1·B

• *Suppose S is a nonempty set. Let V^S denote the set of functions from S to V.*
Define a natural add and scalar multi on V^S, and show that V^S is a vecsp with these defs.

SOLUTION:

- Add on V^S is defined by $(f+g)(x) = f(x) + g(x)$ for any $x \in S$ and $f, g \in V^S$.
- Scalar Multi on V^S is defined by $(\lambda f)(x) = \lambda f(x)$. ☐

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1 *Prove that $-(-v) = v$ for every $v \in V$.*

SOLUTION: $\left.\begin{array}{l} (-(-v)) + (-v) = 0 \\ v + (-v) = 0 \end{array}\right\} \Rightarrow$ By the uniqnes of add inv. ☐

OR. $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$. ☐

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2 *Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.*

SOLUTION: If $a = 0$, then we are done.

Otherwise, $\exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0$. ☐

---

3 *Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.*

SOLUTION:

[*Existns*] Let $x = \frac{1}{3}(w - v)$.

[*Uniqnes*] Suppose $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) − (II) : $3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. ☐

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5 *Show that in the definition of a vector space, the add inv condition can be replaced.*

SOLUTION: Using [1.31]. $0v = 0$ for all $v \in V \iff (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0$. ☐

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6 *Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$.*
Define an add and scalar multi on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess.
The operations of real numbers is as usual. While for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

and (I) $t + \infty = \infty + t = \infty + \infty = \infty$,
(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,
(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of add and scalar multi, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vecsp over $\mathbf{R}$? Explain.

SOLUTION:

No. By Associ: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.

OR. By Distributive properties: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0$. ☐

ENDED

1·C

7 *Prove or give a counterexample: If $\emptyset \neq U \subseteq \mathbf{R}^2$ and U is closed under taking add invs and under add, then U is a subsp of $\mathbf{R}^2$.*

Solution: Let $U = \mathbf{Z}^2, (\mathbf{Z}^*)^2, (\mathbf{Q}^*)^2, \mathbf{Q}^2 \setminus \{0\}$, or $\mathbf{R}^2 \setminus \{0\}$.

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8 *Give an example of $U \subseteq \mathbf{R}^2$ such that U is closed under scalar multi, but U is not a subsp of $\mathbf{R}^2$.*

Solution: Let $U = \{(x,y) \in \mathbf{R}^2 : x = 0 \vee y = 0\}$. Or. Let $U = \{(x,0) \in \mathbf{R}^2\} \cup \{(0,y) \in \mathbf{R}^2\}$.

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9 *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if there exists $p \in \mathbf{N}^+$ such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$? Explain.*

Solution: Denote the set by S.

Suppose $h(x) = \sin\sqrt{2}x + \cos x \in S$, since $\sin\sqrt{2}x, \cos x \in S$.

Assume $\exists\ p \in \mathbf{N}^+$ such that $h(x) = h(x+p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$

$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Contradiction! $\square$

---

11 *Prove that the intersection of every collection of subsps of V is a subsp of V.*

Solution:

Suppose $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collection of subsps of V; here Γ is an arbitrary index set.

We need to prove that $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vectors that are in U_α for each $\alpha \in \Gamma$, is a subsp of V.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under add.

(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under scalar multi.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of V that is closed under add and scalar multi.

Hence $\bigcap_{\alpha\in\Gamma} U_\alpha$ is a subsp of V. $\square$

---

12 *Prove that the union of two subsps of V is a subsp of V if and only if one of the subsps is contained in the other.*

Solution: Suppose U and W are subsps of V.

(a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subsp of V.

(b) Suppose $U \cup W$ is a subsp of V. Suppose $U \not\subseteq W$ and $U \not\supseteq W$ ($U \cup W \neq U$ and W).

Then $\forall a \in U$ but $a \notin W$; $b \in W$ but $b \notin U$. $a + b \in U \cup W$.

$$\left.\begin{array}{l}\text{Consider } a + b \in U \Rightarrow b = (a+b) + (-a) \in U\text{, contradicts!}\\ \text{Consider } a + b \in W \Rightarrow a = (a+b) + (-b) \in W\text{, contradicts!}\end{array}\right\} \Rightarrow U \cup W = U \text{ or } W\text{. Contradicts!}$$

Thus $U \subseteq W$ and $U \supseteq W$. $\square$

---

13 *Prove that the union of three subsps of V is a subsp of V if and only if one of the subsps contains the other two.*

This exercise is not true if we replace $\mathbf{F}$ *with a field containing only two elements.*

Solution: Suppose U_1, U_2, U_3 are subsps of V. Denote $U_1 \cup U_2 \cup U_3$ by $\mathcal{U}$.

(a) Suppose that one of the subsps contains the other two.
Then $\mathcal{U} = U_1, U_2$ or U_3 is a subsp of V.

(b) Suppose that $U_1 \cup U_2 \cup U_3$ is a subsp of V.
By distinct we notice that $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.
Also note that, if $U \cup W = V$ is a vecsp, then in general U and W are not subsps of V.
Hence this literal trick is invalid.

(I) ss

(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.
By applying Problem (12) we conclude that one U_j contains the other two. Thus we are done.

(II) Assume that no U_j is contained in the union of the other two,
and no U_j contains the union of the other two.
Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.
$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3;\ \ v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq \mathcal{U}$.
Note that $W \cap U_1 = \emptyset$, for if $v + \lambda u \in U_1$ then $v + \lambda u - \lambda u = v \in U_1$ while $v \notin U_1$.
又 $W \subseteq U_1 \cup U_2 \cup U_3$. Thus $W \subseteq U_2 \cup U_3$.
$\forall v + \lambda u \in W,\ \exists\, i \in \{2,3\}, v + \lambda u \in U_i$.
Because U_2, U_3 are subsps and hence have at least one element.
If $U_2 = U_3$, then $\mathcal{U} = U_1 \cup U_2$ and by Problem (12) we are done.
Otherwise, $\exists\, \lambda, \mu \in \mathbf{F}$ with $\lambda \neq \mu$ such that $v + \lambda u, v + \mu u \in U_i$ for some $i \in \{2,3\}$.
Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Contradicts. $\square$

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15 *Suppose U is a subsp of V. What is $U + U$?*

Solution:
$$\left.\begin{array}{r}\forall x, y \in U, x + y \in U \Rightarrow U + U \subseteq U \\ \forall x \in U, 0 \in U, x + 0 \in U + U \Rightarrow U \subseteq U + U\end{array}\right\} \Rightarrow U + U = U. \qquad \square$$

---

16 *Suppose U and W are subsps of V. Prove that $U + W = W + U$?*

Solution: $\forall x \in U, y \in W,$
$$\left.\begin{array}{r}x + y = y + x \in W + U \Rightarrow U + W \subseteq W + U \\ y + x = x + y \in U + W \Rightarrow W + U \subseteq U + W\end{array}\right\} \Rightarrow U + W = W + U. \qquad \square$$

---

17 *Suppose V_1, V_2, V_3 are subsps of V. Prove that $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$.*

Solution:

Let $x \in V_1, y \in V_2, z \in V_3$. Denote $(V_1 + V_2) + V_3$ by L, $V_1 + (V_2 + V_3)$ by R.
$$\left.\begin{array}{l}\forall u \in L,\ \exists\, x,y,z,\ u = (x + y) + z = x + (y + z) \in R \Rightarrow L \subseteq R \\ \forall u \in R,\ \exists\, x,y,z,\ u = x + (y + z) = (x + y) + z \in L \Rightarrow R \subseteq L\end{array}\right\} \Rightarrow L = R. \qquad \square$$

---

18 *Does the operation of add on the subsps of V have an additive identity? Which subsps have add invs?*

Solution: Suppose Ω is the additive identity.
For any subsp U of V. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.
Now suppose W is an add inv of $U \Rightarrow U + W = \Omega$.
Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. $\square$

EXAMPLE: *Suppose* $U = \{(x,x,y,y) \in \mathbf{F}^4 : x,y \in \mathbf{F}\}$, $W = \{(x,x,x,y) \in \mathbf{F}^4 : x,y \in \mathbf{F}\}$.
Prove that $U + W = \{(x,x,y,z) \in \mathbf{F}^4 : x,y,z \in \mathbf{F}\}$.

SOLUTION: Let T denote $\{(x,x,y,z) \in \mathbf{F}^4 : x,y,z \in \mathbf{F}\}$.

(a) By def, $U+W = \{(x_1+x_2, x_1+x_2, y_1+x_2, y_1+y_2) \in \mathbf{F}^4 : (x_1,x_1,y_1,y_1) \in U, (x_2,x_2,x_2,y_2) \in W\}$.
$\Rightarrow \forall v \in U+W, \exists t \in T, v = t \Rightarrow U+W \subseteq T$.

(b) $\forall x,y,z \in \mathbf{F}$, let $u = (0,0,y-x,y-x) \in U, w = (x,x,x,-y+x+z) \in W$
$\Rightarrow (x,x,y,z) = u+w \in U+W$. Hence $\forall t \in T, \exists u \in U, w \in W, t = u+w \Rightarrow T \subseteq U+W$. ☐

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21 *Suppose* $U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5 : x,y \in \mathbf{F}\}$.
Find a subsp W of $\mathbf{F}^5$ *such that* $\mathbf{F}^5 = U \oplus W$.

SOLUTION:

(a) Let $W = \{(0,0,z,w,u) \in \mathbf{F}^5 : z,w,u \in \mathbf{F}\}$. Then $W \cap U = \{0\}$.

(b) $\forall x,y,z,w,u \in \mathbf{F}$, let $u = (x,y,x+y,x-y,2x) \in U, w = (0,0,z-x-y,w-x-y,u-2x) \in W$
$\Rightarrow (x,y,z,w,u) = u+w \Rightarrow \mathbf{F}^5 \subseteq U+W$. ☐

---

22 *Suppose* $U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5 : x,y \in \mathbf{F}\}$.
Find three subsps W_1, W_2, W_3 *of* $\mathbf{F}^5$, *none of which equals* $\{0\}$,
such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

SOLUTION:

(1) Let $W_1 = \{(0,0,z,0,0) \in \mathbf{F}^5 : z \in \mathbf{F}\}$. Then $W_1 \cap U = \{0\}$.
Let $U_1 = U \oplus W_1$. Then $U_1 = \{(x,y,z,x-y,2x) \in \mathbf{F}^5 : x,y,z \in \mathbf{F}\}$. (Check it!)

(2) Let $W_2 = \{(0,0,0,w,0) \in \mathbf{F}^5 : w \in \mathbf{F}\}$. Then $W_2 \cap U_1 = \{0\}$.
Let $U_2 = U_1 \oplus W_2$. Then $U_2 = \{(x,y,z,w,2x) \in \mathbf{F}^5 : x,y,z,w \in \mathbf{F}\}$.

(3) Let $W_3 = \{(0,0,0,0,u) \in \mathbf{F}^5 : u \in \mathbf{F}\}$. Then $W_3 \cap U_2 = \{0\}$.
Let $U_3 = U_2 \oplus W_3$. Then $U_3 = \{(x,y,z,w,u) \in \mathbf{F}^5 : x,y,z,w,u \in \mathbf{F}\}$.

Thus $\mathbf{F}^5 = ((U \oplus W_1) \oplus W_2) \oplus W_3$. ☐

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23 *Prove or give a counterexample: If* V_1, V_2, U *are subsps of* V *such that*
$V = V_1 \oplus U$ *and* $V = V_2 \oplus U$, *then* $V_1 = V_2$.

SOLUTION: A counterexample:
$V = \mathbf{F}^2, U = \{(x,x) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_1 = \{(x,0) \in \mathbf{F}^2 : x \in \mathbf{F}\}, V_2 = \{(0,x) \in \mathbf{F}^2 : x \in \mathbf{F}\}$.

---

24 *Let* V_E *denote the set of real-valued even functions on* $\mathbf{R}$
and let V_O *denote the set of real-valued odd functions on* $\mathbf{R}$. *Show that* $\mathbf{R}^\mathbf{R} = V_E \oplus V_O$.

SOLUTION:

(a) $V_E \cap V_O = \{f : f(x) = f(-x) = -f(-x)\} = \{0\}$.

(b) $$\left.\begin{array}{r} f_e \in V_E \Leftrightarrow f_e(x) = f_e(-x) \Leftarrow \text{let } f_e(x) = \dfrac{g(x)+g(-x)}{2} \\ f_o \in V_O \Leftrightarrow f_o(x) = -f_o(-x) \Leftarrow \text{let } f_o(x) = \dfrac{g(x)-g(-x)}{2} \end{array}\right\} \Rightarrow \forall g \in \mathbf{R}^\mathbf{R}, g(x) = f_e(x) + f_o(x).$$ ☐

ENDED

2·A

2 (a) *A list* (v) *of length* 1 *in* V *is linely inde* $\Longleftrightarrow v \neq 0$.
(b) *A list* (v, w) *of length* 2 *in* V *is linely inde* $\Longleftrightarrow \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$.

Solution:

(a) Suppose $v \neq 0$. Then let $av = 0, a \in \mathbf{F}$. Now $a = 0$. Thus (v) is linely inde.
Suppose (v) is linely inde. $av = 0 \Rightarrow a = 0$. Then $v \neq 0$, for if not, $a \neq 0$ while $av = 0$. Contradicts.

(b) Denote the list by (v, w), where $v, w \in V$. If (v, w) is linely inde, let $av + bw = 0 \Rightarrow a = b = 0$.
If, say $v \neq cw \ \forall c \in \mathbf{F}$. Then let $av + bw = 0$, getting $a = b = 0 \Rightarrow (v, w)$ is linely inde.

---

1 *Prove that if* (v_1, v_2, v_3, v_4) *spans* V, *then the list* $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ *also spans V.*

Solution: Assume that $\forall v \in V, \ \exists\, a_1, \dots, a_4 \in \mathbf{F}$,

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4, \text{ letting } b_i = \sum_{r=1}^{i} a_r. \end{aligned}$$

Thus $\forall v \in V, \ \exists\, b_i \in \mathbf{F}, v = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4$.
Hence the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ also spans V. □

---

6 *Suppose* (v_1, v_2, v_3, v_4) *is linearly independent in V.*
Prove that the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ *is also linearly independent.*

Solution: $a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0$
$\Rightarrow a_1 v_1 + (a_2 - a_1) v_2 + (a_3 - a_2) v_3 + (a_4 - a_3) v_4 = 0$
$\Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \Rightarrow a_1 = \cdots = a_4 = 0$. □

---

7 *Prove that if* $(v_1, v_2, \dots, v_m)$ *is a linely independent list of vectors in* V,
then $(5v_1 - 4v_2, v_2, v_3, \dots, v_m)$ *is linely indep.*

Solution: $a_1(5v_1 - 4v_2) + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0$
$\Rightarrow 5a_1 v_1 + (a_2 - 4a_1) v_2 + a_3 v_3 + a_4 v_4 = 0$
$\Rightarrow 5a_1 = a_2 - 4a_1 = a_3 = a_4 = 0 \Rightarrow a_1 = \cdots = a_4 = 0$ □

• *Suppose* $(v_1, \dots, v_m)$ *is a list of vectors in* V. *For* $k \in \{1, \dots, m\}$, *let* $w_k = v_1 + \cdots + v_k$.
(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.
(b) *Show that* $(v_1, \dots, v_m)$ *is linely inde* $\Longleftrightarrow (w_1, \dots, w_m)$ *is linely inde.*

Solution:

(a) Let $\operatorname{span}(v_1, \dots, v_m) = U$. Assume that $\forall v \in U, \ \exists\, a_i \in \mathbf{F}$,
$v = a_1 v_1 + \cdots + a_m v_m = b_1 w_1 + \cdots + b_m w_m = \sum_{j=1}^{m} (\sum_{i=j}^{m} b_i) v_j$
$\Rightarrow b_1 = a_1, \ b_i = a_i - \sum_{r=1}^{i-1} b_r$. Thus $\exists\, b_i \in \mathbf{F}$ such that $v = b_1 w_1 + \cdots + b_m w_m$.
又 Each $w_i \in U \Rightarrow \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

(b) $a_1 w_1 + \cdots + a_m w_m = 0$
$\Rightarrow (a_1 + \cdots + a_m) v_1 + \cdots + (a_i + \cdots + a_m) v_i + \cdots + a_m v_m = 0$

$\Rightarrow a_m = \cdots = (a_m + \cdots + a_i) = \cdots = (a_m + \cdots + a_1) = 0.$ □

---

10 *Suppose $(v_1, \dots, v_m)$ is linely inde in V and $w \in V$.*
(a) *Prove that if $(v_1 + w, \dots, v_m + w)$ is linely depe, then $w \in \operatorname{span}(v_1, \dots, v_m)$.*
(b) *Show that $(v_1, \dots, v_m, w)$ is linely inde $\Longleftrightarrow w \notin \operatorname{span}(v_1, \dots, v_m)$.*

SOLUTION:

(a) Suppose $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0,\ \exists\, a_i \neq 0 \Rightarrow a_1v_1 + \cdots + a_mv_m = 0 = -(a_1 + \cdots + a_m)w.$
Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1v_1 + \cdots + a_mv_m = 0$ while $a_i \neq 0$ for some i, contradicts.
Hence $w \in \operatorname{span}(v_1, \dots, v_m)$.

(b) Suppose $w \in \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1, \dots, v_m, w)$ is linely depe.
Thus have we proven the "$\Rightarrow$" by its contrapositive.
Suppose $w \notin \operatorname{span}(v_1, \dots, v_m)$. Then by [2.23], $(v_1, \dots, v_m, w)$ is linely inde. □

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14 *Prove that V is infinite-dim if and only if there is a sequence $(v_1, v_2, \dots)$ in V such that $(v_1, \dots, v_m)$ is linely inde for every $m \in \mathbf{N}^+$.*

SOLUTION: Similar to [2.16].

Suppose there is a sequence $(v_1, v_2, \dots)$ in V such that $(v_1, \dots, v_m)$ is linely inde for any $m \in \mathbf{N}^+$.
Choose an m. Suppose a linely inde list $(v_1, \dots, v_m)$ spans V.
Then there exists $v_{m+1} \in V$ but $v_{m+1} \notin \operatorname{span}(v_1, \dots, v_m)$. Hence no list spans V. Thus V is infinite-dim.
Conversely it is true as well. For if not, V must be finite-dim, contradicting the assumption. □

---

15 *Prove that $\mathbf{F}^\infty$ is infinite-dim.*

SOLUTION: Let $e_i = (0, \dots, 0, 1, 0, \dots) \in \mathbf{F}^\infty$ for every $m \in \mathbf{N}^+$, where '1' is on the ith entry of e_i.
Suppose $\mathbf{F}^\infty$ is finite-dim. Then let $\operatorname{span}(e_1, \dots, e_m) = V$. But $e_{m+1} \notin \operatorname{span}(e_1, \dots, e_m)$. Contradicts. □

---

16 *Prove that the real vector space of all continuous real-valued functions on the interval $[0,1]$ is infinite-dim.*

SOLUTION: Denote the vecsp by U. Note that for each $m \in \mathbf{N}^+$, $(1, x, \dots, x^m)$ is linely inde.

Because if $a_0, \dots, a_m \in \mathbf{R}$ are such that $a_0 + a_1x + \cdots + a_mx^m = 0,\ \forall x \in [0,1]$, then the poly has infinitely many roots and hence $a_0 = \cdots = a_m = 0$.

Similar to [2.16], U is infinite-dim.

OR. Note that for $a_n = \dfrac{1}{n},\quad a_1 < a_2 < \cdots < a_m,\ \ \forall m \in \mathbf{N}^+.$

Suppose $f_n = \begin{cases} x - \dfrac{1}{n}, & x \in \left[\dfrac{1}{n}, 1\right) \\ 0 & , x \in \left[0, \dfrac{1}{n}\right) \end{cases}$. Then for any m, $f_1(\frac{1}{m}) = \cdots = f_m(\frac{1}{m})$, while $f_{m+1}(\frac{1}{m}) \neq 0$.

Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. Thus by Problem (14), U is infinite-dim.

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17 *Suppose $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$.*
Prove that $(p_0, p_1, \dots, p_m)$ is not linely inde in $\mathcal{P}_m(\mathbf{F})$.

SOLUTION:

Suppose $(p_0, p_1, \dots, p_m)$ is linely inde. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z\ \forall z \in \mathbf{F}$.
But $\forall a_i \in \mathbf{F}, z \neq a_0p_0(z) + \cdots + a_mp_m(z)$, for if not, let $z = 2$, contradicts. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_m) \quad \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has length $m + 1$.
Hence $(p_0, p_1, \dots, p_m)$ is linely depe in $\mathcal{P}_m(\mathbf{F})$.
For if not, notice that the list $(1, z, \dots, z^m)$ spans $\mathcal{P}_m(\mathbf{F})$,

thus by [2.23], $(p_0, p_1, \dots, p_m)$ spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. $\square$

ENDED

2·B

- **NOTE FOR** *linely inde sequence and* [2.34]*:*
  "$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expression.
  If we allow using "infinite list", then we must guarantee that $(v_1, \dots, v_n, \dots)$ is a spanning "list"
  such that for all $v \in V$, there exists a smallest positive integer n such that $v = a_1v_1 + \cdots + a_nv_n$,
  The key point is, how can we guarantee that such a "list" exists?

- **NOTE FOR "$\complement_V U \cap \{0\}$":** "$\complement_V U \cap \{0\}$" is supposed to be a subsp "W" such that $V = U \oplus W$.
  But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cap \{0\} \\ u \pm w \in \complement_V U \cap \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cap \{0\}$. Contradicts.
  To fix this, denote the set $\{W_1, W_2 \dots\}$ by $\mathcal{S}_V U$, where for each W_i, $V = U \oplus W_i$. See also in (1.C.23).

---

1 *Find all vector spaces that have exactly one basis.*

SOLUTION:

---

6 *Suppose (v_1, v_2, v_3, v_4) is a basis of V. Prove that $(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4)$ is also a basis.*

SOLUTION: $\forall v \in V,\ \exists!\, a_1, \dots, a_4 \in \mathbf{F}, v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$.
Assume that $v = b_1(v_1 + v_2) + b_2(v_2 + v_3) + b_3(v_3 + v_4) + b_4v_4$.
Then $v = b_1v_1 + (b_1 + b_2)v_2 + (b_2 + b_3)v_3 + (b_3 + b_4)v_4$.
$\Rightarrow\ \exists!\, b_1 = a_1,\ \ b_2 = a_2 - b_1,\ \ b_3 = a_3 - b_2,\ \ b_4 = a_4 - b_3 \in \mathbf{F}$. $\square$

---

7 *Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subsp of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \in U$, then v_1, v_2 is a basis of U.*

SOLUTION: Let $V = \mathbf{F}^4, v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_3 = (0,0,1,1), v_4 = (0,0,0,1)$.
And $U = \{(x,y,z,0) \in \mathbf{R}^4 : x,y,z \in \mathbf{F}\}$. We have a counterexample.

- *Suppose V is finite-dim and U, W are subsps of V such that $V = U + W$.*
  Prove that there exists a basis of V consisting of vectors in $U \cup W$.

SOLUTION: Let $(u_1, \dots, u_m)$ and $(w_1, \dots, w_n)$ be bases of U and W respectively.
Then $V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$.
Hence, by [2.31], we get a basis of V consisting of vectors in U or W. $\square$

---

8 *Suppose U and W are subsps of V such that $V = U \oplus W$. Suppose also that $(u_1, \dots, u_m)$ is a basis of U and $(w_1, \dots, w_n)$ is a basis of W.*
Prove that $(u_1, \dots, u_m, w_1, \dots, w_n)$ is a basis of V.

SOLUTION: $\forall v \in V,\ \exists!\, a_i, b_i \in \mathbf{F}, v = (a_1u_1 + \cdots + a_mu_m) + (b_1w_1 + \cdots + b_nw_n)$
$\Rightarrow (a_1u_1 + \cdots + a_mu_m) = -(b_1w_1 + \cdots + b_nw_n) \in U \cap W = \{0\}$. Thus $a_1 = \cdots = a_m = b_1 = \cdots = b_n$. $\square$

- OR (9.4) *Suppose V is a real vector space.*
  Show that if $(v_1, \dots, v_n)$ is a basis of V (as a real vector space),
  then $(v_1, \dots, v_n)$ is also a basis of the complexification $V_{\mathbf{C}}$ (as a complex vector space).
  See Section 1B (4e) for the definition of the complexification $V_{\mathbf{C}}$.

Solution: $\forall u + \mathrm{i}v \in V_{\mathbf{C}}$, $\exists!\, u, v \in V, a_i, b_i \in \mathbf{R}$,

$$u + \mathrm{i}v = (a_1 v_1 + \cdots + a_n v_n) + \mathrm{i}(b_1 v_1 + \cdots + b_n v_n) = (a_1 + b_1\mathrm{i})v_1 + \cdots + (a_n + b_n\mathrm{i})v_n$$

$\Rightarrow$ $u + \mathrm{i}v = c_1 v_1 + \cdots + c_n v_n$, $\exists!\, c_i = a_i + b_i\mathrm{i} \in \mathbf{C}$

$\Rightarrow$ By the uniqnes of c_i and [2.29], $(v_1, \ldots, v_n)$ is a basis of $V_{\mathbf{C}}$. □

Ended

2·C

1 *Suppose V is finite-dim and U is a subsp of V such that* $\dim V = \dim U$.

Let $(u_1, \ldots, u_m)$ be a basis of U. Then $n = \dim U = \dim V$. 又 $u_i \in V$.

Then by [2.39], $(u_1, \ldots, u_m)$ is a basis of V. Thus $V = U$.

---

2 *Show that the subsps of* $\mathbf{R}^2$ *are precisely* $\{0\}$, *all lines in* $\mathbf{R}^2$ *containing the origin, and* $\mathbf{R}^2$.

Solution: Suppose U is a subsp of $\mathbf{R}^2$. Let $\dim U = n$.

If $n = 0$, then $U = \{0\}$.

If $n = 1$, then $U = \operatorname{span}(v) = \{\lambda v : \lambda \in \mathbf{F}\}$, for all linely inde $v \in \mathbf{R}^2$.

If $n = 2$, then $U = \mathbf{R}^2$. □

---

3 *Show that the subsps of* $\mathbf{R}^3$ *are precisely* $\{0\}$, *all lines in* $\mathbf{R}^3$ *containing the origin, all planes in* $\mathbf{R}^3$ *containing the origin, and* $\mathbf{R}^3$.

Solution: Suppose U is a subsp of $\mathbf{R}^3$. Let $\dim U = n$.

If $n = 0$, then $U = \{0\}$.

If $n = 1$, then $U = \operatorname{span}(v) = \{\lambda v : \lambda \in \mathbf{F}\}$, for all linely inde $v \in \mathbf{R}^3$.

If $n = 2$, then $U = \operatorname{span}(v, w) = \{\lambda v + \mu w : \lambda, \mu \in \mathbf{F}\}$, for all linely inde $v, w \in \mathbf{R}^3$.

If $n = 3$, then $U = \mathbf{R}^3$. □

---

7 (a) *Let* $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. *Find a basis of* U.

(b) *Extend the basis in (a) to a basis of* $\mathcal{P}_4(\mathbf{F})$.

(c) *Find a subsp* W *of* $\mathcal{P}_4(\mathbf{F})$ *such that* $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution: Suppose $p(z) = az^4 + bz^3 + cz^2 + dz + e$ and $p(2) = p(5) = p(6)$.

$$\text{Then} \begin{cases} p(2) = 16a + 8b + 4c + 2d + e \text{ (I)} \\ p(5) = 625a + 125b + 25c + 5d + e \text{ (II)} \\ p(6) = 1296a + 216b + 36c + 6d + e \text{ (III)} \end{cases}$$

You don't have to compute to know that the dimension of the set of solutions is 3.

(a) A basis: $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.

(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. □

---

9 *Suppose* $(v_1, \ldots, v_m)$ *is linely inde in* V *and* $w \in V$.

Prove that $\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$.

Solution: Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \ldots, v_n + w)$, for each $i = 1, \ldots, m$.

$(v_1, \ldots, v_m)$ is linely inde $\Rightarrow (v_1, v_2 - v_1, \ldots, v_m - v_1)$ is linely inde

$\Rightarrow (v_2 - v_1, \ldots, v_m - v_1)$ is linely inde of length $m - 1$.

又 By the contrapositive of (2.A.10), $w \notin \operatorname{span}(v_1, \ldots, v_m) \Rightarrow (v_1 + w, \ldots, v_m + w)$ is linely inde.

$\therefore m \geq \dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$ $\square$

---

10 *Suppose m is a positive integer and $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $(p_0, p_1, \ldots, p_m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.*

SOLUTION: Using mathematical induction on m.

(i) For $p_0, \deg p_0 = 0 \Rightarrow \operatorname{span}(p_0) = \operatorname{span}(1)$.

(ii) Suppose for $i \geq 1, \operatorname{span}(p_0, p_1, \ldots, p_i) = \operatorname{span}(1, x, \ldots, x^i)$.

Then $\operatorname{span}(p_0, p_1, \ldots, p_i, p_{i+1}) \subseteq \operatorname{span}(1, x, \ldots, x^i, x^{i+1})$.

又 $\deg p_{i+1} = i + 1, \quad p_{i+1}(x) = a_{i+1}x^{i+1} + r_{i+1}(x); \quad a_{i+1} \neq 0, \quad \deg r_{i+1} \leq i.$

$\Rightarrow x^{i+1} = \dfrac{1}{a_{i+1}}(p_{i+1}(x) - r_{i+1}(x)) \in \operatorname{span}(1, x, \ldots, x^i, p_{i+1}) = \operatorname{span}(p_0, p_1, \ldots, p_i, p_{i+1}).$

$\therefore x^{i+1} \in \operatorname{span}(p_0, p_1, \ldots, p_i, p_{i+1}) \Rightarrow \operatorname{span}(1, x, \ldots, x^i, x^{i+1}) \subseteq \operatorname{span}(p_0, p_1, \ldots, p_i, p_{i+1}).$

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \ldots, x^m) = \operatorname{span}(p_0, p_1, \ldots, p_m)$. $\square$

• *Suppose m is a positive integer. For $0 \leq k \leq m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show that $(p_0, \ldots, p_m)$ is a basis of $\mathcal{P}(\mathbf{F})$.*

The basis in this exercise leads to what are called Bernstein polynomials. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on $[0,1]$.

SOLUTION: Using mathematical induction.

(i) $k = 0, 1, 2, p_m(x) = x^m, \;\; p_{m-1}(x) = x^{m-1} - x^m, \;\; p_{m-2}(x) = x^{m-2} + x^m - 2x^{m-1}$.

(ii) $k \geq 2$. Suppose for $p_{m-k}(x), \;\exists\,!\; a_i \in \mathbf{F}, x^{m-k} = p_{m-k}(x) + a_m x^m + \cdots + a_{m-k+1}x^{m-k+1}$.

Then for $p_{m-k-1}(x), \;\exists\,!\; c_i \in \mathbf{F}$,

$$x^{m-k-1} = p_{m-k-1}(x) + C_{k+1}^1(-1)^2 x^{m-k} + \cdots + C_{k+1}^k(-1)^{k+1}x^{m-1} + (-1)^{k-2}x^m$$

$$\Rightarrow c_{m-i} = C_{k+1}^{k+1-i}(-1)^{k-i}.$$

Thus for each $x^i, \;\exists\,!\; b_i \in \mathbf{F}, x^i = b_m p_m(x) + \cdots + b_{m-i}p_{m-i}(x)$

$\Rightarrow \operatorname{span}(x^m, \ldots, x, 1) = \operatorname{span}\underbrace{(p_m, \ldots, p_1, p_0)}_{\text{Basis}}.$ $\square$

• *Suppose V is finite-dim and V_1, V_2, V_3 are subsps of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*

SOLUTION:
$$\left.\begin{array}{l} \dim V_1 + \dim V_2 > 2\dim V - \dim V_3 \geq \dim V \Rightarrow V_1 \cap V_2 \neq \{0\} \\ \dim V_2 + \dim V_3 > 2\dim V - \dim V_1 \geq \dim V \Rightarrow V_2 \cap V_3 \neq \{0\} \\ \dim V_1 + \dim V_3 > 2\dim V - \dim V_2 \geq \dim V \Rightarrow V_1 \cap V_3 \neq \{0\} \end{array}\right\} \Rightarrow V_1 \cap V_2 \cap V_3 \neq \{0\}.$$ $\square$

• *Suppose V is finite-dim and U is a subsp of V with $U \neq V$. Let $n = \dim V, m = \dim U$. Prove that there exist $(n-m)$ subsps of V, say $U_1, \ldots, U_{n-m}$, each of dimension $(n-1)$, such that $\bigcap_{i=1}^{n-m} U_i = U$.*

SOLUTION: Let $(v_1, \ldots, v_m)$ be a basis of U, extend to a basis of V as $(v_1, \ldots, v_m, \ldots, v_n)$.

Define $U_i = \operatorname{span}(v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+i-1}, v_{m+i+1}, \ldots, v_n)$ for each i. Thus we are done.

EXAMPLE: Suppose $\dim V = 6, \dim U = 3$.

$$\underbrace{(\underbrace{v_1, v_2, v_3}_{\text{Basis of U}}, v_4, v_5, v_6)}_{\text{Basis of V}}, \text{ define } \left.\begin{array}{l} U_1 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_5, v_6) \\ U_2 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_4, v_6) \\ U_3 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_4, v_5) \end{array}\right\} \Rightarrow \dim U_i = 6 - 1, \; i = \underbrace{1, 2, 3}_{6-3=3}.$$ $\square$

---

14 *Suppose that $V_1, \dots, V_m$ are finite-dim subsps of V.*

Prove that $V_1 + \cdots + V_m$ is finite-dim and $\dim(V_1 + \cdots + V_m) \le \dim V_1 + \cdots + \dim V_m$.

SOLUTION: Choose a basis $\mathcal{E}_i$ of $V_i \Rightarrow V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim U_i = \operatorname{card} \mathcal{E}_i$.

Then $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$.

又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup E_m) \le \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \le \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.

Thus $\dim(V_1 + \cdots + V_m) \le \dim U_1 + \cdots + \dim U_m$.

COMMENT: $\dim(V_1 + \cdots + V_m) = \dim U_1 + \cdots + \dim U_m \iff V_1 + \cdots + V_m$ is a direct sum.

For each i, $(V_1 + \cdots + V_i) \cap V_{i+1} = \{0\} \iff V_1 + \cdots + V_m$ is a direct sum $\iff$ □

---

17 *Suppose V_1, V_2, V_3 are subsps of a finite-dim vector space, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

SOLUTION:

[*Similar to*] Given three sets A, B and C.

Because $|X \cup Y| = |X| + |Y| - |X \cap Y|$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

Now $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$.

And $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$.

Hence $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$.

Because $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned}
\dim(V_1 + V_2 + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim\left((V_1 + V_2) \cap V_3\right) &\quad (1)\\
&= \dim(V_2 + V_3) + \dim(V_1) - \dim\left((V_2 + V_3) \cap V_1\right) &\quad (2)\\
&= \dim(V_1 + V_3) + \dim(V_2) - \dim\left((V_1 + V_3) \cap V_2\right) &\quad (3)
\end{aligned}$$

Notice that in general, $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$.

For example, $X = \left\{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\right\}, Y = \left\{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\right\}, Z = \left\{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\right\}$.

• **COROLLARY:** *If V_1, V_2 and V_3 are finite-dim vector spaces, then* $\frac{(1) + (2) + (3)}{3}$:

$$\begin{aligned}
\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 &- \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\
&- \frac{\dim\left((V_1 + V_2) \cap V_3\right) + \dim\left((V_1 + V_3) \cap V_2\right) + \dim\left((V_2 + V_3) \cap V_1\right)}{3}.
\end{aligned}$$

The formula above may seem strange because the right side does not look like an integer. □

ENDED

3·A

• **TIPS:** *$T : V \to W$ is linear* $\iff \left|\begin{array}{l} \forall v, u \in V, T(v + u) = Tv + Tu \\ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv) \end{array}\right| \iff T(v + \lambda u) = Tv + \lambda Tu.$

---

3 *Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists\ A_{j,k} \in \mathbf{F}$ such that*

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for any $(x_1, \dots, x_n) \in \mathbf{F}^n$.

Solution:

Let $T(1,0,0,\dots,0,0) = (A_{1,1},\dots,A_{m,1})$,
$T(0,1,0,\dots,0,0) = (A_{1,2},\dots,A_{m,2})$,
$\vdots$
$T(0,0,0,\dots,0,1) = (A_{1,n},\dots,A_{m,n})$.

Note that $(1,0,\dots,0,0),\cdots,(0,0,\dots,0,1)$ is a basis of $\mathbf{F}^n$. Then by [3.5], we are done. ☐

---

4 *Suppose $T \in \mathcal{L}(V,W)$ and $(v_1,\dots,v_m)$ is a list of vectors in V such that $(Tv_1,\dots,Tv_m)$ is linely inde in W. Prove that $(v_1,\dots,v_m)$ is linely inde.*

Solution: Suppose $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$.☐

---

5 *Prove that $\mathcal{L}(V,W)$ is a vector space,*

Solution: Note that $\mathcal{L}(V,W)$ is a subsp of W^V. ☐

---

7 *Show that every linear map from a one-dim vector space to itself is multi by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.*

Solution:

Let u be a nonzero vector in $V \Rightarrow V = \operatorname{span}(u)$.

Because $Tu \in V \Rightarrow Tu = \lambda u$ for some λ.

Suppose $v \in V \Rightarrow v = au,\ \exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. ☐

---

8 *Give an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that $\varphi(av) = a\varphi(v)$ for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.*

Solution:

Define $T(x,y) = \begin{cases} x+y, \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, \qquad \text{otherwise.} \end{cases}$ Or. Define $T(x,y) = \sqrt[3]{(x^3+y^3)}$. ☐

---

9 *Give an example of a function $\varphi : \mathbf{C} \to \mathbf{C}$ such that $\varphi(w+z) = \varphi(w) + \varphi(z)$ for all $w,z \in \mathbf{C}$ but φ is not linear.*
(Here $\mathbf{C}$ is thought of as a complex vector space.)

Solution:

Suppose V_{C} is the complexification of a vector space V. Suppose $\varphi : V_{\mathrm{C}} \to V_{\mathrm{C}}$.

Define $\varphi(u + \mathrm{i}v) = u = \operatorname{Re}(u + \mathrm{i}v)$

Or. Define $\varphi(u + \mathrm{i}v) = v = \operatorname{Im}(u + \mathrm{i}v)$. ☐

• *Prove or give a counterexample:*
If $q \in \mathcal{P}(\mathbf{R})$ and $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by $Tp = q \circ p$, then T is linear.

