简介

这是我个人用于复习的笔记，一本习题补注。由于我个人的复习特点，我把许多对我个人而言没什么复习价值的习题作了省略。为什么我没有用中文？因为我将来要学习的绝大多数数学课本都是全英的，国内目前的专业翻译速度慢、不全面，况且对于专业学习者来说，直接使用英文不会造成任何困扰，并且我不愿意花费额外的时间去翻译，所以我用英文。但我讨厌英文单词的冗长性，这会让我复习起来很不爽，所以我对许多常用词汇适当地作了简写。这份笔记的内容范围和标识说明，我已经在README中写得很清楚，不再赘述。这份笔记尚处于缓慢的编撰进度中。

Goto

1	2	3	4	5	6	7	8	9	10
A	A	A	/	A	A	A	A	A	A
B	B	B	/	B[I]	B	B	B	B	B
/	/	/	/	B[II]	/	/	/	/	/
C	C	C	/	C	C	C	C	/	/
/	/	D	/	/	D	D	D	/	/
/	/	E	/	E*	/	/	/	/	/
/	/	F	/	/	/	F*	/	/	/

Abbreviation Table

def	definition
vec	vector
vecsp	vector space
subsp	subspace
add	addition/additive
multi	multiplication/multiplicative/multiple
assoc	associative/associativity
distr	distributive properties/property
inv	inverse
existns	existence
uniqnes	uniqueness
linely inde	linearly independent/independence
linely dep	linearly dependent/dependence
dim	dimension(al)
inje	injective
surj	surjective
col	column
with resp	with respect
standard basis	std basis
iso	isomorphism/isomorphic
correspd	correspond(ing)
poly	polynomial
eigval	eigenvalue
eigvec	eigenvector
mini poly	minimal polynomial
char poly	characteristic polynomial

1·B

1 *Prove that $\forall v \in V, -(-v) = v$.*

SOLUTION:

$$\left.\begin{aligned} -(-v) + (-v) &= 0 \\ v + (-v) &= 0 \end{aligned}\right\} \Rightarrow \text{By the uniqnes of add inv, we are done.}$$

OR. $-(-v) = (-1)\big((-1)v\big) = \big((-1)(-1)\big)v = 1 \cdot v = v.$ ☐

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2 *Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.*

SOLUTION:

Suppose $a \neq 0, \exists\, a^{-1} \in \mathbf{F}, a^{-1}a = 1$, hence $v = 1 \cdot v = (a^{-1}a)v = a^{-1}(av) = a^{-1} \cdot 0 = 0.$ ☐

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3 *Suppose $v, w \in V$. Explain why $\exists\,!\, x \in V, v + 3x = w$.*

SOLUTION:

[*Existns*] Let $x = \dfrac{1}{3}(w - v)$.

[*Uniqnes*] Suppose $v + 3x_1 = w$,(I) $v + 3x_2 = w$ (II). Then (I) $-$ (II) : $3(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. ☐

OR. $v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \frac{1}{3}(w - v).$ ☐

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5 *Show that in the def of a vecsp, the add inv condition can be replaced by [1.29].*

Hint: *Suppose V satisfies all conds in the def, except we've replaced the add inv cond with [1.29]. Prove that the add inv is true.*

SOLUTION:

Using [1.31]. $0v = 0$ for all $v \in V \Longleftrightarrow (1 + (-1))v = 1 \cdot v + (-1)v = v + (-v) = 0.$ ☐

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6 *Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$.*
Define an add and scalar multi on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess.
The operations of real numbers is as usual. While for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t < 0, \end{cases}$$

(I) $t + \infty = \infty + t = \infty + \infty = \infty$,
(II) $t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty$,
(III) $\infty + (-\infty) = (-\infty) + \infty = 0$.

With these operations of add and scalar multi, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vecsp over $\mathbf{R}$? Explain.

SOLUTION:

Not a vecpsp, since the add and scalar mult is not assoc and distr.
By Assoc: $(a + \infty) + (-\infty) \neq a + (\infty + (-\infty))$.
OR. By Distr: $\infty = (2 + (-1))\infty \neq 2\infty + (-\infty) = \infty + (-\infty) = 0.$ ☐

- **TIPS:** *About the Field $\mathbf{F}$: Many choices.*
  EXAMPLE: $\mathbf{F} = \mathbf{Z}_m = \{K_0, K_1, \dots, K_{m-1}\}, \forall m - 1 \in \mathbf{N}^+$.

ENDED

1·C

7 8 9 11 12 13 15 16 17 18 21 22 23 24

7 *Give a nontrivial $U \subseteq \mathbf{R}^2$,*
U is closed under taking add invs and under add, but is not a subsp of $\mathbf{R}^2$.

Solution: $\big(0 \in U;\ v \in U \Rightarrow -v \in U. \big)$ Let $U = \{0,1\}^2, \mathbf{Z}^2, \mathbf{Q}^2$.

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8 *Give a nontrivial $U \subseteq \mathbf{R}^2$, U is closed under scalar multi, but is not a subsp of $\mathbf{R}^2$.*

Solution: Let $U = \{(x,y) \in \mathbf{R}^2 : x = 0 \vee y = 0\}$.

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9 *A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if $\exists\, p \in \mathbf{N}^+$, $f(x) = f(x+p)$ for all $x \in \mathbf{R}$.*
Is the set of periodic functions $\mathbf{R} \to \mathbf{R}$ a subsp of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution: Denote the set by S.

Suppose $h(x) = \cos x + \sin\sqrt{2}x \in S$, since $\cos x, \sin\sqrt{2}x \in S$.

Assume $\exists\, p \in \mathbf{N}^+$ such that $h(x) = h(x+p), \forall x \in \mathbf{R}$. Let $x = 0 \Rightarrow h(0) = h(\pm p) = 1$.

Thus $1 = \cos p + \sin\sqrt{2}p = \cos p - \sin\sqrt{2}p$

$\Rightarrow \sin\sqrt{2}p = 0,\ \cos p = 1 \Rightarrow p = 2k\pi, k \in \mathbf{Z}$, while $p = \dfrac{m\pi}{\sqrt{2}}, m \in \mathbf{Z}$.

Hence $2k = \dfrac{m}{\sqrt{2}} \Rightarrow \sqrt{2} = \dfrac{m}{2k} \in \mathbf{Q}$. Contradiction! ☐

Or. Because $[\mathrm{I}] : \cos x + \sin\sqrt{2}x = \cos(x+p) + \sin\big(\sqrt{2}x + \sqrt{2}p\big)$. By differentiating twice,

$$[\mathrm{II}] : \cos x + 2\sin\sqrt{2}x = \cos(x+p) + 2\sin\big(\sqrt{2}x + \sqrt{2}p\big).$$

$$\left.\begin{array}{ll} [\mathrm{II}]-[\mathrm{I}] : & \sin\sqrt{2}x = \sin\big(\sqrt{2}x+\sqrt{2}p\big) \\ 2[\mathrm{I}]-[\mathrm{II}] : & \cos x = \cos(x+p) \end{array}\right\} \Rightarrow \text{Let } x = 0,\ p = \frac{m\pi}{\sqrt{2}} = 2k\pi.\ \text{Contradicts.}$$ ☐

• *Suppose U, W, V_1, V_2, V_3 are subsps of V.*

15 $\quad U + U \ni u + w \in U.$ ☐

16 $\quad U + W \ni u + w = w + u \in W + U.$ ☐

17 $\quad (V_1 + V_2) + V_3 \ni (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3).$ ☐

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18 *Does the add on the subsps of V have an add identity? Which subsps have add invs?*

Solution: Suppose Ω is the additive identity.

(a) For any subsp U of V. $\Omega \subseteq U + \Omega = U \Rightarrow \Omega \subseteq U$. Let $U = \{0\}$, then $\Omega = \{0\}$.

(b) Now suppose W is an add inv of $U \Rightarrow U + W = \Omega$.

Note that $U + W \supseteq U, W \Rightarrow \Omega \supseteq U, W$. Thus $U = W = \Omega = \{0\}$. ☐

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11 *Prove that the intersection of every collection of subsps of V is a subsp of V.*

Solution: Suppose $\{U_\alpha\}_{\alpha\in\Gamma}$ is a collection of subsps of V; here Γ is an arbitrary index set.

We show that $\bigcap_{\alpha\in\Gamma} U_\alpha$, which equals the set of vecs that are in U_α for each $\alpha \in \Gamma$, is a subsp of V.

(一) $0 \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Nonempty.

(二) $u, v \in \bigcap_{\alpha\in\Gamma} U_\alpha \Rightarrow u + v \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow u + v \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under add.

(三) $u \in \bigcap_{\alpha\in\Gamma} U_\alpha, \lambda \in \mathbf{F} \Rightarrow \lambda u \in U_\alpha,\ \forall \alpha \in \Gamma \Rightarrow \lambda u \in \bigcap_{\alpha\in\Gamma} U_\alpha$. Closed under scalar multi.

Thus $\bigcap_{\alpha\in\Gamma} U_\alpha$ is nonempty subset of V that is closed under add and scalar multi. ☐

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12 *Suppose U, W are subsps of V. Prove that $U \cup W$ is a subsp of $V \iff U \subseteq W$ or $W \subseteq U$.*

SOLUTION:

(a) Suppose $U \subseteq W$. Then $U \cup W = W$ is a subsp of V.

(b) Suppose $U \cup W$ is a subsp of V. Suppose $U \not\subseteq W$ and $U \not\supseteq W$ $\left(U \cup W \neq U \text{ and } W \right)$.

Then $\forall a \in U \wedge a \notin W, b \in W \wedge b \notin U,\ a + b \in U \cup W$.

$$\left.\begin{array}{l} \text{If } a+b \in U \Rightarrow b = (a+b) + (-a) \in U \text{, contradicts!} \\ \text{If } a+b \in W \Rightarrow a = (a+b) + (-b) \in W \text{, contradicts!} \end{array}\right\} \Rightarrow U \cup W = U \text{ or } W. \text{ Contradicts!}$$

Thus $U \subseteq W$ and $U \supseteq W$. □

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13 *Prove that the union of three subsps of V is a subsp of V if and only if one of the subsps contains the other two.*

This exercise is not true if we replace $\mathbf{F}$ *with a field containing only two elements.*

SOLUTION:

Suppose U_1, U_2, U_3 are subsps of V. Denote $U_1 \cup U_2 \cup U_3$ by $\mathcal{U}$.

(a) Suppose that one of the subsps contains the other two.

Then $\mathcal{U} = U_1, U_2$ or U_3 is a subsp of V.

(b) Suppose that $U_1 \cup U_2 \cup U_3$ is a subsp of V.

Distinctively notice that $A \cup B \cup C = (A \cup B) \cup (B \cup C) = (A \cup C) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.

Also note that, if $U \cup W = V$ is a vecsp, then in general U and W are not subsps of V.

Hence this literal trick is invalid.

(I) If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $\mathcal{U} = U_2 \cup U_3$.

By applying Problem (12) we conclude that one U_j contains the other two. Thus we are done.

(II) Assume that no U_j is contained in the union of the other two,

and no U_j contains the union of the other two.

Say $U_1 \not\subseteq U_2 \cup U_3$ and $U_1 \not\supseteq U_2 \cup U_3$.

$\exists\, u \in U_1 \wedge u \notin U_2 \cup U_3;\ v \in U_2 \cup U_3 \wedge v \notin U_1$. Let $W = \left\{ v + \lambda u : \lambda \in \mathbf{F} \right\} \subseteq \mathcal{U}$.

Note that $W \cap U_1 = \emptyset$, for if $v + \lambda u \in U_1$ then $v + \lambda u - \lambda u = v \in U_1$.

又 $W \subseteq U_1 \cup U_2 \cup U_3$. Thus $W \subseteq U_2 \cup U_3$.

$\forall v + \lambda u \in W, \exists\, i \in \left\{2, 3\right\}, v + \lambda u \in U_i$.

Because U_2, U_3 are subsps and hence have at least one element (zero).

If $U_2 = U_3$, then $\mathcal{U} = U_1 \cup U_2$ and by Problem (12) we are done.

(Note that at least one of U_2, U_3 is not $\left\{0\right\}$, for if not, $U_1 \supseteq U_2 \cup U_3$.)

Otherwise, $\exists$ distinct $\lambda, \mu \in \mathbf{F}, v + \lambda u, v + \mu u \in U_i$ for some $i \in \left\{2, 3\right\}$. (Δ)

Then $u \in U_i$ while $u \notin U_2 \cup U_3$. Contradicts.

EXAMPLE: Let $\mathbf{F} = \mathbf{Z}_2$. TODO

□

• **Example:** *Suppose* $U = \{(x,x,y,y) \in \mathbf{F}^4\}, W = \{(x,x,x,y) \in \mathbf{F}^4\}$.
Prove that $U + W = \{(x,x,y,z) \in \mathbf{F}^4\}$.

Let T denote $\{(x,x,y,z) \in \mathbf{F}^4 : x,y,z \in \mathbf{F}\}$. By def, $U + W \subseteq T$.
And $T \ni (x,x,y,z) \Rightarrow (0,0,y-x,y-x) + (x,x,x,-y+x+z) \in U + W$. Hence $T \subseteq U + W$. $\square$

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21 *Suppose* $U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5\}$. *Find a* W *such that* $\mathbf{F}^5 = U \oplus W$.

Solution: Let $W = \{(0,0,z,w,u) \in \mathbf{F}^5\}$. Then $U \cap W = \{0\}$.
And $\mathbf{F}^5 \ni (x,y,z,w,u) \Rightarrow (x,y,x+y,x-y,2x) + (0,0,z-x-y,w-x-y,u-2x) \in U + W$.

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23 *Give an example of vecsps* V_1, V_2, U *such that* $V_1 \oplus U = V_2 \oplus U$, *but* $V_1 \neq V_2$.

Solution: $V = \mathbf{F}^2, U = \{(x,x) \in \mathbf{F}^2\}, V_1 = \{(x,0) \in \mathbf{F}^2\}, V_2 = \{(0,x) \in \mathbf{F}^2\}$.

• **Tips:** *Suppose* $V_1 \subseteq V_2$ *in Exercise* (23). *Prove or give a counterexample:* $V_1 = V_2$.

Solution:

Because the subset V_1 of vecsp V_2 is closed under add and scalar multi, V_1 is a subspace of V_2.
Suppose W is such that $V_2 = V_1 \oplus W$. Now $V_2 \oplus U = (V_1 \oplus W) \oplus U = (V_1 \oplus U) \oplus W = V_1 \oplus U$.
If $W \neq \{0\}$, then $V_1 \oplus U \subsetneq (V_1 \oplus U) \oplus W$, contradicts. Hence $W = \{0\}, V_1 = V_2$. $\square$

Suppose $V_1 \oplus U_1 = V_2 \oplus U_2, V_1 \subseteq V_2, U_2 \subseteq U_1$. Prove or give a counterexample: $V_1 = V_2$.

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24 *Let* $V_E = \{f \in \mathbf{R}^\mathbf{R} : f \text{ is even}\}, V_O = \{f \in \mathbf{R}^\mathbf{R} : f \text{ is odd}\}$. *Show that* $V_E \oplus V_O = \mathbf{R}^\mathbf{R}$.

Solution: (a) $V_E \cap V_O = \{f \in \mathbf{R}^\mathbf{R} : f(x) = f(-x) = -f(-x)\} = \{0\}$.

(b) $\left| \begin{array}{l} \text{Let } f_e(x) = \frac{1}{2}(g(x) + g(-x)) \Longrightarrow f_e \in V_E \\ \text{Let } f_o(x) = \frac{1}{2}(g(x) - g(-x)) \Longrightarrow f_o \in V_O \end{array} \right\} \Rightarrow \forall g \in \mathbf{R}^\mathbf{R}, g(x) = f_e(x) + f_o(x)$. $\square$

Ended

2·A

1 2 6 10 11 14 16 17 | 4E: 3,14

2 (a) $[P]$ *A list* (v) *of length* 1 *in* V *is linely inde* $\Longleftrightarrow v \neq 0$. $[Q]$
(b) $[P]$ *A list* (v,w) *of length* 2 *in* V *is linely inde* $\Longleftrightarrow \forall \lambda, \mu \in \mathbf{F}, v \neq \lambda w, w \neq \mu v$. $[Q]$

Solution:

(a) $Q \overset{1}{\Rightarrow} P : v \neq 0 \Rightarrow$ if $av = 0$ then $a = 0 \Rightarrow (v)$ linely inde.
$P \overset{2}{\Rightarrow} Q : (v)$ linely inde $\Rightarrow v \neq 0$, for if $v = 0$, then $av = 0 \not\Rightarrow a = 0$.

Or. $\left| \begin{array}{l} \neg Q \overset{3}{\Rightarrow} \neg P : v = 0 \Rightarrow av = 0 \text{ while we can let } a \neq 0 \Rightarrow (v) \text{ is linely dep.} \\ \neg P \overset{4}{\Rightarrow} \neg Q : (v) \text{ linely dep} \Rightarrow av = 0 \text{ while } a \neq 0 \Rightarrow v = 0. \end{array} \right.$

Comment: (1) with (3) and (2) with (4) will do as well. $\square$

(b) $P \overset{1}{\Rightarrow} Q : (v,w)$ linely inde $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow$ no scalar multi.
$Q \overset{2}{\Rightarrow} P :$ no scalar multi $\Rightarrow$ if $av + bw = 0$, then $a = b = 0 \Rightarrow (v,w)$ linely inde.

Or. $\left| \begin{array}{l} \neg P \overset{3}{\Rightarrow} \neg Q : (v,w) \text{ linely dep} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{scalar multi} \\ \neg Q \overset{4}{\Rightarrow} \neg P : \text{scalar multi} \Rightarrow \text{if } av + bw = 0, \text{ then } a \text{ or } b \neq 0 \Rightarrow \text{linely dep.} \end{array} \right.$

Comment: (1) with (3) and (2) with (4) will do as well. $\square$

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1 *Prove that* $[P]\ (v_1, v_2, v_3, v_4)$ *spans* $V \iff (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ *also spans* $V\ [Q]$.

SOLUTION:

Notice that $V = \operatorname{span}(v_1, \dots, v_n) \iff \forall v \in V, \exists\, a_1, \dots, a_n \in \mathbf{F}, v = a_1v_1 + \cdots + a_nv_n$.

Assume that $\forall v \in V, \exists\, a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbf{F}$, (that is, if $\exists\, a_i$, then we are to find b_i, vice versa)

$$\begin{aligned} v &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \\ &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 \\ &= b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4. \end{aligned}$$

Now we can let $b_i = \sum_{r=1}^{i} a_r$ if we are to prove Q with P already assumed;

or let $a_i = b_i - b_{i-1}$ with $b_0 = 0$, if we are to prove P with Q already assumed. □

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6 *Prove that* $[P]\ (v_1, v_2, v_3, v_4)$ *is linely inde*
$\iff [Q]\ (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ *is linely inde.*

SOLUTION:

$$\begin{aligned} P \Rightarrow Q : &\ a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4v_4 = 0 \\ &\Rightarrow a_1v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0 \\ &\Rightarrow a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0 \end{aligned}$$

$$\begin{aligned} Q \Rightarrow P : &\ a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0 \\ &\Rightarrow a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + \cdots + a_4)v_4 = 0 \\ &\Rightarrow a_1 = a_1 + a_2 = a_1 + a_2 + a_3 = a_1 + \cdots + a_4 = 0. \end{aligned}$$ □

• *Suppose* $(v_1, \dots, v_m)$ *is a list of vecs in* V. *For each* k, *let* $w_k = v_1 + \cdots + v_k$.

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

(b) *Show that* $[P]\ (v_1, \dots, v_m)$ *is linely inde* $\iff (w_1, \dots, w_m)$ *is linely inde* $[Q]$.

SOLUTION:

(a) **let** $a_k = \sum\limits_{j=1}^{k} b_j \Leftarrow\ a_1v_1 + \cdots + a_mv_m = b_1w_1 + \cdots + b_mw_m\ \Rightarrow$ **let** $b_1 = a_1,\ b_k = a_k - \sum\limits_{j=1}^{k-1} b_j = \sum\limits_{j=1}^{k} (-1)^{k-j} a_j$.

(b) $P \Rightarrow Q:\ b_1w_1 + \cdots + b_mw_m = 0 = a_1v_1 + \cdots + a_mv_m,$ where $0 = a_k = \sum\limits_{j=1}^{k} b_k$.

$Q \Rightarrow P:\ a_1v_1 + \cdots + a_mv_m = 0 = b_1w_1 + \cdots + b_mw_m = 0,$ where $0 = b_1 = a_1,\ 0 = b_k = \sum\limits_{j=1}^{k} (-1)^{k-j} a_j$

OR. Because $W = \operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

By [2.21](b), a list of length $(m-1)$ spans W, then by [2.23],

$(w_1, \dots, w_m)$ linely dep $\Rightarrow (v_1, \dots, v_m)$ linely dep. Conversely it is true as well. □

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10 *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* V *and* $w \in V$.

Prove that if $(v_1 + w, \dots, v_m + w)$ *is linely depe, then* $w \in \operatorname{span}(v_1, \dots, v_m)$.

SOLUTION:

Suppose $a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0, \exists\, a_i \neq 0 \Rightarrow a_1v_1 + \cdots + a_mv_m = 0 = -(a_1 + \cdots + a_m)w$.

Then $a_1 + \cdots + a_m \neq 0$, for if not, $a_1v_1 + \cdots + a_mv_m = 0$ while $a_i \neq 0$ for some i, contradicts. □

OR. By contrapositive, $w \notin \operatorname{span}(v_1, \dots, v_m)$, similarly. □

OR. $\exists\, j \in \{1, \dots, m\}, v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$ then $v_1 + w = 0$ and we are done.

If $j \geqslant 2$, then $\exists\, a_i \in \mathbf{F}, v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1v_1 + \cdots + a_{j-1}v_{j-1}$.

Where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that $\lambda \neq 0$, for if not, $v_j + \lambda w = v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, contradicts.

Now $w = \lambda^{-1}(a_1v_1 + \cdots + a_{j-1}v_{j-1} - v_j) \Rightarrow w \in \operatorname{span}(v_1, \dots, v_m)$. □

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11 *Suppose $(v_1, \dots, v_m)$ is linely inde in V and $w \in V$.*
Show that $[P]$ $(v_1, \dots, v_m, w)$ is linely inde $\iff w \notin \operatorname{span}(v_1, \dots, v_m)$ $[Q]$.

Solution: $\neg Q \Rightarrow \neg P$: Suppose $w \in \operatorname{span}(v_1, \dots, v_m)$. Then $(v_1, \dots, v_m, w)$ is linely depe.
$\neg P \Rightarrow \neg Q$: Suppose $(v_1, \dots, v_m, w)$ is linely dep. Then by [2.21] $w \in \operatorname{span}(v_1, \dots, v_m)$. □

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14 *Prove that $[P]$ V is infinite-dim $\iff$ $[Q]$ there is a sequence $(v_1, v_2, \dots)$ in V such that $(v_1, \dots, v_m)$ is linely inde for each $m \in \mathbf{N}^+$.*

Solution:

$P \Rightarrow Q$: Suppose V is infinite-dim, so that no list spans V.
Step 1 Pick a $v_1 \neq 0$, (v_1) linely inde.
Step m Pick a $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$, by Problem (10)(b), $(v_1, \dots, v_m)$ is linely inde.
This process recursively defines the desired sequence $(v_1, v_2, \dots)$.

$\neg P \Rightarrow \neg Q$: Suppose V is finite-dim and $V = \operatorname{span}(w_1, \dots, w_m)$.
Let $(v_1, v_2, \dots)$ be a sequence in V, then $(v_1, v_2, \dots, v_{m+1})$ must be linely dep.

Or. $Q \Rightarrow P$: Suppose there is such a sequence.
Choose an m. Suppose a linely inde list $(v_1, \dots, v_m)$ spans V.
(Similar to [2.16]) Then $\exists\, v_{m+1} \in V \backslash \operatorname{span}(v_1, \dots, v_m)$.
Hence no list spans V. Thus V is infinite-dim. □

---

16 *Prove that the vecsp of all continuous functions in $\mathbf{R}^{[0,1]}$ is infinite-dim.*

Solution: Denote the vecsp by U.

Choose an $m \in \mathbf{N}^+$. Suppose $a_0, \dots, a_m \in \mathbf{R}$ are such that $a_0 + a_1 x + \cdots + a_m x^m = 0$, $\forall x \in [0,1]$.
Then the poly has infinitely many roots and hence $a_0 = \cdots = a_m = 0$.
Thus $(1, x, \dots, x^m)$ is linely inde in $\mathbf{R}^{[0,1]}$. Similar to [2.16], U is infinite-dim. □

Or. Note that for $a_n = \dfrac{1}{n}$, $\quad a_1 < a_2 < \cdots < a_m, \quad \forall m \in \mathbf{N}^+$.

Suppose $f_n = \begin{cases} x - \dfrac{1}{n}, & x \in \left(\dfrac{1}{n}, 1\right) \\ 0 & , \ x \in \left[0, \dfrac{1}{n}\right] \end{cases}$ Then for any m, $f_1\left(\dfrac{1}{m}\right) = \cdots = f_m\left(\dfrac{1}{m}\right)$, while $f_{m+1}\left(\dfrac{1}{m}\right) \neq 0$.

Hence $f_{m+1} \notin \operatorname{span}(f_1, \dots, f_m)$. Thus by Problem (14), U is infinite-dim. □

---

17 *Suppose $p_0, p_1, \dots, p_m \in \mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$.*
Prove that $(p_0, p_1, \dots, p_m)$ is not linely inde in $\mathcal{P}_m(\mathbf{F})$.

Solution:

Suppose $(p_0, p_1, \dots, p_m)$ is linely inde. Define $p \in \mathcal{P}_m(\mathbf{F})$ by $p(z) = z\ \forall z \in \mathbf{F}$.
But $\forall a_i \in \mathbf{F}, z \neq a_0 p_0(z) + \cdots + a_m p_m(z)$, for if not, let $z = 2$, contradicts. Thus $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$.
Then $\operatorname{span}(p_0, p_1, \dots, p_m) \subsetneq \mathcal{P}_m(\mathbf{F})$ while the list $(p_0, p_1, \dots, p_m)$ has length $(m+1)$.
Hence $(p_0, p_1, \dots, p_m)$ is linely depe in $\mathcal{P}_m(\mathbf{F})$.
For if not, because $(1, z, \dots, z^m)$ of length $(m+1)$ spans $\mathcal{P}_m(\mathbf{F})$,
thus by [2.23] trivially, $(p_0, p_1, \dots, p_m)$ spans $\mathcal{P}_m(\mathbf{F})$. Contradicts. □

Or. Note that $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(\underbrace{1, z, \dots, z^m}_{\text{of length } (m+1)})$. $(p_0, p_1, \dots, p_m, z)$ of length $(m+2)$ is linely dep.

(See the above) Now $z \notin \operatorname{span}(p_0, p_1, \dots, p_m)$ and hence $(p_0, p_1, \dots, p_m)$ is linely dep. □

Ended

2·B   1 7 8 | 4E: 5, 9

7 *Prove or give a counterexample: If (v_1, v_2, v_3, v_4) is a basis of V and U is a subsp of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then (v_1, v_2) is a basis of U.*

Solution: A counterexample:

Let $V = \mathbf{R}^4$ and e_j be the j^{th} standard basis.

Let $v_1 = e_1, v_2 = e_2, v_3 = e_3 + e_4, v_4 = e_4$. Then $(v_1, \dots, v_4)$ is a basis of $\mathbf{R}^4$.

Let $U = \operatorname{span}(e_1, e_2, e_3) = \operatorname{span}(v_1, v_2, v_3 - v_4)$. Then $v_3 \notin U$ and (v_1, v_2) is not a basis of U. $\square$

• **Note For "$\complement_V U \cap \{0\}$":**

"$\complement_V U \cap \{0\}$" is supposed to be a subsp W such that $V = U \oplus W$.

But if we let $u \in U \backslash \{0\}$ and $w \in W \backslash \{0\}$, then $\left.\begin{array}{l} w \in \complement_V U \cap \{0\} \\ u \pm w \in \complement_V U \cap \{0\} \end{array}\right\} \Rightarrow u \in \complement_V U \cap \{0\}$. Contradicts.

To fix this, denote the set $\{W_1, W_2 \dots\}$ by $\mathcal{S}_V U$, where for each W_i, $V = U \oplus W_i$. See also in (1.C.23).

---

1 *Find all vecsps that have exactly one basis.*

Solution: The trivial vecsp $\{0\}$ will do. Indeed, the only basis of $\{0\}$ is the empty list.

Now consider a field containing only the add identity 0 and the multi identity 1,

and we specify that $1 + 1 = 0$. Hence the vecsp $\{0, 1\}$ will do, the list (1) will be the unique basis.

And more generally, consider $\mathbf{F} = \mathbf{Z}_m$, $\forall m - 1 \in \mathbf{N}^+$. For each $s, t \in \{1, \dots, m\}$,

$\mathbf{F} = \operatorname{span}(K_s) = \operatorname{span}(K_t)$. Hence we fail. Are there other vecsps? Suppose so.

(I) Consider $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Let $(v_1, \dots, v_m)$ be a basis of $V \neq \{0\}$.

While there are infinitely many bases distinct from this one. Hence we fail.

(II) Consider other $\mathbf{F}$. Note that a field contains at least 0 and 1

By ***some theories or facts*** given in the course of Elementary Abstract Algebra, we fail. $\square$

• *Suppose $(v_1, \dots, v_m)$ is a list of vecs in V. For $k \in \{1, \dots, m\}$, let $w_k = v_1 + \cdots + v_k$. Show that $[P]\ B_V = (v_1, \dots, v_m) \iff [Q]\ B_W = (w_1, \dots, w_m)$.*

Solution: Notice that $B_U = (u_1, \dots, u_n) \iff \forall u \in U, \exists!\, a_i \in \mathbf{F}, u = a_1 u_1 + \cdots + a_n u_n$.

$P \Rightarrow Q : \forall v \in V, \exists!\, a_i \in \mathbf{F},\ v = a_1 v_1 + \cdots + a_m v_m \Rightarrow v = b_1 w_1 + \cdots + b_m v_m, \exists!\, b_k = \sum_{j=1}^{k} (-1)^{k-j} a_j$.

$Q \Rightarrow P : \forall v \in V, \exists!\, b_i \in \mathbf{F},\ v = b_1 w_1 + \cdots + b_m w_m \Rightarrow v = a_1 v_1 + \cdots + a_m v_m, \exists!\, a_k = \sum_{j=1}^{k} b_j$. $\square$

• *Suppose V is finite-dim and U, W are subsps of V such that $V = U + W$. Let $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$. Prove that $\exists\, B_V$ consisting of vecs in $U \cup W$.*

Solution:

$V = \operatorname{span}(u_1, \dots, u_m) + \operatorname{span}(w_1, \dots, w_n) = \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$. By [2.31], we get the basis $\square$

---

8 *Suppose U and W are subsps of V such that $V = U \oplus W$. Let $B_U = (u_1, \dots, u_m)$, $B_W = (w_1, \dots, w_n)$. Prove that $B_V = (u_1, \dots, u_m, w_1, \dots, w_n)$.*

Solution:

$\forall v \in V, \exists!\, u \in U, w \in W, v = u + w = (a_1 u_1 + \cdots + a_m u_m) + (b_1 w_1 + \cdots + b_n w_n), \exists!\, a_i, b_i \in \mathbf{F}$

$\Rightarrow (a_1 u_1 + \cdots + a_m u_m) = -(b_1 w_1 + \cdots + b_n w_n) \in U \cap W = \{0\} \Rightarrow a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. $\square$

Ended

• **NOTE FOR** *linely inde sequence and* [2.34]*:*

"$V = \operatorname{span}(v_1, \dots, v_n, \dots)$" is an invalid expression.

If we allow using "infinite list", then we must guarantee that $(v_1, \dots, v_n, \dots)$ is a spanning "list" such that for all $v \in V$, there exists a smallest positive integer n such that $v = a_1 v_1 + \cdots + a_n v_n$,

The key point is, how can we guarantee that such a "list" exists?

2·C

1 7 9 10 14,16 15 17 | 4E: 10, 14, 15, 16

1 (COROLLARY for [2.38,39])

Suppose U *is a subsp of* V *such that* $\dim V = \dim U$. *Then* $V = U$.

---

9 *Suppose* $(v_1, \dots, v_m)$ *is linely inde in* V *and* $w \in V$.

Prove that $\dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$.

SOLUTION: Using the result of Problem (10) and (11) in 2.A.

Note that $v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \dots, v_n + w)$, for each $i = 1, \dots, m$.

$(v_1, \dots, v_m)$ linely inde $\Rightarrow (v_1, v_2 - v_1, \dots, v_m - v_1)$ linely inde $\Rightarrow \underbrace{(v_2 - v_1, \dots, v_m - v_1)}_{\text{of length } (m-1)}$ linely inde.

又 $w \notin \operatorname{span}(v_1, \dots, v_m) \Rightarrow (v_1 + w, \dots, v_m + w)$ is linely inde.

Hence $m \geqslant \dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$. □

---

10 *Suppose* m *is a positive integer and* $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ *are such that each* p_k *has degree* k. *Prove that* $(p_0, p_1, \dots, p_m)$ *is a basis of* $\mathcal{P}_m(\mathbf{F})$.

SOLUTION:

Using mathematical induction on m.

(i) For p_0, $\deg p_0 = 0 \Rightarrow \operatorname{span}(p_0) = \operatorname{span}(1)$.

(ii) Suppose for $i \geqslant 1$, $\operatorname{span}(p_0, p_1, \dots, p_i) = \operatorname{span}(1, x, \dots, x^i)$.

Then $\operatorname{span}(p_0, p_1, \dots, p_i, p_{i+1}) \subseteq \operatorname{span}(1, x, \dots, x^i, x^{i+1})$.

又$\deg p_{i+1} = i + 1$, $\quad p_{i+1}(x) = a_{i+1} x^{i+1} + r_{i+1}(x)$; $\quad a_{i+1} \neq 0$, $\deg r_{i+1} \leqslant i$.

$\Rightarrow x^{i+1} = \dfrac{1}{a_{i+1}} \Big(p_{i+1}(x) - r_{i+1}(x) \Big) \in \operatorname{span}(1, x, \dots, x^i, p_{i+1}) = \operatorname{span}(p_0, p_1, \dots, p_i, p_{i+1})$.

$\therefore x^{i+1} \in \operatorname{span}(p_0, p_1, \dots, p_i, p_{i+1}) \Rightarrow \operatorname{span}(1, x, \dots, x^i, x^{i+1}) \subseteq \operatorname{span}(p_0, p_1, \dots, p_i, p_{i+1})$.

Thus $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, x, \dots, x^m) = \operatorname{span}(p_0, p_1, \dots, p_m)$. □

OR. 用比较系数法. Denote the coefficient of x^i in $p \in \mathcal{P}(\mathbf{F})$ by $\xi_i(p)$.

Suppose $L = a_m p_m(x) + \cdots + a_1 p_1(x) + a_0 p_0(x) = 0 \cdot x^m + \cdots + 0 \cdot x + 0 \cdot 1 = R, \forall x \in \mathbf{F}$.

We use induction on m to show that $a_m = \cdots = a_0 = 0$.

(i) $k = m$, $\xi_m(L) = a_m \xi_m(p_m) = \xi_m(R) = 0$ 又 $\deg p_m = m$, $\xi_m(p_m) \neq 0 \Rightarrow a_m = 0$.

Now $L = a_{m-1} p_{m-1}(x) + \cdots + a_0 p_0(x)$.

(ii) $1 \leqslant k \leqslant m$, $\xi_k(L) = a_k \xi_k(p_k) = \xi_k(R) = 0$ 又 $\deg p_k = k$, $\xi_k(p_k) \neq 0 \Rightarrow a_k = 0$.

Now $L = a_{k-1} p_{k-1}(x) + \cdots + a_0 p_0(x)$. □

- (4E 2.C.10) *Suppose m is a positive integer. For $0 \leqslant k \leqslant m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show that $(p_0, \dots, p_m)$ is a basis of $\mathcal{P}(\mathbf{F})$.*

The basis in this exercise leads to what are called Bernstein polys. You can do a web search to learn how Bernstein polys are used to approximate continuous functions on $[0,1]$.

Solution: Using mathematical induction.

(i) $k = 0, 1, 2$, $p_m(x) = x^m$, $p_{m-1}(x) = x^{m-1} - x^m$, $p_{m-2}(x) = x^{m-2} + x^m - 2x^{m-1}$.

(ii) $k \geqslant 2$. Suppose for $p_{m-k}(x), \exists !\, a_i \in \mathbf{F}, x^{m-k} = p_{m-k}(x) + a_m x^m + \dots + a_{m-k+1}x^{m-k+1}$.

Then for $p_{m-k-1}(x), \exists !\, c_i \in \mathbf{F}$,

$$x^{m-k-1} = p_{m-k-1}(x) + C_{k+1}^{1}(-1)^2 x^{m-k} + \dots + C_{k+1}^{k}(-1)^{k+1}x^{m-1} + (-1)^{k-2}x^m$$

$$\Rightarrow c_{m-i} = C_{k+1}^{k+1-i}(-1)^{k-i}.$$

Thus for each x^i, $\exists !\, b_i \in \mathbf{F}, x^i = b_m p_m(x) + \dots + b_{m-i}p_{m-i}(x)$

$$\Rightarrow \operatorname{span}(x^m, \dots, x, 1) = \operatorname{span}(\underbrace{p_m, \dots, p_1, p_0}_{\text{Basis}}).$$

□

Or. For any $m, k \in \mathbf{N}^+$ such that $k \leqslant m$. Define $p_{k,m}$ by $p_{k,m}(x) = x^k(1-x)^{m-k}$.

Define the statement $S(m)$ by $S(m) : \underbrace{(p_{0,m}, \dots, p_{m,m})}_{\dim \mathcal{P}_m(\mathbf{F}) = m+1}$ is linely inde (and therefore is a basis).

We use induction on to show that $S(m)$ holds for all $m \in \mathbf{N}^+$.

(i) $m = 1$. Suppose $a_0(1-x) + a_1 x = 0, \forall x \in \mathbf{F}$. Then $\begin{cases} x = 0 \Rightarrow a_0 = 0; \\ x = 1 \Rightarrow a_1. \end{cases}$

$m = 2$. Suppose $a_0(1-x)^2 + a_1(1-x)x + a_2x^2, \forall x \in \mathbf{F}$. Then $\begin{cases} x = 0 \Rightarrow a_0 + a_1 = 0; \\ x = 1 \Rightarrow a_2 = 0; \\ x = 2 \Rightarrow a_0 + 2a_1 = 0. \end{cases}$

(ii) $2 \leqslant m$. Assume that $S(m)$ holds.

Suppose $\sum_{k=0}^{m+2} a_k p_{k,m+2}(x) = \sum_{k=0}^{m+2} a_k x^k (1-x)^{m+2-k} = 0, \forall x \in \mathbf{F}$.

While $x = 0 \Rightarrow a_0 = 0$; $x = 1 \Rightarrow a_{m+2} = 0$. Then $\sum_{k=1}^{m+1} a_k x^k(1-x)^{m+2-k} = 0$;

And note that $\sum_{k=1}^{m+1} a_k x^k (1-x)^{m+2-k}$

$$= x(1-x)\sum_{k=1}^{m+1} a_k x^{k-1}(1-x)^{m+1-k}$$

$$= x(1-x)\sum_{k=0}^{m} a_{k+1}x^k(1-x)^{m-k} = x(1-x)\sum_{k=0}^{m} a_{k+1}p_{k,m}(x).$$

Hence $x(1-x)\sum_{k=0}^{m} a_{k+1}p_{k,m}(x) = 0, \forall x \in \mathbf{F} \Rightarrow \sum_{k=0}^{m} a_{k+1}p_{k,m}(x) = 0, \forall x \in \mathbf{F}\backslash\{0,1\}$.

Because $\sum_{k=0}^{m} a_{k+1}p_{k,m}(x)$ has infinitely many zeros. We have $\sum_{k=0}^{m} a_{k+1}p_{k,m}(x) = 0, \forall x \in \mathbf{F}$.

By assumption, $a_1 = \dots = a_m = 0$, while $a_0 = a_{m+2} = 0$,

and also $a_{m+1} = 0$ (because $\sum_{k=0}^{m} a_{k+1}p_{k,m}(x) = a_{m+1}p_{m,m}(x) = a_{m+1}x^m = 0, \forall x \in \mathbf{F}$.)

Thus $(p_{0,m+2}, \dots, p_{m+2,m+2})$ is linely inde and $S(m+2)$ holds.

Since $\forall m \in \mathbf{N}^+, S(m) \Rightarrow S(m+2)$. We have $\left| \begin{array}{c} \forall k \in \mathbf{N}, S(2k+1) \text{ holds} \\ \forall k \in \mathbf{N}^+, S(2k) \text{ holds} \end{array} \right\} \Rightarrow S(m)$ holds.

□

7 (a) *Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.*
(b) *Extend the basis in* (b) *to a basis of $\mathcal{P}_4(\mathbf{F})$.*
(c) *Find a subsp W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.*

SOLUTION: Suppose $p(z) = az^4 + bz^3 + cz^2 + dz + e$ such that $p(2) = p(5) = p(6)$.

$$\text{Then}\left|\begin{array}{r} p(2) = 16a + 8b + 4c + 2d + e \quad (\text{I}) \\ p(5) = 625a + 125b + 25c + 5d + e \quad (\text{II}) \\ p(6) = 1296a + 216b + 36c + 6d + e \quad (\text{III}) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} (\text{II}) - (\text{I}) = 0 \\ (\text{III}) - (\text{II}) = 0 \\ (\text{III}) - (\text{I}) = 0 \end{array}\right.$$

You don't have to compute to know that the dimension of the set of solutions is 3.
(Because $\nexists\, p \in \mathcal{P}_2(\mathbf{F})$ with $1 \leqslant \deg p \leqslant 2, p(2) = p(5) = p(6)$.)
(a) A basis: $1, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.
(b) Extend to a basis of $\mathcal{P}_4(\mathbf{F})$ as $1, z, z^2, (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)$.
(c) Let $W = \operatorname{span}(z, z^2) = \{az + bz^2 : a, b \in \mathbf{F}\}$, so that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$. □

• **TIPS:**

(1) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3))$.
(2) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_3) - \dim(V_2 + (V_1 \cap V_3))$.
(3) $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim(V_3 + (V_1 \cap V_2))$.

For (1). Because $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim(V_1 + (V_2 \cap V_3))$.
And $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$.

• *Suppose V is a 10-dim vecsp and V_1, V_2, V_3 are subsps of V with*
(a) $\dim V_1 = \dim V_2 = \dim V_3 = 7$. *Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*
(b) $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. *Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.*

SOLUTION:
(a) By TIPS , $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 - 2\dim V > 0$.
(b) By TIPS , $\dim(V_1 \cap V_2 \cap V_3) > 2\dim V - \dim(V_2 + V_3) - \dim(V_1 + (V_2 \cap V_3)) \geq 0$. □

• (4E 2.C.16)
Suppose V is finite-dim and U is a subsp of V with $U \neq V$. Let $n = \dim V, m = \dim U$. Prove that $\exists\, (n-m)$ subsps $U_1, \dots, U_{n-m}$, each of dim $(n-1)$, such that $\bigcap_{i=1}^{n-m} U_i = U$.

SOLUTION:
Let $(v_1, \dots, v_m)$ be a basis of U, extend to a basis of V as $(v_1, \dots, v_m, u_1, \dots, v_{n-m})$.
Define $U_i = \operatorname{span}(v_1, \dots, v_m, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-m})$ for each i. Then $U \subseteq U_i$ for each i.
And because $\forall v \in \bigcap_{i=1}^{n-m} U_i, v = v_0 + b_1 u_1 + \cdots + b_{n-m} u_{n-m} \in U_i \Rightarrow b_i = 0$ for each $i \Rightarrow v \in U$.
Hence $\bigcap_{i=1}^{n-m} U_i \subseteq U$. □

EXAMPLE: Suppose $\dim V = 6, \dim U = 3$.

$$(\overbrace{\underbrace{v_1, v_2, v_3}_{\text{Basis of U}}, v_4, v_5, v_6}^{\text{Basis of V}}), \text{define} \left|\begin{array}{l} U_1 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_5, v_6) \\ U_2 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_4, v_6) \\ U_3 = \operatorname{span}(v_1, v_2, v_3) \oplus \operatorname{span}(v_4, v_5) \end{array}\right\} \Rightarrow \dim U_i = 6 - 1,\ i = \underbrace{1, 2, 3}_{6-3=3}.$$

□

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14 *Suppose that $V_1, \dots, V_m$ are finite-dim subsps of V.*
Prove that $V_1 + \cdots + V_m$ is finite-dim and $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$.

SOLUTION:

Choose a basis $\mathcal{E}_i$ of $V_i \Rightarrow V_1 + \cdots + V_m = \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$; $\dim V_i = \operatorname{card} \mathcal{E}_i$.
Then $\dim(V_1 + \cdots + V_m) = \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$.
又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) \leqslant \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$.
Thus $\dim(V_1 + \cdots + V_m) \leqslant \dim V_1 + \cdots + \dim V_m$. □

COMMENT: $\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m \Longleftrightarrow V_1 + \cdots + V_m$ *is a direct sum.*
For each i, $(V_1 + \cdots + V_i) \cap V_{i+1} = \{0\} \Longleftrightarrow V_1 + \cdots + V_m$ is a direct sum
$\Longleftrightarrow (\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{k-1}) \cap \mathcal{E}_k = \emptyset$ for each i 又 $\dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m)$
$\Longleftrightarrow \dim \operatorname{span}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \operatorname{card} \mathcal{E}_1 + \cdots + \operatorname{card} \mathcal{E}_m$
$\Longleftrightarrow \dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m$. □

---

17 *Suppose V_1, V_2, V_3 are subsps of a finite-dim vecsp, then*

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).$$

Explain why you might think and prove the formula above or give a counterexample.

SOLUTION:

[*Similar to*] Given three sets A, B and C.
Because $|X + Y| = |X| + |Y| - |X \cap Y|$; $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.
Now $|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$.
And $|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$.
Hence $|(A \cup B) \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$.

Because $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3) = (V_1 + V_3) + V_2$.

$$\begin{aligned} \dim(V_1 + V_2 + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim\big((V_1 + V_2) \cap V_3\big) && (1) \\ &= \dim(V_2 + V_3) + \dim(V_1) - \dim\big((V_2 + V_3) \cap V_1\big) && (2) \\ &= \dim(V_1 + V_3) + \dim(V_2) - \dim\big((V_1 + V_3) \cap V_2\big) && (3) \end{aligned}$$

Notice that in general, $(X + Y) \cap Z \neq X \cap Z + Y \cap Z$.
For example, $X = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, Y = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, Z = \{(z, z) \in \mathbf{R}^2 : z \in \mathbf{R}\}$.

• **COROLLARY:** *Suppose V_1, V_2 and V_3 are finite-dim vecsps, then* $\dfrac{(1) + (2) + (3)}{3}$:

$$\begin{aligned} \dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 & \\ - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} & \\ - \frac{\dim\big((V_1 + V_2) \cap V_3\big) + \dim\big((V_1 + V_3) \cap V_2\big) + \dim\big((V_2 + V_3) \cap V_1\big)}{3} & . \end{aligned}$$

The formula above may seem strange because the right side does not look like an integer. □

• **TIPS:** Suppose $v_1, \dots, v_n \in V$, $\dim \operatorname{span}(v_1, \dots, v_n) = n$. Then $(v_1, \dots, v_n)$ is a basis of $\operatorname{span}(v_1, \dots, v_n)$.
Notice that $(v_1, \dots, v_n)$ is a spanning list of $\operatorname{span}(v_1, \dots, v_n)$ *of length $n = \dim \operatorname{span}(v_1, \dots, v_n)$.*

---

15 *Suppose V is finite-dim and* $\dim V = n \geqslant 1$.
Prove that $\exists$ *one-dim subsps* $V_1, \dots, V_n$ *of* V *such that* $V = V_1 \oplus \cdots \oplus V_n$.

Solution:

Suppose $B_V = (v_1, \dots, v_n)$. Define V_i by $V_i = \operatorname{span}(v_i)$ for each $i \in \{1, \dots, n\}$.
Then $\forall v \in V, \exists !\, a_i \in \mathbf{F}, v = a_1 v_1 + \cdots + a_n v_n$
$$\Rightarrow \exists !\, u_i \in V_i, v = u_1 + \cdots + u_n \Rightarrow V = V_1 \oplus \cdots \oplus V_n. \qquad \square$$

• **Corollary:**
Suppose W is finite-dim, $\dim W = m$ *and* $w \in W \backslash \{0\}$.
Prove that $\exists B_W = (w_1, \dots, w_m)$ *such that* $w = w_1 + \cdots + w_m$.

[Proof]

By Problem (15), $\exists$ one-dim subsps $W_1, \dots, W_m$ of W such that $W = W_1 \oplus \cdots \oplus W_m$.
Note that $\dim W_i = \dim \operatorname{span}(w_i) = 1 \Rightarrow \forall x_i \in W_i, \exists !\, c_i \in \mathbf{F}, x_i = c_i w_i$.
Suppose $w = x_1 + \cdots + x_m$, where each $x_i = c_i w_i \in W_i$. Then $(x_1, \dots, x_m)$ is also a basis of W. $\square$

Or. Note that $w \neq 0 \Rightarrow m \geqslant 1$. If $m = 1$ then let $w_1 = w$ and we are done. Suppose $m > 1$.
Extend (w) to a basis $(w, w_1, \dots, w_{m-1})$ of W. Let $w_m = w - w_1 - \cdots - w_{m-1}$.
又 $\operatorname{span}(w, w_1, \dots, w_{m-1}) = \operatorname{span}(w_1, \dots, w_m)$. Hence $(w_1, \dots, w_m)$ is also a basis of W. $\square$

• **New Theorem:** *Suppose V is finite-dim with* $\dim V = n$ *and U is a subsp of V with $U \neq V$.*
Prove that $\exists B_V = (v_1, \dots, v_n)$ *such that each* $v_k \notin U$.

Note that $U \neq V \Rightarrow n \geqslant 1$. We will construct B_V via the following process.
Step 1. $\exists v_1 \in V \backslash U \Rightarrow v_1 \neq 0$. If $\operatorname{span}(v_1) = V$ then we stop.
Step k. Suppose $(v_1, \dots, v_{k-1})$ is linely inde in V, each of which belongs to $V \backslash U$.
Note that $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$. And if $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U = V$, then by (1.C.12),
(because $\operatorname{span}(v_1, \dots, v_{k-1}) \nsubseteq U$,) $U \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) \Rightarrow \operatorname{span}(v_1, \dots, v_{k-1}) = V$.
Hence because $\operatorname{span}(v_1, \dots, v_{k-1}) \neq V$, it must be case that $\operatorname{span}(v_1, \dots, v_{k-1}) \cup U \neq V$.
Thus $\exists v_k \in V \backslash U$ such that $v_k \notin \operatorname{span}(v_1, \dots, v_{k-1})$.
By (2.A.11), $(v_1, \dots, v_k)$ is linely inde in V. If $\operatorname{span}(v_1, \dots, v_k) = V$, then we stop.
Because V is finite-dim, this process will stop after n steps. $\square$

Or. If $U = \{0\}$ then we are done. Suppose $\dim U \geqslant 1$.
Let $(u_1, \dots, u_m)$ be a basis of U, extend to a basis $(u_1, \dots, u_n)$ of V.
Then let $B_V = (u_1 - u_k, \dots, u_m - u_k, u_{m+1}, \dots, u_k, \dots, u_n)$. $\square$

Ended

3·A

3 4 5 7 8 10 11 12 13 | 4E: 10, 11, 16

• **Tips:** $T : V \to W$ *is linear* $\Longleftrightarrow \left| \begin{array}{l} (一)\ \forall v, u \in V, T(v+u) = Tv + Tu; \\ (二)\ \forall v, u \in V, \lambda \in \mathbf{F}, T(\lambda v) = \lambda(Tv). \end{array} \right| \Longleftrightarrow T(v + \lambda u) = Tv + \lambda Tu.$

$T \in \mathcal{L}(V, W) \Longleftrightarrow T \in \mathcal{L}(V, \operatorname{range} T)$. And $\{T \in \mathcal{L}(V, W) : \operatorname{range} T \subseteq U\} = \mathcal{L}(V, U)$.

