set $\{W_1, W_2, \cdots\}$ by $\mathcal{S}_V U$, where each $W_i \oplus U = V$. See also in (1.C.23).

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• **Tips 2:** *Supp* $V_1 \subseteq V_2$ *in Exe* (23). *Prove* $V_1 = V_2$.

**Solus:** Becs the subset $V_1$ of vecsp $V_2$ is closd add and scalar multi, $V_1$ is a subspace of $V_2$.
Supp $W$ is suth $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$.
If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, ctradic. Hence $W = \{0\}, V_1 = V_2$. ☐

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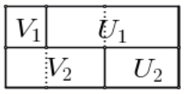
• *Supp* $V_1, V_2, U_1, U_2$ *are vecsps,* $V_1 \oplus U_1 = V_2 \oplus U_2$, $V_1 \subseteq V_2$, $U_2 \subseteq U_1$.
*Prove or give a countexa*: $V_1 = V_2$, $U_1 = U_2$.




**Solus:** Let $U_2 = \{0\}$. Give an exa that each of $V_1, V_2, U_1$ is non0. ☐

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• *Supp the intersec of any two of the vecsps* $U, W, X, Y$ *is* $\{0\}$.
*Give an exa that* $(X \oplus U) \cap (Y \oplus W) \neq \{0\}$.

**Solus:** [ Using notas in Chapter 2. ] Let $B_X = (e_1), B_U = (e_2 - e_1), B_Y = (\ ), B_W = (e_2)$.

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• **Tips 4:** *Let* $V = U + W$, $I = U \cap W$, $U = I \oplus X$, $W = I \oplus Y$. *Prove* $V = I \oplus (X \oplus Y)$.

**Solus:** We show $X \cap Y = U \cap Y = W \cap X = \{0\}$ by ctradic.
$X \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap X \neq \{0\}, I \cap Y \neq \{0\}$.
$U \cap Y = \Delta \neq \{0\} \Rightarrow I = U \cap W \supseteq \Delta \Rightarrow I \cap Y \neq \{0\}$. Simlr for $W \cap X$.
Thus $I + (X + Y) = (I \oplus X) \oplus Y = I \oplus (X \oplus Y)$.
Now we show $V = I + (X + Y)$. $\forall v \in V, v = u + w, \exists\, (u, w) \in U \times W$
$\Rightarrow \exists\, (i_u, x_u) \in I \times X$, $(i_w, y_w) \in I \times Y$, $v = (i_u + i_w) + x_u + y_w \in I + (X + Y)$. ☐

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**12** *Supp* $U, W$ *are subsps of* $V$. *Prove* $U \cup W$ *is a subsp of* $V \iff U \subseteq W$ *or* $W \subseteq U$.

**Solus:** (a) Supp $U \subseteq W$. Then $U \cup W = W$ is a subsp of $V$.
(b) Supp $U \cup W$ is a subsp of $V$. Asum $U \nsubseteq W$, $U \nsupseteq W$ ( $U \cup W \neq U$ and $W$ ).
Then $\forall a \in U \wedge a \notin W, \forall b \in W \wedge b \notin U$, we have $a + b \in U \cup W$.
$a + b \in U \Rightarrow b = (a + b) + (-a) \in U$, ctradic $\Rightarrow W \subseteq U$.
$a + b \in W \Rightarrow a = (a + b) + (-b) \in W$, ctradic $\Rightarrow U \subseteq W$.
Ctradic asum. ☐

---

**13** *Prove the union of three subsps of $V$ is a subsp of $V$ if and only if one of the subsps contains the other two.*

*This exe is not true if we replace* $\mathbf{F}$ *with a field containing only two elems.*

**Solus:** Supp $U_1, U_2, U_3$ are subsps of $V$. Denote $U_1 \cup U_2 \cup U_3$ by $\mathcal{U}$.

(a) Supp that one of the subsps contains the other two. Then $\mathcal{U} = U_1, U_2$ or $U_3$ is a subsp of $V$.

(b) Supp that $U_1 \cup U_2 \cup U_3$ is a subsp of $V$.

Distinctively notice that $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.
Also note that, if $U \cup W = V$ is a vecsp, then in general $U$ and $W$ are not subsps of $V$.
Hence this literal trick is invalid.

(I) If any $U_j$ is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.
By applying Exe (12) we conclude that one $U_j$ contains the other two. Thus done.

(II) Asum no $U_j$ is contained in the union of the other two, and no $U_j$ contains the union of the other two. Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3;\ v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if any $v + \lambda u \in W \cap U_1$ then $v + \lambda u - \lambda u = v \in U_1$.

Now $W \subseteq U_1 \cup U_2 \cup U_3 \Rightarrow W \subseteq U_2 \cup U_3$. $\forall v + \lambda u \in W, v + \lambda u \in U_i, i = 2, 3$.

If $U_2 \subseteq U_3$ or $U_2 \supseteq U_3$, then $\mathcal{U} = U_1 \cup U_i, i = 2, 3$. By Exe (12) done.

Othws, both $U_2, U_3 \neq \{0\}$. Becs $W \subseteq U_2 \cup U_3$ *has at least three elems.*

There must be some $U_i$ that contains at least two elems of $W$.

$\exists$ disti $\lambda_1, \lambda_2 \in \mathbf{F}, v + \lambda_1 u, v + \lambda_2 u \in U_i, i \in \{2, 3\}$.

Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Ctradic. $\square$

**Exa:** Let $\mathbf{F} = \mathbf{Z}_2$. $U_1 = \{u, 0\}, U_2 = \{v, 0\}, U_3 = \{v + u, 0\}$. While $\mathcal{U} = \{0, u, v, v + u\}$ is a subsp.

---

**Ended**

# 2·A

**1** *Prove* $[P]$ $(v_1, v_2, v_3, v_4)$ *spans* $V \iff \big(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\big)$ *also spans* $V$ $[Q]$.

**Solus:** Note that $V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists\, a_1, \dots, a_n \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n$.

Asum $\forall v \in V, \exists\, a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbf{F}$, ( that is, if $\exists\, a_i$, then we are to find $b_i$, vice versa )

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4 \\ &= a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \cdots + a_4)v_4. \end{aligned}$$ □

---

• (4E 3, 14) *Supp* $(v_1, \dots, v_m)$ *is a list in* $V$. *For each* $k$, *let* $w_k = v_1 + \cdots + v_k$.

(a) *Show* $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

(b) *Show* $[P]$ $(v_1, \dots, v_m)$ *is liney indep* $\iff (w_1, \dots, w_m)$ *is liney indep* $[Q]$.

**Solus:**

(a) Asum $a_1 v_1 + \cdots + a_m v_m = b_1 w_1 + \cdots + b_m w_m = b_1 v_1 + \cdots + b_k(v_1 + \cdots + v_k) + \cdots + b_m(v_1 + \cdots + v_m)$.
Then $a_k = b_k + \cdots + b_m$; $\ a_{k+1} = b_{k+1} + \cdots + b_m \Rightarrow b_k = a_k - a_{k+1}$; $b_m = a_m$. Simlr to Exe (1).

(b) $P \Rightarrow Q:\ \ b_1 w_1 + \cdots + b_m w_m = 0 = a_1 v_1 + \cdots + a_m v_m$, where $0 = a_k = b_k + \cdots + b_m$.
$Q \Rightarrow P:\ \ a_1 v_1 + \cdots + a_m v_m = 0 = b_1 w_1 + \cdots + b_m w_m = 0$, where $0 = b_m = a_m$, $0 = b_k = a_k - a_{k+1}$.
Or. By (a), let $W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$. Supp $(v_1, \dots, v_m)$ is liney dep.
By [2.21](b), a list of len $(m-1)$ spans $W$. 又 By [2.23], $(w_1, \dots, w_m)$ liney indep $\Rightarrow m \leqslant m - 1$.
Thus $(w_1, \dots, w_m)$ is liney dep. Now rev the roles of $v$ and $w$. □

---

**2** (a) $[P]$ *A list* $(v)$ *of len* 1 *in* $V$ *is liney indep* $\iff v \neq 0$. $[Q]$
(b) $[P]$ *A list* $(v, w)$ *of len* 2 *in* $V$ *is liney indep* $\iff \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$. $[Q]$

**Solus:** (a) $Q \Rightarrow P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ liney indep.
$P \Rightarrow Q : (v)$ liney indep $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.
$\neg Q \Rightarrow \neg P : v = 0 \Rightarrow av = 0$ while we can let $a \neq 0 \Rightarrow (v)$ is liney dep.
$\neg P \Rightarrow \neg Q : (v)$ liney dep $\Rightarrow av = 0$ while $a \neq 0 \Rightarrow v = 0$.

(b) $P \Rightarrow Q : (v, w)$ liney indep $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.
$Q \Rightarrow P :$ no scalar multi $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v, w)$ liney indep.
$\neg P \Rightarrow \neg Q : (v, w)$ liney dep $\Rightarrow$ if $av + bw = 0$, then $a$ or $b \neq 0 \Rightarrow$ scalar multi.
$\neg Q \Rightarrow \neg P :$ scalar multi $\Rightarrow$ if $av + bw = 0$, then $a$ or $b \neq 0 \Rightarrow$ liney dep. □

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**10** *Supp* $(v_1, \dots, v_m)$ *is liney indep in* $V$ *and* $w \in V$.
*Prove if* $\big(v_1 + w, \dots, v_m + w\big)$ *is linely dep, then* $w \in \operatorname{span}(v_1, \dots, v_m)$.

**Solus:**

Note that $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0 \Rightarrow a_1 v_1 + \cdots + a_m v_m = -(a_1 + \cdots + a_m)w$.
Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1 v_1 + \cdots + a_m v_m = 0$ while $a_i \neq 0$ for some $i$, ctradic.
Or. We prove the ctrapos: Supp $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $a_1 + \cdots + a_m = 0$.
Thus $a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$. Hence $\big(v_1 + w, \dots, v_m + w\big)$ is liney indep. □
Or. $\exists\, j \in \{1, \dots, m\}, v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and done.
If $j \geqslant 2$, then $\exists\, a_i \in \mathbf{F}, v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1 v_1 + \cdots + a_{j-1} v_{j-1}$.
Where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, ctradic.
Now $w = \lambda^{-1}(a_1 v_1 + \cdots + a_{j-1} v_{j-1} - v_j) \Rightarrow w \in \operatorname{span}(v_1, \dots, v_m)$. □

---

**11** *Supp* $(v_1, \dots, v_m)$ *is liney indep in* $V$ *and* $w \in V$.
*Show* $[P]$ $(v_1, \dots, v_m, w)$ *is liney indep* $\iff w \notin \operatorname{span}(v_1, \dots, v_m)$ $[Q]$.

**SOLUS:** Equiv to $(v_1, \dots, v_m, w)$ liney dep $\iff w \in \operatorname{span}(v_1, \dots, v_m)$. Using [2.21]. Obviously. ☐

**NOTE:** (a) Supp $(v_1, \dots, v_m, w)$ is liney indep. Then $(v_1, \dots, v_m)$ liney indep $\iff w \notin \operatorname{span}(v_1, \dots, v_m)$.
(b) Supp $(v_1, \dots, v_m, w)$ is liney dep. Then $(v_1, \dots, v_m)$ liney indep $\iff w \in \operatorname{span}(v_1, \dots, v_m)$.

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**14** *Prove* $[P]$ $V$ *is infinide* $\iff \exists$ *seq* $(v_1, v_2, \dots)$ *in* $V$ *suth each* $(v_1, \dots, v_m)$ *liney indep.* $[Q]$

**SOLUS:**

$P \Rightarrow Q$ : Supp $V$ is infinide, so that no list spans $V$.
**Step 1** Pick a $v_1 \neq 0$, $(v_1)$ liney indep.
**Step m** Pick a $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$, by Exe (11), $(v_1, \dots, v_m)$ is liney indep.
This process recurly defines the desired seq $(v_1, v_2, \dots)$.

$\neg P \Rightarrow \neg Q$ : Supp $V$ is finide and $V = \operatorname{span}(w_1, ..., w_m)$.
Let $(v_1, v_2, \dots)$ be a seq in $V$, then $(v_1, v_2, \dots, v_{m+1})$ must be liney dep.

OR. $Q \Rightarrow P$ : Supp there is such a seq.
Choose an $m$. Supp a liney indep list $(v_1, \dots, v_m)$ spans $V$.
Simlr to [2.16]. $\exists\, v_{m+1} \in V \backslash \operatorname{span}(v_1, \dots, v_m)$. Hence no list spans $V$. ☐

---

**17** *Supp* $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ *suth each* $p_k(2) = 0$.
*Prove* $(p_0, p_1, \dots, p_m)$ *is not liney indep in* $\mathcal{P}_m(\mathbf{F})$.

**SOLUS:**

Supp $(p_0, p_1, \dots, p_m)$ is liney indep. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z$.
NOTICE that $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \dots + a_m p_m(z)$, for if not, let $z = 2$. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has len $(m+1)$.
Hence $(p_0, p_1, \dots, p_m)$ is linely dep. For if not, then becs $(1, z, \dots, z^m)$ of len $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$, by the steps in [2.23] trivially, $(p_0, p_1, \dots, p_m)$ of len $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$. Ctradic. ☐

OR. Note that $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(\underbrace{1, z, \dots, z^m}_{\text{of len } (m+1)})$. Then $(p_0, p_1, \dots, p_m, z)$ of len $(m+2)$ is liney dep.
As shown above, $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$. And hence by [2.21](a), $(p_0, p_1, \dots, p_m)$ is liney dep. ☐

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**ENDED**

# 2·B

- **Tips:** *Supp* $\dim V = n$, *and* $U$ *is a subsp of* $V$ *with* $U \neq V$.
  *Prove* $\exists\, B_V = (v_1, \dots, v_n)$ *suth each* $v_k \notin U$.

  Note that $U \neq V \Rightarrow n \geqslant 1$. We will construct $B_V$ via the following process.
  **Step 1.** $\exists\, v_1 \in V \backslash U \Rightarrow v_1 \neq 0$. If $\operatorname{span}(v_1) = V$ then we stop.
  **Step k.** Supp $(v_1, \dots, v_{k-1})$ is liney indep in $V$, each of which belongs to $V \backslash U$.
  Note that $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$. And if $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U = V$, then by (1.C.12),
  $\big[$ becs $\operatorname{span}(v_1, \dots, v_{k-1}) \not\subseteq U$, $\big]$ $U \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) \Rightarrow \operatorname{span}(v_1, \dots, v_{k-1}) = V$.
  Hence becs $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$, it must be case that $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U \neq V$.
  Thus $\exists\, v_k \in V \backslash U$ suth $v_k \notin \operatorname{span}(v_1, \dots, v_{k-1})$.
  By (2.A.11), $(v_1, \dots, v_k)$ is liney indep in $V$. If $\operatorname{span}(v_1, \dots, v_k) = V$, then we stop.
  Becs $V$ is finide, this process will stop after $n$ steps. $\square$

  Or. Supp $U \neq \{0\}$. Let $B_U = (u_1, \dots, u_m)$. Extend to a bss $(u_1, \dots, u_n)$ of $V$.
  Then let $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$. $\square$

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- (4E 9) *Supp* $(v_1, \dots, v_m)$ *is a list in* $V$. *For* $k \in \{1, \dots, m\}$, *let* $w_k = v_1 + \cdots + v_k$.
  *Show* $[P]\ B_V = (v_1, \dots, v_m) \Longleftrightarrow B_V = (w_1, \dots, w_m)$. $[Q]$

**Solus:** Notice that $B_U = (u_1, \dots, u_n) \Longleftrightarrow \forall u \in U, \exists\,!\, a_i \in \mathbf{F}, u = a_1 u_1 + \cdots + a_n u_n$.
$P \Rightarrow Q: \forall v \in V, \exists\,!\, a_i \in \mathbf{F},\ v = a_1 v_1 + \cdots + a_m v_m \Rightarrow v = b_1 w_1 + \cdots + b_m w_m, \exists\,!\, b_k = a_k - a_{k+1}, b_m = a_m$.
$Q \Rightarrow P: \forall v \in V, \exists\,!\, b_i \in \mathbf{F},\ v = b_1 w_1 + \cdots + b_m w_m \Rightarrow v = a_1 v_1 + \cdots + a_m v_m, \exists\,!\, a_k = \sum_{j=k}^{m} b_j$. $\square$

**Comment:** Or. Using $\big[$3.C Note For [3.30, 32](a)$\big]$.

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- (4E 5) *Supp* $U, W$ *are finide*, $V = U + W$, $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$.
  *Prove* $\exists\, B_V$ *consisting of vecs in* $U \cup W$.

**Solus:** $V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}\big(\overbrace{u_1, \dots, u_m, w_1, \dots, w_n}^{\text{Reduce}}\big)$. By [2.31]. $\square$

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**8** *Supp* $V = U \oplus W$, $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$.
*Prove* $B_V = \big(u_1, \dots, u_m, w_1, \dots, w_n\big)$.

**Solus:** $\forall v \in V, \exists\,!\, u \in U, w \in W \Rightarrow \exists\,!\, a_i, b_i \in \mathbf{F}, v = u + w = \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i$.
Or. $V = \operatorname{span}(u_1, \dots, u_m) \oplus \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}\big(u_1, \dots, u_m, w_1, \dots, w_n\big)$.
Note that $\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0 \Rightarrow \sum_{i=1}^{m} a_i u_i = -\sum_{i=1}^{n} b_i w_i \in U \cap W = \{0\}$. $\square$

---

- **Note For** ***liney indep seq and*** **[2.34]:** "$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expr.
  If we allow using "infini list", then we must assure that $(v_1, \dots, v_n, \dots)$ is a spanning "list"
  suth $\forall v \in V, \exists$ smallest $n \in \mathbf{N}^+$, $v = a_1 v_1 + \cdots + a_n v_n$. Moreover, given a list $(w_1, \cdots, w_n, \cdots)$ in $W$,
  we can prove $\exists\,!\, T \in \mathcal{L}(V, W)$ with each $T v_k = w_k$, which has less restr than [3.5].
  But the key point is, how can we assure that such a "list" exis. TODO: More details.

---

- (9.A.3,4 OR 4E 11) *Supp $V$ is on $\mathbf{R}$, and $v_1, \dots, v_n \in V$. Let $B = (v_1, \dots, v_n)$.*
  (a) *Show $[P]$ $B$ is liney indep in $V \iff B$ is liney indep in $V_{\mathbf{C}}$. $[Q]$*
  (b) *Show $[P]$ $B$ spans $V \iff B$ spans $V_{\mathbf{C}}$. $[Q]$*

  (a) $P \Rightarrow Q:$ Note that each $v_k \in V_{\mathbf{C}}$. $\quad Q \Rightarrow P:$ If $\lambda_k \in \mathbf{R}$ with $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then each $\mathrm{Re}\lambda_k = \lambda_k = 0$.
  $\neg P \Rightarrow \neg Q: \ \exists\, v_j = a_{j-1}v_{j-1} + \cdots + a_1 v_1 \in V_{\mathbf{C}}$.
  $\neg Q \Rightarrow \neg P: \ \exists\, v_j = \lambda_{j-1}v_{j-1} + \cdots + \lambda_1 v_1 \Rightarrow v_j = (\mathrm{Re}\lambda_{j-1})v_{j-1} + \cdots + (\mathrm{Re}\lambda_1)v_1 \in V$.

  (b) $P \Rightarrow Q: \ \forall u + \mathrm{i}v \in V_{\mathbf{C}},\ u, v \in V \Rightarrow \exists\, a_i, b_i \in \mathbf{R}, u + \mathrm{i}v = \sum_{i=1}^{n} (a_i + \mathrm{i}b_i)v_i$.
  $Q \Rightarrow P: \ \forall v \in V, \exists\, a_i + \mathrm{i}b_i \in \mathbf{C},\ v + \mathrm{i}0 = \left(\sum_{i=1}^{n} a_i v_i\right) + \mathrm{i}\left(\sum_{i=1}^{n} b_i v_i\right) \Rightarrow v \in \operatorname{span}(v_1, \dots, v_m)$.
  $\neg Q \Rightarrow \neg P: \ \exists\, v \in V, v \notin \operatorname{span}(B) \Rightarrow v + \mathrm{i}0 \notin \operatorname{span}(B)$ while $v + \mathrm{i}0 \in V_{\mathbf{C}}$.
  $\neg Q \Rightarrow \neg P: \ \exists\, u + \mathrm{i}v \in V_{\mathbf{C}}, u + \mathrm{i}v \notin \operatorname{span}(B) \Rightarrow u$ or $v \notin \operatorname{span}(B)$. Note that $u, v \in V$. ☐

---

**1** *Find all vecsps on whatever $\mathbf{F}$ that have exactly one bss.*

**SOLUS:** The trivial vecsp $\{0\}$ will do. Indeed, the only bss of $\{0\}$ is the empty list ( ).
Now consider the field $\{0, 1\}$ containing only the add id and multi id,
with $1 + 1 = 0$. Then the list $(1)$ is the uniq bss. Now the vecsp $\{0, 1\}$ will do.
**COMMENT:** All vecsp on such $\mathbf{F}$ of dim 1 will do.
Consider other $\mathbf{F}$. Note that this $\mathbf{F}$ contains at least and strictly more than 0 and 1. Failed. ☐

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**ENDED**

# 2·C

**15** *Supp* $\dim V = n \geqslant 1$. *Prove $\exists$ 1-dim subsps $V_1, \dots, V_n$ suth $V = V_1 \oplus \cdots \oplus V_n$.*

**SOLUS:** Supp $B_V = (v_1, \dots, v_n)$. Let each $V_i = \operatorname{span}(v_i)$.
Then $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n \Rightarrow \exists!\, u_i \in V_i, v = u_1 + \cdots + u_n$ ☐

---

- **NOTE FOR Exe (15):** *Supp $v \in V \setminus \{0\}$. Prove $\exists\, B_V = (v_1, \dots, v_n), v = v_1 + \cdots + v_n$.*

**SOLUS:** If $n = 1$ then let $v_1 = v$ and done. Supp $n > 1$.
Extend $(v)$ to a bss $(v, v_1, \dots, v_{n-1})$ of $V$. Let $v_n = v - v_1 - \cdots - v_{n-1}$.
又 $\operatorname{span}(v, v_1, \dots, v_{n-1}) = \operatorname{span}(v_1, \dots, v_n)$. Hence $(v_1, \dots, v_n)$ is also a bss of $V$. ☐

**COMMENT:** Let $B_V = (v_1, \dots, v_n)$ and supp $v = u_1 + \cdots + u_n$, where each $u_i = a_i v_i \in V_i$.
But $(u_1, \dots, u_n)$ might not be a bss, becs there might be some $u_i = 0$.

---

**1** [CORO for [2.38,39]] *Supp $U$ is a subsp of $V$ suth* $\dim V = \dim U$. *Then $V = U$.*

**SOLUS:** Let $B_U = (u_1, \dots, u_m)$. Then $m = \dim V$. 又 $u_i \in V$. By [2.39], $B_V = (u_1, \dots, u_m)$. ☐

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- Let $v_1, \dots, v_n \in V$ and $\dim \operatorname{span}(v_1, \dots, v_n) = n$. Then $(v_1, \dots, v_n)$ is a bss of $\operatorname{span}(v_1, \dots, v_n)$.
  *Notice that $(v_1, \dots, v_n)$ is a spanning list of* $\operatorname{span}(v_1, \dots, v_n)$ *of len* $n = \dim \operatorname{span}(v_1, \dots, v_n)$.

---

**9** *Supp $(v_1, \dots, v_m)$ is liney indep in $V, w \in V$. Prove* $\dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$.

**SOLUS:** Using the result of (2.A.10, 11).
Note that each $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$.
$(v_1, \dots, v_m)$ liney indep $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ liney indep $\Rightarrow \underbrace{(v_2 - v_1, \dots, v_m - v_1)}_{\text{of len } (m-1)}$ liney indep.
又 If $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1 + w, \dots, v_m + w)$ is liney indep.
Hence $m \geqslant \dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$. ☐

• (4E 16) *Supp $V$ is finide, $U$ is a subsp of $V$ with $U \neq V$. Let $n = \dim V, m = \dim U$.*

*Prove $\exists\ (n-m)$ subsps $U_1, \dots, U_{n-m}$, each of dim $(n-1)$, suth $\bigcap\limits_{i=1}^{n-m} U_i = U$.*

**Solus:** Let $B_U = (v_1, \dots, v_m)$, $B_V = (v_1, \dots, v_m, u_1, \dots, u_{n-m})$.

Define each $U_i = \operatorname{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m}) \Rightarrow U \subseteq U_i$.

And becs $\forall v \in \bigcap\limits_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \cdots + b_{n-m} u_{n-m} \in U_i \Rightarrow$ each $b_i = 0 \Rightarrow v \in U$.

Hence $\bigcap\limits_{i=1}^{n-m} U_i \subseteq U$. $\square$

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**14** *Supp $V_1, \dots, V_m$ are finide. Prove* $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$.

**Solus:** For each $V_i$, let $B_{V_i} = \mathcal{E}_i$. Then $V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim V_i = \operatorname{card} \mathcal{E}_i$.

Now $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.

**Coro:** $V_1 + \cdots + V_m$ is direct

$\iff$ For each $k \in \{1, \dots, m-1\}$, $(V_1 \oplus \cdots \oplus V_k) \cap V_{k+1} = \{0\}$, $(\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$

$\iff \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$

$\iff \dim(V_1 \oplus \cdots \oplus V_m) = \dim V_1 + \cdots + \dim V_m$. $\square$

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**17** *Supp $V_1, V_2, V_3$ are subsps of a finide vecsp, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

*Explain why you might think and prove the formula above or give a countexa.*

**Solus:**

[ *Simlr to* ] Given three sets $A, B$ and $C$.

Becs $|X \cup Y| = |X| + |Y| - |X \cap Y|$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

Now $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$.

And $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$.

Hence $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$.

Note that $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned}\dim(V_1 + V_2 + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \quad (1)\\ &= \dim(V_2 + V_3) + \dim(V_1) - \dim((V_2 + V_3) \cap V_1) \quad (2)\\ &= \dim(V_1 + V_3) + \dim(V_2) - \dim((V_1 + V_3) \cap V_2) \quad (3).\end{aligned}$$

Notice that in general, $(X + Y) \cap Z \neq (X \cap Z) + (Y \cap Z)$.

For exa, $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

**Comment:** If $X \subseteq Y$, then $(X + Y) \cap Z = Y \cap Z$; $\dim(X + Y + Z) = \dim(Y) + \dim(Z) - \dim(Y \cap Z)$, and the wrong formual holds. Simlr for $Y \subseteq Z, X \subseteq Z$, and $X, Y \subseteq Z$.

However, it's true that $(X + Y) \cap Z \supseteq (X \cap Z) + (Y \cap Z) = (X + (Y \cap Z)) \cap Z$.

Becs $(X \cap Z) + (Y \cap Z) \ni v = x + y = z_1 + z_2 \in (X + (Y \cap Z)) \cap Z \Rightarrow v \in (X + Y) \cap Z$.

Where $\exists\, x = z_1 \in X \cap Z, y = z_2 \in Y \cap Z$.

**Comment:** $\dim((X + Y) \cap Z) \geqslant \dim(X \cap Z) + \dim(Y \cap Z) - \dim(X \cap Y \cap Z)$.

• **Coro:**
$$\begin{aligned}\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 &- [\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)]/3\\ &- [\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)]/3.\end{aligned}$$

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- **Tips:** Becs $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big)$.
  And $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$. We have (1), and (2), (3) simlr.
  (1) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 + V_3) - \dim\big(V_1 + (V_2 \cap V_3)\big)$.
  (2) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_3) - \dim\big(V_2 + (V_1 \cap V_3)\big)$.
  (3) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim\big(V_3 + (V_1 \cap V_2)\big)$.

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- *Supp $V_1, V_2, V_3$ are subsps of $V$ with*
  (a) $\dim V = 10,\ \dim V_1 = \dim V_2 = \dim V_3 = 7$. *Prove* $V_1 \cap V_2 \cap V_3 \neq \{0\}$.
  By Tips, $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 \underline{- 2\dim V} > 0$.
  (b) $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. *Prove* $V_1 \cap V_2 \cap V_3 \neq \{0\}$.
  By Tips, $\dim(V_1 \cap V_2 \cap V_3) \underline{\underline{\geq 2\dim V}} - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)) \geq 0$. $\square$

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**Ended**

# 3·A

- (3.E.1) *A function $T : V \to W$ is liney $\iff$ The graph of $T$ is a subspace of $V \times W$.*

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- **Tips 1:** $T : V \to W$ is liney $\iff \left| \begin{array}{l} (\text{一})\ \forall v,u \in V, T(v+u) = Tv + Tu; \\ (\text{二})\ \forall v,u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv). \end{array} \right| \iff T(v+\lambda u) = Tv + \lambda Tu.$

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- (9.A.2,6 Or 4E 3.B.33) *Supp that $V, W$ are on $\mathbf{R}$, and $T \in \mathcal{L}(V,W)$. Show*
  (a) $T_C \in \mathcal{L}(V_C, W_C)$. (b) $\operatorname{null}(T_C) = (\operatorname{null} T)_C, \operatorname{range}(T_C) = (\operatorname{range} T)_C$. (c) $T_C$ is inv $\iff T$ is inv.

**Solus:** (a) $T_C\big((u_1 + \mathrm{i}v_1) + (x+\mathrm{i}y)(u_2 + \mathrm{i}v_2)\big) = T(u_1 + xu_2 - yv_2) + \mathrm{i}T(v_1 + xv_2 + yu_2)$
$= T_C(u_1 + \mathrm{i}v_1) + (x+\mathrm{i}y)T_C(u_2 + \mathrm{i}v_2)$.
(b) $u + \mathrm{i}v \in \operatorname{null}(T_C) \iff u, v \in \operatorname{null} T \iff u + \mathrm{i}v \in (\operatorname{null} T)_C$.
$w + \mathrm{i}x \in \operatorname{range}(T_C) \iff w, x \in \operatorname{range} T \iff w + \mathrm{i}x \in (\operatorname{range} T)_C$.
(c) $\forall w, x \in W, \exists!\, u, v \in V, T_C(u + \mathrm{i}v) = w + \mathrm{i}x \iff Tu = w, Tv = x$. Or. By (b). $\square$

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- (9.A.5) *Supp $V$ is on $\mathbf{R}$, and $S, T \in \mathcal{L}(V,W)$. Prove $(S+\lambda T)_C = S_C + \lambda T_C$.*

**Solus:** $(S+\lambda T)_C(u + \mathrm{i}v) = (S+\lambda T)(u) + \mathrm{i}(S+\lambda T)(v)$
$= Su + \mathrm{i}Sv + \lambda(Tu + \mathrm{i}Tv) = (S_C + \lambda T_C)(u + \mathrm{i}v)$. $\square$

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- *Supp $U, V, W$ are on $\mathbf{R}$, $S \in \mathcal{L}(V,W), T \in \mathcal{L}(U,V)$. Prove $(ST)_C = S_C T_C$.*

**Solus:** $\forall u + \mathrm{i}x \in U_C, (ST)_C(u + \mathrm{i}x) = STu + \mathrm{i}STx = S_C(Tu + \mathrm{i}Tx) = (S_C T_C)(u + \mathrm{i}x)$. $\square$

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- (4E 4.3) *Supp $\mathbf{F} = \mathbf{C}$, $\varphi \in \mathcal{L}(V,\mathbf{F})$, $\sigma = \operatorname{Re} \circ \varphi$. Show all $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$.*

**Solus:** $\varphi(v) = \sigma(v) + \mathrm{i}\operatorname{Im}\varphi(v)$. 又 $\operatorname{Re}\varphi(\mathrm{i}v) = \operatorname{Re}(\mathrm{i}\varphi(v)) = -\operatorname{Im}\varphi(v) = \sigma(\mathrm{i}v)$. $\square$

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- **Note For Restr:** *$U$ is a subsp of $V$.*
  (a) $\forall S, T \in \mathcal{L}(V,W), \lambda \in \mathbf{F}$, $(T+\lambda S)|_U = T|_U + \lambda S|_U$.
  (b) $\forall S \in \mathcal{L}(W,X), T \in \mathcal{L}(V,W)$, $(ST)|_U = ST|_U$.

---

- (4E 1.B.7) *Supp $V \neq \emptyset$ and $W$ is a vecsp. Let $W^V = \{f : V \to W\}$.*
  (a) *Define a natural add and scalar multi on $W^V$.* (b) *Prove $W^V$ is a vecsp with these defs.*

**Solus:**
(a) $W^V \ni f + g : x \to f(x) + g(x)$; where $f(x) + g(x)$ is the vec add on $W$.
$W^V \ni \lambda f : x \to \lambda f(x)$; where $\lambda f(x)$ is the scalar multi on $W$.
(b) Commu: $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$.
Assoc: $\big((f+g)+h\big)(x) = \big(f(x)+g(x)\big) + h(x)$
$= f(x) + \big(g(x) + h(x)\big) = \big(f + (g+h)\big)(x)$.
Add Id: $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$.
Add Inv: $(f+g)(x) = f(x) + g(x) = f(x) + \big(-f(x)\big) = 0 = 0(x)$.
Multi Id: $(1f)(x) = 1f(x) = f(x)$. ( Notice that the smallest $\mathbf{F}$ is $\{0,1\}$. )
Distr: $\big(a(f+g)\big)(x) = a(f+g)(x) = a\big(f(x)+g(x)\big)$
$= af(x) + ag(x) = (af)(x) + (ag)(x) = (af+ag)(x)$.
Simlr, $\big((a+b)f\big)(x) = (af+bf)(x)$.
So far, we have used the same properties in $W$. [***If** $W^V$ **is a vecsp, then** $W$ **must be a vecsp.***] $\square$

• **TIPS 2:** $T \in \mathcal{L}(V,W) \iff T \in \mathcal{L}(V, \operatorname{range} T) \iff T \in \mathcal{L}(V,U)$, if range$T$ is a subsp of $U$.
**CORO:** $\{T \in \mathcal{L}(V,W) : \operatorname{range} T \subseteq U\} = \{T \in \mathcal{L}(V,U)\} = \mathcal{L}(V,U)$.

---

**5** *Becs* $\mathcal{L}(V,W) = \{T : V \to W \mid T \text{ is liney}\}$ *is a subsp of* $W^V$, $\mathcal{L}(V,W)$ *is a vecsp.*

---

**3** *Supp* $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. *Prove* $\exists A_{j,k} \in \mathbf{F}$ *suth for any* $(x_1, \dots, x_n) \in \mathbf{F}^n$,

$$T(x_1, \dots, x_n) = \begin{pmatrix} A_{1,1}x_1 + \cdots + A_{1,n}x_n, \\ \vdots \quad\quad \ddots \quad\quad \vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n \end{pmatrix}$$

**SOLUS:**

Let $T(1,0,0,\dots,0,0) = (A_{1,1},\dots,A_{m,1})$, Note that $(1,0,\dots,0,0), \cdots, (0,0,\dots,0,1)$ is a bss of $\mathbf{F}^n$.
$T(0,1,0,\dots,0,0) = (A_{1,2},\dots,A_{m,2})$, Then by [3.5], done. $\square$
$\vdots$
$T(0,0,0,\dots,0,1) = (A_{1,n},\dots,A_{m,n})$.

---

**4** *Supp* $T \in \mathcal{L}(V,W)$, *and* $v_1, \dots, v_m \in V$ *suth* $(Tv_1, \dots, Tv_m)$ *is liney indep in* $W$.
*Prove* $(v_1, \dots, v_m)$ *is liney indep.*

**SOLUS:** Supp $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$. $\square$

---

**7** *Show every liney map from a* 1*-dim vecsp to itself is a multi by some scalar.*
*More precisely, prove if* $\dim V = 1$ *and* $T \in \mathcal{L}(V)$, *then* $\exists \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$.

**SOLUS:** Let $u$ be a non0 vec in $V \Rightarrow V = \operatorname{span}(u)$. Becs $Tu \in V \Rightarrow Tu = \lambda u$ for some $\lambda$.
Supp $v \in V \Rightarrow v = au$, $\exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. $\square$

---

**8** *Give a map* $\varphi : \mathbf{R}^2 \to \mathbf{R}$ *suth* $\forall a \in \mathbf{R}, v \in \mathbf{R}^2, \varphi(av) = a\varphi(v)$ *but* $\varphi$ *is not liney.*

**SOLUS:** Define $T(x,y) = \begin{cases} x+y, & \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, & \text{othws.} \end{cases}$ OR. Define $T(x,y) = \sqrt[3]{(x^3+y^3)}$. $\square$

---

**9** *Give a map* $\varphi : \mathbf{C} \to \mathbf{C}$ *suth* $\forall w,z \in \mathbf{C}, \varphi(w+z) = \varphi(w) + \varphi(z)$ *but* $\varphi$ *is not liney.*

**SOLUS:** Define $\varphi(u + \mathrm{i}v) = u = \operatorname{Re}(u+\mathrm{i}v)$ OR. Define $\varphi(u+\mathrm{i}v) = v = \operatorname{Im}(u+\mathrm{i}v)$. $\square$

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• *Prove if* $q \in \mathcal{P}(\mathbf{R})$ *and* $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ *is defined by* $\underbrace{Tp = q \circ p}_{\text{composition}}$, *then* $T$ *is not liney.*

**SOLUS:** ***Composition and product are not the same in*** $\mathcal{P}(\mathbf{F})$.

NOTICE that $(p \circ q)(x) = p(q(x))$, while $(pq)(x) = p(x)q(x) = q(x)p(x)$.
Becs in general, $[q \circ (p_1 + \lambda p_2)](x) = q(p_1(x) + \lambda p_2(x)) \neq (qp_1)(x) + \lambda(qp_2)(x)$.
**EXA:** Let $q$ be defined by $q(x) = x^2$, then $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$. $\square$

---

**10** *Supp* $U$ *is a subsp of* $V$ *with* $U \neq V$.
*Supp* $S \in \mathcal{L}(U,W)$ *with* $S \neq 0$. *Define* $T : V \to W$ *by* $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \backslash U. \end{cases}$
*Prove* $T$ *is not a liney map on* $V$.

**SOLUS:** Asum $T$ is a liney map. Supp $v \in V \backslash U$, $u \in U$ suth $Su \neq 0$.
Then $v + u \in V \backslash U$, for if not, $v = (v+u) - u \in U$;
while $T(v+u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$. Ctradic. $\square$

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**11** *Supp $U$ is a subsp of $V$ and $S \in \mathcal{L}(U, W)$.*
*Prove $\exists\, T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U$.* ( OR. $\exists\, T \in \mathcal{L}(V, W), T|_U = S$. )
*In other words, every liney map on a subsp of $V$ can be **extended** to a liney map on the entire $V$.*

**SOLUS:** Supp $W$ is suth $V = U \oplus W$. Then $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.
Define $T \in \mathcal{L}(V, W)$ by $T(u_v + w_v) = Su_v$. ☐
OR. [ *Finide Req* ] Define by $T\left(\sum_{i=1}^{m} a_i u_i\right) = \sum_{i=1}^{n} a_i S u_i$. Let $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$. ☐

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**12** *Supp non0 $V$ is finide and $W$ is infinide. Prove $\mathcal{L}(V, W)$ is infinide.*

**SOLUS:** Using (2.A.14).
Let $B_V = (v_1, \dots, v_n)$ be a bss of $V$. Let $(w_1, \dots, w_m)$ be liney indep in $W$ for any $m \in \mathbf{N}^+$.
Define $T_{x,y} : V \to W$ by $T_{x,y}(v_z) = \delta_{z,x} w_y$, $\forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$
$\forall v = \sum_{i=1}^{n} a_i v_i,\ u = \sum_{i=1}^{n} b_i v_i,\ \lambda \in \mathbf{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) w_y = T_{x,y}(v) + \lambda T_{x,y}(u)$.
Linity checked. Now supp $a_1 T_{x,1} + \dots + a_m T_{x,m} = 0$.
Then $(a_1 T_{x,1} + \dots + a_m T_{x,m})(v_x) = 0 = a_1 w_1 + \dots + a_m w_m \Rightarrow a_1 = \dots = a_m = 0$. 又 $m$ arb.
Thus $(T_{x,1}, \dots, T_{x,m})$ is a liney indep list in $\mathcal{L}(V, W)$ for any $x$ and len $m$. Hence by (2.A.14). ☐

---

**13** *Supp $(v_1, \dots, v_m)$ is linely dep in $V$ and $W \neq \{0\}$.*
*Prove $\exists\, w_1, \dots, w_m \in W, \nexists\, T \in \mathcal{L}(V, W)$ suth $Tv_k = w_k, \forall k = 1, \dots, m$.*

**SOLUS:**
We prove by ctradic. By liney dep lemma, $\exists\, j \in \{1, \dots, m\}, v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$.
Supp $a_1 v_1 + \dots + a_m v_m = 0$, where $a_j \neq 0$. Now let $w_j \neq 0$, while $w_1 = \dots = w_{j-1} = w_{j+1} = w_m = 0$.
Define $T \in \mathcal{L}(V, W)$ with each $Tv_k = w_k$. Then $T(a_1 v_1 + \dots + a_m v_m) = 0 = a_1 w_1 + \dots + a_m w_m$.
And $0 = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Ctradic. ☐

OR. We prove the ctrapos: Supp $\forall w_1, \dots, w_m \in W, \exists\, T \in \mathcal{L}(V, W)$, each $Tv_k = w_k$.
Now we show $(v_1, \dots, v_n)$ is liney indep. Supp $\exists\, a_i \in \mathbf{F}, a_1 v_1 + \dots + a_n v_n = 0$.
Choose one $w \in W \backslash \{0\}$. By asum, for $(\overline{a_1} w, \dots, \overline{a_m} w), \exists\, T \in \mathcal{L}(V, W)$, each $Tv_k = \overline{a_k} w$.
Now we have $0 = T\left(\sum_{k=1}^{m} a_k v_k\right) = \sum_{k=1}^{m} a_k T v_k = \sum_{k=1}^{m} a_k \overline{a_k} w = \left(\sum_{k=1}^{m} |a_k|^2\right) w$.
Then $\sum_{k=1}^{m} |a_k|^2 = 0$. Thus $a_1 = \dots = a_m = 0$. Hence $(v_1, \dots, v_n)$ is liney indep. ☐

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• (4E 17) *Supp $V$ is finide. Show all two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*
*A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.*

**SOLUS:** If $\mathcal{E} = \{0\}$, then done. Supp $0 \neq S \in \mathcal{E}$, a two-sided ideal of $\mathcal{L}(V)$. Let $B_V = (v_1, \dots, v_n)$.
Define $R_{x,y} \in \mathcal{L}(V) : v_x \mapsto v_y,\ v_z \mapsto 0\,(z \neq x)$. OR. $R_{x,y} v_z = \delta_{z,x} v_y$. Asum each $R_{x,y} \in \mathcal{E}$.
Then $(R_{1,1} + \dots + R_{n,n}) v_j = v_j \Rightarrow \sum_{r=1}^{n} R_{r,r} = I \Rightarrow \mathcal{L}(V) \ni T = I \circ T = T \circ I \in \mathcal{E}$.
OR. Let each $Tv_j = w_j = A_{1,j} v_1 + \dots + A_{n,j} v_n \Rightarrow T = \sum_{x=1}^{n} \sum_{y=1}^{n} A_{y,x} R_{x,y} \in \mathcal{E}$. Now we prove the asum.
Supp $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \dots + a_n v_n$, where $a_k \neq 0$.
For all $x, y \in \{1, \dots, n\}$, $(R_{k,y} S) v_i = a_k v_y \Rightarrow \left((R_{k,y} S) \circ R_{x,i}\right) v_z = \delta_{z,x} (a_k v_y)$.
Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Now $S \in \mathcal{E} \Rightarrow R_{k,y} S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}$. ☐

**COMMENT:** Not true if infinide. Consider the subsp $X = \{T \in \mathcal{L}(V) : \operatorname{range} T \text{ is finide}\}$.
For any $T \in X$, $\forall E \in \mathcal{L}(V), \operatorname{range} TE \subseteq \operatorname{range} T$; $\operatorname{range} ET = \operatorname{span}(Ew_1, \dots, Ew_n) \Rightarrow TE, ET \in X$.

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- (4E 3.B.32) *Supp* $\dim V = n$. *Supp* $\varphi : \mathcal{L}(V) \to \mathbf{F}$ *is liney.*
  *Show if* $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$, *then* $\varphi = 0$.

**Solus:** Using notas in (4E 17). Using the result in Note For [3.60].

Supp $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \dots, n\}, \varphi(R_{i,j}) \neq 0$. Becs $R_{i,j} = R_{x,j} \circ R_{i,x},\ \forall x = 1, \dots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, becs $R_{i,x} = R_{y,x} \circ R_{i,y},\ \forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0,\ \forall x, y = 1, \dots, n$.

Let $k \neq i, j \neq l$ and then $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Ctradic. □

Or. Note that by (4E 3.A.17), $\exists\, S, T \in \mathcal{L}(V), ST - TS \neq 0$.

Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \operatorname{null}\varphi \neq \{0\}$.

Note that $\forall\, E \in \operatorname{null}\varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \operatorname{null}\varphi$.