Solution: Because in general, $q\,(p_1(x) + \lambda p_2(x)) \neq (q \circ p_1)(x) + \lambda(q \circ p_2)(x)$. ☐

• Or (3.D.16) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Suppose $ST = TS$ for every $S \in \mathcal{L}(V)$. Prove that T is a scalar multi of the identity.*

Solution:

If $V = \{0\}$, then we are done. Now suppose $V \neq \{0\}$.

Assume that (v, Tv) is linely depe for every $v \in V$, then by (2.A.2.(b)), $Tv = \lambda_v v$ for some $\lambda_v \in \mathbf{F}$.

To prove that λ_v is independent of v

(in other words, for any two distinct nonzero vectors v and w in V, we have $\lambda_v \neq \lambda_w$), we discuss in two cases:

$$\left.\begin{array}{l} (-)\ \text{If } (v,w) \text{ is linely inde, } \lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = a_v v + a_w w \\ \qquad\qquad\qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=)\ \text{Otherwise, suppose } w = cv,\ a_w w = Tw = cTv = ca_v v = a_v w \Rightarrow (a_w - a_v)w \end{array}\right\} \Rightarrow a_w = a_v.$$

Now we prove the assumption by contradiction.

Suppose (v, Tv) is linely inde for every nonzero vector $v \in V$.

Fix one v. Extend to $(v, Tv, u_1, \dots, u_n)$ a basis of V.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Contradicts. ☐

OR. Let $(v_1, \dots, v_m)$ be a basis of V.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$. Let $\lambda = \varphi(Tv_1) \in \mathbf{F}$.

For any $v \in V$, define $S_v \in \mathcal{L}(V)$ by $S_v u = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. ☐

---

10 *Suppose U is a subsp of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$).*

Define $T : V \to W$ by $Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \backslash U. \end{cases}$ Prove that T is not a linear map on V.

SOLUTION:

Suppose T is a linear map. And $v \in V\backslash U$, $u \in U$ such that $Su \neq 0$.

Then $v + u \in V\backslash U$, (for if not, $v = (v+u) - u \in U$) while $T(v+u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$.

Hence we get a contradiction. ☐

---

11 *Suppose V is finite-dim. Prove that every linear map on a subsp of V can be extended to a linear map on V. In other words, show that if U is a subsp of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.*

SOLUTION:

Define $T \in \mathcal{L}(V, W)$ by $T(a_1u_1 + \cdots + a_nu_n + a_{n+1}u_{n+1} + \cdots + a_mu_m) = a_1Su_1 + \cdots + a_nSu_n$.

Where we let $(u_1, \dots, u_n)$ be a basis of U, extend to a basis of V as $(u_1, \dots, u_n, \dots, u_m)$. ☐

---

12 *Suppose V is finite-dim with $\dim V > 0$, and W is infinite-dim. Prove that $\mathcal{L}(V, W)$ is infinite-dim.*

SOLUTION:

Let $(v_1, \dots, v_n)$ be a basis of V. Let $(w_1, \dots, w_m)$ be linely inde in W for any $m \in \mathbf{N}^+$.

Define $T_{x,y} \in \mathcal{L}(V, W)$ by $T_{x,y}(v_z) = \delta_{zy}w_y$, $\forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{zy} = \begin{cases} 0, & z \neq y, \\ 1, & z = y. \end{cases}$

Suppose $a_1T_{x,1} + \cdots + a_mT_{x,m} = 0$. Then $(a_1T_{x,1} + \cdots + a_mT_{x,m})(v_x) = 0 = a_1w_1 + \cdots + a_mw_m$.

$\Rightarrow a_1 = \cdots = a_m = 0$. 又 m arbitrary.

Thus $(T_{x,1}, \dots, T_{x,m})$ is a linely inde list in $\mathcal{L}(V, W)$ for any x and length m. Hence by (2.A.14). ☐

---

13 *Suppose $(v_1, \dots, v_m)$ is a linely depe list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $(w_1, \dots, w_m) \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \dots, m$.*

SOLUTION:

We prove by contradiction. By linear dependence lemma, $\exists\, j \in \{1, \dots, m\}$ such that $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$.
Fix j. Let $w_j \neq 0$, while $w_1 = \dots = w_{j-1} = w_{j+1} = w_m = 0$.
Define T by $Tv_k = w_k$ for all k. Suppose $a_1 v_1 + \dots + a_m v_m = 0$ (where $a_j \neq 0$).
Then $T(a_1 v_1 + \dots + a_m v_m) = 0 = a_1 w_1 + \dots + a_m w_m = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. $\square$

OR. We prove the contrapositive:
Suppose for any list $(w_1, \dots, w_m) \in W$, $\exists\, T \in \mathcal{L}(V, W), Tv_k = w_k$ for each w_k.
(We need to) Prove that $(v_1, \dots, v_n)$ is linely inde.
Suppose $\exists\, a_i \in \mathbf{F}, a_1 v_1 + \dots + a_n v_n = 0$. Choose a nonzero $w \in W$.
By assumption, for the list $(\overline{a_1} w, \dots, \overline{a_m} w)$, $\exists\, T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$ for each v_k.
$0 = T(\sum_{k=1}^{m} a_k v_k) = \sum_{k=1}^{m} a_k T v_k = \sum_{k=1}^{m} a_k \overline{a_k} w = (\sum_{k=1}^{m} |a_k|^2) w$. Hence $\sum_{k=1}^{m} |a_k|^2 = 0 \Rightarrow a_k = 0$. $\square$

- *(4E 3.A.16)*
Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION:

Let $(v_1, \dots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.
Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \backslash \{0\}$.
Suppose $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \dots + a_n v_n$, where $a_k \neq 0$.
Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y}(v_x) = v_y, R_{x,y}(v_z) = 0\ (z \neq x)$. Then for any $x, y \in \mathbf{N}^+$,
$(R_{k,y} S)(v_i) = a_k v_y \Rightarrow \left((R_{k,y} S) \circ R_{x,i}\right)(v_x) = a_k v_y,\ \left((R_{k,y} S) \circ R_{x,i}\right)(v_z) = 0\ (z \neq x)$.
Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Denote by $T_{x,y}$.
Getting $(\frac{1}{a_k} T_{1,1} + \dots + \frac{1}{a_k} T_{n,n}) v_j = v_j \Rightarrow \sum_{r=1}^{n} \frac{1}{a_k} T_{r,r} = I$.
又 By assumption, $T_{x,y} \in \mathcal{E} \Rightarrow I \in \mathcal{E}$.
Hence for any $T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. $\square$

ENDED

3·B

2 *Suppose $S, T \in \mathcal{L}(V)$ are such that* $\operatorname{range} S \subseteq \operatorname{null} T$. *Prove that* $(ST)^2 = 0$.

SOLUTION: $TS = 0 \Rightarrow STST = (ST)^2 = 0$. $\square$

---

3 *Suppose $(v_1, \dots, v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m$.*
(a) *The surj of T corresponds to $(v_1, \dots, v_m)$ spanning V.*
(b) *The inje of T corresponds to $(v_1, \dots, v_m)$ being linely inde.*

---

7 *Suppose V is finite-dim with* $2 \leq \dim V$ *and also* $\dim V \leq \dim W$, *if W is finite-dim.*
*Show that $U = \{\, T \in \mathcal{L}(V, W) :$ $\operatorname{null} T \neq \{0\}\}$ *is not a subsp of $\mathcal{L}(V, W)$.*

SOLUTION:

Let $(v_1, \dots, v_n)$ be a basis of V, $(w_1, \dots, w_m)$ be linely inde in W.
(Let $\dim W = m$, if W is finite, otherwise, let $m \in \{n, n+1, \dots\}; 2 \leq n \leq m$).
Define $T_1 \in \mathcal{L}(V, W)$ as $T_1 :\ v_1 \mapsto 0, \quad v_2 \mapsto w_2, \quad v_i \mapsto w_i$.

Define $T_2 \in \mathcal{L}(V,W)$ as $T_2: \quad v_1 \mapsto w_1, \quad v_2 \mapsto 0, \qquad v_i \mapsto w_i, \qquad i = 3, \dots, n.$

Thus $T_1 + T_2 \notin U$. $\square$

Comment: If $\dim V = 0$, then $V = \{0\} = \operatorname{span}()$. $\forall\ T \in \mathcal{L}(V,W), T$ is inje. Hence $U = \emptyset$.

If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0 v_0 = 0$.

If V is infinite-dim, the result is true as well.

---

8 *Suppose W is finite-dim with* $\dim W \geq 2$ *and also* $\dim V \geq \dim W$, *if V is finite-dim. Show that $U = \{\, T \in \mathcal{L}(V,W) : \operatorname{range} T \neq W\}$ is not a subsp of $\mathcal{L}(V,W)$.*

Solution:

Let $(v_1, \dots, v_n)$ be linely inde in V, $(w_1, \dots, w_m)$ be a basis of W.

(Let $n = \dim V$, if V is finite, otherwise we choose $n \in \{m, m+1, \dots\}$; $2 \leq m \leq n$).

Define $T_1 \in \mathcal{L}(V,W)$ as $T_1: \quad v_1 \mapsto 0, \qquad v_2 \mapsto w_2, \qquad v_j \mapsto w_j, \qquad v_{m+i} \mapsto 0.$

Define $T_2 \in \mathcal{L}(V,W)$ as $T_2: \quad v_1 \mapsto w_1, \quad v_2 \mapsto 0, \qquad v_j \mapsto w_j, \qquad v_{m+i} \mapsto 0.$

For each $j = 2, \dots, m;\ i = 1, \dots, n-m$, if V is finite, otherwise let $i \in \mathbf{N}^+$.

Thus $T_1 + T_2 \notin U$. $\square$

Comment: If $\dim W = 0$, then $W = \{0\} = \operatorname{span}()$. $\forall\ T \in \mathcal{L}(V,W), T$ is surj. Hence $U = \emptyset$.

If $\dim W = 1$, then $W = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0 v_0 = 0$.

If W is infinite-dim, the result is true as well.

---

9 *Suppose $T \in \mathcal{L}(V,W)$ is inje and $(v_1, \dots, v_n)$ is linely inde in V. Prove that $(Tv_1, \dots, Tv_n)$ is linely inde in W.*

Solution:

$a_1 Tv_1 + \cdots + a_n Tv_n = 0 = T(\sum\limits_{i=1}^{n} a_i v_i) \iff \sum\limits_{i=1}^{n} a_i v_i = 0 \iff a_1 = \cdots = a_n = 0.$ $\square$

---

10 *Suppose $(v_1, \dots, v_n)$ spans V and $T \in \mathcal{L}(V,W)$. Show that $(Tv_1, \dots, Tv_n)$ spans* $\operatorname{range} T$.

Solution:

(a) $\operatorname{range} T = \{Tv : v \in V\} = \{Tv : v \in \operatorname{span}(v_1, \dots, v_n)\} \Rightarrow Tv_1, \dots, Tv_n \in \operatorname{range} T \Rightarrow$ By [2.7].

Or. $\operatorname{span}(Tv_1, \dots, Tv_n) \ni a_1 Tv_1 + \cdots + a_n Tv_n = T(a_1 v_1 + \cdots + a_n v_n) \in \operatorname{range} T$.

(b) $\forall\, w \in \operatorname{range} T$, $\exists\, v \in V, w = Tv$. ($\exists\ a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n$) $\Rightarrow w = a_1 Tv_1 + \cdots + a_n Tv_n \Rightarrow \square$

---

11 *Suppose $S_1, \dots, S_n$ are inje linear maps and $S_1 S_2 \dots S_n$ makes sence. Prove that $S_1 S_2 \dots S_n$ is inje.*

Solution: $S_1 S_2 \dots S_n(v) = 0 \iff S_2 S_3 \dots S_n(v) = 0 \iff \cdots \iff S_n(v) = 0 \iff v = 0.$ $\square$

---

12 *Suppose that V is finite-dim and that $T \in \mathcal{L}(V,W)$. Prove that there exists a subsp U of V such that $U \cap$* $\operatorname{null} T = \{0\}$ *and* $\operatorname{range} T = \{Tu : u \in U\}$.

Solution:

By [2.34], there exists a subsp U of V such that $V = U \oplus \operatorname{null} T$.

$\forall v \in V,\ \exists!\ w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w+u) = Tu \in \{Tu : u \in U\} \Rightarrow \square$

Comment: V can be infinite-dim. See the above of [2.34].

---

16 *Suppose there exists a linear map on V whose null space and range are both finite-dim. Prove that V is finite-dim.*

Solution:

Denote the linear map by T. Let $(Tv_1, \dots, Tv_n)$ be a basis of $\operatorname{range} T$, $(u_1, \dots, u_m)$ be a basis of $\operatorname{null} T$.

Then for all $v \in V, T(\underbrace{v - a_1 v_1 - \cdots - a_n v_n}_{u \in \operatorname{null} T}) = 0$, where $Tv = a_1 Tv_1 + \cdots + a_n Tv_n$.

$\Rightarrow u = b_1 u_1 + \cdots + b_m u_m \Rightarrow v = a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m$.

Getting $V \subseteq \operatorname{span}(v_1, \dots, v_n, u_1, \dots, u_m)$. Thus V is finite-dim. □

---

17 *Suppose V and W are both finite-dim. Prove that there exists an inje $T \in \mathcal{L}(V, W)$ if and only if* $\dim V \leq \dim W$.

SOLUTION:

(a) Suppose there exists an inje T. Then $\dim V = \dim \operatorname{range} T \leq \dim W$.

(b) Suppose $\dim V \leq \dim W$, letting $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$ be bases of V and W respectively.
Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i,\ i = 1, \dots, n\ (= \dim V)$. □

---

18 *Suppose V and W are both finite-dim. Prove that there exists a surj $T \in \mathcal{L}(V, W)$ if and only if* $\dim V \geq \dim W$.

SOLUTION:

(a) Suppose there exists a surj T. Then $\dim V = \dim W + \dim \operatorname{null} T \Rightarrow \dim W \leq \dim V$.

(b) Suppose $\dim V \geq \dim W$, letting $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$ be bases of V and W respectively.
Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_m v_m + a_{m+1} v_{m+1} + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m$. □

---

19 *Suppose V and W are finite-dim and that U is a subsp of V.
Prove that $\exists\ T \in \mathcal{L}(V, W),\ \operatorname{null} T = U \iff \dim U \geq \dim V - \dim W$.*

SOLUTION:

(a) Suppose $\exists\ T \in \mathcal{L}(V, W),\ \operatorname{null} T = U$. Then $\dim \operatorname{null} T = \dim U \geq \dim V - \dim W$.

(b) Suppose $\underbrace{\dim U}_{m} \geq \underbrace{\dim V}_{m+n} - \underbrace{\dim W}_{p}$ $(\Rightarrow \dim W = p \geq n = \dim V - \dim U)$.
Let $(u_1, \dots, u_m)$ be a basis of U, extend to a basis of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$.
Let $(w_1, \dots, w_p)$ be a basis of W.
Define $T \in \mathcal{L}(V, W)$ by $T(a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m) = a_1 w_1 + \cdots + a_n w_n$. □

• **TIPS:** *Suppose $T \in \mathcal{L}(V, W)$ and $R = (Tv_1, \dots, Tv_n)$ is linely inde in* $\operatorname{range} T$.
(*Let* $\dim \operatorname{range} T = n$, *if* $\operatorname{range} T$ *is finite, otherwise let $n \in \mathbf{N}^+$.*)
By (3.A.4), $L = (v_1, \dots, v_n)$ is linely inde in V.

• **NEW NOTATION:**
Denote $\mathcal{K}_R$ by $\operatorname{span} L$, *if* $\operatorname{range} T$ *is finite-dim, otherwise, denote it by a vecsp in $\mathcal{S}_V \operatorname{null} T$.*
Note that if $\operatorname{range} T$ *is finite-dim, then $\mathcal{K}_{\operatorname{range} T} = \mathcal{K}_R$ for any basis R of* $\operatorname{range} T$.

• **NEW THEOREM:** $\mathcal{K}_R \in \mathcal{S}_V \operatorname{null} T$.
Suppose $\operatorname{range} T$ is finite-dim. Otherwise, we are done immediately.

$$\mathcal{K}_R \oplus \operatorname{null} T = V \Leftarrow \begin{cases} \text{(a) } T(\sum_{i=1}^{n} a_i v_i) = 0 \Rightarrow \sum_{i=1}^{n} a_i T v_i = 0 \Rightarrow a_1 = \cdots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \operatorname{null} T = \{0\}. \\ \text{(b) } \forall v \in V, Tv = \sum_{i=1}^{n} a_i T v_i \Rightarrow Tv - \sum_{i=1}^{n} a_i T v_i = T(v - \sum_{i=1}^{n} a_i v_i) = 0 \\ \quad \Rightarrow v - \sum_{i=1}^{n} a_i v_i \in \operatorname{null} T \Rightarrow v = (v - \sum_{i=1}^{n} a_i v_i) + (\sum_{i=1}^{n} a_i v_i) \Rightarrow \mathcal{K}_R + \operatorname{null} T = V. \end{cases}$$

• **COMMENT:** $\operatorname{null} T \in \mathcal{S}_V \mathcal{K}_R$.

• (4E 3.B.21) *Suppose V is finite-dim, $T \in \mathcal{L}(V, W)$, and U is a subsp of W.
Prove that $\mathcal{K}_U = \{ v \in V : Tv \in U \}$ is a subsp of V*

and $\dim \mathcal{K}_U = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$.

SOLUTION:

For any $u, w \in \mathcal{K}_U$ and $\lambda \in \mathbf{F}, T(u + \lambda w) = Tu + \lambda Tw \in U \Rightarrow \mathcal{K}_U$ is a subsp of V.

Define $S \in \mathcal{L}(\mathcal{K}_U, U)$ as $Rv = Tv$ for all $v \in \mathcal{K}_U$. Hence range $R = U \cap \operatorname{range} T$.

Suppose $Tv = 0$ for some $v \in V$. 又 $0 \in U \Rightarrow Rv = 0$. Thus $\operatorname{null} T \subseteq \operatorname{null} R$. $\square$

---

20 *Suppose* $T \in \mathcal{L}(V, W)$. *Prove that* T *is inje* $\iff \exists\ S \in \mathcal{L}(W, V),\ ST = I \in \mathcal{L}(V)$.

SOLUTION:

(a) Suppose $\exists\ S \in \mathcal{L}(W, V),\ ST = I$. Then if $Tv = 0 \Rightarrow ST(v) = 0 = v$.

(b) Suppose T is inje. Let $R = (Tv_1, \dots, Tv_n)$ be linely inde in range $T \subseteq W$,

where $n = \dim \operatorname{range} T$ if finite-dim, otherwise $n \in \mathbf{N}^+$.

Then $\mathcal{K}_R \oplus \operatorname{null} T = V$. And supose $U \oplus \operatorname{range} T = W$.

Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$ and $Su = 0,\ u \in U$. Thus $ST = I$. $\square$

---

21 *Suppose* $T \in \mathcal{L}(V, W)$. *Prove that* T *is surj* $\iff \exists\ S \in \mathcal{L}(W, V),\ TS = I \in \mathcal{L}(W)$.

SOLUTION:

(a) Suppose $\exists\ S \in \mathcal{L}(W, V),\ TS = I$. Then $\forall\, w \in W, TS(w) = w \in \operatorname{range} T \Rightarrow \operatorname{range} T = W$.

(b) Suppose T is surj. Let $R = (Tv_1, \dots, Tv_n)$ be linely inde in range $T = W$,

where $n = \dim \operatorname{range} T$ if finite-dim, otherwise $n \in \mathbf{N}^+$.

Then $\mathcal{K}_R \oplus \operatorname{null} T = V$. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$. Then $TS = I$. $\square$

---

22 *Suppose* U *and* V *are finite-dim vecsps and* $S \in \mathcal{L}(V, W)$ *and* $T \in \mathcal{L}(U, V)$.
Prove that $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$.

SOLUTION:

Define $R \in \mathcal{L}(\operatorname{null} ST, V)$ by $Ru = Tu$ for all $u \in \operatorname{null} ST \subseteq U$.

$$\left.\begin{aligned} S(Tu) = 0 = S(Ru) &\Rightarrow \operatorname{range} R \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} R \leq \dim \operatorname{null} S \\ Tu = 0 = Ru &\Rightarrow \operatorname{null} R \supseteq \operatorname{null} T \Rightarrow \dim \operatorname{null} R = \dim \operatorname{null} T \end{aligned}\right\} \Rightarrow \square$$

OR. For any $u \in U$, note that $u \in \operatorname{null} ST \iff S(Tu) = 0 \iff Tu \in \operatorname{null} S$.

Thus $\operatorname{null} ST = \mathcal{K}_{\operatorname{null} S \cap \operatorname{range} T} = \{u \in U : Tu \in \operatorname{null} S\}$. By Problem (4E 3B.21),

$\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T) \leq \dim \operatorname{null} T + \dim \operatorname{null} S$. $\square$

COROLLARY:

(1) If T is inje, then $\dim \operatorname{null} T = 0 \Rightarrow \dim \operatorname{null} ST \leq \dim \operatorname{null} S$.

(2) If T is surj, then $\operatorname{range} R = \operatorname{null} S \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.

(3) If S is inje, then $\operatorname{range} R = \{0\} \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} R = \dim \operatorname{null} T$.

---

23 *Suppose* U *and* V *are finite-dim vecsps and* $S \in \mathcal{L}(V, W)$ *and* $T \in \mathcal{L}(U, V)$.
Prove that $\dim \operatorname{range} ST \leq \min \{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.

SOLUTION:

$\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{span}(Su_1, \dots, Su_{\dim \operatorname{range} T})$,

where $\operatorname{span}(u_1, \dots, u_{\dim \operatorname{range} T}) = \operatorname{range} T$.

$\dim \operatorname{range} ST \leq \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leq \dim \operatorname{range} S \Rightarrow \square$

OR. Note that $\operatorname{range}(S|_{\operatorname{range} T}) = \operatorname{range} ST$.

Thus $\dim \operatorname{range} ST = \dim \operatorname{range}(S|_{\operatorname{range} T}) = \dim \operatorname{range} T - \dim \operatorname{null}(S|_{\operatorname{range} T}) \leq \operatorname{range} T$. $\square$

COROLLARY:

(1) If S is inje, then $\dim \operatorname{range} ST = \dim \operatorname{range} T$.
(2) If T is surj, then $\dim \operatorname{range} ST = \dim \operatorname{range} S$.

• (a) *Suppose* $\dim V = 5$ *and* $S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$.
Prove that $\dim \operatorname{range} TS \leq 2$.
(b) *Give an example of* $S, T \in \mathcal{L}(\mathbf{F}^5)$ *with* $ST = 0$ *and* $\dim \operatorname{range} TS = 2$.

SOLUTION:

By Problem (23), $\dim \operatorname{range} TS \leq \min \left\{ \overbrace{\dim \operatorname{range} S}^{5-\dim \operatorname{null} T}, \overbrace{\dim \operatorname{range} T}^{5-\dim \operatorname{null} S} \right\}$.

We show that $\dim \operatorname{range} TS \leq 2$ by contradiction. Assume that $\dim \operatorname{range} TS \geq 3$.
Then $\min\{5 - \dim \operatorname{null} T, 5 - \dim \operatorname{null} S\} \geq 3 \Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leq 2$.
又 $\dim \operatorname{null} ST = 5 \leq \dim \operatorname{null} S + \dim \operatorname{null} T \leq 4$. Contradicts. $\square$

OR. $\left.\begin{array}{l} \dim \operatorname{null} S = 5 - \dim \operatorname{range} S \\ \dim \operatorname{range} TS \leq \dim \operatorname{range} S \end{array}\right\} \Rightarrow \dim \operatorname{null} S \leq 5 - \dim \operatorname{range} TS$.

And $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} TS \leq \dim \operatorname{range} T \leq \dim \operatorname{null} S$.
Thus $\dim \operatorname{range} TS \leq 5 - \dim \operatorname{range} TS \Rightarrow \dim \operatorname{range} TS \leq \frac{5}{2}$. $\square$

EXAMPLE: Let $(v_1, \dots, v_5)$ be a basis of $\mathbf{F}^5$. Define $S, T \in \mathcal{L}(\mathbf{F}^5)$ by:

$$\begin{array}{llll} T: & v_1 \mapsto 0, & v_2 \mapsto 0, & v_i \mapsto v_i \ ; \\ S: & v_1 \mapsto v_4, & v_2 \mapsto v_5, & v_i \mapsto 0 \ ; \qquad i = 3, 4, 5. \end{array}$$

• *Suppose* $\dim V = n$ *and* $S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$.
Prove that $\dim \operatorname{range} TS \leq \left[\frac{n}{2}\right]$.

SOLUTION:

By Problem (23), $\dim \operatorname{range} TS \leq \min \left\{ \overbrace{\dim \operatorname{range} S}^{n-\dim \operatorname{null} T}, \overbrace{\dim \operatorname{range} T}^{n-\dim \operatorname{null} S} \right\}$. We prove by contradiction.

Assume that $\dim \operatorname{range} TS \geq \left[\frac{n}{2}\right] + 1$.

Then $\min\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\} \geq \left[\frac{n}{2}\right] + 1$

$\Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leq n - \left[\frac{n}{2}\right] - 1$.

又 $\dim \operatorname{null} ST = n \leq \dim \operatorname{null} S + \dim \operatorname{null} T \leq 2(n - \left[\frac{n}{2}\right] - 1)$

$\Rightarrow \left[\frac{n}{2}\right] + 1 \leq \frac{n}{2}$. Contradicts. Thus $\dim \operatorname{range} TS \leq \left[\frac{n}{2}\right]$. $\square$

OR. $\dim \operatorname{null} S = n - \dim \operatorname{range} S \leq n - \dim \operatorname{range} TS$.

And $ST = 0 \Rightarrow \dim \operatorname{range} TS \leq \dim \operatorname{range} T \leq \dim \operatorname{null} S \leq n - \dim \operatorname{range} TS$

$\Rightarrow 2 \dim \operatorname{range} TS \leq n \Rightarrow \dim \operatorname{range} TS \leq \frac{n}{2}$

$\Rightarrow \dim \operatorname{range} TS \leq \left[\frac{n}{2}\right]$ (because $\dim \operatorname{range} TS$ is an integer). $\square$

---

24 *Suppose that* W *is finite-dim and* $S, T \in \mathcal{L}(V, W)$.
Prove that $\operatorname{null} S \subseteq \operatorname{null} T \iff \exists\, E \in \mathcal{L}(W)$ *such that* $T = ES$.

SOLUTION:

Suppose $\exists\, E \in \mathcal{L}(W)$ such that $T = ES$. Then $\operatorname{null} T = \operatorname{null} ES \supseteq \operatorname{null} S$.
Suppose $\operatorname{null} S \subseteq \operatorname{null} T$. Let $R = (Sv_1, \dots, Sv_n)$ be a basis of $\operatorname{range} S$

$\Rightarrow (v_1, \dots, v_n)$ is linely inde.

Let $\mathcal{K}_R = \operatorname{span}(v_1, \dots, v_n) \Rightarrow V = \mathcal{K}_R \oplus \operatorname{null} S$.

Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i,\ \ Eu = 0;$ for each $i = 1 \dots, n$ and $u \in \operatorname{null} S$.

Hence $\forall v \in V,\ (\exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} S),\ Tv = a_1Tv_1 + \cdots + a_nTv_n = E(a_1Sv_1 + \cdots + a_nSv_n) \Rightarrow T = ES.$ □

OR. Extend R to a basis $(Sv_1, \dots, Sv_n, w_1, \dots, w_m)$ of W.

Define $E \in \mathcal{L}(W)$ by $E(Sv_k) = Tv_k,\ \ Ew_j = 0$.

Because $\forall\, v \in V,\ \exists\, a_i \in \mathbf{F}, Sv = a_1Sv_1 + \cdots + a_nSv_n$

$$\begin{aligned}&\Rightarrow S\left(v - (a_1v_1 + \cdots + a_nv_n)\right) = 0\\ &\Rightarrow v - (a_1v_1 + \cdots + a_nv_n) \in \operatorname{null} S\\ &\Rightarrow v - (a_1v_1 + \cdots + a_nv_n) \in \operatorname{null} T.\\ &\Rightarrow T\left(v - (a_1v_1 + \cdots + a_nv_n)\right) = 0\end{aligned}$$

Thus $Tv = a_1v_1 + \cdots + a_nv_n$. Hence $E(Sv) = a_1E(Sv_1) + \cdots + a_nE(Sv_n) = a_1Tv_1 + \cdots + a_nTv_n = Tv.$ □

---

25 *Suppose that V is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{range} S \subseteq \operatorname{range} T \iff \exists\ E \in \mathcal{L}(V)$ *such that* $S = TE$.

SOLUTION:

Suppose $\exists\ E \in \mathcal{L}(V)$ such that $S = TE$. Then $\operatorname{range} S = \operatorname{range} TE \subseteq \operatorname{range} T$.

Suppose $\operatorname{range} S \subseteq \operatorname{range} T$. Let $(v_1, \dots, v_m)$ be a basis of V.

Because $\operatorname{range} S \subseteq \operatorname{range} T \Rightarrow Sv_i \in \operatorname{range} T$ for each i. Suppose $u_i \in V$ for each i such that $Tu_i = Sv_i$.

Thus defining $E \in \mathcal{L}(V)$ by $Ev_i = u_i$ for each $i \Rightarrow S = TE$. □

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26 *Prove that the differentiation map $D \in \mathcal{P}(\mathbf{R})$ is surj.*

SOLUTION:

[*Informal Proof*]

Note that $\deg Dx^n = n - 1$.

Because $\operatorname{span}(Dx, Dx^2, \dots) \subseteq \operatorname{range} D$. 又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, \dots) = \operatorname{span}(1, x, \dots) = \mathcal{P}(\mathbf{R})$. □

[*Proper Proof*]

We will recursively define a sequence of polynomials $(p_k)_{k=0}^{\infty}$ where $Dp_k = x^k$.

Because $\dim Dx = (\deg x) - 1 = 0$, we have $Dx = C \in \mathbf{F}$. Define $p_0 = C^{-1}x$. Then $Dp_0 = C^{-1}Dx = 1$.

Suppose we have defined $p_0, \dots, p_n$ such that $Dp_k = x^k$ for each $k \in \{0, \dots, n\}$.

Because $\deg D(x^{n+2}) = n + 1$, we let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_nx^n + \cdots + a_1x + a_0$, where $a_{n+1} \neq 0$.

Then $a_{n+1}^{-1}D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_nDp_n + \cdots + a_1Dp_1 + a_0Dp_0)$

$\Rightarrow x^{n+1} = D\left(a_{n+1}^{-1}(x^{n+2} - a_np_n - \cdots - a_1p_1 - a_0p_0)\right)$.

Thus defining $p_{n+1} = a_{n+1}^{-1}(x^{n+2} - a_np_n - \cdots - a_1p_1 - a_0p_0)$, we have $Dp_{n+1} = x^{n+1}$.

Hence we get the sequence $(p_k)_{k=0}^{\infty}$ by recursion.

Now it suffices to show that D is surj. Let $p = \sum\limits_{k=0}^{\deg p} a_kx^k \in \mathcal{P}(\mathbf{R})$.

Then $D\left(\sum\limits_{k=0}^{\deg p} a_kp_k\right) = \sum\limits_{k=0}^{\deg p} a_kDp_k = \sum\limits_{k=0}^{\deg p} a_kx^k = p$. □

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27 *Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a poly $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.*

SOLUTION:

Define $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$.

Note that $\deg Bx^n = n-1$. Similar to Problem (26), we conclude that B is surj. $\square$

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28 *Suppose $T \in \mathcal{L}(V,W)$ and $(w_1,\dots,w_m)$ is a basis of* range T. *Prove that* $\exists\ \varphi_1,\dots,\varphi_m \in \mathcal{L}(V,\mathbf{F})$ *such that for all* $v \in V, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.

Solution:

Suppose $(v_1,\dots,v_m)$ in V such that $Tv_i = w_i$ for each i.

Then $(v_1,\dots,v_m)$ is linely inde, extend it to a basis of V as $(v_1,\dots,v_m,u_1,\dots,u_n)$.

Note that $\forall v \in V, v = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n,\ \exists!\,a_i,b_i \in \mathbf{F} \Rightarrow Tv = a_1w_1 + \cdots + a_mw_m$.

Define $\varphi_i : V \to \mathbf{F}$ by $\varphi_i(v) = a_iv_i$ for each i. We now check the linearity.

$\forall v,u \in V\ (\exists!\,a_i,b_i,c_i,d_i \in \mathbf{F}), \lambda \in \mathbf{F}, \varphi_i(v+\lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda\varphi(u)$. $\square$

---

29 *Suppose $\varphi \in \mathcal{L}(V,\mathbf{F})\backslash\{0\}$. Suppose $u \in V\backslash$*null φ. *Prove that $V =$* null $\varphi \oplus \{au : a \in \mathbf{F}\}$.

Solution:

(a) $\forall\ v = cu \in \operatorname{null}\varphi \cap \{au : a \in \mathbf{F}\}, \varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$. Hence $\operatorname{null}\varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$.

(b) $$\forall\ v \in V, v = \left(v - \frac{\varphi(v)}{\varphi(u)}u\right) + \frac{\varphi(v)}{\varphi(u)}u.\ \left|\ \begin{array}{l} v - \dfrac{\varphi(v)}{\varphi(u)}u \in \operatorname{null}\varphi \\ \dfrac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array}\right\} \Rightarrow V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}.\ \square$$

This may seems strange. Here we explain why.

$\varphi \neq 0 \Rightarrow \varphi(v_i) = a_i \neq 0$ for each v_i, for some linely inde list $(v_1,\dots,v_k)$.

Fix one v_k. Then $\varphi\left(v_k - \dfrac{a_k}{a_j}v_j\right) = 0$ for each $j = 1,\dots,k-1,k+1,\dots,n$.

Thus span $\left\{v_k - \dfrac{a_k}{a_j}v_j\right\}_{j\neq k} \subseteq \operatorname{null}\varphi$. Hence every vecsp in $\mathcal{S}_V\operatorname{null}\varphi$ is one-dim.

---

30 *Suppose $\varphi_1,\varphi_2 \in \mathcal{L}(V,\mathbf{F})$ and* null $\varphi_1 =$ null $\varphi_2 =$ null φ. *Prove that $\exists\ c \in \mathbf{F}, \varphi_1 = c\varphi_2$*

Solution:

If $\operatorname{null}\varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $u \in V/\operatorname{null}\varphi \Rightarrow \varphi_1(u),\varphi_2(u) \neq 0$.

By Problem (29), $V = \operatorname{null}\varphi \oplus \operatorname{span}(u)$.

Hence for any $v \in V, v = w + a_vu, \exists!\,w \in \operatorname{null}\varphi, a_v \in \mathbf{F}$.

$$\varphi_1(v) = a_v\varphi_1(u),\ \ \varphi_2(v) = a_v\varphi_2(u) \Rightarrow a_v = \frac{\varphi_1(v)}{\varphi_1(u)} = \frac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \frac{\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}.$$

Thus $\varphi_1 = c\varphi_2$. $\square$

• *Suppose V is finite-dim, X is a subsp of V, and Y is a finite-dim subsp of W. Prove that if* $\dim X + \dim Y = \dim V$, *then* $\exists\, T \in \mathcal{L}(V,W)$, null $T = X$ *and* range $T = Y$.

Solution:

Suppose $\dim X + \dim Y = \dim V$. Let $(u_1,\dots,u_n)$ be a basis of X, $R = (w_1,\dots,w_m)$ be a basis of Y.

Extend $(u_1,\dots,u_n)$ to a basis of V as $(u_1,\dots,u_n,v_1,\dots,v_m)$.

Define $T \in \mathcal{L}(V,W)$ by $T(\sum_{i=1}^{m} a_iv_i + \sum_{i=1}^{n} b_iu_i) = \sum_{i=1}^{m} a_iw_i$.

Now we show that null $T = X$ and range $T = Y$

Suppose $v \in V$. Then $\exists!\,a_i,b_j \in \mathbf{F}, v = \sum_{i=1}^{m} a_iv_i + \sum_{i=1}^{n} b_iu_i$.

$$\left.\begin{array}{r} v \in \operatorname{null}T \Rightarrow Tv = 0 \Rightarrow a_1 = \cdots = a_m = 0 \Rightarrow v \in X \\ v \in X \Rightarrow v \in \operatorname{null}T \end{array}\right\} \Rightarrow \operatorname{null}T = X.$$

$$\left.\begin{array}{l} w \in \operatorname{range} T \Rightarrow \exists\, v = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i u_i \in V, Tv = w = \sum_{i=1}^{m} a_i w_i \Rightarrow w \in Y \\ w \in Y \Rightarrow w = a_1 T v_1 + \cdots + a_m T v_m = T(a_1 v_1 + \cdots + a_m v_m) \Rightarrow w \in \operatorname{range} T \end{array}\right\} \Rightarrow \operatorname{range} T = Y. \qquad \square$$

- *Suppose V is finite-dim and $T \in \mathcal{L}(V, W)$. Let $(Tv_1, \dots, Tv_n)$ be a basis of* range T. *Extend $(v_1, \dots, v_n)$ to a basis of V as $(v_1, \dots, v_n, u_1, \dots, u_m)$. Prove or give a counterexample: $(u_1, \dots, u_m)$ is a basis of* null T.

SOLUTION: A counterexample:

Suppose $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$. Then $\operatorname{span}(Tv_1, Tv_2, Tv_3) = \operatorname{span}(w_1)$.

Extend (v_i) to (v_1, v_2, v_3) for each i. But none of $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is a basis of null T. $\square$

COMMENT: $(v_2 - v_1, v_3 - v_1), (v_1 - v_2, v_3 - v_2)$ or $(v_1 - v_3, v_2 - v_3)$ are all bases of null T.

- *Suppose V is finite-dim and $T \in \mathcal{L}(V, W)$. Let $(u_1, \dots, u_m)$ be a basis of* null T. *Extend $(u_1, \dots, u_m)$ to a basis of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$. Prove or give a counterexample: $(Tv_1, \dots, Tv_n)$ spans* range T.

SOLUTION:

$\forall w \in \operatorname{range} T,\ \exists\, v \in V,\ (\exists!\, a_i, b_i \in \mathbf{F}), Tv = T(a_1 v_1 + \cdots + a_n v_n) = w$

$\Rightarrow w \in \operatorname{span}(Tv_1, \dots, Tv_n) \Rightarrow \operatorname{range} T \subseteq \operatorname{span}(Tv_1, \dots, Tv_n)$. $\square$

COMMENT: If T is inje, then $(Tv_1, \dots, Tv_n)$ is a basis of range T.