• *Suppose* $T \in \mathcal{L}(V, W)$. *Prove that* $Tv \neq 0 \Rightarrow v \neq 0$.

Solution: Assume that $v = 0$. Then $Tv = T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$.
Or. $T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$. Contradicts. $\square$

- (4E 1.B.7) *Suppose $V \neq \emptyset$ and W is a vecsp. Let $W^V = \{f : V \to W\}$.*
  (a) *Define a natural add and scalar multi on W^V.*
  (b) *Prove that W^V is a vecsp with these definitions.*

Solution:

(a) $W^V \ni f + g : x \to f(x) + g(y)$; where $f(x) + g(y)$ is the vec add on W.
$W^V \ni \lambda f : x \to \lambda f(x)$; where $\lambda f(x)$ is the scalar multi on W.

(b) Commutativity: $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$.
Associativity: $\big((f+g)+h\big)(x) = \big((f)(x) + (g)(x)\big) + (h)(x)$
$= (f)(x) + \big((g)(x) + (h)(x)\big) = \big(f + (g+h)\big)(x)$.
Additive Identity: $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$.
Additive Inverse: $(f+g)(x) = f(x) + g(x) = f(x) + \big(-f(x)\big) = 0 = 0(x)$.
Distributive Properties:
$\big(a(f+g)\big)(x) = a(f+g)(x) = a\big(f(x)+g(x)\big)$
$= af(x) + ag(x) = (af)(x) + (ag)(x) = (af+ag)(x)$.
Similarly, $\big((a+b)f\big)(x) = (af+bf)(x)$.
So far, we have used the same properties in W.
Which means that ***if** W^V **is a vecsp, then** W **must be a vecsp.***
Multiplication Identity: $(1f)(x) = 1f(x) = f(x)$. (Notice that the smallest $\mathbf{F}$ is $\{0,1\}$.) $\square$

---

5 *Because $\mathcal{L}(V,W) = \{T : V \to W \mid T \text{ is linear}\}$ is a subsp of W^V, $\mathcal{L}(V,W)$ is a vecsp.*

- *Given the fact that $\mathcal{L}(V,W)$ is a vecsp. Prove or give a counterexample: V, W are vecsps.*
  We can guarantee that $\{0\} \subseteq \mathcal{L}(V,W), \{0\} \subseteq V, \{0\} \subseteq W$.
  And by [3.2], the additivity and homogeneity imply that V is closed under add and scalar multi.
  (*We cannot even guarantee that W^V is a vecsp.*)

Solution:

(I) If $W^V = \{0\}$. Then $\mathcal{L}(V,W) = \{0\}$.
And $W = \{0\}$, for if not, $\exists\, w \in W \backslash \{0\}$, define a map f by $f(x) = w, \forall x \in V$.
And V might not be a vecsp. Example:

(II) If W^V is a nonzero vecsp. Then W is a vecsp.
(a) If $\mathcal{L}(V,W) = \{0\}$, then we cannot guarantee that V is a vecsp.
Example:
(b) If not, then $\exists\, T \in \mathcal{L}(V,W)$, $T \neq 0$. Which means $\exists\, v \in V, Tv \neq 0 \Rightarrow v \neq 0$.
Then both W and V have a nonzero element.
(i) If $\exists$ inje $T \in \mathcal{L}(V,W)$, then $T(u+v) = (v+u) \Rightarrow u+v = v+u$. etc. Hence V is a vecsp.
(ii) If not, then we cannot guarantee that V is a vecsp.
Example:

(III) If W^V is not a vecsp, then W is not a vecsp.
Example:

TODO $\square$

---

3 *Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Prove that $\exists A_{j,k} \in \mathbf{F}$ such that for any $(x_1, \dots, x_n) \in \mathbf{F}^n$*

$$T(x_1, \dots, x_n) = \begin{pmatrix} A_{1,1}x_1 + \cdots + A_{1,n}x_n, \\ \vdots \quad \ddots \quad \vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n \end{pmatrix}$$

Solution:

Let $T(1,0,0,\dots,0,0) = (A_{1,1},\dots,A_{m,1})$, Note that $(1,0,\dots,0,0),\cdots,(0,0,\dots,0,1)$ is a basis of $\mathbf{F}^n$.

$T(0,1,0,\dots,0,0) = (A_{1,2},\dots,A_{m,2})$, Then by [3.5], we are done. □

$\vdots$

$T(0,0,0,\dots,0,1) = (A_{1,n},\dots,A_{m,n})$.

---

4 *Suppose $T \in \mathcal{L}(V, W)$, and $v_1, \dots, v_m \in V$ such that $(Tv_1, \dots, Tv_m)$ is linely inde in W. Prove that $(v_1, \dots, v_m)$ is linely inde.*

Solution: Suppose $a_1v_1 + \cdots + a_mv_m = 0$. Then $a_1Tv_1 + \cdots + a_mTv_m = 0$. Thus $a_1 = \cdots = a_m = 0$. □

---

7 *Show that every linear map from a one-dim vecsp to itself is a multi by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then $\exists \lambda \in \mathbf{F}, Tv = \lambda v, \forall v \in V$.*

Solution:

Let u be a nonzero vec in $V \Rightarrow V = \operatorname{span}(u)$. Because $Tu \in V \Rightarrow Tu = \lambda u$ for some λ.

Suppose $v \in V \Rightarrow v = au$, $\exists!\, a \in \mathbf{F}$. Then $Tv = T(au) = \lambda au = \lambda v$. □

---

8 *Give a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that $\forall a \in \mathbf{R}, v \in \mathbf{R}^2, \varphi(av) = a\varphi(v)$ but φ is not linear.*

Solution: Define $T(x,y) = \begin{cases} x+y, \text{if } (x,y) \in \operatorname{span}(3,1), \\ 0, \quad \text{otherwise.} \end{cases}$ Or. Define $T(x,y) = \sqrt[3]{(x^3+y^3)}$. □

---

9 *Give a function $\varphi : \mathbf{C} \to \mathbf{C}$ such that $\forall w, z \in \mathbf{C}, \varphi(w+z) = \varphi(w) + \varphi(z)$ but φ is not linear.* (*Here* $\mathbf{C}$ *is thought of as a complex vecsp.*)

Solution:

Suppose $V_{\mathbf{C}}$ is the complexification of a vecsp V. Suppose $\varphi : V_{\mathbf{C}} \to V_{\mathbf{C}}$.

Define $\varphi(u + iv) = u = \operatorname{Re}(u + iv)$ Or. Define $\varphi(u + iv) = v = \operatorname{Im}(u + iv)$. □

• *Prove that if $q \in \mathcal{P}(\mathbf{R})$ and $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by $\underbrace{Tp = q \circ p}_{\text{composition}}$, then T is not linear.*

Solution: ***Composition and product are not the same in $\mathcal{P}(\mathbf{F})$.***

Because in general, $q \circ (p_1 + \lambda p_2)(x) = q(p_1(x) + \lambda p_2(x)) \neq (q \circ p_1)(x) + \lambda(q \circ p_2)(x)$.

Example: Let q be defined by $q(x) = x^2$, then $q \circ (1 + (-1)) = 0 \neq q(1) + q(-1) = 2$. □

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10 *Suppose U is a subsp of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ with $S \neq 0$ (which means that $\exists u \in U, Su \neq 0$).*

Define $T : V \to W$ by $Tv = \begin{cases} Sv, \text{ if } v \in U, \\ 0, \quad \text{if } v \in V \backslash U. \end{cases}$ Prove that T is not a linear map on V.

Solution:

Suppose T is a linear map. And $v \in V \backslash U$, $u \in U$ such that $Su \neq 0$.

Then $v+u \in V \backslash U$, (for if not, $v = (v+u) - u \in U$) while $T(v+u) = 0 = Tv + Tu = 0 + Su \Rightarrow Su = 0$.

Hence we get a contradiction. □

---

11 *Suppose U is a subsp of V and $S \in \mathcal{L}(U, W)$.*
Prove that $\exists\, T \in \mathcal{L}(V, W), Tu = Su, \forall u \in U$. (OR. $\exists\, T \in \mathcal{L}(V, W), T|_U = S$.)
In other words, every linear map on a subsp of V can be extended to a linear map on the entire V.

SOLUTION: Suppose W is such that $V = U \oplus W$. Then $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.
Define $T \in \mathcal{L}(V, W)$ by $T(u_v + w_v) = Su_v$. □

OR. [*Finite-dim Req*] Define by $T\Big(\sum_{i=1}^{m} a_i u_i\Big) = \sum_{i=1}^{n} a_i S u_i$. Let $B_V = (\overbrace{u_1, \dots, u_n}^{B_U}, \dots, u_m)$. □

---

12 *Suppose nonzero V is finite-dim and W is infinite-dim. Prove that $\mathcal{L}(V, W)$ is infinite-dim.*

SOLUTION:

Let $(v_1, \dots, v_n)$ be a basis of V. Let $(w_1, \dots, w_m)$ be linely inde in W for any $m \in \mathbf{N}^+$.

Define $T_{x,y} : V \to W$ by $T_{x,y}(v_z) = \delta_{z,x} w_y$, $\forall x \in \{1, \dots, n\}, y \in \{1, \dots, m\}$, where $\delta_{z,x} = \begin{cases} 0, & z \neq x, \\ 1, & z = x. \end{cases}$

$\forall v = \sum_{i=1}^{n} a_i v_i,\ u = \sum_{i=1}^{n} b_i v_i,\ \lambda \in \mathbf{F}, T_{x,y}(v + \lambda u) = (a_x + \lambda b_x) v_y = T_{x,y}(v) + \lambda T_{x,y}(u)$.

Linearity checked. Now suppose $a_1 T_{x,1} + \dots + a_m T_{x,m} = 0$.

Then $(a_1 T_{x,1} + \dots + a_m T_{x,m})(v_x) = 0 = a_1 w_1 + \dots + a_m w_m \Rightarrow a_1 = \dots = a_m = 0$. 又 m arbitrary.

Thus $(T_{x,1}, \dots, T_{x,m})$ is a linely inde list in $\mathcal{L}(V, W)$ for any x and length m. Hence by (2.A.14). □

---

13 *Suppose $(v_1, \dots, v_m)$ is linely depe in V and $W \neq \{0\}$.*
Prove that $\exists\, w_1, \dots, w_m \in W, \nexists\, T \in \mathcal{L}(V, W)$ such that $Tv_k = w_k, \forall k = 1, \dots, m$.

SOLUTION:

We prove by contradiction. By linear dependence lemma, $\exists\, j \in \{1, \dots, m\}, v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$.

Fix j. Let $w_j \neq 0$, while $w_1 = \dots = w_{j-1} = w_{j+1} = w_m = 0$.

Define T by $Tv_k = w_k$ for all k. Suppose $a_1 v_1 + \dots + a_m v_m = 0$ (where $a_j \neq 0$).

Then $T(a_1 v_1 + \dots + a_m v_m) = 0 = a_1 w_1 + \dots + a_m w_m = a_j w_j$ while $a_j \neq 0$ and $w_j \neq 0$. Contradicts. □

OR. We prove the contrapositive: Suppose $\forall w_1, \dots, w_m \in W, \exists\, T \in \mathcal{L}(V, W), Tv_k = w_k$ for each w_k.

Now we show that $(v_1, \dots, v_n)$ is linely inde. Suppose $\exists\, a_i \in \mathbf{F}, a_1 v_1 + \dots + a_n v_n = 0$.

Choose one $w \in W \backslash \{0\}$. By assumption, for $(\overline{a_1} w, \dots, \overline{a_m} w), \exists\, T \in \mathcal{L}(V, W), Tv_k = \overline{a_k} w$ for each v_k.

Now we have $0 = T\Big(\sum_{k=1}^{m} a_k v_k\Big) = \sum_{k=1}^{m} a_k T v_k = \sum_{k=1}^{m} a_k \overline{a_k} w = \Big(\sum_{k=1}^{m} |a_k|^2\Big) w$.

Then $\sum_{k=1}^{m} |a_k|^2 = 0 \Rightarrow a_k = 0$ for each k. Hence $(v_1, \dots, v_n)$ is linely inde. □

• (4E 3.A.17)

Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E},\ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

SOLUTION: Let $(v_1, \dots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.

Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$. Let $S \in \mathcal{E} \backslash \{0\}$.

Suppose $Sv_i \neq 0$ and $Sv_i = a_1 v_1 + \dots + a_n v_n$, where $a_k \neq 0$.

Define $R_{x,y} \in \mathcal{L}(V)$ by $R_{x,y}(v_x) = v_y, R_{x,y}(v_z) = 0$ ($z \neq x$). OR. $R_{x,y} v_z = \delta_{z,x} v_y$.

Then $\big(R_{1,1} + \dots + R_{n,n}\big) v_j = v_j \Rightarrow \sum_{r=1}^{n} R_{r,r} = I$. Assume that each $R_{x,y} \in \mathcal{E}$.

Hence $\forall T \in \mathcal{L}(V), I \circ T = T \circ I = T \in \mathcal{E} \Rightarrow \mathcal{E} = \mathcal{L}(V)$. Now we prove the assumption.

Notice that $\forall x, y \in \mathbf{N}^+,\ (R_{k,y} S)(v_i) = a_k v_y \Rightarrow \big((R_{k,y} S) \circ R_{x,i}\big)(v_z) = \delta_{z,x}(a_k v_y)$.

Thus $R_{k,y} S R_{x,i} = a_k R_{x,y}$. Now $S \in \mathcal{E} \Rightarrow R_{k,y} S \in \mathcal{E} \Rightarrow R_{x,y} \in \mathcal{E}$. □

• (4E 3.B.32) *Suppose V is finite-dim with $n = \dim V > 1$.*
Show that if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is linear and $\forall S, T \in \mathcal{L}(V), \varphi(ST) = \varphi(S) \cdot \varphi(T)$, then $\varphi = 0$.

SOLUTION:

Using notations in (4E 3.A.16). Using the result in NOTE FOR [3.60].

Suppose $\varphi \neq 0 \Rightarrow \exists\, i, j \in \{1, \dots, n\}$, $\varphi(R_{i,j}) \neq 0$. Because $R_{i,j} = R_{x,j} \circ R_{i,x},\ \ \forall x = 1, \dots, n$

$\Rightarrow \varphi(R_{i,j}) = \varphi(R_{x,j}) \cdot \varphi(R_{i,x}) \neq 0 \Rightarrow \varphi(R_{x,j}) \neq 0$ and $\varphi(R_{i,x}) \neq 0$.

Again, because $R_{i,x} = R_{y,x} \circ R_{i,y},\ \ \forall y = 1, \dots, n$. Thus $\varphi(R_{y,x}) \neq 0,\ \forall x, y = 1, \dots, n$.

Let $k \neq i, j \neq l$ and then $\varphi(R_{i,j} \circ R_{l,k}) = \varphi(R_{l,k} \circ R_{i,j}) = \varphi(0) = 0 = \varphi(R_{l,k}) \cdot \varphi(R_{i,j})$

$\Rightarrow \varphi(R_{l,k}) = 0$ or $\varphi(R_{i,j}) = 0$. Contradicts. $\square$

OR. Note that by (4E 3.A.16), $\exists\, S, T \in \mathcal{L}(V), ST - TS \neq 0$.

Then $\varphi(ST - TS) = \varphi(S)\varphi(T) - \varphi(T)\varphi(S) = 0 \Rightarrow ST - TS \in \operatorname{null} \varphi \neq \{0\}$.

Note that $\forall\, E \in \operatorname{null} \varphi, T \in \mathcal{L}(V), \varphi(ET) = \varphi(TE) = 0 \Rightarrow ET, TE \in \operatorname{null} \varphi$.

Hence $\operatorname{null} \varphi$ is a nonzero two-sided ideal of $\mathcal{L}(V)$. $\square$

• *Suppose V is finite-dim. $T \in \mathcal{L}(V)$ is such that $\forall S \in \mathcal{L}(V), ST = TS$.*
Prove that $\exists\, \lambda \in \mathbf{F}, T = \lambda I$.

SOLUTION:

If $V = \{0\}$, then we are done. Now suppose $V \neq \{0\}$.

Assume that (v, Tv) is linely depe for every $v \in V$, then by (2.A.2.(b)), $Tv = \lambda_v v$ for some $\lambda_v \in \mathbf{F}$.

To prove that λ_v is independent of v, we discuss in two cases:

$$\left.\begin{array}{l} (-) \text{ If } (v, w) \text{ is linely inde, } \lambda_{v+w}(v + w) = T(v + w) = Tv + Tw = \lambda_v v + \lambda_w w \\ \qquad\qquad \Rightarrow (\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0 \\ (=) \text{ Otherwise, suppose } w = cv, \lambda_w w = Tw = cTv = c\lambda_v v = \lambda_v w \Rightarrow (\lambda_w - \lambda_v)w \end{array}\right\} \Rightarrow \lambda_w = \lambda_v.$$

Now we show the assumption. Assume that (v, Tv) is linely inde for some v. Let $B_V = (v, Tv, u_1, \dots, u_n)$.

Define $S \in \mathcal{L}(V)$ by $S(av + bTv + c_1u_1 + \cdots + c_nu_n) = bv \Rightarrow S(Tv) = v = T(Sv) = 0$. Contradicts. $\square$

OR. Let $(v_1, \dots, v_m)$ be a basis of V.

Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_1) = \cdots = \varphi(v_m) = 1$. Let $\lambda = \varphi(Tv_1) \in \mathbf{F}$.

For any $v \in V$, define $S_v \in \mathcal{L}(V)$ by $S_v u = \varphi(u)v$.

Then $Tv = T(\varphi(v_1)v) = T(S_v v_1) = S_v(Tv_1) = \varphi(Tv_1)v = \lambda v$. $\square$

OR. For each $k \in \{1, \dots, n\}$, define $S_k \in \mathcal{L}(V)$ by $S_k v_j = \begin{cases} v_k, & j = k, \\ 0, & j \neq k. \end{cases}$ OR. $S_k v_j = \delta_{j,k} v_k$

Note that $S_k\left(\sum_{i=1}^n a_i v_i\right) = a_k v_k$. Then $S_k v = v \iff \exists!\, a_k \in \mathbf{F}, v = a_k v_k$.

Hence $S_k(Tv_k) = T(S_k v_k) = Tv_k \Rightarrow Tv_k = a_k v_k$.

Define $A^{(j,k)} \in \mathcal{L}(V)$ by $A^{(j,k)}v_j = v_k, A^{(j,k)}v_k = v_j, A^{(j,k)}v_x = 0, x \neq j, k$.

Then $A^{(j,k)}Tv_j = TA^{(j,k)}v_j = Tv_k = a_k v_k$; $A^{(j,k)}Tv_j = A^{(j,k)}a_j v_j = a_j A^{(j,k)}v_j = a_j v_k$.

Hence $a_k = a_j$. Thus a_k is independent of v_k. $\square$

ENDED

3·B

3 7 8 9 10 11 12 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31
4E: 24, 27, 31, 32, 33

• *Suppose that V and W are real vecsps and $T \in \mathcal{L}(V,W)$.*
Define $T_C : V_C \to W_C$ by $T_C(u + \mathrm{i}v) = Tu + \mathrm{i}Tv$ for all $u, v \in V$.
Show that (a) *T_C is linear,* (b) *T_C is inje $\Longleftrightarrow$ T is inje,* (c) *T_C is surj $\Longleftrightarrow$ T is surj.*

Solution:

(a) $\forall u_1 + \mathrm{i}v_1, u_2 + \mathrm{i}v_2 \in V_C, \lambda \in \mathbf{F}$,

$$T\big((u_1 + \mathrm{i}v_1) + \lambda(u_2 + \mathrm{i}v_2)\big) = T\big((u_1 + \lambda u_2) + \mathrm{i}(v_1 + \lambda v_2)\big) = T(u_1 + \lambda u_2) + \mathrm{i}T(v_1 + \lambda v_2)$$
$$= Tu_1 + \mathrm{i}Tv_1 + \lambda Tu_2 + \mathrm{i}\lambda Tv_2 = T(u_1 + \mathrm{i}v_1) + \lambda T(u_2 + \mathrm{i}v_2).$$

(b) Suppose T_C is inje. Let $T(u) = 0 \Rightarrow T_C(u + \mathrm{i}0) = Tu = 0 \Rightarrow u = 0$.
Suppose T is inje. Let $T_C(u + \mathrm{i}v) = Tu + \mathrm{i}Tv = 0 \Rightarrow Tu = Tv = 0 \Rightarrow u + \mathrm{i}v = 0$.

(c) Suppose T_C is surj. $\forall w \in W, \exists\, u \in V, T(u + \mathrm{i}0) = Tu = w + \mathrm{i}0 = w \Rightarrow$ T is surj.
Suppose T is surj. $\forall w, x \in W, \exists\, u, v \in V, Tu = w, Tv = x$
$\Rightarrow \forall w + \mathrm{i}x \in W_C, \exists\, u + \mathrm{i}v \in V, T(u + \mathrm{i}v) = w + \mathrm{i}x \Rightarrow T_C$ is surj.

---

3 *Suppose $(v_1, \dots, v_m)$ in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1 v_1 + \cdots + z_m v_m$.*
(a) *The surj of T correspds to $(v_1, \dots, v_m)$ spanning V.*
(b) *The inje of T correspds to $(v_1, \dots, v_m)$ being linely inde.*

Comment: Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $Te_k = v_k$.
(a) range $T = \operatorname{span}(v_1, \dots, v_m) = V$; (b) $(v_1, \dots, v_m)$ is linely inde $\Longleftrightarrow$ T is inje.

---

7 *Suppose V is finite-dim with $2 \leqslant \dim V$. And $\dim V \leqslant \dim W = m$, if W is finite-dim.*
Show that $U = \big\{ T \in \mathcal{L}(V,W) : \operatorname{null} T \neq \{0\} \big\}$ is not a subsp of $\mathcal{L}(V,W)$.

Solution: *The set of all inje $T \in \mathcal{L}(V,W)$ is a not subsp either.*

Let $(v_1, \dots, v_n)$ be a basis of V, $(w_1, \dots, w_m)$ be linely inde in W. $\big(2 \leqslant n \leqslant m. \big)$
Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 :\ v_1 \mapsto 0,\quad v_2 \mapsto w_2,\quad v_i \mapsto w_i$.
Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 :\ v_1 \mapsto w_1,\quad v_2 \mapsto 0,\quad v_i \mapsto w_i,\quad i = 3, \dots, n.$
Thus $T_1 + T_2 \notin U$. ☐

Comment: If $\dim V = 0$, then $V = \{0\} = \operatorname{span}(\,)$. $\forall\ T \in \mathcal{L}(V,W), T$ is inje. Hence $U = \emptyset$.
If $\dim V = 1$, then $V = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0 v_0 = 0$.

---

8 *Suppose W is finite-dim with $\dim W \geqslant 2$. And $n = \dim V \geqslant \dim W$, if V is finite-dim.*
Show that $U = \big\{ T \in \mathcal{L}(V,W) : \operatorname{range} T \neq W \big\}$ is not a subsp of $\mathcal{L}(V,W)$.

Solution: *The set of all surj $T \in \mathcal{L}(V,W)$ is not a subspace either.*

Let $(v_1, \dots, v_n)$ be linely inde in V, $(w_1, \dots, w_m)$ be a basis of W. $\big(n \in \{m, m+1, \dots\}; 2 \leqslant m \leqslant n. \big)$
Define $T_1 \in \mathcal{L}(V,W)$ as $T_1 :\ v_1 \mapsto 0,\quad v_2 \mapsto w_2,\quad v_j \mapsto w_j,\quad v_{m+i} \mapsto 0.$
Define $T_2 \in \mathcal{L}(V,W)$ as $T_2 :\ v_1 \mapsto w_1,\quad v_2 \mapsto 0,\quad v_j \mapsto w_j,\quad v_{m+i} \mapsto 0.$
$\big($ For each $j = 2, \dots, m$; $i = 1, \dots, n - m$, if V is finite, otherwise let $i \in \mathbf{N}^+$. $\big)$ Thus $T_1 + T_2 \notin U$. ☐

Comment: If $\dim W = 0$, then $W = \{0\} = \operatorname{span}(\,)$. $\forall\ T \in \mathcal{L}(V,W), T$ is surj. Hence $U = \emptyset$.
If $\dim W = 1$, then $W = \operatorname{span}(v_0)$. Thus $U = \operatorname{span}(T_0)$, where $T_0 v_0 = 0$.

---

11 *Suppose $S_1, \dots, S_n$ are linear and inje. $S_1 S_2 \dots S_n$ makes sence. Prove that $S_1 S_2 \dots S_n$ is inje.*

Solution: $S_1 S_2 \dots S_n(v) = 0 \Longleftrightarrow S_2 S_3 \dots S_n(v) = 0 \Longleftrightarrow \cdots \Longleftrightarrow S_n(v) = 0 \Longleftrightarrow v = 0.$ ☐

9 *Suppose $(v_1,\dots,v_n)$ is linely inde. Prove that $\forall$ inje T, $(Tv_1,\dots,Tv_n)$ is linely inde.*

Solution: $a_1Tv_1+\dots+a_nTv_n=0=T\big(\sum_{i=1}^n a_iv_i\big)\iff\sum_{i=1}^n a_iv_i=0\iff a_1=\dots=a_n=0.$ □

---

10 *Suppose* $\operatorname{span}(v_1,\dots,v_n)=V$. *Show that* $\operatorname{span}(Tv_1,\dots,Tv_n)=\operatorname{range}T$.

Solution:

(a) $\operatorname{range}T=\{Tv:v\in V\}=\{Tv:v\in\operatorname{span}(v_1,\dots,v_n)\}\Rightarrow Tv_1,\dots,Tv_n\in\operatorname{range}T\Rightarrow$ By [2.7].

Or. $\operatorname{span}(Tv_1,\dots,Tv_n)\ni a_1Tv_1+\dots+a_nTv_n=T(a_1v_1+\dots+a_nv_n)\in\operatorname{range}T$.

(b) $\forall w\in\operatorname{range}T,\exists v\in V,w=Tv.\ (\exists a_i\in\mathbf{F},v=a_1v_1+\dots+a_nv_n)\Rightarrow w=a_1Tv_1+\dots+a_nTv_n.$ □

---

16 *Suppose $\exists T\in\mathcal{L}(V)$ such that* $\operatorname{null}T,\operatorname{range}T$ *are finite-dim. Prove that V is finite-dim.*

Solution: Let $B_{\operatorname{range}T}=(Tv_1,\dots,Tv_n),B_{\operatorname{null}T}=(u_1,\dots,u_m)$.

$\forall v\in V,T(v-a_1v_1-\dots-a_nv_n)=0$, letting $Tv=a_1Tv_1+\dots+a_nTv_n$.

$\Rightarrow v-a_1v_1-\dots-a_nv_n=b_1u_1+\dots+b_mu_m$. Hence $V\subseteq\operatorname{span}(v_1,\dots,v_n,u_1,\dots,u_m)$. □

---

17 *Suppose V,W are finite-dim. Prove that $\exists$ inje $T\in\mathcal{L}(V,W)\iff$* $\dim V\leqslant\dim W$.

Solution:

(a) Suppose $\exists$ inje T. Then $\dim V=\dim\operatorname{range}T\leqslant\dim W$.

(b) Suppose $\dim V\leqslant\dim W$. Let $B_V=(v_1,\dots,v_n),B_W=(w_1,\dots,w_m)$.

Define $T\in\mathcal{L}(V,W)$ by $Tv_i=w_i,\ \ i=1,\dots,n\ (=\dim V)$. □

---

18 *Suppose V,W are finite-dim. Prove that $\exists$ surj $T\in\mathcal{L}(V,W)\iff$* $\dim V\geqslant\dim W$.

Solution:

(a) Suppose $\exists$ surj T. Then $\dim V=\dim W+\dim\operatorname{null}T\Rightarrow\dim W\leqslant\dim V$.

(b) Suppose $\dim V\geqslant\dim W$. Let $B_V=(v_1,\dots,v_n),B_W=(w_1,\dots,w_m)$.

Define $T\in\mathcal{L}(V,W)$ by $T(a_1v_1+\dots+a_mv_m+\dots+a_nv_n)=a_1w_1+\dots+a_mw_m$. □

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19 *Suppose V,W are finite-dim, U is a subsp of V.*

Prove that if $\underbrace{\dim U}_{m}\geqslant\underbrace{\dim V}_{m+n}-\underbrace{\dim W}_{p}$, then $\exists T\in\mathcal{L}(V,W)$, $\operatorname{null}T=U$.

Solution:

Let $B_U=(u_1,\dots,u_m),B_V=(u_1,\dots,u_m,v_1,\dots,v_n),B_W=(w_1,\dots,w_p)$.

Define $T\in\mathcal{L}(V,W)$ by $T(a_1v_1+\dots+a_nv_n+b_1u_1+\dots+b_mu_m)=a_1w_1+\dots+a_nw_n$. □

• (4E 3.B.21)

Suppose V is finite-dim, $T\in\mathcal{L}(V,W)$, U is a subsp of W. Let $\mathcal{K}_U=\{v\in V:Tv\in U\}$.
Prove that $\mathcal{K}_U$ is a subsp of V and $\dim\mathcal{K}_U=\dim\operatorname{null}T+\dim(U\cap\operatorname{range}T)$.

Solution:

$\forall u,w\in\mathcal{K}_U,\lambda\in\mathbf{F},T(u+\lambda w)=Tu+\lambda Tw\in U\Rightarrow\mathcal{K}_U$ is a subsp of V.

Define $S\in\mathcal{L}(\mathcal{K}_U,U)$ as $Rv=Tv$ for all $v\in\mathcal{K}_U$. Hence $\operatorname{range}R=U\cap\operatorname{range}T$.

Suppose $\exists v,Tv=0$. 又 $0\in U\Rightarrow Rv=0$. Thus $\operatorname{null}T\subseteq\operatorname{null}R$. □

• **Tips:** *Suppose U is a subsp of V. Prove that $\forall T\in\mathcal{L}(V,W)$,* $U\cap\operatorname{null}T=\operatorname{null}T|_U$.

Solution: Note that $U\cap\operatorname{null}T\subseteq\operatorname{null}T|_U$. On the other hand, suppose $u\in\operatorname{null}T|_U$.

Then $T|_U(u)$ makes sense $\Rightarrow u\in U$. And $T|_U(u)=Tu=0\Rightarrow u\in\operatorname{null}T$. □

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12 *Prove that* $\forall T \in \mathcal{L}(V, W), \exists$ *subsp* U *of* V *such that*

$U \cap \operatorname{null} T = \operatorname{null} T|_U = \{0\}$, $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$.

Which is equivalent to $T|_U : U \to \operatorname{range} T$ *being an iso.*

SOLUTION:

By [2.34] (note that V can be infinite-dim), $\exists$ subsp U of V such that $V = U \oplus \operatorname{null} T$.

$\forall v \in V, \exists!\, w \in \operatorname{null} T, u \in U, v = w + u$. Then $Tv = T(w+u) = Tu \in \{Tu : u \in U\}$. $\square$

- **NEW NOTATION:**

Suppose $T \in \mathcal{L}(V, W)$ and $R = (Tv_1, \dots, Tv_n)$ is linely inde in range T.

Where $n = \dim \operatorname{range} T$ if finite-dim, otherwise $n \in \mathbf{N}^+$.

By (3.A.4), $L = (v_1, \dots, v_n)$ is linely inde in V.

Denote $\mathcal{K}_R$ by span L, if range T is finite-dim, otherwise, denote it by a vecsp in $\mathcal{S}_V \operatorname{null} T$.

Note that if range T is finite-dim, then $\mathcal{K}_R = \operatorname{range} T$ for any basis R of range T.

- **COMMENT:**

If range T is infinite-dim, we cannot write $\mathcal{K}_R = \operatorname{range} T$. For if we do so, we must guarantee that $\forall Tv \in \operatorname{range} T, \exists!\, n \in \mathbf{N}^+, Tv \in \operatorname{span}(Tv_1, \dots, Tv_n)$, where $(Tv_k)_{k=1}^{\infty}$ is linely inde.

So that range $T \subseteq$ span $(Tv_1, \cdots, Tv_n, \cdots)$. This would be invalid, as we have shown before.

- **NEW THEOREM:** $\mathcal{K}_R \in \mathcal{S}_V \operatorname{null} T$. **COMMENT:** $\operatorname{null} T \in \mathcal{S}_V \mathcal{K}_R$.

Suppose range T is finite-dim. Otherwise, we are done immediately.

(a) $T\left(\sum_{i=1}^n a_i v_i\right) = 0 \Rightarrow \sum_{i=1}^n a_i T v_i = 0 \Rightarrow a_1 = \cdots = a_n = 0 \Rightarrow \mathcal{K}_R \cap \operatorname{null} T = \{0\}$.

(b) $\forall v \in V, Tv = \sum_{i=1}^n a_i T v_i \Rightarrow Tv - \sum_{i=1}^n a_i T v_i = T\left(v - \sum_{i=1}^n a_i v_i\right) = 0$

$\Rightarrow v - \sum_{i=1}^n a_i v_i \in \operatorname{null} T \Rightarrow v = \left(v - \sum_{i=1}^n a_i v_i\right) + \left(\sum_{i=1}^n a_i v_i\right) \Rightarrow \mathcal{K}_R + \operatorname{null} T = V$. $\square$

- *Suppose* V *is finite-dim,* $T \in \mathcal{L}(V, W), B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n), B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$.
Prove or give a counterexample: $(u_1, \dots, u_m)$ *is a basis of* $\operatorname{null} T$.

SOLUTION: A counterexample:

Suppose $\dim V = 3, Tv_1 = Tv_2 = Tv_3 = w_1$. Then $\operatorname{span}(Tv_1, Tv_2, Tv_3) = \operatorname{span}(w_1)$.

Extend (v_i) to (v_1, v_2, v_3) for each i. But none of $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is a basis of null T. $\square$

COMMENT: $(v_2 - v_1, v_3 - v_1), (v_1 - v_2, v_3 - v_2)$ or $(v_1 - v_3, v_2 - v_3)$ are all bases of null T.

Always notice that $\mathcal{S}_V \operatorname{span}(v_1, \dots, v_n) = \{U_1, \cdots, \operatorname{null} T, \cdots, U_n, \cdots\}$.

- *Suppose* V *is finite-dim,* X *is a subsp of* V*, and* Y *is a finite-dim subsp of* W.
Prove that if $\dim X + \dim Y = \dim V$*, then* $\exists\, T \in \mathcal{L}(V, W), \operatorname{null} T = X, \operatorname{range} T = Y$.

SOLUTION:

Suppose $\dim X + \dim Y = \dim V$. Let $B_X = (u_1, \dots, u_n), B_Y = (w_1, \dots, w_m), B_V = (u_1, \dots, u_n, v_1, \dots, v_m)$.

Define $T \in \mathcal{L}(V, W)$ by $Tv_i = w_i, Tu_j = 0$. Notice that $\forall v \in V, \exists!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i$.

$v \in \operatorname{null} T \iff Tv = 0 \iff a_1 = \cdots = a_m = 0 \iff v \in X$.

$Y \ni w = a_1 w_1 + \cdots + a_m w_m = a_1 T v_1 + \cdots + a_m T v_m \in \operatorname{range} T$.

OR. range $T = \operatorname{span}(Tv_1, \dots, Tv_m, Tu_1, \dots, Tu_n) = \operatorname{span}(Tv_1, \dots, Tv_m) = \operatorname{span}(w_1, \dots, w_m) = Y$. $\square$

• OR (5.B.4) *Suppose* $P \in \mathcal{L}(V)$ *and* $P^2 = P$. *Prove that* $V = \operatorname{null} P \oplus \operatorname{range} P$.

SOLUTION:

(a) If $v \in \operatorname{null} P \cap \operatorname{range} P \Rightarrow Pv = 0$ and $\exists\, u \in V, v = Pu$. Then $v = Pu = P^2u = Pv = 0$.

(b) Note that $\forall v \in V, v = Pv + (v - Pv)$ and $P(v - Pv) = 0 \Rightarrow v - Pv \in \operatorname{null} P$. $\square$

OR. [*Only in Finite-dim*] Let $(P^2v_1, \dots, P^2v_n)$ be a basis of range P^2. Then $(Pv_1, \dots, Pv_n)$ is linely inde. Let $\mathcal{K} = \operatorname{span}(Pv_1, \dots, Pv_n) \Rightarrow V = \mathcal{K} \oplus \operatorname{null} P^2$. While $\mathcal{K} = \operatorname{range} P = \operatorname{range} P^2$; $\operatorname{null} P = \operatorname{null} P^2$. $\square$

---

20 *Suppose* W *is finite-dim. Prove that* $T \in \mathcal{L}(V, W)$ *is inje* $\Longleftrightarrow \exists\, S \in \mathcal{L}(W, V),\ ST = I_V$.

SOLUTION:

(a) Suppose $\exists\, S \in \mathcal{L}(W, V),\ ST = I$. Then if $Tv = 0 \Rightarrow ST(v) = 0 = v$. OR. $\operatorname{null} T \subseteq \operatorname{null} ST = \{0\}$.

(b) Suppose T is inje. Let $R = B_{\operatorname{range} T} = (Tv_1, \dots, Tv_n)$. Then $\mathcal{K}_R \oplus \operatorname{null} T = V$. Let $U \oplus \operatorname{range} T = W$. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$ and $Su = 0$, where $i \in \{1, \dots, n\}, u \in U$. Thus $ST = I$.

OR. Define $S \in \mathcal{L}(\operatorname{range} T, V)$ by $Sw = T^{-1}w$, *where* T^{-1} *is the inv of* $T \in \mathcal{L}(V, \operatorname{range} T)$. Then extend it to $S \in \mathcal{L}(W, V)$ by (3.A.11). Now $\forall v \in V, STv = T^{-1}Tv = v$. $\square$

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21 *Suppose* W *is finite-dim. Prove that* $T \in \mathcal{L}(V, W)$ *is surj* $\Longleftrightarrow \exists\, S \in \mathcal{L}(W, V),\ TS = I_W$.

SOLUTION:

(a) Suppose $\exists\, S \in \mathcal{L}(W, V),\ TS = I$. Then $\forall\, w \in W, TS(w) = w \in \operatorname{range} T \Rightarrow \operatorname{range} T = W$.

(b) Suppose T is surj. Let $R = B_{\operatorname{range} T} = B_W = (Tv_1, \dots, Tv_n)$. Then $\mathcal{K}_R \oplus \operatorname{null} T = V$. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$. Then $TS = I$.

OR. By Problem (12), $\exists$ subsp U of $V, V = U \oplus \operatorname{null} T, \operatorname{range} T = \{Tu : u \in U\}$. Note that $T|_U : U \to W$ is an iso. Define $S = (T|_U)^{-1}$, where $(T|_U)^{-1} : W \to U$. Then $TS = T \circ (T|_U)^{-1} = T|_U \circ (T|_U)^{-1}$. $\square$

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24 *Suppose* $S, T \in \mathcal{L}(V, W)$. *Prove that* $\operatorname{null} S \subseteq \operatorname{null} T \Longleftrightarrow \exists\, E \in \mathcal{L}(W)$ *such that* $T = ES$.

SOLUTION:

Suppose $\exists\, E \in \mathcal{L}(W)$ such that $T = ES$. Then $\operatorname{null} T = \operatorname{null} ES \supseteq \operatorname{null} S$.

Suppose $\operatorname{null} S \subseteq \operatorname{null} T$. Let $W = \operatorname{range} S \oplus U$.

Define $E \in \mathcal{L}(W)$ by $E(Sv + w) = Tv$ for each Sv and each $w \in U$. Now we check that E is linear.

Because $\forall w_1, w_2 \in W, \exists!\, Sv_1, Sv_2 \in \operatorname{range} S, u_1, u_2 \in U, w_1 = Sv_1 + u_1, w_2 = Sv_2 + u_2$.

Now $E(w_1 + \lambda w_2) = E\big((Sv_1 + \lambda Sv_2) + (u_1 + \lambda u_2)\big) = T(v_1 + \lambda v_2) = Tv_1 + \lambda Tv_2 = Ew_1 + \lambda Ew_2$.

OR. Let $V = \mathcal{K} \oplus U$. Then $S|_{\mathcal{K}} : \mathcal{K} \to \operatorname{range} S$ is an iso.
Now extend $T(S|_{\mathcal{K}})^{-1} \in \mathcal{L}(\operatorname{range} S, W)$ to $E \in \mathcal{L}(W, W)$.

OR. [*Requires that* range S *is Finite-dim*] Let $R = B_{\operatorname{range} S} = (Sv_1, \dots, Sv_n)$. Then $V = \mathcal{K}_R \oplus \operatorname{null} S$.
Define $E \in \mathcal{L}(W)$ by $E(Sv_i) = Tv_i$, $Eu = 0$; for each $i = 1 \dots, n$ and each $u \in \operatorname{null} S$.
Hence $\forall v \in V$, $(\exists!\, a_i \in \mathbf{F}, u \in \operatorname{null} S\,)$, $Tv = a_1Tv_1 + \dots + a_nTv_n = E(a_1Sv_1 + \dots + a_nSv_n) \Rightarrow T = ES$.

OR. [*Requires that* W *is Finite-dim*] Extend R to a basis $(Sv_1, \dots, Sv_n, w_1, \dots, w_m)$ of W.
Define $E \in \mathcal{L}(W)$ by $E(Sv_k) = Tv_k$, $Ew_j = 0$. Because $\forall\, v \in V, \exists\, a_i \in \mathbf{F}, Sv = a_1Sv_1 + \dots + a_nSv_n$.
Now $v - (a_1v_1 + \dots + a_nv_n) \in \operatorname{null} S \Rightarrow v - (a_1v_1 + \dots + a_nv_n) \in \operatorname{null} T \Rightarrow T\big(v - (a_1v_1 + \dots + a_nv_n)\big) = 0$.
Thus $Tv = a_1v_1 + \dots + a_nv_n$. Hence $E(Sv) = a_1E(Sv_1) + \dots + a_nE(Sv_n) = a_1Tv_1 + \dots + a_nTv_n = Tv$ $\square$

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25 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{range} S \subseteq \operatorname{range} T \iff \exists\, E \in \mathcal{L}(V)$ *such that* $S = TE$.

SOLUTION:

Suppose $\exists\, E \in \mathcal{L}(V)$ such that $S = TE$. Then $\operatorname{range} S = \operatorname{range} TE \subseteq \operatorname{range} T$.
Suppose $\operatorname{range} S \subseteq \operatorname{range} T$. Let $(v_1, \dots, v_m)$ be a basis of V.
Note that each $Sv_i \in \operatorname{range} T$. Suppose $u_i \in V$ such that $Tu_i = Sv_i$.
Thus defining $E \in \mathcal{L}(V)$ by $Ev_i = u_i$ for each $i \Rightarrow S = TE$. □

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22 *Suppose U and V are finite-dim vecsps and $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$.*
Prove that $\dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$.

SOLUTION:

Define $R \in \mathcal{L}(\operatorname{null} ST, V)$ by $Ru = Tu$ for all $u \in \operatorname{null} ST \subseteq U$.

$$\left.\begin{aligned} S(Tu) = 0 = S(Ru) &\Rightarrow \operatorname{range} R \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} R \leqslant \dim \operatorname{null} S \\ Tu = 0 = Ru &\Rightarrow \operatorname{null} R \supseteq \operatorname{null} T \Rightarrow \dim \operatorname{null} R = \dim \operatorname{null} T \end{aligned}\right\} \Rightarrow \text{By [3.22], we are done.} \ \square$$

OR. For any $u \in U$, note that $u \in \operatorname{null} ST \iff S(Tu) = 0 \iff Tu \in \operatorname{null} S$.
Thus $\operatorname{null} ST = \mathcal{K}_{\operatorname{null} S \cap \operatorname{range} T} = \{u \in U : Tu \in \operatorname{null} S\}$. By Problem (4E 3B.21),
$\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T) \leqslant \dim \operatorname{null} T + \dim \operatorname{null} S$. □

COROLLARY: (1) If T is inje, then $\dim \operatorname{null} T = 0 \Rightarrow \dim \operatorname{null} ST \leqslant \dim \operatorname{null} S$.
(2) If T is surj, then $\operatorname{range} R = \operatorname{null} S \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} S + \dim \operatorname{null} T$.
(3) If S is inje, then $\operatorname{range} R = \{0\} \Rightarrow \dim \operatorname{null} ST = \dim \operatorname{null} R = \dim \operatorname{null} T$.

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23 *Suppose U and V are finite-dim vecsps and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$.*
Prove that $\dim \operatorname{range} ST \leqslant \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.

SOLUTION:

$\operatorname{range} ST = \{Sv : v \in \operatorname{range} T\} = \operatorname{span}(Su_1, \dots, Su_{\dim \operatorname{range} T})$, where $B_{\operatorname{range} T} = (u_1, \dots, u_{\dim \operatorname{range} T})$.
$\dim \operatorname{range} ST \leqslant \dim \operatorname{range} T$ 又 $\dim \operatorname{range} ST \leqslant \dim \operatorname{range} S$. □

OR. Note that $\operatorname{range} S|_{\operatorname{range} T} = \operatorname{range} ST$.
Thus $\dim \operatorname{range} ST = \dim \operatorname{range} S|_{\operatorname{range} T} = \dim \operatorname{range} T - \dim \operatorname{null} S|_{\operatorname{range} T} \leqslant \operatorname{range} T$. □

COROLLARY: (1) If S is inje, then $\dim \operatorname{range} ST = \dim \operatorname{range} T$.
(2) If T is surj, then $\dim \operatorname{range} ST = \dim \operatorname{range} S$.

• (a) *Suppose* $\dim V = 5$, $S, T \in \mathcal{L}(V)$ *are such that* $ST = 0$. *Prove that* $\dim \operatorname{range} TS \leqslant 2$.
(b) *Let* $\dim V = n$ *in* (a). *Prove that* $\dim \operatorname{range} TS \leqslant \left\lfloor \dfrac{n}{2} \right\rfloor$.
(c) *Give an example of* $S, T \in \mathcal{L}(\mathbf{F}^5)$ *with* $ST = 0$ *and* $\dim \operatorname{range} TS = 2$.

SOLUTION:

(a) By Problem (23), $\dim \operatorname{range} TS \leqslant \min\{\overbrace{\dim \operatorname{range} S}^{5-\dim \operatorname{null} T}, \overbrace{\dim \operatorname{range} T}^{5-\dim \operatorname{null} S}\}$.
We show that $\dim \operatorname{range} TS \leqslant 2$ by contradiction. Assume that $\dim \operatorname{range} TS \geqslant 3$.
Then $\min\{5 - \dim \operatorname{null} T, 5 - \dim \operatorname{null} S\} \geq 3 \Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant 2$.
又 $\dim \operatorname{null} ST = 5 \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \leqslant 4$. Contradicts.

OR. $$\left.\begin{aligned} \dim \operatorname{null} S &= 5 - \dim \operatorname{range} S \\ \dim \operatorname{range} TS &\leqslant \dim \operatorname{range} S \end{aligned}\right\} \Rightarrow \dim \operatorname{null} S \leqslant 5 - \dim \operatorname{range} TS.$$

And $ST = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S \Rightarrow \dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S$. □

(b) By Problem (23), $\dim \operatorname{range} TS \leqslant \min\{\overbrace{\dim \operatorname{range} S}^{n-\dim \operatorname{null} T}, \overbrace{\dim \operatorname{range} T}^{n-\dim \operatorname{null} S}\}$. We prove by contradiction.

Assume that $\dim \operatorname{range} TS \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Then $\min\{n - \dim \operatorname{null} T, n - \dim \operatorname{null} S\} \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$

$\Rightarrow \max\{\dim \operatorname{null} T, \dim \operatorname{null} S\} \leqslant n - \left\lfloor \frac{n}{2} \right\rfloor - 1$.

又 $\dim \operatorname{null} ST = n \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \leqslant 2(n - \left\lfloor \frac{n}{2} \right\rfloor - 1)$

$\Rightarrow \left\lfloor \frac{n}{2} \right\rfloor + 1 \leqslant \frac{n}{2}$. Contradicts. Thus $\dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$. □

OR. $\dim \operatorname{null} S = n - \dim \operatorname{range} S \leqslant n - \dim \operatorname{range} TS$.

And $ST = 0 \Rightarrow \dim \operatorname{range} TS \leqslant \dim \operatorname{range} T \leqslant \dim \operatorname{null} S \leqslant n - \dim \operatorname{range} TS$

$\Rightarrow 2 \dim \operatorname{range} TS \leqslant n \Rightarrow \dim \operatorname{range} TS \leqslant \frac{n}{2}$

$\Rightarrow \dim \operatorname{range} TS \leqslant \left\lfloor \frac{n}{2} \right\rfloor$ (because $\dim \operatorname{range} TS$ is an integer). □

(c) Let $(v_1, \dots, v_5)$ be a basis of $\mathbf{F}^5$. Define $S, T \in \mathcal{L}(\mathbf{F}^5)$ by:

$T: \quad v_1 \mapsto 0, \quad v_2 \mapsto 0, \quad v_i \mapsto v_i \;;$

$S: \quad v_1 \mapsto v_4, \quad v_2 \mapsto v_5, \quad v_i \mapsto 0 \;; \qquad i = 3, 4, 5.$ □

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26 *Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ such that $\forall p \in \mathcal{P}(\mathbf{R}), \deg(Dp) = (\deg p) - 1$. Prove that $D \in \mathcal{P}(\mathbf{R})$ is surj.*

SOLUTION:

[*Informal Proof*] | Note that $\deg Dx^n = n - 1$. Because $\operatorname{span}(Dx, Dx^2, \dots) \subseteq \operatorname{range} D$.
又 By (2.C.10), $\operatorname{span}(Dx, Dx^2, \dots) = \operatorname{span}(1, x, \dots) = \mathcal{P}(\mathbf{R})$.

[*Proper Proof*]

We will recursively define a sequence of polys $(p_k)_{k=0}^{\infty}$ where $Dp_k = x^k$.

(i) Because $\dim Dx = (\deg x) - 1 = 0$, we have $Dx = C \in \mathbf{F}$.
Define $p_0 = C^{-1}x$. Then $Dp_0 = C^{-1}Dx = 1$.

(ii) Suppose we have defined $p_0, \dots, p_n$ such that $Dp_k = x^k$ for each $k \in \{0, \dots, n\}$.
Because $\deg D(x^{n+2}) = n + 1$. Let $D(x^{n+2}) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$, where $a_{n+1} \neq 0$.
Then $a_{n+1}^{-1} D(x^{n+2}) = x^{n+1} + a_{n+1}^{-1}(a_n Dp_n + \cdots + a_1 Dp_1 + a_0 Dp_0)$
$\Rightarrow x^{n+1} = D\big(a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)\big)$.
Thus defining $p_{n+1} = a_{n+1}^{-1}(x^{n+2} - a_n p_n - \cdots - a_1 p_1 - a_0 p_0)$, we have $Dp_{n+1} = x^{n+1}$.

Now we get $(p_k)_{k=0}^{\infty}$ by recursion. Hence $\forall p = \sum_{k=0}^{\deg p} a_k x^k \in \mathcal{P}(\mathbf{R}), \exists\, q = \big(\sum_{k=0}^{\deg p} a_k p_k\big), Dq = p$. □

OR. Let $Dx^0 = 0, Dx^k = p_k$ for all $k \in \mathbf{N}^+$. For any $m \in \mathbf{N}^+$, $(p_1, \dots, p_m)$ is a basis of $\mathcal{P}_{m-1}(\mathbf{R})$.
Because $\forall p' \in \operatorname{range} D, \exists!\, m \in \mathbf{N}, \deg p = m - 1 \Rightarrow \exists!\, a_k \in \mathbf{R}, p' = a_m p_m + \cdots + a_1 p_1$.
Now $Dp = p' = a_m p_m + \cdots + a_1 p_1 = D(a_m x^m + \cdots + a_1 x)$. Thus $\exists\, q \in \mathcal{P}_m(\mathbf{R}), Dq = p$. □

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27 *Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.*

SOLUTION:

Define $B \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $B = 5D^2 + 3D \Rightarrow B(q) = 5q'' + 3q'$.
Note that $\deg Bx^n = n - 1$. Similar to Problem (26), we conclude that B is surj. □

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28 *Suppose* $T \in \mathcal{L}(V, W), B_{\operatorname{range} T} = (w_1, \dots, w_m)$.
Prove that $\exists\, \varphi_1, \dots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ *such that* $\forall v \in V, Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$.