Hence $\operatorname{null}\varphi$ is a non0 two-sided ideal of $\mathcal{L}(V)$. □

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- *Supp* $V$ *is finide,* $T \in \mathcal{L}(V)$ *is suth* $\forall S \in \mathcal{L}(V), ST = TS$. *Prove* $\exists \lambda \in \mathbf{F}, T = \lambda I$.

**Solus:** If $V = \{0\}$, then done. Now supp $V \neq \{0\}$.

Asum $\forall v \in V, (v, Tv)$ is linely dep, then $\exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

To prove $\lambda_v$ is indep of $v$, we discuss in two cases:

$$\left.\begin{array}{l} (-)\text{ If } (v,w) \text{ is liney indep, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \\ \qquad\qquad\qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=)\text{ Othws, supp } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w \end{array}\right\} \Rightarrow \lambda_w = \lambda_v.$$

Now we prove the asum. Asum $\exists\, v \in V, (v, Tv)$ is liney indep. Let $B_V = (v, Tv, u_1, \dots, u_n)$.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Ctradic. □

Or. Let $B_V = (v_1, \dots, v_m)$. Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$.

Supp $v \in V$. Define $S_v \in \mathcal{L}(V)$ by $S_v(u) = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. □

Or. For each $k \in \{1, \dots, n\}$, define $S_k \in \mathcal{L}(V)$ by $S_k v_j = \begin{cases} v_k, & j = k, \\ 0, & j \neq k. \end{cases}$ Or. $S_k v_j = \delta_{j,k} v_k$

Note that $S_k\left(\sum_{i=1}^n a_i v_i\right) = a_k v_k$. Then $S_k v = v \iff \exists\,!\, a_k \in \mathbf{F}, v = a_k v_k$.

Hence $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$.

Define $A^{(j,k)} \in \mathcal{L}(V)$ by $A^{(j,k)}v_j = v_k,\ A^{(j,k)}v_k = v_j,\ A^{(j,k)}v_x = 0, x \neq j, k$.

Then $\left|\begin{array}{r} A^{(j,k)}Tv_j = TA^{(j,k)}v_j = Tv_k = a_k v_k \\ A^{(j,k)}Tv_j = A^{(j,k)}a_j v_j = a_j A^{(j,k)}v_j = a_j v_k \end{array}\right\} \Rightarrow a_k = a_j$. Hence $a_k$ is indep of $v_k$. □

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- **Tips 3:** *Supp* $T \in \mathcal{L}(V, W)$. *Prove* $Tv \neq 0 \Rightarrow v \neq 0$.

**Solus:** Asum $v = 0$. Then $Tv = T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$.

Or. $T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$. Ctradic. □

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• *Given the fact that $\mathcal{L}(V, W)$ is a vecsp. Prove or give a countexa: $V, W$ are vecsps.*

*We can assure that $\{0\} \subseteq \mathcal{L}(V, W), \{0\} \subseteq V, \{0\} \subseteq W$.*

*And by [3.2], the add and homo imply that $V$ is closd add and scalar multi. ( $W^V$ might not be a vecsp. )*

**Solus:**

(I) If $W^V = \{0\}$. Then $\mathcal{L}(V, W) = \{0\}$.

And $W = \{0\}$, for if not, $\exists\, w \in W \backslash \{0\}$, define a map $f$ by $f(x) = w, \forall x \in V$.

And $V$ might not be a vecsp. **Exa:** Let $V = \mathbf{R}$, but with the scalar multi defined by $a \odot v = 0$.

(II) If $W^V$ is a non0 vecsp $\Longleftrightarrow$ $W$ is a non0 vecsp.

(a) If $\mathcal{L}(V, W) = \{0\}$, then by Exa (I), $V$ might not be vecsp.

(b) If not, then $\exists\, T \in \mathcal{L}(V, W), T \neq 0$. Which means $\exists\, v \in V, Tv \neq 0 \Rightarrow v \neq 0$. TODO

Then both $W$ and $V$ have a non0 elem.

(i) If $\exists$ inje $T \in \mathcal{L}(V, W)$, then $T(u + v) = T(v + u) \Rightarrow u + v = v + u$. etc. Hence $V$ is a vecsp.

(ii) If not, then we cannot guarantee that $V$ is a vecsp. Exa: ???

(III) If $W^V$ is not a vecsp $\Longleftrightarrow$ $W$ is not a vecsp.

(a) If $\mathcal{L}(V, W) = \{0\}$, then by Exa (I), $V$ might not be vecsp.

(b) If not. ☐

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**Ended**

# 3·B

**3** *Supp $(v_1, \dots, v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1 v_1 + \cdots + z_m v_m$.*

(a) *The surj of $T$ corres to $(v_1, \dots, v_m)$ spanning $V$.* $\quad$ $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V$.

(b) *The inje of $T$ corres to $(v_1, \dots, v_m)$ being liney indep.* $\quad$ *$(v_1, \dots, v_m)$ liney indep $\Longleftrightarrow$ $T$ inje.*

**Comment:** Let $(e_1, \dots, e_m)$ be std bss of $\mathbf{F}^m$. Then $Te_k = v_k$.

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**7** *Supp $2 \leqslant \dim V = n \leqslant m = \dim W$, if $W$ is finide.*

*Show $U = \{ T \in \mathcal{L}(V, W) : \operatorname{null} T \neq \{0\} \}$ is not a subsp of $\mathcal{L}(V, W)$.*

**Solus:** *The set of all inje $T \in \mathcal{L}(V, W)$ is a not subsp either.*

Let $(v_1, \dots, v_n)$ be a bss of $V$, $(w_1, \dots, w_m)$ be liney indep in $W$. [ $2 \leqslant n \leqslant m.$ ]

Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 : \quad v_1 \mapsto 0, \quad v_2 \mapsto w_2, \quad v_i \mapsto w_i.$

Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 : \quad v_1 \mapsto w_1, \quad v_2 \mapsto 0, \quad v_i \mapsto w_i, \quad i = 3, \dots, n.$

Thus $T_1 + T_2 \notin U$. ☐

**Comment:** If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V, W), T$ is inje. Hence $U = \emptyset$.

If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $\forall v \in V, T_0 v = 0 \Rightarrow T_0 = 0$.

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**8** *Supp $2 \leqslant \dim W = m \leqslant \dim V$, if $V$ is finide.*

*Show $U = \{ T \in \mathcal{L}(V, W) : \operatorname{range} T \neq W \}$ is not a subsp of $\mathcal{L}(V, W)$.*

**Solus:** *The set of all surj $T \in \mathcal{L}(V, W)$ is not a subsp either.* ***Using the generalized version of*** **[3.5].**

Let $(v_1, \dots, v_n)$ be liney indep in $V$, $(w_1, \dots, w_m)$ be a bss of $W$. [ $n \in \{m, m+1, \dots\}; 2 \leqslant m \leqslant n.$ ]

Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 : \quad v_1 \mapsto 0, \quad v_2 \mapsto w_2, \quad v_j \mapsto w_j, \quad v_{m+i} \mapsto 0.$

Define $T_2 \in \mathcal{L}(V, W)$ as $T_2 : \quad v_1 \mapsto w_1, \quad v_2 \mapsto 0, \quad v_j \mapsto w_j, \quad v_{m+i} \mapsto 0.$

( For each $j = 2, \dots, m;\ i = 1, \dots, n - m$, if $V$ is finide, othws let $i \in \mathbf{N}^+$. ) Thus $T_1 + T_2 \notin U$. ☐

**Comment:** If $\dim W = 0$, then $W = \{0\} = \operatorname{span}(\,)$. $\forall\, T \in \mathcal{L}(V, W), T$ is surj. Hence $U = \emptyset$.

If $\dim W = 1$, then $W = \operatorname{span}(w_0)$. Thus $U = \operatorname{span}(T_0)$, where each $T_0 v_i = 0 \Rightarrow T_0 = 0$.

---

**9** *Supp* $(v_1, \ldots, v_n)$ *is liney indep. Prove* $\forall$ *inje* $T$, $(Tv_1, \ldots, Tv_n)$ *is liney indep.*

**Solus:** $a_1 Tv_1 + \cdots + a_n Tv_n = 0 = T\left(\sum_{i=1}^n a_i v_i\right) \iff \sum_{i=1}^n a_i v_i = 0 \iff a_1 = \cdots = a_n = 0.$ $\square$

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**10** *Supp* $\operatorname{span}(v_1, \ldots, v_n) = V$. *Show* $\operatorname{span}(Tv_1, \ldots, Tv_n) = \operatorname{range} T$.

**Solus:** (a) $\operatorname{range} T = \{Tv : v \in \operatorname{span}(v_1, \ldots, v_n)\} \Rightarrow Tv_1, \ldots, Tv_n \in \operatorname{range} T$. By [2.7].

Or. $\operatorname{span}(Tv_1, \ldots, Tv_n) \ni a_1 Tv_1 + \cdots + a_n Tv_n = T(a_1 v_1 + \cdots + a_n v_n) \in \operatorname{range} T$.

(b) $\forall w \in \operatorname{range} T, w = Tv, \exists\, v \in V \Rightarrow \exists\, a_i \in \mathbf{F}, v = \sum_{i=1}^n a_i v_i, w = a_1 Tv_1 + \cdots + a_n Tv_n$. $\square$

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• (4E 3.D.15) *Supp* $T \in \mathcal{L}(V)$ *and* $V = \operatorname{span}(Tv_1, \ldots, Tv_m)$. *Prove* $V = \operatorname{span}(v_1, \ldots, v_m)$.

**Solus:** Becs $V = \operatorname{span}(Tv_1, \ldots, Tv_m) \Rightarrow T$ surj $\Rightarrow T, T^{-1}$ inv.

$\forall v \in V, \exists\, a_i \in \mathbf{F}, v = \sum_{i=1}^m a_i Tv_i \Rightarrow T^{-1} v = \sum_{i=1}^m a_i v_i \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \ldots, v_m)$.

Or. Reduce to a bss $(Tv_{\alpha_1}, \ldots, Tv_{\alpha_k})$, where $k = \dim V$, each $\alpha_i \in \{1, \ldots, m\}$. By (4E 3.D.3). $\square$

---

**11** *Supp* $S_1, \ldots, S_n$ *are liney and inje suth* $S_1 S_2 \cdots S_n$ *makes sense. Prove* $S_1 S_2 \cdots S_n$ *inje.*

**Solus:** $S_1 S_2 \cdots S_n v = 0 \Rightarrow S_1 S_2 \cdots S_{n-1} v = 0 \Rightarrow \cdots \Rightarrow S_1 v = 0 \Rightarrow v = 0.$ $\square$

---

• *Supp* $S, T \in \mathcal{L}(V)$. *Prove or give a countexa*:
(a) $\operatorname{null} S \subseteq \operatorname{null} T \Rightarrow \operatorname{range} T \subseteq \operatorname{range} S$; (b) $\operatorname{range} T \subseteq \operatorname{range} S \Rightarrow \operatorname{null} S \subseteq \operatorname{null} T$.

**Solus:** Let $B_V = (v_1, v_2, v_3)$. Countexas:

(a) Let $S : v_1 \mapsto 0;\ v_2 \mapsto 0;\ v_3 \mapsto v_2.$ $T : v_1 \mapsto 0;\ v_2 \mapsto 0;\ v_3 \mapsto v_3.$ | Then $\operatorname{null} S = \operatorname{null} T$, but $\operatorname{range} T = \operatorname{span}(v_3) \not\subseteq \operatorname{span}(v_2) = \operatorname{null} T$.

(b) Let $S : v_1 \mapsto v_2;\ v_2 \mapsto v_2;\ v_3 \mapsto v_2.$ $T : v_1 \mapsto 0;\ v_2 \mapsto 0;\ v_3 \mapsto v_2.$ | Then $\operatorname{range} T = \operatorname{range} S$, but $\operatorname{null} S = \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_1) \not\subseteq \operatorname{span}(v_1, v_2) = \operatorname{null} T$.

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**16** *Supp* $T \in \mathcal{L}(V)$ *suth* $\operatorname{null} T, \operatorname{range} T$ *are finide. Prove* $V$ *is finide.*

**Solus:** Let $B_{\operatorname{range} T} = (Tv_1, \ldots, Tv_n), B_{\operatorname{null} T} = (u_1, \ldots, u_m)$.

$\forall v \in V, \exists!\, a_i \in \mathbf{F}, T(v - a_1 v_1 - \cdots - a_n v_n) = 0 \Rightarrow \exists!\, b_i \in \mathbf{F}, v - \sum_{i=1}^n a_i v_i = \sum_{i=1}^m b_i u_i$. $\square$

---

**17** *Supp* $V, W$ *are finide. Prove* $\exists$ *inje* $T \in \mathcal{L}(V, W) \iff \dim V \leqslant \dim W$.

**Solus:** (a) Supp $\exists$ inje $T$. Then $\dim V = \dim \operatorname{range} T \leqslant \dim W$.

(b) Supp $\dim V \leqslant \dim W$. Let $B_V = (v_1, \ldots, v_n), B_W = (w_1, \ldots, w_m)$. Define each $Tv_i = w_i$. $\square$

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**18** *Supp* $V, W$ *are finide. Prove* $\exists$ *surj* $T \in \mathcal{L}(V, W) \iff \dim V \geqslant \dim W$.

**Solus:** (a) Supp $\exists$ surj $T$. Then $\dim V = \dim W + \dim \operatorname{null} T \Rightarrow \dim W \leqslant \dim V$.

(b) Supp $\dim V \geqslant \dim W$. Let $B_V = (v_1, \ldots, v_n), B_W = (w_1, \ldots, w_m)$.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_m v_m + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m$. $\square$

---

**19** *Supp* $V, W$ *are finide,* $U$ *is a subsp of* $V$.
*Prove* $\exists\, T \in \mathcal{L}(V, W)$, $\operatorname{null} T = U \iff \underbrace{\dim U}_{m} \geqslant \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_{p}$.

**Solus:**

(a) Supp $\exists\, T \in \mathcal{L}(V, W), \operatorname{null} T = U$. Then $\dim U + \dim \operatorname{range} T = \dim V \leqslant \dim U + \dim W$.

(b) Let $B_U = (u_1, \ldots, u_m), B_V = (u_1, \ldots, u_m, v_1, \ldots, v_n), B_W = (w_1, \ldots, w_p)$. Supp that $p \geqslant n$.

Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m) = a_1 w_1 + \cdots + a_n w_n$. $\square$

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• **TIPS 1:** *Supp $U$ is a subsp of $V$. Then $\forall T \in \mathcal{L}(V,W), U \cap \operatorname{null} T = \operatorname{null} T|_U$.*

• **TIPS 2:** *Supp $T \in \mathcal{L}(V,W)$ and $T|_U$ is inje. Let $V = M + N,\ U = X + Y$.*
*Then $\operatorname{range} T = \operatorname{range} T|_M + \operatorname{range} T|_N = \operatorname{range} T|_X + \operatorname{range} T|_Y$.*
(a) *Show $U = X \oplus Y \iff \operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_Y$.*
(b) *Give an exa suth $V = M \oplus N,\ \operatorname{range} T \neq \operatorname{range} T|_M \oplus \operatorname{range} T|_N$.*

**SOLUS:** Supp $U = X \oplus Y$. Asum for some $v \in V$, there exis two disti pairs $(x_1, y_1), (x_2, y_2)$ in $X \times Y$ suth $Tv = Tx_1 + Ty_1 = Tx_2 + Ty_2$. Becs $\forall v \in X \oplus Y, \exists!\,(x,y) \in X \times Y, v = x + y$.
Now $T(x_1 + y_1) = T(x_2 + y_2) \Longrightarrow x_1 + y_1 = x_2 + y_2 \Longrightarrow x_1 = x_2, y_1 = y_2$. Ctradic.
Thus $\forall Tv \in \operatorname{range} T, \exists!\, Tx \in \operatorname{range} T|_X, Ty \in \operatorname{range} T|_Y, Tv = Tx + Ty$. Convly, becs $T$ is inje. □

**EXA:** Let $B_V = (v_1, v_2, v_3), B_W = (w_1, w_2),\ T: v_1 \mapsto 0,\ v_2 \mapsto w_1,\ v_3 \mapsto w_2$.
Let $B_M = (v_1 - v_2, v_3), B_N = (v_2)$. Then $\operatorname{range} T|_M = \operatorname{span}(w_1, w_2), \operatorname{range} T|_N = \operatorname{span}(w_1)$

**COMMENT:** Also $\operatorname{null} T|_M = \operatorname{null} T|_N = \{0\}$. Hence $\operatorname{null} T \neq \operatorname{null} T|_M \oplus \operatorname{null} T|_N$.

---

**12** *Prove $\forall T \in \mathcal{L}(V,W), \exists$ subsp $U$ of $V$ suth*
$U \cap \operatorname{null} T = \operatorname{null} T|_U = \{0\},\ \operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$.
*Which is equiv to $T|_U : U \to \operatorname{range} T$ being iso.*

**SOLUS:** By [2.34] ( note that $V$ can be infinide ), $\exists$ subsp $U$ of $V$ suth $V = U \oplus \operatorname{null} T$.
$\forall v \in V, \exists!\, w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w + u) = Tu \in \{Tu : u \in U\}$. □

**CORO:** $[P]\quad T|_U : U \to \operatorname{range} T$ *is iso* $\iff U \oplus \operatorname{null} T = V.\quad [Q]$
We have shown $Q \Rightarrow P$. Now we show $P \Rightarrow Q$ to complete the proof.
$\forall v \in V, Tv \in \operatorname{range} T = \operatorname{range} T|_U \Rightarrow \exists!\, u \in U, Tv = Tu \Rightarrow v - u \in \operatorname{null} T$.
Thus $v = (v - u) + u \in U + \operatorname{null} T$. 又 $u \in U \cap \operatorname{null} T \iff T|_U(u) = 0 \iff u = 0$. □

OR. $\neg Q \Rightarrow \neg P$ : Becs $U \oplus \operatorname{null} T \subsetneq V$. We show $\operatorname{range} T \neq \operatorname{range} T|_U$ by ctradic.
Let $X \oplus (U \oplus \operatorname{null} T) = V$. Now $\operatorname{range} T = \operatorname{range} T|_X \oplus \operatorname{range} T|_U$. And $X$ is non0.
Asum $\operatorname{range} T = \operatorname{range} T|_U$. Then $\operatorname{range} T|_X = \{0\}$. While $T|_X$ is inje. Ctradic.
OR. $\operatorname{range} T|_X \subseteq \operatorname{range} T|_U \Rightarrow \forall x \in X, Tx \in \operatorname{range} T|_U, \exists\, u \in U, Tu = Tx \Rightarrow x = 0$.

Also, $\neg P \Rightarrow \neg Q$ : (a) $\operatorname{range} T|_U \subsetneq \operatorname{range} T$; OR (b) $U \cap \operatorname{null} T \neq \{0\}$.
For (a), $\exists\, x \in V \backslash U, Tx \neq 0 \iff x \notin \operatorname{null} T$. Thus $U + \operatorname{null} T \subsetneq V$. For (b), immed. □

**COMMENT:** If $T|_U : U \to \operatorname{range} T$ is iso. Let $R \oplus U = V$. Then $R$ might not be $\operatorname{null} T$.
OR. Extend $B_U$ to $B_V = (u_1, \dots, u_n, r_1, \dots, r_m)$, then $(r_1, \dots, r_m)$ might not be a $B_{\operatorname{null} T}$.

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• **TIPS 3:** Supp $T \in \mathcal{L}(V,W)$ and $U$ is a subsp suth $V = U \oplus \operatorname{null} T$. Let $\operatorname{null} T = X \oplus Y$.
Now $\forall v \in V, \exists!\, u_v \in U, (x_v, y_v) \in X \times Y, v = u_v + x_v + y_v$. Define $i \in \mathcal{L}(V, U \oplus X)$ by $i(v) = u_v + x_v$.
Then $T = T \circ i$. Becs $\forall v \in V, T(v) = T(u_v + x_v + y_v) = T(u_v) = T(u_v + x_v) = T\big(i(v)\big) = (T \circ i)(v)$.

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• **TIPS 4:** Supp $T \in \mathcal{L}(V,W), T \neq 0$. Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
By (3.A.4), $R = (v_1, \dots, v_n)$ is liney indep in $V$. Let $\operatorname{span} R = U$. We will prove $U \oplus \operatorname{null} T = V$.
(a) $T\big(\sum_{i=1}^{n} a_i v_i\big) = 0 \iff \sum_{i=1}^{n} a_i T v_i = 0 \iff a_1 = \cdots = a_n = 0$. Thus $U \cap \operatorname{null} T = \{0\}$.
(b) $Tv = \sum_{i=1}^{n} a_i T v_i \iff v - \sum_{i=1}^{n} a_i v_i \in \operatorname{null} T \iff v = \big(v - \sum_{i=1}^{n} a_i v_i\big) + \big(\sum_{i=1}^{n} a_i v_i\big)$.
Thus $U + \operatorname{null} T = V$. OR. $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$. Using Exe (12). □

**CORO:** Convly, if $U \oplus \operatorname{null} T = V$ and $B_U = (v_1, \dots, v_n)$, then $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$.
Becs $\operatorname{range} T = \operatorname{range} T|_U = \operatorname{span}(Tv_1, \dots, Tv_n)$, 又 $T$ is inje.

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• (4E 27) *Supp* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove* $V = \operatorname{null} P \oplus \operatorname{range} P$.

**Solus:** (a) If $v \in \operatorname{null} P \cap \operatorname{range} P \Rightarrow Pv = 0$, and $\exists\, u \in V, v = Pu$. Then $v = Pu = P^2u = Pv = 0$.
(b) Note that $\forall v \in V, v = Pv + (v - Pv)$ and $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$.
Or. Becs $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P = \dim\big(\operatorname{null} P \oplus \operatorname{range} P\big)$. □

Or. [ *Only in Finide* ] Let $B_{\operatorname{range} P^2} = (P^2v_1, \dots, P^2v_n)$. Then $(Pv_1, \dots, Pv_n)$ is liney indep.
Let $U = \operatorname{span}(Pv_1, \dots, Pv_n) \Rightarrow V = U \oplus \operatorname{null} P^2$. While $U = \operatorname{range} P = \operatorname{range} P^2$; $\operatorname{null} P = \operatorname{null} P^2$. □

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• *Supp* $T \in \mathcal{L}(V), v \in V$, *and* $n \in \mathbf{N}^+$ *suth* $T^{n-1}v \neq 0,\ T^n v = 0$. [See [5.16]]
*Prove* $\big(v, Tv, \dots, T^{n-1}v\big)$ *is liney indep.*

**Solus:** $a_0v + a_1Tv + \cdots + a_{n-1}T^{n-1}v = 0 \Rightarrow a_0T^{n-1}v = 0 \Rightarrow a_0 = 0$. Simlr for $a_1, \dots, a_{n-1}$. □

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• (4E 21) *Supp* $V$ *is finide,* $T \in \mathcal{L}(V, W)$, $Y$ *is a subsp of* $W$. *Let* $\big\{v \in V : Tv \in Y\big\}$.
(a) *Prove* $\big\{v \in V : Tv \in Y\big\}$ *is a subsp of* $V$.
(b) *Prove* $\dim\big\{v \in V : Tv \in Y\big\} = \dim \operatorname{null} T + \dim\big(Y \cap \operatorname{range} T\big)$.

**Solus:** Let $\mathcal{K}_Y = \big\{v \in V : Tv \in Y\big\}$.
(a) $\forall u, w \in \mathcal{K}_Y, [\, Tu, Tw \in Y\,], \lambda \in \mathbf{F}, T(u + \lambda w) = Tu + \lambda Tw \in Y \Longrightarrow \mathcal{K}_Y$ is a subsp of $V$.
(b) Define the range-restr map $R$ of $T$ by $R = T|_{\mathcal{K}_Y} \in \mathcal{L}(\mathcal{K}_Y, Y)$. Now $\operatorname{range} R = Y \cap \operatorname{range} T$.
And $v \in \operatorname{null} T \iff Tv = 0 \in Y \iff Rv = 0 \in \operatorname{range} T \iff v \in \operatorname{null} R$. By [3.22]. □

**Comment:** Now $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = \mathcal{K}_Y$. Where $B_{Y \cap \operatorname{range} T} = (Tv_1, \dots, Tv_m)$.
In particular, $\dim \mathcal{K}_{\operatorname{range} T} = \dim \operatorname{null} T + \dim \operatorname{range} T \Longrightarrow \mathcal{K}_{\operatorname{range} T} = V$.

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• (4E 31) *Supp* $V$ *is finide,* $X$ *is a subsp of* $V$, *and* $Y$ *is a finide subsp of* $W$.
*Prove if* $\dim X + \dim Y = \dim V$, *then* $\exists\, T \in \mathcal{L}(V, W), \operatorname{null} T = X, \operatorname{range} T = Y$.

**Solus:** Let $V = U \oplus X, B_U = (v_1, \dots, v_m)$. Then $\forall v \in V, \exists!\, a_i \in \mathbf{F}, x \in X,\ v = \sum_{i=1}^m a_iv_i + x$.
Let $B_Y = (w_1, \dots, w_m)$. Define $T \in \mathcal{L}(V, W)$ with each $Tv_i = w_i$, $Tx = 0$.
Now $v \in \operatorname{null} T \iff Tv = a_1w_1 + \cdots + a_mw_m = 0 \iff v = x \in X$. Hence $\operatorname{null} T = X$.
And $Y \ni w = a_1w_1 + \cdots + a_mw_m = a_1Tv_1 + \cdots + a_mTv_m \in \operatorname{range} T$. Hence $\operatorname{range} T = Y$.
Or. Notice that $V = U \oplus \operatorname{null} T$. By Exe (12), $\operatorname{range} T = \operatorname{range} T|_U$.
又 $\dim \operatorname{range} T|_U = \dim U = \dim Y$; $\operatorname{range} T \subseteq Y$.
Or. Let $B_X = (x_1, \dots, x_n)$. Now $\operatorname{range} T = \operatorname{span}\big(Tv_1, \dots, Tv_m, Tx_1, \dots, Tx_n\big) = \operatorname{span}(w_1, \dots, w_m) = Y$. □

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**22** *Supp* $U, V$ *are finide,* $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.
*Prove* $\dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$.

**Solus:** We show $\dim \operatorname{null} ST = \dim \operatorname{null} S|_{\operatorname{range} T} + \dim \operatorname{null} T$.
Becs (a) $\underline{\operatorname{range} T|_{\operatorname{null} ST} = \operatorname{range} T \cap \operatorname{null} S = \operatorname{null} S|_{\operatorname{range} T}}$,
(b) $\underline{\operatorname{null} T|_{\operatorname{null} ST} = \operatorname{null} T \cap \operatorname{null} ST = \operatorname{null} T}$. By [3.22] □

Or. Notice that $\underline{u \in \operatorname{null} ST \iff S(Tu) = 0 \iff Tu \in \operatorname{null} S}$.
Thus $\underline{\big\{u \in U : Tu \in \operatorname{null} S\big\} = \mathcal{K}_{\operatorname{null} S \,\cap\, \operatorname{range} T} = \operatorname{null} ST}$.
By Exe (4E 21), $\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim\big(\operatorname{null} S \cap \operatorname{range} T\big)$. □

**Coro:** (1) $T$ surj $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.
(2) $T$ inv $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S,\ \operatorname{null} ST = \operatorname{null} T$.
(3) $S$ inje $\Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} T$.

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**23** *Supp $V$ is finide, $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.*
*Prove* $\dim \operatorname{range} ST \leqslant \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.
COMMENT: If $\dim V = \dim U$. Then $\dim \operatorname{null} ST \geqslant \max\{\dim \operatorname{null} S, \dim \operatorname{null} T\}$.

SOLUS: NOTICE that $\underline{\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{range} S|_{\operatorname{range} T}}$.
Let $\operatorname{range} ST = \operatorname{span}(Su_1, \dots, Su_{\dim \operatorname{range} T})$, where $B_{\operatorname{range} T} = (u_1, \dots, u_{\dim \operatorname{range} T})$.
$\dim \operatorname{range} ST \leqslant \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leqslant \dim \operatorname{range} S$. ☐

OR. $\underline{\dim \operatorname{range} ST = \dim \operatorname{range} S|_{\operatorname{range} T} = \dim \operatorname{range} T - \dim \operatorname{null} S|_{\operatorname{range} T}} \leqslant \operatorname{range} T$. ☐
COMMENT: $\dim \operatorname{range} ST = \dim U - \dim \operatorname{null} ST = \dim \operatorname{range} T|_U - \dim \operatorname{range} T|_{\operatorname{null} ST}$.

CORO: (1) $S|_{\operatorname{range} T}$ inje $\iff \dim \operatorname{range} ST = \dim \operatorname{range} T$.
(2) Let $X \oplus \operatorname{null} S = V$. Then $X \subseteq \operatorname{range} T \iff \operatorname{range} ST = \operatorname{range} S$.
And $T$ is surj $\Rightarrow \operatorname{range} ST = \operatorname{range} S$.

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• (a) *Supp* $\dim V = n$, $ST = 0$ *where* $S, T \in \mathcal{L}(V)$. *Prove* $\dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$.
(b) *Give an exa of such $S, T$ with $n = 5$ and* $\dim \operatorname{range} TS = 2$.

SOLUS: Note that $\dim \operatorname{range} TS \leqslant \min\{\dim \operatorname{range} T, \dim \operatorname{range} S\}$. We prove by ctradic.
Asum $\dim \operatorname{range} TS \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$. Then $\min\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\} \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$
又 $\dim \operatorname{null} ST = n \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \,\Big|\! \Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant \left\lceil \frac{n}{2} \right\rceil - 1$.
Thus $n \leqslant 2\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \Rightarrow \frac{n}{2} \leqslant \left\lceil \frac{n}{2} \right\rceil - 1$. Ctradic. ☐

OR. $\dim \operatorname{null} S = n - \dim \operatorname{range} S \leqslant n - \dim \operatorname{range} TS$. 又 $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S$.
$\dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S \leqslant n - \dim \operatorname{range} TS$. Thus $2 \dim \operatorname{range} TS \leqslant n$. ☐

OR. Becs $\dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$, and $\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n$.
We show $\dim \operatorname{null} TS \geqslant \left\lceil \frac{n}{2} \right\rceil$. Note that $\dim \operatorname{null} S + \dim \operatorname{null} T \geqslant n$.
$\dim \operatorname{null} S + \dim \operatorname{null} T|_{\operatorname{range} S} = \dim \operatorname{null} TS$. If $\dim \operatorname{null} S \geqslant \left\lceil \frac{n}{2} \right\rceil$. Then done.
Othws, $\dim \operatorname{null} S \leqslant \left\lceil \frac{n}{2} \right\rceil - 1 \Rightarrow \dim \operatorname{null} T \geqslant n - \dim \operatorname{null} S \geqslant n - \left\lceil \frac{n}{2} \right\rceil + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1 \geqslant \left\lceil \frac{n}{2} \right\rceil$.
Thus $\dim \operatorname{null} TS \geqslant \max\{\dim \operatorname{null} S, \dim \operatorname{null} T\} = \left\lceil \frac{n}{2} \right\rceil$. ☐

EXA: Define $T: v_1 \mapsto 0,\ v_2 \mapsto 0,\ v_i \mapsto v_i$; $S: v_1 \mapsto v_4,\ v_2 \mapsto v_5,\ v_i \mapsto 0$; $i = 3, 4, 5$.

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**20, 21** (a) *Prove if $ST = I \in \mathcal{L}(V)$, then $T$ is inje and $S$ is surj.*
(b) *Supp $T \in \mathcal{L}(V, W)$. Prove if $T$ is inje, then $\exists$ surj $S \in \mathcal{L}(W, V)$, $ST = I$.*
(c) *Supp $S \in \mathcal{L}(W, V)$. Prove if $S$ is surj, then $\exists$ inje $T \in \mathcal{L}(V, W)$, $ST = I$.*

SOLUS:
(a) $Tv = 0 \Rightarrow S(Tv) = 0 = v$. OR. $\operatorname{null} T \subseteq \operatorname{null} ST = \{0\}$.
$\forall v \in V, ST(v) = v \in \operatorname{range} S$. OR. $V = \operatorname{range} ST \subseteq \operatorname{range} S$.
(b) Define $S \in \mathcal{L}(\operatorname{range} T, V)$ by $Sw = T^{-1}w$, where $T^{-1}$ is the inv of $T \in \mathcal{L}(V, \operatorname{range} T)$.
Then extend to $S \in \mathcal{L}(W, V)$ by (3.A.11). Now $\forall v \in V, STv = T^{-1}Tv = v$.

OR. [ *Req $V$ Finide* ] Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n) \Rightarrow B_V = (v_1, \dots, v_n)$. Let $U \oplus \operatorname{range} T = W$.
Define $S \in \mathcal{L}(W, V)$ with each $S(Tv_i) = v_i$, $Su = 0$ for $u \in U$. Thus $ST = I$.

(c) By Exe (12), $\exists$ subsp $U$ of $W, W = U \oplus \operatorname{null} S$, $\operatorname{range} S = \operatorname{range} S|_U = V$.
Note that $S|_U : U \to V$ is iso. Define $T = (S|_U)^{-1}$, where $(S|_U)^{-1} : V \to U$.
Then $ST = S \circ (S|_U)^{-1} = S|_U \circ (S|_U)^{-1} = I_V$.

OR. [ *Req $V$ Finide* ] Let $B_{\operatorname{range} S} = B_V = (Sw_1, \dots, Sw_n) \Rightarrow \operatorname{span}(w_1, \dots, w_n) \oplus \operatorname{null} S = W$.
Define $T \in \mathcal{L}(V, W)$ by $T(Sw_i) = w_i$. Now $ST(a_1 Sw_1 + \dots + a_n Sw_n) = (a_1 Sw_1 + \dots + a_n Sw_n)$. ☐

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• **TIPS 5:** Supp $S \in \mathcal{L}(U,V)$ is surj. Define $\mathcal{B} \in \mathcal{L}\big(\mathcal{L}(V,W), \mathcal{L}(U,W)\big)$ by $\mathcal{B}(T) = TS$.
Then $\mathcal{B}$ is inje. Becs $\mathcal{B}(T) = TS = 0 \iff T|_{\operatorname{range} S} = 0$. OR. $\operatorname{range} TS = \operatorname{range} T = \{0\}$.

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**24** *Supp* $S \in \mathcal{L}(V,M), T \in \mathcal{L}(V,W)$, *and* $\operatorname{null} S \subseteq \operatorname{null} T$. *Prove* $\exists\, E \in \mathcal{L}(M,W), T = ES$.

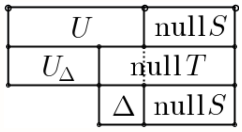
**SOLUS:**

Let $\underline{V = U \oplus \operatorname{null} S}$
$\underline{\Rightarrow S|_U : U \to \operatorname{range} S \text{ is iso.}}$
Extend $T\big(S|_U\big)^{-1}$ to $E \in \mathcal{L}(M,W)$.

$$\begin{tikzcd} \operatorname{range} T & U \arrow[l, "surj\ T"'] \arrow[d, "inv", "S"'] \\ & \operatorname{range} S \arrow[ul, "surj\ E"] \end{tikzcd}$$

OR. Define $E : \operatorname{range} S \to W$ by $\underline{\underline{E : Sv \mapsto Tv.}}$
Extend $E \in \mathcal{L}\big(\operatorname{range} S, W\big)$ to $E \in \mathcal{L}(M,W)$. □

**COMMENT:** Let $\Delta \oplus \operatorname{null} S = \operatorname{null} T$, $U_\Delta \oplus (\Delta \oplus \operatorname{null} S) = V = U_\Delta \oplus \operatorname{null} T$. Redefine $U = U_\Delta \oplus \Delta$.




$$\operatorname{range} S \xleftarrow{S} \begin{array}{l} U_\Delta \xrightarrow{T} \operatorname{range} T \\ \oplus \\ \Delta \xrightarrow{T} \{0\} \end{array}$$

Becs $\Delta = \operatorname{null} T|_U = \operatorname{null} T \cap \operatorname{range}\big(S|_U\big)^{-1}$.
Thus $E = T\big(S|_U\big)^{-1}$ is not inje $\iff \Delta \neq \{0\}$.
In other words, $\operatorname{range} S|_\Delta = \operatorname{null} E$,
while $E|_{\dots} : \operatorname{range} S|_{U_\Delta} \to \operatorname{range} T$ is iso.

**COMMENT:** Let $E_1 \in \mathcal{L}(U_\Delta \oplus \operatorname{null} T, U_\Delta)$, and $E_2$ be an iso of $\operatorname{range} S|_{U_\Delta}$ onto $\operatorname{range} T$.
Define $E_1|_{U_\Delta} = I|_{U_\Delta}$, and $E_2 = T\big(S|_{U_\Delta}\big)^{-1}$. Then $T = E_2 S E_1$.

**CORO:** If $\operatorname{null} S = \operatorname{null} T$. Then $\Delta = \{0\}, U_\Delta = U$. [ *Req W Finide* ] By (3.D.3),
we can extend inje $T\big(S|_U\big)^{-1} \in \mathcal{L}\big(\operatorname{range} S, W\big)$ to inv $E \in \mathcal{L}(M,W)$.

OR. [ *Req* range $S$ *Finide* ] Let $B_{\operatorname{range} S} = (Sv_1, \dots, Sv_n)$. Then $\underline{V = \operatorname{span}(v_1, \dots, v_n) \oplus \operatorname{null} S.}$
Define $E \in \mathcal{L}(\operatorname{range} S, W)$ by $E(Sv_i) = Tv_i$. Extend to $E \in \mathcal{L}(M,W)$.
Hence $\forall v = \sum_{i=1}^n a_i v_i + u \in V$, $\underline{(\exists!\, u \in \operatorname{null} S \subseteq \operatorname{null} T\,), Tv = \sum_{i=1}^n a_i T v_i + 0} = E\Big(\sum_{i=1}^n a_i S v_i + 0\Big)$. □

**CORO:** [ *Req W Finide* ] Supp $\operatorname{null} S = \operatorname{null} T$. We show $\exists$ inv $E \in \mathcal{L}(M,W), T = ES$.
Redefine $E \in \mathcal{L}(M,W)$ by $E(Tv_i) = Sv_i$, $E(w_j) = x_j$, for each $Tv_i$ and $w_j$. Where:
Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m), B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n), B_U = (v_1, \dots, v_m)$.
Now $V = U \oplus \operatorname{null} T = U \oplus \operatorname{null} S \Rightarrow B_{\operatorname{range} S} = \big(Sv_1, \dots, Sv_m\big)$. Let $B_M = \big(Sv_1, \dots, Sv_m, x_1, \dots, x_n\big)$. □

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**25** *Supp* $S \in \mathcal{L}(Y,W), T \in \mathcal{L}(V,W)$, *and* $\operatorname{range} T \subseteq \operatorname{range} S$. *Prove* $\exists\, E \in \mathcal{L}(V,Y), T = SE$.

**SOLUS:** Let $Y = U \oplus \operatorname{null} S$

$\Rightarrow S|_U : U \to \operatorname{range} S$ is iso. Becs $\big(S|_U\big)^{-1} : \overset{\operatorname{range} T \,\subseteq}{\operatorname{range} S} \to U$.
$\underline{\text{Define } E = \big(S|_U\big)^{-1} T = \big(S|_U\big)^{-1}\big|_{\operatorname{range} T} T \in \mathcal{L}(V,U) \subseteq \mathcal{L}(V,Y).}$ □

$$\begin{tikzcd} U_1 \arrow[r, "inv", "S"'] \arrow[d, phantom, "||"] & \operatorname{range} S \arrow[d, phantom, "||"] & \\ \Delta \arrow[r, "inv", "S"'] \arrow[d, phantom, "\oplus"] & \operatorname{range} S|_\Delta \arrow[d, phantom, "\oplus"] & \\ U_{1\Delta} \arrow[r, "inv", "S"'] & \operatorname{range} T & U_2 \arrow[l, "inv", "T"'] \arrow[ll, bend left, "inv\ E|_{U_2}"] \end{tikzcd}$$

**COMMENT:** Let $U_1 = U$. Let $U_2 \oplus \operatorname{null} T = V$.
Let $U_{1\Delta} = \operatorname{range}\big(S|_{U_1}\big)^{-1}\big|_{\operatorname{range} T} \subseteq U_1 = \Delta \oplus U_{1\Delta}$.
OR. Let $U_{1\Delta} = \operatorname{range} E|_{U_2}$. Let $\Delta \oplus \operatorname{range} E|_{U_2} = U_1$.

[ *Req* range $T$ *Finide* ] Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$. Now $B_{U_2} = (v_1, \dots, v_n)$.
$\underline{\text{Let } S(u_i) = Tv_i \text{ for each } Tv_i.}$ Define $E$ with each $Ev_i = u_i, Ex = 0$ for $x \in \operatorname{null} T$. □

**COMMENT:** [ *Req V Finide* ] Note that $\dim U_2 \leqslant \dim U_1 \Longrightarrow \dim \operatorname{null} T = p \geqslant q = \dim \operatorname{null} S$.
Let $B_{\operatorname{null} T} = (x_1, \dots, x_p), B_{\operatorname{null} S} = (y_1, \dots, y_q)$. Redefine $E : v_i \mapsto u_i,\ x_k \mapsto y_k,\ x_j \mapsto 0$,
for each $i \in \{1, \dots, \dim U_2\}, k \in \{1, \dots, \dim \operatorname{null} S\} = K, j \in \{1, \dots, \dim \operatorname{null} T\} \setminus K$.
Note that $(u_1, \dots, u_n)$ is liney indep. Let $X = \operatorname{span}(x_1, \dots, x_q) \oplus \operatorname{span}(v_1, \dots, v_n)$.
Now $E|_X$ is inje, but cannot be re-extend to inv $E \in \mathcal{L}(V,Y)$ suth $T = SE$.

**CORO:** [ *Req V Finide* ] If $\operatorname{range} T = \operatorname{range} S$, then $\dim \operatorname{null} T = \dim \operatorname{null} S = p$.
Redefine $E$ by $Ev_i = u_i$, $Ex_j = y_j$ for each $v_i$ and $x_j$. Then $E \in \mathcal{L}(V,Y)$ is inv. □

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• **Comment:** Supp $S, T \in \mathcal{L}(V, W)$. Then $\operatorname{range} S = \operatorname{range} T \not\Rightarrow \operatorname{null} S, \operatorname{null} T$ iso.
**Exa:** Forwd shift optor on $\mathbf{F}^\infty$ and backwd shift optor on $\{(0, x_1, x_2, \cdots) \in \mathbf{F}^\infty\}$.
While $\operatorname{null} S = \operatorname{null} T \iff E : Sv \mapsto Tv$ and $E^{-1} : Tv \mapsto Sv$ well-defined $\Rightarrow \operatorname{range} S, \operatorname{range} T$ iso.

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• *Supp $S, T \in \mathcal{L}(V, W)$.*
• (3.D.6) *Supp $V$ and $W$ are finide.* $\dim \operatorname{null} S = \dim \operatorname{null} T = n$.
*Prove $S = E_2 T E_1, \exists$ inv $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$.*

**Solus:** Define $E_1 : \quad v_i \mapsto r_i\ ; \quad u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.
Define $E_2 : Tv_i \mapsto Sr_i\ ; \ x_j \mapsto y_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

| | |
|---|---|
| Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m)$; $B_{\operatorname{range} S} = (Sr_1, \dots, Sr_m)$.<br>Let $B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p)$; $B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p)$.<br>Let $B_{\operatorname{null} T} = (u_1, \dots, u_n)$; $B_{\operatorname{null} S} = (s_1, \dots, s_n)$.<br>Thus $B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$; $B'_V = (r_1, \dots, r_m, s_1, \dots, s_n)$. | $\therefore E_1, E_2$ are inv and $S = E_2 T E_1$. |

$\square$

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• (a) *Supp $T = ES$ and $E \in \mathcal{L}(W)$ is inv. Prove* $\operatorname{null} S = \operatorname{null} T$.
(b) *Supp $T = SE$ and $E \in \mathcal{L}(V)$ is inv. Prove* $\operatorname{range} S = \operatorname{range} T$.
(c) *Supp $T = E_2 S E_1$ and $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$ are inv.*
*Prove* $\dim \operatorname{null} S = \dim \operatorname{null} T$.