- OR (5.B.4) *Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V =$ null $P \oplus$ range P.*

SOLUTION:

Let $(P^2 v_1, \dots, P^2 v_n)$ be a basis of range P^2. Then $(Pv_1, \dots, Pv_n)$ is linely inde in V.

$$\left.\begin{array}{l} \text{Let } \mathcal{K} = \operatorname{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \operatorname{null} P^2 \\ \text{又 } \mathcal{K} = \operatorname{range} P = \operatorname{range} P^2;\ \ \operatorname{null} P = \operatorname{null} P^2 \end{array}\right\} \Rightarrow \square$$

OR. (a) Suppose $v \in \operatorname{null} P \cap \operatorname{range} P$.

Then $\exists\, u \in V, v = Pu, Pv = 0 \Rightarrow v = Pu = P^2 u = Pv = 0$. Hence $\operatorname{null} P \cap \operatorname{range} P = \{0\}$.

(b) Note that $v = Pv + (v - Pv)$ and $P^2 v = Pv$ for all $v \in V$.

Then $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$. Hence $V = \operatorname{range} P + \operatorname{null} P$. $\square$

- *Suppose V is finite-dim with* $\dim V > 1$. *Show that if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is a linear map such that $\varphi(ST) = \varphi(S) \cdot \varphi(T)$ for all $S, T \in \mathcal{L}(V)$, then $\varphi = 0$.*

SOLUTION: Using notations in (4E 3.A.16).

Suppose $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \dots, n\},\ \varphi(R_{i,j}) \neq 0$.

Because $R_{i,j} = R_{x,j} \circ R_{i,x},\ \ \forall x = 1, \dots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, because $R_{i,x} = R_{y,x} \circ R_{i,y},\ \ \forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0$ for any $x, y = 1, \dots, n$.

Let $l \neq i, k \neq j$ and then $\varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. $\square$

OR. Note that by (4E 3.A.16), $\exists\, S, T \in \mathcal{L}(V), ST - TS \neq 0$.

Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \operatorname{null} \varphi \neq \{0\}$.

Thus $\forall\, E \in \operatorname{null} \varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \operatorname{null} \varphi$.

Hence null φ is a nonzero two-sided ideal of $\mathcal{L}(V)$. $\square$

- *Suppose that V and W are real vector spaces and $T \in \mathcal{L}(V, W)$.*

Define $T_{\mathbf{C}}: V_{\mathbf{C}} \to W_{\mathbf{C}}$ *by* $T_{\mathbf{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv$ *for all* $u, v \in V$.
(a) *Show that* $T_{\mathbf{C}}$ *is a (complex) linear map from* $V_{\mathbf{C}}$ *to* $W_{\mathbf{C}}$.
(b) *Show that* $T_{\mathbf{C}}$ *is inje* $\Longleftrightarrow$ T *is inje.*
(c) *Show that* $\operatorname{range} T_{\mathbf{C}} = W_{\mathbf{C}} \Longleftrightarrow \operatorname{range} T = W$.

SOLUTION:

(a) $\forall u_1 + \mathrm{i}v_1, u_2 + \mathrm{i}v_2 \in V_{\mathbf{C}}, \lambda \in \mathbf{F}$,

$$T\left((u_1 + \mathrm{i}v_1) + \lambda(u_2 + \mathrm{i}v_2)\right) = T\left((u_1 + \lambda u_2) + \mathrm{i}(v_1 + \lambda v_2)\right) = T(u_1 + \lambda u_2) + \mathrm{i}T(v_1 + \lambda v_2)$$
$$= Tu_1 + \mathrm{i}Tv_1 + \lambda Tu_2 + \mathrm{i}\lambda Tv_2 = T(u_1 + \mathrm{i}v_1) + \lambda T(u_2 + \mathrm{i}v_2).$$

(b) $\left|\begin{array}{l} \text{Suppose } T_{\mathbf{C}} \text{ is inje. Let } T(u) = 0 \Rightarrow T_{\mathbf{C}}(u + \mathrm{i}0) = Tu = 0 \Rightarrow u = 0. \\ \text{Suppose } T \text{ is inje. Let } T_{\mathbf{C}}(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + \mathrm{i}v = 0. \end{array}\right\} \Rightarrow \square$

(c) $\left|\begin{array}{l} \text{Suppose } T_{\mathbf{C}} \text{ is surj. } \forall w \in W,\ \exists\, u \in V, T(u + \mathrm{i}0) = Tu = w + \mathrm{i}0 = w \Rightarrow \text{T is surj.} \\ \text{Suppose } T \text{ is surj. } \forall w, x \in W,\ \exists\, u, v \in V, Tu = w, Tv = x \\ \qquad \Rightarrow \forall w + \mathrm{i}x \in W_{\mathbf{C}},\ \exists\, u + \mathrm{i}v \in V, T(u + \mathrm{i}v) = w + \mathrm{i}x \Rightarrow T_{\mathbf{C}} \text{ is surj.} \end{array}\right\} \Rightarrow \square$

ENDED

3·C

• **NOTE FOR [3.47]:** $LHS = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k} = RHS.$

• **NOTE FOR [3.48]:**

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 1 \times 5 + 2 \times 8 & 1 \times 6 + 2 \times 9 & 1 \times 7 + 2 \times 10 \\ 3 \times 5 + 4 \times 8 & 3 \times 6 + 4 \times 9 & 3 \times 7 + 4 \times 10 \end{pmatrix} = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

• **NOTE FOR [3.49]:**

$$\because [(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$$
$$\therefore (AC)_{\cdot,k} = A_{\cdot,\cdot}C_{\cdot,k} = AC_{\cdot,k}$$

• **EXERCISE 10:**

$$\because [(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$$
$$\therefore (AC)_{j,\cdot} = A_{j,\cdot}C_{\cdot,\cdot} = A_{j,\cdot}C.$$

•*Suppose* $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}$.

(a) For $k = 1, \dots, p$, $\quad (CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^{c} C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$

(b) For $j = 1, \dots, m$, $\quad (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

EXAMPLE:

$$P = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} \in \mathbf{F}^{2,3}.$$
$$P_{\cdot,1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 8\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 21 \\ 47 \end{pmatrix};$$

$$P_{\cdot,2} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} = 6\begin{pmatrix}1\\3\end{pmatrix} + 9\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}24\\54\end{pmatrix};$$

$$P_{\cdot,3} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix} = 7\begin{pmatrix}1\\3\end{pmatrix} + 10\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}27\\61\end{pmatrix};$$

$$P_{1,\cdot} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\8 & 9 & 10\end{pmatrix} = 1\begin{pmatrix}5 & 6 & 7\end{pmatrix} + 2\begin{pmatrix}8 & 9 & 10\end{pmatrix} = \begin{pmatrix}21 & 24 & 27\end{pmatrix};$$

$$P_{2,\cdot} = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\8 & 9 & 10\end{pmatrix} = 3\begin{pmatrix}5 & 6 & 7\end{pmatrix} + 4\begin{pmatrix}8 & 9 & 10\end{pmatrix} = \begin{pmatrix}47 & 54 & 61\end{pmatrix};$$

- **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$

  $\because (Ac)_{j,1} = \sum\limits_{r=1}^{n} A_{j,r}c_{r,1} = [\sum\limits_{r=1}^{n} (A_{\cdot,r}c_{r,1})]_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$

  $\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum\limits_{r=1}^{n} A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ OR. By $(Ac)_{\cdot,1} = Ac_{\cdot,1}$ Using (a) above.

- **EXERCISE 11:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$

  $\because (aC)_{1,k} = \sum\limits_{r=1}^{n} a_{1,r}C_{r,k} = [\sum\limits_{r=1}^{n} a_{1,r}(C_{r,\cdot})]_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$

  $\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum\limits_{r=1}^{n} a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ OR. By $(aC)_{1,\cdot} = a_{1,\cdot}C$. Using (b) above.

- COLUMN-ROW FACTORIZATION (CR Factorization)

  Suppose $A \in \mathbf{F}^{m,n}, A \neq 0$. *Let* $S_c = \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}, \dim S_c = c$.

  And $S_r = \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) \subseteq \mathbf{F}^{1,n}, \dim S_r = r$.

  Prove that $A = CR$. $\exists\, C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,n}$.

SOLUTION: Notice that $A \neq 0 \Rightarrow c, r \geq 1$.

Let $(C_{\cdot,1}, \dots, C_{\cdot,c})$ be a basis of S_c, forming $C \in \mathbf{F}^{m,c}$.

Then for any $A_{\cdot,k}$, $A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$, $\exists!\, R_{1,k}, \dots, R_{c,k} \in \mathbf{F}$.

Hence, by letting $R = \begin{pmatrix}R_{1,1} & \cdots & R_{1,n}\\ \vdots & \ddots & \vdots\\ R_{c,1} & \cdots & R_{c,n}\end{pmatrix}$, we have $A = CR$.

OR. Let $(R_{1,\cdot}, \dots, R_{c,\cdot})$ be a basis of S_r, forming $R \in \mathbf{F}^{c,n}$.

For any $A_{j,\cdot}$, $A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot} = (CR)_{j,\cdot}$, $\exists!\, C_{j,1}, \dots, C_{j,c} \in \mathbf{F}$. Similarly. □

EXAMPLE:

$$A = \begin{pmatrix}10 & 7 & 4 & 1\\26 & 19 & 12 & 5\\46 & 33 & 20 & 7\end{pmatrix} \overset{(1)}{=} \begin{pmatrix}1 & 0\\0 & 1\\2 & 1\end{pmatrix}\begin{pmatrix}10 & 7 & 4 & 1\\26 & 19 & 12 & 5\end{pmatrix} \overset{(2)}{=} \begin{pmatrix}7 & 4\\19 & 12\\33 & 20\end{pmatrix}\begin{pmatrix}2 & -1\\1 & 0\\0 & 1\\-1 & 2\end{pmatrix}$$

(1) Because $\begin{pmatrix}46 & 33 & 20 & 7\end{pmatrix} = 2\begin{pmatrix}10 & 7 & 4 & 1\end{pmatrix} + \begin{pmatrix}26 & 19 & 12 & 5\end{pmatrix}$.

Hence $\dim S_r = 2$. We choose $(A_{1,\cdot}, A_{2,\cdot})$ as the basis.

(2) Because $\begin{pmatrix}10\\26\\46\end{pmatrix} = 2\begin{pmatrix}7\\19\\33\end{pmatrix} - \begin{pmatrix}4\\12\\20\end{pmatrix}$; $\begin{pmatrix}1\\5\\7\end{pmatrix} = 2\begin{pmatrix}4\\12\\20\end{pmatrix} - \begin{pmatrix}7\\19\\33\end{pmatrix}$.

Hence $\dim S_c = 2$. We choose $(A_{\cdot,2}, A_{\cdot,3})$ as the basis.

- COLUMN RANK EQUALS ROW RANK (Using the notation above)

  For any $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

  $\Rightarrow \operatorname{span}(A_{1,\cdot}, \dots, A_{m,\cdot}) = S_r = \operatorname{span}(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leq c = \dim S_c$.

  Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leq r = \dim S_r = \dim S_c^t$. □

• *Suppose $T \in \mathcal{L}(V)$, and $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are bases of V.*
Prove that the following are equi. *Here $A = \mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.*
(a) *T is inje.*
(b) *The cols of $\mathcal{M}(T)$ are linely inde in $\mathbf{F}^{n,1}$.*
(c) *The cols of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.*
(d) *The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.*
(e) *The rows of $\mathcal{M}(T)$ are linely inde in $\mathbf{F}^{1,n}$.*

SOLUTION: T is inje $\iff \dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim \operatorname{range} T$

$\iff (Tu_1, \dots, Tu_n)$ is linely inde in V, and therefore is a basis of V

$\iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n))$ is linely inde, as well as $(A_{\cdot,1}, \dots, A_{\cdot,n})$

$\iff (A_{\cdot,1}, \dots, A_{\cdot,n})$ is a basis of $\mathbf{F}^{n,1}$.

$\Big($ 又 $\dim \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) = \dim \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = n$ $\Big)$

$\iff (A_{1,\cdot}, \dots, A_{n,\cdot})$ is a basis of $\mathbf{F}^{1,n}$. □

• *Suppose A is an m-by-n matrix with $A \neq 0$.*
Prove that the rank of A is 1 if and only if there exist $(c_1, \dots, c_m) \in \mathbf{F}^m$ and $(d_1, \dots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j \cdot d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$.

SOLUTION: Using the notation in CR Factorization.

(a) Suppose $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix}$. ($\exists\, c_j, d_k \in \mathbf{F}, \forall j, k$)

Then $S_c = \operatorname{span}\left(\begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \dots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right) = \operatorname{span}\left(\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right)$.

OR. $S_r = \operatorname{span}\begin{pmatrix} \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \end{pmatrix}, \\ \begin{pmatrix} c_2 d_1 & \cdots & c_2 d_n \end{pmatrix}, \\ \vdots \\ \begin{pmatrix} c_m d_1 & \cdots & c_m d_n \end{pmatrix} \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} \right)$. Hence the rank of A is 1.

(b) Suppose the rank of A is $\dim S_c = \dim S_r = 1$

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}$, $\quad d'_k = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$. ($\forall j, k$)

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$. Letting $d_k = d'_k A_{1,1}$. □

1 *Suppose $T \in \mathcal{L}(V, W)$. Show that with resp to each choice of bases of V and W, the matrix of T has at least* $\dim \operatorname{range} T$ *nonzero entries.*

SOLUTION:

Let $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$ be bases of V and W respectively. We prove by contradiction.
Suppose $A = \mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$ has at most $(\dim \operatorname{range} T - 1)$ nonzero entries.
Then by Pigeon Hole Principle, at least one of $A_{\cdot,k} = 0$.
Thus there are at most $(\dim \operatorname{range} T - 1)$ nonzero vectors in $Tv_1, \dots, Tv_n$.
While $\operatorname{range} T = \operatorname{span}(Tv_1, \dots, Tv_n) \Rightarrow \dim \operatorname{range} T \leq \dim \operatorname{range} T - 1$. We get a contradiction. □

3 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$.*

Prove that there exist a basis of V and a basis of W such that
[letting $A = \mathcal{M}(T)$ with resp to these bases],
$A_{k,k} = 1, A_{i,j} = 0$, *where* $1 \leq k \leq \dim \operatorname{range} T, i \neq j$.

Solution:

Let $R = (Tv_1, \dots, Tv_n)$ be a basis of range T, extend it to the basis of W as $(Tv_1, \dots, Tv_n, w_1, \dots, w_p)$.
Let $\mathcal{K}_R = \operatorname{span}(v_1, \dots, v_n)$. Let $(u_1, \dots, u_m)$ be a basis of null T.
Then $(v_1, \dots, v_n, u_1, \dots, u_m)$ is the basis of V.
Thus $T(v_k) = Tv_k, T(u_j) = 0 \Rightarrow A_{k,k} = 1, A_{i,j}$ for each $k \in \{1, \dots, \dim \operatorname{range} T\}$ and $j \in \{1, \dots, m\}$. ☐

---

4 *Suppose $(v_1, \dots, v_m)$ is a basis of V and W is finite-dim. Suppose $T \in \mathcal{L}(V, W)$.*
Prove that there exists a basis $(w_1, \dots, w_n)$ of W such that
$$[\textit{letting } A = \mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_n))], \quad A_{\cdot,1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \textit{ or } \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Solution: If $Tv_1 = 0$, then we are done. If not then extend (Tv_1). ☐

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5 *Suppose $(w_1, \dots, w_n)$ is a basis of W and V is finite-dim. Suppose $T \in \mathcal{L}(V, W)$.*
Prove that there exists a basis $(v_1, \dots, v_m)$ of V such that
$$[\textit{letting } A = \mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_n))], A_{1,\cdot} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \textit{ or } \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Solution:

Let $(u_1, \dots, u_m)$ be a basis of V. If $A_{1,\cdot} = 0$, then let $v_i = u_i$ for each $i = 1, \dots, n$, we are done.
Otherwise, $\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \end{pmatrix} \neq 0$, choose one $A_{1,k} \neq 0$.
$$\text{Let } v_1 = \frac{u_k}{A_{1,k}}; \qquad v_j = u_{j-1} - A_{1,j-1} v_1 \quad \text{for } j = 2, \dots, k;$$
$$v_i = u_i - A_{1,i} v_1 \qquad \text{for } i = k+1, \dots, n.$$
☐

---

6 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$. Prove that* $\dim \operatorname{range} T = 1$
if and only if there exist a basis of V and a basis of W such that
with resp to these bases, all entries of $A = \mathcal{M}(T)$ equal 1.

Solution: Denote the bases of V and W by $B_V = (v_1, \dots, v_n)$ and $B_W = (w_1, \dots, w_m)$ respectively.

(a) Suppose B_V, B_W are the bases such that all entries of A equal 1.
Then $Tv_i = w_1 + \cdots + w_m$ for all $i = 1, \dots, n$. Hence $\dim \operatorname{range} T = 1$.

(b) Suppose $\dim \operatorname{range} T = 1$. Then $\dim \operatorname{null} T = \dim V - 1$.
Let $(u_2, \dots, u_n)$ be a basis of null T. Extend it to a basis of V as $(u_1, u_2, \dots, u_n)$.
Let $w_1 = Tv_1 - w_2 - \cdots - w_m$. Extend it to B_W the basis of W.
Let $v_1 = u_1, v_i = u_1 + u_i$. Extend it to B_V the basis of V. ☐

---

12 *Give an example of 2-by-2 mtcs A and B such that $AB \neq BA$.*

Solution: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

---

13 *Prove that the distr property holds for matrix add and matrix multi.*
In other words, suppose A, B, C, D, E and F are matrices
whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense.
Explain why $AB + AC$ and $DF + EF$ both make sense and prove that.

Solution: Using [3.36], [3.43].

(a) Left distr: Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Because $[A(B+C)]_{j,k} = \sum_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum_{r=1}^{n} (A_{j,r}B_{r,k} + A_{j,r}C_{r,k})$.
Hence we conclude that $A(B+C) = AB + AC$.

OR. Let $(e_1, \dots, e_M)$ be the standard basis of $\mathbf{F}^M$, where $M = \max\{m, n, p\}$.
Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum_{j=1}^{m} A_{j,k}e_j$ for each $k = 1, \dots, n$. Then $\mathcal{M}(T) = A$.
Similarly, define S, R such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$$\text{Thus } T(S+R) = TS + TR \left| \begin{array}{l} \Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR) \\ \Rightarrow \mathcal{M}(T)[\mathcal{M}(S) + \mathcal{M}(R)] = \mathcal{M}(T)\mathcal{M}(S) + \mathcal{M}(T)\mathcal{M}(R) \\ \Rightarrow A(B+C) = AB + AC. \end{array} \right.$$

$$\text{(b) Right distr: Similarly.} \left| \begin{array}{l} \quad \text{Suppose } \mathcal{M}(T) = D, \mathcal{M}(S) = E, \mathcal{M}(R) = F. \\ \quad \text{Then } (T+S)R = TR + SR \\ \Rightarrow \mathcal{M}((T+S)R) = \mathcal{M}(TR) + \mathcal{M}(SR) \\ \Rightarrow [\mathcal{M}(T) + \mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)\mathcal{M}(R) + \mathcal{M}(S)\mathcal{M}(R) \\ \Rightarrow (D+E)F = DF + EF. \qquad \square \end{array} \right.$$

---

14 *Prove that matrix multi is associ. In other words,*
suppose A, B and C are mtcs whose sizes are such that $(AB)C$ makes sense.
Explain why $A(BC)$ makes sense and prove that $(AB)C = A(BC)$.

SOLUTION:

Because $[(AB)C]_{j,k} = (AB)_{j,\cdot}C_{\cdot,k} = \sum_{s=1}^{n} (A_{j,s}B_{s,\cdot})C_{\cdot,k} = \sum_{s=1}^{n} A_{j,s}(B_{s,\cdot}C_{\cdot,k}) = \sum_{s=1}^{n} A_{j,s}(BC)_{s,k} = A(BC)_{j,k}$
Hence we conclude that $(AB)C = A(BC)$.

OR. Suppose $A \in \mathbf{F}^{m,n}, B \in \mathbf{F}^{n,p}, C \in \mathbf{F}^{p,s}$.
Let $(e_1, \dots, e_M)$ be the standard basis of $\mathbf{F}^M$, where $M = \max\{m, n, p, s\}$.
Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ such that $Te_k = \sum_{j=1}^{m} A_{j,k}e_j$ for each $k = 1, \dots, n$. Then $\mathcal{M}(T) = A$.
Similarly, define S, R such that $\mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$$\begin{aligned} \text{Hence } (TS)R = T(SR) &\Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR)) \\ &\Rightarrow [\mathcal{M}(T)\mathcal{M}(S)]\mathcal{M}(R) = \mathcal{M}(T)[\mathcal{M}(S)\mathcal{M}(R)] \\ &\Rightarrow (AB)C = A(BC). \end{aligned}$$

$\square$

---

15 *Suppose A is an n-by-n matrix and $1 \le j, k \le n$.*
Show that the entry in row j, col k, of A^3
(which is defined to mean AAA) is $\sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.

SOLUTION:

$$(AAA)_{j,k} = (AA)_{j,\cdot}A_{\cdot,k} = \sum_{p=1}^{n} (A_{j,p}A_{p,\cdot})A_{\cdot,k} = \sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}.$$

OR.
$$\begin{aligned} (AAA)_{j,k} &= \sum_{r=1}^{n} (AA)_{j,r}A_{r,k} = \sum_{r=1}^{n} \Big(\sum_{p=1}^{n} A_{j,p}A_{p,r}\Big)A_{r,k} \\ &= \sum_{r=1}^{n} (A_{j,1}A_{1,r}A_{r,k} + \cdots + A_{j,n}A_{n,r}A_{r,k}) \\ &= A_{j,1}\sum_{r=1}^{n} A_{1,r}A_{r,k} + \cdots + A_{j,n}\sum_{r=1}^{n} A_{n,r}A_{r,k} = \sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}. \square \end{aligned}$$

ENDED

3·D

• *Suppose $T \in \mathcal{L}(V,W)$ is inv. Show that T^{-1} is inv and $(T^{-1})^{-1} = T$.*

$$\left.\begin{array}{l} TT^{-1} = (T^{-1})^{-1}T^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = T^{-1}(T^{-1})^{-1} = I \in \mathcal{L}(W) \end{array}\right\} \Rightarrow T = (T^{-1})^{-1}, \text{by the uniqnes of inverse.} \quad \square$$

---

1 *Suppose $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ are both inv linear maps.*
Prove that $ST \in \mathcal{L}(U,W)$ is inv and that $(ST)^{-1} = T^{-1}S^{-1}$.

SOLUTION: $$\left.\begin{array}{l} (ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(V) \end{array}\right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}, \text{by the uniqnes of inverse.} \square$$

---

9 *Suppose V is finite-dim and $S,T \in \mathcal{L}(V)$.*
Prove that ST is inv $\iff$ S and T are inv.

SOLUTION:

Suppose S,T are inv. Then $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$. Hence ST is inv.

Suppose ST is inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T \text{ is inje, } S \text{ is surj.}$$

Notice that V is finite-dim. Hence S,T are inv. $\square$

OR. Suppose ST is inv but S or T is not inv ($\Rightarrow$ not surj and inje).

If S is not inv then $\dim \operatorname{range} S < \dim V$ and by Problem (23) in (3.B),

$\dim \operatorname{range} ST \leq \dim \operatorname{range} S < \dim V$. Thus ST is not surj. Contradicts.

If T is not inv then $\dim \operatorname{range} T < 0$. Similarly, ST is not surj. Contradicts. $\square$

---

10 *Suppose V is finite-dim and $S,T \in \mathcal{L}(V)$. Prove that $ST = I \iff TS = I$.*

SOLUTION:

Suppose $ST = I$. $$\left.\begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T \text{ is inje, } S \text{ is surj.}$$

Notice that V is finite-dim. Thus T,S are inv.

OR. By Problem (9), V is finite-dim and $ST = I$ is inv $\Rightarrow S,T$ are inv.

$S((TS)v) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow S$ is inv.

OR. $ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T$. 又 $S = S \Rightarrow TS = S^{-1}S = I$.

Reversing the roles of S and T, we conclude that $TS = I \Rightarrow ST = I$. $\square$

---

11 *Suppose V is finite-dim and $S,T,U \in \mathcal{L}(V)$ and $STU = I$.*
Show that T is inv and that $T^{-1} = US$.

SOLUTION: Using Problem (9) and (10).

$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I$.

$\Rightarrow U^{-1} = ST, \quad T^{-1} = US, \quad S^{-1} = TU$. $\square$

---

12 *Show that the result in Exercise* 11 *can fail without the hypothesis that V is finite-dim.*

SOLUTION:

Let $V = \mathbf{R}^\infty, S(a_1,a_2,\dots) = (a_2,\dots), T(a_1,\dots) = (0,a_1,\dots), U = I$. Then $STU = I$ but T^{-1} is not inv.

---

13 *Suppose V is finite-dim and $R,S,T \in \mathcal{L}(V)$ are such that RST is surj.*
Prove that S is inje.

SOLUTION: By Problem (1) and (9), Notice that V is finite-dim. Then RST is inv.

$(RST)^{-1} = ((RS)T)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$.

OR. Let $X = (RST)^{-1}$ $\left| \begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje, and therefore is inv.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ is surj, and therefore is inv.} \end{array} \right.$

Thus $S = R^{-1}(RST)T^{-1}$ is inv. $\square$

---

15 *Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multi.*
In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists A \in \mathbf{F}^{m,n}, Tx = Ax, \forall x \in \mathbf{F}^{n,1}$.

SOLUTION:

Let $E_i \in \mathbf{F}^{n,1}$ for each $i = 1, \dots, n$ (where $M = \max\{m, n\}$) be such that $(E_i)_{j,1} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Then $(E_1, \dots, E_n)$ is linely inde and thus is a basis of $\mathbf{F}^{n,1}$.
Similarly, let $(R_1, \dots, R_m)$ be a basis of $\mathbf{F}^{m,1}$.

Suppose $T(E_i) = A_{1,i}R_1 + \cdots + A_{m,i}R_m$ for each $i = 1, \dots, n$. Hence by letting $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$. $\square$

COMMENT: $\mathcal{M}(T) = A$. Conversely it is true as well.

• OR (10.A.2) *Suppose $A, B \in \mathbf{F}^{n,n}$. Prove that $AB = I \iff BA = I$.*

SOLUTION: Using Problem (10) and (15).

Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Then $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.
Thus $AB = I \Leftrightarrow A(Bx) = x \iff T(Sx) = x \Leftrightarrow TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I$.$\square$

• **NOTE FOR [3.60]:** Suppose $(v_1, \dots, v_n)$ is a basis of V and $(w_1, \dots, w_m)$ is a basis of W.

Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{ix} w_j$; $\quad \delta_{ix} = \begin{cases} 0, & i \neq x \\ 1, & i = x \end{cases}$ **COROLLARY:** $E_{l,k}E_{i,j} = \delta_{jl}E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. $(\mathcal{E}^{(j,i)})_{l,k} = \begin{cases} 0, & i \neq k \text{ or } j \neq l \\ 1, & i = k \text{ and } j = l \end{cases}$

Because $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$ are iso. And $T = \mathcal{M}^{-1}\mathcal{M}(T), E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

Hence $\forall T \in \mathcal{L}(V, W),\ \exists! A_{i,j} \in \mathbf{F}(\ \forall i \in \{1, \dots, m\}, j \in \{1, \dots, n\}\ ), \mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)}+ & \cdots & +A_{1,n}\mathcal{E}^{(1,n)} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}\mathcal{E}^{(m,1)}+ & \cdots & +A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff T = \begin{pmatrix} A_{1,1}E_{1,1}+ & \cdots & +A_{1,n}E_{n,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}E_{1,m}+ & \cdots & +A_{m,n}E_{n,m} \end{pmatrix}.$$

$$\therefore \mathcal{L}(V, W) = \operatorname{span} \underbrace{\begin{pmatrix} E_{1,1}, & \cdots & ,E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{1,m}, & \cdots & ,E_{n,m} \end{pmatrix}}_{B}; \quad \mathbf{F}^{m,n} = \operatorname{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & ,\mathcal{E}^{(1,n)}, \\ \vdots & \ddots & \vdots \\ \mathcal{E}^{(m,1)}, & \cdots & ,\mathcal{E}^{(m,n)} \end{pmatrix}}_{B_m}.$$

Hence by [2.42] and [3.61], we conclude that B is a basis of $\mathcal{L}(V, W)$ and that B_M is a basis of $\mathbf{F}^{m,n}$.

∘ *Suppose V is finite-dim and $S \in \mathcal{L}(V)$.*
Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T) = ST$ for $T \in \mathcal{L}(V)$.
(a) *Show that* $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Show that* $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

SOLUTION:

(a) For all $T \in \mathcal{L}(V)$, $ST = 0 \iff \operatorname{range} T \subset \operatorname{null} S$. Thus $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$.

(b) For all $R \in \mathcal{L}(V)$, range $R \subseteq$ range $S \Longleftrightarrow \exists\, T \in \mathcal{L}(V), R = ST$. (By Problem (25) in 3.B)
Thus range $\mathcal{A} = \mathcal{L}(V, \operatorname{range} S)$. $\square$

OR. Using NOTE FOR [3.60].
Let $(w_1, \dots, w_m)$ be a basis of range S, extend it to a basis of V as $(w_1, \dots, w_m, \dots, w_n)$.
Let $v_i \in V$ such that $Sv_i = w_i$ for $m = 1, \dots, m$. Extend $(v_1, \dots, v_m)$ to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.
Define $E_{i,j} \in \mathcal{L}(V)$ by $E_{i,j}(v_x) = \delta_{ix} w_i$.

Thus $S = E_{1,1} + \cdots + E_{m,m}$; $\quad \mathcal{M}\left(S, (v_1, \dots, v_n), (w_1, \dots, w_n)\right) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$.

Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j}(w_x) = \delta_{ix} v_i$.
Let $E_{j,k} R_{i,j} = Q_{i,k}, \quad R_{j,k} E_{i,j} = G_{i,k}$

Because $\forall T \in \mathcal{L}(V), \quad \exists!\, A_{i,j} \in \mathbf{F}(\,\forall i, j = 1, \dots, n\,), \quad T = \begin{pmatrix} A_{1,1}R_{1,1}+ & \cdots & +A_{1,m}R_{m,1}+ & \cdots & +A_{1,n}R_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}R_{1,m}+ & \cdots & +A_{m,m}R_{m,m}+ & \cdots & +A_{m,n}R_{n,m} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{n,1}R_{1,n}+ & \cdots & +A_{n,m}R_{m,n}+ & \cdots & +A_{n,n}R_{n,n} \end{pmatrix}$.

$$\Rightarrow \mathcal{A}(T) = ST = (\sum_{r=1}^{m} E_{r,r})(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i}) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{i,j}Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1}+ & \cdots & +A_{1,m}Q_{m,1}+ & \cdots & +A_{1,n}Q_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}Q_{1,m}+ & \cdots & +A_{m,m}Q_{m,m}+ & \cdots & +A_{m,n}Q_{n,m} \end{pmatrix}.$$

Thus null $\mathcal{A} = \operatorname{span}\begin{pmatrix} R_{1,m+1}, & \cdots & ,R_{n,m+1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ R_{1,n}, & \cdots & ,R_{n,n} \end{pmatrix}$, $\quad$ range $\mathcal{A} = \operatorname{span}\begin{pmatrix} Q_{1,1}, & \cdots & ,Q_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ Q_{1,m}, & \cdots & ,Q_{n,m} \end{pmatrix}$.

Hence (a) $\dim \operatorname{null} \mathcal{A} = n \times (n - m)$; $\quad$ (b) $\dim \operatorname{range} \mathcal{A} = n \times m$. $\square$

COMMENT: Define $\mathcal{B} \in \mathcal{L}\left(\mathcal{L}(V)\right)$ by $\mathcal{B}(T) = TS$ for $T \in \mathcal{L}(V)$.

Similarly, $\mathcal{B}(T) = TS = (\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i})(\sum_{r=1}^{m} E_{r,r}) = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{i,j}G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1}+ & \cdots & +A_{1,m}G_{m,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}G_{1,m}+ & \cdots & +A_{m,m}G_{m,m} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{n,1}G_{1,n}+ & \cdots & +A_{n,m}G_{m,n} \end{pmatrix}$.

• OR (10.A.1) *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \dots, v_n)$ is a basis of V.*
Prove that $\mathcal{M}\left(T, (v_1, \dots, v_n)\right)$ is inv $\Longleftrightarrow$ T is inv.

SOLUTION:

Notice that $\mathcal{M}$ is an iso of $\mathcal{L}(V)$ onto $\mathbf{F}^{n,n}$.
(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I$. $\exists!\, S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$
$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$
$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. □

• OR (10.A.4) *Suppose that $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bases of V.*
Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1, \dots, n$.
Prove that $\mathcal{M}\left(T, (\alpha_1, \dots, \alpha_n)\right) = \mathcal{M}\left(I, (\beta_1, \dots, \beta_n), (\alpha_1, \dots, \alpha_n)\right)$.

SOLUTION:

For ease of notation, let $\mathcal{M}(T, \alpha \to \beta) = \mathcal{M}\left(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)\right)$, $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n))$
Denote $\mathcal{M}(T, \alpha \to \alpha)$ by A and $\mathcal{M}(I, \beta \to \alpha)$ by B.
$\forall k \in \{1, \dots, n\}, Iu_k = u_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. □
OR. Note that $\mathcal{M}(T, \alpha \to \beta)$ is the identity matrix.

$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)\underbrace{\mathcal{M}(T, \alpha \to \beta)}_{=\mathcal{M}(I, \beta \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. □

OR. Note that $\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I$.
$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}[\underbrace{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}_{\mathcal{M}(T, \alpha \to \beta)}] = \mathcal{M}(I, \beta \to \alpha)$. □

COMMENT: Denote $\mathcal{M}(T, \beta \to \beta)$ by A'.
$u_k = Iu_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n,\ \forall\, k \in \{1, \dots, n\}$.
又 $Tu_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = B_{1,k}\beta_1 + \cdots + B_{n,k}\beta_n = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.
OR. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$.

---

16 *Suppose V is finite-dim and $S \in \mathcal{L}(V)$.*
Prove that $\exists\, \lambda \in \mathbf{F}, S = \lambda I \iff ST = TS$ for every $T \in \mathcal{L}(V)$.

SOLUTION: Using the notation and result in $(\circ)$.
Suppose $S = \lambda I$. Then $ST = TS = \lambda T$ for every $T \in \mathcal{L}(V)$. Conversely, if $S = 0$, then we are done.
Suppose $S \neq 0, ST = TS, \forall T \in \mathcal{L}(V)$.
Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}\left(S, (v_1, \dots, v_1)\right) = \mathcal{M}\left(I, (w_1, \dots, w_n), (v_1, \dots, v_n)\right)$.
Then $\forall k \in \{m+1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $n = \dim V = \dim \operatorname{range} S = m$.
Note that $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$. Thus $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \cdots + a_{n,i}v_n)$. Where:
$a_{i,j} = \mathcal{M}\left(I, (w_1, \dots, w_n), (v_1, \dots, v_n)\right)_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n$;
For each j, for all i. Thus $a_{i,i} = a_{k,k} = \lambda, \forall\, k \neq i$.

Hence $w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} = \mathcal{M}\left(\lambda I, (v_1, \dots, v_n)\right) \Rightarrow S = \mathcal{M}^{-1}\left(\mathcal{M}(\lambda I)\right) = \lambda I$. □

• OR (10.A.3) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that T has the same matrix with resp to every basis of V
if and only if T is a scalar multi of the identity operator.

SOLUTION: [Compare with the first solution of Problem (16) in (3.A)]
Suppose $T = \lambda I$ for some $\lambda \in \mathbf{F}$. Then T has the same matrix with resp to every basis of V.
Conversely, if $T = 0$, then we are done; Suppose $T \neq 0$. And v is a nonzero vector in V.
Assume that (v, Tv) is linely inde.
Extend (v, Tv) to a basis of V as $(v, Tv, u_3, \dots, u_n)$. Let $B = \mathcal{M}\left(T, (v, Tv, u_3, \dots, u_n)\right)$.
$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0,\ \forall i \neq 2$.

By assumption, $A = \mathcal{M}(T, (v, w_2, \dots, w_n)) = B$ for any basis $(v, w_2, \dots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0\,(\cdots)$.

$\Rightarrow Tv = w_2$, which is not true if we let $w_2 = u_3, w_3 = Tv, w_j = u_j\ (j = 4, \dots, n)$. Contradicts.

Hence (v, Tv) is linely depe $\Rightarrow \forall v \in V, \exists \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show that λ_v is independent of v, that is,

to show that for any two nonzero distinct vectors $v, w \in V, \lambda_v = \lambda_w$. Thus $T = \lambda I, \exists \lambda \in \mathbf{F}$.

$$\left.\begin{aligned} &(v,w) \text{ is linely inde} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_{v+w}v + \lambda_{v+w}w \\ &\qquad\qquad\qquad\qquad\qquad = \lambda_v v + \lambda_w w = Tv + Tw \Rightarrow \lambda_{v+w} = \lambda_v = \lambda_w \\ &(v,w) \text{ is linely depe}, w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv) \Rightarrow \lambda_v = \lambda_w \end{aligned}\right\} \Rightarrow \square$$

OR. Conversely, denote $\mathcal{M}(T, (u_1, \dots, u_m))$ by A, where the basis $(u_1, \dots, u_m)$ is arbitrary.

Fix one basis $(v_1, \dots, v_m)$ and then $(v_1, \dots, \frac{1}{2}v_k, \dots, v_m)$ is also a basis for any given $k \in \{1, \dots, m\}$.

Fix one k. Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \cdots + A_{k,k}(\frac{1}{2}v_k) + \cdots + A_{m,k}v_m$

$\Rightarrow Tv_k = 2A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + 2A_{m,k}v_m = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{m,k}v_m$.

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1, \dots, m\}$.

Now we show that $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j, k such that $j \neq k$.

Consider the basis $(v'_1, \dots, v'_j, \dots, v'_k, \dots, v'_m)$,

where $v'_j = v_k,\ v_k' = v_j$ and $v'_i = v_i$ for all $i \in \{1, \dots, m\} \setminus \{j, k\}$.