Solution:

Suppose $v_1, \dots, v_m \in V$ such that $Tv_i = w_i$ for each v_i. Then $(v_1, \dots, v_m)$ is linely inde.
Let $B_V = (v_1, \dots, v_m, u_1, \dots, u_n)$. Note that $\forall v \in V, v = \sum_{i=1}^m a_i v_i + \sum_{i=1}^n b_i u_i, \exists\,!\, a_i, b_i \in \mathbf{F}$.
Define $\varphi_i : V \to \mathbf{F}$ by $\varphi_i(v) = a_i v_i$ for each i. We now check the linearity.
$\forall v, u \in V\ (\exists\,!\, a_i, b_i, c_i, d_i \in \mathbf{F}\,), \lambda \in \mathbf{F}, \varphi_i(v + \lambda u) = a_i + \lambda c_i = \varphi(v) + \lambda\varphi(u)$. ☐

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29 *Suppose* $\varphi \in \mathcal{L}(V, \mathbf{F})$. *Suppose* $u \in V \setminus \operatorname{null}\varphi$. *Prove that* $V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}$.

Solution: If $\varphi = 0$ then we are done. Suppose $\varphi \neq 0$.

(a) $\forall\, v = cu \in \operatorname{null}\varphi \cap \{au : a \in \mathbf{F}\}, \varphi(v) = 0 = c\varphi(u) \Rightarrow c = 0$. Hence $\operatorname{null}\varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$.

(b) $$\forall\, v \in V, v = \left(v - \frac{\varphi(v)}{\varphi(u)}u\right) + \frac{\varphi(v)}{\varphi(u)}u. \left|\ \begin{array}{l} v - \dfrac{\varphi(v)}{\varphi(u)}u \in \operatorname{null}\varphi \\[2ex] \dfrac{\varphi(v)}{\varphi(u)}u \in \{au : a \in \mathbf{F}\} \end{array}\right\} \Rightarrow V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}.$$ ☐

Comment: $\varphi \neq 0 \Rightarrow \varphi(v_i) = a_i \neq 0$ for each v_i, for some linely inde list $(v_1, \dots, v_k)$.
Fix one v_k. Then $\forall j \in \{1, \dots, k-1, k+1, \dots, n\}, \operatorname{span}\{a_j v_k - a_k v_j\} \subseteq \operatorname{null}\varphi$.
Hence every vecsp in $\mathcal{S}_V \operatorname{null}\varphi$ is one-dim.

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30 *Suppose* $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbf{F})$ *and* $\operatorname{null}\varphi_1 = \operatorname{null}\varphi_2 = \operatorname{null}\varphi$. *Prove that* $\exists\, c \in \mathbf{F}, \varphi_1 = c\varphi_2$

Solution:

If $\operatorname{null}\varphi = V$, then $\varphi_1 = \varphi_2 = 0$, we are done. Suppose $u \in V \setminus \operatorname{null}\varphi \Rightarrow \varphi_1(u), \varphi_2(u) \neq 0$.
By Problem (29), $V = \operatorname{null}\varphi \oplus \operatorname{span}(u)$. Hence for any $v \in V, v = w + a_v u, \exists\,!\, w \in \operatorname{null}\varphi, a_v \in \mathbf{F}$.
$$\varphi_1(v) = a_v\varphi_1(u),\ \ \varphi_2(v) = a_v\varphi_2(u) \Rightarrow a_v = \frac{\varphi_1(v)}{\varphi_1(u)} = \frac{\varphi_2(v)}{\varphi_2(u)} \Rightarrow \frac{\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(v)}{\varphi_2(v)} = c \in \mathbf{F}.$$ ☐

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31 *Prove that* $\exists\, T_1, T_2 \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^2), \operatorname{null} T_1 = \operatorname{null} T_2$ *and* $T_1 \neq cT_2, \forall c \in \mathbf{F}$.

Solution:

Let $(v_1, \dots, v_5)$ be a basis of $\mathbf{R}^5$, (w_1, w_2) be a basis of $\mathbf{R}^2$. Define $T, S \in \mathcal{L}(V, W)$ by
$$\left.\begin{array}{lll} Tv_1 = w_1, & Tv_2 = w_2, & Tv_3 = Tv_4 = Tv_5 = 0 \\ Sv_1 = w_1, & Sv_2 = 2w_2, & Sv_3 = Sv_4 = Sv_5 = 0 \end{array}\right\} \Rightarrow \operatorname{null} T = \operatorname{null} S.$$
Suppose $T = \lambda S$. Then $w_1 = Tv_1 = \lambda Sv_1 = \lambda w_1 \Rightarrow \lambda = 1$.
While $w_2 = Tv_2 = \lambda Sv_2 = 2\lambda w_2 \Rightarrow \lambda = \frac{1}{2}$. Contradicts. ☐

• **Tips:** *Suppose* $T \in \mathcal{L}(V, W)$ *and* U *is a subsp such that* $V = U \oplus \operatorname{null} T$.
Now $\forall v \in V, \exists\,!\, u_v \in U, w_v \in \operatorname{null} T, v = u_v + w_v$.
Then $T = T \circ i$, *where* $i : V \to U$ *is defined by* $i(v) = u_v$.
Because $\forall v \in V, T(v) = T(u_v + w_v) = T(u_v) = T(i(v)) = (T \circ i)(v)$. ☐

Ended

3·C 1 3 4 5 6 9 10 11 12 13 14 15 | 4E: 16, 17

- **Note For [3.47]:** $LHS = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}(C_{\cdot,k})_{r,1} = (A_{j,\cdot}C_{\cdot,k})_{1,1} = A_{j,\cdot}C_{\cdot,k} = RHS.$

- **Note For [3.48]:**

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6&7\\8&9&10\end{pmatrix} = \begin{pmatrix} \begin{pmatrix}1&2\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} & \begin{pmatrix}1&2\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & \begin{pmatrix}1&2\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix} \\ \begin{pmatrix}3&4\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} & \begin{pmatrix}3&4\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & \begin{pmatrix}3&4\end{pmatrix}\begin{pmatrix}7\\10\end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix}1\times5+2\times8 & 1\times6+2\times9 & 1\times7+2\times10\\ 3\times5+4\times8 & 3\times6+4\times9 & 3\times7+4\times10\end{pmatrix} = \begin{pmatrix}21&24&27\\47&54&61\end{pmatrix}$$

- [4E 3.51] *Suppose* $C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,p}$.

(a) For $k = 1,\dots,p$, $\quad (CR)_{\cdot,k} = CR_{\cdot,k} = C_{\cdot,\cdot}R_{\cdot,k} = \sum_{r=1}^{c} C_{\cdot,r}R_{r,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c}$

Which means that each cols CR is a linear combination of the cols of C.

(b) For $j = 1,\dots,m$, $\quad (CR)_{j,\cdot} = C_{j,\cdot}R = C_{j,\cdot}R_{\cdot,\cdot} = \sum_{r=1}^{c} C_{j,r}R_{r,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$

Which means that each rows CR is a linear combination of the rows of R.

- **Column-Row Factorization** (CR Factorization) *Suppose* $A \in \mathbf{F}^{m,n}, A \neq 0$.

(a) *Let* $S_c = \operatorname{span}(A_{\cdot,1},\dots,A_{\cdot,n}) \subseteq \mathbf{F}^{m,1}, \dim S_c = c$, *the col rank.*

Prove that $\exists\, C \in \mathbf{F}^{m,c}, R \in \mathbf{F}^{c,n}, A = CR$.

(b) *Let* $S_r = \operatorname{span}(A_{1,\cdot},\dots,A_{m,\cdot}) \subseteq \mathbf{F}^{1,n}, \dim S_r = r$, *the row rank.*

Prove that $\exists\, C \in \mathbf{F}^{m,r}, R \in \mathbf{F}^{r,n}, A = CR$.

Solution: Notice that $A \neq 0 \Rightarrow c, r \geqslant 1$.

(a) Let $(C_{\cdot,1},\dots,C_{\cdot,c})$ be a basis of S_c, forming $C \in \mathbf{F}^{m,c}$. Then $\forall k \in \{1,\dots,n\}$,

$A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}, \exists\,!\, R_{1,k},\dots,R_{c,k} \in \mathbf{F}$, forming $R \in \mathbf{F}^{c,n}$. Thus $A = CR$.

(b) Let $(R_{1,\cdot},\dots,R_{r,\cdot})$ be a basis of S_r, forming $R \in \mathbf{F}^{r,n}$. Then $\forall j \in \{1,\dots,m\}$,

$A_{j,\cdot} = C_{j,1}R_{1,\cdot} + \cdots + C_{j,r}R_{r,\cdot} = (CR)_{j,\cdot}, \exists\,!\, C_{j,1},\dots,C_{j,r} \in \mathbf{F}$, forming $C \in \mathbf{F}^{m,r}$. Thus $A = CR$. □

Example:

$$A = \begin{pmatrix}10&7&4&1\\26&19&12&5\\46&33&20&7\end{pmatrix} \overset{(1)}{=} \begin{pmatrix}1&0\\0&1\\2&1\end{pmatrix}\begin{pmatrix}10&7&4&1\\26&19&12&5\end{pmatrix} \overset{(2)}{=} \begin{pmatrix}7&4\\19&12\\33&20\end{pmatrix}\begin{pmatrix}2&-1\\1&0\\0&1\\-1&2\end{pmatrix}$$

(I) $\begin{pmatrix}46&33&20&7\end{pmatrix} = 2\begin{pmatrix}10&7&4&1\end{pmatrix} + \begin{pmatrix}26&19&12&5\end{pmatrix}$.

$\begin{pmatrix}46&33&20&7\end{pmatrix}$ can be uniquely written as a linear combination of $(A_{1,\cdot}, A_{2,\cdot})$.

Hence $\dim S_r = 2$. $(A_{1,\cdot}, A_{2,\cdot})$ is a basis.

(II) $\begin{pmatrix}10\\26\\46\end{pmatrix} = 2\begin{pmatrix}7\\19\\33\end{pmatrix} - \begin{pmatrix}4\\12\\20\end{pmatrix}$; $\begin{pmatrix}1\\5\\7\end{pmatrix} = 2\begin{pmatrix}4\\12\\20\end{pmatrix} - \begin{pmatrix}7\\19\\33\end{pmatrix}$. Hence $\dim S_c = 2$. $(A_{\cdot,2}, A_{\cdot,3})$ is a basis.

• COLUMN RANK EQUALS ROW RANK (Using the notation and result above)

For each $A_{j,\cdot} \in S_r$, $A_{j,\cdot} = (CR)_{j,\cdot} = C_{j,1}R = C_{j,1}R_{1,\cdot} + \cdots + C_{j,c}R_{c,\cdot}$.

For each $A_{\cdot,k} \in S_c$, $A_{\cdot,k} = (CR)_{\cdot,k} = R_{1,k}C_{\cdot,1} + \cdots + R_{c,k}C_{\cdot,c} = (CR)_{\cdot,k}$.

$\Rightarrow \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = S_r = \operatorname{span}(R_{1,\cdot}, \dots, R_{c,\cdot}) \Rightarrow \dim S_r = r \leqslant c = \dim S_c$.

$\Rightarrow \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,m}) = S_r = \operatorname{span}(C_{\cdot,1}, \dots, C_{\cdot,r}) \Rightarrow \dim S_c = c \leqslant r = \dim S_r$.

OR. Apply the result to $A^t \in \mathbf{F}^{n,m} \Rightarrow \dim S_r^t = \dim S_c = c \leqslant r = \dim S_r = \dim S_c^t$. □

• [4E 3.C.17, OR 3.F.32] *Suppose $T \in \mathcal{L}(V)$ and $(u_1, \dots, u_n), (v_1, \dots, v_n)$ are bases of V. Prove that the following are equi.* *Here $A = \mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.*

(a) *T is inje.*

(b) *The cols of $\mathcal{M}(T)$ are linely inde in $\mathbf{F}^{n,1}$.*

(c) *The cols of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.*

(d) *The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.*

(e) *The rows of $\mathcal{M}(T)$ are linely inde in $\mathbf{F}^{1,n}$.*

SOLUTION: Using TIPS in 2.C.

T is inje $\iff \dim V = \dim \operatorname{range} T + \dim \operatorname{null} T = \dim \operatorname{range} T$

$$\Delta \begin{cases} \iff (Tu_1, \dots, Tu_n) \text{ is a basis of } V; \ \dim \operatorname{range} T = \dim \operatorname{span}(\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n)) = n \\ \iff (\mathcal{M}(Tu_1), \dots, \mathcal{M}(Tu_n)) \text{ is a basis of } \mathbf{F}^{n,1}, \text{ as well as } (A_{\cdot,1}, \dots, A_{\cdot,n}) \end{cases}$$

$\left[\text{又} \dim S_c = \dim \operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n}) = \dim \operatorname{span}(A_{1,\cdot}, \dots, A_{n,\cdot}) = \dim S_r = n \right]$

$\iff (A_{1,\cdot}, \dots, A_{n,\cdot})$ is a basis of $\mathbf{F}^{1,n}$. □

Now we show (Δ) properly, that is T is inje $\iff$ The cols of $\mathcal{M}(T)$ are linely inde.

$(a) \Rightarrow (b)$:

Suppose $b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n} = \begin{pmatrix} b_1 A_{1,1} + \cdots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \cdots + b_n A_{n,n} \end{pmatrix} = 0$. Let $u = b_1 u_1 + \cdots + b_n u_n$.

$$\begin{aligned} \text{Then } Tu &= b_1 Tu_1 + \cdots + b_n Tu_n \\ &= b_1(A_{1,1}v_1 + \cdots + A_{n,1}v_n) + \cdots + b_n(A_{1,n}v_1 + \cdots + A_{n,n}v_n) \\ &= (b_1 A_{1,1} + \cdots + b_n A_{1,n})v_1 + \cdots + (b_1 A_{n,1} + \cdots + b_n A_{n,n})v_n \\ &= 0v_1 + \cdots + 0v_n = 0 \\ &\Rightarrow b_1 = \cdots = b_n = 0. \end{aligned}$$

Thus by (2.39), (b) holds.

$(b) \Rightarrow (a)$:

Suppose $u = b_1 u_1 + \cdots + b_n u_n \in \operatorname{null} T$.

Then $Tu = 0 = (b_1 A_{1,1} + \cdots + b_n A_{1,n})v_1 + \cdots + (b_1 A_{n,1} + \cdots + b_n A_{n,n})v_n$.

Thus $b_1 A_{1,1} + \cdots + b_n A_{1,n} = \cdots = b_1 A_{n,1} + \cdots + b_n A_{n,n} = 0$.

Which is equi to $\begin{pmatrix} b_1 A_{1,1} + \cdots + b_n A_{1,n} \\ \vdots \\ b_1 A_{n,1} + \cdots + b_n A_{n,n} \end{pmatrix} = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n} = 0 \Rightarrow b_1 = \cdots = b_n = 0$.

Thus by (2.39), (a) holds. □

• [4E 3.C.16, OR 3.E.11] *Suppose A is an m-by-n matrix with $A \neq 0$. Prove that rank $A = 1 \iff \exists\,(c_1, \dots, c_m) \in \mathbf{F}^m, (d_1, \dots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j \cdot d_k$ for every $j = 1, \dots, m$ and $k = 1, \dots, n$.*

SOLUTION:

Using the notation in CR Factorization.

(a) Suppose $A = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & \ddots & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{pmatrix}$. $(\ \exists\, c_j, d_k \in \mathbf{F}, \forall j, k\ )$

Then $S_c = \left\{ \begin{pmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{pmatrix}, \begin{pmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{pmatrix}, \dots, \begin{pmatrix} c_1 d_n \\ \vdots \\ c_m d_n \end{pmatrix} \right\} = \operatorname{span} \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}$.

OR. $S_r = \operatorname{span} \left\{ \begin{matrix} \begin{pmatrix} c_1 d_1 & \cdots & c_1 d_n \end{pmatrix}, \\ \begin{pmatrix} c_2 d_1 & \cdots & c_2 d_n \end{pmatrix}, \\ \vdots \\ \begin{pmatrix} c_m d_1 & \cdots & c_m d_n \end{pmatrix} \end{matrix} \right\} = \operatorname{span} \left\{ \begin{pmatrix} d_1 & \dots & d_n \end{pmatrix} \right\}$. Hence rank $A = 1$.

OR. Using also the result in [4E 3.51(a)].

Every col of A is a scalar multi of C. Then rank $A \leqslant 1$ 又 rank $A \geqslant 1$ $(\ A \neq 0\ )$.

(b) By CR Factorization, $\exists\, C = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbf{F}^{m,1}, R = \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} \in \mathbf{F}^{1,n}$ such that $A = CR$.

OR. Not using CR Factorization. Suppose rank $A = \dim S_c = \dim S_r = 1$.

Let $c_j = \dfrac{A_{j,1}}{A_{1,1}} = \dfrac{A_{j,2}}{A_{1,2}} = \cdots = \dfrac{A_{j,n}}{A_{1,n}}, \qquad d'_k = \dfrac{A_{1,k}}{A_{1,1}} = \dfrac{A_{2,k}}{A_{2,1}} = \cdots = \dfrac{A_{m,k}}{A_{m,1}}$. $(\ \forall j, k\ )$

$\Rightarrow A_{j,k} = d'_k A_{j,1} = c_j A_{1,k} = c_j d'_k A_{1,1} = c_j d_k$. Letting $d_k = d'_k A_{1,1}$. ☐

---

1 *Suppose $T \in \mathcal{L}(V, W)$. Show that with resp to each choice of bases of V and W, the matrix of T has at least* dim range T *nonzero entries.*

SOLUTION:

Let $B_{\operatorname{null} T} = (v_1, \dots, v_p), B_V = (v_1, \dots, v_n)$. Let $B_W = (w_1, \dots, w_m)$. Denote $\mathcal{M}(T, B_V, B_W)$ by A.

Because at most p of the v_k's can belong to null $T \iff$ at least $n - p = q$ of the v_k's do not.

For $v_k \notin \operatorname{null} T, Tv_k = A_{1,k} w_1 + \cdots + A_{m,k} w_m \neq 0$. Thus col k has at least one nonzero entry.

Since there are $n - p = q$ choices of such k, A has at least $q = \dim \operatorname{range} T$ nonzero entries. ☐

OR. We prove by contradiction.

Suppose A has at most $(\ \dim \operatorname{range} T - 1)$ nonzero entries.

Then by Pigeon Hole Principle, at least one of $A_{\cdot,p+1}, \cdots, A_{\cdot,n}$ equals 0.

Thus there are at most $(\ \dim \operatorname{range} T - 1)$ nonzero vecs in $Tv_{p+1}, \dots, Tv_n$.

While range $T = \operatorname{span}(Tv_{p+1}, \dots, Tv_n) \Rightarrow \dim \operatorname{range} T = \dim \operatorname{span}(Tv_{p+1}, \dots, Tv_n)$. Contradicts. ☐

---

3 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$. Prove that $\exists\, B_V, B_W$ such that* $[$ *letting* $A = \mathcal{M}(T, B_V, B_W)$ $]$ $A_{k,k} = 1, A_{i,j} = 0$, *where* $1 \leqslant k \leqslant \dim \operatorname{range} T, i \neq j$.

SOLUTION:

Let $R = (Tv_1, \dots, Tv_n)$ be a basis of range T, extend to $B_W = (Tv_1, \dots, Tv_n, w_1, \dots, w_p)$.
Let $\mathcal{K}_R = \operatorname{span}(v_1, \dots, v_n)$. Let $(u_1, \dots, u_m)$ be a basis of null T. Then $B_V = (v_1, \dots, v_n, u_1, \dots, u_m)$. □

---

4 *Suppose $B_V = (v_1, \dots, v_m)$ and W is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that $\exists\, B_W = (w_1, \dots, w_n)$, $\mathcal{M}(T, B_V, B_W)^t_{\cdot,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$.

SOLUTION: If $Tv_1 = 0$, then we are done. If not then extend (Tv_1). □

---

5 *Suppose $B_W = (w_1, \dots, w_n)$ and V is finite-dim. Suppose $T \in \mathcal{L}(V,W)$.*
Prove that $\exists\, B_V = (v_1, \dots, v_m)$, $\mathcal{M}(T, B_V, B_W)_{1,\cdot} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$.

SOLUTION:

Let $(u_1, \dots, u_n)$ be a basis of V. Denote $\mathcal{M}(T, (u_1, \dots, u_n), B_W)$ by A.
If $A_{1,\cdot} = 0$, then let $B_V = (u_1, \dots, u_n)$, we are done.
Otherwise, $\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \end{pmatrix} \neq 0$, choose one $A_{1,k} \neq 0$.

$$\text{Let } v_1 = \frac{u_k}{A_{1,k}}; \quad \begin{aligned} v_j &= u_{j-1} - A_{1,j-1} v_1 \quad && \text{for } j = 2, \dots, k; \\ v_i &= u_i - A_{1,i} v_1 && \text{for } i = k+1, \dots, n. \end{aligned}$$

Now because each $u_k \in \operatorname{span}(v_1, \dots, v_n) \Rightarrow V = \operatorname{span}(v_1, \dots, v_n), B_V = (v_1, \dots, v_n)$.

And $Tv_1 = T(\dfrac{u_k}{A_{1,k}}) = \dfrac{1}{A_{1,k}}(A_{1,k} w_1 + \cdots + A_{n,k} w_n) = 1 w_1 + \cdots + \dfrac{A_{n,k}}{A_{1,k}} w_n$.

$$\forall j \in \{2, \dots, k, \underbrace{k+2, \dots, n+1}_{i \in \{k+1, \dots, n\}}\},\ Tv_j = T(u_{j-1} - A_{1,j-1} v_1) = Tu_{j-1} - T(\frac{A_{1,j-1} u_k}{A_{1,k}})$$

$$= A_{1,j-1} w_1 + \cdots + A_{n,j-1} w_n - A_{1,j-1}(1 w_1 + \cdots + \frac{A_{n,k}}{A_{1,k}} w_n) = 0 w_1 + \cdots + (A_{n,j-1} - \frac{A_{1,j-1} A_{n,k}}{A_{1,k}}) w_n.$$ □

---

6 *Suppose V and W are finite-dim and $T \in \mathcal{L}(V,W)$.*
Prove that $\dim \operatorname{range} T = 1 \iff \exists\, B_V, B_W$, *all entries of* $A = \mathcal{M}(T, B_V, B_W)$ *equal* 1.

SOLUTION:

(a) Suppose $B_V = (v_1, \dots, v_n), B_W = (w_1, \dots, w_m)$ are the bases such that all entries of A equal 1.

Then $Tv_i = w_1 + \cdots + w_m$ for all $i = 1, \dots, n$. Because $w_1, \dots, w_n$ is linely inde, $w_1 + \cdots + w_n \neq 0$.

(b) Suppose $\dim \operatorname{range} T = 1$. Then $\dim \operatorname{null} T = \dim V - 1$.
Let $(u_2, \dots, u_n)$ be a basis of null T. Extend it to a basis of V as $(u_1, u_2, \dots, u_n)$.
Let $w_1 = Tv_1 - w_2 - \cdots - w_m$. Extend to a basis of W and we have B_W.
Let $v_1 = u_1, v_i = u_1 + u_i$. Extend to a basis of V and we have B_V. □

OR. Suppose range T has a basis (w).
By (2.C.15 [COROLLARY]), $\exists\, B_W = (w_1, \dots, w_m)$ such that $w = w_1 + \cdots + w_m$.
By (2.C [NEW THEOREM]), $\exists$ a basis $(u_1, \dots, u_n)$ of V such that each $u_k \notin$ null T.
$\forall k \in \{1, \dots, n\}, Tu_k \in \operatorname{range} T = \operatorname{span}(w) \Rightarrow Tu_k = \lambda_k w, \exists\, \lambda_k \in \mathbf{F} \backslash \{0\}$.
Let $v_k = \lambda_k^{-1} u_k \neq 0 \Rightarrow B_V = (v_1, \dots, v_n)$. Hence for each $v_k, Tv_k = w = w_1 + \cdots + w_m$. □

• **NOTE FOR [3.49]:** $\because [(AC)_{\cdot,k}]_{j,1} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}(C_{\cdot,k})_{r,1} = (AC_{\cdot,k})_{j,1}$

$\therefore (AC)_{\cdot,k} = A_{\cdot,\cdot}C_{\cdot,k} = AC_{\cdot,k}$ □

• **EXERCISE 10:** $\because [(AC)_{j,\cdot}]_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}C_{r,k} = (A_{j,\cdot}C)_{1,k}$

$\therefore (AC)_{j,\cdot} = A_{j,\cdot}C_{\cdot,\cdot} = A_{j,\cdot}C.$ □

• **NOTE FOR [3.52]:** $A \in \mathbf{F}^{m,n}, c \in \mathbf{F}^{n,1} \Rightarrow Ac \in \mathbf{F}^{m,1}$

$\because (Ac)_{j,1} = \sum_{r=1}^{n} A_{j,r}c_{r,1} = \left(\sum_{r=1}^{n} (A_{\cdot,r}c_{r,1})\right)_{j,1} = (c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n})_{j,1}$

$\therefore Ac = A_{\cdot,\cdot}c_{\cdot,1} = \sum_{r=1}^{n} A_{\cdot,r}c_{r,1} = c_1A_{\cdot,1} + \cdots + c_nA_{\cdot,n}$ OR. By $(Ac)_{\cdot,1} = Ac_{\cdot,1}$ Using (a) above. □

• **EXERCISE 11:** $a \in \mathbf{F}^{1,n}, C \in \mathbf{F}^{n,p} \Rightarrow aC \in \mathbf{F}^{1,p}$

$\because (aC)_{1,k} = \sum_{r=1}^{n} a_{1,r}C_{r,k} = \left(\sum_{r=1}^{n} a_{1,r}(C_{r,\cdot})\right)_{1,k} = (a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot})_{1,k}$

$\therefore aC = a_{1,\cdot}C_{\cdot,\cdot} = \sum_{r=1}^{n} a_{1,r}C_{r,\cdot} = a_1C_{1,\cdot} + \cdots + a_nC_{n,\cdot}$ OR. By $(aC)_{1,\cdot} = a_{1,\cdot}C$. Using (b) above. □

• *Suppose p is a poly of n variables in $\mathbf{F}$. Prove that $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big)$.*

Where the linear maps $T_1,\dots,T_n$ are such that $p(T_1,\dots,T_n)$ makes sense. See [5.B.16,17,20].

SOLUTION:

Suppose the poly p is defined by $p(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^{n} x_i^{k_i}$.

Note that $\mathcal{M}(T^xS^y) = \mathcal{M}(T)^x\mathcal{M}(S)^y$; $\mathcal{M}(T^x + S^y) = \mathcal{M}(T)^x + \mathcal{M}(S)^y$.

Then $\mathcal{M}\big(p(T_1,\dots,T_n)\big) = \mathcal{M}\Big(\sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^{n} T_i^{k_i}\Big)$

$= \sum_{k_1,\dots,k_n}\alpha_{k_1,\dots,k_n}\prod_{i=1}^{n}\mathcal{M}(T_i^{k_i}) = p\big(\mathcal{M}(T_1),\dots,\mathcal{M}(T_n)\big).$ □

---

13 *Prove that the distr holds for matrix add and matrix multi.*

Suppose A, B, C are matrices such that $A(B+C)$ make sense, we prove the left distr.

SOLUTION:

Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$.

Note that $[A(B+C)]_{j,k} = \sum_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum_{r=1}^{n} (A_{j,r}B_{r,k} + A_{j,r}C_{r,k}) = (AB+AC)_{j,k}$. □

OR. Define T, S, R such that $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$A(B+C) = \mathcal{M}(T(S+R)) \overset{[3.9]}{=} \mathcal{M}(TS+TR) = AB + AC.$

Or $T(S+R) = TS + TR \Rightarrow \mathcal{M}(T(S+R)) = \mathcal{M}(TS+TR) \Rightarrow A(B+C) = AB + AC.$ □

---

14 *Prove that matrix multi is associ.*

Suppose A, B, C are matrices such that $(AB)C$ makes sense, we prove that $(AB)C = A(BC)$.

SOLUTION:

Suppose $A \in \mathbf{F}^{m,n}$ and $B, C \in \mathbf{F}^{n,p}$. We will show that $LHS = [(AB)C]_{j,k} = [A(BC)]_{j,k} = RHS$.

$LHS = (AB)_{j,\cdot}C_{\cdot,k} = \sum_{s=1}^{n} (A_{j,s}B_{s,\cdot})C_{\cdot,k} = \sum_{s=1}^{n} A_{j,s}(B_{s,\cdot}C_{\cdot,k}) = \sum_{s=1}^{n} A_{j,s}(BC)_{s,k} = RHS.$ □

OR. Define T, S, R such that $\mathcal{M}(T) = A, \mathcal{M}(S) = B, \mathcal{M}(R) = C$.

$(AB)C = \mathcal{M}(T(SR)) \overset{[3.9]}{=} \mathcal{M}(TSR) \overset{[3.9]}{=} \mathcal{M}((TS)R) = A(BC).$

OR. $(TS)R = T(SR) \Rightarrow \mathcal{M}((TS)R) = \mathcal{M}(T(SR)) \Rightarrow (AB)C = A(BC).$ □

---

15 *Suppose $A \in \mathbf{F}^{n,n}, j,k \in \{1,\dots,n\}$. Show that $(A^3)_{j,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.*

SOLUTION: $(AAA)_{j,k} = (AA)_{j,\cdot}A_{\cdot,k} = \sum_{p=1}^{n}(A_{j,p}A_{p,\cdot})A_{\cdot,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$.

OR. $(AAA)_{j,k} = \sum_{r=1}^{n}(AA)_{j,r}A_{r,k} = \sum_{r=1}^{n}\big(\sum_{p=1}^{n} A_{j,p}A_{p,r}\big)A_{r,k}$

$= \sum_{r=1}^{n}\left[A_{j,1}(A_{1,r}A_{r,k}) + \cdots + A_{j,n}(A_{n,r}A_{r,k})\right]$

$= A_{j,1}\sum_{r=1}^{n}A_{1,r}A_{r,k} + \cdots + A_{j,n}\sum_{r=1}^{n}A_{n,r}A_{r,k} = \sum_{p=1}^{n}\sum_{r=1}^{n}A_{j,p}A_{p,r}A_{r,k}$. □

• *Prove that the commutativity does not hold in $\mathbf{F}^{m,n}$.*

SOLUTION:

Suppose $\dim V = n, \dim W = m$ and the commutativity holds in $\mathbf{F}^{n,m}$.

$\forall T \in \mathcal{L}(V,W), S \in \mathcal{L}(W,V), \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST)$.

Hence $ST = TS$. Which in general is not true. (See 3.D) □

• [10.A.3, OR 4E 3.D.19] *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that $\forall B_V \neq B'_V,\ \mathcal{M}(T,B_V) = \mathcal{M}(T,B'_V) \iff T = \lambda\mathcal{M}(I), \exists\lambda \in \mathbf{F}$.

SOLUTION: [Compare with the first solution of (3.D.16) in 3.A]

Suppose $T = \lambda I$ for some $\lambda \in \mathbf{F}$. Then $T = \lambda\mathcal{M}(I)$.

Suppose $\forall B_V \neq B'_V,\ \mathcal{M}(T,B_V) = \mathcal{M}(T,B'_V)$. If $T = 0$, then we are done.

Suppose $T \neq 0$, and $v \in V\backslash\{0\}$. Assume that (v, Tv) is linely inde.

Extend (v,Tv) to $B_V = (v, Tv, u_3, \dots, u_n)$. Let $B = \mathcal{M}()(T, B_V)$.

$\Rightarrow Tv = B_{1,1}v + B_{2,1}(Tv) + B_{3,1}u_3 + \cdots + B_{n,1}u_n \Rightarrow B_{2,1} = 1, B_{i,1} = 0,\ \forall i \neq 2$.

By assumption, $A = \mathcal{M}(T,B'_V) = B, \forall B'_V = (v, w_2, \dots, w_n)$. Then $A_{2,1} = 1, A_{i,1} = 0, \forall i \neq 2$.

$\Rightarrow Tv = w_2$, which is not true if we let $w_2 = u_3,\ w_3 = Tv,\ w_j = u_j, \forall j \in \{4,\dots,n\}$. Contradicts.

Hence (v,Tv) is linely depe $\Rightarrow \forall v \in V, \exists\lambda_v \in \mathbf{F}, Tv = \lambda_v v$.

Now we show that λ_v is independent of v, that is, to show that for all $v \neq w \in V\backslash 0, \lambda_v = \lambda_w$.

$$\left.\begin{array}{l}(v,w) \text{ is linely inde} \Rightarrow T(v+w) = \lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w = Tv + Tw \\ (v,w) \text{ is linely depe, } w = cv \Rightarrow Tw = \lambda_w w = \lambda_w cv = c\lambda_v v = T(cv)\end{array}\right\} \Rightarrow T = \lambda I, \exists\lambda \in \mathbf{F}.$$ □

OR. Conversely, denote $\mathcal{M}(T,B_V)$ by A, where $B_V = (u_1,\dots,u_m)$ is arbitrary.

Fix one $B_V = (v_1,\dots,v_m)$ and then $(v_1,\dots,\frac{1}{2}v_k,\dots,v_m)$ is also a basis for any given $k \in \{1,\dots,m\}$.

Fix one k. Now we have $T(\frac{1}{2}v_k) = A_{1,k}v_1 + \cdots + A_{k,k}(\frac{1}{2}v_k) + \cdots + A_{m,k}v_m$

$\Rightarrow Tv_k = 2A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + 2A_{m,k}v_m = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{m,k}v_m$.

Then $A_{j,k} = 2A_{j,k} \Rightarrow A_{j,k} = 0$ for all $j \neq k$. Thus $Tv_k = A_{k,k}v_k, \forall k \in \{1,\dots,m\}$.

Now we show that $A_{k,k} = A_{j,j}$ for all $j \neq k$. Choose j,k such that $j \neq k$.

Consider the basis $B'_V = (v'_1,\dots,v'_j,\dots,v'_k,\dots,v'_m)$,

where $v'_j = v_k,\ v_k' = v_j$ and $v'_i = v_i$ for all $i \in \{1,\dots,m\} \setminus \{j,k\}$.

Remember that $\mathcal{M}(T,B'_V) = \mathcal{M}(T,B_V) = A$.

Hence $T(v'_k) = A_{1,k}v'_1 + \cdots + A_{k,k}v'_k + \cdots + A_{m,k}v'_m = A_{k,k}v'_k = A_{k,k}v_j$, while $T(v'_k) = T(v_j) = A_{j,j}v_j$.

Thus $A_{k,k} = A_{j,j}$. □

ENDED

3·D

1 2 3 4 5 6 8 9 10 11 12 13 15 16 17 18 19 | 4E: 1, 3, 10, 15, 17, 19, 20, 22, 23, 24

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*

$$\left.\begin{array}{l}(Tv_1,\dots,Tv_n) \text{ is a basis of } V \text{ for some basis } (v_1,\dots,v_n) \text{ of } V \iff T \text{ is surj} \\ (Tv_1,\dots,Tv_n) \text{ is a basis of } V \text{ for every basis } (v_1,\dots,v_n) \text{ of } V \iff T \text{ is inje}\end{array}\right\} \iff T \text{ is inv.}$$

• *Suppose $T \in \mathcal{L}(V)$ and $V = \operatorname{span}(Tv_1,\dots,Tv_m)$. Prove that $V = \operatorname{span}(v_1,\dots,v_m)$.*

Solution:

Because $V = \operatorname{span}(Tv_1,\dots,Tv_m) \Rightarrow T$ is surj, 又 V is finite-dim $\Rightarrow T$ is inv $\Rightarrow T^{-1}$ is inv.
$\forall v \in V, \exists a_i \in \mathbf{F}, v = a_1Tv_1+\cdots+a_mTv_m \Rightarrow T^{-1}v = a_1v_1+\cdots+a_mv_m \Rightarrow \operatorname{range} T^{-1} \subseteq \operatorname{span}(v_1,\dots,v_m)$.

Or. Reduce $(Tv_1,\dots,Tv_m)$ to a basis of V as $(Tv_{\alpha_1},\dots,Tv_{\alpha_k})$, where $k = \dim V$ and $\alpha_i \in \{1,\dots,k\}$.
Then $(v_{\alpha_1},\dots,v_{\alpha_k})$ is linely inde of length k, hence is a basis of V, contained in the list $(v_1,\dots,v_m)$. □

• Or (10.A.1) *Suppose $T \in \mathcal{L}(V), B_V = (v_1,\dots,v_n)$. Prove that $\mathcal{M}(T,B_V)$ is inv $\iff T$ is inv.*

Solution: Notice that $\mathcal{M} \in \mathcal{L}(\mathcal{L}(V),\mathbf{F}^{n,n})$ is an iso.

(a) $T^{-1}T = TT^{-1} = I \Rightarrow \mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(T^{-1}) = I \Rightarrow \mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$.

(b) $\mathcal{M}(T)\mathcal{M}(T)^{-1} = \mathcal{M}(T)^{-1}\mathcal{M}(T) = I.\quad \exists!\, S \in \mathcal{L}(V)$ such that $\mathcal{M}(T)^{-1} = \mathcal{M}(S)$
$\Rightarrow \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I = \mathcal{M}(TS) = \mathcal{M}(ST)$
$\Rightarrow \mathcal{M}^{-1}\mathcal{M}(TS) = \mathcal{M}^{-1}\mathcal{M}(ST) = I = TS = ST \Rightarrow S = T^{-1}$. □

• *Suppose $T \in \mathcal{L}(V,W)$ is inv. Show that T^{-1} is inv and $(T^{-1})^{-1} = T$.*

Solution: $\left.\begin{array}{l}TT^{-1} = (T^{-1})^{-1}T^{-1} = I \in \mathcal{L}(V) \\ T^{-1}T = T^{-1}(T^{-1})^{-1} = I \in \mathcal{L}(W)\end{array}\right\} \Rightarrow T = (T^{-1})^{-1}$, by the uniqnes of inverse. □

---

1 *Suppose $T \in \mathcal{L}(U,V), S \in \mathcal{L}(V,W)$ are inv. Prove that ST is inv and $(ST)^{-1} = T^{-1}S^{-1}$.*

Solution: $\left.\begin{array}{l}(ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = I \in \mathcal{L}(W) \\ (T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = I \in \mathcal{L}(V)\end{array}\right\} \Rightarrow (ST)^{-1} = T^{-1}S^{-1}$, by the uniqnes of inv. □

---

2 *Suppose V is finite-dim and* $\dim V > 1$.
Prove that the set of non-inv operators on V is not a subsp of $\mathcal{L}(V)$.
The set of inv operators is not either, although multi identity/inv, and commutativity for vec multi holds.

Solution:

Denote the set by U. Suppose $\dim V = n > 1$. Let $(v_1,\dots,v_n)$ be a basis of V. Define $S,T \in \mathcal{L}(V)$ by
$S(a_1v_1+\cdots+a_nv_n) = a_1v_1, T(a_1v_1+\cdots+a_nv_n) = a_2v_1+\cdots+a_nv_n$. Hence $S+T = I$ is inv. □

Comment: If $\dim V = 1$, then $U = \{0\}$ is a subsp of $\mathcal{L}(V)$.

---

3 *Suppose V is finite-dim, U is a subsp of V, and $S \in \mathcal{L}(U,V)$.*
Prove that $\exists$ inv $T \in \mathcal{L}(V), Tu = Su, \forall u \in U \iff S$ is inje.[*Compare this with (3.A.11).*]

Solution:

(a) $Tu = Su$ for every $u \in U \Rightarrow u = T^{-1}Su \Rightarrow S$ is inje. Or. $\operatorname{null} S = \operatorname{null} T \cap U = \{0\} \cap U = \{0\}$.

(b) Suppose $(u_1,\dots,u_m)$ be a basis of U and S is inje $\Rightarrow (Su_1,\dots,Su_m)$ is linely inde in V.
Extend these to bases of V as $(u_1,\dots,u_m,v_1,\dots,v_n)$ and $(Su_1,\dots,Su_m,w_1,\dots,w_n)$.
Define $T \in \mathcal{L}(V)$ by $T(u_i) = Su_i;\ \ Tv_j = w_j$, for each $i \in \{1,\dots,m\}, j \in \{1,\dots,n\}$. □

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4 *Suppose that W is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{null} S = \operatorname{null} T \,(= U) \iff S = ET,\ \exists$ *inv* $E \in \mathcal{L}(W)$.

SOLUTION:

Define $E \in \mathcal{L}(W)$ by $E(Tv_i) = Sv_i,\ \ E(w_j) = x_j$, for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

$$\left|\begin{array}{l} \text{Let } B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m), \text{ extend to } B_W = (Tv_1, \dots, Tv_m, w_1, \dots, w_n). \\ \text{Let } \mathcal{K} = \operatorname{span}(v_1, \dots, v_m). \text{ 又 } \operatorname{null} S = \operatorname{null} T \Longrightarrow V = \mathcal{K} \oplus \operatorname{null} S \Leftrightarrow \mathcal{K} \in \mathcal{S}_V \operatorname{null} S. \\ \Rightarrow \operatorname{span}(Sv_1, \dots, Sv_m) = \operatorname{range} S \text{ 又 } \dim \operatorname{range} T = \dim \operatorname{range} S = m. \\ \text{Hence } B_{\operatorname{range} S} = (Sv_1, \dots, Sv_m). \text{ Thus we let } B'_W = (Sv_1, \dots, Sv_m, x_1, \dots, x_n). \end{array}\right| \begin{array}{l} \therefore E \text{ is inv} \\ \text{and } S = ET. \end{array}$$

Conversely, $S = ET \Rightarrow \operatorname{null} S = \operatorname{null} ET$.

Then $v \in \operatorname{null} ET \iff ET(v) = 0 \iff Tv = 0 \iff v \in \operatorname{null} T$. Hence $\operatorname{null} ET = \operatorname{null} T = \operatorname{null} S$. ☐

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5 *Suppose that V is finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $\operatorname{range} S = \operatorname{range} T \,(= R) \iff S = TE,\ \exists$ *inv* $E \in \mathcal{L}(V)$.

SOLUTION:

Define $E \in \mathcal{L}(V)$ as $E:\ v_i \mapsto r_i\ ;\quad u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

$$\left|\begin{array}{l} \text{Let } B_R = (Tv_1, \dots, Tv_m);\ B'_R = (Sr_1, \dots, Sr_m) \text{ such that } \forall\, i, Tv_i = Sr_i. \\ \text{Let } B_{\operatorname{null} T} = (u_1, \dots, u_n);\ B_{\operatorname{null} S} = (s_1, \dots, s_n). \\ \text{Thus } B_V = (v_1, \dots, v_m, u_1, \dots, u_n);\ B'_V = (r_1, \dots, r_m, s_1, \dots, s_n). \end{array}\right| \therefore E \text{ is inv and } S = TE.$$

Conversely, $S = TE \Rightarrow \operatorname{range} S = \operatorname{range} TE$.

Then $w \in \operatorname{range} S \iff \exists\, v \in V, Sv = TE(v) = T\big(E(v)\big) = w \in \operatorname{range} T$. Hence $\operatorname{range} S = \operatorname{range} T$. ☐

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6 *Suppose V and W are finite-dim and $S, T \in \mathcal{L}(V, W)$.*
Prove that $S = E_2TE_1, \exists$ inv $E_1 \in \mathcal{L}(V), E_2 \in \mathcal{L}(W)$ $\iff \dim \operatorname{null} S = \dim \operatorname{null} T = n$.

SOLUTION:

Define $E_1:\ \ v_i \mapsto r_i\ ;\quad u_j \mapsto s_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$.

Define $E_2: Tv_i \mapsto Sr_i\ ;\ \ x_j \mapsto y_j;$ for each $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. Where:

$$\left|\begin{array}{l} \text{Let } B_{\operatorname{range} T} = (Tv_1, \dots, Tv_m);\ B_{\operatorname{range} S} = (Sr_1, \dots, Sr_m). \\ \text{Extend to } B_W = (Tv_1, \dots, Tv_m, x_1, \dots, x_p);\ B'_W = (Sr_1, \dots, Sr_m, y_1, \dots, y_p). \\ \text{Let } B_{\operatorname{null} T} = (u_1, \dots, u_n);\ B_{\operatorname{null} S} = (s_1, \dots, s_n). \\ \text{Thus } B_V = (v_1, \dots, v_m, u_1, \dots, u_n);\ B'_V = (r_1, \dots, r_m, s_1, \dots, s_n). \end{array}\right| \begin{array}{l} \therefore E_1, E_2 \text{ are inv} \\ \text{and } S = E_2TE_1. \end{array}$$

Conversely, $S = E_2TE_1 \Rightarrow \dim \operatorname{null} S = \dim \operatorname{null} E_2TE_1$.

$v \in \operatorname{null} E_2TE_1 \iff E_2TE_1(v) = 0 \iff TE_1(v) = 0$. Hence $\operatorname{null} E_2TE_1 = \operatorname{null} TE_1 = \operatorname{null} S$.

又 By (3.B.22.COROLLARY), E is inv $\Rightarrow \dim \operatorname{null} TE_1 = \dim \operatorname{null} T = \dim \operatorname{null} S$. ☐

---

8 *Suppose V is finite-dim and $T: V \to W$ is a **surj** linear map of V onto W.*
Prove that there is a subsp U of V such that $T|_U$ is an iso of U onto W.

SOLUTION:

Let $B_{\operatorname{range} T} = B_W = (w_1, \dots, w_m) \Rightarrow \forall w_i, \exists\,!\, v_i \in V, Tv_i = w_i$. Let $B_{\mathcal{K}} = (v_1, \dots, v_m)$.

Then $\dim \mathcal{K} = \dim W$. Thus $T|_{\mathcal{K}}$ is an iso of $\mathcal{K}$ onto W. ☐

OR. By (3.B.12), there is a subsp U of V such that

$U \cap \operatorname{null} T = \{0\} = \operatorname{null} T|_U, \operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U$. ☐

---

9 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that ST is inv $\iff$ S and T are inv.*

SOLUTION:

Suppose S, T are inv. Then $(ST)(T^{-1}S^{-1}) = (T^{-1}S^{-1})(ST) = I$. Hence ST is inv.

Suppose ST is inv. Let $R = (ST)^{-1} \Rightarrow R(ST) = (ST)R = I$.

$$\left.\begin{array}{l} Tv = 0 \Rightarrow v = R(ST)v = RS(Tv) = 0 \\ \forall v \in V, v = (ST)Rv = S(TRv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T \text{ is inje, } S \text{ is surj. While } V \text{ is finite-dim.}$$ □

OR. Because by (3.B.23), $\dim V = \dim \operatorname{range} ST \leqslant \min\{\operatorname{range} T, \operatorname{range} S\}$. □

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10 *Suppose V is finite-dim and $S, T \in \mathcal{L}(V)$. Prove that $ST = I \iff TS = I$.*

SOLUTION:

Suppose $ST = I$. $\left.\begin{array}{l} Tv = 0 \Rightarrow v = STv = 0 \\ v \in V \Rightarrow v = S(Tv) \in \operatorname{range} S \end{array}\right\} \Rightarrow T$ is inje, S is surj. While V is finite-dim.

OR. By Problem (9), V is finite-dim and $ST = I$ is inv $\Rightarrow S, T$ are inv.

$S\big((TS)v\big) = ST(Sv) = Sv \Rightarrow (TS)v = v \Rightarrow S$ is inv.

OR. $ST = I \Rightarrow S = T^{-1} \Rightarrow S^{-1} = T$. 又 $S = S \Rightarrow TS = S^{-1}S = I$.

Reversing the roles of S and T, we conclude that $TS = I \Rightarrow ST = I$. □

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11 *Suppose V is finite-dim, $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is inv and $T^{-1} = US$.*

SOLUTION: Using Problem (9) and (10). *This result can fail without the hypothesis that V is finite-dim.*

$(ST)U = U(ST) = (US)T = T(US) = S(TU) = (TU)S = I$.

$\Rightarrow U^{-1} = ST, \qquad T^{-1} = US, \qquad S^{-1} = TU.$ □

EXAMPLE: $V = \mathbf{R}^\infty, S(a_1, a_2, \dots) = (a_2, \dots);\ T(a_1, \dots) = (0, a_1, \dots);\ U = I \Rightarrow STU = I$ but T is not inv.

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13 *Suppose V is finite-dim, $R, S, T \in \mathcal{L}(V)$ are such that RST is surj. Prove that S is inje.*

SOLUTION: By Problem (1) and (9), Notice that V is finite-dim. Then RST is inv.

$$\text{Let } X = (RST)^{-1} \left| \left.\begin{array}{l} Tv = 0 \Rightarrow v = X(RSTv) = 0 \Rightarrow T \text{ is inje.} \\ \forall v \in V, v = (RST)Xv \in \operatorname{range} R \Rightarrow R \text{ is surj.} \end{array}\right\} \right. \Rightarrow S = R^{-1}(RST)T^{-1} \text{ is inv.}$$ □

OR. $(RST)^{-1} = \big((RS)T\big)^{-1} = T^{-1}(RS)^{-1} = T^{-1}S^{-1}R^{-1}$. □

---

15 *Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multi.*
In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then $\exists\, A \in \mathbf{F}^{m,n}, Tx = Ax,\ \forall x \in \mathbf{F}^{n,1}$.

SOLUTION:

Let $B_1 = (E_1, \dots, E_n), B_2 = (R_1, \dots, R_m)$ be the standard bases of $\mathbf{F}^{n,1}, \mathbf{F}^{m,1}$.

$\forall k = 1, \dots, n$, suppose $T(E_k) = A_{1,k}R_1 + \cdots + A_{m,k}R_m, \exists\, A_{j,k} \in \mathbf{F}$, forming $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$. □

OR. Let $A = \mathcal{M}(T, B_1, B_2)$. Note that $\mathcal{M}(x, B_1) = x, \mathcal{M}(y, B_2) = y$.

Hence $Tx = \mathcal{M}(Tx, B_2) = \mathcal{M}(T, B_1, B_2)\mathcal{M}(x, B_1) = Ax$, by [3.65]. □

• OR (10.A.2) *Suppose $A, B \in \mathbf{F}^{n,n}$. Prove that $AB = I \iff BA = I$.*

SOLUTION: Using Problem (10) and (15).

Define $T, S \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{n,1})$ by $Tx = Ax, Sx = Bx$ for all $x \in \mathbf{F}^{n,1}$. Then $\mathcal{M}(T) = A, \mathcal{M}(S) = B$.

Thus $AB = I \Leftrightarrow A(Bx) = x \iff T(Sx) = x \Leftrightarrow TS = I \iff ST = I \iff \mathcal{M}(S)\mathcal{M}(T) = BA = I$. □

• **NOTE FOR [3.60]:** Suppose $B_V = (v_1, \dots, v_n)$, $B_W = (w_1, \dots, w_m)$.

Define $E_{i,j} \in \mathcal{L}(V, W)$ by $E_{i,j}(v_x) = \delta_{i,x} w_j$; See (3.A.12). **COROLLARY:** $E_{l,k} E_{i,j} = \delta_{j,l} E_{i,k}$.

Denote $\mathcal{M}(E_{i,j})$ by $\mathcal{E}^{(j,i)}$. And $\left(\mathcal{E}^{(j,i)}\right)_{l,k} = \begin{cases} 0, & i \neq k \vee j \neq l \\ 1, & i = k \wedge j = l \end{cases}$

Because $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$ are iso. And $T = \mathcal{M}^{-1}\mathcal{M}(T)$; $E_{i,j} = \mathcal{M}^{-1}\mathcal{E}^{(j,i)}$.