**Solus:** (a) $v \in \operatorname{null} T \iff Tv = 0 = E(Sv) \iff Sv = 0 \iff v \in \operatorname{null} S$.
(b) $w \in \operatorname{range} T \iff \exists\, v \in V, w = Tv = S(Ev) \iff w \in \operatorname{range} S$.
(c) By the Coro in Exe (22), $\dim \operatorname{null} E_2 S E_1 \overset{E_2}{\underset{\text{inv}}{=\!=}} \dim \operatorname{null} S E_1 \overset{E_1}{\underset{\text{inv}}{=\!=}} \dim \operatorname{null} S = \dim \operatorname{null} T$. $\square$

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**28** *Supp $T \in \mathcal{L}(V, W)$. Let $(Tv_1, \dots, Tv_m)$ be a bss of* $\operatorname{range} T$ *and each $w_i = Tv_i$.*
(a) *Prove $\exists\, \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ suth $\forall v \in V, Tv = \varphi_1(v) w_1 + \cdots + \varphi_m(v) w_m$.*
(b) [4E 3.F.5] $\forall v \in V, \exists\,!\, \varphi_i(v) \in \mathbf{F}, Tv = \varphi_1(v) w_1 + \cdots + \varphi_m(v) w_m$.
*Thus defining each $\varphi_i : V \to \mathbf{F}$. Show each $\varphi_i \in \mathcal{L}(V, \mathbf{F})$.*

**Solus:** *The answer for (b) with (b) itself is the answer for (a).*
(b) $\sum_{i=1}^m \varphi_i(u + \lambda v) w_i = T(u + \lambda v) = Tu + \lambda Tv = \left(\sum_{i=1}^m \varphi_i(u) w_i\right) + \lambda\left(\sum_{i=1}^m \varphi_i(v) w_i\right)$. $\square$
Or. $\forall v \in V, \exists\,!\, a_i \in \mathbf{F}, Tv = a_1 Tv_1 + \cdots + a_m Tv_m$. Let $B_{(\operatorname{range} T)'} = (\psi_1, \dots, \psi_m)$.
Then $[T'(\psi_i)](v) = (\psi_i \circ T)(v) = a_i$. Thus each $\varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$. $\square$
(a) $\operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} T = V \Rightarrow \forall v \in V, \exists\,!\, a_i \in \mathbf{F}, u \in \operatorname{null} T,\ v = \sum_{i=1}^m a_i v_i + u$.
Define $\varphi_i \in \mathcal{L}(V, \mathbf{F})$ by $\varphi_i(v_j) = \delta_{i,j},\ \varphi_i(u) = 0$ for all $u \in \operatorname{null} T$.
Linity: $\forall v, w \in V\ [\exists\,!\, a_i, b_i \in \mathbf{F}], \lambda \in \mathbf{F}, \varphi_i(v + \lambda w) = a_i + \lambda b_i = \varphi(v) + \lambda \varphi(w)$. $\square$

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**29** *Supp $\varphi \in \mathcal{L}(V, \mathbf{F})$. Supp $\varphi(u) \neq 0$. Prove $V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}$.* *By* Tips *(4), immed.*

**Solus:** (a) $v = cu \in \operatorname{null} \varphi \cap \operatorname{span}(u) \Rightarrow c\varphi(u) = 0 \Rightarrow v = 0$. Now $\operatorname{null} \varphi \cap \operatorname{span}(u) = \{0\}$.
(b) For $v \in V$, let $a_v = \varphi(v)$. Then $v = [v - (a_v / a_u) u] + (a_v / a_u) u \Rightarrow V = \operatorname{null} \varphi + \operatorname{span}(u)$. $\square$

---

**30** *Supp $\varphi, \beta \in \mathcal{L}(V, \mathbf{F})$ and* $\operatorname{null} \varphi = \operatorname{null} \beta = \eta$. *Prove $\exists\, c \in \mathbf{F}, \varphi = c\beta$.*

**Solus:** If $\eta = V$, then $\varphi = \beta = 0$, done. Now by Exe (29),
$\varphi(u) \neq 0 \iff V = \operatorname{null} \varphi \oplus \operatorname{span}(u) \iff V = \operatorname{null} \beta \oplus \operatorname{span}(u) \iff \beta(u) \neq 0$.

| | |
|---|---|
| Note that $\forall v \in V, \exists\,!\, u_0 \in \eta,\ a_v \in \mathbf{F}, v = u_0 + a_v u$<br>$\Rightarrow \varphi(u_0 + a_v u) = a_v \varphi(u),\ \beta(u_0 + a_v u) = a_v \beta(u)$. | Let $c = \dfrac{\varphi(u)}{\beta(u)} \in \mathbf{F} \backslash \{0\}$. |

$\square$

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- (4E 3.F.6) *Supp* $\varphi, \beta \in \mathcal{L}(V, \mathbf{F})$. *Prove* $\operatorname{null}\beta \subseteq \operatorname{null}\varphi \iff \varphi = c\beta, \exists\, c \in \mathbf{F}$.
  **Coro:** $\operatorname{null}\varphi = \operatorname{null}\beta \iff \varphi = c\beta,\ \exists\, c \in \mathbf{F}\backslash\{0\}$.

**Solus:** Using Exe (29) and (30).

(a) If $\varphi = 0$, then done. Othws, supp $u \notin \operatorname{null}\varphi \supseteq \operatorname{null}\beta$.
Now $V = \operatorname{null}\varphi \oplus \operatorname{span}(u) = \operatorname{null}\beta \oplus \operatorname{span}(u)$. By [1.C Tips (2)], $\operatorname{null}\varphi = \operatorname{null}\beta$. Let $c = \dfrac{\varphi(u)}{\beta(u)}$.
Or. We discuss in two cases. If $\operatorname{null}\beta = \operatorname{null}\varphi$, or if $\varphi = 0$, then done. Othws,
$\exists\, u' \in \operatorname{null}\varphi \backslash \operatorname{null}\beta$, $\exists\, u \notin \operatorname{null}\varphi \supsetneq \operatorname{null}\beta \Rightarrow V = \operatorname{null}\beta \oplus \operatorname{span}(u') = \operatorname{null}\beta \oplus \operatorname{span}(u)$.
$\forall v \in V, v = w + au = w' + bu', \exists!\, w, w' \in \operatorname{null}\beta$
Thus $\varphi(w + au) = a\varphi(u),\ \beta(w' + bu) = b\beta(u')$. | Let $c = \dfrac{a\varphi(u)}{b\beta(u')} \in \mathbf{F}\backslash\{0\}$. Done.
Notice that by (b) below, we have $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$, ctradic the asum.

(b) If $c = 0$, then $\operatorname{null}\varphi = V \supseteq \operatorname{null}\beta$, done. Othws, becs $v \in \operatorname{null}\beta \iff v \in \operatorname{null}\varphi$. □

Or. By Exe (24), $\operatorname{null}\beta \subseteq \operatorname{null}\varphi \iff \exists\, E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta$. [ If $E$ is inv. Then $\operatorname{null}\beta = \operatorname{null}\varphi$. ]
Now $\exists\, E \in \mathcal{L}(\mathbf{F}), \varphi = E \circ \beta \iff \exists\, c = E(1) \in \mathbf{F}, \varphi = c\beta$. [ $E$ is inv $\iff E(1) \neq 0 \iff c \neq 0$. ] □

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**Ended**

# 3·C

- **Note For [3.30, 32]:** *matrix of span*
  Supp $L_\alpha = (\alpha_1, \dots, \alpha_n)$ and $L_\beta = (\beta_1, \dots, \beta_m)$ are in a vecsp $V$.
  Let each $\alpha_k = A_{1,k}\beta_1 + \cdots + A_{m,k}\beta_m$, forming $A = \mathcal{M}(\operatorname{span} L_\beta \supseteq L_\alpha) \in \mathbf{F}^{m,n}$.
  Which is *the matrix of span*. Then $\begin{pmatrix}\beta_1 & \cdots & \beta_m\end{pmatrix}\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n}\end{pmatrix} = \begin{pmatrix}\alpha_1 & \cdots & \alpha_n\end{pmatrix}$.

  (a) Supp $m = n$. If $(A_{\cdot,1}, \dots, A_{\cdot,n})$ is a bss of $\mathbf{F}^{n,1}$. We show $L_\alpha$ liney indep $\iff L_\beta$ liney indep.
  ($\Leftarrow$) Immed. ($\Rightarrow$) Asum $L_\beta$ is liney dep and $\beta_j = c_1\beta_1 + \cdots + c_{j-1}\beta_{j-1}$. By ctradic. □

  (b) Supp $m \geqslant n$. If $L_\beta$ liney indep. We show $(A_{\cdot,1}, \dots, A_{\cdot,n})$ liney indep $\iff L_\alpha$ liney indep.
  ($\Rightarrow$) Immed. ($\Leftarrow$) By ctradic. □
  **Comment:** $\mathcal{M}(\operatorname{span} L_\beta \supseteq L_\alpha) = \mathcal{M}(I, L_\alpha, L_\beta) \iff L_\alpha, L_\beta$ liney indep $\Rightarrow (A_{\cdot,1}, \dots, A_{\cdot,n})$ liney indep.
  Where $I$ is the id optor retr to $\operatorname{span} L_\alpha \subseteq \operatorname{span} L_\beta$.

  (c) Supp $m < n$. Then $(A_{\cdot,1}, \dots, A_{\cdot,n})$ is liney dep, so is $L_\alpha$.

  Supp $T \in \mathcal{L}(V, W)$ and $B_V = (v_1, \dots, v_m), B_W = (w_1, \dots, w_n)$.
  Then $\mathcal{M}(T, B_V, B_W) = \mathcal{M}\big(\operatorname{span} B_W \supseteq (Tv_1, \dots, Tv_m)\big)$. **Comment:** See also (4E 3.D.23).

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- **NOTE FOR Trspose:** [3.F.33] Define $\mathcal{T} : A \to A^t$. By [3.111], $\mathcal{T}$ is liney. Becs $(A^t)^t = A$. $\mathcal{T}^2 = I$, $\mathcal{T} = \mathcal{T}^{-1} \Rightarrow \mathcal{T}$ is iso of $\mathbf{F}^{m,n}$ onto $\mathbf{F}^{n,m}$. Define $\mathcal{C}_k : A \to A_{\cdot,k}$, $\mathcal{R}_j : A \to A_{j,\cdot}$, $\mathcal{E}_{j,k} : A \to A_{j,k}$.
  Now we show (a) $\underline{\mathcal{T}\mathcal{R}_j = \mathcal{C}_j\mathcal{T}}$, (b) $\underline{\mathcal{T}\mathcal{C}_k = \mathcal{R}_k\mathcal{T}}$, and (c) $\underline{\mathcal{T}\mathcal{E}_{j,k} = \mathcal{E}_{k,j}\mathcal{T}}$.
  So that $\mathcal{T}\mathcal{C}_k\mathcal{T} = \mathcal{R}_k$, $\mathcal{T}\mathcal{R}_j\mathcal{T} = \mathcal{C}_j$, and $\mathcal{T}\mathcal{E}_{j,k}\mathcal{T} = \mathcal{E}_{k,j}$.

  Let $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{pmatrix}$. | Note that $(A_{j,k})^t = A_{j,k} = (A^t)_{k,j}$. Thus (c) holds. And $(A_{\cdot,k})^t = (A_{1,k} \cdots A_{m,k}) = (A^t_{k,1} \cdots A^t_{k,m}) = (A^t)_{k,\cdot}$ $\Longrightarrow$ (b) holds. Simlr for (a).

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- **NOTE FOR [3.48]:**

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}}_{B} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

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- **NOTE FOR [3.47]:** $(AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k}$ ☐
- **NOTE FOR [3.49]:** $\left[(AC)_{\cdot,k}\right]_{j,1} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$ ☐
- **EXE 10:** $\left[(AC)_{j,\cdot}\right]_{1,k} = (AC)_{j,k} = \sum_{r=1}^n A_{j,r}C_{r,k} = \sum_{r=1}^n (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$ ☐
- **COMMENT:** For [3.49], let $B_U = (u_1, \dots, u_p), B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$.
  And $C = \mathcal{M}(T, B_U, B_V) \in \mathbf{F}^{n,p}, A = \mathcal{M}(S, B_V, B_W) \in \mathbf{F}^{m,n}$.
  Then $\mathcal{M}(Tu_k, B_V) = C_{\cdot,k} \Rightarrow \mathcal{M}(S(Tu_k), B_W) = AC_{\cdot,k}$, 又 $\mathcal{M}\big((ST)(u_k), B_W\big) = (AC)_{\cdot,k}$ ☐
  By NOTE FOR Transpose, $(AC)_{j,\cdot} = \left[\left((AC)^t\right)_{\cdot,j}\right]^t = (C^t(A^t)_{\cdot,j})^t = ((A^t)_{\cdot,j})^t C = A_{j,\cdot}C$ ☐

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- **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$. By [4E 3.51(a)], $(Ac)_{\cdot,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ ☐
  OR. $\because$ $(Ac)_{j,1} = \sum_{r=1}^n A_{j,r}c_{r,1} = \left[\sum_{r=1}^n (A_{\cdot,r}c_{r,1})\right]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$
  $\therefore$ $Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^n A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ OR. $(Ac)_{j,1} = (Ac)_{j,\cdot} = A_{j,\cdot}c \in \mathbf{F}$. ☐
  OR. Let $B_V = (v_1, \dots, v_n)$. Now $Ac = \mathcal{M}(Tv, B_W) = \mathcal{M}\big(T(c_1v_1 + \cdots + c_nv_n)\big) = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$. ☐

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- **EXE 11:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$. By [4E 3.51(b)], $(aC)_{1,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ ☐
  OR. $\because$ $(aC)_{1,k} = \sum_{r=1}^n a_{1,r}C_{r,k} = \left[\sum_{r=1}^n a_{1,r}(C_{r,\cdot})\right]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$
  $\therefore$ $aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^n a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ OR. $(aC)_{1,k} = (aC)_{\cdot,k} = aC_{\cdot,k} \in \mathbf{F}$. ☐
  OR. $aC = \left((aC)^t\right)^t = (C^ta^t)^t = \left[a_1^t(C^t)_{\cdot,1} + \cdots + a_n^t(C^t)_{\cdot,n}\right]^t = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$. ☐

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- [4E 3.51] Supp $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}$. [ See also NOTE FOR [3.49] and Exe (10). ]
  (a) For $k = 1, \dots, p$, $(CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^c C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$
  (b) For $j = 1, \dots, m$, $(CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^c C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
- **EXA:** $m = 2, c = 2, p = 3$.

$$(AB)_{\cdot,2} = AB_{\cdot,2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} = A_{\cdot,1}B_{1,2} + A_{\cdot,2}B_{2,2} = 6\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix};$$

$$(AB)_{1,\cdot} = A_{1,\cdot}B = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = A_{1,1}B_{1,\cdot} + A_{1,2}B_{2,\cdot} = 1\begin{pmatrix} 5 & 6 & 7 \end{pmatrix} + 2\begin{pmatrix} 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \end{pmatrix};$$

---

• Column-Row Factoriz (CR Factoriz) *Supp* $A \in \mathbf{F}^{m,n}, A \neq 0$.
*Prove, with p specified below, that* $\exists\, C \in \mathbf{F}^{m,p}, R \in \mathbf{F}^{p,n}, A = CR$.
(a) *Supp* $S_c = \operatorname{span}(A_{\cdot,1}, \cdots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}, \dim S_c = c$, *the col rank. Let* $p = c$.
(b) *Supp* $S_r = \operatorname{span}(A_{1,\cdot}, \cdots, A_{m,\cdot}) \subseteq \mathbf{F}^{1,n}, \dim S_r = r$, *the row rank. Let* $p = r$.

**Solus:** Using [4E 3.51]. Notice that $A \neq 0 \Rightarrow c, r \geqslant 1$.

(a) Reduce to bss $B_C = (C_{\cdot,1}, \cdots, C_{\cdot,c})$, forming $C \in \mathbf{F}^{m,c}$. Then $\forall k \in \{1, \dots, n\}$,
$A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$, $\exists\,!\, R_{1,k}, \cdots, R_{c,k} \in \mathbf{F}$, forming $R \in \mathbf{F}^{c,n}$. Thus $A = CR$.

(b) Reduce to bss $B_R = (R_{1,\cdot}, \cdots, R_{r,\cdot})$, forming $R \in \mathbf{F}^{r,n}$. Then $\forall j \in \{1, \dots, m\}$,
$A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,r}R_{r,\cdot} = (CR)_{j,\cdot}$, $\exists\,!\, C_{j,1}, \dots, C_{j,r} \in \mathbf{F}$, forming $C \in \mathbf{F}^{m,r}$. Thus $A = CR$. $\square$

**Exa:** $$A = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 46 & 33 & 20 & 7 \end{pmatrix} \overset{(\mathrm{I})}{=\!=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix} \overset{(\mathrm{II})}{=\!=} \begin{pmatrix} 7 & 4 \\ 19 & 12 \\ 33 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$$

(I) $\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} = 2\begin{pmatrix} 10 & 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 26 & 19 & 12 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{pmatrix}$, using [4E 3.51(b)].

$\begin{pmatrix} 46 & 33 & 20 & 7 \end{pmatrix} \in \operatorname{span}(A_{1,\cdot}, A_{2,\cdot})$, and $(A_{1,\cdot}, A_{2,\cdot})$ is liney indep. Thus $B_R = (A_{1,\cdot}, A_{2,\cdot})$.

(II) $\begin{pmatrix} 10 \\ 26 \\ 46 \end{pmatrix} = 2\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} - \begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = -\begin{pmatrix} 7 \\ 19 \\ 33 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$. Thus $B_C = (A_{\cdot,2}, A_{\cdot,3})$.

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• Column Rank Equals Row Rank   Using nota and result above.
For each $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$
For each $A_{\cdot,k} \in S_c$, $A_{\cdot,k} = (CR)_{\cdot,k} = CR_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$
$\Rightarrow \operatorname{span}(A_{1,\cdot}, \cdots, A_{n,\cdot}) = S_r = \operatorname{span}(R_{1,\cdot}, \cdots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leqslant c = \dim S_c$.
$\Rightarrow \operatorname{span}(A_{\cdot,1}, \cdots, A_{\cdot,m}) = S_c = \operatorname{span}(C_{\cdot,1}, \cdots, C_{\cdot,r}) \Rightarrow \dim S_c = c \leqslant r = \dim S_r$.
Or. Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leqslant r = \dim S_r = \dim S_c^t$. $\square$

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• (4E 16) *Supp* $A \in \mathbf{F}^{m,n} \backslash \{0\}$. *Prove* $[P]$ $\operatorname{rank} A = 1 \iff \exists\, c_j, d_k \in \mathbf{F}$, *each* $A_{j,k} = c_j \cdot d_k$. $[Q]$

**Solus:**

[ *Using CR Factoriz* ]

$P \Rightarrow Q$ : Immed.

$Q \Rightarrow P$ : Becs $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1d_1 & \cdots & c_1d_n \\ \vdots & \ddots & \vdots \\ c_md_1 & \cdots & c_md_n \end{pmatrix} \Longrightarrow S_r = \operatorname{span}\left\{ \begin{matrix} \begin{pmatrix} \underline{c_1}d_1 & \cdots & \underline{c_1}d_n \end{pmatrix}, \\ \vdots \\ \begin{pmatrix} \underline{c_m}d_1 & \cdots & \underline{c_m}d_n \end{pmatrix} \end{matrix} \right\}$.

Or. $S_c = \operatorname{span}\left\{ \begin{pmatrix} c_1\underline{d_1} \\ \vdots \\ c_m\underline{d_1} \end{pmatrix}, \dots, \begin{pmatrix} c_1\underline{d_n} \\ \vdots \\ c_m\underline{d_n} \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$. $\square$

[ *Not Using CR Factoriz* ]

$Q \Rightarrow P$ : Using [4E 3.51(a)]. Each $A_{\cdot,k} \in \operatorname{span}\left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$. Then $\operatorname{rank} A = \dim S_c \leqslant 1$
又 $A \neq 0 \Rightarrow \dim S_c \geqslant 1$.

$P \Rightarrow Q$ : Becs $\dim S_c = \dim S_r = 1$.

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $\quad d_k' = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$.

$\Rightarrow A_{j,k} = d_k' A_{j,1} = c_j A_{1,k} = c_j d_k' A_{1,1} = c_j d_k$, where $d_k = d_k' A_{1,1}$. $\square$

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• **Tips 1:** *Supp* $T \in \mathcal{L}(V,W)$, $B_V = (v_1,\dots,v_n), B_W = (w_1,\dots,w_m)$.
*Let* $L = (Tv_{\alpha_1},\dots,Tv_{\alpha_k})$, $L_{\mathcal{M}} = (A_{\cdot,\alpha_1},\cdots,A_{\cdot,\alpha_k})$, *where each* $\alpha_i \in \{1,\dots,n\}$.
(a) *Show* $[P]$ $L$ *is liney indep* $\iff$ $L_{\mathcal{M}}$ *is liney indep.* $[Q]$
(b) *Show* $[P]$ $\operatorname{span} L = W \iff \operatorname{span} L_{\mathcal{M}} = \mathbf{F}^{m,1}$. $[Q]$ $\qquad$ [ *Let* $A = \mathcal{M}(T, B_V, B_W)$. ]

**Solus:** (a) Note that $\mathcal{M}\colon Tv_k \to A_{\cdot,k}$ is iso. of $\operatorname{span} L$ onto $\operatorname{span} L_{\mathcal{M}}$. By (3.B.9).
(b) Reduce to liney indep lists. By (a) and (2.39). □

Or. $$c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k} = c_1\big(A_{1,\alpha_1}w_1 + \cdots + A_{m,\alpha_1}w_m\big) + \cdots + c_k\big(A_{1,\alpha_k}w_1 + \cdots + A_{m,\alpha_k}w_m\big)$$
$$= \big(c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k}\big)w_1 + \cdots + \big(c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k}\big)w_m.$$

And $$c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} = c_1\begin{pmatrix}A_{1,\alpha_1}\\ \vdots \\ A_{m,\alpha_1}\end{pmatrix} + \cdots + c_k\begin{pmatrix}A_{1,\alpha_k}\\ \vdots \\ A_{m,\alpha_k}\end{pmatrix} = \begin{pmatrix}c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k}\\ \vdots \\ c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k}\end{pmatrix}.$$

(a) $P \Rightarrow Q$ : Supp $c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} = 0$. Let $v = c_1v_{\alpha_1} + \cdots + c_kv_{\alpha_k}$.
Then $Tv = \big(c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k}\big)w_1 + \cdots + \big(c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k}\big)w_m = 0w_1 + \cdots + 0w_m$.
Now $c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k} = 0$. Then each $c_i = 0 \Rightarrow L_{\mathcal{M}}$ liney indep.

$Q \Rightarrow P$ : Becs $c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k} = 0$. For each $i \in \{1,\dots,m\}$, $c_1A_{i,\alpha_1} + \cdots + c_kA_{i,\alpha_k} = 0$.
Which is equiv to $c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} = 0$. Thus each $c_i = 0 \Rightarrow L$ liney indep.

Or. $\exists\, A_{\cdot,\alpha_j} = c_1A_{\cdot,\alpha_1} + \cdots + c_{j-1}A_{\cdot,\alpha_{j-1}}$
$\iff$ For each $i \in \{1,\dots,m\}$, $A_{i,\alpha_j} = c_1A_{i,\alpha_1} + \cdots + c_{j-1}A_{i,\alpha_{j-1}}$
$\iff Tv_{\alpha_j} = A_{1,\alpha_j}w_1 + \cdots + A_{m,\alpha_j}w_m$
$= \big(c_1A_{1,\alpha_1} + \cdots + c_{j-1}A_{1,\alpha_{j-1}}\big)w_1 + \cdots + \big(c_1A_{m,\alpha_1} + \cdots + c_{j-1}A_{m,\alpha_{j-1}}\big)w_m$
$\iff \exists\, Tv_{\alpha_j} = c_1Tv_{\alpha_1} + \cdots + c_{j-1}Tv_{\alpha_{j-1}}$.

(b) Note that each $\mathcal{M}(Tv_{\alpha_i}) = A_{\cdot,\alpha_i}$
$P \Rightarrow Q$ : Supp each $w_i = Iw_i = J_{1,i}Tv_{\alpha_1} + \cdots + J_{k,i}Tv_{\alpha_k}$.
$\forall a \in \mathbf{F}^{m,1}, \exists\, w = a_1w_1 + \cdots + a_mw_m \in W$, $a = \mathcal{M}(w, B_W)$.
Becs $w = a_1\big(J_{1,1}Tv_{\alpha_1} + \cdots + J_{k,1}Tv_{\alpha_k}\big) + \cdots + a_m\big(J_{1,m}Tv_{\alpha_1} + \cdots + J_{k,m}Tv_{\alpha_k}\big)$
$= \big(a_1J_{1,1} + \cdots + a_mJ_{1,m}\big)Tv_{\alpha_1} + \cdots + \big(a_1J_{k,1} + \cdots + a_mJ_{k,m}\big)Tv_{\alpha_k}$.
Apply $\mathcal{M}$ to both sides, $a = c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k}$, where each $c_i = a_1J_{i,1} + \cdots + a_mJ_{i,m}$.

$Q \Rightarrow P$ : $\forall w \in W, \exists\, a = c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k} \in \mathbf{F}^{m,1}$, $\mathcal{M}(w, B_W) = a$
$\Rightarrow w = \big(c_1A_{1,\alpha_1} + \cdots + c_kA_{1,\alpha_k}\big)w_1 + \cdots + \big(c_1A_{m,\alpha_1} + \cdots + c_kA_{m,\alpha_k}\big)w_m = c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k}$.

$\neg Q \Rightarrow \neg P$ : $\exists\, w \in W, \exists\, a \in \mathbf{F}^{m,1}, \mathcal{M}(w, B_W) = a$, but $\nexists\, (c_1,\dots,c_k) \in \mathbf{F}^k, a = c_1A_{\cdot,\alpha_1} + \cdots + c_kA_{\cdot,\alpha_k}$
$\Rightarrow \nexists\, (c_1,\dots,c_k) \in \mathbf{F}^k$, $w = c_1Tv_{\alpha_1} + \cdots + c_kTv_{\alpha_k}$. For if not, ctradic. □

**Note:** Let $L = (Tv_1,\dots,Tv_n)$, $L_{\mathcal{M}} = (A_{\cdot,1},\cdots,A_{\cdot,n})$.
Then (a*) By [3.B.9, Tips (4)], $T$ is inje $\iff L$ is liney indep, so is $L_{\mathcal{M}}$.
And (b*) $T$ is surj $\iff \operatorname{span} L = W \iff \operatorname{span} L_{\mathcal{M}} = \mathbf{F}^{m,1}$.
**Coro:** $B_{\mathbf{F}^{n,1}} = (A_{\cdot,1},\cdots,A_{\cdot,n}) \iff T$ is inje and surj $\iff B_{\mathbf{F}^{1,n}} = (A_{\cdot,1},\cdots,A_{\cdot,n})$.
**Comment:** If $T$ is inv. Then by (a*, c) or (b*, d), we have another proof of Coro.
Or. If $m = n$. Then by [3.118] and one of (a*, b*, c, d). Yet another proof.
(c) $T$ surj $\iff T'$ inje $\iff \big(T'(\psi_1),\dots,T'(\psi_m)\big)$ liney indep
$\overset{\text{(a)}}{\iff} \big((A^t)_{\cdot,1},\cdots,(A^t)_{\cdot,m}\big)$ liney indep in $\mathbf{F}^{n,1}$, so is $(A_{1,\cdot},\cdots,A_{m,\cdot})$ in $\mathbf{F}^{1,n}$.
(d) $T$ inje $\iff T'$ surj $\iff V' = \operatorname{span}\big(T'(\psi_1),\dots,T'(\psi_m)\big)$
$\overset{\text{(b)}}{\iff} \mathbf{F}^{n,1} = \operatorname{span}\big((A^t)_{\cdot,1},\cdots,(A^t)_{\cdot,m}\big) \iff \mathbf{F}^{1,n} = \operatorname{span}\big(A_{1,\cdot},\cdots,A_{m,\cdot}\big)$.

• **TIPS 2:** *Supp $p$ is a poly of $n$ variables in $\mathbf{F}$. Prove $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big)$.*
*Where the liney maps $T_1,\dots,T_n$ are suth $p(T_1,\dots,T_n)$ makes sense.* See [5.16,17,20].

**SOLUS:** Supp the poly $p$ is defined by $p(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^n x_i^{k_i}$.
Note that $\mathcal{M}(T^xS^y) = \mathcal{M}(T)^x\mathcal{M}(S)^y$; $\mathcal{M}(T^x+S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$.
Then $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = \mathcal{M}\Big(\sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^n T_i^{k_i}\Big)$
$= \sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^n \mathcal{M}\big(T_i^{k_i}\big) = p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big)$. □

• **CORO:** Supp $\tau$ is an algebraic property. Then $\tau$ holds for liney maps $\iff$ $\tau$ holds for matrices.
Supp $\alpha_1,\dots,\alpha_n$ are dist with each $\alpha_k\in\{1,\dots,n\}$.
Now $p(T_1,\dots,T_n) = p(T_{\alpha_1},\dots,T_{\alpha_n}) \iff p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big) = p\big(\mathcal{M}(T_{\alpha_1}),\dots,\mathcal{M}(T_{\alpha_n})\big)$.

---

**13** *Prove the distr holds for matrix add and matrix multi.*
*Supp $A,B,C$ are matrices suth $A(B+C)$ make sense, we prove the left distr.*

**SOLUS:** Supp $A\in\mathbf{F}^{m,n}$ and $B,C\in\mathbf{F}^{n,p}$.
Note that $\big[A(B+C)\big]_{j,k} = \sum_{r=1}^n A_{j,r}(B+C)_{r,k} = \sum_{r=1}^n\big(A_{j,r}B_{r,k}+A_{j,r}C_{r,k}\big) = (AB+AC)_{j,k}$.
OR. Define $T,S,R$ suth $\mathcal{M}(T)=A,\mathcal{M}(S)=B,\mathcal{M}(R)=C$.
$A(B+C) = \mathcal{M}(T(S+R)) \xlongequal{[3.9]} \mathcal{M}(TS+TR) = AB+AC$.
OR. $T(S+R) = TS+TR \Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR) \Rightarrow A(B+C) = AB+AC$. □

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**1** *Supp $T\in\mathcal{L}(V,W)$. Show for each pair of $B_V$ and $B_W$,*
*$A=\mathcal{M}(T,B_V,B_W)$ has at least $n=\dim\operatorname{range}T$ non0 ent.*

**SOLUS:**
Let $U\oplus\operatorname{null}T = V$; $B_U=(v_1,\dots,v_n), B_V=(v_1,\dots,v_m)$.
For each $k\in\{1,\dots,n\}, Tv_k\neq 0 \iff A_{\cdot,k}\neq 0$. Hence every such $A_{\cdot,k}$ has at least one non0 ent. □

OR. We prove by ctradic. Supp $A$ has at most $(n-1)$ non0 ent.
Then by Pigeon Hole Principle, at least one of $A_{\cdot,1},\dots,A_{\cdot,n}$ equals 0.
Thus there are at most $(n-1)$ non0 vecs in $Tv_1,\dots,Tv_n$.
又 $\operatorname{range}T = \operatorname{span}(Tv_1,\dots,Tv_n) \Rightarrow \dim\operatorname{range}T = \dim\operatorname{span}(Tv_1,\dots,Tv_n) \leqslant n-1$. Ctradic. □

---

**6** *Supp $V$ and $W$ are finide and $T\in\mathcal{L}(V,W)$.*
*Prove $\dim\operatorname{range}T = 1 \iff \exists\, B_V,B_W$, all ent of $A=\mathcal{M}\big(T,B_V,B_W\big)$ equal 1.*

**SOLUS:**
(a) Supp $B_V=(v_1,\dots,v_n), B_W=(w_1,\dots,w_m)$ are the bses suth all ent of $A$ equal 1.
Then $Tv_i = w_1+\dots+w_m$ for all $i=1,\dots,n$. Becs $w_1,\dots,w_n$ is liney indep, $w_1+\dots+w_n\neq 0$.

(b) Supp $\dim\operatorname{range}T=1$. Then $\dim\operatorname{null}T=\dim V-1$.
Let $B_{\operatorname{null}T}=(u_2,\dots,u_n)$. Extend to a bss $(u_1,u_2,\dots,u_n)$ of $V$.
Becs $Tv_1\neq 0$. Extend to $(Tv_1,w_2,\dots,w_m)$ a bss of $W$. Let $w_1 = Tv_1-w_2-\dots-w_m$.
Now $B_W=(w_1,\dots,w_m)$. Let $v_1=u_1,\ v_i=u_1+u_i$. Now $B_V=(v_1,\dots,v_n)$. □

OR. Supp $B_{\operatorname{range}T}=(w)$. By [2.C NOTE FOR (15)], $\exists\, B_W=(w_1,\dots,w_m),\ w=w_1+\dots+w_m$.
By [2.C TIPS], $\exists$ a bss $(u_1,\dots,u_n)$ of $V$ suth each $u_k\notin\operatorname{null}T$.
Now each $Tu_k\in\operatorname{range}T=\operatorname{span}(w)\Rightarrow Tu_k=\lambda_k w, \exists\,\lambda_k\in\mathbf{F}\backslash\{0\}$.
Let $v_k=\lambda_k^{-1}u_k\neq 0$, so that each $Tv_k = w = w_1+\dots+w_m$. Thus $B_V=(v_1,\dots,v_n)$ will do. □

---

**3** *Supp $V$ and $W$ are finide and $T \in \mathcal{L}(V,W)$. Prove $\exists\, B_V, B_W$ suth*
*[ letting $A = \mathcal{M}(T, B_V, B_W)$ ] $A_{k,k} = 1, A_{i,j} = 0$, where $1 \leqslant k \leqslant \dim \operatorname{range} T, i \neq j$.*

**Solus:** Let $B_{\operatorname{null} T} = (u_1, \dots, u_m), B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n) \Rightarrow B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$. ☐

**Comment:** Let each $Tv_k = w_k$. Extend $B_{\operatorname{range} T}$ to $B_W = (w_1, \dots, w_n, \dots, w_p)$. See [3.D Note For [3.60]].

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**4** *Supp $B_V = (v_1, \dots, v_m)$ and $W$ is finide. Supp $T \in \mathcal{L}(V,W)$.*
*Prove $\exists\, B_W = (w_1, \dots, w_n)$, $\mathcal{M}\big(T, B_V, B_W\big)_{\cdot,1} = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}^t$ or $\begin{pmatrix}0 & \cdots & 0\end{pmatrix}^t$.*

**Solus:** If $Tv_1 = 0$, then done. If not then extend $(Tv_1)$ to $B_W$. ☐

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**5** *Supp $B_W = (w_1, \dots, w_n)$ and $V$ is finide. Supp $T \in \mathcal{L}(V,W)$.*
*Prove $\exists\, B_V = (v_1, \dots, v_m)$, $\mathcal{M}\big(T, B_V, B_W\big)_{1,\cdot} = \begin{pmatrix}0 & \cdots & 0\end{pmatrix}$ or $\begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}$.*

**Solus:**

Let $(u_1, \dots, u_n)$ be a bss of $V$. Denote $\mathcal{M}\big(T, (u_1, \dots, u_n), B_W\big)$ by $A$.

If $A_{1,\cdot} = 0$, then $B_V = (u_1, \dots, u_n)$ and done. Othws, supp $A_{1,k} \neq 0$.

Let $v_1 = \dfrac{u_k}{A_{1,k}} \Rightarrow Tv_1 = 1w_1 + \dfrac{A_{2,k}}{A_{1,k}} w_2 + \cdots + \dfrac{A_{n,k}}{A_{1,k}} w_n$. | Let $v_{j+1} = u_j - A_{1,j} v_1$ for each $j \in \{1, \dots, k-1\}$. Let $v_i = u_i - A_{1,i} v_1$ for $i \in \{k+1, \dots, n\}$.

Notice that $Tu_i = A_{1,i} w_1 + \cdots + A_{n,i} w_n$. 又 Each $u_i \in \operatorname{span}(v_1, \dots, v_n) = V$. Let $B_V = (v_1, \dots, v_n)$. ☐

Or. Using Exe (4). Let $B_{W'}$ be the $B_V$. Now $\exists\, B_{V'}$ suth $\mathcal{M}(T', B_{W'}, B_{V'})_{\cdot,1} = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}^t$ or $\begin{pmatrix}0 & \cdots & 0\end{pmatrix}^t$.
Which is equiv to $\exists\, B_V$ [Using (3.F.31)] suth $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}$ or $\begin{pmatrix}0 & \cdots & 0\end{pmatrix}$. ☐

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• (10.A.3, Or 4E 3.D.19) *Supp $V$ is finide and $T \in \mathcal{L}(V)$.* [*See also in (3.A).*]
*Prove $\forall B_V \neq B_V'$, $\mathcal{M}\big(T, B_V\big) = \mathcal{M}\big(T, B_V'\big) \Longrightarrow T = \lambda I, \exists\, \lambda \in \mathbf{F}$.*

**Solus:** Supp $\forall B_V \neq B_V'$, $\mathcal{M}(T, B_V) = \mathcal{M}(T, B_V')$. If $T = 0$, then done.
Supp $T \neq 0$, and $v \in V \backslash \{0\}$. Asum $(v, Tv)$ is liney indep.
Extend $(v, Tv)$ to $B_V = (v, Tv, u_3, \dots, u_n)$. Let $B = \mathcal{M}(T, B_V)$.
$\Rightarrow Tv = B_{1,1} v + B_{2,1}(Tv) + B_{3,1} u_3 + \cdots + B_{n,1} u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0, \ \forall i \neq 2$.
By asum, $A = \mathcal{M}(T, B_V') = B, \forall B_V' = (v, w_2, \dots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$.
$\Rightarrow Tv = w_2$, which is not true if $w_2 = u_3,\ w_3 = Tv,\ w_j = u_j, \forall j \in \{4, \dots, n\}$. Ctradic.
Hence $(v, Tv)$ is linely dep $\Rightarrow \forall v \in V, \exists\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.
Now we show $\lambda_v$ is indep of $v$, that is, for all disti $v, w \in V \backslash \{0\}, \lambda_v = \lambda_w$.

$$\left.\begin{array}{l} (v,w) \text{ liney indep} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v,w) \text{ linely dep}, w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \end{array}\right\} \Rightarrow T = \lambda I.$$ ☐

Or. Let $A = \mathcal{M}(T, B_V)$, where $B_V = (u_1, \dots, u_m)$ is arb.
Fix one $B_V = (v_1, \dots, v_m)$ and then $(v_1, \dots, \frac{1}{2} v_k, \dots, v_m)$ is also a bss for any given $k \in \{1, \dots, m\}$.
Fix one $k$. Now we have $T(\frac{1}{2} v_k) = A_{1,k} v_1 + \cdots + A_{k,k}(\frac{1}{2} v_k) + \cdots + A_{m,k} v_m$
$\Rightarrow Tv_k = 2A_{1,k} v_1 + \cdots + A_{k,k} v_k + \cdots + 2A_{m,k} v_m = A_{1,k} v_1 + \cdots + A_{k,k} v_k + \cdots + A_{m,k} v_m$.
Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k} v_k, \forall k \in \{1, \dots, m\}$.
Now we show $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose $j, k$ suth $j \neq k$.
Consider $B_V' = (v_1', \dots, v_j', \dots, v_k', \dots, v_m')$, where $v_j' = v_k,\ v_k' = v_j$ and $v_i' = v_i$ for all $i \in \{1, \dots, m\} \backslash \{j, k\}$.
Now $T(v_k') = A_{1,k} v_1' + \cdots + A_{k,k} v_k' + \cdots + A_{m,k} v_m' = A_{k,k} v_k' = A_{k,k} v_j$, while $T(v_k') = T(v_j) = A_{j,j} v_j$. ☐

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**Ended**

# 3·D

• (3.E.2) *Supp $V_1 \times \cdots \times V_m$ is finide. Prove each $V_j$ is finide.*

**Solus:** For any $k \in \{1, \dots, m\}$, define $S_k \in \mathcal{L}(V_1 \times \cdots \times V_m, V_k)$ by $S_k(v_1, \dots, v_m) = v_k$.
Then $S_k$ is liney map. By [3.22], range $S_k = V_k$ is finide. $\square$

Or. Denote $V_1 \times \cdots \times V_m$ by $U$. Denote $\{0\} \times \cdots \times \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by $U_i$.
We show each $U_i$ is iso to $V_i$. Then $U$ is finide $\Longrightarrow$ its subsp $U_i$ is finide, so is $V_i$.

$$\left.\begin{array}{l}\text{Define } R_i \in \mathcal{L}(V_i, U_i) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0) \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i\end{array}\right\} \Rightarrow \left\{\begin{array}{l} R_i S_j|_{U_j} = \delta_{i,j} I_{U_j}, \\ S_i R_j = \delta_{i,j} I_{V_j}.\end{array}\right.$$ $\square$

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• (3.E.4) *Prove $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.*

**Solus:** Using nota in (3.E.2): $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0);\ S_i : (u_1, \dots, u_m) \mapsto u_i$.
Note that $T(u_1, \dots, u_m) = T(u_1, 0, \dots, 0) + \cdots + T(0, \dots, u_m)$.

$$\left.\begin{array}{l}\text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (TR_1, \dots, TR_m). \\ \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m.\end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ $\square$

---

• (3.E.5) *Prove $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.*

**Solus:** Using nota in (3.E.2): $R_i : u_i \mapsto (0, \dots, u_i, \dots, 0);\ S_i : (u_1, \dots, u_m) \mapsto u_i$.

$$\begin{array}{l|l}\text{Note that } T_i : v \mapsto w_i, & \text{Define } \varphi : T \mapsto (T_1, \dots, T_m) \text{ by } \varphi(T) = (S_1 T, \dots, S_m T). \\ T : v \mapsto (w_1, \dots, w_m). & \text{Define } \psi : (T_1, \dots, T_m) \mapsto T \text{ by } \psi(T_1, \dots, T_m) = R_1 T_1 + \cdots + R_m T_m.\end{array}\ \Bigg\} \Rightarrow \psi = \varphi^{-1}.$$ $\square$

---

**18** *Show $V$ and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.*

**Solus:** Define $\Psi \in \mathcal{L}\big(V, \mathcal{L}(\mathbf{F}, V)\big)$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.
(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Now $\Psi$ inje.
(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi\big(T(1)\big) \in \operatorname{range} \Psi$. $\square$

Or. Define $\Phi \in \mathcal{L}\big(\mathcal{L}(\mathbf{F}, V), V\big)$ by $\Phi(T) = T(1)$.
(a) Supp $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Now $\Phi$ inje.
(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v \in \operatorname{range} \Phi$. $\square$

**Comment:** $\Phi = \Psi^{-1}$. This is a countexa of the stmt that $\mathcal{L}(V, W)$ and $\mathcal{L}(W, V)$ are iso if infinde. See (3.F).

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• (3.E.6) *Supp $m \in \mathbf{N}^+$. Prove $V^m$ and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.*

**Solus:** Using (3.D.18) and (3.E.4), immed. $\square$

Or. Define $T : (v_1, \dots, v_m) \to \varphi$, where $\varphi : (a_1, \dots, a_m) \mapsto a_1 v_1 + \cdots + a_m v_m$.
(a) Supp $T(v_1, \dots, v_m) = 0$. Then $\forall (a_1, \dots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \dots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$
For each $k$, let $a_k = 1, a_j = 0$ for all $j \neq k$. Then each $v_k = 0 \Rightarrow (v_1, \dots, v_m) = 0$. Thus $T$ is inje.
(b) Supp $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \dots, e_m)$ be std bss of $\mathbf{F}^m$. Then $\forall (b_1, \dots, b_n) \in \mathbf{F}^m$,
$\Big[T\big(\psi(e_1), \dots, \psi(e_m)\big)\Big](b_1, \dots, b_m) = b_1 \psi(e_1) + \cdots + b_m \psi(e_m) = \psi\big(b_1 e_1 + \cdots + b_m e_m\big) = \psi(b_1, \dots, b_m)$.
Thus $T\big(\psi(e_1), \dots, \psi(e_m)\big) = \psi$. Hence $T$ is surj. $\square$

---

• *Supp $T \in \mathcal{L}(V)$. Prove $\exists$ inv $R, S \in \mathcal{L}(V)$ suth $T = T_1 + T_2$.*

**Solus:** Let $U \oplus \operatorname{null} T = V,\ W \oplus \operatorname{range} T = V$. Let $S : \operatorname{null} T \to W$ be an iso.

$$\left.\begin{array}{l}\text{Define } T_1 \in \mathcal{L}(V) \text{ by } T_1(u) = \frac{1}{2} Tu, T_1(w) = Sw \\ \text{Define } T_2 \in \mathcal{L}(V) \text{ by } T_2(u) = \frac{1}{2} Tu, T_2(w) = -Sw\end{array}\right\} \Rightarrow T = T_1 + T_2 \text{ and } T_1, T_2 \text{ inv.}$$ $\square$

---

**2** *Supp* $\dim V \geqslant 2$. *The set* $U$ *of non-inv optors on* $V$ *is not a subsp of* $\mathcal{L}(V)$.

*The set of inv optors is not either. Although multi id/inv, and commu for vec multi hold.*

**Solus:** Simlr to (3.B.7 or 8). [ *If* $\dim V = 1$, *then* $U = \{0\}$ *is a subsp of* $\mathcal{L}(V)$. ] □

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• **Tips:** *Supp* $V = U \oplus X = W \oplus X$. *Prove* $U, W$ *are iso.*

**Solus:** $\forall u \in U, \exists!\,(w, x_1) \in W \times X, u = w + x_1$. While $\exists!\,(u', x_2) \in U \times X, w = u' + x_2$.

Now $x_1 = -x_2,\ u = u'$. Thus $\pi : U \to W$ defined by $\pi(u) = w,$ is inje.

$\forall w \in W, \exists!\,(u, x_1) \in U \times X, w = u + x_1$. While $\exists!\,(w', x_2) \in W \times X, u = w' + x_2$.