Remember that $\mathcal{M}(T, (v'_1, \dots, v'_m)) = \mathcal{M}(T, (v_1, \dots, v_m)) = A$.

Hence $T(v'_k) = A_{1,k}v'_1 + \cdots + A_{k,k}v'_k + \cdots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_j$, while $T(v'_k) = T(v_j) = A_{j,j}v_j$.

Thus $A_{k,k} = A_{j,j}$. $\square$

---

17 *Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*

A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Using NOTE FOR [3.60]. Let $(v_1, \dots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.

Then for any $E_{i,j} \in \mathcal{E}$, $(\ \forall x, y = 1, \dots, n\ )$, by assumption, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.

Again, $E_{y,x'}, E_{y',x} \in \mathcal{E}$ for all $x', y', x, y = 1, \dots, n$. Thus $\mathcal{E} = \mathcal{L}(V)$. $\square$

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18 *Show that V and $\mathcal{L}(\mathbf{F}, V)$ are iso vector spaces.*

SOLUTION:

Define $\Psi \in \mathcal{L}\big(V, \mathcal{L}(\mathbf{F}, V)\big)$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.

(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is inje.

(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi(T(1))$. Hence Ψ is surj. $\square$

OR. Define $\Phi \in \mathcal{L}(\mathcal{L}(\mathbf{F}, V), V)$ by $\Phi(T) = T(1)$.

(a) Suppose $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is inje.

(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surj. $\square$

COMMENT: $\Phi = \Psi^{-1}$.

• *Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2+x)p''(x) + 2xp'(x) + p(3), \forall x \in \mathbf{R}$.*

SOLUTION:

Note that $\deg[(x^2 + x)p''(x) + 2xp'(x) + p(3)] = \deg p$.

Define $T_n : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.

As can be easily checked, T_n is an operator.

Because $\deg(T_n p) = \deg p$. If $T_n(p) = 0 \Rightarrow \deg(T_n p) = -\infty$, then $\deg p = -\infty \Rightarrow p = 0$.

Hence T_n is inje and therefore is surj.

For all $q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $m = 0$; if $q \neq 0$, let $m = \deg q$. We have $q \in \mathcal{P}_m(\mathbf{R})$.

Hence $\exists\, p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$.

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19 *Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is inje. $\deg Tp \leq \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.*
(a) *Prove that T is surj.*
(b) *Prove that for every nonzero p, $\deg Tp = \deg p$.*

SOLUTION:

(a) T is inje $\iff T|_{\mathcal{P}_n(\mathbf{R})} : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ is inje for any $n \in \mathbf{N}^+$
$\iff T|_{\mathcal{P}_n(\mathbf{R})}$ is surj for any $n \in \mathbf{N}^+ \iff T$ is surj.

(b) Using mathematical induction.
(i) $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$.
$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$.
(ii) Suppose $\deg f = \deg Tf$ for all $f \in \mathcal{P}_n(\mathbf{R})$. Then suppose $\deg g = n+1, g \in \mathcal{P}_{n+1}(\mathbf{R})$.
Assume that $\deg Tg < \deg g$ ($\Rightarrow \deg Tg \leq n, Tg \in \mathcal{P}_n(\mathbf{R})$).
Then by (a), $\exists f \in \mathcal{P}_n(\mathbf{R}), T(f) = (Tg)$. 又 T is inje $\Rightarrow f = g$.
While $\deg f = \deg Tf = \deg Tg < \deg g$. Contradicts the assumption.
Hence $\deg Tp = \deg p$ for all $p \in \mathcal{P}_{n+1}(\mathbf{R})$.
Thus $\deg Tp = \deg p$ for all $p \in \mathcal{P}(\mathbf{R})$. ☐

• *Suppose $T \in \mathcal{L}(V)$ and $(v_1, \dots, v_m)$ is a list in V such that $(Tv_1, \dots, Tv_m)$ spans V. Prove that $(v_1, \dots, v_m)$ spans V.*

SOLUTION:

Because $V = \operatorname{span}(Tv_1, \dots, Tv_m) \Rightarrow T$ is surj, 又 V is finite-dim $\Rightarrow T$ is inv $\Rightarrow T^{-1}$ is inv.
$\forall v \in V,\ \exists\, a_i \in \mathbf{F}, v = a_1Tv_1 + \cdots + a_nTv_n \Rightarrow T^{-1}v = a_1v_1 + \cdots + a_nv_n \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1, \dots, v_n)$.☐

OR. Reduce $(Tv_1, \dots, Tv_n)$ to a basis of V as $(Tv_{\alpha_1}, \dots, Tv_{\alpha_m})$, where $m = \dim V$ and $\alpha_i \in \{1, \dots, m\}$.
Then $(v_{\alpha_1}, \dots, v_{\alpha_m})$ is linely inde of length m, therefore is a basis of V, contained in the list $(v_1, \dots, v_m)$.☐

---

2 *Suppose V is finite-dim and $\dim V > 1$.*
Prove that the set of non-inv operators on V is not a subsp of $\mathcal{L}(V)$.

SOLUTION: Denote the set by U.

Suppose $\dim V = n > 1$. Let $(v_1, \dots, v_n)$ be a basis of V.
Define $S, T \in \mathcal{L}(V)$ by $S(a_1v_1 + \cdots + a_nv_n) = a_1v_1$ and $T(a_1v_1 + \cdots + a_nv_n) = a_2v_1 + \cdots + a_nv_n$.
Hence $S + T = I$ is inv.
Thus U is not closed under add and therefore is not a subsp. ☐

COMMENT: If $\dim V = 1$, then $U = \{0\}$ is a subsp of $\mathcal{L}(V)$.

---

3 *Suppose V is finite-dim, U is a subsp of V, and $S \in \mathcal{L}(U, V)$.*
Prove that there exists an inv $T \in \mathcal{L}(V, V)$ such that
$Tu = Su$ for every $u \in U$ if and only if S is inje.[*Compare this with (3.A.11).*]

SOLUTION:

(a) $Tu = Su$ for every $u \in U \Rightarrow u = T^{-1}Su \Rightarrow S$ is inje. OR. $\operatorname{null} S = \operatorname{null} T \cap U = \{0\} \cap U = \{0\}$.

(b) Suppose $(u_1, \dots, u_m)$ be a basis of U and S is inje $\Rightarrow (Su_1, \dots, Su_m)$ is linely inde in V.
Extend these to bases of V as $(u_1, \dots, u_m, v_1, \dots, v_n)$ and $(Su_1, \dots, Su_m, w_1, \dots, w_n)$.

Define $T \in \mathcal{L}(V)$ by $T(u_i) = Su_i$; $Tv_j = w_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.

---

4 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{null} S = \operatorname{null} T_{(=U)} \iff S = ET,\ \exists$ *inv* $E \in \mathcal{L}(W)$.

SOLUTION:

Define $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i$, $E(w_j) = x_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $(Tv_1, \dots, Tv_m)$ be a basis of range T, extend it to a basis of W as $(Tv_1, \dots, Tv_m, w_1, \dots, w_n)$.
Let $(u_1, \dots, u_n)$ be a basis of U. Then by (3.B.TIPS), $(v_1, \dots, v_m, u_1, \dots, u_n)$ is a basis of V.
又 $\operatorname{null} S = \operatorname{null} T \Rightarrow V = \operatorname{span}(v_1, \dots, v_m) \oplus \operatorname{null} S \Rightarrow \operatorname{span}(Sv_1, \dots, Sv_m) = \operatorname{range} S$.
And $\dim \operatorname{range} T = \dim \operatorname{range} S = \dim V - \operatorname{null} U = m$. Hence $(Sv_1, \dots, Sv_m)$ is a basis of range S.
Thus we let $(Sv_1, \dots, Sv_m, x_1, \dots, x_n)$ be a basis of W.

$\therefore E$ is inv and $S = ET$.

Conversely, $S = ET \Rightarrow \operatorname{null} S = \operatorname{null} ET$.
Then $v \in \operatorname{null} ET \iff ET(v) = 0 \iff Tv = 0 \iff v \in \operatorname{null} T$. Hence $\operatorname{null} ET = \operatorname{null} T = \operatorname{null} S$. □

---

5 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{range} S = \operatorname{range} T_{(=R)} \iff S = TE,\ \exists$ *inv* $E \in \mathcal{L}(V)$.

SOLUTION:

Define $E \in \mathcal{L}(V)$ as $E: \ v_i \mapsto r_i\ ;\quad u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $(Tv_1, \dots, Tv_m)$ and $(Sr_1, \dots, Sr_m)$ be bases of R such that $\forall\, i, Tv_i = Sr_i$.
Let $(u_1, \dots, u_n)$ and $(s_1, \dots, s_n)$ be bases of null T and null S respectively.
Thus $(v_1, \dots, v_m, u_1, \dots, u_n)$ and $(r_1, \dots, r_m, s_1, \dots, s_n)$ are bases of V.

$\therefore E$ is inv and $S = TE$.

Conversely, $S = TE \Rightarrow \operatorname{range} S = \operatorname{range} TE$.
Then $w \in \operatorname{range} S \iff \exists\, v \in V, Sv = TE(v) = T(E(v)) = w \in \operatorname{range} T$. Hence $\operatorname{range} S = \operatorname{range} T$. □

---

6 *Suppose V and W are finite-dim and $S, T \in \mathcal{L}(V, W)$.* $[\dim \operatorname{null} S = \dim \operatorname{null} T = n]$
Prove that $S = E_2 T E_1,\ \exists$ *inv* $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W) \iff \dim \operatorname{null} S = \dim \operatorname{null} T$.

SOLUTION:

Define $E_1: \ v_i \mapsto r_i\ ;\quad u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.
Define $E_2: Tv_i \mapsto Sr_i\ ;\ x_j \mapsto y_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

Let $(Tv_1, \dots, Tv_m)$ and $(Sr_1, \dots, Sr_m)$ be bases of range T and range S.
Let $(u_1, \dots, u_n)$ and $(s_1, \dots, s_n)$ be bases of null T and null S respectively.
Thus $(v_1, \dots, v_m, u_1, \dots, u_n)$ and $(r_1, \dots, r_m, s_1, \dots, s_n)$ are bases of V.
Extend $(Tv_1, \dots, Tv_m)$ and $(Sr_1, \dots, Sr_m)$ to bases of W as
$(Tv_1, \dots, Tv_m, x_1, \dots, x_p)$ and $(Sr_1, \dots, Sr_m, y_1, \dots, y_p)$.

$\therefore E_1, E_2$ are inv and $S = E_2 T E_1$.

Conversely, $S = E_2 T E_1 \Rightarrow \dim \operatorname{null} S = \dim \operatorname{null} E_2 T E_1$.
$v \in \operatorname{null} E_2 T E_1 \iff E_2 T E_1(v) = 0 \iff TE_1(v) = 0$. Hence $\operatorname{null} E_2 T E_1 = \operatorname{null} TE_1 = \operatorname{null} S$.
又 By (3.B.22.COROLLARY), E is inv $\Rightarrow \dim \operatorname{null} TE_1 = \dim \operatorname{null} T = \dim \operatorname{null} S$. □

---

8 *Suppose V is finite-dim and $T: V \to W$ is a surj linear map of V onto W.*
Prove that there is a subsp U of V such that $T|_U$ is an iso of U onto W.
$T|_U$ is the function whose domain is U, with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.

SOLUTION:

T is surj $\Rightarrow \operatorname{range} T = W \Rightarrow \dim \operatorname{range} T = \dim W = \dim V - \dim \operatorname{null} T$.

Let $(w_1, \dots, w_m)$ be a basis of range $T = W \Rightarrow \forall w_i, \exists!\, v_i \in V, Tv_i = w_i$.
$\Rightarrow (v_1, \dots, v_m)$ is a basis of $\mathcal{K}$. Thus $\dim \mathcal{K} = \dim W$.
Thus $T|_{\mathcal{K}}$ maps a basis of $\mathcal{K}$ to a basis of range $T = W$. Denote $\mathcal{K}$ by U. ☐

OR. By Problem (12) in (3.B), there is a subsp U of V such that
$U \cap \operatorname{null} T = \{0\} = \operatorname{null} T|_U$, $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$. ☐

• *Suppose V and W are finite-dim and U is a subsp of V.*
Let $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}$.
(a) *Show that $\mathcal{E}$ is a subsp of $\mathcal{L}(V, W)$.*
(b) *Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$ and $\dim U$.*
Hint: Define $\Phi : \mathcal{L}(V, W) \to L(U, W)$ by $\Phi(T) = T|_U$. What is $\operatorname{null} \Phi$? What is $\operatorname{range} \Phi$?

SOLUTION:
(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.
(b) Define Φ as in the hint.
Because $T \in \operatorname{null} \Phi \iff \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$.
Hence $\operatorname{null} \Phi = \mathcal{E}$.
Because $S \in \mathcal{L}(U, W) \Rightarrow \exists\, T \in \mathcal{L}(V, W), \Phi(T) = S$, by (3.B.11) $\Rightarrow S \in \operatorname{range} T$.
Hence $\operatorname{range} \Phi = \mathcal{L}(U, W)$.
Thus $\dim \operatorname{null} \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \operatorname{range} \Phi = (\dim V - \dim U) \dim W$. ☐

OR. Extend $(u_1, \dots, u_m)$ a basis of U to $(u_1, \dots, u_m, v_1, \dots, v_n)$ a basis of V. Let $p = \dim W$.
(See NOTE FOR [3.60])

$$\forall\, T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \operatorname{span} \underbrace{\left\{ \begin{matrix} E_{1,1}, & \cdots & , E_{m,1}, \\ \vdots & \ddots & \vdots \\ E_{1,p}, & \cdots & , E_{m,p} \end{matrix} \right\}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

$$又\ W = \operatorname{span} \left\{ \begin{matrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ \vdots & \ddots & \vdots \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{matrix} \right\} \subseteq \mathcal{E}. \text{ Where } \mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}.$$

Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. ☐

ENDED

3·E

2 *Suppose $V_1, \dots, V_m$ are vecsps such that $V_1 \times \cdots \times V_m$ is finite-dim.*
Prove that every V_j is finite-dim.

SOLUTION: Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.
Let $(v_1, \dots, v_M)$ be a basis of U. Note that $\forall u_i \in V_i, \in U_i \subseteq U$, for each i.

$$\left.\begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U) \text{ by } R_i(u_i) = (0, \dots, 0, u_i, 0, \dots, 0). \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \dots, u_i, \dots, u_m) = u_i \end{array}\right\} \Rightarrow S_i|_{U_i} = R_i^{-1}|_{U_i}.$$

Thus U_i and V_i are iso. 又 U_i is a subsp of a finite-dim vecsp U. ☐

---

3 *Give an example of a vecsp V and its two subsps U_1, U_2 such that*
$U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum.

SOLUTION:

NOTE *that at least one of U_1, U_2 must be infinite-dim.*
For if not, $U_1 \times U_2$ is finite-dim and $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$.
And V must be infinite-dim. For if not, both U_1 and U_2 are finite-dim subsps.

Let $V = \mathbf{F}^\infty = U_1, U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$.

$$\left.\begin{array}{l}\text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T\left((x_1, x_2, \cdots), (x, 0, \cdots)\right) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = \left((x_1, x_2, \cdots), (x, 0, \cdots)\right)\end{array}\right\} \Rightarrow S = T^{-1}. \qquad \square$$

---

4 *Suppose $V_1, \ldots, V_m$ are vecsps.*
Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.

SOLUTION: Using the notations in Problem (2).

Note that $T(u_1, \ldots, u_m) = T(u_1, 0, \ldots, 0) + \cdots + T(0, \ldots, u_m)$.

$$\left.\begin{array}{l}\text{Define } \varphi : T \mapsto (T_1, \ldots, T_m) \text{ by } \varphi(T) = (TR_1, \ldots, TR_m). \\ \text{Define } \psi : (T_1, \ldots, T_m) \mapsto T \text{ by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m.\end{array}\right\} \Rightarrow \psi = \varphi^{-1}. \qquad \square$$

---

5 *Suppose $W_1, \ldots, W_m$ are vecsps.*
Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.

SOLUTION: Using the notations in Problem (2).

Note that $Tv = (w_1, \ldots, w_m)$. Define $T_i \in \mathcal{L}(V, W_i)$ by $T_i(v) = w_i$.

$$\left.\begin{array}{l}\text{Define } \varphi : T \mapsto (T_1, \ldots, T_m) \text{ by } \varphi(T) = (S_1 T, \ldots, S_m T). \\ \text{Define } \psi : (T_1, \ldots, T_m) \mapsto T \text{ by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m.\end{array}\right\} \Rightarrow \psi = \varphi^{-1}. \qquad \square$$

---

6 *For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.*

SOLUTION:

Define $T : (v_1, \ldots, v_m) \to \varphi$, where $\varphi : (a_1, \ldots, a_m) \mapsto v$ is defined by $\varphi(a_1, \ldots, a_m) = a_1 v_1 + \cdots + a_m v_m$.

(a) Suppose $T(v_1, \ldots, v_m) = 0$. Then $\forall\ (a_1, \ldots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \ldots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$
$\Rightarrow (v_1, \ldots, v_m) = 0 \Rightarrow T$ is inje.

(b) Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \ldots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $\forall\ (b_1, \ldots, b_n) \in \mathbf{F}^m$,
$[T(\psi(e_1), \ldots, \psi(e_m))](b_1, \ldots, b_m) = b_1 \psi(e_1) + \cdots + b_m \psi(e_m) = \psi(b_1 e_1 + \cdots + b_m e_m) = \psi(b_1, \ldots, b_m)$.
Thus $T(\psi(e_1), \ldots, \psi(e_m)) = \psi$. Hence T is surj. $\square$

---

7 *Suppose $v, x \in V$ (arbitrary) and U and W are subsps of V.*
Suppose $v + U = x + W$. Prove that $U = W$.

SOLUTION:

(a) $\forall u \in U,\ \exists\, w \in W, v + u = x + w$, let $u = 0$, now $v = x + w \Rightarrow v - x \in W$.

(b) $\forall w \in W,\ \exists\, u \in U, v + u = x + w$, let $w = 0$, now $x = v + u \Rightarrow x - v \in U$.

Thus $\pm(v - x) \in U \cap W \Rightarrow \left\{\begin{array}{l} u = (x - v) + w \in W \Rightarrow U \subseteq W \\ w = (v - x) + u \in U \Rightarrow W \subseteq U \end{array}\right. \Rightarrow U = W.$ $\square$

• *Let $U = \left\{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\right\}$. Suppose $A \subseteq \mathbf{R}^3$.*
Prove that A is a translate of $U \iff \exists\, c \in \mathbf{R}, A = \left\{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\right\}$.
[Do it in your mind.]

• *Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $U = \{x \in V : Tx = c\}$ is either $\emptyset$ or is a translate of* $\operatorname{null} T$.

SOLUTION:

If $c \in W$ but $c \notin \operatorname{range} T$, then $U = \emptyset$ and we are done.

Suppose $c \in \operatorname{range} T$, then $\exists\ u \in V, Tu = c \Rightarrow u \in U$.
Suppose $y \in \operatorname{null} T \Rightarrow y + u \in U \iff T(y+u) = Ty + c = c$.
Thus $u + \operatorname{null} T \subseteq U$. Hence $u + \operatorname{null} T = U$,
for if not, suppose $z \notin u + \operatorname{null} T$ but $Tz = c (\Leftrightarrow z \in U)$,
then $\forall w \in \operatorname{null} T, z \neq u + w \Leftrightarrow z - u \notin \operatorname{null} T$.
又 $\tilde{T}(z + \operatorname{null} T) = \tilde{T}(u + \operatorname{null} T) \Rightarrow z + \operatorname{null} T = u + \operatorname{null} T \Rightarrow z - u \in \operatorname{null} T$, contradicts. ☐

COROLLARY: *The set of solutions to a system of linear equations such as [3.28] is either $\emptyset$ or a translate of the null subsp.*

---

8 *Suppose A is a nonempty subset of V.*
Prove that A is a translate of some subsp of $V \iff \lambda v + (1-\lambda)w \in A\ , \forall v, w \in A, \lambda \in \mathbf{F}$.

SOLUTION:

Suppose $A = a + U$, where U is a subsp of V. $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$,
$\lambda(a + u_1) + (1-\lambda)(a + u_2) = a + [\lambda(u_1 - u_2) + u_2] \in A$.
Suppose $\lambda v + (1-\lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $a \in A$ and let $A' = \{x - a : x \in A\}$.
Then $\forall x - a, y - a \in A', \lambda \in \mathbf{F}$,
(I) $\lambda(x-a) = [\lambda x + (1-\lambda)a] - a \in A'$. Then let $\lambda = 2$.
(II) $\lambda(x-a) + (1-\lambda)(y-a) = \frac{1}{2}(x-a) + \frac{1}{2}(y-a) = \frac{1}{2}x + (1-\frac{1}{2})(y) - a \in A'$.
By (I), $2 \times [\frac{1}{2}(x-a) + \frac{1}{2}(y-a)] = (x-a) + (y-a) \in A'$.
Thus A' is a subsp of V. Hence $a + A' = \{(x-a) + a : x \in A\} = A$ is a translate. ☐

---

9 *Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subsps U_1, U_2 of V.*
Prove that the intersection $A_1 \cap A_2$ is either a translate of some subsp of V or is $\emptyset$.

SOLUTION:

Suppose $v + u_1, w + u_2 \in A_1 \cap A_2 \neq \emptyset$. By Problem (8),
$\forall \lambda \in \mathbf{F}, \lambda(v + u_1) + (1-\lambda)(w + u_2) \in A_1$ and A_2. Thus $A_1 \cap A_2$ is a translate of some subsp of V. ☐

---

10 *Prove that the intersection of any collection of translates of subsps of V is either a translate of some subsp or $\emptyset$.*

SOLUTION:

Suppose $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collection of translates of subsps of V, where Γ is an arbitrary index set.
Suppose $x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset$, then by Problem (18), $\forall \lambda \in \mathbf{F}, \lambda x + (1-\lambda)y \in A_\alpha$ for every $\alpha \in \Gamma$.
Thus $\bigcap_{\alpha \in \Gamma} A_\alpha$ is a translate of some subsp of V. ☐

---

11 *Suppose $A = \left\{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^{m} \lambda_i = 1\right\}$, where each $v_i \in V, \lambda_i \in \mathbf{F}$.*
(a) *Prove that A is a translate of some subsp of V*
(b) *Prove that if B is a translate of some subsp of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.*
(c) *Prove that A is a translate of some subsp of V and* $\dim V < m$.

SOLUTION:

(a) By Problem (8), $\forall u, w \in A, \lambda \in \mathbf{F}$, $\exists a_i, b_i \mathbf{F}, \lambda u + (1-\lambda)w = \left(\lambda \sum_{i=1}^{m} a_i + (1-\lambda) \sum_{i=1}^{m} b_i\right) v_i \in A$. ☐
(b) Let $v = \lambda_1 v_+ \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on m by k.
(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.
$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By problem (8), $v \in B$.
(ii) $2 \leq k \leq m$, we assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. ($\forall \lambda_i$ such that $\sum_{i=1}^{k} \lambda_i = 1$)

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. $\forall i = 1, \ldots, k$, $\exists\, \mu_i \neq 1$, fix one such i by ι.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow (\sum_{i=1}^{k+1} \frac{\mu_i}{1-\mu_\iota}) - \frac{\mu_\iota}{1-\mu_\iota} = 1$.

Let $w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k\ terms}$.

Let $\lambda_i = \frac{\mu_i}{1-\mu_\iota}$ for $i = 1, \ldots, \iota - 1$; $\lambda_j = \frac{\mu_{j+1}}{1-\mu_\iota}$ for $j = \iota, \ldots, k$. Then,

$$\left.\begin{aligned} \sum_{i=1}^{k} \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{aligned}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B. \qquad \square$$

(c) Fix a $k \in \{1, \ldots, m\}$. Given $\lambda_i \in \mathbf{F}$ ($i \in \{1, \ldots, m\} \backslash \{k\}$).

Let $\lambda_k = 1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m$

Then $\lambda_1 v_1 + \cdots + \lambda_k v_k + \cdots + \lambda_m v_m = v_k + \sum_{i=1}^{m} \lambda_i (v_i - v_k)$.

Thus $A = v_k + \operatorname{span}(v_1 - v_k, \ldots, v_{k-1} - v_k, v_{k+1} - v_k, \ldots, v_m - v_k)$. $\square$

---

12 *Suppose U is a subsp of V such that V/U is finite-dim.*
Prove that is V is iso to $U \times (V/U)$.

Solution:

Let $(v_1 + U, \ldots, v_n + U)$ be a basis of V/U. Note that

$\forall v \in V,\ \exists\,!\ a_1, \ldots, a_n \in \mathbf{F}, v + U = \sum_{i=1}^{n} a_i (v_i + U) = (\sum_{i=1}^{n} a_i v_i) + U$

$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) = u \in U$ for some $u; v = \sum_{i=1}^{n} a_i v_i + u$.

Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ by $\varphi(v) = (u, \sum_{i=1}^{n} a_i v_i + U)$

and $\psi \in \mathcal{L}(U \times (V/U), V)$ by $\psi(u, w + U) = u + w; w = \sum_{i=1}^{n} b_i v_i + U$.

So that $\psi = \varphi^{-1}$. $\square$

• *Suppose $V = U \oplus W$, $(w_1, \ldots, w_m)$ is a basis of W.*
Prove that $(w_1 + U, \ldots, w_m + U)$ is a basis of V/U.

Solution:

Note that $\forall v \in V, \exists\,!\, u \in U, w \in W, v = u + w$ 又 $\exists\,!\, c_i \in \mathbf{F}$ such that $w = \sum_{i=1}^{m} c_i w_i \Rightarrow v = u + \sum_{i=1}^{m} c_i w_i$.

Thus $v + U = \sum_{i=1}^{m} c_i w_i + U \Rightarrow v + U \in \operatorname{span}(w_1 + U, \ldots, w_m + U) \Rightarrow V/U \subseteq \operatorname{span}(w_1 + U, \ldots, w_m + U)$.

Now suppose $a_1(w_1 + U) + \cdots + a_m(w_m + U) = 0 + U \Rightarrow \sum_{i=1}^{m} a_i w_i \in U$ while $U \cap W = \{0\}$.

Then $\sum_{i=1}^{m} a_i w_i = 0 \Rightarrow a_1 = \cdots = a_m = 0$. $\square$

---

13 *Suppose $(v_1 + U, \ldots, v_m + U)$ is a basis of V/U and $(u_1, \ldots, u_n)$ is a basis of U.*
Prove that $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ is a basis of V.

Solution:

By Problem (12), U and V/U are finite-dim $\Rightarrow U \times (V/U)$ is finite-dim, so is V.

$\dim V = \dim (U \times (V/U)) = \dim U + \dim V/U = m + n$.

Or. Note that $\forall v \in V, v + U = \sum_{i=1}^{m} a_i v_i + U, \exists\,!\, a_i \in \mathbf{F} \Rightarrow U \ni v - \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} b_i v_i, \exists\,!\, b_i \in \mathbf{F}$.

$\Rightarrow v \in \operatorname{span}(v_1, \ldots, v_m, u_1, \ldots, u_n)$.

又 Notice that $(\sum_{i=1}^{m} a_i v_i) + U = 0 + U (\Longleftrightarrow \sum_{i=1}^{m} a_i v_i \in U) \Longleftrightarrow a_1 = \cdots = a_m = 0$.

Hence $\operatorname{span}(v_1,\dots,v_m)\cap U=\{0\}\Rightarrow \operatorname{span}(v_1,\dots,v_m)\oplus U=V$
Thus $(v_1,\dots,v_m,u_1,\dots,u_n)$ is linely inde, so is a basis of V. $\square$

---

14 *Suppose* $U=\{(x_1,x_2,\dots)\in\mathbf{F}^\infty : x_k\neq 0$ *for only finitely many* $k\}$.
(a) *Show that* U *is a subsp of* $\mathbf{F}^\infty$. [*Do it in your mind*]
(b) *Prove that* $\mathbf{F}^\infty/U$ *is infinite-dim.*

Solution:

For $u=(x_1,\dots,x_p,\dots)\in\mathbf{F}^\infty$, denote x_p by $u[p]$. For each $r\in\mathbf{N}^+$.

Define $e_r[p]=\begin{cases}1\,,(p-1)\equiv 0\,(\operatorname{mod} r)\\ 0\,,\text{otherwise}\end{cases}$, simply $e_r=(1,\underbrace{0,\dots,0}_{(p-1)\ times},1,\underbrace{0,\dots,0}_{(p-1)\ times},1,\dots)\in\mathbf{F}^\infty$.

Choose $m\in\mathbf{N}^+$ arbitrarily.

Suppose $a_1(e_1+U)+\cdots+a_m(e_m+U)=(a_1e_1+\cdots+a_me_m)+U=0+U=0$.

$\Rightarrow a_1e_1+\cdots+a_me_m=u$ for some $u\in U$.

Then suppose $u=(x_1,\dots,x_t,0,\dots)\Rightarrow u[t+i]=0,\ \forall i\in\mathbf{N}^+$,

then let $j=s\cdot m!+1\geq t$ ($\exists\, s\in\mathbf{N}^+$) so that $e_1[j]=\cdots=e_m[j]=1,\ u[j+i]=0$.

Now we have: $u[j+i]=(\Sigma_{r=1}^m a_re_r)[j+i]=\Sigma_{r=1}^m a_re_r[s\cdot m!+1+i]=a_{i_1}+\cdots+a_{i_{\tau(i)}}=0$,

$\Rightarrow (\Sigma_{r=1}^m a_re_r)[j+i]=a_{i_1}+\cdots+a_{i_{\tau(i)}}=0.\ (\Delta)$

where $i_1,\dots,i_{\tau(i)}$ are distinct ordered factors of i ($1=i_1\leq\cdots\leq i_{\tau(i)}=i$).

$\Big($ Note that by definition, $e_r[s\cdot m!+1+i]=1\iff s\cdot m!+i\equiv i\equiv 0\,(\operatorname{mod} r)\iff r\mid i.$ $\Big)$

Let $i'=i_{\tau(i)-1}$. Notice that ${i'}_l=i_l,\forall l\in\{1,\dots,\tau(i')\}$; and $\tau(i')=\tau(i)-1$.

Again by (Δ), $(\Sigma_{r=1}^m a_re_r)[j+i']=a_{{i'}_1}+\cdots+a_{{i'}_{\tau(i')}}=a_{i_1}+\cdots+a_{i_{\tau(i)-1}}=0$.

Thus $a_{i_{\tau(i)}}=a_i=0$ for any $i\in\{1,\dots,m\}$.

Hence $(e_1,\dots,e_m)$ is linely inde in$\mathbf{F}^\infty$, so is $(e_1,\dots,e_m,\dots)$, since $m\in\mathbf{N}^+$.

又 $e_i\notin U\Rightarrow(e_1+U,e_2+U,\dots)$ is linely inde in $\mathbf{F}^\infty/U$. By [2.B.14]. $\square$

---

15 *Suppose* $\varphi\in\mathcal{L}(V,\mathbf{F})\backslash\{0\}$. *Prove that* $\dim V/(\operatorname{null}\varphi)=1$.

Solution: By [3.91] (d), $\dim\operatorname{range}\varphi=1=\dim V/(\operatorname{null}\varphi)$. $\square$

• **Note For [3.88, 3.90, 3.91]:**

For any $W\in\mathcal{S}_VU$, because $V=U\oplus W$. $\forall v\in V,\ \exists!\,u_v\in U,w_v\in W,v=u_v+w_v$.

Define $T\in\mathcal{L}(V,W)$ by $T(v)=w_v$. Hence $\operatorname{null}T=U$, $\operatorname{range}T=W$.

Then $\tilde{T}\in\mathcal{L}(V/\operatorname{null}T,W)$ is defined as $\tilde{T}(v+U)=Tv=w_v$.

Thus $\tilde{T}$ is inje (by [3.91(b)]) and surj ($\operatorname{range}\tilde{T}=\operatorname{range}T=W$),

and therefore is an iso. We conclude that V/U and W, namely any vecsp in $\mathcal{S}_V$, are iso.

---

16 *Suppose* $\dim V/U=1$. *Prove that* $\exists\ \varphi\in\mathcal{L}(V,\mathbf{F})$ *such that null* $\varphi=U$.

Solution:

Suppose V_0 is a subsp of V such that $V=U\oplus V_0$. Then V_0 and V/U are iso. $\dim V_0=1$.

Define a linear map $\varphi: v\mapsto\lambda$ by $\varphi(v_0)=1,\varphi(u)=0$, where $v_0\in V_0,u\in U$. $\square$

---

17 *Suppose* V/U *is finite-dim.* W *is a subsp of* V.
(a) *Show that if* $V=U+W$, *then* $\dim W\geq\dim V/U$.

(b) *Suppose* $\dim W = \dim V/U$ *and* $V = U \oplus W$. *Find such* W.

Solution: Let $(w_1, \dots, w_n)$ be a basis of W

(a) $\forall v \in V,\ \exists\ u \in U, w \in W$ such that $v = u + w \Rightarrow v + U = w + U$

Then $V/U \subseteq \operatorname{span}(w_1 + U, \dots, w_n + U) \Rightarrow V/U = \operatorname{span}(w_1 + U, \dots, w_n + U)$.

Hence $\dim V/U = \dim \operatorname{span}(w_1 + U, \dots, w_n + U) \leq \dim W$.

(b) Let $W \in \mathcal{S}_V U$. In other words,

reduce $(w_1{+}U, \dots, w_n{+}U)$ to a basis of V/U as $(w_1{+}U, \dots, w_m{+}U)$ and let $W = \operatorname{span}(w_1, \dots, w_m)$.$\square$

---

18 *Suppose* $T \in \mathcal{L}(V, W)$ *and* U *is a subsp of* V. *Let* π *denote the quotient map.* *Prove that* $\exists\ S \in \mathcal{L}(V/U, W)$ *such that* $T = S \circ \pi$ *if and only if* $U \subseteq \operatorname{null} T$.

Solution:

(a) Define $S \in \mathcal{L}(V/U, W)$ by $S(v + U) = Tv$. We have to check it is *well-defined*.

Suppose $v_1 + U = v_2 + U$, while $v_1 \neq v_2$.

Then $(v_1 - v_2) \in U \Rightarrow S\left((v_1 - v_2) + U\right) = T(v_1 - v_2) = 0 \Rightarrow Tv_1 = Tv_2$. Checked.

(b) Suppose $\exists\ S \in \mathcal{L}(V/U, W),\ T = S \circ \pi$.

Then $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null} T$. $\square$

---

20 *Define* $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ *by* $\Gamma(S) = S \circ \pi \ \big(= \pi'(S)\big)$.

(a) *Prove that* Γ *is linear:* By [3.9] distr properties and [3.6].

(b) *Prove that* Γ *is inje:*

$\Gamma(S) = 0 = S \circ \pi \iff \forall v \in V, S\left(\pi(v)\right) = 0 \iff \forall v + U \in V/U, S(v + U) = 0 \iff S = 0.$

(c) *Prove that* $\operatorname{range} \Gamma\ (= \operatorname{range} \pi'\ ) = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}$*:* By Problem (18). $\square$

Ended

3·F

• By (18) in (3.D) *we know that* $\varphi : V \to \mathcal{L}(\mathbf{F}, V)$ *is an iso. Now we prove that* $(v_1, \dots, v_m)$ *is linely inde* $\iff$ $(\varphi(v_1), \dots, \varphi(v_m))$ *is linely inde.*

Solution:

(a) Suppose $(v_1, \dots, v_m)$ is linely inde and $\vartheta \in \operatorname{span}\ (\varphi(v_1), \dots, \varphi(v_m))$.

Let $\vartheta = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m)$. Then $\vartheta(1) = 0 = a_1v_1 + \cdots + a_mv_m \Rightarrow a_1 = \cdots = a_m = 0$.

Or. Because φ is inje. Suppose $a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) = 0 = \varphi(a_1v_1 + \cdots + a_mv_m)$.

Then $a_1v_1 + \cdots + a_mv_m = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

Thus $(\varphi(v_1), \dots, \varphi(v_m))$ is linely inde.

(b) Suppose $(\varphi(v_1), \dots, \varphi(v_m))$ is linely inde and $v \in \operatorname{span}(v_1, \dots, v_m)$.

Let $v = 0 = a_1v_1 + \cdots + a_mv_m$. Then $\varphi(v) = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) = 0 \Rightarrow a_1 = \cdots = a_m = 0$.

Thus $v_1, \dots, v_m$ is linely inde. $\square$

---

1 *Explain why each linear functional is surj or is the zero map.*

Solution:
$$\left.\begin{aligned} \text{For any } \varphi \in V' \text{ and } \varphi \neq 0, \exists\, v \in V, \text{ such that } \varphi(v) \neq 0.\ \text{(a)} \\ \dim \operatorname{range} \varphi = \dim \mathbf{F} = 1.\ \text{(b)} \end{aligned}\right\} \Rightarrow \square$$

---

4 *Suppose* V *is finite-dim and* U *is a subsp of* V *such that* $U \neq V$. *Prove that* $\exists\, \varphi \in V'$ *and* $\varphi \neq 0$ *such that* $\varphi(u) = 0$ *for every* $u \in U$.

Solution:

Let $(u_1, \dots, u_m)$ be a basis of U, extend to $(u_1, \dots, u_m, u_{m+1}, \dots, u_{m+n})$ a basis of V.

Choose a $k \in \{1, \dots, n\}$. Define $\varphi \in V'$ by $\varphi(u_i) = \begin{cases} 1, & \text{if } i = m + k. \\ 0, & \text{otherwise.} \end{cases}$

OR. Equivalent to proving that $U^0 \neq \{0\}$. By [3.106], $\dim U^0 = \dim V - \dim U > 0$. □

• *Suppose $T \in \mathcal{L}(V, W)$ and $(w_1, \dots, w_m)$ is a basis of* range T.
Hence $\forall v \in V$, $Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$, $\exists!\, \varphi_1(v), \dots, \varphi_m(v)$,
thus defining functions $\varphi_1, \dots, \varphi_m$ from V to $\mathbf{F}$. Show that each $\varphi_i \in V'$.

SOLUTION:

For each w_i, $\exists\, v_i \in V$, $Tv_i = w_i$, getting a linely inde list $(v_1, \dots, v_m)$.

Now we have $Tv = a_1 Tv_1 + \cdots + a_m Tv_m$, $\forall v \in V, \exists!\, a_i \in \mathbf{F}$.