Hence $\forall T \in \mathcal{L}(V, W), \exists ! A_{i,j} \in \mathbf{F} \left(\forall i \in \{1, \dots, m\}, j \in \{1, \dots, n\} \right), \mathcal{M}(T) = A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$.

$$\text{Thus } A = \begin{pmatrix} A_{1,1}\mathcal{E}^{(1,1)}+ & \cdots & +A_{1,n}\mathcal{E}^{(1,n)} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}\mathcal{E}^{(m,1)}+ & \cdots & +A_{m,n}\mathcal{E}^{(m,n)} \end{pmatrix} \iff \begin{pmatrix} A_{1,1}E_{1,1}+ & \cdots & +A_{1,n}E_{n,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}E_{1,m}+ & \cdots & +A_{m,n}E_{n,m} \end{pmatrix} = T.$$

$$\therefore \mathcal{L}(V, W) = \operatorname{span} \underbrace{\begin{pmatrix} E_{1,1}, & \cdots & , E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,m}, & \cdots & , E_{n,m} \end{pmatrix}}_{B}; \quad \mathbf{F}^{m,n} = \operatorname{span} \underbrace{\begin{pmatrix} \mathcal{E}^{(1,1)}, & \cdots & , \mathcal{E}^{(1,n)}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ \mathcal{E}^{(m,1)}, & \cdots & , \mathcal{E}^{(m,n)} \end{pmatrix}}_{B_{\mathcal{M}}}.$$

Hence by [2.42] and [3.61], we conclude that B is a basis of $\mathcal{L}(V, W)$ and that $B_{\mathcal{M}}$ is a basis of $\mathbf{F}^{m,n}$.

• *Suppose V, W are finite-dim, U is a subsp of V.*
Let $\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V, W) : T|_U = 0\}$.
(a) *Show that $\mathcal{E}$ is a subsp of $\mathcal{L}(V, W)$.*
(b) *Find a formula for* $\dim \mathcal{E}$ *in terms of* $\dim V$, $\dim W$ *and* $\dim U$.
Hint: *Define $\Phi : \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. What is $\operatorname{null} \Phi$? What is $\operatorname{range} \Phi$?*

SOLUTION:

(a) $\forall S, T \in \mathcal{E}, \lambda \in \mathbf{F}, \forall u \in U, Su = \lambda Tu = (S + \lambda T)u = 0 \Rightarrow (S + \lambda T) \in \mathcal{E}$.

(b) Define Φ as in the hint.

Because $T \in \operatorname{null} \Phi \iff \Phi(T) = 0 \iff \forall u \in U, Tu = 0 \iff T \in \mathcal{E}$.

Hence $\operatorname{null} \Phi = \mathcal{E}$.

Because $S \in \mathcal{L}(U, W) \Rightarrow \exists T \in \mathcal{L}(V, W), \Phi(T) = S$, by (3.A.11) $\Rightarrow S \in \operatorname{range} T$.

Hence $\operatorname{range} \Phi = \mathcal{L}(U, W)$.

Thus $\dim \operatorname{null} \Phi = \dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim \operatorname{range} \Phi = (\dim V - \dim U) \dim W$. ☐

OR. Extend $(u_1, \dots, u_m)$ a basis of U to $(u_1, \dots, u_m, v_1, \dots, v_n)$ a basis of V. Let $p = \dim W$.
(See NOTE FOR [3.60])

$$\forall\, T \in \mathcal{E}, k \in \{1, \dots, m\}, TE_{k,k} = 0 \Rightarrow \underbrace{\operatorname{span} \left\{ \begin{matrix} E_{1,1}, & \cdots & , E_{m,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{1,p}, & \cdots & , E_{m,p} \end{matrix} \right\}}_{\text{Denote it by R}} \cap \mathcal{E} = \{0\}.$$

$$\text{又 } W = \operatorname{span} \left\{ \begin{matrix} E_{m+1,1}, & \cdots & , E_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ , & & , \\ E_{m+1,p}, & \cdots & , E_{n,p} \end{matrix} \right\} \subseteq \mathcal{E}. \text{ Where } \mathcal{L}(V, W) = R \oplus W \Rightarrow \mathcal{L}(V, W) = R + \mathcal{E}.$$

Then $\dim \mathcal{E} = \dim \mathcal{L}(V, W) - \dim R - \dim(R \cap \mathcal{E}) = (\dim V - \dim U) \dim W$. ☐

∘ *Suppose V is finite-dim and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(T) = ST$.*
(a) *Show that* $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.
(b) *Show that* $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

SOLUTION:

(a) $\forall T \in \mathcal{L}(V), ST = 0 \iff \operatorname{range} T \subseteq \operatorname{null} S$.
Thus $\operatorname{null} \mathcal{A} = \big\{T \in \mathcal{L}(V) : \operatorname{range} T \subseteq \operatorname{null} S\big\} = \mathcal{L}(V, \operatorname{null} S)$.

(b) $\forall R \in \mathcal{L}(V), \operatorname{range} R \subseteq \operatorname{range} S \iff \exists\, T \in \mathcal{L}(V), R = ST$, by (3.B 25).
Thus $\operatorname{range} \mathcal{A} = \big\{R \in \mathcal{L}(V) : \operatorname{range} R \subseteq \operatorname{range} S\big\} = \mathcal{L}(V, \operatorname{range} S)$.

□

OR. Using NOTE FOR [3.60].

Let $B_{\operatorname{range} S} = (\underbrace{w_1, \dots, w_m}_{Sv_i = w_i}), B_{\mathcal{K}} = (v_1, \dots, v_m); (w_1, \dots, w_n), (v_1, \dots, v_n)$ are bases of V.

Define $E_{i,j} \in \mathcal{L}(V)$ by $E_{i,j}(v_x) = \delta_{i,x} w_i$.

Thus $S = E_{1,1} + \cdots + E_{m,m}$; $\quad \mathcal{M}\big(S, (v_1, \dots, v_n), (w_1, \dots, w_n)\big) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$.

Define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j}(w_x) = \delta_{i,x} v_i$.
Let $E_{j,k} R_{i,j} = Q_{i,k}$, $\quad R_{j,k} E_{i,j} = G_{i,k}$.

Because $\forall T \in \mathcal{L}(V),\ \exists\,!\, A_{i,j} \in \mathbf{F},\quad T = \begin{pmatrix} A_{1,1}R_{1,1}+ & \cdots & +A_{1,m}R_{m,1}+ & \cdots & +A_{1,n}R_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}R_{1,m}+ & \cdots & +A_{m,m}R_{m,m}+ & \cdots & +A_{m,n}R_{n,m} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{n,1}R_{1,n}+ & \cdots & +A_{n,m}R_{m,n}+ & \cdots & +A_{n,n}R_{n,n} \end{pmatrix}$.

$$\Rightarrow \mathcal{A}(T) = ST = \Big(\sum_{r=1}^{m} E_{r,r}\Big)\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i}\Big)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} A_{i,j}Q_{j,i} = \begin{pmatrix} A_{1,1}Q_{1,1}+ & \cdots & +A_{1,m}Q_{m,1}+ & \cdots & +A_{1,n}Q_{n,1} \\ + & \cdots & + & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ + & \cdots & + & \cdots & + \\ A_{m,1}Q_{1,m}+ & \cdots & +A_{m,m}Q_{m,m}+ & \cdots & +A_{m,n}Q_{n,m} \end{pmatrix}.$$

Thus $\operatorname{null} \mathcal{A} = \operatorname{span} \begin{pmatrix} R_{1,m+1}, & \cdots & ,R_{n,m+1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ R_{1,n}', & \cdots & ,R_{n,n}' \end{pmatrix}$, $\quad \operatorname{range} \mathcal{A} = \operatorname{span} \begin{pmatrix} Q_{1,1}, & \cdots & ,Q_{n,1}, \\ , & & , \\ \vdots & \ddots & \vdots \\ Q_{1,m}', & \cdots & ,Q_{n,m}' \end{pmatrix}$.

Hence (a) $\dim \operatorname{null} \mathcal{A} = n \times (n - m)$; $\quad$ (b) $\dim \operatorname{range} \mathcal{A} = n \times m$. □

• **Comment:** Define $\mathcal{B} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{B}(T) = TS$. Similarly to Problem $(\circ)$,

(a) $\forall T \in \mathcal{L}(V), TS = 0 \iff \operatorname{range} S \subseteq \operatorname{null} T$.
Thus $\operatorname{null} \mathcal{B} = \{T \in \mathcal{L}(V) : \operatorname{range} S \subseteq \operatorname{null} T\} = \{T \in \mathcal{L}(V) : T|_{\operatorname{range} S} = 0\}$.

(b) $\forall R \in \mathcal{L}(V), \operatorname{null} S \subseteq \operatorname{null} R \iff \exists\, T \in \mathcal{L}(V), R = TS$, by (3.B.24).
Thus $\operatorname{range} \mathcal{B} = \{R \in \mathcal{L}(V) : \operatorname{null} S \subseteq \operatorname{null} R\} = \{R \in \mathcal{L}(V) : R|_{\operatorname{null} S} = 0\}$.

Hence $\dim \operatorname{null} \mathcal{B} = (\dim V - \dim \operatorname{range} S)(\dim V)$;
$\dim \operatorname{range} \mathcal{B} = (\dim V - \dim \operatorname{null} S)(\dim V)$. □

Or. Using Note For [3.60] and the notation in Problem $(\circ)$.

$$\mathcal{B}(T) = TS = \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}R_{j,i}\Big)\Big(\sum_{r=1}^{m} E_{r,r}\Big) = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{i,j}G_{j,i} = \begin{pmatrix} A_{1,1}G_{1,1}+ & \cdots & +A_{1,m}G_{m,1} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{m,1}G_{1,m}+ & \cdots & +A_{m,m}G_{m,m} \\ + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ A_{n,1}G_{1,n}+ & \cdots & +A_{n,m}G_{m,n} \end{pmatrix}.$$

Thus $\operatorname{null} \mathcal{B} = \operatorname{span}\begin{pmatrix} R_{m+1,1}, & \cdots & ,R_{n,1}, \\ \vdots & \ddots & \vdots \\ R_{m+1,n}, & \cdots & ,R_{n,n} \end{pmatrix}$,

$\operatorname{range} \mathcal{B} = \operatorname{span}\begin{pmatrix} G_{1,1}, & \cdots & ,G_{m,1}, \\ \vdots & \ddots & \vdots \\ G_{1,n}, & \cdots & ,G_{m,n} \end{pmatrix}$. Hence (a) $\dim \operatorname{null} \mathcal{B} = n \times (n-m)$;
(b) $\dim \operatorname{range} \mathcal{B} = n \times m$. □

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17 *Suppose V is finite-dim. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.*
A subsp $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal of $\mathcal{L}(V)$ if $TE \in \mathcal{E}, ET \in \mathcal{E}, \ \forall E \in \mathcal{E}, T \in \mathcal{L}(V)$.

Solution: Using Note For [3.60]. Let $(v_1, \dots, v_n)$ be a basis of V. If $\mathcal{E} = 0$, then we are done.
Suppose $\mathcal{E} \neq 0$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$.
Then $\forall E_{i,j} \in \mathcal{E}$, $(\forall x, y = 1, \dots, n)$, by assumption, $E_{j,x}E_{i,j} = E_{i,x} \in \mathcal{E}, E_{i,j}E_{y,i} = E_{y,j} \in \mathcal{E}$.
Again, $E_{y,x'}, E_{y',x} \in \mathcal{E}$ for all $x', y', x, y = 1, \dots, n$. Thus $\mathcal{E} = \mathcal{L}(V)$. □

• Or (10.A.4) *Suppose that $(\beta_1, \dots, \beta_n)$ and $(\alpha_1, \dots, \alpha_n)$ are bases of V.*
Let $T \in \mathcal{L}(V)$ be such that $T\alpha_k = \beta_k, \forall k$. Prove that $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)$
For ease of notation, let $\mathcal{M}(T, \alpha \to \beta) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n))$, $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(T, (\alpha_1, \dots, \alpha_n))$.

Solution:

Denote $\mathcal{M}(T, \alpha \to \alpha)$ by A and $\mathcal{M}(I, \beta \to \alpha)$ by B.
$\forall k \in \{1, \dots, n\}, Iu_k = u_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n = Tv_k = A_{1,k}\alpha_1 + \cdots + A_{n,k}\alpha_n \Rightarrow A = B$. □

Or. Note that $\mathcal{M}(T, \alpha \to \beta) = I$. Hence $\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \beta \to \alpha)\underbrace{\mathcal{M}(T, \alpha \to \beta)}_{=\mathcal{M}(I, \beta \to \beta)} = \mathcal{M}(I, \beta \to \alpha)$. □

Or. Note that $\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta) = \mathcal{M}(T, \alpha \to \beta) = I$.
$\mathcal{M}(T, \alpha \to \alpha) = \mathcal{M}(I, \alpha \to \beta)^{-1}\Big(\underbrace{\mathcal{M}(T, \beta \to \beta)\mathcal{M}(I, \alpha \to \beta)}_{=\mathcal{M}(T, \alpha \to \beta)}\Big) = \mathcal{M}(I, \beta \to \alpha)$. □

Comment: Denote $\mathcal{M}(T, \beta \to \beta)$ by A'.
$u_k = Iu_k = B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n, \ \forall\, k \in \{1, \dots, n\}$.
又 $Tu_k = T(B_{1,k}\alpha_1 + \cdots + B_{n,k}\alpha_n) = B_{1,k}\beta_1 + \cdots + B_{n,k}\beta_n = A'_{1,k}\beta_1 + \cdots + A'_{n,k}\beta_n \Rightarrow A' = B$.
Or. $\mathcal{M}(T, \beta \to \beta) = \mathcal{M}(T, \alpha \to \beta)\mathcal{M}(I, \beta \to \alpha) = B$.

---

16 *Suppose V is finite-dim and $S \in \mathcal{L}(V)$ such that $\forall T \in \mathcal{L}(V), ST = TS$.*
Prove that $\exists\, \lambda \in \mathbf{F}, S = \lambda I$.

SOLUTION: Using the notation and result in ($\circ$).

Suppose $ST = TS$ for every $T \in \mathcal{L}(V)$. If $S = 0$, we are done. Now suppose $S \neq 0$.

Let $S = E_{1,1} + \cdots + E_{m,m} \Rightarrow \mathcal{M}(S, B_{\mathcal{K}}) = \mathcal{M}(I, B_{\text{range}\,S}, B_{\mathcal{K}})$.

Then $\forall k \in \{m+1, \dots, n\}, 0 \neq SR_{k,1} = R_{k,1}S$. Hence $n = \dim V = \dim \operatorname{range} S = m$.

NOTICE that $R_{i,j}S = SR_{i,j} \iff Q_{i,j} = G_{i,j}$. Thus $Q_{i,j}(w_i) = w_j = a_{i,i}v_j = G_{i,j}(a_{1,i}v_1 + \cdots + a_{n,i}v_n)$.

Where $a_{i,j} = \mathcal{M}\big(I, (w_1, \dots, w_n), (v_1, \dots, v_n)\big)_{i,j} \iff w_i = Iw_i = a_{1,i}v_1 + \cdots + a_{n,i}v_n$;

And For each j, for all i. Thus $a_{i,i} = a_{k,k} = \lambda, \forall\, k \neq i$.

Hence $w_i = \lambda v_i \Rightarrow \mathcal{M}(S) = \mathcal{M}\big(\lambda I, (v_1, \dots, v_n)\big) \Rightarrow S = \mathcal{M}^{-1}(\mathcal{M}(\lambda I))\lambda I$. $\square$

---

18 *Show that V and $\mathcal{L}(\mathbf{F}, V)$ are iso vecsps.*

SOLUTION:

Define $\Psi \in \mathcal{L}\big(V, \mathcal{L}(\mathbf{F}, V)\big)$ by $\Psi(v) = \Psi_v$; where $\Psi_v \in \mathcal{L}(\mathbf{F}, V)$ and $\Psi_v(\lambda) = \lambda v$.

(a) $\Psi(v) = \Psi_v = 0 \Rightarrow \forall \lambda \in \mathbf{F}, \Psi_v(\lambda) = \lambda v = 0 \Rightarrow v = 0$. Hence Ψ is inje.

(b) $\forall T \in \mathcal{L}(\mathbf{F}, V)$, let $v = T(1) \Rightarrow T(\lambda) = \lambda v = \Psi_v(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = \Psi\big(T(1)\big)$. Hence Ψ is surj. $\square$

OR. Define $\Phi \in \mathcal{L}\big(\mathcal{L}(\mathbf{F}, V), V\big)$ by $\Phi(T) = T(1)$.

(a) Suppose $\Phi(T) = 0 = T(1) = \lambda T(1) = T(\lambda), \forall \lambda \in \mathbf{F} \Rightarrow T = 0$. Thus Φ is inje.

(b) For any $v \in V$, define $T \in \mathcal{L}(\mathbf{F}, V)$ by $T(\lambda) = \lambda v$. Then $\Phi(T) = T(1) = v$. Thus Φ is surj. $\square$

COMMENT: $\Phi = \Psi^{-1}$.

• *Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{R}), q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$.*

SOLUTION:

Note that $\deg\left[(x^2 + x)p''(x) + 2xp'(x) + p(3)\right] = \deg p$.

Define $T_n : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ by $T_n(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$. Then $T_n \in \mathcal{L}\big(\mathcal{P}_n(\mathbf{R})\big)$.

And note that $T_n(p) = 0 \Rightarrow \deg(T_n p) = -\infty = \deg p \Rightarrow p = 0$. Thus T_n is inv.

$\forall q \in \mathcal{P}(\mathbf{R})$, if $q = 0$, let $m = 0$; if $q \neq 0$, let $m = \deg q$, we have $q \in \mathcal{P}_m(\mathbf{R})$.

Hence $\exists\, p \in \mathcal{P}_m(\mathbf{R}), q(x) = T_m(p) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ for all $x \in \mathbf{R}$. $\square$

---

19 *Suppose $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$ is inje. $\deg Tp \leqslant \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.*
(a) *Prove that T is surj;* (b) *Prove that for every nonzero p, $\deg Tp = \deg p$.*

SOLUTION:

(a) T is inje $\iff \forall n \in \mathbf{N}^+, T|_{\mathcal{P}_n(\mathbf{R})} : \mathcal{P}_n(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ is inje and therefore is inv $\iff T$ is surj.

(b) Using mathematical induction.

(i) $\deg p = 0 \Rightarrow p = C \Rightarrow \deg Tp = \deg p = 0$;

$\deg p = -\infty \Rightarrow p = 0 \Rightarrow \deg Tp = \deg p = -\infty$.

(ii) Assume that $\forall s \in \mathcal{P}_n(\mathbf{R}), \deg s = \deg Ts$.

Suppose $\exists\, r \in \mathcal{P}_{n+1}(\mathbf{R}),\ \deg Tr \leqslant n < \deg r = n+1$.

Then by (a), $\exists\, s \in \mathcal{P}_n(\mathbf{R}),\ T(s) = (Tr)$.

又 T is inje $\Rightarrow s = r$. While $\deg s = \deg Ts = \deg Tr < \deg r$.

Contradicts. Thus $\forall p \in \mathcal{P}_{n+1}(\mathbf{R}), \deg Tp = \deg p$. $\square$

ENDED

3·E

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 20 | 4E: 8, 14

1 *A function $T: V \to W$ is linear $\iff T$ is a subspace of $V \times W$.*

---

2 *Suppose $V_1 \times \cdots \times V_m$ is finite-dim. Prove that each V_j is finite-dim.*

Solution:

For any $k \in \{1, \ldots, m\}$, define $p_k : V_1 \times \cdots \times V_m \to V_k$ by $p_k(v_1, \ldots, v_m) = v_k$.
Then p_k is a surj linear map. By [3.22], range $p_k = V_k$ is finite-dim. $\square$

Or. Denote $V_1 \times \cdots \times V_m$ by U. Denote $\{0\} \times \cdots \{0\} \times V_i \times \{0\} \cdots \times \{0\}$ by U_i.
Let $(v_1, \ldots, v_M)$ be a basis of U. Note that $\forall u_i \in V_i, \in U_i \subseteq U$, for each i.

$$\left.\begin{array}{l} \text{Define } R_i \in \mathcal{L}(V_i, U) \text{ by } R_i(u_i) = (0, \ldots, 0, u_i, 0, \ldots, 0) \\ \text{Define } S_i \in \mathcal{L}(U, V_i) \text{ by } S_i(u_1, \ldots, u_i, \ldots, u_m) = u_i \end{array}\right\} \Rightarrow S_i|_{U_i} = R_i^{-1}|_{U_i}.$$

Thus U_i and V_i are iso. 又 U_i is a subsp of a finite-dim vecsp U. $\square$

---

3 *Give an example of a vecsp V and its two subsps U_1, U_2 such that $U_1 \times U_2$ and $U_1 + U_2$ are iso but $U_1 + U_2$ is not a direct sum.*

Solution: V must be infinite-dim. For if not, both U_1 and U_2 are finite-dim subsps. By [3.76, 3.78].

Note *that at least one of U_1, U_2 must be infinite-dim. And at least one must be infinite-dim???* TODO
For if not, $U_1 \times U_2$ is finite-dim and $\dim(U_1 \times U_2) = \dim(U_1 + U_2) = \dim U_1 + \dim U_2$.
Let $V = \mathbf{F}^\infty = U_1$, $U_2 = \{(x, 0, \cdots) \in \mathbf{F}^\infty : x \in \mathbf{F}\}$.

$$\left.\begin{array}{l} \text{Define } T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2) \text{ by } T\big((x_1, x_2, \cdots), (x, 0, \cdots)\big) = (x, x_1, x_2, \cdots) \\ \text{Define } S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2) \text{ by } S(x, x_1, x_2, \cdots) = \big((x_1, x_2, \cdots), (x, 0, \cdots)\big) \end{array}\right\} \Rightarrow S = T^{-1}. \quad \square$$

---

4 *Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are iso.*

Solution: Using the notation in Problem (2).

Note that $T(u_1, \ldots, u_m) = T(u_1, 0, \ldots, 0) + \cdots + T(0, \ldots, u_m)$.

$$\left.\begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \ldots, T_m) \text{ by } \varphi(T) = (TR_1, \ldots, TR_m). \\ \text{Define } \psi : (T_1, \ldots, T_m) \mapsto T \text{ by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array}\right\} \Rightarrow \psi = \varphi^{-1}. \quad \square$$

---

5 *Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are iso.*

Solution: Using the notation in Problem (2).

Note that $Tv = (w_1, \ldots, w_m)$. Define $T_i \in \mathcal{L}(V, W_i)$ by $T_i(v) = w_i$.

$$\left.\begin{array}{l} \text{Define } \varphi : T \mapsto (T_1, \ldots, T_m) \text{ by } \varphi(T) = (S_1 T, \ldots, S_m T). \\ \text{Define } \psi : (T_1, \ldots, T_m) \mapsto T \text{ by } \psi(T_1, \ldots, T_m) = T_1 S_1 + \cdots + T_m S_m. \end{array}\right\} \Rightarrow \psi = \varphi^{-1}. \quad \square$$

---

6 *For $m \in \mathbf{N}^+$, define V^m by $\underbrace{V \times \cdots \times V}_{m \text{ times}}$. Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are iso.*

Solution:

Define $T : (v_1, \ldots, v_m) \to \varphi$, where $\varphi : (a_1, \ldots, a_m) \mapsto v$ is defined by $\varphi(a_1, \ldots, a_m) = a_1 v_1 + \cdots + a_m v_m$.

(a) Suppose $T(v_1, \ldots, v_m) = 0$. Then $\forall\ (a_1, \ldots, a_n) \in \mathbf{F}^m$, $\varphi(a_1, \ldots, a_m) = a_1 v_1 + \cdots + a_m v_m = 0$
$\Rightarrow (v_1, \ldots, v_m) = 0 \Rightarrow T$ is inje.

(b) Suppose $\psi \in \mathcal{L}(\mathbf{F}^m, V)$. Let $(e_1, \ldots, e_m)$ be the standard basis of $\mathbf{F}^m$. Then $\forall\ (b_1, \ldots, b_n) \in \mathbf{F}^m$,
$\Big[T\big(\psi(e_1), \ldots, \psi(e_m)\big)\Big](b_1, \ldots, b_m) = b_1\psi(e_1) + \cdots + b_m\psi(e_m) = \psi(b_1 e_1 + \cdots + b_m e_m) = \psi(b_1, \ldots, b_m)$.
Thus $T\big(\psi(e_1), \ldots, \psi(e_m)\big) = \psi$. Hence T is surj. $\square$

---

14 *Suppose $U = \{(x_1, x_2, \cdots) \in \mathbf{F}^\infty : x_k \neq 0$ for only finitely many $k\}$.*
(a) *Show that U is a subsp of $\mathbf{F}^\infty$. [Do it in your mind]*
(b) *Prove that $\mathbf{F}^\infty / U$ is infinite-dim.*

SOLUTION: For ease of notation, denote the p^{th} term of $u = (x_1, \cdots, x_p, \cdots) \in \mathbf{F}^\infty$ by $u[p]$.

For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, (p-1) \equiv 0 \pmod r \\ 0, \text{otherwise} \end{cases}$ | simply $e_r = (1, \underbrace{0, \cdots, 0}_{(p-1)\ times}, 1, \underbrace{0, \cdots, 0}_{(p-1)\ times}, 1, \cdots)$.

Choose one $m \in \mathbf{N}^+$. Let $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 + U \Rightarrow \exists\, u \in U, a_1 e_1 + \cdots + a_m e_m = u$.

Suppose $u = (x_1, \cdots, x_L, 0, \cdots)$, where L is the largest such that $u[L] \neq 0$.

Let $s \in \mathbf{N}^+$ be such that $h = s \cdot m! + 1 > L$ and $e_1[h] = \cdots = e_m[h] = 1$.

Note that by definition, $e_r[s \cot m! + 1 + p] = e_r[p+1] = 1 \iff p \equiv 0 \pmod r \iff r \mid p$.

Now for any $p \in \{1, \dots, m\}$, $u[h+p] = \left(\sum_{r=1}^m a_r e_r\right)[p+1] = \sum_{k=1}^{\tau(p)} a_{p_k} = 0$ $\qquad (\Delta)$

where $1 = p_1 \leqslant \cdots \leqslant p_{\tau(p)} = p$ are all the distinct factors of p.

Let $q = p_{\tau(p)-1}$. Notice that $\tau(q) = \tau(p) - 1$ and $q_k = p_k, \forall k \in \{1, \dots, \tau(q)\}$.

Again by (Δ), $\left(\sum_{r=1}^m a_r e_r\right)[h+q] = \sum_{k=1}^{\tau(p)-1} a_{p_k} = 0$. Thus $a_{p_{\tau(p)}} = a_p = 0$ for any $p \in \{1, \dots, m\}$.

Hence $\forall m \in \mathbf{N}^+$, $(e_1, \dots, e_m)$ is linely inde in $\mathbf{F}^\infty$, so is $(e_1 + U, \dots, e_m + U)$ in $\mathbf{F}^\infty / U$. By (2.A.14). □

OR. For each $r \in \mathbf{N}^+$, let $e_r[p] = \begin{cases} 1, \text{if } 2^r \mid p \\ 0, \text{otherwise} \end{cases}$.

Similarly, let $m \in \mathbf{N}^+$ and $a_1(e_1 + U) + \cdots + a_m(e_m + U) = 0 \Rightarrow a_1 e_1 + \cdots + a_m e_m = u \in U$.

Suppose L is the largest such that $u[L] \neq 0$. And l is such that $2^{ml} > L$.

Then $\forall k \in \{1, \dots, m\}, u[2^{ml} + 2^k] = \left(\sum_{r=1}^m a_r e_r\right)[2^k] = a_1 + \cdots + a_k = 0$.

Thus $a_1 = \cdots = a_m = 0$ and $(e_1, \dots, e_m)$ is linely inde. Similarly. □

---

7 *Suppose $v, x \in V$ and U and W are subsps of V. Prove that $v + U = x + W \Rightarrow U = W$.*

SOLUTION:

(a) $\forall u_1 \in U$, $\exists\, w_1 \in W, v + u_1 = x + w_1$, let $u_1 = 0$, now $v = x + w_1' \Rightarrow v - x \in W$.
(b) $\forall w_2 \in W$, $\exists\, u_2 \in U, v + u_2 = x + w_2$, let $w_2 = 0$, now $x = v + u_2' \Rightarrow x - v \in U$.

Thus $\pm(v - x) \in U \cap W \Rightarrow \left\{ \begin{array}{l} u_1 = (x - v) + w_1 \in W \Rightarrow U \subseteq W \\ w_2 = (v - x) + u_2 \in U \Rightarrow W \subseteq U \end{array} \right\} \Rightarrow U = W$. □

- Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$.
Then A is a translate of $U \iff \exists\, c \in \mathbf{R}, A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}$.

- *Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $U = \{x \in V : Tx = c\}$ is either $\emptyset$ or is a translate of* null T.

SOLUTION:

If $c \in W$ but $c \notin \operatorname{range} T$, then $U = \emptyset$, we are done. Now suppose $c \in \operatorname{range} T$ and $x \in U$.

$\forall x + y \in x + \operatorname{null} T$ $(\forall y \in \operatorname{null} T)$, $x + y \in U$. Hence $x + \operatorname{null} T \subseteq U$.

$\forall u \in U, u - x \in \operatorname{null} T \Rightarrow u = x + (u - x)x + \operatorname{null} T$. Hence $U \subseteq x + \operatorname{null} T$. □

COROLLARY: *The set of solutions to a system of linear equations such as [3.28] is either $\emptyset$ or a translate.*

---

8 *Suppose A is a nonempty subset of V.*
Prove that A is a translate of some subsp of $V \iff \lambda v + (1-\lambda)w \in A\ , \forall v, w \in A, \lambda \in \mathbf{F}$.

SOLUTION:

Suppose $A = a + U$. Then $\forall a + u_1, a + u_2 \in A, \lambda \in \mathbf{F}$,
$\lambda(a+u_1) + (1-\lambda)(a+u_2) = a + \big(\lambda(u_1 - u_2) + u_2\big) \in A$.

Suppose $\lambda v + (1-\lambda)w \in A, \forall v, w \in A, \lambda \in \mathbf{F}$. Suppose $a \in A$ and let $A' = \{x - a : x \in A\}$.

Then $0 \in A'$ and $\forall x-a, y-a \in A'$, $(\forall x, y \in A)$, $\lambda \in \mathbf{F}$,
(I) $\lambda(x-a) = [\lambda x + (1-\lambda)a] - a \in A'$.
(II) $\lambda(x-a) + (1-\lambda)(y-a) = \frac{1}{2}(x-a) + \frac{1}{2}(y-a) = \frac{1}{2}x + (1-\frac{1}{2})y - a \in A'$.
OR. By (I), $2 \times [\frac{1}{2}(x-a) + \frac{1}{2}(y-a)] = (x-a) + (y-a) \in A'$.
Thus A' is a subsp of V. Hence $a + A' = \{(x-a) + a : x \in A\} = A$ is a translate. □

OR. Suppose $x-a, y-a \in A', \lambda \in \mathbf{F}$.
Note that $x, a \in A \Rightarrow \lambda x + (1-\lambda)a = 2x - a \in A$. Similarly $2y - a \in A$.
(I) $(x - \frac{1}{2}a) + (y - \frac{1}{2}a) = x + y - a \in A \Rightarrow x + y - 2a = (x-a) + (y-a) \in A'$.
(II) $\lambda(x-a) = (\lambda x + (1-\lambda)a) - a \in A'$.
Thus $-x + A$ is a subsp of V. Hence $A = x + (-x + A)$ is a translate of the subsp $(-x+A)$. □

---

9 *Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subsps U_1, U_2 of V.*
Prove that the intersection $A_1 \cap A_2$ is either a translate of some subsp of V or is $\emptyset$.

SOLUTION:

Suppose $v + u_1, w + u_2 \in A_1 \cap A_2 \neq \emptyset$. By Problem (8),
$\forall \lambda \in \mathbf{F}, \lambda(v+u_1) + (1-\lambda)(w+u_2) \in A_1 \cap A_2$. Thus $A_1 \cap A_2$ is a translate of some subsp of V. □

OR. Let $A_1 = v + U_1, A_2 = w + U_2$. Suppose $x \in (v + U_1) \cap (w + U_2) \neq \emptyset$.
Then $\exists u_1 \in U_1, x = v + u_1 \Rightarrow x - v \in U_1$, $\exists u_2 \in U_2, x = w + u_2 \Rightarrow x - w \in U_2$.
Note that by [3.85], $A_1 = v + U_1 = x + U_1, A_2 = w + U_2 = x + U_2$. We show that $A_1 \cap A_2 = x + (U_1 \cap U_2)$.
(a) $y \in A_1 \cap A_2 \Rightarrow \exists u_1 \in U_1, u_2 \in U_2, y = x + u_1 = x + u_2 \Rightarrow u_1 = u_2 \in U_1 \cap U_2 \Rightarrow y \in x + (U_1 \cap U_2)$.
(b) $y = x + u \in x + (U_1 \cap U_2) = (x + U_1) \cap (x + U_2) \Rightarrow y \in A_1 \cap A_2$. □

---

10 *Prove that the intersection of any collection of translates of subsps of V*
is either a translate of some subsp or $\emptyset$.

SOLUTION:

Suppose $\{A_\alpha\}_{\alpha \in \Gamma}$ is a collection of translates of subsps of V, where Γ is an arbitrary index set.

Suppose $x, y \in \bigcap_{\alpha \in \Gamma} A_\alpha \neq \emptyset$, then by Problem (8), $\forall \lambda \in \mathbf{F}, \lambda x + (1-\lambda)y \in A_\alpha$ for every $\alpha \in \Gamma$.
Thus $\bigcap_{\alpha \in \Gamma} A_\alpha$ is a translate of some subsp of V. □

OR. Let $A_\alpha = w_\alpha + V_\alpha$ for each $\alpha \in \Gamma$. Suppose $x \in \bigcap_{\alpha \in \Gamma} (w_\alpha + V_\alpha) \neq \emptyset$.
Then for each $A_\alpha, \exists v_\alpha \in V_\alpha, x = w_\alpha + v_\alpha \Rightarrow x - w_\alpha \in V_\alpha \Rightarrow A_\alpha = w_\alpha + V_\alpha = x + V_\alpha$.
(a) $y \in \bigcap_{\alpha \in \Gamma} A_\alpha \Rightarrow \forall \alpha \in \Gamma, \exists v_\alpha, y = x + v_\alpha \Rightarrow \forall \alpha, \beta \in \Gamma, v_\alpha = v_\beta \Rightarrow y \in x + \bigcap_{\alpha \in \Gamma} V_\alpha$.
(b) $y = x + v \in x + \bigcap_{\alpha \in \Gamma} V_\alpha = \bigcap_{\alpha \in \Gamma} (x + V_\alpha) \Rightarrow y \in \bigcap_{\alpha \in \Gamma} A_\alpha$. Hence $\bigcap_{\alpha \in \Gamma} A_\alpha = x + \bigcap_{\alpha \in \Gamma} V_\alpha$. □

- **NOTE FOR [3.79, 3.83]:** If $U = \{0\}$, then $v + U = v + \{0\} = \{v\}$, $V/U = V/\{0\} = \{\{v\} : v \in V\}$.

---

11 *Suppose* $A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \sum_{i=1}^{m} \lambda_i = 1\}$, *where each* $v_i \in V, \lambda_i \in \mathbf{F}$.

(a) *Prove that* A *is a translate of some subsp of* V

(b) *Prove that if* B *is a translate of some subsp of* V *and* $\{v_1, \ldots, v_m\} \subseteq B$, *then* $A \subseteq B$.

(c) *Prove that* A *is a translate of some subsp of* V *of dim less than* m.

SOLUTION:

(a) By Problem (8), $\forall u, w \in A, \lambda \in \mathbf{F}, \exists\, a_i, b_i \mathbf{F}$,
$\lambda u + (1-\lambda) w = \left(\lambda \sum_{i=1}^{m} a_i + (1-\lambda) \sum_{i=1}^{m} b_i\right) v_i \in A.$ □

(b) Suppose $B = v + U$, where $v \in V$ and U is a subsp of V. Suppose $\exists\,!\, u_k \in U, v_k = v + u_k \in B$.
Then for all $v = \sum_{i=1}^{m} \lambda_i v_i \in A$, $v = \sum_{i=1}^{m} \lambda_i (v + u_i) = v + \sum_{i=1}^{m} \lambda_i u_i \in v + U = B$. □

OR. Let $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$. To show that $v \in B$, use induction on m by k.

(i) $k = 1, v = \lambda_1 v_1 \Rightarrow \lambda_1 = 1$. 又 $v_1 \in B$. Hence $v \in B$.

$k = 2, v = \lambda_1 v_1 + \lambda_2 v_2 \Rightarrow \lambda_2 = 1 - \lambda_1$. 又 $v_1, v_2 \in B$. By Problem (8), $v \in B$.

(ii) $2 \leqslant k \leqslant m$, we assume that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k \in A \subseteq B$. $\left(\forall \lambda_i \text{ such that } \sum_{i=1}^{k} \lambda_i = 1\right)$

For $u = \mu_1 v_1 + \cdots + \mu_k v_k + \mu_{k+1} v_{k+1} \in A$. $\forall i = 1, \ldots, k$, $\exists\, \mu_i \neq 1$, fix one such i by ι.

Then $\sum_{i=1}^{k+1} \mu_i - \mu_\iota = 1 - \mu_\iota \Rightarrow \left(\sum_{i=1}^{k+1} \dfrac{\mu_i}{1-\mu_\iota}\right) - \dfrac{\mu_\iota}{1-\mu_\iota} = 1.$

$$\text{Let } w = \underbrace{\frac{\mu_1}{1-\mu_\iota} v_1 + \cdots + \frac{\mu_{\iota-1}}{1-\mu_\iota} v_{\iota-1} + \frac{\mu_{\iota+1}}{1-\mu_\iota} v_{\iota+1} + \cdots + \frac{\mu_{k+1}}{1-\mu_\iota} v_{k+1}}_{k \text{ terms}}.$$

Let $\lambda_i = \dfrac{\mu_i}{1-\mu_\iota}$ for $i = 1, \ldots, \iota - 1$; $\lambda_j = \dfrac{\mu_{j+1}}{1-\mu_\iota}$ for $j = \iota, \ldots, k$. Then,

$$\left.\begin{array}{r} \sum_{i=1}^{k} \lambda_i = 1 \Rightarrow w \in B \\ v_\iota \in B \Rightarrow u' = \lambda w + (1-\lambda) v_\iota \in B \end{array}\right\} \Rightarrow \text{Let } \lambda = 1 - \mu_\iota. \text{ Thus } u' = u \in B \Rightarrow A \subseteq B.$$ □

(c) If $m = 1$, then let $A = v_1 + \{0\}$ and we are done.
Choose one $k \in \{1, \ldots, m\}$. Given $\lambda_i \in \mathbf{F}$, where $i \in \{1, \ldots, k-1, k+1, \ldots, m\}$.
Let $\lambda_k = 1 - \lambda_1 - \cdots - \lambda_{k-1} - \lambda_{k+1} - \cdots - \lambda_m$
Then $\lambda_1 v_1 + \cdots + \lambda_k v_k + \cdots + \lambda_m v_m = v_k + \sum_{i=1}^{m} \lambda_i (v_i - v_k)$.
Thus $A = v_k + \operatorname{span}(v_1 - v_k, \ldots, v_{k-1} - v_k, v_{k+1} - v_k, \ldots, v_m - v_k)$. □

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18 *Suppose* $T \in \mathcal{L}(V, W)$ *and* U *is a subsp of* V. *Let* π *denote the quotient map.*
Prove that $\exists\, S \in \mathcal{L}(V/U, W), T = S \circ \pi \iff U \subseteq \operatorname{null} T$.

SOLUTION:

(a) Suppose $U \subseteq \operatorname{null} T$. Define $S \in \mathcal{L}(V/U, W)$ by $S(v + U) = Tv$. Then $S \circ \pi = T$.
Now we show that this map is *well-defined*.
$v_1 + U = v_2 + U \iff (v_1 - v_2) \in U \iff S((v_1 - v_2) + U) = T(v_1 - v_2) = 0 \iff Tv_1 = Tv_2$.

(b) Suppose $\exists\, S, T = S \circ \pi$. Then $\forall u \in U, Tu = S \circ \pi(u) = S(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null} T$. □

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20 *Define* $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ *by* $\Gamma(S) = S \circ \pi$. *Prove that:*

(a) Γ *is linear:* By [3.9] distr and [3.6].

(b) Γ *is inje:* $\Gamma(S) = 0 = S \circ \pi \iff \forall v \in V, S(\pi(v)) = 0 \iff \forall v + U \in V/U, S(v + U) = 0 \iff S = 0$.

(c) $\operatorname{range} \Gamma = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}$: By Problem (18). □

- **NOTE FOR [3.88, 3.90, 3.91]:** Suppose $W \in \mathcal{S}_V U$. Then V/U and W are iso.
  For any $W \in \mathcal{S}_V U$, because $V = U \oplus W$, $\forall v \in V, \exists!\, u_v \in U, w_v \in W$ such that $v = u_v + w_v$.
  Define $T \in \mathcal{L}(V,W)$ by $T(v) = w_v$. Hence $\operatorname{null} T = U$, $\operatorname{range} T = W$, $\operatorname{range} T \oplus \operatorname{null} T = V$.
  Then $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, W)$ is defined by $\tilde{T}(v+U) = Tv = w_v$.
  Now $\pi \circ \tilde{T} = I_{V/U}$, $\tilde{T} \circ \pi = I_W = T|_W$ Hence $\tilde{T}$ is an iso of V/U onto W.
- **COMMENT:** Note that $v = u_v + w_v = (u_v - u') + (w_v' + u')$, where $w_v' \notin W \iff u' \neq 0$.
  Define $S \in \mathcal{L}(V/U, V)$ by $S(v+U) = v$. Hence $\operatorname{null} S = \{0\}$, $\operatorname{range} S \in \mathcal{S}_V U$, $\operatorname{range} S \oplus U = V$.
  Let $E = S \circ \pi$. Now $\operatorname{null} E = \operatorname{null} \pi = U$. Because π is surj 又 $\operatorname{range}(S \circ \pi) \subseteq \operatorname{range} S$. $\operatorname{range} E = \operatorname{range} S$.
  Then $\operatorname{range} E \oplus \operatorname{null} E = V$. NOTICE that $E : V \to \operatorname{range} S$ is a pure *eraser*. Now we explain why:
  EXAMPLE: Suppose $B_V = (v_1, v_2, v_3)$, $U = \operatorname{span}(v_1)$. Then it is uniquely fixed that $\operatorname{range} S = \operatorname{span}(v_2, v_3)$.
  While we might have $\operatorname{range} T = \operatorname{span}(v_2 - 2v_1, v_3) = W$, depending on the choice of W.
  Now $E : v_2 \mapsto v_2;\ \ v_2 - 2v_1 \mapsto v_2$. While $T : v_2 \mapsto v_2 - 2v_1;\ \ v_2 - 2v_1 \mapsto v_2 - 2v_1$.

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12 *Suppose U is a subsp of V such that V/U is finite-dim. Prove that is V is iso to $U \times (V/U)$.*

SOLUTION:

Let $(v_1 + U, \dots, v_n + U)$ be a basis of V/U.
Note that $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^n a_i(v_i + U) = \left(\sum_{i=1}^n a_i v_i\right) + U$
$\Rightarrow (v - a_1 v_1 - \cdots - a_n v_n) \in U \Rightarrow \exists!\, u \in U, v = \sum_{i=1}^n a_i v_i + u$.
Thus define $\varphi \in \mathcal{L}(V, U \times (V/U))$ by $\varphi(v) = (u,\, v + U)$,
and $\psi \in \mathcal{L}(U \times (V/U), V)$ by $\psi(u, v + U) = v + u$, where $\exists!\, a_i \in \mathbf{F}, v = \sum_{i=1}^n a_i v_i + U$. ☐

OR. [*V/U, U and V can be infinite-dim*] Define $S \in \mathcal{L}(V/U, V)$ by $S(v+U) = v$.
By the NOTE FOR [3.88,3.90,3.91], $\operatorname{range} S \oplus U = V$. Thus $\forall v \in V, \exists!\, u \in U, w \in \operatorname{range} S, v = u + w$.
Define $T \in \mathcal{L}(U \times (V/U), V)$ by $T(u, v+U) = u + S(v+U) = u + w = v$. Then T is surj.
And $T(u, v+U) = u + S(v+U) = 0 \Longrightarrow \pi(T(u, v+U)) = v + U = 0$, and $u = -S(v+U) = 0$.
OR. Define $R \in \mathcal{L}(V, U \times (V/U))$ by $R(v) = (u, (w+U))$. Now $R \circ T = I_{U \times (V/U)}$, $T \circ R = I_V$. ☐

- (4E 3.E.14) *Suppose $V = U \oplus W$, $(w_1, \dots, w_m)$ is a basis of W.*
  Prove that $(w_1 + U, \dots, w_m + U)$ is a basis of V/U.

SOLUTION: $\forall v \in V, \exists!\, u \in U, w \in W, v = u + w$. 又 $\exists!\, c_i \in \mathbf{F}, w = \sum_{i=1}^m c_i w_i \Rightarrow v = \sum_{i=1}^m c_i w_i + u$.
Hence $\forall v + U \in V/U, \exists!\, c_i \in \mathbf{F}, v + U = \sum_{i=1}^m c_i w_i + U$. ☐

---

13 *Suppose $(v_1 + U, \dots, v_m + U)$ is a basis of V/U and $(u_1, \dots, u_n)$ is a basis of U.*
Prove that $(v_1, \dots, v_m, u_1, \dots, u_n)$ is a basis of V.

SOLUTION: Notice that $(v_1, \dots, v_m)$ is linely inde.
By Problem (12), U and V/U are finite-dim $\Longrightarrow U \times (V/U)$ is finite-dim, so is V.
$\dim V = \dim(U \times (V/U)) = m + n$. 又 Each $v_i = S(v_i + U)$, where we define $S(v+U) = v$.
Note that $\sum_{i=1}^m a_i v_i \in U \iff \left(\sum_{i=1}^m a_i v_i\right) + U = 0 + U \iff a_1 = \cdots = a_m = 0$.
Hence $\operatorname{span}(v_1, \dots, v_m) \cap U = \{0\} \Rightarrow \operatorname{span}(v_1, \dots, v_m) \oplus U = V$. By (2.B.8), we are done. ☐

OR. Note that $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = \sum_{i=1}^m a_i v_i + U \Rightarrow \exists!\, b_i \in \mathbf{F}, v - \sum_{i=1}^m a_i v_i = \sum_{i=1}^m b_i u_i \in U$
$\Rightarrow \forall v \in V, \exists!\, a_i, b_j \in \mathbf{F}, v = \sum_{i=1}^m a_i v_i + \sum_{j=1}^m b_j u_j$. ☐

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15 *Suppose* $\varphi \in \mathcal{L}(V, \mathbf{F}) \backslash \{0\}$. *Prove that* $\dim V/(\operatorname{null}\varphi) = 1$.

Solution:

By (3.B.29), $\exists\, u \in V, V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}$. By (4E 3.E.14), $(u + \operatorname{null}\varphi)$ is a basis of $V/\operatorname{null}\varphi$.
Or. By [3.91] (d), $\dim \operatorname{range}\varphi = 1 = \dim V/(\operatorname{null}\varphi)$. □

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16 *Suppose* $\dim V/U = 1$. *Prove that* $\exists\, \varphi \in \mathcal{L}(V, \mathbf{F})$ *such that null* $\varphi = U$.

Solution:

Suppose V_0 is a subsp of V such that $V = U \oplus V_0$. Then V_0 and V/U are iso. $\dim V_0 = 1$.
Define $\varphi \in \mathcal{L}(V, \mathbf{F})$ by $\varphi(v_0) = 1, \varphi(u) = 0$, where $v_0 \in V_0, u \in U$. □

Or. Let $(w + U)$ be a basis of V/U. Then $\forall v \in V, \exists!\, a \in \mathbf{F}, v + U = aw + U$.
Define $\varphi : V \to \mathbf{F}$ by $\varphi(v) = a$. Assume that φ is linear.
Then $u \in U \iff u + U = 0w + U \iff \varphi(u) = 0 \iff u \in \operatorname{null}\varphi$. Thus $U = \operatorname{null}\varphi$. □
Now we prove the assumption.
$\forall x, y \in V, \lambda \in \mathbf{F}, \exists!\, a, b \in \mathbf{F}, x + U = aw + U, \lambda y + U = \lambda bw + U \Rightarrow (x + \lambda y) + U = (a + \lambda b)w + U$.
Then $\varphi(x + \lambda y) = a + \lambda b = \varphi(x) + \lambda\varphi(y)$.

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17 *Suppose* V/U *is finite-dim.* W *is a subsp of* V.
(a) *Show that if* $V = U + W$, *then* $\dim W \geqslant \dim V/U$.
(b) *Find a* W *such that* $\dim W = \dim V/U$ *and* $V = U \oplus W$.

Solution: Let $(w_1, \dots, w_n)$ be a basis of W

(a) $\forall v \in V, \exists\, u \in U, w \in W$ such that $v = u + w \Rightarrow v + U = w + U$
And $\exists!\, a_i \in \mathbf{F}, v + U = (a_1 w_1 + \cdots + a_n w_n) + U$. Then $V/U \subseteq \operatorname{span}(w_1 + U, \dots, w_n + U)$.
Hence $\dim V/U = \dim \operatorname{span}(w_1 + U, \dots, w_n + U) \leqslant \dim W$.

(b) Let $W \in \mathcal{S}_V U$. In other words, reduce $(w_1 + U, \dots, w_n + U)$
to a basis $(w_1 + U, \dots, w_m + U)$ of V/U and let $W = \operatorname{span}(w_1, \dots, w_m)$. □

Or. Let $(v_1 + U, \dots, v_m + U)$ be a basis of V/U and define $\tilde{T} \in \mathcal{L}(V/U, V)$ by $\tilde{T}(v_k + U) = v_k$.
Note that $\pi \circ \tilde{T} = I$. By (3.B.20), $\tilde{T}$ is inje. And $(v_1, \dots, v_m)$ is linely inde.
Let $W = \operatorname{range}\tilde{T} = \operatorname{span}(v_1, \dots, v_m)$. Then $\tilde{T} \in \mathcal{L}(V/U, W)$ is an iso. Thus $\dim W = \dim V/U$.
And $\forall v \in V, \exists!\, a_i \in \mathbf{F}, v + U = a_1 v_1 + \cdots + a_m v_m + U$

$$\Rightarrow v - (a_1 v_1 + \cdots + a_m v_m) \in U \Rightarrow \exists!\, w \in W, u \in U, v = w + u.$$ □

Ended

3·F

4 5 6 7 8 9 12 13 15 16 17 18 19 20 21 22 23 24 25 26
28 29 30 31 33 34 35 36 37 | 4E: 5, 6, 8, 17, 23, 24, 25

20, 21 *Suppose* U *and* W *are subsets of* V. *Prove that* $U \subseteq W \iff W^0 \subseteq U^0$.

Solution:

(a) Suppose $U \subseteq W$. Then $\forall \varphi \in W^0, u \in U \subseteq W, \varphi(w) = 0 \Rightarrow \varphi \in U^0$. Thus $W^0 \subseteq U^0$.
(b) Suppose $W^0 \subseteq U^0$. Then $\varphi \in W^0 \Rightarrow \varphi \in U^0$. Hence $\operatorname{null}\varphi \supseteq W \Rightarrow \operatorname{null}\varphi \supseteq U$. Thus $W \supseteq U$.

Or. For a subsp U of V, let $A_U = \{v \in V : \varphi(v) = 0, \forall \varphi \in U^0\} = U$, by Problem (25).
Suppose $W^0 \subseteq U^0$. Then $\forall \varphi \in W^0, v \in A_U, \varphi(v) = 0 \Rightarrow v \in A_W$. Thus $A_U \subseteq A_W$. □

Corollary: $W^0 = U^0 \iff U = W$.