Now $x_1 = -x_2,\ w = w'$. Thus $\pi : U \to W$ defined by $\pi(u) = w,$ is surj. □

**Comment:** Let $V = \mathbf{F}^\infty$. Let $X = \mathbf{F}^\infty, Y = \{(0, x_1, x_2, \cdots) \in \mathbf{F}^\infty\}$. Now $X, Y$ are iso subsps of $V$.

But $\nexists$ iso subsps $M, N$ of $V$, suth $V = M \oplus X = N \oplus Y$.

---

• (3.E.3) *Give an exa of a vecsp* $V$ *and its two subsps* $U_1, U_2$ *suth*

$U_1 \times U_2$ *and* $U_1 + U_2$ *are iso but* $U_1 + U_2$ *is not a direct sum.* [*V must be infinide.*]

**Solus:** Note *that at least one of* $U_1, U_2$ *must be infinide. Both can be infinide.* [*Req Other Courses.*]

Let $V = \mathbf{F}^\infty = U_1,\ U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$. Then $V = U_1 + U_2$ is not a direct sum.

$$\left.\begin{array}{l}\text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T\big((x_1, x_2, \cdots), (x, 0, \cdots)\big) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = \big((x_1, x_2, \cdots), (x, 0, \cdots)\big)\end{array}\right\} \Rightarrow S = T^{-1}.$$ □

---

**3** *Supp* $V$ *and* $W$ *are iso and finide,* $U$ *is a subsp of* $V$, *and* $S \in \mathcal{L}(U, W)$.

*Prove* $\exists$ *inv* $T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U \iff S$ *is inje.* [ *See also* (3.A.11). ]

**Solus:** (a) $\forall u \in U, u = T^{-1}Su \Rightarrow T^{-1}S = I \in \mathcal{L}(U)$. Thus by (3.B.20), $S$ is inje.

Or. $\operatorname{null} S = \operatorname{null} T|_U = \operatorname{null} T \cap U = \{0\}$.

(b) Let $B_U = (u_1, \dots, u_m)$. Then $S$ inje $\Rightarrow (Su_1, \dots, Su_m)$ liney indep.

Extend to $B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_W = (Su_1, \dots, Su_m, w_1, \dots, w_n)$.

Define $T \in \mathcal{L}(V, W)$ by $T(u_i) = Su_i;\ \ Tv_j = w_j$, for each $u_i$ and $v_j$. □

**Exa:** Supp $V, W$ are infinide. Let $V = W = \mathbf{F}^\infty$. Define $S(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$.

Now $S$ is inje. Supp $\exists$ inv $T \in \mathcal{L}(V, W)$ suth $T|_V = S$. Then $T = S$ while $S$ is not surj.

---

**8** *Supp* $T \in \mathcal{L}(V, W)$ *is* ***surj***. *Prove* $\exists$ *subsp* $U$ *of* $V,\ T|_U : U \to W$ *is iso.*

**Solus:** By (3.B.12). Note that $\operatorname{range} T = W$. Or. [ *Req* range $T$ *Finide* ] By [3.B Tips (4)]. □

---

• **Comment:** If $S \in \mathcal{L}(V)$ is iso, $T \in \mathcal{L}(U, W)$ is iso, and $W \subsetneq V$, then $ST = S|_W T$ is merely inje.

---

**9** [Or 1] *Supp* $U, V, W$ *are iso and finide,* $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.

*Prove* $ST$ *is inv* $\iff S, T$ *are inv.*

**Note:** Supp one of $U, V, W$ infinide $\Rightarrow$ all infinde. Then $S, T$ inv $\Longrightarrow ST$ inv.

**Solus:** Supp $S, T$ inv. Then $(ST)(T^{-1}S^{-1}) = I_W, (T^{-1}S^{-1})(ST) = I_U$. Hence $ST$ inv.

Supp $ST$ inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = I_U, (ST)R = I_W$.

$$\begin{array}{l|l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0. & T \text{ inje, } S \text{ surj.} \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S. & \text{又 } \dim U = \dim V = \dim W. \end{array}$$

Or. By (3.B.23), $\dim W = \dim \operatorname{range} ST \leqslant \min\{\operatorname{range} S, \operatorname{range} T\} \Rightarrow S, T$ surj. □

---

- **TIPS:** Supp each $S_k \in \mathcal{L}(V_k, W_k), W_k \subseteq V_{k+1} \Rightarrow S_m \circ S_{m-1} \circ \cdots \circ S_2 \circ S_1$ makes sense.
  (a) By Exe (9), if all $V_k$ finide and iso to each other, then $S_m \circ \cdots \circ S_1$ inje $\Rightarrow$ inv, so are all $S_k$.
  (b) By the ctrapos of (3.B.11), $S_m \circ \cdots \circ S_1$ not inje $\Rightarrow \exists\, S_k$ not inje. Convly not true unless $k = 1$.
  (c) $\operatorname{null} S_1 \subseteq \operatorname{null}(S_2 S_1) \subseteq \cdots \subseteq \operatorname{null}(S_m \cdots S_2 S_1)$; $S_m \circ \cdots \circ S_1$ inje $\Rightarrow$ each $S_k \circ \cdots \circ S_1$ inje.
  Supp each $W_k = V_{k+1}$, for if $W_k \subsetneq V_{k+1}$, then $S_1, S_2$ surj $\not\Rightarrow S_2 \circ S_1 \in \mathcal{L}(V_1, W_2)$ surj.
  (d) Each $S_k$ surj $\Rightarrow S_m \circ \cdots \circ S_1$ surj. Convly not true unless all $V_k$ finide and iso to each other.
  (e) $\operatorname{range} S_m \supseteq \operatorname{range}(S_m S_{m-1}) \supseteq \cdots \supseteq \operatorname{range}(S_m S_{m-1} \cdots S_1)$; $S_m \circ \cdots \circ S_1$ surj $\Rightarrow$ each $S_m \circ \cdots \circ S_k$ surj.

---

**13** *Supp $U, V, W, X$ are iso and finide, $R \in \mathcal{L}(W, X), S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.*
*Supp $RST$ is surj. Prove $S$ is inje.*

**SOLUS:** Using Exe (9). Notice that $U, X$ are finide, so that $RST$ inv.

$$\text{Let } X = (RST)^{-1} \left| \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ inje.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ surj.} \end{array} \right\} \Rightarrow S = R^{-1}(RST)T^{-1}. \qquad \square$$

OR. $(RST)^{-1} = \big((RS)T\big)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. $\square$

---

**10** *Supp $V$ is finide and $S, T \in \mathcal{L}(V)$. Prove $ST = I \iff TS = I$.*

**SOLUS:** Supp $ST = I$. By (3.B 20, 21)(a), $ST = I \Rightarrow T$ inje and $S$ surj. 又 $V$ finide. $S, T$ inv.
OR. By Exe (9), $V$ finide and $ST = I$ inv $\Rightarrow S, T$ inv.
Then $\forall v \in V, S\big((TS)v\big) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow TS = I$.
OR. $S^{-1} = T$ 又 $S = S \Rightarrow TS = S^{-1}S = I$. Rev the roles and done. $\square$

---

**11** *Supp $V$ is finide, $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show $T$ is inv and $T^{-1} = US$.*

**SOLUS:** Using Exe (9) and (10). *This result can fail without the hypo that $V$ is finide.*
$(ST)U = U(ST) = (US)T = I \Rightarrow T^{-1} = US$.
OR. $(ST)U = S(TU) = I \Rightarrow U, S$ inv $\Rightarrow TU = S^{-1}$. 又 $U^{-1} = U^{-1} \Rightarrow T = S^{-1}U^{-1}$. $\square$

**EXA:** $V = \mathbf{R}^\infty, S(a_1, a_2, \cdots) = (a_2, \cdots)$; $T(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots)$; $U = I \Rightarrow STU = I$ but $T$ is not inv.

---

**15** *Prove every liney map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multi.*
*In other words, prove if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists\, A \in \mathbf{F}^{m,n}, Tx = Ax$, $\forall x \in \mathbf{F}^{n,1}$.*

**SOLUS:** Let $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$ be std bses of $\mathbf{F}^{n,1}, \mathbf{F}^{m,1}$.
$\forall k = 1, \dots, n$, $T(E_k) = A_{1,k}R_1 + \cdots + A_{m,k}R_m, \exists\, A_{j,k} \in \mathbf{F}$, forming $A$.
OR. Let $A = \mathcal{M}(T, B_1, B_2)$. Note that $\mathcal{M}(x, B_1) = x, \mathcal{M}(Tx, B_2) = Tx$.
Hence $Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax$, by [3.65]. $\square$

---

- **NOTE FOR [3.62]:** $\mathcal{M}(v) = \mathcal{M}(I, (v), B_V)$. Where $I$ is the id optor restr to $\operatorname{span}(v)$.
- **NOTE FOR [3.65]:** $\mathcal{M}(Tv) = \mathcal{M}(I, (Tv), B_W) = \mathcal{M}(T, B_V, B_W)\mathcal{M}(I, (v), B_V) = \mathcal{M}(T, (v), B_W)$.
  If $v = 0$, then $\operatorname{span}(v) = \operatorname{span}(\,)$, we replace $(v)$ by $B = (\,)$; simlr for $Tv = 0$.

---

- **TIPS:** When using $\mathcal{M}^{-1}$, you must first declare bses and the purpose for using $\mathcal{M}^{-1}$.
  That is, to declare $B_U, B_V, B_W$, $\mathcal{M} : \mathcal{L}(V, W) \mapsto \mathbf{F}^{m,n}$, or $\mathcal{M} : v \mapsto \mathbf{F}^{n,1}$.
  So that $\mathcal{M}^{-1}(AC, B_U, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(C, B_U, B_V)$;
  OR. $\mathcal{M}^{-1}(Ax, B_W) = \mathcal{M}^{-1}(A, B_V, B_W)\mathcal{M}^{-1}(x, B_V)$. Where everything is well-defined.

---

• (4E 22, OR 10.A.1) *Supp $T \in \mathcal{L}(V)$. Prove $\mathcal{M}(T, \alpha \to \beta)$ is inv $\iff$ $T$ itself is inv.*

**SOLUS:** Notice that $\mathcal{M}: T \mapsto \mathcal{M}(T, \alpha \to \beta)$ is iso. And that $\mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(TS)$.

(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(I) = \mathcal{M}(T)\mathcal{M}(T^{-1}) \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I$, $\exists! S \in \mathcal{L}(V)$ suth $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$

$\Rightarrow \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = I = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$

$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. ☐

**CORO:** Supp $A \in \mathbf{F}^{n,n}$. Then $A$ is inv $\iff \exists$ inv $T \in \mathcal{L}(\mathbf{F}^n)$ suth $\mathcal{M}\big(T, (e_1, \dots, e_n), (f_1, \dots, f_n)\big) = A$.

---

• (4E 24, OR 10.A.2) *Supp $A, B \in \mathbf{F}^{n,n}$. Prove $AB = I \iff BA = I$.* [*Using Exe (10, 15).*]

**SOLUS:** Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Now $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.

$AB = I \Leftrightarrow A(Bx) = x \iff T(Sx) = x \Leftrightarrow TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I$.

OR. Becs $\mathcal{M}: \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1}) \to \mathbf{F}^{n,n}$ is iso. $\mathcal{M}^{-1}(AB) = TS = ST = \mathcal{M}^{-1}(BA) = I$. ☐

---

• **NEW NOTA:** For ease of nota, let $\mathcal{M}(T, \alpha \to \beta) = \mathcal{M}\big(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)\big)$.

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• (4E 23, OR 10.A.4) *Supp that $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bses of $V$.
Let $T \in \mathcal{L}(V)$ be suth each $T\alpha_k = \beta_k$. Prove $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)$.*

**SOLUS:**

Denote $\mathcal{M}(T, \alpha \to \alpha)$ by $A$ and $\mathcal{M}(I, \beta \to \alpha)$ by $B$.

$\forall k \in \{1, \dots, n\}, I\beta_k = \beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = T\alpha_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. ☐

OR. Note that $\mathcal{M}(T, \alpha \to \beta) = I$. Hence $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha) \underbrace{\mathcal{M}(T, \alpha \to \beta)}_{= \mathcal{M}(I, \beta \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. ☐

OR. Note that $\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I$.

$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}\Big(\underbrace{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}_{= \mathcal{M}(T, \alpha \to \beta)}\Big) = \mathcal{M}(I, \beta \to \alpha)$. ☐

**COMMENT:** Let $A' = \mathcal{M}(T, \beta \to \beta)$.

$\beta_k = I\beta_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n, \ \forall\, k \in \{1, \dots, n\}$.

又 $T\beta_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.

OR. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$.

---

• **NOTE FOR [3.60]:** Supp $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.

Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{i,x}w_j$. **CORO:** $E_{l,k}E_{i,j} = \delta_{j,l}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. And $\left(\mathcal{E}^{(j,i)}\right)_{l,k} = \begin{cases} 1, & \text{if } (i,j) = (l,k); \\ 0, & \text{othws.} \end{cases}$

Becs $\mathcal{M}: \mathcal{L}(V, W) \to \mathbf{F}^{m,n}$ is iso.

$E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$. By [2.42] and [3.61], $B_{\mathcal{L}(V,W)} = \begin{pmatrix} E_{1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \cdots & , E_{n,m} \end{pmatrix}$; $B_{\mathbf{F}^{m,n}} = \begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & , \mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots & , \mathcal{E}^{(m,n)} \end{pmatrix}$.

• **TIPS:** Let $B_{\operatorname{range} T} = (Tv_1, \dots, Tv_p), B_V = (v_1, \dots, v_p, \dots, v_n)$. Let each $w_k = Tv_k$.

Extend to $B_W = (w_1, \dots, w_p, \dots, w_m)$. Then $T = E_{1,1} + \cdots + E_{p,p}$, $\mathcal{M}(T) = \mathcal{E}^{(1,1)} + \cdots + \mathcal{E}^{(p,p)}$.

---

**17** *Supp $V$ is finide. Show the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

**SOLUS:** If $\mathcal{E} = \{0\}$, then done. Supp $0 \neq T \in \mathcal{E}$, a two-sided ideal of $\mathcal{L}(V)$. Let $w = Tv \neq 0$.

Extend $v = v_1$ to $B_V = (v_1, \dots, v_n) \Rightarrow Tv_1 = a_1v_1 + \cdots + a_nv_n$. Supp $a_k \neq 0$.

Then each $E_{k,y}TE_{x,1} = E_{k,y}\left[a_1E_{x,1} + \cdots + a_kE_{x,k} + \cdots + a_nE_{k,n}\right] = a_kE_{x,y} \in \mathcal{E}$. ☐

---

• (4E 17) *Supp $V$ is finide and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(T) = ST$.*
(a) *Show* $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Show* $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

**Solus:** (a) $\forall T \in \mathcal{L}(V), ST = 0 \iff \operatorname{range} T \subseteq \operatorname{null} S$.
Thus $\operatorname{null} \mathcal{A} = \{T \in \mathcal{L}(V) : \operatorname{range} T \subseteq \operatorname{null} S\} = \mathcal{L}(V, \operatorname{null} S)$.
(b) $\forall R \in \mathcal{L}(V), \operatorname{range} R \subseteq \operatorname{range} S \iff \exists\, T \in \mathcal{L}(V), R = ST$, by (3.B 25).
Thus $\operatorname{range} \mathcal{A} = \{R \in \mathcal{L}(V) : \operatorname{range} R \subseteq \operatorname{range} S\} = \mathcal{L}(V, \operatorname{range} S)$. □

Or. Using Note For [3.60]. Let $B_{\operatorname{range} S} = (\overbrace{w_1, \dots, w_m}^{Sv_i = w_i}), B_U = (v_1, \dots, v_m)$.
Let $(w_1, \dots, w_n), (v_1, \dots, v_n)$ be bses of $V$. Now $S = E_{1,1} + \cdots + E_{m,m}$. $\mathcal{M}(S, v \to w) = \begin{pmatrix} 1 & \ddots & 0 & \ddots & 0 \\ 0 & \ddots & 1 & \ddots & 0 \\ 0 & \ddots & 0 & \ddots & 0 \end{pmatrix}$.
Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j} : w_x \mapsto \delta_{i,x} v_i$. Let $E_{j,k} R_{i,j} = Q_{i,k},\ R_{j,k} E_{i,j} = G_{i,k}$.
Where $E_{i,k} : v_x \mapsto \delta_{i,x} w_k,\ Q_{i,k} : w_x \mapsto \delta_{i,x} w_k,$ and $G_{i,k} : v_x \mapsto \delta_{i,x} v_k$.
For any $T \in \mathcal{L}(V), \exists!\, A_{i,j} \in \mathbf{F}, T = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i} \Longrightarrow \mathcal{M}(T, w \to v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} & \cdots & A_{n,m} \end{pmatrix}$.
$\Longrightarrow \mathcal{A}(T) = ST = \left(\sum_{r=1}^{m} E_{r,r}\right)\left(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j} Q_{j,i}$.

$$\begin{matrix} \mathcal{M}(S, v \to w)\mathcal{M}(T, w \to v) = \mathcal{M}(ST, w) = \\ \mathcal{M}(\mathcal{A}, R \to Q)\mathcal{M}(T, R) = \mathcal{M}\big(\mathcal{A}(T), Q\big) = \end{matrix} \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & A_{m,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

又 $\mathcal{M}(T, R) = \mathcal{M}(T, w \to v)$.
Let $T = I$, we have
$\mathcal{M}(\mathcal{A}, R \to Q) = \mathcal{M}(S, v \to w)$.

$$\operatorname{range} \mathcal{A} = \operatorname{span}\left\{\begin{matrix} Q_{1,1}, & \cdots & , Q_{n,1}, \\ \vdots & \ddots & \vdots \\ Q_{1,m}, & \cdots & , Q_{n,m} \end{matrix}\right\},\ \operatorname{null} \mathcal{A} = \operatorname{span}\left\{\begin{matrix} R_{1,m+1}, & \cdots & , R_{n,m+1}, \\ \vdots & \ddots & \vdots \\ R_{1,n}, & \cdots & , R_{n,n} \end{matrix}\right\}.$$

(a) $\dim \operatorname{null} \mathcal{A} = n \times (n - m)$;
(b) $\dim \operatorname{range} \mathcal{A} = n \times m$. □

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• (4E 10) *Supp $V, W$ are finide, $U$ is a subsp of $V$.*
*Let $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V, W) : T|_U = 0\}$.*
(a) *Show $\mathcal{E}$ is a subsp of $\mathcal{L}(V, W)$.*
(b) *Find a formula for* $\dim \mathcal{E}$ *in terms of* $\dim V$, $\dim W$ *and* $\dim U$.
***Hint***: *Define $\Phi : \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. What is* $\operatorname{null} \Phi$? *What is* $\operatorname{range} \Phi$?

**Solus:**
(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.
(b) Define $\Phi$ as in the hint. $\Phi$ is liney, by [3.A Note For Restriction].
$\Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$. Thus $\operatorname{null} \Phi = \mathcal{E}$.
Extend $S \in \mathcal{L}(U, W)$ to $T \in \mathcal{L}(V, W) \Rightarrow \Phi(T) = S \in \operatorname{range} \Phi$. Thus $\operatorname{range} \Phi = \mathcal{L}(U, W)$.
Thus $\dim \operatorname{null} \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \operatorname{range} \Phi = (\dim V - \dim U) \dim W$. □

Or. Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1, \dots, v_n)$. Let $p = \dim W$. [ See Note For [3.60]. ]

$$\forall\, T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \underbrace{\operatorname{span}\left\{\begin{matrix} E_{1,1}, & \cdots & , E_{m,1}, \\ \vdots & \ddots & \vdots \\ E_{1,p}, & \cdots & , E_{m,p} \end{matrix}\right\}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

$$又\ W = \operatorname{span}\left\{\begin{matrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{matrix}\right\} \subseteq \mathcal{E}.$$

Where $\mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}$.
Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. □

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• *Define* $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ *by* $\mathcal{B}(T) = TS$.
(a) *Show* $\dim \operatorname{null} \mathcal{B} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Show* $\dim \operatorname{range} \mathcal{B} = (\dim V)(\dim \operatorname{range} S)$.

**Solus:** (a) $\forall T \in \mathcal{L}(V), TS = 0 \iff \operatorname{range} S \subseteq \operatorname{null} T$.
Thus $\operatorname{null} \mathcal{B} = \{T \in \mathcal{L}(V) : \operatorname{range} S \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V) : T|_{\operatorname{range} S} = 0\}$.
(b) $\forall R \in \mathcal{L}(V), \operatorname{null} S \subseteq \operatorname{null} R \iff \exists\, T \in \mathcal{L}(V), R = TS$, by (3.B.24).
Thus $\operatorname{range} \mathcal{B} = \{R \in \mathcal{L}(V) : \operatorname{null} S \subseteq \operatorname{null} R\} = \{R \in \mathcal{L}(V) : R|_{\operatorname{null} S} = 0\}$.
Using [3.22] and Exe (4E 10). □

Or. Using Note For [3.60] and nota in Exe (4E 17).

$$\mathcal{B}(T) = TS = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} R_{j,i}\right)\left(\sum_{r=1}^{m} E_{r,r}\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{i,j} G_{j,i} \Longrightarrow \mathcal{M}(TS, v) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} & \cdots & 0 \end{pmatrix}.$$

$$\operatorname{range} \mathcal{B} = \operatorname{span}\begin{Bmatrix} G_{1,1}, & \cdots & , G_{m,1}, \\ \vdots & \ddots & \vdots \\ G_{1,n}, & \cdots & , G_{m,n} \end{Bmatrix}, \quad \operatorname{null} \mathcal{B} = \operatorname{span}\begin{Bmatrix} R_{m+1,1}, & \cdots & , R_{n,1}, \\ \vdots & \ddots & \vdots \\ R_{m+1,n}, & \cdots & , R_{n,n} \end{Bmatrix}.$$

(a) $\dim \operatorname{null} \mathcal{B} = n \times (n - m)$;
(b) $\dim \operatorname{range} \mathcal{B} = n \times m$. □

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**16** *Supp $V$ is finide and $S \in \mathcal{L}(V)$ suth $\forall T \in \mathcal{L}(V), ST = TS$. Prove $\exists\, \lambda \in \mathbf{F}, S = \lambda I$.*

**Solus:** If $S = 0$, done. Now supp $S \neq 0$. [*Using nota in Exe (4E 17). See also in (3.A).*]
Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, B_U) = \mathcal{M}(I, B_{\operatorname{range} S}, B_U)$. Note that $R_{k,1} : w_x \mapsto \delta_{k,x} v_1$.
Then $\forall k \in \{1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1} S$. Hence $\dim \operatorname{null} S = 0,\ \dim \operatorname{range} S = m = n$.
Notice that $G_{i,j} = R_{i,j} S = SR_{i,j} = Q_{i,j}$. Where $G_{i,j} : v_x \mapsto \delta_{i,x} v_j$; $Q_{i,j} : w_x \mapsto \delta_{i,x} w_j$.
For each $w_i, \exists!\, a_{k,i} \in \mathbf{F}, w_i = a_{1,i} v_1 + \cdots + a_{n,i} v_n$. Where $a_{k,i} = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))_{k,i}$.
Then fix one $i$. Now for each $j \in \{1, \dots, n\}, Q_{i,j}(w_i) = w_j = a_{i,i} v_j = G_{i,j}\left(\sum_{k=1}^{n} a_{k,i} v_k\right)$.
Let $\lambda = a_{i,i}$. Hence each $w_j = \lambda v_j$. Now fix one $j$, we have $a_{1,1} v_j = \cdots = a_{n,n} v_j$, then all $a_{i,i}$ are equal.
Thus each $w_j = \lambda v_j \Longrightarrow \mathcal{M}(S, B_U) = \mathcal{M}(\lambda I)$. □

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**Ended**

# 3·E

• **Note For [3.79], def of $v + U$:** Given $v + U$, $v$ is already uniqly determined, as a sort of precond. Even though $v + U = v' + U$, where $v'$ is *purer* than $v$.

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• **Note For [3.85]:** $v + U = w + U \iff v \in w + U,\ w \in v + U$
$\iff v - w \in U \iff (v + U) \cap (w + U) \neq \emptyset$.

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• **NOTE FOR [3.79, 3.83]:**
If $U$ is merely a subset of $V$, then [3.85, 86] do not hold $\Rightarrow V/U$ not a vecsp.
If $V$ is merely a subset of a vecsp of which $U$ is a subsp, then [3,79, 86] do not hold $\Rightarrow V/U$ not a vecsp.
If $U$ is a vecsp but not a subsp of $V$, while $U, V$ are subsps of some vecsp, then everything's alright.
Hence if $V/U$ is a vecsp, then $V, U$ are subsps of some vecsp.
**COMMENT:** Supp $U, V$ are subsps and $U$ is not a subsp of $V$. Note that $V/U = (V+U)/U$.
Supp $v + U \in V/U$. Then $v \in V$, or possibly $v \in V + U$ as well. To avoid this ambiguity, you have to specify the precond, what subsp that $v$ belongs to.

**EXA:** Supp $U + W = V$. Then $V/U = (U+W)/U = W/U$. Let $W \cap U = I, U_I \oplus I = U, W_I \oplus I = W$.
Now $U_I \oplus W_I \oplus I = V$. Thus $W/U = (W_I \oplus I)/U = W_I/U$.
$\forall w_1', w_2' \in W_I$ suth $w_1' + U = w_2' + U \in W_I/U,\ w_1' - w_2' \in U \cap W_I = \{0\} \Rightarrow w_1' = w_2'$.

• ***Trivial Cases*:** If $v \in U$, then $v + U = 0 + U = \{u : u \in U\} = U$. Now $U = 0 \in V/U$.
If $U = \{0\}$, then $v + U = v + \{0\} = \{v\},\ V/U = V/\{0\} = \{\{v\} : v \in V\}$.
If $U = \emptyset$, then $v + U = v + \emptyset = \emptyset,\ V/U = V/\emptyset = \{\emptyset\}$.

• **TIPS 1:** $V$ is a subsp of $U \iff \forall v \in V, v + U = 0 + U = U \iff V/U = \{0\} = \{U\}$.

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• **NOTE FOR [3.88]:** If $U, V$ are subsp of some vecsp $\mathcal{V}$. Define the quot map $\pi \in \mathcal{L}(V, V/U)$.
Then $\pi$ is surj by def, and null $\pi = V \cap U$. Thus if $\mathcal{V}$ is finide, then $\dim V = \dim V/U + \dim (V \cap U)$.
OR. Let $I = V \cap U,\ V_I \oplus I = V$. Becs $V/U = V_I/U$, iso to $V_I$. Now $\dim V = \dim V_I + \dim I$.

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• (4E 8) *Supp* $T \in \mathcal{L}(V, W), w \in \operatorname{range} T$. *Prove* $\{v \in V : Tv = w\} = u + \operatorname{null} T$.

**SOLUS:** Let $\mathcal{K}_w = \{v \in V : Tv = w\}$. [ Not a vecsp. ] Supp $u \in \mathcal{K}_w$. Then $u + \operatorname{null} T \subseteq \mathcal{K}_w$.
And $\forall u' \in \mathcal{K}_w,\ u' - u \in \operatorname{null} T \Rightarrow u' \in u + \operatorname{null} T$. Now $\mathcal{K}_w \subseteq u + \operatorname{null} T$. ☐

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**7** *Supp* $\alpha, \beta \in V$, *and* $U, W$ *are subsps of* $V$. *Prove* $\alpha + U = \beta + W \Rightarrow U = W$.

**SOLUS:** (a) $\alpha \in \alpha + U = \beta + W \Rightarrow \exists\, w \in W, \alpha = \beta + w \Rightarrow \alpha - \beta \in W$.
(b) $\beta \in \beta + W = \alpha + U \Rightarrow \exists\, u \in U, \beta = \alpha + u \Rightarrow \beta - \alpha \in U$.
Now $\beta + U = \alpha + U = \beta + W = \alpha + W$. Thus $\{\alpha + u : u \in U\} = \{\alpha + w : w \in W\} \Rightarrow U = W$.

OR. $\pm(\alpha - \beta) \in U \cap W \Rightarrow \left\{ \begin{array}{l} U \ni u = (\beta - \alpha) + w \in W \Rightarrow U \subseteq W \\ W \ni w = (\alpha - \beta) + u \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W.$ ☐

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**8** *Supp* $A$ *is a nonempty subset of* $V$.
*Prove* $A$ *is a trslate of some subsp of* $V \iff \lambda v + (1-\lambda) w \in A\ , \forall v, w \in A, \lambda \in \mathbf{F}$.

**SOLUS:** (a) Supp $A = a + U$. Then $\lambda(a + u_1) + (1-\lambda)(a + u_2) = a + \big(\lambda(u_1 - u_2) + u_2\big) \in A$.
(b) Supp $\lambda v + (1-\lambda) w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Supp $a \in A$ and let $A' = \{x - a : x \in A\}$.
Then $0 \in A'$ and $\forall (v - a), (w - a) \in A', \lambda \in \mathbf{F}$,
(I) $\lambda(v - a) = [\lambda v + (1-\lambda) a] - a \in A'$.
(II) Becs $\lambda(v - a) + (1-\lambda)(w - a) = [\lambda v + (1-\lambda) w] - a \in A'$.
Let $\lambda = \frac{1}{2}$ here and use (I) above by $\lambda = 2$, we have $(v - a) + (w - a) \in A'$.
OR. Note that $v, a \in A \Rightarrow \lambda v + (1-\lambda) a = 2v - a \in A$. Simlr $2w - a \in A$.
Now $(v - \frac{1}{2}a) + (w - \frac{1}{2}a) = v + w - a \in A \Rightarrow v + w - 2a = (v - a) + (w - a) \in A'$.
Thus $A' = -a + A$ is a subsp of $V$. Hence $a + A' = a + \{x - a : x \in A\} = A$ is a trslate. ☐

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**9** *Supp $A = \alpha + U$ and $B = \beta + W$ for some $\alpha, \beta \in V$ and some subsps $U, W$ of $V$. Prove $A \cap B$ is either a trslate of some subsp of $V$ or is $\emptyset$.*

**Solus:** $\forall \alpha + u, \beta + w \in A \cap B \neq \emptyset, \lambda \in \mathbf{F}, \lambda(\alpha + u) + (1-\lambda)(\beta + w) \in A \cap B$. By Exe (8). $\square$

Or. Let $A = \alpha + U,\ B = \beta + W$. Supp $v \in (\alpha + U) \cap (\beta + W) \neq \emptyset$.

Then $v - \alpha \in U \Rightarrow v + U = \alpha + U = A$, and simlr $v + W = \beta + W = B$.

We show $A \cap B = v + (U \cap W)$. Note that $v + (U \cap W) \subseteq A \cap B$.

And $\forall \gamma = v + u = v + w \in A \cap B \Rightarrow u = w \in U \cap W \Rightarrow \gamma \in v + (U \cap W)$. $\square$

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**10** *Prove the intersec of any collec of trslates of subsps is either a trslate of some subsps or $\emptyset$.*

**Solus:** Supp $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collec of trslates of subsps of $V$, where $\Gamma$ is an index set.

$\forall x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset, \lambda \in \mathbf{F}, \lambda x + (1-\lambda) y \in A_\alpha$ for each $\alpha$. By Exe (8). $\square$

Or. Let each $A_\alpha = w_\alpha + V_\alpha$. Supp $x \in \bigcap_{\alpha \in \Gamma} (w_\alpha + V_\alpha) \neq \emptyset$.

Then $x - w_\alpha \in V_\alpha \Longrightarrow x + V_\alpha = w_\alpha + V_\alpha = A_\alpha$, for each $\alpha$.

We show $\bigcap_{\alpha \in \Gamma} A_\alpha = \bigcap_{\alpha \in \Gamma} (x + V_\alpha) = x + \bigcap_{\alpha \in \Gamma} V_\alpha$.

$y \in \bigcap_{\alpha \in \Gamma} A_\alpha \Longleftrightarrow$ for each $\alpha$, $y = x + v_\alpha \in A_\alpha$

$\Longleftrightarrow$ each $v_\alpha = y - x \in \bigcap_{\alpha \in \Gamma} V_\alpha \Longleftrightarrow y \in x + \bigcap_{\alpha \in \Gamma} V_\alpha$. $\square$

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**11** *Supp $A = \left\{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^{m} \lambda_i = 1\right\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.*

(a) *Prove $A$ is a trslate of some subsp of $V$*

(b) *Prove if $B$ is a trslate of some subsp of $V$ and $\{v_1, \ldots, v_m\} \subseteq B$, then $A \subseteq B$.*

(c) *Prove $A$ is a trslate of some subsp of $V$ of dim $< m$.*

**Solus:** (a) By Exe (8), $\forall u, w \in A, \lambda \in \mathbf{F}, \lambda u + (1-\lambda) w = \left(\lambda \sum_{i=1}^{m} a_i + (1-\lambda) \sum_{i=1}^{m} b_i\right) v_i \in A$.

(b) Supp $B = v + U$, where $v \in V$ and $U$ is a subsp of $V$. <u>Let each $v_k = v + u_k \in B, \exists!\, u_k \in U$.</u>

$\forall w \in A,\ w = \sum_{i=1}^{m} \lambda_i v_i = \sum_{i=1}^{m} \lambda_i (v + u_i) = \underline{\sum_{i=1}^{m} \lambda_i v} + \sum_{i=1}^{m} \lambda_i u_i = v + \sum_{i=1}^{m} \lambda_i u_i \in v + U = B$. $\square$

Or. Let $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$. To show $v \in B$, use induc on $m$ by $k$.

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By Exe (8), $v \in B$.

(ii) $2 \leqslant k < m$. Asum $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. $\left[\forall \lambda_i \text{ suth } \sum_{i=1}^{k} \lambda_i = 1\right]$

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. Fix one $\mu_\iota \neq 1$.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Longrightarrow \left[\sum_{i=1}^{k+1} \dfrac{\mu_i}{1-\mu_\iota}\right] - \dfrac{\mu_\iota}{1-\mu_\iota} = 1$.

$$\text{Let } w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let $\lambda_i = \dfrac{\mu_i}{1-\mu_\iota}$ for $i \in \{1, \ldots, \iota - 1\}$; $\lambda_j = \dfrac{\mu_{j+1}}{1-\mu_\iota}$ for $j \in \{\iota, \ldots, k\}$. Then,

$$\left.\begin{array}{r} \sum_{i=1}^{k} \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \quad \square$$

(c) If $m = 1$, then let $A = v_1 + \{0\}$ and done. Now supp $m \geqslant 2$. Fix one $k \in \{1, \ldots, m\}$.

$$\begin{aligned} A &\ni \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m) v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_m v_m \\ &= v_k + \lambda_1 (v_1 - v_k) + \cdots + \lambda_{k-1} (v_{k-1} - v_k) + \lambda_{k+1} (v_{k+1} - v_k) + \cdots + \lambda_m (v_m - v_k) \\ &\in v_k + \operatorname{span}(v_1 - v_k, \ldots, v_m - v_k). \end{aligned}$$

$\square$

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**18** *Supp* $T \in \mathcal{L}(V, W)$ *and* $U, V$ *are subsps of* $\mathcal{V}$. *Let* $\pi : V \to V/U$ *be the quot map.*
*Prove* $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \cap V = \operatorname{null} \pi \subseteq \operatorname{null} T$.

**SOLUS:** Supp $\operatorname{null}\pi \subseteq \operatorname{null} T$. By (3.B.24), done. OR. Define $S : (v + U) \mapsto Tv$.
$\forall v_1, v_2 \in V$ suth $v_1 + U = v_2 + U \iff v_1 - v_2 \in U \cap V \subseteq \operatorname{null} T \iff Tv_1 = Tv_2$.
Thus $S$ is well-defined. Convly true as well. □

**CORO:** $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ with $S \mapsto S \circ \pi$ is inje, $\operatorname{range}\Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}$.
**COMMENT:** If $T = I_V$. Then $S : v + U \mapsto v$ is not well-defined, unless $U \cap V = \{0\} \subseteq \operatorname{null} I_V$.

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• **NOTE FOR [3.88, 3.90, 3.91]:** Supp $W \oplus U = V$. Then $V/U = W/U$ is iso to $W$. [ Convly not true. ]
Becs $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$. Define $T \in \mathcal{L}(V)$ by $T(v) = w_v$.
Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$, $\operatorname{range} T \oplus \operatorname{null} T = V$.
Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, V)$ is defined by $\tilde{T}(v + U) = \tilde{T}(w'_v + U) = Tw'_v = w_v$. [*See Exa below*]
Now $\pi \circ \tilde{T} = I_{V/U}$, $\tilde{T} \circ \pi|_W = I_W = T|_W$. Hence $\tilde{T} = (\pi|_W)^{-1}$ is iso of $V/U$ onto $W$.

• **EXA:** Let $V = \mathbf{F}^2, B_U = (e_1), B_W = (e_2 - e_1) \Rightarrow U \oplus W = V$.
**SOLUS:** Although $(e_2 - e_1) + U = e_2 + U, \tilde{T}(e_2 + U) = T(e_2) = e_2 - e_1$. Becs $e_2 = e_1 + (e_2 - e_1) \in U \oplus W$.

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**17** *Supp* $V/U$ *is finide. Supp* $W$ *is finide and* $V = U + W$. *Show* $\dim W \geqslant \dim V/U$.

**SOLUS:** Let $Y \oplus (U \cap W) = W$. Then by [1.C TIPS (4)], $V = U \oplus Y$. Note that $V/U$ and $Y$ are iso. □
OR. Let $B_W = (w_1, \dots, w_n)$. Then $V = U + \operatorname{span}(w_1, \dots, w_n)$.
$\forall v \in V, \exists\, u \in U,\ v = u + (a_1 w_1 + \cdots + a_n w_n) \Rightarrow v + U = (a_1 w_1 + \cdots + a_n w_n) + U$. □

**NOTE:** If $\dim W = \dim V/U$. Then $B_{V/U} = (w_1 + U, \dots, w_n + U)$. Supp $v = \sum_{i=1}^{n} a_i w_i \in U \cap W$
$\Rightarrow v + U = 0 = \sum_{i=1}^{n} a_i (w_i + U) \Rightarrow$ each $a_i = 0$. Thus $V = U \oplus W$.

---

**12** *Supp* $U$ *is a subsp of* $V$. *Prove is* $V$ *is iso to* $U \times (V/U)$.

**SOLUS:**

[ *Req* $V/U$ *Finide* ] Let $B_{V/U} = (v_1 + U, \dots, v_n + U)$.
Now $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^{n} a_i v_i + U \Rightarrow v - \sum_{i=1}^{n} a_i v_i \in U \Rightarrow \exists!\, u \in U, v = \sum_{i=1}^{n} a_i v_i + u$.
Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ and $\psi \in \mathcal{L}(U \times (V/U), V)$
by $\varphi(v) = (u,\ \sum_{i=1}^{n} a_i v_i + U)$, and $\psi(u,\ v + U) = \sum_{i=1}^{n} a_i v_i + u$. Then $\psi = \varphi^{-1}$. □

OR. Let $W \oplus U = V$. Define $Tv = u_v, Sv = w_v \Rightarrow \tilde{T} \in \mathcal{L}(V/W, U), \tilde{S} \in \mathcal{L}(V/U, W)$ are iso.
Define $\psi(u, v + U) = u + \tilde{S}(v + U) = u + w_v$. Define $\varphi(v) = (\tilde{T}(v), v + U)$.

$$\left.\begin{aligned}(\psi \circ \varphi)(u_v + w_v) &= \psi(u_v, w_v + U) = u_v + w_v \\ (\varphi \circ \psi)(u, v + U) &= \varphi(u + w_v) = (u, w_v + U)\end{aligned}\right\} \Rightarrow \psi = \varphi^{-1}.$$

OR Becs $\psi$ or $\varphi$ is inje and surj. □

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**13** *Prove* $B_{V/U} = (v_1 + U, \dots, v_m + U), B_U = (u_1, \dots, u_n) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$.

**SOLUS:** $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^{m} a_i v_i + U \Rightarrow \exists!\, b_i \in \mathbf{F}, v - \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{n} b_i u_i \in U$
$\Rightarrow \forall v \in V, \exists!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^{m} a_i v_i + \sum_{j=1}^{n} b_j u_j$. □

OR. $\sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i = 0 \Rightarrow \sum_{i=1}^{m} a_i (v_i + U) = 0 \Rightarrow$ each $a_i = 0 \Rightarrow$ each $b_i = 0$. □

OR. Note that $B = (v_1, \dots, v_m)$ is liney indep, and $[\operatorname{span}(v_1, \dots, v_m) + U] \subseteq V$.
$v \in \operatorname{span} B \cap U \iff v + U = \sum_{i=1}^{m} a_i (v_i + U) = 0 + U \iff v = 0$. Hence $\operatorname{span} B \cap U = \{0\}$.
Becs $\dim[\operatorname{span}(v_1, \dots, v_m) \oplus U] = m + n = \dim V$. Now by (2.B.8). □

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• (4E 14) *Supp* $V = U \oplus W,\ B_W = (w_1, \dots, w_m)$. *Prove* $B_{V/U} = \big(w_1 + U, \dots, w_m + U\big)$.

**Solus:** $\forall v \in V, \exists!\, u \in U, w \in W, v = u + w$. 又 $\exists!\, c_i \in \mathbf{F}, w = \sum_{i=1}^{m} c_i w_i \Rightarrow v = \sum_{i=1}^{m} c_i w_i + u$.
Hence $\forall v + U \in V/U, \exists!\, c_i \in \mathbf{F}, v + U = \sum_{i=1}^{m} c_i w_i + U$. □
Or. Becs $\pi|_W : W \to W/U$ is inv, and $V/U = W/U$. □

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**15** *Supp* $\varphi \in \mathcal{L}(V, \mathbf{F}) \backslash \{0\}$. *Prove* $\dim V/(\operatorname{null}\varphi) = 1$.

**Solus:** By [3.91] (d), $\dim \operatorname{range}\varphi = 1 = \dim V/(\operatorname{null}\varphi)$.
Or. By (3.B.29), $\exists\, u,\ \operatorname{span}(u) \oplus \operatorname{null}\varphi = V$. Then $B_{V/\operatorname{null}\varphi} = (u + \operatorname{null}\varphi)$. □

---

**16** *Supp* $\dim V/U = 1$. *Prove* $\exists\, \varphi \in \mathcal{L}(V, \mathbf{F}), \operatorname{null}\varphi = U$.

**Solus:** Supp $V_0 \oplus U = V$. Then $V_0$ is iso to $V/U,\ \dim V_0 = 1$.
Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. □

Or. Let $B_{V/U} = (w + U)$. Then $\forall v \in V, \exists!\, a \in \mathbf{F}, v + U = aw + U$.
Define $\varphi : V \to \mathbf{F}$ by $\varphi(v) = a$. Then $\varphi(v_1 + \lambda v_2) = a_1 + \lambda a_2 = \varphi(v_1) + \lambda\varphi(v_2)$.
Now $u \in U \iff u + U = 0w + U \iff \varphi(u) = 0$. □

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• *Supp* $U, W$ *are subsps of* $\mathcal{V}$, *and* $X, Y$ *are subsps of* $\mathcal{W}$.
*Supp* $U, X$ *are iso,* $W, Y$ *are iso. Prove or give a countexa*: $U/W$ *and* $X/Y$ *are iso.*

**Solus:** A countexa: Let $\mathcal{V} = \mathcal{W} = \mathbf{F}^2$. Let $U = X = Y = \operatorname{span}(e_1), W = \operatorname{span}(e_2)$.
Then $\dim U/W = \dim U - \dim(U \cap W) = 1 \neq 0 \dim X - \dim(X \cap Y) = \dim X/Y$. □

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• **Tips 2:** *Supp* $U, W$ *are vecsps,* $I = U \cap W$. *Prove* $V = U + W \iff V/I = U/I \oplus W/I$.