Let $(\psi_1, \dots, \psi_m)$ be the dual basis of range T. Then $(T'(\psi_i))(v) = \psi_i \circ T(v) = a_i$.
Thus letting $\varphi_i = \psi_i \circ T$. □

• *Suppose $\varphi, \beta \in V'$. Prove that* $\operatorname{null} \varphi \subseteq \operatorname{null} \beta$ *if and only if $\beta = c\varphi$. $\exists\, c \in \mathbf{F}$.*

SOLUTION: Using (3.B.29, 30)

(a) Suppose $\operatorname{null} \varphi \subseteq \operatorname{null} \beta$. Choose a $u \notin \operatorname{null} \beta$. $V = \operatorname{null} \beta \oplus \{au : a \in \mathbf{F}\}$.

If $\operatorname{null} \varphi = \operatorname{null} \beta$, then let $c = \frac{\beta(u)}{\varphi(u)}$, we are done.

Otherwise, suppose $u' \in \operatorname{null} \beta$, but $u' \notin \operatorname{null} \varphi$, then $V = \operatorname{null} \varphi \oplus \{bu' : b \in \mathbf{F}\}$.

$\forall v \in V, v = w + au = w' + bu'$, $\exists!\, w, w' \in \operatorname{null} \varphi, a, b \in \mathbf{F}$.

Thus $\beta(v) = a\beta(u)$, $\varphi(v) = b\varphi(u')$. Let $c = \frac{a\beta(u)}{b\varphi(u')}$. We are done

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$.

If $c = 0$, then $\operatorname{null} \beta = V \supseteq \operatorname{null} \varphi$, we are done.

Otherwise, $\left.\begin{array}{l} \forall v \in \operatorname{null} \varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \operatorname{null} \varphi \subseteq \operatorname{null} \beta. \\ \forall v \in \operatorname{null} \beta, \beta(v) = 0 = \varphi(v) \Rightarrow \operatorname{null} \beta \subseteq \operatorname{null} \varphi. \end{array}\right\} \Rightarrow \operatorname{null} \varphi = \operatorname{null} \beta$

$\Rightarrow \operatorname{null} \varphi \subseteq \operatorname{null} \beta$. □

---

5 *Prove that $(V_1 \times \cdots \times V_m)'$ and $V'_1 \times \cdots \times V'_m$ are iso.*

SOLUTION: Using notations in (3.E.2).

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to V'_1 \times \cdots \times V'_m \\ \quad \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = (R'_1(T), \dots, R'_m(T)). \\ \text{Define } \psi : V'_1 \times \cdots \times V'_m \to (V_1 \times \cdots \times V_m)' \\ \quad \text{by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m = S'_1(T_1) + \cdots + S'_m(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$ □

• *Suppose $(v_1, \dots, v_n)$ is a basis of V and $(\varphi_1, \dots, \varphi_n)$ is the dual basis of V'.*

$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

---

35 *Prove that $(\mathcal{P}(\mathbf{R}))'$ and $\mathbf{R}^\infty$ are iso.*

SOLUTION:

Define $\theta \in \mathcal{L}((\mathcal{P}(\mathbf{R}))', \mathbf{R}^\infty)$ by $\theta(\varphi) = (\varphi(1), \varphi(x), \dots, \varphi(x^n), \dots)$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall x^k$ in the basis $(1, x, \dots, x^n, \dots)$ of $\mathcal{P}_n(\mathbf{R})$ for any n, $\varphi(x^k) = 0 \Rightarrow \varphi = 0$.

Surj: $\forall\, (a_0, a_1, \dots, a_n, \dots) \in \mathbf{F}^\infty$, let ψ be such that $\psi(x^k) = a_k$ and thus $\theta(\psi) = (a_0, a_1, \dots, a_n, \dots)$.

Hence θ is an iso from $(\mathcal{P}(\mathbf{R}))'$ onto $\mathbf{R}^\infty$. □

---

7 *Suppose m is a positive integer. Show that the dual basis of the basis $(1, x, \dots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \dots, \varphi_m$, where $\varphi_k = \dfrac{p^{(k)}(0)}{k!}$.*

Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

SOLUTION:

For each j and k, $(x^j)^{(k)} = \begin{cases} j(j-1)\dots(j-k+1)\cdot x^{(j-k)}\ , & j \geq k. \\ j(j-1)\dots(j-j+1) = j!\ , & j = k. \\ 0\ , & j \leq k. \end{cases}$ Then $(x^j)^{(k)}(0) = \begin{cases} 0\ , & j \neq k. \\ k!\ , & j = k. \end{cases}$

Thus $\varphi_k = \psi_k$, where $\psi_1, \dots, \psi_m$ is the dual basis of $(1, x, \dots, x_m)$ of $\mathcal{P}_m(\mathbf{R})$. □

---

8 *Suppose m is a positive integer.*

SOLUTION:

(a) By [2.C.10], $B = (1, x-5, \dots, (x-5)^m)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) Let $\varphi_k = \dfrac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \dots, m$. Then $(\varphi_0, \varphi_1, \dots, \varphi_m)$ is the dual basis of B.

---

9 *Suppose $(v_1, \dots, v_n)$ is a basis of V and $(\varphi_1, \cdots, \varphi_n)$ is the correspd dual basis of V'. Suppose $\psi \in V'$. Prove that $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.*

SOLUTION: $\psi(v) = \psi(\sum\limits_{i=1}^{n} a_i v_i) = \sum\limits_{i=1}^{n} a_i \psi(v_i) = \sum\limits_{i=1}^{n} \psi(v_i)\varphi_i(v) = [\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n](v)$. □

COMMENT: For other basis $(u_1, \dots, u_n)$ and the dual basis $(\rho_1, \dots, \rho_n)$, $\psi = \psi(u_1)\rho_1 + \cdots + \psi(u_n)\rho_n$.

---

12 *Show that the dual map of the identity operator on V is the identity operator on V'.*

SOLUTION: $I'(\varphi) = \varphi \circ I = \varphi, \ \forall \varphi \in V'$. □

• *Suppose W is finite-dim and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \iff T = 0$.*

SOLUTION: $T = 0 \Leftrightarrow T'(\varphi) = \varphi \circ T = 0$ for all $\varphi \in V' \Leftrightarrow T' = 0$. □

---

13 *Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ denote the dual basis of the standard basis of $\mathbf{R}^2$ and $\mathbf{R}^3$.*

(a) *Describe the linear functionals $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$*

For any $(x, y, z) \in \mathbf{R}^3$, $(T'(\varphi_1))(x, y, z) = 4x + 5y + 6z$, $(T'(\varphi_2))(x, y, z) = 7x + 8y + 9z$.

(b) *Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.*

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$. □

---

14 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for each $x \in \mathbf{R}$.*

(a) *Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe $T'(\varphi) \in \mathcal{P}(\mathbf{R})'$.*

$(T'(\varphi))(p) = [x^2 p(x) + p''(x)]'(4) = [2xp(x) + x^2 p'(x) + p'''(x)](4) = 8p(4) + 16p'(4) + p'''(4)$.

(b) *Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p(x)\mathrm{d}x$. Evaluate $(T'(\varphi))(x^3)$.*

$(T'(\varphi))(x^3) = \int_0^1 (x^5 + 6x)\mathrm{d}x = \int_0^1 (\frac{1}{6}x^6 + 3x^2)'\mathrm{d}x = \frac{6}{19}$. □

• *Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$. Prove that T is inv $\iff T'$ is inv.*

SOLUTION: By [3.108] and [3.110]. □

---

16 *Suppose V and W are finite-dim. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(V, W)$. Prove that Γ is an iso of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.*

SOLUTION:

V, W are finite-dim $\Rightarrow \dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$. And by [3.101], Γ is linear.

又 Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus T is inje $\Rightarrow T$ is inv. □

---

17 *Suppose $U \subseteq V$. Explain why $U^0 = \{\varphi \in V' : U \subseteq null\varphi\}$.*

SOLUTION: Because for $\varphi \in V'$, $U \subseteq \operatorname{null} \varphi \iff \forall u \in U, \varphi(u) = 0$. By definition in [3.102]. □

---

18 *Suppose V is a vecsp and $U \subseteq V$.*

Then $U = \{0\} \iff \forall \varphi \in V', U \subseteq \operatorname{null}\varphi \iff U^0 = V'$.

---

19 *Suppose* V *is a vecsp and* $U \subseteq V$*. Prove that* $U = V \iff U^0_V = \{0\} = V^0_V$.

Solution:

(a) Suppose $U^0_V = \{0\}$. Then $U = V$.

(b)Suppose $U = V$, then $U^0_V = \{\varphi \in V' : V \subseteq \operatorname{null}\varphi\}$, hence $U^0_V = \{0\}$. □

20, 21 *Suppose* U *and* W *are subsets of* V*. Prove that* $U \subseteq W \iff W^0 \subseteq U^0$.

Solution:

(a) Suppose $U \subseteq W$. Then $\forall w \in W, u \in U, \varphi \in W^0, \varphi(w) = 0 = \varphi(u) \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.

(b) Suppose $W^0 \subseteq U^0$. Then $\varphi \in W^0 \Rightarrow \varphi \in U^0$. Hence $\operatorname{null}\varphi \supseteq W \Rightarrow \operatorname{null}\varphi \supseteq U$. Thus $W \supseteq U$. □

Corollary: $W^0 = U^0 \iff U = W$.

---

22 *Prove that* $(U + W)^0 = U^0 \cap W^0$.

Solution:

(a) $\left.\begin{array}{l} U \subseteq U + W \\ W \subseteq U + W \end{array}\right\} \Rightarrow \left.\begin{array}{l} (U+W)^0 \subseteq U^0 \\ (U+W)^0 \subseteq W^0 \end{array}\right\} \Rightarrow (U+W)^0 \subseteq U^0 \cap W^0.$

(b) $\forall \varphi \in U^0 \cap W^0, \varphi(u+w) = 0$, where $u \in U, w \in W \Rightarrow \varphi \in (U+W)^0$. Thus $(U+W)^0 \supseteq U^0 \cap W^0$. □

---

23 *Prove that* $(U \cap W)^0 = U^0 + W^0$.

Solution:

(a) $\left.\begin{array}{l} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array}\right\} \Rightarrow \left.\begin{array}{l} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array}\right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \supseteq U^0 \cap W^0.$

(b) $\forall \varphi \in U^0, \psi \in W^0$ and $\forall v \in U \cap W$, $(\varphi + \psi)(v) = \varphi(v) + \psi(v) = 0$. Thus $U^0 + W^0 \subseteq (U \cap W)^0$. □

• **Corollary:** *Suppose* $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ *is a collection of subsps of* V.
Then $(\sum_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \bigcap_{\alpha_i \in \Gamma} (V^0_{\alpha_i})$;
And $(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i})^0 = \sum_{\alpha_i \in \Gamma} (V^0_{\alpha_i})$.

---

24 *Suppose* V *is finite-dim and* U *is a subsp of* V.
Prove, using the pattern of [3.104], *that* $\dim U + \dim U^0 = \dim V$.

Solution:

Let $(u_1, \dots, u_m)$ be a basis of U, extend to a basis of V as $(u_1, \dots, u_m, \dots, u_n)$, and let $(\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$ be the dual basis.

(a) Suppose $\varphi \in \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, then $\exists\, a_i \in \mathbf{F}, \varphi = a_{m+1}\varphi_{m+1} + \cdots + a_n\varphi_n$.
For all $u \in U$, $\varphi(u) = 0$. Thus $\varphi \in U^0$, getting $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq U^0$.

(b) Suppose $\varphi \in U^0$, then $\exists\, a_i \in \mathbf{F}, \varphi = a_1\varphi_1 + \cdots + a_m\varphi_m + \cdots + a_n\varphi_n$.
For all $u_i \in U$, $0 = \varphi(u_i) = \sum_{i=1}^{n} \varphi(u_i) = a_i$. Then $\varphi = a_{m+1}\varphi_{m+1} + \cdots + a_n\varphi_n$.
Thus $\varphi \in \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, getting $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \supseteq U^0$.

Hence $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0$, $\dim U^0 = n - m = \dim V - \dim U$. □

---

25 *Suppose* U *is a subsp of* V*. Explain why* $U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\}$.

Solution: Note that $U = \{v \in V : v \in U\}$ is a subsp of V and $\varphi(v) = 0$ for every $\varphi \in U^0 \iff v \in U$. □

---

26 *Suppose V is finite-dim and Ω is a subsp of V'.*
Prove that $\Omega = \{v \in V : \varphi(v) = 0, \forall \varphi \in \Omega\}^0$.

Solution: Using the corollary in Problem (20, 21).

Suppose $U = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}$.

Getting $U^0 = \{v \in V : \forall \varphi \in \Omega, \varphi(v) = 0\}^0$. We need to show that $\Omega = U^0$.

$$\left.\begin{array}{l} \text{(a) } \forall \varphi \in \Omega, v \in U, \varphi(v) = 0 \Rightarrow \varphi \in U^0 \Rightarrow \Omega \subseteq U^0. \\ \text{(b) } v \in U \Leftrightarrow \begin{cases} \forall \varphi \in \Omega, \varphi(v) = 0 \\ \forall \psi \in U^0, \psi(v) = 0 \end{cases} \text{Thus } \Omega \supseteq U^0. \end{array}\right\} \Rightarrow \Box$$

---

27 *Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that* $\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$.

Solution:

By Problem (26), $\operatorname{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \psi(p) = 0, \forall \psi \in \operatorname{span}(\varphi)\}^0$,

Hence $\operatorname{span}(\varphi) = \{p \in \mathcal{P}_5(\mathbf{R}) : \varphi(p) = p(8) = 0\}^0$, 又 $\operatorname{span}(\varphi) = \operatorname{null} T' = (\operatorname{range} T)^0$.

By the corollary in Problem (20, 21), $\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$. $\Box$

28, 29 *Suppose V, W are finite-dim, $T \in \mathcal{L}(V, W)$.*
(a) *Suppose $\exists\, \varphi \in W'$ such that $\operatorname{null} T' = \operatorname{span}(\varphi)$. Prove that $\operatorname{range} T = \operatorname{null} \varphi$.*
(b) *Suppose $\exists\, \varphi \in V'$ such that $\operatorname{range} T' = \operatorname{span}(\varphi)$. Prove that $\operatorname{null} T = \operatorname{null} \varphi$.*

Solution: Using Problem (26), [3.107] and [3.109].

Because $\operatorname{span}(\varphi) = \{v \in V : \forall \psi \in \operatorname{span}(\varphi), \psi(v) = 0\}^0 = \{v \in V : \varphi(v) = 0\}^0 = (\operatorname{null} \varphi)^0$.

(a) $(\operatorname{range} T)^0 = \operatorname{null} T' = \operatorname{span}(\varphi) = (\operatorname{null} \varphi)^0 \Longleftrightarrow \operatorname{range} T = \operatorname{null} \varphi$.

(b) $(\operatorname{null} T)^0 = \operatorname{range} T' = \operatorname{span}(\varphi) = (\operatorname{null} \varphi)^0 \Longleftrightarrow \operatorname{null} T = \operatorname{null} \varphi$. $\Box$

---

31 *Suppose V is finite-dim and $(\varphi_1, \dots, \varphi_n)$ is a basis of V'.*
Show that there exists a basis of V whose dual basis is $(\varphi_1, \dots, \varphi_n)$.

Solution: Using Problem (29) and (30) in (3,B).

$\forall \varphi_i, \operatorname{null} \varphi_i \oplus \{a u_i : a \in \mathbf{F}\} = V$.

Because $\varphi_1, \dots, \varphi_m$ is linely inde. $\operatorname{null} \varphi_i \neq \operatorname{null} \varphi_j$ for each $i, j \in \mathbf{N}^+$ such that $i \neq j$.

Thus $(u_1, \dots, u_m)$ is linely inde, for if not, then $\exists\, i, j$ such that $\operatorname{null} \varphi_i = \operatorname{null} \varphi_j$, contradicts.

又 $\dim V' = m = \dim V$. Then $(u_1, \dots, u_m)$ is a basis of V whose dual basis is $(\varphi_1, \dots, \varphi_n)$. $\Box$.

• *Suppose V is finite-dim and $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following sets are the same.*
(a) $\operatorname{span}(\varphi_1, \dots, \varphi_m)$
(b) $((\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m))^0$
(c) $\{\varphi \in V' : (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m) \subseteq \operatorname{null} \varphi\}$

Solution: By Problem (17), (b) and (c) are equi. By Problem (26) and the corollary in Problem (23),

$$\left.\begin{array}{r} ((\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m))^0 = (\operatorname{null} \varphi_1)^0 + \dots + (\operatorname{null} \varphi_m)^0. \\ \text{又 } \operatorname{span}(\varphi_i) = \{v \in V : \forall \psi \in \operatorname{span}(\varphi_i), \psi(v) = 0\}^0 = (\operatorname{null} \varphi_i)^0. \end{array}\right\} \Rightarrow (a) = (b). \quad \Box$$

Corollary: **30** *Suppose V is finite-dim and $\varphi_1, \dots, \varphi_m$ is a linely inde list in V'.*
Then $\dim((\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m)) = (\dim V) - m$.

---

6 *Define $\Gamma : V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$, where $v_1, \dots, v_m \in V$.*
(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = V \Longleftrightarrow \Gamma$ *is inje.*

(b) *Show that* $(v_1, \dots, v_m)$ *is linely inde* $\Longleftrightarrow \Gamma$ *is surj.*

Solution:

$$\text{(a)}\begin{cases}\text{Suppose } \Gamma \text{ is inje. Then let } \Gamma(\varphi) = 0 \text{, getting } \varphi = 0 \Leftrightarrow \operatorname{null}\varphi = V = \operatorname{span}(v_1, \dots, v_m). \\ \text{Suppose } \operatorname{span}(v_1, \dots, v_m) = V. \text{ Then let } \Gamma(\varphi) = 0 \text{, getting } \varphi(v_i) = 0 \text{ for each } i, \\ \qquad \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m) = V \text{, thus } \varphi = 0, \Gamma \text{ is inje.}\end{cases}$$

$$\text{(b)}\begin{cases}\text{Suppose } \Gamma \text{ is surj. Then let } \Gamma(\varphi_i) = e_i \text{ for each } i \text{, where } (e_1, \dots, e_m) \text{ is the standard basis of } \mathbf{F}^m. \\ \quad \text{Then } (\varphi_1, \dots, \varphi_m) \text{ is linely inde, suppose } a_1v_1 + \dots + a_mv_m = 0, \\ \quad \text{then for each } i \text{, we have } \varphi_i(a_1v_1 + \dots + a_mv_m) = a_i = 0. \text{ Thus } v_1, \dots, v_n \text{ is linely inde.} \\ \text{Suppose } (v_1, \dots, v_m) \text{ is linely inde. Let } (\varphi_1, \dots, \varphi_m) \text{ be the dual basis of } \operatorname{span}(v_1, \dots, v_m). \\ \quad \text{Thus for each } (a_1, \dots, a_m) \in \mathbf{F}^m \text{, we have } \varphi = a_1\varphi_1 + \dots + a_m\varphi_m \text{ so that } \Gamma(\varphi) = (a_1, \dots, a_m). \square\end{cases}$$

• *Define* $\Gamma : V \to \mathbf{F}^m$ *by* $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$, *where* $\varphi_1, \dots, \varphi_m \in V'$.
(c) *Show that* $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \Longleftrightarrow \Gamma$ *is inje.*
(d) *Show that* $(\varphi_1, \dots, \varphi_m)$ *is linely inde* $\Longleftrightarrow \Gamma$ *is surj.*

Solution:

$$\text{(c)}\begin{cases}\text{Suppose } \Gamma \text{ is inje. Then } \Gamma(v) = 0 \Leftrightarrow \forall i, \varphi_i(v) = 0 \Leftrightarrow v \in (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) \Leftrightarrow v = 0. \\ \quad \text{Getting } (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) = \{0\}. \text{ By Problem } (\bullet) \text{ above, } \operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \\ \text{Suppose } \operatorname{span}(\varphi_1, \dots, \varphi_m) = V'. \text{ Again by Problem } (\bullet), (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) = \{0\}. \\ \quad \text{Thus } \Gamma(v) = 0 \Rightarrow \forall i, \varphi_i(v) = 0 \Rightarrow v = 0.\end{cases}$$

$$\text{(d)}\begin{cases}\text{Suppose } (\varphi_1, \dots, \varphi_m) \text{ is linely inde. Then by Problem (31), } (v_1, \dots, v_m) \text{ is linely inde.} \\ \quad \text{Thus for any } (a_1, \dots, a_m) \in \mathbf{F} \text{, by letting } v = \sum_{i=1}^m a_iv_i \text{, then } \varphi_i(v) = a_i \Rightarrow \Gamma(v) = (a_1, \dots, a_m). \\ \text{Suppose } \Gamma \text{ is surj. Let } e_1, \dots, e_m \text{ be a basis of } \mathbf{F}^m. \\ \quad \text{For every } e_i,\ \exists\, v_i \in V \text{ such that } \Gamma(v_i) = (\varphi_1(v_i), \dots, \varphi_m(v)) = e_i, \\ \quad \text{fix } v_i\ (\Rightarrow (v_1, \dots, v_m) \text{ is linely inde). Thus } \varphi_i(v_i) = 1, \varphi_i(v_j) = 0. \\ \quad \text{Hence } (\varphi_1, \dots, \varphi_m) \text{ is the dual basis of the basis } v_1, \dots, \varphi_m \text{ of } \operatorname{span}(v_1, \dots, v_m). \qquad \square\end{cases}$$

---

33 *Suppose* $A \in \mathbf{F}^{m,n}$. *Define* $T : A \to A^t$. *Prove that* T *is an iso of* $\mathbf{F}^{m,n}$ *onto* $\mathbf{F}^{n,m}$

Solution: By [3.111], T is linear. Note that $(A^t)^t = A$.

(a) For any $B \in \mathbf{F}^{n,m}$, let $A = B^t$ so that $T(A) = B$. Thus T is surj.

(b) If $T(A) = 0$ for some $A \in \mathbf{F}^{n,m}$, then $A = 0$. Thus T is inje,
for if not, $\exists\, j, k \in \mathbf{N}^+$ such that $A_{j,k} \neq 0$, then $T(A)_{k,j} \neq 0$, contradicts. $\square$

---

32 *Suppose* $T \in \mathcal{L}(V)$, *and* $(u_1, \dots, u_m)$ *and* $(v_1, \dots, v_m)$ *are bases of* V. *Prove that*
T *is inv* $\Longleftrightarrow$ *The rows of* $\mathcal{M}(T, (u_1, \dots, u_m), (v_1, \dots, v_m))$ *form a basis of* $\mathbf{F}^{1,n}$.

Solution: Note that T is inveritlbe $\Longleftrightarrow T'$ is inv. And $\mathcal{M}(T') = \mathcal{M}(T)^t = A^t$, denote it by B.
Let $(\varphi_1, \dots, \varphi_m)$ be the dual basis of $(v_1, \dots, v_m)$, $(\psi_1, \dots, \psi_m)$ be the dual basis of $(u_1, \dots, u_m)$.

(a) Suppose T is inv, so is T'. Because $T'(\varphi_1), \dots, T'(\varphi_m)$ is linely inde.
Noticing that $T'(\varphi_i) = B_{1,i}\psi_1 + \dots + B_{m,i}\psi_m$.
Thus the cols of B, namely the rows of A, are linely inde (check it by contradiction).

(b) Suppose the rows of A are linely inde, so are the cols of B.
Then $(T'(\varphi_1), \dots, T'(\varphi_m))$ is a basis of range T', namely V'. Thus T' is surj.
Hence T' is inv, so is T. $\square$

---

34 *The double dual space of* V, *denoted by* V'', *is defined to be the dual space of* V'.
In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. *Define* $\Lambda : V \to V''$ *by* $(\Lambda v)(\varphi) = \varphi(v)$.

(a) *Show that Λ is a linear map from V to V''.*
(b) *Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*
(c) *Show that if V is finite-dim, then Λ is an iso from V onto V''.*

Suppose V is finite-dim. Then V and V' are iso, but finding an iso from V onto V' generally requires choosing a basis of V. In contrast, the iso Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

Solution:

(a) $\forall \varphi \in V',\ \forall v, w \in V, a \in \mathbf{F}$, $(\Lambda(v+aw))(\varphi) = \varphi(v+aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.
Thus $\Lambda(v+aw) = \Lambda v + a\Lambda w$. Hence Λ is linear.

(b) $(T''(\Lambda v))(\varphi) = ((\Lambda v) \circ (T'))(\varphi) = (\Lambda v)(T'(\varphi)) = (T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = (\Lambda(Tv))(\varphi)$.
Hence $T''(\Lambda v) = (\Lambda(Tv))$, getting $T'' \circ \Lambda = \Lambda \circ T$.

(c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V'$, $(\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is inje.
又 Because V is finite-dim. $\dim V = \dim V' = \dim V''$. Hence Λ is an iso. ☐

---

36 *Suppose U is a subsp of V. Define $i : U \to V$ by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.*
(a) *Show that $\operatorname{null} i' = U^0$:* *null* $i' = (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$. ☐
(b) *Prove that if V is finite-dim, then* range $i' = U'$*:* $\operatorname{range} i' = (\operatorname{null} i)^0_U = (\{0\})^0_U = U'$. ☐
(c) *Prove that if V is finite-dim, then $\tilde{i'}$ is an iso from V'/U^0 onto U':*

The iso in (c) is natural in that it does not depend on a choice of basis in either vecsp.

Solution: Note that $\tilde{i'} : V'/\operatorname{null} i' \to \operatorname{range} i' \ \Rightarrow\ \tilde{i'} : V'/U^0 \to U'$. By (a), (b) and [3.91(d)]. ☐

---

37 *Suppose U is a subsp of V and π is the quotient map. Thus $\pi' \in \mathcal{L}\left((V/U)', V'\right)$.*
(a) *Show that π' is inje:* *Because π is surj. Use [3.108].* ☐
(b) *Show that $\pi' = U^0$.*
(c) *Conclude that π' is an iso from $(V/U)'$ onto U^0.*

The iso in (c) is natural in that it does not depend on a choice of basis in either vecsp.
In fact, there is no assumption here that any of these vecsps are finite-dim.

Solution: [3.109] is not available. Using (3.E.18), also see (3.E.20).

(b) $\psi \in \operatorname{range} \pi' \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi \iff \operatorname{null} \psi \supseteq U \iff \psi \in U^0$. Hence $\operatorname{range} \pi' = U^0$.
(c) $\psi \in U^0 \iff \operatorname{null} \psi \supseteq U \iff \exists \varphi \in (V/U)', \psi = \varphi \circ \pi = \pi(\varphi)$. Thus π' is surj. And by (a). ☐

Ended

4

- **Note For [4.8]:** *division algorithm for polynomials*

Suppose $p, s \in \mathcal{P}(\mathbf{F})$, with $s \neq 0$. Then $\exists!\, q, r \in \mathcal{P}(\mathbf{F})$ such that $p = sq + r$ and $\deg r < \deg s$. *Another Proof:*
Suppose $\deg p \geq \deg s$. Then $(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of length } \deg s}, \underbrace{s, zs, \cdots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)})$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.
Because $q \in \mathcal{P}(\mathbf{F})$, $\exists!\, a_i, b_j \in \mathbf{F}$,

$$q = a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s$$
$$= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1}}_{r} + s\,\underbrace{(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s})}_{q}.$$

With r, q as defined uniquely above, we are done. ☐

- **Note For [4.11]:** *each zero of a poly corresponds to a degree-one factor;* *Another Proof:*

First suppose $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, $\exists!\, a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.

Then $p(z) = p(z) - p(\lambda) = a_1(z-\lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.
Hence $\forall k \in \{1, \ldots, m\}, z^k - \lambda^k = (z-\lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.
Thus $p(z) = \sum\limits_{j=1}^{m} a_j(z-\lambda)\sum\limits_{i=1}^{k}\lambda^{i-1}z^{k-i} = (z-\lambda)\sum\limits_{j=1}^{m} a_j \sum\limits_{i=1}^{k}\lambda^{i-1}z^{k-i} = (z-\lambda)q(z)$.

- **NOTE FOR [4.13]:** *fundamental theorem of algebra, first version*
  Every nonconst poly with complex coefficients has a zero in $\mathbf{C}$. *Another Proof:*
  For any $w \in \mathbf{C}, k \in \mathbf{N}^+$, by polar coordinates, $\exists\, r \geq 0, \theta \in \mathbf{R}, r(\cos\theta + \mathrm{i}\sin\theta) = w$.
  By De Moivre' theorem, $w^k = [r(\cos\theta + \mathrm{i}\sin\theta)]^k = r^k(\cos k\theta + \mathrm{i}\sin k\theta)$.
  Hence $\left(r^{1/k}(\cos\frac{\theta}{k} + \mathrm{i}\sin\frac{\theta}{k})\right)^k = w$. Thus every complex number has a k^{th} *root*.

  Suppose a nonconst $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.
  Then $|p(z)| \to \infty$ as $|z| \to \infty$ (because $\dfrac{|p(z)|}{|z_m|} \to |c_m|$ as $|z| \to \infty$).
  Thus the continuous function $z \to |p(z)|$ has a global minimum at some point $\zeta \in \mathbf{C}$.
  To show that $p(\zeta) = 0$, assume $p(\zeta) \neq 0$. Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \dfrac{p(z+\zeta)}{p(\zeta)}$.
  The function $z \to |q(z)|$ has a global minimum value of 1 at $z = 0$.
  Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where $k \in \mathbf{N}^+$ is the smallest such that $a_k \neq 0$.
  Let $\beta \in \mathbf{C}$ be such that $\beta^k = -\dfrac{1}{a_k}$.
  There is a const $c > 1$ so that if $t \in (0,1)$, then $|q(t\beta)| \leq |1 + a_k t^k \beta^k| + t^{k+1}c = 1 - t^k(1 - tc)$.
  Now letting $t = 1/(2c)$, we get $|q(t\beta)| < 1$. Contradicts. Hence $p(\zeta) = 0$, as desired. □

- *Prove that if $w, z \in \mathbf{C}$, then $||w| - |z|| \leq |w - z|$.*

SOLUTION:
$$
\begin{aligned}
|w-z|^2 &= (w-z)(\overline{w} - \overline{z}) \\
&= |w|^2 + |z|^2 - (w\overline{z} + \overline{w}z) \\
&= |w|^2 + |z|^2 - (\overline{\overline{w}z} + \overline{w}z) \\
&= |w|^2 + |z|^2 - 2Re(\overline{w}z) \\
&\geq |w|^2 + |z|^2 - 2|\overline{w}z| \\
&= |w|^2 + |z|^2 - 2|w||z| = ||w| - |z||^2.
\end{aligned}
$$

Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides. □

- *Suppose V is on $\mathbf{C}$ and $\varphi \in V'$. Define $\sigma : V \to \mathbf{R}$ by $\sigma(v) = \operatorname{Re}\varphi(v)$ for each $v \in V$. Show that $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$ for all $v \in V$.*

SOLUTION:
Notice that $\varphi(v) = \operatorname{Re}\varphi(v) + \mathrm{i}\operatorname{Im}\varphi(v) = \sigma(v) + \mathrm{i}\operatorname{Im}\varphi(v)$.
又 $\operatorname{Re}\varphi(\mathrm{i}v) = \operatorname{Re}[\mathrm{i}\varphi(v)] = -\operatorname{Im}\varphi(v) = \sigma(\mathrm{i}v)$.
Hence $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$. □

---

2 *Suppose $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

SOLUTION:

$x^m, x^m + x^{m-1} \in U$ but $\deg\left[(x^m + x^{m-1}) - (x^m)\right] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$.

Hence U is not closed under add, and therefore is not a subsp. $\square$

---

3 *Suppose $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : 2 \mid \deg p\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

SOLUTION:

$x^2, x^2 + x \in U$ but $\deg[(x^2 + x) - (x^2)]$ is odd and hence $(x^2 + x) - (x^2) \notin U$.

Thus U is not closed under add, and therefore is not a subsp. $\square$

---

5 *Suppose that $m \in \mathbf{N}, z_1, \ldots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \ldots, w_{m+1} \in \mathbf{F}$. Prove that $\exists\, ! \, p \in \mathcal{P}_m(\mathbf{F})$ such that $p(z_k) = w_k$ for each $k = 1, \ldots, m+1$.*

SOLUTION:

Define $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by $Tq = (q(z_1), \ldots, q(z_m), q(z_{m+1}))$. As can be easily checked, T is linear.

We need to show that T is surj, so that such p exists; and that T is inje, so that such p is unique.

$Tq = 0 \iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0$

$\iff q = 0 \in \mathcal{P}_m(\mathbf{F})$, for if not, q of deg m has at least $m+1$ distinct roots. Contradicts [4.12].

$\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m + 1 = \dim \mathbf{F}^{m+1}$. 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1}$. Hence T is surj. $\square$

---

6 *Suppose $p \in \mathcal{P}_m(\mathbf{C})$ has degree m. Prove that p has m distinct zeros $\iff$ p and its derivative p' have no zeros in common.*

SOLUTION:

(a) Suppose p has m distinct zeros. By [4.14] and $\deg p = m$, let $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$, $\exists\, ! \, c, \lambda_i \in \mathbf{C}$.

For each $j \in \{1, \ldots, m\}$, let $\dfrac{p(z)}{(z - \lambda_j)} = q_j \in \mathcal{P}_{m-1}(\mathbf{C})$, then $p(z) = (z - \lambda_j) q_j(z)$ and $q_j(\lambda_j) \neq 0$.

$p'(z) = (z - \lambda_j) q_j{}'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

(b) To prove the implication on the other direction, we prove the contrapositive:

Suppose p has less than m distinct roots.

We must show that p and its derivative p' have at least one zero in common.

Let λ be a zero of p, then write $p(z) = (z - \lambda)^n q(z)$, $\exists\, ! \, n \in \mathbf{N}^+, q \in \mathcal{P}_{m-n}(\mathbf{C})$.

$p'(z) = (z - \lambda)^n q'(z) + n(z - \lambda)^{n-1} q(z) \Rightarrow p'(\lambda) = 0$, λ is a common root of p' and p. $\square$

---

7 *Prove that every $p \in \mathcal{P}(\mathbf{R})$ of odd degree has a zero.*

SOLUTION:

Using the notation and proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence λ_1 exists. $\square$

OR. Using calculus only.

Suppose $p \in \mathcal{P}_m(\mathbf{F})$, $\deg p = m$, m is odd.

Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1} a_m$ by δ

Write $p(x) = x^m \left(\dfrac{a_0}{x^m} + \dfrac{a_1}{x^{m-1}} + \cdots + \dfrac{a_{m-1}}{x} + a_m \right)$.

Thus $p(x)$ is continuous, and $\lim\limits_{x \to -\infty} p(x) = -\delta\infty$; $\lim\limits_{x \to \infty} p(x) = \delta\infty$.

Hence we conclude that p has at least one real zero. $\square$

---

8 *For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \to \mathbf{R}$ by* $(Tp)(x) = \begin{cases} \dfrac{p(x) - p(3)}{x - 3} & \textit{if } x \neq 3, \\ p'(3) & \textit{if } x = 3 \end{cases}$ *for all $x \in \mathbf{R}$.*
Show that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$ and that $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is a linear map.

SOLUTION:

For $x \neq 3, T(x^n) = \dfrac{x^n - 3^n}{x - 3} = \sum\limits_{i=1}^{n} 3^{i-1} x^{n-i}$.

For $x = 3, T(x^n) = 3^{n-1} \cdot n$. Note that if $x = 3$, then $\sum\limits_{i=1}^{n} 3^{i-1} x^{n-i} = \sum\limits_{i=1}^{n} 3^{n-1} = 3^{n-1} \cdot n$.

Hence for all $x \in \mathbf{R}$ and for all $n \in \mathbf{N}, T(x^n) = \sum\limits_{i=1}^{n} 3^{i-1} x^{n-i} \in \mathcal{P}(\mathbf{R})$.

Because T is linear, we conclude that $Tp \in \mathcal{P}(\mathbf{R})$ for all $p \in \mathcal{P}(\mathbf{R})$.

Now we show that T is linear:

$\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, T(p + \lambda q)(x) = \begin{cases} \dfrac{(p + \lambda q)(x) - (p + \lambda q)(3)}{x - 3} & \text{if } x \neq 3, \\ (p + \lambda q)'(3) & \text{if } x = 3 \end{cases}$ for all $x \in \mathbf{R}$.

Notice that $\begin{cases} (p + \lambda q)(x) - (p + \lambda q)(3) = (p(x) - p(3)) + (\lambda q(x) - \lambda q(3)). \\ (p + \lambda q)'(3) = p'(3) + \lambda q'(3). \end{cases}$

Thus $T(p + \lambda q)(x) = (T(p) + \lambda T(q))(x)$ for all $x \in \mathbf{R}$. □

---

9 *Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\overline{z})}$. Prove that $q \in \mathcal{P}(\mathbf{R})$.*

SOLUTION:

$p(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow p(\overline{z}) = a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0 \Rightarrow \overline{p(\overline{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\overline{z})} = \overline{p(\overline{z})}p(z) = p(\overline{z})\overline{p(\overline{\overline{z}})} = \overline{q(\overline{z})}$.

Hence letting $q(z) = c_m x^m + \cdots + c_1 x + c_0 \;\Rightarrow\; \overline{c_k} = c_k, c_k \in \mathbf{R}$ for each k. □

---

10 *Suppose $m \in \mathbf{N}$ and $p \in \mathcal{P}_m(\mathbf{C})$ such that $p(x_k) \in \mathbf{R}$ for each x_k, where $x_0, x_1, \ldots, x_m \in \mathbf{R}$ are distinct. Prove that $p \in \mathcal{P}(\mathbf{R})$.*

SOLUTION:

Let $p(x_k) = y_k$ for each k. By Problem (5), $\exists!\, q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_k) = y_k$. Hence $p = q$. □

OR. Using the Lagrange Interpolating Polynomial.

Define $q(x) = \sum\limits_{j=0}^{m} \dfrac{(x - x_0)(x - x_1)\cdots(x - x_{j-1})(x - x_{j+1})\cdots(x - x_m)}{(x_j - x_0)(x_j - x_1)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_m)} p(x_j)$.

又 For each j, $x_j, p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R}) \subseteq \mathcal{P}_m(\mathbf{C})$.

Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q - p)(x_k) = 0$ for each $k \in \{0, 1, \ldots, m\}$.