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22 *Suppose U and W are subsps of V. Prove that $(U+W)^0 = U^0 \cap W^0$.*

SOLUTION:

(a) $\left.\begin{array}{c} U \subseteq U+W \\ W \subseteq U+W \end{array}\right\} \Rightarrow \left.\begin{array}{c} (U+W)^0 \subseteq U^0 \\ (U+W)^0 \subseteq W^0 \end{array}\right\} \Rightarrow (U+W)^0 \subseteq U^0 \cap W^0.$

OR. Suppose $\varphi \in (U+W)^0$. Then $\forall u \in U, w \in W, \varphi(u) = \varphi(w) = 0 \Rightarrow \varphi \in U^0 \cap W^0$.

(b) Suppose $\varphi \in U^0 \cap W^0 \subseteq V'$. Then $\forall u \in U, w \in W, \varphi(u+w) = 0 \Rightarrow \varphi \in (U+W)^0$. □

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23 *Suppose U and W are subsets of V. Prove that $(U \cap W)^0 = U^0 + W^0$.*

SOLUTION:

(a) $\left.\begin{array}{c} U \cap W \subseteq U \\ U \cap W \subseteq W \end{array}\right\} \Rightarrow \left.\begin{array}{c} (U \cap W)^0 \supseteq U^0 \\ (U \cap W)^0 \supseteq W^0 \end{array}\right\} \Rightarrow (U \cap W)^0 \supseteq U^0 + W^0 \; [\supseteq U^0 \cap W^0 = (U+W)^0.]$

OR. Suppose $\varphi = \psi + \beta \in U^0 + W^0$. Then $\forall v \in U \cap W, \varphi(v) = (\psi + \beta)(v) = 0 \Rightarrow \varphi \in (U \cap W)^0$.

(b) [*Only in Finite-dim; Requires that U, W are subsps*] Using Problem (22).

$$\begin{aligned} \dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\ &= 2\dim V - \dim U - \dim W - \big(\dim V - \dim(U+W)\big) = \dim V - \dim(U \cap W). \end{aligned}$$

OR. Suppose $\varphi \in (U \cap W)^0$. Let X, Y be such that $V = U \oplus X = W \oplus Y$.
Define $\psi \in U^0, \beta \in W^0$ by $\psi(u+x) = \frac{1}{2}\varphi(x), \beta(w+y) = \frac{1}{2}\varphi(y)$.
$\forall v = u + x = w + y \in V, \varphi(v) = \varphi(x) = \varphi(y)$. Now $\varphi(v) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y) = \psi(v) + \beta(v)$.
Hence $\varphi \in U^0 + W^0$. Now $(U \cap W)^0 \subseteq U^0 + W^0$. □

• **COROLLARY:**

(a) *Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subsets of V. Then $\Big(\bigcap_{\alpha_i \in \Gamma} V_{\alpha_i}\Big)^0 = \sum_{\alpha_i \in \Gamma}\big(V_{\alpha_i}^0\big)$.*

(b) *Suppose $\{V_{\alpha_i}\}_{\alpha_i \in \Gamma}$ is a collection of subsps of V. Then $\Big(\sum_{\alpha_i \in \Gamma} V_{\alpha_i}\Big)^0 = \bigcap_{\alpha_i \in \Gamma}\big(V_{\alpha_i}^0\big)$.*

(c) *Suppose $V = U \oplus W$. Then $V' = U^0 \oplus W^0$. And $U_V' = W^0,\ W_V' = U^0$.*

Where $U_V' = \{\varphi \in V' : \varphi = \varphi \circ \iota\}$. And $\iota \in \mathcal{L}(V, U)$ is defined by $\iota(u_v + w_v) = u_v$.

• (4E 3.F.23) *Suppose $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following sets are the same.*

(a) $\operatorname{span}(\varphi_1, \dots, \varphi_m)$

(b) $\big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big)^0 \overset{(c)}{=\!=} \{\varphi \in V' : (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) \subseteq \operatorname{null}\varphi\}$

SOLUTION: By Problem (17), (c) holds.

By Problem (26) [*May require finite-dim*] and the COROLLARY in Problem (23),

$$\left.\begin{array}{r} \big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big)^0 = (\operatorname{null}\varphi_1)^0 + \cdots + (\operatorname{null}\varphi_m)^0 \\ \operatorname{span}(\varphi_i) = \{v \in V : \forall \psi \in \operatorname{span}(\varphi_i), \psi(v) = 0\}^0 = (\operatorname{null}\varphi_i)^0 \end{array}\right\} \Rightarrow \text{(a)} = \text{(b)}.$$ □

OR. Note that by COROLLARY in Problem (4E 6), for each φ_i, we have
$\forall c \in \mathbf{F} \backslash \{0\}, \psi = c\varphi_i \in \operatorname{span}(\varphi_i) \iff \operatorname{null}\psi = \operatorname{null}\varphi_i \iff \psi \in (\operatorname{null}\psi)^0 = (\operatorname{null}\varphi_i)^0$.
And $0 \in \operatorname{span}(\varphi_i), 0 \in (\operatorname{null}\varphi_i)^0$. Hence $\operatorname{span}(\varphi_i) = (\operatorname{null}\varphi_i)^0$. Similarly. □

OR. [*Only in Finite-dim*] Suppose $\varphi \in V'$. Note that $\dim(\operatorname{null}\varphi)^0 = \dim \operatorname{range}\varphi = \dim \operatorname{span}(\varphi)$.
And because $\forall c \in \mathbf{F}, v \in \operatorname{null}\varphi, c\varphi(v) = 0 \Rightarrow \operatorname{span}(\varphi) \subseteq (\operatorname{null}\varphi)^0$. Similarly. □

COROLLARY: **30** Suppose V is finite-dim and $\varphi_1, \dots, \varphi_m$ is a linely inde list in V'.
Then $\dim\big((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\big) = (\dim V) - m$.

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31 *Suppose V is finite-dim and $B_{V'} = (\varphi_1, \dots, \varphi_n)$. Show that the correspd B_V exists.*

SOLUTION:

Using (3.B.29). Let $\varphi_i(u_i) = 1$ and then $V = \operatorname{null}\varphi_i \oplus \operatorname{span}(u_i)$ for each φ_i.

Suppose $a_1u_1 + \dots + a_nu_n = 0$. Then $0 = \varphi_i(a_1u_1 + \dots + a_nu_n) = a_i$ for each i.

Thus $B_V = (\varphi_1, \dots, \varphi_n)$. And $\varphi_i(u_x) = \delta_{i,x}$. $\square$

OR. For each $k \in \{1, \dots, n\}$, define $\Gamma_k = \{1, \dots, k-1, k+1, \dots, n\}$ and $U_k = \bigcap_{j\in\Gamma_k} \operatorname{null}\varphi_j$.

By Problem (30) OR (4E 2.C.16), $\dim U_k = 1$. Thus $\exists\, u_k \in V, U_k = \operatorname{span}(u_k) \neq 0$.

又 By Problem (30), $(\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_n) = \{0\} = U \cap \operatorname{null}\varphi_k$.

Then if $\varphi_k(u_k) = 0 \Rightarrow u_k \in \operatorname{null}\varphi_k$ while $u_k \in U \Rightarrow u_k \in \{0\}$, contradicts.

Thus $\varphi_k(u_k) \neq 0$. Let $v_k = \big(\varphi_k(u_k)\big)^{-1}u_k \Rightarrow \varphi_k(v_k) = 1$. Now for $j \neq k, u_k \in \operatorname{null}\varphi_j \Rightarrow \varphi_j(v_k) = 0$.

Similarly, suppose $a_1v_1 + \dots + a_nv_n = 0 \Rightarrow a_1 = \dots = a_n = 0$. $B_V = (v_1, \dots, v_n)$. And $\varphi_j(v_k) = \delta_{j,k}$. $\square$

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25 *Suppose U is a subsp of V. Explain why $U = \{v \in V : \varphi(v) = 0, \forall\varphi \in U^0\}$.*

SOLUTION: Note that $U = \{v \in V : v \in U\}$ is a subsp of V; And $v \in U \iff \varphi(v) = 0, \forall\varphi \in U^0$. $\square$

COROLLARY: $U^0 = \{v \in V : \varphi(v) = 0, \forall\varphi \in U^0\}^0$.

COMMENT: $\{v \in V : \varphi(v) = 0, \forall\varphi \in U^0\} = \big((\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) \cap \dots\big)$, where $\varphi_k \in U^0$, always remains a subsp, whether the subset U is a subsp or not.

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26 *Suppose Ω is a subsp of V'. Prove that $\Omega = \{v \in V : \varphi(v) = 0, \forall\varphi \in \Omega\}^0$.*

SOLUTION:

Suppose $U = \{v \in V : \varphi(v) = 0, \forall\varphi \in \Omega\}$, *which is the set of vecs that each $\varphi \in \Omega$ sends to zero in common.*

Then $U^0 = \{v \in V : \varphi(v) = 0, \forall\varphi \in \Omega\}^0$. 又 $U^0 = \{v \in V : \varphi(v) = 0, \forall\varphi \in U^0\}^0$.

Immediately by the COROLLARY in Problem (20,21), we may conclude that $\Omega = U^0$. $\square$

OR. [*Requires Ω finite-dim*] Let $(\varphi_1, \dots, \varphi_m)$ be a basis of Ω. Then by def, $U \subseteq (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)$.

$\forall\varphi \in \Omega, \exists!\, a_i \in \mathbf{F}, \varphi = a_1\varphi_1 + \dots + a_m\varphi_m \Rightarrow \forall v \in (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m), \varphi(v) = 0 \Rightarrow v \in U$.

Hence $(\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) = U$. 又 $\operatorname{span}(\varphi_1, \dots, \varphi_m) = \Omega$. By Problem (23), we are done. $\square$

COROLLARY: For every subsp Ω of V', $\exists!$ subsp U of V such that $\Omega = U^0$.

COMMENT: [*Only in Finite-dim*] Using Problem (31) and the COROLLARY(c) in Problem (22, 23).

Let $B_\Omega = (\varphi_1, \dots, \varphi_m), B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n), B_V = (v_1, \dots, v_m, \dots, v_n)$.

$V' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \overset{\text{(I)}}{=\!=} \operatorname{span}(v_{m+1}, \dots, v_n)^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.

$\Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m) \overset{\text{(II)}}{=\!=} \operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0$; $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \overset{\text{(III)}}{=\!=} \operatorname{span}(v_1, \dots, v_m)^0$.

$\iff U = \operatorname{span}(v_{m+1}, \dots, v_n) = (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m)$. [***Another proof of*** [*3.106*] OR. Problem (24)]

(I) Using the COROLLARY(c), immediately.

(II) NOTICE that each $\operatorname{null}\varphi_k = \operatorname{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n) = U_k$; $\dim U_k = \dim V - 1$.

By (4E 2.C.16), $U = (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_m) = \bigcap_{k=1}^{m} U_k = \operatorname{span}(v_{m+1}, \dots, v_n)$.

Hence $\operatorname{span}(v_{m+1}, \dots, v_n)^0 = U^0 = \Omega = \operatorname{span}(\varphi_1, \dots, \varphi_m)$.

(III) NOTICE that $V' = \Omega \oplus \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = U^0 \oplus \operatorname{span}(v_1, \dots, v_m)^0$.

And that $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) \subseteq \operatorname{span}(v_1, \dots, v_m)^0$.

By the TIPS in (1.C), $\operatorname{span}(\varphi_{m+1}, \dots, \varphi_n) = \operatorname{span}(v_1, \dots, v_m)$.

OR. Similar to (II), let $\Omega = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$, immediately. $\square$

• *Suppose* $T \in \mathcal{L}(V,W), \varphi_k \in V', \psi_k \in W'$.

28 *Prove that* $\operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) \iff \operatorname{range} T = (\operatorname{null} \psi_1) \cap \dots \cap (\operatorname{null} \psi_m)$.

29 *Prove that* $\operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) \iff \operatorname{null} T = (\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m)$.

SOLUTION: Using [3.107], [3.109], Problem (23) and the COROLLARY in Problem (20, 21).

(28) $(\operatorname{range} T)^0 = \operatorname{null} T' = \operatorname{span}(\psi_1, \dots, \psi_m) = \big((\operatorname{null} \psi_1) \cap \dots \cap (\operatorname{null} \psi_m)\big)^0$.

(29) $(\operatorname{null} T)^0 = \operatorname{range} T' = \operatorname{span}(\varphi_1, \dots, \varphi_m) = \big((\operatorname{null} \varphi_1) \cap \dots \cap (\operatorname{null} \varphi_m)\big)^0$. □

COROLLARY: Using the COMMENT in Problem (26).

$\operatorname{null} T = \operatorname{span}(v_1, \dots, v_m) \iff \operatorname{null} T = (\operatorname{null} \varphi_{m+1}) \cap \dots \cap (\operatorname{null} \varphi_n) \iff \operatorname{range} T' = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$.

——Where $B_V = (v_1, \dots, v_m, \dots, v_n) \iff B_{V'} = (\varphi_1, \dots, \varphi_m, \dots, \varphi_n)$.

$\operatorname{range} T = \operatorname{span}(w_1, \dots, w_m) \iff \operatorname{range} T = (\operatorname{null} \psi_{m+1}) \cap \dots \cap (\operatorname{null} \psi_n) \iff \operatorname{null} T' = \operatorname{span}(\psi_{m+1}, \dots, \psi_n)$.

——Where $B_W = (w_1, \dots, w_m, \dots, w_n) \iff B_{W'} = (\psi_1, \dots, \psi_m, \dots, \psi_n)$.

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9 *Let* $B_V = (v_1, \dots, v_n)$, $B_{V'} = (\varphi_1, \cdots, \varphi_n)$. *Then* $\forall \psi \in V', \psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n$.

COROLLARY: *For other* $B'_V = (u_1, \dots, u_n), B'_{V'} = (\rho_1, \dots, \rho_n), \forall \psi \in V', \psi = \psi(u_1)\rho_1 + \dots + \psi(u_n)\rho_n$.

SOLUTION:

$\psi(v) = \psi\Big(\sum_{i=1}^n a_i v_i \Big) = \sum_{i=1}^n a_i \psi(v_i) = \sum_{i=1}^n \psi(v_i)\varphi_i(v) = \big[\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n\big](v)$.

OR. $\big[\psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n\big]\Big(\sum_{i=1}^n a_i v_i \Big) = \psi(v_1)\varphi_1\Big(\sum_{i=1}^n a_i v_i \Big) + \dots + \psi(v_n)\varphi_n\Big(\sum_{i=1}^n a_i v_i \Big)$. □

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13 *Define* $T : \mathbf{R}^3 \to \mathbf{R}^2$ *by* $T(x,y,z) = (4x + 5y + 6z, 7x + 8y + 9z)$.

Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2, \psi_3)$ *denote the dual basis of the standard basis of* $\mathbf{R}^2$ *and* $\mathbf{R}^3$.

(a) *Describe the linear functionals* $T'(\varphi_1), T'(\varphi_2) \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$

For any $(x,y,z) \in \mathbf{R}^3$, $\big(T'(\varphi_1)\big)(x,y,z) = 4x + 5y + 6z$, $\big(T'(\varphi_2)\big)(x,y,z) = 7x + 8y + 9z$.

(b) *Write* $T'(\varphi_1)$ *and* $T'(\varphi_2)$ *as linear combinations of* ψ_1, ψ_2, ψ_3.

$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$, $T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$.

(c) *What is* $\operatorname{null} T'$? *What is* $\operatorname{range} T'$?

$$T(x,y,z) = 0 \iff \begin{cases} 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases} \iff \begin{cases} x + y + z = 0 \\ y = 2z = 0 \end{cases} \iff (x,y,z) \in \operatorname{span}(e_1 - 2e_2 + e_3).$$

Where (e_1, e_2, e_3) is standard basis of $\mathbf{R}^3$.

Let $(e_1 - 2e_2 + e_3, -2e_2, e_3)$ be a basis, with the correspd dual basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Thus $\operatorname{span}(e_1 - 2e_2 + e_3) = \operatorname{null} T \Rightarrow \operatorname{span}(e_1 - 2e_2 + e_3)^0 = \operatorname{span}(\varepsilon_2, \varepsilon_3) = \operatorname{range} T'$.

Note that $\varepsilon_k = \varepsilon_k(e_1)\psi_1 + \varepsilon_k(e_2)\psi_2 + \varepsilon_k(e_3)\psi_3$.

$$\text{And} \left| \begin{array}{l} \varepsilon_2(e_2) = -\frac{1}{2}, \varepsilon_2(e_1) = \varepsilon_2(e_1 - 2e_2 + e_3) + \varepsilon_2(2e_2) - \varepsilon_2(e_3) = 1, \\ \varepsilon_3(e_2) = 0, \varepsilon_3(e_3) = \varepsilon_3(e_1 - 2e_2 + e_3) + \varepsilon_3(2e_2) - \varepsilon_3(e_3) = -1. \end{array} \right.$$

Hence $\varepsilon_2 = \psi_1 - \frac{1}{2}\psi_2$, $\varepsilon_3 = -\psi_1 + \psi_3$. Now $\operatorname{range} T' = \operatorname{span}(\psi_1 - \frac{1}{2}\psi_2, -\psi_1 + \psi_3)$.

OR. $\operatorname{range} T' = \operatorname{span}(T'(\varphi_1), T'(\varphi_2)) = \operatorname{span}(4\psi_1 + 5\psi_2 + 6\psi_3, 7\psi_1 + 8\psi_2 + 9\psi_3)$.

Suppose $T'(x\varphi_1 + y\varphi_2) = (4x + 7y)\varphi_1 + (5x + 8y)\varphi_2 + (6x + 9y)\varphi_3 = 0$.

Then $x + y = 4x + 7y = x = y = 0$. Hence $\operatorname{null} T' = \{0\}$.

OR. $\operatorname{null} T = \operatorname{span}(e_1 - 2e_2 + e_3) \Rightarrow V = \operatorname{span}(-2e_2, e_3) \oplus \operatorname{null} T$.

$\Rightarrow \operatorname{range} T = \{Tx : x \in \operatorname{span}(-2e_2, e_3)\} = \operatorname{span}(T(-2e_2), T(e_3))$

$= \operatorname{span}(-10f_1 - 16f_2, 6f_1 + 9f_2) = \operatorname{span}(f_1, f_2) = \mathbf{R}^2$. Now $\operatorname{null} T' = (\operatorname{range} T)^0 = \{0\}$. □

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24 *Suppose V is finite-dim and U is a subsp of V.*
Prove, using the pattern of [3.104], that $\dim U + \dim U^0 = \dim V$.

Solution:

By Problem (31) and the Comment in Problem (26), $B_U = (v_1, \dots, v_m) \Longleftrightarrow B_{U^0} = (\varphi_{m+1}, \dots, \varphi_n)$. □

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37 *Suppose U is a subsp of V and π is the quotient map. Thus $\pi' \in \mathcal{L}\big((V/U)', V'\big)$.*

(a) *Show that π' is inje*: Because π is surj. Use [3.108].

(b) *Show that* range $\pi' = U^0$: By [3.109](b), range $\pi' = \big(\operatorname{null} \pi\big)^0 = U^0$.

(c) *Conclude that π' is an iso from $(V/U)'$ onto U^0*: Immediately.

Solution: Or. Using (3.E.18), also see (3.E.20).

(a) $\pi'(\varphi) = 0 \Longleftrightarrow \forall v \in V \;(\,\forall v + U \in V\,), \varphi\big(\pi(v)\big) = \varphi(v + U) = 0 \Longleftrightarrow \varphi = 0$.

(b) $\psi \in \operatorname{range} \pi' \Longleftrightarrow \exists\, \varphi \in (V/U)', \psi = \varphi \circ \pi \Longleftrightarrow \operatorname{null} \psi \supseteq U \Longleftrightarrow \psi \in U^0$. Hence range $\pi' = U^0$. □

• *Suppose U is a subsp of V. Prove that $(V/U)'$ and U^0 are iso.* [***Another proof of* [3.106]**]

Solution:

Define $\xi : U^0 \to (V/U)'$ by $\xi(\varphi) = \tilde{\varphi}$, where $\tilde{\varphi} \in (V/U)'$ is defined by $\tilde{\varphi}(v + U) = \varphi(v)$.

We show that ξ is inje and surj.

Inje: $\xi(\varphi) = 0 = \tilde{\varphi} \Rightarrow \forall v \in V \;(\,\forall v + U \in V/U\,), \tilde{\varphi}(v + U) = \varphi(v) = 0 \Rightarrow \varphi = 0$.

Surj: $\Phi \in (V/U)' \Rightarrow \forall u \in U, \Phi(u + U) = \Phi(0 + U) = 0 \Rightarrow U \subseteq \operatorname{null}(\Phi \circ \pi) \Rightarrow \xi(\Phi \circ \pi) = \Phi$. □

Or. Define $\nu : (V/U)' \to U^0$ by $\nu(\Phi) = \Phi \circ \pi$. Now $\nu \circ \xi = I_{U^0}$, $\xi \circ \nu = I_{(V/U)'} \Rightarrow \xi = \nu^{-1}$. □

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4 *Suppose U is a subsp of V and $U \neq V$. Prove that* $\underbrace{\exists\, \varphi \in V' \backslash \{0\}, \varphi(u) = 0 \textit{ for all } u \in U}_{\Longleftrightarrow\, U^0_V \neq \{0\}}$.

Solution:

Let X be such that $V = U \oplus X$. Then $X \neq \{0\}$. Suppose $s \in X$ and $x \neq 0$.

Let Y be such that $X = \operatorname{span}(s) \oplus Y$. Now $V = U \oplus \big(\operatorname{span}(s) \oplus Y\big)$.

Define $\varphi \in V'$ by $\varphi(u + \lambda s + y) = \lambda$. Hence $\varphi \neq 0$ and $\varphi(u) = 0$ for all $u \in U$. □

Or. [*Requires that V is finite-dim*] By [3.106], $\dim U^0 = \dim V - \dim U > 0$. Then $U^0 \neq \{0\}$.

Or. Let $B_V = (\underbrace{u_1, \dots, u_m}_{B_U}, v_1, \dots, v_n)$ with $n \geqslant 1$. Let $B_{V'} = (\psi_1, \dots, \psi_m, \varphi_1, \dots, \varphi_n)$. Let $\varphi = \varphi_i$.

Or. Define $\varphi \in V'$ by $\varphi(u_1) = \cdots = \varphi(u_m) = 0$ and $\varphi(v_1) = \cdots = \varphi(v_n) = 1$. □

Comment: [***Another proof of* [3.108]**]: T is surj $\Longleftrightarrow T'$ is inje.

(a) Suppose T' is inje. Note that $T'(\psi) = 0 \Rightarrow \psi = 0$.

Then $\nexists\, \psi \in W' \backslash \{0\}, \big(T'(\psi)\big)(v) = \psi(Tv) = 0$ for all $w \in \operatorname{range} T$ $(\,\forall v \in V\,)$.

Thus if we assume that range $T \neq W$ then contradicts. Hence range $T = W$.

(b) Suppose T is surj. Then $(\operatorname{range} T)^0 = W^0_W = \{0\} = \operatorname{null} T'$. □

• *Suppose V is a vecsp and U is a subsp of V.*

17 $U^0 = \{\varphi \in V' : U \subseteq \textit{null}\varphi\}$. Noticing $\varphi \in V', U \subseteq \operatorname{null} \varphi \Longleftrightarrow \forall u \in U, \varphi(u) = 0$.

18 $U^0 = V' \Longleftrightarrow \forall \varphi \in V', U \subseteq \operatorname{null} \varphi \Longleftrightarrow U = \{0\}$. [*Which means* $\{0\}^0_V = V'$.]

Or. $U^0 = V' \Longleftrightarrow \dim U^0 = \dim V' = \dim V \Longleftrightarrow \dim U = 0 \Longleftrightarrow U = \{0\}$.

19 $U^0_V = \{0\} = V^0_V \Longleftrightarrow U = V$. By the inverse and contrapositive of Problem (4). Or. By [3.106].

• *Suppose* $V = U \oplus W$. *Define* $\iota : V \to U$ *by* $\iota(u+w) = u$. *Thus* $\iota' \in \mathcal{L}(U', V')$.

(a) *Show that* $\operatorname{null} \iota' = U_U^0 = \{0\}$: $\operatorname{null} \iota' = (\operatorname{range} \iota)_U^0 = U_U^0 = \{0\}$.

(b) *Prove that* $\operatorname{range} \iota' = W_V^0$: $\operatorname{range} \iota' = (\operatorname{null} \iota)_V^0 = W_V^0$.

(c) *Prove that* $\widetilde{\iota'}$ *is an iso from* $U'/\{0\}$ *onto* W^0: By (a), (b) and [3.91](d).

SOLUTION:

(a) $\iota'(\psi) = \psi \circ \iota = 0 \iff U \subseteq \operatorname{null} \psi$.

(b) Note that $W = \operatorname{null}(\iota) \subseteq \operatorname{null}(\psi \circ \iota)$. Then $\psi \circ \iota \in W^0 \Rightarrow \operatorname{range} \iota' \in W^0$.

Suppose $\varphi \in W^0$. Because $\operatorname{null} \iota = W \subseteq \operatorname{null} \varphi$. By TIPS in (3.B), $\varphi = \varphi \circ \iota = \iota'(\varphi)$. □

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36 *Suppose* U *is a subsp of* V. *Define* $i : U \to V$ *by* $i(u) = u$. *Thus* $i' \in \mathcal{L}(V', U')$.

(a) *Show that* $\operatorname{null} i' = U^0$: $\operatorname{null} i' = (\operatorname{range} i)^0 = U^0 \Leftarrow \operatorname{range} i = U$.

(b) *Prove that* $\operatorname{range} i' = U'$: $\operatorname{range} i' = (\operatorname{null} i)_U^0 = \{0\}_U^0 = U'$.

(c) *Prove that* $\widetilde{i'}$ *is an iso from* V'/U^0 *onto* U': By (a), (b) and [3.91](d).

SOLUTION:

(a) $\forall \varphi \in V', i'(\varphi) = \varphi \circ i = \varphi|_U$. Thus $i'(\varphi) = 0 \iff \forall u \in U, \varphi(u) = 0 \iff \varphi \in U^0$.

(b) Suppose $\psi \in U'$. By (3.A.11), $\exists\, \varphi \in V', \varphi|_U = \psi$. Then $i'(\varphi) = \psi$. □

• *Suppose* $T \in \mathcal{L}(V, W)$. *Prove that* $\operatorname{range} T' = (\operatorname{null} T)^0$. [***Another proof of*** **[3.109](b)**]

SOLUTION:

Suppose $\Phi \in (\operatorname{null} T)^0$. Because by (3.B.12), $T|_U : U \to \operatorname{range} T$ is an iso; $V = U \oplus \operatorname{null} T$.

And $\forall v \in V, \exists!\, u_v \in U, w_v \in \operatorname{null} T, v = u_v + w_v$. Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$.

Let $\psi = \Phi \circ (T|_{\operatorname{range} T}^{-1})$. Then $T'(\psi) = \psi \circ T = \Phi \circ (T^{-1}|_{\operatorname{range} T} \circ T|_V)$.

Where $T^{-1}|_{\operatorname{range} T} : \operatorname{range} T \to U$; $\;T : V \to \operatorname{range} T$. Note that $T^{-1}|_{\operatorname{range} T} \circ T|_V = \iota$.

By TIPS in (3.B), $\Phi = \Phi \circ \iota$. Thus $T'(\psi) = \psi \circ T = \Phi \circ \iota = \Phi$. □

• *Suppose* $T \in \mathcal{L}(V, W)$. *Using* [3.108], [3.110].

Now T *is inv* $\iff \left| \begin{array}{c} \operatorname{null} T = \{0\} \iff (\operatorname{null} T)^0 = V' = \operatorname{range} T' \\ \operatorname{range} T = W \iff (\operatorname{range} T)^0 = \{0\} = \operatorname{null} T' \end{array} \right| \iff T'$ *is inv.*

---

15 *Suppose* $T \in \mathcal{L}(V, W)$. *Prove that* $T' = 0 \iff T = 0$.

SOLUTION:

Suppose $T = 0$. Then $\forall \varphi \in W', T'(\varphi) = \varphi \circ T = 0$. Hence $T' = 0$.

Suppose $T' = 0$. Then $\operatorname{null} T' = W' = (\operatorname{range} T)^0$, by [3.107](a).

[*W can be infinite-dim*] By Problem (25),

$\operatorname{range} T = \{w \in W : \varphi(w) = 0, \forall \varphi \in (\operatorname{range} T)^0\} = \{w \in W : \varphi(w) = 0, \forall \varphi \in W'\}$.

Now we prove that if $\forall \varphi \in W', \varphi(w) = 0$, then $w = 0$. So that $\operatorname{range} T = \{0\}$ and we are done.

Assume that $w \neq 0$. Then let U be such that $W = U \oplus \operatorname{span}(w)$.

Define $\psi \in W'$ by $\psi(u + \lambda w) = \lambda$. So that $\psi(w) = 1 \neq 0$. □

OR. [*Only if W is finite-dim*] By [3.106], $\dim \operatorname{range} T = \dim W - \dim(\operatorname{range} T)^0 = 0$. □

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12 NOTICE that $I_{V'} : V' \to V'$. *Now* $\forall \varphi \in V', I_{V'}(\varphi) = \varphi = \varphi \circ I_V = {I_V}'(\varphi)$. *Thus* $I_{V'} = {I_V}'$.

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16 *Suppose V, W are finite-dim. Define Γ by $\Gamma(T) = T'$ for any $T \in \mathcal{L}(V,W)$. Prove that Γ is an iso of $\mathcal{L}(V,W)$ onto $\mathcal{L}(W',V')$.*

SOLUTION: By [3.101], Γ is linear.

Suppose $\Gamma(T) = T' = 0$. By Problem (15), $T = 0$. Thus Γ is inje.

Because V, W are finite-dim. $\dim \mathcal{L}(V,W) = \dim \mathcal{L}(W',V')$. Now Γ inje $\Rightarrow$ inv. □

COMMENT: Let $X = \{T \in \mathcal{L}(V,W) : \operatorname{range} T \text{ is finite-dim}\}$.

Let $Y = \{\mathcal{T} \in \mathcal{L}(W',V') : \operatorname{range} \mathcal{T} \text{ is finite-dim}\}$.

Then $\Gamma|_X$ is an iso of X onto Y, even if V and W are infinite-dim.

The inje of $\Gamma|_X$ is equiv to the inje of Γ, as shown before.

Now we show that $\Gamma|_X$ is surj without the cond that V or W is finite-dim.

Suppose $\mathcal{T} \in Y$. Let $B_{\operatorname{range}\mathcal{T}} = (\varphi_1, \dots, \varphi_m)$, with the correspd $(v_1, \dots, v_m)$. Let $\varphi_k = \mathcal{T}(\psi_k)$.

Let $\mathcal{K}$ be such that $W' = \mathcal{K} \oplus \operatorname{null} \mathcal{T}$. Let $B_{\mathcal{K}} = (\psi_1, \dots, \psi_m)$, with the correspd $(w_1, \dots, w_m)$.

Define $T \in \mathcal{L}(V,W)$ by $Tv_k = w_k, Tu = 0;\ k \in \{1, \dots, m\}, u \in U$.

$\forall \psi \in \operatorname{null} \mathcal{T}, [T'(\psi)](v) = \psi(Tv) = \psi(a_1 w_1 + \cdots + a_p w_p) = 0 = [\mathcal{T}(\psi)](v)$.

$\forall k \in \{1, \dots, m\}, [T'(\psi_k)](v) = \psi_k(Tv) = \psi_k(a_1 w_1 + \cdots + a_m w_m) = a_k = \varphi_k(v) = [\mathcal{T}(\psi)](v)$. □

COMMENT: This is another proof of [3.109(a)]: $\dim \operatorname{range} T = \dim \operatorname{range} T'$.

• (4E 3.F.6) *Suppose $\varphi, \beta \in V'$. Prove that* $\operatorname{null} \varphi \subseteq \operatorname{null} \beta \iff \beta = c\varphi, \exists\, c \in \mathbf{F}$.

COROLLARY: $\operatorname{null} \varphi = \operatorname{null} \beta \iff \beta = c\varphi, \exists\, c \in \mathbf{F} \setminus \{0\}$.

SOLUTION:

Using (3.B.29, 30).

(a) Suppose $\operatorname{null} \varphi \subseteq \operatorname{null} \beta$. Suppose $u \notin \operatorname{null} \beta$, then $u \notin \operatorname{null} \varphi$.

Now $V = \operatorname{null} \beta \oplus \operatorname{span}(u) = \operatorname{null} \varphi \oplus \operatorname{span}(u)$. By TIPS in (1.C), $\operatorname{null} \beta = \operatorname{null} \varphi$. Let $c = \dfrac{\beta(u)}{\varphi(u)}$.

OR. We discuss in two cases. If $\operatorname{null} \varphi = \operatorname{null} \beta$, then we are done.

Otherwise, $\operatorname{null} \beta \neq \operatorname{null} \varphi$. Then $\exists\, u' \in \operatorname{null} \beta \setminus \operatorname{null} \varphi$.

Now $V = \operatorname{null} \varphi \oplus \operatorname{span}(u') = \operatorname{null} \varphi \oplus \operatorname{span}(u)$. $\forall v \in V, v = w + au = w' + bu', \exists!\, w, w' \in \operatorname{null} \varphi$.

Thus $\beta(v) = a\beta(u),\ \varphi(v) = b\varphi(u')$. Let $c = \dfrac{a\beta(u)}{b\varphi(u')}$. We are done.

NOTICE that by (b) below, we have $\operatorname{null} \beta \subseteq \operatorname{null} \varphi,\ u = u'$. Thus contradicts the assumption.

(b) Suppose $\beta = c\varphi$ for some $c \in \mathbf{F}$. If $c = 0$, then $\operatorname{null} \beta = V \supseteq \operatorname{null} \varphi$, we are done.

Otherwise, $\left.\begin{array}{l} \forall v \in \operatorname{null} \varphi, \varphi(v) = 0 = \beta(v) \Rightarrow \operatorname{null} \varphi \subseteq \operatorname{null} \beta \\ \forall v \in \operatorname{null} \beta, \beta(v) = 0 = \varphi(v) \Rightarrow \operatorname{null} \beta \subseteq \operatorname{null} \varphi \end{array}\right\} \Rightarrow \operatorname{null} \varphi = \operatorname{null} \beta$. □

OR. By (3.B.24), $\operatorname{null} \varphi \subseteq \operatorname{null} \beta \iff \exists\, E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi$. (if ***E is inv***, then $\operatorname{null} \varphi = \operatorname{null} \beta$)

Now we show that [P] $\exists\, E \in \mathcal{L}(\mathbf{F}), \beta = E \circ \varphi \iff \exists\, c \in \mathbf{F}, \beta = c\varphi$. [Q].

$[P] \Rightarrow [Q]:$ Let $c = E(1)$. Then $\forall v \in V, \beta(v) = E(\varphi(v)) = \varphi(v)E(1) = c\varphi(v)$. ($\boldsymbol{E(1) \neq 0}$)

$[Q] \Rightarrow [P]:$ Define $E \in \mathcal{L}(\mathbf{F})$ by $E(x) = cx$. Then $\forall v \in V, \beta(v) = c\varphi(v) = E(\varphi(v))$. ($\boldsymbol{c \neq 0}$) □

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5 *Prove that $(V_1 \times \cdots \times V_m)'$ and ${V'}_1 \times \cdots \times {V'}_m$ are iso.* [Using notations in (3.E.2).]

$$\left.\begin{array}{l} \text{Define } \varphi : (V_1 \times \cdots \times V_m)' \to {V'}_1 \times \cdots \times {V'}_m \\ \quad \text{by } \varphi(T) = (T \circ R_1, \dots, T \circ R_m) = ({R'}_1(T), \dots, {R'}_m(T)). \\ \text{Define } \psi : {V'}_1 \times \cdots \times {V'}_m \to (V_1 \times \cdots \times V_m)' \\ \quad \text{by } \psi(T_1, \dots, T_m) = T_1 S_1 + \cdots + T_m S_m = {S'}_1(T_1) + \cdots + {S'}_m(T_m). \end{array}\right\} \Rightarrow \psi = \varphi^{-1}.$$

□

• *In* (3.D.18), $\varphi : V \to \mathcal{L}(\mathbf{F}, V)$ *is an iso. Now we prove that* $[P]\ (v_1, \dots, v_m)$ *is linely inde* $\iff \big(\varphi(v_1), \dots, \varphi(v_m)\big)$ *is linely inde.* $[Q]$

Solution:

$[P] \Rightarrow [Q]$: Notice that φ is inje and by (3.B.9).

Or. Suppose $\vartheta \in \operatorname{span}\big(\varphi(v_1), \dots, \varphi(v_m)\big)$. Let $\vartheta = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m)$.

Then $\vartheta(1) = 0 = a_1 v_1 + \cdots + a_m v_m \Rightarrow a_1 = \cdots = a_m = 0$.

$[Q] \Rightarrow [P]$: Suppose $v \in \operatorname{span}(v_1, \dots, v_m)$. Let $v = 0 = a_1 v_1 + \cdots + a_m v_m$.

Then $\varphi(v) = 0 = a_1\varphi(v_1) + \cdots + a_m\varphi(v_m) \Rightarrow a_1 = \cdots = a_m = 0$. □

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32 *Let* $B_\alpha = (\alpha_1, \dots, \alpha_m), B_\alpha{}' = (\varphi_1, \dots, \varphi_m),\ B_\beta = (v_1, \dots, v_m), B_\beta{}' = (\psi_1, \dots, \psi_m)$.
Prove that $\forall T \in \mathcal{L}(V), T$ *is inv* $\iff$ *the rows of* $A = \mathcal{M}\big(T, B_\alpha, B_\beta\big)$ *form a basis of* $\mathbf{F}^{1,n}$.

Solution: Note that T is inveritlbe $\iff T'$ is inv. And $A^t = \mathcal{M}(T', B_\beta{}', B_\alpha{}')$.

(a) Suppose T is inv, so is T'. Because $\big(T'(\varphi_1), \dots, T'(\varphi_m)\big)$ is linely inde.
Notice that $T'(\varphi_i) = A^t_{1,i}\psi_1 + \cdots + A^t_{m,i}\psi_m$. By the (Δ) part in (4E 3.C.17), the cols of A^t, namely the rows of A, are linely inde.

(b) Suppose the rows of A are linely inde, so are the cols of A^t. Notice that A^t has dim V' cols.
Then $B_{\operatorname{range} T'} = B_{V'} = \big(T'(\varphi_1), \dots, T'(\varphi_m)\big)$. Thus T' is surj. Hence T' is inv, so is T. □

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33 *Suppose* $A \in \mathbf{F}^{m,n}$. *Define* $T : A \to A^t$. *Prove that* T *is an iso of* $\mathbf{F}^{m,n}$ *onto* $\mathbf{F}^{n,m}$

Solution: By [3.111], T is linear. Note that $(A^t)^t = A,\ T \circ T = I$. □

• Define $T \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Tx = xA$, where $A \in \mathbf{F}^{n,n}$, for all $x \in \mathbf{F}^{1,n}$.
Let $B_e = (e_1, \dots, e_n)$ be the standard basis of $\mathbf{F}^{1,n}$, with the dual basis $B_\varphi = (\varphi_1, \dots, \varphi_n)$.
What is $\mathcal{M}(T)$? Because $Te_k = e_k A = \sum_{j=1}^n A_{k,j} e_j = \sum_{j=1}^n A^t_{j,k} e_j$. Now $\mathcal{M}(T) = A^t$.

Note that $A = \mathcal{M}(A, B_e) \in \mathbf{F}^{n,n},\ \mathcal{M}(Te_k) = \mathcal{M}(Te_k, B_e) \in \mathbf{F}^{n,1}$,

$$\mathcal{M}(e_k) = \mathcal{M}(e_k, B_e) \in \mathbf{F}^{n,1},\ \mathcal{M}(e_k A) = \mathcal{M}(e_k A, B_e) \in \mathbf{F}^{n,1}.$$

Now $\mathcal{M}(Te_k) = \mathcal{M}(T)_{\cdot,k} = \mathcal{M}(e_k A) = A^t_{\cdot,k} \Longrightarrow \mathcal{M}(T)\mathcal{M}(e_k) = \mathcal{M}(T)_{\cdot,k} = \mathcal{M}(e_k)\mathcal{M}(A)$.
Then $\mathcal{M}(e_k)\mathcal{M}(A)$ does not make sense. And now??? FIXME: BASIS NOT AGREED

• (4E 3.F.8) *Suppose* $B_V = (v_1, \dots, v_n),\ B_{V'} = (\varphi_1, \dots, \varphi_n)$.

$$\left.\begin{array}{l} \textit{Define } \Gamma : V \to \mathbf{F}^n \textit{ by } \Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)). \\ \textit{Define } \Lambda : \mathbf{F}^n \to V \textit{ by } \Lambda(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n. \end{array}\right\} \Rightarrow \Lambda = \Gamma^{-1}.$$

• (4E 3.F.5) *Suppose* $T \in \mathcal{L}(V, W)$. $B_{\operatorname{range} T} = (w_1, \dots, w_m)$.
Hence $\forall v \in V,\ Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,\ \exists!\, \varphi_1(v), \dots, \varphi_m(v)$,
thus defining $\varphi_i : V \to \mathbf{F}$ *for each* $i \in \{1, \dots, m\}$. *Show that each* $\varphi_i \in V'$.

Solution:

$$\begin{aligned} \forall u, v \in V, \lambda \in \mathbf{F}, T(u + \lambda v) &= \textstyle\sum_{i=1}^m \varphi_i(u + \lambda v) w_i \\ &= Tu + \lambda Tv = \Big(\textstyle\sum_{i=1}^m \varphi_i(u) w_i\Big) + \lambda\Big(\textstyle\sum_{i=1}^m \varphi_i(v) w_i\Big) = \textstyle\sum_{i=1}^m \big(\varphi_i(u) + \lambda\varphi_i(v)\big) w_i. \end{aligned}$$ □

Or. For each $w_i, \exists\, v_i \in V,\ Tv_i = w_i$, then $(v_1, \dots, v_m)$ is linely inde.
Now we have $Tv = a_1 Tv_1 + \cdots + a_m Tv_m,\ \forall v \in V, \exists!\, a_i \in \mathbf{F}$. Let $B_{(\operatorname{range} T)'} = (\psi_1, \dots, \psi_m)$.
Then $\big(T'(\psi_i)\big)(v) = \psi_i \circ T(v) = a_i$. Where $T : V \to \operatorname{range} T;\ \ T' : (\operatorname{range} T)' \to V'$.
Thus for each $i \in \{1, \dots, m\}, \varphi_i = \psi_i \circ T = T'(\psi_i) \in V'$. □

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6 *Define* $\Gamma : V' \to \mathbf{F}^m$ *by* $\Gamma(\varphi) = \big(\varphi(v_1), \dots, \varphi(v_m)\big)$, *where* $v_1, \dots, v_m \in V$.

(a) *Show that* $\operatorname{span}(v_1, \dots, v_m) = V \iff \Gamma$ *is inje.*

(b) *Show that* $(v_1, \dots, v_m)$ *is linely inde* $\iff \Gamma$ *is surj.*

SOLUTION:

(a) NOTICE that $\Gamma(\varphi) = 0 \iff \varphi(v_1) = \cdots = \varphi(v_m) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.

If Γ is inje, then $\Gamma(\varphi) = 0 \iff V = \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$.

If $V = \operatorname{span}(v_1, \dots, v_m)$, then $\Gamma(\varphi) = 0 \iff \operatorname{null}\varphi = \operatorname{span}(v_1, \dots, v_m)$, thus Γ is inje.

(b) Suppose Γ is surj. Then let $\Gamma(\varphi_i) = e_i$ for each i, where $(e_1, \dots, e_m)$ is the standard basis of $\mathbf{F}^m$.

Then by (3.A.4), $(\varphi_1, \dots, \varphi_m)$ is linely inde.

Now $a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow 0 = \varphi_i(a_1 v_1 + \cdots + a_m v_m) = a_i$ for each i.

Suppose $(v_1, \dots, v_m)$ is linely inde. Let $U = \operatorname{span}(\varphi_1, \dots, \varphi_m), B_{U'} = (\varphi_1, \dots, \varphi_m)$.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}^m, \exists!\, \varphi = a_1\varphi_1 + \cdots + a_m\varphi_m$.

Let W be such that $V = U \oplus W$. Now $\forall v \in V, \exists!\, u_v \in U, w_v \in W, v = u_v + w_v$.

Define $\iota \in \mathcal{L}(V, U)$ by $\iota(v) = u_v$. So that $\Gamma(\varphi \circ i -) = (a_1, \dots, a_m)$. □

OR. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$ and let $(\psi_1, \dots, \psi_m)$ be the correspd dual basis.

Define $\Psi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Psi(e_k) = \psi_k$. Then Ψ is an iso.

Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $Te_k = v_k$. Now $T(x_1, \dots, x_m) = T(x_1 e_1 + \cdots + x_m e_m) = x_1 v_1 + \cdots + x_m v_m$.

$\forall \varphi \in V', k \in \{1, \dots, m\}, \big[T'(\varphi)\big](e_k) = \varphi(Te_k) = \varphi(v_k) = \big[\varphi(v_1) \circ \psi_1 + \cdots + \varphi(v_m) \circ \psi_m\big](e_k)$

Now $T'(\varphi) = \varphi(v_1) \circ \psi_1 + \cdots + \varphi(v_m) \circ \psi_m = \Psi\big(\varphi(v_1), \dots, \varphi(v_m)\big) = \Psi\big(\Gamma(\varphi)\big)$. Hence $T' = \Psi \circ \Gamma$.

By (3.B.3), (a) $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V \iff T' = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.

(b) $(v_1, \dots, v_m)$ is linely inde $\iff T$ is inje $\iff T' = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

• (4E 3.F.25) *Define* $\Gamma : V \to \mathbf{F}^m$ *by* $\Gamma(v) = \big(\varphi_1(v), \dots, \varphi_m(v)\big)$, *where* $\varphi_1, \dots, \varphi_m \in V'$.

(c) *Show that* $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff \Gamma$ *is inje.*

(d) *Show that* $(\varphi_1, \dots, \varphi_m)$ *is linely inde* $\iff \Gamma$ *is surj.*

SOLUTION:

(c) NOTICE that $\Gamma(v) = 0 \iff \varphi_1(v) = \cdots = \varphi_m(v) = 0 \iff v \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$.

By Problem (4E 23) and (18), $\operatorname{span}(\varphi_1, \dots, \varphi_m) = V' \iff (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) = \{0\}$.

And $\operatorname{null}\Gamma = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$. Hence Γ inje $\iff \operatorname{null}\Gamma = \{0\} \iff \operatorname{span}(\varphi_1, \dots, \varphi_m) = V'$.

(d) Suppose $(\varphi_1, \dots, \varphi_m)$ is linely inde. Then by Problem (31), $(v_1, \dots, v_m)$ is linely inde.

Thus $\forall (a_1, \dots, a_m) \in \mathbf{F}, \exists!\, v = \sum_{i=1}^m a_i v_i \in V \Rightarrow \varphi_i(v) = a_i, \Gamma(v) = (a_1, \dots, a_m)$. Hence Γ is surj.

Suppose Γ is surj. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbf{F}^m$.

Suppose $v_i \in V$ such that $\Gamma(v_i) = \big(\varphi_1(v_i), \dots, \varphi_m(v_i)\big) = e_i$, for each i.

Then $(v_1, \dots, v_m)$ is linely inde. And $\varphi_j(v_k) = \delta_{j,k}$.

Now $a_1\varphi_1 + \cdots + a_m\varphi_m = 0 \Rightarrow 0(v_i) = a_i$ for each i. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde.

OR. Let $\operatorname{span}(v_1, \dots, v_m) = U$. Then $B_{U'} = (\varphi_1|_U, \dots, \varphi_m|_U)$. Hence $(\varphi_1, \dots, \varphi_m)$ is linely inde. □

OR. Similar to Problem (6), we get $(e_1, \dots, e_m), (\psi_1, \dots, \psi_m)$ and the iso Ψ.

$\forall (x_1, \dots, x_m) \in \mathbf{F}^m, \Gamma'\big(\Psi(x_1, \dots, x_m)\big) = \Gamma'\big(\Psi(x_1 e_1 + \cdots + x_m e_m)\big) = (x_1\psi_1 + \cdots + x_m\psi_m) \circ \Gamma$.

$\forall v \in V, \big[\Gamma'\big(\Psi(x_1, \dots, x_m)\big)\big](v) = \big[x_1\psi_1 + \cdots + x_m\psi_m\big]\big(\Gamma(v)\big) = \big[x_1\varphi_1 + \cdots + x_m\varphi_m\big](v)$.

Now $\Gamma'\big(\Psi(x_1, \dots, x_m)\big) = x_1\varphi_1 + \cdots + x_m\varphi_m$.

Define $\Phi : \mathbf{F}^m \to (\mathbf{F}^m)'$ by $\Phi = \Psi \circ \Gamma$. $\Phi(x_1, \dots, x_m) = x_1\varphi_1 + \cdots + x_m\varphi_m$. Thus by (4E 3.B.3),

(c) the inje of Φ correspds to $(\varphi_1, \dots, \varphi_m)$ spanning V'; 又 $\Phi = \Psi \circ \Gamma$ inje $\iff \Gamma$ inje.

(d) the surj of Φ correspds to $(\varphi_1, \dots, \varphi_m)$ being linely inde; 又 $\Phi = \Psi \circ \Gamma$ surj $\iff \Gamma$ surj. □

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35 *Prove that $\big(\mathcal{P}(\mathbf{F})\big)'$ and $\mathbf{F}^\infty$ are iso.*

Solution:

Define $\theta \in \mathcal{L}\Big(\big(\mathcal{P}(\mathbf{F})\big)', \mathbf{F}^\infty\Big)$ by $\theta(\varphi) = \big(\varphi(1), \varphi(z), \cdots, \varphi(z^n), \cdots\big)$.

Inje: $\theta(\varphi) = 0 \Rightarrow \forall z^k$ in the basis $(1, z, \dots, z^n)$ of $\mathcal{P}_n(\mathbf{F})$ $(\ \forall n\ )$, $\varphi(z^k) = 0 \Rightarrow \varphi = 0$.

$\big[$ Notice that $\forall p \in \mathcal{P}(\mathbf{R}), \exists!\, a_i \in \mathbf{F}, m = \deg p,\ p = a_0 z + a_1 z + \cdots + a_m z^m \in \mathcal{P}_m(\mathbf{F})$. $\big]$

Surj: $\forall \big(a_k\big)_{k=1}^\infty \in \mathbf{F}^\infty$, let ψ be such that $\forall k, \psi(z^k) = a_k$ $\big[$ by [3.5] $\big]$ and thus $\theta(\psi) = \big(a_k\big)_{k=1}^\infty$. □

Comment: Notice that $\mathcal{P}(\mathbf{F})$ and $\mathbf{F}^\infty$ are not iso, so are $\mathcal{P}(\mathbf{F})$ and $\big(\mathcal{P}(\mathbf{F})\big)'$

But if we let $\mathbf{F}^\infty = \big\{(a_1, \cdots, a_n, \underbrace{0, \cdots, 0, \cdots}_{\text{all zero}}) \in \mathbf{F}^\infty \ \big|\ \exists!\, n \in \mathbf{N}^+\big\}$. Then $\mathcal{P}(\mathbf{F})$ and $\mathbf{F}^\infty$ are iso.

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7 *Show that the dual basis of $(1, x, \dots, x^m)$ of $\mathcal{P}_m(\mathbf{R})$ is $(\varphi_0, \varphi_1, \dots, \varphi_m)$, where $\varphi_k(p) = \dfrac{p^{(k)}(0)}{k!}$.*

Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

Solution:

$$\forall j, k \in \mathbf{N},\ \big(x^j\big)^{(k)} = \begin{cases} j(j-1)\dots(j-k+1)\cdot x^{(j-k)}, & j \geq k. \\ j(j-1)\dots(j-j+1) = j! & j = k. \\ 0, & j \leqslant k. \end{cases} \Bigg|\ \text{Then } (x^j)^{(k)}(0) = \begin{cases} 0\ ,\ j \neq k. \\ k!\ ,\ j = k. \end{cases}$$ □

Or. Because $\forall j, k \in \{1, \dots, m\}$ such that $j \neq k$, $\varphi_k(x^j) = \frac{(x^j)^{(k)}(0)}{k!} = \frac{0}{k!} = 0$; $\varphi_k(x^k) = \frac{(x^k)^{(k)}(0)}{k!} = 1$.