**Solus:** (a) Supp $V = U + W$. Then $\forall v + I \in V/I, \exists\,(u_v, w_v) \in U \times W,\ v + I = (u_v + w_v) + I$.
Note that $U/I, W/I \subseteq V/I$. Thus $V/I = U/I + W/I$.
$\forall u + I = w + I \in (U/I) \cap (W/I), \underline{u - w \in I = U \cap W}$
$\underline{\Rightarrow \exists\, w' \in I, u = w + w' \in U \cap W} \Rightarrow u + I = 0 + I = w + I$. Thus $(U/I) \cap (W/I) = \{0\}$.
(b) Supp $V/I = U/I \oplus W/I$. Then $\forall v \in V, v + I = (u + I) + (w + I)$
$\Rightarrow v - u - w \in I = U \cap W \Rightarrow \exists\, x \in U \cap W, v = u + w + x \in U + W$. □

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• **Tips 3:** *Supp* $U, W$ *are subsps of* $V$ *and* $X$ *is a subsp of* $U \cap W$.
*Prove* $U/W$ *and* $\big(U/X\big)/\big(W/X\big)$ *are iso.*

**Solus:** Let $U_X \oplus X = U, W_X \oplus X = W$. Becs $U/W = U_X/W$, and $U/X = U_X/X$.
Define $T \in \mathcal{L}\Big(\big(U_X/X\big)/\big(W/X\big), U_X/W\Big)$ by $T\big((u_x + X) + W/X\big) = u_x + W$.
$\forall u_1, u_2 \in U_X$ suth $(u_1 + X) + W/X = (u_2 + X) + W/X \Rightarrow u_1 - u_2 + X \in W/X$
$\Rightarrow u_1 - u_2 \in X + W$ 又 $u_1, u_2 \in U_X \Rightarrow u_1 - u_2 \in W \Rightarrow u_1 + W = u_2 + W$. Now $T$ is well-defined.
Inje: $\forall u_x \in U_X$ suth $u_x + W = 0 \Rightarrow u_x \in W_X \Rightarrow (u_x + X) \in W_X/X$.
Surj: $\forall u_x \in U_X, u_x + W = T\big((u_x + X) + W/X\big)$. Hence $T$ is iso. □

Or. Define $S \in \mathcal{L}(U_X/X, U_X)$ by $S(u_x + X) = u_x$.
Then $\forall u_1 + X = u_2 + X \in U_X/X,\ u_1 - u_2 \in X$ 又 $u_1, u_2 \in U_X \Rightarrow u_1 = u_2$.
Now $S$ is well-defined. Then $S\Big/_{W}^{(W/X)} = T$ defined above.
Becs $\operatorname{range} S|_{W/X \cap U_X/X} \subseteq W$, and $U_X = \operatorname{range} S \Rightarrow U_X \subseteq \operatorname{range} S + W$. Well-defined. Surj.
For $u_x \in U_X,\ u_x + W = 0 \iff u_x \in U_X \cap W \iff u_x + X \in (U_X \cap W)/X = \operatorname{null} S/_W$. Inje. □

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• Supp $T \in \mathcal{L}(V, W)$, and $U, V$ are subsps of some vecsp, and $X, W$ are subsps of some vecsp.
*Define* $T/^U_X : V/U \to W/X$ *by* $T/^U_X(v+U) = Tv + X$.
(a) *Prove* $T/^U_X$ *is well-defined* $\iff \big(\operatorname{range} T|_{U\cap V}\big)/(X \cap W) = \{0\} \iff \operatorname{range} T|_{U\cap V}$ *is a subsp of* $X \cap W$.
Supp $T/^U_X$ is well-defined, and thus is liney. Define $\pi_U \in \mathcal{L}(V, V/U), \pi_X \in \mathcal{L}(W, W/X)$.
Then $T/^U_X \circ \pi_U = \pi_X \circ T$. Define $T/_X \in \mathcal{L}(V, W/X)$ by $T/_X(v) = Tv + X$.
(b) $\operatorname{range} T/^U_X = \operatorname{range}\big(T/^U_X \circ \pi_U\big) = \operatorname{range}(\pi_X \circ T) = (\operatorname{range} T)/X$.
(c) *Prove* $T/^U_X$ *is surj* $\iff W = \operatorname{range} T + X \cap W$.
(d) *Show* $\operatorname{null} T/^U_X = \big(\operatorname{null} T/_X\big)/U$. (e) $T/^U_X$ *is inje* $\iff \operatorname{null} T/_X \subseteq U$.

**Solus:** (a) For $v, w \in V$. If $v + U = w + U \iff v - w \in U \Rightarrow Tv - Tw \in X \cap W \iff Tv + X = Tw + X$.
Then $\forall u \in V \cap U, Tu \in X \Rightarrow \operatorname{range} T|_{U\cap V} \subseteq X \cap W$. Convly true as well.
(c) Supp $T/^U_X$ is surj. $\forall w \in W, w + X \in W/X \Rightarrow \exists\, v + U \in V/U,\ Tv + X = w + X$
$\Rightarrow w - Tv \in X \cap W \Rightarrow w \in \operatorname{range} T + X \cap W$. Hence $W \subseteq \operatorname{range} T + X \cap W$.
Convly, $W = \operatorname{range} T + X \cap W \Rightarrow (\operatorname{range} T)/X = (\operatorname{range} T + X \cap W)/X = W/X$.
(d) $v + U \in \operatorname{null} T/^U_X \iff Tv \in X \iff v \in \operatorname{null} T/_X \iff v + U \in \big(\operatorname{null} T/_X\big)/U$. ☐

• **Comment:** Supp $T \in \mathcal{L}(V)$. Define $T/U \in \mathcal{L}(V/U)$ by $T/U = T/^U_U$. Then
(a) $T/U$ well-defined $\iff U \cap V$ invard $T$. (b) $\operatorname{range} T/U = \operatorname{range}(\pi \circ T) = (\operatorname{range} T)/U$.
(c) $T/U$ surj $\iff V = \operatorname{range} T + U \cap V$. (d) $\operatorname{null} T/U = \big(\operatorname{null} T/_U\big)/U$. (e) $T/U$ inje $\iff \operatorname{null} T/_U \subseteq U$.

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• (5.A.33) *Supp* $T \in \mathcal{L}(V)$. *Prove* $T/\operatorname{range} T = 0$. By (b) or (d) above, immed.

**Solus:** $v + \operatorname{range} T \in V/\operatorname{range} T \Rightarrow v + \operatorname{range} T \in \operatorname{null}(T/\operatorname{range} T)$. Thus $T/\operatorname{range} T = 0$. ☐

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• (5.A.34) *Supp* $T \in \mathcal{L}(V)$. *Prove* $T/\operatorname{null} T$ *is inje* $\iff \operatorname{null} T \cap \operatorname{range} T = \{0\}$.

**Solus:** Notice that $(T/\operatorname{null} T)(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0 \iff Tu \in \operatorname{null} T \cap \operatorname{range} T$.
Now $T/\operatorname{null} T$ is inje $\iff u + \operatorname{null} T = 0 \iff Tu = 0 \iff \operatorname{null} T \cap \operatorname{range} T = \{0\}$. ☐

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**Ended**

# 3·F

• **NOTE FOR Exe (1):** Every liney functional is either surj or is a zero map.
Which means, for $\varphi \in V'$, $\varphi = 0 \iff \dim \operatorname{span}(\varphi) = 0 \iff \dim \operatorname{range} \varphi = 0$.
And $\varphi \neq 0 \iff \dim \operatorname{span}(\varphi) = 1 \iff \dim \operatorname{range} \varphi = 1$. Thus $\dim \operatorname{span}(\varphi) = \dim \operatorname{range} \varphi$.

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**4** *Supp $U$ is a subsp of $V \neq U$. Prove $U^0 \neq \{0\}$.*

**SOLUS:** Let $X \oplus U = V \Rightarrow X \neq \{0\}$. Supp $s \in X \backslash \{0\}$. Let $Y \oplus \operatorname{span}(s) = X$.
Define $\varphi \in V'$ by $\varphi(u + \lambda s + y) = \lambda$. Hence $\varphi \neq 0$ and $\varphi(u) = 0$ for all $u \in U$. $\square$

OR. [ *Req $V$ Finide* ] By [3.106], $\dim U^0 = \dim V - \dim U > 0$.
OR. Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1 \dots, v_n)$ with $n \geqslant 1$.
Let $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$. Then each $\varphi \in \operatorname{span}(\varphi_1, \dots, \varphi_n)$ will do. $\square$

**CORO:** **19** $U^0 = \{0\} = V^0 \iff U = V$. By the inv and ctrapos of Exe (4).

**COMMENT:** ***Another proof of*** **[3.108]**: $T$ is surj $\iff T'$ is inje.
(a) Supp $T'$ is inje. NOTICE that $\psi \neq 0 \iff T'(\psi) \neq 0 \iff \psi \notin (\operatorname{range} T)^0$.
(b) $T$ is surj $\Rightarrow (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T'$. $\square$

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• **NOTE FOR [3.102] and Exe (18):** For $U = \emptyset$, $U^0 = \{\varphi \in V' : U \subseteq \operatorname{null} \varphi\} = V'$. While $\{0\}_V^0 = V'$.
Not a ctradic becs $\emptyset$ is not a subsp. Now $U^0 = V'$ can be true with $U = \emptyset \neq \{0\}$.

---

**25** *Supp $U$ is a subsp of $V$. Explain why $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.*

**SOLUS:** Asum $\forall \varphi \in U^0, \varphi(v) = 0$ while $v \in V \backslash U$. Then let $\operatorname{span}(v) \oplus U \oplus X = V$.
$\exists\, \varphi \in V', \operatorname{null} \varphi = U \oplus X \Rightarrow \varphi \in U^0$. 又 $\varphi(v) = 0 \Rightarrow 0 \neq v \in \operatorname{null} \varphi \cap \operatorname{span}(v)$. Ctradic. $\square$

**COMMENT:** $X \subseteq W = \{v \in V : \varphi(v) = 0, \forall \varphi \in X^0\}$, the ***promotion*** of the subset $X$ of $V$.

---

• *Supp $U, W$ are subsps of $V$. Prove the promotion of $U \cup W$ is $U + W$.*

**SOLUS:** $(U \cup W)^0 = \{\varphi \in V' : \varphi(u) = \varphi(w) = \varphi(u + w) = 0, \forall u \in U, w \in W\} = (U + W)^0$. $\square$

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• *Supp $X = \{x_1, \dots, x_m\} \subsetneq V$. Prove the promotion of $X$ is* $\operatorname{span}(x_1, \dots, x_m)$.

**SOLUS:** $X^0 = \{\varphi \in V' : \varphi(\lambda x_j + \mu x_k) = 0, \forall j, k \in \{1, \dots, m\}, \lambda, \mu \in \mathbf{F}\} = \operatorname{span}(x_1, \dots, x_m)^0$. $\square$

**COMMENT:** The promotion of every finite subset $X$ of $V$ is the smallest subsp of $V$ containing $X$.

---

**20** *Supp $U, W$ are subsets of $V$. Prove $U \subseteq W \Rightarrow W^0 \subseteq U^0$.*

**SOLUS:** $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(u) = 0 \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$. $\square$

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**21** *Supp $U, W$ are subsps of $V$. Prove $W^0 \subseteq U^0 \Rightarrow U \subseteq W$.*

**SOLUS:** Using Exe (25). Now $v \in U \Rightarrow \forall \varphi \in W^0 \subseteq U^0, \varphi(v) = 0 \Rightarrow v \in W$. $\square$

**NOTE:** $\varphi \in W^0 \iff \underline{\operatorname{null} \varphi \supseteq W \Rightarrow \operatorname{null} \varphi \supseteq U} \iff \varphi \in U^0$. But cannot conclude $W \supseteq U$.

**COMMENT:** (1) If $U$ is merely a subset and $W$ is a subsp. Promote $U$ as $X$, let $W = Y$.
Then $Y^0 = W^0 \subseteq U^0 = X^0 \Rightarrow Y = W \supseteq X \supseteq U$. Still true.
(2) If $W$ is merely a subset and $U$ is a subsp. Promote $W$ as $Y$, let $U = X$. For exa,
Let $W = \{(1, 0), (0, 1)\} \not\supseteq U = \{(x, 0) \in \mathbf{R}^2\}$. Then $Y = \mathbf{R}^2 \supseteq X = U,\ Y^0 = \{0\} \subseteq X^0$.

---

**22** *Supp $U$ and $W$ are subsps of $V$. Prove $\big(U+W\big)^0 = U^0 \cap W^0$.*

**Solus:** (a) $\varphi \in \big(U+W\big)^0 \Rightarrow \forall u \in U, w \in W,$ $\varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$. | $U \subseteq U+W \Rightarrow \big(U+W\big)^0 \subseteq U^0$; $W \subseteq U+W \Rightarrow \big(U+W\big)^0 \subseteq W^0$

(b) $\varphi \in U^0 \cap W^0 \subseteq V' \Rightarrow \forall u \in U, w \in W, \varphi(u+w) = 0 \Rightarrow \varphi \in \big(U+W\big)^0$. □

---

**37** *Supp $U$ is a subsp of $V$ and $\pi$ is the quot map. Thus $\pi' \in \mathcal{L}\big((V/U)', V'\big)$.*

(a) *Show $\pi'$ is inje*: Becs $\pi$ is surj. Use [3.108].

(b) *Show* $\operatorname{range}\pi' = U^0$: By [3.109](b), $\operatorname{range}\pi' = \big(\operatorname{null}\pi\big)^0 = U^0$.

(c) *Conclude that $\pi'$ is iso from $(V/U)'$ onto $U^0$*: Immed.

**Solus:** (a) Or. $\pi'(\varphi) = 0 \iff \forall v \in V\ \big(\ \forall v+U \in V\ \big), \varphi\big(\pi(v)\big) = \varphi(v+U) = 0 \iff \varphi = 0$.

(b) Or. $\psi \in \operatorname{range}\pi' \iff \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null}\psi \supseteq U \iff \psi \in U^0$. □

---

• *Supp $U$ is a subsp of $V$. Prove $(V/U)'$ is iso to $U^0$.* $\big[$ ***Another proof of* [3.106]** $\big]$

**Solus:** Define $\xi : U^0 \to (V/U)'$ by $\xi(\varphi) = \tilde{\varphi}$, where $\tilde{\varphi} \in (V/U)'$ is defined by $\tilde{\varphi}(v+U) = \varphi(v)$.

Inje: $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V\ \big(\ \forall v+U \in V/U\ \big), \tilde{\varphi}(v+U) = \varphi(v) = 0 \Rightarrow \varphi = 0$.

Surj: $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u+U) = \Phi(0+U) = 0 \Rightarrow U \subseteq \operatorname{null}(\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$.

Or. Define $\nu : (V/U)' \to U^0$ by $\nu(\Phi) = \Phi \circ \pi$. Now $\nu \circ \xi = I_{U^0},\ \xi \circ \nu = I_{(V/U)'} \Rightarrow \xi = \nu^{-1}$. □

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**23** *Supp $U$ and $W$ are subsps of $V$. Prove $\big(U \cap W\big)^0 = U^0 + W^0$.*

**Solus:**

(a) $\varphi = \psi + \beta \in U^0 + W^0 \Rightarrow \forall v \in U \cap W,$ $\varphi(v) = (\psi+\beta)(v) = 0 \Rightarrow \varphi \in \big(U \cap W\big)^0$. | Or. $U \cap W \subseteq U \Rightarrow \big(U \cap W\big)^0 \supseteq U^0$; $U \cap W \subseteq W \Rightarrow \big(U \cap W\big)^0 \supseteq W^0$

(b) $\big[$ *Only in Finide; Req $U, W$ Subsps* $\big]$ By Exe (22), $\dim\big(U^0 + W^0\big) = \dim U^0 + \dim W^0 - \dim\big(U^0 \cap W^0\big)$
$= 2\dim V - \dim U - \dim W - \big(\dim V - \dim(U+W)\big) = \dim V - \dim(U \cap W)$. □

Or. $\big[$ *Req $U, W$ Subsps* $\big]$ Let $I = U \cap W$. We show $\big(U \cap W\big)^0 \subseteq U^0 + W^0$.

Define $\chi \in \mathcal{L}(V/I, V/U \times V/W)$ by $\chi : v + I \mapsto (v+U, v+W)$.

Well-defined: $v_1 + I = v_2 + I \in V/I \iff v_1 - v_2 \in I$
$\iff v_1 - v_2 \in U$ and $v_1 - v_2 \in W \Rightarrow (v_1 + U, v_1 + W) = (v_2 + U, v_2 + W)$.

Inje: $(v+U, v+W) = 0 \iff v \in U \cap W = I \iff v + I = 0$.

Surj: $\forall v \in V$ suth $(v+U, v+W) \in V/U \times V/W$, becs $\emptyset \neq (v+U) \cap (v+W) = v + I \in V/I$.

Hence $\chi' \in \mathcal{L}\big((V/U \times V/W)', (V/I)'\big)$ is iso. Now we try finding an iso of $U^0 \times W^0$ onto $(U \cap W)^0$.

By (3.E.4), supp $\xi : (V/U)' \times (V/W)' \to (V/U \times V/W)'$ is iso.

By (c) in Exe (37), supp $\Lambda_1 : U^0 \times W^0 \to (V/U)' \times (V/W)'$ and $\Lambda_2 : (V/I)' \to (U \cap W)^0$ are isos.

Hence $(\Lambda_2 \circ \chi' \circ \xi \circ \Lambda_1) : U^0 \times W^0 \to (U \cap W)^0$ is iso. Now we see how it works:

$\forall (\varphi_U, \varphi_W) \in U^0 \times W^0$, $\operatorname{null}\pi_U \subseteq \operatorname{null}\varphi_U \Rightarrow \exists\, \psi_U \in (V/U)',\ \psi_U \circ \pi_U = \varphi_U$, simlr for $\varphi_W$,

thus $\Lambda_1 : (\varphi_U, \varphi_W) \mapsto (\psi_U, \psi_W)$. Then $\xi : (\psi_U, \psi_W) \mapsto (\psi_U S_U + \psi_W S_W)$, $\big[$ See notas in (3.E.2). $\big]$

Now $(\psi_U S_U + \psi_W S_W) \xmapsto{\chi'} (\psi_U S_U + \psi_W S_W) \circ \chi \xmapsto{\Lambda_2} (\psi_U S_U + \psi_W S_W) \circ \chi \circ \pi_I$,

which sends $v$ to $\psi_U(v+U) + \psi_W(v+W) = (\varphi_U + \varphi_W)(v)$, which is $\varphi_U + \varphi_W$.

Thus $(\Lambda_2 \circ \chi' \circ \xi \circ \Lambda_1)$ is the surj $\Lambda : U^0 \times W^0 \to U^0 + W^0$ defined in [3.77]. □

**Comment:** Not true if $U$ or $W$ is merely a subset. Promote $U \cap W$ as $I$, $U$ as $X$, and $W$ as $Y$.

**Exa:** Let $U = \big\{(x, x+1) \in \mathbf{R}^2\big\}, W = \mathbf{R}^2$. Then $U \cap W = I = U \neq \mathbf{R}^2 = X \cap Y$.

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• **TIPS 1:** *Prove* $V = U \oplus W \iff V' = U^0 \oplus W^0$.

**SOLUS:** $U \cap W = \{0\} \iff (U \cap W)^0 = \{0\}_V^0 = V' = U^0 + W^0$.
$V = U + W \iff (U + W)^0 = V_V^0 = \{0\} = U^0 \cap W^0$. $\square$

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• *Supp* $V = U \oplus W$. *Define* $\iota : V \to U$ *by* $\iota(u + w) = u$. *Thus* $\iota' \in \mathcal{L}(U', V')$.
(a) *Show* $\operatorname{null} \iota' = \{0\}$: $\operatorname{null} \iota' = (\operatorname{range} \iota)_U^0 = U_U^0 = \{0\}$. OR. $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \operatorname{null} \psi$.
(b) *Prove* $\operatorname{range} \iota' = W_V^0$: $\operatorname{range} \iota' = (\operatorname{null} \iota)_V^0 = W_V^0$. Now $\tilde{\iota'}$ is iso from $U'/\{0\}$ onto $W^0$

**SOLUS:** (b) OR. Note that $W = \operatorname{null} \iota \subseteq \operatorname{null}(\psi \circ \iota)$. Then $\psi \circ \iota \in W^0 \Rightarrow \operatorname{range} \iota' \in W^0$.
Supp $\varphi \in W^0$. Becs $\operatorname{null} \iota = W \subseteq \operatorname{null} \varphi$. By [3.B TIPS (3)], $\varphi = \varphi \circ \iota = \iota'(\varphi)$. $\square$

---

• *Supp* $V = U \oplus W$. *Prove* $U^0 = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$, *where* $\iota \in \mathcal{L}(V, W) : u_v + w_v \to w_v$.

**SOLUS:** $\varphi \in U^0 \iff U \subseteq \operatorname{null} \varphi \iff \varphi = \varphi \circ \iota$, by [3.B TIPS (3)]. $\square$

**NOTE:** The nota $W_V' = \{\varphi \in V' : \varphi = \varphi \circ \iota\} = U^0$ is not well-defined [without a bss].
Simply becs $W_V'$ have no info about the given $U$. Here is an informal explanation:
Each liney map $T \in \mathcal{L}(V, W)$ that vanishes on a given nontrivial $U$ has its '$P$'
( though not uniq ) suth '$U \oplus P = V$' with $T : P \mapsto \operatorname{range} T$ being surj.
Hence $\forall W \in \mathcal{S}_V U$, $U^0 = W_V'$. But given nontrivial '$P$', the corres '$U$' is not uniq.
Fix one $W_V'$, then $U^0$ is not uniq, with each $U_k$ not equal to each other while each $U_k^0 = W_V'$.

**EXA:** Let $B_V = (e_1, e_2)$. Let $B_U = (e_1), B_X = (e_2 - e_1), B_Y = (e_2)$.
Then $\iota_X : ae_1 + b(e_2 - e_1) \mapsto b(e_2 - e_1)$, $\iota_Y : ae_1 + be_2 \mapsto be_2$. Now $X_V' = Y_V' = U^0$.
(1) For $V = U \oplus X$, let $B_{U_V'} = (\varphi)$ with $\varphi : e_1 \mapsto 1,\ e_2 - e_1 \mapsto 0 \Rightarrow e_2 \mapsto 1$.
(2) For $V = U \oplus Y$, let $B_{U_V'} = (\psi)$ with $\psi : e_1 \mapsto 1, e_2 \mapsto 0$.
Thus $X^0 = U_V'$ while $Y^0 = U_V' \Rightarrow X^0 = Y^0 \Rightarrow X = Y$, ctradic.
To fix this, we must have a bss of $V'$ as precond, which we'll see in the NOTE FOR Exa (31).

**NOTE:** *Supp $U$ is a subsp of $V$. Then finding the corres subsp in $V'$ firstly req another 'half' $W \in \mathcal{S}_V U$, while finding the corres subsp of $V$ for a subsp of $V'$ must have the another 'half' asumed as precond.*

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**31** *Supp $V$ is finide and $B_{V'} = (\varphi_1, \dots, \varphi_n)$. Show $\exists!\, B_V$ whose dual bss is the $B_{V'}$.*

**SOLUS:** For each $k \in \{1, \dots, n\}$, let $\Gamma_k = \{1, \dots, n\} \setminus \{k\}$. Let each $U_k = \bigcap_{j \in \Gamma} \operatorname{null} \varphi_j$.
By Exe (4E 23), $V' = \operatorname{span}(\varphi_1, \dots, \varphi_n) = (\operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_n)^0 \Rightarrow U_k \cap \varphi_k = \{0\}$.
Thus $\forall x_k \in U_k \setminus \{0\}$, $x_k \notin \operatorname{null} \varphi_k$ while $x_k \in \operatorname{null} \varphi_j$ for all $j \in \Gamma$.
Fix one $x_k$ and let $v_k = [\varphi_k(x_k)]^{-1} x_k \Rightarrow \varphi_k(v_k) = 1$, $\varphi_j(v_k) = 0$ for all $j \neq k$.
Simply for each $v_k$, $\varphi_j(v_k) = \delta_{j,k}$ for all $j \iff$ for each $\varphi_j$, $\varphi_j(v_k) = \delta_{j,k}$ for all $k$.
又 $a_1 v_1 + \cdots + a_n v_n = 0 \Rightarrow$ each $\varphi_k(0) = a_k$.
Now we prove the uniqnes part. Supp the dual bss of $B_V' = (u_1, \dots, u_n)$ is the $B_{V'}$.
For each $k$, we have $\varphi_j(v_k) = \varphi_j(u_k)$ for all $k \Rightarrow v_k - u_k \in \bigcap \operatorname{null} \varphi_j = \{0\}$. $\square$

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• **NOTE FOR Exe (31):** Supp $V$ is finide, and $\Omega$ is a subsp of $V'$ with $B_\Omega = (\varphi_1, \dots, \varphi_m)$.
The '$W$' is not clear when we are to find suth $W_V{}' = \Omega$, becs the another 'half' is undefined.
Extend to $B_{V'} = (\varphi_1, \dots, \varphi_n)$. By Exe (31), $\exists!$ corres $B_V = (v_1, \dots, v_n)$.
Let $B_U = (v_{m+1}, \dots, v_n), B_W = (v_1, \dots, v_m)$. Thus we found the $W$ suth $\Omega = W_V'$,
which is well-defined with $B_V$ as precond.

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• **TIPS 2:** Supp $\varphi_1, \dots, \varphi_m \in V'$. Denote $\left[\operatorname{null} \psi_a \cap \cdots \cap \operatorname{null} \varphi_b\right]$ by $\bigcap_a^b \operatorname{null} \varphi_I$.
Supp $\Omega$ is a subsp of $V'$. Denote $\{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}$ by $C^0\,\Omega$.

If $\Omega$ is infinide, then by def, $\bigcap_{\in \Omega} \operatorname{null} \varphi = C^0\,\Omega$. If $\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$, then $v \in \bigcap_1^m \operatorname{null} \varphi_I \iff$ each $\varphi_k(v) = 0 \iff \forall \varphi = \sum_{i=1}^n a_i \varphi_i \in \Omega, \varphi(v) = 0 \iff v \in C^0\,\Omega$.

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• (4E 23) *Supp $V$ is finide, $\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m) \subseteq V'$. Prove $\Omega = \left(\operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m\right)^0$.*

**SOLUS:** Becs each $\operatorname{span}(\varphi_k) \subseteq (\operatorname{null} \varphi_k)^0$. By NOTE FOR Exe (1) and Exe (23), Immed. □

OR. Reduce to $B_\Omega = (\beta_1, \dots, \beta_p)$. We show $\Omega = (\operatorname{null} \beta_1 \cap \cdots \cap \operatorname{null} \beta_p)^0$, then done by TIPS (3).
Let $B_{V'} = (\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_q)$. By Exe (31), let $B_V = (v_1, \dots, v_p, u_1, \dots, u_q)$.
Define each $\Gamma_k = \{1, \dots, p\} \setminus \{k\}$. Then $\operatorname{null} \beta_k = \operatorname{span}\{v_j\}_{j \in \Gamma_k} \oplus \operatorname{span}(u_1, \dots, u_q)$.
Now $(\operatorname{null} \beta_1 \cap \cdots \cap \operatorname{null} \beta_p) = \operatorname{span}(u_1, \dots, u_q)$. Simlr to (4E 2.C.16).
Supp $\varphi = \sum_{i=1}^p a_i \beta_i + \sum_{j=1}^q b_j \gamma_j \in \operatorname{span}(u_1, \dots, u_q)^0$. Then each $\varphi(u_k) = 0 = b_k$
Thus $\operatorname{span}(u_1, \dots, u_q)^0 \subseteq \operatorname{span}(\beta_1, \dots, \beta_p) = \Omega$. □

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• **TIPS 3:** *Supp each $\varphi_i, \beta_j \in \mathcal{L}(V, W)$. Supp $\operatorname{span}(\varphi_1, \dots, \varphi_m) = \operatorname{span}(\beta_1, \dots, \beta_n)$.
Prove $\operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m = \operatorname{null} \beta_1 \cap \cdots \cap \operatorname{null} \beta_n$.*

**SOLUS:** Becs each $\beta_k \in \operatorname{span}(\varphi_1, \dots, \varphi_m)$.
$\forall v \in \bigcap_1^m \operatorname{null} \varphi_I, \beta_k(v) = 0$. Thus $\bigcap_1^m \operatorname{null} \varphi_I \subseteq \bigcap_1^n \operatorname{null} \beta_I$. Rev the roles and done. □

**NOTE:** Supp $\varphi_j = c_1 \varphi_1 + \cdots + c_{j-1} \varphi_{j-1}$.
Let $N_j \oplus \bigcap_1^{j-1} \operatorname{null} \varphi_I = \operatorname{null} \varphi_j$. Now $\bigcap_1^j \operatorname{null} \varphi_I = \bigcap_1^{j-1} \operatorname{null} \varphi_I \cap \left(\operatorname{null} \varphi_j\right) = \bigcap_1^{j-1} \operatorname{null} \varphi_I$.
Thus $\bigcap_1^m \operatorname{null} \varphi_I = \left[\bigcap_1^{j-1} \operatorname{null} \varphi_I\right] \cap \left[\bigcap_{j+1}^m \operatorname{null} \varphi_I\right]$. Hence $\bigcap_1^n \operatorname{null} \beta_I = \bigcap_1^m \operatorname{null} \varphi_I$.

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**26** *Supp $V$ is finide, $\Omega$ is a subsp of $V'$. Prove $\Omega = \left(C^0\,\Omega\right)^0$.*

**SOLUS:** Let $B_\Omega = (\varphi_1, \dots, \varphi_m)$. By TIPS (2) and Exe (4E 23). □

**EXA:** Immed, $\Omega \subseteq (C^0\,\Omega)^0$. Now we give a countexa for $\Omega \supseteq (C^0\,\Omega)^0$.
Let $V = \left\{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0 \text{ for only finily many } k\right\}$. Then $V' = (\mathbf{F}^\infty)'$.
Let $\Omega = \left\{\varphi \in \operatorname{span}(\varphi_{\alpha_1}, \dots, \varphi_{\alpha_m}) : \exists\, m, \alpha_k \in \mathbf{N}^+\right\} \subsetneq V'$. Then $C^0\,\Omega = \{0\} \Rightarrow (C^0\,\Omega)^0 = V'$.

**CORO:** (1) $C^0 \operatorname{span}(\varphi_1, \dots, \varphi_m) = \operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m$.
(2) Supp $V$ is finide. For every subsp $\Omega$ of $V'$, $\exists!$ subsp $U$ of $V$ suth $\Omega = U^0$.
*This form of $\Omega$ does not depend on a bss and thus is considered more general.*

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• *Supp $\operatorname{span}(\varphi_1, \dots, \varphi_m) \subseteq V'$. Let each $U_k \oplus \operatorname{null} \varphi_k = V$.
Prove or give a countexa: $(U_1 + \cdots + U_m) \oplus \left(\operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m\right) = V$.*

**SOLUS:** Let $V = \mathbf{R}^2$. Define $\varphi_1 = \varphi_2 : (x, y) \mapsto x$. Let $B_{U_1} = (e_1), B_{U_2} = (e_1 + e_2) \Rightarrow U_1 + U_2 = V$.
OR. Let $B_{V'} = (\varphi_1, \varphi_2)$ be corres to the std bss. Let $B_{U_1} = B_{U_2} = (e_1 + e_2) \Rightarrow U_1 + U_2 \subsetneq V$. □

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• **TIPS 4:** Let $B_{U^0} = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_n) \Rightarrow B_V = (v_1, \dots, v_n)$.
We show (a) $B_U = (v_{m+1}, \dots, v_n)$; (b) $U = \operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m$.
(a) Becs $\operatorname{span}(v_{m+1}, \dots, v_n)^0 = \operatorname{span}(\varphi_1, \dots, \varphi_m) = U^0$. Now by Exe (20, 21).
OR. Becs by (b), $U = \bigcap_1^m \operatorname{null} \varphi_I = \operatorname{span}(v_{m+1}, \dots, v_n)$.
(b) Each $\operatorname{null} \varphi_k = \operatorname{span}\{B_V \setminus \{v_k\}\} \Rightarrow \bigcap_1^m \operatorname{null} \varphi_I = \operatorname{span}(v_{m+1}, \dots, v_n)$. Now by (a).
OR. Becs $\operatorname{span}(\varphi_1, \dots, \varphi_m) = U^0 = \left(\operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m\right)^0$. Now by Exe (20, 21). □

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**24** *Prove, using the pattern of [3.104], that* $\dim U + \dim U^0 = \dim V$.

**Solus:** By Tips (4). Or. Let $B_U = (u_1, \dots, u_m), B_V = (u_1, \dots, u_m, v_1, \dots, v_n), B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$.
Supp $\psi = \sum_{i=1}^m a_i\psi_i + \sum_{j=1}^n b_j\varphi_j \in U^0 \Rightarrow$ each $\psi(u_k) = a_k = 0$. Thus $U^0 \subseteq \operatorname{span}(\varphi_1, \dots, \varphi_n)$. □

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• *Supp* $T \in \mathcal{L}(V, W)$, *each* $\varphi_k \in V'$, *and each* $\psi_k \in W'$.

**28** *Prove* $\operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) \iff \operatorname{range} T = (\operatorname{null} \psi_1) \cap \dots \cap (\operatorname{null} \psi_m)$.

**29** *Prove* $\operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \iff \operatorname{null} T = (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m)$.

**Solus:** $(\operatorname{range} T)^0 = \operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) = \big(\operatorname{null} \psi_1 \cap \dots \cap \operatorname{null} \psi_m\big)^0$.
$(\operatorname{null} T)^0 = \operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) = \big(\operatorname{null} \varphi_1 \cap \dots \cap \operatorname{null} \varphi_m\big)^0$. □

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**34** *The double dual space of* $V$, *denoted by* $V''$, *is defined to be the dual space of* $V'$.
*In other words,* $V'' = \mathcal{L}(V', \mathbf{F})$. *Define* $\Lambda : V \to V''$ *by* $(\Lambda v)(\varphi) = \varphi(v)$.

(a) *Show* $\Lambda$ *is a liney map from* $V$ *to* $V''$.

(b) *Show if* $T \in \mathcal{L}(V)$, *then* $T'' \circ \Lambda = \Lambda \circ T$, *where* $T'' = (T')'$.

(c) *Show if* $V$ *is finide, then* $\Lambda$ ***is iso from*** $V$ ***onto*** $V''$.

*Supp V is finide. Then V and V′ are iso, and finding iso from V onto V′ generally req choosing a bss of V. In contrast, the iso Λ from V onto V″ does not req a choice of bss and thus is considered more natural.*

**Solus:** (a) $\big(\Lambda(v + aw)\big)(\varphi) = \varphi(v + aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.
Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence $\Lambda$ is liney.

(b) $\big(T''(\Lambda v)\big)(\varphi) = \big((\Lambda v) \circ T'\big)(\varphi) = (\Lambda v)\big(T'(\varphi)\big)$
$= \big(T'(\varphi)\big)(v) = (\varphi \circ T)(v) = \varphi(Tv) = \big(\Lambda(Tv)\big)(\varphi)$.

(c) $\Lambda v = 0 \Rightarrow \forall \varphi \in V', (\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Inje. Now becs $V$ finide. □

**Comment:** Supp $\Phi \in V''$ and $\Phi \neq 0$. Then $\exists\, \varphi \in V',\ \Phi(\varphi) = 1 \Rightarrow \operatorname{null} \Phi \oplus \operatorname{span}(\varphi) = V'$.
And $\varphi \neq 0 \Rightarrow \exists\, v \in V,\ \varphi(v) = 1, \operatorname{null} \varphi \oplus \operatorname{span}(v) = V$. Becs $\Lambda$ is surj.
Now $\exists\, x \in V, \forall \psi = c\varphi + \rho \in V', \psi(x) = (\Lambda x)(\psi) = \Phi(\psi) = c$.

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**36** *Supp* $U$ *is a subsp of* $V$. *Define* $i : U \to V$ *by* $i(u) = u$. *Thus* $i' \in \mathcal{L}(V', U')$.

(a) *Show* $\operatorname{null} i' = U^0$: $\operatorname{null} i' = (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$.

(b) *Prove* $\operatorname{range} i' = U'$: $\operatorname{range} i' = (\operatorname{null} i)^0_U = \{0\}^0_U = U'$.

(c) *Prove* $\widetilde{i'}$ *is iso from* $V'/U^0$ *onto* $U'$: Immed.

**Solus:** (a) Or. $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$. Thus $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$.
(b) Or. Supp $\psi \in U'$. By (3.A.11), $\exists\, \varphi \in V', \varphi|_U = \psi$. Then $i'(\varphi) = \psi$. □

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• *Supp* $T \in \mathcal{L}(V, W)$. *Prove* $\operatorname{range} T' \supseteq \big(\operatorname{null} T\big)^0$. [ ***Another proof of*** **[3.109](b)** ]

**Solus:** Let $V = U \oplus \operatorname{null} T$. Let $R = (T|_U)^{-1}\big|_{\operatorname{range} T}$. Define $\iota \in \mathcal{L}(V, U)$ by $\iota(u + w) = u$.
$\forall \Phi \in \big(\operatorname{null} T\big)^0$, let $\psi = \Phi \circ R$, then $T'(\psi) = \psi \circ T = \Phi \circ \big(R \circ T|_V\big) = \Phi \circ \iota = \Phi \in \operatorname{range} T'$. □

**Coro:** [3.108] and [3.110] hold without the hypo of finide. Now $T$ inv $\iff T'$ inv.

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**12** Note that $I_V', I_{V'} : V' \to V'$. *For* $\varphi \in V'$, $I_{V'}(\varphi) = \varphi = \varphi \circ I_V = I_V'(\varphi)$. *Thus* $I_{V'} = I_V'$.

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**15** *Supp $T \in \mathcal{L}(V, W)$. Prove $T' = 0 \Rightarrow T = 0$.* **CORO:** If $V, W$ finide, then $\Gamma : T \mapsto T'$ is iso.

**SOLUS:** Supp $T' = 0$. Then $(\operatorname{range} T)^0 = \operatorname{null} T' = W'$.

By Exe (25), $\operatorname{range} T = \{w \in W : \varphi(w) = 0, \forall \varphi \in (\operatorname{range} T)^0 = W'\}$.

Asum $w \neq 0$ suth $\forall \varphi \in W', \varphi(w) = 0$. Let $U \oplus \operatorname{span}(w) = W$.

Define $\psi \in W'$ by $\psi(u + \lambda w) = \lambda \Rightarrow \psi(w) \neq 0$. Ctradic. Hence $\operatorname{range} T = \{0\}$. ☐

OR. [ *Req W Finide* ] By [3.106], $\dim \operatorname{range} T = \dim W - \dim(\operatorname{range} T)^0 = 0$. ☐

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• (4E 8) *Describe the relation of $B_V = (v_1, \dots, v_n)$ and the corres $B_{V'} = (\varphi_1, \dots, \varphi_n)$ using isos.*

**SOLUS:** Define $\Gamma : V \to \mathbf{F}^n$ by $\Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v))$, and $\Gamma^{-1}(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n$. ☐

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**6** *Define $\Gamma : V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$, where $v_1, \dots, v_m \in V$.*

(a) *Show $\operatorname{span}(v_1, \dots, v_m) = V \iff \Gamma$ is inje.*

(b) *Show $(v_1, \dots, v_m)$ is liney indep $\iff \Gamma$ is surj.*

**SOLUS:** Let $(e_1, \dots, e_m)$ be the std bss of $\mathbf{F}^m$.

(a) Becs $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \cdots = \varphi(v_m) = 0 \iff \operatorname{null} \varphi = \operatorname{span}(v_1, \dots, v_m)$. Immed.

(b) Supp $\Gamma$ is surj. Let each $e_k = \Gamma(\varphi_k) \Rightarrow \varphi_k(v_j) = \delta_{j,k}$. Now $a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow$ each $a_k = \varphi_k(0)$.

Supp $(v_1, \dots, v_m)$ is liney indep. Let $U = \operatorname{span}(v_1, \dots, v_m), B_{U'} = (\psi_1, \dots, \psi_m)$. Let $W \oplus U = V$.

Define $\iota : u_v + w_v \mapsto u_v$. Each $\psi_k \circ \iota = \varphi_k \in V' \Rightarrow \varphi_k(v_j) = \psi_k(v_j) = \delta_{j,k} \Rightarrow$ each $e_k = \Gamma(\varphi_k)$. ☐

OR. Let $(\psi_1, \dots, \psi_m)$ be dual bss of the std bss of $\mathbf{F}^m$. Define an iso $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$.

Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $Te_k = v_k$. Now $T(x_1, \dots, x_m) = T(x_1 e_1 + \cdots + x_m e_m) = x_1 v_1 + \cdots + x_m v_m$.

$\forall \varphi \in V', k \in \{1, \dots, m\}, [T'(\varphi)](e_k) = \varphi(Te_k) = \varphi(v_k) = [\varphi(v_1)\psi_1 + \cdots + \varphi(v_m)\psi_m](e_k)$

Now $T'(\varphi) = \varphi(v_1)\psi_1 + \cdots + \varphi(v_m)\psi_m = \Psi(\varphi(v_1), \dots, \varphi(v_m)) = \Psi(\Gamma(\varphi))$. Hence $T' = \Psi \circ \Gamma$.

By (3.B.3), (a) $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V \iff T'$ inje $\iff \Gamma$ inje.

(b) $(v_1, \dots, v_m)$ is liney indep $\iff T$ is inje $\iff T'$ surj $\iff \Gamma$ surj. ☐

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• (4E 25) *Define $\Gamma : V \to \mathbf{F}^m$ by $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$, where $\varphi_1, \dots, \varphi_m \in V'$.*

(c) *Show $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ is inje.*

(d) *Show $(\varphi_1, \dots, \varphi_m)$ is liney indep $\iff \Gamma$ is surj.*

**SOLUS:** Let $(e_1, \dots, e_m)$ be the std bss of $\mathbf{F}^m$.

(c) Becs $\Gamma(v) = 0 \iff \varphi_1(v) = \cdots = \varphi_m(v) = 0 \iff v \in (\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)$.

By Exe (4E 23), $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \operatorname{null} \Gamma = (\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m) = \{0\}$.

(d) Supp $(\varphi_1, \dots, \varphi_m)$ is liney indep. [ *Req Finide* ] Extend to $B_{V'} = (\varphi_1, \dots, \varphi_n)$.

Then by Exe (31), $B_V = (v_1, \dots, v_n)$ and each $\varphi_k(v_j) = \delta_{j,k} \Rightarrow$ each $e_k = \Gamma(\varphi_k)$.

Supp $\Gamma$ is surj. Let each $e_k = \Gamma(v_k) = (\varphi_1(v_k), \dots, \varphi_m(v_k))$.

Now $a_1 \varphi_1 + \cdots + a_m \varphi_m = 0 \Rightarrow$ each $a_k = 0(v_k)$.

OR. Let $U = \operatorname{span}(v_1, \dots, v_m)$. Then $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U) \Rightarrow (\varphi_1, \dots, \varphi_m)$ liney indep. ☐

OR. Let $(\psi_1, \dots, \psi_m)$ be dual bss of the std bss of $\mathbf{F}^m$. Define an iso $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$.

$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'(\Psi(x_1, \dots, x_m)) = (x_1 \psi_1 + \cdots + x_m \psi_m) \circ \Gamma$.

$\forall v \in V, [\Gamma'(\Psi(x_1, \dots, x_m))](v) = [x_1 \psi_1 + \cdots + x_m \psi_m](\varphi_1(v), \dots, \varphi_m(v)) = x_1 \varphi_1(v) + \cdots + x_m \varphi_m(v)$.

Now $\Gamma'(\Psi(x_1, \dots, x_m)) = x_1 \varphi_1 + \cdots + x_m \varphi_m$. Define $\Phi : \mathbf{F}^m \to V'$ by $\Phi = \Gamma' \circ \Psi$. Thus by (3.B.3),

(c) $\Gamma$ inje $\iff \Gamma'$ surj $\iff \Phi$ surj $\iff (\varphi_1, \dots, \varphi_m)$ spanning $V'$.

(d) $\Gamma$ surj $\iff \Gamma'$ inje $\iff \Phi$ inje $\iff (\varphi_1, \dots, \varphi_m)$ being liney indep. ☐

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**9** *Show* $\forall \psi \in V', \psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$*, where* $B_V = (v_1, \dots, v_n), B_{V'} = (\varphi_1, \cdots, \varphi_n)$.

**SOLUS:** $\psi(v) = a_1\psi(v_1) + \cdots + a_n\psi(v_n) = \psi(v_1)\varphi_1(v) + \cdots + \psi(v_n)\varphi_n(v)$. □

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**13** *Define* $T : \mathbf{R}^3 \to \mathbf{R}^2$ *by* $T(x,y,z) = (4x + 5y + 6z, 7x + 8y + 9z)$.

*Let* $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ *denote the dual bss of std bss of* $\mathbf{R}^2$ *and* $\mathbf{R}^3$.

(a) *Describe the liney functionals* $T'(\varphi_1), T'(\varphi_2)$.

For any $(x,y,z) \in \mathbf{R}^3$, $\big(T'(\varphi_1)\big)(x,y,z) = 4x + 5y + 6z$, $\big(T'(\varphi_2)\big)(x,y,z) = 7x + 8y + 9z$.

(b) *Write* $T'(\varphi_1)$ *and* $T'(\varphi_2)$ *as liney combinations of* $\psi_1, \psi_2, \psi_3$.

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3,\ T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$.

(c) *What is* null $T'$? *What is* range $T'$?

$T(x,y,z) = 0 \iff \left\{ \begin{array}{l} 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{array} \right. \iff \left\{ \begin{array}{l} x = z, \\ y = -2z. \end{array} \right.$ | Thus null $T = \operatorname{span}(e_1 - 2e_2 + e_3)$, where $(e_1, e_2, e_3)$ is std bss of $\mathbf{R}^3$.

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a bss, with corres dual bss $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Thus $\operatorname{span}(e_1 - 2e_2 + e_3) = \operatorname{null} T \Rightarrow \operatorname{span}(e_1 - 2e_2 + e_3)^0 = \operatorname{span}(\varepsilon_2, \varepsilon_3) = \operatorname{range} T'$.

Note that $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$.

And $\left| \begin{array}{l} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{array} \right.$

Hence $\varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2,\ \varepsilon_3 = -\psi_1 + \psi_3$. Now $\operatorname{range} T' = \operatorname{span}(\psi_1 - \frac{1}{2}\psi_2,\ -\psi_1 + \psi_3)$.

OR. $\operatorname{range} T' = \operatorname{span}\big(T'(\varphi_1),\ T'(\varphi_2)\big) = \operatorname{span}(4\psi_1 + 5\psi_2 + 6\psi_3,\ 7\psi_1 + 8\psi_2 + 9\psi_3)$.