Then $(q - p)$ has $(m + 1)$ distinct zeros, while $(q - p) \in \mathcal{P}_m(\mathbf{C})$. Hence by [4.12], $q - p = 0 \Rightarrow p = q$. □

---

11 *Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*
(a) *Show that* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.
(b) *Find a basis of $\mathcal{P}(\mathbf{F})/U$.*

SOLUTION:

U is a subsp of $\mathcal{P}(\mathbf{F})$ because $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, pf + \lambda pg = p(f + \lambda g) \in U$.

NOTE: Define $P :\to \mathcal{P}(\mathbf{F})$ by $(Pq)(x) = p(q(x)) = (p \circ q)(x)$ $(\neq p(x)q(x))$. P is not linear.

(a) By [4.8], $\forall f \in \mathcal{P}(\mathbf{F}), \exists!\, q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r)$; $\deg r < \deg p$.
Hence $\forall f \in \mathcal{P}(\mathbf{F}), \exists!\, pq \in U, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), f = (pq) + (r); r \notin U$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F})$. Therefore $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p-1}(\mathbf{F})$ are iso.

OR. $\forall f \in \mathcal{P}(\mathbf{F}), \exists!\, q, r \in \mathcal{P}(\mathbf{F}), f = (p)q + (r);\ \deg r < \deg p$.
Define $R : \mathcal{P}(\mathbf{F}) \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $(Rf)(z) = r(z)$ for each $z \in \mathbf{F}$.
$\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, R(f + \lambda g)(z) = R(f) + \lambda R(g)$.
BECAUSE: $\forall f, g \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}$,

$\exists!\, q_1, r_1 \in \mathcal{P}(\mathbf{F}), f = (p)q_1 + (r_1),\ \deg r_1 < \deg p;$
$\exists!\, q_2, r_2 \in \mathcal{P}(\mathbf{F}), g = (p)q_2 + (r_2),\ \deg r_2 < \deg p;$
$\exists!\, q_3, r_3 \in \mathcal{P}(\mathbf{F}), \lambda g = (p)q_3 + (r_3) = (p)(\lambda q_2) + (\lambda r_2),\ \deg r_3 < \deg p$ and $\deg \lambda r_2 < \deg p$.
$\Rightarrow q_3 = \lambda q_2, r_3 = \lambda r_2$.
$\exists!\, q_0, r_0 \in \mathcal{P}(\mathbf{F}), (f + \lambda g) = (p)q_0 + (r_0)$
$= (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2),\ \deg r_0 < \deg p$ and $\deg(r_1 + \lambda r_2) < \deg p$.
$\Rightarrow q_1 + \lambda q_2 = q_0;\ r_1 + \lambda r_2 = r_0$.

Hence R is linear.
$R(f) = 0 \Longleftrightarrow f = pq, \exists!\, q \in \mathcal{P}(\mathbf{F})$. Thus $\operatorname{null} R = U$.
$\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F})$, let $f = p + r$, then $R(f) = r$. Thus $\operatorname{range} R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.
Finally, by [3.91(d)], $\mathcal{P}(\mathbf{F})/\operatorname{null} R$, namely $\mathcal{P}(\mathbf{F})/U$, and $\operatorname{range} R$, namely $\mathcal{P}_{\deg p-1}(\mathbf{F})$, are iso.

(b) $(1 + U, x + U, \dots, x^{\deg p-1}) + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. ☐

- *Suppose nonconst $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p$, $n = \deg q$. Use* (a)–(c) *below to prove that $\exists!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*
  (a) *Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r, s) = rp + sq$.*
  Show that the linear map T is inje.
  (b) *Show that the linear map T in* (a) *is surj.*
  (c) *Use* (b) *to conclude that $\exists!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*

SOLUTION:

(a) T is linear because $\forall\, r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,
$T\left((r_1, s_1) + \lambda(r_2, s_2)\right) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1, s_1) + \lambda T(r_2, s_2)$.

Suppose $T(r, s) = rp + sq = 0$. Notice that p, q have no zeros in common.
Then $r = s = 0$, for if not, write $\dfrac{q(z)}{r(z)} = \dfrac{p(z)}{s(z)}$, while for any zero λ of q, $\dfrac{q(\lambda)}{r(z)} = 0 \neq \dfrac{p(\lambda)}{s(z)}$. ☐

(b) $\dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) = \dim \mathcal{P}_{n-1}(\mathbf{C}) + \dim \mathcal{P}_{m-1}(\mathbf{C}) = n + m = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.
又 T is inje. Hence $\dim \operatorname{range} T = \dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) - \dim \operatorname{null} T = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$.
Thus $\operatorname{range} T = \mathcal{P}m + n - 1 \Rightarrow T$ is surj, and therefore is an iso.

(c) Immediately. ☐

ENDED

5.A

[1]: 31; [2]: 1, 2, 3, 15, 21; [3]: 23, (2E Ch5.20), (4E.5.A.37), 4, 5; [4]: 6, (4E.5.A.17, 18) OR 16, (4E.5.A.15); [5]: 7, 8, (4E.5.A.8), 22, 9, 10; [6]: 11, 12, 14, 30, 13, (4E.5.A.11); [7]: 17, (4E.5.A.16), 18; [8]: 19, 20, 24; [9]: 24′, 25, 26, 27, 28; [10]: (4E.5.A.39), 29; [11]: 32, (4E.5.A.35), (4E.5.A.38) OR 35, 36; [12] 32, 34.

- **NOTE FOR [5.6]:**
  More generally, suppose we do not know whether V is finite-dim. Then $(a) \Longleftrightarrow (b)$.
  Suppose (a) λ is an eigval of T with an eigvec v. Then $(T - \lambda I)v = 0$.

Hence we get (b), $(T-\lambda I)$ is not inje. And then (d), $(T-\lambda I)$ is not inv.
But $(d) \Rightarrow (b)$ fails (because S is not inv $\Longleftrightarrow$ S is not inje *or* S is not surj).

---

31 *Suppose V is finite-dim and $v_1, \dots, v_m \in V$. Prove that $(v_1, \dots, v_m)$ is linely inde $\Longleftrightarrow \exists\, T \in \mathcal{L}(V)$, $v_1, \dots, v_m$ are eigvecs of T correspd to distinct eigvals.*

SOLUTION:

Suppose $(v_1, \dots, v_m)$ is linely inde, extend it to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.
Define $T \in \mathcal{L}(V)$ by $Tv_k = kv_k$ for each $k \in \{1, \dots, m, \dots, n\}$. Conversely by [5.10]. ☐

---

1 *Suppose $T \in \mathcal{L}(V)$ and U is a subsp of V.*
(a) *Prove that if $U \subseteq \operatorname{null} T$, then U is invar under T.* $\forall u \in U \subseteq \operatorname{null} T, Tu = 0 \in U$. ☐
(b) *Prove that if $\operatorname{range} T \subseteq U$, then U is invar under T.* $\forall u \in U, Tu \in \operatorname{range} T \subseteq U$. ☐

• *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*
(a) *Prove that $\operatorname{null}(T-\lambda I)$ is invar under S for any $\lambda \in \mathbf{F}$.*
(b) *Prove that $\operatorname{range}(T-\lambda I)$ is invar under S for any $\lambda \in \mathbf{F}$.*

SOLUTION: Note that $ST = TS \Rightarrow (T-\lambda I)S = S(T-\lambda I)$.
(a) Suppose $v \in \operatorname{null}(T-\lambda I)$, then $(T-\lambda I)(Sv) = S(T-\lambda I)v = S(0) = 0$.
Hence $Sv \in \operatorname{null}(T-\lambda I)$ and therefore $\operatorname{null}(T-\lambda I)$ is invar under S.
(b) Suppose $v \in \operatorname{range}(T-\lambda I)$, therefore $\exists\, u \in V, (T-\lambda I)u = v$.
Then $Sv = S(T-\lambda I)u = (T-\lambda I)(Su) \in \operatorname{range}(T-\lambda I)$.
Hence $Sv \in \operatorname{range}(T-\lambda I)$ and therefore $\operatorname{range}(T-\lambda I)$ is invar under S. ☐

COMMENT: Reversing the roles of S and T, letting $\lambda = 0$, we conclude that
$\operatorname{null} S$ and $\operatorname{range} S$ is invar under T, which will be shown in Problem (2) and (3) below.

• *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*
2 *Show that $W = \operatorname{null} S$ is invar under T.* $\forall u \in W, Su = 0 \Rightarrow TSu = 0 = STu \Rightarrow Tu \in W$. ☐
3 *Show that $U = \operatorname{range} S$ is invar under T.* $\forall w \in U, \exists\, v \in V, Sv = w, STv = TSv = Tw \in U$. ☐

---

15 *Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is inv.*
(a) *Prove that T and $S^{-1}TS$ have the same eigvals.*
(b) *What is the relationship between the eigvecs of T and the eigvecs of $S^{-1}TS$?*

SOLUTION:

Suppose λ is an eigval of T with an eigvec v.
Then $S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v$.
Thus λ is also an eigval of $S^{-1}TS$ with an eigvec $S^{-1}v$.

Suppose λ is an eigval of $S^{-1}TS$ with an eigvec v.
Then $S(S^{-1}TS)v = TSv = \lambda Sv$.
Thus λ is also an eigval of T with an eigvec Sv. ☐

OR. Note that $S(S^{-1}TS)S^{-1} = T$. Hence every eigval of $S^{-1}TS$ is an eigval of $S(S^{-1}TS)S^{-1} = T$.
And every eigvec v of $S^{-1}TS$ is $S^{-1}v$, every eigvec u of T is Su. ☐

---

21 *Suppose $T \in \mathcal{L}(V)$ is inv.*
(a) *Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that λ is an eigval of $T \Longleftrightarrow \frac{1}{\lambda}$ is an eigval of T^{-1}.*

(b) *Prove that T and T^{-1} have the same eigvecs.*

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec v.
Then $T^{-1}Tv = \lambda T^{-1}v = v \Rightarrow T^{-1}v = \frac{1}{\lambda}v$. Hence $\frac{1}{\lambda}$ is an eigval of T^{-1}.

(b) Suppose $\frac{1}{\lambda}$ is an eigval of T^{-1} with an eigvec v.
Then $TT^{-1}v = v = \frac{1}{\lambda}Tv \Rightarrow Tv = \lambda v$. Hence λ is an eigval of T.
OR. Note that $(T^{-1})^{-1} = T$ and $\frac{1}{\frac{1}{\lambda}} = \lambda$. □

---

23 *Suppose V is finite-dim, $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigvals.*

SOLUTION:

Suppose λ is an eigval of ST with an eigvec v. Then $T(STv) = \lambda Tv = TS(Tv)$.

If $Tv \neq 0$, then λ is an eigval of TS.

Otherwise, $\lambda = 0$, ($v \neq 0, \lambda v = 0 = STv$), then T is not inv

$\Rightarrow TS$ is not inv $\Rightarrow (TS - 0I)$ is not inv $\Rightarrow \lambda = 0$ is an eigval of TS.

Reversing the roles of T and S, we conclude that ST and TS have the same eigvals. □

• (*2E Ch5.20*)

Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigvals and $S \in \mathcal{L}(V)$ has the same eigvecs (but might not with the same eigvals). Prove that $ST = TS$.

Let $n = \dim V$. For each $j \in \{1, \dots, n\}$, let v_j be an eigvec with eigval λ_j of T and α_j of S.

Then $(v_1, \dots, v_n)$ is a basis of V. Because $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$ for each j. Hence $ST = TS$. □

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*

Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(S) = TS$ for each $S \in \mathcal{L}(V)$.

Prove that the set of eigvals of T equals the set of eigvals of $\mathcal{A}$.

(a) Suppose $v_1, \dots, v_m$ are all linely inde eigvecs of T

with correspd eigvals $\lambda_1, \dots, \lambda_m$ respectively (possibly with repetitions).

Extend to a basis of V as $(v_1, \dots, v_m, \dots, v_n)$.

Then for each $k \in \{1, \dots, m\}$, $\operatorname{span}(v_k) \subseteq \operatorname{null}(T - \lambda_k I)$.

Define $S_k \in \mathcal{L}(V)$ by $S_k(v_j) = v_k$ for each $j \in \{1, \dots, n\}$,

so that $\operatorname{range} S_k = \operatorname{span}(v_k)$ for each $k \in \{1, \dots, m\}$, then $\mathcal{A}(S_k) = TS_k = \lambda_k S_k$.

Thus the eigvals of T are eigvals of $\mathcal{A}$.

(b) Suppose $\lambda_1, \dots, \lambda_m$ are all eigvals of $\mathcal{A}$ with eigvecs $S_1, \dots, S_m$ respectively.

Then for each $k \in \{1, \dots, m\}$, $\exists\, v \in V, 0 \neq u = S_k(v) \in V \Rightarrow Tu = (TS_k)v = (\lambda_k S_k)v = \lambda_k u$.

Thus the eigvals of $\mathcal{A}$ are eigvals of T. $\square$

Or.

(*a*) *Suppose λ is an eigval of T with an eigvec v.*

Let $v_1 = v$ and extend to a basis $(v_1, \dots, v_m)$ of V.

Define $S \in \mathcal{L}(V)$ by $Sv_1 = v_1,\ Sv_k = 0$ for $k \geq 2$.

Then $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$.

Hence $(T - \lambda I)S = 0 \Rightarrow TS = \lambda S$ while $S \neq 0$. Thus λ is also an eigval of $\mathcal{A}$.

(*b*) *Suppose λ is an eigval of $\mathcal{A}$ with an eigvec S. Then $(T - \lambda I)S = 0$ while $S \neq 0$.*

Hence $(T - \lambda I)$ is not inje. Thus λ is also an eigval of T. $\square$

Comment: *Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(S) = ST, \forall S \in \mathcal{L}(V)$. Then the eigvals of $\mathcal{B}$ are not the eigvals of T.*

---

4 *Suppose $T \in \mathcal{L}(V)$ and $V_1, \dots, V_m$ are invar subsps of V under T.*

Prove that $V_1 + \cdots + V_m$ is invar under T.

Solution: *For each $i = 1, \dots, m$, $\forall v_i \in V_i, Tv_i \in V_i$*

Hence $\forall v = v_1 + \cdots + v_m \in V_1 + \cdots + V_m, Tv = Tv_1 + \cdots + Tv_m \in V_1 + \cdots + V_m$. $\square$

---

5 *Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection*

Solution:

of subsps of V invar under T is invar under T.

Suppose $\{V_\alpha\}_{\alpha\in\Gamma}$ is a collection of subsps of V invar under T; here Γ is an arbitrary index set.

We need to prove that $\bigcap_{\alpha\in\Gamma} V_\alpha$, which equals the set of vectors

that are in V_α for each $\alpha \in \Gamma$, is invar under T.

For each $\alpha \in \Gamma$, $\forall v_\alpha \in V_\alpha, Tv_\alpha \in V_i$.

Hence $\forall v \in \bigcap_{\alpha\in\Gamma} V_\alpha, Tv \in V_\alpha, \forall \alpha \in \Gamma \Rightarrow Tv \in \bigcap_{\alpha\in\Gamma} V_\alpha$. Thus $\bigcap_{\alpha\in\Gamma} V_\alpha$ is invar under T. $\square$

---

6 *Prove or give a counterexample:*

Solution:

If V is finite-dim and U is a subsp of V that is invar under every operator on V,

then $U = \{0\}$ or $U = V$.

Notice that V might be $\{0\}$. In this case we are done. Suppose $\dim V \geq 1$. We prove by contrapositive:

Suppose $U \neq \{0\}$ and $U \neq V$, then $\exists\, T \in \mathcal{L}(V)$ such that U is not invar under T.

Let W be such that $V = U \oplus W$.

Let $(u_1, \dots, u_m)$ be a basis of U and $(w_1, \dots, w_n)$ be a basis of W.

Hence $(u_1, \dots, u_m, w_1, \dots, w_n)$ is a basis of V.

Define $T \in \mathcal{L}(V)$ by $T(a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n) = b_1w_1 + \cdots + b_nw_n$. $\square$

- *Suppose* $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.

  (a) (OR (9.11)) $\lambda \in \mathbf{R}$. *Prove that* λ *is an eigval of* $T \iff \lambda$ *is an eigval of* T_{C}.

  (b) (OR Problem (16)) $\lambda \in \mathbf{C}$. *Prove that* λ *is an eigval of* $T_{\mathrm{C}} \iff \overline{\lambda}$ *is an eigval of* T_{C}.

(a) *Suppose* $v \in V$ *is an eigvec correspd to the eigval* λ.

Then $Tv = \lambda v \Rightarrow T_{\mathrm{C}}(v + \mathrm{i}0) = Tv + \mathrm{i}T0 = \lambda v$.

Thus λ *is an eigval of* T.

Suppose $v + \mathrm{i}u \in V_{\mathrm{C}}$ *is an eigvec correspd to the eigval* λ.

Then $T_{\mathrm{C}}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \Rightarrow Tv = \lambda v, Tu = \lambda u$. (*Note that* v *or* u *might be zero*).

Thus λ *is an eigval of* T_{C}.

(b) *Suppose* λ *is an eigval of* T_{C} *with an eigvec* $v + \mathrm{i}u$.

Let $(v_1, \dots, v_n)$ *be a basis of* V. *Write* $v = \sum_{i=1}^{n} a_i v_i, u = \sum_{i=1}^{n} b_i v_i$, *where* $a_i, b_i \in \mathbf{R}$.

Then $T_{\mathrm{C}}(v + \mathrm{i}u) = Tv + \mathrm{i}Tu = \lambda v + \mathrm{i}\lambda u = \lambda \sum_{i=1}^{n} (a_i + \mathrm{i}b_i) v_i$. *Conjugating two sides, we have:*

$$\overline{T_{\mathrm{C}}(v + \mathrm{i}u)} = \overline{Tv + \mathrm{i}Tu} = \overline{Tv} - \mathrm{i}\overline{Tu} = Tv - \mathrm{i}Tu = T_{\mathrm{C}}(\overline{v + \mathrm{i}u}) = \overline{\lambda \sum_{i=1}^{n} (a_i + \mathrm{i}b_i) v_i} = \overline{\lambda} \sum_{i=1}^{n} (a_i - \mathrm{i}b_i) v_i.$$

Hence $\overline{\lambda}$ *is an eigval of* T_{C}. *To prove the other direction, notice that* $\overline{\overline{\lambda}} = \lambda$. □

- *Suppose* V *is finite-dim*, $T \in \mathcal{L}(V)$, *and* $\lambda \in \mathbf{F}$.

  Show that λ *is an eigval of* $T \iff \lambda$ *is an eigval of the dual operator* $T' \in \mathcal{L}(V')$.

(a) *Suppose* λ *is an eigval of* T *with an eigvec* v.

Then $(T - \lambda I_V)$ *is not inv.* 又 V *is finite-dim.*

Thus by [3.108, 110], [3.101] *and Problem* (12) *in* (3.F), $(T - \lambda I_V)' = T' - \lambda I_{V'}$ *is not inv.*

Hence λ *is an eigval of* T'.

(b) *Suppose* λ *is an eigval* T' *with an eigvec* ψ. *Then* $T'(\psi) = \psi \circ T = \lambda \psi$.

又 $\psi \neq 0 \Rightarrow \exists\, v \in V$ *such that* $\psi(v) \neq 0$. *Note that* $\psi(Tv) = \lambda \psi(v)$.

Thus $\lambda = \dfrac{\psi(Tv)}{\psi(v)} \Rightarrow Tv = \dfrac{\psi(Tv)}{\psi(v)} v = \lambda v$. *Hence* λ *is an eigval of* T. □

7 *Suppose* $T \in \mathcal{L}(\mathbf{R}^2)$ *is defined by* $T(x, y) = (-3y, x)$. *Find the eigvals of* T.

Solution:

Suppose $\lambda \in \mathbf{R}$ *and* $(x, y) \in \mathbf{R}^2 \backslash \{0\}$ *such that* $T(x, y) = (-3y, x) = \lambda(x, y)$. *Then* $-3y = \lambda x$ *and* $x = \lambda y$.

Thus $-3y = \lambda^2 y \Rightarrow \lambda^2 = -3$, *ignoring the possibility of* $y = 0$ (*because if* $y = 0$, *then* $x = 0$).

Hence the set of solution for this equation is $\emptyset$, *and therefore* T *has no eigvals in* $\mathbf{R}$. $\square$

---

8 *Define* $T \in \mathcal{L}(\mathbf{F}^2)$ *by* $T(w,z) = (z,w)$. *Find all eigvals and eigvecs of* T.

Solution:

Suppose $\lambda \in \mathbf{F}$ *and* $(w,z) \in \mathbf{F}^2$ *such that* $T(w,z) = (z,w) = \lambda(w,z)$. *Then* $z = \lambda w$ *and* $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, *ignoring the possibility of* $z = 0$ ($z = 0 \Rightarrow w = 0$).

Hence $\lambda_1 = -1$ *and* $\lambda_2 = 1$ *are all eigvals of* T. *For* $\lambda_1 = -1, z = -w, w = -z$; *For* $\lambda_2 = 1, z = w$.

Thus the set of all eigvecs is $\{(z,-z),(z,z) : z \in \mathbf{F} \wedge z \neq 0\}$. $\square$

- *Suppose* $P \in \mathcal{L}(V)$ *is such that* $P^2 = P$.

  Prove that if λ *is an eigval of* P, *then* $\lambda = 0$ *or* $\lambda = 1$.

Solution: (*See also at* (3.B), *just below Problem* (25), *where* (5.B.4) *is answered.*)

Suppose λ *is an eigval with an eigvec* v. *Then* $P(Pv) = Pv \Rightarrow \lambda^2 v = \lambda v$. *Thus* $\lambda = 1$ *or* 0. $\square$

---

22 *Suppose* $T \in \mathcal{L}(V)$ *and* $\exists$ *nonzero vectors* u, w *in* V *such that* $Tu = 3w$ *and* $Tw = 3u$.

Prove that 3 *or* -3 *is an eigval of* T.

Solution:

Comment: $Tu = 3w, Tw = 3u \Rightarrow T(Tu) = 9u \Rightarrow T^2$ *has an eigval* 9.

$Tu = 3w, Tw = 3u \Rightarrow T(u+w) = 3(u+w), T(u-w) = 3(w-u) = -3(u-w)$. $\square$

---

9 *Define* $T \in \mathcal{L}(\mathbf{F}^3)$ *by* $T(z_1,z_2,z_3) = (2z_2, 0, 5z_3)$.

Find all eigvals and eigvecs of T.

Solution:

Suppose λ *is an eigval of* T *with an eigvec* $(z_1,z_2,z_3) \in \mathbf{F}^3$.

Then $T(z_1,z_2,z_3) = (2z_2,0,5z_3) = \lambda(z_1,z_2,z_3)$. *Thus* $2z_2 = \lambda z_1, \quad 0 = \lambda z_2, \quad 5z_3 = \lambda z_3$.

We discuss in two cases:

For $\lambda = 0$, $z_2 = z_3 = 0$ *and* z_1 *can be arbitrary* ($z_1 \neq 0$).

For $\lambda \neq 0$, $z_2 = 0 = z_1$, *and* z_3 *can be arbitrary* ($z_3 \neq 0$), *then* $\lambda = 5$.

The set of all eigvecs is $\{(0,0,z),(z,0,0) : z \in \mathbf{F} \wedge z \neq 0\}$. $\square$

---

10 *Define* $T \in \mathcal{L}(\mathbf{F}^n)$ *by* $T(x_1,x_2,x_3,\dots,x_n) = (x_1,2x_2,3x_3,\dots,nx_n)$

Solution:

(*a*) *Find all eigvals and eigvecs of* T.

(*b*) *Find all invar subsps of* V *under* T.

(a) *Suppose* $v = (x_1, x_2, x_3, \dots, x_n)$ *is an eigvec of* T *with an eigval* λ.

Then $Tv = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$.

Hence $1, \dots, n$ *are eigvals of* T.

And $\{(0, \dots, 0, x_\lambda, 0, \dots, 0) \in \mathbf{F}^n : \lambda = 1, \dots, n,\ x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$ *is the set of all eigvecs of* T.

(b) *Let* $V_\lambda = \{(0, \dots, 0, x_\lambda, 0, \dots, 0) \in \mathbf{F}^n : x_\lambda \in \mathbf{F} \wedge x_\lambda \neq 0\}$. *Then* $V_1, \dots, V_n$ *are invar under* T.

Hence by Problem (4), *every sum of* $V_1, \dots, V_n$ *is a invar subsp of* V *under* T. ☐

---

11 *Define* $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ *by* $Tp = p'$. *Find all eigvals and eigvecs of* T.

SOLUTION:

Note that in general, $\deg p' < \deg p$ ($\deg 0 = -\infty$).

Suppose λ *is an eigval of* T *with an eigvec* p.

Suppose $\lambda \neq 0$. *Then* $\deg \lambda p > \deg p'$ *while* $\lambda p \neq p'$. *Contradicts. Thus* $\lambda = 0$.

Therefore $\deg \lambda p = -\infty = \deg p \Rightarrow p$ *is a nonzero const poly.*

Hence the set of all eigvecs is $\{C : C \in \mathbf{R} \wedge C \neq 0\} = \mathcal{P}_0(\mathbf{R}) \backslash \{0\}$. ☐

---

12 *Define* $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ *by* $(Tp)(x) = xp'(x)$ *for all* $x \in \mathbf{R}$.

SOLUTION:

Find all eigvals and eigvecs of T.

Suppose λ *is an eigval of* T *with an eigvec* p, *then* $(Tp)(x) = xp'(x) = \lambda p(x)$.

Let $p = a_0 + a_1 x + \cdots + a_n x^n$.

Then $xp'(x) = a_1 x + 2a_2 x^2 + \cdots + n a_n x^n = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n$.

Similar to Problem (10), $0, 1, \dots, n$ *are eigvals of* T.

The set of all eigvecs of T *is* $\left\{cx^\lambda : \lambda = 0, 1, \dots, n,\ c \in \mathbf{F} \wedge c \neq 0\right\}$. ☐

---

30 *Suppose* $T \in \mathcal{L}(\mathbf{R}^3)$ *and* $-4, 5, \sqrt{7}$ *are eigvals of* T.

SOLUTION:

Prove that $\exists\, x \in \mathbf{R}^3$ *such that* $Tx - 9x = (-4, 5, \sqrt{7})$.

Because 9 *is not an eigval. Hence* $(T - 9I)$ *is surj.* ☐

---

14 *Suppose* $V = U \oplus W$, *where* U *and* W *are nonzero subsps of* V.

SOLUTION:

Define $P \in \mathcal{L}(V)$ *by* $P(u + w) = u$ *for each* $u \in U$ *and each* $w \in W$.

Find all eigvals and eigvecs of P.

Suppose λ *is an eigval of* P *with an eigvec* $(u + w)$.

Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$. *By* [1.44] *and* $V = U \oplus W$, $(\lambda - 1)u = \lambda w = 0$.

Thus if $\lambda = 1$, *then* $w = 0$; *if* $\lambda = 0$, *then* $u = 0$.

Hence the eigvals of P *are* 0 *and* 1, *the set of all eigvecs in* P *is* $U \cup W$. ☐

---

13 *Suppose* V *is finite-dim,* $T \in \mathcal{L}(V)$, *and* $\lambda \in \mathbf{F}$.

Solution:

Prove that $\exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}$ *and* $(T - \alpha I)$ *is inv.*

Let $\alpha_k \in \mathbf{F}$ *be such that* $|\alpha_k - \lambda| = \dfrac{1}{1000 + k}$ *for each* $k = 1, \ldots, \dim V + 1$.

Note that each $T \in \mathcal{L}(V)$ *has at most* $\dim V$ *distinct eigvals.*

Hence $\exists\, k = 1, \ldots, \dim V + 1$ *such that* α_k *is not an eigval of* T *and therefore* $(T - \alpha_k I)$ *is inv.* ☐

• *Suppose* V *is finite-dim,* $T \in \mathcal{L}(V)$, *and* $\lambda \in \mathbf{F}$.

Prove that $\exists\, \delta > 0$ *such that* $(T - \alpha I)$ *is inv for all* $\alpha \in \mathbf{F}$ *such that* $0 < |\alpha - \lambda| < \delta$.

If T *has no eigvals, then* $(T - \alpha I)$ *is inje for all* $\alpha \in \mathbf{F}$ *and we are done.*

Let $\delta > 0$ *be such that, for each eigval* λ_k, $\lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.

So that for all $\alpha \in \mathbf{F}$ *such that* $0 < |\alpha - \lambda| < \delta$, $(T - \alpha I)$ *is not inje.* ☐

---

17 *Give an example of an operator on* $\mathbf{R}^4$ *that has no (real) eigvals.*

Solution:

Solution:

Define $T \in \mathcal{L}(\mathbf{R}^4)$ *by* $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$. *Where* (e_1, e_2, e_3, e_4) *is the standard basis of* $\mathbf{R}^4$.

Suppose λ *is an eigval of* T *with an eigvec* (x, y, z, w).

Then $T(x, y, z, w) = \lambda(x, y, z, w) \Rightarrow \begin{cases} (1-\lambda)x + y + z + w = 0 \\ -x + (1-\lambda)y - z - w = 0 \\ 3x + 8y + (11-\lambda)z + 5w = 0 \\ 3x - 8y - 11z + (5-\lambda)w = 0 \end{cases}$

This linear equation has no solutions.

(*You can type it on* `https://zh.numberempire.com/equationsolver.php` *to check.*)

Or. *Define* $T \in \mathcal{L}(\mathbf{R}^4)$ *by* $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Suppose λ *is an eigval of* T *with an eigvec* (x, y, z, w).

Then $T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \Rightarrow \begin{cases} -y = \lambda x \\ x = \lambda y \\ -w = \lambda z \\ z = \lambda w \end{cases} \Rightarrow \begin{cases} -xy = \lambda^2 xy \\ -zw = \lambda^2 zw \end{cases}$

If $xy \neq 0$ *or* $zw \neq 0$, *then* $\lambda^2 = -1$, *we fail.*

Otherwise, $xy = 0 \Rightarrow x = y = 0$, *for if* $x \neq 0$, *then* $\lambda = 0 \Rightarrow x = 0$, *contradicts.*

Similarly, $y = z = w = 0$. *Then we fail.*

Thus T *has no eigvals.* ☐

• TODO *Suppose* $(v_1, \ldots, v_n)$ *is a basis of* V *and* $T \in \mathcal{L}(V)$, $\mathcal{M}(T, (v_1, \ldots, v_n)) = A$.

Prove that if λ *is an eigval of* T, *then* $|\lambda| \leq n \max\left\{ \left|A_{j,k}\right| : 1 \leq j, k \leq n \right\}$.

First we show that $|\lambda| = n \max\left\{ \left|A_{j,k}\right| : 1 \leq j, k \leq n \right\}$ *for some cases.*

Consider $A = \begin{pmatrix} k & \cdots & k \\ \vdots & \ddots & \vdots \\ k & \cdots & k \end{pmatrix}$. *Then* nk *is an eigval of* T *with an eigvec* $v_1 + \cdots + v_n$.

Now we show that if $|\lambda| \neq n \max\left\{\left|A_{j,k}\right| : 1 \leq j,k \leq n\right\}$, *then* $|\lambda| < n \max\left\{\left|A_{j,k}\right| : 1 \leq j,k \leq n\right\}$.

---

18 *Show that the forward shift operator* $T \in \mathcal{L}(\mathbf{F}^\infty)$

SOLUTION:

defined by $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$ *has no eigvals.*

Suppose λ *is an eigval of* T *with an eigvec* $(z_1, z_2, \dots)$.

Then $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$.

Thus $\lambda z_1 = 0, \lambda z_2 = z_1, \dots, \lambda z_k = z_{k-1}, \dots$.

Let $\lambda = 0$, *then* $\lambda z_2 = z_1 = 0 = \lambda z_k = z_{k-1}$, *therefore* $(z_1, z_2, \dots) = 0$ *is not an eigvec.*

Suppose $\lambda \neq 0$. *Then* $\lambda z_1 = 0 \Rightarrow z_1 = 0 \Rightarrow z_2 = 0 = z_k$ *for all* $k \in \mathbf{N}^+$.

And then $(z_1, z_2, \dots) = 0$ *is not an eigvec. Hence* T *has no eigvals.* □

---

19 *Suppose* $n \in \mathbf{N}^+$. *Define* $T \in \mathcal{L}(\mathbf{F}^n)$ *by*

SOLUTION:

$$T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n).$$

In other words, the entries of $\mathcal{M}(T)$ *with resp to the standard basis are all* 1*'s.*

Find all eigvals and eigvecs of T.

Suppose λ *is an eigval of* T *with an eigvec* $(x_1, \dots, x_n)$.

Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.

Thus $\lambda x_1 = \cdots = \lambda x_n = x_1 + \cdots + x_n$.

For $\lambda = 0$, $x_1 + \cdots + x_n = 0$.

For $\lambda \neq 0$, $x_1 = \cdots = x_n$ *and then* $\lambda x_k = n x_k$ *for each* k.

Hence $0, n$ *are eigvecs of* T.

And the set of all eigvecs of T *is* $\{(x_1, \dots, x_n) \in \mathbf{F}^n : x_1 + \cdots + x_n = 0 \vee x_1 = \cdots = x_n\}$. □

---

20 *Define the backward shift operator* $S \in \mathcal{L}(\mathbf{F}^\infty)$ *by* $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.

SOLUTION:

(a) *Show that every element of* $\mathbf{F}$ *is an eigval of* S.

(b) *Find all eigvecs of* S.

Suppose λ *is an eigval of* S *with an eigvec* $(z_1, z_2, \dots)$.

Then $S(z_1, z_2, z_3 \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$.

Thus $\lambda z_1 = z_2, \lambda z_2 = z_3, \dots, \lambda z_k = z_{k+1}, \dots$.

For $\lambda = 0$, $\lambda z_1 = z_2 = 0 = \lambda z_2 = z_3 = \cdots = z_k$ *for all* k.

While z_1 *can be arbitrary, so that* $(z_1, 0, \dots)$ *is an eigvec with* $z_1 \neq 0$.

For $\lambda \neq 0$, $\lambda^k z_1 = \lambda^{k-1} z_2 = \cdots = \lambda z_k = z_{k+1}$ *for all* k.

Then $(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots)$ *is an eigvec with* $z_1 \neq 0$.

Hence (a) *each element of* $\lambda \in \mathbf{F}$ *is an eigval of* T.

And (b) *the set of all eigvecs of* T *is* $\left\{(z_1, \lambda z_1, \lambda^2 z_1, \dots, \lambda^k z_1, \dots) \in \mathbf{F}^\infty : \lambda \in \mathbf{F}, z_1 \neq 0\right\}$ □

---

24 *Suppose* $A \in \mathbf{F}^{n,n}$. *Define* $T \in \mathcal{L}(\mathbf{F}^n)$ *by* $Tx = Ax$,

where elements of $\mathbf{F}^n$ *are thought of as n-by-1 col vectors.*

(a) *Suppose the sum of the entries in each row of* A *equals* 1.

Prove that 1 *is an eigval of* T.

(b) *Suppose the sum of the entries in each col of* A *equals* 1.

Prove that 1 *is an eigval of* T.

Solution:

(a) *Suppose* λ *is an eigval of* T *with an eigvec* $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $Tx = Ax = \begin{pmatrix} \sum\limits_{c=1}^{n} A_{1,c} x_c \\ \vdots \\ \sum\limits_{c=1}^{n} A_{n,c} x_c \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. *While* $\sum\limits_{r=1}^{n} A_{R,c} = 1$ *for each* $R = 1, \dots, n$.

Thus if we let $x_1 = \cdots = x_n$, *then* $\lambda = 1$, *and hence is an eigval of* T.

(b) *Suppose* λ *is an eigval of* T *with an eigvec* $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $Tx = Ax = \begin{pmatrix} \sum\limits_{r=1}^{n} A_{1,r} x_r \\ \vdots \\ \sum\limits_{r=1}^{n} A_{n,r} x_r \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. *While* $\sum\limits_{r=1}^{n} A_{r,C} = 1$ *for each* $C = 1, \dots, n$.

$$\begin{aligned} \text{Thus } \sum_{r=1}^{n} (Ax)_{r,\cdot} &= \sum_{r=1}^{n} (Ax)_{r,1} \\ &= \sum_{c=1}^{n} (A_{1,c} + \cdots + A_{n,c}) x_c = \sum_{c=1}^{n} x_c = \lambda \begin{pmatrix} x_1 \\ + \\ \vdots \\ + \\ x_n \end{pmatrix}. \end{aligned}$$

Hence $\lambda = 1$, *for all* x *such that* $\sum\limits_{c=1}^{n} x_{c,1} \neq 0$. □

Or. *Prove that* $(T - I)$ *is not inv, so that we can conclude* $\lambda = 1$ *is an eigval.*

Because $(T-I)x = (A - \mathcal{M}(I))\,x = \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^{n} A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

Then $y_1 + \cdots + y_n = \sum_{r=1}^{n}\sum_{c=1}^{n}(A_{r,c}x_c - x_r) = \sum_{c=1}^{n} x_c \sum_{r=1}^{n} A_{r,c} - \sum_{r=1}^{n} x_r = 0.$

Thus range $(T-I) \subseteq \{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbf{F}^n : y_1 + \cdots + y_n = 0\}$. *Hence* $(T-I)$ *is not surj.* □

• *Suppose* $A \in \mathbf{F}^{n,n}$. *Define* $T \in \mathcal{L}(\mathbf{F}^n)$ *by* $Tx = xA$,

where elements of $\mathbf{F}^n$ *are thought of as* 1*-by-*n *row vectors.*

(a) *Suppose the sum of the entries in each col of* A *equals* 1.

Prove that 1 *is an eigval of* T.

(b) *Suppose the sum of the entries in each row of* A *equals* 1.

Prove that 1 *is an eigval of* T.

(a) *Suppose* λ *is an eigval of* T *with an eigvec* $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.

Then $Tx = xA = \begin{pmatrix} \sum_{r=1}^{n} x_r A_{r,1} & \cdots & \sum_{r=1}^{n} x_r A_{r,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. *While* $\sum_{r=1}^{n} A_{r,C} = 1$ *for each* $C = 1, \ldots, n$.

Thus if we let $x_1 = \cdots = x_n$, *then* $\lambda = 1$, *hence is an eigval of* T.

(b) *Suppose* λ *is an eigval of* T *with an eigvec* $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.

Then $Tx = xA = \begin{pmatrix} \sum_{c=1}^{n} x_c A_{c,1} & \cdots & \sum_{c=1}^{n} x_c A_{c,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. *While* $\sum_{c=1}^{n} A_{R,c} = 1$ *for each* $R = 1, \ldots, n$.