Thus $\frac{p^{(k)}(0)}{k!}$ act exactly the same as φ_k on the same basis $(1, \dots, x^m)$, hence is just another def of φ_k□

Example: Suppose $m \in \mathbf{N}^+$. By [2.C.10], $B = \big(1, x-5, \dots, (x-5)^m\big)$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

Let $\varphi_k = \dfrac{p^{(k)}(5)}{k!}$ for each $k = 0, 1, \dots, m$. Then $(\varphi_0, \varphi_1, \dots, \varphi_m)$ is the dual basis of B.

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34 *The double dual space of V, denoted by V'', is defined to be the dual space of V'.*
In other words, $V'' = \mathcal{L}(V', \mathbf{F})$. Define $\Lambda : V \to V''$ by $(\Lambda v)(\varphi) = \varphi(v)$.

(a) *Show that Λ is a linear map from V to V''.*

(b) *Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.*

(c) *Show that if V is finite-dim, then Λ* ***is an iso from V onto V''****.*

Suppose V is finite-dim. Then V and V' are iso, and finding an iso from V onto V' generally requires choosing a basis of V. In contrast, the iso Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

Solution:

(a) $\forall \varphi \in V', v, w \in V, a \in \mathbf{F}$, $\big(\Lambda(v + aw)\big)(\varphi) = \varphi(v + aw) = \varphi(v) + a\varphi(w) = (\Lambda v)(\varphi) + a(\Lambda w)(\varphi)$.

Thus $\Lambda(v + aw) = \Lambda v + a\Lambda w$. Hence Λ is linear.

(b) $\big(T''(\Lambda v)\big)(\varphi) = \big((\Lambda v) \circ T'\big)(\varphi) = (\Lambda v)\big(T'(\varphi)\big)$

$= \big(T'(\varphi)\big)(v) = (\varphi \circ T)(v) = \varphi(Tv) = \big(\Lambda(Tv)\big)(\varphi)$.

Hence $T''(\Lambda v) = \big(\Lambda(Tv)\big) \Rightarrow T'' \circ \Lambda = \Lambda \circ T$.

(c) Suppose $\Lambda v = 0$. Then $\forall \varphi \in V'$, $(\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0$. Thus Λ is inje.

又 Because V is finite-dim. $\dim V = \dim V' = \dim V''$. Hence Λ is an iso. □

Ended

- **Tips:** *Suppose $p \in \mathcal{P}(\mathbf{F}), \deg p \leqslant m$ and p has at least $(m+1)$ distinct zeros.*
  Then by the contrapositive of [4.12], 又 $\deg p = m$, we conclude that $m < 0$. Hence $p = 0$.

  Or. We show that if p has at least m distinct zeros, then either $p = 0$ or $\deg p \geqslant m$.
  If $p = 0$ then we are done. If not, then suppose p has exactly n distinct zeros $\lambda_1, \dots, \lambda_n$.
  Because $\exists ! \alpha_i \geqslant 1, q \in \mathcal{P}(\mathbf{F})$, and $q \neq 0$, such that $p(z) = \left[(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_n)^{\alpha_n}\right] q(z)$. □
- **Comment:** *Notice that by [4.17], some term of the poly factorization might not in the form $(x-\lambda_k)^{\alpha_k}$.*

- **Note For [4.7]:** *the uniqness of coeffs of polys* [*Another proof*]

  If a poly had two different sets of coeffs, then subtracting the two representations would give a poly with some nonzero coeffs but infinitely many zeros. By Tips . □

- **Note For [4.8]:** *division algorithm for polys* [*Another proof*]

  Suppose $\deg p \geqslant \deg s$. Then $\left(\underbrace{1, z, \dots, z^{\deg s - 1}}_{\text{of length } \deg s}, \underbrace{s, zs, \cdots, z^{\deg p - \deg s} s}_{\text{of length } (\deg p - \deg s + 1)} \right)$ is a basis of $\mathcal{P}_{\deg p}(\mathbf{F})$.

  Because $q \in \mathcal{P}(\mathbf{F}), \exists ! a_i, b_j \in \mathbf{F}$,
  $q = a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1} + b_0 s + b_1 zs + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} s$
  $= \underbrace{a_0 + a_1 z + \cdots + a_{\deg s - 1} z^{\deg s - 1}}_{r} + s \underbrace{\left(b_0 + b_1 z + \cdots + b_{\deg p - \deg s} z^{\deg p - \deg s} \right)}_{q}$. Note that r, q are unique. □

- **Note For [4.11]:** *each zero of a poly corresponds to a degree-one factor;* [*Another proof*]

  First suppose $p(\lambda) = 0$. Write $p(z) = a_0 + a_1 z + \cdots + a_m z^m, \exists ! a_0, a_1, \dots, a_m \in \mathbf{F}$ for all $z \in \mathbf{F}$.
  Then $p(z) = p(z) - p(\lambda) = a_1(z-\lambda) + \cdots + a_m(z^m - \lambda^m)$ for all $z \in \mathbf{F}$.
  Hence $\forall k \in \{1, \dots, m\}, z^k - \lambda^k = (z-\lambda)(z^{k-1}\lambda^0 + z^{k-2}\lambda^1 + \cdots + z^{k-(j+1)}\lambda^j + \cdots + z\lambda^{k-2} + z^0\lambda^{k-1})$.
  Thus $p(z) = \sum_{j=1}^m a_j (z-\lambda) \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z-\lambda) \sum_{j=1}^m a_j \sum_{i=1}^k \lambda^{i-1} z^{k-i} = (z-\lambda) q(z)$. □

- **Note For [4.13]:** *Every nonconst poly with complex coefficients has a zero in* $\mathbf{C}$. [*Another proof*]

  For any $w \in \mathbf{C}, k \in \mathbf{N}^+$, by polar coordinates, $\exists r \geqslant 0, \theta \in \mathbf{R}, r(\cos\theta + \mathrm{i}\sin\theta) = w$.
  By De Moivre' theorem, $w^k = [r(\cos\theta + \mathrm{i}\sin\theta)]^k = r^k(\cos k\theta + \mathrm{i}\sin k\theta)$.
  Hence $\left(r^{1/k}(\cos\frac{\theta}{k} + \mathrm{i}\sin\frac{\theta}{k})\right)^k = w$. Thus every complex number has a k^{th} *root*.

  Suppose a nonconst $p \in \mathcal{P}(\mathbf{C})$ with highest-order nonzero term $c_m z_m$.
  Then $|p(z)| \to \infty$ as $|z| \to \infty$ $\left(\text{because } \dfrac{|p(z)|}{|z_m|} \to |c_m| \text{ as } |z| \to \infty \right)$.
  Thus the continuous function $z \to |p(z)|$ has a global minimum at some point $\zeta \in \mathbf{C}$.
  To show that $p(\zeta) = 0$, assume $p(\zeta) \neq 0$. Define $q \in \mathcal{P}(\mathbf{C})$ by $q(z) = \dfrac{p(z+\zeta)}{p(\zeta)}$.
  The function $z \to |q(z)|$ has a global minimum value of 1 at $z = 0$.
  Write $q(z) = 1 + a_k z^k + \cdots + a_m z^m$, where $k \in \mathbf{N}^+$ is the smallest such that $a_k \neq 0$.
  Let $\beta \in \mathbf{C}$ be such that $\beta^k = -\dfrac{1}{a_k}$.
  There is a const $c > 1$ so that if $t \in (0,1)$, then $|q(t\beta)| \leqslant |1 + a_k t^k \beta^k| + t^{k+1} c = 1 - t^k(1 - tc)$.
  Now letting $t = 1/(2c)$, we get $|q(t\beta)| < 1$. Contradicts. Hence $p(\zeta) = 0$, as desired. □

• (4E 4 2) *Prove that if $w, z \in \mathbf{C}$, then $\big||w| - |z|\big| \leqslant |w - z|$.*

SOLUTION:

$$\begin{aligned} |w-z|^2 &= (w-z)(\overline{w}-\overline{z}) \\ &= |w|^2 + |z|^2 - (w\overline{z} + \overline{w}z) \\ &= |w|^2 + |z|^2 - (\overline{\overline{w}z} + \overline{w}z) \\ &= |w|^2 + |z|^2 - 2Re(\overline{w}z) \\ &\geqslant |w|^2 + |z|^2 - 2|\overline{w}z| \\ &= |w|^2 + |z|^2 - 2|w||z| = \big||w| - |z|\big|^2. \end{aligned}$$

OR. $\left.\begin{aligned} |w| = |w - z + z| \leqslant |w - z| + |z| \Rightarrow |w| - |z| \leqslant |w - z| \\ |z| = |z - w + w| \leqslant |z - w| + |w| \Rightarrow |z| - |w| \leqslant |w - z| \end{aligned}\right\}$

Geometric interpretation: The length of each side of a triangle is greater than or equal to the difference of the lengths of the two other sides.

$\square$

• (4E 4 3) *Suppose $\mathbf{F} = \mathbf{C}$, $\varphi \in V'$. Define $\sigma : V \to \mathbf{R}$ by $\sigma(v) = \operatorname{Re}\varphi(v)$ for each $v \in V$. Show that $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$ for all $v \in V$.*

SOLUTION: Notice that $\varphi(v) = \operatorname{Re}\varphi(v) + \mathrm{i}\operatorname{Im}\varphi(v) = \sigma(v) + \mathrm{i}\operatorname{Im}\varphi(v)$.
又 $\operatorname{Re}\varphi(\mathrm{i}v) = \operatorname{Re}\big(\mathrm{i}\varphi(v)\big) = -\operatorname{Im}\varphi(v) = \sigma(\mathrm{i}v)$. Hence $\varphi(v) = \sigma(v) - \mathrm{i}\sigma(\mathrm{i}v)$. $\square$

---

4 *Suppose $m, n \in \mathbf{N}^+$ with $m \leqslant n$, $\lambda_1, \dots, \lambda_m \in \mathbf{F}$. Prove that $\exists\, p \in \mathcal{P}(\mathbf{F})$, $\deg p = n$, the zeros of p are $\lambda_1, \dots, \lambda_m$.*

SOLUTION: Let $p(z) = (z - \lambda_1)^{n-(m-1)}(z - \lambda_2)\cdots(z - \lambda_m)$. $\square$

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5 *Suppose $m \in \mathbf{N}$, and $z_1, \dots, z_{m+1}$ are distinct in $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$. Prove that $\exists!\, p \in \mathcal{P}_m(\mathbf{F})$, $p(z_k) = w_k$ for each $k \in \{1, \dots, m+1\}$.*

SOLUTION:

Define $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by $Tq = \big(q(z_1), \dots, q(z_m), q(z_{m+1})\big)$. Moreover, T is linear.

We now show that T is surj, so that such p exists; and that T is inje, so that such p is unique.
Inje: $Tq = 0 \iff q(z_1) = \cdots = q(z_m) = q(z_{m+1}) = 0 \iff q = 0$, by TIPS .
Surj: $\dim \operatorname{range} T = \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T = m + 1 = \dim \mathbf{F}^{m+1}$ 又 $\operatorname{range} T \subseteq \mathbf{F}^{m+1} \Rightarrow T$ is surj. $\square$

OR. Let $p_1 = 1, p_k(z) = \prod_{i=1}^{k-1}(z - z_i) = (z - z_1)\cdots(z - z_{k-1})$ for each $k \in \{2, \dots, m+1\}$.
By (2.C.10), $B_p = (p_1, \dots, p_{m+1})$ is a basis of $\mathcal{P}_m(\mathbf{F})$. Let $B_e = (e_1, \dots, e_{m+1})$ be the std basis of $\mathbf{F}^{m+1}$.
NOTICE that $Tp_1 = (1, \dots, 1), Tp_k = \Big(\prod_{i=1}^{k-1}(z_1 - z_i), \dots, \underbrace{\prod_{i=1}^{k-1}(z_j - z_i)}_{j^{th}\text{ entry}}, \dots, \prod_{i=1}^{k-1}(z_{m+1} - z_i)\Big)$.

And that $\prod_{i=1}^{k-1}(z_j - z_i) = 0 \iff j \leqslant k - 1$, because $z_1, \dots, z_{m+1}$ are distinct.

$$\text{Thus } \mathcal{M}(T, B_P, B_E) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & A_{2,2} & 0 & \cdots & 0 \\ 1 & A_{3,2} & A_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{m+1,2} & A_{m+1,3} & \cdots & A_{m+1,m+1} \end{pmatrix}.$$

Where $A_{j,k} = \prod_{i=1}^{k-1}(z_j - z_i) \neq 0$ for all $j > k - 1 \geqslant 1$. The rows of $\mathcal{M}(T)$ is linely inde.
By (4E 3.C.17) 又 $\dim \mathcal{P}_m(\mathbf{F}) = \dim \mathbf{F}^{m+1}$; OR By (3.F.32); T is inv. $\square$

---

2 *Suppose $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

SOLUTION: $x^m, x^m + x^{m-1} \in U$ but $\deg\big[(x^m + x^{m-1}) - (x^m)\big] \neq m \Rightarrow (x^m + x^{m-1}) - (x^m) \notin U$. $\square$

---

3 *Suppose $m \in \mathbf{N}^+$. Is the set $U = \{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : 2 \mid \deg p\}$ a subsp of $\mathcal{P}(\mathbf{F})$?*

SOLUTION: $x^2, x^2 + x \in U$ but $\deg\left[(x^2+x) - (x^2)\right]$ is odd and hence $(x^2+x) - (x^2) \notin U$. $\square$

---

6 *Suppose nonzero $p \in \mathcal{P}_m(\mathbf{F})$ has degree m. Prove that*
$[P]$ p has m distinct zeros $\Longleftrightarrow$ p and its derivative p' have no zeros in common $[Q]$.

SOLUTION:

(a) Suppose p has m distinct zeros. And $\deg p = m$. By [4.14], $\exists!\, c, \lambda_i \in \mathbf{R}, p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$.
If $m = 0$, then $p = c \neq 0 \Rightarrow p$ has no zeros, and $p' = 0$, we are done.
If $m = 1$, then $p(z) = c(z - \lambda_1)$, and $p' = c$ has no zeros, we are done.
For each $j \in \{1, \dots, m\}$, let $q_j \in \mathcal{P}_{m-1}(\mathbf{F})$ be such that $p(z) = (z - \lambda_j)q_j \Rightarrow q_j(\lambda_j) \neq 0$.
Now $p'(z) = (z - \lambda_j){q_j}'(z) + q_j(z) \Rightarrow p'(\lambda_j) = q_j(\lambda_j) \neq 0$, as desired.

OR. To prove $[P] \Rightarrow [Q]$, we prove $\neg[Q] \Rightarrow \neg[P]$:
Suppose $p(z) = (z-\lambda)q(z),\ p'(z) = (z-\lambda)r(z)$.
又 $p'(z) = (z-\lambda)q'(z) + q(z)$. Now $p'(\lambda) = q(\lambda) = 0 \Rightarrow p(z) = (z-\lambda)^2 s(z)$.
Hence p has strictly less than m distinct zeros.

(b) To prove $[Q] \Rightarrow [P]$, we prove $\neg[P] \Rightarrow \neg[Q]$:
Because nonzero $p \in \mathcal{P}_m(\mathbf{F})$, we suppose $\lambda_1, \dots, \lambda_M$ are the distinct zeros of p, where $M < m$.
By Pigeon Hole Principle, $\exists\, \lambda_k$ such that $p(z) = (z - \lambda_k)^2 q(z)$ for some $q \in \mathcal{P}(\mathbf{F})$.
Hence $p'(z) = 2(z-\lambda_k)q(z) + (z-\lambda_k)^2 q'(z) \Rightarrow p'(\lambda_k) = 0 = p(\lambda_k)$. $\square$

---

7 *Prove that every $p \in \mathcal{P}(\mathbf{R})$ of odd degree has a zero.*

SOLUTION:

Using the notation and proof of [4.17]. $\deg p = 2M + m$ is odd $\Rightarrow m$ is odd. Hence λ_1 exists. $\square$

OR. Using calculus only.
Suppose $p \in \mathcal{P}_m(\mathbf{F}), \deg p = m, m$ is odd.
Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then $a_m \neq 0$. Denote $|a_m|^{-1} a_m$ by δ
Write $p(x) = x^m \left(\dfrac{a_0}{x^m} + \dfrac{a_1}{x^{m-1}} + \cdots + \dfrac{a_{m-1}}{x} + a_m \right)$.
Thus $p(x)$ is continuous, and $\lim\limits_{x \to -\infty} p(x) = -\delta\infty$; $\lim\limits_{x \to \infty} p(x) = \delta\infty$.
Hence we conclude that p has at least one real zero. $\square$

---

9 *Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by $q(z) = p(z)\overline{p(\overline{z})}$. Prove that $q \in \mathcal{P}(\mathbf{R})$.*

SOLUTION:

NOTICE that by [4.5], $\overline{z}^n = \overline{z^n}$.
Suppose $q(z) = a_n z^n + \cdots + a_1 z + a_0 \Rightarrow q(\overline{z}) = a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0 \Rightarrow \overline{q(\overline{z})} = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}$.

Note that $q(z) = p(z)\overline{p(\overline{z})} = \overline{p(\overline{z})}p(z) = \overline{p(\overline{z})\overline{p(\overline{\overline{z}})}} = \overline{q(\overline{z})}$. Hence for each $a_k, \overline{a_k} = a_k \Rightarrow a_k \in \mathbf{R}$. $\square$

OR. Suppose $p(z) = a_m z^m + \cdots + a_1 z + a_0$. Now $\overline{p(\overline{z})} = \overline{a_m} z^m + \cdots + \overline{a_1} z + \overline{a_0}$.
NOTICE that $q(z) = p(z)\overline{p(\overline{z})} = \sum_{k=0}^{2} m \left(\sum_{i+j=k} a_i \overline{a_j} \right) z^k$.
NOTICE that by [4.5], $z - \overline{z} = 2(\operatorname{Im} z) \Rightarrow z = \overline{z} + 2(\operatorname{Im} z)$. So that $z = \overline{z} \Longleftrightarrow \operatorname{Im} z = 0 \Longleftrightarrow z \in \mathbf{R}$.
Now for each $k \in \{0, \dots, 2m\}$, $\overline{\sum_{i+j=k} a_i \overline{a_j}} = \sum_{i+j=k} \overline{a_i \overline{a_j}} = \sum_{i+j=k} a_j \overline{a_i} = \sum_{i+j=k} a_i \overline{a_j} \in \mathbf{R}$. $\square$

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8 *For $p \in \mathcal{P}(\mathbf{R})$, define $Tp : \mathbf{R} \to \mathbf{R}$ by* $(Tp)(x) = \begin{cases} \dfrac{p(x)-p(3)}{x-3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}$

Show that (a) $Tp \in \mathcal{P}(\mathbf{R})$ *for all* $p \in \mathcal{P}(\mathbf{R})$ *and that* (b) $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ *is linear.*

SOLUTION:

(a) For $x \neq 3, T(x^n) = \dfrac{x^n - 3^n}{x-3} = \sum\limits_{i=1}^{n} 3^{i-1}x^{n-i}$. For $x = 3, T(x^n) = 3^{n-1} \cdot n$.

Note that if $x = 3$, then $\sum_{i=1}^{n} 3^{i-1}x^{n-i} = \sum_{i=1}^{n} 3^{n-1} = 3^{n-1} \cdot n$.

Hence $T(x^n) = \sum_{i=1}^{n} 3^{i-1}x^{n-i} \Rightarrow T(x^n) \in \mathcal{P}(\mathbf{R})$.

(b) Now we show that T is linear: $\forall p, q \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}$,

$$T(p+\lambda q)(x) = \left\{ \begin{array}{ll} \dfrac{(p+\lambda q)(x) - (p+\lambda q)(3)}{x-3}, & \text{if } x \neq 3, \\ (p+\lambda q)'(3), & \text{if } x = 3 \end{array} \right| = \big[T(p) + \lambda T(q)\big](x) \text{ for all } x \in \mathbf{R}.$$ □

OR. (a) Note that $\exists!\, q \in \mathcal{P}(\mathbf{R}), p(x) - p(3) = (x-3)q(z) \Rightarrow q(x) = \dfrac{p(x)-p(3)}{x-3}$.

$p'(x) = \big(p(x) - p(3)\big)' = \big((x-3)q(x)\big)' = q(x) + (x-3)q'(x)$.

Hence $p'(3) = q(3)$. Now $Tp = q \in \mathcal{P}(\mathbf{R})$.

(b) $\forall p_1, p_2 \in \mathcal{P}(\mathbf{R}), \lambda \in \mathbf{R}, \exists!\, q_1, q_2 \in \mathcal{P}(\mathbf{R})$,

$p_1(x) - p_1(3) = (x-3)q_1(x)$ and $p_2(x) - p_2(3) = (x-3)q_2(x)$.

By (a), $Tp_1 = q_1, Tp_2 = q_2$. Note that $(p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x-3)(q_1 + \lambda q_2)(x)$.

Hence by the uniqnes of $q_1 + \lambda q_2$ for $p_1 + \lambda p_2$, we must have $T(p_1 + \lambda p_2) = q_1 + \lambda q_2$. □

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11 *Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.*

(a) *Show that* $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) *Find a basis of* $\mathcal{P}(\mathbf{F})/U$.

SOLUTION: NOTICE that $pq \neq p \circ q$, see (4E 3.A.10).

U is a subsp of $\mathcal{P}(\mathbf{F})$ because $\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, ps_1 + \lambda ps_2 = p(s_1 + \lambda s_2) \in U$.

If $\deg p = 0$, then $U = \mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F})/U = \{0\}$, with the unique basis $()$. Suppose $\deg p \geqslant 1$.

(a) By [4.8], $\forall s \in \mathcal{P}(\mathbf{F}), \exists!\, r \in \mathcal{P}_{\deg p - 1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F})\ [\,\exists!\, pq \in U\,], s = (p)q + (r)$.

Thus $\mathcal{P}(\mathbf{F}) = U \oplus \mathcal{P}_{\deg p-1}(\mathbf{F})$. By the NOTE FOR [3.91] in (3.E), $\mathcal{P}(\mathbf{F})/U$ and $\mathcal{P}_{\deg p-1}(\mathbf{F})$ are iso.

OR. Define $R : \mathcal{P}(\mathbf{F}) \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ by $R(s) = r$ for all $s \in \mathcal{P}(\mathbf{F})$ We show that R is linear.

$\forall s_1, s_2 \in \mathcal{P}(\mathbf{F}), \lambda \in \mathbf{F}, \exists!\, r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q_1, q_2 \in \mathcal{P}(\mathbf{F}), s_1 = (p)q_1 + (r_1);\ s_2 = (p)q_2 + (r_2)$.

又 $\exists!\, r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), q \in \mathcal{P}(\mathbf{F}), (s_1 + \lambda s_2) = (p)q + (r) = (p)(q_1 + \lambda q_2) + (r_1 + \lambda r_2)$.

Note that $r_1, r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F}) \Rightarrow r_1 + \lambda r_2 \in \mathcal{P}_{\deg p-1}(\mathbf{F})$.

OR Note that $\deg(r_1 + \lambda r_2) \leqslant \max\{\deg r_1, \deg(\lambda r_2)\} \leqslant \max\{\deg r_1, \deg r_2\} < \deg p$.

By the uniqnes part of [4.8], $s = s_1 + \lambda s_2;\ r = r_1 + \lambda r_2$. Thus $R(s_1 + \lambda s_2) = R(s_1) + \lambda R(s_2)$.

Because $Rs = 0 \iff s = pq, \exists!\, q \in \mathcal{P}(\mathbf{F}) \iff s \in U$. And $\forall r \in \mathcal{P}_{\deg p-1}(\mathbf{F}), Rr = r$.

Now $\operatorname{null} R = U$, $\operatorname{range} R = \mathcal{P}_{\deg p-1}(\mathbf{F})$.

Hence $\tilde{R} : \mathcal{P}(\mathbf{F})/U \to \mathcal{P}_{\deg p-1}(\mathbf{F})$ is defined by $\tilde{R}(s+U) = Rs$. By [3.91(d)], $\tilde{R}$ is an iso.

(b) For each $k \in \{0, 1, \dots, \deg p - 1\}, \tilde{R}(z^k + U) = R(z^k) = z^k \Rightarrow \tilde{R}^{-1}(z^k) = z^k + U$.

Thus $(1 + U, z + U, \dots, z^{\deg p-1} + U)$ can be a basis of $\mathcal{P}(\mathbf{F})/U$. □

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10 *Suppose $m \in \mathbf{N}, p \in \mathcal{P}_m(\mathbf{C})$ is such that $p(x_k) \in \mathbf{R}$ for each of distinct $x_0, x_1, \dots, x_m \in \mathbf{R}$. Prove that $p \in \mathcal{P}(\mathbf{R})$.*

SOLUTION:

By TIPS and Problem (5), $\exists!\, q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_k) = p(x_k)$. Hence $p = q$. $\square$

OR. Using the Lagrange Interpolating Polynomial.

Define $q(x) = \displaystyle\sum_{j=0}^{m} \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)} p(x_j)$.

又 Each x_j, $p(x_j) \in \mathbf{R} \Rightarrow q \in \mathcal{P}_m(\mathbf{R})$. Notice that $q(x_k) = 1 \cdot p(x_k) \Rightarrow (q-p)(x_k) = 0$ for each x_k.
Then $(q-p)$ has $(m+1)$ zeros, while $(q-p) \in \mathcal{P}_m(\mathbf{C})$. By TIPS , $q - p = 0 \Rightarrow p = q \in \mathcal{P}(\mathbf{R})$. $\square$

- (4E 4 13) *Suppose nonconst $p, q \in \mathcal{P}(\mathbf{C})$ have no zeros in common. Let $m = \deg p$, $n = \deg q$. Define $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by $T(r,s) = rp + sq$. Prove that T is an iso.*
  COROLLARY: *$\exists!\, r \in \mathcal{P}_{n-1}(\mathbf{C}), s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.*

SOLUTION:

T is linear because $\forall\, r_1, r_2 \in \mathcal{P}_{n-1}(\mathbf{C}), s_1, s_2 \in \mathcal{P}_{m-1}(\mathbf{C}), \lambda \in \mathbf{F}$,
$T\big((r_1,s_1) + \lambda(r_2,s_2)\big) = T(r_1 + \lambda r_2, s_1 + \lambda s_2) = (r_1 + \lambda r_2)p + (s_1 + \lambda s_2)q = T(r_1,s_1) + \lambda T(r_2,s_2)$.

Let $\lambda_1, \dots, \lambda_M$ and $\mu_1, \dots, \mu_N$ be the distinct zeros of p and q respectively. NOTICE that $M \leqslant m, N \leqslant n$.
Note that the contrapositive of [4.13], $M = 0 \iff m = 0 \Rightarrow s = 0 \iff r = 0 \Leftarrow n = 0 \iff N = 0$.
Now suppose $M, N \geqslant 1$. We show that $s = 0$. Showing $r = 0$ is almost the same.
Write $p(z) = a(z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}$. $(\ \exists!\, \alpha_j \geqslant 1, a \in \mathbf{F}.\ )$ Let $\max\{\alpha_1, \dots, \alpha_M\} = A$.
For each $D \in \{0, 1, \dots, A-1\}$, let $I_{D,\alpha} = \{\gamma_{D,1}, \dots, \gamma_{D,J}\}$ be such that each $\alpha_{\gamma_{D,j}} \geqslant D + 1$.
Note that $I_{A-1,\alpha} \subseteq \cdots \subseteq I_{0,\alpha} = \{1, \dots, M\}$. Because $rp + sq = 0 \Rightarrow (rp+sq)^{(k)} = 0$ for all $k \in \mathbf{N}^+$.
We use induction by D to show that $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$ for each $D \in \{0, \dots, A-1\}$.
NOTICE that $p^{(D)}(\lambda_\gamma) = 0$ for each $D \in \{0, \dots, A-1\}$ and each $\lambda_\gamma \in I_{D,\alpha}$. (Δ)

(i) $D = 0$. $(rp+sq)(\lambda_{\gamma_{0,j}}) = (sq)(\lambda_{\gamma_{0,j}}) = s(\lambda_{\gamma_{0,j}}) = 0$.
  $D = 1$. $(rp+sq)'(\lambda_{\gamma_{1,j}}) = (r'p + rp')(\lambda_{\gamma_{1,j}}) + (s'q + sq')(\lambda_{\gamma_{1,j}}) = (s'q)(\lambda_{\gamma_{1,j}}) = s'(\lambda_{\gamma_{1,j}}) = 0$.

(ii) $2 \leqslant D \leqslant A-1$. Assume that $s^{(d)}(\lambda_{\gamma_{d,j}}) = 0$ for each $d \in \{1, \dots, D-1\}$ and each $\lambda_{\gamma_{d,j}} \in I_{d,\alpha}$.
  $\Big($ Because $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}.\ \Big)$ (Δ)

  Now
$$\begin{aligned}[rp+sq]^{(D)}(\lambda_{\gamma_{D,j}}) &= \big[C_D^D r^{(D)} p^{(0)} + \cdots + C_D^d r^{(d)} p^{(D-d)} + \cdots + C_D^0 r^{(0)} p^{(D)}\big](\lambda_{\gamma_{D,j}}) \\ &\quad + \big[C_D^D s^{(D)} q^{(0)} + \cdots + C_D^d s^{(d)} q^{(D-d)} + \cdots + C_D^0 s^{(0)} q^{(D)}\big](\lambda_{\gamma_{D,j}}) \\ &= \big[C_D^D s^{(D)} q^{(0)}\big](\lambda_{\gamma_{D,j}}).\end{aligned}$$
  Where each $\lambda_{\gamma_{D,j}} \in I_{D,\alpha} \subseteq I_{D-1,\alpha}$.

  Hence $s^{(D)}(\lambda_{\gamma_{D,j}}) = 0$. The assumption holds for all $D \in \{0, \dots, A-1\}$.

NOTICE that $\forall k = \{0, \dots, A-2\}, s^{(k)}$ and $s^{(k+1)}$ have zeros $\{\lambda_{\gamma_{k+1,1}}, \dots, \lambda_{\gamma_{k+1,J}}\}$ in common.
Now $\forall D \in \{1, A-1\}, s = s^{(0)}, \dots, s^{(D)}$ have zeros $\{\lambda_{\gamma_{D,1}}, \dots, \lambda_{\gamma_{D,J}}\}$ in common.
Thus $\forall D \in \{0, A-1\}$, $s(z)$ is divisible by $(z - \lambda_{\gamma_{D,1}})^{\alpha_{\gamma_{D,1}}} \cdots (z - \lambda_{\gamma_{D,J}})^{\alpha_{\gamma_{D,J}}}$.
Hence we write $s(z) = \Big((z-\lambda_1)^{\alpha_1} \cdots (z-\lambda_M)^{\alpha_M}\Big) s_0(z)$, while $\deg s \leqslant m - 1 < m = \alpha_1 + \cdots + \alpha_M$.
Thus by TIPS , $s = 0$. Following the same pattern, we conclude that $r = 0$.
Hence T is inje. And $\dim\big(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})\big) = \dim \mathcal{P}_{m+n-1}(\mathbf{C}) \Rightarrow T$ is surj. Thus T is an iso. $\square$

COMMENT: We now prove the statement that marked by (Δ) above.

L1: *Prove that* $\forall p, q \in \mathcal{P}(\mathbf{F}), k \in \mathbf{N}^+, (pq)^{(k)} = C_k^k p^{(k)} q^{(0)} + \cdots + C_k^j p^{(j)} q^{(k-j)} + \cdots + C_k^0 p^{(0)} q^{(k)}$.

SOLUTION:

We use induction by $k \in \mathbf{N}^+$.

(i) $k = 1$. $(pq)^{(1)} = pq = C_1^1 p^{(1)} q^{(0)} + C_1^0 p^{(0)} q^{(1)}$.

(ii) $k \geqslant 2$. Assume that for $(pq)^{(k-1)} = C_{k-1}^{k-1} p^{(k-1)} q^{(0)} + \cdots + C_{k-1}^{j} p^{(j)} q^{(k-1-j)} + \cdots + C_{k-1}^{0} p^{(0)} q^{(k-1)}$.

Now $(pq)^{(k)} = \left((pq)^{(k-1)}\right)' = \left(\sum_{j=0}^{k-1} C_{k-1}^j p^{(j)} q^{(k-j-1)}\right)' = \sum_{j=0}^{k-1} \left[C_{k-1}^j \left(p^{(j+1)} q^{(k-j-1)} + p^{(j)} q^{(k-j)}\right)\right]$.

$$= \left[C_{k-1}^0 \left(p^{(1)} q^{(k-1)} + \boxed{p^{(0)} q^{(k)}}\right)\right] + \left[C_{k-1}^1 \left(p^{(2)} q^{(k-2)} + p^{(1)} q^{(k-1)}\right)\right]$$
$$+ \cdots + \left[C_{k-1}^{j-2} \left(\underline{p^{(j-1)} q^{(k-j+1)}} + p^{(j-2)} q^{(k-j+2)}\right)\right] + \left[C_{k-1}^{j-1} \left(\underline{p^{(j)} q^{(k-j)}} + \underline{p^{(j-1)} q^{(k-j+1)}}\right)\right]$$
$$+ \left[C_{k-1}^{j} \left(p^{(j+1)} q^{(k-j-1)} + \underline{p^{(j)} q^{(k-j)}}\right)\right] + \left[C_{k-1}^{j+1} \left(p^{(j+2)} q^{(k-j-2)} + p^{(j+1)} q^{(k-j-1)}\right)\right]$$
$$+ \cdots + \left[C_{k-1}^{k-2} \left(\underline{p^{(k-1)} q^{(1)}} + p^{(k-2)} q^{(2)}\right)\right] + \left[C_{k-1}^{k-1} \left(\boxed{p^{(k)} q^{(0)}} + \underline{p^{(k-1)} q^{(1)}}\right)\right].$$

Hence $(pq)^{(k)} = C_k^0 p^{(0)} q^{(k)} + \cdots + \left[C_{k-1}^j + C_{k-1}^{j-1}\right] \left(p^{(j)} q^{(k-j)}\right) + \cdots + C_k^k p^{(k)} q^{(0)}$. □

L2: *Suppose* $p(z) = (z - \lambda)^\alpha q(z)$ *and* $\alpha \in \mathbf{N}^+$. *Prove that* $p^{(\alpha-1)}(\lambda) = 0$.

SOLUTION:

Suppose $p \in \mathcal{P}(\mathbf{F})$. Write $p(z) = (z - \lambda)^A q(z)$, where $A \in \mathbf{N}^+, q(\lambda) \neq 0$.

We use induction to show that for all $\alpha \in \{1, \dots, A\}, p^{(\alpha-1)}(\lambda) = 0$.

(i) $\alpha = 1$. $p^{(0)}(\lambda) = 0$.

(ii) $2 \leqslant \alpha \leqslant A$. Assume that $p^{(a-2)}(\lambda) = 0$ for all $a \in \{1, \dots, \alpha\}$.

NOTICE that $p(z) = (z - \lambda)^{\alpha-1} q_{\alpha-1}(z) = (z - \lambda)^\alpha q_\alpha(z)$, where $q_\alpha(z) = (z - \lambda) q_{\alpha-1}(z)$.

Because $p^{(\alpha-1)}(z) = \Big[C_{\alpha-1}^{\alpha-1} (z - \lambda)^0 q_{\alpha-1}(z) + \cdots + C_{\alpha-1}^{k} (z - \lambda)^{\alpha-1-k} q_{\alpha-1-k}(z)$

$+ \cdots + C_{\alpha-1}^{0} (z - \lambda)^{\alpha-1} {q_{\alpha-1}}^{(\alpha-1)}(z)\Big]$. Now $p^{(\alpha-1)}(\lambda) = C_{\alpha-1}^{\alpha-1} q_{\alpha-1}(\lambda) = 0$. □

ENDED

5.A

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 | 2E: Ch5.20 | 4E: 8, 11, 15, 16, 17, 36, 37, 38, 39

- **NOTE FOR [5.6]:**

More generally, suppose we do not know whether V is finite-dim. We show that $(a) \iff (b)$.

Suppose (a) λ is an eigval of T with an eigvec v. Then $(T - \lambda I)v = 0$.

Hence we get (b), $(T - \lambda I)$ is not inje. And then (d), $(T - \lambda I)$ is not inv.

But $(d) \Rightarrow (b)$ fails, because S is not inv $\iff$ S is not inje OR S is not surj.

- **TIPS:** For $T_1, \dots, T_m \in \mathcal{L}(V)$:

(a) Suppose $T_1, \dots, T_m$ are all inje. Then $(T_1 \circ \cdots \circ T_m)$ is inje.

(b) Suppose $(T_1 \circ \cdots \circ T_m)$ is not inje. Then at least one of $T_1, \dots, T_m$ is not inje.

(c) At least one of $T_1, \dots, T_m$ is not inje $\not\Rightarrow$ $(T_1 \circ \cdots \circ T_m)$ is not inje.

EXAMPLE: In infinite-dim only. Let $V = \mathbf{F}^\infty$.

$$\left.\begin{array}{l}\text{Let } S \text{ be the backward shift (surj but not inje)} \\ \text{Let } T \text{ be the forward shift (inje but not surj)}\end{array}\right\} \Rightarrow \text{Then } ST = I.$$ □

• **NOTE FOR [5.2]:** Suppose $T \in \mathcal{L}(V)$. Then U is an invar subsp of V under $T \iff \operatorname{range} T|_U \subseteq U$.

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invar subsp of V under T.*
Prove that there exists an invar subsp W of dimension $\dim V - dim U$.

SOLUTION:

Using the NOTE FOR [3.88,90,91]. Define the eraser S. Now $V = \operatorname{range} S \oplus U$.
Define E_1 by $E_1(u+w) = u$. Define E_2 by $E_2(u+w) = w$. $(\ E_2 = S \circ \pi.\ )$
Note that $T - TE_1 = T(I - E_1) = TE_2$. And $\operatorname{null} TE_2 = \operatorname{null} T \oplus U$, $\operatorname{range} T = \operatorname{range} TE_2 \oplus U$.
Because $\dim \operatorname{null} TE_2 \geqslant \dim U \Rightarrow \dim \operatorname{range} TE_2 \leqslant \dim V - \dim U$.
Let $B_U = (u_1, \dots, u_n)$, $B_{\operatorname{range} TE_2} = (v_1, \dots, v_m) \Rightarrow B_V = (v_1, \dots, v_m, u_1, \dots, u_n, \dots, u_p)$.
Let $X = \operatorname{span}(v_1, \dots, v_m, u_{\alpha_1}, \dots, u_{\alpha_{p-\dim U}})$. Where $\alpha_1, \dots, \alpha_{p-\dim U} \in \{1, \dots, p\}$ are distinct.
Then $\dim X = \dim V - \dim U$. $[\ \operatorname{range} TE_2 \subseteq\ ]$ X is invar under TE_2, by Problem (1)(b).
We have $x \in X \Rightarrow TE_2(x) \in X \Rightarrow Tx - TE_1(x) \in X \Rightarrow Tx \in X$. Hence X is invar under T. $\square$
$($ Note that $E_1(x) \in \operatorname{span}(u_{\beta_1}, \dots, u_{\beta_t})$, where $\beta_1, \dots, \beta_t \in \{\alpha_1, \dots, \alpha_{p-\dim U}\}$ and each $u_{\beta_i} \in U.\ )$
COMMENT: Conversely, by reversing the roles of U and W, we conclude that it is true as well.

• *Suppose $T \in \mathcal{L}(V)$ and U is an invar subsp of V under T.*
Suppose $\lambda_1, \dots, \lambda_m$ are the distinct eigvals of T correspd eigvecs $v_1, \dots, v_m$.

• **TIPS 1:** *Prove that $v_1 + \cdots + v_m \in U \iff$ each $v_k \in U$.*

SOLUTION:

Suppose each $v_k \in U$. Then because U is a subsp, $v_1 + \cdots + v_m \in U$.
Define the statement $P(k)$: if $v_1 + \cdots + v_k \in U$, then each $v_j \in U$. We use induction on m.
(i) For $k = 1$, $v_1 \in U$.
(ii) For $2 \leqslant k \leqslant m$. Assume that $P(k-1)$ holds. Suppose $v = v_1 + \cdots + v_k \in U$.
Then $Tv = \lambda_1 v_1 + \cdots + \lambda_k v_k \in U \Longrightarrow Tv - \lambda_k v = (\lambda_1 - \lambda_k)v_1 + \cdots + (\lambda_{k-1} - \lambda_k)v_{k-1} \in U$.
For each $j \in \{1, \dots, k-1\}, \lambda_j - \lambda_k \neq 0 \Rightarrow (\lambda_j - \lambda_k)v_j = v_j'$ is an eigvec of T correspd λ_j.
By assumption, each $v_j' \in U$. Thus $v_1, \dots, v_{k-1} \in U$. So that $v_k = v - v_1 - \cdots - v_{k-1} \in U$. $\square$

• **TIPS 2:** *If $\dim V = m$. Prove that $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m)$, where $E_k = \operatorname{span}(v_k)$.*

SOLUTION:

Because $V = E_1 \oplus \cdots \oplus E_m$. $\forall u \in U, \exists!\, e_j \in E_j, u = e_1 + \cdots + e_m$.
If $e_j \neq 0$, then e_j is an eigvec correspd λ_j. Otherwise $e_j = 0 \in U$. By (TIPS 1), each nonzero $e_j \in U$.
Thus $u \in (U \cap E_1) + \cdots + (U \cap E_m) = U$. Because each $(U \cap E_j) \subseteq E_j$.
For each $k \in \{2, \dots, n\}$, $\big((U \cap E_1) + \cdots + (U \cap E_{k-1})\big) \bigcap (U \cap E_k) \subseteq (E_1 + \cdots + E_{k-1}) \cap E_k = \{0\}$ $\square$

• **TIPS 3:** *Suppose W is a nonzero invar subsp of V under T. If $\dim V = m \geqslant 1$.*
Prove that $W = \operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ for some distinct $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$.

SOLUTION:

Each $\operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$ is invar under T.
By (TIPS 2), $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m)$. Because each $\dim E_k = 1$, $U \cap E_k = \{0\}$ or E_k.
There must be at least one k such that $E_k = U \cap E_k$, for if not, $U = \{0\}$ since $V = E_1 \oplus \cdots \oplus E_m$.
Let $\alpha_1, \dots, \alpha_A \in \{1, \dots, m\}$ be all the distinct indices for which $E_k = U \cap E_k$.
Thus $U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_m) = E_{\alpha_1} \oplus \cdots E_{\alpha_A} = \operatorname{span}(v_{\alpha_1}, \dots, v_{\alpha_A})$. $\square$

---

1 *Suppose $T \in \mathcal{L}(V)$ and U is a subsp of V.*

(a) *If $U \subseteq \operatorname{null} T$, then U is invar under T.* $\forall u \in U \subseteq \operatorname{null} T, Tu = 0 \in U$. $\square$

(b) *If $\operatorname{range} T \subseteq U$, then U is invar under T.* $\forall u \in U, Tu \in \operatorname{range} T \subseteq U$. $\square$

• *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*

(a) *Prove that* $\operatorname{null}(T - \lambda I)$ *is invar under S for any $\lambda \in \mathbf{F}$.*

(b) *Prove that* $\operatorname{range}(T - \lambda I)$ *is invar under S for any $\lambda \in \mathbf{F}$.*

SOLUTION:

Note that $ST = TS \Rightarrow (T - \lambda I)S = S(T - \lambda I)$.

(a) $(T - \lambda I)(v) = 0 \Rightarrow (T - \lambda I)(Sv) = \big(S(T - \lambda I)\big)(v) = 0$.

(b) $(T - \lambda I)(u) = v \in \operatorname{range}(T - \lambda I) \Rightarrow Sv = \big(S(T - \lambda I)\big)(u) = (T - \lambda I)(Su) \in \operatorname{range}(T - \lambda I)$. $\square$

• *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$.*

2 *Show that $W = \operatorname{null} T$ is invar under S.* $\forall u \in W, Tu = 0 \Rightarrow STu = 0 = TSu \Rightarrow Su \in W$. $\square$

3 *Show that $U = \operatorname{range} T$ is invar under S.* $\forall w \in U, \exists v \in V, Tv = w, TSv = STv = Sw \in U$. $\square$

• *Suppose $T \in \mathcal{L}(V)$ and $V_1, \dots, V_m$ are invar subsps of V under T.*

4 $\forall v_i \in V_i, Tv_i \in V_i \Rightarrow \forall v = v_1 + \dots + v_m \in V_1 + \dots + V_m, Tv = Tv_1 + \dots + Tv_m \in V_1 + \dots + V_m$. $\square$

5 $\forall v \in \bigcap\limits_{i=1}^{m} V_i, Tv \in V_i, \forall i \in \{1, \dots, m\} \Rightarrow Tv \in \bigcap\limits_{i=1}^{m} V_i$. Thus $\bigcap\limits_{i=1}^{m} V_i$ is invar under T. $\square$

---

6 *Suppose U is an invar subsp of V under each $T \in \mathcal{L}(V)$. Show that $U = \{0\}$ or $U = V$.*

SOLUTION: If $V = \{0\}$. Then we are done. Suppose $V \neq \{0\}$. We show the contrapositive:

Suppose $U \neq \{0\}$ and $U \neq V$. Prove that $\exists\, T \in \mathcal{L}(V)$ such that U is not invar under T.

Let W be such that $V = U \oplus W$. Define $T \in \mathcal{L}(V)$ by $T(u + w) = w$. $\square$

• **TIPS:** Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is the counterclockwise rotation by the angle $\theta \in \mathbf{R}$.

Define $\mathcal{C} \in \mathcal{L}(\mathbf{R}^2, \mathbf{C})$ by $\mathcal{C}(a, b) = a + ib = r(\cos\alpha + \mathrm{i}\sin\alpha) \Rightarrow a = r\cos\alpha, b = r\sin\alpha$, where $r = a^2 + b^2$.

Then $(\cos\theta + \mathrm{i}\sin\theta)(a + ib) = r\big(\cos(\alpha + \theta) + \mathrm{i}\sin(\alpha + \theta)\big) = \mathcal{C}^{-1}T(a, b)$.

Hence $T(a, b) = \big(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta\big)$. Now $\mathcal{M}(T) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

EXAMPLE: OR **7** Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find all eigvals of T.

NOTICE that $\mathcal{M}(T) = \begin{pmatrix} \cos 90^\circ & -3\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$. By [5.8](a), we conclude that T has no eigvals.

OR. Suppose λ is an eigval with an eigvec (x, y). Then $(\lambda x, \lambda y) = (-3y, x) \Rightarrow -3y = \lambda^2 y \Rightarrow \lambda^2 = -3$.

$\big[$ Ignoring the possibility of $y = 0$, because $x = 0 \Longleftrightarrow y = 0$. $\big]$ $\square$

---

8 *Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigvals and eigvecs.*

SOLUTION: Suppose λ is an eigval with an eigvec (w, z). Then $z = \lambda w$ and $w = \lambda z$.

Thus $z = \lambda^2 z \Rightarrow \lambda^2 = 1$, ignoring the possibility of $z = 0$ ($z = 0 \Longleftrightarrow w = 0$).

Hence $\lambda_1 = -1$ and $\lambda_2 = 1$ are all the eigvals of T. And $T(z, z) = (z, z), T(z, -z) = (-z, z)$.

又 $\dim \mathbf{F}^2 = 2$. Thus the set of all eigvecs is $\{(z, z), (z, -z) : z \neq 0\}$. $\square$

---

9 *Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigvals and eigvecs.*

SOLUTION: Suppose λ is an eigval with an eigvec (z_1, z_2, z_3).
Then $(2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$. We discuss in two cases:
For $\lambda = 0$, $z_2 = z_3 = 0$ and z_1 can be arbitrary ($z_1 \neq 0$).
For $\lambda \neq 0$, $z_2 = 0 = z_1$, and z_3 can be arbitrary ($z_3 \neq 0$), then $\lambda = 5$.
The set of all eigvecs is $\{(0,0,w),(w,0,0) : w \neq 0\}$. ☐

---

10 *Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$*
(a) *Find all eigvals and eigvecs;* (b) *Find all invar subsps of V under T.*

SOLUTION:
(a) Suppose $x = (x_1, x_2, x_3, \dots, x_n)$ is an eigvec with an eigval λ.
Then $Tx = \lambda v = (x_1, 2x_2, 3x_3, \dots, nx_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$.
Hence $1, \dots, n$ of length $\dim \mathbf{F}^n$ are all the eigvals.
And $\{(0, \dots, 0, x_k, 0, \dots, 0) \in \mathbf{F}^n : x_k \neq 0, k = 1, \dots, n\}$ is the set of all eigvecs.
(b) Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbf{F}^n$. Let $V_k = \operatorname{span}(e_k)$. Then $V_1, \dots, V_n$ are invar under T.
Hence by (TIPS 3), every sum of $V_1, \dots, V_n$ is a invar subsp of V under T. ☐

---

18 *Define the forward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ by $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$.*
Show that T has no eigvals.

SOLUTION: Suppose λ is an eigval of T with an eigvec $(z_1, z_2, \dots)$.
Then $T(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (0, z_1, z_2, \dots)$. Thus $\lambda z_1 = 0, \lambda z_k = z_{k-1}$.
If $\lambda = 0$, then $\lambda z_2 = z_1 = 0 = \dots = z_k \Rightarrow (z_1, z_2, \dots) = 0 \Longrightarrow 0$ is not an eigval.
If $\lambda \neq 0$, then $\lambda z_1 = 0 \Rightarrow z_1 = \dots = z_k = 0 \Longrightarrow \lambda$ is not an eigval. Now no $\lambda \in \mathbf{F}$ is an eigval ☐

---

19 *Suppose $n \in \mathbf{N}^+$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$.*
In other words, the entries of $\mathcal{M}(T)$ with resp to the standard basis are all 1's.
Find all eigvals and eigvecs of T.