Supp $T'(x\varphi_1 + y\varphi_2) = (4x + 7y)\psi_1 + (5x + 8y)\psi_2 + (6x + 9y)\psi_3 = 0$.
Then $x + y = 4x + 7y = x = y = 0$. Hence $\operatorname{null} T' = \{0\}$.

OR. $\operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \operatorname{span}(-2e_2,\ e_3) \oplus \operatorname{null} T$.

$\Rightarrow \operatorname{range} T = \{Tx : x \in \operatorname{span}(-2e_2,\ e_3)\} = \operatorname{span}\big(T(-2e_2),\ T(e_3)\big)$

$= \operatorname{span}(-10f_1 - 16f_2,\ 6f_1 + 9f_2) = \operatorname{span}(f_1, f_2) = \mathbf{R}^2$. Now $\operatorname{null} T' = (\operatorname{range} T)^0 = \{0\}$.

OR. For any $A, B \in \mathbf{R}$, asum $(x,y,z)$ is suth $A = 4x + 5y + 6z,\ B = 7x + 8y + 9z$.

By computing $x = z + 4/3(b - a),\ y = -2z + (7a - 4b)/3,\ z = z$. *An exa for (4E 3.E.8).*

Hence $(x,y,z)$ exis $\Rightarrow (A, B) \in \operatorname{range} T$. Now $T$ surj $\Rightarrow T'$ inje. □

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**ENDED**

## Exes about Sequences and Number Theory before Chapter 4

• (2.A.16) *Prove the vecsp of all continuous functions in* $\mathbf{R}^{[0,1]}$ *is infinide.*

**Solus:** Denote the vecsp by $U$.

Choose one $m \in \mathbf{N}^+$. Supp $a_0, \dots, a_m \in \mathbf{R}$ are suth $p(x) = a_0 + a_1 x + \cdots + a_m x^m = 0$, $\forall x \in [0,1]$.

Then $p$ has infily many roots and hence each $a_k = 0$, othws $\deg p \geqslant 0$, ctradic [4.12].

Thus $(1, x, \dots, x^m)$ is liney indep in $\mathbf{R}^{[0,1]}$. Simlr to [2.16], $U$ is infinide. $\square$

Or. Note that $\dfrac{1}{1} > \dfrac{1}{2} > \cdots > \dfrac{1}{m}$, $\forall m \in \mathbf{N}^+$. Supp $f_m = \begin{cases} x - \frac{1}{m}, & x \in \left(\frac{1}{m}, 1\right] \\ 0, & x \in \left[0, \frac{1}{m}\right] \end{cases}$

Then $f_1\left(\frac{1}{m}\right) = \cdots = f_m\left(\frac{1}{m}\right) = 0 \neq f_{m+1}\left(\frac{1}{m}\right)$. Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. By (2.A.14). $\square$

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• (3.F.35) *Prove* $\big(\mathcal{P}(\mathbf{F})\big)'$ *is iso to* $\mathbf{F}^\infty$.

**Solus:** Define $\theta \in \mathcal{L}\big[\big(\mathcal{P}(\mathbf{F})\big)', \mathbf{F}^\infty\big]$ by $\theta(\varphi) = \big(\varphi(1), \varphi(z), \cdots, \varphi(z^m), \cdots\big)$.

Notice that $\forall p \in \mathcal{P}(\mathbf{R}), \exists\,!\, c_i \in \mathbf{F}, m = \deg p,\ p(z) = c_0 + c_1 z + \cdots + c_m z^m \in \mathcal{P}_m(\mathbf{F})$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall p \in \mathcal{P}(\mathbf{F}), \varphi(p) = c_0\varphi(1) + c_1\varphi(z) + \cdots + c_m\varphi(z^m) = 0$.

Surj: Define $\psi_x(p) = x_0 c_0 + \cdots + x_m c_m$ for any $x = (x_0, x_1, \cdots) \in \mathbf{F}^\infty$. Now each $\psi(z^k) = x_k$.

$\forall p, q \in \mathcal{P}(\mathbf{F})$, supp $\deg p = m \geqslant n = \deg q$, [*which is why we do not write* $(p + \lambda q)$.]

$\psi_x(p+q) = x_0(a_0 + b_0) + x_n(a_n + b_n) + x_{n+1}a_{n+1} + \cdots + x_m a_m = \psi_x(p) + \psi_x(q)$. $\square$

**Comment:** $\mathcal{P}(\mathbf{F})$ is not iso to $\mathbf{F}^\infty$, so is $\mathcal{P}(\mathbf{F})$ to $\big(\mathcal{P}(\mathbf{F})\big)'$. But $\mathcal{P}(\mathbf{F})$ is iso to $\mathbf{F}^{\mathbf{N}}$, *which the 'U' in (3.E.14).*

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• (3.E.14) *Supp* $U = \big\{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0$ *for only finily many* $k\big\}$. *Denote it by* $\mathbf{F}^{\mathbf{N}}$.

(a) *Show* $U$ *is a subsp of* $\mathbf{F}^\infty$. [*Do it in your mind*] (b) *Prove* $\mathbf{F}^\infty / U$ *is infinide.*

**Solus:** *For ease of nota, denote the* $p^{th}$ *term of* $u = (x_1, \cdots, x_p, \cdots) \in \mathbf{F}^\infty$ *by* $u[p]$.

For each $r \in \mathbf{N}^+$, let $e_r[k] = \begin{cases} 1, & (k-1) \equiv 0 \pmod r \\ 0, & \text{othws} \end{cases}$ simply $e_r = (1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \underbrace{0, \cdots, 0}_{(r-1)}, 1, \cdots)$.

For $m \in \mathbf{N}^+$. Let $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 + U \Rightarrow \exists\, u \in U, a_1 e_1 + \cdots + a_m e_m = u$.

Supp $u = (x_1, \cdots, x_L, 0, \cdots)$, where $L$ is the largest suth $u[L] \neq 0$.

Let $s \in \mathbf{N}^+$ be suth $h = s \cdot m! + 1 > L$, and $e_1[h] = \cdots = e_m[h] = 1$.

Notice that for any $p, r \in \{1, \dots, m\}$, $e_r[s \cdot m! + 1 + p] = e_r[p+1] = 1 \Longleftrightarrow p \equiv 0 \pmod r \Longleftrightarrow r \,|\, p$.

Let $1 = p_1 \leqslant \cdots \leqslant p_{\tau(p)} = p$ be the disti factors of $p$. Moreover, $r \,|\, p \Longleftrightarrow r = p_k$ for some $k$.

Now $u[h+p] = 0 = \sum_{r=1}^{m} a_r e_r[p+1] = \sum_{k=1}^{\tau(p)} a_{p_k}$.

Let $q = p_{\tau(p)-1}$. Then $\tau(q) = \tau(p) - 1$, and each $q_k = p_k$. Again, $\sum_{r=1}^{m} a_r e_r[h+q] = 0 = \sum_{k=1}^{\tau(p)-1} a_{p_k}$.

Thus $a_{p_{\tau(p)}} = a_p = 0$ for all $p \in \{1, \dots, m\} \Rightarrow (e_1, \dots, e_m)$ is liney indep in $\mathbf{F}^\infty$. $\square$

Or. For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, \text{if } 2^r \,|\, p \\ 0, \text{othws} \end{cases}$ Simlr, let $m \in \mathbf{N}^+$ and $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0$ $\Rightarrow a_1 e_1 + \cdots + a_m e_m = u \in U$.

Supp $L$ is the largest suth $u[L] \neq 0$. And $l$ is suth $2^{ml} > L$. Then for each $k \in \{1, \dots, m\}$,

$u[2^{ml} + 2^k] = 0 = \sum_{r=1}^{m} a_r e_r[2^k] = a_1 + \cdots + a_k$. Thus each $a_k = 0$. Simlr. $\square$

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**Ended**

## Exes about Polys before Chapter 4

- (1.C.9) *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if $\exists\, p \in \mathbf{N}^+$, $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions $\mathbf{R} \to \mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$ ? Explain.*

**Solus:** Denote the set by $S$.

Supp $h(x) = \cos x + \sin\sqrt{2}x \in S$, since $\cos x, \sin\sqrt{2}x \in S$.

Asum $\exists\, p \in \mathbf{N}^+$ suth $h(x) = h(x+p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$

$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Ctradic! $\square$

Or. Becs $\cos x + \sin\sqrt{2}x = \cos(x+p) + \sin\left(\sqrt{2}x + \sqrt{2}p\right)$. By diff twice,

$$\cos x + 2\sin\sqrt{2}x = \cos(x+p) + 2\sin\left(\sqrt{2}x + \sqrt{2}p\right).$$

$$\left.\begin{aligned} \sin\sqrt{2}x &= \sin\left(\sqrt{2}x + \sqrt{2}p\right) \\ \cos x &= \cos(x+p) \end{aligned}\right\} \Rightarrow \text{Let } x = 0,\ p = \frac{m\pi}{\sqrt{2}} = 2k\pi.\ \text{Ctradic.}$$ $\square$

---

- (1.C.24) *Let $V_E = \left\{ f \in \mathbf{R}^{\mathbf{R}} : f \text{ is even} \right\}, V_O = \left\{ f \in \mathbf{R}^{\mathbf{R}} : f \text{ is odd} \right\}$. Show $V_E \oplus V_O = \mathbf{R}^{\mathbf{R}}$.*

**Solus:** (a) $V_E \cap V_O = \left\{ f \in \mathbf{R}^{\mathbf{R}} : f(x) = f(-x) = -f(-x) \right\} = \{0\}$.

$$\text{(b)} \left| \begin{aligned} &\text{Let } f_e(x) = \frac{1}{2}\Big[g(x) + g(-x)\Big] \Longrightarrow f_e \in V_E \\ &\text{Let } f_o(x) = \frac{1}{2}\Big[g(x) - g(-x)\Big] \Longrightarrow f_o \in V_O \end{aligned} \right\} \Rightarrow \forall g \in \mathbf{R}^{\mathbf{R}},\ g(x) = f_e(x) + f_o(x).$$ $\square$

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- (2.C.7) (a) *Let $U = \left\{ p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6) \right\}$. Find a bss of $U$.*
  (b) *Extend the bss in (a) to a bss of $\mathcal{P}_4(\mathbf{F})$, and find a $W$ suth $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.*

**Solus:** Using (2.C.10).

Notice that $\nexists\, p \in \mathcal{P}(\mathbf{F})$ of deg 1 and 2, while $p \in U$. Thus $\dim U \leqslant \dim \mathcal{P}_4(\mathbf{F}) - 2 = 3$.

(a) Consider $B = \big(1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

Let $a_0 + a_3(z-2)(z-5)(z-6) + a_4 z(z-2)(z-5)(z-6) = 0 \Rightarrow a_0 = a_3 = a_4 = 0$.

Thus the list $B$ is liney indep in $U$. Now $\dim U \geqslant 3 \Rightarrow \dim U = 3$. Thus $B_U = B$.

(b) Extend to a bss of $\mathcal{P}_4(\mathbf{F})$ as $\big(1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\big)$.

Let $W = \operatorname{span}(z, z^2) = \left\{ az + bz^2 : a, b \in \mathbf{F} \right\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. $\square$

---

- **Note For (2.C.10):** For each nonC $p \in \operatorname{span}(1, z, \dots, z^m)$, $\exists$ smallest $m \in \mathbf{N}^+$, which is $\deg p$.

(a) If $p_0, p_1, \dots, p_m$ are suth all $a_{k,k} \neq 0$, and

$p_0 = a_{0,0}$, each $p_k = a_{0,k} + a_{1,k}z + \cdots + a_{k,k}z^k$.

Then the upper-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m} \\ 0 & a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix}$.

(b) If $p_0, p_1, \dots, p_m$ are suth all $a_{k,k} \neq 0$, and

$p_0 = a_{0,0} + \cdots + a_{m,0}x^m$, each $p_k = a_{k,k}x^k + \cdots + a_{m,k}x^m$.

Then the lower-trig $\mathcal{M}\big(I, (p_0, p_1, \dots, p_m), (1, z, \dots, z^m)\big) = \begin{pmatrix} a_{0,0} & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,0} & a_{m,1} & \cdots & a_{m,m} \end{pmatrix}$.

**Comment:** Define $\xi_k(p)$ by the coeff of $z^k$ in $p \in \mathcal{P}_m(\mathbf{F})$.

Then $\mathcal{M}\big(\xi_k, (1, z, \dots, z^m), (1)\big) = \mathcal{E}^{(1,k)} \in \mathbf{F}^{1,m+1}$.

---

• (2.C.10) *Supp* $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ *are suth each* $\deg p_k = k$.
*Prove* $(p_0, p_1, \dots, p_m)$ *is a bss of* $\mathcal{P}_m(\mathbf{F})$.

**Solus:** Using induc on $m$.

(i) $k = 1$. $\deg p_0 = 0$; $\deg p_1 = 1 \Rightarrow \operatorname{span}(p_0, p_1) = \operatorname{span}(1, x)$.

(ii) $1 \leqslant k \leqslant m-1$. Asum $\operatorname{span}(p_0, p_1, \dots, p_k) = \operatorname{span}(1, x, \dots, x^k)$.

Then $\operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \subseteq \operatorname{span}(1, x, \dots, x^k, x^{k+1})$.

又 $\deg p_{k+1} = k+1$, $p_{k+1}(x) = a_{k+1}x^{k+1} + r_{k+1}(x)$; $a_{k+1} \neq 0$, $\deg r_{k+1} \leqslant k$.

$\Rightarrow x^{k+1} = \dfrac{1}{a_{k+1}}\Big(p_{k+1}(x) - r_{k+1}(x)\Big) \in \operatorname{span}(1, x, \dots, x^k, p_{k+1}) = \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

$\therefore x^{k+1} \in \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1}) \Rightarrow \operatorname{span}(1, x, \dots, x^k, x^{k+1}) \subseteq \operatorname{span}(p_0, p_1, \dots, p_k, p_{k+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m)$. $\square$

Or. By comparing coeffs. Denote the coeff of $x^k$ in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Supp $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is liney indep.

**Step 1.** For $k = m$, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m$, $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.

Now $L = a_{m-1}p_{m-1}(x) + \cdots + a_0 p_0(x)$.

**Step k.** For $0 \leqslant k \leqslant m$, we have $a_m = \cdots = a_{k+1} = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = 0$, then done. Othws, we have $L = a_{k-1}p_{k-1}(x) + \cdots + a_0 p_0(x)$. $\square$

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• **Tips:** *Supp* $m \in \mathbf{N}^+$, $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ *are suth the lowest term of each* $p_k$ *is of deg* $k$.
*Prove* $(p_0, p_1, \dots, p_m)$ *is a bss of* $\mathcal{P}_m(\mathbf{F})$.

**Solus:** Using induc on $m$.

Let each $p_k$ be defined by $p_k(x) = a_{k,k}x^k + \cdots + a_{m,k}x^m$, where $a_{k,k} \neq 0$.

(i) $k = 1$. $p_m(x) = a_{m,m}x^m$; $p_{m-1}(x) = a_{m-1,m-1}x^{m-1} + a_{m,m-1}x^m \Longrightarrow \operatorname{span}(x^m, x^{m-1}) = \operatorname{span}(p_m, p_{m-1})$.

(ii) $1 \leqslant k \leqslant m-1$. Asum $\operatorname{span}(x^m, \dots, x^{m-k}) = \operatorname{span}(p_m, \dots, p_{m-k})$.

Then $\operatorname{span}(p_m, \dots, p_{m-(k+1)}) \subseteq \operatorname{span}(x^m, \dots, x^{m-(k+1)})$.

又 $p_{m-(k+1)}$ has the form $a_{m-(k+1),m-(k+1)}x^{m-(k+1)} + r_{m-(k+1)}(x)$;
where the lowest term of $r_{m-(k+1)} \in \mathcal{P}_m(\mathbf{F})$ is of deg $(m-k)$.

$$\Rightarrow x^{m-(k+1)} = \frac{1}{a_{m-(k+1),m-(k+1)}}\Big(p_{m-(k+1)}(x) - r_{m-(k+1)}(x)\Big) \in \operatorname{span}(x^m, \dots, x^{m-k}, p_{m-(k+1)}) = \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)}).$$

$\therefore x^{m-(k+1)} \in \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$

$\Rightarrow \operatorname{span}(x^m, \dots, x^{m-k}, x^{m-(k+1)}) \subseteq \operatorname{span}(p_m, \dots, p_{m-k}, p_{m-(k+1)})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}(p_m, \dots, p_1, p_0)$. $\square$

Or. By comparing coeffs. Denote the coeff of $x^k$ in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_k(p)$.

Supp $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We show $a_m = \cdots = a_0 = 0$ via the following process. So that $(p_0, p_1, \dots, p_m)$ is liney indep.

**Step 1.** For $k = 0$, $\xi_0(L) = a_0 \xi_0(p_0) = \xi_0(R) = 0$ 又 $\deg p_0 = 0$, $\xi_0(p_0) \neq 0 \Rightarrow a_0 = 0$.

Now $L = a_1 p_1(x) + \cdots + a_m p_m(x)$.

**Step k.** For $0 \leqslant k \leqslant m$, we have $a_{k-1} = \cdots = a_0 = 0$.

Now $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now if $k = m$, then done. Othws, we have $L = a_{k+1}p_{k+1}(x) + \cdots + a_m p_m(x)$. $\square$

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• **NOTE FOR [2.11]:** ***Good definition for a general term always aviods undefined behaviours.***
If $\deg p = 0$, then $p(z) = a_0 \neq 0$, but $\underline{\text{not literally } a_0 z^0}$, by which if $p$ is defined, then it comes to $0^0$.
To make it clear, we $\underline{\text{specify that } \textit{in } \mathcal{P}(\mathbf{F}),\, a_0 z^0 = a_0, \text{ where } z^0 \text{ appears just for nota conveni.}}$
Becs by def, the term $a_0 z^0$ in a poly only represents the const term of the poly, which is $a_0$.
For conveni, we asum $z^0 = 1$ in formula deduction and poly def. Absolutely without $0^0$.

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• (4E 2.C.10) *Supp $m$ is a positive integer. For $0 \leqslant k \leqslant m$, let $p_k(x) = x^k(1-x)^{m-k}$.*
*Show $(p_0, \dots, p_m)$ is a bss of $\mathcal{P}_m(\mathbf{F})$.*

**SOLUS:** *We may see $p_0 = 1$ and $p_m(x) = x^m$, from the expansion below, by the* NOTE FOR *[2.11] above.*

Note that each $p_k(x) = \sum_{j=0}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j = \underbrace{(-1)^0 \cdot x^k \cdot 1^0}_{\text{of deg k}} + \underbrace{\sum_{j=1}^{m-k} C_{m-k}^{j}(-1)^j \cdot x^{j+k} \cdot 1^j}_{\text{of deg m; denote it by } q_k(x)}$.
And, each $q_k \in \operatorname{span}(x^{k+1}, \dots, x^m)$. Using TIPS above. ☐

OR. Simlr to the TIPS above. We will recurly prove each $x^{m-k} \in \operatorname{span}(p_m, \dots, p_{m-k})$.
(i) $k = 1$. $p_m(x) = x^m \in \operatorname{span}(p_m)$; $\quad p_{m-1}(x) = x^{m-1} - x^m \Rightarrow x^{m-1} \in \operatorname{span}(p_{m-1}, p_m)$.
(ii) $k \in \{1, \dots, m-1\}$. Supp for each $j \in \{0, \dots, k\}$, we have $x^{m-j} \in \operatorname{span}(p_{m-j}, \dots, p_m), \exists\, !\, a_m \in \mathbf{F}$.
Note that $x^{m-(k+1)} = p_{m-(k+1)}(x) + \sum_{j=1}^{k+1} C_{k+1}^{j}(-1)^{j+1} x^{m-(k+1)+j} \in \operatorname{span}(p_{m-(k+1)}, x^{m-k}, \dots, x^m)$.
Thus $x^{m-(k+1)} \in \operatorname{span}(p_{m-(k+1)}, p_{m-k}, \dots, p_m)$. ☐

**COMMENT:** The base step and the induc step can be indep.

OR. For any $m, k \in \mathbf{N}^+$ suth $k \leqslant m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k(1-x)^{m-k}$.
Define the stmt $S(m)$ by $S(m) : (p_{0,m}, \dots, p_{m,m})$ is liney indep ( and therefore is a bss ).
We use induc on to show $S(m)$ holds for all $m \in \mathbf{N}^+$.
(i) $m = 0$. $p_{0,0} = 1$, and $ap_{0,0} = 0 \Rightarrow a = 0$.
$m = 1$. Let $a_0(1-x) + a_1 x = 0, \forall x \in \mathbf{F}$. Then take $x = 1, x = 0 \Rightarrow a_1 = a_0 = 0$.
(ii) $1 \leqslant m$. Asum $S(m)$ and $S(m-1)$ holds. Now we show $S(m+1)$ holds.
Supp $\sum_{k=0}^{m+1} a_k p_{k,m+1}(x) = \sum_{k=0}^{m+1} a_k\left[x^k(1-x)^{m+1-k}\right] = 0, \forall x \in \mathbf{F}$.
Now $a_0(1-x)^{m+1} + \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k} + a_{m+1}x^{m+1} = 0, \forall x \in \mathbf{F}$.
While $x = 0 \Rightarrow a_0 = 0$; and $x = 1 \Rightarrow a_{m+1} = 0$.
Then $0 = \sum_{k=1}^{m} a_k x^k (1-x)^{m+1-k}$
$= x(1-x)\sum_{k=1}^{m} a_k x^{k-1}(1-x)^{m-k}$, note that $m - k = (m-1) - (k-1)$
$= x(1-x)\sum_{k=0}^{m-1} a_{k+1} x^k (1-x)^{m-1-k} = x(1-x)\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x)$.
Hence $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0, \forall x \in \mathbf{F}\backslash\{0,1\}$. Which has infily many zeros.
Moreover, $\sum_{k=0}^{m-1} a_{k+1} p_{k,m-1}(x) = 0$. By asum, $a_1 = \cdots = a_{m-1} = a_m = 0$.
Thus $(p_{0,m+1}, \dots, p_{m+1,m+1})$ is liney indep and $S(m+1)$ holds. ☐

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• (4E 3.D.20) *Supp $q \in \mathcal{P}(\mathbf{R})$. Prove $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2+x)p''(x) + 2xp'(x) + p(3)$.*

**SOLUS:** Note that $\deg\left[(x^2+x)p''(x) + 2xp'(x) + p(3)\right] = \deg p$.
Define $T_n \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ by $T_n(p) = (x^2+x)p''(x) + 2xp'(x) + p(3)$.
And note that $T_n(p) = 0 \Rightarrow \deg T_n(p) = -\infty = \deg p \Rightarrow p = 0$. Thus $T_n$ is inv.
$\forall q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $n = 0$; if $q \neq 0$, let $n = \deg q$, we have $q \in \mathcal{P}_n(\mathbf{R})$.
Now $\exists\, p \in \mathcal{P}_n(\mathbf{R}), q(x) = T_n(p) = (x^2+x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$. ☐

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• (3.D.19) *Supp* $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ *is inje. And* $\deg Tp \leqslant \deg p$ *for every non0* $p \in \mathcal{P}(\mathbf{R})$.
(a) *Prove* $T$ *is surj.* (b) *Prove for every non0* $p$, $\deg Tp = \deg p$.

**Solus:** (a) $T$ is inje $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ is inje, so is inv $\iff T$ is surj.

(b) Using induc.

(i) $\deg p = -\infty \geqslant \deg Tp \iff p = 0 = Tp$. And $\deg p = 0 \geqslant \deg Tp \iff p = C \neq 0$.

(ii) Asum $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$. We show $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$ by ctradic.
Supp $\exists\, r \in \mathcal{P}_{n+1}(\mathbf{R}),\ \deg Tr \leqslant n < n+1 = \deg r$. By (a), $\exists\, s \in \mathcal{P}_n(\mathbf{R}),\ T(s) = (Tr)$.
又 $T$ is inje $\Rightarrow s = r$. While $\deg s = \deg Ts = \deg Tr < \deg r$. Ctradic. $\square$

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• (3.B.26) *Supp* $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ *and* $\forall p, \deg(Dp) = (\deg p) - 1$. *Prove* $D \in \mathcal{P}(\mathbf{R})$ *is surj.*

**Solus:** $\big[$ $D$ might not be $D : p \mapsto p'$. $\big]$ Notice that the following proof is wrong:
Becs $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) \subseteq \operatorname{range} D$, and $\deg Dx^n = n-1$.
又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, Dx^3, \cdots) = \operatorname{span}(1, x, x^2, \cdots) = \mathcal{P}(\mathbf{R})$.

$\underline{\text{Let } D(C) = 0, Dx^k = p_k}$ of deg $(k-1)$, for all $C \in \mathcal{P}_0(\mathbf{R})$ and each $k \in \mathbf{N}^+$. Notice that $\mathbf{R} \neq \mathcal{P}_0(\mathbf{R})$.
Becs $B_{\mathcal{P}_m(\mathbf{R})} = (p_1, \dots, p_m, p_{m+1})$. And for all $p \in \mathcal{P}(\mathbf{R}), \exists\,!\, m = \deg p \in \mathbf{N}^+$.
So that $\exists\,!\, a_i \in \mathbf{R}, p = \sum_{i=1}^{m+1} a_i p_i \Rightarrow \exists\, q = \sum_{i=1}^{m+1} a_i x^i, Dq = p$. $\square$

Or. We will $\underline{\text{recurly define a seq of polys } (p_k)_{k=0}^{\infty} \text{ where } Dp_0 = 1, Dp_k = x^k}$ for each $k \in \mathbf{N}^+$.
So that $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), Dq = p, \exists\, q = \sum_{k=0}^{\deg p} a_k p_k$.

(i) Becs $\deg Dx = (\deg x) - 1 = 0, Dx = C \in \mathbf{F}\backslash\{0\}$. Let $p_0 = C^{-1}x \Rightarrow Dp_0 = C^{-1}Dx = 1$.

(ii) Supp we have defined $Dp_0 = 1, Dp_k = x^k$ for each $k \in \{1, \dots, n\}$. Becs $\deg D(x^{n+2}) = n+1$.
Let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$, with $a_{n+1} \neq 0$.
Then $a_{n+1}^{-1} D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_n Dp_n + \cdots + a_1 Dp_1 + a_0 Dp_0)$
$\Rightarrow x^{n+1} = D\Big[\underline{a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)}\Big]$. Thus defining $p_{n+1}$, so that $Dp_{n+1} = x^{n+1}$. $\square$

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• *Supp* $V = \mathbf{R}^{\mathbf{R}}$ *and* $U = \{f \in \mathbf{R}^{\mathbf{R}} : f(x_1) = \cdots = f(x_m) = 0\}$ *is a subsp of* $V$,
*with each* $x_k \in \mathbf{R}$. *Prove* $\forall W \in \mathcal{S}_V U, \dim W = m$. ***Hint***: *Find an iso from* $V/U$ *onto* $\mathbf{R}^m$.

**Solus:** Define $T \in \mathcal{L}(V/U, \mathbf{R}^m)$ by $T(f+U) = (f(x_1), \dots, f(x_m))$.
$\forall f_1 + U = f_2 + U \in V/U, f_1 - f_2 \in U \Rightarrow f_1(x_k) = f_2(x_k)$. Now $T$ is well-defined.
Inje: Each $f(x_k) = 0 \Rightarrow f + U = 0$. Let $S = T \circ \pi \Rightarrow \tilde{S} = T$. Then $S$ is surj, so is $T$. $\square$

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• (3.F.7) *Show the dual bss of* $(1, x, \dots, x^m)$ *of* $\mathcal{P}_m(\mathbf{R})$ *is* $(\varphi_0, \varphi_1, \dots, \varphi_m)$, *where* $\varphi_k(p) = \dfrac{p^{(k)}(0)}{k!}$.

**Solus:** *The uniqnes of dual bss is guaranteed by* [3.5].

$$\text{For } j,k \in \mathbf{N},\ (x^j)^{(k)} = \begin{cases} j(j-1)\dots(j-k+1)\cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\dots(j-j+1) = j! & j = k. \\ 0, & j \leqslant k. \end{cases} \Bigg|\ \Rightarrow (x^j)^{(k)}(0) = \begin{cases} 0\ , & j \neq k. \\ k!\ , & j = k. \end{cases} \quad \square$$

**Exa:** By [2.C.10], $B_m = (1, 7x-5, \dots, (7x-5)^m)$ is a bss of $\mathcal{P}_m(\mathbf{R})$. Let each $\varphi_k = \dfrac{p^{(k)}(5/7)}{7 \cdot k!}$.

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**Ended**

- **Tips:** *Supp $p \in \mathcal{P}_n(\mathbf{F})$ has at least $n+1$ disti zeros. Then by the ctrapos of [4.12], $\deg p < 0 \Rightarrow p = 0$.*

  Or. We show if $p \in \mathcal{P}(\mathbf{F})$ has at least $m$ disti zeros, then either $p = 0$ or $\deg p \geqslant m$.

  If $p = 0$ then done. If not, then supp $p$ has exactly $m$ disti zeros $\lambda_1, \dots, \lambda_m$.

  Becs $\exists!\, \alpha_i \geqslant 1, q \in \mathcal{P}(\mathbf{F})$, and $q \neq 0$, suth $p(z) = \left[(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_m)^{\alpha_m}\right] q(z)$. □
- **Comment:** Notice *that by [4.17], some term of the poly factoriz might not be in the form $(x-\lambda_k)^{\alpha_k}$.*

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- **Note For [4.7]:** *the uniqnes of coeffs of polys* [***Another proof***]

  If a poly had two different sets of coeffs, then subtracting the two exprs would give a poly with some non0 coeffs but infily many zeros. By Tips. □

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- **Note For [4.8]:** *div algo for polys* [***Another proof***]

  Supp $\deg p \geqslant \deg s$. Then $\left(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of len } \deg s}, \overbrace{s, zs, \cdots, z^{\deg p - \deg s} s}^{\text{of len } (\deg p - \deg s + 1)}\right)$ is a bss of $\mathcal{P}_{\deg p}(\mathbf{F})$.

  Becs $q \in \mathcal{P}(\mathbf{F}), \exists!\, a_i, b_j \in \mathbf{F}$,

  $$q = a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s$$
  $$= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1}}_{r} + s \underbrace{\left(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s}\right)}_{q}.$$ Note that $r, q$ are uniq. □

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- **Note For [4.11]:** *each zero of a poly corres to a deg-one factor;* [***Another proof***]

  First supp $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m, \exists!\, a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.

  Then $p(z) = p(z) - p(\lambda) = a_1(z-\lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.

  Hence $\forall k \in \{1, \dots, m\}, z^k - \lambda^k = (z-\lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.

  Thus $p(z) = \sum_{j=1}^{m} a_j (z-\lambda) \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z-\lambda) \sum_{j=1}^{m} a_j \sum_{i=1}^{k} \lambda^{i-1} z^{k-i} = (z-\lambda) q(z)$. □

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- (4E 2) *Prove if $w, z \in \mathbf{C}$, then $\big||w| - |z|\big| \leqslant |w - z|$.*

**Solus:** $|w-z|^2 = (w-z)(\overline{w} - \overline{z}) = |w|^2 + |z|^2 - 2Re(w\overline{z}) \geqslant |w|^2 + |z|^2 - 2|w\overline{z}| = \big||w| - |z|\big|^2$.

Or. $|w| = |w - z + z| \leqslant |w - z| + |z| \Rightarrow |w| - |z| \leqslant |w - z|$.

$|z| = |z - w + w| \leqslant |z - w| + |w| \Rightarrow |z| - |w| \leqslant |w - z|$. □

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**5** *Supp $m \in \mathbf{N}$, and $z_1, \dots, z_{m+1}$ are disti in $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$.*
*Prove $\exists!\, p \in \mathcal{P}_m(\mathbf{F}), p(z_k) = w_k$ for each $k \in \{1, \dots, m+1\}$.*

**Solus:**

Define $T \in \mathcal{L}(\mathcal{P}_m(\mathbf{F}), \mathbf{F}^{m+1})$ by $Tq = (q(z_1), \dots, q(z_m), q(z_{m+1}))$.

Becs $Tq = 0 \Longleftrightarrow q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0 \Longleftrightarrow q = 0$, by Tips. Now $T$ iso. Immed. □

Or. Let $p_1 = 1$, $p_k(z) = \prod_{i=1}^{k-1}(z - z_i) = (z - z_1) \cdots (z - z_{k-1})$ for each $k \in \{2, \dots, m+1\}$.

By (2.C.10), $B_p = (p_1, \dots, p_{m+1})$ is a bss of $\mathcal{P}_m(\mathbf{F})$. Let $B_e = (e_1, \dots, e_{m+1})$ be the std bss of $\mathbf{F}^{m+1}$.

Now $Tp_1 = (1, \dots, 1)$, $Tp_k = \left(\prod_{i=1}^{k-1}(z_1 - z_i), \dots, \prod_{i=1}^{k-1}(z_j - z_i), \dots, \prod_{i=1}^{k-1}(z_{m+1} - z_i)\right)$;

And $\prod_{i=1}^{k-1}(z_j - z_i) = 0 \Longleftrightarrow j \leqslant k-1$, becs $z_1, \dots, z_{m+1}$ are disti.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & A_{2,2} & 0 & \cdots & 0 \\ 1 & A_{3,2} & A_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{m+1,2} & A_{m+1,3} & \cdots & A_{m+1,m+1} \end{pmatrix} = \mathcal{M}(T, B_p, B_e).$$ Where $A_{j,k} = \prod_{i=1}^{k-1}(z_j - z_i) \neq 0$ for all $j > k-1 \geqslant 1$.

Now the rows $\mathcal{M}(T)$ are liney indep. By (4E 3.C.17) Or (3.F.32). □

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**6** *Supp non0 $p \in \mathcal{P}_m(\mathbf{F})$ has deg $m$. Prove*
*$[P]$ $p$ has $m$ disti zeros $\iff$ $p$ and its deri $p'$ have no common zeros. $[Q]$*

**Solus:** (a) Supp $p$ of deg $m$ has $m$ disti zeros. By [4.14], $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$.
If $m = 0$, then $p = c \neq 0 \Rightarrow p$ has no zeros, and $p' = 0$, done.
If $m = 1$, then $p(z) = c(z-\lambda_1)$, and $p' = c$ has no zeros, done.
For each $j \in \{1, \dots, m\}$, let $q_j(z-\lambda_j) = p(z) \Rightarrow q_j(\lambda_j) \neq 0$.
Now $p'(z) = (z-\lambda_j)q_j{}'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$.
Or. $\neg Q \Rightarrow \neg P$ : Supp $p(z) = (z-\lambda)q(z)$, $p'(z) = (z-\lambda)r(z)$.
Becs $p'(z) = (z-\lambda)q'(z) + q(z) \Rightarrow p'(\lambda) = q(\lambda) = 0 \Rightarrow q(z) = (z-\lambda)s(z)$.
Now $p(z) = (z-\lambda)^2 s(z)$. Hence $p$ has strictly less than $m$ disti zeros.

(b) $\neg P \Rightarrow \neg Q$ : Becs $0 \neq p \in \mathcal{P}_m(\mathbf{F})$. Supp all disti zeros are $\lambda_1, \dots, \lambda_M$, with $M < m$.
By Pigeon Hole Principle, $(z-\lambda_k)^2 q(z) = p(z)$ for some $\lambda_k \Rightarrow p'(\lambda_k) = 0 = p(\lambda_k)$. □

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**7** *Prove every $p \in \mathcal{P}(\mathbf{R})$ of odd deg has a zero.*

**Solus:** Using the nota and proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence $\lambda_1$ exis. □

Or. Supp $p \in \mathcal{P}(\mathbf{R})$ of odd deg $m$. Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$.
Write $p(x) = x^m\left(\frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \cdots + \frac{a_{m-1}}{x} + a_m\right) \Rightarrow p(x)$ continuous. Let $\delta = |a_m|^{-1}a_m$.
Then $\lim\limits_{x\to-\infty} p(x) = -\delta\infty$; $\lim\limits_{x\to\infty} p(x) = \delta\infty \Rightarrow p$ has at least one real zero. □

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**8** *Supp $p \in \mathcal{P}(\mathbf{R})$. Define $Tp : \mathbf{R} \to \mathbf{R}$ by* $(Tp)(x) = \begin{cases} \dfrac{p(x) - p(3)}{x-3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}$
*Show* (a) $Tp \in \mathcal{P}(\mathbf{R})$; (b) $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$.

**Solus:**

(a) For $x \neq 3$, $T(x^n) = \frac{x^n - 3^n}{x-3} = \sum_{i=1}^n 3^{i-1}x^{n-i}$.
For $x = 3$, $T(x^n) = n3^{n-1} = \sum_{i=1}^n 3^{n-1} = \sum_{i=1}^n 3^{i-1}x^{n-i}$. Now each $T(x^n) = \sum_{i=1}^n 3^{i-1}x^{n-i} \in \mathcal{P}(\mathbf{R})$.

(b) $T(p+\lambda q)(x) = \left\{\begin{array}{ll} \dfrac{(p+\lambda q)(x) - (p+\lambda q)(3)}{x-3}, & \text{if } x \neq 3, \\ (p+\lambda q)'(3), & \text{if } x = 3 \end{array}\right| = \big[T(p) + \lambda T(q)\big](x)$ for all $x \in \mathbf{R}$. □

Or. (a) Becs $\exists!\, q \in \mathcal{P}(\mathbf{R}), p(x) - p(3) = (x-3)q(x)$. For $x \neq 3$, $q(x) = \frac{p(x)-p(3)}{x-3}$.
$p'(x) = \big(p(x) - p(3)\big)' = q(x) + (x-3)q'(x)$. For $x = 3$, $p'(3) = q(3)$. Now $Tp = q$.

(b) Let $q_k(x)(x-3) = p_k(x) - p_k(3)$. Now by (a), $Tp_k = q_k$.
Then $(p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x-3)(q_1 + \lambda q_2)(x)$. By the uniqnes of $q_1 + \lambda q_2$. □

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**11** *Supp $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*
(a) *Show* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$; (b) *Find a bss of* $\mathcal{P}(\mathbf{F})/U$.

**Solus:** *Note that $pq \neq p \circ q$, see (4E 3.A.10).* Let $\deg p = m$ as precond.
If $\deg p = 0$, then $U = \mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F})/U = \{0 + U\}$, with the uniq bss $()$. Supp $\deg p \geqslant 1$.

(a) Becs $\forall s \in \mathcal{P}(\mathbf{F}), \exists!\, r \in \mathcal{P}_{m-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}) \Rightarrow \exists!\, pq \in U$, $s = (p)q + (r) \Rightarrow \mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{m-1}(\mathbf{F})$.
By [3.E Note For [3.88, 90, 91]] Or Define $R(s) = r \Rightarrow \operatorname{null} R = U$, and $R$ surj. Immed.

(b) Let $(1, z, \dots, z^{m-1})$ be a bss of $\mathcal{P}_{m-1}(\mathbf{F})$. By (4E 3.E.14) Or $\widetilde{R}^{-1} : \mathcal{P}_{m-1}(\mathbf{F}) \to \mathcal{P}(\mathbf{F})/U$, immed. □

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**9** *Supp $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\bar{z})}$. Prove $q \in \mathcal{P}(\mathbf{R})$.*

**Solus:** By [4.5], $\bar{z}^n = \overline{z^n}$. For any $f(z) = a_n z^n + \cdots + a_1 z + a_0$, $\overline{f(\bar{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.
Becs $q(z) = p(z)\overline{p(\bar{z})} = \overline{p(\bar{z})} p(z) = \overline{q(\bar{z})}$. Each $c_k = \overline{c_k} \Rightarrow c_k \in \mathbf{R}$. ☐

Or. Becs $q(z) = p(z)\overline{p(\bar{z})} = \sum_{k=0}^{2n} \left( \sum_{i+j=k} c_i \overline{c_j} \right) z^k$. For each $k \in \{0, \dots, 2n\}$,
$\overline{\sum_{i+j=k} c_i \overline{c_j}} = \sum_{i+j=k} \overline{c_i \overline{c_j}} = \sum_{i+j=k} c_j \overline{c_i} = \sum_{i+j=k} c_i \overline{c_j} \in \mathbf{R}$. ☐

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**10** *Supp disti $x_0, x_1, \dots, x_m \in \mathbf{R}$, and $p \in \mathcal{P}_m(\mathbf{C})$ suth each $p(x_k) \in \mathbf{R}$. Prove $p \in \mathcal{P}(\mathbf{R})$.*

**Solus:** By Tips and Exe (5), $\exists!\, q \in \mathcal{P}_m(\mathbf{R})$ suth $q(x_k) = p(x_k)$. Hence $p = q$. ☐

Or. Define $q(x) = \sum_{j=0}^{m} \dfrac{(x - x_0)(x - x_1) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_m)}{(x_j - x_0)(x_j - x_1) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_m)} p(x_j)$.
又 Each $x_j$, $p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R})$. Becs each $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q - p)(x_k) = 0$.
$(q - p)$ has $(m + 1)$ zeros. By Tips, $q - p = 0 \Rightarrow p = q \in \mathcal{P}(\mathbf{R})$. ☐

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• (4E 13) *Supp nonC $p, q \in \mathcal{P}(\mathbf{C})$ have no common zeros. Let $m = \deg p$, $n = \deg q$. Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r, s) = rp + sq$. Prove $T$ is inje.*
**Coro:** *$\exists!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ suth $rp + sq = 1$.*

**Solus:** Immed, $T$ is liney. Supp $T(r, s) = rp + sq = 0$.
Then $rp = -sq$. Becs $p, q$ are coprime $\Rightarrow p \mid s$, while $\deg s \leqslant m - 1 \Rightarrow s = 0 \Rightarrow r = 0$. ☐

Or. Let $\lambda_1, \dots, \lambda_M$ and $\mu_1, \dots, \mu_N$ be the disti zeros of $p$ and $q$ respectly. Notice that $M \leqslant m, N \leqslant n$.
By the ctrapos of [4.13], $M = 0 \iff m = 0 \Rightarrow s = 0 \iff r = 0 \Leftarrow n = 0 \iff N = 0$.
Now supp $M, N \geqslant 1$. We show $s = 0$. Simlr for $r = 0$. Or. $s = 0 \Rightarrow r = 0$.
Write $p(z) = a(z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_M)^{\alpha_M}$. ( $\exists!\, \alpha_j \geqslant 1, a \in \mathbf{F}$. ) Let $\max\{\alpha_1, \dots, \alpha_M\} = A$.
For each $D \in \{0, 1, \dots, A - 1\}$, let $I_{>D} = \{I_{D,1}, \dots, I_{D,J_D}\}$ be suth each $\alpha[I_{D,j}] = \alpha_{I_{D,j}} \geqslant D + 1$.
Now $\{M\} = I_{>A-1} \subseteq \cdots \subseteq I_{>0} = \{1, \dots, M\}$. Becs $rp + sq = 0 \Rightarrow (rp + sq)^{(k)} = 0$ for all $k \in \mathbf{N}^+$.
We use induc by $D$ to show $s^{(D)}(\lambda[I_{D,j}]) = 0$ for each $D \in \{0, \dots, A - 1\}$.
Notice that $p^{(D)}(\lambda[I_{D,j}]) = 0$ for each $D \in \{0, \dots, A - 1\}$ and each $I_{D,j} \in I_{>D}$. (L2)
(i) $D = 0$. Each $(rp + sq)(\lambda[I_{0,j}]) = (sq)(\lambda[I_{0,j}]) = s(\lambda[I_{0,j}]) = 0$. Where $q(\lambda[I_{0,j}]) = 0$.
$D = 1$. Each $(r'p + rp')(\lambda[I_{1,j}]) + (s'q + sq')(\lambda[I_{1,j}]) = (s'q)(\lambda[I_{1,j}]) = s'(\lambda[I_{1,j}]) = 0$.
Where $p'(\lambda[I_{1,j}]) = 0$, and each $I_{1,j} \subseteq I_{0,j} \Rightarrow s(\lambda[I_{1,j}]) = 0$.
(ii) $2 \leqslant D \leqslant A - 1$. Asum $s^{(d)}(\lambda[I_{d,j}]) = 0$ for each $d \in \{0, 1, \dots, D - 1\}$ and each $\lambda[I_{d,j}] \in I_{>d}$.

$$\begin{aligned} \text{Each } [rp + sq]^{(D)}(\lambda[I_{D,j}]) &= [C_D^D r^{(D)} p^{(0)} + \cdots + C_D^d r^{(d)} p^{(D-d)} + \cdots + C_D^0 r^{(0)} p^{(D)}](\lambda[I_{D,j}]) \qquad \text{(L1)} \\ &\quad + [C_D^D s^{(D)} q^{(0)} + \cdots + C_D^d s^{(d)} q^{(D-d)} + \cdots + C_D^0 s^{(0)} q^{(D)}](\lambda[I_{D,j}]) \\ &= [C_D^D s^{(D)} q^{(0)}](\lambda[I_{D,j}]). \text{ Where each } \lambda[I_{D,j}] \in I_{>D} \subseteq I_{D-1,\alpha}. \end{aligned}$$

Hence $s^{(D)}(\lambda[I_{D,j}]) = 0$. The asum holds for all $D \in \{0, \dots, A - 1\}$.
Notice that $\forall k = \{0, \dots, A - 2\}, s^{(k)}$ and $s^{(k+1)}$ have zeros $\{\lambda[I_{k+1,1}], \dots, \lambda[I_{k+1,J_{k+1}}]\}$ in common.
Now $\forall D \in \{1, \dots, A - 1\}, s = s^{(0)}, \dots, s^{(D)}$ have zeros $\{\lambda[I_{D,1}], \dots, \lambda[I_{D,J_D}]\}$ in common.
Thus $s(z)$ is divisible by $(z - \lambda[I_{D,1}])^{\alpha[I_{D,1}]} \cdots (z - \lambda[I_{D,J_D}])^{\alpha[I_{D,J_D}]}$, for each $D \in \{0, \dots, A - 1\}$.
Hence $s(z) = [(z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_M)^{\alpha_M}] s_0(z)$, while $\deg s < m = \alpha_1 + \cdots + \alpha_M$. Now by Tips. ☐

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**L1** *Prove* $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}$.