Thus $\sum_{c=1}^{n} (xA)_{\cdot,c} = \sum_{c=1}^{n} (xA)_{1,c} = \sum_{c=1}^{n} (A_{c,1} + \cdots + A_{c,n})x_c = \sum_{c=1}^{n} x_c = \lambda \begin{pmatrix} x_1 & + & \cdots & + & x_n \end{pmatrix}$.

Hence $\lambda = 1$, *for all* x *such that* $\sum_{r=1}^{n} x_{1,r} \neq 0$. □

Or. Prove that $(T-I)$ *is not inv, so that we can conclude* $\lambda = 1$ *is an eigval.*

Because $(T-I)x = x\,(A - \mathcal{M}(I)) == \begin{pmatrix} \sum_{c=1}^{n} x_c A_{c,1} - x_1 & \cdots & \sum_{c=1}^{n} x_c A_{c,n} - x_n \end{pmatrix} = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}$.

Then $y_1 + \cdots + y_n = \sum_{c=1}^{n}\sum_{r=1}^{n}(x_r A_{r,c} - x_c) = \sum_{r=1}^{n} x_r \sum_{c=1}^{n} A_{r,c} - \sum_{c=1}^{n} x_c = 0.$

Thus range $(T-I) \subseteq \{ \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}. \in \mathbf{F}^n : y_1 + \cdots + y_n = 0\}$. *Hence* $(T-I)$ *is not surj.* □

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25 *Suppose* $T \in \mathcal{L}(V)$ *and* u, w *are eigvecs of* T

Solution:

such that $u + w$ *is also an eigvec of* T.

Prove that u *and* w *are eigvecs of* T *correspd to the same eigval.*

Suppose $\lambda_1, \lambda_2, \lambda_0$ *are eigvals of* T *correspd to* $u, w, u + w$ *respectively.*

Then $T(u+w) = \lambda_0(u+w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$.

Notice that $u, w, u+w$ *are nonzero.*

If (u, w) *is linely depe, then let* $w = cu$, *therefore*

$$\lambda_2 cu = Tw = cTu = \lambda_1 cu \qquad \Rightarrow \lambda_2 = \lambda_1.$$

$$\lambda_0(u+w) = T(u+w) = \lambda_1 u + \lambda_1 cu = \lambda_1(u+w) \qquad \Rightarrow \lambda_0 = \lambda_1.$$

Otherwise, $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. □

---

26 *Suppose* $T \in \mathcal{L}(V)$ *is such that every nonzero vector in* V *is an eigvec of* T.

SOLUTION:

Prove that T *is a scalar multi of the identity operator.*

Because $\forall v \in V, \exists! \lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Then for any two distinct nonzero vectors $v, w \in V$,

$T(v+w) = \lambda_{v+w}(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$.

If (v, w) *is linely inde, then let* $w = cv$, *therefore*

$$\lambda_v cv = cTv = Tw = \lambda_w w \qquad \Rightarrow \lambda_w = \lambda_v.$$

$$\lambda_{v+w}(v+w) = T(v+w) = Tv + Tw = \lambda_v(v+cv) \Rightarrow \lambda_{v+w} = \lambda_v.$$

Otherwise, $\lambda_v = \lambda_{v+w} = \lambda_w$. □

27, 28 *Suppose* V *is finite-dim and* $k \in \{1, \dots, \dim V - 1\}$.

SOLUTION:

Suppose $T \in \mathcal{L}(V)$ *is such that every subsp of* V *of dim* k *is invar under* T.

Prove that T *is a scalar multi of the identity operator.*

We prove the contrapositive:

If $T \neq \lambda I, \forall \lambda \in \mathbf{F}$, *then* $\exists$ *a subsp* U *of* V *such that* $\dim U = k$, *and* U *is invar under* T.

By Problem (26), $\exists\, v \in V$ *and* $v \neq 0$ *such that* v *is not an eigvec of* T.

Thus (v, Tv) *is linely inde. Extend to a basis of* V *as* $(v, Tv, u_1, \dots, u_n)$.

Let $U = \operatorname{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$ *is not an invar subsp of* V *under* T.

OR. Suppose $0 \neq v = v_1 \in V$ *and extend to a basis of* V *as* $(v_1, \dots, v_n)$.

Suppose $Tv_1 = c_1 v_1 + \cdots + c_n v_n$, $\exists! c_i \in \mathbf{F}$.

Consider a k-$\dim$ *subsp* $U = \operatorname{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$,

where $\alpha_j \in \{2, \dots, n\}$ *for each* j, *and* $\alpha_1, \dots, \alpha_{k-1}$ *are distinct.*

Because every subsp such U *is invar.*

Thus $Tv_1 = c_1 v_1 + \cdots + c_n v_n \in U$

$\Rightarrow c_2 = \cdots = c_n = 0$,

for if not, for each $c_i \neq 0$, *choose* U_i *such that* $\alpha_j \in \{\underbrace{2, \dots, i-1, i+1, \dots, n}_{\text{length } (n-2)}\}$ *for each* j, *hence for* $Tv_1 = c_1v_1 + \cdots + c_{i-1}v_{i-1} + c_{i+1}v_{i+1} + \cdots + c_nv_n \in U_i$, *we conclude that* $c_i = 0$.

$\Rightarrow Tv_1 = c_1v_1$, 又 $v_1 = v \in V$ *is arbitrary* $\Rightarrow T = \lambda I$ *for some* λ. □

• *Suppose* V *is finite-dim and* $T \in \mathcal{L}(V)$. *Prove that*
T *has an eigval* $\Longleftrightarrow \exists$ *a subsp* U *of* V
such that $\dim U = \dim V - 1$, U *is invar under* T.

(*a*) *Suppose* λ *is an eigval of* T *with an eigvec* v.

(*If* $\dim V = 1$, *then* $U = \{0\}$ *and we are done.*)

Extend $v_1 = v$ *to a basis of* V *as* $(v_1, v_2 \ldots, v_n)$.

Step 1 *If* $\exists\, w_1 \in \operatorname{span}(v_2, \ldots, v_n)$ *such that* $0 \neq Tw_1 \in \operatorname{span}(v_1)$,

then extend $w_1 = \alpha_{1,1}$ *to a basis of* $\operatorname{span}(v_2, \ldots, v_n)$ *as* $(\alpha_{1,1}, \ldots, \alpha_{1,n-1})$.

Otherwise, we stop at step 1.

⋮

Step k *If* $\exists\, w_k \in \operatorname{span}(\alpha_{k-1,2}, \ldots, \alpha_{k-1,n-k+1})$ *such that* $0 \neq Tw_k \in \operatorname{span}(v_1, w_1, \ldots, w_{k-1})$,

then extend $w_k = \alpha_{k,1}$ *to a basis of* $\operatorname{span}(\alpha_{k-1,2}, \ldots, \alpha_{k-1,n-k+1})$ *as* $(\alpha_{k,1}, \ldots, \alpha_{k,n-k})$.

Otherwise, we stop at step k.

⋮

Finally, we stop at step m, *thus we get* $(v_1, w_1, \ldots, w_{m-1})$ *and* $(\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1})$,

$\operatorname{range} T|_{\operatorname{span}(w_1,\ldots,w_{m-1})} = \operatorname{span}(v_1, w_1, \ldots, w_{m-2}) \Rightarrow \dim \operatorname{null} T|_{\operatorname{span}(w_1,\ldots,w_{m-1})} = 0$,

$\operatorname{span}(\underbrace{v_1, w_1, \ldots, w_{m-1}}_{\textit{length} \dim m})$ *and* $\operatorname{span}(\underbrace{\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1}}_{\textit{length} \dim (n-m)})$ *are invar under* T.

Let $U = \operatorname{span}(\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1}) \oplus \operatorname{span}(v_1, w_1, \ldots, w_{m-2})$ *and we are done.* □

COMMENT: *Both* $\operatorname{span}(v_2, \ldots, v_n)$ *and* $\operatorname{span}(\alpha_{m-1,2}, \ldots, \alpha_{m-1,n-m+1}) \oplus \operatorname{span}(w_1, \ldots, w_{m-1})$ *are in* $\mathcal{S}_V \operatorname{span}(v_1)$.

(*b*) *Suppose* U *is an invar subpsace of* V *under* T *with* $\dim U = m = \dim V - 1$.

(*If* $m = 0$, *then* $\dim V = 1$ *and we are done*).

Let $(u_1, \ldots, u_m)$ *be a basis of* U, *extend to a basis of* V *as* $(u_0, u_1, \ldots, u_m)$.

We discuss in cases:

For $Tu_0 \in U$, *then* $\operatorname{range} T = U$ *so that* T *is not surj* $\Longleftrightarrow \operatorname{null} T \neq \{0\} \Longleftrightarrow 0$ *is an eigval of* T.

For $Tu_0 \notin U$, *then* $Tu_0 = a_0u_0 + a_1u_1 + \cdots + a_mu_m$.

(1) *If* $Tu_0 \in \operatorname{span}(u_0)$, *then we are done.*

(2) *Otherwise, if* range $T|_U = U$, *then* $Tu_0 = a_0 u_0$ *and we are done;*

otherwise, $T|_U : U \to U$ *is not surj* ($\Rightarrow$ *not inje*), *suppose* range $T|_U \neq \{0\}$

(*Suppose* range $T|_U = \{0\}$. *If* $\dim U = 0$ *then we are done.*

Otherwise $\exists\, u \in U \backslash \{0\}, Tu = 0$ *and we are done.*)

then $\exists\, u \in U \backslash \{0\}, Tu = 0$, *we are done.* $\square$

---

29 *Suppose* $T \in \mathcal{L}(V)$ *and* range T *is finite-dim.*

SOLUTION:

Prove that T *has at most* $1 + \dim \operatorname{range} T$ *distinct eigvals.*

Let $\lambda_1, \dots, \lambda_m$ *be the distinct eigvals of* T *and let* $v_1, \dots, v_m$ *be the correspd eigvecs.*

(*Because* range T *is finite-dim. Let* $(v_1, \dots, v_n)$ *be a list of all the linely inde eigvecs of* T, *so that the correspd eigvals are finite.*)

For every $\lambda_k \neq 0, T(\frac{1}{\lambda_k} v_k) = v_k$. *And if* $T = T - 0I$ *is not inje, then* $\exists\,!\lambda_A = 0$ *and* $Tv_A = \lambda_A v_A = 0$.

Thus for $\lambda_k \neq 0, \forall\, k$, $(Tv_1, \dots, Tv_m)$ *is a linely inde list of length* m *in* range T.

And for $\lambda_A = 0$, *there is a linely inde list of length at most* $(m-1)$ *in* range T.

Hence, by [2.23], $m \leq \dim \operatorname{range} T + 1$. $\square$

---

32 *Suppose that* $\lambda_1, \dots, \lambda_n$ *are distinct real numbers.*

SOLUTION:

Prove that $(e^{\lambda_1}x, \dots, e^{\lambda_n}x)$ *is linely inde in* $\mathbf{R}^{\mathbf{R}}$.

HINT: *Let* $V = \operatorname{span}(e^{\lambda_1}x, \dots, e^{\lambda_n}x)$, *and define an operator* $D \in \mathcal{L}(V)$ *by* $Df = f'$.

Find eigvals and eigvecs of D.

Define V *and* $D \in \mathcal{L}(V)$ *as in* HINT. *Then because for each* $k, D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.

Thus $\lambda_1, \dots, \lambda_n$ *are distinct eigvals of* D. *By* [5.10], $(e^{\lambda_1}x, \dots, e^{\lambda_n}x)$ *is linely inde in* $\mathbf{R}^{\mathbf{R}}$. $\square$

- *Suppose* $\lambda_1, \dots, \lambda_n$ *are distinct positive numbers.*

Prove that $(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$ *is linely inde in* $\mathbf{R}^{\mathbf{R}}$.

Let $V = \operatorname{span}\left(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\right)$. *Define* $D \in \mathcal{L}(V)$ *by* $Df = f'$.

Then because $D\left(\cos(\lambda_k x)\right) = -\lambda_k \sin(\lambda_k x)$. 又 $D\left(\sin(\lambda_k x)\right) = \lambda_k \cos(\lambda_k x)$.

Thus $D^2\left(\cos(\lambda_k x)\right) = -\lambda_k^2 \cos(\lambda_k x)$.

Notice that $\lambda_1, \dots, \lambda_n$ *are distinct* $\Rightarrow -\lambda_1^2, \dots, -\lambda_n^2$ *are distinct.*

Hence $-\lambda_1^2, \dots, -\lambda_n^2$ *are distinct eigvals of* D^2

with the correspd eigvecs $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ *respectively.*

And then $(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$ *is linely inde in* $\mathbf{R}^{\mathbf{R}}$. $\square$

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is a subsp of V invar under T.*

The quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v+U) = Tv + U \text{ for each } v \in V.$$

(a) *Show that the definition of T/U makes sense*

(which requires using the condition that U is invar under T)

and show that T/U is an operator on V/U.

(b) (OR Problem 35) *Show that each eigval of T/U is an eigval of T.*

(a) *Suppose $v+U = w+U$ ($\iff v-w \in U$).*

Then because U is invar under T, $T(v-w) \in U \iff Tv+U = Tw+U$.

Hence the definition of T/U makes sense.

Now we show that T/U is linear.

$\forall v+U, w+U \in V/U, \lambda \in \mathbf{F}, (T/U)\left((v+U)+\lambda(w+U)\right)$

$$= T(v+\lambda w) + U = (Tv+U) + \lambda(Tw+U)$$
$$= (T/U)(v+U) + \lambda(T/U)(w).$$

(b) *Suppose λ is an eigval of T/U with an eigvec $v+U$.*

Then $(T/U)(v+U) = \lambda(v+U) = Tv+U = \lambda v + U \Rightarrow (T-\lambda I)v \in U$.

If $(T-\lambda I)v = 0 \Rightarrow Tv = \lambda v$, then we are done.

Otherwise, then $(T|_U - \lambda I) : U \to U$ is inv,

hence $\exists!\, w \in U, (T|_U - \lambda I)(w) = (T-\lambda I)v \Rightarrow T(v-w) = \lambda(v-w)$.

Note that $v - w \neq 0$ (for if not, $v \in U \Rightarrow v+U = 0+U$ is not an eigvec). $\square$

---

36 *Prove or give a counterexample:*

SOLUTION:

The result of (b) in Exercise 35 is still true if V is infinite-dim.

A counterexample:

Consider $V = \operatorname{span}(1, e^x, e^{2x}, \dots)$ in $\mathbf{R}^{\mathbf{R}}$, and a subsp $U = \operatorname{span}(e^x, e^{2x}, \dots)$ of V.

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then $\operatorname{range} T = U$ is invar under T.

Consider $(T/U)(1+U) = e^x + U = 0$

$\Rightarrow 0$ is an eigval of T/U but is not an eigval of T

($\operatorname{null} T = \{0\}$, for if not, $\exists f \in V\backslash\{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, contradicts). $\square$

---

33 *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{range} T) = 0$.*

SOLUTION:

$\forall v + \operatorname{range} T \in V/\operatorname{range} T, v + \operatorname{range} T \in \operatorname{null}\ (T/(\operatorname{range} T))$

$\Rightarrow \operatorname{null}\ (T/(\operatorname{range} T)) = V/\operatorname{range} T \Rightarrow T/(\operatorname{range} T)$ *is a zero map.* □

---

34 *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{null} T)$ is inje $\Longleftrightarrow (\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$.*

SOLUTION:

(*a*) *Suppose $T/(\operatorname{null} T)$ is inje.*

Then $(T/(\operatorname{null} T))(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0$

$\Longleftrightarrow Tu \in \operatorname{null} T$ 又 $Tu \in \operatorname{range} T \Longleftrightarrow u + \operatorname{null} T = 0 \Longleftrightarrow u \in \operatorname{null} T \Longleftrightarrow Tu = 0.$

Thus $(\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$.

(*b*) *Suppose* $(\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$.

Then $(T/(\operatorname{null} T))(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0$

$\Longleftrightarrow Tu \in \operatorname{null} T$ 又 $Tu \in \operatorname{range} T \Longleftrightarrow Tu = 0 \Longleftrightarrow u \in \operatorname{null} T \Longleftrightarrow u + \operatorname{null} T = 0.$

Thus $T/(\operatorname{null} T)$ is inje. □

ENDED

5.B: I [*See* 5.B: II *below.*]

COMMENT: 下面是第 5 章 *B* 节。为了照顾 *5.B* 节两版过大的差距，特别将 *5.B* 补注分成 I 和 II 两部分。
又考虑到第 4 版中 *5.B* 节的「本征值与极小多项式」与「奇维度实向量空间的本征值」
(相当一部分是从原第 3 版 *8.C* 节挪过来的) 是对原第 3 版「多项式作用于算子」与
「本征值的存在性」(也即第 3 版 *5.B* 前半部分) 的极大扩充，这一扩充也大大改变了
原第 3 版后半部分的「上三角矩阵」这一小节，故而将第 4 版 *5.B* 节放在第 3 版 *5.B* 节前面。

I 部分除了覆盖第 4 版 *5.B* 节和第 3 版 *5.B* 节前半部分与之相关的所有习题，
还会覆盖第 4 版 *5.A* 节末。
II 部分除了覆盖第 3 版 *5.B* 节后半部分「上三角矩阵」这一小节，还会覆盖第 4 版 *5.C* 节；
并且，下面 **5.C** 还会覆盖第 4 版 *5.D* 节。

「注：[*8.40*] OR (*4E 5.22*) —— *mini poly;*
[*8.44,8.45*] OR (*4E 5.25,5.26*) —— *how to find the mini poly;*
[*8.49*] OR (*4E 5.27*) —— *eigvals are the zeros of the mini poly;*
[*8.46*] OR (*4E 5.29*) —— $q(T) = 0 \Leftrightarrow q$ *is a poly multi of the mini poly.*」

[1]: (4E.5.A.33), 13; [2]: (4E.5.B.25, 26, 27, 28, 22); [3]: 6, (4E.5.B.10, 23, 21), 19; [4]: (4E 5.B.13, 14); [5]: (4E.5.B.20, 24), 10; [6]: 1, 2, 7, 3, (4E.5.A.32); [7]: 8, (4E 5.B.12, 3, 8); [8]: (4E.5.B.11), 5, (4E.5.B.7); [9]: 11, 12, (4E.5.B.17, 18); [10]: 18 OR (4E.5.B.15), (4E.5.B.9), (4E.5.B.16); [11]: (2E Ch5.24), (4E.5.B.29).

- *Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.*
  (*a*) *Prove that T is inje $\Longleftrightarrow T^m$ is inje.*
  (*b*) *Prove that T is surj $\Longleftrightarrow T^m$ is surj.*

(*a*) *Suppose T^m is inje. Then* $Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0.$ □

Suppose T is inje.

Then $T^m v = T(T^{m-1}v) = 0$

$$\Rightarrow T^{m-1}v = 0 = T(T^{m-2}v) \Rightarrow \cdots$$
$$\Rightarrow T^2 v = TTv = 0$$
$$\Rightarrow Tv = 0 \Rightarrow v = 0. \quad \square$$

(b) *Suppose T^m is surj.* $\forall u \in V, \exists v \in V, T^m v = u = Tw$, *let* $w = T^{m-1}v$. $\square$

Suppose T is surj.

Then $\forall u \in V, \exists v \in V, T\underline{(v)} = u$

$$\Rightarrow \exists v_2 \in V, Tv_2 = \underline{v}, T^2\underline{(v_2)} = u$$
$$\vdots$$
$$\Rightarrow \exists v_k \in V, Tv_k = \underline{v_{k-1}}, T^k\underline{(v_k)} = u$$
$$\vdots$$
$$\Rightarrow \exists v_{m-1} \in V, Tv_{m-1} = \underline{v_{m-2}}, T^{m-1}\underline{(v_{m-1})} = u$$
$$\Rightarrow \exists v_m \in V, Tv_m = \underline{v_{m-1}}, T^{m-1}(Tv_m) = u. \quad \square$$

Solution: **Note For [5.17]:** *Suppose* $T \in \mathcal{L}(V)$ *and* $p \in \mathcal{P}(\mathbf{F})$.

Prove that $\operatorname{null} p(T)$ *and* $\operatorname{range} p(T)$ *are invar under T.*

Using the commutativity in [5.10].

(a) *Suppose* $u \in \operatorname{null} p(T)$. *Then* $p(T)u = 0$.

Thus $p(T)(Tu) = (p(T)T)u\,(Tp(T))\,u = T(p(T)u) = 0$. *Hence* $Tu \in \operatorname{null} p(T)$. $\square$

(b) *Suppose* $u \in \operatorname{range} p(T)$. *Then* $\exists v \in V$ *such that* $u = p(T)v$.

Thus $Tu = T(p(T)v) = p(T)(Tv) \in \operatorname{range} p(T)$. $\square$

Solution: **Note For [5.21]:** *Every operator on a finite-dim nonzero complex vector space has an eigval.*

Suppose V is a finite-dim complex vector space of dim $n > 0$ *and* $T \in \mathcal{L}(V)$.

Choose a nonzero $v \in V$. $(v, Tv, T^2v, \dots, T^n v)$ *of length* $n+1$ *is linely depe.*

Suppose $a_0 I + a_1 T + \cdots + a_n T^n = 0$. *Then* $\exists a_j \neq 0$.

Thus $\exists$ *nonconst p of smallest degree* ($\deg p > 0$) *such that* $p(T)v = 0$.

Because $\exists \lambda \in \mathbf{C}$ *such that* $p(\lambda) = 0 \Rightarrow \exists q \in \mathcal{P}(\mathbf{C}), p(z) = (z-\lambda)q(z), \forall z \in \mathbf{C}$.

Thus $0 = p(T)v = (T - \lambda I)(q(T)v)$. *By the minimality of* $\deg p$ *and* $\deg q < \deg p$, $q(T)v \neq 0$.

Then $(T - \lambda I)$ *is not inje. Thus λ is an eigval of T with eigvec* $q(T)v$.

Solution: **Example:** *an operator on a complex vector space with no eigvals*

Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ *by* $(Tp)(z) = zp(z)$.

Suppose $p \in \mathcal{P}(\mathbf{C})$ *is a nonzero poly. Then* $\deg Tp = \deg p + 1$, *and thus* $Tp \neq \lambda p$, $\forall \lambda \in \mathbf{C}$.

Hence T has no eigvals. Because $\mathcal{P}(\mathbf{C})$ *is infinite-dim, this example does not contradict the result above.*

---

13 *Suppose V is a complex vector space and* $T \in \mathcal{L}(V)$ *has no eigvals.*

Solution:

Prove that every subsp of V invar under T is either $\{0\}$ *or infinite-dim.*

Solution: *Suppose U is a finite-dim nonzero invar subsp on* $\mathbf{C}$. *Then by* [5.21], $T|_U$ *has an eigval.* $\square$

Solution: **Tips:** *For* $T_1, \dots, T_m \in \mathcal{L}(V)$:

(a) *Suppose* $T_1, \dots, T_m$ *are all inje. Then* $(T_1 \circ \cdots \circ T_m)$ *is inje.*

(b) *Suppose* $(T_1 \circ \cdots \circ T_m)$ *is not inje. Then at least one of* $T_1, \dots, T_m$ *is not inje.*

(c) *At least one of* $T_1, \dots, T_m$ *is not inje* $\not\Rightarrow (T_1 \circ \cdots \circ T_m)$ *is not inje.*

EXAMPLE: On infinite-dim only. Let $V = \mathbf{F}^\infty$.

Let S be the backward shift (surj but not inje), T be the forward shift (inje but not surj). Then $ST = I$. $\square$

---

16 *Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}(\mathcal{P}_{\dim V}(\mathbf{C})), V)$ by $S(p) = p(T)v$. Prove [5.21].*

SOLUTION:

Because $\dim \mathcal{P}_{\dim V}(\mathbf{C})) = \dim V + 1$. Then S is not inje. Hence $\exists\, 0 \neq p \in \mathcal{P}_{\dim V}(\mathbf{C})), p(T)v = 0$.

Using [4.14], write $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$. Apply T to both sides: $p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.

Thus at least one of $(T - \lambda_j I)$ is not inje (because $p(T)$ is not inje). $\square$

---

17 *Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}\left(\mathcal{P}_{(\dim V)^2}(\mathbf{C})), \mathcal{L}(V)\right)$ by $S(p) = p(T)$. Prove [5.21]*

SOLUTION:

Because $\dim \mathcal{P}_{(\dim V)^2}(\mathbf{C})) = (\dim V)^2 + 1$. Then S is not inje. Hence $\exists\, 0 \neq p \in \mathcal{P}_{(\dim V)^2}(\mathbf{C})), p($ 0.

Using [4.14], write $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$. Applying T, we have $0 = p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.

Thus $(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0 \Rightarrow \exists\, j, (T - \lambda_j)$ is not inje. $\square$

COMMENT: *$\exists$ monic $q \in \operatorname{null} S \neq \{0\}$ of smallest degree, $S(q) = q(T) = 0$, then q is the mini poly.*

SOLUTION:

NOTE FOR [8.40]: *def for mini poly*

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that $\exists\,!$ monic poly $p \in \mathcal{P}(\mathbf{F})$ of smallest degree, $p(T) = 0$. Moreover, $\deg p \leq \dim V$.

SOLUTION *OR Another Proof* :

[Existns Part] *We use induction on $\dim V$.*

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, we are done.

(ii) Suppose $\dim V \geq 1$.

Assume that $\dim V > 0$ and that the desired result is true for all operators on all vecsps of smaller dim.
Let $u \in V, u \neq 0$. The list $(u, Tu, \ldots, T^{\dim V} u)$ of length $(1 + \dim V)$ is linely depe.
Then $\exists\,!\, T^m$ of smallest degree such that $T^m u \in \operatorname{span}(u, Tu, \ldots, T^{m-1}u)$.
Thus $\exists\, c_j \in \mathbf{F}, c_0 u + c_1 Tu + \cdots + c_{m-1} T^{m-1} u + T^m u = 0$.
Define q by $q(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m$.
Then $0 = T^k(q(T)u) = q(T)(T^k u), \forall k \in \{1, \ldots, m-1\} \subseteq \mathbf{N}$.
Because $(u, Tu, \ldots, T^{m-1}u)$ is linely inde.
Thus $\dim \operatorname{null} q(T) \geq m \Rightarrow \dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leq \dim V - m$.

Let $W = \operatorname{range} q(T)$.
By assumption, $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ and $\deg s \leq \dim W$, so that $s(T|_W) = 0$.
Hence $\forall v \in V, ((sq)(T))(v) = s(T)(q(T)v) = 0$.
Thus sq is a monic poly such that $\deg sq \leq \dim V$ and $(sq)(T) = 0$.

[Uniqnes Part]

Let $p, q \in \mathcal{P}(\mathbf{F})$ be monic polys of smallest degree such that $p(T) = q(T) = 0$

$\Rightarrow (p-q)(T) = 0$ 又 $\deg(p-q) < \deg p$.

If $p - q = a_m z^m + \cdots + a_1 z_1 + a_0 \neq 0$, then $\frac{1}{a_m}(p-q)$ is a monic poly of smaller degree than p.

Hence contradicts the minimality of $\deg p$. Thus $p - q = 0$ and we are done. □

• (4E 5.31, 4E 5.B.25 and 26) *mini poly of restriction operator and mini poly of quotient operator*

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invar subsp of V under T.
Let p be the mini poly of T.
(a) *Prove that p is a poly multi of the mini poly of $T|_U$.*
(b) *Prove that p is a poly multi of the mini poly of T/U.*
(c) *Prove that (mini poly of $T|_U$) $\times$ (mini poly of T/U) is a poly multi of p.*
(d) *Prove that the set of eigvals of T equals the union of the set of eigvals of $T|_U$ and the set of eigvals of T/U.*

(a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46].□

(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v+U) = p(T)v + U = 0.$ □

(c) *Suppose r is the mini poly of $T|_U$, s is the mini poly of T/U.*

Because $\forall v \in V, s(T/U)(v+U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.

又 $\forall u \in U, r(T|_U)u = r(T)u = 0.$

Thus $\forall v \in V$ but $v \notin U, (rs)(T)v = r(s(T)v) = 0.$

And $\forall u \in U, (rs)(T)u = r(s(T)u) = 0$ (because $s(T)u = s(T|_U)u \in U$).

Hence $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0.$ □

(d) *By [8.49], immediately.* □

• (4E 5.B.27)

Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, and $T \in \mathcal{L}(V)$.
Prove that the mini poly p of $T_{\mathbf{C}}$ equals the mini poly q of T.

SOLUTION:

$$\left.\begin{array}{l} \forall u + \mathrm{i}0 \in V_{\mathbf{C}}, p(T_{\mathbf{C}})(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p \text{ is a poly multi of } q. \\ q(T) = 0 \Rightarrow \forall u + \mathrm{i}v \in V_{\mathbf{C}}, q(T_{\mathbf{C}})(u + \mathrm{i}v) = q(T)u + \mathrm{i}q(T)v = 0 \Rightarrow q \text{ is a poly multi of } p. \end{array}\right\} \Rightarrow \square$$

• (4E 5.B.28)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.
Prove that the mini poly p of $T' \in \mathcal{L}(V')$ equals the mini poly q of T.

SOLUTION:

$$\left.\begin{array}{r} \forall \varphi \in V', p(T')\varphi = \varphi \circ (p(T)) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \operatorname{null} \varphi \Rightarrow p(T) = 0 \\ \Rightarrow p(T) = 0 \Rightarrow p \text{ is a poly multi of } q. \\ q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ (q(T)) = q(T')\varphi = 0 \Rightarrow q \text{ is a poly multi of } p. \end{array}\right\} \Rightarrow \square$$

• (4E 5.32) *Suppose $T \in \mathcal{L}(V)$ and p is the mini poly.*
Prove that T is not inje $\iff$ the const term of p is 0.

T is not inje $\iff$ 0 is an eigval of T $\iff$ 0 is a zero of p $\iff$ the const term of p is 0. □

OR. Because $p(0) = (z-0)(z-\lambda_1)\cdots(z-\lambda_m) = 0 \Rightarrow T(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0$

又 *p is the mini poly $\Rightarrow q$ define by $q(z) = (z-\lambda_1)\cdots(z-\lambda_m)$ is such that $q(T) \neq 0$.*

Hence $0 = p(T) = Tq(T) \Rightarrow T$ is not inje.

Conversely, suppose $(T - 0I)$ is not inje, then 0 is a zero of p, so that the const term is 0. □

• (4E 5.B.22)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$. Prove that T is inv $\Longleftrightarrow I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$.
Denote the mini poly by p, where for all $z \in \mathbf{F}, p(z) = a_0 + a_1 z + \cdots + z^m$.

Notice that V is finite-dim. T is inv $\Longleftrightarrow$ T is inje $\Longleftrightarrow p(0) \neq 0$.

Hence $p(T) = 0 = a_0 I + a_1 T + \cdots + T^m$, where $a_0 \neq 0$ and $m \leq \dim V$. □

---

6. *Suppose $T \in \mathcal{L}(V)$ and U is a subsp of V invar under T.*
Solution:

Prove that U is invar under $p(T)$ for every poly $p \in \mathcal{P}(\mathbf{F})$.

$\forall u \in U, Tu \in U \Rightarrow \forall u \in U, Iu, Tu, T(Tu), \dots, T^m u \in U \Rightarrow \forall u \in U, (a_0 I + a_1 T + \cdots + a_m T^m)u \in U$. □

- (*4E 5.B.10, 5.B.23*)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$ and p is the mini poly with degree m. Suppose $v \in V$.

(a) Prove that $\operatorname{span}(v, Tv, \dots, T^{m-1}v) = \operatorname{span}(v, Tv, \dots, T^{j-1}v)$ for some $j \leq m$.

(b) Prove that $\operatorname{span}(v, Tv, \dots, T^{m-1}v) = \operatorname{span}(v, Tv, \dots, T^{m-1}v, \dots, T^n v)$.

Comment: *By Note For [8.40], j has an upper bound $m - 1$, m has an upper bound $\dim V$.*

Write $p(z) = a_0 + a_1 z + \cdots + z^m$ ($m \leq \dim V$). If $v = 0$, then we are done. Suppose $v \neq 0$.

(a) Suppose $j \in \mathbf{N}^+$ is the smallest such that $T^j v \in \operatorname{span}(v, Tv, \dots, T^{j-1}v) = U_0$. Then $j \leq m$.

Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1} T^{j-1} v$. And because $T(T^k v) = T^{k+1} \in U_0$. U_0 is invar under T.

By Problem (6), $\forall k \in \mathbf{N}$, $T^{j+k} v = T^k (T^j v) \in U_0$.

Thus $U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}v, \dots, T^n v)$ for all $n \geq j - 1$. Let $n = m - 1$ and we are done.

(b) Let $U = \operatorname{span}(v, Tv, \dots, T^{m-1}v)$.

By (a), $U = U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}, \dots, T^{m-1}, \dots, T^n)$ for all $n \geq m - 1$. □

- (*4E 5.B.21*)

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.

Prove that the mini poly p has degree at most $1 + \dim \operatorname{range} T$.
If $\dim \operatorname{range} T < \dim V - 1$, then this result gives a better upper bound for the degree of mini poly.
Solution:

If T is inje, then $\operatorname{range} T = V$ and we are done. Now choose $0 \neq v \in \operatorname{null} T$, then $Tv + 0 \cdot v = 0$.

1 is the smallest positive integer such that $T^1 v \in \operatorname{span}(v, \dots, T^0 v)$. Define q by $q(z) = z \Rightarrow q(T)v = 0$.

Let $W = \operatorname{range} q(T) = \operatorname{range} T$. $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ of smallest degree ($\deg s \leq \dim W$), $s(T|_W) = 0$.

Hence sq is the mini poly (see NOTE FOR $[8.40]$ *) and* $\deg(sq) = \deg s + \deg q \leq \dim \operatorname{range} T + 1$. $\square$

---

19 *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.*

SOLUTION:

Prove that $\dim \mathcal{E}$ *equals the degree of the mini poly of T.*

Because the list $(I, T, \dots, T^{(\dim V)^2})$ of length $\dim \mathcal{L}(V) + 1$ *is linely depe in* $\dim \mathcal{L}(V)$.

Suppose $m \in \mathbf{N}^+$ is the smallest such that $T^m = a_0 I + \dots + a_{m-1} T^{m-1}$.

Then q defined by $q(z) = z^m - a_{m-1} z^{m-1} - \dots - a_0$ is the mini poly (see $[8.40]$).

For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k(T^m) \in \operatorname{span}(I, T, \dots, T^{m-1}) = U$.

Hence $\operatorname{span}(I, T, \dots, T^{(\dim V)^2}) = \operatorname{span}(I, T, \dots, T^{(\dim V)^2 - 1}) = U$.

Note that by the minimality of m, the list $(I, T, \dots, T^{m-1})$ is linely inde.

Thus $\dim U = m = \dim \operatorname{span}(I, T, \dots, T^{(\dim V)^2 - 1}) = \dim \operatorname{span}(I, T, \dots, T^n)$ *for all $m < n \in \mathbf{N}^+$.*

Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.

(a) *Suppose $p(T) = 0$.* 又 $\deg p \leq m - 1 \Rightarrow p = 0$. *Then φ is inje.*

(b) $\forall S = a_0 I + a_1 T + \dots + a_{m-1} T^{m-1} \in \mathcal{E}$, *define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by*

$p(z) = a_0 + a_1 z + \dots + a_{m-1} z^{m-1} \Rightarrow \varphi(p) = S$. *Then φ is surj.*

Hence $\mathcal{E}$ and $\mathcal{P}_{m-1}(\mathbf{F})$ are iso. 又 $\dim \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. $\square$

• (4E 5.B.13)

Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$ is defined by

$$q(z) = a_0 + a_1 z + \dots + a_n z^n, \text{ where } a_n \neq 0, \text{ for all } z \in \mathbf{F}.$$

Denote the mini poly of T by p defined by

$$p(z) = c_0 + c_1 z + \dots + c_{m-1} z^{m-1} + z^m \text{ for all } z \in \mathbf{F}.$$

Prove that $\exists!\, r \in \mathcal{P}(\mathbf{F})$ such that $q(T) = r(T)$, $\deg r < \deg p$.

If $\deg q < \deg p$, *then we are done.*

If $\deg q = \deg p$, *notice that* $p(T) = 0 = c_0 I + c_1 T + \dots + c_{m-1} T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0 I - c_1 T - \dots - c_{m-1} T^{m-1},$$

define r by $r(z) = q(z) + [-a_m z^m + a_m(-c_0 - c_1 z - \dots - c_{m-1} z^{m-1})]$

$$= (a_0 - a_m c_0) + (a_1 - a_m c_1) z + \dots + (a_{m-1} - a_m c_{m-1}) z^{m-1},$$

hence $r(T) = 0$, $\deg r < m$ *and we are done.*

Now suppose $\deg q \geq \deg p$. *We use induction on* $\deg q$.

(i) $\deg q = \deg p$, *then the desired result is true, as shown above.*

(ii) $\deg q > \deg p$, *assume that the desired result is true for* $\deg q = n$.

Suppose $f \in \mathcal{P}(\mathbf{F})$ *such that* $f(z) = b_0 + b_1 z + \cdots + b_n z^n + b_{n+1} z^{n+1}$.
Apply the assumption to g *defined by* $g(z) = b_0 + b_1 z + \cdots + b_n z^n$,
getting s *defined by* $s(z) = d_0 + d_1 z + \cdots + d_{m-1} z^{m-1}$.
Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1} T^{n+1} = s(T) + b_{n+1} T^{n+1}$.
Apply the assumption to t *defined by* $t(z) = z^n$,
getting δ *defined by* $\delta(z) = c_0{}' + c_1{}' z + \cdots + c_{m-1}{}' z^{m-1}$.
Thus $t(T) = T^n = c_0{}' + c_1{}' z + \cdots + c_{m-1}{}' z^{m-1} = \delta(T)$.
又 $\operatorname{span}(v, Tv, \dots, T^{m-1}v)$ *is invar under* T.
Hence $\exists!\, k_j \in \mathbf{F}, T^{n+1} = T(T^n) = k_0 + k_1 z + \cdots + k_{m-1} z^{m-1}$.
And $f(T) = s(T) + b_{n+1}(k_0 + k_1 T + \cdots + k_{m-1} T^{m-1})$
$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1) z + \cdots + (d_{m-1} + k_{m-1}) z^{m-1} = h(T)$, *thus defining* h. □

- (4E 5.B.14) *Suppose* V *is finite-dim,* $T \in \mathcal{L}(V)$ *has mini poly* p
defined by $p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m$, $a_0 \neq 0$.
Find the mini poly of T^{-1}.