SOLUTION:
Suppose λ is an eigval of T with an eigvec $(x_1, \dots, x_n)$.
Then $T(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$.
Thus $\lambda x_1 = \dots = \lambda x_n = x_1 + \dots + x_n$.

$$\left.\begin{array}{r} \text{For } \lambda = 0,\ x_1 + \dots + x_n = 0 \\ \text{For } \lambda \neq 0,\ x_1 = \dots = x_n \Longrightarrow \lambda x_k = nx_k \end{array}\right\} \Rightarrow 0, n \text{ are the eigvals of } T.$$

And the set of all eigvecs of T is $\{(x_1, \dots, x_n) \in \mathbf{F}^n \setminus \{0\} : x_1 + \dots + x_n = 0 \vee x_1 = \dots = x_n\}$. ☐

---

20 *Define the backward shift operator $S \in \mathcal{L}(\mathbf{F}^\infty)$ by $S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$.*
(a) *Show that every element of $\mathbf{F}$ is an eigval of S;* (b) *Find all eigvecs of S.*

SOLUTION:
Suppose λ is an eigval of S with an eigvec $(z_1, z_2, \dots)$.
Then $S(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots) = (z_2, z_3, \dots)$. Thus for each $k \in \mathbf{N}^+, \lambda z_k = z_{k+1}$.
If $\lambda = 0$, then $\lambda z_1 = z_2 = \dots = z_k = 0$ for all k, while z_1 can be nonzero. Thus 0 is an eigval.
If $\lambda \neq 0$, then $\lambda^k z_1 = \lambda^{k-1} z_2 = \dots = \lambda z_k = z_{k+1}$, let $z_1 \neq 0 \Longrightarrow (1, \lambda, \lambda^2, \dots, \lambda^k, \dots)$ is an eigvec.
Now each $\lambda \in \mathbf{F}$ is an eigval of T, with the correspd eigvecs in $\operatorname{span}((1, \lambda, \lambda^2, \dots, \lambda^k, \dots))$. ☐

---

11 *Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigvals and eigvecs.*

SOLUTION:

Note that $\forall p \in \mathcal{P}(\mathbf{R}) \backslash \{0\}, \deg p' < \deg p$. And $\deg 0 = -\infty$. Suppose λ is an eigval with an eigvec p.
Assume that $\lambda \neq 0$. Then $\deg \lambda p > \deg p'$ while $\lambda p = p'$. Contradicts. Thus $\lambda = 0$.
Therefore $\deg \lambda p = -\infty = \deg p' \Rightarrow p \in \mathcal{P}_0(\mathbf{R})$. Hence the eigvecs are all the nonzero consts. $\square$

---

12 *Define $T \in \mathcal{L}\big(\mathcal{P}_n(\mathbf{R})\big)$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigvals and eigvecs.*

SOLUTION:

Suppose λ is an eigval of T with an eigvec p, then $(Tp)(x) = xp'(x) = \lambda p(x)$.
Let $p = a_0 + a_1 x + \cdots + a_n x^n$. Then $xp'(x) = a_1 x + 2a_2 x^2 + \cdots + n a_n x^n = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n$.
Define $S \in \mathcal{L}(\mathbf{F}^{n+1}, \mathcal{P}_n(\mathbf{R}))$ by $S(a_0, a_1, \dots, a_n) = a_0 + a_1 x + \cdots + a_n x^n$.
Then $(S^{-1}TS)(a_0, a_1, \dots, a_n) = (0 \cdot a_0, 1 \cdot a_1, 2 \cdot a_2, \dots, n \cdot a_n)$. Thus $0, 1, \dots, n$ are the eigvals of $S^{-1}TS$.
By Problem (15), $0, 1, \dots, n$ are the eigvals of T. The set of all eigvecs is $\{cx^\lambda : c \neq 0, \lambda = 0, 1, \dots, n\}$. $\square$

• *Suppose V is finite-dim, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$.*

13 *Prove that $\forall \lambda \in \mathbf{F}, \exists\, \alpha \in \mathbf{F}, |\alpha - \lambda| < \frac{1}{1000}$, $(T - \alpha I)$ is inv.*

SOLUTION:

Let $\alpha_k \in \mathbf{F}$ be such that $|\alpha_k - \lambda| = \frac{1}{1000+k}$ for each $k = 1, \dots, \dim V + 1$.
Note that each $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigvals.
Hence $\exists\, k = 1, \dots, \dim V + 1$ such that α_k is not an eigval of T and therefore $(T - \alpha_k I)$ is inv. $\square$

• (4E 5.A.11) *Prove that $\exists\, \delta > 0$ such that $(T - \alpha I)$ is inv for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.*

SOLUTION:

If T has no eigvals, then $(T - \alpha I)$ is inje for all $\alpha \in \mathbf{F}$ and we are done.
Suppose $\lambda_1, \dots, \lambda_m$ are all the distinct eigvals of T.
Let $\delta > 0$ be such that, for each eigval $\lambda_k, \lambda_k \notin (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$.
So that for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$, $(T - \alpha I)$ is not inje. $\square$

OR. Let $\delta = \min\{|\lambda - \lambda_k| : k \in \{1, \dots, m\}, \lambda_k \neq \lambda\}$.
Then $\delta > 0$ and each $\lambda_k \neq \alpha$ $\big[\iff (T - \alpha I)$ is inv $\big]$ for all $\alpha \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$. $\square$

• (5.B.4 OR 4E 3.B.27) *Suppose λ is an eigval of $P \in \mathcal{L}(V), P^2 = P$. Prove that $\lambda = 0$ or $\lambda = 1$.*

SOLUTION: Suppose λ is an eigval with an eigvec v. Then $P(Pv) = Pv \Rightarrow \lambda^2 v = \lambda v$. Thus $\lambda = 1$ or 0. $\square$

---

14 *Suppose $V = U \oplus W$, where U and W are nonzero subsps of V.
Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$.
Find all eigvals and eigvecs of P.*

SOLUTION:

Suppose λ is an eigval of P with an eigvec $(u + w)$.
Then $P(u + w) = u = \lambda u + \lambda w \Rightarrow (\lambda - 1)u + \lambda w = 0$.
OR. Note that $P|_{\operatorname{range} P} = I|_{\operatorname{range} P} \iff P^2 = P$. By (4E 5.A.8), 1 and 0 are the eigvals.
By [1.44], $(\lambda - 1)u = \lambda w = 0$, hence $\lambda = 0 \iff u = 0$, and $\lambda = 1 \iff w = 0$.
Thus $Pu = u, Pw = 0$. Hence the eigvals are 0 and 1, the set of all eigvecs of P is $U \cup W$. $\square$

---

15 *Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is inv.*

(a) *Prove that T and $S^{-1}TS$ have the same eigvals.*

(b) *What is the relationship between the eigvecs of T and the eigvecs of $S^{-1}TS$?*

SOLUTION:

(a) λ is an eigval of T with an eigvec $v \Rightarrow S^{-1}TS(\underline{S^{-1}v}) = S^{-1}Tv = S^{-1}(\lambda v) = \underline{\lambda S^{-1}v}$.

λ is an eigval of $S^{-1}TS$ with an eigvec $v \Rightarrow S(S^{-1}TS)v = T\underline{Sv} = \underline{\lambda Sv}$.

OR. Note that $S(S^{-1}TS)S^{-1} = T$. Hence every eigval of $S^{-1}TS$ is an eigval of $S(S^{-1}TS)S^{-1} = T$.

OR. $Tv = \lambda v \iff (TS)(u) = \lambda Su \iff (S^{-1}TS)(u) = \lambda u$. Where $v = Su$.

$(S^{-1}TS)(u) = \lambda u \iff (S^{-1}T)(v) = \lambda S^{-1}v \iff Tv = \lambda v$. Where $u = S^{-1}v$.

(b) Because λ is an eigval of $T \iff \lambda$ is an eigval of $S^{-1}TS$.

(See [5.36].) Now $E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\}$; $E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}$. ☐

---

17 *Give an example of an operator on $\mathbf{R}^4$ that has no real eigvals.*

SOLUTION:

Let (e_1, e_2, e_3, e_4) be the standard basis of $\mathbf{R}^4$.

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $\mathcal{M}(T, (e_1, e_2, e_3, e_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 8 & 11 & 5 \\ 3 & -8 & -11 & 5 \end{pmatrix}$.

Suppose λ is an eigval of T with an eigvec (x, y, z, w). Then we get $\begin{cases} (1-\lambda)x + y + z + w = 0, \\ -x + (1-\lambda)y - z - w = 0, \\ 3x + 8y + (11-\lambda)z + 5w = 0, \\ 3x - 8y - 11z + (5-\lambda)w = 0. \end{cases}$

This set of linear equations has no solutions.

[You can type it on https://zh.numberempire.com/equationsolver.php to check.]

OR. Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$.

Suppose λ is an eigval of T with an eigvec (x, y, z, w).

Then $T(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w) = (-y, x, -w, z) \implies \begin{cases} -y = \lambda x, x = \lambda y \implies -xy = \lambda^2 xy \\ -w = \lambda z, z = \lambda w \implies -zw = \lambda^2 zw \end{cases}$

If $xy \neq 0$ or $zw \neq 0$, then $\lambda^2 = -1$, we fail.

Otherwise, $xy = 0 \Rightarrow x = y = 0$, for if $x \neq 0$, then $\lambda = 0 \Rightarrow x = 0$, contradicts.

Similarly, $y = z = w = 0$. Then we fail. Thus T has no eigvals. ☐

• (4E 5.A.16) *Suppose $B_V = (v_1, \dots, v_n), T \in \mathcal{L}(V), \mathcal{M}(T, (v_1, \dots, v_n)) = A$. Prove that if λ is an eigval of T, then $|\lambda| \leqslant n \max\{|A_{j,k}| : 1 \leqslant j, k \leqslant n\}$.*

SOLUTION:

Suppose v is an eigval of T correspd to λ. Let $v = c_1 v_1 + \cdots + c_n v_n$.

Because $\lambda c_1 v_1 + \cdots + \lambda c_n v_n = c_1 T v_1 + \cdots + c_n T v_n = \sum_k^n c_k \big(\sum_j^n A_{j,k} v_j \big)$.

We have $\lambda c_j = \sum_{k=1}^n c_k A_{j,k} \implies |\lambda||c_j| = \sum_{k=1}^n |c_k||A_{j,k}|$ for each $j \in \{1, \dots, n\}$

Let $|c_j| = \max\{|c_1|, \dots, |c_n|\}$. Note that $|c_j| \neq 0$, for if not, $c_1 = \cdots = c_n = 0 \Rightarrow v = 0$, contradicts.

Let $M = \max\{|A_{j,k}| : 1 \leqslant j, k \leqslant n\}$. Note that for each j, $\sum_{k=1}^n |A_{j,k}| \leqslant \sum_{k=1}^n M = nM$.

Thus $|\lambda||c_j| = \sum_{k=1}^n |c_k||A_{j,k}| \implies |\lambda| \leqslant \sum_{k=1}^n |A_{j,k}| \dfrac{|c_k|}{|c_j|} \leqslant \sum_{k=1}^n |A_{j,k}| \leqslant nM$. ☐

• (4E 5.A.15) *Suppose $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$.*
Show that λ is an eigval of $T \iff \lambda$ is an eigval of the dual operator $T' \in \mathcal{L}(V')$.

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec v.
Let U be invar such that $V = \operatorname{span}(v) \oplus U$ $\big[$ by (4E 5.A.39) $\big]$.
Define $\psi \in V'$ by $\psi(cv + u) = c$.
Now $\big[T'(\psi)\big](cv+u) = \psi\big(cv + Tu\big) = \lambda cv = \lambda\psi(cv+u)$. Hence $T'(\psi) = \lambda\psi$.

(b) Suppose λ is an eigval T' with an eigvec ψ. Then $T'(\psi) = \psi \circ T = \lambda\psi$.
Note that $\psi \neq 0, \psi(Tv) = \lambda\psi(v)$ Thus $\exists\, v \in V\backslash\{0\}, Tv = \dfrac{\psi(Tv)}{\psi(v)} v = \lambda v.$ ☐

OR. [*Only in Finite-dim*] Using [5.6], (4E 3.F.17), [3.101] and (3.F.12).
λ is an eigval of $T \iff (T - \lambda I_V)$ is not inv
$\iff (T-\lambda I_V)' = T' - \lambda I_{V'}$ is not inv $\iff \lambda$ is an eigval of T'. ☐

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24 *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Tx = Ax$.*
(a) *Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigval of T.*
(b) *Suppose the sum of the entries in each col of A equals 1. Prove that 1 is an eigval of T.*

SOLUTION:

Suppose λ is an eigval of T with an eigvec x. Then $Tx = Ax = \begin{pmatrix} \sum_{k=1}^n A_{1,k}x_k \\ \vdots \\ \sum_{k=1}^n A_{n,k}x_k \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

(a) Suppose $\sum_{r=1}^n A_{R,c} = 1$ for each $R \in \{1, \dots, n\}$.
Then if we let $x_1 = \cdots = x_n$, then $\lambda = 1$, and hence is an eigval of T.

(b) Suppose $\sum_{r=1}^n A_{r,C} = 1$ for each $C \in \{1, \dots, n\}$.
Then $\sum_{r=1}^n (Ax)_{r,\cdot} = \sum_{r=1}^n (Ax)_{r,1} = \sum_{c=1}^n (A_{1,c} + \cdots + A_{n,c})x_c = \sum_{c=1}^n x_c = \lambda(x_1 + \cdots + x_n)$.
Hence $\lambda = 1$ for all $x \in \mathbf{F}^{n,1}$ such that $\sum_{c=1}^n x_{c,1} \neq 0$. ☐

OR. We show that $(T - I)$ is not inv, so that $\lambda = 1$ is an eigval.
Because $(T-I)x = (A-I)x = \begin{pmatrix} \sum_{r=1}^n A_{1,r}x_r - x_1 \\ \vdots \\ \sum_{r=1}^n A_{n,r}x_r - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.
Then $y_1 + \cdots + y_n = \sum_{r=1}^n \sum_{c=1}^n (A_{r,c}x_c - x_r) = \sum_{c=1}^n x_c \sum_{r=1}^n A_{r,c} - \sum_{r=1}^n x_r = 0$.
Thus range $(T-I) \subseteq \Big\{ \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}^t \in \mathbf{F}^{n,1} : y_1 + \cdots + y_n = 0 \Big\}$. Hence $(T - I)$ is not surj. ☐

OR. Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbf{F}^{n,1}$. Define $\psi \in (\mathbf{F}^{n,1})'$ by $\psi(e_k) = 1$.
Thus $\big(\psi \circ (T-I)\big)(e_k) = \psi\Big(\big(\sum_{j=1}^n A_{j,k}e_j \big) - e_k \Big) = \big(\sum_{j=1}^n A_{j,k} \big) - 1 = 0$.
Which means that $\psi \circ (T - I) = 0$. 又 $\psi \neq 0$. Hence $(T-I)$ is not inje. ☐

OR. Define $S \in \mathcal{L}(\mathbf{F}^{n,1})$ by $Sx = A^t x$. Because the rows of A^t are the cols of A.
Now by (a), 1 is an eigval of S. Let $(\varphi_1, \dots, \varphi_n)$ be the dual basis of $(e_1, \dots, e_n)$.
Define $\Phi \in \mathcal{L}\Big(\mathbf{F}^{n,1}, (\mathbf{F}^{1,n})'\Big)$ by $\Phi(e_k) = \varphi_k$. Note that $\mathcal{M}(T') = A^t$.
Now $(\Phi^{-1}T'\Phi)(e_k) = (\Phi^{-1}T')(\varphi_k) = \Phi^{-1}\big(\sum_{j=1}^n A_{k,j}\varphi_j \big) = \sum_{j=1}^n A_{k,j}e_j = A^t e_k = Se_k$.
Thus 1 is an eigval of $S = \Phi^{-1}T'\Phi$, so of T', $\big[$ by Problem (15) $\big]$, so of T, $\big[$ by (4E 5.A.15) $\big]$. ☐

• *Suppose $A \in \mathbf{F}^{n,n}$. Define $T \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Tx = xA$.*

(a) *Suppose the sum of the entries in each col of A equals 1. Prove that 1 is an eigval of T.*

(b) *Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigval of T.*

SOLUTION:

Suppose λ is an eigval with an eigvec x. Then $\begin{pmatrix} \sum_{r=1}^{n} x_r A_{r,1} & \cdots & \sum_{r=1}^{n} x_r A_{r,n} \end{pmatrix} = \lambda \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$.

(a) Suppose $\sum_{r=1}^{n} A_{r,C} = 1$ for each $C \in \{1, \dots, n\}$.

Thus if $x_1 = \cdots = x_n$, then $\lambda = 1$, hence is an eigval of T.

(b) Suppose $\sum_{c=1}^{n} A_{R,c} = 1$ for each $R \in \{1, \dots, n\}$.

Thus $\sum_{c=1}^{n} (xA)_{\cdot,c} = \sum_{c=1}^{n} (A_{c,1} + \cdots + A_{c,n}) x_c = \sum_{c=1}^{n} x_c = \lambda(x_1 + \cdots + x_n)$.

Hence $\lambda = 1$, for all x such that $\sum_{r=1}^{n} x_{1,r} \neq 0$. □

OR. We show that $(T - I)$ is not inv, so that $\lambda = 1$ is an eigval.

Because $(T - I)x = x(A - \mathcal{M}(I)) = \begin{pmatrix} \sum_{c=1}^{n} x_c A_{c,1} - x_1 & \cdots & \sum_{c=1}^{n} x_c A_{c,n} - x_n \end{pmatrix} = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}$.

Then $y_1 + \cdots + y_n = \sum_{c=1}^{n} \sum_{r=1}^{n} (x_r A_{r,c} - x_c) = \sum_{r=1}^{n} x_r \sum_{c=1}^{n} A_{r,c} - \sum_{c=1}^{n} x_c = 0$.

Thus range $(T - I) \subseteq \left\{ \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix} \in \mathbf{F}^{1,n} : y_1 + \cdots + y_n = 0 \right\}$. Hence $(T - I)$ is not surj. □

OR. Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbf{F}^{1,n}$. Define $\psi \in (\mathbf{F}^{n,1})'$ by $\psi(e_k) = 1$.

Because $Te_k = e_k A = \begin{pmatrix} A_{k,1} & \cdots & A_{k,n} \end{pmatrix} = \sum_{i=1}^{n} A_{k,i} e_i$. **COROLLARY:** $\mathcal{M}(T) = A^t$.

$(\psi \circ (T - I))(e_k) = \left(\sum_{i=1}^{n} A_{k,i} \right) - 1 = 0$. Then $\psi \circ (T - I) = 0$. 又 $\psi \neq 0$. $(T - I)$ is not inje. □

OR. Define $S \in \mathcal{L}(\mathbf{F}^{1,n})$ by $Sx = xA^t$. Because the rows of A are the cols of A^t.

Now by (a), 1 is an eigval of S. Let $(\varphi_1, \dots, \varphi_n)$ be the dual basis of $(e_1, \dots, e_n)$.

Define $\Phi \in \mathcal{L}\left(\mathbf{F}^{1,n}, (\mathbf{F}^{1,n})'\right)$ by $\Phi(e_k) = \varphi_k$. Because $[T'(\varphi_k)](e_j) = \varphi_k\left(\sum_{i=1}^{n} A_{j,i} e_i \right) = A_{j,k}$.

By (3.F.9), $T'(\varphi_k) = \sum_{j=1}^{n} A_{j,k} \varphi_j$. **COROLLARY:** $\mathcal{M}(T') = A = \mathcal{M}(T)^t$. FIXME: $\mathcal{M}(T) e_k = A^t e_k = e_k A$

Now $(\Phi^{-1} T' \Phi)(e_k) = (\Phi^{-1} T')(\varphi_k) = \Phi^{-1}\left(\sum_{j=1}^{n} A_{j,k} \varphi_j \right) = \sum_{j=1}^{n} A_{j,k} e_j = e_k A^t = Se_k$.

Thus 1 is an eigval of $S = \Phi^{-1} T' \Phi$, so of T', [by Problem (15)], so of T, [by (4E 5.A.15)]. □

• *Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$.*

(a) [OR (9.11)] *$\lambda \in \mathbf{R}$. Prove that λ is an eigval of $T \iff \lambda$ is an eigval of $T_{\mathbf{C}}$.*

(b) [OR **16** OR [9.16]] *$\lambda \in \mathbf{C}$. Prove that λ is an eigval of $T_{\mathbf{C}} \iff \bar{\lambda}$ is an eigval of $T_{\mathbf{C}}$.*

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec v.

Then $Tv = \lambda v \Longrightarrow T_{\mathbf{C}}(v + \mathrm{i}0) = Tv + \mathrm{i}T0 = \lambda v$. Thus λ is an eigval of $T_{\mathbf{C}}$.

Suppose λ is an eigval of $T_{\mathbf{C}}$ with an eigvec $v + \mathrm{i}u$.

Then $T_{\mathbf{C}}(v + \mathrm{i}u) = \lambda v + \mathrm{i}\lambda u \Longrightarrow Tv = \lambda v, Tu = \lambda u$. Thus λ is an eigval of T.

(Note that $v + \mathrm{i}u$ is nonzero $\iff$ at least one of v, u is nonzero).

(b) Suppose λ is an eigval of $T_{\mathbf{C}}$ with an eigvec $v + \mathrm{i}u$. Then $T_{\mathbf{C}}(v + \mathrm{i}u) = Tv + \mathrm{i}Tu = \lambda(v + \mathrm{i}u)$.

Note that $\overline{T_{\mathbf{C}}(v + \mathrm{i}u)} = \overline{Tv + \mathrm{i}Tu} = Tv - \mathrm{i}Tu = T_{\mathbf{C}}(v - \mathrm{i}u) = T_{\mathbf{C}}(\overline{v + \mathrm{i}u})$.

And that $\overline{\lambda(v + \mathrm{i}u)} = \bar{\lambda}v - \mathrm{i}\bar{\lambda}u = \bar{\lambda}(v - \mathrm{i}u) = \bar{\lambda}(\overline{v + \mathrm{i}u})$.

Hence $\bar{\lambda}$ is an eigval of $T_{\mathbf{C}}$. To prove the other direction, notice that $\bar{\bar{\lambda}} = \lambda$. □

OR. Suppose $\lambda = a + \mathrm{i}b$ is an eigval of $T_{\mathbf{C}}$ with an eigvec $v + \mathrm{i}u$.

Because $T_{\mathbf{C}}(v + \mathrm{i}u) = \lambda(v + \mathrm{i}u) = (av - bu) + \mathrm{i}(au + bv) = Tv + \mathrm{i}Tu \Longrightarrow Tv = av - bu, Tu = au + bv$.

Now $T_{\mathbf{C}}(\overline{v + \mathrm{i}u}) = Tv - \mathrm{i}Tu = (av - bu) - \mathrm{i}(au + bv) = (a - \mathrm{i}b)(v - \mathrm{i}u) = \bar{\lambda}(\overline{v - \mathrm{i}u})$. Similarly □

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21 *Suppose $T \in \mathcal{L}(V)$ is inv.*

(a) *Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that λ is an eigval of $T \iff \lambda^{-1}$ is an eigval of T^{-1}.*

(b) *Prove that T and T^{-1} have the same eigvecs.*

SOLUTION: (a) $Tv = \lambda v \iff v = \lambda T^{-1} v \iff \lambda^{-1} v = T^{-1} v$. Where $v \neq 0$.

(b) NOTICE that T is inv $\Longrightarrow$ 0 is not an eigval of T or T^{-1}. By (a), immediately. $\square$

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22 *Suppose $T \in \mathcal{L}(V)$ and $\exists$ nonzero vecs u, w in V such that $Tu = 3w,\ Tw = 3u$. Prove that 3 or -3 is an eigval of T.*

SOLUTION: $T(u+w) = 3(u+w)$, $T(u-w) = 3(w-u) = -3(u-w)$. Note that $u - w \neq 0$ or $u + w \neq 0$.

OR. $T(Tu) = 9u \Rightarrow T^2 - 9 = (T-3I)(T+3I)$ is not injective $\Rightarrow$ 3 or -3 is an eigval. $\square$

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23 *Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigvals.*

SOLUTION: Suppose λ is an eigval of ST with an eigvec v. Then $T(STv) = \lambda Tv = TS(Tv)$.

If $Tv = 0$ (while $v \neq 0$), then T is not inje $\Rightarrow$ $(TS - 0I)$ and $(ST - 0I)$ are not inje.

Thus $\lambda = 0$ is an eigval of ST and TS with the same eigvec v.

Otherwise, $Tv \neq 0$, then λ is an eigval of TS. Reversing the roles of T and S. $\square$

• (2E 20) *Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigvals and $S \in \mathcal{L}(V)$ has the same eigvecs (but might not with the same eigvals). Prove that $ST = TS$.*

SOLUTION: Let $n = \dim V$. For each $j \in \{1, \dots, n\}$, let v_j be an eigvec with eigval λ_j of T and α_j of S.

Then $B_V = (v_1, \dots, v_n)$. Because $(ST)v_j = \alpha_j \lambda_j v_j = (TS)v_j$ for each j. Hence $ST = TS$. $\square$

• (4E 5.A.37) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by $\mathcal{A}(S) = TS$ for each $S \in \mathcal{L}(V)$. Prove that the set of eigvals of T equals the set of eigvals of $\mathcal{A}$.*

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec $v = v_1$. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.

Note that $\operatorname{span}(v) \subseteq \operatorname{null}(T - \lambda I)$. Define $S \in \mathcal{L}(V)$ by $S(v_j) = v$ for each $j \in \{1, \dots, n\}$.

OR. Define $S \in \mathcal{L}(V)$ by $Sv_1 = v_1,\ Sv_j = 0$ for $j \geqslant 2$. Then $(T - \lambda I)Sv_1 = 0 = (T - \lambda I)Sv_k = 0$.

Then $(T - \lambda I)S = 0$. Thus $\mathcal{A}(S) = TS = \lambda S$ while $S \neq 0$. Hence λ is an eigval of $\mathcal{A}$.

(b) Suppose λ is an eigval of $\mathcal{A}$ with an eigvec S.

Then $\exists\, v \in V, 0 \neq u = S(v) \in V \Rightarrow Tu = (TS)v = (\lambda S)v = \lambda u$. Thus λ is an eigval T.

OR. Because $TS - \lambda S = (T - \lambda I)S = 0 \Rightarrow \{0\} \subsetneq \operatorname{range} S \subseteq \operatorname{null}(T - \lambda I)$. $(T - \lambda I)$ is not inje. $\square$

COMMENT: If $\mathcal{A}(S) = ST, \forall S \in \mathcal{L}(V)$. Then the eigvals of $\mathcal{A}$ are not the eigvals of T.

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25 *Suppose $T \in \mathcal{L}(V)$ and u, w are eigvecs of T such that $u + w$ is also an eigvec of T. Prove that u and w correspd to the same eigval.*

SOLUTION: Suppose $\lambda_1, \lambda_2, \lambda_0$ are eigvals of T with eigvecs to $u, w, u + w$ respectively.

Then $T(u+w) = \lambda_0(u+w) = Tu + Tw = \lambda_1 u + \lambda_2 w \Rightarrow (\lambda_0 - \lambda_1)u = (\lambda_2 - \lambda_0)w$.

If (u, w) is linely depe, then let $w = cu$, therefore $\lambda_2 cu = Tw = cTu = \lambda_1 cu \Rightarrow \lambda_2 = \lambda_1$.

Otherwise, (u, w) is linely inde. Then $\lambda_0 - \lambda_1 = \lambda_2 - \lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_0$. $\square$

OR. Assume that $\lambda_1 \neq \lambda_2$. Then (u, w) is linely inde. Thus $\lambda_0 - \lambda_1 = \lambda_0 - \lambda_2$. Contradicts$\square$

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26 *Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vec in V is an eigvec of T.*
Prove that T is a scalar multi of the identity operator.

SOLUTION: If $\dim V = 0, 1$ then we are done. Suppose $\dim V \geqslant 2$.

Because $\forall v \in V, \exists ! \lambda_v \in \mathbf{F}, Tv = \lambda_v v$. For any two distinct nonzero vecs $v, w \in V$,
$T(v+w) = \lambda_{v+w}(v+w) = Tv + Tw = \lambda_v v + \lambda_w w \Rightarrow (\lambda_{v+w} - \lambda_v)v = (\lambda_w - \lambda_{v+w})w$. □

OR. For any two nonzero vecs $u, v \in V, u, v$ are eigvecs.
If $u + v \neq 0$, then $u + v$ is also an eigvec. Otherwise, $u + v = 0$, then $Tu = -Tv = \lambda u = -\lambda v$.
Thus by Problem (25), $\forall u, v \in V, Tu = \lambda u, Tv = \lambda v \Rightarrow \forall v \in V, Tv = \lambda v$. □

27, 28 *Suppose V is finite-dim and $k \in \{1, \dots, \dim V - 1\}$.*
Suppose $T \in \mathcal{L}(V)$ is such that every subsp of V of dim k is invar under T.
Prove that T is a scalar multi of the identity operator.

SOLUTION: If $\dim V \leqslant 1$ then we are done. Suppose $\dim V \geqslant 2$.

We prove the contrapositive: *If T is not a scalar multi of I. Then $\exists$ subsp U of dim k not invar under T.*
By Problem (26), $\exists\, v \in V$ and $v \neq 0$ such that v is not an eigvec of T.
Thus (v, Tv) is linely inde. Extend to $B_V = (v, Tv, u_1, \dots, u_n)$.
Let $U = \operatorname{span}(v, u_1, \dots, u_{k-1}) \Rightarrow U$ is not an invar subsp of V under T. □

OR. Suppose $0 \neq v = v_1 \in V$. Extend to $B_V = (v_1, \dots, v_n)$. Suppose $Tv_1 = c_1 v_1 + \cdots + c_n v_n, \exists ! c_i \in \mathbf{F}$.
Consider a k-dim subsp $U = \operatorname{span}(v_1, v_{\alpha_1}, \dots, v_{\alpha_{k-1}})$. Where $\alpha_1, \dots, \alpha_{k-1} \in \{2, \dots, n\}$ are distinct.
Because every subsp such U is invar. $Tv_1 = c_1 v_1 + \cdots + c_n v_n \in U \Longrightarrow c_2 = \cdots = c_n = 0$.
For if not, $\exists\, c_i \neq 0$, let $W = \operatorname{span}(v_1, v_{\beta_1}, \dots, v_{\beta_{k-1}})$, where each $\beta_j \in \{2, \dots, i-1, i+1, \dots, n\}$.
Hence $Tv_1 = c_1 v_1$. Because $v_1 = v \in V$ is arbitrary. We conclude that $T = \lambda I$ for some $\lambda \in \mathbf{F}$. □

OR. For each $k \in \{1, \dots, \dim V - 1\}$, define $P(k)$: if every subsp of dim k is invar, then $T = \lambda I$.
(i) If every subsp of dim 1 is invar, then by Problem (26), $T = \lambda I$. Thus $P(1)$ holds.
(ii) Assume that $P(k)$ holds for $k \in \{1, \dots, \dim V - 1\}$. And every subsp of dim $k+1$ is invar.
Let U be a subsp of dim k. If $\dim U = \dim V - 1$ then extend B_U to B_V and we are done.
Suppose $\dim U \in \{1, \dots, \dim V - 2\}$. Choose two linely inde vecs $v, w \notin U$.
Because $U \oplus \operatorname{span}(v)$ and $U \oplus \operatorname{span}(w)$ of dim $k+1$ are invar.
Suppose $u \in U$. Let $Tu = a_1 u_1 + bv = a_2 u_2 + cw, \exists ! u_1, u_2 \in U, a_1, a_2, b, c \in \mathbf{F}$.
Now $a_1 u_1 - a_2 u_2 = cw - bv \in U \cap \operatorname{span}(v) = \{0\} \Rightarrow b = c = 0$. Thus $Tu \in U$.
Because $P(k)$ holds, we conclude that $T = \lambda I$. Thus $P(k+1)$ holds. □

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29 *Suppose $T \in \mathcal{L}(V)$ and* range T *is finite-dim.*
Prove that T has at most $1 + \dim \operatorname{range} T$ distinct eigvals.

SOLUTION:

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigvals of T with correspd eigvecs $v_1, \dots, v_m$.
(Because range T is finite-dim. The correspd eigvals are finite.)
Then $(v_1, \dots, v_m)$ linely inde $\Longrightarrow (\lambda_1 v_1, \dots, \lambda_m v_m)$ linely inde, if each $\lambda_k \neq 0$.
Otherwise, $\exists ! \lambda_k = 0$. Now $(\lambda_1 v_1, \dots, \lambda_{k-1} v_{k-1}, \lambda_{k+1} v_{k+1}, \dots, \lambda_m v_m)$ is linely inde.
Hence, by [2.23], $m - 1 \leqslant \dim \operatorname{range} T$. □

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30 *Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5, \sqrt{7}$ are eigvals. Prove that $\exists\, x, Tx - 9x = (-4, 5, \sqrt{7})$.*

SOLUTION: T has $\dim \mathbf{R}^3$ eigvals not including $9 \Rightarrow (T - 9I)$ is inv. $x = (T - 9I)^{-1}(-4, 5, \sqrt{7})$. □

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31 *Suppose V is finite-dim, and $v_1, \dots, v_m \in V$. Prove that*
$(v_1, \dots, v_m)$ is linely inde $\iff$ $v_1, \dots, v_m$ are eigvecs of some T correspd to distinct eigvals.

SOLUTION: Suppose $(v_1, \dots, v_m)$ is linely inde. Let $B_V = (v_1, \dots, v_m, \dots, v_n)$.
Define $T \in \mathcal{L}(V)$ by $Tv_k = k \cdot v_k$ for each $k \in \{1, \dots, m, \dots, n\}$. Conversely by [5.10]. $\square$

• *Suppose $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ are distinct.*
(a) **32** *Prove that $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linely inde in $\mathbf{R}^{\mathbf{R}}$.*
HINT: *Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$. Define $D \in \mathcal{L}(V)$ by $Df = f'$. Find eigvals and eigvecs of D.*
(b) [4E 36] *Show that $\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$ is linely inde in $\mathbf{R}^{\mathbf{R}}$.*

SOLUTION:
(a) Define V and $D \in \mathcal{L}(V)$ as in HINT. Then because for each k, $D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$.
Thus $\lambda_1, \dots, \lambda_n$ are distinct eigvals of D. By [5.10], $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linely inde in $\mathbf{R}^{\mathbf{R}}$. $\square$
(b) Let $V = \operatorname{span}\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$. Define $D \in \mathcal{L}(V)$ by $Df = f'$.
Then because $D\big(\cos(\lambda_k x)\big) = -\lambda_k \sin(\lambda_k x)$. 又 $D\big(\sin(\lambda_k x)\big) = \lambda_k \cos(\lambda_k x)$.
Thus $D^2\big(\cos(\lambda_k x)\big) = -\lambda_k^2 \cos(\lambda_k x)$.
Notice that $\lambda_1, \dots, \lambda_n$ are distinct $\Longrightarrow -\lambda_1^2, \dots, -\lambda_n^2$ are distinct. And $\dim V = n$.
Hence $-\lambda_1^2, \dots, -\lambda_n^2$ are all the eigvals of D^2 with correspd eigvecs $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$.
And then $\big(\cos(\lambda_1 x), \dots, \cos(\lambda_n x)\big)$ is linely inde in $\mathbf{R}^{\mathbf{R}}$. $\square$

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33 *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{range} T) = 0$.*

SOLUTION: $v + \operatorname{range} T \in V/\operatorname{range} T \Longrightarrow v + \operatorname{range} T \in \operatorname{null}\big(T/(\operatorname{range} T)\big)$. Hence $T/(\operatorname{range} T) = 0$. $\square$

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34 *Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{null} T)$ is inje $\iff$ $(\operatorname{null} T) \bigcap (\operatorname{range} T) = \{0\}$.*

SOLUTION: NOTICE that $\big(T/(\operatorname{null} T)\big)(u + \operatorname{null} T) = Tu + \operatorname{null} T = 0 \iff Tu \in (\operatorname{null} T) \bigcap (\operatorname{range} T)$.
Now $T/(\operatorname{null} T)$ is inje $\iff u + \operatorname{null} T = 0 \iff Tu = 0 \iff (\operatorname{null} T) \bigcap (\operatorname{range} T) = \{0\}$. $\square$

• *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invar subsp of V under T.*
Define $T/U : V/U \to V/U$ by $(T/U)(v + U) = Tv + U$ for each $v \in V$.
(a) *Show that T/U is well-defined and is linear.* *Requires that U is invar under T.*
(b) [OR **35**] *Show that each eigval of T/U is an eigval of T.*

SOLUTION:
(a) $v + U = w + U \iff v - w \in U \Longrightarrow T(v - w) \in U \iff Tv + U = Tw + U$.
Hence T/U is well-defined. Now we show that T/U is linear.
$(T/U)\big((v + U) + \lambda(w + U)\big) = T(v + \lambda w) + U = (T/U)(v + U) + \lambda(T/U)(w)$. Checked.
(b) Suppose λ is an eigval of T/U with an eigvec $v + U$. Then $Tv + U = \lambda v + U \Rightarrow (T - \lambda I)v = u \in U$.
If $u = 0 \Rightarrow Tv = \lambda v$, then we are done. Otherwise, we discuss in two cases.
If $(T - \lambda I)|_U$ is inv. Then $\exists!\, w \in U, (T - \lambda I)(w) = u = (T - \lambda I)v \Rightarrow T(v + w) = \lambda(v + w)$.
Note that $v + w \neq 0$, for if not, $v \in U \Rightarrow v + U = 0$, contradicts. Thus λ is an eigval of T.
If $(T - \lambda I)|_U$ is not inv. Then because V is finite-dim, $(T - \lambda I)|_U$ is not inje,
so that $\exists\, w \in \operatorname{null}(T - \lambda I)|_U, w \neq 0, (T - \lambda I)w = 0 \Rightarrow Tw = \lambda w$. $\square$

OR. Let $B_U = (u_1, \dots, u_m)$. Then $\big((T - \lambda I)v, (T - \lambda I)u_1, \dots, (T - \lambda I)u_m\big)$ is linely inde in U.
So that $a_0(T - \lambda I)v + a_1(T - \lambda I)u_1 + \dots + a_m(T - \lambda I)u_m = 0, \exists\, a_0, a_1, \dots, a_m \in \mathbf{F}$ with some $a_i \neq 0$.
Let $w = a_0 v + a_1 u_1 + \dots + a_m u_m \Longrightarrow Tw = \lambda w$. Note that $w \neq 0$, for if not, $a_0 v \in U$, each $a_i = 0$. $\square$

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36 *Prove or give a counterexample: The result in Exercise 35 is still true if V is infinite-dim.*

SOLUTION: A counterexample:

Consider $V = \{ f \in \mathbf{R}^{\mathbf{R}} : \exists\,!\, m \in \mathbf{N}, f \in \operatorname{span}(1, e^x, \dots, e^{mx}) \}$. Note that V is infinite-dim.

And a subsp $U = \{ f \in \mathbf{R}^{\mathbf{R}} : \exists\,!\, m \in \mathbf{N}^+, f \in \operatorname{span}(e^x, \dots, e^{mx}) \}$.

Define $T \in \mathcal{L}(V)$ by $Tf = e^x f$. Then range $T = U$ is invar under T.

Consider $(T/U)(1+U) = e^x + U = 0 \Longrightarrow 0$ is an eigval of T/U but is not an eigval of T.

$\big[$ null $T = \{0\}$, for if not, $\exists f \in V \backslash \{0\}, (Tf)(x) = e^x f(x) = 0, \forall x \in \mathbf{R} \Rightarrow f = 0$, contradicts. $\big]$ $\square$

- (4E 5.A.39) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
  Prove that T has an eigval $\Longleftrightarrow \exists$ an invar subsp U under T of dimension $\dim V - 1$.

SOLUTION:

(a) Suppose λ is an eigval of T with an eigvec v. (If $\dim V = 1$, then $U = \{0\}$ and we are done.)

Extend $v_1 = v$ to $B_V = (v_1, v_2 \dots, v_n)$.

Step 1. If $\exists\, w_1 \in \operatorname{span}(v_2, \dots, v_n)$ such that $0 \neq Tw_1 \in \operatorname{span}(v_1)$.

Then extend $w_1 = \alpha_{1,2}$ to a basis of $\operatorname{span}(v_2, \dots, v_n)$ as $(\alpha_{1,2}, \dots, \alpha_{1,n})$.

Otherwise, we stop at step 1.

Step 2. If $\exists\, w_2 \in \operatorname{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$ such that $0 \neq Tw_2 \in \operatorname{span}(v_1, w_1)$.

Then extend $w_2 = \alpha_{2,3}$ to a basis of $\operatorname{span}(\alpha_{1,3}, \dots, \alpha_{1,n})$ as $(\alpha_{2,3}, \dots, \alpha_{2,n})$.

Otherwise, we stop at step 2.

Step k. If $\exists\, w_k \in \operatorname{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$ such that $0 \neq Tw_k \in \operatorname{span}(v_1, w_1, \dots, w_{k-1})$,

Then extend $w_k = \alpha_{k,k+1}$ to a basis of $\operatorname{span}(\alpha_{k-1,k+1}, \dots, \alpha_{k-1,n})$ as $(\alpha_{k,k+1}, \dots, \alpha_{k,n})$.

Otherwise, we stop at step k.

Finally, we stop at step m, thus we get $(v_1, w_1, \dots, w_{m-1})$ and $(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})$,

range $T|_{\operatorname{span}(w_1,\dots,w_{m-1})} = \operatorname{span}(v_1, w_1, \dots, w_{m-2}) \Rightarrow \dim \operatorname{null} T|_{\operatorname{span}(w_1,\dots,w_{m-1})} = 0$,

$\underbrace{\operatorname{span}(v_1, w_1, \dots, w_{m-1})}_{\dim m}$ and $\underbrace{\operatorname{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n})}_{\dim (n-m)}$ are invar under T.

Let $U = \operatorname{span}(\alpha_{m-1,m}, \dots, \alpha_{m-1,n}) \oplus \operatorname{span}(v_1, w_1, \dots, w_{m-2})$ and we are done. $\square$

COMMENT: Both $\operatorname{span}(v_2, \dots, v_n)$ and $U \oplus \operatorname{span}(w_{m-1})$ are in $\mathcal{S}_V \operatorname{span}(v_1)$.

If $T|_U$ is inv, then by the similar algorithm, we can extend U to an invar subsp.

OR. Note that $\dim \operatorname{null}(T - \lambda I) \geqslant 1$. And $\dim \operatorname{range}(T - \lambda I) \leqslant \dim V - 1$.

Let $B_{\operatorname{range}(T-\lambda I)} = (w_1, \dots, w_m)$, $B_V = (w_1, \dots, w_m, u_1, \dots, u_n)$.

If $m = \dim V - 1$. $\big[\Longleftrightarrow n = 0. \big]$ Then range $(T - \lambda I)$ is an invar subsp of dim $\dim V - 1$.

Otherwise, choose $k \in \{1, \dots, n\}$ and then let $U = \operatorname{span}(w_1, \dots, w_m, u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n)$.

By Problem (1)(b), U is invar under $(T - \lambda I)$. Now $u \in U \Rightarrow (T - \lambda I)(u) \in U \Rightarrow Tu \in U$.

(b) Suppose U is an invar subsp under T of dim $m = \dim V - 1$. (If $m = 0$, then we are done.)

Let $B_U = (u_1, \dots, u_m), B_V = (u_0, u_1, \dots, u_m)$. We discuss in cases:

(I) If $Tu_0 \in U$, then range $T = U$ so that T is not surj $\Longleftrightarrow$ null $T \neq \{0\} \Longleftrightarrow 0$ is an eigval of T.

(II) If $Tu_0 \notin U$, then $Tu_0 = a_0 u_0 + a_1 u_1 + \cdots + a_m u_m$.

If range $T|_U = U \Longleftrightarrow a_1 = \cdots = a_m = 0 \Longleftrightarrow Tu_0 \in \operatorname{span}(u_0)$ then we are done.

Otherwise, $T|_U : U \to U$ is not surj, so is not inje. Thus 0 is an eigval of $T|_U$, so of T. $\square$

OR. Consider $T/U \in \mathcal{L}(V/U)$. Because $\dim V/U = 1$. $\exists \lambda \in \mathbf{F}, T/U = \lambda I$. By Problem (35). $\square$

ENDED

5.B: I [*See 5.B: II below.*]

COMMENT: 下面，为了照顾原书 5.B 节两版过大的差距，特别将此节补注分成 I 和 II 两部分。
又考虑到第 4 版中 5.B 节的「本征值与极小多项式」与「奇维度实向量空间的本征值」
(相当一部分是从原第 3 版 8.C 节挪过来的) 是对原第 3 版「多项式作用于算子」与
「本征值的存在性」(也即第 3 版 5.B 前半部分) 的极大扩充，这一扩充也大大改变了
原第 3 版后半部分的「上三角矩阵」这一小节，故而将第 4 版 5.B 节放在第 3 版前面。

I 部分除了覆盖第 4 版 5.B 节全部和第 3 版 5.B 节前半部分与之相关的所有习题，
还会覆盖第 4 版 5.A 节末。
II 部分除了覆盖第 3 版 5.B 节后半部分「上三角矩阵」这一小节，还会覆盖第 4 版 5.C 节；
并且，下面 5.C 还会覆盖第 4 版 5.D 节。

「注：[8.40] OR (4E 5.22) —— mini poly;
[8.44,8.45] OR (4E 5.25,5.26) —— how to find the mini poly;
[8.49] OR (4E 5.27) —— eigvals are the zeros of the mini poly;
[8.46] OR (4E 5.29) —— $q(T) = 0 \Leftrightarrow q$ is a poly multi of the mini poly.」

1 2 3 5 6 7 8 10 11 12 13 18 19 | 2E Ch5.24
4E: 5.A.32, 5.A.33; 3, 7, 8, 9, 10, 11, 12, 13, 14, 15,
16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29.

- (4E 5.A.33) *Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.*
  (a) *Prove that T is inje $\Longleftrightarrow T^m$ is inje.*
  (b) *Prove that T is surj $\Longleftrightarrow T^m$ is surj.*

SOLUTION:

(a) Suppose T^m is inje. Then $Tv = 0 \Rightarrow T^{m-1}Tv = T^m v = 0 \Rightarrow v = 0$.
Suppose T is inje. Then $T^m v = T^{m-1}v = \cdots = T^2 v = Tv = v = 0$.

(b) Suppose T^m is surj. $\forall u \in V, \exists v \in V, T^m v = u = Tw$, let $w = T^{m-1}v$.
Suppose T is surj. Then $\forall u \in V, \exists v_1, \dots, v_m \in V, T(v_1) = T^2 v_2 = \cdots = T^m v_m = u$. □

- **NOTE FOR [5.17]:**
  Suppose $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{F})$. Prove that null $p(T)$ *and* range $p(T)$ *are invar under T.*

SOLUTION: Using the commutativity in [5.10].

(a) Suppose $u \in \operatorname{null} p(T)$. Then $p(T)u = 0$.
Thus $p(T)(Tu) = \big(p(T)T\big)u(Tp(T))u = T\big(p(T)u\big) = 0$. Hence $Tu \in \operatorname{null} p(T)$. □

(b) Suppose $u \in \operatorname{range} p(T)$. Then $\exists v \in V$ such that $u = p(T)v$.
Thus $Tu = T\big(p(T)v\big) = p(T)(Tv) \in \operatorname{range} p(T)$. □

- **NOTE FOR [5.21]:** *Every operator on a finite-dim nonzero complex vecsp has an eigval.*

Suppose V is a finite-dim complex vecsp of dim $n > 0$ and $T \in \mathcal{L}(V)$.
Choose a nonzero $v \in V$. $(v, Tv, T^2 v, \dots, T^n v)$ of length $n+1$ is linely depe.
Suppose $a_0 I + a_1 T + \cdots + a_n T^n = 0$. Then $\exists a_j \neq 0$.
Thus $\exists$ nonconst p of smallest degree ($\deg p > 0$) *such that* $p(T)v = 0$.
Because $\exists \lambda \in \mathbf{C}$ such that $p(\lambda) = 0 \Rightarrow \exists q \in \mathcal{P}(\mathbf{C}), p(z) = (z - \lambda)q(z), \forall z \in \mathbf{C}$.
Thus $0 = p(T)v = (T - \lambda I)\big(q(T)v\big)$. By the minimality of $\deg p$ and $\deg q < \deg p$, $q(T)v \neq 0$.
Then $(T - \lambda I)$ is not inje. Thus λ is an eigval of T with eigvec $q(T)v$.

- **EXAMPLE:** *an operator on a complex vecsp with no eigvals*

Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by $(Tp)(z) = zp(z)$.

Suppose $p \in \mathcal{P}(\mathbf{C})$ is a nonzero poly. Then $\deg Tp = \deg p + 1$, and thus $Tp \neq \lambda p$, $\forall \lambda \in \mathbf{C}$.
Hence T has no eigvals.

---

13 *Suppose V is a complex vecsp and $T \in \mathcal{L}(V)$ has no eigvals.*
Prove that every subsp of V invar under T is either $\{0\}$ or infinite-dim.

Solution: Suppose U is a finite-dim nonzero invar subsp on $\mathbf{C}$. Then by [5.21], $T|_U$ has an eigval. □

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16 *Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}\big(\mathcal{P}_{\dim V}(\mathbf{C}), V\big)$ by $S(p) = p(T)v$. Prove [5.21].*

Solution:

Because $\dim \mathcal{P}_{\dim V}(\mathbf{C}) = \dim V + 1$. Then S is not inje. Hence $\exists\, 0 \neq p \in \mathcal{P}_{\dim V}(\mathbf{C}), p(T)v = 0$.
Using [4.14], write $p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$. Apply T to both sides: $p(T) = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$.
Thus at least one of $(T-\lambda_j I)$ is not inje (because $p(T)$ is not inje). □

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17 *Suppose $0 \neq v \in V$. Define $S \in \mathcal{L}\big(\mathcal{P}_{(\dim V)^2}(\mathbf{C}), \mathcal{L}(V)\big)$ by $S(p) = p(T)$. Prove [5.21].*

Solution:

Because $\dim \mathcal{P}_{(\dim V)^2}(\mathbf{C}) = (\dim V)^2 + 1$. Then S is not inje.
Hence $\exists\, p \in \mathcal{P}_{(\dim V)^2}(\mathbf{C}) \backslash \{0\}, 0 = S(p) = p(T) = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$, where $c \neq 0$.
Thus $(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0 \Longrightarrow \exists\, j, (T-\lambda_j)$ is not inje. □

Comment: $\exists$ monic $q \in \operatorname{null} S \neq \{0\}$ of smallest degree, $S(q) = q(T) = 0$, then q is the *mini poly.*

• **Note For [8.40]:** *def for mini poly*

Suppose V is finite-dim and $T \in \mathcal{L}(V)$.
Suppose $M_T^0 = \{p_j\}_{j\in\Gamma}$ is the set of all monic poly that give 0 whenever T is applied.
Prove that $\exists\,! \, p_k \in M_T^0, \deg p_k = \min\{\deg p_j\}_{j\in\Gamma} \leqslant \dim V$.

Solution: Or. Another Proof :

[*Existns Part*] We use induction on $\dim V$.

(i) If $\dim V = 0$, then $I = 0 \in \mathcal{L}(V)$ and let $p = 1$, we are done.

(ii) Suppose $\dim V \geqslant 1$.

Assume that $\dim V > 0$ *and that the desired result is true for all operators on all vecsps of smaller dim.*
Let $u \in V, u \neq 0$. The list $(u, Tu, \dots, T^{\dim V}u)$ of length $(1 + \dim V)$ is linely depe.
Then $\exists\,! \, T^m$ of smallest degree such that $T^m u \in \operatorname{span}(u, Tu, \dots, T^{m-1}u)$.
Thus $\exists\, c_j \in \mathbf{F}, c_0 u + c_1 Tu + \cdots + c_{m-1}T^{m-1}u + T^m u = 0$.
Define q by $q(z) = c_0 + c_1 z + \cdots + c_{m-1}z^{m-1} + z^m$.
Then $0 = T^k\big(q(T)u\big) = q(T)(T^k u), \forall k \in \{1, \dots, m-1\} \subseteq \mathbf{N}$.
Because $(u, Tu, \dots, T^{m-1}u)$ is linely inde.
Thus $\dim \operatorname{null} q(T) \geq m \Rightarrow \dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leqslant \dim V - m$.

Let $W = \operatorname{range} q(T)$.
By *assumption*, $\exists\, s \in M_T^0$ of smallest degree (and $\deg s \leqslant \dim W$,) so that $s(T|_W) = 0$.
Hence $\forall v \in V, \big((sq)(T)\big)(v) = s(T)\big(q(T)v\big) = 0$.
Thus $sq \in M_T^0$ and $\deg sq \leqslant \dim V$.

[*Uniqnes Part*]

Suppose $p, q \in M_T^0$ are of the smallest degree. Then $(p-q)(T) = 0$. 又 $\deg\,(p-q) = m < \min\{\deg p_j\}_{j\in\Gamma}$.
Hence $p - q = 0$, for if not, $\exists\,! \, c \in \mathbf{F}, c(p-q) \in M_T^0$. Contradicts. □

• (4E 5.31, 4E 5.B.25 and 26) *mini poly of restriction operator and mini poly of quotient operator*
Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and U is an invar subsp of V under T.
Let p be the mini poly of T.
(a) *Prove that p is a poly multi of the mini poly of $T|_U$.*
(b) *Prove that p is a poly multi of the mini poly of T/U.*
(c) *Prove that (mini poly of $T|_U$) $\times$ (mini poly of T/U) is a poly multi of p.*
(d) *Prove that the set of eigvals of T equals*
the union of the set of eigvals of $T|_U$ and the set of eigvals of T/U.