**Solus**: We use induc by $k \in \mathbf{N}^+$. (i) $k = 1$. $(pq)^{(1)} = (pq)' = C_1^1 p^{(1)} q^{(0)} + C_1^0 p^{(0)} q^{(1)}$. (ii) $k \geqslant 2$.

Asum for $(pq)^{(k-1)} = C_{k-1}^{k-1} p^{(k-1)} q^{(0)} + \cdots + C_{k-1}^j p^{(j)} q^{(k-1-j)} + \cdots + C_{k-1}^0 p^{(0)} q^{(k-1)}$.

Now $(pq)^{(k)} = \left((pq)^{(k-1)}\right)' = \left(\sum_{j=0}^{k-1} C_{k-1}^j p^{(j)} q^{(k-j-1)}\right)' = \sum_{j=0}^{k-1} \left[C_{k-1}^j \left(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)}\right)\right]$.

$$\begin{aligned} &= \left[C_{k-1}^0 \left(p^{(1)} q^{(k-1)} + \boxed{p^{(0)} q^{(k)}}\right)\right] + \left[C_{k-1}^1 \left(p^{(2)} q^{(k-2)} + p^{(1)} q^{(k-1)}\right)\right] \\ &\quad + \cdots + \left[C_{k-1}^{j-2} \left(\underline{p^{(j-1)} q^{(k-j+1)}} + p^{(j-2)} q^{(k-j+2)}\right)\right] + \left[C_{k-1}^{j-1} \left(\underline{\underline{p^{(j)} q^{(k-j)}}} + \underline{p^{(j-1)} q^{(k-j+1)}}\right)\right] \\ &\quad + \left[C_{k-1}^{j} \left(p^{(j+1)} q^{(k-j-1)} + \underline{\underline{p^{(j)} q^{(k-j)}}}\right)\right] + \left[C_{k-1}^{j+1} \left(p^{(j+2)} q^{(k-j-2)} + p^{(j+1)} q^{(k-j-1)}\right)\right] \\ &\quad + \cdots + \left[C_{k-1}^{k-2} \left(\underline{p^{(k-1)} q^{(1)}} + p^{(k-2)} q^{(2)}\right)\right] + \left[C_{k-1}^{k-1} \left(\boxed{p^{(k)} q^{(0)}} + \underline{p^{(k-1)} q^{(1)}}\right)\right]. \end{aligned}$$

Hence $(pq)^{(k)} = C_k^0 p^{(0)} q^{(k)} + \cdots + \left[C_{k-1}^j + C_{k-1}^{j-1}\right] \left(p^{(j)} q^{(k-j)}\right) + \cdots + C_k^k p^{(k)} q^{(0)}$. □

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**L2** *Supp* $\alpha \in \mathbf{N}^+$ *suth* $p(z) = (z - \lambda)^\alpha q(z)$. *Prove* $p^{(\alpha-1)}(\lambda) = 0$.

**Solus**: $\left[(z-\lambda)^\alpha q(z)\right]^{(\alpha-1)} = \sum_{j=1}^{\alpha-1} C_{\alpha-1}^j \left[(z-\lambda)^\alpha\right]^j q^{(\alpha-1-j)}$. Immed. □

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**Ended**

# 5.A

- **NOTE FOR [5.6]:** If $V$ is infinide. Then (a) $\Longleftrightarrow$ (b) $\Rightarrow$ (d), while (b) $\nRightarrow$ (c), and (b) $\nRightarrow$ (d).
- **COMMENT:** $\lambda$ not an eigval of $T \Longleftrightarrow T - \lambda I$ inje $\Longleftrightarrow$ inv, if finide.

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- *Supp $V$ is finide, $T \in \mathcal{L}(V)$, and $U$ is invarsp of $V$ under $T$.*
  *Prove or give a countexa: there exis invarsp of dimension* $\dim V - \dim U$.

**SOLUS:**

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- *Supp $T \in \mathcal{L}(V)$ and $U$ is invarsp of $V$ under $T$.*
  *Supp $\lambda_1, \dots, \lambda_m$ are the disti eigvals of $T$ corres eigvecs $v_1, \dots, v_m$.*
- **TIPS 1:** *Prove $v_1 + \cdots + v_m \in U \Longleftrightarrow$ each $v_k \in U$.*

**SOLUS:** Supp each $v_k \in U$. Then becs $U$ is a subsp, $v_1 + \cdots + v_m \in U$.
Convly, consider the stmt $P(k)$ : if $v_1 + \cdots + v_k \in U$, then each $v_j \in U$.
(i) For $k = 1$, $v_1 \in U$, $P(1)$ holds.
(ii) For $2 \leqslant k \leqslant m$. Asum $P(k-1)$ holds. Supp $v = v_1 + \cdots + v_k \in U$.
Then $Tv = \lambda_1 v_1 + \cdots + \lambda_k v_k \in U \Longrightarrow Tv - \lambda_k v = (\lambda_1 - \lambda_k)v_1 + \cdots + (\lambda_{k-1} - \lambda_k)v_{k-1} \in U$.
For each $j \in \{1, \dots, k-1\}$, $\lambda_j - \lambda_k \neq 0 \Rightarrow (\lambda_j - \lambda_k)v_j = v_j'$ is an eigvec of $T$ corres $\lambda_j$.
By asum, each $v_j' \in U$. Thus $v_1, \dots, v_{k-1} \in U$. So that $v_k = v - v_1 - \cdots - v_{k-1} \in U$. $\square$

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- **TIPS 2:** *Supp* $\dim V = m \Rightarrow B_V = (v_1, \dots, v_m)$. *Let each* $E_k = \operatorname{span}(v_k)$.
  *Prove* $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m)$.

**SOLUS:** Becs $V = E_1 \oplus \cdots \oplus E_m \Rightarrow \forall v \in U, \exists!\, c_j \in E_j, v = c_1 v_1 + \cdots + c_m v_m$.
For each $j$, $c_j \neq 0 \Rightarrow c_j v_j$ eigvec corres $\lambda_j$. Othws $c_j v_j = 0 \in U$.
By TIPS (1), each $c_j v_j \in U$. Thus $v \in (U \cap E_1) \oplus \cdots \oplus (U \cap E_m) = U$. $\square$

**CORO:** Becs each $\dim E_j = 1 \Rightarrow (U \cap E_j) = E_j$ or $\{0\}$. Let $E_{k_1}, \dots, E_{k_M}$ be all suth each $E_{k_j} = U \cap E_{k_j}$.

- **TIPS 3:** *Supp $U$ is a non0 invarsp of $V$ under $T$. Let* $\dim V = m$. *Then* $U = \operatorname{span}(v_{k_1}, \dots, v_{k_M})$.

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**2, 3** *Supp $S, T \in \mathcal{L}(V)$ suth $ST = TS$. Prove* $\operatorname{null} T, \operatorname{range} T$ *invard $S$.*

**SOLUS:** (a) $Tv = 0 \Rightarrow TSv = STv = 0$. (b) $Tu = v \Rightarrow Sv = STu = TSu \in \operatorname{range} T$. $\square$

**CORO:** For any $\lambda, \mu \in \mathbf{F}$, $\operatorname{null}(T - \lambda I)$, $\operatorname{range}(T - \lambda I)$ is invard $(S - \mu I)$.

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**6** *Supp $U$ is invarsp of non0 $V$ under any $T \in \mathcal{L}(V)$. Show $U = V$ or $\{0\}$.*

**SOLUS:** We show the ctrapos: *Supp $U \neq \{0\}$ and $U \neq V$. Prove $\exists\, T \in \mathcal{L}(V)$, $U$ is not invard $T$.*
Let $W \oplus U = V$. Define $T \in \mathcal{L}(V)$ by $T(u + w) = w$. $\square$

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- (4E 8 OR 5.B.4) *Supp $\lambda$ is eigval of $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove $\lambda = 0$ or $1$.*

**SOLUS:** $v \neq 0$, $Pv = \lambda v = \lambda^2 v = P(Pv)$. Thus $\lambda = 1$ or $0$. $\square$

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**14** *Supp $V = U \oplus W$, and $U, W$ non0. Define $P(u + w) = u$. Find all eigvals and eigvecs.*

**SOLUS:** Supp $u + w \neq 0$ and $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$.
Becs $(\lambda - 1)u = \lambda w = 0$. Now $\lambda = 0 \Longleftrightarrow u = 0$, and $\lambda = 1 \Longleftrightarrow w = 0$. Thus $Pu = u, Pw = 0$. $\square$

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• **TIPS 4:** Supp $T \in \mathcal{L}(\mathbf{R}^2)$ is the countclockwise rotation by the angle $\theta \in \mathbf{R}$.
Define $\mathcal{C} \in \mathcal{L}(\mathbf{R}^2, \mathbf{C})$ by $\mathcal{C}(a,b) = a+ib = r(\cos\alpha + i\sin\alpha) \Rightarrow a = r\cos\alpha, b = r\sin\alpha$, where $r = a^2+b^2$.
Then $(\cos\theta + i\sin\theta)(a+ib) = r(\cos(\alpha+\theta) + i\sin(\alpha+\theta)) = \mathcal{C}^{-1}T(a,b)$.
Hence $T(a,b) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$. Now $\mathcal{M}(T) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

**EXA:** OR **7** Supp $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x,y) = (-3y, x)$. Find all eigvals of $T$.

NOTICE that $\mathcal{M}(T) = \begin{pmatrix} \cos 90^\circ & -3\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$. By [5.8](a), we conclude that $T$ has no eigvals.

OR. Supp $\lambda$ is eigval with eigvec $(x,y)$. Then $(\lambda x, \lambda y) = (-3y, x) \Rightarrow -3y = \lambda^2 y \Rightarrow \lambda^2 = -3$.
$\big[$ Ignoring the possibility of $y = 0$, becs $x = 0 \Longleftrightarrow y = 0$. $\big]$ ☐

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**10** *Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$.*
(a) *Find all eigvals and eigvecs;* (b) *Find all invarsps of $V$ under $T$.*

**SOLUS:** Let $(e_1, \dots, e_n)$ be the std bss of $\mathbf{F}^n$.
(a) The eigvals are $\{1, \dots, n\}$ of len $\dim \mathbf{F}^n$. Let each $E_k = \operatorname{span}(e_k)$.
The set of all eigvecs is $(E_1 \cup \cdots \cup E_n) \setminus \{0\}$.
(b) Let each $V_k = \operatorname{span}(e_k) \Rightarrow V_k$ invard $T$. Then every sum of $V_1, \dots, V_n$ is invard $T$. ☐

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**18** *Define $T \in \mathcal{L}(\mathbf{F}^\infty)$ by $T(z_1, z_2, \cdots) = (0, z_1, z_2, \cdots)$. Show $T$ has no eigvals.*

**SOLUS:** Supp $z_k \neq 0$ and $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$. Thus $\lambda z_1 = 0$, $\lambda z_k = z_{k-1}$.
$(-)$ $\lambda = 0 \Rightarrow \lambda z_2 = z_1 = 0 = \cdots = z_k \Rightarrow (z_1, z_2, \dots) = 0$. Not an eigval.
$(=)$ $\lambda \neq 0 \Rightarrow \lambda z_1 = 0 \Rightarrow z_1 = \cdots = z_k = 0$. Not an eigval. ☐

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**19** *Supp $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.*
*In other words, the ent of $\mathcal{M}(T)$ wrto the std bss are all 1's. Find all eigvals and eigvecs of $T$.*

**SOLUS:** Supp $x_k \neq 0$ and $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.
Then (I) $\lambda = 0 \Rightarrow x_1 + \cdots + x_n = 0$; (II) $\lambda \neq 0 \Rightarrow x_1 = \cdots = x_n \Rightarrow \lambda x_k = nx_k$.

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**20** *Define $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \cdots) = (z_2, z_3, \cdots)$.*
*Show every elem of $\mathbf{F}$ is an eigval of $S$, and find all eigvecs of $S$.*

**SOLUS:** Supp $z_k \neq 0$ and $S(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$. Then each $\lambda z_k = z_{k+1}$.
(I) $\lambda = 0 \Rightarrow$ each$z_k = \cdots = z_2 = \lambda z_1 = 0$. Let $z_1 \neq 0 \Rightarrow E(0, S) = \operatorname{span}(e_1)$.
(II) $\lambda \neq 0 \Rightarrow \lambda^k z_1 = \lambda^{k-1} z_2 = \cdots = \lambda z_k = z_{k+1}$, let $z_1 \neq 0 \Rightarrow E(\lambda, S) = \operatorname{span}[(1, \lambda^1, \cdots, \lambda^k, \cdots)]$. ☐

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**11** *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigvals and eigvecs.*

**SOLUS:** For $0 \neq p \in \mathcal{P}(\mathbf{R})$, $\deg p' < \deg p$. And $\deg 0 = -\infty$. Supp $p' = \lambda p$.
Asum $\lambda \neq 0$. Then $\deg \lambda p = \deg p' < \deg \lambda p$, ctradic. Thus $\lambda = 0$.
Therefore $\deg \lambda p = -\infty = \deg p' \Rightarrow p \in \mathcal{P}_0(\mathbf{R})$. ☐

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**12** *Define $T \in \mathcal{L}(\mathcal{P}_n(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigvals and eigvecs.*

**SOLUS:** Supp $p \neq 0$ and $(Tp)(x) = xp'(x) = \lambda p(x)$. Define an iso $S(a_0, a_1, \dots, a_n) = a_0 + a_1 x + \cdots + a_n x^n$.
Let $p = S(a_0, a_1, \dots, a_n) \Rightarrow xp'(x) = S(a_1, 2a_2, \dots, na_n) = (\lambda a_0, \lambda a_1, \dots, \lambda a_n)$.
Now $S^{-1}TS : (x_0, x_1, \dots, x_n) \mapsto (0x_0, 1x_1, 2x_2, \dots, nx_n)$. Simlr to Exe (10). ☐

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• *Supp $V$ is finide, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$.*

**13** *Prove $\exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}$ suth $(T - \alpha I)$ is inv.*

**Solus:** Let each $|\alpha_k - \lambda| = \frac{1}{1000+k}$, where $k \in \{1, \ldots, \underline{\dim V + 1}\}$. Then $\exists\, \alpha_k$ not an eigval. ☐

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• (4E 11) *Prove $\exists\, \delta > 0$ suth $(T - \alpha I)$ is inv for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$.*

**Solus:** If $T$ has no eigvals, then $(T - \alpha I)$ is inje for all $\alpha \in \mathbf{F}$, done.

Supp $\lambda_1, \ldots, \lambda_m$ are all the disti eigvals of $T$.

Let $\delta > 0$ be suth, for each eigval $\lambda_k, \lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.

So that for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta, (T - \alpha I)$ is not inje. ☐

Or. Let $\delta = \min\{|\lambda - \lambda_k| : k \in \{1, \ldots, m\}, \lambda_k \neq \lambda\}$.

Then $\delta > 0$ and each $\lambda_k \neq \alpha$ $\left[\iff (T - \alpha I) \text{ is inv}\right]$ for all $\alpha \in \mathbf{F}$ suth $0 < |\alpha - \lambda| < \delta$. ☐

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**15** *Supp $T \in \mathcal{L}(V)$. Supp $S \in \mathcal{L}(V)$ is inv.*

(a) *Prove $T$ and $S^{-1}TS$ have the same eigvals.*

(b) *Describe the relationship between the eigvecs of $T$ and the eigvecs of $S^{-1}TS$.*

**Solus:** (a) $\lambda$ is an eigval of $T$ with an eigvec $v \Rightarrow S^{-1}TS(\underline{S^{-1}v}) = S^{-1}Tv = S^{-1}(\lambda v) = \underline{\lambda S^{-1}v}$.

$\lambda$ is an eigval of $S^{-1}TS$ with an eigvec $v \Rightarrow S(S^{-1}TS)v = T\underline{Sv} = \underline{\lambda Sv}$.

Or. Note that $S(S^{-1}TS)S^{-1} = T$. Every eigval of $S^{-1}TS$ is an eigval of $S(S^{-1}TS)S^{-1} = T$.

Or. $Tv = \lambda v \iff TSu = \lambda Su \iff (S^{-1}TS)u = \lambda u$. Where $v = Su$.

$(S^{-1}TS)u = \lambda u \iff S^{-1}Tv = \lambda S^{-1}v \iff Tv = \lambda v$. Where $u = S^{-1}v$.

(b) Becs $\lambda$ is eigval of $T \iff$ of $S^{-1}TS$.

Now $E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\}$; $E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}$. ☐

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• (4E 15) *Show $\lambda$ is an eigval of $T \iff \lambda$ is an eigval of the dual optor $T' \in \mathcal{L}(V')$.*

**Solus:** $\left[\, \textit{Req Finide} \,\right]$ $T - \lambda I_V$ not inv $\iff (T - \lambda I_V)' = T' - \lambda I_{V'}$ not inv. ☐

(a) Supp $\lambda$ is eigval with $v$. Let $U$ be invar with $U \oplus \operatorname{span}(v) = V$, by Exe (4E 39).

Define $\psi \in V'$ by $\psi(cv + u) = c$. Then $[T'(\psi)](cv + u) = \psi(c\lambda v + Tu) = \lambda c = \lambda\psi(cv + u)$.

(b) A countexa: Let $T$ be the forwd shift optor on $V = \mathbf{F}^\infty$. No eigvals for $T$, by Exe (18).

Define $\psi \in V'$ by $\psi(x_1, x_2, \cdots) = x_1$. Then $[T'(\psi)](x_1, x_2, \cdots) = \psi(0, x_1, x_2, \cdots) = 0$. ☐

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• *Supp $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$.*

(a) [(4E 17) Or (9.11)] *$\lambda \in \mathbf{R}$. Prove $\lambda$ is an eigval of $T \iff \lambda$ is an eigval of $T_{\mathrm{C}}$.*

(b) [**16** Or [9.16]] *$\lambda \in \mathbf{C}$. Prove $\lambda$ is an eigval of $T_{\mathrm{C}} \iff \overline{\lambda}$ is an eigval of $T_{\mathrm{C}}$.*

**Solus:** (a) $Tv = \lambda v \Rightarrow T_{\mathrm{C}}(v + \mathrm{i}0) = \lambda v$. $T_{\mathrm{C}}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \Rightarrow Tv = \lambda v, Tu = \lambda u$.

(b) Supp $T_{\mathrm{C}}(v + \mathrm{i}u) = Tv + \mathrm{i}Tu = \lambda(v + \mathrm{i}u)$.

Becs $\overline{T_{\mathrm{C}}(v + \mathrm{i}u)} = \overline{Tv + \mathrm{i}Tu} = Tv - \mathrm{i}Tu = T_{\mathrm{C}}(v - \mathrm{i}u) = T_{\mathrm{C}}(\overline{v + \mathrm{i}u})$.

And $\overline{\lambda(v + \mathrm{i}u)} = \overline{\lambda}v - \mathrm{i}\overline{\lambda}u = \overline{\lambda}(v - \mathrm{i}u) = \overline{\lambda}(\overline{v + \mathrm{i}u})$. ☐

Or. Supp $\lambda = a + ib$ is eigval of $T_{\mathrm{C}}$ with $v + \mathrm{i}u$.

Becs $T_{\mathrm{C}}(v + \mathrm{i}u) = \lambda(v + \mathrm{i}u) = (\underline{av - bu}) + \mathrm{i}(\underline{au + bv}) = \underline{Tv} + \mathrm{i}\underline{Tu}$.

Now $T_{\mathrm{C}}(\overline{v + \mathrm{i}u}) = Tv - \mathrm{i}Tu = (av - bu) - \mathrm{i}(au + bv) = (a - \mathrm{i}b)(v - \mathrm{i}u) = \overline{\lambda}(\overline{v + \mathrm{i}u})$. ☐

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**21** *Supp* $T \in \mathcal{L}(V)$ *is inv.* *Then 0 is not eigval of $T$ or $T^{-1}$.*

(a) *Supp* $\lambda \in \mathbf{F}$ *with* $\lambda \neq 0$. *Prove* $\lambda$ *is eigval of* $T \iff \lambda^{-1}$ *is eigval of* $T^{-1}$.

(b) *Prove* $T, T^{-1}$ *have the same eigvecs.*

**SOLUS:** $Tv = \lambda v \iff v = \lambda T^{-1} v \iff \lambda^{-1} v = T^{-1} v$. Where $v \neq 0$. □

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**23** *Supp* $S, T \in \mathcal{L}(V)$. *Prove* $ST$ *and* $TS$ *have the same eigvals.*

**SOLUS:** Supp $v \neq 0$ and $STv = \lambda v \Rightarrow T(STv) = \lambda Tv = TS(Tv)$.

If $Tv = 0$, then $T$ not inje, so are $TS, ST$. An eigval of $TS, ST$ with the same $v$.

Othws, $\lambda$ is eigval of $TS$. Rev the roles in asum. □

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• (2E 20) *Supp* $T \in \mathcal{L}(V)$ *has* $n = \dim V$ *disti eigvals and* $S \in \mathcal{L}(V)$ *has the same eigvecs but might not with the same eigvals.* *Prove* $ST = TS$.

**SOLUS:** Let each $\lambda_j v_j = Tv_j, \mu_j v_j = Sv_j$. Where $\mu_1, \dots, \mu_n$ might have repeti.

Becs $B_V = (v_1, \dots, v_n)$. Each $(ST)v_j = \mu_j \lambda_j v_j = (TS)v_j \Rightarrow ST = TS$. □

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• (4E 37) *Supp* $V$ *is finide,* $T \in \mathcal{L}(V)$.
*Define* $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ *by* $\mathcal{A}(S) = TS$ *for each* $S \in \mathcal{L}(V)$.
*Prove the set of eigvals of* $T$ *equals the set of eigvals of* $\mathcal{A}$.

**SOLUS:** (a) For $v \neq 0$ and $Tv = \lambda v$. Let $v_1 = v \Rightarrow B_V = (v_1, \dots, v_n)$.

Define $S \in \mathcal{L}(V) : v_j \mapsto v$, OR $v_j \mapsto \delta_{1,j} v_1$. Then each $(T - \lambda I)Sv_j = 0$.

Thus $(T - \lambda I)S = 0 \Rightarrow \mathcal{A}(S) = TS = \lambda S$ with $S \neq 0$.

(b) Supp $S \neq 0$ and $TS = \lambda S$. Then $\exists\, v \in V \backslash \operatorname{null} S$. Let $u = Sv \Rightarrow Tu = TSv = \lambda Sv = \lambda u$.

OR. $TS - \lambda S = (T - \lambda I)S = 0 \Rightarrow \{0\} \neq \operatorname{range} S \subseteq \operatorname{null}(T - \lambda I) \Rightarrow (T - \lambda I)$ not inje. □

**COMMENT:** If $\mathcal{A}(S) = ST, \forall S \in \mathcal{L}(V)$. Then the eigvals of $\mathcal{A}$ are not the eigvals of $T$.

---

**26** *Supp* $T \in \mathcal{L}(V)$ *is suth* $\forall v \in V, \exists!\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$. *Prove* $T = \lambda I$.

**SOLUS:** Supp $V$ non0. Becs $\forall v \in V, \exists!\, \lambda_v \in \mathbf{F}, Tv = \lambda_v v$. For any disti non0 $v, w \in V$,

$T(v + w) = \lambda_{v+w}(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$. □

OR. For any non0 $u, v \in V, u, v$ are eigvecs. If $u + v \neq 0$, then $u + v$ is also eigvec.

Othws done. By Exe (25), $\forall u, v \in V, Tu = \lambda u, Tv = \lambda v \Rightarrow \forall v \in V, Tv = \lambda v$. □

---

**27, 28** *Supp* $\dim V > 1$, $k \in \{1, \dots, \dim V - 1\}$.
*Supp every subsp dim* $k$ *is invard a* $T \in \mathcal{L}(V)$. *Prove* $T = \lambda I$.

**SOLUS:** We prove the ctrapos. Supp $\exists\, v \in V \backslash \{0\}$ not eigvec.

Then $(v, Tv)$ liney indep $\Rightarrow B_V = (v, Tv, u_1, \dots, u_n)$. Let $U = \operatorname{span}(v, u_1, \dots, u_{k-1})$. □

OR. Supp $v = v_1 \in V \backslash \{0\} \Rightarrow B_V = (v_1, \dots, v_n)$. Let $Tv_1 = c_1 v_1 + \dots + c_n v_n$.

Let $B_U = (v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$. Becs every such $U$ invar. Now $Tv_1 \in U \Rightarrow Tv_1 = c_1 v_1$.

By Exe (26), done. $\big[$For $0 \neq c_j \in \{c_2, \dots, c_n\}$, let $B_W = (v_1, v_{\beta_1}, \dots, v_{\beta_{k-1}})$ with each $\beta_i \neq j$.$\big]$ □

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**29** *Supp* $T \in \mathcal{L}(V)$, $\operatorname{range} T$ *is finide.* *Prove* $T$ *has at most* $1 + \dim \operatorname{range} T$ *disti eigvals.*

**SOLUS:** Becs $\operatorname{range} T$ finide $\Rightarrow$ not too many. Let $\lambda_1, \dots, \lambda_m$ be the disti eigvals of $T$ with corres $v_1, \dots, v_m$.

Then $(v_1, \dots, v_m)$ liney indep $\Rightarrow (\lambda_1 v_1, \dots, \lambda_m v_m)$ liney indep, if each $\lambda_k \neq 0$. Othws,

$\exists!\, \lambda_k = 0$. Now $\{\lambda_j v_j : j \neq k\}$ liney indep. By [2.23], $m - 1 \leqslant \dim \operatorname{range} T$. □

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• *Supp* $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ *are disti.*

(a) **32** *Prove* $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ *is liney indep in* $\mathbf{R}^{\mathbf{R}}$.

(b) [4E 36] *Show* $\big(\cos\lambda_1 x, \dots, \cos\lambda_n x\big)$ *is liney indep in* $\mathbf{R}^{\mathbf{R}}$.

**Solus:** (a) Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.
Then becs each $\lambda_k e^{\lambda_k x} = D(e^{\lambda_k x})$. Now $\lambda_1, \dots, \lambda_n$ are disti eigvals of $D$. By [5.10]. ☐

(b) Define $V, D$ simlr. Becs $D(\cos\lambda_k x) = -\lambda_k \sin\lambda_k x$. 又 $D(\sin\lambda_k x) = \lambda_k \cos\lambda_k x$.
Thus $D^2(\cos\lambda_k x) = -\lambda_k^2 \cos\lambda_k x$. Now $-\lambda_1^2, \dots, -\lambda_n^2$ are disti eigvals of $D^2$. Simlr. ☐

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**24** *Supp* $A \in \mathbf{F}^{n,n}$. *Define* $T \in \mathcal{L}(\mathbf{F}^{n,1})$ *by* $Tx = Ax$. *Prove* 1 *is eigval of* $T$ *if*:
(a) *the sum of the ent in each row of* $A$ *equals* 1. (b) *each col of* $A$.

**Solus:** Supp $x \neq 0$ and $Ax = \big(A_{j,1}x_1 + \dots + A_{j,n}x_n\big)_{j=1}^n = \lambda(x_j)_{j=1}^n = \lambda x$.

(a) Supp $A_{R,1} + \dots + A_{R,n} = 1$. Let $x_1 = \dots = x_n$. Immed.

(b) Supp $A_{1,C} + \dots + A_{n,C} = 1$. Then $\big[\sum_{R=1}^n A_{R,\cdot}\big]x = \sum_{k=1}^n \big(A_{1,k} + \dots + A_{n,k}\big)x_k$.
Now each $(Ax)_{R,1} = (x)_{R,1} = (\lambda x)_{R,1}$. Thus for $x$ with $\sum_{k=1}^n x_k \neq 0, \lambda = 1$ is the corres eigval. ☐

Or. Becs $(T - I)x = (A - I)x = \big((A_{j,1}x_1 + \dots + A_{j,n}x_n) - x_j\big)_{j=1}^n = (y_j)_{j=1}^n$.
Now $y_1 + \dots + y_n = \sum_{j=1}^n \sum_{k=1}^n \big(A_{j,k}x_k - x_j\big) = \sum_{k=1}^n x_k\big[\sum_{j=1}^n A_{j,k}\big] - \sum_{j=1}^n x_j = 0$.
Thus $\operatorname{range}(T - I) \subseteq \big\{(y_1, \dots, y_n) : y_1 + \dots + y_n = 0\big\}$. Now $(T - I)$ is not inv. ☐

Or. Let $(e_1, \dots, e_n)$ be the std bss of $\mathbf{F}^{n,1}$. Define $\psi \in \big(\mathbf{F}^{n,1}\big)'$ with each $\psi(e_k) = 1$.
Becs $Ae_k = A_{\cdot,k} = \sum_{j=1}^n A_{j,k}e_j \Rightarrow \psi(T - I)e_k = \psi\big(\sum_{j=1}^n A_{j,k}e_j - e_k\big) = \sum_{j=1}^n A_{j,k} - 1 = 0$.
Thus $\psi(T - I) = 0 \Rightarrow (T - I)$ not inje. ☐

Or. Define $S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Sx = A^t x$. Becs the rows of $\mathcal{M}(S) = A^t$ are the cols of $\mathcal{M}(T) = A$.
Let $(\varphi_1, \dots, \varphi_n)$ be the dual bss of $(e_1, \dots, e_n)$. Define $\Phi \in \mathcal{L}\big[\mathbf{F}^{n,1}, (\mathbf{F}^{1,n})'\big]$ by $\Phi(e_k) = \varphi_k$.
Now $(\Phi^{-1}T'\Phi)e_k = (\Phi^{-1}T')\varphi_k = \Phi^{-1}\big(\sum_{j=1}^n A^t_{j,k}\varphi_j\big) = \sum_{j=1}^n A^t_{j,k}e_j = A^t e_k = Se_k$.
Becs by (a), 1 is eigval of $S = \Phi^{-1}T'\Phi$. So of $T'$, by Exe (15). So of $T$, by Exe (4E 15). ☐

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• *Supp* $A \in \mathbf{F}^{n,n}$. *Define* $T \in \mathcal{L}(\mathbf{F}^{1,n})$ *by* $Tx = xA$. *Prove* 1 *is eigval of* $T$ *if*:
(a) *the sum of the ent in each col of* $A$ *equals* 1. (b) *each row of* $A$.

**Solus:** Supp $x \neq 0$ and $xA = \big(x_1A_{1,k} + \dots + x_nA_{n,k}\big)_{k=1}^n = \lambda(x_k)_{k=1}^n = \lambda x$.

(a) Supp $A_{1,C} + \dots + A_{n,C} = 1$. Let $x_1 = \dots = x_n$. Immed.

(b) Supp $A_{R,1} + \dots + A_{R,n} = 1$. Then $\sum_{C=1}^n xA_{\cdot,C} = \sum_{j=1}^n \big(A_{j,1} + \dots + A_{j,n}\big)x_j$.
Now each $(xA)_{1,C} = (x)_{1,C} = (\lambda x)_{1,C}$. Thus for $x$ suth $\sum_{k=1}^n x_k \neq 0, \lambda = 1$ is the corres eigval. ☐

Or. Becs $(T - I)x = x(A - I) = \big((x_1A_{1,k} + \dots + x_nA_{n,k}) - x_k\big)_{k=1}^n = (y_k)_{k=1}^n$.
Now $y_1 + \dots + y_n = \sum_{k=1}^n \sum_{j=1}^n (x_jA_{j,k} - x_k) = \sum_{j=1}^n x_j\big[\sum_{k=1}^n A_{j,k}\big] - \sum_{k=1}^n x_k = 0$.
Thus $\operatorname{range}(T - I) \subseteq \big\{(y_1, \dots, y_n) : y_1 + \dots + y_n = 0\big\}$. Now $(T - I)$ is not inv. ☐

Or. Define $(e_1, \dots, e_n)$ and $\psi(e_k) = 1$ simlr in Exe (24). Becs $e_jA = A_{j,\cdot} = \sum_{k=1}^n A_{j,k}e_k$.
Now $\psi(T - I)e_j = \sum_{k=1}^n A_{j,k} - 1 = 0 \Rightarrow \psi \circ (T - I) = 0$. Simlr. ☐

Or. Define $S \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Sx = xA^t$. Notice that $\mathcal{M}(S) \neq A$ and $\mathcal{M}(T) \neq A^t$. [*Noted by* AI.]
Let $(\varphi_1, \dots, \varphi_n)$ be the dual bss. Define $\Phi$ by $\Phi(e_k) = \varphi_k$.
Becs $\big[T'(\varphi_k)\big](e_j) = \varphi_k\big(\sum_{i=1}^n A_{j,i}e_i\big) = A_{j,k}$. By (3.F.9), $T'(\varphi_k) = \sum_{j=1}^n A_{j,k}\varphi_j$.
Now $(\Phi^{-1}T'\Phi)e_k = (\Phi^{-1}T')\varphi_k = \Phi^{-1}\big(\sum_{j=1}^n A_{j,k}\varphi_j\big) = \sum_{j=1}^n A_{j,k}e_j = e_kA^t = Se_k$. Simlr. ☐

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- $(4E\ 16)$ *Supp $B_V = (v_1, \dots, v_n), T \in \mathcal{L}(V)$, and $\lambda$ is eigval.*
*Let $A_M$ be the max of all ent of $A = \mathcal{M}\big(T, B_V\big)$. Prove $|\lambda| \leqslant A_M \cdot \dim V$.*

**Solus:** Supp $\lambda$ is eigval with to $v$. Let $v = c_1 v_1 + \cdots + c_n v_n$.
Becs $\lambda c_1 v_1 + \cdots + \lambda c_n v_n = c_1 T v_1 + \cdots + c_n T v_n = \sum_{k=1}^{n} c_k \Big[ \sum_{j=1}^{n} A_{j,k} v_j \Big] = \sum_{j=1}^{n} \Big[ \sum_{k=1}^{n} c_k A_{j,k} \Big] v_j$.
Thus $\lambda c_j = \sum_{k=1}^{n} c_k A_{j,k} \Rightarrow$ each $|\lambda||c_j| = \sum_{k=1}^{n} |c_k| \big|A_{j,k}\big|$. Let $|c_M| = \max\{|c_1|, \dots, |c_n|\}$.
Becs $v \neq 0 \Rightarrow |c_M| \neq 0$. Now $|\lambda||c_M| = \sum_{k=1}^{n} |c_k| \big|A_{M,k}\big| \Rightarrow |\lambda| \leqslant \sum_{k=1}^{n} |A_{M,k}| \leqslant nM$. □

---

**35** *Supp $V$ is finide, $T \in \mathcal{L}(V)$, and $U$ is invard $T$. Show $\lambda$ is eigval of $T/U \Rightarrow$ of $T$.*

**Solus:**

Supp $v + U \neq 0$ and $Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v = u \in U$. If $u = 0$, done. Othws, two cases.
If $(T - \lambda I)\big|_U$ inje $\Rightarrow$ surj. Then $(T - \lambda I)v = u = (T - \lambda I)\big|_U(w), \exists\, w \in U \Rightarrow T(v + w) = \lambda(v + w)$.
If $(T - \lambda I)\big|_U = T|_U - \lambda I_U$ not inje. Then $\lambda$ is eigval of $T|_U \Rightarrow$ of $T$. □

Or. Let $B_U = (u_1, \dots, u_m) \Rightarrow \big(Tv - \lambda v,\ Tu_1 - \lambda u_1, \cdots, Tu_m - \lambda u_m\big)$ of len $(m+1)$ liney dep in $U$.
So that $a_0(T - \lambda I)v + a_1(T - \lambda I)u_1 + \cdots + a_m(T - \lambda I)u_m = 0, \exists\, a_k \neq 0$.
Then $Tw = \lambda w$, where $w = a_0 v + a_1 u_1 + \cdots + a_m u_m \neq 0 \Leftarrow w \notin U \Leftarrow v \notin U$. □

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**36** *Give a countexa: The result in Exercise 35 is still true if $V$ is infinide.*

**Solus:** Let $V = \big\{ f \in \mathbf{R}^{\mathbf{R}} : \exists\,!\, m \in \mathbf{N}, f \in \operatorname{span}(1, e^x, \dots, e^{mx}) \big\}$.
Let $U = \big\{ f \in \mathbf{R}^{\mathbf{R}} : \exists\,!\, m \in \mathbf{N}^+, f \in \operatorname{span}(e^x, \dots, e^{mx}) \big\}$.
Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then range $T = U$ invard inje $T$.
Note that $(T/U)(1 + U) = e^x + U = 0$. While $0$ is not an eigval of $T$. □

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- $(4E\ 39)$ *Supp $T \in \mathcal{L}(V)$, $V$ is finide. Prove $\exists$ eigval of $T \Longleftrightarrow \exists$ invarsp of dim* $\dim V - 1$.

**Solus:** (a) Supp $\lambda$ is eigval with $v$. Becs $\dim \operatorname{null}(T - \lambda I) \geqslant 1 \Longleftrightarrow \dim \operatorname{range}(T - \lambda I) \leqslant \dim V - 1 = N$.
Let $B_{\operatorname{range}(T - \lambda I)} = (w_1, \dots, w_m)$, $B_V = (w_1, \dots, w_m, u_1, \dots, u_n)$, $B_U = (w_1, \dots, w_m, u_1, \dots, u_{N-m})$.
Becs $U$ invard $(T - \lambda I)$. Now $u \in U \Rightarrow (T - \lambda I)u \in U \Rightarrow Tu \in U$.
(b) Supp $U$ is invarspd $T$ with $\dim U = \dim V - 1 \Rightarrow \dim V/U = 1$. By (3.A.7) and Exe (35)□

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**Ended**

# 5.B: I

下面，为了照顾原书两版过大的差距，特别将此节分成 (I), (II) 两部分。考虑到本节 4E「本征值与极小多项式」和「奇维度实向量空间的本征值」(相当一部分是从 3E 的 8.C 节挪过来的) 是对 3E「多项式作用于算子」和「本征值的存在性」(也即本节 3E 前半部分) 的极大扩充，这一扩充也大大改变了本节 3E 后半部分的「上三角矩阵」，故而将 4E 放在 3E 前面。

(I) 除了覆盖本节 4E 全部和 3E 前半部分与之相关的所有习题，还会覆盖上节 4E 末。

(II) 除了覆盖本节 3E 后半部分「上三角矩阵」，还会覆盖下节 4E；并且，下节还会覆盖下下节 4E。

- $(\text{4E 5.A.33})$ *Supp $T \in \mathcal{L}(V), m \in \mathbf{N}^+$. Prove $T$ inje $\overset{(a)}{\iff}$ $T^m$ inje, and $T$ surj $\overset{(b)}{\iff}$ $T^m$ surj.*

**Solus:** (a) $T^m$ inje $\Rightarrow Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$.
$T$ inje $\Rightarrow T^m v = T^{m-1}v = \cdots = T^2 v = Tv = v = 0$.
(b) $T^m$ surj $\Rightarrow \forall u \in V, \exists\, v \in V \Rightarrow \exists\, w = T^{m-1}v,\ T^m v = u = Tw$.
$T$ surj $\Rightarrow \forall u \in V, \exists\, v_1, \dots, v_m \in V,\ T(v_1) = T^2 v_2 = \cdots = T^m v_m = u$. ☐

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- **Note For [5.17]:** *Prove* $\operatorname{null} p(T)$ *and* $\operatorname{range} p(T)$ *are invard* $T$.

**Solus:** (a) $u \in \operatorname{null} p(T) \iff p(T)u = 0 \Rightarrow p(T)Tu = Tp(T)u = 0 \iff Tu \in \operatorname{null} p(T)$.
(b) $u \in \operatorname{range} p(T) \Rightarrow \exists\, v \in V,\ u = p(T)v \Rightarrow Tu = p(T)Tv \in \operatorname{range} p(T)$. ☐

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**13** *Supp $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$ has no eigvals. Prove every invarsp either $\{0\}$ or infinide.*

**Solus:** Supp $U$ is a finide non0 invarsp. Then by [5.21], $\exists$ eigval of $T|_U$, so of $T$. ☐

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- *Supp non0 $v \in V$. Prove [5.21] using the given map below.*

**16** *Define $S : \mathcal{P}_{\dim V}(\mathbf{C}) \to V$ by $S(p) = p(T)v$. Then $S$ not inje $\Rightarrow \exists$ non0 $p \in \operatorname{null} S$.*
**17** *Define $S : \mathcal{P}_{\dim V^2}(\mathbf{C}) \to \mathcal{L}(V)$ by $S(p) = p(T)$. Then $S$ not inje $\Rightarrow \exists$ non0 $p \in \operatorname{null} S$.*

**Solus:** Let $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow (T - \lambda_1 I) \cdots (T - \lambda_m I)$ not inje. ☐

**Note:** $\exists$ monic $q \in \operatorname{null} S|_W$ of smallest deg, $q(T) = 0$, then $q$ is the *min poly.*

---

- **Note For [4E 5.22]:** *Supp $V$ finide, $T \in \mathcal{L}(V)$.*
*Prove $\exists\,!$ monic $p \in \mathcal{P}_{\dim V}(\mathbf{F})$ of smallest deg, suth $p(T) = 0$.*

**Solus:** Using induc on $\dim V$. (i) $\dim V = 0$. Let $p = 1 \Rightarrow p(T) = T = 0$.
(ii) Asum for each $U$ of smaller dim, $\forall S \in \mathcal{L}(U), \exists$ *monic* $s \in \mathcal{P}_{\dim U}(\mathbf{F})$, *suth* $s(S) = 0$.
Let $u \in V \backslash \{0\}$. Then $(Iu, Tu, \dots, T^{\dim V}u)$ liney dep
$\Rightarrow \exists$ smallest $m \in \mathbf{N}^+$ suth $c_0 Iu + c_1 Tu + \cdots + c_{m-1}T^{m-1}u + T^m u = 0$. Thus define $q \in \mathcal{P}_m(\mathbf{F})$.
Notice that $q(T)(T^k u) = 0 \Rightarrow \operatorname{span}(Iu, Tu, \dots, T^{m-1}u) \subseteq \operatorname{null} q(T)$.
Hence $\dim \operatorname{null} q(T) \geqslant m \iff \dim \operatorname{range} q(T) \leqslant \dim V - m$.
Becs $\operatorname{range} q(T)$ invard $T$, $S = T\big|_{\operatorname{range} q(T)} \in \mathcal{L}(\operatorname{range} q(T))$. Now by asum,
$\exists$ monic $s \in \mathcal{P}_{\dim V - m}(\mathbf{F})$, suth $s(S) = 0 \Rightarrow (sq)(T) = 0$. [*The remaining part is obvious.*] ☐

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- $(\text{4E 5.31, 4E 25, 26})$ *min poly of restr optor and min poly of quot optor*
*Supp $V$ is finide, $T \in \mathcal{L}(V)$, with $U$ invarsp, and $p$ be min poly.*
(a) *Prove $p$ is a multi of the min of $T|_U$.* (b) *of the min of $T/U$.*
(c) *Prove [min poly of $T|_U$] $\times$ [min poly of $T/U$] is a poly multi of $p$.*
(d) *Prove the set of eigvals of $T$ is the set of eigvals of $T|_U$ or of $T/U$.* By [8.49], immed.

**Solus:** (a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46].
(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v + U) = p(T)v + U = 0$.
(c) Supp $r$ is the min of $T|_U$. Then $\forall u \in U,\ r(T|_U)u = r(T)u = 0$.
Supp $s$ is the min of $T/U$. Then $\forall v \in V,\ s(T/U)(v + U) = s(T)v + U = 0$.
Thus $\forall v \in V, (rs)(T)v = r(T)\big(s(T)v\big) = 0 \Rightarrow (rs)(T) = 0$. ☐

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• (4E 27) *Supp* $\mathbf{F} = \mathbf{R}$, *$V$ is finide, and $T \in \mathcal{L}(V)$. Prove the min $p$ of $T_{\mathrm{C}}$ is the min $q$ of $T$.*

**Solus:** (a) $\forall u + \mathrm{i}0 \in V_{\mathrm{C}}, p(T_{\mathrm{C}})(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p$ is a multi of $q$.