Notice that V *is finite-dim. Then* $p(0) = a_0 \neq 0 \Rightarrow 0$ *is not a zero of* $p \Rightarrow T - 0I = T$ *is inv.*
Then $p(T) = a_0 I + a_1 T + \cdots + T^m = 0$. *Apply* T^{-m} *to both sides,*
$a_0 (T^{-1})^m + a_1 (T^{-1})^{m-1} + \cdots + a_{m-1} T^{-1} + I = 0$.
Define q *by* $q(z) = z^m + \dfrac{a_1}{a_0} z^{m-1} + \cdots + \dfrac{a_{m-1}}{a_0} z + \dfrac{1}{a_0}$ *for all* $z \in \mathbf{F}$.

We now show that $(T^{-1})^k \notin \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$
for every $k \in \{1, \dots, m-1\}$ *by contradiction, so that* q *is exactly the mini poly of* T^{-1}.
Suppose $(T^{-1})^k \in \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$.
Then let $(T^{-1})^k = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. *Apply* T^k *to both sides,*
getting $I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T$, *hence* $T^k \in \operatorname{span}(I, T, \dots, T^{k-1})$.
Thus f *defined by* $f(z) = z^k + \dfrac{b_1}{b_0} z^{k-1} + \cdots + \dfrac{b_{k-1}}{b_0} z - \dfrac{1}{b_0}$ *is a poly multi of* p.
While $\deg f < \deg p$. *Contradicts.* □

- **NOTE FOR [8.49]:** *Suppose* V *is a finite-dim complex vecsp,* $T \in \mathcal{L}(V)$.
By [4.14], the mini poly has the form $(z - \lambda_1) \cdots (z - \lambda_m)$,
where $\lambda_1, \dots, \lambda_m$ *is a list of all eigvals of* T, *possibly with repetitions.*

SOLUTION: COMMENT: *A nonzero poly has at most as many distinct zeros as its degree (see [4.12]).*
Thus by the upper bound for the deg of mini poly given in NOTE FOR [8.40], *and by [8.49,]*

we can ***give an alternative proof of*** $[\mathbf{5.13}]$.

• **NOTICE**: (*See also 4E 5.B.20,24*)

Suppose $\alpha_1, \dots, \alpha_n$ *are all the distinct eigvals of* T,

and therefore are all the distinct zeros of the mini poly.

Also, the mini poly of T *is a poly multi of, but not equal to,* $(z-\alpha_1)\cdots(z-\alpha_n)$.

If we define q *by* $q(z) = (z-\alpha_1)^{\dim V-(n-1)} \cdots (z-\alpha_n)^{\dim V-(n-1)}$,

then q *is a poly multi of the char poly* (*see* $[8.34]$ *and* $[8.26]$)

(*Because* $\dim V > n$ *and* $n-1>0$, $n[\dim V-(n-1)] > \dim V$.)

The char poly has the form $(z-\alpha_1)^{\gamma_1}\cdots(z-\alpha_n)^{\gamma_n}$, *where* $\gamma_1+\cdots+\gamma_n = \dim V$.

The mini poly has the form $(z-\alpha_1)^{\delta_1}\cdots(z-\alpha_n)^{\delta_n}$, *where* $0 \le \delta_1+\cdots+\delta_n \le \dim V$.

---

10 *Suppose* $T \in \mathcal{L}(V)$, λ *is an eigval of* T *with an eigvec* v.

SOLUTION:

Prove that for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.

Suppose p *is defined by* $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ *for all* $z \in \mathbf{F}$. *Because for any* $n \in \mathbf{N}^+$, $T^n v = \lambda^n v$.

Thus $p(T)v = a_0 v + a_1 T v + \cdots + a_m T^m v = a_0 v + a_1 \lambda v + \cdots + a_m \lambda^m v = p(\lambda)v$. $\square$

• **COMMENT**: *For any* $p \in \mathcal{P}(\mathbf{F})$ *such that* $p(z) = (z-\lambda_1)^{\alpha_1}\cdots(z-\lambda_m)^{\alpha_m}$, *the result is true as well.*

Now we prove that $(T-\lambda_1 I)^{\alpha_1}\cdots(T-\lambda_m I)^{\alpha_m} v = (\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_m)^{\alpha_m} v$.

Define q_i *by* $q_i(z) = (z-\lambda_i)^{\alpha_i}$ *for all* $z \in \mathbf{F}$.

Because $(a+b)^n = a^n + C_n^1 a^{n-1} b + \cdots + C_n^k a^{n-k} b^k + \cdots + C_n^n b^n$.

Let $a = z, b = \lambda_i, n = \alpha_i$, *so we can write* $q_i(z)$ *in the form* $a_0 + a_1 z + \cdots + a_m z^m$.

Hence $q_i(T)v = q_i(\lambda)v \Rightarrow (T-\lambda_i I)^{\alpha_i} v = (\lambda-\lambda_i)^{\alpha_i} v$.

Then for each $k \in \{2, \dots, m\}$, $(T-\lambda_{k-1}I)^{\alpha_{k-1}}(T-\lambda_k I)^{\alpha_k} v$

$$\begin{aligned} &= q_{k-1}(T)(q_k(T)v) \\ &= q_{k-1}(T)(q_k(\lambda)v) \\ &= q_{k-1}(\lambda)(q_k(\lambda)v) \\ &= (\lambda-\lambda_{k-1})^{\alpha_{k-1}}(\lambda-\lambda_k)^{\alpha_k} v. \end{aligned}$$

So that $(T-\lambda_1 I)^{\alpha_1}\cdots(T-\lambda_m I)^{\alpha_m} v$

$$\begin{aligned} &= q_1(T)\,(q_2(T)(\dots(q_m(T)v)\dots)) \\ &= q_1(\lambda)(q_2(\lambda)\,(\dots(q_m(\lambda)v)\dots)) \\ &= (\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_m)^{\alpha_m} v. \end{aligned}$$ $\square$

---

1 *Suppose $T \in \mathcal{L}(V)$ and $\exists\, n \in \mathbf{N}^+$ such that $T^n = 0$.*
SOLUTION:

Prove that $(I - T)$ is inv and $(I-T)^{-1} = I + T + \cdots + T^{n-1}$.

Note that $1 - x^n = (1-x)(1 + x + \cdots + x^{n-1})$.

$$\left.\begin{aligned} (I-T)(1 + T + \cdots + T^{n-1}) = I - T^n = I \\ (1 + T + \cdots + T^{n-1})(I-T) = I - T^n = I \end{aligned}\right\} \Rightarrow (I-T)^{-1} = 1 + T + \cdots + T^{n-1}. \qquad \square$$

---

2 *Suppose $T \in \mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I) = 0$.*
SOLUTION:

Suppose λ is an eigval of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Suppose v is an eigvec correspd to λ. Then for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.

Hence $0 = (T-2I)(T-3I)(T-4I)v = (\lambda-2)(\lambda-3)(\lambda-4)v$ *while* $v \neq 0 \Rightarrow \lambda = 2$ *or* $\lambda = 3$ *or* $\lambda = 4$. $\square$

OR. Because $(T-2I)(T-3I)(T-4I) = 0$ *is not inje. By TIPS.* $\square$

---

7 *Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigval of $T^2 \Longleftrightarrow 3$ or -3 is an eigval of T.*
SOLUTION:

SOLUTION: *COMMENT: Note that V can be infinite-dim. See also in (5.A.22).*

(a) *Suppose 9 is an eigval of T^2. Then* $(T^2 - 9I)v = (T-3I)(T+3I)v = 0$ *for some v. By TIPS.*

(b) *Suppose 3 or -3 is an eigval of T with an eigvec v. Then* $Tv = \pm 3v \Rightarrow T^2 v = T(Tv) = 9v$ $\square$

---

3 *Suppose $T \in \mathcal{L}(V)$, $T^2 = I$ and -1 is not an eigval of T. Prove that $T = I$.*
SOLUTION:

$T^2 - I = (T+I)(T-I)$ *is not inje,* 又 -1 *is not an eigval of $T \Rightarrow$ By TIPS.* $\square$

OR. Note that $v = [\frac{1}{2}(I-T)v] + [\frac{1}{2}(I+T)v]$ *for all $v \in V$.*

And $(I - T^2)v = (I-T)(I+T)v = 0$ *for all $v \in V$,*

$$\left.\begin{aligned} (I+T)(\tfrac{1}{2}(I-T)v) = \tfrac{1}{2}(I-T^2)v = 0 \Rightarrow \tfrac{1}{2}(I-T)v \in \operatorname{null}(I+T) \\ (I-T)(\tfrac{1}{2}(I+T)v) = \tfrac{1}{2}(I-T^2)v = 0 \Rightarrow \tfrac{1}{2}(I+T)v \in \operatorname{null}(I-T) \end{aligned}\right\} \Rightarrow V = \operatorname{null}(I+T) + \operatorname{null}(I-T).$$

又 -1 *is not an eigval of $T \Rightarrow (I+T)$ is inje* $\Rightarrow \operatorname{null}(I+T) = \{0\}$.

Hence $V = \operatorname{null}(I-T) \Rightarrow \operatorname{range}(I-T) = \{0\}$. *Thus* $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$. $\square$

• (4E 5.A.32) *Suppose $T \in \mathcal{L}(V)$ has no eigvals and $T^4 = I$. Prove that $T^2 = -I$.*

Because $T^4 - I = (T^2 - I)(T^2 + I) = 0$ *is not inje* $\Rightarrow (T^2 - I)$ *or* $(T^2 + I)$ *is not inje.*

又 T *has no eigvals* $\Rightarrow (T^2 - I) = (T - I)(T + I)$ *is inje,*

for if not, $(T - I)$ *or* $(T + I)$ *is not inje. Contradicts.*

Hence $T^2 + I = 0 \in \mathcal{L}(V)$, *for if not,* $\exists\, v \in V, (T^2+I)v \neq 0$ *while* $(T^2-I)\big((T^2+I)v\big) = 0$. *Contradicts.*□

Or. *Note that* $v = [\frac{1}{2}(I - T^2)v] + [\frac{1}{2}(I + T^2)v]$ *for all* $v \in V$.

And $(I - T^4)v = (I - T^2)(I + T^2)v = 0$ *for all* $v \in V$,

$$\left.\begin{array}{l}(I + T^2)(\frac{1}{2}(I - T^2)v) = 0 \Rightarrow \frac{1}{2}(I - T^2)v \in \operatorname{null}(I + T^2) \\ (I - T^2)(\frac{1}{2}(I + T^2)v) = 0 \Rightarrow \frac{1}{2}(I + T^2)v \in \operatorname{null}(I - T^2)\end{array}\right\} \Rightarrow V = \operatorname{null}(I+T^2) + \operatorname{null}(I-T^2).$$

又 T *has no eigvals* $\Rightarrow (I - T^2)$ *is inje* $\Rightarrow \operatorname{null}(I - T^2) = \{0\}$.

Hence $V = \operatorname{null}(I+T^2) \Rightarrow \operatorname{range}(I+T^2) = \{0\}$. *Thus* $I+T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$.□

---

8 (Or 4E 5.A.31) *Give an example of* $T \in \mathcal{L}(\mathbf{R}^2)$ *such that* $T^4 = -I$.

Simply by computing: $p(z) = z^4 + 1 = (z^2 + \mathrm{i})(z^2 - \mathrm{i}) = (z + \mathrm{i}^{1/2})(z - \mathrm{i}^{1/2})(z - (-\mathrm{i})^{1/2})(z + (-\mathrm{i})^{1/2})$.

Note that $\mathrm{i}^{1/2} = \dfrac{\sqrt{2}}{2} + \mathrm{i}\dfrac{\sqrt{2}}{2}$, $(-\mathrm{i})^{1/2} = \dfrac{\sqrt{2}}{2} - \mathrm{i}\dfrac{\sqrt{2}}{2}$.

Hence $T = \pm(\pm\mathrm{i})^{1/2}$.

Define T *by* $T(x, y) = (\dfrac{\sqrt{2}}{2}x - \dfrac{\sqrt{2}}{2}y, \dfrac{\sqrt{2}}{2}x + \dfrac{\sqrt{2}}{2}y)$.

$$\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix} \Rightarrow \mathcal{M}(T)^4 = \mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathcal{M}(I). \square$$

$$\left(\text{Using} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}. \right)$$

• (*4E 5.B.12 See also at 5.A.9*)

Define $T \in \mathcal{L}(\mathbf{F}^n)$ *by* $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$. *Find the mini poly.*

$T(x_1, \dots, 0) =$ *By (5.A.9) and [8.49],* $1, 2, \dots, n$ *are zeros of the mini poly of* T.

(又 *Each eigvals of* T *corresponds to exact one-dim subsp of* $\mathbf{F}^n$.)

Define a poly q *by* $q(z) = (z - 1)(z - 2) \cdots (z - n)$, *for all* $z \in \mathbf{F}$. (*Then* q *is the char poly of* T.)

Because $q(T)e_j = [\,(T - I) \cdots (T - (j - 1)I)(T - (j + 1)I) \cdots (T - nI)\,]\,(T - jI)e_j = 0$ *for each* j,

where $(e_1, \dots, e_n)$ *is the standard basis. Thus* $\forall v \in \mathbf{F}^n, q(T)v = 0$. *Hence* q *is the mini poly of* T. □

• *Suppose* $n \in \mathbf{N}^+$. *Define* $T \in \mathcal{L}(\mathbf{F}^n)$ *by* $T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$.

[*See also at (5.A.19)*] *Find the mini poly of* T.

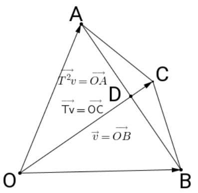
Because n *and* 0 *are all eigvals of* T, 又 *For all* $e_k, Te_k = e_1 + \cdots + e_n$; $T^2 e_k = n(e_1 + \cdots + e_n)$.

Hence $T^2e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T-n)$. *Thus* $z(z-n)$ *is the mini poly of* T. $\square$

- (*4E 5.B.8*)

  Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ *is the operator of counterclockwise rotation by the angel* θ*, where* $x \in \mathbf{R}^+$. *Find the mini poly of* T.

If $\theta = \pi$*, then* $T(w,z) = (-w,-z), T^2 = I$ *and the mini poly is* $z+1$.
If $2\pi|\theta$*, then* $T = I$ *and the mini poly is* $z-1$.
Now suppose (v, Tv) *is linely inde.*
Then $\operatorname{span}(v, Tv) = \mathbf{R}^2$.
Suppose the mini poly p *is defined by* $p(z) = z^2 + bz + c$ *for all* $z \in \mathbf{R}$.

Because 

$$\begin{cases} Tv = \dfrac{|\vec{v}|}{2L}(T^2v + v) \Rightarrow T = \dfrac{|\vec{v}|}{2L}(T^2 + I) \\[2ex] L = |\vec{v}|\cos\theta \Rightarrow \dfrac{|\vec{v}|}{2L} = \dfrac{1}{2\cos\theta} \end{cases}$$

Hence $p(T) = T^2 - 2\cos\theta T + I = 0$. $z^2 - 2\cos\theta z + 1$ *is the mini poly of* T. $\square$

- (*4E 5.B.11*)

  Suppose V *is a two-dim vecsp,* $T \in \mathcal{L}(V)$*, and the matrix of* T *with resp to some basis of* V *is* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

  (a) *Show that* $T^2 - (a+d)T + (ad-bc)I = 0$.

  (b) *Show that the mini poly of* T *equals*

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad-bc) & \text{otherwise.} \end{cases}$$

(a) *Suppose the basis is* (v, w). *Because* $\begin{cases} Tv = av + bw \Rightarrow (T-aI)v = bw, \text{ then apply } (T-dI) \text{ to both sides.} \\ Tw = cv + dw \Rightarrow (T-dI)w = cv, \text{ then apply } (T-aI) \text{ to both sides.} \end{cases}$

Hence $(T-aI)(T-dI) = bcI \Rightarrow T^2 - (a+d)T + (ad-bc)I = 0$.

(b) *If* $b = c = 0$ *and* $a = d$. *Then* $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. *Thus* $T = aI$. *Hence the mini poly is* $z - a$.

Otherwise, by (a), $z^2 - (a+d)z + (ad-bc)$ *is a poly multi of the mini poly.*
Now we prove that $T \notin \operatorname{span}(I)$*, so that then the mini poly of* T *has exactly degree* 2.
(*At least one of the assumption of* (*I*),(*II*) *below is true.*)
(*I*) *Suppose* $a = d$*, then* $Tv = av + bw \notin \operatorname{span}(v), Tw = cv + aw \notin \operatorname{span}(w)$.
(*II*) *Suppose at most one of* b, c *is not* 0. *If* $b = 0$*, then* $Tw \notin \operatorname{span}(w)$*; If* $c = 0$*, then* $Tv \notin \operatorname{span}(v)$. $\square$

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5 *Suppose* $S, T \in \mathcal{L}(V)$, S *is inv, and* $p \in \mathcal{P}(\mathbf{F})$. *Prove that* $p(TS) = S^{-1}p(ST)S$.

Solution:

We prove $(TS)^m = S^{-1}(ST)^m S$ *for each* $m \in \mathbf{N}$ *by induction.*

(i) $m = 0, 1$. $TS^0 = I = S^{-1}(ST)^0 S$; $TS = S^{-1}(ST)S$.

(ii) $m > 1$. *Assume that* $(TS)^m = S^{-1}(ST)^m S$.

$$\text{Then } (TS)^{m+1} = (TS)^m(TS) = S^{-1}(ST)^m STS = S^{-1}(ST)^{m+1}S.$$

$$\begin{aligned}\text{Hence } \forall p \in \mathcal{P}(\mathbf{F}), p(TS) &= a_0(TS)^0 + a_1(TS) + \cdots + a_m(TS)^m \\ &= a_0[S^{-1}(ST)^0S] + a_1[S^{-1}(ST)S] + \cdots + a_m[S^{-1}(ST)^mS] \\ &= S^{-1}[a_0(ST)^0 + a_1(ST) + \cdots + a_m(ST)^m]S = S^{-1}p(ST)S. \quad \square\end{aligned}$$

- (*4E 5.B.7*)

  (*a*) *Give an example of $S, T \in \mathcal{L}(\mathbf{F}^2)$ such that*

  the mini poly of ST does not equal the mini poly of TS.

  (*b*) *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that if S or T is inv,*

  then the mini poly of ST equals the mini poly of TS.

(*a*) *Define S by $S(x,y) = (x,x)$. Define T by $T(x,y) = (0,y)$.*

Then $ST(x,y) = 0$, $TS(x,y) = (0,x)$ for all $(x,y) \in \mathbf{F}^2$.

Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.

Hence the mini poly of ST does not equal to the mini poly of TS.

(*b*) *Denote the mini poly of ST by p, and the mini poly TS by q.*

Suppose S is inv.

$$\left.\begin{aligned} &p(ST) = 0 = Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a poly multi of } q. \\ &q(TS) = 0 = S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a poly multi of } p. \end{aligned}\right\} \Rightarrow p = q.$$

Reversing the roles of S and T, we conclude that if T is inv, then $p = q$ as well. $\square$

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11 *Suppose* $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{C})$, *and* $\alpha \in \mathbf{C}$.

Solution:

Prove that α is an eigval of $p(T) \iff \alpha = p(\lambda)$ for some eigval λ of T.

(*a*) *Suppose α is an eigval of $p(T) \Leftrightarrow (p(T) - \alpha I)$ is not inje.*

Write $p(z) - \alpha = c(z - \lambda_1)\cdots(z - \lambda_m) \Rightarrow p(T) - \alpha I = c(T - \lambda_1 I)\cdots(T - \lambda_m I)$.

By Tips, $\exists\, (T - \lambda_j I)$ not inje. Thus $p(\lambda_j) - \alpha = 0$. $\square$

(*b*) *Suppose $\alpha = p(\lambda)$ and λ is an eigval of T with an eigvec v. Then $p(T)v = p(\lambda)v = \alpha v$.* $\square$

Or. Define q by $q(z) = p(z) - \alpha$. λ is a zero of q.

Because $q(T)v = (p(T) - \alpha I)v = q(\lambda)v = (p(\lambda) - \alpha)v = 0$.

Hence $q(T)$ is not inje $\Rightarrow (p(T) - \alpha I)$ is not inje. $\square$

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12 (

Solution: *Or 4E.5.B.6*) *Give an example of an operator on* $\mathbf{R}^2$

that shows the result above does not hold if $\mathbf{C}$ *is replaced with* $\mathbf{R}$.

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(w,z) = (-z, w)$.

By Problem (4E 5.B.11), $\mathcal{M}(T, ((1,0),(0,1))) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ the mini poly of T is $z^2 + 1$.

Define p by $p(z) = z^2$. Then $p(T) = T^2 = -I$. Thus $p(T)$ has eigval -1.

While $\nexists\, \lambda \in \mathbf{R}$ such that $-1 = p(\lambda) = \lambda^2$. $\square$

- ($4E$ 5.B.17)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and p is the mini poly of T. Suppose $\lambda \in \mathbf{F}$.

Show that the mini poly of $(T - \lambda I)$ is the poly q defined by $q(z) = p(z + \lambda)$.

$q(T - \lambda I) = 0 \Rightarrow$ *q is poly multi of the mini poly of $(T - \lambda I)$.*

Suppose the degree of the mini poly of $(T - \lambda I)$ is n, and the degree of the mini poly of T is m.

By definition of mini poly,

n is the smallest such that $(T - \lambda I)^n \in \operatorname{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{n-1})$;

m is the smallest such that $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.

又 $T^k \in \operatorname{span}(I, T, \dots, T^{k-1}) \Longleftrightarrow (T - \lambda)^k \in \operatorname{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{k-1})$.

Thus $n = m$. 又 q is monic. By the uniqnes of mini poly. $\square$

- ($4E$ 5.B.18)

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and p is the mini poly of T. Suppose $\lambda \in \mathbf{F} \backslash \{0\}$.

Show that the mini poly of λT is the poly q defined by $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.

$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow$ *q is a poly multi of the mini poly of λT.*

Suppose the degree of the mini poly of λT is n, and the degree of the mini poly of T is m.

By definition of mini poly,

n is the smallest such that $(\lambda T)^n \in \operatorname{span}(\lambda I, \lambda T, \dots, (\lambda T)^{n-1})$;

m is the smallest such that $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.

又 $(\lambda T)^k \in \operatorname{span}(\lambda I, \lambda T, \dots, (\lambda T)^{k-1}) \Longleftrightarrow T^k \in \operatorname{span}(I, T \dots, T^{k-1})$.

Thus $n = m$. 又 q is monic. By the uniqnes of mini poly. $\square$

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18 (**SOLUTION:** *OR 4E 5.B.15*)

Suppose V is a finite-dim complex vector space with $\dim V > 0$ and $T \in \mathcal{L}(V)$.

Define $f : \mathbf{C} \to \mathbf{R}$ by $f(\lambda) = \dim \operatorname{range}(T - \lambda I)$. Prove that f is not a continuous function.

Note that V is finite-dim.

Let λ_0 be an eigval of T. Then $(T - \lambda_0 I)$ is not surj. Hence $\dim \operatorname{range}(T - \lambda_0 I) < \dim V$.

Because T has finitely many eigvals. There exist a sequence of number $\{\lambda_n\}$ such that $\lim\limits_{n \to \infty} \lambda_n = \lambda_0$.

And λ_n *is not an eigval of* T *for each* $n \Rightarrow \dim \operatorname{range}(T - \lambda_n I) = \dim V \neq \dim \operatorname{range}(T - \lambda_0 I)$.

Thus $f(\lambda_0) \neq \lim_{n \to \infty} f(\lambda_n)$. $\square$

- (*4E 5.B.9*)

Suppose $T \in \mathcal{L}(V)$ *is such that with resp to some basis of* V,
all entries of the matrix of T *are rational numbers.*
Explain why all coefficients of the mini poly of T *are rational numbers.*

Let $(v_1, \dots, v_n)$ *denote the basis such that* $\mathcal{M}(T, (v_1, \dots, v_n))_{j,k} = A_{j,k} \in \mathbf{Q}$ *for all* $j, k = 1, \dots, n$.

Denote $\mathcal{M}\left(v_j, (v_1, \dots, v_n)\right)$ *by* x_j *for each* v_j.

Suppose p *is the mini poly of* T *and* $p(z) = z^m + \cdots + c_1 z + c_0$. *Now we show that each* $c_j \in \mathbf{Q}$.

Note that $\forall s \in \mathbf{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbf{Q}^{n,n}$ *and* $T^s v_k = A^s_{1,k} v_1 + \cdots + A^s_{n,k} v_n$ *for all* $k \in \{1, \dots, n\}$.

$$\text{Thus} \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \cdots + c_1 A + c_0 I)x_1 = \sum_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,1} x_j = 0; \\ \qquad \vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \cdots + c_1 A + c_0 I)x_n = \sum_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly,} \begin{cases} (A^m + \cdots + c_1 A + c_0 I)_{1,1} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,1} = 0; \\ \qquad \vdots \qquad \ddots \qquad \vdots \\ (A^m + \cdots + c_1 A + c_0 I)_{1,n} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,n} = 0; \end{cases}$$

Hence we get a system of n^2 *linear equations in* m *unknowns* $c_0, c_1, \dots, c_{m-1}$.

We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbf{Q}$. $\square$

- *OR* (*4E 5.B.16*), *OR* (*8.C.18*)

Suppose $a_0, \dots, a_{n-1} \in \mathbf{F}$. *Let* T *be the operator on* $\mathbf{F}^n$ *such that*

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \text{ with resp to the standard basis } (e_1, \dots, e_n).$$

Show that the mini poly of T *is* p *defined by* $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$.

*$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the mini poly of some operator. Hence a formula or an algorithm that could produce exact eigvals for each operator on each $\mathbf{F}^n$ could then produce exact zeros for each poly [by 8.36(b)]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an operator.*

Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ *is linely inde.* 又 *The deg of mini poly is at most* n.

$T^n e_1 = \cdots = T^{n-k}e_{1+k} = \cdots = Te_n = -a_0 e_1 - a_1 e_2 - a_2 e_3 - \cdots - a_{n-1}e_n$
$= (-a_0 I - a_1 T - a_2 T^2 - \cdots - a_{n-1}T^{n-1})e_1$. *Thus* $p(T)e_1 = 0 = p(T)e_j$ *for each* $e_j = T^{j-1}e_1$.□

• *Eigenvalues on Odd-Dimensional Real Vector Spaces* (*Eigvals on Odd-dim Real Vecsps*)

• *Even-Dimensional Null Space*

Suppose $\mathbf{F} = \mathbf{R}$, V *is finite-dim,* $T \in \mathcal{L}(V)$ *and* $b, c \in \mathbf{R}$ *with* $b^2 < 4c$.

Prove that $\dim \operatorname{null}(T^2 + bT + cI)$ *is an even number.*

Denote $\operatorname{null}(T^2 + bT + cI)$ *by* R. *Then* $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.

Suppose λ *is an eigval of* T_R *with an eigvec* $v \in R$.

Then $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \left((\lambda + b)^2 + c - \dfrac{b^2}{4}\right)v$.

Because $c - \dfrac{b^2}{4} > 0$ *and we have* $v = 0$. *Thus* T_R *has no eigvals.*

Let U *be an invar subsp of* R *that has the largest, even dim among all invar subsps.*

Assume that $U \neq R$. *Then* $\exists\, w \in R$ *but* $w \notin U$. *Let* W *be such that* $(w, T|_R w)$ *is a basis of* W.

Because $T|_R^2 w = -bT|_R w - cw \in W$. *Hence* W *is an invar subsp of dim* 2.

Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, *where* $U \cap W = \{0\}$,

for if not, because $w \notin U, T|_R w \in U$,

$U \cap W$ *is invar under* $T|_R$ *of one dim (impossible because* $T|_R$ *has no eigvecs).*

Hence $U + W$ *is even-dim invar subsp under* $T|_R$, *contradicting the maximality of* $\dim U$.

Thus the assumption was incorrect. Hence $R = \operatorname{null}(T^2 + bT + cI) = U$ *has even dim.* □

• *Operators On Odd-Dimensional Vector Spaces Have Eigenvalues*

(a) *Suppose* $\mathbf{F} = \mathbf{C}$. *Then by* [5.21], *we are done.*

(b) *Suppose* $\mathbf{F} = \mathbf{R}$, V *is finite-dim, and* $\dim V = n \neq 0$ *is an odd number.*

Let $T \in \mathcal{L}(V)$ *and the mini poly is* p. *Prove that* T *has an eigval.*

(i) *If* $n = 1$, *then we are done.*

(ii) *Suppose* $n \geq 3$. *Assume that every operator, on odd-dim vecsps of dim less than* n, *has an eigval.*

If p *is a poly multi of* $(x - \lambda)$ *for some* $\lambda \in \mathbf{R}$, *then by* [8.49] λ *is an eigval of* T *and we are done.*

Now suppose $b, c \in \mathbf{R}$ *such that* $b^2 < 4c$ *and* p *is a poly multi of* $x^2 + bx + c$ (*see* [4.17]).

Then $\exists\, q \in \mathcal{P}(\mathbf{R})$ *such that* $p(x) = q(x)(x^2 + bx + c)$ *for all* $x \in \mathbf{R}$.

Now $0 = p(T) = (q(T))\,(T^2 + bT + cI)$*, which means that* $q(T)|_{\operatorname{range}(T^2+bT+cI)} = 0$.

Because $\deg q < \deg p$ *and* p *is the mini poly of* T*, hence* $\operatorname{range}(T^2 + bT + cI) \neq V$.

又 $\dim V$ *is odd and* $\dim \operatorname{null}(T^2 + bT + cI)$ *is even (by our previous result).*

Thus $\dim V - \dim \operatorname{null}(T^2 + bT + cI) = \dim \operatorname{range}(T^2 + bT + cI)$ *is odd.*

By [5.18], $\operatorname{range}(T^2 + bT + cI)$ *is an invar subsp of* V *under* T *that has odd dim less than* n.

Our induction hypothesis now implies that $T|_{\operatorname{range}(T^2+bT+cI)}$ *has an eigval.*

By mathematical induction. □

• (*2E Ch5.24*)

Suppose $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$ *has no eigvals.*

Prove that every invar subsp of V *under* T *is even-dim.*

Suppose U *is such a subsp. Then* $T|_U \in \mathcal{L}(U)$*. We prove by contradiction.*

If $\dim U$ *is odd, then* $T|_U$ *has an eigval and so is* T*, so that* $\exists$ *invar subsp of* 1 *dim, contradicts.* □

• (*4E 5.B.29*)

Show that every operator on a finite-dim vecsp of dim ≥ 2

has an invar subsp of dim 2.

Exercise (4E 5.C.6) will give an improvement of this result when $\mathbf{F} = \mathbf{C}$.

Using induction on $\dim V$.

(i) $\dim V = 2$*, we are done.*

(ii) $\dim V > 2$*. Assume that the desired result is true for vecsp of smaller dim.*

Suppose p *is the mini poly of degree* m *and* $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.

If $T = \lambda I$ ($\Leftrightarrow m = 1 \vee m = -\infty$), *then we are done.* ($m \neq 0$ *because* $\dim V \neq 0$.)

Now define a q *by* $q(z) = (z - \lambda_1)(z - \lambda_2)$.

By assumption, $T|_{\operatorname{null} q(T)}$ *has an invar subsp of dim* 2. □

Ended

5.B: II

• (*4E 5.C.1*) *Prove or give a counterexample:*

If $T \in \mathcal{L}(V)$ *and* T^2 *has an upper-trig matrix, then* T *has an upper-trig matrix.*

Solution:

• $(4E\ 5.C.2)$ *Suppose A and B are upper-trig mtcs of the same size,*
with $\alpha_1, \dots, \alpha_n$ on the diag of A and $\beta_1, \dots, \beta_n$ on the diag of B.
(a) *Show that $A + B$ is an upper-trig matrix with $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ on the diag.*
(b) *Show that AB is an upper-trig matrix with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diag.*

Solution:

• $(4E\ 5.C.3)$
Suppose $T \in \mathcal{L}(V)$ is inv and $B = (v_1, \dots, v_n)$ is a basis of V such that
$\mathcal{M}(T, B) = A$ is upper trig, with $\lambda_1, \dots, \lambda_n$ on the diag.
Show that the matrix of $\mathcal{M}(T^{-1}, B) = A^{-1}$ is also upper trig, with $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ on the diag.

Solution:

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9 $(4E\ 5.C.7)$
Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $v \in V$.
(a) *Prove that $\exists\,!$ monic poly p_v of smallest degree such that $p_v(T)v = 0$.*
(b) *Prove that the mini poly of T is a poly multi of p_v.*

Solution:

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14 $(\text{OR } 4E\ 5.C.4)$ *Give an operator T such that with resp to some basis,*
$\mathcal{M}(T)_{k,k} = 0$ for each k, while T is inv.

Solution:

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15 $(\text{OR } 4E\ 5.C.5)$ *Give an operator T such that with resp to some basis,*
$\mathcal{M}(T)_{k,k} \neq 0$ for each k, while T is not inv.

Solution:

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20 $(\text{OR } 4E\ 5.C.6)$
Suppose $\mathbf{F} = \mathbf{C}$, V is finite-dim, and $T \in \mathcal{L}(V)$.
Prove that if $k \in \{1, \dots, \dim V\}$, then V has a k dim subsp invar under T.

Solution:

• $(4E\ 5.C.8)$ *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\exists\, v \in V \backslash \{0\}$ such that $T^2v + 2Tv = -2v$.*
(a) *Prove that if $\mathbf{F} = \mathbf{R}$, then $\nexists$ a basis of V with resp to which T has an upper-trig matrix.*

(*b*) *Prove that if* $\mathbf{F} = \mathbf{C}$ *and* A *is an upper-trig matrix that equals the matrix of* T *with resp to some basis of* V, *then* $-1 + \mathrm{i}$ *or* $-1 - \mathrm{i}$ *appears on the diag of* A.

Solution:

- (4E 5.C.9) *Suppose* $B \in \mathbf{F}^{n,n}$ *with complex entries.*
  Prove that $\exists$ *inv* $A \in \mathbf{F}^{n,n}$ *with complex entries such that* $A^{-1}BA$ *is an upper-trig matrix.*

Solution:

- (4E 5.C.10) *Suppose* $T \in \mathcal{L}(V)$ *and* $(v_1, \dots, v_n)$ *is a basis of* V.
  Show that the following are equi.
  (*a*) *The matrix of* T *with resp to* $(v_1, \dots, v_n)$ *is lower trig.*
  (*b*) $\operatorname{span}(v_k, \dots, v_n)$ *is invar under* T *for each* $k = 1, \dots, n$.
  (*c*) $Tv_k \in \operatorname{span}(v_k, \dots, v_n)$ *for each* $k = 1, \dots, n$.

Solution:

- (4E 5.C.11) *Suppose* $\mathbf{F} = \mathbf{C}$ *and* V *is finite-dim.*
  Prove that if $T \in \mathcal{L}(V)$, *then* T *has a lower-trig matrix with resp to some basis.*

Solution:

- (4E 5.C.12)
  Suppose V *is finite-dim,* $T \in \mathcal{L}(V)$ *has an upper-trig matrix with resp to some basis, and* U *is a subsp of* V *that is invar under* T.
  (*a*) *Prove that* $T|_U$ *has an upper-trig matrix with resp to some basis of* U.
  (*b*) *Prove that* T/U *has an upper-trig matrix with resp to some basis of* V/U.

Solution:

- (4E 5.C.13) *Suppose* V *is finite-dim,* $T \in \mathcal{L}(V)$. *Suppose* U *is an invar subsp of* V *under* T *such that* $T|_U$ *has an upper-trig matrix and also* T/U *has an upper-trig matrix.*
  Prove that T *has an upper-trig matrix.*

Solution:

- (4E 5.C.14) *Suppose* V *is finite-dim and* $T \in \mathcal{L}(V)$.
  Prove that T *has an upper-trig matrix* $\Longleftrightarrow$ T' *has an upper-trig matrix.*

Solution:

Ended

5.C

Ended

5.E* (4E)

1 *Give an example of two commuting operators $S, T \in \mathbf{F}^4$ such that there is an invar subsp of $\mathbf{F}^4$ under S but not under T and an invar subsp of $\mathbf{F}^4$ under T but not under S.*

Solution:

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2 *Suppose $\mathcal{E}$ is a subset of $\mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagable.*
Prove that $\exists$ a basis of V with resp to which every element of $\mathcal{E}$ has a diag matrix $\iff$ every pair of elements of $\mathcal{E}$ commutes.

This exercise extends [5.76], which considers the case in which $\mathcal{E}$ contains only two elements.

For this exercise, $\mathcal{E}$ may contain any number of elements, and $\mathcal{E}$ may even be an infinite set.

Solution:

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3 *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.*

(a) Prove that $\operatorname{null} p(S)$ is invar under T.

(b) Prove that $\operatorname{range} p(S)$ is invar under T.

See Note For [5.17] for the special case $S = T$.

Solution:

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4 *Prove or give a counterexample:*
A diag matrix A and an upper-trig matrix B of the same size commute.

Solution:

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5 *Prove that a pair of operators on a finite-dim vecsp commute $\iff$ their dual operators commute.*

Solution:

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6 *Suppose V is a finite-dim complex vecsp and $S, T \in \mathcal{L}(V)$ commute.*
Prove that $\exists\, \alpha, \lambda \in \mathbf{C}$ *such that* $\text{range}\,(S - \alpha I) + \text{range}\,(T - \lambda I) \neq V$.

Solution:

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7 *Suppose V is a complex vecsp, $S \in \mathcal{L}(V)$ is diagable, and T commutes with S.*
Prove that $\exists$ basis B of V such that S has a diag matrix with resp to B
and T has an upper-trig matrix with resp to B.

Solution:

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8 *Suppose $m = 3$ in Example* [5.72]
and D_x, D_y are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbf{R}^2)$ from that example.
Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ with resp to which D_x and D_y each have an upper-trig matrix.

Solution:

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9 *Suppose V is a finite-dim nonzero complex vecsp.*
Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.
(a) *Prove that $\exists\, v \in V$ is an eigvec for every element of $\mathcal{E}$.*
(b) *Prove that $\exists$ a basis of V with resp to which every element of $\mathcal{E}$ has an upper-trig matrix.*

Solution:

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10 *Give an example of two commuting operators S, T on a finite-dim real vecsp such that*
$S + T$ has a eigval that does not equal an eigval of S plus an eigval of T
and ST has a eigval that does not equal an eigval of S times an eigval of T.

Solution:

Ended