SOLUTION:
(a) $p(T) = 0 \Rightarrow \forall u \in U, p(T)u = 0 \Rightarrow p(T|_U) = 0 \Rightarrow$ By [8.46]. □
(b) $p(T) = 0 \Rightarrow \forall v \in V, p(T)v = 0 \Rightarrow p(T/U)(v + U) = p(T)v + U = 0$. □
(c) Suppose r is the mini poly of $T|_U$, s is the mini poly of T/U.
Because $\forall v \in V, s(T/U)(v + U) = s(T)v + U = 0$. So that $\forall v \in V$ but $v \notin U, s(T)v \in U$.
又 $\forall u \in U, r(T|_U)u = r(T)u = 0$.
Thus $\forall v \in V$ but $v \notin U, (rs)(T)v = r(s(T)v) = 0$.
And $\forall u \in U, (rs)(T)u = r(s(T)u) = 0$ (because $s(T)u = s(T|_U)u \in U$).
Hence $\forall v \in V, (rs)(T)v = 0 \Rightarrow (rs)(T) = 0$. □
(d) By [8.49], immediately. □

• (4E 5.B.27) *Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, and $T \in \mathcal{L}(V)$.*
Prove that the mini poly p of T_C equals the mini poly q of T.

SOLUTION:
(a) $\forall u + i0 \in V_C, p(T_C)(u) = p(T)u = 0 \Rightarrow \forall u \in V, p(T)u = 0 \Rightarrow p$ is a poly multi of q.
(b) $q(T) = 0 \Rightarrow \forall u + iv \in V_C, q(T_C)(u + iv) = q(T)u + iq(T)v = 0 \Rightarrow q$ is a poly multi of p. □

• (4E 5.B.28) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that the mini poly p of $T' \in \mathcal{L}(V')$ equals the mini poly q of T.

SOLUTION:
(a) $\forall \varphi \in V', p(T')\varphi = \varphi \circ (p(T)) = 0 \Rightarrow \forall \varphi \in V', p(T) \in \operatorname{null} \varphi \Rightarrow p(T) = 0, p$ is a poly multi of q.
(b) $q(T) = 0 \Rightarrow \forall \varphi \in V', \varphi \circ (q(T)) = q(T')\varphi = 0 \Rightarrow q(T) = 0, q$ is a poly multi of p. □

• (4E 5.32) *Suppose $T \in \mathcal{L}(V)$ and p is the mini poly.*
Prove that T is not inje $\Longleftrightarrow$ the const term of p is 0.

SOLUTION:
T is not inje $\Longleftrightarrow$ 0 is an eigval of $T \Longleftrightarrow$ 0 is a zero of $p \Longleftrightarrow$ the const term of p is 0. □

OR. Because $p(0) = (z - 0)(z - \lambda_1) \cdots (z - \lambda_m) = 0 \Rightarrow T(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0$
又 p is the mini poly $\Rightarrow q$ define by $q(z) = (z - \lambda_1) \cdots (z - \lambda_m)$ is such that $q(T) \neq 0$.
Hence $0 = p(T) = Tq(T) \Rightarrow T$ is not inje.
Conversely, suppose $(T - 0I)$ is not inje, then 0 is a zero of p, so that the const term is 0. □

• (4E 5.B.22)
Suppose V is finite-dim, $T \in \mathcal{L}(V)$. Prove that T is inv $\Longleftrightarrow I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$.

SOLUTION: Denote the mini poly by p, where for all $z \in \mathbf{F}, p(z) = a_0 + a_1 z + \cdots + z^m$.
Notice that V is finite-dim. T is inv $\Longleftrightarrow T$ is inje $\Longleftrightarrow p(0) \neq 0$.

Hence $p(T) = 0 = a_0 I + a_1 T + \cdots + T^m$, where $a_0 \neq 0$ and $m \leqslant \dim V$. $\square$

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6 *Suppose $T \in \mathcal{L}(V)$ and U is a subsp of V invar under T.*
Prove that U is invar under $p(T)$ for every poly $p \in \mathcal{P}(\mathbf{F})$.

SOLUTION:

$\forall u \in U, Tu \in U \Rightarrow Iu, Tu, T(Tu), \dots, T^m u \in U \Longrightarrow \forall a_k \in \mathbf{F}, (a_0 I + a_1 T + \cdots + a_m T^m)u \in U.$ $\square$

• (4E 5.B.10, 23) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$ and p is the mini poly with degree m. Suppose $v \in V$.*
(a) *Prove that* $\operatorname{span}(v, Tv, \dots, T^{m-1}v) = \operatorname{span}(v, Tv, \dots, T^{j-1}v)$ *for some $j \leqslant m$.*
(b) *Prove that* $\operatorname{span}(v, Tv, \dots, T^{m-1}v) = \operatorname{span}(v, Tv, \dots, T^{m-1}v, \dots, T^n v)$.

SOLUTION:

COMMENT: By NOTE FOR[8.40], j has an upper bound $m-1$, m has an upper bound $\dim V$.

Write $p(z) = a_0 + a_1 z + \cdots + z^m$ ($m \leqslant \dim V$). If $v = 0$, then we are done. Suppose $v \neq 0$.

(a) Suppose $j \in \mathbf{N}^+$ is the smallest such that $T^j v \in \operatorname{span}(v, Tv, \dots, T^{j-1}v) = U_0$. Then $j \leqslant m$.
Write $T^j v = c_0 v + c_1 Tv + \cdots + c_{j-1} T^{j-1} v$. And because $T(T^k v) = T^{k+1} \in U_0$. U_0 is invar under T.
By Problem (6), $\forall k \in \mathbf{N},\ T^{j+k} v = T^k (T^j v) \in U_0$.
Thus $U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}v, \dots, T^n v)$ for all $n \geqslant j-1$. Let $n = m-1$ and we are done.

(b) Let $U = \operatorname{span}(v, Tv, \dots, T^{m-1}v)$.
By (a), $U = U_0 = \operatorname{span}(v, Tv, \dots, T^{j-1}, \dots, T^{m-1}, \dots, T^n)$ for all $n \geqslant m-1$. $\square$

• (4E 5.B.21) *Suppose V is finite-dim and $T \in \mathcal{L}(V)$.*
Prove that the mini poly p has degree at most $1 + \dim \operatorname{range} T$.
If $\dim \operatorname{range} T < \dim V - 1$, *then this result gives a better upper bound for the degree of mini poly.*

SOLUTION:

If T is inje, then range $T = V$ and we are done. Now choose $0 \neq v \in \operatorname{null} T$, then $Tv + 0 \cdot v = 0$.
1 is the smallest positive integer such that $T^1 v \in \operatorname{span}(v, \dots, T^0 v)$. Define q by $q(z) = z \Rightarrow q(T)v = 0$.
Let $W = \operatorname{range} q(T) = \operatorname{range} T$. $\exists$ monic $s \in \mathcal{P}(\mathbf{F})$ of smallest degree ($\deg s \leqslant \dim W$), $s(T|_W) = 0$.
Hence sq is the mini poly (see NOTE FOR[8.40]) and $\deg(sq) = \deg s + \deg q \leqslant \dim \operatorname{range} T + 1$. $\square$

---

19 *Suppose V is finite-dim,* $\dim V > 1, T \in \mathcal{L}(V)$. *Prove that* $\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V)$.

SOLUTION: If $\forall S \in \mathcal{L}(V), \exists p \in \mathcal{P}(\mathbf{F}), S = p(T)$. Then by [5.20], $\forall S_1, S_2 \in \mathcal{L}(V), S_1 S_2 = S_2 S_1$.
Note that $\dim \geqslant 2$. By (3.A.14), $\exists S_1, S_2 \in \mathcal{L}(V), S_1 S_2 \neq S_2 S_1$. Contradicts. $\square$

• *Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Let $\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}$.*
Prove that $\dim \mathcal{E}$ equals the degree of the mini poly of T.

SOLUTION:

Because the list $(I, T, \dots, T^{(\dim V)^2})$ of length $\dim \mathcal{L}(V) + 1$ is linely depe in $\dim \mathcal{L}(V)$.
Suppose $m \in \mathbf{N}^+$ is the smallest such that $T^m = a_0 I + \cdots + a_{m-1} T^{m-1}$.
Then q defined by $q(z) = z^m - a_{m-1} z^{m-1} - \cdots - a_0$ is the mini poly (see [8.40]).
For any $k \in \mathbf{N}^+$, $T^{m+k} = T^k (T^m) \in \operatorname{span}(I, T, \dots, T^{m-1}) = U$.
Hence $\operatorname{span}(I, T, \dots, T^{(\dim V)^2}) = \operatorname{span}(I, T, \dots, T^{(\dim V)^2 - 1}) = U$.
Note that by the minimality of m, $(I, T, \dots, T^{m-1})$ is linely inde.

Thus $\dim U = m = \dim \operatorname{span}(I, T, \dots, T^{(\dim V)^2 - 1}) = \dim \operatorname{span}(I, T, \dots, T^n)$ for all $m < n \in \mathbf{N}^+$.

Define $\varphi \in \mathcal{L}(\mathcal{P}_{m-1}(\mathbf{F}), \mathcal{E})$ by $\varphi(p) = p(T)$.

(a) Suppose $p(T) = 0$. 又 $\deg p \leqslant m - 1 \Rightarrow p = 0$. Then φ is inje.

(b) $\forall S = a_0 I + a_1 T + \dots + a_{m-1} T^{m-1} \in \mathcal{E}$, define $p \in \mathcal{P}_{m-1}(\mathbf{F})$ by

$p(z) = a_0 + a_1 z + \dots + a_{m-1} z^{m-1} \Rightarrow \varphi(p) = S$. Then φ is surj.

Hence $\mathcal{E}$ and $\mathcal{P}_{m-1}(\mathbf{F})$ are iso. 又 $\dim \mathcal{P}_{m-1}(\mathbf{F}) = m = \dim U$. □

• (4E 5.B.13) *Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$ is defined by*

$$q(z) = a_0 + a_1 z + \dots + a_n z^n, \text{ where } a_n \neq 0, \text{ for all } z \in \mathbf{F}.$$

Denote the mini poly of T by p defined by

$$p(z) = c_0 + c_1 z + \dots + c_{m-1} z^{m-1} + z^m \text{ for all } z \in \mathbf{F}.$$

Prove that $\exists !\, r \in \mathcal{P}(\mathbf{F})$ such that $q(T) = r(T), \deg r < \deg p$.

Solution:

If $\deg q < \deg p$, then we are done.

If $\deg q = \deg p$, notice that $p(T) = 0 = c_0 I + c_1 T + \dots + c_{m-1} T^{m-1} + T^m$

$$\Rightarrow T^m = -c_0 I - c_1 T - \dots - c_{m-1} T^{m-1},$$

define r by $r(z) = q(z) + \left[-a_m z^m + a_m (-c_0 - c_1 z - \dots - c_{m-1} z^{m-1}) \right]$

$$= (a_0 - a_m c_0) + (a_1 - a_m c_1) z + \dots + (a_{m-1} - a_m c_{m-1}) z^{m-1},$$

hence $r(T) = 0, \deg r < m$ and we are done.

Now suppose $\deg q \geqslant \deg p$. We use induction on $\deg q$.

(i) $\deg q = \deg p$, then the desired result is true, as shown above.

(ii) $\deg q > \deg p$, assume that the desired result is true for $\deg q = n$.

Suppose $f \in \mathcal{P}(\mathbf{F})$ such that $f(z) = b_0 + b_1 z + \dots + b_n z^n + b_{n+1} z^{n+1}$.

Apply the assumption to g defined by $g(z) = b_0 + b_1 z + \dots + b_n z^n$,

getting s defined by $s(z) = d_0 + d_1 z + \dots + d_{m-1} z^{m-1}$.

Thus $g(T) = s(T) \Rightarrow f(T) = g(T) + b_{n+1} T^{n+1} = s(T) + b_{n+1} T^{n+1}$.

Apply the assumption to t defined by $t(z) = z^n$,

getting δ defined by $\delta(z) = c_0{}' + c_1{}' z + \dots + c_{m-1}{}' z^{m-1}$.

Thus $t(T) = T^n = c_0{}' + c_1{}' z + \dots + c_{m-1}{}' z^{m-1} = \delta(T)$.

又 $\operatorname{span}(v, Tv, \dots, T^{m-1} v)$ is invar under T.

Hence $\exists !\, k_j \in \mathbf{F}, T^{n+1} = T(T^n) = k_0 + k_1 z + \dots + k_{m-1} z^{m-1}$.

And $f(T) = s(T) + b_{n+1}(k_0 + k_1 T + \dots + k_{m-1} T^{m-1})$

$\Rightarrow f(T) = (d_0 + k_0) + (d_1 + k_1) z + \dots + (d_{m-1} + k_{m-1}) z^{m-1} = h(T)$, thus defining h. □

• (4E 5.B.14) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$ has mini poly p*

$$\text{defined by } p(z) = a_0 + a_1 z + \dots + a_{m-1} z^{m-1} + z^m, a_0 \neq 0.$$

Find the mini poly of T^{-1}.

Solution:

Notice that V is finite-dim. Then $p(0) = a_0 \neq 0 \Rightarrow 0$ is not a zero of $p \Rightarrow T - 0I = T$ is inv.

Then $p(T) = a_0 I + a_1 T + \dots + T^m = 0$. Apply T^{-m} to both sides,

$a_0 (T^{-1})^m + a_1 (T^{-1})^{m-1} + \dots + a_{m-1} T^{-1} + I = 0$.

Define q by $q(z) = z^m + \dfrac{a_1}{a_0} z^{m-1} + \dots + \dfrac{a_{m-1}}{a_0} z + \dfrac{1}{a_0}$ for all $z \in \mathbf{F}$.

We now show that $(T^{-1})^k \notin \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$

for every $k \in \{1, \dots, m-1\}$ by contradiction, so that q is exactly the mini poly of T^{-1}.

Suppose $(T^{-1})^k \in \operatorname{span}(I, T^{-1}, \dots, (T^{-1})^{k-1})$.

Then let $(T^{-1})^k = b_0 I + b_1 T^{-1} + \cdots + b_{k-1} T^{k-1}$. Apply T^k to both sides,

getting $I = b_0 T^k + b_1 T^{k-1} + \cdots + b_{k-1} T$, hence $T^k \in \operatorname{span}(I, T, \dots, T^{k-1})$.

Thus f defined by $f(z) = z^k + \dfrac{b_1}{b_0} z^{k-1} + \cdots + \dfrac{b_{k-1}}{b_0} z - \dfrac{1}{b_0}$ is a poly multi of p.

While $\deg f < \deg p$. Contradicts. □

- **NOTE FOR [8.49]:**

Suppose V is a finite-dim complex vecsp and $T \in \mathcal{L}(V)$.

By [4.14], the mini poly has the form $(z - \lambda_1) \cdots (z - \lambda_m)$,

*where $\lambda_1, \dots, \lambda_m$ are all the eigvals of T, **possibly with repetitions**.*

- **COMMENT:**

A nonzero poly has at most as many distinct zeros as its degree (see [4.12]).

Thus by the upper bound for the deg of mini poly given in NOTE FOR*[8.40], and by [8.49,]*

we can ***give an alternative proof of*** **[5.13]**.

- NOTICE (See also 4E 5.B.20,24)

Suppose $\alpha_1, \dots, \alpha_n$ are all the distinct eigvals of T,

and therefore are all the distinct zeros of the mini poly.

Also, the mini poly of T is a poly multi of, but not equal to, $(z - \alpha_1) \cdots (z - \alpha_n)$.

If we define q by $q(z) = (z - \alpha_1)^{\dim V - (n-1)} \cdots (z - \alpha_n)^{\dim V - (n-1)}$,

then q is a poly multi of the char poly (see [8.34] and [8.26])

(Because $\dim V > n$ and $n - 1 > 0$, $n[\dim V - (n-1)] > \dim V$.)

The char poly has the form $(z - \alpha_1)^{\gamma_1} \cdots (z - \alpha_n)^{\gamma_n}$, where $\gamma_1 + \cdots + \gamma_n = \dim V$.

The mini poly has the form $(z - \alpha_1)^{\delta_1} \cdots (z - \alpha_n)^{\delta_n}$, where $0 \leqslant \delta_1 + \cdots + \delta_n \leqslant \dim V$.

---

10 *Suppose $T \in \mathcal{L}(V)$, λ is an eigval of T with an eigvec v.*
Prove that for any $p \in \mathcal{P}(\mathbf{F})$, $p(T)v = p(\lambda)v$.

SOLUTION:

Suppose p is defined by $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbf{F}$. Because for any $n \in \mathbf{N}^+$, $T^n v = \lambda^n v$.

Thus $p(T)v = a_0 v + a_1 T v + \cdots + a_m T^m v = a_0 v + a_1 \lambda v + \cdots + a_m \lambda^m v = p(\lambda) v$. □

COMMENT: For any $p \in \mathcal{P}(\mathbf{F})$ such that $p(z) = (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_m)^{\alpha_m}$, the result is true as well.

Now we prove that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v = (\lambda - \lambda_1)^{\alpha_1} \cdots (\lambda - \lambda_m)^{\alpha_m} v$.

Define q_i by $q_i(z) = (z - \lambda_i)^{\alpha_i}$ for all $z \in \mathbf{F}$.

Because $(a+b)^n = a^n + C_n^1 a^{n-1} b + \cdots + C_n^k a^{n-k} b^k + \cdots + C_n^n b^n$.

Let $a = z, b = \lambda_i, n = \alpha_i$, so we can write $q_i(z)$ in the form $a_0 + a_1 z + \cdots + a_m z^m$.

Hence $q_i(T)v = q_i(\lambda)v \Rightarrow (T - \lambda_i I)^{\alpha_i} v = (\lambda - \lambda_i)^{\alpha_i} v$.

Then for each $k \in \{2, \dots, m\}$, $(T - \lambda_{k-1} I)^{\alpha_{k-1}} (T - \lambda_k I)^{\alpha_k} v$

$$
\begin{aligned}
&= q_{k-1}(T)(q_k(T)v) \\
&= q_{k-1}(T)(q_k(\lambda)v) \\
&= q_{k-1}(\lambda)(q_k(\lambda)v) \\
&= (\lambda - \lambda_{k-1})^{\alpha_{k-1}} (\lambda - \lambda_k)^{\alpha_k} v.
\end{aligned}
$$

So that $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_m I)^{\alpha_m} v$

$$
\begin{aligned}
&= q_1(T)\Big(q_2(T)\Big(\dots\big(q_m(T)v\big)\dots\Big)\Big) \\
&= q_1(\lambda)\big(q_2(\lambda)\big(\dots\big(q_m(\lambda)v\big)\dots\big)\big) \\
&= (\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_m)^{\alpha_m}v.
\end{aligned}
$$
$\square$

1 *Suppose $T \in \mathcal{L}(V)$ and $\exists\, n \in \mathbf{N}^+$ such that $T^n = 0$.*
Prove that $(I - T)$ is inv and $(I-T)^{-1} = I + T + \cdots + T^{n-1}$.

Solution: Note that $1 - x^n = (1-x)(1 + x + \cdots + x^{n-1})$.

$$\left.\begin{array}{l}(I-T)(1+T+\cdots+T^{n-1}) = I - T^n = I \\ (1+T+\cdots+T^{n-1})(I-T) = I - T^n = I\end{array}\right\} \Rightarrow (I-T)^{-1} = 1 + T + \cdots + T^{n-1}. \qquad \square$$

---

2 *Suppose $T \in \mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I) = 0$.*
Suppose λ is an eigval of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Solution:

Suppose v is an eigvec correspd to λ. Then for any $p \in \mathcal{P}(\mathbf{F}), p(T)v = p(\lambda)v$.
Hence $0 = (T-2I)(T-3I)(T-4I)v = (\lambda-2)(\lambda-3)(\lambda-4)v$ while $v \neq 0 \Rightarrow \lambda = 2, 3$ or 4. $\square$

Comment: Note that $(T-2I)(T-3I)(T-4I) = 0$ is not inje, so that $2, 3, 4$ are eigvals of T.
But it doesn't mean that all the eigvals of T are exactly $2, 3, 4$.

---

7 [See 5.A.22] *Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigval of $T^2 \Longleftrightarrow 3$ or -3 is an eigval of T.*

Solution:

(a) Suppose λ is an eigval of T with an eigvec v.
Then $(T-3I)(T+3I)v = (\lambda-3)(\lambda+3)v = 0 \Rightarrow \lambda = \pm 3$.

(b) Suppose 3 or -3 is an eigval of T with an eigvec v. Then $Tv = \pm 3v \Rightarrow T^2 v = T(Tv) = 9v$ $\square$

Or. 9 is an eigval of $T^2 \Longleftrightarrow (T^2 - 9I) = (T-3I)(T+3I)$ is not inje $\Longleftrightarrow \pm 3$ is an eigval. $\square$

---

3 *Suppose $T \in \mathcal{L}(V), T^2 = I$ and -1 is not an eigval of T. Prove that $T = I$.*

Solution:

$T^2 - I = (T+I)(T-I)$ is not inje, 又 -1 is not an eigval of $T \Longrightarrow$ By Tips . $\square$

Or. Note that $\forall v \in V, v = [\frac{1}{2}(I-T)v] + [\frac{1}{2}(I+T)v]$.

$$\left.\begin{array}{l}(I+T)\big((I-T)v\big) = 0 \Longrightarrow (I-T)v \in \operatorname{null}(I+T) \\ (I-T)\big((I+T)v\big) = 0 \Longrightarrow (I+T)v \in \operatorname{null}(I-T)\end{array}\right\} \Rightarrow V = \operatorname{null}(I+T) + \operatorname{null}(I-T).$$

又 -1 is not an eigval of $T \Longleftrightarrow (I+T)$ is inje $\Longleftrightarrow \operatorname{null}(I+T) = \{0\}$.
Hence $V = \operatorname{null}(I-T) \Rightarrow \operatorname{range}(I-T) = \{0\}$. Thus $I - T = 0 \in \mathcal{L}(V) \Rightarrow T = I$. $\square$

• (4E 5.A.32) *Suppose $T \in \mathcal{L}(V)$ has no eigvals and $T^4 = I$. Prove that $T^2 = -I$.*

Solution:

Because $T^4 - I = (T^2 - I)(T^2 + I) = 0$ is not inje $\Rightarrow (T^2 - I)$ or $(T^2 + I)$ is not inje.
又 T has no eigvals $\Rightarrow (T^2 - I) = (T-I)(T+I)$ is inje. Hence $T^2 + I = 0 \in \mathcal{L}(V)$, for if not, $\exists\, v \in V, (T^2 + I)v \neq 0$ while $(T^2 - I)\big((T^2+I)v\big) = 0$ but $(T^2 - I)$ is inje. Contradicts.
Or. $\forall v \in V, 0 = (T^2 - I)(T^2 + I)v \Longleftrightarrow 0 = (T^2 + I)v$. Hence $T^2 + I = 0$. $\square$

Or. Note that $\forall v \in V, v = \left[\frac{1}{2}(I - T^2)v\right] + \left[\frac{1}{2}(I + T^2)v\right]$.

$$\left.\begin{array}{l}(I+T^2)\big((I-T^2)v\big) = 0 \Longrightarrow (I-T^2)v \in \operatorname{null}(I+T^2) \\ (I-T^2)\big((I+T^2)v\big) = 0 \Longrightarrow (I+T^2)v \in \operatorname{null}(I-T^2)\end{array}\right\} \Rightarrow V = \operatorname{null}(I+T^2) + \operatorname{null}(I-T^2).$$

又 T has no eigvals $\Longleftrightarrow (I - T^2)$ is inje $\Longleftrightarrow \operatorname{null}(I - T^2) = \{0\}$.
Hence $V = \operatorname{null}(I + T^2) \Rightarrow \operatorname{range}(I + T^2) = \{0\}$. Thus $I + T^2 = 0 \in \mathcal{L}(V) \Rightarrow T^2 = -I$. $\square$

---

8 $\big[$OR (4E 5.A.31)$\big]$ *Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$.*

SOLUTION:

Define $\mathrm{i} \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y) = (-y,x)$. Just like $\mathrm{i} : \mathbf{C} \to \mathbf{C}$ defined by $\mathrm{i}(x+\mathrm{i}y) = -y + \mathrm{i}x$.

Define $\mathrm{i}^n \in \mathcal{L}(\mathbf{R}^2)$ by $\mathrm{i}(x,y) = \Big(\mathrm{Re}\big(\mathrm{i}^n x + \mathrm{i}^{n+1}y\big), \mathrm{Im}\big(\mathrm{i}^n x + \mathrm{i}^{n+1}y\big)\Big)$.

$T^4 + I = (T^2 + \mathrm{i}I)(T^2 - \mathrm{i}I) = (T + \mathrm{i}^{1/2}I)(T - \mathrm{i}^{1/2}I)(T - (-\mathrm{i})^{1/2}I)(T + (-\mathrm{i})^{1/2}I)$.

Note that $\mathrm{i}^{1/2} = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}$, $(-\mathrm{i})^{1/2} = \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}$. Hence $T = \pm(\pm\mathrm{i})^{1/2}I$.

Let $T = \mathrm{i}^{1/2}I$ defined by $\mathrm{i}^{1/2}(x,y) = \Big(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\Big)$. ☐

OR. Because $\mathcal{M}(T^4) = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ -\sin(-\pi) & \cos(-\pi) \end{pmatrix}$. Using $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}$.

We define $T \in \mathcal{L}(\mathbf{R}^2)$ such that $\mathcal{M}(T) = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix}$. ☐

• (4E 5.B.12) *Find the mini poly of T defined in (5.A.10).*

SOLUTION: By (5.A.9) and [8.40, 8.49], $1, 2, \dots, n$ are all the zeros of the mini poly of T. ☐

• (4E 5.B.3) *Find the mini poly of T defined in (5.A.19).*

SOLUTION:

If $n = 1$ then 1 is the only eigval of T, and $(z-1)$ is the mini poly.

Because n and 0 are all the eigvals of T, 又 $\forall k \in \{1, \dots, n\}, Te_k = e_1 + \dots + e_n$; $T^2 e_k = n(e_1 + \dots + e_n)$.

Hence $T^2 e_k = n(Te_k) \Rightarrow T^2 = nT \Rightarrow T^2 - nT = T(T-n) = 0$. Thus $\big(z(z-n)\big)$ is the mini poly. ☐

• (4E 5.B.8) *Find the mini poly of T. Where $T \in \mathcal{L}(\mathbf{R}^2)$ is the operator of counterclockwise rotation by θ, where $\theta \in \mathbf{R}^+$.*

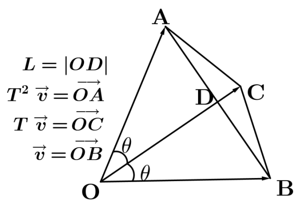
SOLUTION:

If $\theta = \pi + 2k\pi$, then $T(w,z) = (-w,-z), T^2 = I$ and the mini poly is $z+1$.

If $\theta = 2k\pi$, then $T = I$ and the mini poly is $z-1$.

Otherwise (v, Tv) is linely inde. Then $\mathrm{span}(v, Tv) = \mathbf{R}^2$. Note that $\nexists\, b \in \mathbf{F}, T - bI = 0$.

Thus suppose the mini poly p is defined by $p(z) = z^2 + bz + c$ for all $z \in \mathbf{R}$.

Because 

$$Tv = \frac{|\vec{v}|}{2L}(T^2 v + v) \Rightarrow T = \frac{|\vec{v}|}{2L}(T^2 + I)$$

$$L = |\vec{v}|\cos\theta \Rightarrow \frac{|\vec{v}|}{2L} = \frac{1}{2\cos\theta}$$

Hence $p(T) = T^2 - 2\cos\theta\, T + I = 0$ and $z^2 - 2\cos\theta\, z + 1$ is the mini poly of T. ☐

OR. Let (e_1, e_2) be the standard basis of $\mathbf{R}^2$. We use the pattern shown in [8.44].

Because $Te_1 = \cos\theta\, e_1 + \sin\theta\, e_2$, $T^2 e_1 = \cos 2\theta\, e_1 + \sin 2\theta\, e_2$.

Thus $ce_1 + bTe_1 = -T^2 e_1 \iff \begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix}\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$. Now $\det = \sin\theta \neq 0, c = 1, b = 2\cos\theta$. ☐

OR. $\mathcal{M}\big(T, (e_1, e_2)\big) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. By (4E 5.B.11), the mini poly is $(z \pm 1)$ or $(z^2 - 2\cos\theta\, z + 1)$. ☐

• (4E 5.B.11) *Suppose V is a two-dim vecsp, $T \in \mathcal{L}(V)$, and the matrix of T with resp to some basis of V is* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) *Show that $T^2 - (a+d)T + (ad-bc)I = 0$.*

(b) *Show that the mini poly of T equals*

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad-bc) & \text{otherwise.} \end{cases}$$

SOLUTION:

(a) Suppose the basis is (v, w). Because $\begin{cases} Tv = av + bw \Rightarrow (T - aI)v = bw, \text{ then apply } (T - dI) \text{ to both sid} \\ Tw = cv + dw \Rightarrow (T - dI)w = cv, \text{ then apply } (T - aI) \text{ to both sic} \end{cases}$

Hence $(T - aI)(T - dI) = bcI \Rightarrow T^2 - (a+d)T + (ad-bc)I = 0$.

(b) If $b = c = 0$ and $a = d$. Then $\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathcal{M}(I)$. Thus $T = aI$. Hence the mini poly is $z - a$.

Otherwise, by (a), $z^2 - (a+d)z + (ad-bc)$ is a poly multi of the mini poly.

Now we prove that $T \notin \operatorname{span}(I)$, so that then the mini poly of T has exactly degree 2.

(At least one of the assumption of (I),(II) below is true.)

(I) Suppose $a = d$, then $Tv = av + bw \notin \operatorname{span}(v)$, $Tw = cv + aw \notin \operatorname{span}(w)$.

(II) Suppose at most one of b, c is not 0. If $b = 0$, then $Tw \notin \operatorname{span}(w)$; If $c = 0$, then $Tv \notin \operatorname{span}(v)$. □

• *Suppose $S, T \in \mathcal{L}(V)$, S is inv, and $p \in \mathcal{P}(\mathbf{F})$. Prove that $Sp(TS) = p(ST)S$.*

SOLUTION:

We prove $S(TS)^m = (ST)^m S$ for each $m \in \mathbf{N}$ by induction.

(i) If $m = 0, 1$. Then $S(TS)^0 = I = (ST)^0 S$; $S(TS)^1 = (ST)S$.

(ii) If $m > 1$. Assume that $S(TS)^m = (ST)^m S$.

Then $S(TS)^{m+1} = S(TS)^m(TS) = (ST)^m STS = (ST)^{m+1}S$.

Hence $\forall p \in \mathcal{P}(\mathbf{F})$, $Sp(TS) = \sum_{k=1}^m a_k S(TS)^k = \sum_{k=1}^m a_k p(ST)^k S = \left[\sum_{k=1}^m a_k (TS)^k \right] S$. □

COMMENT: $p(TS) = S^{-1}p(ST)S$, $p(ST) = Sp(TS)S^{-1}$.

COROLLARY: **5** Because S is inv, $T \in \mathcal{L}(V)$ is arbitrary $\iff R = ST$ is arbitrary.

Hence $\forall R \in \mathcal{L}(V)$, inv $S \in \mathcal{L}(V)$, $p(S^{-1}RS) = S^{-1}p(R)S$.

• (4E 5.B.7) *Suppose $S, T \in \mathcal{L}(V)$. Let p, q be the mini polys of ST, TS respectively.*

(a) *If $V = \mathbf{F}^2$. Give an example such that $p \neq q$;* (b) *If S or T is inv. Prove that $p = q$.*

SOLUTION:

(a) Define S by $S(x, y) = (x, x)$. Define T by $T(x, y) = (0, y)$.

Then $ST(x, y) = 0$, $TS(x, y) = (0, x)$ for all $(x, y) \in \mathbf{F}^2$. Thus $ST = 0 \neq TS$ and $(TS)^2 = 0$.

Hence the mini poly of ST does not equal to the mini poly of TS.

(b) Suppose S is inv. Because p, q are monic.

$$\left. \begin{array}{l} p(ST) = 0 = Sp(TS)S^{-1} \Rightarrow p(TS) = 0, p \text{ is a poly multi of } q \\ q(TS) = 0 = S^{-1}q(ST)S \Rightarrow q(ST) = 0, q \text{ is a poly multi of } p \end{array} \right\} \Rightarrow p = q.$$

Reversing the roles of S and T, we conclude that if T is inv, then $p = q$ as well. □

---

11 *Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$, and $\alpha \in \mathbf{C}$.*

Prove that α is an eigval of $p(T) \iff \alpha = p(\lambda)$ for some eigval λ of T.

SOLUTION:

(a) Suppose α is an eigval of $p(T) \Leftrightarrow (p(T) - \alpha I)$ is not inje.

Write $p(z) - \alpha = c(z - \lambda_1) \cdots (z - \lambda_m) \Rightarrow p(T) - \alpha I = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$.
By TIPS , $\exists\,(T - \lambda_j I)$ not inje. Thus $p(\lambda_j) - \alpha = 0$.

(b) Suppose $\alpha = p(\lambda)$ and λ is an eigval of T with an eigvec v. Then $p(T)v = p(\lambda)v = \alpha v$. ☐

OR. Define q by $q(z) = p(z) - \alpha$. λ is a zero of q.
Because $q(T)v = \big(p(T) - \alpha I\big)v = q(\lambda)v = \big(p(\lambda) - \alpha\big)v = 0$.
Hence $q(T)$ is not inje $\Rightarrow \big(p(T) - \alpha I\big)$ is not inje. ☐

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12 [OR (4E.5.B.6)] *Give an example of an operator on* $\mathbf{R}^2$ *that shows the result above does not hold if* $\mathbf{C}$ *is replaced with* $\mathbf{R}$.

SOLUTION:

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(w,z) = (-z, w)$.
By Problem (4E 5.B.11), $\mathcal{M}\big(T, ((1,0),(0,1))\big) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ the mini poly of T is $z^2 + 1$.
Define p by $p(z) = z^2$. Then $p(T) = T^2 = -I$. Thus $p(T)$ has eigval -1.
While $\nexists\,\lambda \in \mathbf{R}$ such that $-1 = p(\lambda) = \lambda^2$. ☐

• (4E 5.B.17) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, $\lambda \in \mathbf{F}$, and p is the mini poly of T. Show that the mini poly of $(T - \lambda I)$ is the poly q defined by $q(z) = p(z + \lambda)$.*

SOLUTION:

$q(T - \lambda I) = 0 \Rightarrow q$ is poly multi of the mini poly of $(T - \lambda I)$.
Suppose the degree of the mini poly of $(T - \lambda I)$ is n, and the degree of the mini poly of T is m.
By definition of mini poly,
n is the smallest such that $(T - \lambda I)^n \in \operatorname{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{n-1})$;
m is the smallest such that $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.
又 $T^k \in \operatorname{span}(I, T, \dots, T^{k-1}) \iff (T - \lambda)^k \in \operatorname{span}(I, (T - \lambda I), \dots, (T - \lambda I)^{k-1})$.
Thus $n = m$. 又 q is monic. By the uniqnes of mini poly. ☐

• (4E 5.B.18) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, $\lambda \in \mathbf{F}\backslash\{0\}$, and p is the mini poly of T. Show that the mini poly of λT is the poly q defined by $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.*

SOLUTION:

$q(\lambda T) = \lambda^{\deg p} p(T) = 0 \Rightarrow q$ is a poly multi of the mini poly of λT.
Suppose the degree of the mini poly of λT is n, and the degree of the mini poly of T is m.
By definition of mini poly,
n is the smallest such that $(\lambda T)^n \in \operatorname{span}(\lambda I, \lambda T, \dots, (\lambda T)^{n-1})$;
m is the smallest such that $T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$.
又 $(\lambda T)^k \in \operatorname{span}(\lambda I, \lambda T, \dots, (\lambda T)^{k-1}) \iff T^k \in \operatorname{span}(I, T \dots, T^{k-1})$.
Thus $n = m$. 又 q is monic. By the uniqnes of mini poly. ☐

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18 [OR (4E 5.B.15)] *Suppose V is a finite-dim complex vecsp with* $\dim V > 0$ *and* $T \in \mathcal{L}(V)$. *Define $f : \mathbf{C} \to \mathbf{R}$ by $f(\lambda) = \dim \operatorname{range}(T - \lambda I)$. Prove that f is not a continuous function.*

SOLUTION: Note that V is finite-dim.

Let λ_0 be an eigval of T. Then $(T - \lambda_0 I)$ is not surj. Hence $\dim \operatorname{range}(T - \lambda_0 I) < \dim V$.
Because T has finitely many eigvals. There exist a sequence of number $\{\lambda_n\}$ such that $\lim\limits_{n\to\infty} \lambda_n = \lambda_0$.

And λ_n is not an eigval of T for each $n \Rightarrow \dim \operatorname{range}(T - \lambda_n I) = \dim V \neq \dim \operatorname{range}(T - \lambda_0 I)$.
Thus $f(\lambda_0) \neq \lim_{n\to\infty} f(\lambda_n)$. $\square$

• (4E 5.B.9) *Suppose $T \in \mathcal{L}(V)$ is such that with resp to some basis of V, all entries of the matrix of T are rational numbers. Explain why all coefficients of the mini poly of T are rational numbers.*

Solution:

Let $(v_1, \dots, v_n)$ denote the basis such that $\mathcal{M}(T, (v_1, \dots, v_n))_{j,k} = A_{j,k} \in \mathbf{Q}$ for all $j, k = 1, \dots, n$.
Denote $\mathcal{M}(v_j, (v_1, \dots, v_n))$ by x_j for each v_j.
Suppose p is the mini poly of T and $p(z) = z^m + \cdots + c_1 z + c_0$. Now we show that each $c_j \in \mathbf{Q}$.
Note that $\forall s \in \mathbf{N}^+, \mathcal{M}(T^s) = \mathcal{M}(T)^s = A^s \in \mathbf{Q}^{n,n}$ and $T^s v_k = A^s_{1,k} v_1 + \cdots + A^s_{n,k} v_n$ for all $k \in \{1, \dots, n\}$.

$$\text{Thus} \begin{cases} \mathcal{M}(p(T)v_1) = (A^m + \cdots + c_1 A + c_0 I)x_1 = \sum_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,1} x_j = 0; \\ \qquad \vdots \\ \mathcal{M}(p(T)v_n) = (A^m + \cdots + c_1 A + c_0 I)x_n = \sum_{j=1}^{n} (A^m + \cdots + c_1 A + c_0 I)_{j,n} x_j = 0; \end{cases}$$

$$\text{More clearly,} \begin{cases} (A^m + \cdots + c_1 A + c_0 I)_{1,1} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,1} = 0; \\ \qquad \vdots \quad \ddots \quad \vdots \\ (A^m + \cdots + c_1 A + c_0 I)_{1,n} = \cdots = (A^m + \cdots + c_1 A + c_0 I)_{n,n} = 0; \end{cases}$$

Hence we get a system of n^2 linear equations in m unknowns $c_0, c_1, \dots, c_{m-1}$.
We conclude that $c_0, c_1, \dots, c_{m-1} \in \mathbf{Q}$. $\square$

• [Or (4E 5.B.16), Or (8.C.18)] *Suppose $a_0, \dots, a_{n-1} \in \mathbf{F}$. Let T be the operator on $\mathbf{F}^n$ such that*

$$\mathcal{M}(T) = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ 0 & & & 1 & -a_{n-1} \end{pmatrix}, \textit{ with resp to the standard basis } (e_1, \dots, e_n).$$

Show that the mini poly of T is p defined by $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$.

*$\mathcal{M}(T)$ is called the **companion matrix** of the poly above. This exercise shows that every monic poly is the mini poly of some operator. Hence a formula or an algorithm that could produce exact eigvals for each operator on each $\mathbf{F}^n$ could then produce exact zeros for each poly [by 8.36(b)]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigvals of an operator.*

Solution: Note that $(e_1, Te_1, \dots, T^{n-1}e_1)$ is linely inde. 又 The deg of mini poly is at most n.

$T^n e_1 = \cdots = T^{n-k} e_{1+k} = \cdots = Te_n = -a_0 e_1 - a_1 e_2 - a_2 e_3 - \cdots - a_{n-1} e_n$
$= (-a_0 I - a_1 T - a_2 T^2 - \cdots - a_{n-1} T^{n-1}) e_1$. Thus $p(T)e_1 = 0 = p(T)e_j$ for each $e_j = T^{j-1} e_1$. $\square$

• Eigenvalues On Odd-Dimensional Real Vector Spaces

• Even-Dimensional Null Space

Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dim, $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$. Prove that $\dim \operatorname{null}(T^2 + bT + cI)$ is an even number.

Solution:

Denote $\operatorname{null}(T^2 + bT + cI)$ by R. Then $T|_R + bT|_R + cI_R = (T + bT + cI)|_R = 0 \in \mathcal{L}(R)$.
Suppose λ is an eigval of T_R with an eigvec $v \in R$.

Then $0 = (T|_R^2 + bT|_R + cI_R)(v) = (\lambda^2 + \lambda b + c)v = \big((\lambda + b)^2 + c - \frac{b^2}{4}\big)v$.

Because $c - \frac{b^2}{4} > 0$ and we have $v = 0$. Thus T_R has no eigvals.

Let U be an invar subsp of R that has the largest, even dim among all invar subsps.

Assume that $U \neq R$. Then $\exists\, w \in R$ but $w \notin U$. Let W be such that $(w, T|_R w)$ is a basis of W.

Because $T|_R^2 w = -bT|_R w - cw \in W$. Hence W is an invar subsp of dim 2.

Thus $\dim(U + W) = \dim U + 2 - \dim(U \cap W)$, where $U \cap W = \{0\}$,

for if not, because $w \notin U, T|_R w \in U$,

$U \cap W$ is invar under $T|_R$ of one dim (impossible because $T|_R$ has no eigvecs).

Hence $U + W$ is even-dim invar subsp under $T|_R$, contradicting the maximality of $\dim U$.

Thus the assumption was incorrect. Hence $R = \operatorname{null}(T^2 + bT + cI) = U$ has even dim. □

• Operators On Odd-Dimensional Vector Spaces Have Eigenvalues

(a) *Suppose* $\mathbf{F} = \mathbf{C}$. Then by [5.21], we are done.

(b) *Suppose* $\mathbf{F} = \mathbf{R}$, *V is finite-dim, and* $\dim V = n$ *is an odd number.*
Let $T \in \mathcal{L}(V)$ *and the mini poly is* p. *Prove that T has an eigval.*

Solution:

(i) If $n = 1$, then we are done.

(ii) Suppose $n \geqslant 3$. Assume that every operator, on odd-dim vecsps of dim less than n, has an eigval.

If p is a poly multi of $(x - \lambda)$ for some $\lambda \in \mathbf{R}$, then by [8.49] λ is an eigval of T and we are done.

Now suppose $b, c \in \mathbf{R}$ such that $b^2 < 4c$ and p is a poly multi of $x^2 + bx + c$ (see [4.17]).

Then $\exists\, q \in \mathcal{P}(\mathbf{R})$ such that $p(x) = q(x)(x^2 + bx + c)$ for all $x \in \mathbf{R}$.

Now $0 = p(T) = \big(q(T)\big)(T^2 + bT + cI)$, which means that $q(T)|_{\operatorname{range}(T^2+bT+cI)} = 0$.

Because $\deg q < \deg p$ and p is the mini poly of T, hence $\operatorname{range}(T^2 + bT + cI) \neq V$.

又 $\dim V$ is odd and $\dim \operatorname{null}(T^2 + bT + cI)$ is even (by our previous result).

Thus $\dim V - \dim \operatorname{null}(T^2 + bT + cI) = \dim \operatorname{range}(T^2 + bT + cI)$ is odd.

By [5.18], $\operatorname{range}(T^2 + bT + cI)$ is an invar subsp of V under T that has odd dim less than n.

Our induction hypothesis now implies that $T|_{\operatorname{range}(T^2+bT+cI)}$ has an eigval.

By mathematical induction. □

• (2E Ch5.24) *Suppose* $\mathbf{F} = \mathbf{R}, T \in \mathcal{L}(V)$ *has no eigvals.*
Prove that every invar subsp of V under T is even-dim.

Solution:

Suppose U is such a subsp. Then $T|_U \in \mathcal{L}(U)$. We prove by contradiction.

If $\dim U$ is odd, then $T|_U$ has an eigval and so is T, so that $\exists$ invar subsp of 1 dim, contradicts. □

• (4E 5.B.29) *Show that every operator on a finite-dim vecsp of dim* ≥ 2 *has a 2-dim invar subsp.*

Solution:

Using induction on $\dim V$.

(i) $\dim V = 2$, we are done.

(ii) $\dim V > 2$. Assume that the desired result is true for vecsp of smaller dim.

Suppose p is the mini poly of degree m and $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.

If $T = \lambda I$ ($\Leftrightarrow m = 1 \vee m = -\infty$), then we are done. ($m \neq 0$ because $\dim V \neq 0$.)

Now define a q by $q(z) = (z - \lambda_1)(z - \lambda_2)$.

By assumption, $T|_{\operatorname{null} q(T)}$ has an invar subsp of dim 2. □

ENDED

5.B: II

9 14 15 20 | 4E: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 11, 12, 13, 14

- (4E 5.C.1) *Prove or give a counterexample:*
  If $T \in \mathcal{L}(V)$ and T^2 has an upper-trig matrix, then T has an upper-trig matrix.

SOLUTION:

- (4E 5.C.2) *Suppose A and B are upper-trig matrices of the same size,*
  with $\alpha_1, \dots, \alpha_n$ on the diag of A and $\beta_1, \dots, \beta_n$ on the diag of B.
  (a) *Show that $A + B$ is an upper-trig matrix with $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ on the diag.*
  (b) *Show that AB is an upper-trig matrix with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diag.*

SOLUTION:

- (4E 5.C.3)
  Suppose $T \in \mathcal{L}(V)$ is inv and $B = (v_1, \dots, v_n)$ is a basis of V such that
  $\mathcal{M}(T, B) = A$ is upper trig, with $\lambda_1, \dots, \lambda_n$ on the diag.
  Show that the matrix of $\mathcal{M}(T^{-1}, B) = A^{-1}$ is also upper trig, with $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ on the diag.

SOLUTION:

---

9 [4E 5.C.7] *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $v \in V$.*
(a) *Prove that $\exists\,!$ monic poly p_v of smallest degree such that $p_v(T)v = 0$.*
(b) *Prove that the mini poly of T is a poly multi of p_v.*

SOLUTION:

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14 [OR (4E 5.C.4)] *Give an operator T such that with resp to some basis,*
$\mathcal{M}(T)_{k,k} = 0$ for each k, while T is inv.

SOLUTION:

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15 [OR (4E 5.C.5)] *Give an operator T such that with resp to some basis,*
$\mathcal{M}(T)_{k,k} \neq 0$ for each k, while T is not inv.

SOLUTION:

---

20 [OR (OR 4E 5.C.6)]
Suppose $\mathbf{F} = \mathbf{C}$, V is finite-dim, and $T \in \mathcal{L}(V)$.
Prove that if $k \in \{1, \dots, \dim V\}$, then V has a k dim subsp invar under T.

SOLUTION:

- (4E 5.C.8) *Suppose V is finite-dim, $T \in \mathcal{L}(V)$, and $\exists\, v \in V \backslash \{0\}$ such that $T^2 v + 2Tv = -2v$.*
  (a) *Prove that if $\mathbf{F} = \mathbf{R}$, then $\nexists$ a basis of V with resp to which T has an upper-trig matrix.*
  (b) *Prove that if $\mathbf{F} = \mathbf{C}$ and A is an upper-trig matrix that equals the matrix of T*
  with resp to some basis of V, then $-1 + \mathrm{i}$ or $-1 - \mathrm{i}$ appears on the diag of A.

SOLUTION:

- (4E 5.C.9) *Suppose* $B \in \mathbf{F}^{n,n}$ *with complex entries.*
  Prove that $\exists$ *inv* $A \in \mathbf{F}^{n,n}$ *with complex entries such that* $A^{-1}BA$ *is an upper-trig matrix.*

Solution:

- (4E 5.C.10) *Suppose* $T \in \mathcal{L}(V)$ *and* $(v_1, \dots, v_n)$ *is a basis of* V.
  Show that the following are equi.
  (a) *The matrix of* T *with resp to* $(v_1, \dots, v_n)$ *is lower trig.*
  (b) $\operatorname{span}(v_k, \dots, v_n)$ *is invar under* T *for each* $k = 1, \dots, n$.
  (c) $Tv_k \in \operatorname{span}(v_k, \dots, v_n)$ *for each* $k = 1, \dots, n$.

Solution:

- (4E 5.C.11) *Suppose* $\mathbf{F} = \mathbf{C}$ *and* V *is finite-dim.*
  Prove that if $T \in \mathcal{L}(V)$, *then* T *has a lower-trig matrix with resp to some basis.*

Solution:

- (4E 5.C.12)
  Suppose V *is finite-dim,* $T \in \mathcal{L}(V)$ *has an upper-trig matrix with resp to some basis, and* U *is a subsp of* V *that is invar under* T.
  (a) *Prove that* $T|_U$ *has an upper-trig matrix with resp to some basis of* U.
  (b) *Prove that* T/U *has an upper-trig matrix with resp to some basis of* V/U.

Solution:

- (4E 5.C.13) *Suppose* V *is finite-dim,* $T \in \mathcal{L}(V)$. *Suppose* U *is an invar subsp of* V *under* T *such that* $T|_U$ *has an upper-trig matrix and also* T/U *has an upper-trig matrix.*
  Prove that T *has an upper-trig matrix.*

Solution:

- (4E 5.C.14) *Suppose* V *is finite-dim and* $T \in \mathcal{L}(V)$.
  Prove that T *has an upper-trig matrix* $\iff$ T' *has an upper-trig matrix.*

Solution:

Ended

5.C

xxxx

Ended

5.E* (4E) 1 2 3 4 5 6 7 8 9 10

1 *Give an example of two commuting operators* $S, T \in \mathbf{F}^4$ *such that there is an invar subsp of* $\mathbf{F}^4$ *under* S *but not under* T *and an invar subsp of* $\mathbf{F}^4$ *under* T *but not under* S.

Solution:

2 *Suppose $\mathcal{E}$ is a subset of $\mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagable.*
Prove that $\exists$ a basis of V with resp to which
every element of $\mathcal{E}$ has a diag matrix $\Longleftrightarrow$ every pair of elements of $\mathcal{E}$ commutes.

This exercise extends [5.76], which considers the case in which $\mathcal{E}$ contains only two elements.

For this exercise, $\mathcal{E}$ may contain any number of elements, and $\mathcal{E}$ may even be an infinite set.

SOLUTION:

---

3 *Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.*
(a) *Prove that* $\operatorname{null} p(S)$ *is invar under T.*
(b) *Prove that* $\operatorname{range} p(S)$ *is invar under T.*
See NOTE FOR[*5.17*] *for the special case $S = T$.*

SOLUTION:

---

4 *Prove or give a counterexample:*
A diag matrix A and an upper-trig matrix B of the same size commute.

SOLUTION:

---

5 *Prove that a pair of operators on a finite-dim vecsp commute $\Longleftrightarrow$ their dual operators commute.*

SOLUTION:

---

6 *Suppose V is a finite-dim complex vecsp and $S, T \in \mathcal{L}(V)$ commute.*
Prove that $\exists\, \alpha, \lambda \in \mathbf{C}$ such that $\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V$.

SOLUTION:

---

7 *Suppose V is a complex vecsp, $S \in \mathcal{L}(V)$ is diagable, and T commutes with S.*
Prove that $\exists$ basis B of V such that S has a diag matrix with resp to B
and T has an upper-trig matrix with resp to B.

SOLUTION:

---

8 *Suppose $m = 3$ in Example [5.72]*
and D_x, D_y are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbf{R}^2)$ from that example.
Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ with resp to which D_x and D_y each have an upper-trig matrix.

SOLUTION:

---

9 *Suppose V is a finite-dim nonzero complex vecsp.*
Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.
(a) *Prove that $\exists\, v \in V$ is an eigvec for every element of $\mathcal{E}$.*
(b) *Prove that $\exists$ a basis of V with resp to which every element of $\mathcal{E}$ has an upper-trig matrix.*

SOLUTION:

---

10 *Give an example of two commuting operators S, T on a finite-dim real vecsp such that*
$S + T$ has a eigval that does not equal an eigval of S plus an eigval of T
and ST has a eigval that does not equal an eigval of S times an eigval of T.

SOLUTION:

Ended