(b) $q(T) = 0 \Rightarrow \forall u + \mathrm{i}v \in V_{\mathrm{C}}, q(T_{\mathrm{C}})(u + \mathrm{i}v) = q(T)u + \mathrm{i}q(T)v = 0 \Rightarrow q$ is a multi of $p$. □

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• (4E 28) *Supp $V$ is finide and $T \in \mathcal{L}(V)$. Prove the min $p$ of $T'$ equals the min $q$ of $T$.*

**Solus:** (a) $\forall \varphi \in V', p(T')(\varphi) = \varphi p(T) = 0 \Rightarrow p(T) \in \operatorname{null} \varphi$. Thus $p(T) = 0$.

(b) $q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi q(T) = q(T')(\varphi) = 0$. Thus $q(T') = 0$. □

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• [4E 5.32] *Supp $T \in \mathcal{L}(V)$ with the min $p$. Prove $T$ not inje $\Longleftrightarrow$ the const term of $p$ is 0.*

**Solus:** $T$ not inje $\Longleftrightarrow$ 0 is eigval of $T \Longleftrightarrow$ 0 is zero of $p \Longleftrightarrow$ 0 the const term of $p$. □

Or. Supp $p(0) = (z - 0)(z - \lambda_1) \cdots (z - \lambda_m) = 0 \Rightarrow T(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0$.

又 $p$ is the min $\Rightarrow q(T) = (T - \lambda_1) \cdots (T - \lambda_m) \neq 0$. Now $0 = p(T) = Tq(T) \Rightarrow T$ not inje. □

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• (4E 22) *Supp $V$ is finide, $T \in \mathcal{L}(V)$. Prove $T$ is inv $\Longleftrightarrow I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$.*

**Solus:** Let $p(z) = a_0 + a_1 z + \cdots + z^m$ be the min. Becs $T$ inv $\Longleftrightarrow$ inje $\Longleftrightarrow p(0) = a_0 \neq 0$. By $p(T) = 0$. □

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• (4E 10, 23) *Supp $V$ is finide, $T \in \mathcal{L}(V)$, with the min $p$ of deg $m$.*
*Supp non0 $v \in V$. Let each* $\operatorname{span}^k = \operatorname{span}(v, Tv, \dots, T^k v)$.
(a) *Prove* $\exists j \in \{1, \dots, m\}$, $\operatorname{span}^{j-1} = \operatorname{span}^{m-1}$.
(b) *Prove* $\operatorname{span}^{m-1} = \operatorname{span}^n$ *for all* $n \geqslant m - 1$.

**Solus:**

**Comment:** By Note For [8.40], $j$ has an upper bound $m - 1$, $m$ has an upper bound $\dim V$.

Let $p(z) = a_0 + a_1 z + \cdots + z^m$. If $v = 0$, then done. Supp $v \neq 0$.

(a) Supp $j \in \mathbf{N}^+$ is the smallest suth $T^j v \in \operatorname{span}(v, Tv, \dots, T^{j-1}v) = U_0$. Then $j \leqslant m$.

Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1} T^{j-1} v$. And becs $T(T^k v) = T^{k+1} \in U_0$. $U_0$ is invard $T$.

By Exe (6), $\forall k \in \mathbf{N}$, $T^{j+k} v = T^k(T^j v) \in U_0$.

Thus $U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}v, \dots, T^n v)$ for all $n \geqslant j - 1$. Let $n = m - 1$ and done.

(b) Let $U = \operatorname{span}(v, Tv, \dots, T^{m-1}v)$.

By (a), $U = U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}, \dots, T^{m-1}, \dots, T^n)$ for all $n \geqslant m - 1$. □

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• (4E 21) *Supp $V$ finide, $T \in \mathcal{L}(V)$. Prove the min $p$ has deg at most* $1 + \dim \operatorname{range} T$.

*If* $\dim \operatorname{range} T < \dim V - 1$, *then this result gives a better upper bound for the deg of min poly.*

**Solus:**

If $T$ inje, then $\operatorname{range} T = V$ and done. Supp non0 $v \in \operatorname{null} T$. Then $Tv = 0$.

1 is the smallest positive integer suth $T^1 v \in \operatorname{span}(v)$. Define $q(z) = z \Rightarrow q(T)v = 0$.

Let $W = \operatorname{range} q(T) = \operatorname{range} T$. $\exists$ monic $s \in \mathcal{P}_{\dim W}(\mathbf{F})$ of smallest deg, suth $s(T|_W) = 0$.

Hence $sq$ is the min poly, and $\deg(sq) = \deg s + \deg q \leqslant \dim \operatorname{range} T + 1$. □

---

**19** *Supp $V$ is finide,* $\dim V > 1, T \in \mathcal{L}(V)$. *Prove* $\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V)$.

**Solus:** If $\forall S \in \mathcal{L}(V), \exists p \in \mathcal{P}(\mathbf{F}), S = p(T)$. Then by [5.20], $\forall S_1, S_2 \in \mathcal{L}(V), S_1 S_2 = S_2 S_1$.

Note that $\dim \geqslant 2$. By (3.A.14), $\exists S_1, S_2 \in \mathcal{L}(V), S_1 S_2 \neq S_2 S_1$. Ctradic. □

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• (4E 19) *Supp $V$ is finide and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.*
*Prove $\dim \mathcal{E}$ equals the deg of the min poly of $T$.*

**Solus:**

Becs the list $(I, T, \dots, T^{\dim V^2})$ of len $\dim \mathcal{L}(V) + 1$ is linely dep in $\dim \mathcal{L}(V)$.
Supp $m \in \mathbf{N}^+$ is the smallest suth $T^m = a_0 I + \cdots + a_{m-1} T^{m-1}$.
Then $q$ defined by $q(z) = z^m - a_{m-1} z^{m-1} - \cdots - a_0$ is the min poly ( see [8.40] ).
For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k(T^m) \in \operatorname{span}(I, T, \dots, T^{m-1}) = U$.
Hence $\operatorname{span}(I, T, \dots, T^{\dim V^2}) = \operatorname{span}(I, T, \dots, T^{\dim V^2 - 1}) = U$.
Note that by the min of $m$, $(I, T, \dots, T^{m-1})$ is liney indep.
Thus $\dim U = m = \dim \operatorname{span}(I, T, \dots, T^{\dim V^2 - 1}) = \dim \operatorname{span}(I, T, \dots, T^n)$ for all $m < n \in \mathbf{N}^+$.
Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.
(a) Supp $p(T) = 0$. 又 $\deg p \leqslant m - 1 \Rightarrow p = 0$. Then $\varphi$ is inje.
(b) $\forall S = a_0 I + a_1 T + \cdots + a_{m-1} T^{m-1} \in \mathcal{E}$, define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by
$p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} \Rightarrow \varphi(p) = S$. Then $\varphi$ is surj.
Hence $\mathcal{E}$ and $\mathcal{P}_{m-1}(\mathbf{F})$ are iso. 又 $\dim \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. □

---

• (4E 5.B.13) *Supp $T \in \mathcal{L}(V)$ and $q(z) = a_0 + a_1 z + \cdots + a_n z^n$.*
*Denote the min poly of $T$ by $p(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m$.*
*Prove $\exists!\, r \in \mathcal{P}(\mathbf{F})$ suth $q(T) = r(T), \deg r < \deg p$.*

**Solus:**

If $\deg q < \deg p$, then done.
If $\deg q = \deg p$, notice that $p(T) = 0 = c_0 I + c_1 T + \cdots + c_{m-1} T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0 I - c_1 T - \cdots - c_{m-1} T^{m-1},$$

define $r$ by $r(z) = q(z) + \left[ -a_m z^m + a_m(-c_0 - c_1 z - \cdots - c_{m-1} z^{m-1}) \right]$

$$= (a_0 - a_m c_0) + (a_1 - a_m c_1) z + \cdots + (a_{m-1} - a_m c_{m-1}) z^{m-1},$$

hence $r(T) = 0, \deg r < m$ and done.
Now supp $\deg q \geqslant \deg p$. We use induc on $\deg q$.
(i) $\deg q = \deg p$, then the desired result is true, as shown above.
(ii) $\deg q > \deg p$, asum the desired result is true for $\deg q = n$.
Supp $f \in \mathcal{P}(\mathbf{F})$ suth $f(z) = b_0 + b_1 z + \cdots + b_n z^n + b_{n+1} z^{n+1}$.
Apply the asum to $g$ defined by $g(z) = b_0 + b_1 z + \cdots + b_n z^n$,
getting $s$ defined by $s(z) = d_0 + d_1 z + \cdots + d_{m-1} z^{m-1}$.
Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1} T^{n+1} = s(T) + b_{n+1} T^{n+1}$.
Apply the asum to $t$ defined by $t(z) = z^n$,
getting $\delta$ defined by $\delta(z) = {c_0}' + {c_1}' z + \cdots + {c_{m-1}}' z^{m-1}$.
Thus $t(T) = T^n = {c_0}' + {c_1}' z + \cdots + {c_{m-1}}' z^{m-1} = \delta(T)$.
又 $\operatorname{span}(v, Tv, \dots, T^{m-1} v)$ is invard $T$.
Hence $\exists!\, k_j \in \mathbf{F}, T^{n+1} = T(T^n) = k_0 + k_1 z + \cdots + k_{m-1} z^{m-1}$.
And $f(T) = s(T) + b_{n+1}(k_0 + k_1 T + \cdots + k_{m-1} T^{m-1})$
$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1) z + \cdots + (d_{m-1} + k_{m-1}) z^{m-1} = h(T)$, thus defining $h$. □

---

• (4E 14) *Supp $V$ is finide, $T \in \mathcal{L}(V)$ has min $p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m$.*
*Find the min poly of $T^{-1}$.*

**Solus:**

Notice that $V$ is finide. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is inv.
Then $p(T) = a_0 I + a_1 T + \cdots + T^m = 0$. Apply $T^{-m}$ to both sides,
$a_0 (T^{-1})^m + a_1 (T^{-1})^{m-1} + \cdots + a_{m-1} T^{-1} + I = 0$.
Define $q$ by $q(z) = z^m + \frac{a_1}{a_0} z^{m-1} + \cdots + \frac{a_{m-1}}{a_0} z + \frac{1}{a_0}$ for all $z \in \mathbf{F}$.

We now show $(T^{-1})^k \notin \operatorname{span}(I, T^{-1}, \ldots, (T^{-1})^{k-1})$
for every $k \in \{1, \ldots, m-1\}$ by ctradic, so that $q$ is exactly the min poly of $T^{-1}$.
Supp $(T^{-1})^k \in \operatorname{span}(I, T^{-1}, \ldots, (T^{-1})^{k-1})$.
Then let $(T^{-1})^k = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. Apply $T^k$ to both sides,
getting $I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T$, hence $T^k \in \operatorname{span}(I, T, \ldots, T^{k-1})$.

Thus $f$ defined by $f(z) = z^k + \frac{b_1}{b_0} z^{k-1} + \cdots + \frac{b_{k-1}}{b_0} z - \frac{1}{b_0}$ is a poly multi of $p$.
While $\deg f < \deg p$. Ctradic. $\square$

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- **NOTE FOR [8.49]:** Supp $\mathbf{F} = \mathbf{C}$, $V$ is finide, $T \in \mathcal{L}(V)$, and $p$ is the min.
Let $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$, where $\lambda_1, \ldots, \lambda_m$ are all the eigvals of $T$, possibly with repeti.
**COMMENT:** $\big[$ ***Another proof of*** **[5.13]** $\big]$ Besc the min poly of $T$ has at most $\dim V$ dist zeros. $\square$
- **CORO:** Supp $\alpha_1, \ldots, \alpha_n$ are all the disti eigvals of $T$.
**NOTE:** the min of $T$ is a multi of, but not equal to, $(z - \alpha_1) \cdots (z - \alpha_n)$.
Let $q(z) = (z - \alpha_1)^{\dim V - (n-1)} \cdots (z - \alpha_n)^{\dim V - (n-1)}$. Then $q$ is a multi of the ch.
$\big[$ Becs $\dim V > n$ and $n - 1 \geqslant 0$, $n \times [\dim V - (n-1)] > \dim V$. $\big]$
Form for ch: $(z - \alpha_1)^{\gamma_1} \cdots (z - \alpha_n)^{\gamma_n}$, where $\gamma_1 + \cdots + \gamma_n = \dim V$.
Form for min: $(z - \alpha_1)^{\delta_1} \cdots (z - \alpha_n)^{\delta_n}$, where $0 \leqslant \delta_1 + \cdots + \delta_n \leqslant \dim V$.

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**10** *Supp $T \in \mathcal{L}(V)$, $\lambda$ is eigval of $T$ with $v$. Prove if $p \in \mathcal{P}(\mathbf{F})$, then $p(T)v = p(\lambda)v$.*

**SOLUS:** Define $p(z) = a_0 + a_1 z + \cdots + a_m z^m$. Becs for each $k \in \mathbf{N}^+$, $T^k v = \lambda^k v$, and $T^0 v = v$.
Now $p(T)v = a_0 v + a_1 \lambda v + \cdots + a_m \lambda^m v = p(\lambda)v$. $\square$

**CORO:** $p(T) = c(T - \lambda_1)^{\alpha_1} \cdots (T - \lambda_m)^{\alpha_m} = c(\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m}$.

---

**1** *Supp $T \in \mathcal{L}(V)$ and $\exists\, n \in \mathbf{N}^+$ suth $T^n = 0$.*
*Prove $(I - T)$ is inv and $(I - T)^{-1} = I + T + \cdots + T^{n-1}$.*

**SOLUS:** Note that $1 - x^n = (1 - x)(1 + x + \cdots + x^{n-1})$.

$$\left.\begin{aligned} (I - T)(1 + T + \cdots + T^{n-1}) = I - T^n = I \\ (1 + T + \cdots + T^{n-1})(I - T) = I - T^n = I \end{aligned}\right\} \Rightarrow (I - T)^{-1} = 1 + T + \cdots + T^{n-1}.$$ $\square$

---

**2** *Supp $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Prove the eigvals are $2, 3, 4$.*

**SOLUS:**

Supp $v$ is an eigvec corres to $\lambda$. Then for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.
Hence $0 = (T - 2I)(T - 3I)(T - 4I)v = (\lambda - 2)(\lambda - 3)(\lambda - 4)v$ while $v \neq 0 \Rightarrow \lambda = 2, 3$ or $4$. $\square$

---

**3** *Supp $T \in \mathcal{L}(V)$, $T^2 = I$ and $-1$ is not eigval. Prove $T = I$.*

**SOLUS:**

$T^2 - I = (T + I)(T - I)$ is not inje, 又 $-1$ is not an eigval of $T \Longrightarrow$ By TIPS. $\square$

OR. Note that $\forall v \in V, v = [\frac{1}{2}(I - T)v] + [\frac{1}{2}(I + T)v]$.

$$\left.\begin{array}{l}(I+T)\big((I-T)v\big)=0 \Longrightarrow (I-T)v\in \operatorname{null}(I+T)\\ (I-T)\big((I+T)v\big)=0 \Longrightarrow (I+T)v\in \operatorname{null}(I-T)\end{array}\right\}\Rightarrow V=\operatorname{null}(I+T)+\operatorname{null}(I-T).$$

又 $-1$ is not an eigval of $T \iff (I+T)$ is inje $\iff \operatorname{null}(I+T)=\{0\}$.
Hence $V=\operatorname{null}(I-T)\Rightarrow \operatorname{range}(I-T)=\{0\}$. Thus $I-T=0\in\mathcal{L}(V)\Rightarrow T=I$. □

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- (4E 5.A.32) *Supp $T\in\mathcal{L}(V)$ has no eigvals and $T^4=I$. Prove $T^2=-I$.*

**Solus:**

Becs $T^4-I=(T^2-I)(T^2+I)=0$ is not inje $\Rightarrow (T^2-I)$ or $(T^2+I)$ is not inje.
又 $T$ has no eigvals $\Rightarrow (T^2-I)=(T-I)(T+I)$ is inje. Hence $T^2+I=0\in\mathcal{L}(V)$, for if not, $\exists v\in V,(T^2+I)v\neq 0$ while $(T^2-I)\big((T^2+I)v\big)=0$ but $(T^2-I)$ is inje. Ctradic.
Or. $\forall v\in V, 0=(T^2-I)(T^2+I)v \iff 0=(T^2+I)v$. Hence $T^2+I=0$. □

Or. Note that $\forall v\in V, v=\left[\frac{1}{2}(I-T^2)v\right]+\left[\frac{1}{2}(I+T^2)v\right]$.

$$\left.\begin{array}{l}(I+T^2)\big((I-T^2)v\big)=0 \Longrightarrow (I-T^2)v\in \operatorname{null}(I+T^2)\\ (I-T^2)\big((I+T^2)v\big)=0 \Longrightarrow (I+T^2)v\in \operatorname{null}(I-T^2)\end{array}\right\}\Rightarrow V=\operatorname{null}(I+T^2)+\operatorname{null}(I-T^2).$$

又 $T$ has no eigvals $\iff (I-T^2)$ is inje $\iff \operatorname{null}(I-T^2)=\{0\}$.
Hence $V=\operatorname{null}(I+T^2)\Rightarrow \operatorname{range}(I+T^2)=\{0\}$. Thus $I+T^2=0\in\mathcal{L}(V)\Rightarrow T^2=-I$. □

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**8** [Or (4E 5.A.31)] *Give an exa of $T\in\mathcal{L}(\mathbf{R}^2)$ suth $T^4=-I$.*

**Solus:**

Define $\mathrm{i}\in\mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y)=(-y,x)$. Just like $\mathrm{i}:\mathbf{C}\to\mathbf{C}$ defined by $\mathrm{i}(x+\mathrm{i}y)=-y+\mathrm{i}x$.
Define $\mathrm{i}^n\in\mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y)=\Big(\operatorname{Re}(\mathrm{i}^n x+\mathrm{i}^{n+1}y),\operatorname{Im}(\mathrm{i}^n x+\mathrm{i}^{n+1}y)\Big)$.
$T^4+I=(T^2+\mathrm{i}I)(T^2-\mathrm{i}I)=(T+\mathrm{i}^{1/2}I)(T-\mathrm{i}^{1/2}I)(T-(-\mathrm{i})^{1/2}I)(T+(-\mathrm{i})^{1/2}I)$.
Note that $\mathrm{i}^{1/2}=\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}$, $(-\mathrm{i})^{1/2}=\frac{\sqrt{2}}{2}-\mathrm{i}\frac{\sqrt{2}}{2}$. Hence $T=\pm(\pm\mathrm{i})^{1/2}I$.
Let $T=\mathrm{i}^{1/2}I$ defined by $\mathrm{i}^{1/2}(x,y)=\left(\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y,\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y\right)$. □

Or. Becs $\mathcal{M}(T^4)=\begin{pmatrix}\cos(-\pi) & \sin(-\pi)\\ -\sin(-\pi) & \cos(-\pi)\end{pmatrix}$. Using $\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}^n=\begin{pmatrix}\cos n\alpha & \sin n\alpha\\ -\sin n\alpha & \cos n\alpha\end{pmatrix}$.

We define $T\in\mathcal{L}(\mathbf{R}^2)$ suth $\mathcal{M}(T)=\begin{pmatrix}\cos(-\pi/4) & \sin(-\pi/4)\\ -\sin(-\pi/4) & \cos(-\pi/4)\end{pmatrix}$. □

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- (4E 5.B.12) *Find the min poly of $T$ defined in (5.A.10).*

**Solus:** By (5.A.9) and [8.40, 8.49], $1,2,\dots,n$ are all the zeros of the min poly of $T$. □

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- (4E 5.B.3) *Find the min poly of $T$ defined in (5.A.19).*

**Solus:**

If $n=1$ then 1 is the only eigval of $T$, and $(z-1)$ is the min poly.
Becs $n$ and 0 are all the eigvals of $T$, 又 $\forall k\in\{1,\dots,n\}, Te_k=e_1+\cdots+e_n$; $T^2e_k=n(e_1+\cdots+e_n)$.
Hence $T^2e_k=n(Te_k)\Rightarrow T^2=nT\Rightarrow T^2-nT=T(T-n)=0$. Thus $\big(z(z-n)\big)$ is the min poly. □

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- (4E 5.B.8) *Find the min poly of $T\in\mathcal{L}(\mathbf{R}^2)$, the countclockws rotat optor by $\theta\in\mathbf{R}^+$.*

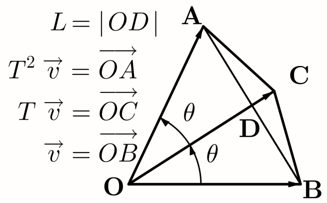
**Solus:**

If $\theta=\pi+2k\pi$, then $T(w,z)=(-w,-z), T^2=I$ and the min poly is $z+1$.
If $\theta=2k\pi$, then $T=I$ and the min poly is $z-1$.

Othws $(v, Tv)$ is liney indep. Then $\operatorname{span}(v, Tv) = \mathbf{R}^2$. Note that $\nexists\, b \in \mathbf{F}, T - bI = 0$.

Thus supp the min $p(z) = z^2 + bz + c$.

Becs




$$Tv = \frac{|\vec{v}|}{2L}(T^2v + v) \Rightarrow T = \frac{|\vec{v}|}{2L}(T^2 + I)$$
$$L = |\vec{v}|\cos\theta \Rightarrow \frac{|\vec{v}|}{2L} = \frac{1}{2\cos\theta}$$

Hence $p(T) = T^2 - 2\cos\theta\, T + I = 0$ and $z^2 - 2\cos\theta\, z + 1$ is the min poly of $T$. ☐

Or. Let $(e_1, e_2)$ be the std bss of $\mathbf{R}^2$. We use the pattern shown in [8.44].

Becs $Te_1 = \cos\theta\, e_1 + \sin\theta\, e_2,\ T^2e_1 = \cos 2\theta\, e_1 + \sin 2\theta\, e_2$.

Thus $ce_1 + bTe_1 = -T^2e_1 \iff \begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix}\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$. Now $\det = \sin\theta \neq 0, c = 1, b = 2\cos\theta$. ☐

Or. $\mathcal{M}(T, (e_1, e_2)) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. By (4E 11), the min poly is $(z \pm 1)$ or $(z^2 - 2\cos\theta\, z + 1)$. ☐

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- (4E 11) *Supp $V$ is 2-dim, $T \in \mathcal{L}(V)$, and $\mathcal{M}(T, B_V) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.*
  (a) *Show $T^2 - (a+d)T + (ad - bc)I = 0$.*
  (b) *Show the min poly of $T$ equals*
$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad - bc) & \text{othws.} \end{cases}$$

**Solus:**

(a) Supp the bss is $(v, w)$. Becs $\begin{cases} Tv = av + bw \Rightarrow (T - aI)v = bw, \text{ then apply } (T - dI) \text{ to both sides.} \\ Tw = cv + dw \Rightarrow (T - dI)w = cv, \text{ then apply } (T - aI) \text{ to both sides.} \end{cases}$

Hence $(T - aI)(T - dI) = bcI \Rightarrow T^2 - (a+d)T + (ad - bc)I = 0$.

(b) If $b = c = 0$ and $a = d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the min poly is $z - a$.

Othws, by (a), $z^2 - (a+d)z + (ad - bc)$ is a poly multi of the min poly.

Now we prove that $T \notin \operatorname{span}(I)$, so that then the min poly of $T$ has exactly deg 2.

( At least one of the asum of (I),(II) below is true. )

(I) Supp $a = d$, then $Tv = av + bw \notin \operatorname{span}(v), Tw = cv + aw \notin \operatorname{span}(w)$.

(II) Supp at most one of $b, c$ is not 0. If $b = 0$, then $Tw \notin \operatorname{span}(w)$; If $c = 0$, then $Tv \notin \operatorname{span}(v)$. ☐

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- *Supp $S, T \in \mathcal{L}(V)$, $S$ is inv, and $p \in \mathcal{P}(\mathbf{F})$. Prove $Sp(TS) = p(ST)S$.*

**Solus:**

We prove $S(TS)^m = (ST)^m S$ for each $m \in \mathbf{N}$ by induc.

(i) If $m = 0, 1$. Then $S(TS)^0 = I = (ST)^0 S$; $S(TS)^1 = (ST)S$.

(ii) If $m > 1$. Asum $S(TS)^m = (ST)^m S$.

Then $S(TS)^{m+1} = S(TS)^m(TS) = (ST)^m STS = (ST)^{m+1}S$.

Hence $\forall p \in \mathcal{P}(\mathbf{F}), Sp(TS) = \sum_{k=1}^{m} a_k S(TS)^k = \sum_{k=1}^{m} a_k p(ST)^k S = \left[\sum_{k=1}^{m} a_k (TS)^k\right] S$. ☐

**Comment:** $p(TS) = S^{-1}p(ST)S,\ p(ST) = Sp(TS)S^{-1}$.

**Coro:** **5** Becs $S$ is inv, $T \in \mathcal{L}(V)$ is arb $\iff R = ST$ is arb.

Hence $\forall R \in \mathcal{L}(V)$, inv $S \in \mathcal{L}(V), p(S^{-1}RS) = S^{-1}p(R)S$.

---

- (4E 5.B.7) *Supp $S, T \in \mathcal{L}(V)$. Let $p, q$ be the min polys of $ST, TS$ respectly.*
  (a) *If $V = \mathbf{F}^2$. Give an exa suth $p \neq q$;* (b) *If $S$ or $T$ is inv. Prove $p = q$.*

**SOLUS:**

(a) Define $S$ by $S(x,y) = (x,x)$. Define $T$ by $T(x,y) = (0,y)$.
Then $ST(x,y) = 0,\ TS(x,y) = (0,x)$ for all $(x,y) \in \mathbf{F}^2$. Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.
Hence the min poly of $ST$ does not equal to the min poly of $TS$.

(b) Supp $S$ is inv. Becs $p, q$ are monic.
$$\left.\begin{array}{l} p(ST) = 0 = Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a poly multi of } q \\ q(TS) = 0 = S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a poly multi of } p \end{array}\right\} \Rightarrow p = q.$$
Reversing the roles of $S$ and $T$, we conclude that if $T$ is inv, then $p = q$ as well. □

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**11** *Supp* $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{C})$, *and* $\alpha \in \mathbf{C}$.
*Prove* $\alpha$ *is an eigval of* $p(T) \iff \alpha = p(\lambda)$ *for some eigval* $\lambda$ *of* $T$.

**SOLUS:**

(a) Supp $\alpha$ is an eigval of $p(T) \Leftrightarrow (p(T) - \alpha I)$ is not inje.
Write $p(z) - \alpha = c(z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow p(T) - \alpha I = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.
By TIPS, $\exists\, (T - \lambda_j I)$ not inje. Thus $p(\lambda_j) - \alpha = 0$.

(b) Supp $\alpha = p(\lambda)$ and $\lambda$ is an eigval of $T$ with an eigvec $v$. Then $p(T)v = p(\lambda)v = \alpha v$. □

OR. Define $q$ by $q(z) = p(z) - \alpha$. $\lambda$ is a zero of $q$.
Becs $q(T)v = (p(T) - \alpha I)v = q(\lambda)v = (p(\lambda) - \alpha)v = 0$.
Hence $q(T)$ is not inje $\Rightarrow (p(T) - \alpha I)$ is not inje. □

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**12** [4E 6] *Give a* $T \in \mathcal{L}(\mathbf{R}^2)$ *that shows the result above does not hold if* $\mathbf{C}$ *is replaced with* $\mathbf{R}$.

**SOLUS:**

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(w,z) = (-z, w)$.
By Exe (4E 5.B.11), $\mathcal{M}(T, ((1,0),(0,1))) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ the min poly of $T$ is $z^2 + 1$.
Define $p$ by $p(z) = z^2$. Then $p(T) = T^2 = -I$. Thus $p(T)$ has eigval $-1$.
While $\nexists \lambda \in \mathbf{R}$ suth $-1 = p(\lambda) = \lambda^2$. □

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• (4E 17) *Supp* $V$ *is finide,* $p$ *is min of* $T \in \mathcal{L}(V)$, *and* $\lambda \in \mathbf{F}$.
*Show* $q(z) = p(z + \lambda)$ *is min of* $(T - \lambda I)$.

**SOLUS:**

$q(T - \lambda I) = 0 \Rightarrow q$ is poly multi of the min poly of $(T - \lambda I)$.
Supp the deg of the min poly of $(T - \lambda I)$ is $n$, and the deg of the min poly of $T$ is $m$.
By definition of min poly,
$n$ is the smallest suth $(T - \lambda I)^n \in \operatorname{span}(I, (T - \lambda I), \ldots, (T - \lambda I)^{n-1})$;
$m$ is the smallest suth $T^m \in \operatorname{span}(I, T, \ldots, T^{m-1})$.
又 $T^k \in \operatorname{span}(I, T, \ldots, T^{k-1}) \iff (T - \lambda)^k \in \operatorname{span}(I, (T - \lambda I), \ldots, (T - \lambda I)^{k-1})$.
Thus $n = m$. 又 $q$ is monic. By the uniqnes of min poly. □

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• (4E 18) *Supp* $V$ *is finide,* $p$ *is min of* $T \in \mathcal{L}(V), \lambda \neq 0$. *Show* $q(z) = \lambda^{\deg p} p(z/\lambda)$ *is min of* $\lambda T$.

**SOLUS:**

$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow q$ is a poly multi of the min poly of $\lambda T$.
Supp the deg of the min poly of $\lambda T$ is $n$, and the deg of the min poly of $T$ is $m$.

By definition of min poly,
$n$ is the smallest suth $(\lambda T)^n \in \operatorname{span}(\lambda I, \lambda T, \dots, (\lambda T)^{n-1})$;
$m$ is the smallest suth $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.
又 $(\lambda T)^k \in \operatorname{span}(\lambda I, \lambda T, \dots, (\lambda T)^{k-1}) \Longleftrightarrow T^k \in \operatorname{span}(I, T \dots, T^{k-1})$.
Thus $n = m$. 又 $q$ is monic. By the uniqnes of min poly. $\square$

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**18** [4E 15] *Supp $V$ is a finide complex vecsp with $\dim V > 0$ and $T \in \mathcal{L}(V)$.*
*Define $f : \mathbf{C} \to \mathbf{R}$ by $f(\lambda) = \dim \operatorname{range}(T - \lambda I)$. Prove $f$ is not continuous.*

**Solus:** Note that $V$ is finide.
Let $\lambda_0$ be an eigval of $T$. Then $(T - \lambda_0 I)$ is not surj. Hence $\dim \operatorname{range}(T - \lambda_0 I) < \dim V$.
Becs $T$ has finily many eigvals. There exis a seq of number $\{\lambda_n\}$ suth $\lim\limits_{n\to\infty} \lambda_n = \lambda_0$.
And $\lambda_n$ is not an eigval of $T$ for each $n \Rightarrow \dim \operatorname{range}(T - \lambda_n I) = \dim V \neq \dim \operatorname{range}(T - \lambda_0 I)$.
Thus $f(\lambda_0) \neq \lim\limits_{n\to\infty} f(\lambda_n)$. $\square$

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• (4E 5.B.9) *Supp $T \in \mathcal{L}(V)$ is suth wrto some bss of $V$,*
*all ent of the matrix of $T$ are rational numbers.*
*Explain why all coeffs of the min poly of $T$ are rational numbers.*

**Solus:**
Let $(v_1, \dots, v_n)$ denote the bss suth $\mathcal{M}(T, (v_1, \dots, v_n))_{j,k} = A_{j,k} \in \mathbf{Q}$ for all $j, k = 1, \dots, n$.
Denote $\mathcal{M}(v_j, (v_1, \dots, v_n))$ by $x_j$ for each $v_j$.
Supp $p$ is the min poly of $T$ and $p(z) = z^m + \dots + c_1 z + c_0$. Now we show each $c_j \in \mathbf{Q}$.
Note that $\forall s \in \mathbf{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbf{Q}^{n,n}$ and $T^s v_k = A^s_{1,k} v_1 + \dots + A^s_{n,k} v_n$ for all $k \in \{1, \dots, n\}$.

$$\text{Thus} \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \dots + c_1 A + c_0 I)x_1 = \sum\limits_{j=1}^{n} (A^m + \dots + c_1 A + c_0 I)_{j,1} x_j = 0; \\ \qquad\vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \dots + c_1 A + c_0 I)x_n = \sum\limits_{j=1}^{n} (A^m + \dots + c_1 A + c_0 I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly,} \begin{cases} (A^m + \dots + c_1 A + c_0 I)_{1,1} = \dots = (A^m + \dots + c_1 A + c_0 I)_{n,1} = 0; \\ \qquad\qquad \vdots \quad \ddots \quad \vdots \\ (A^m + \dots + c_1 A + c_0 I)_{1,n} = \dots = (A^m + \dots + c_1 A + c_0 I)_{n,n} = 0; \end{cases}$$

Hence we get a system of $n^2$ liney equations in $m$ unknowns $c_0, c_1, \dots, c_{m-1}$.
We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbf{Q}$. $\square$

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• [Or (4E 5.B.16), Or (8.C.18)] *Supp $a_0, \dots, a_{n-1} \in \mathbf{F}$. Let $T$ be the optor on $\mathbf{F}^n$ suth*

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ wrto the std bss } (e_1, \dots, e_n).$$

*Show the min poly of $T$ is $p$ defined by $p(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + z^n$.*
*$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the min poly of some optor. Hence a formula or an algo that could produce exact eigvals for each optor on each $\mathbf{F}^n$ could then produce exact zeros for each poly [ by 8.36(b) ]. Thus there is no such formula or algo. However, efficient numeric methods exis for obtaining very good approximations for the eigvals of an optor.*

**Solus:** Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ is liney indep. 又 The deg of min poly is at most $n$.

$T^n e_1 = \cdots = T^{n-k} e_{1+k} = \cdots = Te_n = -a_0 e_1 - a_1 e_2 - a_2 e_3 - \cdots - a_{n-1} e_n$
$= \big( -a_0 I - a_1 T - a_2 T^2 - \cdots - a_{n-1} T^{n-1} \big) e_1$. Thus $p(T)e_1 = 0 = p(T)e_j$ for each $e_j = T^{j-1} e_1$. □

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• EIGENVALUES ON ODD-DIMENSIONAL REAL VECTOR SPACES

• EVEN-DIMENSIONAL NULL SPACE

*Supp* $\mathbf{F} = \mathbf{R}$*, $V$ is finide, $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$.*
*Prove* $\dim \operatorname{null}(T^2 + bT + cI)$ *is an even number.*

**SOLUS:**

Denote $\operatorname{null}(T^2 + bT + cI)$ by $R$. Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.
Supp $\lambda$ is an eigval of $T_R$ with an eigvec $v \in R$.
Then $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \big((\lambda + b)^2 + c - \frac{b^2}{4}\big)v$.
Becs $c - \frac{b^2}{4} > 0$ and we have $v = 0$. Thus $T_R$ has no eigvals.
Let $U$ be invarsp of $R$ that has the largest, even dim among all invarsps.
Asum $U \neq R$. Then $\exists\, w \in R$ but $w \notin U$. Let $W$ be suth $(w, T|_R w)$ is a bss of $W$.
Becs $T|_R^2 w = -bT|_R w - cw \in W$. Hence $W$ is invarsp of dim 2.
Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,
for if not, becs $w \notin U, T|_R w \in U$,
$U \cap W$ is invard $T|_R$ of one dim ( impossible becs $T|_R$ has no eigvecs ).
Hence $U + W$ is even-dim invarsp under $T|_R$, ctradic the max of $\dim U$.
Thus the asum was incorrect. Hence $R = \operatorname{null}(T^2 + bT + cI) = U$ has even dim. □

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• OPERATORS ON ODD-DIMENSIONAL VECTOR SPACES HAVE EIGENVALUES

(a) *Supp* $\mathbf{F} = \mathbf{C}$. Then by [5.21], done.
(b) *Supp* $\mathbf{F} = \mathbf{R}$*, $V$ is finide, and* $\dim V = n$ *is an odd number.*
*Let $T \in \mathcal{L}(V)$ and the min poly is $p$. Prove $T$ has an eigval.*

**SOLUS:**

(i) If $n = 1$, then done.
(ii) Supp $n \geqslant 3$. Asum every optor, on odd-dim vecsps of dim less than $n$, has an eigval.
If $p$ is a poly multi of $(x - \lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] $\lambda$ is an eigval of $T$ and done.
Now supp $b, c \in \mathbf{R}$ suth $b^2 < 4c$ and $p$ is a poly multi of $x^2 + bx + c$ (see [4.17]).
Then $\exists\, q \in \mathcal{P}(\mathbf{R})$ suth $p(x) = q(x)(x^2 + bx + c)$ for all $x \in \mathbf{R}$.
Now $0 = p(T) = \big(q(T)\big)(T^2 + bT + cI)$, which means that $q(T)|_{\operatorname{range}(T^2+bT+cI)} = 0$.
Becs $\deg q < \deg p$ and $p$ is the min poly of $T$, hence $\operatorname{range}(T^2 + bT + cI) \neq V$.
又 $\dim V$ is odd and $\dim \operatorname{null}(T^2 + bT + cI)$ is even ( by our previous result ).
Thus $\dim V - \dim \operatorname{null}(T^2 + bT + cI) = \dim \operatorname{range}(T^2 + bT + cI)$ is odd.
By [5.18], $\operatorname{range}(T^2 + bT + cI)$ is invarsp of $V$ under $T$ that has odd dim less than $n$.
Our induc hypo now implies that $T|_{\operatorname{range}(T^2+bT+cI)}$ has an eigval.

By induc. □

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• (2E 24) *Supp* $\mathbf{F} = \mathbf{R}$*, $T \in \mathcal{L}(V)$ has no eigvals. Prove every invarsp $U$ is even-dim.*

**SOLUS:** Asum $\dim U$ is odd. Then $\exists$ eigval of $T|_U$, so of $T \Rightarrow \exists$ 1-dim invarsp, ctradic. □

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• (4E 29) *Supp $V$ is finide,* $\dim V = n \geqslant 2$*, $T \in \mathcal{L}(V)$. Show $T$ has a 2-dim invarsp.*

**Solus:** (i) $n = 2$, done.

(ii) $n > 2$. Asum the desired result is true for $V$ of smaller dim.

Supp $p$ is the min of deg $m$ and $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.

If $T = \lambda I \iff p = 0$ or $\deg p = 1$, then done. $\big[\ m \neq 0$ becs $\dim V \neq 0.\ \big]$

Now define $q(z) = (z - \lambda_1)(z - \lambda_2)$. By asum, $T\big|_{\operatorname{null} q(T)}$ has invarsp of dim 2. $\square$

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**Ended**

# 5.B: II

• (4E 5.C.1) *Supp $T^2$ has up-trig matrix. Give a countexa: $T$ has up-trig matrix.*

**Solus:**

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• (4E 5.C.2) *Supp $A$ and $B$ are up-trig matrices of the same size, with $\alpha_1, \dots, \alpha_n$ on the diag of $A$ and $\beta_1, \dots, \beta_n$ on the diag of $B$.*
(a) *Show $A + B$ up-trig with $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ on the diag.*
(b) *Show $AB$ up-trig with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diag.*

**Solus:**

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• (4E 5.C.3) *Supp $T$ inv, $B_V = (v_1, \dots, v_n)$, $\mathcal{M}(T) = A$ is up-trig, with $\lambda_1, \dots, \lambda_n$ on diag. Show $A^{-1}$ is also up-trig, with $\lambda_1^{-1}, \dots, \lambda_n^{-1}$ on diag.*

**Solus:**

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**9** [4E 5.C.7] *Supp $V$ is finide, and $v \in V$.*
(a) *Prove $\exists\,!$ monic $p_v$ of smallest deg suth $p_v(T)v = 0$.*
(b) *Prove the min poly of $T$ is a poly multi of $p_v$.*

**Solus:**

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**14** [Or (4E 5.C.4)] *Give an inv $T$ and a $B_V$ suth each $\mathcal{M}(T)_{k,k} = 0$.*

**Solus:**

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**15** [Or (4E 5.C.5)] *Give a non-inv $T$ and a $B_V$ suth each $\mathcal{M}(T)_{k,k} \neq 0$.*

**Solus:**

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**20** [Or (4E 5.C.6)] *Supp $\mathbf{F} = \mathbf{C}$, $V$ is finide, and $k \in \{1, \dots, \dim V\}$.*
*Prove $V$ has a $k$-dim subsp invard $T$.*

**Solus:**

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• (4E 5.C.8) *Supp $V$ is finide, and $\exists\, v \in V \setminus \{0\}$ suth $T^2 v + 2Tv = -2v$.*
(a) *Supp $\mathbf{F} = \mathbf{R}$. Prove $\nexists\, B_V$ suth $\mathcal{M}(T)$ up-trig.*
(b) *Supp $\mathbf{F} = \mathbf{C}$, and $\exists\, B_V$ suth $A = \mathcal{M}(T)$ up-trig. Prove $-1 + \mathrm{i}$ or $-1 - \mathrm{i}$ on the diag of $A$.*

**Solus:**

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• (4E 5.C.9) *Supp $B \in \mathbf{F}^{n,n}$ with complex ent.*
*Prove $\exists$ inv $A \in \mathbf{F}^{n,n}$ with complex ent suth $A^{-1}BA$ is up-trig.*

**Solus:**

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• (4E 5.C.10) *Supp* $B_V = (v_1, \dots, v_n)$. *Show the following are equi*:
(a) $\mathcal{M}(T, B_V)$ *lower trig.* (b) *Each* $\operatorname{span}(v_k, \dots, v_n)$ *invard* $T$. (c) *Each* $Tv_k \in \operatorname{span}(v_k, \dots, v_n)$.

**Solus:**

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• (4E 5.C.11) *Supp* $\mathbf{F} = \mathbf{C}$, $V$ *is finide. Prove* $\exists\, B_V$ *suth* $\mathcal{M}(T)$ *low-trig.*

**Solus:**

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• (4E 5.C.12) *Supp* $V$ *is finide,* $U$ *invarspd* $T$, *and* $\mathcal{M}(T)$ *is up-trig for some* $B_V$.
(a) *Prove* $\mathcal{M}(T|_U)$ *up-trig for some* $B_U$. (b) *Prove* $\mathcal{M}(T/U)$ *up-trig for some* $B_{V/U}$.

**Solus:**

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• (4E 5.C.13) *Supp* $V$ *is finide,* $U$ *invarspd* $T$ *suth* $T|_U, T/U$ *up-trig. Prove* $T$ *up-trig.*

**Solus:**

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• (4E 5.C.14) *Supp* $V$ *is finide. Prove* $T$ *up-trig* $\iff T'$ *up-trig.*

**Solus:**

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**Ended**

# 5.C

XXXX

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**Ended**

# 5.E* [4E]

**1** *Give commu optors* $S, T \in \mathbf{F}^4$ *suth* $\exists$ *invarspd* $S$ *but not* $T$ *and* $\exists$ *invarspd* $T$ *but not* $S$.

**Solus:**

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**2** *Supp* $\mathcal{E}$ *is a subset of* $\mathcal{L}(V)$ *and every elem of* $\mathcal{E}$ *is diag.*
*Prove* $\exists\, B_V$ *suth each elem of* $\mathcal{E}$ *diag* $\iff$ *each pair of elems of* $\mathcal{E}$ *commu.*

**Solus:**

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**3** *Supp* $S, T \in \mathcal{L}(V)$ *are suth* $ST = TS$. *Supp* $p \in \mathcal{P}(\mathbf{F})$.
(a) *Prove* $\operatorname{null} p(S)$ *is invard* $T$. (b) *Prove* $\operatorname{range} p(S)$ *is invard* $T$.

**Solus:**

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**4** *Prove or give a countexa*: *A diag matrix* $A$ *and up-trig matrix* $B$ *of the same size commu.*

**Solus:**

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**5** *Prove a pair of optors on a finide vecsp commu* $\iff$ *their dual optors commu.*

**Solus:**

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**6** *Supp $V$ is a finide complex vecsp and $S, T \in \mathcal{L}(V)$ commu.*
*Prove $\exists\, \alpha, \lambda \in \mathbf{C}$ suth* $\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V$.

**SOLUS:**

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**7** *Supp $V$ is a complex vecsp, $S \in \mathcal{L}(V)$ is diag, and $T$ commu with $S$.*
*Prove $\exists\, B_V$ suth $S$ diag $T$ up-trig.*

**SOLUS:**

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**8** *Supp $m = 3$ in [5.72] and $D_x, D_y$ are the commu partial diff optors on $\mathcal{P}_3(\mathbf{R}^2)$ from [5.72].*
*Find a bss of $\mathcal{P}_3(\mathbf{R}^2)$ suth $D_x$ and $D_y$ each up-trig.*

**SOLUS:**

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**9** *Supp $V$ is a finide non0 complex vecsp.*
*Supp that $\mathcal{E} \subseteq \mathcal{L}(V)$ is suth $S$ and $T$ commu for all $S, T \in \mathcal{E}$.*
(a) *Prove $\exists$ eigvec $v \in V$ for every elem of $\mathcal{E}$.*
(b) *Prove $\exists$ a bss of $V$ wrto which every elem of $\mathcal{E}$ has up-trig matrix.*

**SOLUS:**

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**10** *Give commu optors $S, T$ on a finide real vecsp suth*
*$S + T$ has a eigval that does not equal an eigval of $S$ plus an eigval of $T$*
*and $ST$ has a eigval that does not equal an eigval of $S$ times an eigval of $T$.*

**SOLUS:**

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**ENDED**